Springer Proceedings in Mathematics & Statistics

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Lie Theory and Its Applications in Physics

Varna, Bulgaria, June 2015

Springer

Springer Proceedings in Mathematics & Statistics

Volume 191

Springer Proceedings in Mathematics & Statistics

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Vladimir Dobrev
Editor

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Varna, Bulgaria, June 2015

Springer

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ISSN 2194-1009 ISSN 2194-1017 (electronic)
Springer Proceedings in Mathematics & Statistics
ISBN 978-981-10-2635-5 ISBN 978-981-10-2636-2 (eBook)
DOI 10.1007/978-981-10-2636-2

Library of Congress Control Number: 2016955072

Mathematics Subject Classification (2010): 17Bxx, 22Exx, 120Gxx, 17A70, 18E25, 14Qxx, 37Kxx, 81Txx, 81Rxx, 83Cxx, 83Fxx, 60Bxx, 46Fxx, 37N20, 81Q60

Printed on acid-free paper

This Springer imprint is published by Springer Nature
The registered company is Springer Nature Singapore Pte Ltd.
The registered company address is: 152 Beach Road, #22-06/08 Gateway East, Singapore 189721, Singapore

Preface

The workshop series 'Lie Theory and Its Applications in Physics' is designed to serve the community of theoretical physicists, mathematical physicists and mathematicians working on mathematical models for physical systems based on geometrical methods and in the field of Lie theory.

The series reflects the trend towards a geometrisation of the mathematical description of physical systems and objects. A geometric approach to a system yields in general some notion of symmetry which is very helpful in understanding its structure. Geometrisation and symmetries are meant in their widest sense, i.e., representation theory, algebraic geometry, number theory infinite-dimensional Lie algebras and groups, superalgebras and supergroups, groups and quantum groups, noncommutative geometry, symmetries of linear and nonlinear PDE, special functions, functional analysis. This is a big interdisciplinary and interrelated field.

The first three workshops were organized in Clausthal (1995, 1997, 1999), the 4th was part of the 2nd Symposium 'Quantum Theory and Symmetries' in Cracow (2001), the 5th, 7–10th were organized in Varna (2003, 2007, 2009, 2011, 2013), the 6th was part of the 4th Symposium 'Quantum Theory and Symmetries' in Varna (2005), but has its own volume of proceedings.

The 11th Workshop of the series (LT-11) was organized by the Institute of Nuclear Research and Nuclear Energy of the Bulgarian Academy of Sciences (BAS) in June 2015 (15–21), at the Guest House of BAS near Varna on the Bulgarian Black Sea Coast.

The overall number of participants was 76 and they came from 21 countries.

The scientific level was very high as can be judged by the speakers. The *plenary speakers* were: Luigi Accardi (Rome), Loriano Bonora (Trieste), Branko Dragovich (Belgrade), Malte Henkel (Nancy), Stefan Hollands (Leipzig), Evgeny Ivanov (Dubna), Toshiyuki Kobayashi (Tokyo), Zohar Komargodski (Weizmann), Ivan Penkov (Bremen), Birgit Speh (Cornell U.), Ivan Todorov (Sofia), Joris Van Der Jeugt (Ghent), Joseph A. Wolf (Berkeley), Milen Yakimov (Louisiana SU), George Zoupanos (Athens).

The topics covered the most modern trends in the field of the workshop: Symmetries in String Theories and Gravity Theories, Conformal Field Theory, Integrable Systems, Representation Theory, Supersymmetry, Quantum Groups, Vertex Algebras, Application of Symmetry to Probability, Dynamical Symmetries.

There is some similarity with the topics of preceding workshops, however, the comparison shows how certain topics evolve and that new structures were found and used. For the present workshop we mention more emphasis on: representation theory, on conformal field theories, integrable systems, vertex algebras, number theory, higher-dimensional unified theories.

The International Organizing Committee was: Vladimir Dobrev (Sofia) and H.-D. Doebner (Clausthal) in collaboration with G. Rudolph (Leipzig).

The Local Organizing Committee was: Vladimir Dobrev (Chairman), L.K. Anguelova, V.I. Doseva, A.Ch. Ganchev, D.T. Nedanovski, T.V. Popov, D.R. Staicova, M.N. Stoilov, N.I. Stoilova, S.T. Stoimenov.

Acknowledgments

We express our gratitude to the

– Institute of Nuclear Research and Nuclear Energy
– Abdus Salam International Centre for Theoretical Physics

for financial help. We thank the Bulgarian Academy of Sciences for providing its Guest House which contributed very much to the stimulating and pleasant atmosphere during the workshop. We thank the publisher, Springer Japan, represented by Ms. Chino Hasebe (Executive Editor in Mathematics, Statistics, Business, Economics, Computer Science) and Mr. Masayuki Nakamura (Editorial Department), for assistance in the publication. Last but not least, I thank the members of the Local Organizing Committee who, through their efforts, made the workshop run smoothly and efficiently.

Sofia, Bulgaria Vladimir Dobrev
May 2016

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Part I
Plenary Talks

$*$–Lie Algebras Canonically Associated to Probability Measures on $\mathbb{R}$ with All Moments

Luigi Accardi, Abdessatar Barhoumi, Yun Gang Lu and Mohamed Rhaima

Abstract In the paper Accardi et al.: Identification of the theory of orthogonal polynomials in d–indeterminates with the theory of 3–diagonal symmetric interacting Fock spaces on $\mathbb{C}^d$, submitted to: IDA–QP (Infinite Dimensional Anal. Quantum Probab. Related Topics), [1], it has been shown that, with the natural definitions of morphisms and isomorphisms (that will not be recalled here) the category of orthogonal polynomials in a finite number of variables is isomorphic to the category of symmetric interacting fock spaces (IFS) with a 3–diagonal structure. Any IFS is canonically associated to a $*$–Lie algebra (commutation relations) and a $*$–Jordan algebra (anti–commutation relations). In this paper we continue the study of these algebras, initiated in Accardi et al. An Information Complexity index for Probability Measures on $\mathbb{R}$ with all moments, submitted to: IDA–QP (Infinite Dimensional Anal. Quantum Probab. Related Topics), [2], in the case of polynomials in one variable, refine the definition of **information complexity index of a probability measure** on the real line, introduced there, and prove that the $*$–Lie algebra canonically associated to the probability measures of complexity index $(0, K, 1)$, defining finite–dimensional approximations, in the sense of Jacobi sequences, of the Heisenberg algebra, coincides with the algebra of all $K \times K$ complex matrices.

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V. Dobrev (ed.), *Lie Theory and Its Applications in Physics*,
Springer Proceedings in Mathematics & Statistics 191,
DOI 10.1007/978-981-10-2636-2_1

Keywords Interacting fock space · Quantum decomposition of a classical random variable · Information complexity index

AMS Subject Classification Primary 60J65 · Secondary 60J45 · 60J51 · 60H40

1 Introduction

Let $\mu \in \text{Prob}_\infty(\mathbb{R})$, the family of probability measures on $\mathbb{R}$ with all moments. The quantum decomposition of the classical random variable with distribution μ (see [3]), implies that to every such a measure one can naturally associate different algebraic structures, in particular a $*$–Lie algebra and a Jordan algebra structure.

It follows that any classification of these algebraic structures induces a classification of the corresponding probability measures on $\mathbb{R}$.

In the paper [2] we have started this classification program with the study of the $*$–Lie algebra associated to a generic $\mu \in \text{Prob}_\infty(\mathbb{R})$. A necessary condition for the finite dimensionality of this Lie algebra is that, starting from a certain index K, the principal Jacobi sequence of μ is the solution of a difference equation of finite order (see Theorem 1 in Sect. 4.1).

This condition is not sufficient. In fact, if the order of the difference equation is ≥ 3, then the $*$–Lie algebra associated to μ is infinite dimensional (see Theorem 7 in Sect. 5.4). This produces a new class of infinite dimensional $*$–Lie algebras, canonically associated to probability measures and not previously considered in the literature.

Motivated by Theorem 1 and by Kolmogorov's idea [4] to define the complexity of a sequence as the minimal length of a program that generates it, we have introduced a *complexity index* on $\text{Prob}_\infty(\mathbb{R})$ that defines a hierarchy among the probability measures on $\mathbb{R}$, based on their complexity.

In fact, if the principal Jacobi sequence of a measure satisfies a difference equation of the form $(\partial^n \omega)_m = 0$, the entire information of the sequence (ω_k) can be condensed in the n real parameters characterizing the solutions of this difference equation (see [5]).

The simplest complexity index is given by a pair of natural integers (n, K) depending only on the principal Jacobi sequence (ω_n) of μ, where n is the minimum natural integer such that, for any $m \geq K + 1$, the finite difference equation mentioned above begins to hold.

For example, the complexity class defined by the index $(0, K)$ consists of those measures whose principal Jacobi sequence is constant starting from the index K. If this constant is equal to 0 one finds, when K varies, all the measures with finite support. If it is >0, one finds the *semi–circle–arcsine class*, called in this way for reasons explained below (see Sect. 5.1.4).

It is interesting to notice that the semi–circle law (the *Gaussian* for free independence) is in the class $(0, 0)$, the arcsine law (the *Gaussian* for monotone independence) is in the class $(0, 1)$, and the class $(0, 2)$ naturally appears in the study of central limits of quantum random walks in the sense of Konno (see [6, 7]). The structure of the measures in the classes $(0, K)$, with $K \geq 3$ is not known at the moment.

The class $(1, 0)$ includes the mean zero Gaussians (the unique symmetric measures in this class) and the Poisson. The corresponding $*$–Lie algebra is the Heisenberg algebra. The class $(2, 0)$ includes the three non–standard Meixner classes and the corresponding $*$–Lie algebra is $sl(2, \mathbb{R})$. Starting with $n \geq 3$, the $*$–Lie algebra of the class $(n, 0)$ is infinite–dimensional, and these are the new classes we referred to in the beginning of this section.

In the case of measures with finite support, the connection between Lie algebras and orthogonal polynomials has been studied, from a point of view different from the present one, by several authors (see the paper by Jafarov, Stoilova and Van der Jeugt [8] for references).

2 $*$–Lie and $*$–Jordan Algebras Canonically Associated to Interacting Fock Spaces (IFS)

The notion of Interacting Fock Spaces (IFS) was introduced in [2] in the more general framework of Hilbert modules. Here we recall from [1] a variant of this notion for pre–Hilbert spaces.

For any pair of pre–Hilbert spaces $(H, \langle \cdot , \cdot \rangle_H)$, $(K, \langle \cdot , \cdot \rangle_K)$, denote $\mathcal{L}_a((H, \langle \cdot , \cdot \rangle_H), (K, \langle \cdot , \cdot \rangle_K))$, or simply, when no confusion is possible, $\mathcal{L}_a(H, K)$, the space of all adjointable pre–Hilbert space maps $A : H \to K$, such that there exists a linear map $A^* : K \to H$ satisfying

$$\langle f, Ag \rangle_K = \langle A^* f, g \rangle_H \quad ; \quad \forall g \in H \ , \ \forall f \in K$$

If $H = K$ $\mathcal{L}(K, \langle \cdot , \cdot \rangle_K)$ has a natural structure of $*$–algebra and we simply write $\mathcal{L}_a(K)$.

Definition 1 Let V be a vector space. An **interacting Fock space on** V is a pair:

$$\left\{(H_n, \langle \cdot , \cdot \rangle_n)_{n \in \mathbb{N}}), a^+\right\} \tag{1}$$

such that:

- $\left(H_n, \langle \cdot , \cdot \rangle_n\right)_{n \in \mathbb{N}}$ is a sequence of pre–Hilbert spaces with

$$H_0 =: \mathbb{C} \cdot \Phi_0 \quad ; \quad \|\Phi_0\| = 1$$

  Φ_0 is called the **vacuum or Fock vector**;

- denoting $\langle \cdot , \cdot \rangle$ the unique pre–Hilbert space scalar product on the vector space direct sum of the family $\big(H_n\big)_{n\in\mathbb{N}}$ which makes this direct sum

$$H := \bigoplus_{n\in\mathbb{N}} (H_n, \langle \cdot , \cdot \rangle_n) \tag{2}$$

an orthogonal sum, the linear operator

$$a^+ : V \to \mathcal{L}_a \left((H_n, \langle \cdot , \cdot \rangle_n)_{n\in\mathbb{N}} \right)$$

satisfies the following conditions:

$$H_{n+1} = \text{lin-span } \left\{ \left\{ a^+(V) H_n \right\} \right\} \quad ; \quad \forall n \in \mathbb{N} \tag{3}$$

For each $v \in V$, one fixes a choice of adjoint $a^*(v)$, denoted $a^-(v)$ (or simply a_v) so that

$$a(v)\Phi_0 = 0 \text{ Fock prescription} \qquad ; \qquad \forall v \in V \tag{4}$$

The operators $a^+(v)$ $(f \in V)$ are called **creators** and their adjoints $a^-(v)$ – **annihilators**. The spaces $\big(H_n\big)_{n\in\mathbb{N}}$ are called the $n-$**particle spaces**, if $n = 0$ one speaks of the vacuum space.

Definition 1 implies that, for all $u, v \in V$, $a_u^+ a_v$ and $a_v a_u^+$ are homogeneous linear operators on $\mathcal{H}$ of degree zero, i.e.

$$a_u^+ a_v(\mathcal{H}_n) \; , \; a_v a_u^+(\mathcal{H}_n) \subseteq \mathcal{H}_n \qquad ; \qquad \forall n \in \mathbb{N}$$

Then one can associate to the pairs (a_u^+, a_v):
(i) The smallest $*$–Lie algebra containing all the a_v and the a_u^+, with brackets given by the usual commutator

$$[a_v, a_u^+] := a_v a_u^+ - a_u^+ a_v$$

(ii) The smallest $*$–Jordan algebra containing (T, T^+), with brackets given by the usual anti–commutator

$$\{a_v, a_u^+\} := a_v a_u^+ + a_u^+ a_v$$

Many $*$–Lie and $*$–Jordan algebras that play an important role in physics arise in this way.

3 Notations on Orthogonal Polynomials

The assignment of a probability distribution $\mu \in \text{Prob}_\infty(\mathbb{R})$ (the space of probability measures on $\mathbb{R}$ with all moments), allows to identify the multiplication operator

$(Xf)(x) := xf(x)$ with a real valued classical random variable with all moments. In this identification, the $*$–algebra $\mathcal{P}$, of complex polynomials in a single indeterminate, with the pointwise operations (the involution is complex conjugacy) is identified with the $*$–algebra of complex valued polynomial functions of X. The identity

$$\langle P, Q\rangle := \int_{\mathbb{R}} \overline{P(x)} Q(x)\mu(dx) \tag{5}$$

defines a pre–scalar product $\langle \cdot\ ,\ \cdot \rangle$ hence a pre–Hilbert algebra structure on $\mathcal{P}$. The **normalized** orthogonal polynomials $\tilde{\Phi}_n$ are defined inductively, in terms of the **monic** orthogonal polynomials Φ_n, as follows:

$$\tilde{\Phi}_0 := 1$$

$$P_{0]} := \tilde{\Phi}_0 \tilde{\Phi}_0^*$$

where for any pre–Hilbert space $\mathcal{K}$ and any unit vector $\xi \in \mathcal{K}$, we use the notation

$$\xi^*(\eta) := \langle \xi, \eta\rangle \quad ; \quad \forall \eta \in \mathcal{K} \tag{6}$$

and, having defined the pairs $(\tilde{\Phi}_m, P_{m]})$ $(m \in \{1, \ldots, n\})$, the next pair is defined by

$$\Phi_{n+1} := X^{n+1} - P_{n]}(X^{n+1}) \tag{7}$$

$$\tilde{\Phi}_{n+1} := \begin{cases} \frac{\Phi_{n+1}}{\|\Phi_{n+1}\|}, & \text{if } \|\Phi_{n+1}\| \neq 0 \\ \Phi_{n+1}, & \text{if } \|\Phi_{n+1}\| = 0 \end{cases} \tag{8}$$

$$P_{n+1]} := \sum_{j \in \{1,\ldots,n+1\}} \tilde{\Phi}_j \tilde{\Phi}_j^* \tag{9}$$

By construction one has,

$$\|\tilde{\Phi}_n\| = 1 \text{ or } 0 \quad ; \quad \forall n \in \mathbb{N}$$

$$\mathcal{P} = \bigoplus_{n\in\mathbb{N}} \mathbb{C} \cdot \tilde{\Phi}_n \tag{10}$$

$$\mathcal{P}_{n]} := \{P \in \mathcal{P} \ :\ \text{degree}(P) \le n\}$$

Define the creation, annihilation, preservation (CAP) operators respectively by

$$a^+ = \sum_{n\in\mathbb{N}} \sqrt{\omega}_{n+1} \tilde{\Phi}_{n+1} \tilde{\Phi}_n^* \tag{11}$$

$$a^- = (a^+)^* = \sum_{n\in\mathbb{N}} \sqrt{\omega_n}\tilde{\Phi}_n\tilde{\Phi}^*_{n+1} \tag{12}$$

$$a^0 = \sum_{n\in\mathbb{N}} \alpha_n\tilde{\Phi}_n\tilde{\Phi}^*_n \tag{13}$$

where (α_n) and (ω_n) are the Jacobi sequences of μ and the $\tilde{\Phi}_n$ are defined by (8). The basic property of the sequence (ω_n) is:

$$\omega_n \geq 0 \quad ; \quad \forall n \in \mathbb{N} \quad ; \quad \omega_n = 0 \Rightarrow \omega_p = 0 \quad , \quad \forall p \geq n \tag{14}$$

With the above notations, **the quantum decomposition of the classical random variable** X (see [3]) is

$$X = a^+ + a^0 + a^-$$

Defining the number operator, associated to the orthogonal gradation (10), by

$$\Lambda := \sum_{n\in\mathbb{N}} n\tilde{\Phi}_n\tilde{\Phi}^*_n = \Lambda^+ \tag{15}$$

the commutations relations $[a^-, a^+]$ are

$$[a^-, a^+] = \omega_{\Lambda+1} - \omega_\Lambda =: \sum(\omega_{n+1} - \omega_n)\tilde{\Phi}_n\tilde{\Phi}^*_n \tag{16}$$

where, for any function $F : n \in \mathbb{N} \to F_n \in \mathbb{C}$,

$$F_\Lambda := \sum_{n\in\mathbb{N}} F_n\tilde{\Phi}_n\tilde{\Phi}^*_n$$

The anti–commutations relations $[a^-, a^+]$ are

$$\{a^-, a^+\} = (\omega_{\Lambda+1} + \omega_\Lambda)$$

4 $*$–Lie Algebras Canonically Associated to μ

Definition 2 The $*$–Lie algebra generated by the adjointable operators on the pre–Hilbert space $\mathcal{P}$

$$a^+ \quad , \quad a^-$$

i.e. the smallest $*$–Lie algebra containing these operators, will be denoted $\mathcal{L}^0_X$ (or simply $\mathcal{L}$).

Our goal is to describe the structure of $\mathcal{L}^0_X$.

Remark The natural ∗–Lie algebra $\mathcal{L}_X$, canonically associated to μ and generated by all the CAP operators of μ

$$\{a^+ \ , \ a^0 \ , \ a^-\}$$

Thus to restrict one's attention to $\mathcal{L}_X^0$, as we will do in the present paper, is equivalent to consider only symmetric measures.

Definition 3 The left–shift operator T and the difference operators $\partial^{(k)}$ ($k \in \mathbb{N}^* := \mathbb{N} \setminus \{0\}$) are defined on the space of complex valued sequences respectively by:

$$(TF)_n := F_{n-1} \qquad ; \qquad \forall n \in \mathbb{N} \tag{17}$$

with the convention

$$F_h = 0 \qquad ; \qquad \forall h < 0 \tag{18}$$

$$(\partial^{(k)} F)_n := F_n - (T^k F)_n = F_n - F_{n-k} \quad ; \quad \forall n \in \mathbb{N} \tag{19}$$

Remark We will use the notation

$$\partial F_n := \partial^{(1)} F_n = F_n - F_{n-1} \tag{20}$$

Remark Using the basis (10), one extends the operators T, $\partial^{(k)}$, ∂ to linear operators on the pre–Hilbert space $\mathcal{P}$, still denoted with the same symbols, by the prescriptions:

$$Z\left(\sum_{n=0}^{\infty} F_n \tilde{\Phi}_n\right) := \sum_{n=0}^{\infty} (ZF)_n \tilde{\Phi}_n \qquad ; \qquad Z \in \{T, \partial^{(k)}, \partial\}$$

Lemma 1 *For any $k \in \mathbb{N}$ and any function $F : n \in \mathbb{N} \to F_n \in \mathbb{C}$, one has:*

$$[a^{+k}, F_\Lambda] = -\partial^{(k)} F_\Lambda a^{+k} \qquad ; \qquad [a^k, F_\Lambda] = a^k \partial^{(k)} \bar{F}_\Lambda \tag{21}$$

where $\bar{F}$ denotes the complex conjugate of F.

Proof See [2].

4.1 The Dimension of $\mathcal{L}_X^0$

From this section on, we use the identification: $\mathcal{L}_X^0 \equiv \mathcal{L}_X$. It is clear that, if $\dim(L^2(\mathbb{R}, \mu)) < +\infty$ then also $\mathcal{L}_X^0$ will be finite–dimensional. Therefore the problem to distinguish between finite and infinite dimensional $\mathcal{L}_X^0$ is non–trivial only if $\dim(L^2(\mathbb{R}, \mu)) = +\infty$, and this is the case if and only if

$$\omega_n > 0 \qquad ; \qquad \forall n \in \mathbb{N}^* \tag{22}$$

or equivalently

$$\|\Phi_n\| \neq 0 \quad ; \quad \forall n \in \mathbb{N} \tag{23}$$

On the other hand, independently of a^0, $\mathcal{L}_X^0$ must contain the $*$–Lie algebra generated by

$$a^- \ , \ a^+ \ , \ [a^-, a^+] = \partial\omega_\Lambda$$

Therefore, a sufficient condition for $\mathcal{L}_X$ to be infinite dimensional is that this algebra is infinite dimensional. For a symmetric random variable, i.e. $a^0 = 0$, this condition is also necessary.

Theorem 1 *Under the assumption (23), for a random variable X with principal Jacobi sequence (ω_m), a necessary condition for $\mathcal{L}_X^0$ to be finite dimensional is that there exists $n, K \in \mathbb{N}$ such that*

$$(\partial^n\omega)_m = 0 \quad ; \quad \forall m \geq K + 1 \tag{24}$$

Proof See [2].

5 Indices of Information Complexity

Theorem 1 suggests that, among all the probability measures $\mu \in \mathrm{Prob}_\infty(\mathbb{R})$, the *simplest ones* are those whose principal Jacobi sequence (ω_n) satisfies a difference equation of the form (24).

In this section we give a quantitative formulation of this intuition.
We do not assume that $\omega_n > 0$ for each $n \in \mathbb{N}$.

Definition 4 The **index of information complexity** (or simply **complexity index**) of a probability measure $\mu \in \mathrm{Prob}_\infty(\mathbb{R})$, with principal Jacobi sequence (ω_n), is the pair $C(\mu) \in \mathbb{N}^2$ defined as follows:

$$C(\mu) := \begin{cases} (k, K), \text{ if } k = \min\{n \in \mathbb{N} \ : \ (\partial^{n+1}\omega)_m = 0 \ , \ \forall m \geq K + 1\} \\ +\infty, \quad \text{if no pair } (n, K) \in \mathbb{N}^2 \text{ with the above property exists} \end{cases} \tag{25}$$

Remark Notice that the relation

$$\mu \sim \nu \iff C(\mu) = C(\nu) \quad ; \quad \mu, \nu \in \mathrm{Prob}_\infty(\mathbb{R}) \tag{26}$$

is an equivalence relation and that it involves only the principal Jacobi sequence $\omega \equiv (\omega_n)$.

5.1 The Case $C(\mu) = (0, K)$ $(K \in \mathbb{N})$

According to Definition 4 a probability measure $\mu \in \mathrm{Prob}_\infty(\mathbb{R})$ with principal Jacobi sequence (ω_m) belongs to the information complexity class $(0, K)$ $(K \in \mathbb{N})$ if

$$(\partial\omega)_m = 0 \quad ; \quad \forall m \geq K + 1 \tag{27}$$

and K is smallest number with respect to the property (27).

Theorem 2 *Let $\mu \in Prob_\infty(\mathbb{R})$ be a probability measure with principal Jacobi sequence (ω_n) and information complexity $C(\mu) = (0, K)$, $(K \in \mathbb{N})$. Then, with the convention that*

$$x \leq 0 \Rightarrow \{1, \ldots, x\} := \emptyset \tag{28}$$

and for K as in (27), one of the following alternatives takes place:
(i) $|supp(\mu)| = K - 2$ and (ω_n) has the form

$$\omega_n = \begin{cases} arbitrary\ > 0 \ if\, n \in \{1, \ldots, K-1\} \\ 0, \qquad\qquad\quad if\, n \geq K \end{cases} \tag{29}$$

Moreover all probability measures $\mu \in Prob_\infty(\mathbb{R})$ with principal Jacobi sequence satisfying (29) are in this class.
(ii) $|supp(\mu)| = \infty$ and (ω_n) has the form

$$\omega_n = \begin{cases} arbitrary,\ > 0\ ,\ if\, n \in \{1, \ldots, K\} \\ \omega > 0,\ , \qquad\quad if\, n \geq K+1 \end{cases} \tag{30}$$

Proof See [2].

In the following we discuss some examples of measures in this class.

5.1.1 The Case $C(\mu) = (0, 0)$, $\omega = 0$: The δ–Measures

If $C(\mu) = (0, 0)$ and $\omega = 0$, then $\partial\omega_n = 0$ for all $n \geq 1$. In particular

$$\partial\omega_1 = \omega_1 - \omega_0 = \omega_1 = \mu(X^2) - \mu(X)^2 = 0$$

and this condition characterizes the δ–**measures**, i.e. those μ such that

$$\mu = \delta_c \quad \text{for some } c \in \mathbb{R}$$

5.1.2 The Case $C(\mu) = (0, 0)$, $\omega > 0$: The Semi–circle Laws

In this case, if μ has infinite support, $\omega_n =: a > 0$ for all $n \geq 1$ and it is known (see e.g. the table in [9]) that this class coincides with the class of **Semi–circle** distributions.

The associated commutation relations are trivial and the algebra generated by a^+ and a is abelian. Thus, from the point of view of the canonical quantum decomposition, the semi–circle laws are the *most commutative* among all probability measures.

5.1.3 The Case $C(\mu) = (0, 1)$, $\omega > 0$: The Arcsine Laws

In this case, if μ has infinite support,

$$\omega_n = \begin{cases} a > 0, & \text{if } n = 1 \\ 0 < b \neq a, & \text{if } n \geq 2 \end{cases}$$

and it is known (see e.g. the table in [9]) that this class coincides with the class of **Arcsine laws**.

5.1.4 The Case $C(\mu) = (0, K)$, $\omega = b > 0$, $K \geq 2$: The Extended Semi–circle–arcsine Laws

It is clear that the structure of the measures in this class is a natural extension of the semi–circle and arcsine laws.

5.2 *The Case $C(\mu) = (1, K)$*

According to Definition 4 the probability measures belonging to the information complexity class $(1, K)$ are those for which there exists $K \in \mathbb{N}$ such that

$$\partial^2 \omega_n = 0 \quad ; \quad \forall n \geq K + 1 \tag{31}$$

and both the exponent 2 and the number K are the smallest ones with respect to property (31).

Theorem 3 *The class of probability measures $\mu \in Prob_\infty(\mathbb{R})$ with information complexity $C(\mu) = (1, K)$, $(K \in \mathbb{N})$ is characterized by the fact that their principal Jacobi sequence (ω_n) has the following structure:*
there exists $b \in \mathbb{R}_+^*$ and $c \in \mathbb{R}$ such that, with the convention (28), (ω_n) has the form

$$\omega_n = \begin{cases} arbitrary \ > 0, & if\, n \in \{1, \dots, K\} \\ bn + c > 0, & if\, n \geq K + 1 \end{cases} \tag{32}$$

In particular, if c is positive, then it can be arbitrary while, if negative, it must satisfy

$$|c| < b(K + 1) \tag{33}$$

Proof See [2].

5.2.1 The Case $C_\omega(\mu) = (1, 0)$: Gaussian and Poisson

For the measures in the complexity class $C(\mu) = (1, 0)$, $\omega_n = bn + c$ for all $n \geq 1$ with $b \in \mathbb{R}_+^*,\ c \in \mathbb{R}_+$.

In particular, for $c = 0$, $\omega_n = bn > 0$ for all $n \geq 1$ and it is known (see e.g. the table in [9]) that this class includes both the Gaussian distribution with mean 0 and variance b and the Poisson distribution with intensity b (see [9]).

5.2.2 The ∗–Lie Algebra of the Class $C_\omega(\mu) = (1, 0)$ is The Heisenberg Algebra

Theorem 4 *Let* $\mu \in Prob_\infty(\mathbb{R})$ *be a probability measure with principal Jacobi sequence* (ω_n) *and information complexity* $C(\mu) = (1, 0)$*, then the* 3*–dimensional linear space* $\mathcal{L}_X^0$ *generated by the operators*

$$\left\{a^-, a^+, \partial\omega_\Lambda\right\}$$

is a ∗*–Lie algebra isomorphic to the Heisenberg Lie algebra.*

Proof If $C(\mu) = (1, 0)$ then, from Theorem 3, (ω_n) has the form $bn+c$ for all $n \geq 1$. Therefore $\omega_\Lambda - \omega_{\Lambda-1} = b \cdot 1$ and the commutation relations become

$$[a^-, a^+] = b \cdot 1 \qquad ; \qquad [a^-, b \cdot 1] = [a^+, b \cdot 1] = 0$$

which are the defining relations of the Heisenberg ∗–Lie algebra.

5.3 *The Classes* $C_\omega(\mu) = (2, K)$

According to Definition 4 the probability measures belonging to the information complexity class $(2, K)$ are those for which there exists $K, n \in \mathbb{N}$ such that

$$\partial^3\omega_n = 0 \qquad ; \qquad \forall n \geq K + 1 \tag{34}$$

and both the exponent 3 and the number K are minimal with respect to the property (36).

Theorem 5 *The class of probability measures on $\mathbb{R}$ with information complexity $C(\mu) = (2, K)$, $(K \in \mathbb{N})$ is characterized by the fact that their principal Jacobi sequence (ω_n) has the following structure:*
there exists $b, c, d \in \mathbb{R}$ such that $b > 0$ and, with the convention (28) (ω_n) has the form

$$\omega_n = \begin{cases} arbitrary > 0, & if\, n \in \{1, \ldots, K\} \\ bn^2 + cn + d > 0, & if\, n \geq K + 1 \end{cases} \tag{35}$$

In particular, if c, d are positive, then they can be arbitrary while, if one of them is negative, then their choice is constrained by the fact that the right hand side of (35) must be strictly positive.

Proof If $C(\mu) = (2, K)$, $(K \in \mathbb{N})$, then we know that there exists $K \in \mathbb{N}$ such that

$$\partial^3 \omega_n = 0 \quad ; \quad \forall n \geq K + 1 \tag{36}$$

Therefore, there exists $b, c, d \in \mathbb{R}$ such that

$$\omega_n = bn^2 + cn + d > 0 \quad ; \quad \forall n \geq K + 1 \tag{37}$$

In particular, since $\omega_n > 0$ for each n, one must have $b > 0$. Conversely, given a triple b, c, d such that $bn^2 + cn + d > 0$ for all $n \geq K + 1$, for any choice of the strictly positive numbers $\omega_0, \ldots, \omega_K$, by Favard Lemma, the sequence $(\omega_n)_{n \in \mathbb{N}}$ defines a unique symmetric state on $\mathcal{P}$. The remaining statements are clear.

5.3.1 The Case $C(\mu) = (2, 0)$

In this case $(\omega_n)_n$ has the form $bn^2 + cn + d$ for all $n \geq 1$, $(b \in \mathbb{R}_+^*,\ c \in \mathbb{R}_-)$. In particular, if $d = 0$ then $\omega_n = bn^2 + cn > 0$ for all $n \geq 1$ and it is known that this class coincides with the class of non–standard (i.e. neither Gaussian nor Poisson) Meixner distributions (see [9]).

5.3.2 The ∗–Lie Algebra of the Class $C_\omega(\mu) = (2, 0)$ is $sl(2, \mathbb{R})$

Theorem 6 *Let $\mu \in Prob_\infty(\mathbb{R})$ be a probability measure with principal Jacobi sequence (ω_n) and information complexity $C(\mu) = (2, 0)$, then the 3–dimensional linear space $\mathcal{L}_X^0$ generated by the operators*

$$\left\{a^-, a^+, \partial\omega_\Lambda\right\} \tag{38}$$

is a ∗–Lie algebra isomorphic to a (necessarily trivial) central extension of the $sl(2, \mathbb{R})$ Lie algebra.

Proof If $C(\mu) = (2, 0)$ then from Theorem 5 (ω_n) has the form $\omega_n = bn^2 + cn + d$ for all $n \geq 1$, with $b > 0$. This implies that

$$\partial\omega_\Lambda = 2b\Lambda + c \qquad ; \qquad \partial^2\omega_\Lambda = 2b$$

Hence, from Lemma 1 we have the commutation relations

$$[a^-, a^+] = \partial\omega_\Lambda = 2b\Lambda + c$$

$$[a^-, \partial\omega_\Lambda] = [a^-, 2b\Lambda + c] = 2b[a^-, \Lambda] = 2ba^-\partial\Lambda = 2ba^-$$

Consequently

$$[a^+, \partial\omega_\Lambda] = -2ba^+$$

where elements of $\mathbb{R}$ are identified to multiples of a central element of the Lie algebra. The statement then follows from the definition of the $sl(2, \mathbb{R})$ ∗–Lie algebra and from the known fact that all its central extensions are trivial.

5.4 The Case $C(\mu) = (3, 0)$

Theorem 7 *Let $\mu \in Prob_\infty(\mathbb{R})$ be a probability measure with principal Jacobi sequence (ω_n) and information complexity $C(\mu) = (3, 0)$, then the Lie algebra $\mathcal{L}_X^0$ generated by the operators*

$$\{a^-, a^+, \partial\omega_\Lambda\}$$

is an ∞–dimensional ∗–Lie algebra.

Proof See [2].

6 Refinement of the Information Complexity Index

A refinement of the complexity index can be obtained by considering, in each class (n, K), those measures such that the initial *segment* $(\omega_1, \ldots, \omega_K)$ also satisfies a difference equation (possibly of order different from n). This allows to introduce a complexity hierarchy also in the class of finitely supported measures.

In particular the class $(0, K, 1)$ is defined by the condition

$$\omega_n = \begin{cases} \text{arbitrary } 0\ , & \text{if } n < K \\ \omega \geq 0\ , & \text{if } n \geq K \end{cases}$$

In the following we discuss the structure of the $*$–Lie algebra associated to this class.

6.1 Algebras Associated to the Class $(0, K, 1)$, with $\omega = 0$

This class is characterized by the condition

$$\omega_n = \begin{cases} \omega\ , & \text{if } n < K \\ 0\ , & \text{if } n \geq K \end{cases}$$

In this case

$$\|a^{+n}\Phi_0\|^2 = \omega_k! = 0\ , \quad \text{for } n \geq K\ ;$$

$$\|\Phi_{K-1}\|^2 = \|a^{+(K-1)}\Phi_0\|^2 = \omega_{K-1}! = \omega^{K-1}$$

One has

$$\mathcal{P} = \bigoplus_{n\in\{0,1,\ldots,K-1\}} \mathbb{C}\cdot\Phi_n \oplus \mathcal{N} \sim \mathbb{C}^K \oplus \mathcal{N}$$

where $\mathcal{N}$ denotes the sub–space of $\mathcal{P}$ consisting of zero–norm vectors.

$$a^{+K}\Phi_n = \begin{cases} \Phi_{n+1}\ , & \text{if } n < K \\ 0\ , & \text{if } n \geq K \end{cases}$$

In this case, the associated $*$–Lie algebra has dimensions at most K^2.

For the associated Jordan $*$–algebra one finds

$$\{a^-, a^+\}\mathcal{P}_n = 2\omega \quad ; \quad \forall\, n \geq K$$

Thus the Jordan $*$–algebra, canonically associated to this class of probability measures, is a K–dimensional generalization of the Fermi algebra, which corresponds to the case $K = 1$.

6.2 Algebras Associated to the Class $(0, K, 1)$, with $\omega = 0$ and $K > 0$

The class $(0, K, 1)$ is the sub–class of the class $(0, K)$ defined by the additional condition that the non–zero ω_n satisfy a first order difference equation. Equivalently:

$$\omega_n = cn + a \quad ; \quad c > 0 \qquad (c > |a| \text{ if } a < 0)\ ,\ \forall n \in \{1, \ldots, K-1\}$$

Theorem 8 *The $*$–Lie algebra generated by the creator and annihilator of any measure in the class* $(0, K, 1)$ *with* $\omega = 0$ *(i.e.* $\omega_n = 0$ *for* $n \geq K$*) coincides with the $*$–Lie algebra* $M_K(\mathbb{C})$*, of all* $K \times K$ *complex matrices.*

Proof Let μ be any measure in the class $(0, K, 1)$. Denote by a^- and a^+ its creation and annihilation operators and $\mathcal{H}$ the $*$–Lie algebra generated by them. In the following, for any linear adjointable operator Z on $\mathcal{P}$, we use the notation $Z \tilde{\in} \mathcal{H}$ to mean that $Z = Z' + N$, where N is an operator whose range is contained in the zero–norm vectors. Under our assumptions, up to zero–norm vectors, one has

$$[a^-, a^+]\Phi_n = \begin{cases} c\Phi_n\ ,\ \text{if } n < K-1 \\ -a^+a^-\Phi_{K-1} = -(c(K-1)+a)\Phi_{K-1}\ ,\ \text{if } n = K-1 \\ = 0\ ,\ \text{if } n \geq K \end{cases}$$

or equivalently

$$[a^-, a^+] = cP_{K-2]} - (c(K-1)+a)\tilde{\Phi}_{K-1}\tilde{\Phi}^*_{K-1} =: L_1 \tilde{\in} \mathcal{H} \tag{39}$$

where $\tilde{\Phi}_n$ denotes the n–th normalized orthogonal polynomial of μ. (Thus the $*$–Lie algebra generated by a^- and a^+ on the quotient space of non–zero–norm polynomials of degree $\leq K-2$, can be considered as a $(K-2)$*–th order approximation* of the Heisenberg $*$–Lie algebra).

In order to compute the commutator

$$[a^+, L_1] = [a^+, cP_{K-2]} - \omega_{K-1}\tilde{\Phi}_{K-1}\tilde{\Phi}^*_{K-1}] =$$

$$= c[a^+, P_{K-2]}] - \omega_{K-1}[a^+, \tilde{\Phi}_{K-1}\tilde{\Phi}^*_{K-1}]$$

let us compute separately $[a^+, P_{K-2]}]$ and $[a^+, \tilde{\Phi}_{K-1}\tilde{\Phi}^*_{K-1}]$.

To this goal notice that the relations

$$a^-\tilde{\Phi}_n = \sqrt{\omega_n}\tilde{\Phi}_{n-1} \quad ; \quad a^+\tilde{\Phi}_n = \sqrt{\omega_{n+1}}\tilde{\Phi}_{n+1} \tag{40}$$

imply that

$$[a^+, \tilde{\Phi}_n\tilde{\Phi}^*_n] = a^+\tilde{\Phi}_n\tilde{\Phi}^*_n - \tilde{\Phi}_n\tilde{\Phi}^*_n a^+ = \sqrt{\omega_{n+1}}\tilde{\Phi}_{n+1}\tilde{\Phi}^*_n - (a^-\tilde{\Phi}_n\tilde{\Phi}^*_n)^*$$

$$= \sqrt{\omega_{n+1}}\tilde{\Phi}_{n+1}\tilde{\Phi}^*_n - (\sqrt{\omega_n}\tilde{\Phi}_{n-1}\tilde{\Phi}^*_n)^* = \sqrt{\omega_{n+1}}\tilde{\Phi}_{n+1}\tilde{\Phi}^*_n - \sqrt{\omega_n}\tilde{\Phi}_n\tilde{\Phi}^*_{n-1}$$

i.e.

$$[a^+, \tilde{\Phi}_n\tilde{\Phi}^*_n] = \sqrt{\omega_{n+1}}\tilde{\Phi}_{n+1}\tilde{\Phi}^*_n - \sqrt{\omega_n}\tilde{\Phi}_n\tilde{\Phi}^*_{n-1} \tag{41}$$

Taking the adjoint and changing sign, one has:

$$[a^-, \tilde{\Phi}_n \tilde{\Phi}_n^*] = \sqrt{\omega_n}\tilde{\Phi}_{n-1}\tilde{\Phi}_n^* - \sqrt{\omega_{n+1}}\tilde{\Phi}_n\tilde{\Phi}_{n+1}^* \tag{42}$$

Recalling that

$$P_{K-2]} = \sum_{n=0}^{K-2} \tilde{\Phi}_h \tilde{\Phi}_h^*$$

one has

$$[a^+, P_{K-2]}] = \sum_{n=0}^{K-2}[a^+, \tilde{\Phi}_n\tilde{\Phi}_n^*] = \sum_{n=0}^{K-2}(\sqrt{\omega_{n+1}}\tilde{\Phi}_{n+1}\tilde{\Phi}_n^* - \sqrt{\omega_n}\tilde{\Phi}_n\tilde{\Phi}_{n-1}^*)$$

Similarly, since $\tilde{\Phi}_K$ has zero norm, up to norm–zero operators, one has

$$[a^+, \tilde{\Phi}_{K-1}\tilde{\Phi}_{K-1}^*] = \sqrt{\omega_K}\tilde{\Phi}_K\tilde{\Phi}_{K-1}^* - \sqrt{\omega_{K-1}}\tilde{\Phi}_{K-1}\tilde{\Phi}_{K-2}^* \equiv$$

$$\equiv -\sqrt{\omega_{K-1}}\tilde{\Phi}_{K-1}\tilde{\Phi}_{K-2}^*$$

hence

$$[a^+, L_1] = [a^+, cP_{K-2]} - \omega_{K-1}\tilde{\Phi}_{K-1}\tilde{\Phi}_{K-1}^*] =$$

$$= c[a^+, P_{K-2]}] - \omega_{K-1}[a^+, \tilde{\Phi}_{K-1}\tilde{\Phi}_{K-1}^*] =$$

$$= c\sum_{n=0}^{K-2}(\sqrt{\omega_{n+1}}\tilde{\Phi}_{n+1}\tilde{\Phi}_n^* - \sqrt{\omega_n}\tilde{\Phi}_n\tilde{\Phi}_{n-1}^*) + \omega_{K-1}\sqrt{\omega_{K-1}}\tilde{\Phi}_{K-1}\tilde{\Phi}_{K-2}^*$$

$$= \sum_{n=0}^{K-2}(c\sqrt{\omega_{n+1}}\tilde{\Phi}_{n+1}\tilde{\Phi}_n^* - c\sqrt{\omega_n}\tilde{\Phi}_n\tilde{\Phi}_{n-1}^*) + \omega_{K-1}\sqrt{\omega_{K-1}}\tilde{\Phi}_{K-1}\tilde{\Phi}_{K-2}^*$$

$$= c\sqrt{\omega_{K-1}}\tilde{\Phi}_{K-1}\tilde{\Phi}_{K-2}^* - c\sqrt{\omega_0}\tilde{\Phi}_1\tilde{\Phi}_{-1}^*) + \omega_{K-1}\sqrt{\omega_{K-1}}\tilde{\Phi}_{K-1}\tilde{\Phi}_{K-2}^*$$

$$= c\sqrt{\omega_{K-1}}\,(1 + \omega_{K-1})\,\tilde{\Phi}_{K-1}\tilde{\Phi}_{K-2}^* \mathrel{\tilde{\in}} \mathcal{H}$$

because $\tilde{\Phi}_{-1} := 0$. Therefore also $\tilde{\Phi}_{K-1}\tilde{\Phi}_{K-2}^* =: L_2 \mathrel{\tilde{\in}} \mathcal{H}$, hence $L_2^* = \tilde{\Phi}_{K-2}\tilde{\Phi}_{K-1}^* \mathrel{\tilde{\in}} \mathcal{H}$. Taking commutator, we find

$$[L_2, L_2^*] = [\tilde{\Phi}_{K-1}\tilde{\Phi}_{K-2}^*\,,\ \tilde{\Phi}_{K-2}\tilde{\Phi}_{K-1}^*]$$

$$= \tilde{\Phi}_{K-1}\tilde{\Phi}_{K-1}^* - \tilde{\Phi}_{K-2}\tilde{\Phi}_{K-2}^* =: L_3 \mathrel{\tilde{\in}} \mathcal{H} \tag{43}$$

Therefore

$$L_1 + \omega_{K-1} L_3 = \sum_{n=0}^{K-2} \tilde{\Phi}_h \tilde{\Phi}_h^* - \omega_{K-1} \tilde{\Phi}_{K-1} \tilde{\Phi}_{K-1}^* + \omega_{K-1} (\tilde{\Phi}_{K-1} \tilde{\Phi}_{K-1}^* - \tilde{\Phi}_{K-2} \tilde{\Phi}_{K-2}^*)$$

$$= \sum_{n=0}^{K-2} \tilde{\Phi}_h \tilde{\Phi}_h^* - \omega_{K-1} \tilde{\Phi}_{K-2} \tilde{\Phi}_{K-2}^* =: L_4 \mathrel{\tilde{\in}} \mathcal{H}$$

hence also

$$[L_4, L_2] = \left[\sum_{n=0}^{K-2} \tilde{\Phi}_h \tilde{\Phi}_h^* - \omega_{K-1} \tilde{\Phi}_{K-2} \tilde{\Phi}_{K-2}^* \, , \; \tilde{\Phi}_{K-1} \tilde{\Phi}_{K-2}^* \right]$$

$$= -\tilde{\Phi}_{K-1} \tilde{\Phi}_{K-2}^* + \omega_{K-1} \tilde{\Phi}_{K-2} \tilde{\Phi}_{K-2}^* = -L_2 + \omega_{K-1} \tilde{\Phi}_{K-2} \tilde{\Phi}_{K-2}^* \mathrel{\tilde{\in}} \mathcal{H} \tag{44}$$

that implies $\tilde{\Phi}_{K-2} \tilde{\Phi}_{K-2}^* \mathrel{\tilde{\in}} \mathcal{H}$. The combination of (43) and (44) implies that $\tilde{\Phi}_{K-1} \tilde{\Phi}_{K-1}^* \mathrel{\tilde{\in}} \mathcal{H}$. In conclusion, for $m = 0, 1, 2$:

$$\tilde{\Phi}_{K-m} \tilde{\Phi}_{K-(m-1)}^* \quad ; \quad \tilde{\Phi}_{K-(m-1)} \tilde{\Phi}_{K-m}^* \quad ; \quad \tilde{\Phi}_{K-m} \tilde{\Phi}_{K-m}^* \mathrel{\tilde{\in}} \mathcal{H}$$

Suppose by induction that, for every $0 \le m \le n \le K$, one has

$$\tilde{\Phi}_{K-m} \tilde{\Phi}_{K-(m-1)}^* \quad ; \quad \tilde{\Phi}_{K-(m-1)} \tilde{\Phi}_{K-m}^* \quad ; \quad \tilde{\Phi}_{K-m} \tilde{\Phi}_{K-m}^* \mathrel{\tilde{\in}} \mathcal{H}$$

It follows that, for every $0 \le m \le n \le K$, one has also

$$[a^+, \tilde{\Phi}_{K-n} \tilde{\Phi}_{K-n}^*] = \sqrt{\omega_{K-(n-1)}} \tilde{\Phi}_{K-(n-1)} \tilde{\Phi}_{K-n}^* - \sqrt{\omega_{K-n}} \tilde{\Phi}_{K-n} \tilde{\Phi}_{K-(n+1)}^* \mathrel{\tilde{\in}} \mathcal{H}$$

and since, by the induction assumption

$$\sqrt{\omega_{K-(n-1)}} \tilde{\Phi}_{K-(n-1)} \tilde{\Phi}_{K-n}^* \mathrel{\tilde{\in}} \mathcal{H}$$

this implies that

$$\sqrt{\omega_{K-n}} \tilde{\Phi}_{K-n} \tilde{\Phi}_{K-(n+1)}^* \mathrel{\tilde{\in}} \mathcal{H}$$

Since in our case $\omega_{K-n} \neq 0$, this is equivalent to

$$\tilde{\Phi}_{K-n} \tilde{\Phi}_{K-(n+1)}^* \, , \; \tilde{\Phi}_{K-(n+1)} \tilde{\Phi}_{K-n}^* \mathrel{\tilde{\in}} \mathcal{H}$$

which implies

$$[\tilde{\Phi}_{K-n} \tilde{\Phi}_{K-(n+1)}^* \, , \; \tilde{\Phi}_{K-(n+1)} \tilde{\Phi}_{K-n}^*] = \tilde{\Phi}_{K-n} \tilde{\Phi}_{K-n}^* - \tilde{\Phi}_{K-(n+1)} \tilde{\Phi}_{K-(n+1)}^* \mathrel{\tilde{\in}} \mathcal{H}$$

Since, by the induction assumption $\tilde{\Phi}_{K-n}\tilde{\Phi}^*_{K-n}$ is in $\mathcal{H}$, it follows that $\tilde{\Phi}_{K-(n+1)}\tilde{\Phi}^*_{K-(n+1)} \tilde{\in} \mathcal{H}$. Therefore, by induction, one has

$$\tilde{\Phi}_n\tilde{\Phi}^*_{n-1} \ , \ \tilde{\Phi}_{n-1}\tilde{\Phi}^*_n \ , \ \tilde{\Phi}_n\tilde{\Phi}^*_n \tilde{\in} \mathcal{H} \quad ; \quad \forall n \in \{0, 1, \dots, K-1\} \tag{45}$$

with the convention that $\tilde{\Phi}_{-1} := 0$. Denote $\mathcal{L}$ the $*$–Lie sub–algebra of $\mathcal{H}$ generated by the set (45). Since

$$[\tilde{\Phi}_{n-1}\tilde{\Phi}^*_n, \tilde{\Phi}_{m-1}\tilde{\Phi}^*_m] = \tilde{\Phi}_{n-1}\tilde{\Phi}^*_n\tilde{\Phi}_{m-1}\tilde{\Phi}^*_m - \tilde{\Phi}_{m-1}\tilde{\Phi}^*_m\tilde{\Phi}_{n-1}\tilde{\Phi}^*_n =$$

$$= \delta^*_{n+1,m}\tilde{\Phi}_{n-1}\tilde{\Phi}^*_{n+1} - \delta^*_{m,n-1}\tilde{\Phi}_{n-2}\tilde{\Phi}^*_n$$

$$= \begin{cases} 0 \ , \ \text{if} \ , \ m \notin \{n-1, n+1\} \\ \tilde{\Phi}_{n-1}\tilde{\Phi}^*_{n+1} \ , \ \text{if} \ , \ m = n+1 \\ -\tilde{\Phi}_{n-2}\tilde{\Phi}^*_n \ , \ \text{if} \ , \ m = n-1 \end{cases}$$

Suppose by induction that, for given $2 < h < K$, one has

$$\left\{ \Phi_m\tilde{\Phi}^*_n \ : \ |m-n| \leq h \ , \ 0 \leq m, n \leq K \right\} \tilde{\subseteq} \mathcal{L} \tag{46}$$

and notice that, if $|m-n| = h$, the one can always suppose that $n = m + h$ up to exchange of m and n. Under this assumption:

$$[\tilde{\Phi}_{m-1}\tilde{\Phi}^*_m, \tilde{\Phi}_m\tilde{\Phi}^*_n] = \tilde{\Phi}_{m-1}\tilde{\Phi}^*_m\tilde{\Phi}_m\tilde{\Phi}^*_n - \tilde{\Phi}_m\tilde{\Phi}^*_n\tilde{\Phi}_{m-1}\tilde{\Phi}^*_m = \tilde{\Phi}_{m-1}\tilde{\Phi}^*_n \tilde{\in} \mathcal{L}$$

Since $n - (m-1) = n - m + 1 = h + 1$, it follows by induction that $\mathcal{L}$ contains all the matrix units $e_{m,n} := \tilde{\Phi}_m\tilde{\Phi}^*_n$ of $M_K(\mathbb{C})$, hence $\mathcal{L} = \mathcal{H} = M_K(\mathbb{C})$.

Acknowledgements LA acknowledges support by the RSF grant 14-11-00687, Steklov Mathematical Institute.

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Special Conformal Transformations and Contact Terms

Loriano Bonora

Abstract In this contribution I construct the Ward identity of special conformal transformations in momentum space and discuss some of its consequences on conformal field theory correlators. I show a few examples of covariant correlators in dimension 2 and 3 dimensions and in particular of those made of pure contact terms. I discuss in some detail the odd parity correlator in 3d and its connection with the gravitational Chern–Simons theory in 3d.

1 Introduction

Correlators in conformal field theories can be formulated both in configuration space and, via Fourier transform, in momentum space. In the first form they may happen to be singular at coincident insertion points and in need of regularization. In coordinate space they are therefore simply distributions. In the simplest cases such distributions have been studied and can be found in textbooks. But in general the correlators of CFT are very complicated expressions and their regularization has to be carried out from scratch. It is often convenient to do it in momentum space, [1] via Fourier transform, and regularize the Fourier transform of the relevant correlators. This procedure produces various types of terms, which we refer to as *non-local, partially local and local terms*. Local terms, a.k.a contact terms, are represented by polynomials of the external momenta in momentum space, or by delta functions and derivatives of delta functions in configuration space. The unregularized correlators will be referred to as *bare* correlators; they are ordinary regular functions at non-coincident points and are classified as non-local in the previous classification. While regularizing the latter one usually produces not only local terms, but also intermediate ones, which are product of bare functions and delta functions or derivatives thereof. These are referred to as partially local.

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V. Dobrev (ed.), *Lie Theory and Its Applications in Physics*,
Springer Proceedings in Mathematics & Statistics 191,
DOI 10.1007/978-981-10-2636-2_2

Many general results are known nowadays about bare correlators in CFT, [2, 3]. But a complete analysis of the contact terms permitted by conformal symmetry in various dimensions is still lacking. In this contribution I would like to argue that such an analysis is possible and can be conveniently carried out in momentum space. The basic tool for this analysis is the special conformal transformation Ward identity in momentum space. The paper is intended to be an introduction to the subject and is mostly pedagogical. I start with some basic definitions about the conformal algebra in momentum space. Then I formulate the Ward Identities of special conformal transformations in momentum space and their consistency conditions, which lead to the corresponding cohomology, or K-cohomology. Finally I show a few examples of covariant correlators in 2 and 3 dimensions and in particular those made of pure contact terms. I discuss in some detail the odd parity correlator in 3d and its connection with the gravitational Chern–Simons theory in 3d.

2 The Conformal Algebra and SCT's

In this section we briefly introduce the conformal transformations in d dimensions, in particular the special conformal (SCT) ones, which are the main subject of this presentation. The conformal group is made of the usual Poincaré transformation plus dilatations $x^\mu \to \lambda x^\mu$, with generator D, and special conformal transformations with generator K_μ. A special conformal transformation (SCT)

$$x^{'\mu} = \frac{x^\mu + b^\mu x^2}{1 + 2b\cdot x + b^2 x^2} \approx x^\mu + b^\mu x^2 - 2b\cdot x\, x^\mu,$$

for b^μ small, can be seen as a diffeomorphism $x^\mu \to x^\mu + \xi^\mu$ where $\xi^\mu = b^\mu x^2 - 2b\cdot x\, x^\mu$. Introducing a metric $\eta_{\mu\nu}$, this implies a transformation $\eta_{\mu\nu} \to \eta_{\mu\nu} + \delta_\xi \eta_{\mu\nu}$, where

$$\delta_\xi \eta_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu = -4b\cdot x\, \eta_{\mu\nu} \tag{1}$$

which is a Weyl rescaling. On the other hand the square line element

$$dx^{'2} \to x^2\,(1 - 4b\cdot x)$$

which confirms that SFT's are Weyl rescaling, because this can be viewed as a transformation $\eta_{\mu\nu} \to \eta_{\mu\nu}(1 - 4b\cdot x)$.

The conformal generators are

$$\begin{aligned}
P_\mu &= -i\partial\mu \\
D &= -ix^\mu\,\partial\mu \\
L_{\mu\nu} &= i(x_\mu\partial_\nu - x_\nu\partial_\mu) \\
K_\mu &= -i(2x_\mu x^\nu\partial_\nu - x^2\partial\mu)
\end{aligned}$$

They form the Lie algebra

$$
\begin{aligned}
&[L_{\mu\nu}, L_{\lambda\rho}] = i\left(\eta_{\mu\lambda}L_{\nu\rho} - \eta_{\mu\rho}L_{\nu\lambda} - \eta_{\nu\lambda}L_{\mu\rho} + \eta_{\nu\rho}L_{\mu\lambda}\right)\\
&[P^{\mu}, P^{\nu}] = 0\\
&[L_{\mu\nu}, P_{\lambda}] = i\left(\eta_{\mu\lambda}P_{\nu} - \eta_{\nu\lambda}P_{\mu}\right)\\
&[P^{\mu}, D] = iP^{\mu}\\
&[K^{\mu}, D] = -iK^{\mu}\\
&[P^{\mu}, K^{\nu}] = 2i\eta^{\mu\nu}D + 2iL^{\mu\nu}\\
&[K^{\mu}, K^{\nu}] = 0\\
&[L^{\mu\nu}, D] = 0\\
&[L^{\mu\nu}, K^{\lambda}] = i\eta^{\lambda\mu}K^{\nu} - i\eta^{\lambda\nu}K^{\mu}
\end{aligned}
\tag{2}
$$

which is isomorphic to the Lie algebra of SO(d,2).

2.1 Momentum Space Algebra

If we Fourier transform the generators of the conformal algebra we get (a tilde represents the transformed generator and $\tilde{\partial} = \frac{\partial}{\partial k}$)

$$
\begin{aligned}
&\tilde{P}_{\mu} = -k_{\mu}\\
&\tilde{D} = i(d + k^{\mu}\tilde{\partial}_{\mu})\\
&\tilde{L}_{\mu\nu} = i(k_{\mu}\tilde{\partial}_{\nu} - k_{\nu}\tilde{\partial}_{\mu})\\
&\tilde{K}_{\mu} = 2d\,\tilde{\partial}_{\mu} + 2k_{\nu}\tilde{\partial}^{\nu}\tilde{\partial}_{\mu} - k_{\mu}\tilde{\Box}
\end{aligned}
$$

Notice that $\tilde{P}_{\mu}$ is a multiplication operator and $\tilde{K}_{\mu}$ is a quadratic differential operator. The Leibniz rule does not hold for $\tilde{K}_{\mu}$ and $\tilde{P}_{\mu}$ with respect to the ordinary product. However it does hold for the convolution product:

$$
\tilde{K}_{\mu}(\tilde{f} \star \tilde{g}) = (\tilde{K}_{\mu}\tilde{f}) \star \tilde{g} + \tilde{f} \star (\tilde{K}_{\mu}\,\tilde{g})
$$

where $(\tilde{f} \star \tilde{g})(k) = \int dp\, f(k-p)g(p)$.

Nevertheless these generators form a closed algebra under commutation

$$
\begin{aligned}
&[\tilde{D}, \tilde{P}_{\mu}] = i\tilde{P}_{\mu}\\
&[\tilde{D}, \tilde{K}_{\mu}] = i\tilde{K}_{\mu}\\
&[\tilde{K}_{\mu}, \tilde{K}_{\nu}] = 0\\
&[\tilde{K}_{\mu}, \tilde{P}_{\nu}] = i(\eta_{\mu\nu}\tilde{D} - \tilde{L}_{\mu\nu})\\
&[\tilde{K}_{\lambda}, \tilde{L}_{\mu\nu}] = i(\eta_{\lambda\mu}K_{\nu} - \eta_{\lambda\nu}K_{\mu})
\end{aligned}
$$

$$[\tilde{P}_\lambda, \tilde{L}_{\mu\nu}] = i(\eta_{\lambda\mu} P_\nu - \eta_{\lambda\nu} P_\mu)$$
$$[\tilde{L}_{\mu\nu}, \tilde{L}_{\lambda\rho}] = i(\eta_{\nu\lambda}\tilde{L}_{\mu\rho} + \eta_{\mu\rho}\tilde{L}_{\nu\lambda} - \eta_{\mu\lambda}\tilde{L}_{\nu\rho} - \eta_{\nu\rho}\tilde{L}_{\mu\lambda}$$

One should however remember that they do not generate infinitesimal transformation in momentum space.

Our purpose is to use this formulation in momentum space to study the cohomology of SCT's, referred to as K-cohomology, and in particular the polynomial K-cohomology. As explained in the introduction polynomials in momentum space represent contact terms in field theory and the latter are important in two respects, as action terms and as anomalies. To arrive at the cohomology corresponding to a given symmetry one needs the Ward identities of that symmetry. So the next step is to formulate the Ward identities of SCT's (the WI's of the scaling transformation is rather trivial and is understood to be always satisfied).

2.2 Ward Identities for SCT's

Since currents and energy-momentum tensor will play the main role in the sequel, we start with their transformation properties under SCT's

$$i[K_\lambda, J_\mu] = \left(2(d-1)x_\lambda + 2x_\lambda\, x\cdot\partial - x^2\partial_\lambda\right) J_\mu + 2\left(x^\alpha J_\alpha \eta_{\lambda\mu} - x_\mu J_\lambda\right) \tag{3}$$

$$\begin{aligned} i[K_\lambda, T_{\mu\nu}] &= \left(2dx_\lambda + 2x_\lambda\, x\cdot\partial - x^2\partial_\lambda\right) T_{\mu\nu} \\ &\quad + 2\left(x^\alpha T_{\alpha\nu}\eta_{\lambda\mu} + x^\alpha T_{\mu\alpha}\eta_{\lambda\nu} - x_\mu T_{\lambda\nu} - x_\nu T_{\mu\lambda}\right) \end{aligned} \tag{4}$$

In momentum representation they are given by

$$\tilde{K}_\mu \tilde{J}_\lambda(k) = (-2\tilde{\partial}_\mu - 2k\cdot\tilde{\partial}\,\tilde{\partial}_\mu + k_\mu\tilde{\Box})\tilde{J}_\lambda + 2(\tilde{\partial}^\alpha \tilde{J}_\alpha \eta_{\mu\lambda} - \tilde{\partial}_\lambda \tilde{J}_\mu) \tag{5}$$

$$\begin{aligned} \tilde{K}_\mu \tilde{T}_{\lambda\rho}(k) &= (-2k\cdot\tilde{\partial}\,\tilde{\partial}_\mu + k_\mu\tilde{\Box})\tilde{T}_{\lambda\rho} \\ &\quad + 2(\tilde{\partial}^\alpha \tilde{T}_{\alpha\rho}\eta_{\mu\lambda} - \tilde{\partial}_\lambda \tilde{T}_{\mu\rho} + \tilde{\partial}^\alpha \tilde{T}_{\lambda\alpha}\eta_{\mu\rho} - \tilde{\partial}_\rho \tilde{T}_{\lambda\mu}) \end{aligned} \tag{6}$$

where $\tilde{T}_{\mu\nu}(k)$, $\tilde{J}_\mu(k)$ denote the Fourier transforms of $T_{\mu\nu}(x)$, $J_\mu(x)$, respectively.

In order to formulate Ward Identities (WI) on correlators let us couple $T_{\mu\nu}$ to an external source $h_{\mu\nu}$ (this will eventually be identified with the background metric fluctuation: $g_{\mu\nu} \approx \eta_{\mu\nu} + h_{\mu\nu}$), [6]. The generating function of connected Green functions is

$$W[h_{\mu\nu}] = \sum_{n=1}^{\infty} \frac{i^{n+1}}{2^n n!} \int \prod_{i=1}^{n} dx_i\;\, h^{\mu_i\nu_i}(x_i)\, \langle 0|\mathcal{T}\{T_{\mu_1\nu_1}(x_1)\ldots T_{\mu_n\nu_n}(x_n)\}|0\rangle_c,$$

In order for W to be invariant under SCT's the external source $h_{\mu\nu}$ must transform as $\delta_b h_{\mu\nu} = [b^\lambda K_\lambda(x), h_{\mu\nu}(x)] \equiv [b\cdot K(x), h_{\mu\nu}(x)]$, where

$$\begin{aligned} & i[K_\lambda(x), h_{\mu\nu}(x)] \\ & = \left(2x_\lambda\, x\cdot\partial - x^2\partial_\lambda\right) h_{\mu\nu} + 2\left(x^\alpha h_{\alpha\nu}\eta_{\lambda\mu} + x^\alpha h_{\mu\alpha}\eta_{\lambda\nu} - x_\mu h_{\lambda\nu} - x_\nu h_{\mu\lambda}\right) \end{aligned} \tag{7}$$

Invariance of $W[h]$ leads to

$$\begin{aligned} 0 = \delta_b W &= \int d^d x \frac{\delta W}{\delta h^{\mu\nu}} \delta h^{\mu\nu} = \int d^d x \, [b\cdot K, h^{\mu\nu}(x)] \langle\!\langle T_{\mu\nu}(x) \rangle\!\rangle \\ &= -\int d^d x \, h^{\mu\nu}(x) [b\cdot K, \langle\!\langle T_{\mu\nu}(x) \rangle\!\rangle] = 0 \end{aligned} \tag{8}$$

where

$$\begin{aligned} \langle\!\langle T_{\mu\nu}(x) \rangle\!\rangle = 2 \frac{\delta W[h]}{\delta h^{\mu\nu}(x)} &= \frac{1}{n!} \sum_{n=1}^{\infty} \int dx_1 \ldots \int dx_n h^{\mu_1\nu_1}(x_1) \ldots h^{\mu_n\nu_n}(x_n) \\ & \times \langle 0 | \mathcal{T}\{T_{\mu_1\nu_1}(x_1) \ldots T_{\mu_n\nu_n}(x_n)\} | 0 \rangle_c \end{aligned} \tag{9}$$

Differentiating twice (8) with respect $h_{\mu\nu}$ and integrating by parts we get

$$(b\cdot K(x) + b\cdot K(y)) \langle 0 | \mathcal{T} T_{\mu\nu}(x) T_{\lambda\rho}(y) \rangle | 0 \rangle = 0 \tag{10}$$

Differentiating three times (8)

$$(b\cdot K(x) + b\cdot K(y) + b\cdot K(z)) \langle 0 | \mathcal{T} T_{\mu\nu}(x) T_{\lambda\rho}(y) T_{\alpha\beta}(z) \rangle | 0 \rangle = 0 \tag{11}$$

In both equations it is understood that the Lorentz part of $b\cdot K(x)$ acts on the indices $\mu\nu$ only, $b\cdot K(y)$ on the indices $\lambda\rho$ and $b\cdot K(z)$ on $\alpha\beta$ alone.

Due to translational invariance we can set $y = 0$ in (10) and $z = 0$ in (11). These equations become

$$b\cdot K(x) \langle 0 | \mathcal{T} T_{\mu\nu}(x) T_{\lambda\rho}(0) \rangle | 0 \rangle = 0 \tag{12}$$

and

$$(b\cdot K(x) + b\cdot K(y)) \langle 0 | \mathcal{T} T_{\mu\nu}(x) T_{\lambda\rho}(y) T_{\alpha\beta}(0) \rangle | 0 \rangle = 0 \tag{13}$$

In these equations $K_\mu(\cdot)$ is understood as the differential operator at the RHS's of (3), (4). So far the results are classical. But we know that a SCT produces a conformal factor $\sim b\cdot x$. Therefore the RHS of (10) and (11) may no vanish if we take the trace of the e.m. tensor:

$$\begin{aligned} &(b\cdot K(x) + b\cdot K(y)) \langle 0 | \mathcal{T} T^\mu_\mu(x) T_{\lambda\rho}(y) \rangle | 0 \rangle = \mathcal{A}_{\lambda\rho}(x, y) \\ &(b\cdot K(x) + b\cdot K(y) + b\cdot K(z)) \langle 0 | \mathcal{T} T^\mu_\mu(x) T_{\lambda\rho}(y) T_{\alpha\beta}(z) \rangle | 0 \rangle = \mathcal{A}_{\lambda\rho\alpha\beta}(x, y, z) \end{aligned} \tag{14}$$

The RHS's are linear in b. They are unintegrated anomalies. Using translational invariance we can set

$$b\cdot K(x)\langle 0|\mathcal{T}T^{\mu}_{\mu}(x)T_{\lambda\rho}(0)\rangle|0\rangle = \mathcal{A}_{\lambda\rho}(x) \tag{15}$$

$$(b\cdot K(x) + b\cdot K(y))\langle 0|\mathcal{T}T^{\mu}_{\mu}(x)T_{\lambda\rho}(y)T_{\alpha\beta}(0)\rangle|0\rangle = \mathcal{A}_{\lambda\rho\alpha\beta}(x,y) \tag{16}$$

As is well known the above anomalies have to satisfy consistency conditions, which we are going to derive next.

Coupling the current $J_\mu(x)$ with a background gauge field $A^\mu(x)$, it is easy to derive similar WI's also for current correlators.

2.3 Consistency Conditions

Let us start again from $W[h]$ and perform two SCT's on a row. We get

$$\begin{aligned}
\delta_{b_2}\delta_{b_1}W &= \delta_{b_2}\int d^dx\,\frac{\delta W}{\delta h^{\mu\nu}(x)}\delta_{b_1}h^{\mu\nu}(x)\\
&= \int d^dy\int d^dx\,\Bigg\{\frac{\delta^2 W}{\delta h^{\mu\nu}(x)\delta h^{\lambda\rho}(y)}\delta_{b_1}h^{\mu\nu}(x)\delta_{b_2}h^{\lambda\rho}(y)\\
&\quad + \frac{\delta W}{\delta h^{\mu\nu}(x)}\frac{\delta\delta_{b_1}h^{\mu\nu}(x)}{\delta h^{\lambda\rho}(y)}\delta_{b_2}h^{\lambda\rho}(y)\Bigg\}\\
&= \int d^dy\int d^dx\,\Bigg\{[b_1\cdot K(x), h^{\mu\nu}(x)][b_2\cdot K(y), h^{\lambda\rho}(y)]\frac{\delta^2 W}{\delta h^{\mu\nu}(x)\delta h^{\lambda\rho}(y)}\\
&\quad + \frac{\delta W}{\delta h^{\mu\nu}(x)}[b_1\cdot K(x), \delta(x-y)][b_2\cdot K(y), h^{\mu\nu}(y)]\Bigg\}\\
&= \int d^dy\int d^dx\,\Bigg\{[b_1\cdot K(x), h^{\mu\nu}(x)][b_2\cdot K(y), h^{\mu\nu}(y)]\frac{\delta^2 W}{\delta h^{\mu\nu}(x)\delta h^{\lambda\rho}(y)}\\
&\quad - \left[b_1\cdot K(x), \frac{\delta W}{\delta h^{\mu\nu}(x)}\right]\delta(x-y)\left[b_2\cdot K(y), h^{\mu\nu}(y)\right]\Bigg\}
\end{aligned}$$

after integration by parts in x. Integrating over y and integrating again by parts one finally gets

$$\begin{aligned}
\delta_{b_2}\delta_{b_1}W = \int d^dy\int d^dx\,\Bigg\{&[b_1\cdot K(x), h^{\mu\nu}(x)][b_2\cdot K(y), h^{\mu\nu}(y)]\,\times\\
&\times\frac{\delta^2 W}{\delta h^{\mu\nu}(x)\delta h^{\lambda\rho}(y)} + \left[b_1\cdot K(x), \left[b_2\cdot K(x), h^{\mu\nu}(x)\right]\right]\frac{\delta W}{\delta h^{\mu\nu}(x)}\Bigg\}
\end{aligned} \tag{17}$$

Making the transformations in reverse order and taking the difference one gets

$$0 = \delta_{b_2}\delta_{b_1}W - \delta_{b_1}\delta_{b_2}W = \int d^d x\, h^{\mu\nu}(x) \left\{ \left[b_1 \cdot K(x), \left[b_2 \cdot K(x), \frac{\delta W}{\delta h^{\mu\nu}(x)} \right] \right] - \left[b_2 \cdot K(x), \left[b_1 \cdot K(x), \frac{\delta W}{\delta h^{\mu\nu}(x)} \right] \right] \right\} \tag{18}$$

This is equivalent to promoting b to an anticommuting parameter and writing

$$\int d^d x\, h^{\mu\nu}(x) \left[b \cdot K(x), \left[b \cdot K(x), \frac{\delta W}{\delta h^{\mu\nu}(x)} \right] \right] = 0 \tag{19}$$

In fact differentiating (18) with respect to b_1^μ and b_2^ν and (19) first with respect to b^μ and then wrt to b^ν one gets the same result. From now on we will use the second formulation, i.e. b anticommuting.

Differentiating (19) wrt to h several times one gets the consistency conditions for (10) and (11). For instance

$$\begin{aligned} & b \cdot K(x)\, b \cdot K(x) \langle 0 | \mathcal{T} T_{\mu\nu}(x) T_{\lambda\rho}(y) \rangle |0\rangle \;+ \\ & + b \cdot K(y)\, b \cdot K(y) \langle 0 | \mathcal{T} T_{\mu\nu}(x) T_{\lambda\rho}(y) \rangle |0\rangle = 0 \end{aligned}$$

The RHS is strictly 0 even in the quantum theory. Due to translational invariance we can rewrite this equation as

$$b \cdot K(x)\, b \cdot K(x) \langle 0 | \mathcal{T} T_{\mu\nu}(x) T_{\lambda\rho}(0) \rangle |0\rangle = 0 \tag{20}$$

and (14) becomes the consistency condition

$$b \cdot K(x)\, \mathcal{A}_{\lambda\rho}(x) = b \cdot K(x)\, \mathcal{A}_{\lambda\rho}(x) = 0 \tag{21}$$

We can Fourier transform this equation and obtain

$$b \cdot \tilde{K}(k)\, \tilde{\mathcal{A}}_{\lambda\rho}(k) = 0 \tag{22}$$

where $\tilde{K}(k)$ is given by Eq. (6).

3 Examples

We consider now a few simple examples of the approach outlined above. Here we limit ourselves 0-cocycles (i.e. invariants) of the K-cohomology. The analysis of 1-cocycles, i.e. anomalies, requires additional tools and will not be considered here.

In momentum representation the CFT correlators must be annihilated by $b\cdot\tilde{K}$. For instance the 2-pt function of a scalar field of weight Δ is $\sim (k^2)^{\Delta-\frac{d}{2}}$ and

$$\tilde{K}_\mu (k^2)^{\Delta-\frac{d}{2}} = (2\Delta - d)\cdot 0 \cdot (k^2)^{\Delta-\frac{d}{2}-1} = 0 \tag{23}$$

in any dimension. A less trivial, but still simple, example is the 2-pt function of two currents in 3d

$$\langle \tilde{J}_i(k)\tilde{J}_j(-k)\rangle = \frac{\delta_{ij}k^2 - k_i k_j}{|k|} \tag{24}$$

Working out the expression

$$\begin{aligned} &\Big(2(b\cdot\tilde{\partial}) - (b\cdot k\tilde{\Box} - 2k\cdot\tilde{\partial}\, b\cdot\tilde{\partial})\Big)\langle \tilde{J}_i(k)\tilde{J}_j(-k)\rangle \\ &+2(b^l\partial_i - b_i\tilde{\partial}^l)\langle \tilde{J}_l(k)\tilde{J}_j(-k)\rangle + 2(b^l\partial_j - b_j\tilde{\partial}^l)\langle \tilde{J}_l(k)\tilde{J}_l(-k)\rangle \end{aligned} \tag{25}$$

one can check that it is 0.

The 2-pt function of the energy momentum tensor in 3d has three possible (conserved) tensorial structures, which are given by the expression

$$\Big\langle T_{\mu\nu}(k)\, T_{\rho\sigma}(-k)\Big\rangle = -\frac{i\,\tau}{|k|}\left(k_\mu k_\nu - \eta_{\mu\nu}k^2\right)\left(k_\rho k_\sigma - \eta_{\rho\sigma}k^2\right)$$

$$-\frac{i\,\tau'}{|k|}\Big[\left(k_\mu k_\rho - \eta_{\mu\rho}k^2\right)\left(k_\nu k_\sigma - \eta_{\nu\sigma}k^2\right) + \mu\leftrightarrow\nu\Big] \tag{26}$$

$$+\frac{\kappa}{192\pi}\Big[\epsilon_{\mu\rho\tau}k^\tau\left(k_\nu k_\sigma - \eta_{\nu\sigma}k^2\right) + \epsilon_{\mu\sigma\tau}k^\tau\left(k_\nu k_\rho - \eta_{\nu\rho}k^2\right) + \mu\leftrightarrow\nu\Big] \tag{27}$$

where τ, τ', κ are (model-dependent) constants, [4, 5].

Let us show that these structures satisfy the SCT Ward identities. We have

$$\begin{aligned} b\cdot\tilde{K}\,\frac{k_\mu k_\nu k_\lambda k_\rho}{|k|} &= (d-3)\frac{b_\mu k_\nu k_\lambda k_\rho + k_\mu b_\nu k_\lambda k_\rho + k_\mu k_\nu b_\lambda k_\rho + k_\mu k_\nu k_\lambda b_\rho}{|k|} \\ &\quad -(d-3)b\cdot k\,\frac{k_\mu k_\nu k_\lambda k_\rho}{|k|^3} \end{aligned} \tag{28}$$

$$b\cdot\tilde{K}\,\frac{k_\mu k_\nu k^2}{|k|} = (d-3)(b_\mu k_\nu + k_\mu b_\nu)|k| + (d-3)\frac{b\cdot k}{|k|}k_\mu k_\nu \tag{29}$$

and

$$b\cdot\tilde{K}\,|k|^3 = 3(d-3)b\cdot k\,|k| \tag{30}$$

Therefore the even (nonlocal) tensorial structures (26) satisfy the SCT WI.

The third tensorial structure in 3d is parity-odd, traceless and local

$$\langle \tilde{T}_{\mu\nu}(k)\tilde{T}_{\lambda\rho}(-k)\rangle \sim \epsilon_{\mu\lambda\sigma}k^{\sigma}\left(k_{\nu}k_{\rho}-\eta_{\nu\rho}k^{2}\right)+\begin{pmatrix}\mu\leftrightarrow\nu\\ \lambda\leftrightarrow\rho\end{pmatrix}\equiv \tilde{F}_{\mu\nu\lambda\rho}(k) \tag{31}$$

Acting on it with $b\cdot K$ we find

$$\begin{aligned}
b\cdot K\ \tilde{F}_{\mu\nu\lambda\rho} &= \left(-2k\cdot\tilde{\partial}\, b\cdot\tilde{\partial}+b\cdot k\tilde{\square}\right)\tilde{F}_{\mu\nu\lambda\rho}+2(b_{\mu}\tilde{\partial}^{\tau}-b^{\tau}\tilde{\partial}_{\mu})\tilde{F}_{\tau\nu\lambda\rho}\\
&\quad +2(b_{\nu}\tilde{\partial}^{\tau}-b^{\tau}\tilde{\partial}_{\nu})\tilde{F}_{\mu\tau\lambda\rho}\\
&= -2(d-2)\, b\cdot k\ \epsilon_{\mu\lambda\sigma}k^{\sigma}\eta_{\nu\rho}-2b^{\sigma}\epsilon_{\sigma\mu\lambda}(k_{\nu}k_{\rho}-\eta_{\nu\rho}k^{2})\\
&\quad +2(d-2)k^{\sigma}\epsilon_{\sigma\mu\lambda}b_{\nu}k_{\rho}+2\,b^{\tau}k^{\sigma}\epsilon_{\lambda\tau\sigma}(k_{\rho}\eta_{\mu\nu}+k_{\nu}\eta_{\mu\rho})\\
&\quad +4b^{\tau}k^{\sigma}\epsilon_{\tau\lambda\sigma}k_{\mu}\eta_{\nu\rho}+\begin{pmatrix}\mu\leftrightarrow\nu\\ \lambda\leftrightarrow\rho\end{pmatrix}
\end{aligned} \tag{32}$$

This vanishes thanks to the identities

$$b^{\sigma}\epsilon_{\sigma\mu\lambda}k_{\nu}-b_{\nu}\epsilon_{\tau\mu\lambda}k^{\tau}+b^{\tau}\epsilon_{\tau\lambda\sigma}k^{\sigma}\eta_{\mu\nu}-b^{\tau}\epsilon_{\tau\mu\sigma}k^{\sigma}\eta_{\nu\lambda}=0 \tag{33}$$

$$b^{\sigma}\epsilon_{\sigma\mu\lambda}k^{2}+b^{\sigma}b^{\sigma}\epsilon_{\sigma\lambda\tau}k_{\mu}k^{\tau}-b^{\sigma}\epsilon_{\sigma\mu\tau}k_{\tau}k_{\lambda}-b\cdot k\, k^{\tau}\epsilon_{\tau\mu\lambda}=0 \tag{34}$$

which are consequences of

$$\eta_{\mu\nu}\epsilon_{\lambda\rho\sigma}-\eta_{\mu\lambda}\epsilon_{\nu\rho\sigma}+\eta_{\mu\rho}\epsilon_{\nu\lambda\sigma}-\eta_{\mu\sigma}\epsilon_{\nu\lambda\rho}=0$$

Therefore also the parity-odd structure satisfies the SCT Ward identity. Actually the two terms in the RHS of (31) are separately invariant under a SCT. What determines the relative $-$ sign is the em tensor conservation.

4 Massive Fermions and Chern–Simons Theory in 3d

The examples of CFT correlators we have met before (31) were polynomials of the coordinates divided by powers of the relative distances between the insertion points (or their Fourier transforms). Equation (31) represents a new kind of correlator, which corresponds in momentum space to a polynomial of the momenta. By Fourier anti-transforming it we get,

$$F_{\mu\nu\lambda\rho}(x,y)\sim \epsilon_{\mu\lambda\sigma}\partial^{\sigma}\left(\partial_{\nu}\partial_{\rho}-\eta_{\nu\rho}\square\right)\delta^{(3)}(x-y)+\begin{pmatrix}\mu\leftrightarrow\nu\\ \lambda\leftrightarrow\rho\end{pmatrix} \tag{35}$$

This expression is completely localized in coordinate space, that is made solely of delta functions and derivative of delta functions. Such expressions are called *contact* *terms*. The previous ones, like the even parity structures in $3d$, are *nonlocal terms*. It is interesting to dwell on (31) and (35) for several reasons. These formulas are a 2-point correlator of the e.m. tensor, which has been derived only on the basis of conformal symmetry properties. One question we may ask is whether, like in other cases, this correlator can be obtained from the regularization of a bare one. Another question is whether this may come from some free matter field theory, as it often happens in other cases. The answer is negative for both questions. So it is legitimate to ask: what is the conformal theory that supports such correlator? Well, in a sense (31) can indeed be obtained from a free field theory, but not in the usual way, and in another sense there is a theory that supports such correlators, but it is not free. Let us see how.

Consider the theory of a massive fermion in $3d$, minimally coupled to a metric $g_{\mu\nu} \approx \eta_{\mu\nu} + h_{\mu\nu}$:

$$S[g] = \int d^3x\, e \left[i\bar{\psi} E^{\mu}_{a} \gamma^{a} \nabla_{\mu} \psi - m \bar{\psi}\psi \right], \tag{36}$$
$$\nabla_{\mu} = \partial_{\mu} + \frac{1}{2}\omega_{\mu bc}\Sigma^{bc}, \qquad \Sigma^{bc} = \frac{1}{4}\left[\gamma^{b}, \gamma^{c}\right].$$

The corresponding energy momentum tensor

$$T_{\mu\nu} = \frac{i}{4}\bar{\psi}\left(\gamma_{\mu} \overset{\leftrightarrow}{\nabla}_{\nu} + \gamma_{\nu} \overset{\leftrightarrow}{\nabla}_{\mu}\right)\psi. \tag{37}$$

is covariantly conserved on shell as a consequence of the diffeomorphism invariance of the action.

$$\nabla^{\mu} T_{\mu\nu}(x) = 0 \tag{38}$$

The presence of a mass term breaks parity. From (7), the lowest term of the effective action in an expansion in $h_{\mu\nu}$ comes from the two-point function of the e.m. tensor. So let us compute the two-point function of the e.m. tensor in this theory with the Feynman diagram technique. The corresponding contribution comes from the bubble diagram (one graviton entering and one graviton exiting with momentum k, one fermionic loop):

$$\tilde{T}_{\mu\nu\lambda\rho}(k) = \tag{39}$$
$$= \frac{1}{64}\int \frac{d^3p}{(2\pi)^3}\left[\mathrm{Tr}\left(\frac{1}{\not{p} - m}(2p-k)_{\mu}\gamma_{\nu} \frac{1}{\not{p} - \not{k} - m}(2p-k)_{\lambda}\gamma_{\rho}\right) + \begin{pmatrix} \mu \leftrightarrow \nu \\ \lambda \leftrightarrow \rho \end{pmatrix}\right]$$

Working out the calculations involved (which requires also subtracting a divergent term) yields

$$\langle T_{\mu\nu}(k)\, T_{\lambda\rho}(-k)\rangle_{\text{P-odd}} = \frac{\kappa_g(k^2/m^2)}{192\pi}\, \epsilon_{\sigma\nu\rho}\, k^\sigma \left(k_\mu k_\lambda - k^2 \eta_{\mu\lambda}\right) + \begin{pmatrix} \mu \leftrightarrow \nu \\ \lambda \leftrightarrow \rho \end{pmatrix} \tag{40}$$

with

$$\kappa_g(k^2/m^2) = \frac{3m}{k^2}\left(2m + \frac{k^2 - 4m^2}{|k|} \arctan \frac{|k|}{2m}\right) \quad , \qquad |k| \equiv \sqrt{-k^2} \tag{41}$$

It is worth recalling that (40) is conserved and traceless.

Now let us take the IR limit of κ_g, i.e. the limit in which the energy $|k| = \sqrt{k^2}$ becomes much smaller than the mass $|m|$. We get

$$\kappa_{IR} = \lim_{\frac{|k|}{m} \to 0} \kappa_g = \kappa = \frac{m}{|m|} \tag{42}$$

Therefore we recover the form of (31) with a precise coefficient in front, which is the same as in (27) with $\kappa = \pm 1$. It is remarkable that also in the UV there exists a similar limit, [8].

Now let us Fourier anti-transform (31)

$$\begin{aligned} \langle T_{\mu\nu}(x) T_{\lambda\rho}(y)\rangle_{\text{P-odd}} = \frac{\kappa}{192\pi} \epsilon_{\mu\lambda\sigma} \partial^\sigma \left(\partial_\nu \partial_\rho - \eta_{\nu\rho} \Box\right) \delta^{(3)}(x-y) \; + \\ + \begin{pmatrix} \mu \leftrightarrow \nu \\ \lambda \leftrightarrow \rho \end{pmatrix} \end{aligned} \tag{43}$$

Saturating it with $h^{\mu\nu}(x)$ and $h^{\lambda\rho}(y)$ and integrating over x and y (according to the formula (2.2), one gets

$$\frac{\kappa}{192\pi} \int \epsilon_{\mu\lambda\sigma} \left(\partial^\sigma h^{\mu\nu} \partial_\nu \partial_\rho h^{\lambda\rho} - \partial^\sigma h^{\mu\nu} \Box h^\lambda_\nu\right) \tag{44}$$

This represents, to lowest order of approximation, the 3d CS action. It can in fact be obtained from

$$CS = -\frac{\kappa}{96\pi} \int d^3x\, \epsilon^{\mu\nu\lambda} \left(\partial_\mu \omega_\nu^{ab} \omega_{\lambda ba} + \frac{2}{3} \omega_{\mu a}{}^b \omega_{\nu b}{}^c \omega_{\lambda c}{}^a\right) \tag{45}$$

by expanding the spin connection ω in terms of $h_{\mu\nu}$, [7].

5 Comments

In this paper I have defined K-cohomology, and discussed some of its 0-cocycles, i.e. correlators that satisfy the WI of special conformal transformations. It is interesting to find out that there are correlators made out only of contact terms, that is corresponding to local action terms. I have shown the well-known example of 3d, where there exists a two-point function of the e.m. tensor, which is of this type, and corresponds to the lowest order expansion of the gravitational CS action. What is not so well-known, perhaps, is that the higher order terms of the CS action correspond to three, four, ... -point functions of the e.m. tensor. However these correlators are not included in the usual classification of the conformal correlators, because the latter are only required to be naively conserved, i.e. in momentum representation they are required to be transverse to the total momentum, or in configuration space to divergenceless. Such a requirement is totally adequate for the bare correlators, but not for correlators containing contact terms, such as (31). For the latter the usual requirement of transversality is only adequate for two-point functions, not for higher order ones. For instance for a three-point e.m. tensor correlator, its divergence does not vanish but satisfies an equation that involves also the two-point correlators, and so on, [6]. To be more concrete we show the example of 2- and 3-point function for a current J^a_μ. Their conservation laws takes the form

$$k^\mu \tilde{J}^{ab}_{\mu\nu}(k) = 0 \tag{46}$$

$$-iq^\mu \tilde{J}^{abc}_{\mu\nu\lambda}(k_1, k_2) + f^{abd}\tilde{J}^{dc}_{\nu\lambda}(k_2) + f^{acd}\tilde{J}^{db}_{\lambda\nu}(k_1) = 0 \tag{47}$$

where $q = k_1 + k_2$ and $\tilde{J}^{ab}_{\mu\nu}(k)$ and $\tilde{J}^{abc}_{\mu\nu\lambda}(k_1, k_2)$ are Fourier transform of the 2- and 3-point functions, respectively. A similar relation holds for the e.m. tensor. This part of the research program on conformal correlators is still largely unexplored, [8].

Acknowledgements I would like to thank Bruno Lima de Souza for allowing me to use common research material. I would like to thank Vladimir Dobrev, Ivan Todorov and Nikolai Nikolov for hospitality, discussions and suggestions.

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On Nonlocal Modified Gravity and Its Cosmological Solutions

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Abstract During hundred years of General Relativity (GR), many significant gravitational phenomena have been predicted and discovered. General Relativity is still the best theory of gravity. Nevertheless, some (quantum) theoretical and (astrophysical and cosmological) phenomenological difficulties of modern gravity have been motivation to search more general theory of gravity than GR. As a result, many modifications of GR have been considered. One of promising recent investigations is Nonlocal Modified Gravity. In this article we present a brief review of some nonlocal gravity models with their cosmological solutions, in which nonlocality is expressed by an analytic function of the d'Alembert-Beltrami operator $\Box$. Some new results are also presented.

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DOI 10.1007/978-981-10-2636-2_3

1 Introduction

General relativity (GR) was formulated one hundred years ago and is also known as Einstein theory of gravity. GR is regarded as one of the most profound and beautiful physical theories with great phenomenological achievements and nice theoretical properties. It has been tested and quite well confirmed in the Solar system, and it has been also used as a theoretical laboratory for gravitational investigations at other spacetime scales. GR has important astrophysical implications predicting existence of black holes, gravitational lensing and gravitational waves.[1] In cosmology, it predicts existence of about 95 % of additional new kind of matter, which makes dark side of the universe. Namely, if GR is the gravity theory for the universe as a whole and if the universe is homogeneous and isotropic with the flat Friedmann-Lemaître-Robertson-Walker (FLRW) metric at the cosmic scale, then it contains about 68 % of *dark energy*, 27 % of *dark matter*, and only about 5 % of *visible matter* [2].

Despite of some significant phenomenological successes and many nice theoretical properties, GR is not complete theory of gravity. For example, attempts to quantize GR lead to the problem of nonrenormalizability. GR also contains singularities like the Big Bang and black holes. At the galactic and large cosmic scales GR predicts new forms of matter, which are not verified in laboratory conditions and have not so far seen in particle physics. Hence, there are many attempts to modify General relativity. Motivations for its modification usually come from quantum gravity, string theory, astrophysics and cosmology (for a review, see [22, 60, 63]). We are mainly interested in cosmological reasons to modify Einstein theory of gravity, i.e. to find such extension of GR which will not contain the Big Bang singularity and offer another possible description of the universe acceleration and large velocities in galaxies instead of mysterious dark energy and dark matter. It is obvious that physical theory has to be modified when it contains a singularity. Even if it happened that dark energy and dark matter really exist it is still interesting to know is there a modified gravity which can imitate the same or similar effects. Hence, adequate gravity modification can reduce role and rate of the dark matter/energy in the universe.

Any well founded modification of the Einstein theory of gravity has to contain general relativity and to be verified at least on the dynamics of the Solar system. In other words, it has to be a generalization of the general theory of relativity. Mathematically, it should be formulated within the pseudo-Riemannian geometry in terms of covariant quantities and take into account equivalence of the inertial and gravitational mass. Consequently, the Ricci scalar R in gravity Lagrangian $\mathscr{L}_g$ of the Einstein-Hilbert action should be replaced by an adequate function which, in general, may contain not only R but also some scalar covariant constructions which are possible in the pseudo-Riemannian geometry. However, we do not know what is here adequate function and there are infinitely many possibilities for its construction. Unfortunately, so far there is no guiding theoretical principle which could make appropriate choice between all possibilities. In this context the Einstein–Hilbert action is the simplest

[1] While we prepared this contribution, the discovery of gravitational waves was announced [1].

one, i.e. it can be viewed as realization of the principle of simplicity in construction of $\mathscr{L}_g$.

One of promising modern approaches towards more complete theory of gravity is its nonlocal modification. Motivation for nonlocal modification of general relativity can be found in string theory which is nonlocal theory and contains gravity. We present here a brief review and some new results of nonlocal gravity with related bounce cosmological solutions. In particular, we pay special attention to models in which nonlocality is expressed by an analytic function of the d'Alembert operator $\Box = \frac{1}{\sqrt{-g}}\partial_\mu \sqrt{-g} g^{\mu\nu}\partial_\nu$ like nonlocality in string theory. In these models, we are mainly interested in nonsingular bounce solutions for the cosmic scale factor $a(t)$.

In Sect. 2 we mention a few different approaches to nonlocal modified gravity. Section 3 contains rather general modified action with an analytic nonlocality and with corresponding equations of motion. Cosmological equations for the FLRW metric is presented in Sect. 4. Cosmological solutions for constant scalar curvature are considered separately in Sect. 5. Some new examples of nonlocal models and related Ansätze are introduced in Sect. 6. At the and a few remarks are also noticed.

2 Nonlocal Modified Gravity

We consider here nonlocal modified gravity. Usually a nonlocal modified gravity model contains an infinite number of spacetime derivatives in the form of a power series expansion with respect to the d'Alembert operator $\Box = \frac{1}{\sqrt{-g}}\partial_\mu \sqrt{-g} g^{\mu\nu}\partial_\nu$. In this article, we are mainly interested in nonlocality expressed in the form of an analytic function $\mathscr{F}(\Box) = \sum_{n=0}^{\infty} f_n \Box^n$, where coefficients f_n should be determined from various theoretical and phenomenological conditions. Some conditions are related to the absence of tachyons and ghosts.

Before to proceed with this analytic nonlocality it is worth to mention some other interesting nonlocal approaches. For approaches containing $\Box^{-1}$ one can see, e.g., [26, 27, 42, 43, 45–47, 61, 66, 67] and references therein. For nonlocal gravity with $\Box^{-1}$ see also [8, 58]. Many aspects of nonlocal gravity models have been considered, see e.g. [16–18, 20, 36, 59] and references therein.

Our motivation to modify gravity in an analytic nonlocal way comes mainly from string theory, in particular from string field theory (see the very original effort in this direction in [3]) and p-adic string theory [15, 38–40, 65]. Since strings are one-dimensional extended objects, their field theory description contains spacetime nonlocality expressed by some exponential functions of d'Alembert operator $\Box$.

At classical level analytic non-local gravity has proven to alleviate the singularity of the Black-hole type because the Newtonian potential appears regular (tending to a constant) on a universal basis at the origin [9, 11, 41]. Also there was significant success in constructing classically stable solution for the cosmological bounce [11, 13, 48, 51, 55].

Analysis of perturbations revealed a natural ability of analytic non-local gravities to accommodate inflationary models. In particular, the Starobinsky inflation was studied in details and new predictions for the observable parameters were made [24, 53]. Moreover, in the quantum sector infinite derivative gravity theories improve renormalization, see e.g. while the unitarity is still preserved [53, 56, 57] (note that just a local quadratic curvature gravity was proven to be renormalizable while being non-unitary [64]).

3 Modified GR with Analytical Nonlocality

To better understand nonlocal modified gravity itself, we investigate it here without presence of matter. Models of nonlocal gravity which we mainly investigate are given by the following action

$$S = \int d^4x\sqrt{-g}\left(\frac{M_P^2}{2}R - \Lambda + \frac{\lambda}{2}P(R)\mathscr{F}(\Box)Q(R)\right), \tag{1}$$

where R is the scalar curvature, Λ is the cosmological constant, $\mathscr{F}(\Box) = \sum_{n=0}^{\infty} f_n\Box^n$ is an analytic function of the d'Alembert–Beltrami operator $\Box = \nabla^\mu\nabla_\mu$ where ∇_μ is the covariant derivative. The Planck mass M_P is related to the Newtonian constant G as $M_P^2 = \frac{1}{8\pi G}$ and P, Q are scalar functions of the scalar curvature. The spacetime dimensionality $D = 4$ and our signature is $(-, +, +, +)$. λ is a constant and can be absorbed in the rescaling of $\mathscr{F}(\Box)$. However, it is convenient to remain λ and recover GR in the limit $\lambda \to 0$.

Note that to have physically meaningful expressions one should introduce the scale of nonlocality using a new mass parameter M. Then the function $\mathscr{F}$ would be expanded in Taylor series as $\mathscr{F}(\Box) = \sum_{n=0}^{\infty} \bar{f}_n\Box^n/M^{2n}$ with all barred constants dimensionless. For simplicity we shall keep $M^2 = 1$. We shall also see later that analytic function $\mathscr{F}(\Box) = \sum_{n=0}^{\infty} f_n\Box^n$, has to satisfy some conditions, in order to escape unphysical degrees of freedom like ghosts and tachyons, and to have good behavior in quantum sector (see [9, 10, 41]).

Varying the action (1) by substituting

$$g_{\mu\nu} \to g_{\mu\nu} + h_{\mu\nu} \tag{2}$$

to the linear order in $h_{\mu\nu}$, removing the total derivatives and integrating from time to time by parts one gets

$$\delta S = \int d^4x\sqrt{-g}\frac{h^{\mu\nu}}{2}\Big[-\mathscr{G}_{\mu\nu}\Big], \tag{3}$$

where

$$\mathscr{G}_{\mu\nu} \equiv M_P^2 G_{\mu\nu} + g_{\mu\nu}\Lambda - \frac{\lambda}{2} g_{\mu\nu} P\mathscr{F}(\Box)Q + \lambda(R_{\mu\nu} - K_{\mu\nu})V - \frac{\lambda}{2}\sum_{n=1}^{\infty} f_n$$
$$\times \sum_{l=0}^{n-1}\left(P_\mu^{(l)} Q_\nu^{(n-l-1)} + P_\nu^{(l)} Q_\mu^{(n-l-1)} - g_{\mu\nu}(g^{\rho\sigma} P_\rho^{(l)} Q_\sigma^{(n-l-1)} + P^{(l)} Q^{(n-l)})\right) = 0 \tag{4}$$

presents equations of motion for gravitational field $g_{\mu\nu}$ in the vacuum. In (4) $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$ is the Einstein tensor,

$$K_{\mu\nu} = \nabla_\mu \nabla_\nu - g_{\mu\nu}\Box, \qquad V = P_R \mathscr{F}(\Box) Q + Q_R \mathscr{F}(\Box) P,$$

where the subscript R indicates the derivative w.r.t. R (as many times as it is repeated) and

$$P^{(l)} = \Box^l P, \;\; P_\rho^{(l)} = \partial_\rho \Box^l P \text{ with the same for } Q, \;\; P_R, \;\; \dots$$

In the case of gravity with matter, the full equations of motion are $\mathscr{G}_{\mu\nu} = T_{\mu\nu}$, where $T_{\mu\nu}$ is the energy-momentum tensor. Thanks to the integration by parts there is always the symmetry of an exchange $P \leftrightarrow Q$.

When $\lambda = 0$ in (4) we recognize the Einstein's GR equation with the cosmological constant Λ. If $f_n = 0$ for $n \geq 1$ then (4) corresponds to equations of motion of an $f(R)$ theory.

4 Cosmological Equations for FLRW Metric

We use the FLRW metric

$$ds^2 = -dt^2 + a^2(t)\left(\frac{dr^2}{1 - kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2\right)$$

and look for some cosmological solutions. In the FLRW metric the Ricci scalar curvature is

$$R = 6\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2}\right)$$

and

$$\Box = -\partial_t^2 - 3H\partial_t,$$

where $H = \frac{\dot{a}}{a}$ is the Hubble parameter. We use natural system of units in which speed of light $c = 1$.

Due to symmetries of the FLRW spacetime, in (4) there are only two linearly independent equations. They are: trace and 00, i.e. when indices $\mu = \nu = 0$.

The trace equation and 00-equation, respectively, are

$$M_P^2 R - 4\Lambda + 2\lambda P \mathscr{F}(\Box) Q - \lambda(R + 3\Box)V$$
$$-\lambda \sum_{n=1}^{\infty} f_n \sum_{l=0}^{n-1} \left(g^{\rho\sigma} \partial_\rho \Box^l P \, \partial_\sigma \Box^{n-l-1} Q + 2\Box^l P \, \Box^{n-l} Q\right) = 0, \tag{5}$$

$$M_p^2 G_{00} - \Lambda + \frac{\lambda}{2} P \mathscr{F}(\Box) Q + \lambda(R_{00} - \nabla_0 \nabla_0 - \Box)V - \frac{\lambda}{2} \sum_{n=1}^{\infty} f_n$$
$$\times \sum_{l=1}^{n-1} \left(2\partial_0 \Box^l P \, \partial_0 \Box^{n-l-1} Q + g^{\rho\sigma} \partial_\rho \Box^l P \, \partial_\sigma \Box^{n-l-1} Q + \Box^l P \, \Box^{n-l} Q\right) = 0. \tag{6}$$

5 Cosmological Solutions for Constant Scalar Curvature *R*

When R is a constant then P and Q are also some constants and we have that $\Box R = 0$, $\mathscr{F}(\Box) = f_0$. The corresponding equations of motion (5) and (6) contain solutions as in the local case. However, metric perturbations at the background $R = const.$ can give nontrivial cosmic structure due to nonlocality.

Let $R = R_0 = constant \neq 0$. Then

$$6\Big(\frac{\ddot{a}}{a} + \big(\frac{\dot{a}}{a}\big)^2 + \frac{k}{a^2}\Big) = R_0. \tag{7}$$

The change of variable $b(t) = a^2(t)$ transforms (7) into equation

$$3\ddot{b} - R_0 b = -6k. \tag{8}$$

Depending on the sign of R_0, the following solutions of Eq. (8) are

$$b(t) = \frac{6k}{R_0} + \sigma e^{\sqrt{\frac{R_0}{3}}\, t} + \tau e^{-\sqrt{\frac{R_0}{3}}\, t}, \quad R_0 > 0,$$
$$b(t) = \frac{6k}{R_0} + \sigma \cos\sqrt{\frac{-R_0}{3}}\, t + \tau \sin\sqrt{\frac{-R_0}{3}}\, t, \quad R_0 < 0, \tag{9}$$

where σ and τ are some constant coefficients.

Substitution $R = R_0$ into equations of motion (5) and (6) yields, respectively,

$$M_p^2 R_0 - 4\Lambda + 2\lambda P f_0 Q - \lambda R_0 V_0 = 0, \tag{10}$$

$$M_p^2 G_{00} - \Lambda + \frac{\lambda}{2} P f_0 Q + \lambda R_{00} V_0 = 0, \tag{11}$$

where $V_0 = f_0(P_R Q + Q_R P)|_{R=R_0}$ and $G_{00} = R_{00} + \frac{R_0}{2}$.

Combining Eqs. (10) and (11) one obtains

$$M_p^2 R_0 - 4\Lambda + 2\lambda P f_0 Q - \lambda R_0 V_0 = 0, \tag{12}$$

$$4R_{00} + R_0 = 0. \tag{13}$$

Equation (12) connects some parameters of the nonlocal model (1) in the algebraic form with respect to R_0, while (13) implies a condition on the parameters σ, τ, k and R_0 in solutions (9). Namely, R_{00} is related to function $b(t)$ as

$$R_{00} = -\frac{3\ddot{a}}{a} = \frac{3}{4}\frac{(\dot{b})^2 - 2b\ddot{b}}{b^2}. \tag{14}$$

Replacing R_{00} in (13) by (14) and using different solutions for $b(t)$ in (9) we obtain

$$\begin{aligned} 9k^2 &= R_0^2\sigma\tau, \qquad R_0 > 0, \\ 36k^2 &= R_0^2(\sigma^2 + \tau^2), \qquad R_0 < 0. \end{aligned} \tag{15}$$

5.1 Case: $R_0 > 0$

- Let $k = 0$. From $9k^2 = R_0^2\sigma\tau$ follows that at least one of σ and τ has to be zero. Thus there is possibility for an exponential solution for $a(t)$ and $a(t) = 0$. Taking $\tau = 0$ and $\sigma = a_0^2$ one has

$$b(t) = a_0^2\, e^{\sqrt{\frac{R_0}{3}}\, t}. \tag{16}$$

- If $k = +1$ one can find φ such that $\sigma + \tau = \frac{6}{R_0}\cosh\varphi$ and $\sigma - \tau = \frac{6}{R_0}\sinh\varphi$. Moreover, we obtain

$$\begin{aligned} b(t) &= \frac{12}{R_0}\cosh^2\frac{1}{2}\left(\sqrt{\frac{R_0}{3}}t + \varphi\right), \\ a(t) &= \sqrt{\frac{12}{R_0}}\cosh\frac{1}{2}\left(\sqrt{\frac{R_0}{3}}t + \varphi\right). \end{aligned} \tag{17}$$

- If $k=-1$ one can transform $b(t)$ and $a(t)$ to

$$\begin{aligned} b(t) &= \tfrac{12}{R_0}\sinh^2\tfrac{1}{2}\left(\sqrt{\tfrac{R_0}{3}}t+\varphi\right), \\ a(t) &= \sqrt{\tfrac{12}{R_0}}\left|\sinh\tfrac{1}{2}\left(\sqrt{\tfrac{R_0}{3}}t+\varphi\right)\right|. \end{aligned} \tag{18}$$

5.1.1 Case: $R = 12\gamma^2$

This is a special case of R_0, which simplifies the above expressions and yields de Sitter-like cosmological solutions.

- $k=0$:

$$b(t) = a_0^2\, e^{2\,\gamma\, t}, \quad a(t) = a_0\, e^{\gamma\, t}. \tag{19}$$

- $k=+1$:

$$\begin{aligned} b(t) &= \frac{1}{\gamma^2}\cosh^2\left(\gamma\, t+\frac{\varphi}{2}\right), \\ a(t) &= \frac{1}{|\gamma|}\cosh\left(\gamma\, t+\frac{\varphi}{2}\right). \end{aligned} \tag{20}$$

- $k=-1$:

$$\begin{aligned} b(t) &= \frac{1}{\gamma^2}\sinh^2\left(\gamma\, t+\frac{\varphi}{2}\right), \\ a(t) &= \frac{1}{|\gamma|}\left|\sinh\left(\gamma\, t+\frac{\varphi}{2}\right)\right|. \end{aligned} \tag{21}$$

5.2 *Case:* $R_0 < 0$

- When $k=0$ then $\sigma=\tau=0$, and consequently $b(t)=0$.
- If $k=-1$ one can define φ by $\sigma=\frac{-6}{R_0}\cos\varphi$ and $\tau=\frac{-6}{R_0}\sin\varphi$, and rewrite $b(t)$ and $a(t)$ as

$$\begin{aligned} b(t) &= \frac{-12}{R_0}\cos^2\frac{1}{2}\left(\sqrt{-\frac{R_0}{3}}t-\varphi\right), \\ a(t) &= \sqrt{\frac{-12}{R_0}}\left|\cos\frac{1}{2}(\sqrt{-\frac{R_0}{3}}t-\varphi)\right|. \end{aligned} \tag{22}$$

- In the last case $k = +1$, by the same procedure as for $k = -1$, one can transform $b(t)$ to expression

$$b(t) = \frac{12}{R_0} \sin^2 \frac{1}{2}\Big(\sqrt{-\frac{R_0}{3}}t - \varphi\Big), \tag{23}$$

  which is not positive and hence yields no solution.

5.3 Case: $R_0 = 0$

The case $R_0 = 0$ can be considered as limit of $R_0 \to 0$ in both cases $R_0 > 0$ and $R_0 < 0$. When $R_0 > 0$ there is condition $9k^2 = R_0^2\sigma\tau$ in (15). From this condition, $R_0 \to 0$ implies $k = 0$ and arbitrary values of constants σ and τ. The same conclusion obtains when $R_0 < 0$ with condition $36k^2 = R_0^2(\sigma^2 + \tau^2)$. In both these cases there is Minkowski solution with $b(t) = constant > 0$ and consequently $a(t) = constant > 0$, see (9).

6 Some Models and Related Ansätze for Cosmological Solutions

6.1 Nonlocal Gravity Model Quadratic in R

Nonlocal gravity model which is quadratic in R was given by the action [11, 12]

$$S = \int d^4x\sqrt{-g}\Big(\frac{R - 2\Lambda}{16\pi G} + R\mathscr{F}(\Box)R\Big). \tag{24}$$

This model is important because it is ghost free and has some nonsingular bounce solutions, which can be regarded as a solution of the Big Bang cosmological singularity problem.

The corresponding equations of motion can be easily obtained from (5) and (6). To evaluate related equations of motion, the following Ansätze were used:

- Linear Ansatz: $\Box R = rR + s$, where r and s are constants.
- Quadratic Ansatz: $\Box R = qR^2$, where q is a constant.
- Qubic Ansatz: $\Box R = CR^3$, where C is a constant.
- Ansatz $\Box^n R = c_n R^{n+1}$, $n \geq 1$, where c_n are constants.

These Ansätze make some constraints on possible solutions, but simplify formalism to find a particular solution (see [29] and references therein).

6.1.1 Linear Ansatz and Nonsingular Bounce Cosmological Solutions

Using Ansatz $\Box R = rR + s$ a few nonsingular bounce solutions for the scale factor are found: $a(t) = a_0 \cosh\Big(\sqrt{\frac{\Lambda}{3}}t\Big)$ (see [11, 12]), $a(t) = a_0 e^{\frac{1}{2}\sqrt{\frac{\Lambda}{3}}t^2}$ (see [48, 49]) and $a(t) = a_0(\sigma e^{\lambda t} + \tau e^{-\lambda t})$ [30]. The first two consequences of this Ansatz are

$$\Box^n R = r^n(R + \frac{s}{r}), \; n \geq 1, \qquad \mathscr{F}(\Box)R = \mathscr{F}(r)R + \frac{s}{r}(\mathscr{F}(r) - f_0), \tag{25}$$

which considerably simplify nonlocal term.

Generalization of the above quadratic model in the form of nonlocal term $R^p \mathscr{F}(\Box)R^q$, where p and q are some natural numbers, was recently considered in [28]. Here cosmological solution for the scale factor has the form $a(t) = a_o\, e^{-\gamma\, t^2}$.

6.2 *Gravity Model with Nonlocal Term $R^{-1}\mathscr{F}(\Box)R$*

This model was introduced in [31] and its action may be written in the form

$$S = \int d^4x \sqrt{-g}\Big(\frac{R}{16\pi G} + R^{-1}\mathscr{F}(\Box)R\Big), \tag{26}$$

where $\mathscr{F}(\Box) = \sum_{n=0}^{\infty} f_n \Box^n$ and $f_0 = -\frac{\Lambda}{8\pi G}$ plays role of the cosmological constant.

The nonlocal term $R^{-1}\mathscr{F}(\Box)R$ in (26) is invariant under transformation $R \to CR$. This nonlocal term does not depend on the magnitude of scalar curvature R, but on its spacetime dependence, and in the FLRW case is relevant only dependence of R on time t. Term $f_0 = -\frac{\Lambda}{8\pi G}$ is completely determined by the cosmological constant Λ, which according to ΛCDM model is small and positive energy density of the vacuum. Coefficients $f_i,\; i \in \mathbb{N}$ can be estimated from other conditions, including agreement with dynamics the Solar system. In comparison to the model quadratic in R (24), complete Lagrangian of this model remains to be linear in R and in such sense is simpler nonlocal modification than (24).

In this model are also used the above Ansätze. Especially quadratic Ansatz $\Box R = qR^2$, where q is a constant, is effective to consider power-law cosmological solutions, see [31–33, 37].

6.3 *Some New Models and Ansätze*

It is worth to consider some particular examples of action (1) when $P = Q = (R + R_0)^m$, i.e.

$$S = \int \Big(\frac{1}{16\pi G} R - \Lambda + \frac{\lambda}{2}(R+R_0)^m \mathscr{F}(\Box)(R+R_0)^m\Big)\sqrt{-g}\, d^4x, \tag{27}$$

where $R_0 \in \mathbb{R}$, $m \in \mathbb{Q}$, and which have scale factor solution as

$$a(t) = A\, t^n e^{\gamma t^2}, \quad \gamma \in \mathbb{R}. \tag{28}$$

To this end we consider the Ansatz

$$\Box(R+R_0)^m = p(R+R_0)^m, \tag{29}$$

where p is a constant and $\Box$ is the d'Alembert operator in FLRW metric.

From Ansatz (29) and scalar curvature R for $k = 0$, we get the following system of equations:

$$\begin{aligned}
&72m(1+2m-3n)n^2(-1+2n)^2 = 0,\\
&36n(-1+2n)(-np+2n^2p+mR_0-mnR_0+12m\gamma+48mn\gamma-72mn^2\gamma) = 0,\\
&12n(-1+2n)(pR_0+12p\gamma+48np\gamma-6mR_0\gamma+312m\gamma^2-192m^2\gamma^2-288mn\gamma^2) = 0,\\
&pR_0^2+24pR_0\gamma+96npR_0\gamma+144p\gamma^2+576np\gamma^2+3456n^2p\gamma^2+96mR_0\gamma^2+\\
&+288mnR_0\gamma^2+1152m\gamma^3+8064mn\gamma^3+13824mn^2\gamma^3 = 0,\\
&96\gamma^2(pR_0+12p\gamma+48np\gamma+6mR_0\gamma+24m\gamma^2+96m^2\gamma^2+432mn\gamma^2) = 0,\\
&2304\gamma^4(p+12m\gamma) = 0.
\end{aligned} \tag{30}$$

The system of Eq. (30) has 5 solutions:

1. $p = -12m\gamma,\ n = 0,\ R_0 = -12\gamma,\ m = \frac{1}{2}$
2. $p = -12m\gamma,\ n = \frac{2m+1}{3},\ R_0 = -28\gamma,\ m = \frac{1}{2}$
3. $p = -12m\gamma,\ n = 0,\ R_0 = -4\gamma,\ m = 1$
4. $p = -12m\gamma,\ n = \frac{1}{2},\ R_0 = -16\gamma,\ m = 1$
5. $p = -12m\gamma,\ n = \frac{1}{2},\ R_0 = -36\gamma,\ m = -\frac{1}{4}$

We shall now shortly consider each of the above cases.

6.3.1 Case 1: $a(t) = A\, e^{\gamma T^2}$, $m = \frac{1}{2}$

Here Ansatz is $\Box\sqrt{R+R_0} = p\sqrt{R+R_0}$, where $R_0 = -12\gamma$, $p = -6\gamma$ and γ is a parameter. The scale factor is $a(t) = A\, e^{\gamma t^2}$.

The first consequences of this Ansatz are

$$\begin{aligned}
\Box^\ell\sqrt{R+R_0} &= p^\ell\sqrt{R+R_0}, \quad \ell \geq 0,\\
\mathscr{F}(\Box)\sqrt{R+R_0} &= \mathscr{F}(p)\sqrt{R+R_0},\\
R(t) &= 12\gamma(1+4\gamma t^2).
\end{aligned}$$

Relevant action is

$$S = \int \Big(\frac{1}{16\pi G} R - \Lambda + \frac{\lambda}{2} \sqrt{R - 12\gamma}\, \mathscr{F}(\Box) \sqrt{R - 12\gamma} \Big) \sqrt{-g}\; d^4x. \tag{31}$$

Equations of motion follow from (5) and (6), where $P = Q = \sqrt{R - 12\gamma}$. Straightforward calculation gives cosmological solution $a(t) = A\, e^{\gamma\, t^2}$ with conditions:

$$\mathscr{F}(p) = \frac{\gamma - 4\pi G\Lambda}{16\gamma\pi G\lambda}, \quad \mathscr{F}'(p) = \frac{4\pi G\Lambda - 3\gamma}{192\gamma^2\pi G\lambda}, \quad p = -6\gamma.$$

6.3.2 Case 2: $a(t) = A\, t^{2/3} e^{\gamma\, t^2}$, $m = \frac{1}{2}$

In this case the Ansatz is $\Box\sqrt{R + R_0} = p\sqrt{R + R_0}$, where R_0 and p are real constants.

The first consequences of this Ansatz are

$$\Box^\ell \sqrt{R + R_0} = p^\ell \sqrt{R + R_0}, \;\; \ell \geq 0,$$
$$\mathscr{F}(\Box)\sqrt{R + R_0} = \mathscr{F}(p)\sqrt{R + R_0}.$$

For scale factor $a(t) = A\, t^{2/3} e^{\gamma\, t^2}$ the Ansatz $\Box\sqrt{R + R_0} = p\sqrt{R + R_0}$ is satisfied if and only if $R_0 = -28\gamma$ and $p = -6\gamma$.

Direct calculation shows that

$$R(t) = 44\gamma + \frac{4}{3} t^{-2} + 48\gamma^2 t^2,$$
$$\Box^\ell \sqrt{R - 28\gamma} = (-6\gamma)^\ell \sqrt{R - 28\gamma}, \quad \ell \geq 0,$$
$$\mathscr{F}(\Box)\sqrt{R - 28\gamma} = \mathscr{F}(-6\gamma)\sqrt{R - 28\gamma},$$
$$\dot{R} = 96\gamma^2 t - \frac{8}{3} t^{-3}.$$

The related action is

$$S = \int \Big(\frac{1}{16\pi G} R - \Lambda + \frac{\lambda}{2} \sqrt{R - 28\gamma}\, \mathscr{F}(\Box) \sqrt{R - 28\gamma} \Big) \sqrt{-g}\; d^4x. \tag{32}$$

The corresponding trace and 00 equations of motion are satisfied under conditions:

$$\mathscr{F}(p) = -\frac{1}{8\pi G\lambda}, \quad \mathscr{F}'(p) = 0, \quad \gamma = \frac{4}{7}\pi G\Lambda, \quad p = -6\gamma.$$

6.3.3 Case 3: $a(t) = A\, e^{\gamma T^2}$, $m = 1$

In this case $\Box(R - 4\gamma) = -12\gamma(R - 4\gamma)$, what is an example of already above considered linear Ansatz. The corresponding action is

$$S = \int \Big(\frac{1}{16\pi G} R - \Lambda + \frac{\lambda}{2}(R - 4\gamma)\mathscr{F}(\Box)(R - 4\gamma)\Big)\sqrt{-g}\, d^4x. \tag{33}$$

Equations of motion have cosmological solution $a(t) = A\, e^{\gamma t^2}$ under conditions:

$$\mathscr{F}(p) = -\frac{1}{512\pi G\lambda\gamma}, \quad \mathscr{F}'(p) = 0, \quad p = -12\gamma, \quad \gamma = 8\pi G\Lambda.$$

6.3.4 Case 4: $a(t) = A\sqrt{t} e^{\gamma t^2}$, $m = 1$

This case is quite similar to the previous one. Now Ansatz is $\Box(R - 16\gamma) = -12\gamma(R - 16\gamma)$ and action

$$S = \int \Big(\frac{1}{16\pi G} R - \Lambda + \frac{\lambda}{2}(R - 16\gamma)\mathscr{F}(\Box)(R - 16\gamma)\Big)\sqrt{-g}\, d^4x. \tag{34}$$

Scale factor $a(t) = A\sqrt{t} e^{\gamma t^2}$ is solution of equations of motion if the following conditions are satisfied:

$$\mathscr{F}(p) = -\frac{1}{320\pi G\lambda\gamma}, \quad \mathscr{F}'(p) = 0, \quad p = -12\gamma, \quad \gamma = 8\pi G\Lambda.$$

6.3.5 Case 5: $a(t) = A\sqrt{t}\, e^{\gamma t^2}$, $m = -\frac{1}{4}$

According to the Ansatz, in this case $p = 3\gamma$, $n = \frac{1}{2}$, $R_0 = -36\gamma$. However the action

$$S = \int \Big(\frac{1}{16\pi G} R - \Lambda + \frac{\lambda}{2}\sqrt{R - 36\gamma}\,\mathscr{F}(\Box)\sqrt{R - 36\gamma}\Big)\sqrt{-g}\, d^4x. \tag{35}$$

has no solution $a(t) = A\sqrt{t}\, e^{\gamma t^2}$ for the Ansatz $\Box(R + R_0)^m = p(R + R_0)^m$, $m = -\frac{1}{4}$.

7 Concluding Remarks

In this paper we presented a brief review of nonlocal modified gravity, where nonlocality is realized by an analytic function of the d'Alembert operator $\Box$. Considered models are presented by actions, their equations of motion, related Ansätze and some cosmological solutions for the scale factor $a(t)$. A few new models are introduced, and they deserve to be further investigated, especially Case 1 and Case 2 in Sect. 6.

Many details on (1) and its extended versions can be found in [9, 10, 13, 49–51]. Perturbations and physical excitations of the equations of motion of action (24) around the de Sitter background are considered in [34, 35], respectively. As some recent developments in nonlocal modified gravity, see [21, 25, 41, 44, 53, 68].

Notice that nonlocal cosmology is related also to cosmological models in which matter sector contains nonlocality (see, e.g. [4, 6, 7, 19, 38, 39, 52]). String field theory and p-adic string theory models have played significant role in motivation and construction of such models. One particular aspect in which non-local models prove important is the ability to resolve the Null Energy Condition obstacle [5] common to many models of generalized gravity. In short, that is an ability to construct a healthy model which has sum of energy and pressure of the matter positive and thereby avoids ghosts in the spectrum alongside with a nonsingular space-time structure [23].

Nonsingular bounce cosmological solutions are very important (as reviews on bouncing cosmology, see e.g. [14, 62]) and their progress in nonlocal gravity may be a further step towards cosmology of the cyclic universe [54].

Acknowledgements Work on this paper was partially supported by Ministry of Education, Science and Technological Development of the Republic of Serbia, grant No 174012. B.D. thanks Prof. Vladimir Dobrev for invitation to participate and give a talk on nonlocal gravity, as well as for hospitality, at the XI International Workshop "Lie Theory and its Applications in Physics", 15–21 June 2015, Varna, Bulgaria. B.D. also thanks a support of the ICTP - SEENET-MTP project PRJ-09 "Cosmology and Strings" during preparation of this article. AK is supported by the FCT Portugal fellowship SFRH/BPD/105212/2014 and in part by FCT Portugal grant UID/MAT/00212/2013 and by RFBR grant 14-01-00707.

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Kinetics of Interface Growth: Physical Ageing and Dynamical Symmetries

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Abstract Dynamical symmetries and their Lie algebra representations, relevant for the non-equilibrium kinetics of growing interfaces are discussed. Physical consequences are illustrated in the ageing of the $1D$ Glauber-Ising and Arcetri models.

1 Introduction

Theories of the effective long-time and long-distance behaviour of strongly interacting many-body systems have raised a considerable amount of conceptual, computational and experimental challenges. Modern formulations are always almost cast in the framework of a renormalisation group, which usually allows to identify a small number of 'relevant' physical scaling operators. Much insight has been obtained through the study of paradigmatic systems, where the specific formulation of models often allow to formulate question in such a way that the predictions derived from general theoretical schemes can be brought to explicit tests, either through numerical simulations and occasionally exact solution and, under favourable circumstances, even through experiments [1, 10, 22, 39].

Here, we shall concentrate on the long-time and large-distance behaviour in the kinetics of growing interfaces. Interfaces are grown on a substrate, onto which particle are allowed to deposit, according to certain microscopic rules. The interface separates those particles which are already absorbed, from empty space, and is described in terms of a possibly time-dependent height variable $h_i(t)$, attached to each site i of the substrate. The set of all heights $h_i(t)$ at a given time t is an interface configuration $\{h\}$. In Fig. 1, one such adsorption event is illustrated.

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V. Dobrev (ed.), *Lie Theory and Its Applications in Physics*,
Springer Proceedings in Mathematics & Statistics 191,
DOI 10.1007/978-981-10-2636-2_4

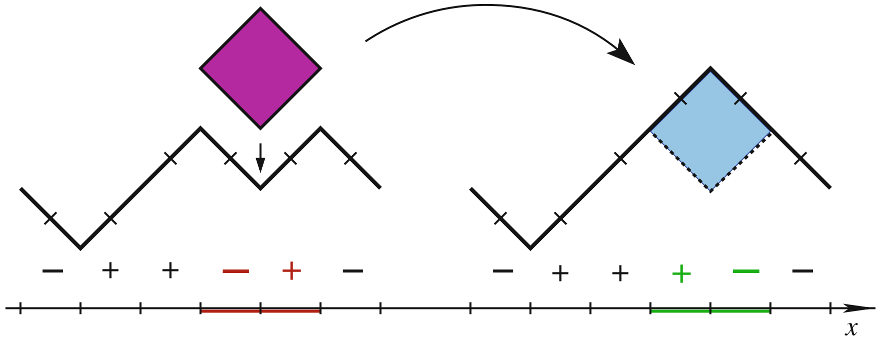

Fig. 1 Schematic evolution of an interface, described in terms of a time-dependent height configuration. Upon adsorption of a particle, the height configuration evolves locally. Below the heights, the local slope is also indicated. The adsorption process corresponds to a biased exchange reaction $-+ \longrightarrow +-$ of a TASEP between the slopes on two neighbouring links

In a coarse-grained description, most of the 'details' of the precise microscopic rules which govern an adsorption event will not enter into the long-time and large-distance behaviour, they are 'irrelevant' in the renormalisation-group sense. For example, the interface shown in Fig. 1 has the property that the height differences between nearest neighbours may only take the values $h_{i+1}(t) - h_i(t) = \pm 1$. If such an RSOS-condition is used to select admissible adsorption events, one can show that a coarse-grained description, in the continuum limit describes the height function $h = h(t, \mathbf{r})$ as a solution of the *Kardar-Parisi-Zhang (*KPZ*) equation* [27]

$$\partial_t h = \nu \nabla^2 h + \frac{\mu}{2} (\nabla h)^2 + \eta \tag{1}$$

where ∇ is the spatial gradient, η is a centred gaussian white noise, with co-variance

$$\left\langle \eta(t, \mathbf{r})\eta(t', \mathbf{r}')\right\rangle = 2\nu T \delta(t - t')\delta(\mathbf{r} - \mathbf{r}') \tag{2}$$

and where ν, μ, T are material-dependent parameters. On the other hand, if the RSOS-constraint is not imposed, the continuum equation obtained is (1) with $\mu = 0$, which is known as the *Edwards-Wilkinson (*EW*) equation* [13].

The exponents used to describe the interface are conventionally defined as follows. One is mainly interested in the fluctuations around the spatially averaged height[1] $\overline{h}(t) := L^{-d} \sum_{\mathbf{r}\in\Lambda} h(t, \mathbf{r})$, where the sum runs over the lattice sites. All analysis is built around the *Family-Viscek scaling* [14] of the interface width, where $\langle . \rangle$ denotes an average over many independent samples.

$$w^2(t; L) := \frac{1}{L^d} \sum_{\mathbf{r}\in\Lambda} \left\langle \left(h(t, \mathbf{r}) - \overline{h}(t)\right)\right\rangle^2 = L^{2\alpha} f_w\left(tL^{-z}\right) \sim \tag{3}$$
$$\sim \begin{cases} t^{2\beta} & ; \text{ if } tL^{-z} \ll 1 \\ L^{2\alpha} & ; \text{ if } tL^{-z} \gg 1 \end{cases}$$

[1]Throughout, we consider a hyper-cubic lattice $\Lambda \subset \mathbb{Z}^d$ with L^d sites.

Here, $\alpha = \beta z$ is the roughness exponent, β the growth exponent and $z > 0$ the dynamical exponent and $\langle . \rangle$ denotes an average over many independent samples black(under the same thermodynamic conditions). Physically, one says that the interface is *rough* when $\beta > 0$ and *smooth* if $\beta \leq 0$. Throughout, the $L \to \infty$ limit will be taken and the initial state is always a flat, uncorrelated substrate. Relaxational properties of the interface are characterised by the two-time correlations and (linear) responses

$$C(t,s;\mathbf{r}) := \left\langle \left(h(t,\mathbf{r}) - \left\langle \overline{h}(t)\right\rangle\right)\left(h(s,\mathbf{0}) - \left\langle \overline{h}(s)\right\rangle\right)\right\rangle = s^{-b} F_C\left(\frac{t}{s}; \frac{\mathbf{r}}{s^{1/z}}\right) \quad (4)$$

$$R(t,s;\mathbf{r}) := \left.\frac{\delta\left\langle h(t,\mathbf{r}) - \overline{h}(t)\right\rangle}{\delta j(s,\mathbf{0})}\right|_{j=0} = \left\langle h(t,\mathbf{r})\widetilde{h}(s,\mathbf{0})\right\rangle = $$
$$= s^{-1-a} F_R\left(\frac{t}{s}; \frac{\mathbf{r}}{s^{1/z}}\right) \quad (5)$$

where j is an external field conjugate to h and spatial translation-invariance is assumed. The long-time dynamical scaling is formulated by analogy with the ageing as it occurs in simple magnets [5, 10, 22]. Generalised Family-Viscek forms are expected in the long-time limit, for the waiting time s and the observation time t, where not only $t, s \gg \tau_{\text{micro}}$, but also $t - s \gg \tau_{\text{micro}}$ is required (τ_{micro} is a microscopic reference time). Some entries of a dictionary between the ageing of simple magnets and interface growth are listed in Table 1.

In (5), we quote a result from Janssen-de Dominicis theory expressing the response function as a correlator of the height scaling operator $h(t, \mathbf{r})$ with the associated

Table 1 Analogies between the critical dynamics in magnets and growing interfaces. The average $\langle . \rangle_c$ denotes a connected correlator. Some models, with the equilibrium hamiltonian for magnets, are defined through their kinetic equations

	Magnets	Interfaces
Order parameter/height	$\phi(t,\mathbf{r})$	$h(t,\mathbf{r})$
Width/variance	$\left\langle(\phi(t,\mathbf{r}) - \langle\phi(t,\mathbf{r})\rangle)^2\right\rangle \sim$ $\sim t^{-2\beta/(\nu z)}$	$w^2(t) = \langle(h(t,\mathbf{r}) - \overline{h}(t))^2\rangle \sim$ $\sim t^{2\beta}$
Autocorrelator	$C(t,s) = \langle\phi(t,\mathbf{r})\phi(s,\mathbf{r})\rangle_c$	$C(t,s) = \langle h(t,\mathbf{r})h(s,\mathbf{r})\rangle_c$
Autoresponse	$R(t,s) =$ $\delta\langle\phi(t,\mathbf{r})\rangle/\delta h(s,\mathbf{r})\vert_{h=0}$	$R(t,s) =$ $\delta\langle h(t,\mathbf{r})\rangle/\delta j(s,\mathbf{r})\vert_{j=0}$
Models		
Gaussian field/EW	$\mathcal{H}[\phi] = -\frac{1}{2}\int \mathrm{d}\mathbf{r}\,(\nabla\phi)^2$	
	$\partial_t\phi = D\nabla^2\phi + \eta$	$\partial_t h = \nu\nabla^2 h + \eta$
Ising model/KPZ	$\mathcal{H}[\phi] =$ $-\frac{1}{2}\int \mathrm{d}\mathbf{r}\,[(\nabla\phi)^2 + \frac{g}{2}\phi^4]$	
	$\partial_t\phi = D(\nabla^2\phi + g\phi^3) + \eta$	$\partial_t h = \nu\nabla^2 h + \frac{\mu}{2}(\nabla h)^2 + \eta$

Table 2 Exponents of the ageing of growing interfaces

Model	d	z	β	a	b	λ_C	λ_R	Ref.
KPZ	1	3/2	1/3	$-1/3$	$-2/3$	1	1	[23, 27, 29]
	2	1.61(2)	0.2415(15)	0.30(1)	$-0.483(3)$	1.97(3)	2.04(3)	[33]
	2	1.61(2)	0.241(1)		-0.483	1.91(6)		[18]
	2	1.627(4)	0.229(6)					[28]
EW	<2	2	$(2-d)/4$	$d/2-1$	$d/2-1$	d	d	
	2	2	0(log)	0	0	2	2	[13, 36]
	>2	2	0	$d/2-1$	$d/2-1$	d	d	
Arcetri **1**								
$T=T_c$	<2	2	$(2-d)/4$	$d/2-1$	$d/2-1$	$3d/2-1$	$3d/2-1$	
	2	2	0(log)	0	0	2	2	[24]
	>2	2	0	$d/2-1$	$d/2-1$	d	d	
$T<T_c$	d	2	1/2	$d/2-1$	-1	$d/2-1$	$d/2-1$	

response scaling operator $\widetilde{h}(s,\mathbf{r}')$. This will be needed for an analysis of the dynamical symmetries of $R(t,s)$ below.

Turning to the exponents defined in (4), (5), one notes that $b=-2\beta$ [11, 26], but the relationship of a to other exponents seems to depend on the universality class. For example, one finds $a=b$ in the EW-universality class [36], and $1+a=b+2/z$ in the $1D$ KPZ-class [23]. The exponents λ_C, λ_R of the autocorrelator and the autoresponse, respectively, are defined from the asymptotics $F_{C,R}(y,\mathbf{0})\sim y^{-\lambda_{C,R}/z}$ as $y\to\infty$. A rigorous bound states that $\lambda_C\geq(d+zb)/2$ [24].

Concerning the values of the exponents λ_C, λ_R, an important difference arises between simple magnets and growing interfaces, notably for those in the KPZ universality class. In simple magnets, with so-called with a non-conserved order parameter and with disordered initial conditions, renormalisation-group studies strongly indicate that λ_C, λ_R are independent of those describing the stationary state [5, 39]. In contrast, for the KPZ class, for dimensions $d<2$ it was shown that $\lambda_C=d$, to all orders in perturbation-theory [29], but this analysis breaks down for $d\geq 2$ [39]. This is because of a strong-coupling fixed point, not reachable by a perturbative analysis, and analysed through the non-perturbative renormalisation-group [28]. In Table 2, we list values of these exponents, either exact results or simulational estimates.

Remarkably, in recent years several new experiments on interface growth have been carried out, which furnish several non-trivial examples in the $1D$ KPZ universality class. For a list of the measured values of the exponents, see [24].

This work is organised as follows. In Sect. 2, we shall define the recently introduced exactly solvable 'Arcetri models'. In Sect. 3, we recall some elements of the theory of local scale-invariance (LSI), which in particular permits to predict the shape of the scaling functions defined above. It has turned out that the usual way of extending global scale-invariance to a more local scaling, as so successfully used in the study of conformal invariance at equilibrium phase transitions, is not always flexible enough to take into account what is going on far from a stationary state. Technically, this requires to consider more general representations. In Sect. 4, we present such extensions, first for the conformal algebra and then give the extensions also for LSI. Applications to the first Arcetri model and the $1D$ Glauber-Ising model will be given. Section 5 gives our conclusions.

2 The Arcetri Models: Exact Solution

Important recent work on the exact solution of the $1D$ KPZ-equation relates the probability distribution $\mathcal{P}(h)$ of the fluctuation $h - \overline{h}$ with the extremal value statistics of the largest eigenvalue of random matrices, see [6, 37]. Here, we rather look for different universality classes with exactly solvable members, not as straightforward to treat as the EW-equation but still not confined to $d = 1$ dimensions. Inspiration comes from the well-studied *spherical model* of a ferromagnet [3, 30]. Therein, the traditional Ising spin variables $\sigma_i = \pm 1$, attached to the sites i of a lattice with $\mathcal{N}$ sites, are replaced by 'spherical spins' $S_i \in \mathbb{R}$ and subject to the constraint $\sum_i \langle S_i^2 \rangle = \mathcal{N}$. A conventional nearest-neighbour interaction leads to an exactly solvable model, which undergoes a non-mean-field phase transition in $2 < d < 4$ dimensions [3, 30]. The relaxational properties can be likewise analysed exactly [35].

How can one find an useful analogy with growing interfaces? Considering Fig. 1, we see that the slopes $h_{i+1}(t) - h_i(t) = \pm 1$ might be viewed as analogues of Ising spins. Then at least in $d = 1$ dimensions, from the KPZ-equation one has for the local slope $u = \nabla h$ the (noisy) Burgers equation. A 'spherical model variant' of the KPZ-universality class might be found by relaxing the RSOS-constraints $u_i = \pm 1$ to a 'spherical constraint' $\sum_i u_i^2 = \mathcal{N}$ [24]. More precisely, this leads to the variants:

1. Start from the Burgers equation and replace its non-linearity as follows

$$\partial_t u = \nu \nabla^2 u + \mu u \nabla u + \nabla \eta \;\; \mapsto \;\; \partial_t u = \nu \nabla^2 u + \mathfrak{z}(t) u + \nabla \eta \tag{6}$$

with a Lagrange multiplier $\mathfrak{z}(t)$. Its value is determined by the spherical constraint $\sum \langle u^2 \rangle = \mathcal{N}$, where the sums runs over all sites of the lattice [24]. The variance (2) of the gaussian white noise $\eta(t, \mathbf{r})$ defines the 'temperature' T.

2. Treat the non-linearity of the Burgers equation as follows

$$\partial_t u = \nu \nabla^2 u + \mu u \nabla u + \nabla \eta \;\; \mapsto \;\; \partial_t u = \nu \nabla^2 u + \mathfrak{z}(t) \nabla u + \nabla \eta \tag{7}$$

and find the Lagrange multiplier $\mathfrak{z}(t)$ from the constraint $\sum \langle u^2 \rangle = \mathcal{N}$ [12].

3. Finally, start directly from the KPZ equation, and replace

$$\partial_t h = \nu \nabla^2 h + \frac{1}{2}\mu \left(\nabla h\right)^2 + \eta \;\; \mapsto \;\; \partial_t h = \nu \nabla^2 h + \mathfrak{z}(t) \nabla h + \eta \tag{8}$$

where $\mathfrak{z}(t)$ is to be found from $\sum \langle (\nabla h)^2 \rangle = \mathcal{N}$ [12].

Equations (6)–(8) would define the first, second and third *Arcetri models*,[2] respectively. However, it turns out that Eqs. (7) and (8) lead to undesirable properties of the height and slope profiles in the stationary state, as well as to internal inconsistencies [12]. Therefore, a more careful definition is required.

In one spatial dimension, the slope profile $u(t,r) = 1 - 2\varrho(t,r)$ has an interesting relationship with the dynamics of interacting particles, of density $\varrho(t,r)$. In the KPZ universality class, $u(t,r) = \pm 1$ from the RSOS-constraint. Then denote by $\bullet$ an occupied site with $\varrho = 1 \Leftrightarrow u = +1$ and by $\circ$ an empty site with $\varrho = 0 \Leftrightarrow u = -1$. The interface growth process leads to the only admissible reaction $\bullet\circ \longrightarrow \circ\bullet$, between neighbouring sites, see Fig. 1. This is a *totally asymmetric exclusion process* (TASEP), see [11, 17, 31]. For the Arcetri model(s), the exact RSOS-constraint is relaxed to the mean 'spherical constraint' $\left\langle \sum_r u(t,r)^2 \right\rangle \stackrel{!}{=} \mathcal{N}$. Hence, the noise-averaged,[3] and spatially averaged, particle-density $\bar{\rho}(t)$ becomes [12]

$$\bar{\rho}(t) := \frac{1}{\mathcal{N}} \sum_r \langle \varrho(t,r) \rangle \stackrel{!}{=} \frac{1}{\mathcal{N}} \sum_r \left\langle \varrho(t,r)^2 \right\rangle \geq 0 \tag{9}$$

where the equality follows from the constraint. Notably, the non-averaged density variable $\varrho(t,r) \in \mathbb{R}$ has no physical meaning, but the constraint (9) ensures that the measurable disorder-averaged observables take physically reasonable values.

2.1 First Arcetri Model

On a hypercubic lattice of $\mathcal{N} = N^d$ sites, in Fourier space the slopes $\widehat{u}_a(t,\mathbf{p}) = \mathrm{i} \sin\left(\frac{2\pi}{N}\right) \widehat{h}(t,\mathbf{p})$ are related to the heights, hence the disordered, uncorrelated initial state is specified by $\left\langle \widehat{h}(0,\mathbf{p}) \right\rangle = N^d H_0 \delta_{\mathbf{p},\mathbf{0}}$ and $\left\langle \widehat{h}(0,\mathbf{p}) \widehat{h}(0,\mathbf{q}) \right\rangle = N^d H_1 \delta_{\mathbf{p}+\mathbf{q},\mathbf{0}}$, with $H_1 = H_0^2$. From (6) and using the definition

$$g(t) := \exp\left(-2 \int_0^t \mathrm{d}t' \, \mathfrak{z}(t') \right) \tag{10}$$

the spherical constraint can be cast into the form of a Volterra integral equation

$$H_1 f(t) + 2\nu T \int_0^t \mathrm{d}\tau \, g(\tau) f(t-\tau) = d g(t) \tag{11}$$

with the kernel $f(t) = \frac{d}{4\nu t} e^{-4\nu t} I_1(4\nu t) \left(I_0(4\nu t) \right)^{d-1}$ and the I_n are modified Bessel functions. This is readily solved in terms of Laplace transformations, viz. $\overline{g}(p) =$

[2]The name comes from the location of the Galileo Galilei Institute of Physics, near to the village of Arcetri (Florence, Italy), where these models were invented in spring 2014.

[3]Here, the average is both over 'thermal' as well as over 'initial' noise.

$H_1\overline{f}(p)\left[d-2T\overline{f}(p)\right]^{-1}$. The location of the singularity defines the '*critical temperature*' [24]

$$\frac{1}{T_c(d)} = \frac{2}{d}\overline{f}(0) = \int_0^\infty \mathrm{d}t\, \frac{\exp(-dt)}{2t}\, I_1(t) I_0(t)^{d-1} = \begin{cases} 2 & ;\ \text{if } d=1 \\ 2\pi/(\pi-2) & ;\ \text{if } d=2 \\ 9.53099\ldots & ;\ \text{if } d=3 \end{cases} \tag{12}$$

such that $T_c(d) > 0$ for all $d > 0$. Interesting long-time scaling behaviour is found whenever $T \le T_c(d)$. In particular, one has the long-time asymptotic behaviour $g(t) \sim t^{F}$, where

$$F = \begin{cases} d/2-1 & ;\ \text{if } T = T_c(d) \text{ and } 0<d<2 \\ 0 & ;\ \text{if } T = T_c(d) \text{ and } 2<d \\ -(d/2+1) & ;\ \text{if } T < T_c(d) \end{cases} \tag{13}$$

This implies that for $t \to \infty$, $\mathfrak{z}(t) \simeq -\frac{F}{2t}$. Height correlators and responses (4) and (5) read, where $F_{\mathbf{r}}(\tau) := \prod_{a=1}^{d} e^{2\nu\tau} I_{r_a}(2\nu\tau)$ and the Heaviside function $\Theta(\tau)$

$$C(t,s;\mathbf{r}) = \frac{H_1}{\sqrt{g(t)g(s)}} F_{\mathbf{r}}(t+s) + \frac{2\nu T}{\sqrt{g(t)g(s)}} \int_0^s \mathrm{d}\tau\, g(\tau) F_{\mathbf{r}}(t+s-2\tau) \tag{14}$$

$$R(t,s;\mathbf{r}) = \Theta(t-s)\sqrt{\frac{g(s)}{g(t)}}\, F_{\mathbf{r}}(t-s) \tag{15}$$

Straightforward calculation verify the non-equilibrium dynamical scaling of simple ageing, as expressed in (4) and (5), with the exponents listed in Table 2 [24].

Considering the correlators and responses of the slope $\mathbf{u}(t,\mathbf{r})$, it can be shown that in these variables the first Arcetri model is identical to (i) the $p=2$ spherical spin glass [9] with $T = 2T_{\mathrm{SG}}$ and (ii) the statistics of the gap of the largest eigenvalues of gaussian unitary matrices [15].

2.2 Second and Third Arcetri Models

Since the equations of motion (7) and (8) do not conserve parity, it is preferable to separate into an even part $a(t,r) = a(t,-r)$ and an odd part $b(t,r) = -b(t,-r)$. Formally, $u(t,r) := a(t,r) + \mathrm{i}b(t,r)$ obeys Eq. (7) of the second Arcetri model and $h(t,r) := a(t,r) + \mathrm{i}b(t,r)$ obeys Eq. (8) of the third Arcetri model. In both cases, the Lagrange multiplier $\mathfrak{z}(t) \in \mathrm{i}\mathbb{R}$ [12]. The spherical constraint is quite distinct from (11). For instance, in the second model with initially uncorrelated slopes, we find [12]

$$\mathcal{J}_0(4\nu t, 2Z(t)) + 2\nu T \int_0^t \mathrm{d}\tau\, e^{4\nu\tau)} \mathcal{J}_2(4\nu(t-\tau), 2Z(t) - 2Z(\tau)) = e^{4\nu t} \tag{16}$$

where $Z(t) := \int_0^t \mathrm{d}\tau\,\mathfrak{z}(\tau)$, $\mathcal{J}_0(A,Z) = I_0(\sqrt{A^2+Z^2}\,)$ and $\mathcal{J}_2(A,Z) = \partial_Z^2 \mathcal{J}_0(A,Z)$. For temperature $T=0$, we find the long-time asymptotics $Z(t) \simeq \sqrt{t \ln(4\pi\nu t)}$. [Analogous results hold in the third model.] The logarithmic factor in $Z(t)$ leads to a breaking of dynamical scaling. For example, the equal-time slope correlator

$$\begin{aligned} C_n(t) &= \langle a(t,n)a(t,0) + b(t,n)b(t,0)\rangle \;\simeq \\ &\simeq\; \exp\left[-\left(\frac{n}{\sqrt{32\nu t}}\right)^2\right]\cos\left[\frac{n}{\sqrt{2t/\ln 4\nu\pi t}}\right] \end{aligned} \tag{17}$$

displays *two* marginally different length scales. The two-time slope autocorrelator

$$C(t,s) \simeq e^{-y^2/32} \;\; ; \;\; \text{with } y := \tfrac{t-s}{s}\sqrt{\ln 4\pi\nu s} \;\; \text{fixed} \tag{18}$$

shows logarithmic *sub-ageing*, distinct from simple ageing (4), but known from the kinetic $T=0$ spherical model with a conserved order-parameter ('model B') [4, 8].

3 Dynamical Scaling Far from Equilibrium and Symmetries

In order to prepare the discussion of dynamical symmetries of the Arcetri models in Sect. 4, we now recall several known results on the dynamical symmetries of non-equilibrium systems. Much of this discussion is based on analogies with conformal invariance at $2D$ equilibrium critical points. Working in complex coordinates $z = x + \mathrm{i}y$, the basic representation of the conformal algebra generators is $\ell_n - z^{n+1}\partial_z - \Delta(n+1)z^n$ [7] with the *conformal weight* $\Delta \in \mathbb{R}$, which obey the commutator $[\ell_n, \ell_m] = (n-m)\ell_{n+m}$ for $n, m \in \mathbb{Z}$. Writing the Laplace operator $\mathcal{S} := 4\partial_z\partial_{\bar{z}}$, and provided $\Delta = 0$, the commutator

$$[\mathcal{S}, \ell_n]\,\phi(z,\bar{z}) = -(n+1)z^n\mathcal{S}\phi(z,\bar{z}) - 4\Delta(n+1)nz^{n-1}\partial_{\bar{z}}\phi(z,\bar{z}) \tag{19}$$

expresses the conformal invariance of the space of solutions of $\mathcal{S}\phi = 0$.

An analogue for dynamical scaling, with dynamical exponent $z=2$, is the *Schrödinger-Virasoro algebra* $\mathfrak{sv}(d)$ algebra [19, 20], with generators

$$\begin{aligned} X_n &= -t^{n+1}\partial_t - \frac{n+1}{2}t^n\mathbf{r}\cdot\nabla_{\mathbf{r}} - \frac{\mathcal{M}}{4}(n+1)nt^{n-1}\mathbf{r}^2 - \frac{n+1}{2}xt^n \\ Y_m^{(j)} &= -t^{m+1/2}\partial_j - \left(m+\frac{1}{2}\right)t^{m-1/2}\mathcal{M}r_j \\ M_n &= -t^n\mathcal{M} \quad ; \qquad R_n^{(jk)} = -t^n\left(r_j\partial_k - r_k\partial_j\right) \;=\; -R_n^{(kj)} \end{aligned} \tag{20}$$

where $\partial_j := \partial/\partial r_j$ and $\nabla_{\mathbf{r}} = (\partial_1, \ldots, \partial_d)^{\mathrm{T}}$. The non-vanishing commutators are

$$
\begin{aligned}
\left[X_n, X_{n'}\right] &= (n-n')X_{n+n'} &, \quad \left[X_n, Y_m^{(j)}\right] &= \left(\frac{n}{2}-m\right) Y_{n+m}^{(j)} \\
\left[X_n, M_{n'}\right] &= -n' M_{n+n'} &, \quad \left[X_n, R_{n'}^{(jk)}\right] &= -n' R_{n+n'}^{(jk)} \\
\left[Y_m^{(j)}, Y_{m'}^{(k)}\right] &= \delta^{j,k} \left(m-m'\right) M_{m+m'} \quad , && \\
\left[R_n^{(jk)}, Y_m^{(\ell)}\right] &= \delta^{j,\ell}\, Y_{n+m}^{(k)} - \delta^{k,\ell}\, Y_{n+m}^{(j)} && \\
\left[R_n^{(jk)}, R_{n'}^{(\ell i)}\right] &= \delta^{j,i} R_{n+n'}^{(\ell k)} - \delta^{k,\ell} R_{n+n'}^{(ji)} + \delta^{k,i} R_{n+n'}^{(j\ell)} - \delta^{j,\ell} R_{n+n'}^{(ik)} &&
\end{aligned} \tag{21}
$$

with integer indices $n, n' \in \mathbb{Z}$, half-integer indices $m, m' \in \mathbb{Z} + \frac{1}{2}$ and $i, j, k, \ell \in \{1, \ldots, d\}$. It was already known to Jacobi and Lie that the maximal finite-dimensional sub-algebra of $\mathfrak{sv}(d)$, namely the thoroughly-analysed *Schrödinger algebra* $\mathfrak{sch}(d) := \left\langle X_{0,\pm1}, Y_{\pm1/2}^{(j)}, M_0, R_0^{(jk)} \right\rangle_{j,k=1,\ldots d}$, leaves the solution space of free-particle motion or of the free diffusion equation invariant. For our purposes, we want to use symmetries as $\mathfrak{sch}(d)$ to derive Ward identities for co-variant n-point functions $\langle \phi_1(t_1, \mathbf{r}_1) \ldots \phi_n(t_n, \mathbf{r}_n) \rangle$. Since the generator $M_0 \in \mathfrak{sch}(d)$ is central, the *Bargman superselection rule* [2]

$$
(\mathcal{M}_1 + \cdots + \mathcal{M}_n) \; \langle \phi_1(t_1, \mathbf{r}_1) \ldots \phi_n(t_n, \mathbf{r}_n) \rangle = 0 \tag{22}
$$

follows, where the ϕ_j are scaling operators of the physical theory. This feature distinguishes Schrödinger-invariance from conformal invariance [19, 22].

The importance of (22) appears if one recalls that models of non-equilibrium statistical mechanics are often specified via a *stochastic* Langevin equation, viz.

$$
2\mathcal{M}\partial_t \phi = \nabla_{\mathbf{r}} \cdot \nabla_{\mathbf{r}} \phi - \frac{\delta \mathcal{V}[\phi]}{\delta \phi} + \eta \tag{23}
$$

and a Ginzburg–Landau potential $\mathcal{V}[\phi]$. In the context of Janssen-de Dominicis theory [39], this can be recast as the variational equation of motion of a dynamic functional $\mathcal{J}[\phi, \widetilde{\phi}] = \mathcal{J}_0[\phi, \widetilde{\phi}] + \mathcal{J}_b[\widetilde{\phi}]$ where the term $\mathcal{J}_0[\phi, \widetilde{\phi}]$ contains the deterministic terms coming from the Langevin equation and $\mathcal{J}_b[\widetilde{\phi}]$ contains the stochastic terms generated by averaging over the thermal noise and the initial condition. In this context, the two-time linear response function (spatial arguments are suppressed)

$$
R(t,s) = \left.\frac{\delta \langle \phi(t) \rangle}{\delta h(s)}\right|_{h=0} = \int \mathcal{D}\phi \mathcal{D}\widetilde{\phi}\; \phi(t) \widetilde{\phi}(s) e^{-\mathcal{J}[\phi, \widetilde{\phi}]} = \left\langle \phi(t) \widetilde{\phi}(s) \right\rangle \tag{24}
$$

is expressed as a correlator with the associated response operator $\widetilde{\phi}$.

Theorem ([34]) *Consider the functional* $\mathcal{J}[\phi, \widetilde{\phi}] = \mathcal{J}_0[\phi, \widetilde{\phi}] + \mathcal{J}_b[\widetilde{\phi}]$. *If* $\mathcal{J}_0$ *is Galilei-invariant with non-vanishing masses such that (22) holds, then all responses and correlators reduce to averages only involving* $\mathcal{J}_0[\phi, \widetilde{\phi}]$.

Proof Define the average $\langle X \rangle_0 = \int \mathcal{D}\phi \mathcal{D}\widetilde{\phi}\; X[\phi] e^{-\mathcal{J}_0[\phi, \widetilde{\phi}]}$ with respect to the functional $\mathcal{J}_0[\phi, \widetilde{\phi}]$. For illustration, consider merely $R(t,s)$. From (24)

$$R(t,s) = \left\langle \phi(t)\widetilde{\phi}(s)e^{-\mathcal{J}_b[\widetilde{\phi}]}\right\rangle_0 = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \left\langle \phi(t)\widetilde{\phi}(s)\mathcal{J}_b[\widetilde{\phi}]^k\right\rangle_0 = \left\langle \phi(t)\widetilde{\phi}(s)\right\rangle_0$$

The superselection rule (22) in the last step implies that only the $k = 0$ term is kept. Hence the response $R(t,s) = R_0(t,s)$ does not depend explicitly on the noise.

q.e.d.

Hence a symmetry analysis of systems described by Langevin equations (23) reduces to the symmetries of its 'deterministic part', with the noise $\eta \mapsto 0$. Physicists' conventions require that 'physical masses' $\mathcal{M}_i \geq 0$. One must define a formal 'complex conjugate' ϕ^* of the scaling operator ϕ, such that its mass $\mathcal{M}^* := -\mathcal{M} \leq 0$ becomes negative. This rôle is played by $\widetilde{\phi}$.

Example 1 The *ageing algebra* $\mathfrak{age}(d) := \langle X_{0,1}, Y^{(j)}_{\pm\frac{1}{2}}, M_0, R^{(jk)}_0 \rangle$ with $j,k = 1,\ldots,d$ does *not* include time-translations X_{-1}. Starting from the representation (20), the generators X_n now read [21, 34]

$$X_n = -t^{n+1}\partial_t - \frac{n+1}{2}t^n\mathbf{r}\cdot\nabla_{\mathbf{r}} - \frac{n+1}{2}xt^n - n(n+1)\xi t^n - \frac{n(n+1)}{4}\mathcal{M}t^{n-1}\mathbf{r}^2 \tag{25}$$

The Bargman rule (22) still holds. The Schrödinger operator is $\mathcal{S} = 2\mathcal{M}\partial_t - \partial_r^2 + 2\mathcal{M}t^{-1}\left(x + \xi - \frac{d}{2}\right)$ and commutes with all generators of $\mathfrak{age}(d)$, up to

$$[\mathcal{S}, X_0] = -\mathcal{S} \quad , \quad [\mathcal{S}, X_1] = -2t\mathcal{S} \tag{26}$$

without any constraint, neither on x nor on ξ [38]. Remarkably, *each non-equilibrium scaling operator ϕ must at least be characterised by two distinct, independent scaling dimensions, here labelled x, ξ.*

An explicit example for this is given by the $1D$ *kinetic Ising model with Glauber dynamics*. The model's configurations $\sigma = \{\sigma_1, \ldots, \sigma_{\mathcal{N}}\}$ of Ising spins $\sigma_i = \pm 1$ evolve in discrete time, according to a Markov process with the Glauber rates [16]

$$P\left(\sigma_i(t+1) = \pm 1\right) = \frac{1}{2}\left[1 \pm \tanh\left(\frac{1}{T}\left(\sigma_{i-1}(t) + \sigma_{i+1}(t) + h_i(t)\right)\right)\right] \tag{27}$$

where $h_i(t)$ is a time-dependent external field and T is the temperature. The exact solution gives at $T = 0$ in the scaling limit $t,s \to \infty$ with t/s kept fixed, the autocorrelator and autoresponse (independently of the initial conditions) are

$$C(t,s) = 2/\pi \arctan\sqrt{2[t/s-1]^{-1}} \;\; ; \;\; R(t,s) = (2\pi^2)^{-1/2}s^{-1}(t/s-1)^{-1/2}. \tag{28}$$

We read off the scaling dimensions $x = \frac{1}{2}$, $\xi = 0$ for the magnetisation and $\widetilde{x} = 0$, $\widetilde{\xi} = \frac{1}{4}$ for the response operator [21, 25].

4 Representations and Invariant Equations

We now give several extensions of the representations discussed so far. The basic new fact, first observed in [32], is compactly best stated for the conformal algebra.

Proposition 1 ([25]) *Let γ be a constant and $g(z)$ a non-constant function. Then*

$$\ell_n = -z^{n+1}\partial_z - n\gamma z^n - g(z)z^n \tag{29}$$

obey the conformal algebra $[\ell_n, \ell_m] = (n-m)\ell_{n+m}$ for all $n, m \in \mathbb{Z}$.

Proposition 2 ([25]) *If $\phi(z)$ is a quasi-primary scaling operator under the representation (29) of the conformal algebra $\langle \ell_{\pm 1,0}\rangle$, its co-variant two-point function is given by, up to normalisation and with $\Gamma_i(z) :=$
$= z^{\gamma_i} \exp\left(-\int_1^z \mathrm{d}\zeta\, \frac{g(\zeta)}{\zeta}\right)$*

$$\langle \phi_1(z_1)\phi_2(z_2)\rangle = \delta_{\gamma_1,\gamma_2}\,(z_1-z_2)^{-\gamma_1-\gamma_2}\Gamma_1(z_1)\Gamma_2(z_2). \tag{30}$$

Proof Re-write $\langle \phi_1(z_1)\phi_2(z_2)\rangle = \Gamma_1(z_1)\Gamma_2(z_2)\Psi(z_1,z_2)$ and show that $\Psi(z_1,z_2)$ obeys the Ward identities of standard conformal invariance, where γ_i play the rôles of conformal weights. q.e.d.

Proposition 3 ([25]) *If one replaces in the representation (20) the generator X_n as follows, with $n \in \mathbb{Z}$*

$$\begin{aligned} X_n = -t^{n+1}\partial_t - \frac{n+1}{2}t^n \mathbf{r}\cdot\nabla_{\mathbf{r}} - \frac{n+1}{2}xt^n \; - \\ - n(n+1)\xi t^n - \Xi(t)t^n - \frac{n(n+1)}{4}\mathcal{M}t^{n-1} \end{aligned} \tag{31}$$

where x, ξ are constants and $\Xi(t)$ is an arbitrary (non-constant) function, then the commutators (21) of the Lie algebra $\mathfrak{sv}(d)$ are still satisfied.

This result was first obtained, for the Schrödinger algebra $\mathfrak{sch}(d)$, by Minic, Vaman and Wu [32]. They also take the dependence on the mass $\mathcal{M}$ in $u(t)$ into account and write down terms up to order $\mathrm{O}(1/\mathcal{M})$ and $\mathrm{O}(1)$. The representation (25) of $\mathfrak{age}(d)$ is a special case, with arbitrary ξ, but with $\Xi(t) = 0$. Explicit two- and three-point functions, co-variant under either $\mathfrak{sch}(d)$ or $\mathfrak{age}(d)$, are derived in [32].

Proposition 4 ([25]) *Consider the representation (20), with the generators X_n replaced by (31), of either $\mathfrak{age}(d)$, or $\mathfrak{sch}(d)$. The invariant Schrödinger operator is*

$$\mathcal{S} = 2\mathcal{M}\partial_t - \nabla_{\mathbf{r}}^2 + 2\mathcal{M}u(t), \quad u(t) = (x+\xi-d/2)t^{-1} + \Xi(t)t^{-1} \tag{32}$$

such that a solution of $\mathcal{S}\phi = 0$ is mapped onto another solution of the same equation. For the algebra $\mathfrak{age}(d)$, there is no restriction, neither on x, nor on ξ, nor on $\Xi(t)$. For the algebra $\mathfrak{sch}(d)$, one has the additional condition $x = \frac{d}{2} - 2\xi$.

Proof For brevity, restrict to $d = 1$ and reproduce (26). First look at $\mathfrak{age}(1)$. Consideration of X_0 gives $t\dot{u}(t) + u - \dot{\Xi}(t) = 0$ and considering X_1 gives $x + \xi - \frac{1}{2} + \Xi(t) + t\dot{\Xi}(t) - 2tu(t) - t^2\dot{u}(t) = 0$, where the dot denotes the derivative with respect to t. The second relation can be simplified to $x + \xi - \frac{1}{2} + \Xi(t) - tu(t) = 0$ which gives the assertion. Going over to $\mathfrak{sch}(1)$, the condition $[\mathcal{S}, X_{-1}] = 0$ leads to $\xi/t^2 + \dot{\Xi}(t)/t - \Xi(t)/t^2 - \dot{u}(t) = 0$. This is only compatible with the result found before for $\mathfrak{age}(1)$, if $\xi = -x - \xi + \frac{1}{2}$, as asserted. q.e.d.

Example 2 These results have an immediate application in the first Arcetri models, discussed in Sect. 2. In the continuum limit, the slopes $u_a(t, \mathbf{r}) = \partial h(t, \mathbf{r})/\partial r_a$ satisfy a Langevin equation $\partial_t u_a(t, \mathbf{r}) = \nabla_{\mathbf{r}}^2 u_a(t, \mathbf{r}) + \mathfrak{z}(t)u_a(t, \mathbf{r}) + \frac{\partial}{\partial r_a}\eta(t, \mathbf{r})$, analogous to (6). Because of the theorem in Sect. 2, we can compare the deterministic part of this with the invariant Schrödinger operator (32). Clearly, for the first Arcetri model with $T \leq T_c(d)$, one has $\Xi(t) = 0$ and $\frac{d}{2} - \digamma = x + \xi$, using the definition (10) of $g(t)$ and recalling the values (13) of the universal exponent $\digamma$.

The second and third Arcetri model do not obey simple ageing. It remains open if their long-time behaviour can be cast into a simple local sub-ageing scaling form.

5 Conclusions

The kinetics of growing interfaces furnish paradigmatic examples for case studies of extended dynamic scaling. Scaling operators in non-equilibrium dynamical scaling are characterised by at least *two* distinct and independent scaling dimensions, x and ξ. These arise from new representations of the Schrödinger and ageing algebras. Explicit examples include the exactly solved Glauber-Ising and Arcetri models.

Acknowledgements Je remercie chaleureusement X. Durang et S. Stoimenov pour le plaisir constant de nos coopérations et les organisateurs du LT-11 pour leur gentille invitation. Ce travail a reçu du support financier par PHC Rila et par le Collège Doctoral franco-allemand Nancy-Leipzig-Coventry (Systèmes complexes à l'équilibre et hors équilibre) de l'UFA-DFH.

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News on SU(2|1) Supersymmetric Mechanics

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Abstract We report on a recent progress in exploring the $SU(2|1)$ supersymmetric quantum mechanics. Our focus is on the harmonic $SU(2|1)$ superspace formalism which provides a superfield description of the multiplet $(\mathbf{4}, \mathbf{4}, \mathbf{0})$ and its "mirror" version. We present the σ-model and Wess–Zumino type actions for these multiplets, in both the superfield and the component approaches. An interesting new feature as compared to the flat $\mathcal{N} = 4$, $d = 1$ case is the absence of the explicit $SU(2|1)$ invariant Wess–Zumino term for the ordinary $(\mathbf{4}, \mathbf{4}, \mathbf{0})$ multiplet and yet the existence of such a term for the mirror multiplet. The superconformal subclass of the $SU(2|1)$ invariant $(\mathbf{4}, \mathbf{4}, \mathbf{0})$ actions is also described. Its main distinguishing features are the "trigonometric" realization of the $d = 1$ conformal group $SO(2, 1)$ and the oscillator-type potential terms in the component actions.

1 Introduction

In [1], we started a systematic study of a new type of $\mathcal{N} = 4$ supersymmetric quantum mechanics (SQM) based on the worldline realizations of the supergroup $SU(2|1)$. It can be treated as a deformation of the standard $\mathcal{N} = 4$ SQM by an intrinsic mass parameter m. The idea to consider such a deformation was motivated by the growing interest in theories with a curved rigid supersymmetry (see, e.g., [2]). In the subsequent papers [3–5], the study of the deformed $SU(2|1)$ mechanics was continued.

We proceeded from the universal way of constructing supersymmetric theories, *viz* the superfield approach. The real and complex $SU(2|1)$ superspaces were defined in [1] as appropriate cosets of the supergroup $SU(2|1)$ (and its central extension). It was shown that all off-shell multiplets of flat $\mathcal{N} = 4$, $d = 1$ supersymmetry have

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V. Dobrev (ed.), *Lie Theory and Its Applications in Physics*,
Springer Proceedings in Mathematics & Statistics 191,
DOI 10.1007/978-981-10-2636-2_5

their $SU(2|1)$ analogs. For instance, the $SU(2|1)$ analog of the $(\mathbf{1}, \mathbf{4}, \mathbf{3})$ multiplet is described by a superfield which "lives" on the real $SU(2|1)$ superspace, and it yields the "weak supersymmetry" models of Ref. [6]. The supergroup $SU(2|1)$ possesses also invariant complex chiral supercosets which are carriers of the chiral multiplets $(\mathbf{2}, \mathbf{4}, \mathbf{2})$. We also showed that the $(\mathbf{2}, \mathbf{4}, \mathbf{2})$ multiplet can be generalized [3] in such a way that the Lagrangian of the super Kähler oscillator [7, 8] can be constructed on its basis. The superconformal $D(2, 1; \alpha)$ invariant SQM models in the $SU(2|1)$ superspace formulation were studied in [4]. Their characteristic feature is that they naturally yield the *trigonometric* realization of the $d = 1$ conformal group $SO(2, 1)$ [9, 10].

In the contribution to the proceedings of the previous Workshop from this series [11], we reviewed the superfield approach and $SU(2|1)$ SQM models based on deformed analogs of the standard $\mathcal{N} = 4$, $d = 1$ superspaces. Here, we describe the deformed SQM models in the framework of the harmonic $SU(2|1)$ superspace [5], which is a deformation of the $\mathcal{N} = 4$, $d = 1$ harmonic superspace [12]. We consider the $(\mathbf{4}, \mathbf{4}, \mathbf{0})$ multiplet, as well as its "mirror" $(\mathbf{4}, \mathbf{4}, \mathbf{0})$ counterpart, and construct the general σ-model and WZ (Wess–Zumino) type Lagrangians for both multiplets. It is shown that an external $SU(2|1)$ invariant WZ term can be defined only for the mirror multiplet $(\mathbf{4}, \mathbf{4}, \mathbf{0})$, in a crucial distinction from the flat $\mathcal{N} = 4$, $d = 1$ case. The general expressions for the relevant supercharges, in both the classical and quantum cases, as well as the explicit spectrum of the corresponding Hamiltonian for a few simple models can be found in [5]. We also present here the superconformal subclass of the $(\mathbf{4}, \mathbf{4}, \mathbf{0})$ $SU(2|1)$ invariant actions. As in the case of superconformal actions for the multiplets $(\mathbf{1}, \mathbf{4}, \mathbf{3})$ and $(\mathbf{2}, \mathbf{4}, \mathbf{2})$ [4], when constructing the superconformal $(\mathbf{4}, \mathbf{4}, \mathbf{0})$ actions, we capitalize on the notable property of the conformal superalgebra $D(2, 1; \alpha)$ to be a closure of its two $su(2|1)$ subalgebras related to each other via the reflection of the corresponding intrinsic mass parameter.

2 Harmonic $SU(2|1)$ Superspace

2.1 Superalgebra

The basic relations of the central-extended superalgebra $\widehat{su}(2|1)$ are as follows:

$$
\begin{aligned}
&\{Q^i, \bar{Q}_j\} = 2m \left(I^i_j - \delta^i_j F\right) + 2\delta^i_j H\,, \qquad \left[I^i_j, I^k_l\right] = \delta^k_j I^i_l - \delta^i_l I^k_j\,,\\
&\left[I^i_j, \bar{Q}_l\right] = \frac{1}{2}\,\delta^i_j \bar{Q}_l - \delta^i_l \bar{Q}_j\,, \qquad \left[I^i_j, Q^k\right] = \delta^k_j Q^i - \frac{1}{2}\,\delta^i_j Q^k,\\
&\left[F, \bar{Q}_l\right] = -\frac{1}{2}\,\bar{Q}_l\,, \qquad \left[F, Q^k\right] = \frac{1}{2}\,Q^k. \qquad (1)
\end{aligned}
$$

Here, the generators I^i_j and F form the internal symmetry group $SU(2)_{\rm int} \times U(1)_{\rm int}$. The central charge generator H is identified with the time-translation generator becoming the Hamiltonian in the relevant SQM models. The limit $m = 0$ yields the standard flat $\mathcal{N} = 4$, $d = 1$ Poincaré superalgebra.

Using the notations

$$
\begin{aligned}
&Q^1 \equiv Q^+, \quad Q^2 \equiv Q^-, \quad \bar{Q}_1 \equiv \bar{Q}^-, \quad \bar{Q}_2 \equiv -\bar{Q}^+, \\
&I^{++} \equiv I_2^1, \quad I^{--} \equiv I_1^2, \quad I^0 \equiv I_1^1 - I_2^2 = 2I_1^1,
\end{aligned} \tag{2}
$$

we can rewrite the (anti)commutation relations of the superalgebra $\widehat{su}(2|1)$ as

$$
\begin{aligned}
&\{Q^-, \bar{Q}^+\} = mI^0 - 2H + 2mF, \qquad \{Q^+, \bar{Q}^-\} = mI^0 + 2H - 2mF, \\
&\{Q^\pm, \bar{Q}^\pm\} = \mp 2mI^{\pm\pm}, \qquad \left[I^0, I^{\pm\pm}\right] = \pm 2I^{\pm\pm}, \quad \left[I^{++}, I^{--}\right] = I^0, \\
&\left[I^0, \bar{Q}^\pm\right] = \pm \bar{Q}^\pm, \qquad \left[I^{++}, \bar{Q}^-\right] = \bar{Q}^+, \qquad \left[I^{--}, \bar{Q}^+\right] = \bar{Q}^-, \\
&\left[I^0, Q^\pm\right] = \pm Q^\pm, \qquad \left[I^{++}, Q^-\right] = Q^+, \qquad \left[I^{--}, Q^+\right] = Q^-, \\
&\left[F, \bar{Q}^\pm\right] = -\frac{1}{2}\bar{Q}^\pm, \qquad \left[F, Q^\pm\right] = \frac{1}{2}Q^\pm.
\end{aligned} \tag{3}
$$

We can also add, to the superalgebra $\widehat{su}(2|1)$, the automorphism group $SU(2)_{\rm ext}$ with the generators $\{T^0, T^{++}, T^{--}\}$ which rotate the supercharges in the precisely same way as the internal $SU(2)_{\rm int}$ generators $\{I^0, I^{++}, I^{--}\}$ do. For consistency, the $SU(2)_{\rm ext}$ generators should rotate, in the same way, the indices of the $SU(2)_{\rm int}$ generators I^i_j, so these two $SU(2)$ groups form a semi-direct product

$$
[T, I] \propto I. \tag{4}
$$

2.2 Harmonic $SU(2|1)$ Superspace as a Coset Superspace

We introduce the following harmonic coset of the extended supergroup:

$$
\frac{\{H, Q^\pm, \bar{Q}^\pm, F, I^{\pm\pm}, I^0, T^{\pm\pm}, T^0\}}{\{F, I^{++}, I^0, I^{--} - T^{--}, T^0\}} \sim \left(t_{(A)}, \theta^\pm, \bar{\theta}^\pm, w_i^\pm\right) =: \zeta_H\,. \tag{5}
$$

It is a deformation of the standard "flat" $\mathcal{N} = 4\,, d = 1$ harmonic superspace [12].

We can consider the harmonic superspace (5) as an extension of the $SU(2|1)$ superspace $SU(2|1)$ [1] by harmonic variables $w_i^\pm$ satisfying

$$
w^{+i}w_i^- = 1\,. \tag{6}
$$

Such an extension will be referred to as the *central* basis of the harmonic $SU(2|1)$ superspace, while the parametrization (5) as the *analytic* basis.

In the central basis of the harmonic $SU(2|1)$ superspace,

$$\zeta_C = \left(t, \theta_i, \bar{\theta}^j, w_i^{\pm}\right), \tag{7}$$

the world-line supergroup $SU(2|1)$ is realized by the following transformations:

$$\delta\theta_i = \epsilon_i + 2m\,\bar{\epsilon}^k\theta_k\theta_i\,, \qquad \delta\bar{\theta}^j = \bar{\epsilon}^j - 2m\,\epsilon_k\bar{\theta}^k\bar{\theta}^j, \qquad \delta t = i\left(\epsilon_k\bar{\theta}^k + \bar{\epsilon}^k\theta_k\right),$$
$$\delta w_i^+ = -m\left(1 - m\,\bar{\theta}^l\theta_l\right)\left(\bar{\theta}^k\epsilon^j + \theta^k\bar{\epsilon}^j\right)w_k^+ w_j^+ w_i^-\,, \qquad \delta w_i^- = 0\,. \tag{8}$$

The explicit relation with the analytic basis coordinates (5) is given by

$$\theta^i w_i^- = \theta^-, \qquad \theta^i w_i^+ = \theta^+\left(1 + m\,\bar{\theta}^+\theta^- - m\,\bar{\theta}^-\theta^+\right),$$
$$\bar{\theta}^k w_k^- = \bar{\theta}^-, \qquad \bar{\theta}^k w_k^+ = \bar{\theta}^+\left(1 + m\,\bar{\theta}^+\theta^- - m\,\bar{\theta}^-\theta^+\right),$$
$$t = t_{(A)} + i\left(\bar{\theta}^-\theta^+ + \bar{\theta}^+\theta^-\right). \tag{9}$$

Then the coordinates $\left\{t_{(A)}, \bar{\theta}^{\pm}, \theta^{\pm}, w_i^{\pm}\right\}$ transform as

$$\delta\theta^+ = \epsilon^+ + m\,\bar{\theta}^+\theta^+\epsilon^-, \quad \delta\bar{\theta}^+ = \bar{\epsilon}^+ - m\,\bar{\theta}^+\theta^+\bar{\epsilon}^-,$$
$$\delta\theta^- = \epsilon^- + 2m\,\bar{\epsilon}^-\theta^-\theta^+, \quad \delta\bar{\theta}^- = \bar{\epsilon}^- + 2m\,\epsilon^-\bar{\theta}^-\bar{\theta}^+,$$
$$\delta t_{(A)} = 2i\left(\epsilon^-\bar{\theta}^+ + \theta^+\bar{\epsilon}^-\right),$$
$$\delta w_i^+ = -m\left(\bar{\theta}^+\epsilon^+ + \theta^+\bar{\epsilon}^+\right)w_i^-\,, \qquad \delta w_i^- = 0\,, \tag{10}$$

where

$$\epsilon^{\pm} = \epsilon^i w_i^{\pm}\,, \qquad \bar{\epsilon}^{\pm} = \bar{\epsilon}^i w_i^{\pm}\,. \tag{11}$$

The analytic subspace closed under the $SU(2|1)$ transformation is defined as the set

$$\zeta_A := \left(t_{(A)}, \bar{\theta}^+, \theta^+, w_i^{\pm}\right). \tag{12}$$

One can define the analytic subspace integration measure

$$d\zeta_{(A)}^{--} := dw\,dt_{(A)}\,d\bar{\theta}^+ d\theta^+, \tag{13}$$

which is invariant under the supersymmetry transformations (10). The corresponding full integration measure $d\zeta_H$ in the analytic basis can be written as

$$d\zeta_H := dw\,dt_{(A)}\,d\bar{\theta}^- d\theta^- d\bar{\theta}^+ d\theta^+\left(1 + m\,\bar{\theta}^+\theta^- - m\,\bar{\theta}^-\theta^+\right), \tag{14}$$

and it transforms as

$$\delta\left(d\zeta_H\right) = d\zeta_H\Big[-\,m\left(\bar{\theta}^-\epsilon^+ + \theta^-\bar{\epsilon}^+\right)\left(1 - m\,\bar{\theta}^+\theta^- + m\,\bar{\theta}^-\theta^+\right)$$
$$-\,m\left(\bar{\theta}^+\epsilon^- + \theta^+\bar{\epsilon}^-\right)\Big]. \tag{15}$$

One can check that there is no way to achieve the $SU(2|1)$ invariance of this measure: no a scalar factor can be picked up to compensate the non-zero variation (15).

2.3 Covariant Derivatives and $SU(2|1)$ Harmonic Analyticity

The $SU(2|1)$ covariant derivatives in the analytic basis (5) were defined in [5]. The (anti)commutation relations among them mimic those of the superalgebra (3):

$$
\begin{aligned}
&\{\bar{\mathcal{D}}^{+}, \mathcal{D}^{-}\} = m\mathcal{D}^{0} - 2m\tilde{F} + 2i\mathcal{D}_{(A)}\,,\\
&\{\bar{\mathcal{D}}^{-}, \mathcal{D}^{+}\} = m\mathcal{D}^{0} + 2m\tilde{F} - 2i\mathcal{D}_{(A)}\,,\\
&\{\mathcal{D}^{\pm}, \bar{\mathcal{D}}^{\pm}\} = \mp 2m\mathcal{D}^{\pm\pm}, \quad \left[\mathcal{D}^{++}, \mathcal{D}^{--}\right] = \mathcal{D}^{0}, \quad \left[\mathcal{D}^{0}, \mathcal{D}^{\pm\pm}\right] = \pm 2\mathcal{D}^{\pm\pm},\\
&\left[\mathcal{D}^{++}, \mathcal{D}^{-}\right] = \mathcal{D}^{+}, \qquad \left[\mathcal{D}^{--}, \mathcal{D}^{+}\right] = \mathcal{D}^{-}, \qquad \left[\mathcal{D}^{0}, \mathcal{D}^{\pm}\right] = \pm\mathcal{D}^{\pm},\\
&\left[\mathcal{D}^{++}, \bar{\mathcal{D}}^{-}\right] = \bar{\mathcal{D}}^{+}, \qquad \left[\mathcal{D}^{--}, \bar{\mathcal{D}}^{+}\right] = \bar{\mathcal{D}}^{-}, \qquad \left[\mathcal{D}^{0}, \bar{\mathcal{D}}^{\pm}\right] = \pm\bar{\mathcal{D}}^{\pm},
\end{aligned}
\tag{16}
$$

$$
\tilde{F}\,\mathcal{D}^{\pm} = -\frac{1}{2}\,\mathcal{D}^{\pm}, \qquad \tilde{F}\,\bar{\mathcal{D}}^{\pm} = \frac{1}{2}\,\bar{\mathcal{D}}^{\pm}. \tag{17}
$$

Here, $\tilde{F}$ is a matrix part of the $U(1)_{\text{int}}$ generator F. The harmonic derivative $\mathcal{D}^{--}$, together with $\mathcal{D}^{++}$ and $\mathcal{D}^{0}$, form an $SU(2)$ algebra. To define the analyticity conditions, it is enough to explicitly know the covariant derivatives $\mathcal{D}^{+}$, $\bar{\mathcal{D}}^{+}$ and $\mathcal{D}^{++}$:

$$
\begin{aligned}
\mathcal{D}^{++} &= \left(1 + m\,\bar{\theta}^{+}\theta^{-} - m\,\bar{\theta}^{-}\theta^{+}\right)^{-1}\partial^{++} + 2i\,\theta^{+}\bar{\theta}^{+}\partial_{(A)} - 2m\,\theta^{+}\bar{\theta}^{+}\tilde{F}\\
&\quad + \theta^{+}\frac{\partial}{\partial\theta^{-}} + \bar{\theta}^{+}\frac{\partial}{\partial\bar{\theta}^{-}}\,,\\
\mathcal{D}^{+} &= \frac{\partial}{\partial\theta^{-}} + m\,\bar{\theta}^{-}\mathcal{D}^{++},\\
\bar{\mathcal{D}}^{+} &= -\,\frac{\partial}{\partial\bar{\theta}^{-}} - m\,\theta^{-}\mathcal{D}^{++}.
\end{aligned}
\tag{18}
$$

The explicit expressions for the rest of covariant derivatives are given in [5].

The spinor derivatives $\mathcal{D}^{+}$, $\bar{\mathcal{D}}^{+}$, together with $\mathcal{D}^{++}$ and $\mathcal{D}^{0}$, form the so-called CR (“Cauchy–Riemann”) structure [13]

$$
\begin{aligned}
&\left\{\mathcal{D}^{+}, \bar{\mathcal{D}}^{+}\right\} = -\,2m\,\mathcal{D}^{++}, \quad \left\{\mathcal{D}^{+}, \mathcal{D}^{+}\right\} = \left\{\bar{\mathcal{D}}^{+}, \bar{\mathcal{D}}^{+}\right\} = 0\,,\\
&\left[\mathcal{D}^{++}, \mathcal{D}^{+}\right] = \left[\mathcal{D}^{++}, \bar{\mathcal{D}}^{+}\right] = 0\,,\\
&\left[\mathcal{D}^{0}, \mathcal{D}^{+}\right] = \mathcal{D}^{+}\,, \quad \left[\mathcal{D}^{0}, \bar{\mathcal{D}}^{+}\right] = \bar{\mathcal{D}}^{+}\,, \quad \left[\mathcal{D}^{0}, \mathcal{D}^{++}\right] = 2\,\mathcal{D}^{++}\,.
\end{aligned}
\tag{19}
$$

By commuting this set with $\mathcal{D}^{--}$, one can restore the whole algebra (18). Worthy of note is the first relation in (19). It implies that in the $SU(2|1)$ case, with $m^2 \neq 0$, the Grassmann harmonic analyticity is necessarily accompanied by the bosonic harmonic analyticity. No such a property is exhibited by the harmonic formalism in the flat $\mathcal{N}=4, d=1$ case [12].

2.4 Harmonic $SU(2|1)$ Superfields

The passive odd transformation of the harmonic superfields in the analytic basis $\Phi\,(\zeta_H)$ can be written as

$$\delta\Phi = -\,m\Big[2\left(\bar\theta^+\epsilon^- - \theta^+\bar\epsilon^-\right)\tilde F + \left(\bar\theta^+\epsilon^- + \theta^+\bar\epsilon^-\right)\mathcal{D}^0 \\ + \left(\bar\theta^-\epsilon^- + \theta^-\bar\epsilon^-\right)\mathcal{D}^{++}\Big]\Phi. \tag{20}$$

The superfields Φ are assumed to have definite $U(1)$ charges, $\tilde F\Phi = \kappa\Phi\,,\ \mathcal{D}^0\Phi = q\,\Phi$.

We can write the general σ-model-type action as

$$S = \int dt\,\mathcal{L} = \int d\zeta_H\,K\left(\Phi_1, \Phi_2, \ldots \Phi_N\right). \tag{21}$$

Here K is an arbitrary function of superfields $\Phi_1, \Phi_2, \ldots \Phi_N$. It satisfies the following restrictions:

$$\tilde F K\left(\Phi_1, \Phi_2, \ldots \Phi_N\right) = \mathcal{D}^0 K\left(\Phi_1, \Phi_2, \ldots \Phi_N\right) = 0\,, \tag{22}$$

which are implied by the requirement of $SU(2|1)$ invariance of the action (21) (modulo a total derivative in the variation of the integrand).

The analytic superfields with the harmonic $U(1)$ charge $+q$ are defined by the constraints:

$$\mathcal{D}^+\varphi^{+q} = \bar{\mathcal{D}}^+\varphi^{+q} = 0 \quad\Rightarrow\quad \mathcal{D}^{++}\varphi^{+q} = 0\,. \tag{23}$$

In contrast to the standard case [12], the Grassmann analyticity conditions in the $SU(2|1)$ case lead to the harmonic analyticity condition. This is a consequence of the first relation in (19).

3 The Multiplet (4, 4, 0)

The $SU(2|1)$ multiplet $(\mathbf{4}, \mathbf{4}, \mathbf{0})$ is described by an analytic harmonic superfield q^{+a} satisfying the analyticity constraints (23). Here $a = 1, 2$ is the doublet index of the "Pauli-Gürsey" $SU(2)$ symmetry. Equation (23) yield the following solution for q^{+a}:

$$q^{+a}\left(\zeta_A\right) = x^{ia} w_i^+ + \theta^+ \psi^a + \bar{\theta}^+ \bar{\psi}^a - 2i\theta^+ \bar{\theta}^+ \dot{x}^{ia} w_i^- \,. \tag{24}$$

The superfield q^{+a} and its components are transformed as

$$\begin{aligned}
&\delta q^{+a} = -\, m \left(\bar{\theta}^+ \epsilon^- + \theta^+ \bar{\epsilon}^-\right) q^{+a} \quad \Rightarrow \\
&\delta x^{ia} = -\, \epsilon^i \psi^a - \bar{\epsilon}^i \bar{\psi}^a, \\
&\delta \bar{\psi}_a = 2i\epsilon_k \dot{x}^k_a - m\, \epsilon_k x^k_a\,, \quad \delta \psi^a = 2i\bar{\epsilon}^k \dot{x}^a_k + m\, \bar{\epsilon}^k x^a_k\,.
\end{aligned} \tag{25}$$

3.1 The σ-Model Actions

The most general σ-model type invariant action is written as:

$$S\left(q^{\pm a}\right) = \int d\zeta_H\, L\left(q^{+a} q^-_a\right), \quad q^{-a} = \mathcal{D}^{--} q^{+a}\,, \tag{26}$$

and, in components, yields:

$$\begin{aligned}
\mathcal{L} = G\left[\dot{x}^{ia}\dot{x}_{ia} + \frac{i}{2}\left(\bar{\psi}_a\dot{\psi}^a - \dot{\bar{\psi}}_a\psi^a\right) + \frac{m}{2}\,\psi^a\bar{\psi}_a\right] - \frac{i}{2}\,\dot{x}^{ia}\partial_{ic}G\left(\psi_a\bar{\psi}^c + \psi^c\bar{\psi}_a\right) \\
-\frac{\Delta_x G}{16}\left(\bar{\psi}\right)^2(\psi)^2 + \frac{m}{2}\,x^2 G'\,\psi^a\bar{\psi}_a - \frac{m^2}{4}\,x^2\, G\,,
\end{aligned} \tag{27}$$

where

$$\begin{aligned}
&\partial_{ia} = \partial/\partial x^{ia}, \quad \Delta_x = \epsilon^{ik}\epsilon^{ab}\partial_{ia}\partial_{kb}\,, \quad x^2 = x^{ia}x_{ia}\,, \\
&G\left(x^2\right) = \Delta_x L\left(\frac{1}{2}x^2\right).
\end{aligned} \tag{28}$$

The free model corresponds to the choice $L^{\text{free}}\left(q^{+a}q^-_a\right) = \frac{1}{4}q^{+a}q^-_a$ and $G = 1$.

3.2 The Absence of WZ Type Actions

The most general Wess–Zumino (WZ) action [12] is given by the integral over the analytic subspace

$$S_{\text{WZ}}(q^{+a}) = -\frac{i}{2}\int d\zeta_A^{--}\, L^{++}\left(q^{+a}, w_i^{\pm}\right). \tag{29}$$

Since the analytic superfield (24) is not deformed by the parameter m, this action coincides with the non-deformed WZ action for the multiplet (**4**, **4**, **0**) given in [12].

The action (29) can be shown to respect no $SU(2|1)$ invariance for any choice of L^{++}. This just means the absence of the independent WZ action for the multiplet $(\mathbf{4},\mathbf{4},\mathbf{0})$ in the case of $SU(2|1)$ supersymmetry.[1]

4 Mirror (4, 4, 0) Multiplet

In the flat case ($m = 0$), the full automorphism group of the $\mathcal{N} = 4\,, d = 1$ Poincaré superalgebra is

$$SU(2) \times SU'(2). \tag{30}$$

The multiplet $(\mathbf{4},\mathbf{4},\mathbf{0})$ has its "mirror" cousin for which two commuting $SU(2)$ automorphism groups of the $\mathcal{N} = 4\,,\ d = 1$ superalgebra switch their roles. Since these $SU(2)$ groups enter the game in the entirely symmetric way, there is a full equivalence between these two types of the $(\mathbf{4},\mathbf{4},\mathbf{0})$ supermultiplet - in the sense that the actions including only one sort of such multiplets are indistinguishable from each other.[2] In the $SU(2|1)$ deformed case the symmetry between the two former automorphism $su(2)$ algebras of the flat superalgebra proves to be broken: one of these $su(2)$ becomes $su(2)_{\rm int} \subset \widehat{su}(2|1)$, while only one $U(1)$ generator F from the group $SU'(2)$ is inherited by the $\widehat{su}(2|1)$ superalgebra. So one can expect an essential difference between the $(\mathbf{4},\mathbf{4},\mathbf{0})$ multiplets of two sorts in the $SU(2|1)$ case.

Let us consider the mirror $(\mathbf{4},\mathbf{4},\mathbf{0})$ multiplet [14, 15] in the framework of the harmonic $SU(2|1)$ superspace. The superfield describing the mirror $SU(2|1)$ multiplet $(\mathbf{4},\mathbf{4},\mathbf{0})$ is $\left(Y^A\right)^\dagger = \bar{Y}_A, A = 1, 2$, constrained by

$$\begin{aligned}
&\bar{\mathcal{D}}^+ Y^A = \mathcal{D}^+ \bar{Y}^A = 0\,, \quad \mathcal{D}^+ Y^A = -\bar{\mathcal{D}}^+ \bar{Y}^A\,, \quad \mathcal{D}^{++} Y^A = \mathcal{D}^{++} \bar{Y}^A = 0 \quad \Rightarrow \\
&m\tilde{F}\,\bar{Y}^A = -\frac{m}{2}\,\bar{Y}^A, \qquad m\tilde{F}\,Y^A = \frac{m}{2}\,Y^A.
\end{aligned} \tag{31}$$

Note that the $SU(2)$ group acting on the index A is a sort of the Pauli-Gürsey group *commuting* with $SU(2|1)$. The solution of the constraints (31) is

$$\begin{aligned}
Y^A\left(\zeta^{(3)}_{(A)}\right) &= y^A - \theta^+\psi^{iA}w_i^- + \theta^-\psi^{iA}w_i^+ - 2i\,\theta^-\bar{\theta}^+\dot{y}^A + 2i\,\theta^-\theta^+\dot{\bar{y}}^A \\
&\quad + m\,\theta^-\bar{\theta}^+y^A + m\,\theta^-\theta^+\bar{y}^A - 2i\,\theta^-\theta^+\bar{\theta}^+\dot{\psi}^{iA}w_i^-\,, \\
\bar{Y}^A\left(\bar{\zeta}^{(3)}_{(A)}\right) &= \bar{y}^A - \bar{\theta}^+\psi^{iA}w_i^- + \bar{\theta}^-\psi^{iA}w_i^+ - 2i\,\theta^+\bar{\theta}^-\dot{\bar{y}}^A + 2i\,\bar{\theta}^+\bar{\theta}^-\dot{y}^A \\
&\quad - m\,\theta^+\bar{\theta}^-\bar{y}^A - m\,\bar{\theta}^+\bar{\theta}^-y^A - 2i\,\bar{\theta}^-\theta^+\bar{\theta}^+\dot{\psi}^{iA}w_i^-\,.
\end{aligned} \tag{32}$$

The superfields Y^A, $\bar{Y}^A$ and their components transform as

[1]In the case of non-zero $U(1)$ charge, $\tilde{F}q^{+a} = \kappa(\sigma_3)^a{}_b q^{+b}$, there appears a bosonic WZ term for x^{ia} in the σ-model action, with the strength $\sim \kappa$.

[2]As opposed to the actions including both sorts simultaneously.

$$\delta Y^A = -\,m\left(\bar\theta^+\epsilon^- - \theta^+\bar\epsilon^-\right)Y^A, \qquad \delta\bar Y^A = m\left(\bar\theta^+\epsilon^- - \theta^+\bar\epsilon^-\right)\bar Y^A \qquad \Rightarrow$$
$$\delta y^A = -\,\epsilon_i\psi^{iA}, \qquad \delta\bar y^A = -\,\bar\epsilon_i\psi^{iA},$$
$$\delta\psi^{iA} = \bar\epsilon^i\left(2i\dot y^A - m y^A\right) - \epsilon^i\left(2i\dot{\bar y}^A + m\bar y^A\right). \tag{33}$$

We observe that the field content of Y^A is just $(\mathbf{4},\mathbf{4},\mathbf{0})$, but the $SU(2)$ assignment of the involved fields is different from that of the previous $(\mathbf{4},\mathbf{4},\mathbf{0})$ multiplet. The harmonic superfield q^{+a} describing the multiplet $(\mathbf{4},\mathbf{4},\mathbf{0})$ contains the bosonic field x^{ia} and the fermionic field $\psi^{i'a}$. Here we have the opposite realizations of $SU(2)$ groups on the fermionic and bosonic fields, respectively.

4.1 The σ-Model Actions

One can write the general $SU(2|1)$ invariant action in terms of the function $\tilde L$ as

$$\tilde S\left(Y,\bar Y\right) = \int dt\,\tilde{\mathcal L} = \int d\zeta_H\,\tilde L\left(Y,\bar Y\right), \tag{34}$$
$$\text{with}\quad m\left(y^B\partial_B - \bar y^B\bar\partial_B\right)\tilde L\left(y,\bar y\right) = 0\,.$$

Then the general component Lagrangian reads

$$\begin{aligned}\tilde{\mathcal L} = &\left[2\,\dot y^A\dot{\bar y}_A + \frac{i}{2}\,\psi^{iA}\dot\psi_{iA} - \frac{i}{2}\,\psi^{iA}\psi_{iC}\left(\dot y^C\partial_A + \dot{\bar y}^C\bar\partial_A\right)\right.\\ &\left.+\,\frac{1}{48}\,\psi^{iA}\psi^k_A\psi^B_i\psi_{kB}\,\Delta_y\right]\tilde G - \,im\left(\dot y^A\bar y_A - y^A\dot{\bar y}_A\right)\tilde G\\ &+2im\left(\dot y^A\partial_A\tilde L - \dot{\bar y}^A\bar\partial_A\tilde L\right) - m\,\psi^{iA}\psi^B_i\partial_A\bar\partial_B\tilde L\\ &+\frac{m}{4}\,\psi^{iA}\psi_{iC}\left(y^C\partial_A\tilde G - \bar y^C\bar\partial_A\tilde G\right) + \frac{m^2}{2}\,y^A\bar y_A\,\tilde G\\ &-m^2\left(y^A\partial_A\tilde L + \bar y^A\bar\partial_A\tilde L\right),\end{aligned} \tag{35}$$

where

$$\tilde G := \Delta_y\tilde L\,, \qquad \Delta_y = -\,2\,\epsilon^{AB}\partial_A\bar\partial_B\,, \qquad \partial_A = \frac{\partial}{\partial y^A}\,, \quad \bar\partial_B = \frac{\partial}{\partial\bar y^B}\,. \tag{36}$$

4.2 Wess–Zumino Term

For the mirror $(\mathbf{4},\mathbf{4},\mathbf{0})$ multiplet, the independent WZ term can be constructed as an integral over the analytic superspace

$$\tilde{S}_{\mathrm{WZ}}\left(Y,\bar{Y}\right) = -\gamma\int d\zeta^{--}_{(A)}\left(\bar{\theta}^{+}\bar{\mathcal{D}}^{+} + \theta^{+}\mathcal{D}^{+}\right)f\left(Y,\bar{Y}\right). \tag{37}$$

For ensuring the $SU(2|1)$ invariance, we need to impose the analyticity condition (23) on the Lagrangian density in (37)

$$\left(\bar{\theta}^{+}\bar{\mathcal{D}}^{+} + \theta^{+}\mathcal{D}^{+}\right)f\left(Y,\bar{Y}\right). \tag{38}$$

It amounts to the following condition on f:

$$\Delta_{y}f = 0\,. \tag{39}$$

In addition, the requirement of $SU(2|1)$ invariance implies a new constraint on f at $m \neq 0$:

$$m\tilde{F}f\left(Y,\bar{Y}\right) = 0 \qquad \Rightarrow \qquad m\left(y^{B}\partial_{B} - \bar{y}^{B}\bar{\partial}_{B}\right)f\left(y,\bar{y}\right) = 0\,. \tag{40}$$

In the limit $m = 0$, the matrix generator $\tilde{F}$ becomes an external automorphism generator and the condition (40) is satisfied trivially, without imposing any constraints on $f\left(Y_{A},\bar{Y}^{B}\right)$.

The component Lagrangian corresponding to the action (37) reads

$$\begin{aligned}\tilde{\mathcal{L}}_{\mathrm{WZ}} = 2\gamma\Big\{ & i\left(\dot{y}^{A}\,\partial_{A}f - \dot{\bar{y}}^{A}\,\bar{\partial}_{A}f\right) - \frac{m}{2}\left(y^{A}\,\partial_{A}f + \bar{y}^{A}\bar{\partial}_{A}f\right)\\ & - \frac{1}{2}\,\psi^{iA}\psi^{B}_{i}\,\partial_{A}\bar{\partial}_{B}f\Big\}.\end{aligned} \tag{41}$$

Employing the conditions (39), (40), one can directly check that this Lagrangian is indeed invariant under the supersymmetry transformations (33).

5 Superconformal Models

5.1 *The Superalgebra $D(2,1;\alpha)$ as a Closure of Its Two $su(2|1)$ Subalgebras*

The most general $\mathcal{N} = 4$, $d = 1$ superconformal group is $D(2,1;\alpha)$ [16, 17]:

$$\begin{aligned}&\{Q_{\alpha ii'}, Q_{\beta jj'}\} = 2\left(\epsilon_{ij}\epsilon_{i'j'}T_{\alpha\beta} + \alpha\,\epsilon_{\alpha\beta}\epsilon_{i'j'}J_{ij} - (1{+}\alpha)\,\epsilon_{\alpha\beta}\epsilon_{ij}L_{i'j'}\right), \\ &\left[T_{\alpha\beta}, Q_{\gamma ii'}\right] = -i\,\epsilon_{\gamma(\alpha}Q_{\beta)ii'}\,, \qquad \left[T_{\alpha\beta}, T_{\gamma\delta}\right] = i\left(\epsilon_{\alpha\gamma}T_{\beta\delta} + \epsilon_{\beta\delta}T_{\alpha\gamma}\right), \\ &\left[J_{ij}, Q_{\alpha ki'}\right] = -i\,\epsilon_{k(i}Q_{\alpha j)i'}\,, \qquad \left[J_{ij}, J_{kl}\right] = i\left(\epsilon_{ik}J_{jl} + \epsilon_{jl}J_{ik}\right), \\ &\left[L_{i'j'}, Q_{\alpha ik'}\right] = -i\,\epsilon_{k'(i'}Q_{\alpha ij')}\,, \qquad \left[L_{i'j'}, L_{k'l'}\right] = i\left(\epsilon_{i'k'}L_{j'l'} + \epsilon_{j'l'}L_{i'k'}\right).\end{aligned} \tag{42}$$

Here, $Q_{\alpha i i'}$ are eight supercharges and the bosonic subalgebra is

$$su(2) \oplus su(2)' \oplus so(2,1) \equiv \{J_{ik}\} \oplus \{L_{i'k'}\} \oplus \{T_{\alpha\beta}\}. \tag{43}$$

Switching α as $\alpha \leftrightarrow -(\alpha+1)$ amounts to switching $SU(2)$ generators as $J_{ik} \leftrightarrow L_{i'k'}$. At $\alpha = -1, 0$, the superalgebra $D(2,1;\alpha)$ is reduced to

$$D\,(2,1;\alpha) \cong PSU(1,1|2) \rtimes SU(2)_{\rm ext}\,. \tag{44}$$

How to implement $D(2,1;\alpha)$ in the $SU(2|1)$ superspaces? The crucial property allowing to do this is the existence of *two* different subalgebras $su(2|1) \subset D(2,1;\alpha)$, so that the latter is a closure of these two. These subalgebras are defined by the following relations

$$\begin{aligned}
&(\mathrm{I}).\quad \{Q^i, \bar{Q}_j\} = 2m(\mu)\, I^i_j + 2\delta^i_j [H(\mu) - m(\mu)\, F]\,, && (45)\\
&m(\mu) := -\alpha\mu\,,\quad H(\mu) := \mathcal{H} + \mu F\,,\quad \mathcal{H} = \hat{H} + \frac{\mu^2}{4}\hat{K}\,, &&\\
&(\hat{H}, \hat{K}) \in so(2,1)\,,\quad F \in su(2)',\quad I^i_j \in su(2)\,, &&\\
&(\mathrm{II}).\quad \{S^i, \bar{S}_j\} = 2m(-\mu)\, I^i_j + 2\delta^i_j [H(-\mu) - m(-\mu)\, F]. && (46)
\end{aligned}$$

Here, $Q_i := -\left(Q_{1i1'} + \frac{i}{2}\,\mu\, Q_{2i1'}\right)$, $S_i := -\left(Q_{1i1'} - \frac{i}{2}\,\mu\, Q_{2i1'}\right)$. The remaining $D\,(2,1;\alpha)$ generators appear in the anticommutators of S and Q.

The subgroup $SU(2|1)$ corresponding to the $su(2|1)$ subalgebra (I) is identified with the manifest superisometry of the $SU(2|1)$ superspace; then the second $SU(2|1)$ subgroup is realized on the superspace coordinates and superfields as a hidden symmetry. The most salient feature of the relevant realizations of $D(2,1;\alpha)$ is the *trigonometric* form of the realization of the $d=1$ bosonic conformal generators:

$$\hat{H} = \frac{i}{2}\,[1 + \cos\mu t\,]\,\partial_t\,,\quad \hat{K} = \frac{2i}{\mu^2}\,[1 - \cos\mu t\,]\,\partial_t\,,\quad \hat{D} = \frac{i}{\mu}\sin\mu t\,\partial_t\,. \tag{47}$$

The basic constraints for both types of the multiplet $(\mathbf{4},\mathbf{4},\mathbf{0})$ considered in the previous sections are $D\,(2,1;\alpha)$ covariant for any α.

The superconformal subclasses of the general $SU(2|1)$ actions are singled out by requiring them to be even functions of μ, in accord with the above structure of $D\,(2,1;\alpha)$ as a closure of two $su(2|1)$ subalgebras with $\pm\mu$.

5.2 *Superconformal Actions for the Multiplet* (4, 4, 0)

After redefining the fermionic fields as

$$\psi^a \to \psi^a e^{\frac{i}{2}\mu t},\qquad \bar{\psi}^a \to \bar{\psi}^a e^{-\frac{i}{2}\mu t},\qquad m = -\,\alpha\mu\,, \tag{48}$$

the transformations (25) are modified as

$$
\begin{aligned}
&\delta x^{ia} = -\,\epsilon^{i}\psi^{a}e^{\frac{i}{2}\mu t} - \bar{\epsilon}^{i}\bar{\psi}^{a}e^{-\frac{i}{2}\mu t},\\
&\delta\bar{\psi}_{a} = \left(2i\epsilon_{k}\dot{x}^{k}_{a} + \alpha\mu\,\epsilon_{k}x^{k}_{a}\right)e^{\frac{i}{2}\mu t}, \qquad \delta\psi^{a} = \left(2i\bar{\epsilon}^{k}\dot{x}^{a}_{k} - \alpha\mu\,\bar{\epsilon}^{k}x^{a}_{k}\right)e^{-\frac{i}{2}\mu t}
\end{aligned} \tag{49}
$$

These transformations correspond to the subalgebra (45) of $D\,(2, 1; \alpha)$. The second type of transformations, corresponding to the subalgebra (46), can be found via replacing $\mu \to -\mu$ in (49).

The superconformal superfield actions for the multiplet (**4**, **4**, **0**) are written as

$$
S^{(\alpha)}_{\rm sc}(q^2) = \int d\zeta_H\, L^{(\alpha)}_{\rm sc}\left(q^2\right), \tag{50}
$$

where the superfield function $L^{(\alpha)}_{\rm sc}\left(q^2\right)$ is given by

$$
\begin{aligned}
L^{(\alpha)}_{\rm sc}\left(q^2\right) = \begin{cases} \frac{\alpha^2}{4(1+\alpha)}\left(q^2\right)^{\frac{1}{\alpha}} & \text{for } \alpha \neq -1, 0, \\ \frac{1}{4}\left(q^2\right)^{-1}\ln\left(q^2\right) & \text{for } \alpha = -1, \end{cases} \quad &\Rightarrow \\
\Rightarrow\; G\left(x^{ia}\right) = \left(x^{ia}x_{ia}\right)^{\frac{1-\alpha}{\alpha}}&
\end{aligned} \tag{51}
$$

The relevant trigonometric superconformal component Lagrangian is given, for any $\alpha \neq 0$, by

$$
\begin{aligned}
\mathcal{L}^{(\alpha)}_{\rm sc} = \Big[\dot{x}^{ia}\dot{x}_{ia} &+ \frac{i}{2}\left(\bar{\psi}_a\dot{\psi}^a - \dot{\bar{\psi}}_a\psi^a\right) - \frac{i}{2}\left(\psi_a\bar{\psi}^c + \psi^c\bar{\psi}_a\right)\dot{x}^{ia}\partial_{ic} \\
&- \frac{1}{16}\left(\bar{\psi}\right)^2(\psi)^2\,\Delta_x - \frac{\alpha^2\mu^2}{4}\,x^{ia}x_{ia}\Big]\left(x^{ia}x_{ia}\right)^{\frac{1-\alpha}{\alpha}}.
\end{aligned} \tag{52}
$$

It is a deformation of the parabolic ($m = 0$) superconformal Lagrangian by the oscillator term [10].

5.3 Superconformal Actions for the Mirror Multiplet (4, 4, 0)

Passing to the new basis (45) of $su(2|1)$ implies the following field redefinitions (cf. (48)):

$$
y^A \to y^A e^{-\frac{i}{2}\mu t}, \qquad \bar{y}^A \to \bar{y}^A e^{\frac{i}{2}\mu t}, \qquad m = -\,\alpha\mu\,. \tag{53}
$$

They bring the transformations (33) to the form:

$$\delta y^A = -\,\epsilon_i \psi^{iA} e^{\frac{i}{2}\mu t}, \qquad \delta \bar{y}^A = -\,\bar{\epsilon}_i \psi^{iA} e^{-\frac{i}{2}\mu t},$$
$$\delta \psi^{iA} = \bar{\epsilon}^i \left[2i\dot{y}^A + (1+\alpha)\,\mu\, y^A\right] e^{-\frac{i}{2}\mu t} - \epsilon^i \left[2i\dot{\bar{y}}^A - (1+\alpha)\,\mu\, \bar{y}^A\right] e^{\frac{i}{2}\mu t}. \tag{54}$$

These transformations, together with those in which the replacement $\mu \to -\mu$ is made, once again generate superconformal $D(2, 1; \alpha)$ transformations.

The superconformal superfield actions are written as

$$\tilde{S}\left(Y, \bar{Y}\right) = \int d\zeta_H\, \tilde{L}\left(Y, \bar{Y}\right), \tag{55}$$

where $\tilde{L}$ is chosen to be

$$\tilde{L}^{(\alpha)}_{\rm sc}\left(Y, \bar{Y}\right) = \begin{cases} -\frac{(1+\alpha)^2}{2\alpha}\left(Y^A \bar{Y}_A\right)^{-\frac{1}{1+\alpha}} & \text{for } \alpha \neq -1, 0, \\ -\frac{1}{2}\left(Y^A \bar{Y}_A\right)^{-1} \ln\left(Y^A \bar{Y}_A\right) & \text{for } \alpha = 0, \end{cases}$$
$$\Rightarrow \tilde{G} = \left(y^A \bar{y}_A\right)^{-\frac{2+\alpha}{1+\alpha}}. \tag{56}$$

The component superconformal Lagrangian for the mirror multiplet (**4**, **4**, **0**) reads

$$\tilde{\mathcal{L}}^{(\alpha)}_{\rm sc} = \Big[2\,\dot{y}^A \dot{\bar{y}}_A + \frac{i}{2}\,\psi^{iA}\dot{\psi}_{iA} - \frac{i}{2}\,\psi^{iA}\psi_{iC}\left(\dot{y}^C \partial_A + \dot{\bar{y}}^C \bar{\partial}_A\right)$$
$$+ \frac{1}{48}\,\psi^{iA}\psi^k_A \psi^B_i \psi_{kB}\,\Delta_y - \frac{(1+\alpha)^2\,\mu^2}{2}\, y^A \bar{y}_A\Big]\left(y^A \bar{y}_A\right)^{-\frac{2+\alpha}{1+\alpha}}. \tag{57}$$

The trigonometric superconformal models of both (**4**, **4**, **0**) multiplets are equivalent to each other, up to the substitutions

$$x^{ia} \leftrightarrow y^{i'A}, \qquad \psi^{i'a} \leftrightarrow \psi^{iA} \qquad \alpha \leftrightarrow -\,(1+\alpha)\,. \tag{58}$$

The interchange $\alpha \leftrightarrow -\,(1+\alpha)$ amounts to permuting the $SU(2)$ and $SU'(2)$ generators in $D\,(2, 1; \alpha)$. So such an interchange is an automorphism of the superconformal superalgebra.

In the special case $\alpha = -1$ for Y^A, $\bar{Y}^B$ ($\alpha = 0$ for q^{+a}), superconformal action can be obtained via a standard trick described in [10].

6 Summary and Outlook

In this contribution, we reviewed the construction of the $d = 1$ harmonic superspace approach to the supergroup $SU(2|1)$ as a deformation of the flat $\mathcal{N} = 4$, $d = 1$ supersymmetry and presented the general superfield and component actions for the $SU(2|1)$ multiplet (**4**, **4**, **0**) and its "mirror" version. We also explained how to select the superconformal subclass of general $SU(2|1)$ invariant actions of these multiplets.

In contrast to the flat harmonic superspace, there is no direct equivalence between the two types of the (**4**, **4**, **0**) supermultiplet. One of the manifestations of this non-equivalence is the non-existence of the $SU(2|1)$ invariant WZ action for $q^{\pm a}$ and the existence of such an action for the mirror multiplet. On the other hand, the superconformal models of both (**4**, **4**, **0**) multiplets are equivalent to each other.

As some particular research prospects, it is worth to mention the construction of SQM models for the $SU(2|1)$ multiplet (**3**, **4**, **1**) which should have a natural description in the analytic harmonic superspace, the construction of the multi-particle SQM models involving various types of the $SU(2|1)$ multiplets, and generalizations of the harmonic superspace approach to higher-rank deformed $d = 1$ supersymmetries, e.g., $SU(2|2)$, which can be viewed as a deformation of the flat $\mathcal{N} = 8, d = 1$ supersymmetry [1]. It would be also interesting to generalize, to the $SU(2|1)$ case, some important notions of the flat $\mathcal{N} = 4, d = 1$ supersymmetry, such as the semi-dynamical spin multiplets [18], the gauging procedure in the $\mathcal{N} = 4$ SQM models [19], etc. There also remains the problem of recovering $SU(2|1)$ SQM models through the direct dimensional reduction from the higher-dimensional theories with the curved analogs of the Poincaré supersymmetry. Recently, a few $SU(2|1)$ SQM models were reobtained in this way [20–22] and used for clarifying some properties of the "parent" theories.

Acknowledgements Evgeny Ivanov thanks the organizers of the 11-th International Workshop "Lie Theory and Its Applications in Physics" and, especially, Vladimir Dobrev for the kind hospitality in Varna. The authors acknowledge a partial support from the RFBR grant 15-02-06670 and a grant of the Heisenberg - Landau program.

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Intrinsic Sound of Anti-de Sitter Manifolds

Toshiyuki Kobayashi

Abstract As is well-known for compact Riemann surfaces, eigenvalues of the Laplacian are distributed discretely and most of eigenvalues vary viewed as functions on the Teichmüller space. We discuss a new feature in the Lorentzian geometry, or more generally, in pseudo-Riemannian geometry. One of the distinguished features is that L^2-eigenvalues of the Laplacian may be distributed densely in $\mathbb{R}$ in pseudo-Riemannian geometry. For three-dimensional anti-de Sitter manifolds, we also explain another feature proved in joint with F. Kassel [Adv. Math. 2016] that there exist countably many L^2-eigenvalues of the Laplacian that are stable under any small deformation of anti-de Sitter structure. Partially supported by Grant-in-Aid for Scientific Research (A) (25247006), Japan Society for the Promotion of Science.

Keywords Laplacian · Locally symmetric space · Lorentzian manifold · Spectral analysis · Clifford–Klein form · Reductive group · Discontinuous group

MSC Classification: Primary 22E40 · 22E46 · 58J50: Secondary 11F72 · 53C35

1 Introduction

Our "common sense" for music instruments says:

"shorter strings produce a higher pitch than longer strings",

"thinner strings produce a higher pitch than thicker strings".

Let us try to "hear the sound of pseudo-Riemannian locally symmetric spaces". Contrary to our "common sense" in the Riemannian world, we find a phenomenon

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V. Dobrev (ed.), *Lie Theory and Its Applications in Physics*,
Springer Proceedings in Mathematics & Statistics 191,
DOI 10.1007/978-981-10-2636-2_6

that compact three-dimensional anti-de Sitter manifolds have "intrinsic sound" which is stable under any small deformation. This is formulated in the framework of spectral analysis of anti-de Sitter manifolds, or more generally, of pseudo-Riemannian locally symmetric spaces X_Γ. In this article, we give a flavor of this new topic by comparing it with the flat case and the Riemannian case.

To explain briefly the subject, let X be a pseudo-Riemannian manifold, and Γ a discrete isometry group acting properly discontinuously and freely on X. Then the quotient space $X_\Gamma := \Gamma\backslash X$ carries a pseudo-Riemannian manifold structure such that the covering map $X \to X_\Gamma$ is isometric. We are particularly interested in the case where X_Γ is a pseudo-Riemannian locally symmetric space, see Sect. 3.2.

Problems we have in mind are symbolized in the following diagram:

	existence problem	deformation *versus* rigidity
Geometry	Does cocompact Γ exist? (Sect. 4.1)	Higher Teichmüller theory *versus* rigidity theorem (Sect. 4.2)
Analysis	Does L^2-spectrum exist? (Problem 1)	Whether L^2-eigenvalues vary or not (Problem 2)

2 A Program

In [5, 6, 12] we initiated the study of "spectral analysis on pseudo-Riemannian locally symmetric spaces" with focus on the following two problems:

Problem 1 Construct eigenfunctions of the Laplacian Δ_{X_Γ} on X_Γ. Does there exist a nonzero L^2-eigenfunction?

Problem 2 Understand the behaviour of L^2-eigenvalues of the Laplacian Δ_{X_Γ} on X_Γ under small deformation of Γ inside G.

Even when X_Γ is compact, the existence of countably many L^2-eigenvalues is already nontrivial because the Laplacian Δ_{X_Γ} is not elliptic in our setting. We shall discuss in Sect. 2.2 for further difficulties concerning Problems 1 and 2 when X_Γ is non-Riemannian.

We may extend these problems by considering *joint* eigenfunctions for "invariant differential operators" on X_Γ rather than the single operator Δ_{X_Γ}. Here by "invariant differential operators on X_Γ" we mean differential operators that are induced from G-invariant ones on $X = G/H$. In Sect. 7, we discuss Problems 1 and 2 in this general formulation based on the recent joint work [6, 7] with F. Kassel.

2.1 Known Results

Spectral analysis on a pseudo-Riemannian locally symmetric space $X_\Gamma = \Gamma\backslash X = \Gamma\backslash G/H$ is already deep and difficult in the following special cases:

(1) (noncommutative harmonic analysis on G/H) $\Gamma = \{e\}$.

In this case, the group G acts unitarily on the Hilbert space $L^2(X_\Gamma) = L^2(X)$ by translation $f(\cdot) \mapsto f(g^{-1}\cdot)$, and the irreducible decomposition of $L^2(X)$ (*Plancherel-type formula*) is essentially equivalent to the spectral analysis of G-invariant differential operators when X is a semisimple symmetric space. Noncommutative harmonic analysis on semisimple symmetric spaces X has been developed extensively by the work of Helgason, Flensted-Jensen, Matsuki–Oshima–Sekiguchi, Delorme, van den Ban–Schlichtkrull among others as a generalization of Harish Chandra's earlier work on the regular representation $L^2(G)$ for group manifolds.

(2) (automorphic forms) H is compact and Γ is arithmetic.

If H is a maximal compact subgroup of G, then $X_\Gamma = \Gamma\backslash G/H$ is a Riemannian locally symmetric space and the Laplacian Δ_{X_Γ} is an elliptic differential operator. Then there exist infinitely many L^2-eigenvalues of Δ_{X_Γ} if X_Γ is compact by the general theory for compact Riemannian manifolds (see Fact 1). If furthermore Γ is irreducible, then Weil's local rigidity theorem [18] states that nontrivial deformations exist only when X is the hyperbolic plane $SL(2,\mathbb{R})/SO(2)$, in which case compact quotients X_Γ have a classically-known deformation space modulo conjugation, i.e., their Teichmüller space. Viewed as a function on the Teichmüller space, L^2-eigenvalues vary analytically [1, 20], see Fact 11.

Spectral analysis on X_Γ is closely related to the theory of automorphic forms in the Archimedean place if Γ is an arithmetic subgroup.

(3) (abelian case) $G = \mathbb{R}^{p+q}$ with $H = \{0\}$ and $\Gamma = \mathbb{Z}^{p+q}$.

We equip $X = G/H$ with the standard flat pseudo-Riemannian structure of signature (p, q) (see Example 1). In this case, G is abelian, but $X = G/H$ is non-Riemannian. This is seemingly easy, however, spectral analysis on the $(p+q)$-torus $\mathbb{R}^{p+q}/\mathbb{Z}^{p+q}$ is much involved, as we shall observe a connection with Oppenheim's conjecture (see Sect. 5.2).

2.2 Difficulties in the New Settings

If we try to attack a problem of spectral analysis on $\Gamma\backslash G/H$ in the more general case where H is noncompact and Γ is infinite, then new difficulties may arise from several points of view:

(1) Geometry. The G-invariant pseudo-Riemannian structure on $X = G/H$ is not Riemannian anymore, and discrete groups of isometries of X do not always act properly discontinuously on such X.

(2) Analysis. The Laplacian Δ_X on X_Γ is not an elliptic differential operator. Furthermore, it is not clear if Δ_X has a self-adjoint extension on $L^2(X_\Gamma)$.
(3) Representation theory. If Γ acts properly discontinuously on $X = G/H$ with H noncompact, then the volume of $\Gamma\backslash G$ is infinite, and the regular representation $L^2(\Gamma\backslash G)$ may have infinite multiplicities. In turn, the group G may not have a good control of functions on $\Gamma\backslash G$. Moreover $L^2(X_\Gamma)$ is not a subspace of $L^2(\Gamma\backslash G)$ because H is noncompact. All these observations suggest that an application of the representation theory of $L^2(\Gamma\backslash G)$ to spectral analysis on X_Γ is rather limited when H is noncompact.

Point (1) creates some underlying difficulty to Problem 2: we need to consider locally symmetric spaces X_Γ for which proper discontinuity of the action of Γ on X is preserved under small deformations of Γ in G. This is nontrivial. This question was first studied by the author [9, 11]. See [4] for further study. An interesting aspect of the case of noncompact H is that there are more examples where nontrivial deformations of compact quotients exist than for compact H (*cf*. Weil's local rigidity theorem [18]). Perspectives from Point (1) will be discussed in Sect. 4.

Point (2) makes Problem 1 nontrivial. It is not clear if the following well-known properties in the *Riemannian* case holds in our setting in the *pseudo-Riemannian* case.

Fact 1 *Suppose M is a compact Riemannian manifold.*

(1) *The Laplacian Δ_M extends to a self-adjoint operator on $L^2(M)$.*
(2) *There exist infinitely many L^2-eigenvalues of Δ_M.*
(3) *An eigenfunction of Δ_M is infinitely differentiable.*
(4) *Each eigenspace of Δ_M is finite-dimensional.*
(5) *The set of L^2-eigenvalues is discrete in $\mathbb{R}$.*

Remark 1 We shall see that the third to fifth properties of Fact 1 may fail in the pseudo-Riemannian case, e.g., Example 6 for (3) and (4), and $M = \mathbb{R}^{2,1}/\mathbb{Z}^3$ (Theorem 7).

In spite of these difficulties, we wish to reveal a mystery of spectral analysis of pseudo-Riemannian locally homogeneous spaces $X_\Gamma = \Gamma\backslash G/H$. We shall discuss self-adjoint extension of the Laplacian in the pseudo-Riemannian setting in Theorem 13, and the existence of countable many L^2-eigenvalues in Theorems 8, 12 and 13.

3 Pseudo-Riemannian Manifolds

3.1 Laplacian on Pseudo-Riemannian Manifolds

A *pseudo-Riemannian manifold M* is a smooth manifold endowed with a smooth, nondegenerate, symmetric bilinear tensor g of signature (p, q) for some $p, q \in \mathbb{N}$.

(M, g) is a Riemannian manifold if $q = 0$, and is a Lorentzian manifold if $q = 1$. The metric tensor g induces a Radon measure $d\mu$ on X, and the divergence div. Then the Laplacian

$$\Delta_M := \text{div grad},$$

is a differential operator of second order which is a symmetric operator on the Hilbert space $L^2(X, d\mu)$.

Example 1 Let (M, g) be the standard flat pseudo-Riemannian manifold:

$$\mathbb{R}^{p,q} := (\mathbb{R}^{p+q}, dx_1^2 + \cdots + dx_p^2 - dx_{p+1}^2 - \cdots - dx_{p+q}^2).$$

Then the Laplacian takes the form

$$\Delta_{\mathbb{R}^{p,q}} = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \cdots - \frac{\partial^2}{\partial x_{p+q}^2}.$$

In general, Δ_M is an elliptic differential operator if (M, g) is Riemannian, and is a hyperbolic operator if (M, g) is Lorentzian.

3.2 Homogeneous Pseudo-Riemannian Manifolds

A typical example of pseudo-Riemannian manifolds X with "large" isometry groups is semisimple symmetric spaces, for which the infinitesimal classification was accomplished by M. Berger in 1950s. In this case, X is given as a homogeneous space G/H where G is a semisimple Lie group and H is an open subgroup of the fixed point group $G^\sigma = \{g \in G : \sigma g = g\}$ for some involutive automorphism σ of G. In particular, $G \supset H$ are a pair of reductive Lie groups.

More generally, we say G/H is a *reductive homogeneous space* if $G \supset H$ are a pair of real reductive algebraic groups. Then we have the following:

Proposition 1 *Any reductive homogeneous space $X = G/H$ carries a pseudo-Riemannian structure such that G acts on X by isometries.*

Proof By a theorem of Mostow, we can take a Cartan involution θ of G such that $\theta H = H$. Then $K := G^\theta$ is a maximal compact subgroup of G, and $H \cap K$ is that of H. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the corresponding Cartan decomposition of the Lie algebra $\mathfrak{g}$ of G. Take an $\mathrm{Ad}(G)$-invariant nondegenerate symmetric bilinear form $\langle\ ,\ \rangle$ on $\mathfrak{g}$ such that $\langle\ ,\ \rangle|_{\mathfrak{k}\times\mathfrak{k}}$ is negative definite, $\langle\ ,\ \rangle|_{\mathfrak{p}\times\mathfrak{p}}$ is positive definite, and $\mathfrak{k}$ and $\mathfrak{p}$ are orthogonal to each other. (If G is semisimple, then we may take $\langle\ ,\ \rangle$ to be the Killing form of $\mathfrak{g}$.)

Since $\theta H = H$, the Lie algebra $\mathfrak{h}$ of H is decomposed into a direct sum $\mathfrak{h} = (\mathfrak{h} \cap \mathfrak{k}) + (\mathfrak{h} \cap \mathfrak{p})$, and therefore the bilinear form $\langle\ ,\ \rangle$ is nondegenerate when restricted to $\mathfrak{h}$. Then $\langle\ ,\ \rangle$ induces an $\mathrm{Ad}(H)$-invariant nondegenerate symmetric bilinear form

$\langle\ ,\ \rangle_{\mathfrak{g}/\mathfrak{h}}$ on the quotient space $\mathfrak{g}/\mathfrak{h}$, with which we identify the tangent space $T_o(G/H)$ at the origin $o = eH \in G/H$. Since the bilinear form $\langle\ ,\ \rangle_{\mathfrak{g}/\mathfrak{h}}$ is $\mathrm{Ad}(H)$-invariant, the left translation of this form is well-defined and gives a pseudo-Riemannian structure g on G/H of signature $(\dim \mathfrak{p}/\mathfrak{h} \cap \mathfrak{p}, \dim \mathfrak{k}/\mathfrak{h} \cap \mathfrak{k})$. By the construction, the group G acts on the pseudo-Riemannian manifold $(G/H, g)$ by isometries. $\square$

3.3 Pseudo-Riemannian Manifolds with Constant Curvature, Anti-de Sitter Manifolds

Let $Q_{p,q}(x) := x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_{p+q}^2$ be a quadratic form on $\mathbb{R}^{p+q}$ of signature (p, q), and we denote by $O(p, q)$ the indefinite orthogonal group preserving the form $Q_{p,q}$. We define two hypersurfaces $M_\pm^{p,q}$ in $\mathbb{R}^{p+q}$ by

$$M_\pm^{p,q} := \{x \in \mathbb{R}^{p+q} : Q_{p,q}(x) = \pm 1\}.$$

By switching p and q, we have an obvious diffeomorphism

$$M_+^{p,q} \simeq M_-^{q,p}.$$

The flat pseudo-Riemannian structure $\mathbb{R}^{p,q}$ (Example 1) induces a pseudo-Riemannian structure on the hypersurface $M_+^{p,q}$ of signature $(p-1, q)$ with constant curvature 1, and that on $M_-^{p,q}$ of signature $(p, q-1)$ with constant curvature -1.

The natural action of the group $O(p, q)$ on $\mathbb{R}^{p,q}$ induces an isometric and transitive action on the hypersurfaces $M_\pm^{p,q}$, and thus they are expressed as homogeneous spaces:

$$M_+^{p,q} \simeq O(p,q)/O(p-1,q), \quad M_-^{p,q} \simeq O(p,q)/O(p,q-1),$$

giving examples of pseudo-Riemannian homogeneous spaces as in Proposition 1.

The *anti-de Sitter space* $\mathrm{AdS}^n = M_-^{n-1,2}$ is a model space for n-dimensional Lorentzian manifolds of constant negative sectional curvature, or *anti-de Sitter n-manifolds*. This is a Lorentzian analogue of the real hyperbolic space H^n. For the convenience of the reader, we list model spaces of Riemannian and Lorentzian manifolds with constant positive, zero, and negative curvatures.

Riemannian manifolds with constant curvature:

$S^n = M_+^{n+1,0} \simeq O(n+1)/O(n)$	: standard sphere,
$\mathbb{R}^n$	: Euclidean space,
$H^n = M_-^{n,1} \simeq O(1,n)/O(n)$	: hyperbolic space,

Lorentzian manifolds with constant curvature:

$$\begin{array}{lll} \mathrm{dS}^n = M_+^{n,1} \simeq O(n,1)/O(n-1,1) & & : \text{de Sitter space}, \\ \mathbb{R}^{n-1,1} & & : \text{Minkowski space}, \\ \mathrm{AdS}^n = M_-^{n-1,2} \simeq O(2,n-1)/O(1,n-1) & & : \text{anti-de Sitter space}, \end{array}$$

4 Discontinuous Groups for Pseudo-Riemannian Manifolds

4.1 Existence Problem of Compact Clifford–Klein Forms

Let H be a closed subgroup of a Lie group G, and $X = G/H$, and Γ a discrete subgroup of G. If H is compact, then the double coset space $\Gamma\backslash G/H$ becomes a C^∞-manifold for any torsion-free discrete subgroup Γ of G. However, we have to be careful for noncompact H, because not all discrete subgroups acts properly discontinuously on G/H, and $\Gamma\backslash G/H$ may not be Hausdorff in the quotient topology. We illustrate this feature by two general results:

Fact 2 (1) (Moore's ergodicity theorem [15]) *Let G be a simple Lie group, and Γ a lattice. Then Γ acts ergodically on G/H for any noncompact closed subgroup H. In particular, $\Gamma\backslash G/H$ is non-Hausdorff.*

(2) (Calabi–Markus phenomenon [2, 8]) *Let G be a reductive Lie group, and Γ an infinite discrete subgroup. Then $\Gamma\backslash G/H$ is non-Hausdorff for any reductive subgroup H with* $\mathrm{rank}_{\mathbb{R}}\, G = \mathrm{rank}_{\mathbb{R}}\, H$.

In fact, determining which groups act properly discontinuously on reductive homogeneous spaces G/H is a delicate problem, which was first considered in full generality by the author; we refer to [13, Sect. 3.2] for a survey.

Suppose now a discrete subgroup Γ acts properly discontinuously and freely on $X = G/H$. Then the quotient space

$$X_\Gamma := \Gamma\backslash X \simeq \Gamma\backslash G/H$$

carries a C^∞-manifold structure such that the quotient map $p : X \to X_\Gamma$ is a covering, through which X_Γ inherits any G-invariant local geometric structure on X. We say Γ is a *discontinuous group for X* and X_Γ is a *Clifford–Klein form* of $X = G/H$.

Example 2 (1) If $X = G/H$ is a reductive homogeneous space, then any Clifford–Klein form X_Γ carries a pseudo-Riemannian structure by Proposition 1.

(2) If $X = G/H$ is a semisimple symmetric space, then any Clifford–Klein form $X_\Gamma = \Gamma\backslash G/H$ is a pseudo-Riemannian locally symmetric space, namely, the (local) geodesic symmetry at every $p \in X_\Gamma$ with respect to the Levi-Civita connection is locally isometric.

By *space forms*, we mean pseudo-Riemannian manifolds of constant sectional curvature. They are examples of pseudo-Riemannian locally symmetric spaces. For simplicity, we shall assume that they are geodesically complete.

Example 3 Clifford–Klein forms of $M_+^{p+1,q} = O(p+1,q)/O(p,q)$ (respectively, $M_-^{p,q+1} = O(p,q+1)/O(p,q)$) are pseudo-Riemannian space forms of signature (p,q) with positive (respectively, negative) curvature. Conversely, any (geodesically complete) pseudo-Riemannian space form of signature (p,q) is of this form as far as $p \neq 1$ for positive curvature or $q \neq 1$ for negative curvature.

A general question for reductive homogeneous spaces G/H is:

Question 1 Does compact Clifford–Klein forms of G/H exist?

or equivalently,

Question 2 Does there exist a discrete subgroup Γ of G acting cocompactly and properly discontinuously on G/H?

This question has an affirmative answer if H is compact by a theorem of Borel. In the general setting where H is noncompact, the question relates with a "global theory" of pseudo-Riemannian geometry: *how local pseudo-Riemannian homogeneous structure affects the global nature of manifolds*? A classic example is *space form problem* which asks the global properties (e.g. compactness, volume, fundamental groups, etc.) of a pseudo-Riemannian manifold of constant curvature (local property). The study of discontinuous groups for $M_+^{p+1,q}$ and $M_-^{p,q+1}$ shows the following results in pseudo-Riemannian space forms of signature (p,q):

Fact 3 *Space forms of positive curvature are*

(1) *always closed if* $q = 0$*, i.e., sphere geometry in the Riemannian case;*
(2) *never closed if* $p \geq q > 0$*, in particular, if* $q = 1$ *(de Sitter geometry in the Lorentzian case [2]).*

The phenomenon in the second statement is called the *Calabi–Markus phenomenon* (see Fact 2 (2) in the general setting).

Fact 4 *Compact space forms of negative curvature exist*

(1) *for all dimensions if* $q = 0$*, i.e., hyperbolic geometry in the Riemannian case;*
(2) *for odd dimensions if* $q = 1$*, i.e., anti-de Sitter geometry in the Lorentzian case;*
(3) *for* $(p,q) = (4m, 3)$ $(m \in \mathbb{N})$ *or* $(8,7)$.

See [13, Sect. 4] for the survey of the space form problem in pseudo-Riemannian geometry and also of Question 1 for more general G/H.

A large and important class of Clifford–Klein forms X_Γ of a reductive homogeneous space $X = G/H$ is constructed as follows (see [8]).

Definition 1 A quotient $X_\Gamma = \Gamma \backslash X$ of X by a discrete subgroup Γ of G is called *standard* if Γ is contained in some reductive subgroup L of G acting properly on X.

If a subgroup L acts properly on G/H, then any discrete subgroup of Γ acts properly discontinuously on G/H. A handy criterion for the triple (G, H, L) of reductive groups such that L acts properly on G/H is proved in [8], as we shall recall below. Let $G = K \exp \overline{\mathfrak{a}_+} K$ be a Cartan decomposition, where $\mathfrak{a}$ is a maximal abelian subspace of $\mathfrak{p}$ and $\overline{\mathfrak{a}_+}$ is the dominant Weyl chamber with respect to a fixed positive system $\Sigma^+(\mathfrak{g}, \mathfrak{a})$. This defines a map $\mu : G \to \overline{\mathfrak{a}_+}$ (*Cartan projection*) by

$$\mu(k_1 e^X k_2) = X \quad \text{for } k_1, k_2 \in K \text{ and } X \in \mathfrak{a}.$$

It is continuous, proper and surjective. If H is a reductive subgroup, then there exists $g \in G$ such that $\mu(gHg^{-1})$ is given by the intersection of $\overline{\mathfrak{a}_+}$ with a subspace of dimension $\operatorname{rank}_{\mathbb{R}} H$. By an abuse of notation, we use the same H instead of gHg^{-1}. With this convention, we have:

Properness Criterion 5 *([8]) L acts properly on G/H if and only if $\mu(L) \cap \mu(H) = \{0\}$.*

By taking a lattice Γ of such L, we found a family of pseudo-Riemannian locally symmetric spaces X_Γ in [8, 13]. The list of symmetric spaces admitting standard Clifford–Klein forms of finite volume (or compact forms) include $M_-^{p,q+1} = O(p, q+1)/O(p, q)$ with (p, q) satisfying the conditions in Fact 4. Further, by applying Properness Criterion 5, Okuda [16] gave examples of pseudo-Riemannian locally symmetric spaces $\Gamma\backslash G/H$ of infinite volume where Γ is isomorphic to the fundamental group $\pi_1(\Sigma_g)$ of a compact Riemann surface Σ_g with $g \geq 2$.

For the construction of stable spectrum on X_Γ (see Theorems 10 and 12 (2) below), we introduced in [6, Sect. 1.6] the following concept:

Definition 2 A discrete subgroup Γ of G acts *strongly properly discontinuously* (or *sharply*) on $X = G/H$ if there exists $C, C' > 0$ such that for all $\gamma \in \Gamma$,

$$d(\mu(\gamma), \mu(H)) \geq C\|\mu(\gamma)\| - C'.$$

Here $d(\cdot, \cdot)$ is a distance in $\mathfrak{a}$ given by a Euclidean norm $\|\cdot\|$ which is invariant under the Weyl group of the restricted root system $\Sigma(\mathfrak{g}, \mathfrak{a})$. We say the positive number C is *the first sharpness constant* for Γ.

If a reductive subgroup L acts properly on a reductive homogeneous space G/H, then the action of a discrete subgroup Γ of L is strongly properly discontinuous ([6, Example 4.10]).

4.2 Deformation of Clifford–Klein Forms

Let G be a Lie group and Γ a finitely generated group. We denote by $\operatorname{Hom}(\Gamma, G)$ the set of all homomorphisms of Γ to G topologized by pointwise convergence. By taking a finite set $\{\gamma_1, \cdots, \gamma_k\}$ of generators of Γ, we can identify $\mathrm{Hom}(\Gamma, G)$ as a subset of the direct product $G \times \cdots \times G$ by the inclusion:

$$\mathrm{Hom}(\Gamma, G) \hookrightarrow G \times \cdots \times G, \quad \varphi \mapsto (\varphi(\gamma_1), \cdots, \varphi(\gamma_k)). \tag{1}$$

If Γ is finitely presentable, then $\mathrm{Hom}(\Gamma, G)$ is realized as a real analytic variety via (1).

Suppose G acts continuously on a manifold X. We shall take $X = G/H$ with noncompact closed subgroup H later. Then not all discrete subgroups act properly discontinuously on X in this general setting. The main difference of the following definition of the author [9] in the general case from that of Weil [18] is a requirement of proper discontinuity.

$$\begin{aligned} R(\Gamma, G; X) := \{\varphi \in \mathrm{Hom}(\Gamma, G) : & \ \varphi \text{ is injective,} \\ & \text{and } \varphi(\Gamma) \text{ acts properly discontinuously and freely on } G/H\}. \end{aligned} \tag{2}$$

Suppose now $X = G/H$ for a closed subgroup H. Then the double coset space $\varphi(\Gamma)\backslash G/H$ forms a family of manifolds that are locally modelled on G/H with parameter $\varphi \in R(\Gamma, G; X)$. To be more precise on "parameter", we note that the conjugation by an element of G induces an automorphism of $\mathrm{Hom}(\Gamma, G)$ which leaves $R(\Gamma, G; X)$ invariant. Taking these unessential deformations into account, we define the *deformation space* (*generalized Teichmüller space*) as the quotient set

$$\mathcal{T}(\Gamma, G; X) := R(\Gamma, G; X)/G.$$

Example 4 (1) Let Γ be the surface group $\pi_1(\Sigma_g)$ of genus $g \geq 2$, $G = PSL(2, \mathbb{R})$, $X = H^2$ (two-dimensional hyperbolic space). Then $\mathcal{T}(\Gamma, G; X)$ is the classical Teichmüller space, which is of dimension $6g - 6$.

(2) $G = \mathbb{R}^n$, $X = \mathbb{R}^n$, $\Gamma = \mathbb{Z}^n$. Then $\mathcal{T}(\Gamma, G; X) \simeq GL(n, \mathbb{R})$ (see (4) below).

(3) $G = SO(2, 2)$, $X = \mathrm{AdS}^3$, and $\Gamma = \pi_1(\Sigma_g)$. Then $\mathcal{T}(\Gamma, G; X)$ is of dimension $12g - 12$ (see [6, Sect. 9.2] and references therein).

Remark 2 There is a natural isometry between $X_{\varphi(\Gamma)}$ and $X_{\varphi(g\Gamma g^{-1})}$. Hence, the set $\mathrm{Spec}_d(X_{\varphi(\Gamma)})$ of L^2-eigenvalues is independent of the conjugation of $\varphi \in R(\Gamma, G; X)$ by an element of G. By an abuse of notation we shall write $\mathrm{Spec}_d(X_{\varphi(\Gamma)})$ for $\varphi \in \mathcal{T}(\Gamma, G; X)$ when we deal with Problem 2 of Sect. 2.

5 Spectrum on $\mathbb{R}^{p,q}/\mathbb{Z}^{p+q}$ and Oppenheim Conjecture

This section gives an elementary but inspiring observation of spectrum on flat pseudo-Riemannian manifolds.

5.1 *Spectrum of* $\mathbb{R}^{p,q}/\varphi(\mathbb{Z}^{p+q})$

Let $G = \mathbb{R}^n$ and $\Gamma = \mathbb{Z}^n$. Then the group homomorphism $\varphi : \Gamma \to G$ is uniquely determined by the image $\varphi(\mathbf{e}_j)$ $(1 \le j \le n)$ where $\mathbf{e}_1, \ldots, \mathbf{e}_n \in \mathbb{Z}^n$ are the standard basis, and thus we have a bijection

$$\mathrm{Hom}(\Gamma, G) \overset{\sim}{\leftarrow} M(n, \mathbb{R}), \quad \varphi_g \leftarrow\!\shortmid g \tag{3}$$

by $\varphi_g(\mathbf{m}) := g\mathbf{m}$ for $\mathbf{m} \in \mathbb{Z}^n$, or equivalently, by $g = (\varphi_g(\mathbf{e}_1), \ldots, \varphi_g(\mathbf{e}_n))$.

Let $\sigma \in \mathrm{Aut}(G)$ be defined by $\sigma(\mathbf{x}) := -\mathbf{x}$. Then $H := G^\sigma = \{0\}$ and $X := G/H \simeq \mathbb{R}^n$ is a symmetric space. The discrete group Γ acts properly discontinuously on X via φ_g if and only if $g \in GL(n, \mathbb{R})$. Moreover, since G is abelian, G acts trivially on $\mathrm{Hom}(\Gamma, G)$ by conjugation, and therefore the deformation space $\mathcal{T}(\Gamma, G; X)$ identifies with $R(\Gamma, G; X)$. Hence we have a natural bijection between the two subsets of (3):

$$\mathcal{T}(\Gamma, G; X) \overset{\sim}{\leftarrow} GL(n, \mathbb{R}). \tag{4}$$

Fix $p, q \in \mathbb{N}$ such that $p + q = n$, and we endow $X \simeq \mathbb{R}^n$ with the standard flat indefinite metric $\mathbb{R}^{p,q}$ (see Example 1). Let us determine $\mathrm{Spec}_d(X_{\varphi_g(\Gamma)}) \simeq \mathrm{Spec}_d(\mathbb{R}^{p,q}/\varphi_g(\mathbb{Z}^n))$ for $g \in GL(n, \mathbb{R}) \simeq \mathcal{T}(\Gamma, G; X)$.

For this, we define a function on $X = \mathbb{R}^n$ by

$$f_{\mathbf{m}}(\mathbf{x}) := \exp(2\pi\sqrt{-1}\,{}^t\mathbf{m}g^{-1}\mathbf{x}) \qquad (\mathbf{x} \in \mathbb{R}^n)$$

for each $\mathbf{m} \in \mathbb{Z}^n$ where $\mathbf{x}$ and $\mathbf{m}$ are regarded as column vectors. Clearly, $f_{\mathbf{m}}$ is $\varphi_g(\Gamma)$-periodic and defines a real analytic function on $X_{\varphi_g(\Gamma)}$. Furthermore, $f_{\mathbf{m}}$ is an eigenfunction of the Laplacian $\Delta_{\mathbb{R}^{p,q}}$:

$$\Delta_{\mathbb{R}^{p,q}} f_{\mathbf{m}} = -4\pi^2 Q_{g^{-1}I_{p,q}{}^t g^{-1}}(\mathbf{m}) f_{\mathbf{m}},$$

where, for a symmetric matrix $S \in M(n, \mathbb{R})$, Q_S denotes the quadratic form on $\mathbb{R}^n$ given by

$$Q_S(\mathbf{y}) := {}^t\mathbf{y}S\mathbf{y} \qquad \text{for } \mathbf{y} \in \mathbb{R}^n.$$

Since $\{f_{\mathbf{m}} : \mathbf{m} \in \mathbb{Z}^n\}$ spans a dense subspace of $L^2(X_{\varphi_g(\Gamma)})$, we have shown:

Proposition 2 *For any* $g \in GL(n, \mathbb{R}) \simeq \mathcal{T}(\Gamma, G; X)$,

$$\mathrm{Spec}_d(X_{\varphi_g(\Gamma)}) = \{-4\pi^2 Q_{g^{-1}I_{p,q}{}^t g^{-1}}(\mathbf{m}) : \mathbf{m} \in \mathbb{Z}^n\}.$$

Here are some observation in the $n = 1, 2$ cases.

Example 5 Let $n = 1$ and $(p, q) = (1, 0)$. Then $\mathrm{Spec}_d(X_{\varphi_g(\Gamma)}) = \{-4\pi^2 m^2/g^2 : m \in \mathbb{Z}\}$ for $g \in \mathbb{R}^\times \simeq GL(1, \mathbb{R})$ by Proposition 2. Thus the smaller the period $|g|$ is, the larger the absolute value of the eigenvalue $|-4\pi^2 m^2/g^2|$ becomes for each fixed

$m \in \mathbb{Z} \setminus \{0\}$. This is thought of as a mathematical model of a music instrument for which shorter strings produce a higher pitch than longer strings (see Introduction).

Example 6 Let $n = 2$ and $(p, q) = (1, 1)$. Take $g = I_2$, so that $\varphi_g(\Gamma) = \mathbb{Z}^2$ is the standard lattice. Then the L^2-eigenspace of the Laplacian $\Delta_{\mathbb{R}^{1,1}/\mathbb{Z}^2}$ for zero eigenvalue contains $W := \{\psi(x - y) : \psi \in L^2(\mathbb{R}/\mathbb{Z})\}$. Since W is infinite-dimensional and $W \not\subset C^\infty(\mathbb{R}^2/\mathbb{Z}^2)$, the third and fourth statements of Fact 1 fail in this pseudo-Riemannian setting.

By the explicit description of $\mathrm{Spec}_d(X_{\varphi(\Gamma)})$ for all $\varphi \in \mathcal{T}(\Gamma, G; X)$ in Proposition 2, we can also tell the behaviour of $\mathrm{Spec}_d(X_{\varphi(\Gamma)})$ under deformation of Γ by φ. Obviously, any constant function on $X_{\varphi(\Gamma)}$ is an eigenfunction of the Laplacian $\Delta_{X_{\varphi(\Gamma)}} = \Delta_{\mathbb{R}^{p,q}}/\varphi(\mathbb{Z}^{p+q})$ with eigenvalue zero. We see that this is the unique stable L^2-eigenvalue in the flat compact manifold:

Corollary 1 (non-existence of stable eigenvalues) *Let $n = p + q$ with $p, q \in \mathbb{N}$. For any open subset V of $\mathcal{T}(\Gamma, G; X)$,*

$$\bigcap_{\varphi \in V} \mathrm{Spec}_d(X_{\varphi(\Gamma)}) = \{0\}.$$

5.2 Oppenheim's Conjecture and Stability of Spectrum

In 1929, Oppenheim [17] raised a question about the distribution of an indefinite quadratic forms at integral points. The following theorem, referred to as Oppenheim's conjecture, was proved by Margulis (see [14] and references therein).

Fact 6 (Oppenheim's conjecture) *Suppose $n \geq 3$ and Q is a real nondegenerate indefinite quadratic form in n variables. Then either Q is proportional to a form with integer coefficients (and thus $Q(\mathbb{Z}^n)$ is discrete in $\mathbb{R}$), or $Q(\mathbb{Z}^n)$ is dense in $\mathbb{R}$.*

Combining this with Proposition 2, we get the following.

Theorem 7 *Let $p + q = n$, $p \geq 2$, $q \geq 1$, $G = \mathbb{R}^n$, $X = \mathbb{R}^{p,q}$ and $\Gamma = \mathbb{Z}^n$. We define an open dense subset U of $\mathcal{T}(\Gamma, G; X) \simeq GL(n, \mathbb{R})$ by*

$$\begin{aligned} U := \{\, & g \in GL(n, \mathbb{R}) : g^{-1} I_{p,q}\, {}^t g^{-1} \text{ is not proportional} \\ & \text{to an element of } M(n, \mathbb{Z}) \,\} \end{aligned}$$

Then the set $\mathrm{Spec}_d(X_{\varphi(\Gamma)})$ of L^2-eigenvalues of the Laplacian is dense in $\mathbb{R}$ if and only if $\varphi \in U$.

Thus the fifth statement of Fact 1 for compact Riemannian manifolds do fail in the pseudo-Riemannian case.

6 Main Results—Sound of Anti-de Sitter Manifolds

6.1 Intrinsic Sound of Anti-de Sitter Manifolds

In general, it is not clear whether the Laplacian Δ_M admits infinitely many L^2-eigenvalues for compact pseudo-Riemannian manifolds. For anti-de Sitter 3-manifolds, we proved in [6, Theorem 1.1]:

Theorem 8 *For any compact anti-de Sitter 3-manifold M, there exist infinitely many L^2-eigenvalues of the Laplacian Δ_M.*

In the abelian case, it is easy to see that compactness of X_Γ is necessary for the existence of L^2-eigenvalues:

Proposition 3 *Let $G = \mathbb{R}^{p+q}$, $X = \mathbb{R}^{p,q}$, $\Gamma = \mathbb{Z}^k$, and $\varphi \in R(\Gamma, G; X)$. Then $\operatorname{Spec}_d(X_{\varphi(\Gamma)}) \neq \emptyset$ if and only if $X_{\varphi(\Gamma)}$ is compact, or equivalently, $k = p + q$.*

However, anti-de Sitter 3-manifolds M admit infinitely many L^2-eigenvalues even when M is of infinite-volume (see [6, Theorem 9.9]):

Theorem 9 *For any finitely generated discrete subgroup Γ of $G = SO(2,2)$ acting properly discontinuously and freely on $X = \mathrm{AdS}^3$,*

$$\operatorname{Spec}_d(X_\Gamma) \supset \{l(l-2) : l \in \mathbb{N}, l \geq 10C^{-3}\}$$

where $C \equiv C(\Gamma)$ is the first sharpness constant of Γ.

The above L^2-eigenvalues are stable in the following sense:

Theorem 10 (stable L^2-eigenvalues) *Suppose that $\Gamma \subset G = SO(2,2)$ and $M = \Gamma \backslash \mathrm{AdS}^3$ is a compact standard anti-de Sitter 3-manifold. Then there exists a neighbourhood $U \subset \operatorname{Hom}(\Gamma, G)$ of the natural inclusion with the following two properties:*

$$U \subset R(\Gamma, G; \mathrm{AdS}^3), \tag{5}$$

$$\# \left(\bigcap_{\varphi \in U} \operatorname{Spec}_d(X_\Gamma) \right) = \infty. \tag{6}$$

The first geometric property (5) asserts that a small deformation of Γ keeps proper discontinuity, which was conjectured by Goldman [3] in the AdS^3 setting, and proved affirmatively in [11]. Theorem 10 was proved in [6, Corollary 9.10] in a stronger form (e.g., without assuming "standard" condition).

Figuratively speaking, Theorem 10 says that compact anti-de Sitter manifolds have “intrinsic sound” which is stable under any small deformation of the anti-de Sitter structure. This is a new phenomenon which should be in sharp contrast to the abelian case (Corollary 1) and the Riemannian case below:

Fact 11 (see [20, Theorem 5.14]) *For a compact hyperbolic surface, no eigenvalue of the Laplacian above* $\frac{1}{4}$ *is constant on the Teichmüller space.*

We end this section by raising the following question in connection with the flat case (Theorem 7):

Question 3 Suppose M is a compact anti-de Sitter 3-manifold. Find a geometric condition on M such that $\mathrm{Spec}_d(M)$ is discrete.

7 Perspectives and Sketch of Proof

The results in the previous section for anti-de Sitter 3-manifolds can be extended to more general pseudo-Riemannian locally symmetric spaces of higher dimension:

Theorem 12 ([6, Theorem 1.5]) *Let* X_Γ *be a standard Clifford–Klein form of a semisimple symmetric space* $X = G/H$ *satisfying the rank condition*

$$\text{rank } G/H = \text{rank } K/H \cap K. \tag{7}$$

Then the following holds.

(1) *There exists an explicit infinite subset* I *of joint* L^2*-eigenvalues for all the differential operators on* X_Γ *that are induced from* G*-invariant differential operators on* X.
(2) *(stable spectrum) If* Γ *is contained in a simple Lie group L of real rank one acting properly on* $X = G/H$*, then there is a neighbourhood* $V \subset \mathrm{Hom}(\Gamma, G)$ *of the natural inclusion such that for any* $\varphi \in V$*, the action* $\varphi(\Gamma)$ *on* X *is properly discontinuous and the set of joint* L^2*-eigenvalues on* $X_{\varphi(\Gamma)}$ *contains the infinite set* I.

Remark 3 We do not require X_Γ to be of finite volume in Theorem 12.

Remark 4 It is plausible that for a general locally symmetric space $\Gamma\backslash G/H$ with G reductive, no nonzero L^2-eigenvalue is stable under nontrivial small deformation unless the rank condition (7) is satisfied. For instance, suppose $\Gamma = \pi_1(\Sigma_g)$ with

$g \geq 2$ and $R(\Gamma, G; X) \neq \emptyset$. (Such semisimple symmetric space $X = G/H$ was recently classified in [16].) Then we expect the rank condition (7) is equivalent to the existence of an open subset U in $R(\Gamma, G; X)$ such that

$$\# \left(\bigcap_{\varphi \in U} \mathrm{Spec}_d(X_{\varphi(\Gamma)}) \right) = \infty.$$

It should be noted that not all L^2-eigenvalues of compact anti-de Sitter manifolds are stable under small deformation of anti-de Sitter structure. In fact, we proved in [7] that there exist also countably many *negative* L^2-eigenvalues that are NOT stable under deformation, whereas the countably many stable L^2-eigenvalues that we constructed in Theorem 9 are all positive. More generally, we prove in [7] the following theorem that include both stable and unstable L^2-eigenvalues:

Theorem 13 *Let G be a reductive homogeneous space and L a reductive subgroup of G such that $H \cap L$ is compact. Assume that the complexification $X_{\mathbb{C}}$ is $L_{\mathbb{C}}$-spherical. Then for any torsion-free discrete subgroup Γ of L, we have:*

(1) *the Laplacian Δ_{X_Γ} extends to a self-adjoint operator on $L^2(X_\Gamma)$;*
(2) $\# \mathrm{Spec}_d(X_\Gamma) = \infty$ *if X_Γ is compact.*

By "$L_{\mathbb{C}}$-spherical" we mean that a Borel subgroup $L_{\mathbb{C}}$ has an open orbit in $X_{\mathbb{C}}$. In this case, a reductive subgroup L acts transitively on X by [10, Lemma 5.1].

Here are some examples of the setting of Theorem 13, taken from [13, Corollary 3.3.7].

Examples for Theorem 13 include Table 1 (ii) for all $n \in \mathbb{N}$, whereas we need $n \in 2\mathbb{N}$ in Theorem 12 for the rank condition (7).

Table 1 Triple (*G, H, L*) satisfying the condition of Theorem 13

	G	H	L
(i)	$SO(2n, 2)$	$SO(2n, 1)$	$U(n, 1)$
(ii)	$SO(2n, 2)$	$U(n, 1)$	$SO(2n, 1)$
(iii)	$SU(2n, 2)$	$U(2n, 1)$	$Sp(n, 1)$
(iv)	$SU(2n, 2)$	$Sp(n, 1)$	$U(2n, 1)$
(v)	$SO(4n, 4)$	$SO(4n, 3)$	$Sp(1) \times Sp(n, 1)$
(vi)	$SO(8, 8)$	$SO(8, 7)$	$Spin(8, 1)$
(vii)	$SO(8, \mathbb{C})$	$SO(7, \mathbb{C})$	$Spin(7, 1)$
(viii)	$SO(4, 4)$	$Spin(4, 3)$	$SO(4, 1) \times SO(3)$
(ix)	$SO(4, 3)$	$G_2(\mathbb{R})$	$SO(4, 1) \times SO(2)$

The idea of the proof for Theorem 12 is to take an average of a (nonperiodic) eigenfunction on X with rapid decay at infinity over Γ-orbits as a generalization of Poincaré series. Geometric ingredients of the convergence (respectively, nonzeroness) of the generalized Poincaré series include "counting Γ-orbits" stated in Lemma 1 below (respectively, the Kazhdan–Margulis theorem, cf. [6, Proposition 8.14]). Let $B(o, R)$ be a "pseudo-ball" of radius $R > 0$ centered at the origin $o = eH \in X = G/H$, and we set

$$N(x, R) := \#\{\gamma \in \Gamma : \gamma \cdot x \in B(o, R)\}.$$

Lemma 1 ([6, Corollary 4.7])

(1) *If Γ acts properly discontinuously on X, then $N(x, R) < \infty$ for all $x \in X$ and $R > 0$.*
(2) *If Γ acts strongly properly discontinuously on X, then there exists $A_x > 0$ such that*
$$N(x, R) \leq A_x \exp(\frac{R}{C}) \quad \textit{for all } R > 0,$$
where C is the first sharpness constant of Γ.

The key idea of Theorem 13 is to bring branching laws to spectral analysis [10, 12], namely, we consider the restriction of irreducible representations of G that are realized in the space of functions on the homogeneous space $X = G/H$ and analyze the G-representations when restricted to the subgroup L. Details will be given in [7].

Acknowledgements This article is based on the talk that the author delivered at the eleventh International Workshop: Lie Theory and its Applications in Physics in Varna, Bulgaria, 15–21, June, 2015. The author is grateful to Professor Vladimir Dobrev for his warm hospitality.

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Sphere Partition Functions and the Kähler Metric on the Conformal Manifold

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Abstract We discuss marginal operators in $\mathcal{N} = 2$ Superconformal Field Theories in four dimensions. These operators are necessarily exactly marginal and they lead to a manifold, $\mathcal{M}$, of Superconformal Field Theories. The space $\mathcal{M}$ is argued to be a Kähler manifold. We further argue that upon a stereographic projection of R^4 to S^4, the partition function Z_{S^4} measures the Kähler potential. These results are established by a careful study of the interplay between conformal anomalies and the space $\mathcal{M}$.

1 The Superconformal Algebra

A special class of Quantum Field Theories (QFTs) are those that have no intrinsic length scale. This happens when the correlation length of the corresponding theory on the lattice diverges. In addition, such theories often arise when we take generic QFTs and scale the distances to be much larger or much smaller than the typical inverse mass scales. Of course, one may sometimes encounter gapped theories at long distances, but there are also many examples in which one finds nontrivial theories in this way.

In general, we are interested here in QFTs which are invariant under the Poincaré group of R^4. The Poincaré group consists of rotations in $SO(4)$ (generated by $M_{\mu\nu}$, with $(\mu, \nu = 1, \ldots, 4)$) and translations (generated by P_μ). If the theory has no intrinsic length scale then the Poincaré group is enhanced by adding the generator of dilations, Δ. Oftentimes, the symmetry is further enhanced to $SO(5, 1)$, which includes the original Poincaré generators, the dilation Δ, and the so-called special conformal transformations K_μ.

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V. Dobrev (ed.), *Lie Theory and Its Applications in Physics*,
Springer Proceedings in Mathematics & Statistics 191,
DOI 10.1007/978-981-10-2636-2_7

The commutation relations are

$$
\begin{aligned}
\left[\Delta, P_\mu\right] &= P_\mu \ , \\
\left[\Delta, K_\mu\right] &= -K_\mu \ , \\
\left[K_\mu, P_\nu\right] &= 2\left(\delta_{\mu\nu}\Delta - iM_{\mu\nu}\right) \ , \\
\left[M_{\mu\nu}, P_\rho\right] &= i\left(\delta_{\mu\rho}P_\nu - \delta_{\nu\rho}P_\mu\right) \ , \\
\left[M_{\mu\nu}, K_\rho\right] &= i\left(\delta_{\mu\rho}K_\nu - \delta_{\nu\rho}K_\mu\right) \ , \\
\left[M_{\mu\nu}, M_{\rho\sigma}\right] &= i\left(\delta_{\mu\rho}M_{\nu\sigma} + \delta_{\nu\sigma}M_{\mu\rho} - \delta_{\nu\rho}M_{\mu\sigma} - \delta_{\mu\sigma}M_{\nu\rho}\right) \ .
\end{aligned}
$$

They can be realized by the differential operators acting on R^4:

$$
\begin{aligned}
M_{\mu\nu} &= -i\left(x_\mu\partial_\nu - x_\nu\partial_\mu\right) \ , \\
P_\mu &= -i\partial_\mu \ , \\
K_\mu &= i\left(2x_\mu x.\partial - x^2\partial_\mu\right) \ , \\
\Delta &= x.\partial \ .
\end{aligned}
$$

A primary operator, $\Phi(x)$, is, by definition, an operator that is annihilated by K_μ when placed at the origin:

$$[K_\mu, \Phi(0)] = 0 \ . \tag{1}$$

The origin of R^4 is fixed by rotations and dilations. Therefore we can characterise $\Phi(0)$ by its quantum numbers under rotations and dilations. In $d = 4$ the group of rotations, $SO(4)$, is locally just $SU(2) \times SU(2)$ and hence a primary operator is labeled by $(j_1, j_2; \Delta)$.[1] We will be only interested in unitary theories, where the allowed representations of the conformal algebra do not have negative-norm states.

It is sometimes the case that the conformal field theory has primary operators of dimension 4,

$$[\Delta, O_I(x)] = i(x.\partial + 4)O_I(x) \ .$$

If we add such operators to the action with couplings λ^I then we get

$$S \to S + \sum_I \lambda^I \int d^4x O_I(x) \ . \tag{2}$$

A simple example is the free conformal field theory in $d = 4$ to which we can add a quartic interaction. The coupling λ^I is dimensionless but in general there may be a nontrivial beta function

$$\beta^I \equiv \frac{d\lambda^I}{d\log\mu} = (\beta^{(1)})^I_{JK}\lambda^J\lambda^K + \cdots \ . \tag{3}$$

[1]If there is a global symmetry $\mathcal{G}$, then the primary operators would furnish some representations of $\mathcal{G}$.

Therefore, conformal symmetry is broken at second order in λ. (If we add to the action an operator of $\Delta \neq 4$ then conformal symmetry is already broken at first order in the coupling constant.)

Under some special circumstances it may happen that $\beta^I = 0$ as a function of λ^I. Then we say that the deformation (2) is exactly marginal. We therefore have a manifold of conformal field theories, $\mathcal{M}$, with coordinates $\{\lambda^I\}$. This manifold has a natural Riemannian structure given by the Zamolodchikov metric

$$\langle O_I(\infty) O_J(0) \rangle_{\{\lambda^I\}} = g_{IJ}(\lambda^I) \ . \tag{4}$$

One situation in which exactly marginal operators are common is in Superconformal Field Theories (SCFTs). The conformal algebra is enlarged by adding $\mathcal{N}$ Poincaré supercharges $Q^i_\alpha, \bar{Q}_{i\dot{\alpha}}$ and $\mathcal{N}$ superconformal supercharges $S_i^\alpha, \bar{S}^{i\dot{\alpha}}$ $(i = 1, .., \mathcal{N})$. In addition, we must add the R-symmetry group, $U(\mathcal{N})$, whose generators are $R^i_{\ j}$. This furnishes the superalgebra $SU(2, 2|\mathcal{N})$.

We do not list all the commutation relations. They can be found in [5]. All we need to know for our purposes is summarised below.

Our main interest in this note lies in $\mathcal{N} = 2$ theories. The maximally supersymmetric theory with $\mathcal{N} = 4$ would be a special case. The R-symmetry group in $\mathcal{N} = 2$ theories is $SU(2)_R \times U(1)_R$. We denote the $U(1)_R$ charge by r.

- It is consistent to impose at the origin, in addition to (1),

$$[S_i^\alpha, \Phi(0)] = [\bar{S}^{i\dot{\alpha}}, \Phi(0)] = 0 \ .$$

  (The quantum numbers of $\Phi(0)$ are omitted.) Such operators are called superconformal primaries. In every unitary representation the operators with the lowest eigenvalues of Δ are superconformal primaries.
- If one further imposes

$$[\bar{Q}_{i\dot{\alpha}}, \Phi(0)] = 0 \ , \tag{5}$$

  one obtains a short representation (such representations may or may not exist in a given model). The operator $\Phi(0)$ satisfying (5) is called a chiral primary.[2] Chiral primary operators are necessarily $SU(2)_R$ singlets and they obey a relationship between their $U(1)_R$ charge and their scaling dimension

$$\Delta = r \ .$$

- Marginal operators that preserve $\mathcal{N} = 2$ supersymmetry are necessarily the descendants of chiral primary operators with $\Delta = r = 2$. We can upgrade the formula (2) to a superspace formula

$$S \to S + \lambda^I \int d^4x \, d^4\theta \Phi_I(x, \theta) + \bar{\lambda}^{\bar{I}} \int d^4x \, d^4\bar{\theta} \bar{\Phi}_{\bar{I}}(x, \bar{\theta}) \ . \tag{6}$$

[2] We henceforth assume chiral primary operators carry no spin, see [4] for a discussion.

which shows that $\mathcal{N} = 2$ supersymmetry is indeed preserved. We denote the dimension 4 descendant of Φ_I by O_I. Therefore, (6) is just

$$S \to S + \lambda^I \int d^4x\, O_I(x) + \bar{\lambda}^{\bar{I}} \int d^4x\, \bar{O}_{\bar{I}}(x)\ . \tag{7}$$

The Zamolodchikov metric is defined by the two-point function $\langle O_I(\infty)\bar{O}_{\bar{J}}(0)\rangle$. (This is proportional to $\langle \Phi_I(\infty)\bar{\Phi}_{\bar{J}}(0)\rangle$.)

It remains to argue that for the deformations (6) the beta function $\beta^I = 0$ identically. The argument is along the lines of [11]. There is a scheme in which the superpotential is not renormalized. Then if the beta function is nonzero it has to be reflected by a D-term in the action $\int d^4x\, d^8\theta \mathcal{O}$ with $\mathcal{O}$ some real primary operator. But since the λ^I are classically dimensionless, $\Delta(\mathcal{O}) = 0$ in the original fixed point. Therefore, $\mathcal{O}$ has to be the unit operator and the deformation $\int d^4x\, d^8\theta \mathcal{O}$ is therefore trivial. This proves that $\beta^I = 0$.

The $\{\lambda^I, \bar{\lambda}^{\bar{I}}\}$ are therefore coordinates on the manifold $\mathcal{M}$ of $\mathcal{N} = 2$ SCFTs. In the next section we will argue that $\mathcal{M}$ is a Kähler manifold, i.e. the Zamolodchikov metric (4) satisfies

$$g_{I\bar{J}} = \partial_I \partial_{\bar{J}} K(\lambda^I, \bar{\lambda}^{\bar{I}})\ . \tag{8}$$

Then we will argue that the Kähler potential can be extracted from the S^4 partition function and we will use supersymmetric localization to compute it in some simple $\mathcal{N} = 2$ SCFTs. Large parts of the discussion in the next two sections follows [10].

2 Conformal Anomalies and the Zamolodchikov Metric

Let us for a moment consider the definition (4) more carefully, and in arbitrary dimension. The Zamolodchikov metric on the conformal manifold is given by

$$\langle O_I(x) O_J(0)\rangle_\lambda = \frac{g_{IJ}(\lambda)}{x^{2d}}\ , \tag{9}$$

where $0 \neq x \in R^d$. In momentum space the two-point function (9) takes the form

$$\langle O_I(p) O_J(-p)\rangle_\lambda \sim g_{IJ}(\lambda) \begin{cases} p^d & d = 2n+1 \\ p^{2n} \log\left(\frac{\mu^2}{p^2}\right) & d = 2n \end{cases} \tag{10}$$

with $n \in \mathbb{N}$. Thus, if we rescale μ the even-dimensional result will change by a polynomial in p^2 (delta function in position space). It follows that the separated points correlation function is covariant under such rescaling while the coincident points correlation function is not covariant. The appearance of such a logarithm in conformal field theories signifies a conformal anomaly, which manifests itself as a non-vanishing contribution to the trace of the stress-energy tensor. By promoting the exactly marginal couplings λ^I to spacetime dependent background fields $\lambda^I(x)$, such that they act as sources of the exactly marginal operators $O_I(x)$, one can detect a contribution to the trace anomaly of the schematic form

$$T^\mu_\mu \supset g_{IJ}\lambda^I \Box^{\frac{d}{2}} \lambda^J \ . \tag{11}$$

The precise action of the derivatives in the formula above will be determined below.

The trace anomaly $\langle T^\mu_\mu \rangle$ can be derived from the variation of the free energy, $\delta_\sigma \log Z$, under an infinitesimal Weyl rescaling,

$$\delta_\sigma \gamma_{\mu\nu} = 2\delta\sigma\, \gamma_{\mu\nu} \ , \tag{12}$$

where $\gamma_{\mu\nu}$ is the spacetime metric. $\delta_\sigma \log Z$ must be local in $\gamma_{\mu\nu}$, $\delta\sigma$ and λ, and its form is constrained by the Wess–Zumino consistency condition, which is simply the statement that Weyl transformations commute; $\delta_\sigma\delta_{\sigma'} \log Z = \delta_{\sigma'}\delta_\sigma \log Z$. It also needs to be invariant under coordinate transformations in spacetime, and under coordinate transformations in the conformal manifold. If we have some symmetry that is respected by the class of regulators we consider (supersymmetry for example), we will require $\delta_\sigma \log Z$ to preserve this symmetry as well.

In addition, $\delta_\sigma \log Z$ is defined only up to terms that can be written as $\delta_\sigma W$ for some local functional W (which also needs to respect the symmetry constraints described above), as such terms can be removed by adding local counterterms to the free energy (in other words these terms can be removed by choosing an appropriate regulator and therefore they do not contribute to the anomaly, which cannot be removed with any choice of regulator). Thus, in order to find the allowed form of the anomaly one needs to solve a cohomology problem.

In four dimensional CFTs, the local functional that produces the Weyl variation of (10) is[3]

$$\begin{aligned} \delta_\sigma \log Z \ \supset \ & \frac{1}{192\pi^2} \int d^4x \sqrt{\gamma}\, \delta\sigma \Big(g_{IJ} \hat{\Box}\lambda^I \hat{\Box}\lambda^J \\ & - 2\, g_{IJ} \partial_\mu \lambda^I \left(R^{\mu\nu} - \frac{1}{3}\gamma^{\mu\nu} R \right) \partial_\nu \lambda^J \Big) \ , \end{aligned} \tag{14}$$

where coordinate invariance in $\mathcal{M}$ requires introducing a connection

$$\hat{\Box}\lambda^I = \Box\lambda^I + \Gamma^I_{JK} \partial^\mu \lambda^J \partial_\mu \lambda^K \ , \tag{15}$$

---

[3]The normalization conventions from now on will be such that the exactly marginal deformation is

$$S \to S + \frac{1}{\pi^2}\lambda^I \int d^4x\, O_I(x) \ . \tag{13}$$

The convention we use for $R_{\mu\nu\rho\sigma}$ is $[\nabla_\mu, \nabla_\nu]V_\rho = R_{\mu\nu\rho\sigma}V^\sigma$.

and the Wess–Zumino consistency condition forces this connection to be the Christoffel connection on $\mathcal{M}$:

$$\Gamma^I_{JK} = g^{IR}\left(\partial_K g_{RJ} + \partial_J g_{RK} - \partial_R g_{JK}\right) \ . \tag{16}$$

The anomaly (14) needs to be added to the well-known conformal anomalies[4]:

$$\delta_\sigma \log Z \supset \frac{1}{16\pi^2}\int d^4x\,\sqrt{\gamma}\,\delta\sigma\left(c\,C^{\mu\nu\rho\sigma}C_{\mu\nu\rho\sigma} - aE_4\right) \ , \tag{17}$$

which do not depend on the coordinates λ^I.

Let us now discuss the $\mathcal{N}=2$ superconformal manifold. We will assume that the superconformal theory is regulated in a way that preserves diffeomorphism invariance and $\mathcal{N}=2$ supersymmetry, i.e. we assume that the physics at coincident points is supersymmetric and diffeomorphism invariant.[5] The assumption above constrains the way the anomaly and the allowed counterterms can depend on the parameters of the theory and on the spacetime geometry. A convenient way to implement these constraints is to derive the anomaly and the counterterms as supergravity invariants that are constructed from supergravity multiplets. For this sake the parameters of the theory and of the geometry need to be embedded into supergravity multiplets.

According to Eq. (6) the exactly marginal operators are integrals over half superspace of chiral and antichiral superfields with $\Delta = r = 2$. Thus, the corresponding couplings need to be realized as bottom components of chiral and antichiral superfields, Λ^I and $\bar{\Lambda}^{\bar{I}}$, with $\Delta = r = 0$.[6] In addition, the Weyl variation $\delta\sigma$ is embedded in the bottom component of the chiral Weyl superfield $\delta\Sigma$ (see, e.g. [13] for details) and the integration measure $\sqrt{\gamma}$ is promoted to the density measure superfield E. In terms of these superfields, the supersymmetrization of the anomaly (14) is given by the superspace integral

$$\delta_\Sigma \log Z \supset \frac{1}{192\pi^2}\int d^4x\, d^4\theta\, d^4\bar{\theta}\, E(\delta\Sigma + \delta\bar{\Sigma})K(\Lambda^I, \bar{\Lambda}^{\bar{I}})\,. \tag{18}$$

When this integral is expanded in components, one finds (among many other terms) the anomaly (14) with

$$g_{I\bar{J}} = \partial_I\partial_{\bar{J}}K \ . \tag{19}$$

[4] E_4 denotes the Euler density and $C_{\mu\nu\rho\sigma}$ is the Weyl tensor.

[5] Note that we cannot assume that the coincident points physics is conformal invariant since this would contradict (10).

[6] Note that $\Lambda^I(x,\theta) = \lambda^I$ (with constant λ^I) is consistent with the supersymmetry variations of a chiral multiplet, and that substituting this in $\int d^4x\, d^4\theta\, \Lambda^I(x,\theta)\Phi_I(x,\theta)$ + c.c. one gets Eq. (6) back. After constructing the anomaly and counterterms in terms of the superfields $\Lambda^I(x,\theta)$, $\bar{\Lambda}^{\bar{I}}(x,\bar{\theta})$ we substitute the constant background values. We do a similar thing with the geometry background parameters.

We therefore conclude that for $\mathcal{N}=2$ SCFTs, the Zamolodchikov metric is Kähler. This statement, which is true also for $\mathcal{N}=1$ SCFTs, was proven in [1] using superconformal Ward identities.

Expanding (18) in components while keeping only the bottom component of Λ^I, $\bar{\Lambda}^{\bar{I}}$ and the metric background (setting the auxiliary fields in the gravity multiplet to zero) one ends up with the following anomaly:

$$
\begin{aligned}
&\delta_\Sigma \log Z \supset \frac{1}{96\pi^2}\int d^4x\sqrt{\gamma}\Big\{\delta\sigma\mathcal{R}_{I\bar{K}J\bar{L}}\nabla^\mu\lambda^I\,\nabla_\mu\lambda^J\,\nabla^\nu\bar{\lambda}^{\bar{K}}\,\nabla_\nu\bar{\lambda}^{\bar{L}} \qquad (20)\\
&+\delta\sigma g_{I\bar{J}}\left(\hat{\Box}\lambda^I\,\hat{\Box}\bar{\lambda}^{\bar{J}}-2\Big(R^{\mu\nu}-\frac{1}{3}R\,\gamma^{\mu\nu}\Big)\nabla_\mu\lambda^I\,\nabla_\nu\bar{\lambda}^{\bar{J}}\right)\\
&+\frac{1}{2}K\,\Box^2\delta\sigma+\frac{1}{6}K\,\nabla^\mu R\,\nabla_\mu\delta\sigma+K\,\left(R^{\mu\nu}-\frac{1}{3}\gamma^{\mu\nu}R\right)\nabla_\mu\nabla_\nu\delta\sigma\\
&-2\,g_{I\bar{J}}\,\nabla^\mu\lambda^I\,\nabla^\nu\bar{\lambda}^{\bar{J}}\,\nabla_\mu\nabla_\nu\delta\sigma+i\,g_{I\bar{J}}\left(\hat{\nabla}^\mu\hat{\nabla}^\nu\lambda^I\,\nabla_\nu\bar{\lambda}^{\bar{J}}-\hat{\nabla}^\mu\hat{\nabla}^\nu\bar{\lambda}^{\bar{J}}\,\nabla_\nu\lambda^I\right)\nabla_\mu\delta a\\
&-\frac{i}{2}\Big(\hat{\nabla}_I\hat{\nabla}_J K\,\nabla^\mu\lambda^I\nabla_\mu\lambda^J-\hat{\nabla}_{\bar{I}}\hat{\nabla}_{\bar{J}}K\,\nabla^\mu\bar{\lambda}^{\bar{I}}\nabla_\mu\bar{\lambda}^{\bar{J}}+\nabla_I K\,\hat{\Box}\lambda^I\\
&-\nabla_{\bar{I}}K\,\hat{\Box}\bar{\lambda}^{\bar{I}}\Big)\Box\delta a+i\left(R^{\mu\nu}-\frac{1}{3}R\,\gamma^{\mu\nu}\right)\left(\nabla_I K\,\nabla_\mu\lambda^I-\nabla_{\bar{I}}K\,\nabla_\mu\bar{\lambda}^{\bar{I}}\right)\nabla_\nu\delta a\Big\}
\end{aligned}
$$

where, as before, the hats denote covariant derivatives with respect to coordinate transformations in the conformal manifold, $\mathcal{R}_{I\bar{J}K\bar{L}}=\partial_I\partial_{\bar{J}}g_{K\bar{L}}-g^{M\bar{N}}\,\partial_I g_{K\bar{L}}\,\partial_{\bar{J}}\,g_{M\bar{N}}$, and $\delta\sigma+i\delta a$ is the bottom component of $\delta\Sigma$. Note that the anomaly (14) appears in the second line.

Setting λ, $\bar{\lambda}$ to constants, we remain with a non-vanishing contribution:

$$
\begin{aligned}
\delta_\Sigma \log Z \;&\supset\; \frac{1}{96\pi^2}K(\lambda,\bar{\lambda})\int d^4x\sqrt{\gamma}\left(\frac{1}{2}\Box^2\delta\sigma+\frac{1}{6}\nabla^\mu R\,\nabla_\mu\delta\sigma\right.\\
&\left.+\left(R^{\mu\nu}-\frac{1}{3}\gamma^{\mu\nu}R\right)\nabla_\mu\nabla_\nu\delta\sigma\right) \qquad (21)\\
&=\;\delta_\sigma\left(\frac{1}{96\pi^2}K(\lambda,\bar{\lambda})\int d^4x\sqrt{\gamma}\left[\frac{1}{8}E_4-\frac{1}{12}\Box R+f(\lambda,\bar{\lambda})C^2\right]\right),
\end{aligned}
$$

where $f(\lambda,\bar{\lambda})$ is an arbitrary function on $\mathcal{M}$, E_4 is the Euler density and $C_{\mu\nu\rho\sigma}$ is the Weyl tensor.

Note that this expression is not cohomologically trivial. The second line in (21) is written as a Weyl variation of a local term, but this is not a supersymmetric local term. Thus, this contribution cannot be removed with a supersymmetric regulator. In the next section we will show that, as a result of this term, the sphere partition function has a universal (i.e. regularization independent) content - it computes the Kähler potential on the superconformal manifold.

3 Sphere Partition Functions

Any conformal field theory on R^d can be placed on S^d using the stereographic projection. Since this map is a conformal transformation we can obtain correlation functions in R^d from the corresponding correlation functions in S^d by applying the inverse map. The sphere is compact and therefore the theory on the sphere is free from infrared divergences. Since the sphere is locally equivalent to R^d, the ultraviolet divergences on the sphere are the same as in flat space.

In the recent years the exact computation of some supersymmetric observables in $\mathcal{N}=2$ theories on S^4 became possible due to the technique of supersymmetric localization in which the path integral is reduced to a finite dimensional integral. In particular, sphere partition functions for Lagrangian $\mathcal{N}=2$ theories (not necessarily conformal) can be computed exactly, including all perturbative and instanton contributions [15]. In this section we will show that the sphere partition function for $\mathcal{N}=2$ SCFTs computes the Kähler potential on the superconformal manifold. This was proved in [7, 8, 10]. Here, we will follow [10], in which this statement was derived from the anomaly (21).

According to Eq. (21), the sphere partition function, when regulated in a supersymmetry preserving fashion, contains the contribution[7]:

$$\log Z_{S^4} \supset \frac{1}{96\pi^2} K(\lambda, \bar{\lambda}) \int_{S^4} d^4x \sqrt{\gamma} \left(\frac{1}{8} E_4 - \frac{1}{12} \Box R \right) = \frac{1}{12} K(\lambda, \bar{\lambda}) \; . \tag{22}$$

An additional contribution comes from the usual a-anomaly. Together, the two contributions give

$$Z_{S^4} = \left(\frac{r}{r_0} \right)^{-4a} e^{K(\lambda,\bar{\lambda})/12} \; , \tag{23}$$

where r is the radius of the sphere and r_0 a scheme dependent scale. Thus, the sphere partition function computes the Kähler potential on the superconformal manifold. This is reminiscent of a known result in two-dimensional theories. For $d=2$, $\mathcal{N}=(2,2)$ SCFTs the Zamolodchikov metric is Kähler and the sphere partition function, which has been computed using localization in [3, 6], computes the Kähler potential on the superconformal manifold [9, 12].

Note that the Kähler potential is defined up to a holomorphic ambiguity,

$$K(\lambda, \bar{\lambda}) \to K(\lambda, \bar{\lambda}) + F(\lambda) + \bar{F}(\bar{\lambda}) \; . \tag{24}$$

This ambiguity in $\log Z_{S^4}$ is due to the existence of a supersymmetric counterterm that depends on an arbitrary holomorphic function of λ^I. This counterterm can be constructed from the supergravity invariant

[7] We dropped the Weyl tensor since it vanishes on the conformally flat sphere.

$$\int d^4x\, d^4\theta\, \mathcal{E} F(\Lambda)\left(\Xi - W^{\alpha\beta}W_{\alpha\beta}\right) + \text{c.c.} \tag{25}$$

Here $\mathcal{E}$ is a chiral density superfield. The chiral superfields Ξ and $W_{\alpha\beta}$ can be found in [13]. In the sphere geometry background, and with the substitution $\Lambda^I(x,\theta) = \lambda^I$, this evaluates to $F(\lambda) + \bar{F}(\bar{\lambda})$ (up to a numerical coefficient). This counterterm was first constructed from $\mathcal{N} = 2$ supergravity in [8].

As mentioned above, (22) cannot be removed by an $\mathcal{N} = 2$ supersymmetric counterterm and therefore the sphere partition function has a universal meaning in $\mathcal{N} = 2$ SCFTs. If we only assume that the regularization scheme preserves $\mathcal{N} = 1$ supersymmetry we would have a counterterm that depends on a general function of λ and $\bar{\lambda}$.[8] Thus, $\mathcal{N} = 1$ supersymmetry of the regulator is not enough to give a universal meaning to Z_{S^4}. For the same reason the λ-dependence of the sphere partition function of $\mathcal{N} = 1$ SCFTs or of non-supersymmetric CFTs is regularization scheme dependent. The only universal contribution to the sphere partition function of a non-supersymmetric CFT is the contribution due to the conformal anomaly a, which is independent of the exactly marginal couplings.

As an example for the computation of the Zamolodchikov metric using Eq. (23), consider an $SU(2)$ gauge theory with 4 hypermultiplets in the fundamental representation. This theory is superconformal, with one exactly marginal parameter $\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2}$, where g is the Yang–Mills coupling and θ is the theta angle. The sphere partition function can be computed using localization, and one finds:

$$Z_{S^4}(\tau,\bar{\tau}) = \int_{-\infty}^{\infty} da\, e^{-4\pi\,\mathrm{Im}\tau\, a^2}(2a)^2 \frac{H(2ia)H(-2ia)}{[H(ia)H(-ia)]^4} |Z_{\text{inst}}(a,\tau)|^2 \,, \tag{26}$$

where $H(x)$ is given in terms of the Barnes G-function by $H(x) = G(1+x)G(1-x)$, and Z_{inst} is Nekrasov's instanton partition function on the Omega background [14]. By expanding the integrand in powers of g^2 we can compute Z_{S^4} to any order in perturbation theory. We can also include instanton corrections, up to any instanton number. It is then straightforward to compute the Zamolodchikov metric via $g_{\tau\bar{\tau}} = \partial_\tau \partial_{\bar{\tau}} \log Z_{S^4}$.[9] The perturbative result for the metric is:

$$g_{\tau\bar{\tau}} = \frac{3}{8}\frac{1}{(\mathrm{Im}\tau)^2} - \frac{135\zeta(3)}{32\pi^2}\frac{1}{(\mathrm{Im}\tau)^4} + \frac{1575\zeta(5)}{64\pi^3}\frac{1}{(\mathrm{Im}\tau)^5} + \mathcal{O}\left(\frac{1}{(\mathrm{Im}\tau)^6}\right) . \tag{27}$$

The first two terms in this result were checked against an explicit, two-loop, Feynman diagrams computation in [2]. The one-instanton correction for the perturbative result is given by

[8] See Sect. 4 of [7].

[9] Here we dropped the factor of 12.

$$g^{\text{1-inst}}_{\tau\bar{\tau}} = \cos\theta\, e^{-\frac{8\pi^2}{g^2}} \left(\frac{3}{8}\frac{1}{(\mathrm{Im}\tau)^2} + \frac{3}{16\pi}\frac{1}{(\mathrm{Im}\tau)^3} - \frac{135\zeta(3)}{32\pi^2}\frac{1}{(\mathrm{Im}\tau)^4} + \mathcal{O}\left(\frac{1}{(\mathrm{Im}\tau)^5}\right)\right) . \tag{28}$$

Acknowledgements E.G. and Z.K. are supported by the ERC STG grant 335182, by the Israel Science Foundation under grant 884/11, by the United States-Israel Bi-national Science Foundation (BSF) under grant 2010/629, by the Israel Science Foundation center for excellence grant (grant no. 1989/14) and also by the I-CORE Program of the Planning and Budgeting Committee.

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Real Group Orbits on Flag Ind-Varieties of $\mathrm{SL}(\infty, \mathbb{C})$

Mikhail V. Ignatyev, Ivan Penkov and Joseph A. Wolf

Abstract We consider the complex ind-group $G = \mathrm{SL}(\infty, \mathbb{C})$ and its real forms $G^0 = \mathrm{SU}(\infty, \infty)$, $\mathrm{SU}(p, \infty)$, $\mathrm{SL}(\infty, \mathbb{R})$, $\mathrm{SL}(\infty, \mathbb{H})$. Our main object of study are the G^0-orbits on an ind-variety G/P for an arbitrary splitting parabolic ind-subgroup $P \subset G$, under the assumption that the subgroups $G^0 \subset G$ and $P \subset G$ are aligned in a natural way. We prove that the intersection of any G^0-orbit on G/P with a finite-dimensional flag variety G_n/P_n from a given exhaustion of G/P via G_n/P_n for $n \to \infty$, is a single $(G^0 \cap G_n)$-orbit. We also characterize all ind-varieties G/P on which there are finitely many G^0-orbits, and provide criteria for the existence of open and closed G^0-orbits on G/P in the case of infinitely many G^0-orbits.

Keywords Homogeneous ind-variety · Real group orbit · Generalized flag

AMS Subject Classification: 14L30 · 14M15 · 22F30 · 22E65

1 Introduction

This study has its roots in linear algebra. Witt's Theorem claims that, given any two subspaces V_1, V_2 of a finite-dimensional vector space V endowed with a nondegenerate bilinear or Hermitian form, the spaces V_1 and V_2 are isometric within V (i.e., one

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V. Dobrev (ed.), *Lie Theory and Its Applications in Physics*,
Springer Proceedings in Mathematics & Statistics 191,
DOI 10.1007/978-981-10-2636-2_8

is obtained from the other via an isometry of V) if and only if V_1 and V_2 are isometric. When V is a Hermitian space, this is a statement about the orbits of the unitary group $U(V)$ on the complex grassmannian $\mathrm{Gr}(k, V)$, where $k = \dim V_1 = \dim V_2$. More precisely, the orbits of $U(V)$ on $\mathrm{Gr}(k, V)$ are parameterized by the possible signatures of a, possibly degenerate, Hermitian form on a k-dimensional space of V.

A general theory of orbits of a real form G^0 of a semisimple complex Lie group G on a flag variety G/P was developed by the third author in [22, 24]. This theory has become a standard tool in semisimple representation theory and complex algebraic geometry. For automorphic forms and automorphic cohomology we mention [9, 21, 26]. For double fibration transforms and similar applications to representation theory see [14, 28]. For the structure of real group orbits and cycle spaces with other applications to complex algebraic geometry see, for example, [2, 3, 8, 9, 15–17, 24–28]. Finally, applications to geometric quantization are indicated by [19, 20].

The purpose of the present paper is to initiate a systematic study of real group orbits on flag ind-varieties or, more precisely, on ind-varieties of generalized flags. The study of the classical simple ind-groups like $\mathrm{SL}(\infty, \mathbb{C})$ arose from studying stabilization phenomena for classical algebraic groups. By now, the classical ind-groups, their Lie algebras, and their representations have grown to a separate subfield in the vast field of infinite-dimensional Lie groups and Lie algebras. In particular, it was seen in [5] that the ind-varieties G/P for classical ind-groups G consist of generalized flags (rather than simply of flags) which are, in general, infinite chains of subspaces subject to two delicate conditions, see Sect. 2.3 below.

Here we restrict ourselves to the ind-group $G = \mathrm{SL}(\infty, \mathbb{C})$ and its real forms G^0. We study G^0-orbits on an arbitrary ind-variety of generalized flags G/P, and establish several foundational results in this direction. Our setting assumes a certain alignment between the subgroups $G^0 \subset G$ and $P \subset G$.

Our first result is the fact that any G^0-orbit in G/P, when intersected with a finite-dimensional flag variety G_n/P_n from a given exhaustion of G/P via G_n/P_n for $n \to \infty$, yields a single G_n^0-orbit for $G_n^0 = G^0 \cap G_n$. This means that the mapping

$$\{G_n^0\text{-orbits on } G_n/P_n\} \to \{G_{n+1}^0\text{-orbits on } G_{n+1}/P_{n+1}\}$$

is injective. Using this feature, we are able to answer the following questions.

1. When are there finitely many G^0-orbits on G/P?
2. When is a given G^0-orbit on G/P closed?
3. When is a given G^0-orbit on G/P open?

The answers depend on the type of real form and not only on the parabolic subgroup $P \subset G$. For instance, if $P = B$ is an upper-triangular Borel ind-subgroup of $\mathrm{SL}(\infty, \mathbb{C})$ (B depends on a choice of an ordered basis in the natural representation of $\mathrm{SL}(\infty, \mathbb{C})$), then G/B has no closed $\mathrm{SU}(\infty, \infty)$-orbit and has no open $\mathrm{SL}(\infty, \mathbb{R})$-orbit.

We see the results of this paper only as a first step in the direction of understanding the structure of G/P as a G^0-ind-variety for all real forms of all classical ind-groups G (and all splitting parabolic subgroups $P \subset G$). Substantial work lies ahead.

2 Background

In this section we review some basic facts about finite-dimensional real group orbits. We then discuss the relevant class of infinite-dimensional Lie groups and the corresponding real forms and flag ind-varieties.

2.1 Finite-Dimensional Case

Let V be a finite-dimensional complex vector space. Recall that a *real structure* on V is an antilinear involution τ on V. The set $V^0 = \{v \in V \mid \tau(v) = v\}$ is a *real form* of V, i.e., V^0 is a real vector subspace of V such that $\dim_{\mathbb{R}} V^0 = \dim_{\mathbb{C}} V$ and the $\mathbb{C}$-linear span $\langle V^0\rangle_{\mathbb{C}}$ coincides with V. A real form V^0 of V defines a unique real structure τ on V such that V^0 is the set of fixed point of τ. A *real form* of a complex finite-dimensional Lie algebra $\mathfrak{g}$ is a real Lie subalgebra $\mathfrak{g}^0$ of $\mathfrak{g}$ such that $\mathfrak{g}^0$ is a real form of $\mathfrak{g}$ as a complex vector space.

Let G be a complex semisimple connected algebraic group, and G^0 be a *real form* of G, i.e., G^0 is a real closed algebraic subgroup of G such that its Lie algebra $\mathfrak{g}^0$ is a real form of the Lie algebra $\mathfrak{g}$ of G. Let P be a parabolic subgroup of G, and $X = G/P$ be the corresponding flag variety. The group G^0 naturally acts on X. In [22] the third author proved the following facts about the G^0-orbit structure of X, see [22, Theorems 2.6, 3.3, 3.6, Corollary 3.4] (here we use the usual differentiable manifold topology on X).

Theorem 2.1

(i) *Each G^0-orbit is a real submanifold of X.*
(ii) *The number of G^0-orbits on X is finite.*
(iii) *The union of the open G^0-orbits is dense in X.*
(iv) *There is a unique closed orbit Ω on X.*
(v) *The inequality* $\dim_{\mathbb{R}} \Omega \geq \dim_{\mathbb{C}} X$ *holds.*

Here is how this theorem relates to Witt's Theorem in the case of a Hermitian form. Let V be an n-dimensional complex vector space and $G = \mathrm{SL}(V)$. Fix a nondegenerate Hermitian form ω of signature $(p, n-p)$ on the vector space V and denote by $G^0 = \mathrm{SU}(V, \omega)$ the group of all linear operators on V of determinant 1 which preserve ω. Then G^0 is a real form of G. Given $k \leq n$, the group G naturally acts on the grassmannian $X = \mathrm{Gr}(k, V)$ of all k-dimensional complex subspaces of V. To each $U \in X$ one can assign its signature (a, b, c), where the restricted form $\omega|_U$ has rank $a + b$ with a positive squares and b negative ones, c equals the dimension of the intersection of U and its orthogonal complement, and $a + b + c = k$. By Witt's Theorem, two subspaces U_1, $U_2 \in X$ belong to the same G^0-orbit if and only if their signatures coincide. Set $l = \min\{p,\ n-p\}$. Then one can verify the following formula for the number $|X/G^0|$ of G^0-orbits on X:

$$|X/G^0| = \begin{cases} (-k^2 - 2l^2 - n^2 + 2kn + 2ln + k + n + 2)/2, & \text{if } n - l \leq k, \\ (l+1)(2k - l + 2)/2, & \text{if } l \leq k \leq n - l, \\ (k+1)(k+2)/2, & \text{if } k \leq l. \end{cases}$$

Furthermore, a G^0-orbit of a subspace $U \in X$ is open if and only if the restriction of ω to U is nondegenerate, i.e., if $c = 0$. Therefore, the number of open orbits equals $\min\{k+1, l+1\}$. There is a unique closed G^0-orbit Ω on X, and it consists of all k-dimensional subspaces of V such that $c = \min\{k, l\}$ (the condition $c = \min\{k, l\}$ maximizes the nullity of the form $\omega|_U$ for k-dimensional subspaces $U \subset V$). In particular, if $k = p \leq n - p$, then Ω consists of all totally isotropic[1] k-dimensional complex subspaces of V. See [22] for more details in this latter case.

2.2 *The Ind-Group* $\mathrm{SL}(\infty, \mathbb{C})$ *and Its Real Forms*

In the rest of the paper, V denotes a fixed countable-dimensional complex vector space with fixed basis $\mathcal{E}$. We fix an order on $\mathcal{E}$ via the ordered set $\mathbb{Z}_{>0}$, i.e., $\mathcal{E} = \{\epsilon_1, \epsilon_2, \ldots\}$. Let V_* denote the span of the dual system $\mathcal{E}^* = \{\epsilon_1^*, \epsilon_2^*, \ldots\}$. By definition, the group $\mathrm{GL}(V, \mathcal{E})$ is the group of invertible $\mathbb{C}$-linear transformations on V that keep fixed all but finitely many elements of $\mathcal{E}$. It is not difficult to verify that $\mathrm{GL}(V, \mathcal{E})$ depends only on the pair (V, V_*) but not on $\mathcal{E}$. Clearly, any operator from $\mathrm{GL}(V, \mathcal{E})$ has a well-defined determinant. By $\mathrm{SL}(V, \mathcal{E})$ we denote the subgroup of $\mathrm{GL}(V, \mathcal{E})$ of all operators with determinant 1. In the sequel $G = \mathrm{SL}(V, \mathcal{E})$ and we also write $\mathrm{SL}(\infty, \mathbb{C})$ instead of G.

Express the basis $\mathcal{E}$ as a union $\mathcal{E} = \bigcup \mathcal{E}_n$ of nested finite subsets. Then V is exhausted by the finite-dimensional subspaces $V_n = \langle \mathcal{E}_n \rangle_{\mathbb{C}}$, i.e., $V = \varinjlim V_n$. To each linear operator φ on V_n one can assign the operator $\widetilde{\varphi}$ on V_{n+1} such that $\widetilde{\varphi}(x) = \varphi(x)$ for $x \in V_n$, $\widetilde{\varphi}(\epsilon_m) = \epsilon_m$ for $\epsilon_m \in \mathcal{E} \setminus \mathcal{E}_n$. This gives embeddings $\mathrm{SL}(V_n) \hookrightarrow \mathrm{SL}(V_{n+1})$, so that $G = \mathrm{SL}(V, \mathcal{E}) = \varinjlim \mathrm{SL}(V_n)$. In what follows we consider this exhaustion of G fixed, and set $G_n = \mathrm{SL}(V_n)$.

Recall that an *ind-variety* over $\mathbb{R}$ or $\mathbb{C}$ (resp., an *ind-manifold*) is an inductive limit of algebraic varieties (resp., of manifolds): $Y = \varinjlim Y_n$. Below we always assume that Y_n form an ascending chain

$$Y_1 \hookrightarrow Y_2 \hookrightarrow \ldots \hookrightarrow Y_n \hookrightarrow Y_{n+1} \hookrightarrow \ldots,$$

where $Y_n \hookrightarrow Y_{n+1}$ are closed embeddings. Any ind-variety or ind-manifold is endowed with a topology by declaring a subset $U \subset Y$ open if $U \cap Y_n$ is open for all n in the corresponding topologies. A *morphism* $f\colon Y = \varinjlim Y_n \to Y' = \varinjlim Y'_n$ is a map induced by a collection of morphisms $\{f_n\colon Y_n \to Y'_{i_n}\}_{n \geq 1}$ for $i_1 < i_2 < \cdots$, such that the restriction of f_{n+1} to Y_n coincides with f_n for all $n \geq 1$. A morphism

[1] In what follows we use the terms `isotropic` and `totally isotropic` as synonyms.

$f: Y \to Y'$ is an *isomorphism* if there exists a morphism $g: Y' \to Y$ for which $f \circ g = \mathrm{id}_{Y'}$ and $g \circ f = \mathrm{id}_Y$, where id is a morphism induced by the collection of the identity maps.

A *locally linear algebraic ind-group* is an ind-variety $\mathcal{G} = \bigcup \mathcal{G}_n$ such that all $\mathcal{G}_n$ are linear algebraic groups and the inclusions are group homomorphisms. In what follows we write *ind-group* for brevity. Clearly, G is an ind-group. By an *ind-subgroup* of G we understand a subgroup of G closed in the direct limit Zariski topology. By definition, a real ind-subgroup $\mathcal{G}^0$ of G is called a *real form* of G, if G can be represented as an increasing union $G = \bigcup \mathcal{G}_n$ of its finite-dimensional Zariski closed subgroups such that $\mathcal{G}_n$ is a semi-simple algebraic group and $\mathcal{G}^0 \cap \mathcal{G}_n$ is a real form of $\mathcal{G}_n$ for each n. Below we recall the classification of real forms of G due to A. Baranov [1].

Fix a real structure τ on V such that $\tau(e) = e$ for all $e \in \mathcal{E}$. Then each V_n is τ-invariant. Denote by $\mathrm{GL}(V_n, \mathbb{R})$ (resp., by $\mathrm{SL}(V_n, \mathbb{R})$) the group of invertible (resp., of determinant 1) operators on V_n defined over $\mathbb{R}$. Recall that a linear operator on a complex vector space with a real structure is *defined over* $\mathbb{R}$ if it commutes with the real structure, or, equivalently, if it maps the real form to itself. For each n, the map $\varphi \mapsto \widetilde{\varphi}$ gives an embedding $\mathrm{SL}(V_n, \mathbb{R}) \hookrightarrow \mathrm{SL}(V_{n+1}, \mathbb{R})$, so the direct limit $G^0 = \varinjlim \mathrm{SL}(V_n, \mathbb{R})$ is well defined. We denote this real form of G by $\mathrm{SL}(\infty, \mathbb{R})$.

Fix a nondegenerate Hermitian form ω on V. Suppose that its restriction $\omega_n = \omega|_{V_n}$ is nondegenerate for all n, and that $\omega(\epsilon_m, V_n) = 0$ for $\epsilon_m \in \mathcal{E} \setminus \mathcal{E}_n$. Denote by p_n the dimension of a maximal ω_n-positive definite subspace of V_n, and put $q_n = \dim V_n - p_n$. Let $\mathrm{SU}(p_n, q_n)$ be the subgroup of G_n consisting of all operators preserving the form ω_n. For each n, the map $\varphi \mapsto \widetilde{\varphi}$ induces an embedding $\mathrm{SU}(p_n, q_n) \hookrightarrow \mathrm{SU}(p_{n+1}, q_{n+1})$, so we have a direct limit $G^0 = \varinjlim \mathrm{SU}(p_n, q_n)$. If there exists p such that $p_n = p$ for all sufficiently large n (resp., if $\lim_{n\to\infty} p_n = \lim_{n\to\infty} q_n = \infty$), then we denote this real form of G by $\mathrm{SU}(p, \infty)$ (resp., by $\mathrm{SU}(\infty, \infty)$).

Finally, fix a *quaternionic structure* J on V, i.e., an antilinear automorphism of V such that $J^2 = -\mathrm{id}_V$. Assume that the complex dimension of V_n is even for $n \geq 1$, and that the restriction J_n of J to V_n is a quaternionic structure on V_n. Furthermore, suppose that

$$J(\epsilon_{2i-1}) = -\epsilon_{2i}, \ \ J(\epsilon_{2i}) = \epsilon_{2i-1}$$

for $i \geq 1$. Let $\mathrm{SL}(V_n, \mathbb{H})$ be the subgroup of G_n consisting of all linear operators commuting with J_n, then, for each n, the map $\varphi \mapsto \widetilde{\varphi}$ induces an embedding of the groups $\mathrm{SL}(V_n, \mathbb{H}) \hookrightarrow \mathrm{SL}(V_{n+1}, \mathbb{H})$, and we denote the direct limit by $G^0 = \mathrm{SL}(\infty, \mathbb{H}) = \varinjlim \mathrm{SL}(V_n, \mathbb{H})$. This group is also a real form of G.

The next result is a corollary of [1, Theorem 1.4] and [6, Corollary 3.2].

Theorem 2.2 *If* $G = \mathrm{SL}(\infty, \mathbb{C})$, *then* $\mathrm{SL}(\infty, \mathbb{R})$, $\mathrm{SU}(p, \infty)$, $0 \leq p < \infty$, $\mathrm{SU}(\infty, \infty)$, $\mathrm{SL}(\infty, \mathbb{H})$ *are all real forms of G up to isomorphism. These real forms are pairwise non-isomorphic as ind-groups.*

2.3 Flag Ind-Varieties of the Ind-Group G

Recall some basic definitions from [5]. A *chain* of subspaces in V is a linearly ordered (by inclusion) set $\mathcal{C}$ of distinct subspaces of V. We write $\mathcal{C}'$ (resp., $\mathcal{C}''$) for the subchain of $\mathcal{C}$ of all $F \in \mathcal{C}$ with an immediate successor (resp., an immediate predecessor). Also, we write $\mathcal{C}^\dagger$ for the set of all pairs $(F',\ F'')$ such that $F'' \in \mathcal{C}''$ is the immediate successor of $F' \in \mathcal{C}'$.

A *generalized flag* is a chain $\mathcal{F}$ of subspaces in V such that $\mathcal{F} = \mathcal{F}' \cup \mathcal{F}''$ and $V \setminus \{0\} = \bigcup_{(F',F'')\in\mathcal{F}^\dagger} F'' \setminus F'$. Note that each nonzero vector $v \in V$ determines a unique pair $(F'_v, F''_v) \in \mathcal{F}^\dagger$ such that $v \in F''_v \setminus F'_v$. If $\mathcal{F}$ is a generalized flag, then each of $\mathcal{F}'$ and $\mathcal{F}''$ determines $\mathcal{F}$, because if $(F', F'') \in \mathcal{F}^\dagger$, then $F' = \bigcup_{G''\in\mathcal{F}'',\ G''\subsetneq F''} G''$, $F'' = \bigcap_{G'\in\mathcal{F}',\ G'\supsetneq F'} G'$ (see [5, Proposition 3.2]). We fix a linearly ordered set $(A, \preceq)$ and an isomorphism of ordered sets $A \to \mathcal{F}^\dagger\colon a \mapsto (F'_\alpha, F''_\alpha)$, so that $\mathcal{F}$ can be written as $\mathcal{F} = \{F'_\alpha, F''_\alpha,\ \alpha \in A\}$. We will write $\alpha \prec \beta$ if $\alpha \preceq \beta$ and $\alpha \neq \beta$ for $\alpha, \beta \in A$.

A generalized flag $\mathcal{F}$ is called *maximal* if it is not properly contained in another generalized flag. This is equivalent to the condition that $\dim F''_v/F'_v = 1$ for all nonzero vectors $v \in V$. A generalized flag is called a *flag* if the set of all proper subspaces of $\mathcal{F}$ is isomorphic as a linearly ordered set to a subset of $\mathbb{Z}$.

We say that a generalized flag $\mathcal{F}$ is *compatible* with a basis $E = \{e_1,\ e_2, \ldots\}$ of V if there exists a surjective map $\sigma\colon\ E \to A$ such that every pair $(F'_\alpha, F''_\alpha) \in \mathcal{F}^\dagger$ has the form $F'_\alpha = \langle e \in E \mid \sigma(e) \prec \alpha\rangle_{\mathbb{C}}$, $F''_\alpha = \langle e \in E \mid \sigma(e) \preceq \alpha\rangle_{\mathbb{C}}$. By [5, Proposition 4.1], every generalized flag admits a compatible basis. A generalized flag $\mathcal{F}$ is *weakly compatible* with E if $\mathcal{F}$ is compatible with a basis L of V such that the set $E \setminus (E \cap L)$ is finite. Two generalized flags $\mathcal{F}, \mathcal{G}$ are *E-commensurable* if both of them are weakly compatible with E and there exist an isomorphism of ordered sets $\phi\colon \mathcal{F} \to \mathcal{G}$ and a finite-dimensional subspace $U \subset V$ such that

(i) $\phi(F) + U = F + U$ for all $F \in \mathcal{F}$;

(ii) $\dim \phi(F) \cap U = \dim F \cap U$ for all $F \in \mathcal{F}$.

Given a generalized flag $\mathcal{F}$ compatible with E, denote by $X = X_{\mathcal{F},E} = \mathcal{F}\ell(\mathcal{F}, E)$ the set of all generalized flags in V, which are E-commensurable with F.

To endow X with an ind-variety structure, fix an exhaustion $E = \bigcup E_n$ of E by its finite subsets and denote $\mathcal{F}_n = \{F \cap \langle E_n\rangle_{\mathbb{C}},\ F \in \mathcal{F}\}$. Given $\alpha \in A$, denote

$$d'_{\alpha,n} = \dim F'_\alpha \cap \langle E_n\rangle_{\mathbb{C}} = |\{e \in E_n \mid \sigma(e) \prec \alpha\}|,$$
$$d''_{\alpha,n} = \dim F''_\alpha \cap \langle E_n\rangle_{\mathbb{C}} = |\{e \in E_n \mid \sigma(e) \preceq \alpha\}|,$$

where $|\cdot|$ stands for cardinality. We define X_n to be the projective varieties of flags in $\langle E_n\rangle_{\mathbb{C}}$ of the form $\{U''_\alpha, U''_\alpha,\ \alpha \in A\}$, where U'_α, U''_α are subspaces of $\langle E_n\rangle_{\mathbb{C}}$ of dimensions $d'_{\alpha,n}$, $d''_{\alpha,n}$ respectively, $U'_\alpha \subset U''_\alpha$ for all $\alpha \in A$, and $U''_\alpha \subset U'_\beta$ for all $\alpha \prec \beta$. (If A is infinite, there exist infinitely many $\alpha, \beta \in A$ such that $U''_\alpha = U'_\beta$.)

Define an embedding $\iota_n : X_n \to X_{n+1} : \{U''_\alpha, U''_\alpha,\ \alpha \in A\} \mapsto \{W''_\alpha, W''_\alpha,\ \alpha \in A\}$ by

$$\begin{aligned} W'_\alpha &= U'_\alpha \oplus \langle e \in E_{n+1} \setminus E_n \mid \sigma(e) \prec \alpha \rangle_{\mathbb{C}}, \\ W''_\alpha &= U''_\alpha \oplus \langle e \in E_{n+1} \setminus E_n \mid \sigma(e) \preceq \alpha \rangle_{\mathbb{C}}. \end{aligned} \tag{1}$$

Then ι_n is a closed embedding of algebraic varieties, and there exists a bijection from X to the inductive limit of this chain of morphisms, see [5, Proposition 5.2] or [10, Sect. 3.3]. This bijection endows X with an ind-variety structure which is independent on the chosen filtration $\bigcup E_n$ of the basis E. We will explain this bijection in more detail in Sect. 3.

From now on we suppose that the linear span of E_n coincides with V_n and V_* coincides with the span of the dual system $E^* = \{e_1^*, e_2^*, \ldots\}$. We assume also that the inclusion $G_n \hookrightarrow G_{n+1}$ induced by this exhaustion of E coincides with the inclusion $\varphi \mapsto \widetilde{\varphi}$ defined above, i.e., that $\langle E_{n+1} \setminus E_n \rangle_{\mathbb{C}} = \langle \mathcal{E}_{n+1} \setminus \mathcal{E}_n \rangle_{\mathbb{C}}$. Denote by H the ind-subgroup of $G = \mathrm{SL}(\infty, \mathbb{C})$ of all operators from G which are diagonal in E; H is called a *splitting Cartan subgroup* of G (in fact, H is a Cartan subgroup of G in terminology of [7]). We define a *splitting Borel* (resp., *parabolic*) *subgroup* of G to be and ind-subgroup of G containing H such that its intersection with G_n is a Borel (resp., parabolic) subgroup of G_n. Note that if P is a splitting parabolic subgroup of G and $P_n = P \cap G_n$, then $G/P = \bigcup G_n/P_n$ is a locally projective ind-variety, i.e., an ind-variety exhausted by projective varieties. One can easily check that the group G naturally acts on X. Given a generalized flag $\mathcal{F}$ in V which is compatible with E, denote by $P_{\mathcal{F}}$ the stabilizer of $\mathcal{F}$ in G. For the proof of the following theorem, see [5, Proposition 6.1, Theorem 6.2].

Theorem 2.3 *Let $\mathcal{F}$ be a generalized flag compatible with E, $X = \mathcal{F}\ell(\mathcal{F}, E)$ and $G = \mathrm{SL}(\infty, \mathbb{C})$.*
(i) *The group $P_{\mathcal{F}}$ is a parabolic subgroup of G containing H, and the map $\mathcal{F} \mapsto P_{\mathcal{F}}$ is a bijection between generalized flags compatible with E and splitting parabolic subgroups of G.*
(ii) *The ind-variety X is in fact G-homogeneous, and the map $g \mapsto g \cdot \mathcal{F}$ induces an isomorphism of ind-varieties $G/P_{\mathcal{F}} \cong X$.*
(iii) *$\mathcal{F}$ is maximal if and only if $P_{\mathcal{F}}$ is a splitting Borel subgroup of G.*

Example 2.4 (i) A first example of generalized flags is provided by the flag $\mathcal{F} = \{\{0\} \subset F \subset V\}$, where F is a proper nonzero subspace of V. If F is compatible with E, then we can assume that $F = \langle \sigma \rangle_{\mathbb{C}}$ for some subset σ of E. In this case the ind-variety X is called an *ind-grassmannian*, and is denoted by $\mathrm{Gr}(F, E)$. If $k = \dim F$ is finite, then a flag $\{\{0\} \subset F' \subset V\}$ is E-commensurable with $\mathcal{F}$ if and only if $\dim F = k$, hence $\mathrm{Gr}(F, E)$ depends only on k, and we denote it by $\mathrm{Gr}(k, V)$. Similarly, if $k = \mathrm{codim}_V F$ is finite, then $\mathrm{Gr}(F, E)$ depends only on E and k (but not on F) and is isomorphic to $\mathrm{Gr}(k, V_*)$: an isomorphism $\mathrm{Gr}(F, E) \to \{F \subset V_* \mid \dim F = k\} = \mathrm{Gr}(k, V_*)$ is induced by the map $\mathrm{Gr}(F, E) \ni U \mapsto U^{\#} = \{\phi \in V_* \mid \phi(x) = 0 \text{ for all } x \in U\}$. Finally, if F is both infinite dimensional and infinite codimensional, then $\mathrm{Gr}(F, E)$ depends on F and E, but all such ind-varieties are isomorphic and denoted by $\mathrm{Gr}(\infty)$, see [18] or [10, Sect. 4.5] for the details. Clearly, in each case one has $\mathcal{F}' = \{\{0\} \subset F\}$, $\mathcal{F}'' = \{F \subset V\}$.

(ii) Our second example is the generalized flag $\mathcal{F} = \{\{0\} = F_0 \subset F_1 \subset \ldots\}$, where $F_i = \langle e_1, \ldots, e_i\rangle_{\mathbb{C}}$ for all $i \geq 1$. This clearly is a flag. A flag $\widetilde{\mathcal{F}} = \{\{0\} = \widetilde{F}_0 \subset \widetilde{F}_1 \subset \ldots\}$ is E-commensurable with $\mathcal{F}$ if and only if $\dim F_i = \dim \widetilde{F}_i$ for all i, and $F_i = \widetilde{F}_i$ for large enough i. The flag $\mathcal{F}$ is maximal, and $\mathcal{F}' = \mathcal{F}$, $\mathcal{F}'' = \mathcal{F} \setminus \{0\}$.

(iii) Put $\mathcal{F} = \{\{0\} = F_0 \subset F_1 \subset F_2 \subset \ldots \subset F_{-2} \subset F_{-1} \subset V\}$, where $F_i = \langle e_1,\ e_3, \ldots, e_{2i-1}\rangle_{\mathbb{C}}$, $F_{-i} = \langle\{e_j,\ j \text{ odd}\} \cup \{e_{2j},\ j > i\}\rangle_{\mathbb{C}}$ for $i \geq 1$. This generalized flag is clearly not a flag, and is maximal. Here $\mathcal{F}' = \mathcal{F} \setminus V$, $\mathcal{F}'' = \mathcal{F} \setminus \{0\}$. Note also that $\widetilde{\mathcal{F}} \in X = \mathcal{F}\ell(\mathcal{F}, E)$ does not imply that $\widetilde{F}_i = F_i$ for i large enough. For example, let $\widetilde{F}_1 = \mathbb{C}e_2$,

$$\widetilde{F}_i = \langle e_2,\ e_3,\ e_5,\ e_7, \ldots,\ e_{2i-1}\rangle_{\mathbb{C}}$$

for $i > 1$, and $\widetilde{F}_{-i} = \langle\{e_j,\ j \text{ odd},\ j \geq 3\} \cup \{e_2\} \cup \{e_{2j}, j > i\}\rangle_{\mathbb{C}},\ i \geq 1$, then $\widetilde{\mathcal{F}} \in X$, but $\widetilde{F}_i \neq F_i$ for all i.

Remark 2.5 In all above examples $X = G/P_{\mathcal{F}}$, where $P_{\mathcal{F}}$ is the stabilizer of $\mathcal{F}$ in G. The ind-grassmannians in (i) are precisely the ind-varieties $G/P_{\mathcal{F}}$ for maximal splitting parabolic ind-subgroups $P_{\mathcal{F}} \subset G$. The ind-variety $\mathcal{F}\ell(\mathcal{F}, E)$, where $\mathcal{F}$ is the flag in (ii), equals $G/P_{\mathcal{F}}$ where $P_{\mathcal{F}}$ is the upper-triangular Borel ind-subgroup in the realization of G as $\mathbb{Z}_{>0} \times \mathbb{Z}_{>0}$-matrices.

3 G^0-Orbits as Ind-Manifolds

In this section, we establish a basic property of the orbits on G/P of a real form G^0 of $G = \mathrm{SL}(\infty, \mathbb{C})$. Precisely, we prove that the intersection of a G^0-orbit with X_n is a single orbit. Consequently, each G^0-orbit is an infinite-dimensional real ind-manifold.

We start by describing explicitly the bijection $X \to \varinjlim X_n$ mentioned in Sect. 2.3. Let $\mathcal{F}$ be a generalized flag in V compatible with the basis E, and $X = \mathcal{F}\ell(\mathcal{F}, E)$ be the corresponding ind-variety of generalized flags. Recall that we consider X as the inductive limit of flag varieties X_n, where the embeddings $\iota_n \colon X_n \hookrightarrow X_{n+1}$ are defined in the previous subsection. Put $E'_m = \{e_1,\ e_2, \ldots,\ e_m\}$ and $\mathcal{V}_m = \langle E'_m\rangle_{\mathbb{C}}$. The construction of ι_n can be reformulated as follows.

The dimensions of the spaces of the flag $\mathcal{F} \cap \mathcal{V}_m$ form a sequence of integers

$$0 = d_{m,0} < d_{m,1} < \ldots < d_{m,\,s_m-1} < d_{m,\,s_m} = \dim \mathcal{V}_m = m.$$

Let $\mathcal{F}\ell(d_m, \mathcal{V}_m)$ be the flag variety of type $d_m = (d_{m,1}, \ldots, d_{m,s_m-1})$ in $\mathcal{V}_m$. Since either $s_{m+1} = s_m$ or $s_{m+1} = s_m + 1$, there is a unique j_m such that $d_{m+1,i} = d_{m,i} + 1$ for $0 \leq i < j_m$ and $d_{m+1,j_m} > d_{m,j_m}$. Then, for $j_m \leq i < s_m$, $d_{m+1,i} = d_{m,i} + 1$ in case $s_{m+1} = s_m$, and $d_{m+1,i} = d_{m,i-1} + 1$ in case $s_{m+1} = s_m + 1$. In other words,

$j_m \le s_m$ is the minimal nonnegative integer for which there is $\alpha \in A$ with

$$\dim F''_\alpha \cap \mathcal{V}_{m+1} = \dim F''_\alpha \cap \mathcal{V}_m + 1.$$

Now, for each m we define an embedding $\xi_m \colon \mathcal{F}\ell(d_m, \mathcal{V}_m) \hookrightarrow \mathcal{F}\ell(d_{m+1}, \mathcal{V}_{m+1})$: given a flag $\mathcal{G}_m = \{\{0\} = G_0^m \subset G_1^m \subset \ldots \subset G_{s_m}^m = V_m\} \in \mathcal{F}\ell(d_m, \mathcal{V}_m)$, we set $\xi_m(\mathcal{G}_m) = \mathcal{G}_{m+1} = \{\{0\} = G_0^{m+1} \subset G_1^{m+1} \subset \ldots \subset G_{s_{m+1}}^{m+1} = V_{m+1}\} \in \mathcal{F}\ell(d_{m+1}, \mathcal{V}_{m+1})$, where

$$G_i^{m+1} = \begin{cases} G_i^m, & \text{if } 0 \le i < j_m, \\ G_i^m \oplus \mathbb{C}e_{m+1}, & \text{if } j_m \le i \le s_{m+1} \text{ and } s_{m+1} = s_m, \\ G_{i-1}^m \oplus \mathbb{C}e_{m+1}, & \text{if } j_m \le i \le s_{m+1} \text{ and } s_{m+1} = s_m + 1. \end{cases} \tag{2}$$

For any $\mathcal{G} \in X$ we choose a positive integer $m_\mathcal{G}$ such that $\mathcal{F}$ and $\mathcal{G}$ are compatible with bases containing $\{e_i \mid i \ge m_\mathcal{G}\}$, and $\mathcal{V}_{m_\mathcal{G}}$ contains a subspace which makes these generalized flags E-commensurable. In addition, we can assume that $m_\mathcal{F} \le m_\mathcal{G}$ for all $\mathcal{G} \in X$ (in fact, we can set $m_\mathcal{F} = 1$ because $\mathcal{F}$ is compatible with E). Let $m_\mathcal{F} \le m_1 < m_2 < \ldots$ be an arbitrary sequence of integer numbers. For $n \ge 1$, denote $E_n = E'_{m_n}$, $V_n = \mathcal{V}_{m_n}$. Then $X_n = \mathcal{F}\ell(d_{m_n}, \mathcal{V}_{m_n})$ and, according to (1), $\iota_n = \xi_{m_{n+1}-1} \circ \xi_{m_{n+1}-2} \circ \ldots \circ \xi_{m_n}$. The bijection $X \to \varinjlim X_n$ from Sect. 2.3 now has the form $\mathcal{G} \mapsto \varinjlim \mathcal{G}_n$, where $\mathcal{G}_n = \{F \cap V_n,\ F \in \mathcal{G}\}$ for n such that $m_n \ge m_\mathcal{G}$. By a slight abuse of notation, in the sequel we will denote the canonical embedding $X_n \hookrightarrow X$ by the same letter ι_n.

Let G^0 be a real form of $G = \mathrm{SL}(\infty, \mathbb{C})$ (see Theorem 2.2). The group $G_n = \mathrm{SL}(V_n)$ naturally acts on X_n, and the map ι_n is equivariant: $g \cdot \iota_n(x) = \iota_n(g \cdot x)$, $g \in G_n \subset G_{n+1}$, $x \in X_n$. Put also $G_n^0 = G^0 \cap G_n$. Then G_n^0 is a real form of G_n. For the rest of the paper we fix some specific assumptions on V_n for different real forms. We now describe these assumptions case by case.

Let $G^0 = \mathrm{SU}(p, \infty)$ or $\mathrm{SU}(\infty, \infty)$. Recall that the restriction ω_n of the fixed nondegenerate Hermitian form ω to V_n is nondegenerate. From now on, we assume that if $e \in E_{n+1} \setminus E_n$, then e is orthogonal to V_n with respect to ω_{n+1}. Next, let $G^0 = \mathrm{SL}(\infty, \mathbb{R})$. Here we assume that m_n is odd for each $n \ge 1$, and that $\langle E_n \rangle_\mathbb{R}$ is a real form of V_n. Finally, for $G^0 = \mathrm{SL}(\infty, \mathbb{H})$, we assume that m_n is even for all $n \ge 1$ and that $J(e_{2i-1}) = -e_{2i}$, $J(e_{2i}) = e_{2i-1}$ for all i. These additional assumptions align the real form G^0 with the flag variety X.

Our main result in this section is as follows.

Theorem 3.1 *If $\iota_n(X_n)$ has nonempty intersection with a G_{n+1}^0-orbit, then that intersection is a single G_n^0-orbit.*

Proof The proof goes case by case.

<u>Case $G^0 = \mathrm{SU}(\infty, \infty)$</u>. (The proof for $G^0 = \mathrm{SU}(p, \infty)$, $0 \le p < \infty$, is completely similar.) Pick two flags

$$\mathcal{A} = \{\{0\} = A_0 \subset A_1 \subset \ldots \subset A_{s_{m_n}} = V_n\},$$
$$\mathcal{B} = \{\{0\} = B_0 \subset B_1 \subset \ldots \subset B_{s_{m_n}} = V_n\}$$

in X_n such that $\widetilde{\mathcal{A}} = \iota_n(\mathcal{A})$ and $\widetilde{\mathcal{B}} = \iota_n(\mathcal{B})$ belong to a given G^0_{n+1}-orbit.

Put

$$\widetilde{\mathcal{A}} = \{\{0\} = \widetilde{A}_0 \subset \widetilde{A}_1 \subset \ldots \subset \widetilde{A}_{s_{m_{n+1}}} = V_{n+1}\},$$
$$\widetilde{\mathcal{B}} = \{\{0\} = \widetilde{B}_0 \subset \widetilde{B}_1 \subset \ldots \subset \widetilde{B}_{s_{m_{n+1}}} = V_{n+1}\}.$$

There exists $\widetilde{\varphi} \in \mathrm{SU}(\omega_{n+1}, V_{n+1})$ satisfying $\widetilde{\varphi}(\widetilde{\mathcal{A}}) = \widetilde{\mathcal{B}}$, i.e., $\widetilde{\varphi}(\widetilde{A}_i) = \widetilde{B}_i$ for all i from 0 to $s_{m_{n+1}}$. To prove the result, we must construct an isometry $\varphi\colon V_n \to V_n$ satisfying $\varphi(\mathcal{A}) = \mathcal{B}$. Of course, one can scale φ to obtain an isometry of determinant 1. By Huang's extension of Witt's Theorem [11, Theorem 6.2], such an isometry exists if and only if A_i and B_i are isometric for all i from 1 to s_{m_n}, and

$$\dim(A_i \cap A_j^{\perp, V_n}) = \dim(B_i \cap B_j^{\perp, V_n}) \tag{3}$$

for all $i < j$ from 1 to s_{m_n}. (Here $U^{\perp, V_n}$ denotes the ω_n-orthogonal complement within V_n of a subspace $U \subset V_n$.) Pick i from 1 to s_{m_n}. Since e_{n+1} is orthogonal to V_n and $\widetilde{\varphi}$ establishes an isometry between $\widetilde{A}_i$ and $\widetilde{B}_i$, the first condition is satisfied. So it remains to prove (3).

To do this, denote $C_n = \langle E_{n+1} \setminus E_n \rangle_{\mathbb{C}}$. Since C_n is orthogonal to V_n, for given subspaces $U \subset V_n$, $W \subset C_n$ one has $(U \oplus W)^{\perp, V_{n+1}} = U^{\perp, V_n} \oplus W^{\perp, C_n}$. Hence, if $\widetilde{A}_k = A_k \oplus W_k$, $\widetilde{B}_k = B_k \oplus W_k$ for $k \in \{i, j\}$ and some subspaces of $W_i, W_j \subset C_n$, then

$$\widetilde{A}_i \cap \widetilde{A}_j^{\perp, V_{n+1}} = (A_i \oplus W_i) \cap (A_j^{\perp, V_n} \oplus W_j^{\perp, C_n}) = (A_i \cap A_j^{\perp, V_n}) \oplus (W_i \cap W_j^{\perp, C_n}),$$

and the similar equality holds for $\widetilde{B}_i \cap \widetilde{B}_j^{\perp, V_{n+1}}$. The result follows.

<u>CASE $G^0 = \mathrm{SL}(\infty, \mathbb{R})$</u>. Here we first prove that if $\mathcal{A}$ and $\mathcal{B}$ are flags in $\mathcal{V}_n$, $\widetilde{\mathcal{A}}$ and $\widetilde{\mathcal{B}}$ are their images in $\mathcal{V}_{n+1}$ under the map ξ_n, and there exists $\varphi \in \mathrm{GL}(\mathcal{V}_{n+1}, \mathbb{R})$ satisfying $\varphi(\widetilde{\mathcal{A}}) = \widetilde{\mathcal{B}}$, then there exists an operator $\nu \in \mathrm{GL}(\mathcal{V}_n, \mathbb{R})$ such that $\nu(\mathcal{A}) = \mathcal{B}$.

Consider first the case when $\varphi(e_{n+1}) \notin \mathcal{V}_n$. Denote $\varphi(e_{n+1}) = v + te_{n+1}$, $v \in \mathcal{V}_n$, $t \in \mathbb{R}$, $t \neq 0$. Then $t^{-1}\varphi \in \mathrm{GL}(\mathcal{V}_{n+1}, \mathbb{R})$ maps $\widetilde{\mathcal{A}}$ to $\widetilde{\mathcal{B}}$, so we can assume that $t = 1$, i.e., $\varphi(e_{n+1}) = v + e_{n+1}$. Since

$$\varphi(A_{j_n} \oplus \mathbb{C}e_{n+1}) = \varphi(\widetilde{A}_{j_n}) = \widetilde{B}_{j_n} = B_{j_n} \oplus \mathbb{C}e_{n+1},$$

the vector v belongs to B_i for all $i \geq j_n$. Let $\psi \in \mathrm{GL}(\mathcal{V}_{n+1}, \mathbb{R})$ be defined by

$$\psi(x + se_{n+1}) = x + s(e_{n+1} - v),\ x \in \mathcal{V}_n,\ s \in \mathbb{C}.$$

Clearly, $\psi(\varphi(e_{n+1})) = e_{n+1}$.

If $i < j_n$ and $x \in A_i$, then $\varphi(x) \in B_i \subset \mathcal{V}_n$, so $\psi(\varphi(x)) = \varphi(x) \in B_i$. If $i \geq j_n$ and $x \in A_r$, where $r = i$ for $s_{n+1} = s_n$ and $r = i - 1$ for $s_{n+1} = s_n + 1$, then we put $\varphi(x) = y + se_{n+1}$, $y \in B_i$, $s \in \mathbb{C}$. One has

$$\psi(\varphi(x)) = \psi(y + se_{n+1}) = y + s(e_{n+1} - v) \in B_r \oplus \mathbb{C}e_{n+1} = \widetilde{B}_i.$$

In both cases the operator $\psi \circ \varphi$ maps $\widetilde{A}_i$ to $\widetilde{B}_i$ for all i from 0 to s_{n+1}. Hence we may assume without loss of generality that $\varphi(e_{n+1}) = e_{n+1}$. Then the operator $\nu = \pi \circ \varphi|_{\mathcal{V}_n}$, where $\pi \colon \mathcal{V}_{n+1} \to \mathcal{V}_n$ is the projection onto $\mathcal{V}_n$ along $\mathbb{C}e_{n+1}$, is invertible, is defined over $\mathbb{R}$, and maps each A_i to B_i, $0 \leq i \leq s_n$, as required.

Suppose now that $\varphi(e_{n+1}) = b \in \mathcal{V}_n$. In this case $s_{n+1} = s_n$ because the condition

$$\varphi(A_{j_n-1} \oplus \mathbb{C}e_{n+1}) = \varphi(\widetilde{A}_{j_n}) = \widetilde{B}_{j_n} = B_{j_n-1} \oplus \mathbb{C}e_{n+1}$$

contradicts the equality $s_{n+1} = s_n + 1$. Arguing as above, we see that $b \in B_i$ for all $i \geq j_n$. If $\varphi^{-1}(e_{n+1}) = a \notin \mathcal{V}_n$, then one can construct ν as in the case when $b \notin \mathcal{V}_n$ with φ^{-1} instead of φ. Therefore, we may assume that $a \in A_i$ for all $i \geq j_n$. Let U^0 be an $\mathbb{R}$-subspace of $\mathcal{V}_n^0$ such that $\mathcal{V}_n^0 = U^0 \oplus \mathbb{R}b$, then $\mathcal{V}_n = U \oplus \mathbb{C}b$, where $U = \mathbb{C} \otimes_{\mathbb{R}} U^0$. If $a,\ b$ are linearly independent, we choose U^0 so that $a \in U^0$. Define ν as follows: if $\varphi(x) = y + sb + re_{n+1}$, $x \in \mathcal{V}_n$, $y \in U$, $s, r \in \mathbb{C}$, then put $\nu(x) = y + (s + r)b$. One can easily check that ν satisfies all required conditions.

Now we are ready to prove the result for $G^0 = \mathrm{SL}(\infty, \mathbb{R})$. Namely, let $\mathcal{A}, \mathcal{B} \in X_n$, and $\varphi \in \mathrm{SL}(V_n, \mathbb{R})$ satisfy $\varphi(\iota_n(\mathcal{A})) = \iota_n(\mathcal{B})$, then φ belongs to $\mathrm{GL}(\mathcal{V}_{m_{n+1}}, \mathbb{R})$. Hence there exists $\nu' \in \mathrm{GL}(\mathcal{V}_{m_{n+1}-1}, \mathbb{R})$ which maps $\xi_{m_{n+1}-2} \circ \ldots \circ \xi_{m_n}(\mathcal{A})$ to $\xi_{m_{n+1}-2} \circ \ldots \circ \xi_{m_n}(\mathcal{B})$ because $\iota_n = \xi_{m_{n+1}-1} \circ \xi_{m_{n+1}-2} \circ \ldots \circ \xi_{m_n}$. Continuing this process, we see that there exists an operator $\nu'' \in \mathrm{GL}(V_n, \mathbb{R})$ such that $\nu''(\mathcal{A}) = \mathcal{B}$. Since V_n is odd-dimensional, one can scale ν'' to obtain a required operator $\nu \in \mathrm{SL}(V_n, \mathbb{R})$.

<u>Case $G^0 = \mathrm{SL}(\infty, \mathbb{H})$</u>. Let $\mathcal{A}, \mathcal{B}$ be two flags in $\mathcal{V}_{2n}$ and $\widetilde{\mathcal{A}} = \xi_{2n+1} \circ \xi_{2n}(\mathcal{A})$, $\widetilde{\mathcal{B}} = \xi_{2n+1} \circ \xi_{2n}(\mathcal{B})$. Let $\varphi \in \mathrm{SL}(\mathcal{V}_{2n+2}, \mathbb{H})$ satisfy $\varphi(\widetilde{\mathcal{A}}) = \widetilde{\mathcal{B}}$. Our goal is to construct $\nu \in \mathrm{SL}(\mathcal{V}_{2n}, \mathbb{H})$ such that $\nu(\mathcal{A}) = \mathcal{B}$. Then, repeated application of this procedure will imply the result.

For simplicity, denote $e = e_{2n+1}$, $e' = e_{2n+2}$. Recall that $J(e) = -e'$, $J(e') = e$, and note that $b = \varphi(e) \in \mathcal{V}_{2n}$ if and only if $b' = \varphi(e') \in \mathcal{V}_{2n}$, because $\mathcal{V}_{2n}$ is J-invariant and φ commutes with J.

First, suppose that both b and b' do not belong to $\mathcal{V}_{2n}$. The vector b admits a unique representation in the form $b = v + te + t'e'$ for $v \in \mathcal{V}_{2n}$, $t,\ t' \in \mathbb{C}$. Then

$$b' = \varphi(e') = \varphi(-J(e)) = -J(\varphi(e)) = -J(b) = v' - \bar{t}'e + \bar{t}e',$$

where $v' = -J(v) \in \mathcal{V}_{2n}$. Set

$$T = \begin{pmatrix} t & -\bar{t}' \\ t' & \bar{t} \end{pmatrix},\ d = \det T = |t|^2 + |t'|^2 \in \mathbb{R}_{>0}.$$

Let $\psi \in \mathrm{GL}(\mathcal{V}_{2n+2})$ be the operator defined by $\psi(x) = x$, $x \in \mathcal{V}_{2n}$,

$$\psi(e) = -d^{-1}(\bar{t}(v + e) - t'(v' + e')),$$
$$\psi(e') = -d^{-1}(\bar{t}'(v + e) + t(v' + e')).$$

It is easy to see that ψ commutes with J, $\det \psi = \det T^{-1} \in \mathbb{R}_{>0}$, and $\psi(b) = e$, $\psi(b') = e'$. Furthermore, one can check that $\nu = \pi \circ \psi \circ \varphi|_{\mathcal{V}_{2n}} \colon \mathcal{V}_{2n} \to \mathcal{V}_{2n}$ commutes with J and maps $\mathcal{A}$ to $\mathcal{B}$, where $\pi\colon \mathcal{V}_{2n+2} \to \mathcal{V}_{2n}$ is the projection onto $\mathcal{V}_{2n}$ along $\mathbb{C}e \oplus \mathbb{C}e'$. Since $\det \nu \in \mathbb{R}_{>0}$, one can scale ν to obtain an operator from $\mathrm{SL}(\mathcal{V}_{2n}, \mathbb{H})$, as required.

Second, suppose that $b, b' \in \mathcal{V}_{2n}$. If $a = \varphi^{-1}(e)$ and $a' = \varphi^{-1}(e')$ do not belong to $\mathcal{V}_{2n}$, one can argue as in the first case with φ^{-1} instead of φ, so we may assume without loss of generality that $a, a' \in \mathcal{V}_{2n}$. (Note that if $a,\ a',\ b,\ b'$ are linearly dependent, then $\mathbb{C}a \oplus \mathbb{C}a' = \mathbb{C}b \oplus \mathbb{C}b'$.) In this case, denote by U a J-invariant subspace of $\mathcal{V}_{2n}$ spanned by some basic vectors e_i such that $\mathcal{V}_{2n} = U \oplus \mathbb{C}b \oplus \mathbb{C}b'$. (If $a,\ a',\ b,\ b'$ are linearly independent, we choose U such that $a, a' \in U$.) Define ν by the following rule: if $\varphi(x) = y + sb + sb' + re + r'e'$, $x \in \mathcal{V}_{2n}$, $y \in U$, $s,\ s',\ r,\ r' \in \mathbb{C}$, then $\nu(x) = y + (s + r)b + (s' + r')b'$. One can check that $\det \nu = \det \varphi = 1$, ν commutes with J (so $\nu \in \mathrm{SL}(\mathcal{V}_{2n}, \mathbb{H})$) and maps each A_i, $0 \le i \le s_n$, to B_i. Thus, ν satisfies all required conditions. □

The following result is an immediate corollary of this theorem.

Corollary 3.2 *Let Ω be a G^0-orbit on X, and $\Omega_n = \iota_n^{-1}(\Omega) \subset X_n$. Then*
(i) *Ω_n is a single G_n^0-orbit;*
(ii) *Ω is an infinite-dimensional real ind-manifold.*

Proof (i) Suppose $\mathcal{A}, \mathcal{B} \in \Omega_n$. Then there exists $m \ge n$ such that images of $\mathcal{A}$ and $\mathcal{B}$ under the morphism $\iota_{m-1} \circ \iota_{m-2} \circ \ldots \circ \iota_n$ belong to the same G_m^0-orbit. Applying Theorem 3.1 subsequently to $\iota_{m-1}, \iota_{m-2}, \ldots, \iota_n$, we see that $\mathcal{A}$ and $\mathcal{B}$ belong to the same G_n^0-orbit.

(ii) By definition, $\Omega = \varinjlim \Omega_n$. Next, (i) implies that Ω is a real ind-manifold. By Theorem 2.1 (v), we have $\dim_{\mathbb{R}} \Omega_n \ge \dim_{\mathbb{C}} X_n$. Since $\lim_{n\to\infty} \dim_{\mathbb{C}} X_n = \infty$, we conclude that Ω is infinite dimensional. □

4 Case of Finitely Many G^0-Orbits

We give now a criterion for $X = \mathcal{F}\ell(\mathcal{F}, E)$ to have a finite number of G^0-orbits, and observe that, if this is the case, the degeneracy order on the G^0-orbits in X coincides with that on the G_n^0-orbits in X_n for large enough n. Recall that the *degeneracy order* on the orbits is the partial order $\Omega \leq \Omega' \iff \Omega \subseteq \overline{\Omega}'$.

A generalized flag $\mathcal{F}$ is *finite* if it consists of finitely many (possibly infinite-dimensional) subspaces. We say that a generalized flag $\mathcal{F}$ has *finite type* if it consists of finitely many subspaces of V each of which has either finite dimension or finite codimension in V. A finite type generalized flag is clearly a flag. An ind-variety $X = \mathcal{F}\ell(\mathcal{F}, E)$ is *of finite type* if $\mathcal{F}$ is of finite type (equivalently, if any $\widetilde{\mathcal{F}} \in X$ is of finite type).

Proposition 4.1 *For* $G^0 = \mathrm{SU}(\infty, \infty)$, $\mathrm{SL}(\infty, \mathbb{R})$ *and* $\mathrm{SL}(\infty, \mathbb{H})$*, the number of* G^0*-orbits on* X *is finite if and only if* X *is of finite type. For* $G^0 = \mathrm{SU}(p, \infty)$*,* $0 < p < \infty$*, the number of* G^0*-orbits on* X *is finite if and only if* $\mathcal{F}$ *is finite. For* $G^0 = \mathrm{SU}(0, \infty)$*, the number of* G^0*-orbits on* X *equals* 1.

Proof
<u>CASE $G^0 = \mathrm{SU}(\infty, \infty)$</u>. First consider the case $X = \mathrm{Gr}(F, E)$, where F is a subspace of V. Clearly, X is of finite type if and only if $\dim F < \infty$ or $\mathrm{codim}_V F < \infty$. Note that for ind-grassmannians, the construction of ι_n from (1) is simply the following. Given n, let W_{n+1} be the span of $E_{n+1} \setminus E_n$, and U_{n+1} be a fixed $(k_{n+1} - k_n)$-dimensional subspace of W_{n+1}, where $k_i = \dim F \cap V_i$. Then the embedding $\iota_n \colon X_n = \mathrm{Gr}(k_n, V_n) \to X_{n+1} = \mathrm{Gr}(k_{n+1}, V_{n+1})$ has the form $\iota_n(A) = A \oplus U_{n+1}$ for $A \in X_n$.

Recall that if $\mathrm{codim}_V F = k$, then the map

$$U \mapsto U^\# = \{\phi \in V_* \mid \phi(x) = 0 \text{ for all } x \in U\}$$

induces an isomorphism $\mathrm{Gr}(F, E) \to \{F' \subset V_* \mid \dim F' = k\} = \mathrm{Gr}(k, V_*)$; we denote this isomorphism by D. To each operator $\psi \in \mathrm{GL}(V, E)$ one can assign the linear operator ψ_* on V_* acting by $(\psi_*(\lambda))(x) = \lambda(\psi(x))$, $\lambda \in V_*$, $x \in V$. This defines an isomorphism $\mathrm{SL}(V, E) \to \mathrm{SL}(V_*, E^*)$, and D becomes a G-equivariant isomorphism of ind-varieties. Hence, for X of finite type, we can consider only the case when $\dim F = k$.

If $\dim F = k$, then X consists of all k-dimensional subspaces of V. Pick $A, B \in X$. There exists n such that $X_n = \mathrm{Gr}(k, V_n)$ and $A,\ B \in \iota_n(X_n)$. Witt's Theorem shows that, for each $m \geq n$, A and B belong to the same G_m^0-orbit if and only if the signatures of the forms $\omega_m|_A$ and $\omega_m|_B$ coincide. Since $\omega_m|_{A,B} = \omega|_{A,B}$, we conclude A and B belong to the same G^0-orbit if and only if their signatures coincide. Thus, the number of G^0-orbits on X is finite.

On the other hand, if $\dim F = \mathrm{codim}_V F = \infty$, then

$$\lim_{n\to\infty} k_n = \lim_{n\to\infty} (\dim V_n - k_n) = \infty.$$

In this case, the number of possible signatures of the restriction of ω_n to a k_n-dimensional subspace tends to infinity, hence the number of G_n^0-orbits tends to infinity. By Theorem 3.1, the number of G^0-orbits on X is infinite.

Now, consider the general case $X = \mathcal{F}\ell(\mathcal{F}, E)$. Let $\mathcal{F}$ be of finite type. Then $\mathcal{F} = \mathcal{A} \cup \mathcal{B}$ where $\mathcal{A}$ and $\mathcal{B}$ are finite type subflags of $\mathcal{F}$ consisting of finite-dimensional and finite-codimensional subspaces from $\mathcal{F}$ respectively. Note that $\mathcal{A}$ and $\mathcal{B}$ are compatible with the basis E, hence there exists N such that if $n \geq N$, then $A \subseteq V_n$ for all $A \in \mathcal{A}$ and $\operatorname{codim}_{V_n}(B \cap V_n) = \operatorname{codim}_V B$ for all $B \in \mathcal{B}$. Set

$$\mathcal{A} = \{A_1 \subset A_2 \subset \ldots \subset A_k\},$$
$$\mathcal{B} = \{B_1 \subset B_2 \subset \ldots \subset B_l\},$$

and $a_i = \dim A_i$, $1 \leq i \leq k$, $b_i = \operatorname{codim}_V B_i$, $1 \leq i \leq l$.

Denote by $s(U)$ the signature of $\omega|_U$ for a finite-dimensional subspace $U \subset V$. According to [11, Theorem 6.2], to check that the number of G^0-orbits on X is finite, it is enough to prove that all of the following sets are finite:

$$\begin{aligned}
S_A &= \{s(A) \mid A \subset V_n,\ n \geq N,\ \dim A = a_i \text{ for some } i\},\\
S_B &= \{s(B) \mid B \subset V_n,\ n \geq N.\ \operatorname{codim}_{V_n} B = b_i \text{ for some } i\},\\
P_A &= \{\dim A \cap A_0^{\perp, V_n} \mid A,\ A_0 \subset V_n,\ n \geq N,\\
&\quad \dim A = a_i,\ \dim A_0 = a_j \text{ for some } i < j\},\\
P_B &= \{\dim B \cap B_0^{\perp, V_n} \mid B,\ B_0 \subset V_n,\ n \geq N,\\
&\quad \operatorname{codim}_{V_n} B = b_i,\ \operatorname{codim}_{V_n} B_0 = b_j \text{ for some } i < j\},\\
P_{AB} &= \{\dim A \cap B^{\perp, V_n} \mid A,\ B \subset V_n,\ n \geq N,\\
&\quad \dim A = a_i,\ \operatorname{codim}_{V_n} B = b_j \text{ for some } i, j\}.
\end{aligned}$$

The finiteness of S_A and P_A is obvious. In particular, this implies that the number of G^0-orbits on $\mathcal{F}\ell(\mathcal{A}, E)$ is finite. Applying the map $U \mapsto U^\#$ described above, we see that the number of G^0-orbits on $\mathcal{F}\ell(\mathcal{B}, E)$ is finite. Consequently, the sets S_B and P_B are finite. Finally, since $\omega_n = \omega|_{V_n}$ is nondegenerate for each n, we see that if $B \subset V_n$ and $\operatorname{codim}_{V_n} B = b_i$ for some i, then $\dim B^{\perp, V_n} = \operatorname{codim}_{V_n} B = b_i$. Hence P_{AB} is finite. Thus, if $\mathcal{F}$ is of finite type then the number of G^0-orbits on $\mathcal{F}\ell(\mathcal{F}, E)$ is finite.

On the other hand, suppose that $\mathcal{F}$ is not of finite type. If there is a space $F \in \mathcal{F}$ with $\dim F = \operatorname{codim}_V F = \infty$, then we are done, because the map

$$X \to \operatorname{Gr}(F, E) \colon \mathcal{G} \mapsto \text{the subspace in } \mathcal{G} \text{ corresponding to } F$$

is a G-equivariant epimorphism of ind-varieties, and the number of G^0-orbits on the ind-grassmannian $\operatorname{Gr}(F, E)$ is infinite by the above.

If all $F \in \mathcal{F}$ are of finite dimension or finite codimension, there exist subspaces $F_n \in \mathcal{F}$ of arbitrarily large dimension or arbitrarily large codimension. In the former case the statement follows from the fact that the number of possible signatures of such spaces tends to infinity, and in the latter case the statement gets reduced to the former one via the map $U \mapsto U^{\#}$.

CASE $G^0 = \mathrm{SU}(p, \infty), 0 < p < \infty$. First suppose that $\mathcal{F}$ is finite, i.e., $|\mathcal{F}| = N < \infty$. Given $n \geq 1$, denote $S_n = \{s(A) \mid A \subset V_n\}$ and $P_n = \{\dim A \cap B^{\perp, V_n} \mid A \subset B \subset V_n\}$. Let $s(A) = (a, b, c)$ for some subspace A of V_n. Then, clearly, $a \leq p$ and $c \leq p$, hence $|S_n| \leq p^2$. On the other hand, if $A \subset B$ are subspaces of V_n then $A^{\perp, V_n} \supset B^{\perp, V_n}$, so $A \cap B^{\perp, V_n} \subset A \cap A^{\perp, V_n}$. But $\dim A \cap A^{\perp, V_n} = c \leq p$. Thus $|P| \leq p$. Now [11, Theorem 6.2] shows that the number of G_n^0-orbits on X_n is less or equal to $N|S_n|N^2|P_n| \leq N^3 p^3$. Hence, by Theorem 3.1, the number of G^0-orbits on X is finite.

Now suppose that $\mathcal{F}$ is infinite. In this case, given $m \geq 1$, there exists n such that the length of each flag from X_n is not less than m, the positive index of $\omega|_{V_n}$ (i.e., the dimension of a maximal positive definite subspace of V_n) equals p, and $\mathrm{codim}_{V_n} F_m \geq p$, where $\mathcal{F}_n = \{F_1 \subset \ldots \subset F_m \subset \ldots \subset V_n\}$. It is easy to check that the number of G_n^0-orbits on X_n is not less than m. Consequently, by Theorem 3.1, the number of G^0-orbits on X is not less than m. The proof for $\mathrm{SU}(p, \infty)$, $p > 0$, is complete.

CASE $G^0 = \mathrm{SU}(0, \infty)$. Evident.

CASE $G^0 = \mathrm{SL}(\infty, \mathbb{R})$. First, let $X = \mathrm{Gr}(F, V)$ for a subspace $F \subset V$ compatible with E. If $\dim F = k < \infty$, then X consists of all k-dimensional subspaces of V. We claim that the number of G^0-orbits on X equals $k + 1$. Indeed, pick $A, B \in X$ and $n \geq k + 1$ such that $A, B \in \iota_n(X_n)$ (recall that $\dim V_n = 2n - 1$). Clearly, if A and B belong to the same G^0-orbit, then

$$\dim A \cap \tau(A) = \dim B \cap \tau(B). \tag{4}$$

Since $n \geq k + 1$ and V_n is τ-stable, $\dim A \cap \tau(A)$ can be an arbitrary integer number from 0 to k, hence the number of G^0-orbits on X is at least $k + 1$.

On the other hand, suppose that (4) is satisfied. Let A', B' be complex subspaces of A, B respectively such that $A = A' \oplus (A \cap \tau(A))$ and $B = B' \oplus (B \cap \tau(B))$. Clearly, $A' \cap \tau(A') = B' \cap \tau(B') = 0$. Furthermore, it is easy to see that

$$A + \tau(A) = (A \cap \tau(A)) \oplus (A' \oplus \tau(A')),$$
$$B + \tau(B) = (B \cap \tau(B)) \oplus (B' \oplus \tau(B')).$$

For simplicity, set $A_\tau = A + \tau(A)$, $A'_\tau = A' \oplus \tau(A')$, $A^\tau = A \cap \tau(A)$, and define B_τ, B'_τ, B^τ similarly. Then $A_\tau = A^\tau \oplus A'_\tau$, $B_\tau = B^\tau \oplus B'_\tau$. Note that all these subspaces are defined over $\mathbb{R}$. By [12, Lemma 2.1], the $\mathrm{SL}(A'_\tau, \mathbb{R})$-orbit of A' is open in the corresponding grassmannian. Furthermore, there are two open $\mathrm{SL}(A'_\tau, \mathbb{R})$-orbits on this grassmannian, and their union is a single $\mathrm{GL}(A'_\tau, \mathbb{R})$-orbit. Hence there exists an operator $\psi \colon A_\tau \to B_\tau$ which is defined over $\mathbb{R}$ and maps A'_τ, A', A^τ to B'_τ, B', B^τ respectively. Since A_τ and B_τ are defined over $\mathbb{R}$ (i.e., are τ-invariant), there exist τ-invariant complements A_0, B_0 of A_τ, B_τ in V_n. Thus one can extend ψ to an operator $\nu \in \mathrm{GL}(V_n, \mathbb{R})$ such that $\nu(A) = B$. Finally, since $\dim V_n$ is odd, we can scale ν to obtain an operator from $\mathrm{SL}(V_n, \mathbb{R})$ which maps A to B, as required.

At the contrary, assume that $\dim F = \infty$. As it was shown above, given $m \geq n$, two finite-dimensional spaces $A, B \in V_n$ belong to the same G_m^0-orbit if and only if $\dim A \cap \tau(A) = \dim B \cap \tau(B)$, so the number of G_m^0-orbits on the grassmannian of k_n-subspaces of V_n equals $k_n + 1$. But we have $\lim_{n\to\infty} k_n = \infty$, so the number of G^0-orbits on X is infinite by Theorem 3.1.

Now, consider the general case $X = \mathcal{F}\ell(\mathcal{F}, E)$. We claim that, given a type $d = (d_1, \ldots, d_r)$, there exists a number $u(d)$ such that the number of G_n^0-orbits on the flag variety $\mathcal{F}\ell(d, V_n)$ is less or equal than $u(d)$, i.e., this upper bound depends only on d, but not on the dimension of V_n. To prove this, denote by $K_n = \mathrm{SO}(V_n)$ the subgroup of G_n preserving the bilinear form

$$\beta_n(x, y) = \sum_{i=1}^{\dim V_n} x_i y_i,\ x = \sum_{i=1}^{\dim V_n} x_i e_i,\ y = \sum_{i=1}^{\dim V_n} y_i \in V_n.$$

By Matsuki duality [4], there exists a one-to-one correspondence between the set of K_n-orbits and the set of G_n^0-orbits on $\mathcal{F}\ell(d, V_n)$. Hence our claim follows immediately from (3), because [11, Theorem 6.2] holds for nondegenerate symmetric bilinear forms.

Finally, suppose that $\mathcal{F}$ is of finite type. Let $\mathcal{A}, \mathcal{B}, N$ be as for $\mathrm{SU}(\infty, \infty)$. Note that the form β_n is nondegenerate, hence the β_n-orthogonal complement to a subspace $B \subset V_n$ is of dimension $\operatorname{codim}_{V_n} B$. Arguing as for $\mathrm{SU}(\infty, \infty)$ and applying our remark about Matsuki duality, we conclude that there exists a number $u(\mathcal{F})$ such that the number of G_n^0-orbits on X_n is less or equal to $u(\mathcal{F})$ for every $n \geq N$. It follows from Theorem 3.1 that the total number of G^0-orbits on X is also less or equal to $u(\mathcal{F})$. Finally, if $\mathcal{F}$ is not of finite type, then, as in the case of $\mathrm{SU}(\infty, \infty)$, one can use G-equivariant projections from X onto ind-grassmannians to show that the number of G^0-orbits on the ind-variety X is infinite. The proof for $G^0 = \mathrm{SL}(\infty, \mathbb{R})$ is complete.

<u>CASE $G^0 = \mathrm{SL}(\infty, \mathbb{H})$</u>. Denote by κ_n an antisymmetric bilinear form on V_n defined by

$$\kappa_n(e_{2i-1}, e_{2i}) = 1,\ \kappa_n(e_{2i}, e_{2i-1}) = -1,\ \kappa_n(e_i, e_j) = 0 \text{ for } |i - j| > 1.$$

Let K_n be the subgroup of G_n preserving this form. Then $K_n \cap G_n^0$ is a maximal compact subgroup of G_n^0 (see, e.g., [9]), so, by duality, given d, there exists a bijection between the set of K_n-orbits and the set of G_n^0-orbits on the flag variety $\mathcal{F}\ell(d, V_n)$. Since K_n is isomorphic to $\mathrm{Sp}_{\dim V_n}(\mathbb{C})$, we can argue as for $\mathrm{SL}(\infty, \mathbb{R})$ to complete the proof. $\square$

Example 4.2 Let $X = \mathrm{Gr}(k, V)$ for $k < \infty$. Then

$$|X/G^0| = \begin{cases} (p+1)(2k-p+2)/2 & \text{for } G^0 = \mathrm{SU}(p,\infty),\ p \le k, \\ (k+1)(k+2)/2 & \text{for } G^0 = \mathrm{SU}(p,\infty),\ k \le p, \\ (k+1)(k+2)/2 & \text{for } G^0 = \mathrm{SU}(\infty,\infty), \\ k+1 & \text{for } G^0 = \mathrm{SL}(\infty,\mathbb{R}), \\ [k/2]+1 & \text{for } G^0 = \mathrm{SL}(\infty,\mathbb{H}). \end{cases}$$

For $\mathrm{SU}(p,\infty)$ and $\mathrm{SU}(\infty,\infty)$, this follows from the formula for the number of $\mathrm{SU}(p, n-p)$-orbits on a finite-dimensional grassmannian, see Sect. 2.1. For $\mathrm{SL}(\infty,\mathbb{R})$, this was proved in Proposition 4.1; the proof for $\mathrm{SL}(\infty,\mathbb{H})$ is similar to the case of $\mathrm{SL}(\infty,\mathbb{R})$.

As a corollary of Theorem 3.1, we describe the degeneracy order on the set X/G^0 of G^0-orbits on an arbitrary ind-variety $X = \mathcal{F}\ell(F, E)$ of finite type. By definition, $\Omega \le \Omega' \iff \Omega \subseteq \overline{\Omega}'$. We define the partial order on the set X_n/G_n^0 of G_n^0-orbits on X_n in a similar way.

Corollary 4.3 *Suppose the number of G^0-orbits on $X = \mathcal{F}\ell(\mathcal{F}, E)$ is finite. Then there exists N such that X/G^0 is isomorphic as partially ordered set to X_n/G_n^0 for each $n \ge N$.*

Proof Given a G^0-orbit Ω on X, there exists n such that $\Omega \cap \iota_n(X_n)$ is nonempty. Since there are finitely many G^0-orbits on X, there exists N such that $\Omega \cap \iota_N(X_N)$ is nonempty for each orbit Ω. By Theorem 3.1, given $n \ge N$ and a G^0-orbit Ω on X, there exists a unique G_n^0-orbit Ω_n on X_n such that $\iota_n^{-1}(\Omega \cap \iota_n(X_n)) = \Omega_n$. Hence, the map

$$\alpha_n \colon X/G^0 \to X_n/G_n^0,\ \ \Omega \mapsto \Omega_n$$

is well defined for each $n \ge N$. It is clear that this map is bijective. It remains to note that, by the definition of the topology on X, a G^0-orbit Ω is contained in the closure of a G^0-orbit Ω' if and only if Ω_n is contained in the closure of Ω_n' for all $n \ge N$. Thus, α_n is in fact an isomorphism of the partially ordered sets X/G^0 and X_n/G_n^0 for each $n \ge N$. □

5 Open and Closed Orbits

In this section we provide necessary and sufficient conditions for a given G^0-orbit on $X = \mathcal{F}\ell(\mathcal{F}, E)$ to be open or closed. We also prove that X has both an open and a closed orbit if and only if the number of orbits is finite.

First, consider the case of open orbits. Pick any n. Recall [13, 23] that the G_n^0-obit of a flag $\mathcal{A} = \{A_1 \subset A_k \subset \ldots \subset A_k\} \in X_n$ is open if and only if

$$\text{for } G^0 = \mathrm{SU}(p,\infty) \text{ or } \mathrm{SU}(\infty,\infty)\colon \text{ all } A_i\text{'s are nondegenerate with respect to } \omega;$$
$$\text{for } G^0 = \mathrm{SL}(\infty,\mathbb{R})\colon \text{for all } i,\ j, \dim A_i \cap \tau(A_j) \text{ is minimal,}$$
$$\text{i.e., equals } \max\{\dim A_i + \dim A_j - \dim V_n,\ 0\};$$
$$\text{for } G^0 = \mathrm{SL}(\infty,\mathbb{H})\colon \text{for all } i,\ j, \dim A_i \cap J(A_j) \text{ is minimal in the above sense.}$$

Note that, for any two generalized flags $\mathcal{F}_1$ and $\mathcal{F}_2$ in X, there is a canonical identification of $\mathcal{F}_1$ and $\mathcal{F}_2$ as linearly ordered sets. For a space $F \in \mathcal{F}_1$, we call the image of F under this identification the space in $\mathcal{F}_2$ *corresponding* to F.

Fix an antilinear operator μ on V. A point $\mathcal{G}$ in $X = \mathcal{F}\ell(\mathcal{F}, E)$ is *in general position with respect to* μ if $F \cap \mu(H)$ does not properly contain $\widetilde{F} \cap \mu(\widetilde{H})$ for all $F,\ H \in \mathcal{G}$ and all $\widetilde{\mathcal{G}} \in X$, where $\widetilde{F},\ \widetilde{H}$ are the spaces in $\widetilde{\mathcal{G}}$ corresponding to F, H respectively. A similar definition can be given for flags in X_n. Note that, for $G^0 = \mathrm{SL}(\infty,\mathbb{R})$ or $\mathrm{SL}(\infty,\mathbb{H})$, the G_n^0-orbit of $\mathcal{A} \in X_n$ is open if and only if $\mathcal{A}$ is in general position with respect to τ or J respectively.

With the finite-dimensional case in mind, we give the following

Definition 5.1 A generalized flag $\mathcal{G}$ is *nondegenerate* if

$$\text{for } G^0 = \mathrm{SU}(p,\infty) \text{ or } \mathrm{SU}(\infty,\infty)\colon$$
$$\text{each } F \in \mathcal{G} \text{ is nondegenerate with respect to } \omega;$$
$$\text{for } G^0 = \mathrm{SL}(\infty,\mathbb{R}) \text{ or } \mathrm{SL}(\infty,\mathbb{H})\colon$$
$$\mathcal{G} \text{ is in general position with respect to } \tau \text{ or } J \text{ respectively.}$$

Remark 5.2 A generalized flag being nondegenerate with respect to ω can be thought of as being "in general position with respect to ω". Therefore, all conditions in Definition 5.1 are clearly analogous.

Proposition 5.3 *The G^0-orbit Ω of $\mathcal{G} \in X$ is open if and only if $\mathcal{G}$ is nondegenerate.*

Proof By the definition of the topology on X, Ω is open if and only if $\Omega_n = \iota_n^{-1}(\Omega \cap \iota_n(X_n))$ is open for each n.

First, suppose $G^0 = \mathrm{SU}(p,\infty)$ or $\mathrm{SU}(\infty,\infty)$. To prove the claim in this case, it suffices to show that $A \in \mathcal{G}$ is nondegenerate with respect to ω if and only if $\omega|_{A\cap V_n}$ is nondegenerate for all n for which $m_n \geq m_{\mathcal{G}}$. This is straightforward. Indeed, if A is degenerate, then there exists $v \in A$ such that $\omega(v, w) = 0$ for all $w \in A$. Let $v \in V_{n_0}$ for some n_0 with $m_{n_0} \geq m_{\mathcal{G}}$. Then $\omega|_{A\cap V_{n_0}}$ is degenerate. On the other hand, if $v \in A \cap V_n$ is orthogonal to all $w \in A \cap V_n$ for some n such that $m_n \geq m_{\mathcal{G}}$, then v is orthogonal to all $w \in A$ because e_m is orthogonal to V_n for $m > n$. The result follows.

Second, consider the case $G^0 = \mathrm{SL}(\infty,\mathbb{R})$. Suppose Ω is open, so Ω_n is open for each n satisfying $m_n \geq m_{\mathcal{G}}$. Assume $\mathcal{G} \in X$ is not nondegenerate. Then there exist $\widetilde{\mathcal{G}} \in X$ and $A,\ B \in \mathcal{G}$ such that $\widetilde{A} \cap \tau(\widetilde{B}) \subsetneq A \cap \tau(B)$, where $\widetilde{A},\ \widetilde{B}$ are the subspaces in $\widetilde{\mathcal{G}}$ corresponding to $A,\ B$ respectively. Let $v \in (A \cap \tau(B)) \setminus (\widetilde{A} \cap \tau(\widetilde{B}))$, and n be such that $v \in V_n$. Since V_n is τ-invariant, we have $v \in (A_n \cap \tau(B_n)) \setminus (\widetilde{A}_n \cap \tau(\widetilde{B}_n))$

where $A_n = A \cap V_n,\ B_n = B \cap V_n,\ \widetilde{A}_n = \widetilde{A} \cap V_n,\ \widetilde{B}_n = \widetilde{B} \cap V_n$. This means that $\widetilde{A}_n \cap \tau(\widetilde{B}_n)$ is properly contained in $A_n \cap \tau(B_n)$. Hence, $\mathcal{G}_n$ is not in general position with respect to $\tau|_{V_n}$, which contradicts the condition that Ω_n is open.

Now, assume that Ω_n is not open for some n with $m_n \geq m_{\mathcal{G}}$. This means that there exist $A_n,\ B_n \in \mathcal{G}_n = \iota_n^{-1}(\mathcal{G})$ and $\widetilde{\mathcal{G}}_n \in X_n$ so that $A_n \cap \tau(B_n)$ properly contains $\widetilde{A}_n \cap \tau(\widetilde{B}_n)$, where $\widetilde{A}_n$ and $\widetilde{B}_n$ are the respective subspaces in $\widetilde{\mathcal{G}}_n$ corresponding to A_n and B_n. Since $\tau(e_{n+1}) = e_{n+1}$, the space $A_{n+1} \cap \tau(B_{n+1})$ properly contains $\widetilde{A}_{n+1} \cap \tau(\widetilde{B}_{n+1})$, where A_{n+1}, B_{n+1}, $\widetilde{A}_{n+1}$, $\widetilde{B}_{n+1}$ are the respective images of A_n, B_n, $\widetilde{A}_n$, $\widetilde{B}_n$ under the embedding $X_n \hookrightarrow X_{n+1}$. Repeating this procedure, we see that $\mathcal{G}$ is not nondegenerate. The result follows.

The case $G^0 = \mathrm{SL}(\infty, \mathbb{H})$ can be considered similarly. □

We say that two generalized flags *have the same type* if there is an automorphism of V transforming one into the other. Clearly, two E-commensurable generalized flags always have the same type. On the other hand, it is clearly not true that two generalized flags having the same type are $\widetilde{E}$-commensurable for some basis $\widetilde{E}$.

It turns out that, for $G^0 = \mathrm{SU}(p, \infty)$ and $\mathrm{SU}(\infty, \infty)$, the requirement for the existence of an open orbit on an ind-variety of the form $\mathcal{F}\ell(\mathcal{F}, E)$ imposes no restriction on the type of the flag $\mathcal{F}$. More precisely, we have

Corollary 5.4 *If $G^0 = \mathrm{SU}(p, \infty)$, $0 \leq p < \infty$, then X always has an G^0-open orbit. If $G^0 = \mathrm{SU}(\infty, \infty)$, then there exist a basis $\widetilde{E}$ of V and a generalized flag $\widetilde{\mathcal{F}}$ such that $\mathcal{F}$ and $\widetilde{\mathcal{F}}$ are of the same type and $\widetilde{X} = \mathcal{F}\ell(\widetilde{\mathcal{F}}, \widetilde{E})$ has an open G^0-orbit.*

Proof For $\mathrm{SU}(p, \infty)$, let n be a positive integer such that the positive index of $\omega|_{V_n}$ equals p. Let $\mathcal{G}_n \in X_b$ be a flag in V_n consisting of nondegenerate subspaces (i.e., the G_n^0-orbit of $\mathcal{G}_n$ is open in X_n). Denote by g a linear operator from G_n such that $g(\mathcal{F}_n) = \mathcal{G}_n$, where $\mathcal{F}_n = \iota_n^{-1}(\mathcal{F}) \in X_n$. Then, clearly, $g(\mathcal{F})$ belongs to X and is nondegenerate. Therefore the G^0-orbit of $g(\mathcal{F})$ on X is open.

Now consider the case $G^0 = \mathrm{SU}(\infty, \infty)$. Let $\widetilde{E}$ be an ω-orthogonal basis of V. Fix a bijection $E \to \widetilde{E}$. This bijection defines an automorphism $V \to V$. Denote by $\widetilde{\mathcal{F}}$ the generalized flag consisting of the images of subspaces from $\mathcal{F}$ under this isomorphism. Then $\widetilde{\mathcal{F}}$ and $\mathcal{F}$ are of the same type, and each space in $\widetilde{\mathcal{F}}$ is nondegenerate as it is spanned by a subset of $\widetilde{E}$. Thus the G^0-orbit of $\widetilde{\mathcal{F}}$ on $\widetilde{X}$ is open. □

Remark 5.5 Of course, in general an ind-variety $\widetilde{X} = \mathcal{F}\ell(\widetilde{\mathcal{F}}, \widetilde{E})$ having an open $\mathrm{SU}(\infty, \infty)$-orbit does not equal a given $X = \mathcal{F}\ell(\mathcal{F}, E)$.

The situation is different for $G^0 = \mathrm{SL}(\infty, \mathbb{R})$. While an ind-grassmannian $\mathrm{Gr}(F, E)$ has an open orbit if and only if either $\dim F < \infty$ or $\operatorname{codim}_V F < \infty$, an ind-variety of the form $\widetilde{X} = \mathcal{F}\ell(\widetilde{\mathcal{F}}, \widetilde{E})$, where $\widetilde{\mathcal{F}}$ has the same type as the flag $\mathcal{F}$ from Example 2.4 (ii), cannot have an open orbit as long as the basis $\widetilde{E}$ satisfies $\tau(\widetilde{e}) = \widetilde{e}$ for all $\widetilde{e} \in \widetilde{E}$. Indeed, suppose $\widehat{\mathcal{F}} = \{\{0\} = \widehat{F}_0 \subset \widehat{F}_1 \subset \ldots\} \in \widetilde{X}$. As we pointed out in Example 2.4 (ii), there exists N such that $\widehat{F}_n = \widetilde{F}_n = \langle \widetilde{e}_1, \ldots, \widetilde{e}_n \rangle_{\mathbb{C}}$ for $n \geq N$. Pick n so that $m_n \geq \max\{2N, \widetilde{m}_{\widehat{\mathcal{F}}}\}$, where $\widetilde{m}_{\widehat{\mathcal{F}}}$ is an integer such that $\widehat{\mathcal{F}}$

and $\widehat{\mathcal{F}}$ are compatible with respect to bases containing $\{\widetilde{e}_i,\ i \geq \widetilde{m}_{\widetilde{\mathcal{F}}}\}$. Then the flag $\widehat{\mathcal{F}}_n = \iota_n^{-1}(\widehat{\mathcal{F}}) \in \widetilde{X}_n$ contains the subspace $\widehat{F}_N$ which is defined over $\mathbb{R}$. Thus, $\widehat{\mathcal{F}}_n$ is not in general position with respect to $\tau|_{V_n}$, so the G_n^0-orbit of $\widehat{\mathcal{F}}_n$ in $\widetilde{X}_n$ is not open. Consequently, the G^0-orbit of $\widehat{\mathcal{F}}$ in $\widetilde{X}$ is not open.

Let now $\widetilde{X} = \mathcal{F}\ell(\widetilde{\mathcal{F}}, \widetilde{E})$ where $\widetilde{\mathcal{F}}$ is a generalized flag having the same type as the generalized flag $\mathcal{F}$ from Example 2.4 (iii). Recall that

$$\mathcal{F} = \{\{0\} = F_0 \subset F_1 \subset F_2 \subset \ldots \subset F_{-2} \subset F_{-1} \subset V\},$$

where $F_i = \langle e_1,\ e_3, \ldots, e_{2i-1}\rangle_{\mathbb{C}}$, $F_{-i} = \langle \{e_j,\ j \text{ odd}\} \cup \{e_{2j},\ j > i\}\rangle_{\mathbb{C}}$ for $i \geq 1$. We claim that $\widetilde{X}$ also cannot have an open orbit. Indeed, assume $\widehat{\mathcal{F}}$ is $\widetilde{E}$-commensurable to $\widetilde{\mathcal{F}}$. Then $\widehat{\mathcal{F}}$ is compatible with a basis $\widehat{E}$ of V such that $\widehat{E} \setminus \widetilde{E}$ is finite. This means that there exists $\widetilde{e} \in \widetilde{E}$ and a finite-dimensional subspace $F \in \widehat{\mathcal{F}}$ with $\widetilde{e} \in F$. Now, pick n so that $F \subset V_n$ and $m_n \geq \max\{2 \dim F,\ \widetilde{m}_{\widetilde{\mathcal{F}}}\}$. Then $F \cap \tau(F) \neq 0$, so $\widehat{\mathcal{F}}_n = \iota_n^{-1}(\widehat{\mathcal{F}}) \in X_n$ is not in general position with respect to $\tau|_{V_n}$.

Finally, let $G^0 = \mathrm{SL}(\infty, \mathbb{H})$. In this case, clearly, an ind-grassmannian $\mathrm{Gr}(F, E)$ may or may not have an open orbit. A similar argument as for $\mathrm{SL}(\infty, \mathbb{R})$ shows that if $\mathcal{F}$ is as in Example 2.4 (ii), then $\widetilde{X}$ cannot have an open orbit. Surprisingly, for $G^0 = \mathrm{SL}(\infty, \mathbb{H})$ and X as in Example 2.4 (iii), $\widetilde{X}$ may have an open orbit. Consider first the case of $X = \mathcal{F}\ell(\mathcal{F}, E)$ itself. It it easy to check that if $\dim V_n = n$ then $\mathcal{F}_n$ is in general position with respect to $J|_{V_n}$ for each n, so the orbit of $\mathcal{F}$ is open. On the other hand, if $\widetilde{\mathcal{F}}$ and $\mathcal{F}$ have the same type and each $2n$-dimensional subspace in $\widetilde{\mathcal{F}}$ is spanned by the vectors $\widetilde{e}_1,\ \widetilde{e}_2,\ \widetilde{e}_5,\ \widetilde{e}_6, \ldots,\ \widetilde{e}_{4n-3},\ \widetilde{e}_{4n-2}$, then $\widetilde{X}$ does not have an open orbit because each generalized flag $\widetilde{E}$-commensurable to $\widetilde{F}$ contains a finite-dimensional subspace F such that $F \cap J(F) \neq \{0\}$.

We now turn our attention to closed orbits. The conditions for an orbit to be closed are based on the same idea for each of the real forms, but (as was the case for open orbits) the details differ.

Suppose $G^0 = \mathrm{SU}(\infty, \infty)$ or $\mathrm{SU}(p, \infty)$. We call a generalized flag $\mathcal{G}$ in X *pseudo-isotropic* if $F \cap H^{\perp,V}$ is not properly contained in $\widetilde{F} \cap \widetilde{H}^{\perp,V}$ for all $F,\ H \in \mathcal{G}$ and all $\widetilde{\mathcal{G}} \in X$, where $\widetilde{F},\ \widetilde{H}$ are the subspaces in $\widetilde{\mathcal{G}}$ corresponding to $F,\ H$ respectively. A similar definition can be given for flags in X_n. An isotropic generalized flag, as defined in [5], is always pseudo-isotropic, but the converse does not hold. In the particular case when the generalized flag $\mathcal{G}$ is of the form $\{\{0\} \subset F \subset V\}$, $\mathcal{G}$ is pseudo-isotropic if and only if the kernel of the form $\omega|_F$ is maximal over all E-commensurable flags of the form $\{\{0\} \subset \widetilde{F} \subset V\}$.

Next, suppose $G^0 = \mathrm{SL}(\infty, \mathbb{R})$. A generalized flag $\mathcal{G}$ in X is *real* if $\tau(F) = F$ for all $F \in \mathcal{G}$. This condition turns out to be equivalent to the following condition: $F \cap \tau(H)$ is not properly contained in $\widetilde{F} \cap \tau(\widetilde{H})$ for all $F,\ H \in \mathcal{G}$ and all $\widetilde{\mathcal{G}} \in X$, where $\widetilde{F},\ \widetilde{H}$ are the subspaces in $\widetilde{\mathcal{G}}$ corresponding to $F,\ H$ respectively.

Finally, suppose $G^0 = \mathrm{SL}(\infty, \mathbb{H})$. We call a generalized flag $\mathcal{G}$ in X *pseudo-quaternionic* if $F \cap J(H)$ is not properly contained in $\widetilde{F} \cap J(\widetilde{H})$ for all $F,\ H \in \mathcal{G}$ and all $\widetilde{\mathcal{G}} \in X$, where $\widetilde{F},\ \widetilde{H}$ are the subspaces in $\widetilde{\mathcal{G}}$ corresponding to $F,\ H$ respectively. If $\mathcal{G}$ is *quaternionic*, i.e., if $J(F) = F$ for each $F \in \mathcal{G}$, then $\mathcal{G}$ is clearly pseudo-quaternionic, but the converse does not hold. If the generalized flag $\mathcal{G}$ is of the form $\{\{0\} \subset F \subset V\}$, then $\mathcal{G}$ is pseudo-quaternionic if and only if $\operatorname{codim}_F(F \cap J(F)) \leq 1$.

Proposition 5.6 *The G^0-orbit Ω of $\mathcal{G} \in X$ is closed if and only if*

$$\mathcal{G} \textit{ is pseudo-isotropic for } G^0 = \mathrm{SU}(\infty, \infty) \textit{ and } \mathrm{SU}(p, \infty);$$
$$\mathcal{G} \textit{ is real for } G^0 = \mathrm{SL}(\infty, \mathbb{R});$$
$$\mathcal{G} \textit{ is pseudo-quaternionic for } G^0 = \mathrm{SL}(\infty, \mathbb{H}).$$

Proof First consider the finite-dimensional case, where there is a unique closed G_n^0-orbit on X_n (see Theorem 2.1). For all real forms the conditions of the proposition applied to finite-dimensional flags in V_n are easily checked to be closed conditions on points of X_n. Therefore, the G_n^0-orbit of a flag in V_n is closed if and only if this flag satisfies the conditions of the proposition at the finite level.

Let $G^0 = \mathrm{SU}(\infty, \infty)$ or $\mathrm{SU}(p, \infty)$. Suppose Ω is closed, so Ω_n is closed for each n satisfying $m_n \geq m_{\mathcal{G}}$. Assume $\mathcal{G}$ is not pseudo-isotropic. Then there exist $\widetilde{\mathcal{G}} \in X$ and $A,\ B \in \mathcal{G}$ such that $\widetilde{A} \cap \widetilde{B}^{\perp,V} \supsetneq A \cap B^{\perp,V}$, where $\widetilde{A},\ \widetilde{B}$ are the subspaces in $\widetilde{\mathcal{G}}$ corresponding to $A,\ B$ respectively. Let $v \in (\widetilde{A} \cap \widetilde{B}^{\perp,V}) \setminus (A \cap B^{\perp,V})$, and n be such that $v \in V_n$ and $m_n \geq m_{\mathcal{G}}$. Then $v \in (\widetilde{A}_n \cap \widetilde{B}_n^{\perp,V_n}) \setminus (A_n \cap B_n^{\perp,V_n})$, where $A_n = A \cap V_n,\ B_n = B \cap V_n,\ \widetilde{A}_n = \widetilde{A} \cap V_n,\ \widetilde{B}_n = \widetilde{B} \cap V_n$, because $B^{\perp,V} \cap V_n = B_n^{\perp,V_n}$. This means that $A_n \cap B_n^{\perp,V_n}$ is properly contained in $\widetilde{A}_n \cap \widetilde{B}_n^{\perp,V_n}$. Hence $\mathcal{G}_n$ is not pseudo-isotropic, which contradicts the condition that Ω_n is closed.

Now, assume that Ω_n is not closed for some n with $m_n \geq m_{\mathcal{G}}$. This means that there exist $A_n,\ B_n \in \mathcal{G}_n = \iota_n^{-1}(\mathcal{G})$ and $\widetilde{\mathcal{G}}_n \in X_n$ such that $A_n \cap B_n^{\perp,V_n}$ is properly contained in $\widetilde{A}_n \cap \widetilde{B}_n^{\perp,V_n}$, where $\widetilde{A}_n,\ \widetilde{B}_n$ are the subspaces in $\widetilde{\mathcal{G}}_n$ corresponding to $A_n,\ B_n$ respectively. Since each $e \in E_{n+1} \setminus E_n$ is orthogonal to V_n, $A_{n+1} \cap B_{n+1}^{\perp,V_{n+1}}$ is properly contained in $\widetilde{A}_{n+1} \cap \widetilde{B}_{n+1}^{\perp,V_{n+1}}$, where $A_{n+1},\ B_{n+1},\ \widetilde{A}_{n+1},\ \widetilde{B}_{n+1}$ are the respective images of $A_n,\ B_n,\ \widetilde{A}_n,\ \widetilde{B}_n$ under the embedding $X_n \hookrightarrow X_{n+1}$. Repeating this procedure, we see that $\mathcal{G}$ is not pseudo-isotropic. The result follows.

Let $G^0 = \mathrm{SL}(\infty, \mathbb{R})$. As above, given n, denote $\mathcal{G}_n = \iota_n^{-1}(\mathcal{G})$. Note that, given $F \in \mathcal{G}$, $\tau(F) = F$ if and only if F_n is defined over $\mathbb{R}$, i.e., $\tau(F_n) = F_n$ where $F_n = F \cap V_n$, because V_n is τ-invariant. The G_n^0-orbit Ω_n of $\mathcal{G}_n$ is closed if and only if each subspace in $\mathcal{G}_n$ is defined over $\mathbb{R}$. Hence if $\tau(F) = F$ for all $F \in \mathcal{G}$, then Ω_n is closed for each n (so Ω is closed), and vice versa.

The proof for $G^0 = \mathrm{SL}(\infty, \mathbb{H})$ is similar to the case of $\mathrm{SU}(\infty, \infty)$ and is based on the following facts: if A is a subspace of V, then $J(A) \cap V_n = J(A \cap V_n)$ for all n; the subspace $\langle E_{n+1} \setminus E_n \rangle_{\mathbb{C}}$ is J-invariant for all n. □

Corollary 5.7 *If $G^0 = \mathrm{SU}(p, \infty)$ for $0 \leq p < \infty$, or $\mathrm{SL}(\infty, \mathbb{R})$, then $X = \mathcal{F}\ell(\mathcal{F}, E)$ always has a closed orbit.*

Proof For $G^0 = \mathrm{SU}(p, \infty)$ one can argue as in the proof of Corollary 5.4. For $G^0 = \mathrm{SL}(\infty, \mathbb{R})$, the G^0-orbit of the generalized flag $\mathcal{F}$ is closed because $\tau(e) = e$ for all basic vectors $e \in E$. □

Let $G^0 = \mathrm{SU}(\infty, \infty)$. Obviously, the ind-grassmannian $\mathrm{Gr}(F, E)$ may have or may not have a closed orbit. If $\widetilde{X} = \mathcal{F}\ell(\widetilde{\mathcal{F}}, \widetilde{E})$, where $\widetilde{\mathcal{F}}$ is a generalized flag having the same type as the generalized flag $\mathcal{F}$ from Example 2.4 (ii) and $\widetilde{E}$ satisfies all required conditions, then $\widetilde{X}$ does not have a closed orbit. Indeed, assume $\widehat{\mathcal{F}} \in \widetilde{X}$, then $\widehat{F}_n$ contains V_k for certain n and k. The form $\omega|_{V_k}$ is nondegenerate, hence $\widehat{F}_n$ is not isotropic. There exists an isotropic subspace I of V of dimension $n = \dim \widehat{F}_n$ containing $\widehat{F}_n \cap \widehat{F}_n^{\perp, V}$, and it is easy to see that there exists $\widehat{\mathcal{F}}_0 \in \widetilde{X}$ such that I is the subspace of $\widehat{\mathcal{F}}_0$ corresponding to $\widehat{F}_n$. Thus, $\widehat{\mathcal{F}}$ is not pseudo-isotropic.

Now, suppose $\mathcal{F}$ is as in Example 2.4 (iii). Here $\widetilde{X}$ may or may have not a closed orbit. For example, assume that $\widetilde{E}$ is an ω-orthogonal basis of V. Then each $\widehat{\mathcal{F}} \in \widetilde{X}$ contains a nonisotropic finite-dimensional subspace, and, arguing as in the previous paragraph, we see that $\widehat{\mathcal{F}}$ is not pseudo-isotropic. On the other hand, suppose that $e_{2i-1} = e'_{2i-1} + e'_{2i}$ and $e_{2i} = e'_{2i-1} - e'_{2i}$ for all i, where $\{e'_1,\ e'_2, \ldots\}$ is an ω-orthogonal basis with $\omega(e'_{2i-1}, e'_{2i-1}) = -\omega(e_{2i}, e_{2i}) = 1$. In this case, one can easily check that $\mathcal{F}$ is pseudo-isotropic, so its G^0-orbit in X is closed.

Finally, let $G^0 = \mathrm{SL}(\infty, \mathbb{H})$. Here, in all three cases (i), (ii), (iii) of Example 2.4, if $\widetilde{X} = \mathcal{F}\ell(\widetilde{\mathcal{F}}, \widetilde{E})$ for a generalized flag $\widetilde{\mathcal{F}}$ having the same type as $\mathcal{F}$, then $\widetilde{X}$ may or may not have a closed orbit. Consider, for instance, case (ii). The flag $\mathcal{F}$ itself is pseudo-quaternionic, so its G^0-orbit in X is closed. On the other hand, if each $(4n+2)$-dimensional subspace in $\widetilde{\mathcal{F}}$ is spanned by $\{e_i,\ i \leq 4n\} \cup \{e_{4n+1},\ e_{4n+3}\}$, then $\widetilde{X}$ does not have a closed orbit.

If $G^0 = \mathrm{SU}(p, \infty)$, then, by Corollaries 5.4 and 5.7, X always has an open and a closed orbit. Combining our results on the existence of open and closed orbits, we now obtain the following corollary for all other real forms.

Corollary 5.8 *For a given real form G^0 of $G = \mathrm{SL}(\infty, \mathbb{C})$, $G^0 \neq \mathrm{SU}(p, \infty)$, $0 < p < \infty$, an ind-variety of generalized flags $X = \mathcal{F}\ell(\mathcal{F}, E)$ has both an open and a closed G^0-orbits if, and only if, there are only finitely many G^0-orbits on X.*

Proof If X has finitely many G^0-orbits, then the existence of an open orbit is obvious, and the existence of a closed orbit follows immediately from Corollary 4.3.

Assume that X has both an open and a closed G^0-orbit. Let $G^0 = \mathrm{SU}(\infty, \infty)$. Fix a nondegenerate generalized flag $\mathcal{H} \in X$ (lying on an open G^0-orbit). Suppose that there exists a subspace $F \in \mathcal{H}$ satisfying $\dim F = \operatorname{codim}_V F = \infty$. Since X has a closed G^0-orbit, there exists a pseudo-isotropic generalized flag $\widetilde{\mathcal{H}} \in X$. Let $\widetilde{F}$ be the subspace in $\widetilde{\mathcal{H}}$ corresponding to F. Since $\mathcal{H}$ and $\widetilde{\mathcal{H}}$ are E-commensurable to $\mathcal{F}$, there exists n such that $F = A \oplus B$ and $\widetilde{F} = \widetilde{A} \oplus B$, where A, $\widetilde{A}$ are subspaces of V_n and B is the span of a certain infinite subset of $E \setminus E_n$; in particular, B is a subspace of $\overline{V}_n = \langle E \setminus E_n \rangle_{\mathbb{C}}$.

The restriction of ω to B is nondegenerate, because V_n and $\overline{V}_n$ are orthogonal. This implies that $B^{\perp, \overline{V}_n} \cap B = \{0\}$. But $\widetilde{F}^{\perp, V} = \widetilde{A}^{\perp, V_n} \oplus B^{\perp, \overline{V}_n}$, hence $\widetilde{F} \cap \widetilde{F}^{\perp, V} = \widetilde{A} \cap \widetilde{A}^{\perp, V_n}$. Clearly, if $B \neq \overline{V}_n$, then $B^{\perp, \overline{V}_n} \neq \{0\}$. In this case, there exists $v \in \overline{V}_n \setminus B$ contained in $\widetilde{F}$, and one can easily construct a generalized flag $\widehat{\mathcal{H}} \in X$ such that $\widetilde{F} \cap \widetilde{F}^{\perp} \subsetneq \widehat{F} \cap \widehat{F}^{\perp, V}$, where $\widehat{F}$ is the subspace in $\widehat{\mathcal{H}}$ corresponding to $\widetilde{F}$, a contradiction. Thus, $B = \overline{V}_n$, but this contradicts the condition $\operatorname{codim}_V \widetilde{F} = \infty$.

We conclude that $\mathcal{H} = \mathcal{A} \cup \mathcal{B}$, where each subspace in $\mathcal{A}$ (resp., in $\mathcal{B}$) is of finite dimension (resp., of finite codimension). Assume that $\mathcal{F}$ is not finite, then at least one of the generalized flags $\mathcal{A}$ and $\mathcal{B}$ is infinite. Suppose $\mathcal{A}$ is infinite. (The case when $\mathcal{B}$ is infinite can be considered using the map $U \mapsto U^{\#}$.) Let n be such that $\mathcal{H}$ and $\widetilde{\mathcal{H}}$ are compatible with bases containing $E \setminus E_n$. Let F be a subspace in $\mathcal{A}$ such that F does not belong to V_n. Then, arguing as in the previous paragraph, one can show that $\widetilde{\mathcal{H}}$ cannot be pseudo-isotropic, a contradiction.

Now, let $G^0 = \mathrm{SL}(\infty, \mathbb{R})$. Suppose that $\mathcal{H} \in X$ is in general position with respect to τ, and $\widetilde{\mathcal{H}} \in X$ is real. As above, pick n so that $\mathcal{H}$ and $\widetilde{\mathcal{H}}$ are compatible with bases of V containing $E \setminus E_n$. Suppose for a moment that there exists a subspace $F \in \mathcal{H}$ such that $F \not\subset V_n$, then $F = A \oplus B$, where A is a subspace of V_n, and B is a nonzero subspace of $\overline{V}_n$ spanned by a subset of $E \setminus E_n$. Similarly, the corresponding subspace $\widetilde{F} \in \widetilde{\mathcal{H}}$ has the form $\widetilde{F} = \widetilde{A} \oplus B$, where $\tau(\widetilde{A}) = \widetilde{A}$ and $\tau(B) = B$. Suppose also that $B \neq \overline{V}_n$, then there exist $e \in E \cap B$ and $e' \in (E \setminus E_n) \setminus B$. Let $B' \subset \overline{V}_n$ be spanned by $((E \cap B) \setminus \{e\}) \cup \{e + ie'\}$. It is easy to check that there exists $\widehat{\mathcal{H}} \in X$ such that the subspace $\widehat{F} \in \widehat{\mathcal{H}}$ corresponding to F has the form $A \oplus B'$. Thus, $F \cap \tau(F)$ properly contains $\widehat{F} \cap \tau(\widehat{F})$, a contradiction. It remains to note that if $\mathcal{F}$ is not of finite type, then such a subspace F always exists (if necessary, after applying the map $U \mapsto U^{\#}$).

Finally, let $G^0 = \mathrm{SL}(\infty, \mathbb{H})$. Suppose that $\mathcal{H} \in X$ is in general position with respect to J, and $\widetilde{\mathcal{H}} \in X$ is pseudo-quaternionic. As above, pick n so that $\mathcal{H}$ and $\widetilde{\mathcal{H}}$ are compatible with bases of V containing $E \setminus E_n$. Suppose for a moment that there exists a subspace $F \in \mathcal{H}$ such that $F \not\subset V_n$, then $F = A \oplus B$, where A is a subspace of V_n, and B is a nonzero subspace of $\overline{V}_n = \langle E \setminus E_n \rangle_{\mathbb{C}}$ spanned by a subset of $E \setminus E_n$. The corresponding subspace $\widetilde{F} \in \widetilde{\mathcal{H}}$ has the form $\widetilde{F} = \widetilde{A} \oplus B$, where $\widetilde{A}$ is a subspace of V_n. Suppose also that $\dim B \geq 2$ and $\mathrm{codim}_{\overline{V}_n} B \geq 2$. There exist a subspace $B' \subset \overline{V}_n$ and $\widehat{\mathcal{H}} \in X$ such that the subspace $\widehat{F} \in \widehat{\mathcal{H}}$ corresponding to F has the form $A \oplus B'$, and $B' \cap J(B')$ is either properly contains or is properly contained in $B \cap J(B)$. Thus, either $\mathcal{H}$ is not in general position with respect to J, or $\widetilde{\mathcal{H}}$ is not pseudo-quaternionic, a contradiction. It remains to note that if $\mathcal{F}$ is not of finite type, then such a subspace F always exists (possibly, after applying the map $U \mapsto U^{\#}$). $\square$

Acknowledgements The first author was supported in part by the Russian Foundation for Basic Research through grants no. 14–01–97017 and 16–01–00154, by the Dynasty Foundation and by the Ministry of Science and Education of the Russian Federation, project no. 204. A part of this work was done at the Oberwolfach Research Institute for Mathematics (program "Oberwolfach Leibniz Fellows") and at Jacobs University Bremen, and the first author thanks these institutions for their hospitality. The second and third authors thank Professor V.K. Dobrev for the invitation to speak at the XI International Workshop "Lie Theory and its Applications in Physics" in Varna, 15–21 June 2015. The second author acknowledges continued partial support by the DFG through Priority Program SPP 1388 and grant PE 980/6–1. The third author acknowledges partial support from the Dickson Emeriti Professorship at the University of California and from a Simons Foundation Collaboration Grant for Mathematicians.

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Derived Functors and Intertwining Operators for Principal Series Representations of $SL_2(\mathbb{R})$

Raul Gomez and Birgit Speh

Abstract We consider the principal series representations I_ν induced from a character ν of the upper triangular matrices B and its realization on the Frechet space of C^∞-sections of a line bundle over G/B. Its continuous dual is denoted by I_ν^*. Let $N \subset B$ be the nilpotent subgroup whose diagonal entries are 1 and denote by $\mathfrak{n}$ its Lie algebra. We determine $H^0(\mathfrak{n}, I_\nu^*)$ and $H^1(\mathfrak{n}, I_\nu^*)$ and conclude that space of the intertwining operators $T : I_\nu \to I_{-\nu}$ is 2 dimensional for some integral parameter, otherwise it is one dimensional. The intertwining operators are identified with distributions. We show that for certain parameters the support of this distribution is a point, i.e. that the intertwining operator is a differential intertwining operator.

1 Introduction

In this note we revisit the well known theory of intertwining operators for principal series representations for $G = SL(2, \mathbb{R})$. We consider intertwining operators as a special case of symmetry breaking operators for principal series representations $I_\nu \to I_{-\nu}$ and analyze them using mostly geometry and homological algebra instead of analysis. This leads to a different perspective of a theory developed almost 50 years ago and to some new insights. These ideas is also essential in determining the invariant trilinear functionals on tensor products of principal series representations [5].

We do not consider the usual realization of the principal series representations on a Banach or Hilbert space, but instead we take the representation space of the induced representation to be the space of C^∞-sections of the G-equivariant vector bundle $G \times_B (\chi_\nu, \mathbb{C}) \to G/B$, so that $I_\nu^\infty \simeq I_\nu$ is the Fréchet globalization having moderate growth in the sense of Casselman–Wallach [10]. Here $B = MAN$ is the

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V. Dobrev (ed.), *Lie Theory and Its Applications in Physics*,
Springer Proceedings in Mathematics & Statistics 191,
DOI 10.1007/978-981-10-2636-2_9

Borel subgroup of upper triangular matrices, $\chi_\nu = \epsilon \otimes e^\nu$ is a character of B and ϵ is a character of the center of G. The representation is denoted by $I_{\nu,\epsilon}$. It is called spherical if ϵ is trivial on the center of G and is denoted by I_ν. The parametrization is chosen so that the spherical representation I_ν is reducible if and only if $|\nu|$ is an odd integer. For an odd positive integer i the principal series representations I_{-i} contains a finite-dimensional representation $F(i)$ as the unique subrepresentation.

The dual space, the space of tempered distributions, is also a $U(\mathfrak{g})$-module, and a realization $I^*_{\epsilon,\nu}$ of the contragredient representation of $I_{\epsilon,\nu}$.

The results of W. Casselman and N. Wallach [10] imply that to compute the dimension of the space intertwining operators it suffices to determine the MA-modules $H_0(\mathfrak{n}, I_{\epsilon,\nu})$, respectively $H^0(\mathfrak{n}, I^*_{\epsilon,\nu})$.

To determine the $\mathfrak{n}$ cohomology of $I^*_{\epsilon,\nu}$, respectively the $\mathfrak{n}$-homology of $I_{\epsilon,\nu}$, we proceed as follows:

There is a stratification of G/B by orbits of N; one orbit $NwB \simeq \mathbb{R}$ is open and dense and one orbit is a closed point $eB \simeq \mathbb{O}$ and we get an exact sequence to $\mathfrak{n}$-modules.

$$0 \to \mathcal{S}(\mathbb{R}) \to I_{\epsilon,\nu} \to \mathcal{S}_{\mathbb{O}}(\mathbb{R}) \to 0$$

On the dual side we have an exact sequence

$$0 \to \mathcal{S}^*_{\mathbb{O}}(\mathbb{R}) \to I^*_{\epsilon,\nu} \to \mathcal{S}^*(\mathbb{R}) \to 0$$

and so obtain a long exact sequence in $\mathfrak{n}$-cohomology

$$0 \leftarrow \mathcal{S}^*_{\mathbb{O}}(\mathbb{R})^{\mathfrak{n}} \leftarrow (I^*_{\epsilon,\nu})^{\mathfrak{n}} \leftarrow \mathcal{S}^*(\mathbb{R})^{\mathfrak{n}} \leftarrow H^1(\mathfrak{n}, \mathcal{S}^*_{\mathbb{O}}(\mathbb{R})) \leftarrow H^1(\mathfrak{n}, I^*_{\epsilon,\nu}) \leftarrow \cdots$$

We show that $\mathcal{S}^*_{\mathbb{O}}(\mathbb{R})$ is isomorphic to the restriction of a Verma module to $\mathfrak{b}$. The dimension of $H^0(\mathfrak{n}, \mathcal{S}^*(\mathbb{R}))= (\mathcal{S}^*(\mathbb{R}))^{\mathfrak{n}}$ is one and thus to determine the cohomology it suffices to compute the $\mathfrak{n}$–cohomology of a Verma module as well as the connection homomorphism

$$H^0(\mathfrak{n}, \mathcal{S}^*(\mathbb{R})) \leftarrow H^1(\mathfrak{n}, \mathcal{S}^*_{\mathbb{O}}(\mathbb{R})).$$

Lastly, we analyze the action of the diagonal matrices A on the cohomology. Thus we obtain

Theorem 1 *Let $I_{\varepsilon,\nu}$ be a principal series representation.*

1. *If ν is not a negative integer then*

$$dim\, H_0(\mathfrak{n}, I_{\varepsilon,\nu}) = 2 \quad and \quad dim\, H_1(\mathfrak{n}, I_{\varepsilon,\nu}) = 0$$

2. *If $\nu \equiv +\varepsilon \bmod 2$ is a negative integer, then*

$$dim\, H_0(\mathfrak{n}, I_{\varepsilon,\nu}) = 2 \quad and \quad dim\, H_1(\mathfrak{n}, I_{\varepsilon,\nu}) = 0$$

3. *If* $\nu \equiv \varepsilon + 1 \, mod \, 2$ *is a negative integer, then*

$$dim\, H_0(\mathfrak{n}, I_{\varepsilon,\nu}) = 3 \quad and \quad dim\, H_1(\mathfrak{n}, I_{\varepsilon,\nu}) = 1$$

Corollary 2

1. *Under the assumptions (2) the support of both distributions in* $(I^*_{\varepsilon,\nu})^N$ *is the identity.*
2. *Under the assumptions (3) the support of 2 distributions in* $(I^*_{\varepsilon,\nu})^N$ *is at the identity and one distribution has support on* G/B.

After analyzing the action of A on the cohomology we conclude

Theorem 3 *Let* $I_{\varepsilon,\nu}$ *be a principal series representation. Then*

1. *If* ν *is not a non positive integer then*

$$dim\, Hom_G(I_{\varepsilon,\nu}, I_{\varepsilon,-\nu}) = 1$$

and the intertwining operator is an integral operator.

2. *If* $\nu \equiv \varepsilon + k \; mod \; 2$ *then*

$$dim\, Hom_G(I_{\varepsilon,\nu}, I_{\varepsilon,-\nu}) = 1.$$

If ν *is a non positive integer then the intertwining operator is a differential operator.*

3. *If* $\nu = \varepsilon + k - 1 \; mod \; 2$ *is a nonpositive integer then*

$$dim\, Hom_G(I_{\varepsilon,\nu}, I_{\varepsilon,-\nu}) = 2$$

One intertwining operator is an integral operator and the other intertwining operator is a differential operator.

The article is organized as follows:

1. Notation and generalities
2. I_ν and I^*_ν as a $U(\mathfrak{n})$–modules
3. The $\mathfrak{n}$-cohomology and $\mathfrak{n}$-homology
4. The main theorem
5. Application to intertwining operators
6. Closing Remarks

2 Notation and Generalities

In this section we establish the notations and recall some well known results about the Casselman-Wallach-model of principal series representations of $SL(2, \mathbb{R})$.

2.1 Let G be the special linear group $SL(2,\mathbb{R})$. The Lie algebra of any subgroup of G is denoted by the corresponding lower case Gothic letter and the enveloping algebras of a Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$ by $U(\mathfrak{h})$. We choose the usual basis

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}$$

of $\mathfrak{g}$.

We fix a Borel group B of upper triangular matrices. Let $N = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$ its nilpotent subgroup with diagonal entries 1. The connected component of the group of diagonal matrices is denoted by A, $M = (-1)^i \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $K = SO(2)$. Then $B = MAN$, $G = KAN$. We denote by $\bar{N}$ the transpose of N. Then $\bar{N}B$ is dense in the group G.

2.2 The space G/B is isomorphic to the projective space

$$\mathbb{P}^1 = \{[x:y] | (x,y) \in \mathbb{R}^2 - (0,0)\}.$$

The group N acts on $\mathbb{P}^1$ by $[x:y] \to [x+ny:y]$. So we have 2 orbits: the closed orbit: $[x:0]$ and the open orbit $[x:y]$, $y \neq 0$.

2.3 We denote by α the positive root. A character of B is determined by a character of $B/N = MA$

$$\epsilon\chi_\nu : MA \to \mathbb{C}$$

where

$$\epsilon\chi_\nu : ma \to \epsilon(m) \mathrm{e}^{(\frac{(\nu)}{2}\alpha(log(a))}$$

Here $\nu \in \mathbb{C}$ and the character ϵ can be identified with an element in $\mathbb{Z}_2$. If ϵ is trivial we simplify the notation and write only χ_ν for the character of MA.

We consider the principal series representation

$$I_\nu = \mathrm{ind}_B^G \chi_{\nu+1}.$$

in its Casselman-Wallach realization as continuous representation [10] acting on the Frechet space

$$I_\nu = \{f \in C^\infty(G) \,|\, f(nag) = \chi_{\nu+1}(a) f(g) \text{ for } g \in G,\ b \in B\}$$

We also consider the dual representation I_ν^* acting on the space I_ν^* of tempered distributions on I_ν.

2.4 The character χ_ν defines a linear functional on $\mathfrak{b}$ and hence on $U(\mathfrak{b})$ which we denote by the same letter. We define the Verma module

$$M(\nu) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \chi_{\nu-1}.$$

Recall that for positive integral parameter ν the Verma module has a composition series of length 2 with the finite dimensional representation as the irreducible quotient and subrepresentation $M(-\nu-2)$. Otherwise the modules are irreducible [6].

3 A Filtration of the Representations I_ν and I_ν^*

In this section we use the N-orbits G/B by N to define a filtration of I_ν and I_ν^* by $\mathfrak{n}$-modules and to analyze it.

3.1 We consider the restriction of the representation I_ν to $\mathfrak{n}$ and define a filtration of I_ν^* as follows:

For $\nu \in \mathfrak{a}^*$ we have the $U(\mathfrak{n})$–module

$$\mathbb{U}_\nu = \{f \in I_\nu \,|\, f \text{ and all its derivatives vanish on the closed orbit } [x:0]\},$$

It is isomorphic as a $U(\mathfrak{n})$-module to $\mathbb{U}$ under $f \mapsto f|_K$.

We define $\mathbb{W}_\nu$ by the exact sequence

$$0 \to \mathbb{U}_\nu \to I_\nu \to \mathbb{W}_\nu \to 0.$$

and obtain an exact sequence of $U(\mathfrak{n})$-modules. On the dual side we have the exact sequence

$$0 \to \mathbb{W}_\nu^* \to I_\nu^* \to \mathbb{U}_\nu^* \to 0.$$

3.2 Let $w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and define maps

$$T_\nu^0 : I_\nu \to C^\infty(\mathbb{R})$$

and

$$T_\nu^\infty : I_\nu \to C^\infty(\mathbb{R})$$

by

$$T_\nu^0(f)(x) = \left(\pi\left(\begin{bmatrix} 1 & x \\ & 1 \end{bmatrix}\right) f\right)(w),$$

and

$$T_\nu^\infty(f)(x) = \left(\pi\left(\begin{bmatrix} 1 & \\ y & 1 \end{bmatrix}\right) f\right)(w),$$

respectively.

Lemma 1

1. *$T^\infty_\nu(U_\nu) = \mathcal{S}(\mathbb{R})$, where $\mathcal{S}(\mathbb{R})$ is the Schwartz space for $\mathbb{R}$.*
2. *If $x \neq 0$, then*
$$T^\infty_\nu(f)(1/x) = |x|^{\nu+1} T^0_\nu(f)(x). \tag{1}$$

Proof The first part follows immediately from the definition of T^0_ν.

For the second part, observe that

$$\begin{bmatrix} 1 & y \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & \frac{y}{y^2+1} \\ & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{1+y^2}} & \\ 0 & \sqrt{1+y^2} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{1+y^2}} & \frac{-y}{\sqrt{1+y^2}} \\ \frac{y}{\sqrt{1+y^2}} & \frac{1}{\sqrt{1+y^2}} \end{bmatrix}$$

and

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & \\ x & 1 \end{bmatrix} = \begin{bmatrix} 1 & \frac{-x}{x^2+1} \\ & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{1+x^2}} & \\ 0 & \sqrt{1+x^2} \end{bmatrix} \begin{bmatrix} \frac{x}{\sqrt{1+x^2}} & \frac{-1}{\sqrt{1+x^2}} \\ \frac{1}{\sqrt{1+x^2}} & \frac{x}{\sqrt{1+x^2}} \end{bmatrix}.$$

Now observe that if we set $y = 1/x$, then

$$\begin{bmatrix} 1 & 1/x \\ & 1 \end{bmatrix} = \begin{bmatrix} 1 & \frac{x}{x^2+1} \\ & 1 \end{bmatrix} \begin{bmatrix} \frac{x}{\sqrt{1+x^2}} & \\ 0 & \frac{\sqrt{1+x^2}}{x} \end{bmatrix} \begin{bmatrix} \frac{x}{\sqrt{1+x^2}} & \frac{-1}{\sqrt{1+x^2}} \\ \frac{1}{\sqrt{1+x^2}} & \frac{x}{\sqrt{1+x^2}} \end{bmatrix}.$$

It follows immediately that

$$T^\infty_\nu(f)(1/x) = |x|^{\nu+\rho} T^0_{\varepsilon,\nu}(f)(x). \qquad \square$$

Corollary 4 *$\mathbb{U}^*_\nu$ is isomorphic to the tempered distributions on $\mathbb{R}$.*

3.3 We analyze next the module $\mathbb{W}_\nu$.

Lemma 2 *The map*
$$f \mapsto (f(e), Ff(e), F^2 f(e), \ldots)$$

induces an isomorphism between $\mathbb{W}_\nu$ and

$$\prod_{k=0}^{\infty} \mathbb{C} = \{(a_0, a_1, \ldots) \mid a_k \in \mathbb{C} \text{ for all } k \geq 0\},$$

where the last space is endowed with the projective limit topology.

Proof Any linear functional on $\mathbb{W}'_\nu$ corresponds to a linear functional on $I_{\varepsilon,\nu}$ that vanishes on U_ν. Under the map $f \mapsto T^\infty_\nu(f)$, this corresponds to distributions on $\mathbb{R}$ supported at the origin. From well-known results from functional analysis, we know that this space of distributions is the inductive limit generated by derivatives of the delta-functional 0. For $F = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}$

$$T_\nu^\infty(Ff)(y) = \frac{d}{dy} T_{\varepsilon,\nu}^\infty(f)(y),$$

we see that this space correspond to the inductive limit generated by the linear functionals $F^k\delta_e$, where δ_e is the delta-function at the identity. The lemma now follows immediately. $\square$

Lemma 3 *For all* $f \in I_\nu$,

$$(F^kEf)(e) = -k(\nu+k)f^{k-1}f(e)$$

and

$$(F^kHf)(e) = (\nu+2k+1)F^kf(e).$$

Proof We will prove this formulas by induction on k. For $k=0$ the formulas are trivially true. Assume that the formulas are valid for all $j \leq k$. Then

$$\begin{aligned}
(F^{k+1}Ef)(e) &= (F^kEF)f(e) - (F^kHf)(e) \\
&= -k(\nu+k)(F^kF)f(e) - (\nu+2k+1)(F^kf)(e) \\
&= [-k^2 - k\nu - \nu - 2k - 1]F^kf(e) \\
&= -[\nu(k+1) + (k+1)^2]F^kf(e) \\
&= -[(k+1)(\nu+(k+1)]F^kf(e).
\end{aligned}$$

Similarly,

$$\begin{aligned}
(F^{k+1}Hf)(e) &= (F^kHF)f(e) + 2(F^{k+1})(e) \\
&= (\nu+2k+1)(F^{k+1})f(e) + 2(F^{k+1})(e) \\
&= (\nu+2(k+1)+1)(F^{k+1})f(e).
\end{aligned}$$

Corollary 5 *As a* $U(\mathfrak{n})$*-module*

$$\mathbb{W}_\nu^* = M(-\nu)_{|U(\mathfrak{n})}.$$

4 The $\mathfrak{n}$-Cohomology and $\mathfrak{n}$-Homology

In this section we give a definition of $H^*(\mathfrak{n}, I_\nu^*)$ and $H_*(\mathfrak{n}, I_\nu)$. For details see [2], Sect. I.

4.1 Let H be a (connected) real Lie group and let V be a $\mathfrak{h}$-module. For $n \in \mathbb{N}$ let

$$C^n = C^n(\mathfrak{h}, V) = Hom(\wedge \mathfrak{h}^n, V)$$

and $d : C^n \to C^{n+1}$ defined by

$$\begin{aligned} df(X_0, X_1, \ldots, X_n) = &= \sum_i (-1)^i X_i f(X_0, \ldots, \hat{X}_i, \ldots, X_n) \\ &+ \sum_{i<j} f([X_i, X_j], X_0, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots.., X_n). \end{aligned}$$

Similarly we define the homology with coefficients in an $\mathfrak{h}$–module by the complex

$$B^n = B^n(\mathfrak{h}, W) = \wedge^n \mathfrak{h} \otimes W$$

and δ by

$$\begin{aligned} \delta(X_1 \wedge \cdots \wedge X_n \otimes w) = &\sum_{1 \le j < k \le n} (-1)^{j+k-1} [X_j, X_k] \wedge X_1 \wedge \cdots \hat{X}_j \cdots \hat{X}_k \cdots \wedge X_n \otimes w \\ &+ \sum_{1 \le l \le n} (-1)^l \wedge X_1 \wedge \cdots \hat{X}_l \cdots \wedge X_n \otimes X_l w, \end{aligned}$$

4.2

Example 1 The Schwarz space $\mathcal{S}(U) = \mathbb{U}_\nu$ is a smooth N-module. Given $f \in \mathbb{U}_\nu$, set

$$J_\nu(f) = \int_{\mathbb{R}} f\left(w \begin{bmatrix} 1 & x \\ & 1 \end{bmatrix}\right) dx = \int_{\mathbb{R}} T^0_\nu(f)(x)\, dx.$$

Then J_ν induces an isomorphism between $(\mathbb{U}_\nu)_{\mathfrak{n}}$ and $\mathbb{C}$. On the other hand we may consider $[J_\nu] \in (\mathbb{U}^*_\nu)^N = H^0(\mathfrak{n}, \mathbb{U}^*_\nu)$. Furthermore,

$$H_1(\mathfrak{n}, \mathbb{U}_\nu) = H^1(\mathfrak{n}, \mathbb{U}^*_\nu) = 0 \qquad \text{for all } \nu \in \mathfrak{a}^*$$

Example 2 $\mathbb{W}_\nu$ and $\mathbb{W}^*_\nu$ are a $\mathfrak{n}$-modules. If ν is not an strictly negative integer, then the Verma module $\mathbb{W}^*_\nu = M(-\nu)$ is irreducible and the map $f \mapsto f(e)$ induces an isomorphism between $(\mathbb{W}_\nu)_{\mathfrak{n}}$ and $\mathbb{C}$ and thus $H_0(\mathfrak{n}, \mathbb{W}_\nu) = \mathbb{C}$. Furthermore the class of the δ_e–distribution is nontrivial in $H^0(\mathfrak{n}, \mathbb{W}^*_\nu)$. In this case,

$$H_1(\mathfrak{n}, \mathbb{W}_\nu) = 0 = H^1(\mathfrak{n}, \mathbb{W}^*_\nu)$$

On the other hand, if ν is an strictly negative integer then the Verma module $\mathbb{W}^*_\nu = M(-\nu)$ is reducible. For a nontrivial generator F of $\overline{\mathfrak{n}}$, the map $f \mapsto (f(e), (F^{-\nu} f)(e))$ induces an isomorphism between $(\mathbb{W}_\nu)_{\mathfrak{n}}$ and $\mathbb{C}^2$. In this case the classes of the δ_e–distribution and of $F^{-\nu}\delta_e$ are spanning $H^2(\mathfrak{n}, \mathbb{W}^*_\nu)$. Furthermore

$\dim H_1(\mathfrak{n}, \mathbb{W}_\nu) = 1$ and this space is generated by an element $f \in I_\nu$ such that $F^{-\nu-1}f(e) \neq 0$ and $F^j f(e) = 0$ for all $j \neq -\nu - 1$.

4.3 The $\mathfrak{n}$–modules I_ν^* and I_ν are smooth and nuclear. Furthermore, I_ν is also a Frechet space. We define

$$\mathrm{Tor}_N^n(I_\nu, \mathbb{C}) = H_n(\mathfrak{n}, I_\nu)$$

and

$$\mathrm{Ext}_N^n(\mathbb{C}, I_\nu^*) = H^n(\mathfrak{n}, I_\nu^*).$$

Note that

$$H_n(\mathfrak{n}, I_\nu)^* = H^n(\mathfrak{n}, I_\nu^*)$$

The short exact sequences

$$0 \to \mathbb{W}_\nu^* \to I_\nu^* \to \mathbb{U}_\nu^* \to 0$$

and

$$0 \to \mathbb{U}_\nu \to I_\nu \to \mathbb{W}_\nu \to 0$$

of $U(\mathfrak{n})$–modules induce the long exact sequences

$$0 \leftarrow H^1(\mathfrak{n}, \mathbb{U}_\nu^*) \leftarrow H^1(\mathfrak{n}, I_\nu^*) \leftarrow H^1(\mathfrak{n}, \mathbb{W}_\nu^*) \leftarrow (\mathbb{U}_\nu^*)^{\mathfrak{n}} \leftarrow (I_\nu^*)^{\mathfrak{n}} \leftarrow (\mathbb{W}_\nu^*)^{\mathfrak{n}} \leftarrow 0$$

and

$$0 \to H_1(\mathfrak{n}, \mathbb{U}_\nu) \to H_1(\mathfrak{n}, I_\nu) \to H_1(\mathfrak{n}, \mathbb{W}_\nu) \to (\mathbb{U}_\nu)_{\mathfrak{n}} \to (I_\nu)_{\mathfrak{n}} \to (\mathbb{W}_\nu)_{\mathfrak{n}} \to 0.$$

The sequences are dual to each other.

5 The Main Theorem

In this section we state and prove the main theorem.

5.1 First we determine the $\mathfrak{n}$-cohomology of I_ν^*.

Proposition 1 *Let I_ν be a spherical principal series representation. Then*

1. *If ν is not a negative integer then*

$$dim\ H_0(\mathfrak{n}, I_\nu) = 2 \quad and \quad dim\ H_1(\mathfrak{n}, I_\nu) = 0$$

2. *If ν is a non zero even negative integer, then*

$$dim\, H_0(\mathfrak{n}, I_\nu) = 2 \quad and \quad dim\, H_1(\mathfrak{n}, I_\nu) = 0$$

3. *If ν is a non odd negative integer, then*

$$dim\, H_0(\mathfrak{n}, I_\nu) = 3 \quad and \quad dim\, H_1(\mathfrak{n}, I_\nu) = 1$$

Remark For the equivalent statement for the p-adic general linear group $GL(2, \mathbb{Q}_p)$ see [3].

Proof of Proposition 1: Suppose first that ν is not a negative integer. Then by the previous section dim $(\mathbb{W}_\nu)_\mathfrak{n}$ = dim $(\mathbb{U}_\nu)_\mathfrak{n} = 1$ and the sequence

$$0 \to (\mathbb{U}_\nu)_\mathfrak{n} \to (I_\nu)_\mathfrak{n} \to (\mathbb{W}_\nu)_\mathfrak{n} \to 0$$

is exact. In particular,

$$\dim\, (I_\nu)_\mathfrak{n} = 2.$$

and

$$H_1(\mathfrak{n}, I_\nu) = 0$$

Now suppose that ν is a negative integer. Then we have the exact sequence

$$0 \to H_1(\mathfrak{n}, I_\nu) \to H_1(\mathfrak{n}, \mathbb{W}_\nu) \to (\mathbb{U}_\nu)_\mathfrak{n} \to (I_\nu)_\mathfrak{n} \to (\mathbb{W}_\nu)_\mathfrak{n} \to 0.$$

and so we have to understand the connection homomorphism $\mathbb{C} = H_1(\mathfrak{n}, \mathbb{W}_\nu) \to (\mathbb{U}_\nu)_\mathfrak{n} = \mathbb{C}$. Let $f \in \mathbb{I}_\nu$ be such that

$$T_\nu^\infty f(y) = y^{k-1}$$

in a neighborhood of 0. Then $0 \neq [f] \in H_1(\mathfrak{n}, W_\nu)$. To compute the action of the connecting homomorphism, we use Eq. (1) to get the following identity in a neighborhood of ∞ for x:

$$(1/x)^{k-1} = T_\nu^\infty f(1/x) = |x|^{-k+1} T_\nu^0 f(x).$$

In other words, if $|x| \gg 0$, then

$$T_\nu^0 f(x) = \operatorname{sgn}(x)^{-k+1}.$$

Now, it's straightforward to check that

$$\begin{aligned} J_{-k}(\partial f) &= \int_{-\infty}^{\infty} \left(\frac{d}{dx} T_\nu^0 f(x) \right)(x)\, dx \\ &= \lim_{x \to \infty} \operatorname{sgn}(x)^{-k+1} - \operatorname{sgn}(-x)^{-k+1} \\ &= 1 - (-1)^{-k+1}. \end{aligned}$$

Since J_ν defines an isomorphism between $\mathbb{U}_\nu$ and $\mathbb{C}$ we conclude that The connecting homomorphism ∂ is trivial if an only if

$$0 \equiv k-1 \mod 2.$$

Thus if $\nu = -k$ is a negative integer then

$$\dim I_k = \begin{cases} 3 \text{ if } 0 \equiv k-1 \mod 2 \\ 2 \text{ if } 0 \equiv k \mod 2 \end{cases}$$

5.2 Similarly we determine the $\mathfrak{n}$-cohomology for the principal representations $I_{\varepsilon,\nu}$ for nontrivial ε and prove

Theorem 6 *Let $I_{\varepsilon,\nu}$ be a principal series representation. Then*

1. *If ν is not a negative integer then*

$$dim\ H_0(\mathfrak{n}, I_{\varepsilon,\nu}) = 2 \quad and \quad dim\ H_1(\mathfrak{n}, (I_{\varepsilon,\nu}) = 0$$

2. *If $\nu \equiv \varepsilon + k$ mod 2, then*

$$dim\ H_0(\mathfrak{n}, I_{\varepsilon\nu}) = 2 \quad and \quad dim\ H_1(\mathfrak{n}, (I_{\varepsilon,\nu}) = 0$$

3. *If $\nu \equiv \varepsilon + k - 1$ mod 2 is a non odd negative integer, then*

$$dim\ H_0(\mathfrak{n}, I_{\varepsilon,\nu}) = 3 \quad and \quad dim\ H_1(\mathfrak{n}, (I_{\varepsilon,\nu}) = 1$$

Since the $\mathfrak{n}$–cohomology is finite dimensional we can conclude

Corollary 7 *Let $I_{\varepsilon,\nu}$ be a principal series representation.*

1. *Under the assumptions of Theorem 6 (2) the support of both distributions in $(I^*_{\varepsilon,\nu})^N$ is the identity.*
2. *Under the assumptions of Theorem 6 (3) the support of 2 distributions in $(I^*_{\varepsilon,\nu})^N$ is at the identity and one distribution has support on G/B.*

6 Application to Intertwining Operators

We first determine the MA-module $H_0(\mathfrak{n}, I_\nu)$. Then we review the connection between the Jacquet module $H_0(\mathfrak{n}, I_\nu)$ and intertwining operators between principal series representations in their Casselman-Wallach realization, to complete the proof of Corollary I. 3. For another approach see Chap. VII. of [7].

6.1 The diagonal matrices A act on $\mathfrak{n}$ and hence on $H^0(\mathfrak{n}, I_\nu)$. An eigenvector V_μ transforming according to the character χ_μ defines an intertwining operator

$$\mathcal{V}_\mu : I_\nu \to I_\mu$$

There is always a linear functional in $H^0(\mathfrak{n}, I_\nu)^*$ with Eigenvalue ν corresponding to the delta distribution at the identity, It corresponds to the identity intertwining operator.

6.2 Since A normalizes $\mathfrak{n}$ we consider $(I_\nu)_\mathfrak{n}$ and hence $(I_\nu^*)^\mathfrak{n}$ as an A–module. If ν is nonsingular then [4] implies that the action of A on $H_0(\mathfrak{n}, I_\nu)$ is semi simple. If ν is not a negative integer, zero or if $\nu \equiv \varepsilon + k$ mod 2, then A acts by the characters χ_ν and $\chi_{-\nu}$ on $H_0(\mathfrak{n}, I_{\varepsilon,\nu})$.

6.3 Suppose now that I_ν is a spherical principal series representation and $\nu = 0$. Let

$$\tilde{U}_{0,\nu} = \{f \in I_{0,\nu} \,|\, f(e) = 0\}$$

and observe, that, if we assume that Re$\nu > -1$, then the integral

$$J_{0,\nu}(f) = \int_R f\left(w \begin{bmatrix} 1 & x \\ & 1 \end{bmatrix}\right) dx$$

is absolutely convergent. On the other hand, if $1_\nu \in I_{0,\nu}$ is the element such that $1_\nu|_K \equiv 1$, then for Re$\nu > 0$

$$J_{0,\nu}(1_\nu) = \int_R (1 + x^2)^{-\frac{\nu+1}{2}} dx = B(\nu/2, 1/2),$$

where

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x + y)}$$

is the Beta function. Let

$$\begin{aligned} K_{0,\nu} &= J_{0,\nu} - J_{0,\nu}(1_\nu)\delta_e \\ &= J_{0,\nu} - B(\nu/2, 1/2)\delta_e. \end{aligned}$$

Then it is straightforward to check that $K_{0,\nu}$ is well defined for Re$\nu > 0$. Furthermore, since $I_{0,\nu} = \tilde{U}_{0,\nu} \oplus \langle 1_\nu \rangle$, the map $K_{0,\nu}$ has holomorphic continuation to Re$\nu > -1$

and

$$(I'_{0,\nu})^{\mathfrak{n}} = \mathrm{Span}_{\mathbb{C}}\{\delta_e, K_{0,\nu}\}.$$

Now, observe that

$$H \cdot \delta_e = -(\nu+1)\delta_e$$

and

$$\begin{aligned} H \cdot K_{0,\nu} &= (\nu-1)J_{0,\nu} + (\nu+1)B(\nu/2, 1/2)\delta_e \\ &= (\nu-1)K_{0,\nu} + 2\nu B(\nu/2, 1/2)\delta_e, \end{aligned}$$

that is, the action of H on $(I'_{0,\nu})^{\mathfrak{n}}$ corresponds to the matrix

$$\begin{bmatrix} -\nu-1 & 2\nu B(\nu/2, 1/2) \\ & \nu-1 \end{bmatrix}.$$

Taking the limit as $\nu \to 0$, we get

$$H \leftrightarrow \begin{bmatrix} -1 & 4 \\ & -1 \end{bmatrix}.$$

In particular, this implies that although $\dim(I'_{0,0})^{\mathfrak{n}} = 2$, we have

$$\dim \mathrm{Hom}_G(I_{0,0}, I_{0,0}) = 1.$$

and thus we have another proof of the well known fact that $I_{0,0}$ is irreducible.

7 Closing Remarks

Remark on analytic continuation: A. Knapp and E. Stein [8] introduced in 1967 intertwining operators for principal series of Lie groups realized on the Hilbert Space of L^2–sections of a vector bundle $\mathcal{V}_\nu$. These operators were initially defined only for parameters in a region in the positive Weyl chamber then a normalized operator was defined and analytically continued to all continuous parameters. The main tool was L^2-harmonic analysis. The normalized linear functional

$$\mathcal{A}_{\varepsilon,\nu} f(e) = \frac{1}{\Gamma(\frac{\varepsilon+\nu-1}{2})} \int_{G/B} \chi_\nu(p^{-1}) f(pg) dg$$

defines an intertwining operator. This operator coincides on the C^∞-vectors with the operator defined by A. Knapp and E. Stein, since we have multiplicity one for the operators and the actions coincide on the minimal K-types.

Remark on differential intertwining operators: Differential intertwining operators were discovered around 1974 by B. Kostant [9]. Differential intertwining operators for rank one principal series representations were considered by B. Boe and D. Collingwood [1]. It is known which of these differential intertwining operators are residues of integral intertwining operators and which ones define truly new additional operators.

Remark on N-cohomology for p-adic representations: For the p-adic principal series representations we can proceed as in the real case. In this case the functions in the Schwartz space on $\mathbb{Q}_p$ have compact support and $\mathcal{S}(\mathbb{O})$ is the trivial representation. Its N-homology is concentrated in degree 0. The cohomology is also concentrated in degree 0. and we always have a unique nontrivial integral intertwining operator. See also the argument in Bump's book.

Acknowledgements Research by B. Speh partially supported by NSF grant DMS-1500644.

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Hyperlogarithms and Periods in Feynman Amplitudes

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Abstract The role of hyperlogarithms and multiple zeta values (and their generalizations) in Feynman amplitudes is being gradually recognized since the mid 1990s. The present lecture provides a concise introduction to a fast developing subject that attracts the interests of a wide range of specialists – from number theorists to particle physicists.

1 Introduction

Observable quantities in particle physics: scattering amplitudes, anomalous magnetic moments, are typically expressed in perturbation theory as (infinite) sums of *Feynman amplitudes* – integrals over internal position or momentum variables corresponding to *Feynman graphs* (ordered by the number of internal vertices or by the number of loops – the first Betti number of a graph). Whenever these integrals are divergent (which is often the case) one writes them as Laurent expansions in a (small) regularization parameter ε. (In the commonly used dimensional regularization ε is half of the deviation of spacetime dimension from four: $2\varepsilon = 4 - D$. We shall encounter in Sect. 2 a more general regularization with similar properties.) It was observed – first as an unexpected curiosity in more advanced calculations (beyond one loop), then in a more systematic study – that the resulting integrals involved interesting numbers like values of the zeta function at odd integers. Such numbers, first studied by Euler, but then forgotten for over two centuries, attracted independently, at about

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V. Dobrev (ed.), *Lie Theory and Its Applications in Physics*,
Springer Proceedings in Mathematics & Statistics 191,
DOI 10.1007/978-981-10-2636-2_10

the same time the interest of mathematicians who defined the $\mathbb{Q}$-algebra $\mathcal{P}$ of *periods*. According to the elementary definition of Kontsevich and Zagier [29] periods are complex numbers whose real and imaginary parts are given by absolutely convergent integrals of rational differential forms:

$$I = \int_{\Sigma} \frac{P}{Q} dx_1 \dots dx_n \qquad (\in \mathcal{P}), \tag{1}$$

where P and Q are polynomials with rational coefficients and the integration domain Σ is given by polynomial inequalities again with rational coefficients.

Remarkably, the set of periods is denumerable – they form a tiny (measure zero) part of the complex numbers but they suffice to answer all questions in particle physics. More precisely, it has been proven [6] that for rational ratios of invariants and masses all Laurent coefficients of dimensionally regularized Euclidean Feynman amplitudes are periods. Brown [17] announces a similar result for convergent "generalized Feynman amplitudes" (that include the residues of primitively divergent graphs) without specifying the regularization procedure.

Amplitudes are, in general, functions of the external variables – coordinates or momenta – and of the masses of "virtual particles" associated with internal lines. Just like the numbers – periods (that appear as special values of these functions) the resulting family of functions, the iterated integrals [21], has attracted independently the interest of mathematicians. Here belong the hyperlogarithms which possess a rich algebraic structure and appear in a large class of Feynman amplitudes, in particular, in conformally invariant massless theories.

The topic has become the subject of specialized conferences and research semesters.[1] The present lecture is addressed, by contrast, to a mixed audience of mathematicians and theoretical physicists working in a variety of different domains. Its aim is to introduce the basic notions and to highlight some recent trends in the subject. We begin in Sect. 2 with a shortcut from the early Euler's work on zeta to the amazing appearance of his alternating ("ϕ-function") series in the calculation of the electron (anomalous) magnetic moment. Section 3 reviews the appearance of periods as residues of primitively divergent Feynman amplitudes. Section 4 introduces the double shuffle algebra of hyperlogarithms appearing inter alia in the calculation of position space conformal 4-point amplitudes. In Sect. 5 we introduce implicitly the formal multiple zeta values (MZV) defined by the double shuffle relations including "divergent words" and setting $\zeta(1) = 0$. The generating series $L(z)$ and Z "$= L(1)$" are used to write down the monodromy around the possible singularities at $z = 0$ and $z = 1$ of the multipolylogarithms. We also display the periods of the "zig-zag diagrams" of Broadhurst and Kreimer and of the six-loop graph where a double zeta value ($\zeta(3, 5)$) first appears. In Sect. 6 we define the "multiple Deligne values"

[1] To cite a few: "Loops and Legs in Quantum Field Theory" Bi-annual Workshop taking place (since 2008) in various towns in Germany; Durham Workshop: "Polylogarithms as a Bridge between Number Theory and Particle Physics" [43]; Research Trimester "Multiple Zeta Values, Multiple Polylogarithms, and Quantum Field Theory", ICMAT, Madrid, 2014, [39, 40].

(involving N^{th} roots of one) and provide a superficial glance at motivic zeta values [4, 13, 14, 17] using them (following [13]) to derive the Zagier formula for the dimensions of the weight spaces of (motivic) MZVs. Finally, in Sect. 7 we give an outlook (and references to) items not treated in the text: single valued and elliptic hyperlogarithms and give, in particular, a glimpse on the recent work of Francis Brown [17] that views the "motivic Feynman periods" as a representation of a "cosmic Galois group" revealing hidden structures of Feynman amplitudes.

2 From Euler's Alternating Series to the Electron Magnetic Moment

Euler's interest in the zeta function and its alternating companion $\phi(s)$,

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad \phi(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} \ \left(= (1 - 2^{1-s})\, \zeta(s) \text{ for } s > 1\right) \tag{2}$$

was triggered by the *Basel problem* [42] (posed by Pietro Mengoli in mid 17 century): to find a closed form expression for $\zeta(2)$. Euler discovered the non-trivial answer, $\zeta(2) = \frac{\pi^2}{6}$, in 1734 and ten years later found an expression for all $\zeta(2n), n = 1, 2, \ldots,$ as a rational multiple of π^{2n}. An elementary (Euler's style) derivation of the first few relations uses the expansion of $\cot(z)$ in simple poles (see [7]):

$$\begin{aligned} z \cot(z) &= 1 - 2\, z^2 \sum_{n=1}^{\infty} \frac{1}{n^2\, \pi^2 - z^2} = 1 - 2 \sum_{n=1}^{\infty} \zeta(2n) \left(\frac{z^2}{\pi^2}\right)^n \\ &= \frac{1 - \frac{z^2}{2} + \frac{z^4}{4!} - \ldots}{1 - \frac{z^2}{3!} + \frac{z^4}{5!} \ldots} \Longrightarrow \zeta(2) = \frac{\pi^2}{6}, \quad \zeta(4) = \frac{\pi^4}{90}, \ldots. \end{aligned} \tag{3}$$

Euler tried to extend the result to odd zeta values but it did not work [22]. (We still have no proof that $\frac{\zeta(3)}{\pi^3}$ is irrational.) Trying to find polynomial relations among zeta values Euler was led by the *stuffle product*

$$\zeta(m)\, \zeta(n) = \zeta(m, n) + \zeta(n, m) + \zeta(n + m) \tag{4}$$

to the concept of multiple zeta values (MZVs):

$$\zeta(n_1, \ldots, n_d) = \sum_{0<k_1<\ldots<k_d} \frac{1}{k_1^{n_1} \ldots k_d^{n_d}}. \tag{5}$$

The alternating series $\phi(s)$ (2) (alias the Dirichlet *eta function*[2]) provide faster convergence in a larger domain. While $\zeta(s)$ has a pole for $s = 1$, we have

$$\phi(1) = \ln 2. \tag{6}$$

Applying the stuffle relation for $\phi^2(1)$:

$$\phi^2(1)(= (\ln 2)^2) = 2\phi(1, 1) + \zeta(2) \tag{7}$$

where

$$\phi(m, n) = \sum_{0<k<\ell} \frac{(-1)^{k+\ell}}{k^m \ell^n} < 0 \tag{8}$$

Euler expressed $\zeta(2)$ in terms of a much faster converging series (eventually guessing and then deriving $\zeta(2) = \frac{\pi^2}{6}$ – see Sect. 3 of [3]):

$$\zeta(2) = \phi(1)^2 - 2\phi(1, 1) = (\ln 2)^2 + \sum_{n=1}^{\infty} \frac{1}{n^2 2^n}$$

and computed it up to six digits ($\zeta(2) \approx 1.644934$).

Remarkably, it is the same ϕ-function which enters the g-factor of the magnetic moment of the electron $\boldsymbol{\mu} = g\,\frac{e}{2m}\,\boldsymbol{s}$ – probably the most precisely measured and calculated quantity in physics [28]. Up to third order in $\frac{\alpha}{\pi}$, where $\alpha = \frac{e^2}{4\pi\hbar c}$ is the fine structure constant, the anomalous magnetic moment $a_e = \frac{g-2}{2}$ is given by [30, 35]:

$$\begin{aligned} a_e = {} & \frac{1}{2}\,\frac{\alpha}{\pi} + \left[\phi(3) - 6\,\phi(1)\,\phi(2) + \phi(2) + \frac{197}{2^4\,3^2}\right]\left(\frac{\alpha}{\pi}\right)^2 \\ & + \left[\frac{2}{3^2}\,(83\,\phi(2)\,\phi(3) - 43\,\phi(5)) - \frac{50}{3}\,\phi(1, 3) + \frac{13}{5}\,\phi(2)^2\right. \\ & \left. + \frac{278}{3}\left(\frac{\phi(3)}{3^2} - 12\,\phi\,(1)\,\phi(2)\right) + \frac{34202}{3^3\,5}\,\phi(2) + \frac{28259}{2^5\,3^4}\right]\left(\frac{\alpha}{\pi}\right)^3 + \dots . \end{aligned} \tag{9}$$

Schwinger's 1947 calculation of the first term $\left(\frac{\alpha}{2\pi}\right)$, contributing a 10^{-3} correction to Dirac's magnetic moment, won him (together with Tomonaga and Feynman) the 1965 Nobel Prize in physics. The second term (of order $\left(\frac{\alpha}{\pi}\right)^2$) was finally correctly calculated only 10 years later (by Peterman and Sommerfield). If Schwinger's work amounted to computing a single 1-loop (triangular) graph with 3 internal lines each, the 2-loop calculation involved 7 graphs, each new loop adding three additional lines

[2]Nowadays the term is usually associated with the *Dedekind η-function* $\eta(\tau) = e^{i\frac{\pi\tau}{12}} \prod_{n=1}^{\infty}(1 - q^n)$, $q = e^{2\pi i\tau}$, defined on the upper half plane τ.

(and as many new integrations in the Schwinger parameters – see [27, 28, 37]). The three-loop calculation involving 72 graphs was completed first numerically (comparing with partial analytic results) by Kinoshita in 1995 and then fully analytically by Laporte and Remiddi a year later. The accuracy of both experimental measurement and theoretical computation (going nowadays beyond 4 loops!) is improving and the results match in a record of 12 significant digits (with uncertainty a part in a trillion):

$$a_e = 1.159652\,180\,91\,(\pm 26) \times 10^{-3} \text{ (experiment)}$$
$$a_e = 1.159652\,181\,13\,(\pm 86) \times 10^{-3} \text{ (theoretical).}$$

In the words of "a spectator" in the "tennis match between theory and experiment" [27] "20-year-long experiments are matched by 30-year-long calculations".

It is hard to overestimate the beauty and the significance of a formula like (9) given the precision with which it is confirmed experimentally. The perturbative expansion is likely to be divergent but is believed to be asymptotic. Individual terms have a meaning of their own, both as special exactly known numbers and as measured quantity (the higher powers of $\frac{\alpha}{\pi}$ provide at least a hundred times smaller contribution).

The ϕ-function appearing in (9) had a more than passing interest for Euler. In a 1740 paper (5 years after publishing his discovery of the formulas for $\zeta(2n)$, $n = 1, 2, \ldots, 6$) he wrote $\zeta(n) = N\pi^n$ indicating that for n-even, N is rational while for n odd he conjectures that N is a function of $\ln 2$ (Sect. 6 of [3]) – a natural conjecture in view of (6). In another paper of 1749 (of his Berlin period) after playing with some divergent series Euler proposes the functional equation for $\phi(s)$ writing: "I shall hazard the following conjecture:

$$\frac{\phi(1-s)}{\phi(s)} = -\frac{\Gamma(s)(2^s - 1)\cos\frac{\pi s}{2}}{(2^{s-1} - 1)\,\pi^s} \tag{10}$$

is true for all s." From here and from (2) the functional equation for $\zeta(s)$, proven by Riemann 110 years later (in 1859), follows immediately. Euler then admits that his earlier conjecture about odd zeta values went astray: "I have already observed that $\phi(n)$ can only be computed for even n. When n is odd all my efforts have been useless up to now." (Sect. 7 of [3].) Euler last returned to the topic in 1772. (He continued doing mathematics – and dictating papers – already completely blind until his death in 1783.)

3 Residues of Primitively Divergent Amplitudes

Let Γ be a connected graph with finite sets $\mathcal{E}$ of *edges* (internal lines) and *vertices* $\mathcal{V}$, such that each edge $e \in \mathcal{E}$ is incident with a pair of different vertices $(v_i, v_j, v_i \neq v_j$ – we do not allow for tadpoles). To each such graph we make correspond a position space Feynman integrand

$$G_\Gamma(\mathbf{x}) = \prod_{e \in \mathcal{E}} G_e(x_{ij}), \quad x_{ij} = x_i - x_j \ (x_i = (x_i^\alpha, \alpha = 1, \ldots, D))$$

$$1 \le i, j \le V, \quad \mathbf{x} = (x^1, \ldots, x^N), \quad N = D(V-1) \tag{11}$$

where i, j label the vertices v_i, v_j, incident with the edge e, $V = |\mathcal{V}|$ is the number of vertices, D is the spacetime dimension ($D = 4, 6, \ldots$). We are in fact just interested in the case $D = 4$. Each *propagator* $G_e(x)$ is assumed to be locally integrable away from the origin. In the Euclidean picture, to be used below (in which square intervals are given by $x^2 = \sum_\alpha (x^\alpha)^2$) the integrands (11) are actually smooth bounded (usually going to zero) at infinity functions away from the *large diagonal* ($x_i = x_j$ for some $i \neq j$). In Minkowski space G_e is, generally, singular on the light cone $x^2 = 0$. The integrand (11) is said to be *ultraviolet* (UV) *convergent* if it is locally integrable (at the diagonal) and hence gives rise to a (unique, tempered) distribution in $\mathbb{R}^N$. Otherwise, it is called (UV) divergent.

A (proper) subgraph γ of Γ is defined to contain a proper subset of vertices of Γ together with the adjacent half edges and to contain every edge in Γ incident with a pair of vertices v_1, v_2 of γ. A divergent integrand G_Γ is said to be *primitively divergent* if for any (connected) subgraph $\gamma \subset \Gamma$ the corresponding integrand G_γ is convergent. In a massless quantum field theory (QFT) in which every propagator $G_e(x)$ is a rational homogeneous function of x,

$$G_e(x) = \frac{p_e(x)}{(x^2)^{\mu_e}}, \quad \mu_e \in \mathbb{N}, \quad p_e(\lambda x) = \lambda^\nu p_e(x) \text{ for } \lambda > 0 \tag{12}$$

($\nu \le 2\mu_e$), there are simple convergence criteria in terms of homogeneity degrees [31].

Proposition-Definition 1 *A homogeneous density $G(\mathbf{x})\, d^N x$ is convergent if its homogeneity degree is (strictly) positive. Otherwise, for*

$$G(\lambda \mathbf{x})\, d^N \lambda x = \lambda^{-\kappa} G(x)\, d^N x \quad (d^N x = dx^1 \ldots dx^N), \quad \kappa \ge 0 \tag{13}$$

it is called superficially divergent with degree of superficial divergence κ.

In a (massless) *scalar* QFT, in which all polynomials $p_e(x)$ are constants, one proves that superficially divergent amplitudes are divergent. For more general spin tensor fields whose propagators have polynomial numerators a superficially divergent amplitude may, in fact, turn out to be convergent (see Sec. 5.2 of [31]).

The following proposition (cf. [31]) serves as a definition of both the *residue* $\operatorname{Res} G$ and of renormalized primitively divergent amplitude $G^\rho(\mathbf{x})$.

Proposition 2 *If $G(\mathbf{x})$ is primitively divergent then for any non-zero smooth semi-norm $\rho(x)$ on $\mathbb{R}^N$ there exists a distribution* $\operatorname{Res} G$ *with support at the origin such that*

$$(\rho(\mathbf{x}))^{2\varepsilon} G(\mathbf{x}) - \frac{1}{\varepsilon} \operatorname{Res} G(\mathbf{x}) = G^{\rho}(\mathbf{x}) + O(\varepsilon) \tag{14}$$

where $G^{\rho}(\mathbf{x})$ is a distribution valued extension of $G(\mathbf{x})$ to $\mathbb{R}^N$. The calculation of the distribution Res G *can be reduced to the case $\kappa = 0$ of a logarithmically divergent amplitude by using the identity*

$$(\operatorname{Res} G)(\mathbf{x}) = \frac{(-1)^{\kappa}}{\kappa!} \partial_{i_1} \ldots \partial_{i_\kappa} \operatorname{Res}(x^{i_1} \ldots x^{i_\kappa} G(\mathbf{x})) \tag{15}$$

where summation is assumed (from 1 *to N) over the repeated indices $i_1, \ldots, i_\kappa$. For a (reduced) G that is homogeneous of degree $-N$ we have*

$$\operatorname{Res} G(\mathbf{x}) = \operatorname{res} G\, \delta(\mathbf{x}) \quad \textit{(whenever } \partial_i(x^i G) = 0 \textit{ for } x \neq 0). \tag{16}$$

Here the numerical residue res G *is given by an integral over the (compact) projective space $\mathbb{P}^{N-1}$:*

$$\operatorname{res} G = \frac{1}{\pi^{N/2}} \int G(\mathbf{x}) \sum_{i=1}^{N} (-1)^{i-1} x^i \, dx^1 \wedge \ldots \wedge \widehat{dx^i} \wedge \ldots \wedge dx^N \tag{17}$$

(the hat over dx^i means that it should be omitted).

We note that for D (and hence N) even $N-1$ is odd so that the space $\mathbb{P}^{N-1}$ is orientable.

Schnetz (Definition-Theorem 2.7 of [35]) gives six equivalent expressions for the *"period of a graph"* (i.e. for the residue of the corresponding amplitude). The statement, formulated for the massless φ^4 theory in $D = 4$ is actually valid for any homogeneous of degree $-N$ $(= D(1 - V)$ – i.e. logarithmic) primitively divergent amplitude. In particular, the residue of a position space integrand $G(x_1, \ldots, x_V)$ in a scalar QFT can be written as an $(N - D)$ dimensional integral

$$\operatorname{res} G = \int \left(\prod_{i=2}^{V-1} \frac{d^D x_i}{\pi^{D/2}} \right) G(e, x_2, \ldots, x_{V-1}, 0) \tag{18}$$

where e is any (D-dimensional) unit vector $e^2 = 1$. For $D > 2$ the Schwinger parameter representation gives a still lower, $(L - 1)$-dimensional, projective integral representation for the residue. (For a 4-point graph in the φ^4 theory the number of internal lines L is related to the number of vertices V by $L = 2(V - 1)$ so that $L - 1 < D(V - 2)$ for $D > 2$.)

In the important special case of a (massless) φ^4 theory in $D = 4$ Schnetz [35, 36] associates with each 4-point graph Γ (i.e. a graph with 4 external half edges incident to four different vertices) a *completed* 4*-regular vacuum graph* $\overline{\Gamma}$. He proves (Proposition 2.6 and Theorem 2.7 of [35]) that all primitive 4-point graphs with the same primitive vacuum completion have the same residues. Moreover, there is a simple criterion allowing to tell when a 4-regular vacuum graph is primitive: namely, if the only way to split it by a four edge cut is by splitting off one vertex. (See examples in Sect. 5.)

4 Conformal 4-Point Functions and Hyperlogarithms

Each primitively divergent 4-point Feynman amplitude in a (classically) conformally invariant QFT defines (upon integration over the internal vertices) a conformally covariant, locally integrable function away from the small diagonal $x_1 = \cdots = x_4$. On the other hand, every four points, $x_1, \ldots, x_4$ can be confined by a conformal transformation to a 2-plane (by sending, say, a point to infinity and another to the origin). Then one can represent each *Euclidean* point x_i by a complex number z_i so that

$$x_{ij}^2 = |z_{ij}|^2 = (z_i - z_j)(\bar{z}_i - \bar{z}_j). \tag{19}$$

In order to make the map $x \to z$ explicit we fix a unit vector $e \in \mathbb{R}^4$ and let n be a variable unit vector orthogonal to e parametrizing a 2-sphere $\mathbb{S}^2$. Then any Euclidean 4-vector x can be written (in spherical coordinates) in the form

$$x = r(\cos\rho\, e + \sin\rho\, n)\,, \quad e^2 = 1 = n^2\,,\ en = 0\,,\ r \geq 0\,,\ 0 \leq \rho \leq \pi. \tag{20}$$

We make correspond to the vector (20) a complex number z such that

$$z = r\, e^{i\rho} \to x^2 (= r^2) = z\bar{z}\,, \quad (x - e)^2 = |1 - z|^2 = (1 - z)(1 - \bar{z}); \tag{21}$$

$$\int_{n \in \mathbb{S}^2} \frac{d^4 x}{\pi^2} = |z - \bar{z}|^2\, \frac{d^2 z}{\pi} \left(\int_{\mathbb{S}^2} \delta(x)\, d^4 x = \delta(z)\, d^2 z \right). \tag{22}$$

The 4-point amplitude with four distinct external vertices in the φ^4 theory has scale dimension 12 (in mass or inverse length units) and can be written in the form

$$G(x_1, \ldots, x_4) = \frac{f(u, v)}{\prod\limits_{i<j} x_{ij}^2} = \frac{F(z)}{\prod\limits_{i<j} |z_{ij}|^2}. \tag{23}$$

Here the (positive real) variables u, v; and the (complex) z are conformally invariant cross ratios

$$u = \frac{x_{12}^2\, x_{34}^2}{x_{13}^2\, x_{24}^2} = z\bar{z}\,, \quad v = \frac{x_{14}^2\, x_{23}^2}{x_{13}^2\, x_{24}^2} = |1 - z|^2\,, \quad z = \frac{z_{12}\, z_{34}}{z_{13}\, z_{24}}. \tag{24}$$

The cross ratios $z, \bar{z}$ are the simplest realizations of the argument of a hyperlogarithmic function whose graded Hopf algebra we proceed to define [10, 12].

Let $\sigma_0 = 0, \sigma_1, \dots, \sigma_N$ be distinct complex numbers corresponding to an alphabet $X = \{e_0, \dots, e_N\}$. Let X^* be the set of (finite) words w in this alphabet, including the empty word $\emptyset$. The hyperlogarithm $L_w(z)$ is an iterated integral [12, 21] defined recursively in any simply connected open subset U of the punctured complex plane $D = \mathbb{C} \backslash \Sigma$, $\Sigma = \{\sigma_0, \dots, \sigma_N\}$ by the *unipotent differential equations*[3]

$$\frac{d}{dz} L_{w\sigma}(z) = \frac{L_w(z)}{z - \sigma}, \quad \sigma \in \Sigma, \quad L_\emptyset = 1, \tag{25}$$

and the initial condition

$$L_w(0) = 0 \text{ for } w \neq 0^n = (\underbrace{0, \dots, 0}_{n}), \quad L_{0^n} = \frac{(\ln z)^n}{n!}. \tag{26}$$

There is a correspondence between iterated integrals and multiple power series; setting $n_i' = n_i - 1$, $k_i' = k_i - 1$ (and assuming $\sigma_i \neq 0$ for $1 \leq i \leq d$) we find

$$(-1)^d L_{0^{n_0}\,\sigma_1\, 0^{n_1'} \dots \sigma_d\, 0^{n_d'}}(z) = \tag{27}$$

$$\sum_{\substack{k_0 \geq 0, k_i \geq n_i \text{ for } i=1,\dots,d \\ k_0+\dots+k_d = n_0+\dots+n_d}} (-1)^{k_0+n_0} \prod_{i=1}^{d} \binom{k_i'}{n_i'} L_{0^{k_0}}(z)\, Li_{k_1 \dots k_r}\left(\frac{\sigma_2}{\sigma_1}, \dots, \frac{\sigma_d}{\sigma_{d-1}}, \frac{z}{\sigma_d}\right)$$

where

$$Li_{k_1 \dots k_r}(z_1, \dots, z_r) = \sum_{0 < m_1 < \dots < m_r} \frac{z_1^{m_1} \dots z_r^{m_r}}{m_1^{k_1} \dots m_r^{k_r}}. \tag{28}$$

The number of letters $|w| = n_0 + \dots + n_d$ of a word w defines its *weight*, while the number d of non-zero letters is its *depth*. The product $L_w L_{w'}$ of two hyperlogarithms of weights $|w|, |w'|$ and depths d, d' can be expanded in hyperlogarithms of weight $|w| + |w'|$ and depth $d + d'$, since the product of simplices can be expanded into a sum of higher dimensional simplices. In fact, the set X^* of words can be equipped with a commutative *shuffle product* $w \sqcup\!\sqcup\, w'$ defined recursively by

$$\emptyset \sqcup\!\sqcup\, w = w (= w \sqcup\!\sqcup\, \emptyset), \quad au \sqcup\!\sqcup\, bv = a(u \sqcup\!\sqcup\, bv) + b(au \sqcup\!\sqcup\, v) \tag{29}$$

where u, v, w are (arbitrary) words while a, b are letters (note that the empty word is *not* a letter). Denote by O_Σ the ring of regular functions on D:

[3] We use, following [13, 36], concatenation to the right. Other authors [8, 24] use the opposite convention.

$$O_\Sigma = \mathbb{C}\left[z, \left(\frac{1}{z-\sigma_\alpha}\right)_{\alpha=0,1,\dots,N}\right]. \tag{30}$$

Extending by O_Σ-linearity the correspondence $w \to L_w$ one proves that it defines a homomorphism of shuffle algebras $O_\Sigma \otimes \mathbb{C}\langle X\rangle \to \mathcal{L}_\Sigma$ where $\mathcal{L}_\Sigma$ is the O_Σ span of L_w, $w \in X^*$. The commutativity of the shuffle product is reflected in the identity

$$L_{u\sqcup v} = L_u\, L_v\ (= L_v\, L_u). \tag{31}$$

If the shuffle relations are suggested by the expansion of products of iterated integrals, the product of series expansions of type (28) suggests another commutative *stuffle product*. We illustrate the corresponding rule on the example of the product of a depth one and a depth two factors:

$$\begin{aligned} Li_{n_1,n_2}(z_1,z_2)\, Li_{n_3}(z_3) = {} & Li_{n_1,n_2,n_3}(z_1,z_2,z_3) + Li_{n_1,n_3,n_2}(z_1,z_3,z_2) \\ & + Li_{n_3,n_1,n_2}(z_3,z_1,z_2) + Li_{n_1,n_2+n_3}(z_1,z_2z_3) \\ & + Li_{n_1+n_3,n_2}(z_1z_3,z_2). \end{aligned} \tag{32}$$

(The corresponding product of words $u = z_1\, 0^{n_1'}\, z_2\, 0^{n_2'}$, $v = z_3\, o^{n_3'}$ is denoted by $u * v$.) Clearly, the stuffle product also respects the weight but only filters the depth (there are terms of depth two and three in the right hand side of (32) always not exceeding the total depth – three – of the left hand side). The shuffle and stuffle products give a number of relations among hyperlogarithms of the same weight. The monodromies of the multivalued hyperlogarithms around the possible singularities for $z = \sigma_\alpha \in \Sigma$ provide more (not easy to find) such relations. The bialgebra structure of hyperlogarithms introduced by Goncharov [26] (see also Theorem 3.8 of [13] and Sect. 5.3 of [24]) allow to reduce the calculation of monodromies and discontinuities of higher weight hyperlogarithms to those of simple logarithms (see e.g. [1]). Here we shall just reproduce the coproduct for the special case of the classical polylogarithm:

$$\Delta\, Li_n(z) = Li_n(z) \otimes 1 + \sum_{k=0}^{n-1} \frac{(\ln z)^k}{k!} \otimes Li_{n-k}(z), \tag{33}$$

the natural logarithm appearing as primitive element,

$$\Delta\, L_\sigma(z) = L_\sigma(z) \otimes 1 + 1 \otimes L_\sigma(z)\,,\ L_\sigma(z) = \ln\left(1-\frac{z}{\sigma}\right) = -Li_1\left(\frac{z}{\sigma}\right) \tag{34}$$

$$\text{for } \sigma \neq 0\,, \quad L_0(z) = \ln z.$$

In order to apply (33) to the specialization to $z = 1$, $Li_n(1) = \zeta(n)$ for n even we need to quotient the algebra of hyperlogarithms by $\zeta(2)$ or, better, by $\ln(-1) = i\pi$ ($= \sqrt{-6\,\zeta(2)}$) introducing the Hopf algebra $\mathcal{H}$:

$$\mathcal{H} := \mathcal{L}_\Sigma / i\pi\, \mathcal{L}_\Sigma \quad \text{so that} \quad \mathcal{L}_\Sigma = \mathcal{H}[i\pi]. \tag{35}$$

(Otherwise the relation $\zeta(4) = \frac{2}{5}\,\zeta^2(2)$ would not be respected by the coproduct Δ satisfying, according to (33), $\Delta\zeta(n) = \zeta(n) \otimes 1 + 1 \otimes \zeta(n)$.)

The coaction Δ is extended to $\mathcal{L}_\Sigma$ by

$$\Delta : \mathcal{L}_\Sigma \to \mathcal{H} \otimes \mathcal{L}_\Sigma\,, \quad \Delta(i\pi) = 1 \otimes i\pi. \tag{36}$$

The asymmetry of the coproduct is reflected on its relation to differentiation and to discontinuity $\mathrm{dsic}_\sigma = M_\sigma - 1$ (where M_σ stands for the monodromy around $z = \sigma$):

$$\Delta\left(\frac{\partial}{\partial z} F\right) = \left(\frac{\partial}{\partial z} \otimes \mathrm{id}\right) \Delta F\,, \quad \Delta(\mathrm{disc}_\sigma\, F) = (\mathrm{id} \otimes \mathrm{disc}_\sigma)\Delta F. \tag{37}$$

(One easily verifies, for instance, that both sides of (37) give the same result for $F = Li_2(z)$.) This allows us to consider $\mathcal{L}_\Sigma$ as a *differential graded Hopf algebra*.

The resulting structure allows to read off the symmetry properties of hyperlogarithms from the simpler properties of ordinary logarithms, as illustrated in Example 25 of [24] which begins with a derivation of the inversion formula for the dilog:

$$Li_2\left(\frac{1}{x}\right) = i\pi \ln x - Li_2(x) - \frac{1}{2}\ln^2 x + 2\zeta(2).$$

5 Multiple Zeta Values and Feynman Periods

The *multiple zeta values* (MZVs) are the values of the hyperlogarithms (28) at arguments equal to one:

$$\zeta(n_1, \ldots, n_d) = Li_{n_1 \ldots n_d}(1, \ldots, 1) = (-1)^d\, \zeta_{10^{n'_1} \ldots 10^{n'_d}} \quad (n'_i = n_i - 1) \tag{38}$$

(cf. (27)). The corresponding series is convergent for $n_d > 1$. In order to recover the known relations among MZVs of the same weight one needs along with the shuffle and stuffle products of *convergent words* also a relation involving multiplication with the "divergent word" e_1 (in the case of a 2-letter alphabet, $\Sigma = \{0, 1\}$, $X = \{e_0, e_1\}$):

$$\zeta_{u ш v} = \zeta_u\, \zeta_v = \zeta_{u*v}\,, \quad \zeta(e_1 ш w - e_1 * w) = 0 \tag{39}$$

for all convergent words u, v and w. We note that the divergent words (with $n_d = 1$) in the last Eq. (39) cancel out. For instance, setting $(n) = -e_1\, e_0^{n-1}$, we find

$$\zeta((1) ш (n) - (1) * (n)) = \sum_{i=1}^{n-1} \zeta(i, n+1-i) - \zeta(n+1) = 0, \quad n \geq 2 \tag{40}$$

a relation known to Euler. (Already for $n = 2$ the resulting formula, $\zeta(1, 2) = \zeta(3)$, is nontrivial.) Setting $(-\zeta_1 =)\zeta(1) = 0$ and using the relations (39) one can write the *generating* series Z of MZVs (also called *Drinfeld's associator*) in terms of multiple commutators of e_0, e_1:

$$Z = L(1) = 1 + \zeta(2)[e_0, e_1] + \zeta(3)\,[[e_0, e_1], e_0 + e_1] + \ldots,$$

$$\text{for } L(z) = 1 + \ln z\, e_0 + \ln(1 - z)\, e_1 + \sum_{|w|\geq 2} L_w(z)\, w \quad (w \in X^*, X = \{e_0, e_1\}). \tag{41}$$

(The limit of $L(z)$ for $z \to 1$ in the expression for Z involves a regularization so that the divergent series for $-\zeta_1 = \lim_{z\to 1} \sum_{n=1}^{\infty} \frac{z^n}{n}$ is substituted by 0.) The generating series (41) allow to express in a compact form the monodromy of the multipolylogarithms $L_w(z)$ around the (possible) singularities at $z = 0$ and $z = 1$:

$$\begin{aligned} \mathcal{M}_0 L(z) &= e^{2\pi i e_0} L(z), \\ \mathcal{M}_1 L(z) &= Z\, e^{2\pi i e_1} Z^{-1} L(z). \end{aligned} \tag{42}$$

(In writing Z^{-1} we observe that any formal power series starting with 1 is invertible.)

There are infinitely many primitive vacuum graphs in the (massless) φ^4 theory. They are completions of 4-point graphs with increasing number of loops $\ell = 3, 4, \ldots$ Their periods, up to $\ell = 6$, are MZVs of weight not exceeding $2\ell - 3$. Broadhurst and Kreimer [9] discovered a remarkable sequence of *zig-zag graphs* whose periods are rational multiples of $\zeta(2\ell - 3)$. Their completions are n-vertex vacuum graphs $\overline{\Gamma}_n^{(2)}$ $(n = \ell + 2)$ which admit a *hamiltonian cycle* that passes through all vertices in consecutive order in such a way that each vertex i is also connected with $i \pm 2$ mod n (see Fig. 1a for the $n = 8$ graph).

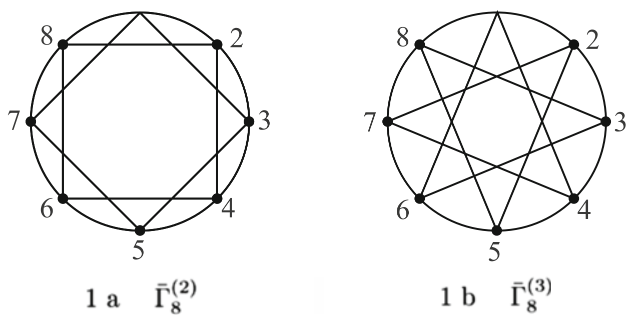

Fig. 1 Eight point vacuum completions of six-loop 4-point graphs in φ^4 : Per $(\overline{\Gamma}_8^{(2)}) = 24\,\zeta(9)$, Per $(\overline{\Gamma}_8^{(3)}) = 32 P_{3,5}$

Their periods depend on the parity of ℓ:

$$\begin{aligned}\text{Per}\left(\overline{\Gamma}^{(2)}_{\ell+2}\right) &= \frac{4-4^{3-\ell}}{\ell}\binom{2\ell-2}{\ell-1}\zeta(2\ell-3) \quad \text{for} \quad \ell = 3, 5, \ldots \\ &= \frac{4}{\ell}\binom{2\ell-2}{\ell-1}\zeta(2\ell-3) \quad \text{for} \quad \ell = 4, 6, \ldots \end{aligned} \tag{43}$$

(a result conjectured in [9] and proven in [18]). The 8-vertex graph on Fig. 1b also admits a hamiltonian cycle but with vertices i connected with $i \pm 3 \mod 8$. Its period, computed numerically in [9], is the first that involves a double zeta value:

$$\text{Per}\left(\overline{\Gamma}^{(2)}_{8}\right) = 32\, P_{3,5} = 288\left\{\frac{2}{5}\left[29\,\zeta(8) - 12\,\zeta(3,5)\right] - 9\,\zeta(3)\,\zeta(5)\right\}. \tag{44}$$

(The notation $P_{3,5}$ conforms with that of Brown [17].) The first Feynman period, not expressible as a rational linear combination of MZVs was identified at 7 loops by E. Panzer [32] (following suggestions by Broadhurst and Schnetz) in 2014, as rational linear combination of hyperlogarithms at sixth roots of 1 (called *multiple Deligne values* in [8]). The 9-vertex vacuum completion of the graph in question is of type $F_9^{(3)}$: it again admits a hamiltonian cycle with hords joining vertices congruent mod 3 (as in the graph $\overline{\Gamma}^{(3)}_{8}$ displayed on Fig. 1b).

6 Generalized and Motivic MZVs

Remarkably, MZVs ζ_w labeled by words in the $(N+1)$-point alphabet $X = \{e_0, e_1, \ldots, e_N\}$ corresponding to $\Sigma = \{0, 1, \lambda, \ldots, \lambda^{N-1}\}$ where λ is a primitive N^{th} root of unity again close a *double shuffle* (i.e. a shuffle and a stuffle) *algebra* and hence represent a natural generalization of the classical MZVs. In particular, the Euler ϕ-function corresponds to $\Sigma = \{0, 1, -1\}$:

$$\phi(n) = L_{-10^{n'}}(1) = -Li_n(-1) \qquad (n' = n-1).$$

Given the many relations these generalized MZVs ζ_w satisfy, the question arises to find a basis of such periods independent over the rationals. This question is wide open even for the classical MZVs (for which w is a word in the two letter alphabet corresponding to $\Sigma = \{0, 1\}$). We know the relations coming from (39) but have no proof that there are no more relations for weights $|w| > 4$. If we denote by d_n the dimension of the space of MZVs of weight $|w| = n$ we only know that $d_1 = 0$, $d_2 = d_3 = d_4 = 1$. (The reader is invited to verify – using the relations (39) and (40) – that all MZVs of weight 4 are integer multiples of $\zeta(1,3) = \frac{\pi^4}{360}$ – see Eq. (B.8) of [38].) We do not even have a proof that $\zeta(5)$ is irrational. A way to get around the resulting (difficult!) problem amounts to substitute the real MZVs by some abstract objects as *formal MZVs* (defined by the relations (39) – see [34, 39]) and *motivic zeta values* [8, 13] whose application for calculating the dimensions d_n of the motivic MZVs of weight n we proceed to sketch.

Consider the *concatenation algebra* $\mathcal{C}$ defined as the free algebra over the rational numbers $\mathbb{Q}$ on the countable alphabet $\{f_3, f_5, \ldots\}$. The algebra of *motivic* MZVs is identified (non-canonically) with the algebra $\mathcal{C}[f_2]$ of polynomials in a single variable f_2 with coefficients in $\mathcal{C}$:

$$\mathcal{C}[f_2] = \mathcal{C} \otimes_{\mathbb{Q}} \mathbb{Q}[f_2], \quad \mathcal{C} = \mathbb{Q}\langle f_3, f_5, \ldots \rangle. \tag{45}$$

The algebra $\mathcal{C}[f_2]$ is graded by the weight (the sum of indices of f_i) and it is straightforward to compute the dimensions d_n of the weight spaces $\mathcal{C}[f_2]_n$. Indeed, the generating *(Hilbert-Poincaré)* series for the dimensions $d_n^{\mathcal{C}}$ of the weight n subspaces of $\mathcal{C}$ is given by

$$\sum_{n\geq 0} d_n^{\mathcal{C}}\, t^n = \frac{1}{1-t^3-t^5-\ldots} = \frac{1-t^2}{1-t^2-t^3}, \tag{46}$$

while the generating series of $\mathbb{Q}[f_2]$ is $(1-t^2)^{-1}$. Multiplying the two series we obtain the dimensions d_n of the weight spaces of (motivic) MZVs conjectured by Don Zagier:

$$\sum_{n\geq 0} d_n\, t^n = \frac{1}{1-t^2-t^3} \Rightarrow d_0 = 1, d_1 = 0, d_2 = 1, d_{n+2} = d_n + d_{n-1}. \tag{47}$$

The concatenation algebra $\mathcal{C}$ can be equipped with a Hopf algebra structure (with f_i as primitive elements) with the *deconcatenation coproduct* $\Delta : \mathcal{C} \to C \otimes C$ given by

$$\Delta(f_{i_1} \ldots f_{i_r}) = 1 \otimes f_{i_1} \ldots f_{i_r} + f_{i_1 \ldots i_r} \otimes 1 + \sum_{k=1}^{r-1} f_{i_1} \ldots f_{i_k} \otimes f_{i_{k+1}} \ldots f_{i_r}. \tag{48}$$

The coproduct can be extended to the trivial comodule $\mathcal{C}[f_2]$ (45) by setting

$$\Delta\mathcal{C}[f_2] \to \mathcal{C} \otimes \mathcal{C}[f_2], \quad \Delta(f_2) = 1 \otimes f_2 \tag{49}$$

(and assuming that f_2 commutes with f_{odd}). There exists a surjective *period map* of $\mathcal{C}[f_2]$ onto the space $\mathcal{Z}$ of real MZVs. The *main conjecture* in the theory of MZVs says that the period map is also injective, – i.e., it defines an isomorphism of graded algebras. If true it would imply that the infinite sequence of numbers $\pi, \zeta(3), \zeta(5), \ldots$ are transcendentals algebraically independent over the rationals [41]. It would also give $\dim Z_n = d_n$. Presently, we only know that

$$\dim Z_n \leq d_n \quad (\dim Z_n = d_n \text{ for } n \leq 4). \tag{50}$$

For weights $n \leq 7$ one can express all MZVs in terms of (products of) simple (depth 1) zeta values. For $n \geq 8$ this is no longer possible (as illustrated by the presence of $\zeta(3,5)$ in (44)). Brown [14] has established that the *Hoffman elements* $\zeta(n_1, \ldots, n_d)$ with $n_i \in \{2,3\}$ form a basis of motivic zeta values for all weights (see also [22, 41]).

7 Outlook

We shall sketch in this concluding section three complementary lines of development in the topic of our review.

The first goes in the direction of further specializing the class of functions and associated numbers (periods) appearing in the Feynman amplitudes of interest. Euclidean picture conformal amplitudes are *singlevalued* (as argued in [23, 25]). Knowing the monodromy (42) one can construct a shuffle algebra of *single valued hyperlogarithms* [10, 11] which belong to the tensor product $\overline{\mathcal{L}}_\Sigma \otimes \mathcal{L}_\Sigma$ of hyperlogarithms and their complex conjugates (see also [38, 39] for lightened reviews). The resulting functions and numbers, [15, 36], are also encountered in superstring calculations (for a review see [20]).

A second trend proceeds to considering massive Feynman amplitudes as well as higher order massless amplitudes which requires extending the family of functions (and periods) of interest. The new functions appearing in the *sunrise* (or *sunset*) graph in two and four dimensions are *elliptic hyperlogarithms* (for recent reviews and further references – see [2, 5, 33]). Modular forms and associated L-functions are also expected to play a role [7, 16, 19].

We do not touch another lively development championed by Goncharov and a group of physicists who also proceed to extending the mathematical tools – using, in particular, *cluster algebras* – in order to describe multileg amplitudes in $N = 4$ supersymmetric Yang-Mills theory (for a review and references – see [40]).

A third approach attempts to reveal structures common to all Feynman amplitudes. Brown [17] gives a new meaning of the notion of *cosmic Galois group* (a term introduced by Cartier in 1998) *of motivic periods*: it is associated with the family of graphs with a fixed number of external lines and a fixed maximal number of different masses. Thus $\mathcal{C}_{4,0}$ is the cosmic Galois group associated with the 4-point amplitudes in a massless (say φ^4-) theory). Every Feynman amplitude of this class defines canonically a motivic period that gives rise to a finite dimensional representation of $\mathcal{C}_{4,0}$. One can associate a weight to motivic periods that generalizes the weight of MZVs. Brown proves that the space of motivic Feynman periods of a given type (say, the type (4, 0) above) of weight not exceeding k is finite dimensional (Theorem 5.2 of [17]). This theorem allows to predict the type of periods of a given weight in amplitudes of any order. An illustration of what this means is the observation by Schnetz [35] that the combination $\frac{2}{5}\,[29\zeta(8) - 12\zeta(3,5)]$ of the period of the six loop graph corresponding to $\overline{\Gamma}_8^{(3)}$ (see (44)) also appears in a 7 loop period (multiplied by $252\zeta(3)$ – see Eq. (6.2) of [17]). Brown remarks that the motivic version of the anomalous magnetic moment of the electron $a = \frac{g-2}{2}$ (Sect. I) also displays some compatibility with the action of the cosmic Galois group on periods.

Acknowledgements It is a pleasure to thank V. Dobrev for the invitation to Varna and for his careful editorial work. I thank IHES and the Theoretical Physics Department of CERN for hospitality during the preparation of this text. The author's work is supported in part by Grant DFNI T02/6 of the Bulgarian National Science Foundation.

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The Parastatistics Fock Space and Explicit Infinite-Dimensional Representations of the Lie Superalgebra $\mathfrak{osp}(2m+1|2n)$

N.I. Stoilova and J. Van der Jeugt

Abstract The defining triple relations of m pairs of parafermion operators $f_i^\pm$ and n pairs of paraboson operators $b_j^\pm$ with relative parafermion relations can be considered as defining relations for the Lie superalgebra $\mathfrak{osp}(2m+1|2n)$ in terms of $2m+2n$ generators. As a consequence of this the parastatistics Fock space of order p corresponds to an infinite-dimensional unitary irreducible representation $\mathfrak{V}(p)$ of $\mathfrak{osp}(2m+1|2n)$, with lowest weight $(-\frac{p}{2},\ldots,-\frac{p}{2}|\frac{p}{2},\ldots,\frac{p}{2})$. An explicit construction of the representations $\mathfrak{V}(p)$ is given for any m and n, as well as the computation of matrix elements of the $\mathfrak{osp}(2m+1|2n)$ generators.

1 Introduction

Standard quantum mechanics considers two types of particles, bosons $B_j^\pm$ ($[a,b]=ab-ba$)

$$[B_j^-, B_l^+] = \delta_{jl}, \qquad [B_j^-, B_l^-] = [B_j^+, B_l^+] = 0, \tag{1}$$

and fermions $F_i^\pm$ ($\{a,b\}=ab+ba$)

$$\{F_i^-, F_k^+\} = \delta_{ik}, \qquad \{F_i^-, F_k^-\} = \{F_i^+, F_k^+\} = 0, \tag{2}$$

and the corresponding quantum statistics, Bose-Einstein and Fermi-Dirac statistics. The n-boson Fock space with vacuum vector $|0\rangle$ satisfies

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V. Dobrev (ed.), *Lie Theory and Its Applications in Physics*, Springer Proceedings in Mathematics & Statistics 191,
DOI 10.1007/978-981-10-2636-2_11

$$\langle 0|0\rangle = 1, \qquad B_j^-|0\rangle = 0, \quad (B_j^\pm)^\dagger = B_j^\mp \tag{3}$$

and the other orthogonal normalized basis vectors are defined by

$$|k_1,\ldots,k_n\rangle = \frac{(B_1^+)^{k_1}\cdots(B_n^+)^{k_n}}{\sqrt{k_1!\cdots k_n!}}|0\rangle, \qquad k_1,\ldots,k_n \in \mathbb{Z}_+. \tag{4}$$

Similarly, the m-fermion Fock space is defined by

$$\langle 0|0\rangle = 1, \qquad F_i^-|0\rangle = 0, \qquad (F_i^\pm)^\dagger = F_i^\mp \qquad (i=1,\ldots,m). \tag{5}$$

and the basis vectors are as follows

$$|\theta_1,\ldots,\theta_m\rangle = (F_1^+)^{\theta_1}\cdots(F_m^+)^{\theta_m}|0\rangle, \qquad \theta_1,\ldots,\theta_m \in \{0,1\}. \tag{6}$$

Bose-Einstein and Fermi-Dirac statistics were generalized by Green [3] in 1953. He has shown that tensor fields can be quantized with creation and annihilation operators $b_j^\pm$ (parabosons), which satisfy the triple relations

$$[\{b_j^\xi, b_k^\eta\}, b_l^\epsilon] = (\epsilon-\xi)\delta_{jl}b_k^\eta + (\epsilon-\eta)\delta_{kl}b_j^\xi, \tag{7}$$

whereas for spinor fields he has introduced parafermions $f_j^\pm$ postulating the commutation relations

$$[[f_j^\xi, f_k^\eta], f_l^\epsilon] = \frac{1}{2}(\epsilon-\eta)^2\delta_{kl}f_j^\xi - \frac{1}{2}(\epsilon-\xi)^2\delta_{jl}f_k^\eta, \tag{8}$$

where $j,k,l \in \{1,2,\ldots\}$ and $\eta,\epsilon,\xi \in \{+,-\}$ (or, in the algebraic expressions, $\eta,\epsilon,\xi \in \{+1,-1\}$). The paraboson Fock space $V(p)$ is the Hilbert space with vacuum vector $|0\rangle$, defined by means of

$$\begin{aligned} &\langle 0|0\rangle = 1, \qquad b_j^-|0\rangle = 0, \qquad (b_j^\pm)^\dagger = b_j^\mp, \\ &\{b_j^-, b_k^+\}|0\rangle = p\,\delta_{jk}\,|0\rangle, \end{aligned} \tag{9}$$

and by irreducibility under the action of the algebra spanned by the elements b_j^+, b_j^-, subject to (7). In the same way, the parafermion Fock space $W(p)$ is the Hilbert space with unique vacuum vector $|0\rangle$, defined by

$$\begin{aligned} &\langle 0|0\rangle = 1, \qquad f_j^-|0\rangle = 0, \qquad (f_j^\pm)^\dagger = f_j^\mp, \\ &[f_j^-, f_k^+]|0\rangle = p\,\delta_{jk}\,|0\rangle, \end{aligned} \tag{10}$$

and by irreducibility under the action of the algebra spanned by the elements f_j^+, f_j^-, subject to (8). In both cases the parameter p is known as the order of the corresponding para system. For $p = 1$ the paraboson (parafermion) Fock space coincides with the boson (fermion) Fock space. The paraboson and parafermion Fock spaces can in principle be constructed by the so-called Green ansatz [3]. However the explicit construction of these para Fock spaces has been an open problem for many years because of the difficulties of finding a proper basis of an irreducible constituent of a p-fold tensor product [4]. In recent papers [8, 12, 13], these problems of giving complete constructions of the paraboson and parafermion Fock spaces were solved. The solutions rely on the facts that paraboson and parafermion statistics are incorporated into algebraic structures. More precisely, a finite set of parafermions $f_j^{\pm}, i = 1, 2, \ldots, m$ subject to the parafermion relations (8) defines the Lie algebra $\mathfrak{so}(2m+1)$ by means of generators and relations [7, 11]. The Fock space $W(p)$ is the unitary irreducible representation of $\mathfrak{so}(2m+1)$ with lowest weight $(-\frac{p}{2}, -\frac{p}{2}, \ldots, -\frac{p}{2})$. In a similar way, n paraboson operators $b_j^{\pm}$ subject to (7) are generating elements of the orthosymplectic Lie superalgebra $\mathfrak{osp}(1|2n)$ [2]. The Fock space $V(p)$ is the unitary irreducible representation of $\mathfrak{osp}(1|2n)$ with lowest weight $(\frac{p}{2}, \frac{p}{2}, \ldots, \frac{p}{2})$. If one considers an infinite number of parafermions (parabosons) the creation and annihilation operators generate the infinite-dimensional algebra $\mathfrak{so}(\infty)$ (superalgebra $\mathfrak{osp}(1|\infty)$) [13].

In the case of a mixed system consisting of parafermions $f_j^{\pm}$ and parabosons $b_j^{\pm}$ the relative commutation relations among paraoperators were studied by Greenberg and Messiah [4]. They have shown that there can exist at most four types of relative commutation relations: straight commutation, straight anticommutation, relative paraboson, and relative parafermion relations and the most interesting case is the latter one. Palev [10] proved that m parafermions $f_j^{\pm}$ (8) and n parabosons $b_j^{\pm}$ (7) with relative parafermion relations generate the orthosymplectic Lie superalgebra $\mathfrak{osp}(2m+1|2n)$. Therefore the parastatistics Fock space corresponds to an infinite-dimensional unitary representation of $\mathfrak{osp}(2m+1|2n)$. For its explicit construction, the techniques developed in [8, 12] can be applied, namely the branching $\mathfrak{osp}(2m+1|2n) \supset \mathfrak{gl}(m|n)$, an induced module construction, a basis description for the covariant tensor representations of $\mathfrak{gl}(m|n)$ [14], Clebsch-Gordan coefficients of $\mathfrak{gl}(m|n)$ [14], and the method of reduced matrix elements.

In Sect. 2, we define m parafermions $f_j^{\pm}$ (8) and n parabosons $b_j^{\pm}$ (7) with relative parafermion relations and the parastatistics Fock space $\mathfrak{V}(p)$. In Sect. 3, we consider the important relation between parastatistics operators and the Lie superalgebra $\mathfrak{osp}(2m+1|2n)$, and give a description of $\mathfrak{V}(p)$ in terms of representations of $\mathfrak{osp}(2m+1|2n)$. The rest of this section is devoted to the analysis of the representations $\mathfrak{V}(p)$ for $\mathfrak{osp}(2m+1|2n)$ and to the matrix elements for any m and n. These matrix elements were recently computed [15]. We conclude the paper with some final remarks.

2 The Parastatistics Algebra and Its Fock Space $\mathfrak{V}(p)$

Consider a system of m pairs of parafermions $f_i^\pm \equiv c_i^\pm$, $i = 1, \ldots, m$ and n pairs of parabosons $b_j^\pm \equiv c_{m+j}^\pm$, $j = 1, \ldots, n$ with relative parafermion relations among them. The defining triple relations for such a system are given by

$$\begin{aligned} &[\![[\![c_j^+, c_k^-]\!], c_l^+]\!] = 2\delta_{kl} c_j^+, \quad [\![[\![c_j^+, c_k^+]\!], c_l^+]\!] = 0, \\ &[\![c_j^-, [\![c_k^+, c_l^-]\!]]\!] = 2\delta_{jk} c_l^-, \quad [\![[\![c_j^-, c_k^-]\!], c_l^-]\!] = 0 \end{aligned} \tag{11}$$

or

$$\begin{gathered} [\![[\![c_j^\xi, c_k^\eta]\!], c_l^\epsilon]\!] = -2\delta_{jl}\delta_{\epsilon,-\xi}\epsilon^{\langle l\rangle}(-1)^{\langle k\rangle\langle l\rangle} c_k^\eta + 2\epsilon^{\langle l\rangle}\delta_{kl}\delta_{\epsilon,-\eta} c_j^\xi, \\ \xi, \eta, \epsilon = \pm \text{ or } \pm 1; \quad j, k, l = 1, \ldots, n+m, \end{gathered} \tag{12}$$

where

$$[\![a, b]\!] = ab - (-1)^{\deg(a)\deg(b)} ba \tag{13}$$

and

$$\deg(c_i^\pm) \equiv \langle i\rangle = \begin{cases} 0 & \text{if } \ j = 1, \ldots, m \\ 1 & \text{if } \ j = m+1, \ldots, n+m. \end{cases} \tag{14}$$

In the case $j, k, l = 1, \ldots, m$ (12) reduces to (8) and in the case $j, k, l = m + 1, \ldots, m + n$ (12) reduces to (7).

The parastatistics Fock space $\mathfrak{V}(p)$ is the Hilbert space with vacuum vector $|0\rangle$, defined by means of $(j, k = 1, 2, \ldots, m+n)$

$$\begin{aligned} &\langle 0|0\rangle = 1, \qquad c_j^-|0\rangle = 0, \qquad (c_j^\pm)^\dagger = c_j^\mp, \\ &[\![c_j^-, c_k^+]\!]|0\rangle = p\delta_{jk}|0\rangle, \end{aligned} \tag{15}$$

and by irreducibility under the action of the algebra spanned by the elements c_j^+, c_j^-, $j = 1, \ldots, m+n$, subject to (12). The parameter p is referred to as the order of the parastatistics system.

In 1982 Palev [10] proved the following theorem.

Theorem 1 (Palev) *The Lie superalgebra generated by* $2m$ *even elements* $f_i^\pm \equiv c_i^\pm$ *($i = 1, \ldots, m$) and* $2n$ *odd elements* $b_j^\pm \equiv c_{m+j}^\pm$ *($j = 1, \ldots, n$) subject to the relations (12) is the orthosymplectic Lie superalgebra* $\mathfrak{osp}(2m+1|2n)$*. The Fock space* $\mathfrak{V}(p)$ *is the unitary irreducible representation of* $\mathfrak{osp}(2m+1|2n)$ *with lowest weight* $(-\frac{p}{2}, \ldots, -\frac{p}{2}|\frac{p}{2}, \ldots, \frac{p}{2})$.

Constructing a basis for the parastatistics Fock space $\mathfrak{V}(p)$ for general (integer) p-values turns out to be a difficult problem, for which we describe the solution in the rest of the paper.

3 The Lie Superalgebras $\mathfrak{osp}(2m+1|2n)$ and a Class of $\mathfrak{osp}(2m+1|2n)$ Explicit Representations

The orthosymplectic Lie superalgebra $B(m|n) \equiv \mathfrak{osp}(2m+1|2n)$ [5] consists of $(2m+2n+1 \times 2m+2n+1)$ matrices of the form

$$\begin{pmatrix} a & b & u & x & x_1 \\ c & -a^t & v & y & y_1 \\ -v^t & -u^t & 0 & z & z_1 \\ y_1^t & x_1^t & z_1^t & d & e \\ -y^t & -x^t & -z^t & f & -d^t \end{pmatrix}, \tag{16}$$

with a any $(m \times m)$-matrix, b and c antisymmetric $(m \times m)$-matrices, u and v $(m \times 1)$-matrices, x, y, x_1, y_1 $(m \times n)$-matrices, z and z_1 $(1 \times n)$-matrices, d any $(n \times n)$-matrix, and e and f symmetric $(n \times n)$-matrices. The even elements have $x = y = x_1 = y_1 = 0, z = z_1 = 0$ and the odd elements are those with $a = b = c = 0$, $u = v = 0$, $d = e = f = 0$. Denote the row and column indices running from 1 to $2m+2n+1$ and by e_{ij} the matrix with zeros everywhere except a 1 on position (i, j). The Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{osp}(2m+1|2n)$ is the subspace of diagonal matrices with basis $h_i = e_{ii} - e_{i+m,i+m}$ $(i = 1, \ldots, m)$, $h_{m+i} = e_{2m+1+j,2m+1+j} - e_{2m+1+n+j,2m+1+n+j}$ $(j = 1, \ldots, n)$. Denote by ϵ_i $(i = 1, \ldots, m)$, δ_j $(j = 1, \ldots, n)$ the dual basis of $\mathfrak{h}^*$.

Introducing the following multiples of the even vectors with roots $\pm\epsilon_j$ $(j = 1, \ldots, m)$

$$\begin{aligned} c_j^+ = f_j^+ &= \sqrt{2}(e_{j,2m+1} - e_{2m+1,j+m}), \\ c_j^- = f_j^- &= \sqrt{2}(e_{2m+1,j} - e_{j+m,2m+1}), \end{aligned} \tag{17}$$

and of the odd vectors with roots $\pm\delta_j$ $(j = 1, \ldots, n)$

$$\begin{aligned} c_{m+j}^+ = b_j^+ &= \sqrt{2}(e_{2m+1,2m+1+n+j} + e_{2m+1+j,2m+1}), \\ c_{m+j}^- = b_j^- &= \sqrt{2}(e_{2m+1,2m+1+j} - e_{2m+1+n+j,2m+1}), \end{aligned} \tag{18}$$

it is easy to verify that these operators satisfy the triple relations (12).

The operators c_j^+ are positive root vectors, and the c_j^- are negative root vectors.

We are interested in the construction of the parastatistics Fock space $\mathfrak{V}(p)$ defined by (15). It is straightforward to see that

$$[c_i^-, c_i^+] = -2h_i \ (i = 1, \ldots, m), \quad \text{and} \quad \{c_{m+j}^-, c_{m+j}^+\} = 2h_{m+j} \ (j = 1, \ldots, n). \tag{19}$$

Therefore indeed Theorem 1 holds.

In general the representations $\mathfrak{V}(p)$ can be constructed using an induced module procedure (see [15] for more details). The relevant subalgebras of $\mathfrak{osp}(2m+1|2n)$ are as follows.

Proposition 1 *A basis for the even subalgebra* $\mathfrak{so}(2m+1) \oplus \mathfrak{sp}(2n)$ *of* $\mathfrak{osp}(2m+1|2n)$ *is given by*

$$[c_i^\xi, c_k^\eta],\ c_l^\epsilon\ (i,k,l = 1,\ldots,m); \quad \{c_{m+j}^\xi, c_{m+s}^\eta\}\ (j,s = 1,\ldots,n,\ \xi,\eta = \pm). \tag{20}$$

The elements

$$[\![c_j^+, c_k^-]\!] \qquad (j,k = 1,\ldots,m+n) \tag{21}$$

constitute a basis for the subalgebra $\mathfrak{u}(m|n)$.

Note that with the notation $\frac{1}{2}[\![c_j^+, c_k^-]\!] \equiv E_{jk}$, the triple relations (12) imply the relations

$$[\![E_{ij}, E_{kl}]\!] = \delta_{jk}E_{il} - (-1)^{deg(E_{ij})deg(E_{kl})}\delta_{li}E_{kj}. \tag{22}$$

Therefore, the elements $[\![c_j^+, c_k^-]\!]$ form, up to a factor 2, the standard basis elements of $\mathfrak{u}(m|n)$ or $\mathfrak{gl}(m|n)$.

The subalgebra $\mathfrak{u}(m|n)$ can be extended to a parabolic subalgebra $\mathcal{P}$ of $\mathfrak{osp}(2m+1|2n)$

$$\mathcal{P} = \text{span}\{c_j^-, [\![c_j^+, c_k^-]\!], [\![c_j^-, c_k^-]\!] \mid j,k = 1,\ldots,m+n\}. \tag{23}$$

Because of the fact that $[\![c_j^-, c_k^+]\!]|0\rangle = p\,\delta_{jk}\,|0\rangle$, with $[c_i^-, c_i^+] = -2h_i$ $(i = 1,\ldots,m,)$ and $\{c_{m+j}^-, c_{m+j}^+\} = 2h_{m+j}$ $(j = 1,\ldots,n)$, the space spanned by $|0\rangle$ is a trivial one-dimensional $\mathfrak{u}(m|n)$ module $\mathbb{C}|0\rangle$ of weight $(-\frac{p}{2},\ldots,-\frac{p}{2}|\frac{p}{2},\ldots,\frac{p}{2})$. As $c_j^-|0\rangle = 0$, the $\mathfrak{u}(m|n)$ module $\mathbb{C}|0\rangle$ can be extended to a one-dimensional $\mathcal{P}$ module. The induced $\mathfrak{osp}(2m+1|2n)$ module $\overline{\mathfrak{V}}(p)$ is defined by

$$\overline{\mathfrak{V}}(p) = \text{Ind}_{\mathcal{P}}^{\mathfrak{osp}(2m+1|2n)}\mathbb{C}|0\rangle. \tag{24}$$

This is an $\mathfrak{osp}(2m+1|2n)$ representation with lowest weight $(-\frac{p}{2},\ldots,-\frac{p}{2}|\ \frac{p}{2},\ldots,\frac{p}{2})$. By the Poincaré-Birkhoff-Witt theorem [6], a basis for $\overline{\mathfrak{V}}(p)$ is given by

$$\begin{aligned}
&(c_1^+)^{k_1}\cdots(c_{m+n}^+)^{k_{m+n}}([\![c_1^+, c_2^+]\!])^{k_{12}}([\![c_1^+, c_3^+]\!])^{k_{13}}\cdots\\
&\quad\cdots([\![c_{m+n-1}^+, c_{m+n}^+]\!])^{k_{m+n-1,m+n}}|0\rangle,\\
&\quad k_1,\ldots,k_{m+n}, k_{12}, k_{13}\ldots, k_{m-1,m}, k_{m+1,m+2}, k_{m+1,m+3}\ldots,\\
&\quad k_{m+n-1,m+n} \in \mathbb{Z}_+,\\
&\quad k_{1,m+1}, k_{1,m+2}\ldots, k_{1,m+n}, k_{2,m+1},\ldots, k_{m,m+n} \in \{0,1\}.
\end{aligned} \tag{25}$$

In general $\overline{\mathfrak{V}}(p)$ is not an irreducible representation of $\mathfrak{osp}(2m+1|2n)$. Let $M(p)$ be the maximal nontrivial submodule of $\overline{\mathfrak{V}}(p)$. Then the irreducible module, corresponding to the parastatistics Fock space, is

$$\mathfrak{V}(p) = \overline{\mathfrak{V}}(p)/M(p). \tag{26}$$

Now the aim is to determine the vectors belonging to $M(p)$, and thus find the structure of $\mathfrak{V}(p)$, and to compute the matrix elements of the algebra generators.

For this purpose, let us first consider the character of $\overline{\mathfrak{V}}(p)$: this is a formal infinite series of terms $\nu x_1^{j_1} x_2^{j_2} \ldots x_m^{j_m} y_1^{j_{m+1}} y_2^{j_{m+2}} \ldots y_n^{j_{m+n}}$, where the exponents carry a weight $(j_1, \ldots, j_m | j_{m+1}, \ldots, j_{m+n})$ of $\overline{\mathfrak{V}}(p)$ and ν is the dimension of this weight space. The vacuum vector $|0\rangle$ of $\overline{\mathfrak{V}}(p)$, of weight $(-\frac{p}{2}, \ldots, -\frac{p}{2} | \frac{p}{2}, \ldots, \frac{p}{2})$, yields a term $x_1^{-\frac{p}{2}} \ldots x_m^{-\frac{p}{2}} y_1^{\frac{p}{2}} \ldots y_n^{\frac{p}{2}}$ in the character $\mathrm{char}\overline{\mathfrak{V}}(p)$ and from the basis vectors (25) it follows that

$$\mathrm{char}\overline{\mathfrak{V}}(p) = \frac{(x_1)^{-p/2} \cdots (x_m)^{-p/2} (y_1)^{p/2} \cdots (y_n)^{p/2} \prod_{i,j}(1 + x_i y_j)}{\prod_i (1 - x_i) \prod_{i<k}(1 - x_i x_k) \prod_j (1 - y_j) \prod_{j<l}(1 - y_j y_l)}. \tag{27}$$

Such expressions can be expanded in terms of supersymmetric Schur functions, valid for general m and n.

Proposition 2 (Cummins and King) *Consider two sets of variables*

$$(\mathbf{x}) = (x_1, x_2, \ldots, x_m), \qquad (\mathbf{y}) = (y_1, y_2, \ldots, y_n).$$

Then [1]

$$\frac{\prod_{i,j}(1 + x_i y_j)}{\prod_i (1 - x_i) \prod_{i<k}(1 - x_i x_k) \prod_j (1 - y_j) \prod_{j<l}(1 - y_j y_l)} = \sum_{\lambda \in \mathcal{H}} s_\lambda(x_1, \ldots, x_m | y_1, \ldots, y_n) = \sum_{\lambda \in \mathcal{H}} s_\lambda(\mathbf{x}|\mathbf{y}). \tag{28}$$

In the right hand side, the sum is over all partitions λ satisfying the so called hook condition $\lambda_{m+1} \leq n$ ($\lambda \in \mathcal{H}$), and $s_\lambda(\mathbf{x}|\mathbf{y})$ is the supersymmetric Schur function [9] defined by

$$s_\lambda(\mathbf{x}|\mathbf{y}) = \sum_\tau s_{\lambda/\tau}(\mathbf{x}) s_{\tau'}(\mathbf{y}) = \sum_{\sigma,\tau} c^\lambda_{\sigma\tau} s_\sigma(\mathbf{x}) s_{\tau'}(\mathbf{y}),$$

where $\ell(\sigma) \leq m$, $\ell(\tau') \leq n$ and $|\lambda| = |\sigma| + |\tau|$. Herein, some standard notation [9] is used: for a partition λ, $\ell(\lambda)$ is the length of λ and $|\lambda|$ its weight; τ' is the partition conjugate to τ; $c^\lambda_{\sigma\tau}$ are the Littlewood-Richardson coefficients; and $s_\nu(\mathbf{x})$ is the ordinary Schur function.

Now it is well known that the characters of the irreducible covariant $\mathfrak{u}(m|n)$ tensor representations $V([\Lambda^\lambda])$ are given by such supersymmetric Schur functions $s_\lambda(x|y)$ ($\lambda \in \mathcal{H}$). The relation between the partitions $\lambda = (\lambda_1, \lambda_2, \ldots)$, $\lambda_{m+1} \leq n$ and the highest weights $\Lambda^\lambda \equiv [\mu]^r \equiv [\mu_{1r}, \ldots, \mu_{mr} | \mu_{m+1,r} \ldots, \mu_{rr}]$ ($r = m + n$) of the irreducible covariant $\mathfrak{u}(m|n)$ tensor representations is known [16]:

$$\begin{aligned} &\mu_{ir} = \lambda_i, \quad 1 \le i \le m, \\ &\mu_{m+i,r} = \max\{0, \lambda'_i - m\}, \quad 1 \le i \le n, \end{aligned} \tag{29}$$

where λ' is the partition conjugate [9] to λ. Therefore the formula (28) gives the branching to $\mathfrak{u}(m|n)$ of the $\mathfrak{osp}(2m+1|2n)$ representation $\overline{\mathfrak{V}}(p)$. This also gives a possibility to label the basis vectors of $\overline{\mathfrak{V}}(p)$. For each irreducible covariant $\mathfrak{u}(m|n)$ tensor representations one can use the Gelfand-Zetlin basis (GZ) [14] and the union of all these GZ basis is then the basis for $\overline{\mathfrak{V}}(p)$. In such a way the new basis of $\overline{\mathfrak{V}}(p)$ consists of vectors of the form

$$|\mu) \equiv |\mu)^r = \left|\begin{array}{llllll} \mu_{1r} \;\cdots & \mu_{m-1,r} & \mu_{mr} & \mu_{m+1,r} & \cdots\; \mu_{r-1,r} & \mu_{rr} \\ \mu_{1,r-1} \;\cdots & \mu_{m-1,r-1} & \mu_{m,r-1} & \mu_{m+1,r-1} & \cdots\; \mu_{r-1,r-1} & \\ \vdots \quad\quad \vdots & \vdots & \vdots & \vdots & \iddots & \\ \mu_{1,m+1} \;\cdots & \mu_{m-1,m+1} & \mu_{m,m+1} & \mu_{m+1,m+1} & & \\ \mu_{1m} \;\cdots & \mu_{m-1,m} & \mu_{mm} & & & \\ \mu_{1,m-1} \;\cdots & \mu_{m-1,m-1} & & & & \\ \vdots \quad \iddots & & & & & \\ \mu_{11} & & & & & \end{array}\right), \tag{30}$$

which satisfy the conditions

$$\begin{aligned} &1.\ \mu_{ir} \in \mathbb{Z}_+ \text{ are fixed and } \mu_{jr} - \mu_{j+1,r} \in \mathbb{Z}_+,\ j \ne m,\ 1 \le j \le r-1, \\ &\quad \mu_{mr} \ge \#\{i : \mu_{ir} > 0,\ m+1 \le i \le r\}; \\ &2.\ \mu_{ip} - \mu_{i,p-1} \equiv \theta_{i,p-1} \in \{0,1\}, \quad 1 \le i \le m;\ m+1 \le p \le r; \\ &3.\ \mu_{mp} \ge \#\{i : \mu_{ip} > 0,\ m+1 \le i \le p\}, \quad m+1 \le p \le r; \\ &4.\ \text{if } \ \mu_{m,m+1} = 0, \text{ then } \theta_{mm} = 0; \\ &5.\ \mu_{ip} - \mu_{i+1,p} \in \mathbb{Z}_+, \quad 1 \le i \le m-1;\ m+1 \le p \le r-1; \\ &6.\ \mu_{i,j+1} - \mu_{ij} \in \mathbb{Z}_+ \text{ and } \mu_{i,j} - \mu_{i+1,j+1} \in \mathbb{Z}_+, \\ &\quad 1 \le i \le j \le m-1 \text{ or } m+1 \le i \le j \le r-1. \end{aligned} \tag{31}$$

Note that the last m lines of the triangular GZ-array correspond to a GZ-pattern of $\mathfrak{gl}(m)$, whereas the last n columns correspond to a GZ-pattern for $\mathfrak{gl}(n)$. The conditions above follow from the correspondence between a highest weight in partition notation and its coordinates, see (29), and from the fact that for covariant representations, the decomposition from $\mathfrak{u}(m|n)$ to $\mathfrak{u}(m|n-1)$ is governed by

$$s_\lambda(\mathbf{x}|\mathbf{y}) = \sum_\sigma s_\sigma(\mathbf{x}|y_1, \ldots, y_{n-1})\, y_n^{|\lambda|-|\sigma|}. \tag{32}$$

In this last expression, the sum is over all partitions σ such that $\lambda - \sigma$ is a vertical strip [9]. That actually explains why the $\theta_{i,p}$'s in (31) take values in $\{0,1\}$.

Now the task is to give the explicit action of the generating elements $c_i^\pm$ (12) of $\mathfrak{osp}(2m+1|2n)$. For this purpose, we introduce the following notations:

$$|\mu) \equiv |\mu)^r = \left| \begin{matrix} [\mu]^r \\ |\mu)^{r-1} \end{matrix} \right),$$

$([\mu]^r) = (\mu_{1r}, \mu_{2r}, \ldots, \mu_{rr})$ and $([\mu]^r_{\pm k}) = (\mu_{1r}, \ldots, \mu_{kr} \pm 1, \ldots, \mu_{rr})$. Then

Proposition 3 *The explicit actions of the Lie superalgebra generators $c_j^{\pm}$ on a basis of $\overline{\mathfrak{V}}(p)$ are as follows:*

$$c_j^+|\mu) = \sum_{k,\mu'} \left(\begin{matrix} [\mu]^r \\ |\mu)^{r-1} \end{matrix} ; \begin{matrix} 10\cdots 00 \\ 10\cdots 0 \\ \cdots \\ 0 \end{matrix} \left| \begin{matrix} [\mu]^r_{+k} \\ |\mu')^{r-1} \end{matrix} \right. \right) \times G_k([\mu]^r) \left| \begin{matrix} [\mu]^r_{+k} \\ |\mu')^{r-1} \end{matrix} \right), \tag{33}$$

$$c_j^-|\mu) = \sum_{k,\mu'} \left(\begin{matrix} [\mu]^r_{-k} \\ |\mu')^{r-1} \end{matrix} ; \begin{matrix} 10\cdots 00 \\ 10\cdots 0 \\ \cdots \\ 0 \end{matrix} \left| \begin{matrix} [\mu]^r \\ |\mu)^{r-1} \end{matrix} \right. \right) \times G_k([\mu]^r_{-k}) \left| \begin{matrix} [\mu]^r_{-k} \\ |\mu')^{r-1} \end{matrix} \right). \tag{34}$$

The first factor in the right hand sides of (33) and (34) is a $\mathfrak{u}(m|n)$ Clebsch-Gordan coefficient (CGC) given by formulae (4.9)–(4.17) in [14], and the second factor is a reduced matrix element. *The reduced matrix elements G_k ($k = 1, \ldots, m+n = r$) are given by:*

$$G_k(\mu_{1r}, \mu_{2r}, \ldots, \mu_{rr}) = \\ \left(-\frac{(\mathcal{E}_m(\mu_{kr}+m-n-k)+1)\prod_{j\neq k=1}^{m}(\mu_{kr}-\mu_{jr}-k+j)}{\prod_{j\neq \frac{k}{2}=1}^{\lfloor m/2 \rfloor}(\mu_{kr}-\mu_{2j,r}-k+2j)(\mu_{kr}-\mu_{2j,r}-k+2j+1)} \right)^{1/2} \\ \times \prod_{j=1}^{n} \left(\frac{\mu_{kr}+\mu_{m+j,r}+m-j-k+2}{\mu_{kr}+\mu_{m+j,r}+m-j-k+2-\mathcal{E}_{m+\mu_{m+j,r}}} \right)^{1/2} \tag{35}$$

for $k \le m$ and k even;

$$G_k(\mu_{1r}, \mu_{2r}, \ldots, \mu_{rr}) = (p-\mu_{kr}+k-1)^{1/2} \times \\ \frac{(\mathcal{O}_m(\mu_{kr}+m-n-k)+1)^{1/2}\left(\prod_{j\neq k=1}^{m}(\mu_{kr}-\mu_{jr}-k+j)\right)^{1/2}}{\left(\prod_{j\neq \frac{k+1}{2}=1}^{\lceil m/2 \rceil}(\mu_{kr}-\mu_{2j-1,r}-k+2j-1)(\mu_{kr}-\mu_{2j-1,r}-k+2j)\right)^{1/2}} \\ \times \prod_{j=1}^{n} \left(\frac{\mu_{kr}+\mu_{m+j,r}+m-j-k+2}{\mu_{kr}+\mu_{m+j,r}+m-j-k+2-\mathcal{O}_{m+\mu_{m+j,r}}} \right)^{1/2} \tag{36}$$

for $k \le m$ and k odd. The remaining expressions for $k = 1, 2, \ldots, n$ are

$$
\begin{aligned}
&G_{m+k}(\mu_{1r},\mu_{2r},\ldots,\mu_{rr}) = (-1)^{\mu_{m+k+1,r}+\mu_{m+k+2,r}+\ldots+\mu_{rr}} \\
&\times \big((\mathcal{O}_{\mu_{m+k,r}}(\mu_{m+k,r}-k+n)+1)(\mathcal{E}_{m+\mu_{m+k,r}}(p+\mu_{m+k,r}+m-k)+1)\big)^{1/2} \\
&\times \left(\frac{\prod_{j=1}^{\lfloor m/2\rfloor}(\mathcal{E}_{m+\mu_{m+k,r}}(\mu_{2j,r}+\mu_{m+k,r}-2j-k+m+1)+1)}{\prod_{j=1}^{\lceil m/2\rceil}(\mathcal{E}_{m+\mu_{m+k,r}}(\mu_{2j-1,r}+\mu_{m+k,r}-2j-k+m+1)+1)}\right)^{1/2} \\
&\times \left(\frac{\prod_{j=1}^{\lceil m/2\rceil}(\mathcal{O}_{m+\mu_{m+k,r}}(\mu_{2j-1,r}+\mu_{m+k,r}-2j-k+m+2)+1)}{\prod_{j=1}^{\lfloor m/2\rfloor}(\mathcal{O}_{m+\mu_{m+k,r}}(\mu_{2j,r}+\mu_{kr}-2j-k+m)+1)}\right)^{1/2} \\
&\times \prod_{j\neq k=1}^{n}\left(\frac{\mu_{m+j,r}-\mu_{m+k,r}-j+k}{\mu_{m+j,r}-\mu_{m+k,r}-j+k-\mathcal{O}_{\mu_{m+j,r}-\mu_{m+k,r}}}\right)^{1/2}. \qquad (37)
\end{aligned}
$$

Herein $\mathcal{E}$ and $\mathcal{O}$ are the even and odd functions defined by

$$
\begin{aligned}
&\mathcal{E}_j = 1 \text{ if } j \text{ is even and 0 otherwise,}\\
&\mathcal{O}_j = 1 \text{ if } j \text{ is odd and 0 otherwise;} \qquad (38)
\end{aligned}
$$

where obviously $\mathcal{O}_j = 1-\mathcal{E}_j$, but it is still convenient to use both notations. Also, note that products such as $\prod_{j\neq k=1}^{s}$ means "the product over all j-values running from 1 to s, but excluding $j=k$". The notation $\lfloor a\rfloor$ (resp. $\lceil a\rceil$) refers to the *floor* (resp. *ceiling*) of a, i.e. the largest integer not exceeding a (resp. the smallest integer greater than or equal to a).

Now, taking into account the general conditions (31), the only factor in the right hand sides of (35)–(37) that may become zero appears in (36) and is

$$p-\mu_{kr}+k-1 \qquad (k\leq m \text{ and } k \text{ odd}).$$

For $k=1$ this factor is $(p-\mu_{1r})$, and μ_{1r} is the largest integer in the first row of the GZ-pattern (30) (which is also the first part of the partition λ, see (29)). Starting from the vacuum vector, with a GZ-pattern consisting of all zeros, one can raise the entries in the GZ-pattern by applying the operators c_j^+. However, when μ_{1r} has reached the value p it can no longer be increased. As a consequence, all vectors $|\mu)$ with $\mu_{1r}>p$ belong to the submodule $M(p)$. This gives the structure of $\mathfrak{V}(p)$.

Theorem 2 *An orthonormal basis for the space $\mathfrak{V}(p)$ is given by the vectors $|\mu)$, see (30) and (31), with $\mu_{1r}\leq p$. The action of the Cartan algebra elements of $\mathfrak{osp}(2m+1|2n)$ is:*

$$
\begin{aligned}
h_k|\mu) &= \left(-\frac{p}{2}+\sum_{j=1}^{k}\mu_{jk}-\sum_{j=1}^{k-1}\mu_{j,k-1}\right)|\mu), \quad k=1,\ldots,m;\\
h_k|\mu) &= \left(\frac{p}{2}+\sum_{j=1}^{k}\mu_{jk}-\sum_{j=1}^{k-1}\mu_{j,k-1}\right)|\mu), \quad k=m+1,\ldots,r. \qquad (39)
\end{aligned}
$$

The action of the operators $c_j^{\pm}$, $j = 1, \dots, r$ is given by (33) and (34), where the CGCs are found in [14] (see formulae (4.9)–(4.17)) and the reduced matrix elements are given by (35)–(37).

4 Summary and Conclusion

In the present paper we have constructed the Fock spaces $\mathfrak{V}(p)$ of m parafermions and n parabosons with relative parafermion relations among them, which are the unitary irreducible representations of $\mathfrak{osp}(2m+1|2n)$ with lowest weight of the form $(-\frac{p}{2}, \dots, -\frac{p}{2}|\frac{p}{2}, \dots, \frac{p}{2})$. The subalgebra $\mathfrak{u}(m|n)$ of $\mathfrak{osp}(2m+1|2n)$, generated by all supercommutators of the parafermions and parabosons, and its covariant tensor representations play a crucial role in the analysis. For each irreducible covariant $\mathfrak{u}(m|n)$ tensor representation the known Gelfand-Zetlin basis follows the decomposition $\mathfrak{u}(m|n) \supset \mathfrak{u}(m|n-1) \supset \dots \supset \mathfrak{u}(m|1) \supset \mathfrak{u}(m) \supset \mathfrak{u}(m-1) \supset \dots \supset \mathfrak{u}(1)$.

The real interest is in such quantum systems (mixed systems of parafermions and parabosons) with infinite degrees of freedom ($m \to \infty$ and $n \to \infty$). It is clear that the GZ-basis used here cannot be used for such a purpose: as $m \to \infty$ in (30), there is no longer control over n. In order to investigate such systems one should construct the irreducible covariant tensor representations of $\mathfrak{u}(n|n)$ in another Gelfand-Zetlin basis, namely following the decomposition $\mathfrak{u}(n|n) \supset \mathfrak{u}(n|n-1) \supset \mathfrak{u}(n-1|n-1) \dots \supset \mathfrak{u}(2|2) \supset \mathfrak{u}(2|1) \supset \mathfrak{u}(1|1) \supset \mathfrak{u}(1)$. We hope to report this result soon.

Acknowledgements The authors were supported by Joint Research Project "Representation theory of Lie (super)algebras and generalized quantum statistics" in the framework of an international collaboration programme between the Research Foundation - Flanders (FWO) and the Bulgarian Academy of Sciences. NIS was partially supported by Bulgarian NSF grant DFNI T02/6.

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Stepwise Square Integrable Representations: The Concept and Some Consequences

Joseph A. Wolf

Abstract There are some new developments on Plancherel formula and growth of matrix coefficients for unitary representations of nilpotent Lie groups. These have several consequences for the geometry of weakly symmetric spaces and analysis on parabolic subgroups of real semisimple Lie groups, and to (infinite dimensional) locally nilpotent Lie groups. Many of these consequences are still under development. In this note I'll survey a few of these new aspects of representation theory for nilpotent Lie groups and parabolic subgroups.

1 Introduction

There is a well developed theory of square integrable representations of nilpotent Lie groups [17]. It is based on the general representation theory of Kirillov [12] for connected nilpotent real Lie groups. A connected simply connected Lie group N with center Z is called *square integrable* if it has unitary representations π whose coefficients $f_{u,v}(x) = \langle u, \pi(x)v\rangle$ satisfy $|f_{u,v}| \in \mathcal{L}^2(N/Z)$. If N has one such square integrable representation then there is a certain polynomial function $P(\gamma)$ on the linear dual space $\mathfrak{z}^*$ of the Lie algebra of Z that is key to harmonic analysis on N. Here $P(\gamma)$ is the Pfaffian of the antisymmetric bilinear form on $\mathfrak{n}/\mathfrak{z}$ given by $b_\lambda(x, y) = \lambda([x, y])$ where $\gamma = \lambda|_{\mathfrak{z}}$. The square integrable representations of N are certain easily-constructed representations π_γ where $\gamma \in \mathfrak{z}^*$ with $P(\gamma) \neq 0$, Plancherel almost irreducible unitary representations of N are square integrable, and up to an explicit constant $|P(\gamma)|$ is the Plancherel density of the unitary dual $\widehat{N}$ at π_λ. This theory has some interesting analytic consequences [26].

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V. Dobrev (ed.), *Lie Theory and Its Applications in Physics*,
Springer Proceedings in Mathematics & Statistics 191,
DOI 10.1007/978-981-10-2636-2_12

More recently there was a serious extension of that theory [27]. Under certain conditions, the nilpotent Lie group N has a decomposition into subgroups that have square integrable representations, and the Plancherel formula then is synthesized explicitly in terms of the Plancherel formulae of those subgroups. In particular the extended theory applies to nilradicals of minimal parabolic subgroups [27]. With a minor technical adjustment it has just been extended to nilradicals of arbitrary real parabolics [32]. The consequences include explicit Plancherel and Fourier inversion formulas. Applications include analysis on minimal parabolic subgroups [28] and, more generally, on maximal amenable subgroups of parabolics [32], They also include analysis on commutative spaces, i.e. on Gelfand pairs [31]. We sketch some of these developments. Due to constraints of time and space we pass over many aspects of operator theory and orbit geometry, for example those described in [2–4], related to stepwise square integrable representations.

In Sect. 2 we recall the basic facts [17], with a few extensions, on square integrable representations of nilpotent Lie groups. In Sect. 3 we recall the concept and main results for stepwise square integrable nilpotent Lie group.

In Sect. 4 we show how nilradicals of minimal parabolic subgroups have the required decomposition for stepwise square integrability. This is a construction based on concept of strongly orthogonal restricted roots.

In Sect. 5 we indicate the consequences for homogeneous compact nilmanifolds, and in Sect. 6 we mention the application to analysis on commutative nilmanifolds.

In Sect. 7 we start the extension of stepwise square integrability results from the nilradical N of a minimal parabolic $P = MAN$ to various subgroups that contain N. This section concentrates on the subgroup MN and takes advantage of principal orbit theory. That gives a sharp simplification to the Plancherel and Fourier Inversion formulae. In Sect. 8 we look at P and its subgroup AN. They are not unimodular, so we introduce the Dixmier–Pukánszky operator D whose semi-invariance balances that of the modular function. It is a key point for the Plancherel and Fourier Inversion formulae.

Sections 9 and 10 are a short discussion of work in progress on the extension of results from minimal parabolics to parabolics in general. There are two places where matters diverge from the minimal parabolic case. First, there is a technical adjustment to the definition of stepwise square integrable representation, caused by the fact that in the non-minimal case the restricted roots need not form a root system. Second, again for technical reasons, the explicit Plancherel Formula only comes through for the maximal amenable subgroups UAN of G, and not for all of the parabolic.

This work was partially supported by a Simons Foundation grant and by the award of a Dickson Emeriti Professorship. It expands a talk at the 11-th International Workshop “Lie Theory and Its Applications in Physics” in Varna. My thanks to Prof. Vladimir Dobrev and the others on the organizing committee for hospitality at that Workshop.

2 Square Integrable Representations

Let G be a unimodular locally compact group with center Z, and let π be an irreducible unitary representation. We associate the central character $\chi_\pi \in \widehat{Z}$ by $\pi(z) = \chi_\pi(x) \cdot 1$ for $z \in Z$. Consider a matrix coefficient $f_{u,v} : x \mapsto \langle u, \pi(x)v\rangle$. Then $|f_{u,v}|$ is a well defined function on G/Z. Fix Haar measures μ_G on G, μ_Z on Z and $\mu_{G/Z}$ on G/Z such that $d\mu_G = d\mu_Z\, d\mu_{G/Z}$. The following results are well known.

Theorem 2.1 *The following conditions on $\pi \in \widehat{G}$ are equivalent.*

(1) There exist nonzero $u, v \in \mathcal{H}_\pi$ with $|f_{u,v}| \in \mathcal{L}^2(G/Z)$.
(2) $|f_{u,v}| \in \mathcal{L}^2(G/Z)$ for all $u, v \in \mathcal{H}_\pi$.
(3) π is a discrete summand of the representation $\mathrm{Ind}_Z^G(\chi_\pi)$.

Theorem 2.2 *If the conditions of* Theorem 2.1 *are satisfied for an irreducible $\pi \in \widehat{G}$, then there is a number* $\deg \pi > 0$ *such that*

$$\int_{G/Z} f_{u,v}(x)\overline{f_{u',v'}(x)} d\mu_{G/Z}(xZ) = \tfrac{1}{\deg \pi}\langle u, u'\rangle\overline{\langle vv'\rangle} \tag{1}$$

for all $u, u', v, v' \in \mathcal{H}_\pi$. If $\pi_1, \pi_2 \in \widehat{G}$ are inequivalent and satisfy the conditions of Theorem 2.1*, and $\chi_{\pi_1} = \chi_{\pi_2}$, then*

$$\int_{G/Z} \langle u, \pi_1(x)v\rangle\overline{\langle u', \pi_2(x)v'\rangle} d\mu_{G/Z}(xZ) = 0 \tag{2}$$

for all $u, v \in \mathcal{H}_{\pi_1}$ and all $u', v' \in \mathcal{H}_{\pi_2}$.

The main results of [17] shows exactly how this works for nilpotent Lie groups.

Theorem 2.3 *Let N be a connected simply connected Lie group with center Z, $\mathfrak{n}$ and $\mathfrak{z}$ their Lie algebras, and $\mathfrak{n}^*$ the linear dual space of $\mathfrak{n}$. Let $\lambda \in \mathfrak{n}^*$ and let π_λ denote the irreducible unitary representation attached to* $\mathrm{Ad}^*(N)\lambda$ *by the Kirillov theory* [12]*. Then the following conditions are equivalent.*

(1) π_λ satisfies the conditions of Theorem 2.1.
(2) The coadjoint orbit $\mathrm{Ad}^*(N)\lambda = \{\nu \in \mathfrak{n}^* \mid \nu|_\mathfrak{z} = \lambda|_\mathfrak{z}$.
(3) The bilinear form $b_\lambda(x, y) = \lambda([x, y])$ on $\mathfrak{n}/\mathfrak{z}$ is nondegenerate.
(4) The universal enveloping algebra $\mathcal{U}(\mathfrak{z})$ is the center of $\mathcal{U}(\mathfrak{n})$.

The Pfaffian polynomial $\mathrm{Pf}\,(b_\lambda)$ *is a polynomial function $P(\lambda|_\mathfrak{z})$ on $\mathfrak{z}^*$, and the set of representations π_λ for which these conditions hold, is parameterized by the set* $\{\gamma \in \mathfrak{z}^* \mid P(\gamma) \neq 0\}$ *(which is empty or Zariski open in $\mathfrak{z}^*$).*

We will say that the connected simply connected Lie group N is *square integrable* if there exists $\lambda \in \mathfrak{n}^*$ such that $P(\lambda|_\mathfrak{z}) \neq 0\}$. For convenience we will sometimes write $P(\lambda)$ for $P(\lambda|_\mathfrak{z})$ and π_γ for π_λ where $\gamma = \lambda|_\mathfrak{z}$.

Theorem 2.4 *Let N be a square integrable connected simply connected Lie group with center Z. Then Plancherel measure on $\widehat{N}$ is concentrated on $\{\pi_\lambda \mid P(\lambda) \neq 0\}$, and there the Plancherel measure is given by the measure $|P(\lambda)d\lambda|$ on $\mathfrak{z}^*$ and the formal degree* $\deg \pi_\lambda = |P(\lambda|_{\mathfrak{z}})|$.

Given $\gamma \in \mathfrak{z}^*$ with $P(\gamma) \neq 0$ and a Schwartz class $(\mathcal{C}(N))$ function f on N we write $\mathcal{O}(\gamma)$ for the co-adjoint orbit $\mathrm{Ad}^*(N)\gamma = \gamma + \mathfrak{z}^\perp$, f_γ for the restriction of $f \cdot \exp$ to $\mathcal{O}(\gamma)$, and $\widehat{f_\gamma}$ for the Fourier transform of f_γ on $\mathcal{O}(\gamma)$.

Theorem 2.5 *Let N be a square integrable connected simply connected Lie group with center Z and $f \in \mathcal{C}(N)$. If $\gamma \in \mathfrak{z}^*$ with $P(\gamma) \neq 0$ then the distribution character of π_γ is given by*

$$\Theta_{\pi_\gamma}(f) = \mathrm{trace} \int_N f(x)\pi_\gamma(x)d\mu_G(x) = c^{-1}|P(\gamma)|^{-1} \int_{\nu \in \mathcal{O}(\gamma)} \widehat{f_\gamma}\, d\nu \tag{3}$$

where $c = d!2^d$ and $d = \dim(\mathfrak{n}/\mathfrak{z})/2$ and $d\nu$ is ordinary Lebesgue measure on the affine space $\mathcal{O}(\gamma)$. The Fourier Inversion formula for N is

$$f(x) = c \int_{\mathfrak{z}^*} \Theta_\gamma(r_x f)|P(\gamma)|\, d\gamma \text{ where } (r_x f)(y) = f(yx) \text{ (right translate)}. \tag{4}$$

There also are multiplicity results on $\mathcal{L}^2(N/\Gamma)$ where N is square integrable and Γ is a discrete co-compact subgroup, but they are the same as in the stepwise square integrable case, so we postpone their description.

3 Stepwise Square Integrability

In order to go beyond square integrable nilpotent groups, we suppose that the connected simply connected nilpotent Lie group decomposes as

$$\begin{aligned} &N = L_1 L_2 \dots L_{m-1} L_m \text{ where} \\ &\quad \text{(a) each } L_r \text{ has unitary representations with coeff in } \mathcal{L}^2(L_r/Z_r), \\ &\quad \text{(b) } N_r := L_1 L_2 \dots L_r \text{ is normal in } N \text{ with } N_r = N_{r-1} \rtimes L_r\,, \\ &\quad \text{(c) } [\mathfrak{l}_r, \mathfrak{z}_s] = 0 \text{ and } [\mathfrak{l}_r, \mathfrak{l}_s] \subset \mathfrak{v} \text{ for } r > s \text{ with } \mathfrak{l}_r = \mathfrak{z}_r + \mathfrak{v}_r \\ &\qquad \text{where } \mathfrak{n} = \mathfrak{s} + \mathfrak{v},\ \ \mathfrak{s} = \textstyle\bigoplus \mathfrak{z}_r \text{ and } \mathfrak{v} = \bigoplus \mathfrak{v}_r\,. \end{aligned} \tag{5}$$

We will use the following notation.

$$\begin{aligned}
&\text{(a) } d_r = \tfrac{1}{2}\dim(\mathfrak{l}_r/\mathfrak{z}_r) \text{ so } \tfrac{1}{2}\dim(\mathfrak{n}/\mathfrak{s}) = d_1 + \cdots + d_m\,, \\
&\qquad\qquad \text{and } c = 2^{d_1+\cdots+d_m} d_1! d_2! \dots d_m! \\
&\text{(b) } b_{\lambda_r} : (x, y) \mapsto \lambda([x, y]) \text{ viewed as a bilinear form on } \mathfrak{l}_r/\mathfrak{z}_r \\
&\text{(c) } S = Z_1 Z_2 \dots Z_m = Z_1 \times \cdots \times Z_m \text{ where } Z_r \text{ is the center of } L_r \\
&\text{(d) } P: \text{ polynomial } P(\lambda) = \mathrm{Pf}\,(b_{\lambda_1})\mathrm{Pf}\,(b_{\lambda_2}) \dots \mathrm{Pf}\,(b_{\lambda_m}) \text{ on } \mathfrak{s}^* \\
&\text{(e) } \mathfrak{t}^* = \{\lambda \in \mathfrak{s}^* \mid P(\lambda) \neq 0\} \\
&\text{(f) } \pi_\lambda \in \widehat{N} \text{ for } \lambda \in \mathfrak{s}^* \text{ with } P(\lambda) \neq 0, \text{ irreducible unitary representation} \\
&\qquad\qquad \text{of } N = L_1 L_2 \dots L_m \text{ constructed as follows.}
\end{aligned} \tag{6}$$

Start with the representation $\pi_{\lambda_1} \in \widehat{N_1}$ specified by $\lambda_1 \in \mathfrak{z}_1^*$ with $\mathrm{Pf}\,(b_{\lambda_1}) \neq 0$. Choose an invariant polarization $\mathfrak{p}_1' \subset \mathfrak{n}_2$ for the linear functional $\lambda_1' \in \mathfrak{n}_2^*$ that agrees with λ_1 on $\mathfrak{n}_1$ and vanishes on $\mathfrak{l}_2$. Since L_r centralizes S_{r-1}, $\mathrm{ad}^*(\mathfrak{l}_2)(\lambda_1')|_{\mathfrak{z}_1+\mathfrak{l}_2} = 0$, so $\mathfrak{p}_1' = \mathfrak{p}_1 + \mathfrak{l}_2$ where $\mathfrak{p}_1$ is an invariant polarization for $\lambda_1 \in \mathfrak{n}_1^*$. The associated representations are $\pi'_{\lambda_1} \in \widehat{N_2}$ and $\pi_{\lambda_1} \in \widehat{N_1}$. Note that $N_2/P_1' = N_1/P_1$, so the representation spaces $\mathcal{H}_{\pi'_{\lambda_1}} = \mathcal{L}^2(N_2/P_1') = \mathcal{L}^2(N_1/P_1) = \mathcal{H}_{\pi_{\lambda_1}}$. In other words, π'_{λ_1} extends π_{λ_1} to a unitary representation of N_2 on the same Hilbert space $\mathcal{H}_{\pi_{\lambda_1}}$, and $d\pi_{\lambda_1'}(\mathfrak{z}_2) = 0$. Now the Mackey Little Group method gives us

Lemma 3.1 *The irreducible unitary representations of N_2, whose restrictions to N_1 are multiples of π_{λ_1}, are the $\pi'_{\lambda_1}\widehat{\otimes}\gamma$ where $\gamma \in \widehat{L_2} = \widehat{N_2/N_1}$.*

Given $\lambda_2 \in \mathfrak{z}_2^*$ with $\mathrm{Pf}\,(b_{\lambda_2}) \neq 0$ we have $\pi_{\lambda_2} \in \widehat{L_2}$ with coefficients in $\mathcal{L}^2(L_2/Z_2)$. In the notation of Lemma 3.1 we define

$$\pi_{\lambda_1+\lambda_2} \in \widehat{N_2} \text{ by } \pi_{\lambda_1+\lambda_2} = \pi'_{\lambda_1}\widehat{\otimes}\pi_{\lambda_2}\,. \tag{7}$$

Proposition 3.2 *The coefficients $f_{z,w}(xy) = \langle z, \pi_{\lambda_1+\lambda_2}(xy)w\rangle$ of $\pi_{\lambda_1+\lambda_2}$ belong to $\mathcal{L}^2(N_2/S_2)$, in fact satisfy $||f_{z,w}||^2_{\mathcal{L}^2(N_r/S_r)} = \frac{||z||^2||w||^2}{\deg(\pi_{\lambda_1})\dots\deg(\pi_{\lambda_r})}$.*

Proposition 3.2 starts a recursion using $N_{r+1} = N_r \rtimes L_{r+1}$. We fix nonzero $\lambda_i \in \mathfrak{z}_i^*$ for $1 \leqq i \leqq r+1$, and we start with the representation $\pi_{\lambda_1+\cdots+\lambda_r}$ constructed step by step from the square integrable representations $\pi_{\lambda_i} \in \widehat{L_i}$ for $1 \leqq i \leqq r$. The representation space $\mathcal{H}_{\pi_{\lambda_1+\cdots+\lambda_r}} = \mathcal{H}_{\pi_{\lambda_1}}\widehat{\otimes}\cdots\widehat{\otimes}\mathcal{H}_{\pi_{\lambda_r}}$. The coefficients of $\pi_{\lambda_1+\cdots+\lambda_r}$ have absolute value in $\mathcal{L}^2(N_r/S_r)$. They satisfy

$$||f_{z,w}||^2_{\mathcal{L}^2(N_r/S_r)} = \frac{||z||^2||w||^2}{\deg(\pi_{\lambda_1})\dots\deg(\pi_{\lambda_r})}\,. \tag{8}$$

Then $\pi_{\lambda_1+\cdots+\lambda_r}$ extends to a representation $\pi'_{\lambda_1+\cdots+\lambda_r}$ of L_{r+1} on the same Hilbert space $\mathcal{H}_{\pi_{\lambda_1+\cdots+\lambda_r}}$, and it satisfies $d\pi'_{\lambda_1+\cdots+\lambda_r}(\mathfrak{z}_{r+1}) = 0$. As in Lemma 3.1,

Lemma 3.3 *The irreducibles $\pi \in \widehat{N_{r+1}}$, whose restrictions to N_r are multiples of $\pi_{\lambda_1+\cdots+\lambda_r}$, are the $\pi'_{\lambda_1+\cdots+\lambda_r}\widehat{\otimes}\gamma$ where $\gamma \in \widehat{L_{r+1}} = \widehat{N_{r+1}/N_r}$.*

As in Proposition 3.2, define $\pi_{\lambda_1+\cdots+\lambda_{r+1}} = \pi'_{\lambda_1+\cdots+\lambda_r}\widehat{\otimes}\pi_{\lambda_{r+1}}$. Then

Proposition 3.4 *The coefficients $f_{z,w}(x_1 \ldots x_{r+1}) = \langle z, \pi_{\lambda_1+\cdots+\lambda_{r+1}}(x_1 x_2 \cdots x_{r+1}) w\rangle$ of $\pi_{\lambda_1+\cdots+\lambda_{r+1}}$ belong to $\mathcal{L}^2(N_{r+1}/S_{r+1})$, in fact satisfy*

$$||f_{z,w}||^2_{\mathcal{L}^2(N_{r+1}/S_{r+1})} = \frac{||z||^2||w||^2}{\deg(\pi_{\lambda_1})\ldots\deg(\pi_{\lambda_{r+1}})}\,.$$

Since $\deg \pi_{\lambda_r} = |\mathrm{Pf}\,(b_{\lambda_r})|$, Proposition 3.4 is the recursion step for our construction. Passing to the end case $r+1=m$ we see that Plancherel measure is concentrated on $\{\pi_\lambda \mid \lambda \in \mathfrak{t}^*\}$. Using (5)(c) to see that conjugation by elements of L_s has no effect on the Pf (b_{λ_r}) for $r < s$, we arrive at

Theorem 3.5 *Let N be a connected simply connected nilpotent Lie group that satisfies (5). Then Plancherel measure for N is concentrated on $\{\pi_\lambda \mid \lambda \in \mathfrak{t}^*, P(\lambda) \neq 0\}$. If $\lambda \in \mathfrak{t}^*$, $P(\lambda) \neq 0$ and $u, v \in \mathcal{H}_{\pi_\lambda}$, then the coefficient $f_{u,v}(x) = \langle u, \pi_\nu(x)v\rangle$ satisfies*

$$||f_{u,v}||^2_{\mathcal{L}^2(N/S)} = ||u||^2||v||^2/|P(\lambda)|\,. \tag{9}$$

The distribution character $\Theta_{\pi_\lambda} : f \mapsto$ trace $\int_G f(x)\pi(x)dx$ of π_λ is given by

$$\Theta_{\pi_\lambda}(f) = c^{-1}|P(\lambda)|^{-1}\int_{\mathcal{O}(\lambda)} \widehat{f_\lambda}(\xi)d\nu_\lambda(\xi) \text{ for } f \in \mathcal{C}(N) \tag{10}$$

where $\mathcal{C}(N)$ is the Schwartz space, $\mathcal{O}(\lambda) = \mathrm{Ad}^*(N)\lambda = \mathfrak{s}^\perp + \lambda$, f_λ is the lift $f_\lambda(\xi) = f(\exp(\xi))$, $\widehat{f_\lambda}$ is its classical Fourier transform, and $d\nu_\lambda$ is the translate of normalized Lebesgue measure from $\mathfrak{s}^\perp$ to $\mathrm{Ad}^*(N)\lambda$. Further,

$$f(x) = c\int_{\mathfrak{t}^*} \Theta_{\pi_\lambda}(r_x f)|P(\lambda)|d\lambda \text{ for } f \in \mathcal{C}(N). \tag{11}$$

Definition 3.6 The representations π_λ of (6)(f) are the *stepwise square integrable* representations of N relative to (5). $\diamondsuit$

The left action $(l(x)f)(g) = f(x^{-1}g)$ and the right action $(r(y)f)(g) = f(gy)$ of N on functions carries over to coefficients of π as $l(x)r(y)f_{u,v} = f_{\pi(x)u,\pi(y)v}$. If $\pi = \pi_\lambda$ stepwise square integrable, $u, v \in \mathcal{H}_{\pi_\lambda}$ are C^∞ vectors, and if Φ and Ψ belong to the universal enveloping algebra $\mathcal{U}(\mathfrak{n})$, then $l(\Phi)r(\Psi)f_{u,v} = f_{d\pi(\Psi)u,d\pi(\Phi)v}$ is just another coefficient, C^∞ and $\mathcal{L}^2(N/S)$. If $\zeta_\lambda \in \widehat{S}$ is the quasicentral character of π_λ it follows that $f_{u,v}$ belongs to the relative Schwartz space $\mathcal{C}(N/S, \zeta_\lambda)$. In particular it follows that $|f_{u,v}| \in \mathcal{L}^p(N/S)$ for all $p \geqq 1$. Taking Schwartz class wave packets over S of coefficient functions of stepwise square integrable representations of N one can express the Plancherel formula of Theorem 3.5 in terms of coefficient functions.

4 Nilradicals of Minimal Parabolics

Fix a real simple Lie group G, an Iwasawa decomposition $G = KAN$, and a minimal parabolic subgroup $Q = MAN$ in G. Let $m = \mathrm{rank}_{\mathbb{R}} G = \dim_{\mathbb{R}} A$. As usual, write $\mathfrak{k}$ for the Lie algebra of K, $\mathfrak{a}$ for the Lie algebra of A, and $\mathfrak{n}$ for the Lie algebra of N. Complete $\mathfrak{a}$ to a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. Then $\mathfrak{h} = \mathfrak{t} + \mathfrak{a}$ with $\mathfrak{t} = \mathfrak{h} \cap \mathfrak{k}$. Now we have root systems

$$\begin{aligned}
&\Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})\text{: roots of } \mathfrak{g}_{\mathbb{C}} \text{ relative to } \mathfrak{h}_{\mathbb{C}} \text{ (ordinary roots),}\\
&\Delta(\mathfrak{g}, \mathfrak{a})\text{: roots of } \mathfrak{g} \text{ relative to } \mathfrak{a} \text{ (restricted roots),}\\
&\Delta_0(\mathfrak{g}, \mathfrak{a}) = \{\alpha \in \Delta(\mathfrak{g}, \mathfrak{a}) \mid 2\alpha \notin \Delta(\mathfrak{g}, \mathfrak{a})\} \text{ (nonmultipliable).}
\end{aligned} \tag{12}$$

Here $\Delta(\mathfrak{g}, \mathfrak{a})$ and $\Delta_0(\mathfrak{g}, \mathfrak{a})$ are root systems in the usual sense. Any positive root system $\Delta^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}}) \subset \Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ defines positive systems

$$\begin{aligned}
&\Delta^+(\mathfrak{g}, \mathfrak{a}) = \{\gamma|_{\mathfrak{a}} \mid \gamma \in \Delta^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}}) \text{ and } \gamma|_{\mathfrak{a}} \neq 0\},\\
&\Delta_0^+(\mathfrak{g}, \mathfrak{a}) = \Delta_0(\mathfrak{g}, \mathfrak{a}) \cap \Delta^+(\mathfrak{g}, \mathfrak{a}).
\end{aligned} \tag{13}$$

We can (and do) choose $\Delta^+(\mathfrak{g}, \mathfrak{h})$ so that

$$\begin{aligned}
&\mathfrak{n} \text{ is the sum of the positive restricted root spaces and}\\
&\text{if } \gamma \in \Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}}) \text{ and } \gamma|_{\mathfrak{a}} \in \Delta^+(\mathfrak{g}, \mathfrak{a}) \text{ then } \gamma \in \Delta^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}}).
\end{aligned} \tag{14}$$

Two roots are called *strongly orthogonal* if their sum and their difference are not roots. Then they are orthogonal. The Kostant cascade construction is

$$\begin{aligned}
&\beta_1 \in \Delta^+(\mathfrak{g}, \mathfrak{a}) \text{ is a maximal positive restricted root and}\\
&\beta_{r+1} \in \Delta^+(\mathfrak{g}, \mathfrak{a}) \text{ is a maximum among the roots of } \Delta^+(\mathfrak{g}, \mathfrak{a})\\
&\text{that are orthogonal to all } \beta_i \text{ with } i \leqq r
\end{aligned} \tag{15}$$

Then the β_r are mutually strongly orthogonal. Each $\beta_r \in \Delta_0^+(\mathfrak{g}, \mathfrak{a})$, and β_1 is unique because $\Delta(\mathfrak{g}, \mathfrak{a})$ is irreducible. For $1 \leqq r \leqq m$ define

$$\begin{aligned}
&\Delta_1^+ = \{\alpha \in \Delta^+(\mathfrak{g}, \mathfrak{a}) \mid \beta_1 - \alpha \in \Delta^+(\mathfrak{g}, \mathfrak{a})\} \text{ and}\\
&\Delta_{r+1}^+ = \{\alpha \in \Delta^+(\mathfrak{g}, \mathfrak{a}) \setminus (\Delta_1^+ \cup \cdots \cup \Delta_r^+) \mid \beta_{r+1} - \alpha \in \Delta^+(\mathfrak{g}, \mathfrak{a})\}.
\end{aligned} \tag{16}$$

Lemma 4.1 *If* $\alpha \in \Delta^+(\mathfrak{g}, \mathfrak{a})$*, either* $\alpha \in \{\beta_1, \ldots, \beta_m\}$ *or* α *belongs to just one* Δ_r^+.

Lemma 4.2 $\Delta_r^+ \cup \{\beta_r\} = \{\alpha \in \Delta^+ \mid \alpha \perp \beta_i$ *for* $i < r$ *and* $\langle \alpha, \beta_r \rangle > 0\}$. *In particular,* $[\mathfrak{l}_r, \mathfrak{l}_s] \subset \mathfrak{l}_t$ *where* $t = \min\{r, s\}$.

Lemma 4.1 shows that the Lie algebra $\mathfrak{n}$ of N is the direct sum of its subspaces

$$\mathfrak{l}_r = \mathfrak{g}_{\beta_r} + \sum\nolimits_{\Delta_r^+} \mathfrak{g}_\alpha \text{ for } 1 \leqq r \leqq m \tag{17}$$

and Lemma 4.2 shows that $\mathfrak{n}$ has an increasing foliation by ideals

$$\mathfrak{n}_r = \mathfrak{l}_1 + \mathfrak{l}_2 + \cdots + \mathfrak{l}_r \text{ for } 1 \leqq r \leqq m. \tag{18}$$

Now we will see that the corresponding group level decomposition $N = L_1 L_2 \ldots L_m$ and the semidirect product decompositions $N_r = N_{r-1} \rtimes L_r$ satisfy (5). Denote

$$\begin{aligned} &s_{\beta_r} \text{ is the Weyl group reflection in } \beta_r \text{ and} \\ &\sigma_r : \Delta(\mathfrak{g}, \mathfrak{a}) \to \Delta(\mathfrak{g}, \mathfrak{a}) \text{ by } \sigma_r(\alpha) = -s_{\beta_r}(\alpha). \end{aligned} \tag{19}$$

Note that $\sigma_r(\beta_s) = -\beta_s$ for $s \neq r$, $+\beta_s$ if $s = r$. If $\alpha \in \Delta_r^+$ we still have $\sigma_r(\alpha) \perp \beta_i$ for $i < r$ and $\langle \sigma_r(\alpha), \beta_r \rangle > 0$. If $\sigma_r(\alpha) < 0$ then $\beta_r - \sigma_r(\alpha) > \beta_r$ contradicting maximality of β_r. Thus, using Lemma 4.2, $\sigma_r(\Delta_r^+) = \Delta_r^+$.

Lemma 4.3 *If $\alpha \in \Delta_r^+$ then $\alpha + \sigma_r(\alpha) = \beta_r$. (It is possible that $\alpha = \sigma_r(\alpha) = \frac{1}{2}\beta_r$ when $\frac{1}{2}\beta_r$ is a root.). If $\alpha, \alpha' \in \Delta_r^+$ and $\alpha + \alpha' \in \Delta(\mathfrak{g}, \mathfrak{a})$ then $\alpha + \alpha' = \beta_r$.*

Lemma 4.4 *Let $\mathfrak{n}$ be a nilpotent Lie algebra, $\mathfrak{z}$ its center, and $\mathfrak{v}$ a vector space complement to $\mathfrak{z}$ in $\mathfrak{n}$. Suppose that $\mathfrak{v} = \mathfrak{u} + \mathfrak{u}'$, $\mathfrak{u} = \sum \mathfrak{u}_a$ and $\mathfrak{u}' = \sum \mathfrak{u}'_a$, and $\mathfrak{z} = \sum \mathfrak{z}_b$ with* $\dim \mathfrak{z}_b = 1$ *in such a way that* (i) *each* $[\mathfrak{u}_a, \mathfrak{u}_a] = 0 = [\mathfrak{u}'_a, \mathfrak{u}'_a]$, (ii) *if $a_1 \neq a_2$ then $[\mathfrak{u}_{a_1}, \mathfrak{u}'_{a_2}] = 0$ and* (iii) *for each a there is a nondegenerate pairing $\mathfrak{u}_a \otimes \mathfrak{u}'_a \to \mathfrak{z}_{b_a}$, by $u \otimes u' \mapsto [u, u']$. Then $\mathfrak{n}$ is a direct sum of Heisenberg algebras $\mathfrak{z}_{b_a} + \mathfrak{u}_a + \mathfrak{u}'_a$ and the commutative algebra that is the sum of the remaining $\mathfrak{z}_b$.*

Now one runs through a number of special situations: (1) If $\mathfrak{g}$ is the split real form of $\mathfrak{g}_{\mathbb{C}}$ then each L_r has square integrable representations. (2) If $\mathfrak{g}$ is simple but not absolutely simple then each L_r has square integrable representations. (3) If G is the quaternion special linear group $SL(n; \mathbb{H})$ then L_1 has square integrable representations. (4) If G is the group E_{6,F_4} of collineations of the Cayley projective plane then L_1 has square integrable representations. (5) The group L_1 has square integrable representations. (6) If $\mathfrak{g}$ is absolutely simple then each L_r has square integrable representations. Putting these together, Theorem 3.5 applies to nilradicals of minimal parabolic subgroups:

Theorem 4.5 *Let G be a real reductive Lie group, $G = KAN$ an Iwasawa decomposition, $\mathfrak{l}_r$ and $\mathfrak{n}_r$ the subalgebras of $\mathfrak{n}$ defined in* (17) *and* (18)*, and L_r and N_r the corresponding analytic subgroups of N. Then the L_r and N_r satisfy* (5)*. In particular, Plancherel measure for N is concentrated on $\{\pi_\lambda \mid \lambda \in \mathfrak{t}^*\}$. If $\lambda \in \mathfrak{t}^*$, and if u and v belong to the representation space $\mathcal{H}_{\pi_\lambda}$ of π_λ, then the coefficient $f_{u,v}(x) = \langle u, \pi_\lambda(x)v \rangle$ satisfies $||f_{u,v}||^2_{\mathcal{L}^2(N/S)} = \frac{||u||^2||v||^2}{|P(\lambda)|}$. The distribution character Θ_{π_λ} of π_λ satisfies $\Theta_{\pi_\lambda}(f) = c^{-1}|P(\lambda)|^{-1} \int_{\mathcal{O}(\lambda)} \widehat{f_\lambda}(\xi) d\nu_\lambda(\xi)$ for $f \in \mathcal{C}(N)$. Here $\mathcal{C}(N)$ is the Schwartz space, $\mathcal{O}(\lambda)$ is the coadjoint orbit* $\mathrm{Ad}^*(N)\lambda = \mathfrak{s}^\perp + \lambda$, *$f_\gamma$ is the lift $f_\gamma(\xi) = f(\exp(\xi))$ to $\mathfrak{s}^\perp + \lambda$, $\widehat{f_\gamma}$ is its classical Fourier transform, and $d\nu_\lambda$ is the translate of normalized Lebesgue measure from $\mathfrak{s}^\perp$ to* $\mathrm{Ad}^*(N)\lambda$. *The Plancherel formula on N is $f(x) = c \int_{\mathfrak{t}^*} \Theta_{\pi_\lambda}(r_x f)|P(\lambda)| d\lambda$ for $f \in \mathcal{C}(N)$.*

5 Compact Nilmanifolds

Here are the basic facts on discrete uniform (i.e. co-compact) subgroups of connected simply connected nilpotent Lie groups. See [21, Chap. 2] for an exposition.

Proposition 5.1 *The following are equivalent.*

- *N has a discrete subgroup Γ with N/Γ compact.*
- *$N \cong N_{\mathbb{R}}$ where $N_{\mathbb{R}}$ is the group of real points in a unipotent linear algebraic group defined over the rational number field $\mathbb{Q}$.*
- *$\mathfrak{n}$ has a basis $\{\xi_j\}$ for which the coefficients $c_{i,j}^k$ in $[\xi_i, \xi_j] = \sum c_{i,j}^k \xi_k$ are rational numbers.*

Under those conditions let $\mathfrak{n}_{\mathbb{Q}}$ denote the rational span of $\{\xi_j\}$ and let $\mathfrak{n}_{\mathbb{Z}}$ be the integral span. Then $\exp(\mathfrak{n}_{\mathbb{Z}})$ generates a discrete subgroup $N_{\mathbb{Z}}$ of $N = N_{\mathbb{R}}$ and $N_{\mathbb{R}}/N_{\mathbb{Z}}$ is compact. Conversely, if Γ is a discrete co-compact subgroup of N then the $\mathbb{Z}$-span of $\exp^{-1}(\Gamma)$ is a lattice in $\mathfrak{n}$ for which any generating set $\{\xi_j\}$ is a basis of $\mathfrak{n}$ such that the coefficients $c_{i,j}^k$ in $[\xi_i, \xi_j] = \sum c_{i,j}^k \xi_k$ are rational numbers.

The conditions of Proposition 5.1 hold for the nilpotent groups studied in Sect. 4; there one can choose the basis $\{\xi_j\}$ of $\mathfrak{n}$ so that the $c_{i,j}^k$ are integers.

The basic facts on square integrable representations that occur in compact quotients N/Γ, as described in [17, Theorem 7], are

Proposition 5.2 *Let N be a connected simply connected nilpotent Lie group that has square integrable representations, and let Γ a discrete co-compact subgroup. Let Z be the center of N and normalize the volume form on $\mathfrak{n}/\mathfrak{z}$ by normalizing Haar measure on N so that $N/Z\Gamma$ has volume 1. Let P be the corresponding Pfaffian polynomial on $\mathfrak{z}^*$. Note that $\Gamma \cap Z$ is a lattice in Z and $\exp^{-1}(\Gamma \cap Z)$ is a lattice (denote it Λ) in $\mathfrak{z}$. That defines the dual lattice Λ^* in $\mathfrak{z}^*$. Then a square integrable representation π_λ occurs in $\mathcal{L}^2(N/\Gamma)$ if and only if $\lambda \in \Lambda^*$, and in that case π_λ occurs with multiplicity $|P(\lambda)|$.*

Definition 5.3 Let $N = N_{\mathbb{R}}$ be defined over $\mathbb{Q}$ as in Proposition 5.1, so we have a fixed rational form $N_{\mathbb{Q}}$. We say that a connected Lie subgroup $L \subset N$ is *rational* if $L \cap N_{\mathbb{Q}}$ is a rational form of L, in other words if $\mathfrak{l} \cap \mathfrak{n}_{\mathbb{Q}}$ contains a basis of $\mathfrak{l}$. We say that a decomposition (5) is *rational* if the subgroups L_r and N_r are rational. $\diamond$

The following is immediate from this definition.

Lemma 5.4 *Let N be defined over $\mathbb{Q}$ as in* Proposition 5.1 *with rational structure defined by a discrete co-compact subgroup Γ. If the decomposition* (5) *is rational then each $\Gamma \cap Z_r$ in Z_r, each $\Gamma \cap L_r$ in L_r, each $\Gamma \cap S_r$ in S_r, and each $\Gamma \cap N_r$ in N_r, is a discrete co-compact subgroup defining the same rational structure as the one defined by its intersection with $N_{\mathbb{Q}}$.*

Now assume that N and Γ satisfy the rationality conditions of Lemma 5.4. Then for each r, $Z_r \cap \Gamma$ is a lattice in the center Z_r of L_r, and $\Lambda_r := \log(Z_r \cap \Gamma)$ is a lattice in its Lie algebra $\mathfrak{z}_r$. That defines the dual lattice Λ_r^* in $\mathfrak{z}_r^*$. We normalize the Pfaffian polynomials on the $\mathfrak{z}_r^*$, and thus the polynomial P on $\mathfrak{s}^*$, by requiring that the $N_r/(S_r \cdot (N_r \cap \Gamma))$ have volume 1.

Theorem 5.5 *Let $\lambda \in \mathfrak{t}^*$. Then a stepwise square integrable representation π_λ of N occurs in $\mathcal{L}^2(N/\Gamma)$ if and only if each $\lambda_r \in \Lambda_r^*$, and in that case the multiplicity of π_λ on $\mathcal{L}^2(N/\Gamma)$ is $|P(\lambda)|$.*

6 Commutative Spaces

A commutative space $X = G/K$, or equivalently a Gelfand pair (G, K), consists of a locally compact group G and a compact subgroup K such that the convolution algebra $\mathcal{L}^1(K\backslash G/K)$ is commutative. When G is a connected Lie group it is equivalent to say that the algebra $\mathcal{D}(G, K)$ of G-invariant differential operators on G/K is commutative. We say that the commutative space G/K is a commutative nilmanifold if it is a nilmanifold in the sense that some nilpotent analytic subgroup N of G acts transitively. When G/K is connected and simply connected it follows that N is the nilradical of G, that N acts simply transitively on G/K, and that G is the semidirect product group $N \rtimes K$, so that $G/K = (N \rtimes K)/K$. In this section we study the role of square integrability and stepwise square integrability for commutative nilmanifolds $G/K = (N \rtimes K)/K$.

The cases where G/K and (G, K) are *irreducible* in the sense that $[\mathfrak{n}, \mathfrak{n}]$ (which must be central) is the center of $\mathfrak{n}$ and K acts irreducibly on $\mathfrak{n}/[\mathfrak{n}, \mathfrak{n}]$, have been classified by E.B. Vinberg [22, 23]. See [26, Sect. 13.4B] for the Lie algebra structure $\mathfrak{v} \times \mathfrak{v} \to \mathfrak{z}$. The classification of commutative nilmanifolds is based on Vinberg's work and was completed by O. Yakimova in [34, 35].

It turns out that almost all commutative manifolds correspond to nilpotent groups that are square integrable. The exceptions are those with a certain direct factor, and in those cases the nilpotent group is stepwise square integrable in two steps, so in those cases the Plancherel formula follows directly from the general result above. See [31] for the details.

7 Minimal Parabolics: Subgroup *MN*

Fix an Iwasawa decomposition $G = KAN$ for a simple Lie group G and the minimal parabolic subgroup $Q = MAN$. As usual, write $\mathfrak{k}$ for the Lie algebra of K, $\mathfrak{a}$ for the Lie algebra of A, $\mathfrak{m}$ for the Lie algebra of M, and $\mathfrak{n}$ for the Lie algebra of N. Complete $\mathfrak{a}$ to a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. Then we have root systems $\Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$,

$\Delta(\mathfrak{g}, \mathfrak{a})$ and $\Delta_0(\mathfrak{g}, \mathfrak{a})$ described in (12). M is the centralizer of A in K. Write ^{0} for identity component; then $Q^0 = M^0 AN$.

Recall the Pf-nonsingular set $\mathfrak{t}^* = \{\lambda \in \mathfrak{s}^* \mid \mathrm{Pf}\,(b_\lambda) \neq 0\}$ of (6)(e); so $\mathrm{Ad}^*(M)\mathfrak{t}^* = \mathfrak{t}^*$. Further, if $\lambda \in \mathfrak{t}^*$ and $c \neq 0$ then $c\lambda \in \mathfrak{t}^*$, in fact $\mathrm{Pf}\,(b_{c\lambda}) = c^{\dim(\mathfrak{n}/\mathfrak{s})/2}\,\mathrm{Pf}\,(b_\lambda)$.

Fix an M-invariant inner product (μ, ν) on $\mathfrak{s}^*$. So $\mathrm{Ad}^*(M)$ preserves each sphere $\mathfrak{s}_t^* = \{\lambda \in \mathfrak{s}^* \mid (\lambda, \lambda) = t^2\}$. Two orbits $\mathrm{Ad}^*(M)\mu$ and $\mathrm{Ad}^*(M)\nu$ are of the *same orbit type* if the isotropy subgroups M_μ and M_ν are conjugate, and an orbit is *principal* if all nearby orbits are of the same type. Since M and $\mathfrak{s}_t^*$ are compact, there are only finitely many orbit types of M on $\mathfrak{s}_t^*$, there is only one principal orbit type, and the union of the principal orbits forms a dense open subset of $\mathfrak{s}_t^*$ whose complement has codimension $\geqq 2$. See [5, Chap. 4, Sect. 3] for a complete treatment of this material, or [10, Part II, Chap. 3, Sect. 1] modulo references to [5], or [18, Chap. 5] for a basic treatment, still with some references to [5].

The action of M on $\mathfrak{s}^*$ commutes with dilation so the structural results on the $\mathfrak{s}_t$ also hold on $\mathfrak{s}^* = \bigcup_{t>0} \mathfrak{s}_t^*$. Define the Pf-nonsingular principal orbit set as follows:

$$\mathfrak{u}^* = \{\lambda \in \mathfrak{t}^* \mid \mathrm{Ad}^*(M)\lambda \text{ is a principal } M\text{-orbit on } \mathfrak{s}^*\}. \tag{20}$$

Now principal orbit set $\mathfrak{u}^*$ is a dense open set with complement of codimension $\geqq 2$ in $\mathfrak{s}^*$. If $\lambda \in \mathfrak{u}^*$ and $c \neq 0$ then $c\lambda \in \mathfrak{u}^*$ with isotropy $M_{c\lambda} = M_\lambda$. If $\lambda \in \mathfrak{u}_t^* := \mathfrak{u}^* \cap \mathfrak{s}_t^*$, so $\mathrm{Ad}^*(M)\lambda$ is a Pf-nonsingular principal orbit of M on the sphere $\mathfrak{s}_t^*$, then $\mathrm{Ad}^*(M^0)\lambda$ is a principal orbit of M^0 on $\mathfrak{s}_t^*$. Principal orbit isotropy subgroups of compact connected linear groups are studied in [11] and the possibilities for the isotropy $(M^0)_\lambda$ are essentially known. The following lets us go from $(M^0)_\lambda$ to M_λ.

Proposition 7.1 ([28]) *Suppose that G is connected and linear. Then $M = FZ_G M^0$ where Z_G is the center of G, $F = (\exp(i\mathfrak{a}) \cap K)$ is an elementary abelian 2-group, and $Ad^*(F)$ acts trivially on $\mathfrak{s}^*$. If $\lambda \in \mathfrak{u}^*$ then the isotropy $M_\lambda = FZ_G(M^0)_\lambda$.*

Thus the groups M_λ are specified by the work of W.-C. and W.-Y. Hsiang [11].

Given $\lambda \in \mathfrak{u}^*$ the stepwise square integrable representation $\pi_\lambda \in \widehat{N}$ one proves that the Mackey obstruction $\varepsilon \in H^2(M_\lambda; U(1))$ is trivial, and in fact that π_λ extends to a unitary representation $\pi_\lambda^\dagger$ of $N \rtimes M_\lambda$ on the representation space of π_λ.

Each $\lambda \in \mathfrak{u}^*$ now defines classes

$$\mathcal{E}(\lambda) := \left\{\pi_\lambda^\dagger \otimes \gamma \mid \gamma \in \widehat{M_\lambda}\right\},\ \ \mathcal{F}(\lambda) := \left\{\mathrm{Ind}_{NM_\lambda}^{NM}(\pi_\lambda^\dagger \otimes \gamma) \mid \pi_\lambda^\dagger \otimes \gamma \in \mathcal{E}(\lambda)\right\} \tag{21}$$

of irreducible unitary representations of $N \rtimes M_\lambda$ and NM. The Mackey little group method, plus the fact that the Plancherel density on $\widehat{N}$ is polynomial on $\mathfrak{s}^*$, and $\mathfrak{s}^* \setminus \mathfrak{u}^*$ has measure 0 in $\mathfrak{t}^*$, gives us

Proposition 7.2 *Plancherel measure for NM is concentrated on $\bigcup_{\lambda \in \mathfrak{u}^*} \mathcal{F}(\lambda)$, equivalence classes of irreducible representations $\eta_{\lambda,\gamma} := \mathrm{Ind}_{NM_\lambda}^{NM}(\pi_\lambda^\dagger \otimes \gamma)$ such that $\pi_\lambda^\dagger \otimes \gamma \in \mathcal{E}(\lambda)$ and $\lambda \in \mathfrak{u}^*$. Further*

$$\eta_{\lambda,\gamma}|_N = \left(\mathrm{Ind}\,_{NM_\lambda}^{NM}(\pi_\lambda^\dagger \otimes \gamma)\right)\Big|_N = \int_{M/M_\lambda} (\dim\gamma)\,\pi_{\mathrm{Ad}^*(m)\lambda}\,d(mM_\lambda).$$

There is a Borel section σ to $\mathfrak{u}^* \to \mathfrak{u}^*/\mathrm{Ad}^*(M)$ that picks out an element in each M-orbit so that M has the same isotropy subgroup at each of those elements. In other words in each M-orbit on $\mathfrak{u}^*$ we measurably choose an element $\lambda = \sigma(\mathrm{Ad}^*(M)\lambda)$ such that those isotropy subgroups M_λ are all the same. Let us denote

$$M_\diamond\text{: isotropy subgroup of } M \text{ at } \sigma(\mathrm{Ad}^*(M)\lambda) \text{ for every } \lambda \in \mathfrak{u}^* \tag{22}$$

We replace M_λ by $M_\diamond$, independent of $\lambda \in \mathfrak{u}^*$, in Proposition 7.2. That lets us assemble to representations of Proposition 7.2 for a Plancherel Formula, as follows. Since M is compact, we have the Schwartz space $\mathcal{C}(NM)$ just as in the discussion of $\mathcal{C}(N)$ between (6) and Theorem 3.5, except that the pullback $\exp^*\mathcal{C}(NM) \neq \mathcal{C}(\mathfrak{n}+\mathfrak{m})$. The same applies to $\mathcal{C}(NA)$ and $\mathcal{C}(NAM)$.

Proposition 7.3 *Let $f \in \mathcal{C}(NM)$ and write $(f_m)(n) = f(nm) = ({}_n f)(m)$ for $n \in N$ and $m \in M$. The Plancherel density at* $\mathrm{Ind}\,_{NM_\diamond}^{NM}(\pi_\lambda^\dagger \otimes \gamma)$ *is* $(\dim\gamma)|\mathrm{Pf}\,(b_\lambda)|$ *and the Plancherel Formula for NM is*

$$f(nm) = c\int_{\mathfrak{u}^*/\mathrm{Ad}^*(M)} \sum_{\mathcal{F}(\lambda)} \mathrm{trace}\,\eta_{\lambda,\gamma}({}_n f_m)\cdot \dim(\gamma)\cdot|\mathrm{Pf}\,(b_\lambda)|d\lambda$$

where $c = 2^{d_1+\cdots+d_m} d_1!d_2!\dots d_m!$, *from* (6)*, as in* Theorem 3.5.

8 Minimal Parabolics: MAN and AN

Let G be a separable locally compact group of type I. Then [14, Sect. 1] the Plancherel formula for G has form

$$f(x) = \int_{\widehat{G}} \mathrm{trace}\,\pi(D(r(x)f))d\mu_G(\pi) \tag{23}$$

where D is an invertible positive self adjoint operator on $L^2(G)$, conjugation-semi-invariant of weight equal to the modular function δ_G, and μ is a positive Borel measure on the unitary dual $\widehat{G}$. If G is unimodular then D is the identity and (23) reduces to the usual Plancherel formula. The point is that semi-invariance of D compensates any lack of unimodularity. See [14, Sect. 1] for a detailed discussion. $D \otimes \mu$ is unique (up to normalization of Haar measures) and one tries to find a "best" choice of D. Given any such pair (D, μ) we refer to D as a *Dixmier–Pukánszky Operator* on G and to μ as the associated *Plancherel measure* on $\widehat{G}$. We will construct a Dixmier–Pukánszky Operator from the Pfaffian polynomial $\mathrm{Pf}\,(b_\lambda)$.

Let δ_{AN} and δ_Q denote the modular functions on AN and on $Q = MAN$. As M is compact and $\mathrm{Ad}_Q(N)$ is unipotent on $\mathfrak{p}$, they are determined by their restrictions to A, where they are given by $\delta(\exp(\xi)) = \exp(\mathrm{trace}\,(\mathrm{ad}(\xi)))$ with $\xi = \log a \in \mathfrak{a}$.

Lemma 8.1 *Let $\xi \in \mathfrak{a}$. Then $\frac{1}{2}(\dim \mathfrak{l}_r + \dim \mathfrak{z}_r) \in \mathbb{Z}$ for $1 \leqq r \leqq m$ and*
(i) *the trace of* $\mathrm{ad}(\xi)$ *on* $\mathfrak{l}_r$ *is* $\frac{1}{2}(\dim \mathfrak{l}_r + \dim \mathfrak{z}_r)\beta_r(\xi)$*,*
(ii) *the trace of* $\mathrm{ad}(\xi)$ *on* $\mathfrak{n}$ *and on* $\mathfrak{p}$ *is* $\frac{1}{2}\sum_r(\dim \mathfrak{l}_r + \dim \mathfrak{z}_r)\beta_r(\xi)$*,*
(iii) *the determinant of* $\mathrm{Ad}(\exp(\xi))$ *on* $\mathfrak{n}$ *and on* $\mathfrak{p}$ *is* $\prod_r \exp(\beta_r(\xi))^{\frac{1}{2}(\dim \mathfrak{l}_r + \dim \mathfrak{z}_r)}$*,*
(iv) $\delta_Q(man) = \prod_r \exp(\beta_r(\log a))^{\frac{1}{2}(\dim \mathfrak{l}_r + \dim \mathfrak{z}_r)}$ *and* $\delta_{AN} = \delta_Q|_{AN}$*.*

Now compute

Lemma 8.2 *Let $\xi \in \mathfrak{a}$ and $a = \exp(\xi) \in A$. Then* $\mathrm{ad}(\xi)\mathrm{Pf} = \left(\frac{1}{2}\sum_r \dim(\mathfrak{l}_r/\,\mathfrak{z}_r)\beta_r(\xi)\right)\mathrm{Pf}$ *and* $\mathrm{Ad}(a)\mathrm{Pf} = \left(\prod_r \exp(\beta_r(\xi))^{\frac{1}{2}\dim(\mathfrak{l}_r/\dim \mathfrak{z}_r)}\right)\mathrm{Pf}$*.*

At this point it is convenient to introduce some notation and definitions.

Definition 8.3 The algebra $\mathfrak{s}$ is the *quasi-center* of $\mathfrak{n}$. The polynomial function $\mathrm{Det}_{\mathfrak{s}^*}(\lambda) := \prod_r(\beta_r(\lambda))^{\dim \mathfrak{g}_{\beta_r}}$ on $\mathfrak{s}^*$ is the *quasi-center determinant.*

For $\xi \in \mathfrak{a}$ and $a = \exp(\xi) \in A$ compute $(\mathrm{Ad}(a)\mathrm{Det}_{\mathfrak{s}^*})(\lambda) = \mathrm{Det}_{\mathfrak{s}^*}(\mathrm{Ad}^*(a^{-1})(\lambda)) = \prod_r(\beta_r(\mathrm{Ad}(a^{-1})^*\lambda))^{\dim \mathfrak{g}_{\beta_r}} = \prod_r(\beta_r(\exp(\beta_r(\xi))\lambda))^{\dim \mathfrak{g}_{\beta_r}}$. In other words,

Lemma 8.4 *Let $a = \exp(\xi) \in A$. Then* $\mathrm{Ad}(a)\mathrm{Det}_{\mathfrak{s}^*} = \left(\prod_r \exp(\beta_r(\xi))^{\dim \mathfrak{z}_r}\right)\mathrm{Det}_{\mathfrak{s}^*}$ *where $\xi = \log a \in \mathfrak{a}$.*

Combining Lemmas 8.1, 8.2 and 8.4 we have

Proposition 8.5 *The product* $\mathrm{Pf}\cdot\mathrm{Det}_{\mathfrak{s}^*}$ *is an* $\mathrm{Ad}(MAN)$*-semi-invariant (and thus* $\mathrm{Ad}(AN)$*-semi-invariant) polynomial on $\mathfrak{s}^*$ of degree* $\frac{1}{2}(\dim \mathfrak{n} + \dim \mathfrak{s})$ *and of weight equal to the respective modular functions of Q and AN.*

From $\mathfrak{n} = \mathfrak{v} + \mathfrak{s}$ we have $N = VS$ where $V = \exp(\mathfrak{v})$ and $S = \exp(\mathfrak{s})$. Now define

$$D: \text{ Fourier transform of } \mathrm{Pf}\cdot\mathrm{Det}_{\mathfrak{s}^*}, \text{ acting on the } S \text{ variable of } N = VS. \tag{24}$$

Theorem 8.6 *The operator D of (24) is an invertible self-adjoint differential operator of degree* $\frac{1}{2}(\dim \mathfrak{n} + \dim \mathfrak{s})$ *on $L^2(MAN)$ with dense domain $\mathcal{C}(MAN)$, and it is* $\mathrm{Ad}(MAN)$*-semi-invariant of weight equal to the modular function δ_{MAN}. In other words $|D|$ is a Dixmier–Pukánszky Operator on MAN with domain equal to the space of rapidly decreasing C^∞ functions. This applies as well to AN.*

Since $\lambda \in \mathfrak{t}^*$ has nonzero projection on each summand $\mathfrak{z}_r^*$ of $\mathfrak{s}^*$, and $a \in A$ acts by the positive real scalar $\exp(\beta_r(\log(a)))$ on $\mathfrak{z}_r$,

$$A_\lambda = \exp(\{\xi \in \mathfrak{a} \mid \text{each } \beta_r(\xi) = 0\}), \text{ independent of } \lambda \in \mathfrak{t}^*. \tag{25}$$

Because of this independence, and using $\mathfrak{a}_\diamond = \{\xi \in \mathfrak{a} \mid \text{ each } \beta_r(\xi) = 0\}$, we define

$$A_\diamond = A_\lambda \text{ for any (and thus for all) } \lambda \in \mathfrak{t}^*. \tag{26}$$

Lemma 8.7 *If* $\lambda \in \sigma(\mathfrak{u}^*)$ *then the stabilizer* $(MA)_\lambda = M_\diamond A_\diamond$.

There is no problem with the Mackey obstruction:

Lemma 8.8 *Let* $\lambda \in \sigma(\mathfrak{u}^*)$. *Recall the extension* (before (21)) $\pi_\lambda^\dagger$ *of* π_λ *to* $NM_\diamond$. *Then* $\pi_\lambda^\dagger$ *extends to* $\widetilde{\pi_\lambda} \in \widehat{NM_\diamond A_\diamond}$ *with the same representation space as* π_λ.

When $\lambda \in \sigma(\mathfrak{u}^*)$, $\widehat{A_\diamond}$ consists of the unitary characters $\exp(i\phi) : a \mapsto e^{i\phi(\log a)}$ with $\phi \in \mathfrak{a}_\diamond^*$. The representations of Q corresponding to λ are the

$$\pi_{\lambda,\gamma,\phi} := \mathrm{Ind}_{NM_\diamond A_\diamond}^{NMA}(\widetilde{\pi_\lambda} \otimes \gamma \otimes \exp(i\phi)) \text{ where } \gamma \in \widehat{M_\diamond} \text{ and } \phi \in \mathfrak{a}_\diamond^*\,. \tag{27}$$

$\mathrm{Ad}^*(A)$ fixes γ because A centralizes M, and it fixes ϕ because A is commutative, so

$$\pi_{\lambda,\gamma,\phi} \cdot \mathrm{Ad}((ma)^{-1}) = \pi_{\mathrm{Ad}^*(ma)\lambda,\gamma,\phi} \tag{28}$$

Proposition 8.9 *Plancherel measure for* Q *is concentrated on the set of all* $\pi_{\lambda,\gamma,\phi}$ *for* $\lambda \in \sigma(\mathfrak{u}^*)$, $\gamma \in \widehat{M_\diamond}$ *and* $\phi \in \mathfrak{a}_\diamond^*$. *The equivalence class of* $\pi_{\lambda,\gamma,\phi}$ *depends only on* $(\mathrm{Ad}^*(MA)\lambda, \gamma, \phi)$.

Representations of AN are the case $\gamma = 1$. In effect, let π'_λ denote the obvious extension $\widetilde{\pi_\lambda}|_{AN}$ of the stepwise square integrable representation π_λ from N to $NA_\diamond$ where $\widetilde{\pi_\lambda}$ is given by Lemma 8.8. Denote

$$\pi_{\lambda,\phi} = \mathrm{Ind}_{NA_\diamond}^{NA}(\pi'_\lambda \otimes \exp(i\phi)) \text{ where } \lambda \in \mathfrak{u}^* \text{ and } \phi \in \mathfrak{a}_\diamond^*. \tag{29}$$

Corollary 8.10 *Plancherel measure for* AN *is concentrated on the set of all* $\pi_{\lambda,\phi}$ *for* $\lambda \in \mathfrak{u}^*$ *and* $\phi \in \mathfrak{a}_\diamond^*$. *The equivalence class of* $\pi_{\lambda,\phi}$ *depends only on* $(\mathrm{Ad}^*(MA)\lambda, \phi)$.

A result of C.C. Moore implies

Lemma 8.11 *The Pf-nonsingular principal orbit set* $\mathfrak{u}^*$ *is a finite union of open* $\mathrm{Ad}^*(MA)$*-orbits.*

Let $\{\mathcal{O}_1, \ldots, \mathcal{O}_v\}$ denote the (open) $\mathrm{Ad}^*(MA)$-orbits on $\mathfrak{u}^*$. Denote $\lambda_i = \sigma(\mathcal{O}_i)$, so $\mathcal{O}_i = \mathrm{Ad}^*(MA)\lambda_i$ and $(MA)_{\lambda_i} = M_\diamond A_\diamond$ for $1 \leqq i \leqq v$. Then Proposition 8.9 becomes

Theorem 8.12 *Plancherel measure for* MAN *is concentrated on the set* (*of equivalence classes of*) *unitary representations* $\pi_{\lambda_i,\gamma,\phi}$ *for* $1 \leqq i \leqq v$, $\gamma \in \widehat{M_\diamond}$ *and* $\phi \in \mathfrak{a}_\diamond^*$.

The Plancherel Formula (or Fourier Inversion Formula) for MAN is

Theorem 8.13 *Let* $Q = MAN$ *be a minimal parabolic subgroup of the real reductive Lie group* G. *Given* $\pi_{\lambda,\gamma,\phi} \in \widehat{MAN}$ *as described in* (27) *let* $\Theta_{\pi_{\lambda,\gamma,\phi}} : h \mapsto \mathrm{trace}\, \pi_{\lambda,\gamma,\phi}(h)$ *denote its distribution character. Then* $\Theta_{\pi_{\lambda,\gamma,\phi}}$ *is a tempered distribution. If* $f \in \mathcal{C}(MAN)$ *then*

$$f(x) = c \sum_{i=1}^{v} \sum_{\gamma \in \widehat{M_\diamond}} \int_{\mathfrak{a}_\diamond^*} \Theta_{\pi_{\lambda_i,\gamma,\phi}}(D(r(x)f))|\mathrm{Pf}\,(b_{\lambda_i})| \dim \gamma \; d\phi$$

where $c > 0$ *depends on normalizations of Haar measures.*

The Plancherel Theorem for NA follows similar lines. For the main computation in the proof of Theorem 8.13 we omit M and γ. That gives

$$\int_{\mathfrak{a}_\diamond^*} \mathrm{trace}\, \pi_{\lambda_0,\phi}(Dh)\, d\phi = \int_{\mathrm{Ad}^*(A)\lambda_0} \mathrm{trace}\, \pi_\lambda(h)|\mathrm{Pf}\,(b_\lambda)|d\lambda \tag{30}$$

In order to go from an $\mathrm{Ad}^*(A)\lambda_0$ to an integral over $\mathfrak{u}^*$ we use M to parameterize the space of $\mathrm{Ad}^*(A)$-orbits on $\mathfrak{u}^*$. If $\lambda \in \mathfrak{u}^*$ one proves $\mathrm{Ad}^*(A)\lambda \cap \mathrm{Ad}^*(M)\lambda = \{\lambda\}$. That leads to

Proposition 8.14 *Plancherel measure for* NA *is concentrated on the equivalence classes of representations* $\pi_{\lambda,\phi} = \mathrm{Ind}\,_{NA_\diamond}^{NA}(\pi'_\lambda \otimes \exp(i\phi))$ *where* $\lambda \in S_i := \mathrm{Ad}^*(M)\lambda_i$*,* $1 \leqq i \leqq v$*,* π'_λ *extends* π_λ *from* N *to* $NA_\diamond$ *and* $\phi \in \mathfrak{a}_\diamond^*$*. Representations* $\pi_{\lambda,\phi}$ *and* $\pi_{\lambda',\phi'}$ *are equivalent if and only if* $\lambda' \in \mathrm{Ad}^*(A)\lambda$ *and* $\phi' = \phi$*. Further,* $\pi_{\lambda,\phi}|_N = \int_{a \in A/A_\diamond} \pi_{\mathrm{Ad}^*(a)\lambda} da$*.*

Theorem 8.15 *Let* $Q = MAN$ *be a minimal parabolic subgroup of the real reductive Lie group* G*. If* $\pi_{\lambda,\phi} \in \widehat{AN}$ *let* $\Theta_{\pi_{\lambda,\phi}} : h \mapsto \mathrm{trace}\, \pi_{\lambda,\phi}(h)$ *denote its distribution character. Then* $\Theta_{\pi_{\lambda,\phi}}$ *is a tempered distribution. If* $f \in \mathcal{C}(AN)$ *then*

$$f(x) = c \sum_{i=1}^{v} \int_{\lambda \in \mathrm{Ad}*(M)\lambda_i} \int_{\mathfrak{a}_\diamond^*} \mathrm{trace}\, \pi_{\lambda,\phi}(D(r(x)f))|\mathrm{Pf}\,(b_\lambda)|d\lambda d\phi.$$

where $c > 0$ *depends on normalizations of Haar measures.*

9 Parabolic Subgroups in General: The Nilradical

In Sects. 7 and 8 we studied minimal parabolic subgroups $Q = MAN$ in simple Lie groups, along with certain of their subgroups MN and AN. This section and the next form a glance at more general parabolics. This material is taken from [32], which is a work in progress, and is limited to the part that I've written down. We start with the structure of the nilradical.

The condition (c) of (5) does not always hold for nilradicals of parabolic subgroups. In this section and the next we weaken (5) to

$$
\begin{aligned}
&N = L_1 L_2 \dots L_{m-1} L_m \text{ where}\\
&\qquad \text{(a) each} L_r \text{ has unitary representations with coefficients in } L^2(L_r/Z_r),\\
&\qquad \text{(b) each } N_r := L_1 L_2 \dots L_r = N_{r-1} \rtimes L_r \text{ semidirect,}\\
&\qquad \text{(c) if } r \geqq s \text{ then } [\mathfrak{l}_r, \mathfrak{z}_s] = 0.
\end{aligned}
\tag{31}
$$

The conditions of (31) are sufficient to construct stepwise square integrable representations, but are not always sufficient to compute the Pfaffian that is the Plancherel density. So we refer to (5) as the *strong computability condition* and make use of the *weak computability condition*

$$
\text{Let } \mathfrak{l}_r = \mathfrak{l}'_r \oplus \mathfrak{l}''_r \text{ where } \mathfrak{l}''_r \subset \mathfrak{z}_r \text{ and } \mathfrak{v}_r \subset \mathfrak{l}'_r; \text{ then } [\mathfrak{l}_r, \mathfrak{l}_s] \subset \mathfrak{l}''_s + \mathfrak{v}_s \text{ for } r > s. \tag{32}
$$

where we retain $\mathfrak{l}_r = \mathfrak{z}_r + \mathfrak{v}_r$ and $\mathfrak{n} = \mathfrak{s} + \mathfrak{v}$.

Consider an arbitrary parabolic subgroup of G. It contains a minimal parabolic $Q = MAN$. Let Ψ denote the set of simple roots for the positive system $\Delta^+(\mathfrak{g}, \mathfrak{a})$. Then the parabolic subgroups of G that contain Q are in one to one correspondence with the subsets $\Phi \subset \Psi$, say $Q_\Phi \leftrightarrow \Phi$, as follows. Denote $\Psi = \{\psi_i\}$ and set

$$
\begin{aligned}
\Phi^{red} &= \left\{\alpha = \sum\nolimits_{\psi_i \in \Psi} n_i \psi_i \in \Delta(\mathfrak{g}, \mathfrak{a}) \mid n_i = 0 \text{ whenever } \psi_i \notin \Phi\right\}\\
\Phi^{nil} &= \left\{\alpha = \sum\nolimits_{\psi_i \in \Psi} n_i \psi_i \in \Delta^+(\mathfrak{g}, \mathfrak{a}) \mid n_i > 0 \text{ for some } \psi_i \notin \Phi\right\}.
\end{aligned}
\tag{33}
$$

On the Lie algebra level, $\mathfrak{q}_\Phi = \mathfrak{m}_\Phi + \mathfrak{a}_\Phi + \mathfrak{n}_\Phi$ where

$$
\begin{aligned}
&\mathfrak{a}_\Phi = \{\xi \in \mathfrak{a} \mid \psi(\xi) = 0 \text{ for all } \psi \in \Phi\} = \Phi^\perp,\\
&\mathfrak{m}_\Phi + \mathfrak{a}_\Phi \text{ is the centralizer of } \mathfrak{a}_\Phi \text{ in } \mathfrak{g}, \text{ so } \mathfrak{m}_\Phi \text{ has root system } \Phi^{red}, \text{ and}\\
&\mathfrak{n}_\Phi = \sum\nolimits_{\alpha \in \Phi^{nil}} \mathfrak{g}_\alpha, \text{ nilradical of } \mathfrak{q}_\Phi, \text{ sum of the positive } \mathfrak{a}_\Phi\text{-root spaces.}
\end{aligned}
\tag{34}
$$

Since $\mathfrak{n} = \sum_r \mathfrak{l}_r$, as given in (17) and (18) we have

$$
\mathfrak{n}_\Phi = \sum\nolimits_r (\mathfrak{n}_\Phi \cap \mathfrak{l}_r) = \sum\nolimits_r \left((\mathfrak{g}_{\beta_r} \cap \mathfrak{n}_\Phi) + \sum\nolimits_{\Delta_r^+} (\mathfrak{g}_\alpha \cap \mathfrak{n}_\Phi)\right). \tag{35}
$$

As $\mathrm{ad}(\mathfrak{m})$ is irreducible on each restricted root space, if $\alpha \in \{\beta_r\} \cup \Delta_r^+$ then $\mathfrak{g}_\alpha \cap \mathfrak{n}_\Phi$ is 0 or all of $\mathfrak{g}_\alpha$.

Lemma 9.1 *Suppose* $\mathfrak{g}_{\beta_r} \cap \mathfrak{n}_\Phi = 0$. *Then* $\mathfrak{l}_r \cap \mathfrak{n}_\Phi = 0$.

Lemma 9.2 *Suppose* $\mathfrak{g}_{\beta_r} \cap \mathfrak{n}_\Phi \neq 0$. *Define* $J_r \subset \Delta_r^+$ *by* $\mathfrak{l}_r \cap \mathfrak{n}_\Phi = \mathfrak{g}_{\beta_r} + \sum_{J_r} \mathfrak{g}_\alpha$. *Decompose* $J_r = J'_r \cup J''_r$ *where* $J'_r = \{\alpha \in J_r \mid \sigma_r\alpha \in J_r\}$ *and* $J''_r = \{\alpha \in J_r \mid \sigma_r\alpha \notin J_r\}$. *Then* $\mathfrak{g}_{\beta_r} + \sum_{J''_r} \mathfrak{g}_\alpha$ *belongs to a single* $\mathfrak{a}_\Phi$*-root space in* $\mathfrak{n}_\Phi$, *i.e.* $\alpha|_{\mathfrak{a}_\Phi} = \beta_r|_{\mathfrak{a}_\Phi}$, *for every* $\alpha \in J''_r$.

Lemma 9.3 *Suppose* $\mathfrak{l}_r \cap \mathfrak{n}_\Phi \neq 0$. *Then the algebra* $\mathfrak{l}_r \cap \mathfrak{n}_\Phi$ *has center* $\mathfrak{g}_{\beta_r} + \sum_{J_r''} \mathfrak{g}_\alpha$, *and* $\mathfrak{l}_r \cap \mathfrak{n}_\Phi = (\mathfrak{g}_{\beta_r} + \sum_{J_r''} \mathfrak{g}_\alpha) + (\sum_{J_r'} \mathfrak{g}_\alpha))$. *Further,* $\mathfrak{l}_r \cap \mathfrak{n}_\Phi = \left(\sum_{J_r''} \mathfrak{g}_\alpha\right) \oplus \left(\mathfrak{g}_{\beta_r} + \left(\sum_{J_r'} \mathfrak{g}_\alpha\right)\right)$ *direct sum of ideals.*

It will be convenient to define sets of simple $\mathfrak{a}_\Phi$-roots

$$\Psi_1 = \Psi \text{ and } \Psi_{s+1} = \{\psi \in \Psi \mid \langle \psi, \beta_i \rangle = 0 \text{ for } 1 \leqq i \leqq s\}. \tag{36}$$

Note that Ψ_r is the simple root system for $\{\alpha \in \Delta^+(\mathfrak{g}, \mathfrak{a}) \mid \alpha \perp \beta_i \text{ for } i < r\}$.

Lemma 9.4 *If* $r > s$ *then* $[\mathfrak{l}_r \cap \mathfrak{n}_\Phi\ ,\ \mathfrak{g}_{\beta_s} + \sum_{J_s''} \mathfrak{g}_\alpha] = 0$.

For our dealings with arbitrary parabolics it is not sufficient to consider linear functionals on $\sum_r \mathfrak{g}_{\beta_r}$. Instead we have to look at linear functionals on $\sum_r \left(\mathfrak{g}_{\beta_r} + \sum_{J_r''} \mathfrak{g}_\alpha\right)$. of the form $\lambda = \sum \lambda_r$ where $\lambda_r \in \mathfrak{g}_{\beta_r}^*$ such that b_{λ_r} is nondegenerate on $\sum_r \sum_{J_r'} \mathfrak{g}_\alpha$. We know that (5)(c) holds for the nilradical of the minimal parabolic $\mathfrak{q}$ that contains $\mathfrak{q}_\Phi$. In view of Lemma 9.4 it follows that $b_\lambda(\mathfrak{l}_r, \mathfrak{l}_s) = \lambda([\mathfrak{l}_r, \mathfrak{l}_s] = 0$ for $r > s$. For this particular type of λ, the bilinear form b_λ has kernel $\sum_r \left(\mathfrak{g}_{\beta_s} + \sum_{J_s''} \mathfrak{g}_\alpha\right)$ and is nondegenerate on $\sum_r \sum_{J_r'} \mathfrak{g}_\alpha$. Then $N_\Phi = (L_1 \cap N_\Phi)(L_2 \cap N_\Phi) \ldots (L_m \cap N_\Phi)$ satisfies the first two conditions of (5). That is enough to carry out the construction of stepwise square integrable representations π_λ of N_Φ, but one needs to do more to deal with Pfaffian polynomials as in (5)(c) and (32).

Let $I_1 = \{i \mid \beta_i|_{\mathfrak{a}_\Phi} = \beta_{q_1}|_{\mathfrak{a}_\Phi}\}$ where q_1 is the first index of (5) with $\beta_{q_1}|_{\mathfrak{a}_\Phi} \neq 0$. Next, $I_2 = \{i \mid \beta_i|_{\mathfrak{a}_\Phi} = \beta_{q_2}|_{\mathfrak{a}_\Phi}\}$ where q_2 is the first index of (5) such that $q_2 \notin I_1$ and $\beta_{q_2}|_{\mathfrak{a}_\Phi} \neq 0$. Continuing as long as possible, $I_k = \{i \mid \beta_i|_{\mathfrak{a}_\Phi} = \beta_{q_k}|_{\mathfrak{a}_\Phi}\}$ where q_k is the first index of (5) such that $q_k \notin (I_1 \cup \cdots \cup I_{k-1})$ and $\beta_{q_k}|_{\mathfrak{a}_\Phi} \neq 0$. Then $I_1 \cup \cdots \cup I_\ell$ consists of all the indices i for which $\beta_i|_{\mathfrak{a}_\Phi} \neq 0$. For $1 \leqq j \leqq \ell$ define

$$\mathfrak{l}_{\Phi,j} = \sum\nolimits_{i \in I_j} (\mathfrak{l}_i \cap \mathfrak{n}_\Phi) = \left(\sum\nolimits_{i \in I_j} \mathfrak{l}_i\right) \cap \mathfrak{n}_\Phi \text{ and } \mathfrak{l}_{\Phi,j}^\dagger = \sum\nolimits_{k \geqq j} \mathfrak{l}_{\Phi,k}\,. \tag{37}$$

Lemma 9.5 *If* $k \geqq j$ *then* $[\mathfrak{l}_{\Phi,k}, \mathfrak{l}_{\Phi,j}] \subset \mathfrak{l}_{\Phi,j}$. *For each index* j, $\mathfrak{l}_{\Phi,j}$ *and* $\mathfrak{l}_{\Phi,j}^\dagger$ *are subalgebras of* $\mathfrak{n}_\Phi$ *and* $\mathfrak{l}_{\Phi,j}$ *is an ideal in* $\mathfrak{l}_{\Phi,j}^\dagger$.

Lemma 9.6 *If* $k > j$ *then* $[\mathfrak{l}_{\Phi,k}\ ,\ \mathfrak{l}_{\Phi,j}] \cap \sum_{i \in I_j} \mathfrak{g}_{\beta_i} = 0$.

In the notation of Lemma 9.2, if $r \in I_j$ then

$$\mathfrak{l}_r \cap \mathfrak{n}_\Phi = \mathfrak{l}_r' + \mathfrak{l}_r'' \text{ where } \mathfrak{l}_r' = \mathfrak{g}_{\beta_r} + \sum\nolimits_{J_r'} \mathfrak{g}_\alpha \text{ and } \mathfrak{l}_r'' = \sum\nolimits_{J_r''} \mathfrak{g}_\alpha\,. \tag{38}$$

For $1 \leqq j \leqq \ell$ define

$$\mathfrak{z}_{\Phi,j} = \sum\nolimits_{i \in I_j} (\mathfrak{g}_{\beta_i} + \mathfrak{l}_i'') \tag{39}$$

and decompose

$$\mathfrak{l}_{\Phi,j} = \mathfrak{l}'_{\Phi,j} + \mathfrak{l}''_{\Phi,j} \text{ where } \mathfrak{l}'_{\Phi,j} = \sum\nolimits_{i\in I_j} \mathfrak{l}'_i \text{ and } \mathfrak{l}''_{\Phi,j} = \sum\nolimits_{i\in I_j} \mathfrak{l}''_i\,. \tag{40}$$

Lemma 9.7 *Recall* $\mathfrak{l}^{\dagger}_{\Phi,j} = \sum_{k\geqq j} \mathfrak{l}_{\Phi,k}$ *from* (37). *For each* j, *both* $\mathfrak{z}_{\Phi,j}$ *and* $\mathfrak{l}''_{\Phi,j}$ *are central ideals in* $\mathfrak{l}^{\dagger}_{\Phi,j}$, *and* $\mathfrak{z}_{\Phi,j}$ *is the center of* $\mathfrak{l}_{\Phi,j}$.

Decompose

$$\mathfrak{n}_{\Phi} = \mathfrak{z}_{\Phi} + \mathfrak{v}_{\Phi} \text{ where } \mathfrak{z}_{\Phi} = \sum_j \mathfrak{z}_{\Phi,j}\ ,\ \mathfrak{v}_{\Phi} = \sum_j \mathfrak{v}_{\Phi,j} \text{ and } \mathfrak{v}_{\Phi,j} = \sum_{i\in I_j}\sum_{\alpha\in J'_i} \mathfrak{g}_{\alpha}\,. \tag{41}$$

Then Lemma 9.7 gives us (32) for the $\mathfrak{l}_{\Phi,j}$: $\mathfrak{l}_{\Phi,j} = \mathfrak{l}'_{\Phi,j} \oplus \mathfrak{l}''_{\Phi,j}$ with $\mathfrak{l}''_{\Phi,j} \subset \mathfrak{z}_{\Phi,j}$ and $\mathfrak{v}_{\Phi,j} \subset \mathfrak{l}'_{\Phi,j}$.

Lemma 9.8 *For generic* $\lambda_j \in \mathfrak{z}^*_{\Phi,j}$ *the kernel of* b_{λ_j} *on* $\mathfrak{l}_{\Phi,j}$ *is just* $\mathfrak{z}_{\Phi,j}$, *in other words* b_{λ_j} *is nondegenerate on* $\mathfrak{v}_{\Phi,j} \simeq \mathfrak{l}_{\Phi,j}/\mathfrak{z}_{\Phi,j}$. *In particular* $L_{\Phi,j}$ *has square integrable representations.*

Theorem 9.9 *Let* G *be a real reductive Lie group and* Q *a real parabolic subgroup. Express* $Q = Q_{\Phi}$ *in the notation of* (33) *and* (34). *Then its nilradical* N_{Φ} *has decomposition* $N_{\Phi} = L_{\Phi,1}L_{\Phi,2}\dots L_{\Phi,\ell}$ *that satisfies the conditions of* (5) *and* (32) *as follows. The center* $Z_{\Phi,j}$ *of* $L_{\Phi,j}$ *is the analytic subgroup for* $\mathfrak{z}_{\Phi,j}$ *and*

$$\begin{aligned}
&\text{(a) } \textit{each } L_{\Phi,j} \textit{ has unitary representations with coefficients in } L^2(L_{\Phi,j}/Z_{\Phi,j})\\
&\text{(b) } \textit{each } N_{\Phi,j} := L_{\Phi,1}L_{\Phi,2}\dots L_{\Phi,j} \textit{ is a normal subgroup of } N_{\Phi}\\
&\qquad\qquad\qquad\qquad \textit{with } N_{\Phi,j} = N_{\Phi,j-1} \rtimes L_{\Phi,j} \textit{ semidirect,}\\
&\text{(c) } [\mathfrak{l}_{\Phi,k}, \mathfrak{z}_{\Phi,j}] = 0 \textit{ and } [\mathfrak{l}_{\Phi,k}, \mathfrak{l}_{\Phi,j}] \subset \mathfrak{v}_{\Phi,j} + \mathfrak{l}''_{\Phi,j} \textit{ for } k > j.
\end{aligned} \tag{42}$$

In particular N_{Φ} *has stepwise square integrable representations relative to the decomposition* $N_{\Phi} = L_{\Phi,1}L_{\Phi,2}\dots L_{\Phi,\ell}$.

10 Amenable Subgroups of Semisimple Lie Groups

In this section we apply the results of Sect. 9 to certain important subgroups of the parabolic $Q_{\Phi} = M_{\Phi}A_{\Phi}N_{\Phi}$, specifically its amenable subgroups $A_{\Phi}N_{\Phi}$, $U_{\Phi}N_{\Phi}$ and $U_{\Phi}A_{\Phi}N_{\Phi}$ where U_{Φ} is a maximal compact subgroup of M_{Φ}.

The theory of the group $U_{\Phi}N_{\Phi}$ goes exactly as in Sect. 7. When $N_{\Phi} = L_{\Phi,1}L_{\Phi,2}\dots L_{\Phi,\ell}$ is weakly invariant we can proceed more or less as in [28]. The argument, but not the final result, will make use of

Definition 10.1 The decomposition $N_\Phi = L_{\Phi,1}L_{\Phi,2}\dots L_{\Phi,\ell}$ of Theorem 9.9 is *invariant* if each $\mathrm{ad}(\mathfrak{m}_\Phi)\mathfrak{z}_{\Phi,j} = \mathfrak{z}_{\Phi,j}$, equivalently if each $\mathrm{Ad}(M_\Phi)\mathfrak{z}_{\Phi,j} = \mathfrak{z}_{\Phi,j}$, in other words whenever $\mathfrak{z}_{\Phi,j} = \mathfrak{g}_{[\Phi,\beta_{j_0}]}$. The decomposition $N_\Phi = L_{\Phi,1}L_{\Phi,2}\dots L_{\Phi,\ell}$ is *weakly invariant* if each $\mathrm{Ad}(U_\Phi)\mathfrak{z}_{\Phi,j} = \mathfrak{z}_{\Phi,j}$. ◇

Set

$$\mathfrak{r}^*_\Phi = \{\lambda \in \mathfrak{s}^*_\Phi \mid P(\lambda) \neq 0 \text{ and } \mathrm{Ad}(U_\Phi)\lambda \text{ is a principal } U_\Phi\text{-orbit on } \mathfrak{s}^*_\Phi\}. \tag{43}$$

Then $\mathfrak{r}^*_\Phi$ is dense, open and U_Φ-invariant in $\mathfrak{s}^*_\Phi$. By definition of principal orbit the isotropy subgroups of U_Φ at the various points of $\mathfrak{r}^*_\Phi$ are conjugate, and we take a measurable section σ to $\mathfrak{r}^*_\Phi \to U_\Phi\backslash\mathfrak{r}^*_\Phi$ on whose image all the isotropy subgroups are the same,

$$U'_\Phi : \text{ isotropy subgroup of } U_\Phi \text{ at } \sigma(U_\Phi(\lambda)), \text{ independent of } \lambda \in \mathfrak{r}^*_\Phi\,. \tag{44}$$

The principal isotropy subgroups U'_Φ are pinned down in [11]. Given $\lambda \in \mathfrak{r}^*_\Phi$ and $\gamma \in \widehat{U'_\Phi}$ let $\pi^\dagger_\lambda$ denote the extension of π_λ to a representation of $U'_\Phi N_\Phi$ on the space of π_λ and define

$$\pi_{\lambda,\gamma} = \mathrm{Ind}\,_{U'_\Phi N_\Phi}^{U_\Phi N_\Phi}(\gamma \otimes \pi^\dagger_\lambda). \tag{45}$$

The first result in this setting, as in [28, Proposition 3.3], is

Theorem 10.2 *Suppose that $N_\Phi = L_{\Phi,1}L_{\Phi,2}\dots L_{\Phi,\ell}$ as in (31). Then the Plancherel density on $\widehat{U_\Phi N_\Phi}$ is concentrated on the representations $\pi_{\lambda,\gamma}$ of (45), the Plancherel density at $\pi_{\lambda,\gamma}$ is $(\dim\gamma)|P(\lambda)|$, and the Plancherel Formula for $U_\Phi N_\Phi$ is*

$$f(un) = c\int_{\mathfrak{r}^*_\Phi/\mathrm{Ad}^*(U_\Phi)} \sum\nolimits_{\gamma\in\widehat{U'_\Phi}} \mathrm{trace}\,\mathrm{Ind}\,_{U'_\Phi N_\Phi}^{U_\Phi N_\Phi} r_{un}(f)\cdot\dim(\gamma)\cdot|P(\lambda)|d\lambda$$

where $c = 2^{d_1+\dots+d_\ell}d_1!d_2!\dots d_\ell!$ as in (6).

Recall the notion of amenability. A *mean* on a locally compact group H is a linear functional μ on $L^\infty(H)$ of norm 1 and such that $\mu(f) \geqq 0$ for all real-valued $f \geqq 0$. H is *amenable* if it has a left-invariant mean. Solvable groups and compact groups are amenable, as are extensions of amenable groups by amenable subgroups. In particular $E_\Phi := U_\Phi A_\Phi N_\Phi$ and its closed subgroups are amenable.

We need a technical condition [16, p. 132]. Let H be the group of real points in a linear algebraic group whose rational points are Zariski dense, let A be a maximal $\mathbb{R}$-split torus in H, let $Z_H(A)$ denote the centralizer of A in H, and let H_0 be the algebraic connected component of the identity in H. Then H is *isotropically connected* if $H = H_0 \cdot Z_H(A)$. More generally we will say that a subgroup $H \subset G$ is *isotropically connected* if the algebraic hull of $\mathrm{Ad}_G(H)$ is isotropically connected.

Proposition 10.2 [16, Theorem 3.2]. *The groups $E_\Phi := U_\Phi A_\Phi N_\Phi$ are maximal amenable subgroups of G. They are isotropically connected and self-normalizing.*

The various $\Phi \subset \Psi$ are mutually non-conjugate. An amenable subgroup $H \subset G$ is contained in some E_Φ if and only if it is isotropically connected.

The isotropy subgroups are the same at every $\lambda \in \mathfrak{t}^*_\Phi$,

$$A'_\Phi : \text{ isotropy subgroup of } A_\Phi \text{ at } \lambda \in \mathfrak{r}^*_\Phi \,. \tag{46}$$

Given a stepwise square integrable representation π_λ where $\lambda \in \mathfrak{s}^*_\Phi$, write $\pi^\dagger_\lambda$ for the extension of π_λ to a representation of $A'_\Phi N_\Phi$ on the same Hilbert space. That extension exists because the Mackey obstruction vanishes. The representations of $A'_\Phi N_\Phi$ corresponding to π_λ are the

$$\pi_{\lambda,\phi} := \mathrm{Ind}\,^{A_\Phi N_\Phi}_{A'_\Phi N_\Phi}(\exp(i\phi) \otimes \pi^\dagger_\lambda) \text{ where } \phi \in \mathfrak{a}'_\Phi \,. \tag{47}$$

Note also that

$$\pi_{\lambda,\phi} \cdot \mathrm{Ad}(an) = \pi_{\mathrm{Ad}^*(a)\lambda,\phi} \text{ for } a \in A_\Phi \text{ and } n \in N_\Phi \,. \tag{48}$$

The resulting formula $f(x) = \int_{\widehat{H}} \mathrm{trace}\, \pi(D(r(x)f))d\mu_H(\pi)$, $H = A_\Phi N_\Phi$, is

Theorem 10.3 *Let $Q_\Phi = M_\Phi A_\Phi N_\Phi$ be a parabolic subgroup of the real reductive Lie group G. Given $\pi_{\lambda,\phi} \in \widehat{A_\Phi N_\Phi}$ as described in (47), its distribution character $\Theta_{\pi_{\lambda,\phi}} : h \mapsto \mathrm{trace}\, \pi_{\lambda,\phi}(h)$ is a tempered distribution. If $f \in \mathcal{C}(A_\Phi N_\Phi)$ then*

$$f(x) = c \int_{(\mathfrak{a}'_\Phi)^*} \left(\int_{\mathfrak{s}^*_\Phi/\mathrm{Ad}^*(A_\Phi)} \Theta_{\pi_{\lambda,\phi}}(D(r(x)f))|\mathrm{Pf}\,(b_\lambda)|d\lambda \right) d\phi$$

where $c = 2^{d_1+\cdots+d_\ell} d_1!d_2!\ldots d_\ell!$.

The representations of $U_\Phi A_\Phi N_\Phi$ corresponding to π_λ are the

$$\pi_{\lambda,\phi,\gamma} := \mathrm{Ind}\,^{U_\Phi A_\Phi N_\Phi}_{U'_\Phi A'_\Phi N_\Phi}(\gamma \otimes \exp(i\phi) \otimes \pi^\dagger_\lambda) \text{ where } \phi \in \mathfrak{a}'_\Phi \text{ and } \gamma \in \widehat{U'_\Phi} \,. \tag{49}$$

Combining Theorems 10.2 and 10.3 we arrive at

Theorem 10.4 *Let $Q_\Phi = M_\Phi A_\Phi N_\Phi$ be a parabolic subgroup of the real reductive Lie group G and decompose $N_\Phi = L_{\Phi,1} L_{\Phi,2} \ldots L_{\Phi,\ell}$ as in (31). Then the Plancherel density on $\widehat{U_\Phi A_\Phi N_\Phi}$ is concentrated on the $\pi_{\lambda,\phi,\gamma}$ of (49), the Plancherel density at $\pi_{\lambda,\phi,\gamma}$ is $(\dim \gamma)|P(\lambda)|$, the distribution character $\Theta_{\pi_{\lambda,\phi,\gamma}} : h \mapsto \mathrm{trace}\, \pi_{\lambda,\phi,\gamma}(h)$ is tempered, and if $f \in \mathcal{C}(U_\Phi A_\Phi N_\Phi)$ then*

$$f(x) = c \sum_{\widehat{U'_\Phi}} \int_{(\mathfrak{a}'_\Phi)^*} \left(\int_{\mathfrak{s}^*_\Phi/\mathrm{Ad}^*(U_\Phi A_\Phi)} \Theta_{\pi_{\lambda,\phi,\gamma}}(D(r(x)f))\, \deg(\gamma)\, |\mathrm{Pf}\,(b_\lambda)|d\lambda \right) d\phi$$

where $c = 2^{d_1+\cdots+d_\ell} d_1!d_2!\ldots d_\ell!$.

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Higher-Dimensional Unified Theories with Continuous and Fuzzy Coset Spaces as Extra Dimensions

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Abstract We first briefly review the Coset Space Dimensional Reduction (CSDR) programme and present the results of the best model so far, based on the $\mathcal{N}=1$, $d=10$, E_8 gauge theory reduced over the nearly-Kähler manifold $SU(3)/U(1)\times U(1)$. Then, we present the adjustment of the CSDR programme in the case that the extra dimensions are considered to be fuzzy coset spaces and then, the best model constructed in this framework, too, which is the trinification GUT, $SU(3)^3$.

1 Introduction

During the last decades, unification of the fundamental interactions has focused the interest of theoretical physicists. This has led to the rise of very interesting and well-established approaches. Important and appealing are the ones that elaborate extra dimensions. A consistent framework in this approach is superstring theories [1] with the Heterotic String [2] (defined in ten dimensions) being the most promising, due to the possibility that in principle could lead to experimentally testable predictions. More specifically, the compactification of the 10-dimensional spacetime and the dimensional reduction of the $E_8\times E_8$ initial gauge theory lead to phenomenologically interesting Grand Unified theories (GUTs), containing the SM gauge group.

A few years before the development of the superstring theories, another important framework aiming at the same direction was employed, that is the dimensional reduction of higher-dimensional gauge theories. Pioneers in this field were Forgacs-

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V. Dobrev (ed.), *Lie Theory and Its Applications in Physics*,
Springer Proceedings in Mathematics & Statistics 191,
DOI 10.1007/978-981-10-2636-2_13

Manton and Scherk-Schwartz studying the Coset Space Dimensional Reduction (CSDR) [3–5] and Scherk-Schwarz group manifold reduction [6], respectively. In both of these approaches, the higher-dimensional gauge fields are unifying the gauge and scalar fields, while the 4-dimensional theory contains the surviving components after the procedure of the dimensional reduction. Moreover, in the CSDR scheme, the inclusion of fermionic fields in the initial theory leads to Yukawa couplings in the 4-dimensional theory. Furthermore, upgrading the higher-dimensional gauge theory to $\mathcal{N} = 1$ supersymmetric, i.e. grouping the gauge and fermionic fields of the theory into the same vector supermultiplet, is a way to unify further the fields of the initial theory, in certain dimensions [7, 8]. A very remarkable achievement of the CSDR scheme is the possibility of obtaining chiral theories in four dimensions [9, 10].

The above context of the CSDR adopted some very welcome suggestions coming from the superstring theories (specifically from the Heterotic String [2]), that is the dimensions of the space-time and the gauge group of the higher-dimensional supersymmetric theory. In addition, taking into account the fact that the superstring theories are consistent only in ten dimensions, the following important issues have to be addressed, (a) distinguish the extra dimensions from the four observable ones by considering an appropriate compactification of the metric and (b) determine the resulting 4-dimensional theory. Additionally, a suitable choice of the compactification manifolds could result into $\mathcal{N} = 1$ supersymmetry, aiming for a chance to be led to realistic GUTs.

Aiming at the preservation of an $\mathcal{N} = 1$ supersymmetry after the dimensional reduction, Calabi–Yau (CY) spaces serve as suitable compact internal manifolds [11]. However, the emergence of the moduli stabilization problem, led to the study of flux compactification, in the context of which a wider class of internal spaces, called manifolds with $SU(3)$-structure, was suggested. In this class of manifolds, a non-vanishing, globally defined spinor is admitted. This spinor is covariantly constant with respect to a connection with torsion, versus the CY case, where the spinor is constant with respect to the Levi-Civita connection. Here, we consider the nearly-Kähler manifolds, that is an interesting class of $SU(3)$-structure manifolds [12–15]. The class of homogeneous nearly-Kähler manifolds in six dimensions consists of the non-symmetric coset spaces $G_2/SU(3)$, $Sp(4)/(SU(2) \times U(1))_{non-max}$ and $SU(3)/U(1) \times U(1)$ and the group manifold $SU(2) \times SU(2)$ [15] (see also [12–14]). It is worth mentioning that 4-dimensional theories which are obtained after the dimensional reduction of a 10-dimensional $\mathcal{N} = 1$ supersymmetric gauge theory over non-symmetric coset spaces, contain supersymmetry breaking terms [16, 17], contrary to CY spaces.

Another very interesting framework which admits a description of physics at the Planck scale is non-commutative geometry [18–38]. Regularizing quantum field theories, or even better, building finite ones are the features that render it as a promising framework. On the other hand, the construction of quantum field theories on non-commutative spaces is a difficult task and, furthermore, problematic ultraviolet features have emerged [21] (see also [22, 23]. However, non-commutative geometry is an appropriate framework to accommodate particle models with non-commutative gauge theories [24] (see also [25–27]).

It is remarkable that the two frameworks (superstring theories and non-commutative geometry) found contact, after the realization that, in M-theory and open String theory, the effective physics on D-branes can be described by a non-commutative gauge theory [28, 29], if a non-vanishing background antisymmetric field is present. Moreover, the type IIB superstring theory (and others related with type IIB with certain dualities) in its conjectured non-perturbative formulation as a matrix model [30], is a non-commutative theory. In the framework of non-commutative geometry, Seiberg and Witten [29] contributed the most with their study (map between commutative and non-commutative gauge theories) based on which notable developments [31, 32] were achieved and afterwards a non-commutative version of the SM was constructed [33]. Unfortunately, such extensions fail to solve the main problem of the SM, which is the presence of many free parameters.

A very interesting development in the framework of the non-commutative geometry is the programme in which the extra dimensions of higher-dimensional theories are considered to be non-commutative (fuzzy) [34–38]. This programme overcomes the ultraviolet/infrared problematic behaviours of theories defined in non-commutative spaces. A very welcome feature of such theories is that they are renormalizable, versus all known higher-dimensional theories. This aspect of the theory was examined from the 4-dimensional point of view too, using spontaneous symmetry breakings which mimic the results of the dimensional reduction of a higher-dimensional gauge theory with non-commutative (fuzzy) extra dimensions. In addition, another interesting feature is that in theories constructed in this programme, there is an option of choosing the initial higher-dimensional gauge theory to be abelian. Then, non-abelian gauge theories result in lower dimensions in the process of the dimensional reduction over fuzzy coset spaces. Finally, the important problem of chirality in this framework has been addressed by applying an orbifold projection on a $\mathcal{N} = 4$ SYM theory. After the orbifolding, the resulting theory is an $\mathcal{N} = 1$ supersymmetric, chiral $SU(3)^3$.

2 The Coset Space Dimensional Reduction of a D-Dimensional YMD Lagrangian

An obvious and crude way to realize a dimensional reduction of a higher-dimensional gauge theory is to demand that all the fields of the theory are independent of the extra coordinates (trivial reduction) and therefore the Lagrangian is independent, too. A much more elegant way is to allow for a non-trivial dependence considering that a symmetry transformation on the fields by an element that belongs in the isometry group S of the compact coset space $B = S/R$ formed by the extra dimensions is a gauge transformation (symmetric fields). Therefore, the a priori consideration of the Lagrangian as gauge invariant, renders it independent of the extra coordinates. The above way of getting rid of the extra dimensions is the basic concept of the CSDR scheme [3–5].

Let us now consider the action of the D-dimensional YM theory with gauge symmetry G, coupled to fermions defined on M^D with metric g^{MN}

$$A = \int d^4x d^d y \sqrt{-g} \left[-\frac{1}{4} Tr(F_{MN} F_{K\Lambda}) g^{MK} g^{N\Lambda} + \frac{i}{2} \bar{\psi} \Gamma^M D_M \psi \right], \qquad (1)$$

where $D_M = \partial_M - \theta_M - A_M$, with $\theta_M = \frac{1}{2}\theta_{MN\Lambda}\Sigma^{N\Lambda}$ the spin connection of M^D and $F_{MN} = \partial_M A_N - \partial_N A_M - [A_M, A_N]$, where $M, \mathcal{N} = 1 \dots D$ and A_M, ψ are D-dimensional symmetric fields. The fermions can be accommodated in any representation F of G, unless an additional symmetry, e.g. supersymmetry, is considered.

Let ξ_A^α, ($A = 1, \dots, dimS$ and $\alpha = dimR + 1, \dots, dimS$ the curved index) be the Killing vectors which generate the symmetries of S/R and W_A, the gauge transformation associated with ξ_A. The following constraint equations for scalar ϕ, vector A_α and spinor ψ fields on S/R, derive from the definition of the symmetric fields, that is the $S-$transformations of the fields are gauge transformations

$$\delta_A \phi = \xi_A^\alpha \partial_\alpha \phi = D(W_A)\phi, \qquad (2)$$

$$\delta_A A_\alpha = \xi_A^\beta \partial_\beta A_\alpha + \partial_\alpha \xi_A^\beta A_\beta = \partial_\alpha W_A - [W_A, A_\alpha], \qquad (3)$$

$$\delta_A \psi = \xi_A^\alpha \partial_\alpha \psi - \frac{1}{2} G_{Abc} \Sigma^{bc} \psi = D(W_A)\psi, \qquad (4)$$

where W_A depend only on internal coordinates y and $D(W_A)$ represents a gauge transformation in the corresponding representation where the fields belong. Solving the above constraints (2)–(4), we result with [3, 4] the unconstrained 4-dimensional fields, as well as with the remaining 4-dimensional gauge symmetry.

We proceed by analysing the constraints on the fields in the theory. We start with the gauge field A_M on M_D, which splits into its components as (A_μ, A_α) corresponding to M^4 and S/R, respectively. Solving the corresponding constraint, (3), we obtain the following information: First, the 4-dimensional gauge field, A_μ is completely independent of the coset space coordinates and second, the 4-dimensional gauge fields commute with the generators of the subgroup R in G. This means that the surviving gauge symmetry, H, is the subgroup of G that commutes with R, that is the centralizer of R in G, i.e. $H = C_G(R_G)$. The $A_\alpha(x, y) \equiv \phi_\alpha(x, y)$, transform as scalars in the 4-dimensional theory and $\phi_\alpha(x, y)$ act as intertwining operators connecting induced representations of R acting on G and S/R. In order to find the representation in which the scalars are accommodated in the 4-dimensional theory, we have to decompose G according to the embedding

$$G \supset R_G \times H, \qquad adjG = (adjR, 1) + (1, adjH) + \sum (r_i, h_i), \qquad (5)$$

and S under R

$$S \supset R, \qquad adjS = adjR + \sum s_i . \qquad (6)$$

Therefore, we conclude that for every pair r_i, s_i, where r_i and s_i are identical irreducible representations of R, there remains a scalar (Higgs) multiplet which transforms under the representation h_i of H. All other scalar fields vanish.

As far as the spinors are concerned [4, 9, 10, 39], the analysis of the corresponding constraint, (4), is quite similar. Again, solving the constraint, one finds that the spinors in the 4-dimensional theory are independent of the coset coordinates and act as intertwining operators connecting induced representations of R in $SO(d)$ and in G. In order to obtain the representation of H, where the fermions are accommodated in the resulting 4-dimensional theory, one has to decompose the initial representation F of G under the $R_G \times H$,

$$G \supset R_G \times H\,, \qquad F = \sum (r_i, h_i), \tag{7}$$

and the spinor of $SO(d)$ under R

$$SO(d) \supset R\,, \qquad \sigma_d = \sum \sigma_j\,. \tag{8}$$

Therefore, for each pair r_i and σ_i, where r_i and σ_i are identical irreducible representations, there exists a multiplet, h_i of spinor fields in the 4-dimensional theory. As for the chirality of the surviving fermions, if one begins with Dirac fermions in the higher-dimensional theory it is impossible to result with chiral fermions in the 4-dimensional theory. Further requirements have to be imposed in order to result with chiral fermions in the 4-dimensional theory. Indeed, imposing the Weyl condition in the chiral representations of an even higher-dimensional initial theory, one is led to a chiral theory in four dimensions. This is not the case in an odd higher-dimensional initial theory, in which Weyl condition cannot be imposed. The most interesting case is the $D = 2n + 2$ even higher dimensional initial theory, in which starting with fermions in the adjoint representation the Weyl condition leads to two sets of chiral fermions with the same quantum numbers under H of the 4-dimensional theory. This doubling of the fermionic spectrum can be eliminated after imposing the Majorana condition. The two conditions are compatible when $D = 4n + 2$, which is the case of our interest.

Now, let us move on and determine the 4-dimensional effective action. The first and very important step is to compactify the space M^D to $M^4 \times S/R$, with S/R a compact coset space. After the compactification, the metric will be transformed to

$$g^{MN} = \begin{pmatrix} \eta^{\mu\nu} & 0 \\ 0 & -g^{ab} \end{pmatrix}, \tag{9}$$

where $\eta^{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$ and g^{ab} is the metric of the coset. Inserting the (9) into the initial action and taking into account the constraints of the fields, we obtain

$$A = C \int d^4x \left[-\frac{1}{4} F^t_{\mu\nu} F^{t\mu\nu} + \frac{1}{2} (D_\mu \phi_\alpha)^t (D^\mu \phi^\alpha)^t + V(\phi) + \frac{i}{2} \bar{\psi} \Gamma^\mu D_\mu \psi - \right.$$
$$\left. - \frac{i}{2} \bar{\psi} \Gamma^a D_a \psi \right] , \tag{10}$$

where $D_\mu = \partial_\mu - A_\mu$ and $D_a = \partial_a - \theta_a - \phi_a$, with $\theta_a = \frac{1}{2}\theta_{abc}\Sigma^{bc}$ the connection of the space and C is the volume of the space. The potential $V(\phi)$ is given by the following expression

$$V(\phi) = -\frac{1}{4} g^{ac} g^{bd} Tr(f^C_{ab}\phi_C - [\phi_a, \phi_b])(f^D_{cd}\phi_D - [\phi_c, \phi_d]), \tag{11}$$

where, $A = 1, \ldots, dimS$ and f's are the structure constants appearing in the commutators of the Lie algebra of S. Considering the constraints of the fields, (2)–(3), one finds that the scalar fields ϕ_a have to obey the following equation:

$$f^D_{ai}\phi_D - [\phi_a, \phi_i] = 0 , \tag{12}$$

where the ϕ_i are the generators of R_G. Consequently, some fields will be filtered out, while others will survive the reduction and will be identified as the genuine Higgs fields. The potential $V(\phi)$, written down in terms of the surviving scalars (the Higgs fields), is a quartic polynomial which is invariant under the 4-dimensional gauge group, H. Then, it follows the determination of the minimum of the potential and the finding of the remaining gauge symmetry of the vacuum [40–42]. In general, this is a rather difficult procedure. However, there is a case in which one could obtain the result of the spontaneous breaking of the gauge group H very easily, whether the following criterion is satisfied. Whenever S has an isomorphic image S_G in G, then the 4-dimensional gauge group H breaks spontaneously to a subgroup K, where K is the centralizer of S_G in the gauge group of the initial, higher-dimensional, theory, G [4, 40–42]. This can be illustrated in the following scheme,

$$\begin{array}{ccccc} G & \supset & S_G & \times & K \\ & & \cup & & \cap \\ G & \supset & R_G & \times & H \end{array} \tag{13}$$

In addition, the potential of the resulting 4-dimensional gauge theory is always of spontaneous symmetry breaking form, when the coset space is symmetric.[1] A negative result in this case is that, after the dimensional reduction, the fermions end up being supermassive, as in the Kaluza–Klein theory.

Let us now summarize a few results coming from the dimensional reduction of the $\mathcal{N} = 1$, E_8 SYM over the nearly-Kähler manifold $SU(3)/U(1) \times U(1)$. The 4-dimensional gauge group will be derived by the following decomposition of E_8

[1] A coset space is called symmetric when $f^c_{ab} = 0$.

under $R = U(1) \times U(1)$

$$E_8 \supset E_6 \times SU(3) \supset E_6 \times U(1)_A \times U(1)_B\,. \tag{14}$$

Satisfying the above criterion, the surviving gauge group in four dimensions is

$$H = C_{E_8}(U(1)_A \times U(1)_B) = E_6 \times U(1)_A \times U(1)_B\,. \tag{15}$$

The surviving scalars and fermions in four dimensions are obtained by the decomposition of the adjoint representation of E_8, that is the 248, under $U(1)_A \times U(1)_B$. Applying the CSDR rules, one obtains the resulting 4-dimensional theory, which is an $\mathcal{N} = 1$, E_6 GUT, with $U(1)_A$, $U(1)_B$ global symmetries. The potential is fully determined after a lengthy calculation and can be found in Ref. [17]. Subtracting the $F-$ and $D-$ terms contributing to this potential, one can determine also scalar masses and trilinear scalar terms, which can be identified with the scalar part of the soft supersymmetry breaking sector of the theory. In addition, the gaugino obtains a mass, which receives a contribution from the torsion, contrary to the rest soft supersymmetry breaking terms. The imortant point to note is that the CSDR leads to the soft supersymmetry breaking sector without any additional assumption.

In order to break further the E_6 GUT, one has to employ the Wilson flux mechanism. Due to the space limitation we cannot describe here the mechanism and its application in the present case. The details can be found in Ref. [14]. The resulting theory is a softly broken $\mathcal{N} = 1$, chiral $SU(3)^3$ theory which can break further to an extension of the MSSM.

3 Fuzzy Spaces

3.1 The Fuzzy Sphere

In order to introduce the non-commutative space of the fuzzy sphere, we are going to begin with the familiar ordinary sphere S^2 and extend it to its fuzzy version. The S^2 may be considered as a manifold embedded into the 3-dimensional Euclidean space, $\mathbb{R}^3$. This embedding allows us to specify the algebra of the functions on S^2 through $\mathbb{R}^3$, by imposing the constraint

$$\sum_{a=1}^{3} x_a^2 = R^2\,, \tag{16}$$

where x_a are the coordinates of $\mathbb{R}^3$ and R is the radius of S^2. The isometry group of S^2 is a global $SO(3)$, which is generated by the three angular momentum operators, $L_a = -i\epsilon_{abc}x_b\partial_c$, due to the isomorphism $SO(3) \simeq SU(2)$.

If we write the three operators L_a in terms of the spherical coordinates θ, ϕ, the generators are expressed as $L_a = -i\xi_a^\alpha \partial_\alpha$, where the greek index, α, denotes the spherical coordinates and ξ_a^α are the components of the Killing vector fields which generate the isometries of the sphere.[2]

The spherical harmonics, $Y_{lm}(\theta, \phi)$, are the eigenfunctions of the operator

$$L^2 = -R^2 \triangle_{S^2} = -R^2 \frac{1}{\sqrt{g}} \partial_a (g^{ab} \sqrt{g} \partial_b) . \tag{17}$$

Acting with L^2 on $Y_{lm}(\theta, \phi)$, one obtains its eigenvalues,

$$L^2 Y_{lm} = l(l+1) Y_{lm} , \tag{18}$$

where l is a non-negative integer. In addition, the $Y_{lm}(\theta, \phi)$ obey the orthogonality condition

$$\int \sin\theta d\theta d\phi Y_{lm}^\dagger Y_{l'm'} = \delta_{ll'} \delta_{mm'} . \tag{19}$$

Since $Y_{lm}(\theta, \phi)$ form a complete and orthogonal set of functions, any function on S^2 can be expanded on this set

$$a(\theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} a_{lm} Y_{lm}(\theta, \phi) , \tag{20}$$

where a_{lm} are complex coefficients. Alternatively, spherical harmonics can also be expressed in terms of the coordinates of $\mathbb{R}^3$, x_a, as

$$Y_{lm}(\theta, \phi) = \sum_{\mathbf{a}} f_{a_1 \dots a_l}^{lm} x^{a_1 \dots a_l} , \tag{21}$$

where $f_{a_1 \dots a_l}^{lm}$ is an $l-$rank (traceless) symmetric tensor.

Let us now make the extension of S^2 to its fuzzy version. Fuzzy sphere is a typical case of a non-commutative space, meaning that the algebra of functions is not commutative, as it is on S^2, with l having an upper limit. Therefore, due to this truncation, one obtains a finite dimensional (non-commutative) algebra, in particular l^2 dimensional. Thus, it is natural to consider the truncated algebra as a matrix algebra and it is consistent to consider the fuzzy sphere as a matrix approximation of the S^2.

According to the above, it follows that we are able to expand N-dimensional matrices on a fuzzy sphere as

$$\hat{a} = \sum_{l=0}^{N-1} \sum_{m=-l}^{l} a_{lm} \hat{Y}_{lm} , \tag{22}$$

[2]The S^2 metric can be expressed in terms of the Killing vectors as $g^{\alpha\beta} = \frac{1}{R^2} \xi_a^\alpha \xi_a^\beta$.

where $\hat{Y}_{lm}$ are spherical harmonics of the fuzzy sphere, which are now given by

$$\hat{Y}_{lm} = R^{-l} \sum_{\mathbf{a}} f^{lm}_{a_1 \ldots a_l} \hat{X}^{a_1} \cdots \hat{X}^{a_l} , \tag{23}$$

where

$$\hat{X}_a = \frac{2R}{\sqrt{N^2 - 1}} \lambda_a^{(N)} , \tag{24}$$

where $\lambda_a^{(N)}$ are the $SU(2)$ generators in the N-dimensional representation and $f^{lm}_{a_1 \ldots a_l}$ is the same tensor that we met in (21). The $\hat{Y}_{lm}$ also satisfy the orthonormality condition

$$\mathrm{Tr}_N \left(\hat{Y}^\dagger_{lm} \hat{Y}_{l'm'} \right) = \delta_{ll'} \delta_{mm'} . \tag{25}$$

Moreover, there is a relation between the expansion of a function, (20), and that of a matrix, (22) on the original and the fuzzy sphere, respectively

$$\hat{a} = \sum_{l=0}^{N-1} \sum_{m=-l}^{l} a_{lm} \hat{Y}_{lm} \quad \rightarrow \quad a = \sum_{l=0}^{N-1} \sum_{m=-l}^{l} a_{lm} Y_{lm}(\theta, \phi) . \tag{26}$$

The above relation is obviously a map from matrices to functions. Since we introduced the fuzzy sphere as a truncation of the algebra of functions on S^2, considering the same a_{lm} was just the most natural choice. Of course, the choice of the map is not unique, since it is not obligatory to consider the same expansion coefficients a_{lm}. The above is a $1:1$ mapping given by [43],

$$a(\theta, \phi) = \sum_{lm} \mathrm{Tr}_N (\hat{Y}^\dagger_{lm} \hat{a}) Y_{lm}(\theta, \phi) . \tag{27}$$

The matrix trace is mapped to an integral over the sphere:

$$\frac{1}{N} \mathrm{Tr}_N \quad \rightarrow \quad \frac{1}{4\pi} \int d\Omega . \tag{28}$$

Summing up, the fuzzy sphere is a matrix approximation of the ordinary sphere, S^2. The truncation of the algebra of the functions results to loss of commutativity, ending up with a non-commutative algebra, that of matrices, $\mathrm{Mat}(N; \mathbb{C})$. Therefore, the fuzzy sphere, S_N, is the non-commutative manifold with $\hat{X}_a$ being the coordinate functions. As given by (24), $\hat{X}_a$ are $N \times N$ hermitian matrices produced by the generators of $SU(2)$ in the N-dimensional representation. Obviously they have to obey both the condition

$$\sum_{a=1}^{3} \hat{X}_a \hat{X}_a = R^2 , \tag{29}$$

which is the analogue of (16), and the commutation relations

$$[\hat{X}_a, \hat{X}_b] = i\alpha\epsilon_{abc}\hat{X}_c\,, \quad \alpha = \frac{2R}{\sqrt{N^2-1}}\,. \tag{30}$$

Equivalently, one can consider the algebra on S_N being described by the antihermitian matrices

$$X_a = \frac{\hat{X}_a}{i\alpha R}\,, \tag{31}$$

also satisfying the modified relations (29), (30)

$$\sum_{a=1}^{3} X_a X_a = -\frac{1}{\alpha^2}\,, \quad [X_a, X_b] = C_{abc}X_c\,, \tag{32}$$

where $C_{abc} = \frac{1}{R}\epsilon_{abc}$.

Let us proceed by briefly mentioning the differential calculus on the fuzzy sphere, which is 3-dimensional and $SU(2)$ covariant. The derivations of a function f, along X_a are

$$e_a(f) = [X_a, f]\,, \tag{33}$$

and consequently, the Lie derivative on f is

$$\mathcal{L}_a f = [X_a, f]\,, \tag{34}$$

where $\mathcal{L}_a$ obeys the Leibniz rule and the commutation relation of $\mathfrak{su}(2)$

$$[\mathcal{L}_a, \mathcal{L}_b] = C_{abc}\mathcal{L}_c\,. \tag{35}$$

Working on the framework of differential forms, θ^a are the $1-$forms dual to the vector fields e_a, namely $\langle e_a, \theta^b\rangle = \delta_a^b$. Therefore, the exterior derivative, d, acting on a function f, gives

$$df = [X_a, f]\theta^a\,, \tag{36}$$

and the action of the Lie derivative on the $1-$forms θ^b gives

$$\mathcal{L}_a\theta^b = C_{abc}\theta^c\,. \tag{37}$$

The Lie derivative obeys the Leibniz law, therefore its action on any $1-$form $\omega = \omega_a\theta^a$ gives

$$\mathcal{L}_b\omega = \mathcal{L}_b(\omega_a\theta^a) = [X_b, \omega_a]\theta^a - \omega_a C^a_{bc}\theta^c\,, \tag{38}$$

where we have applied (34), (37). Therefore, one obtains the result

$$(\mathcal{L}_b\omega)_a = [X_b, \omega_a] - \omega_c C^c_{ba} \,. \tag{39}$$

After having stated the differential geometry of fuzzy sphere, one could extend the study of the differential geometry of $M^4 \times S^2_N$, which is the product of Minkowski space and fuzzy sphere with fuzziness level $N-1$. For example, any $1-$form A of $M^4 \times S^2_N$ can be expressed in terms of M^4 and S^2_N, that is

$$A = A_\mu dx^\mu + A_a\theta^a \,, \tag{40}$$

where A_μ, A_a depend on both x^μ and X_a coordinates.

Furthermore, instead of functions on the fuzzy sphere, one can examine the case of spinors [34]. Moreover, although we do not include them in the present review, studies of the differential geometry of other higher-dimensional fuzzy spaces (e.g. fuzzy CP^M) have been done [34].

3.2 Gauge Theory on the Fuzzy Sphere

Let us consider [44] a field $\phi(X_a)$ on the fuzzy sphere, depending on the powers of the coordinates, X_a. The infinitesimal transformation of $\phi(X_a)$ is

$$\delta\phi(X) = \lambda(X)\phi(X)\,, \tag{41}$$

where $\lambda(X)$ is the parameter of the gauge transformation. If $\lambda(X)$ is an antihermitian function of X_a, the (41) is an infinitesimal (abelian) $U(1)$ transformation. On the other hand, if $\lambda(X)$ is valued in Lie$(U(P))$, that is the algebra of $P \times P$ hermitian matrices, then the (41) is infinitesimal (non-abelian), $U(P)$. Naturally, it holds that $\delta X_a = 0$, which ensures the invariance of the covariant derivatives under a gauge transformation. Therefore, in the non-commutative case, left multiplication by a coordinate is not a covariant operation, that is

$$\delta(X_a\phi) = X_a\lambda(X)\phi\,, \tag{42}$$

and in general it holds that

$$X_a\lambda(X)\phi \neq \lambda(X)X_a\phi\,. \tag{43}$$

Motivated by the non-fuzzy gauge theory, one may introduce the covariant coordinates ϕ_a, such that

$$\delta(\phi_a\phi) = \lambda\phi_a\phi\,, \tag{44}$$

which holds if

$$\delta(\phi_a) = [\lambda, \phi_a]\,. \tag{45}$$

Usual (non-fuzzy) gauge theory also guides one to define

$$\phi_a \equiv X_a + A_a \,, \tag{46}$$

with the A_a being interpreted as the gauge potential of the non-commutative theory. Therefore, the covariant coordinates ϕ_a are the non-commutative analogue of the covariant derivative of ordinary gauge theories. From (46), (45) one is led to the transformation of A_a, that is

$$\delta A_a = -[X_a, \lambda] + [\lambda, A_a] \,, \tag{47}$$

a form that encourages the interpretation of A_a as a gauge field. In correspondence with the non-fuzzy gauge theory, one proceeds with defining a field strength tensor, F_{ab}, as

$$F_{ab} \equiv [X_a, A_b] - [X_b, A_a] + [A_a, A_b] - C^c_{ab} A_c = [\phi_a, \phi_b] - C^c_{ab}\phi_c \tag{48}$$

It can be proven that the transformation of the above field strength tensor is covariant:

$$\delta F_{ab} = [\lambda, F_{ab}] \,. \tag{49}$$

4 Ordinary Fuzzy Dimensional Reduction and Gauge Symmetry Enhancement

Let us now proceed by performing a simple (trivial) dimensional reduction in order to demonstrate the structure we sketched in the previous section. Starting with a higher-dimensional theory on $M^4 \times (S/R)_F$, with gauge group $G = U(P)$, we determine the produced 4-dimensional theory after performing the reduction and finally we make comments on the results. Let $(S/R)_F$ be a fuzzy coset, e.g. the fuzzy sphere, S^2_N. The action is

$$\mathcal{S}_{YM} = \frac{1}{4g^2} \int d^4x k \mathrm{Tr} \mathrm{tr}_G F_{MN} F^{MN} \,, \tag{50}$$

with tr_G the trace of the gauge group G and $k\mathrm{Tr}$[3] denotes the integration over $(S/R)_F$, i.e. the fuzzy coset which is described by $N \times N$ matrices. F_{MN} is the higher-dimensional field strength tensor, which is composed of both 4-dimensional spacetime and extra-dimensional parts, i.e. $(F_{\mu\nu}, F_{\mu a}, F_{ab})$. The components of F_{MN} in the extra (non-commutative) directions, are expressed in terms of the covariant coordinates ϕ_a, as follows

[3]In general, k is a parameter related to the size of the fuzzy coset space. In the case of the fuzzy sphere, k is related to the radius of the sphere and the integer l.

$$F_{\mu a} = \partial_\mu \phi_a + [A_\mu, \phi_a] = D_\mu \phi_a$$
$$F_{ab} = [X_a, A_b] - [X_b, A_a] + [A_a, A_b] - C^c_{ba} A_{ac} \,.$$

Putting the above equations in (50), the action takes the form

$$\mathcal{S}_{YM} = \int d^4x \mathrm{Trtr}_G \left(\frac{k}{4g^2} F^2_{\mu\nu} + \frac{k}{2g^2} (D_\mu \phi_a)^2 - V(\phi) \right) , \tag{51}$$

where $V(\phi)$ denotes the potential, derived from the kinetic term of F_{ab}, that is

$$\begin{aligned} V(\phi) &= -\frac{k}{4g^2} \mathrm{Trtr}_G \sum_{ab} F_{ab} F_{ab} \\ &= -\frac{k}{4g^2} \mathrm{Trtr}_G \left([\phi_a, \phi_b][\phi^a, \phi^b] - 4C_{abc} \phi^a \phi^b \phi^c + 2R^{-2} \phi^2 \right) . \end{aligned} \tag{52}$$

It is natural to consider (51) as an action of a 4-dimensional theory. Let $\lambda(x^\mu, X^a)$ be the gauge parameter that appears in an infinitesimal gauge transformation of G. This transformation can be interpreted as a M^4 gauge transformation. We write

$$\lambda(x^\mu, X^a) = \lambda^I(x^\mu, X^a) \mathcal{T}^I = \lambda^{h,I}(x^\mu) \mathcal{T}^h \mathcal{T}^I \,, \tag{53}$$

where $\mathcal{T}^I$ denote the hermitian generators of the gauge group $U(P)$. $\lambda^I(x^\mu, X^a)$ are the $N \times N$ antihermitian matrices, therefore they can be expressed as $\lambda^{I,h}(x^\mu)\mathcal{T}^h$, where $\mathcal{T}^h$ are the antihermitian generators of $U(N)$ and $\lambda^{I,h}(x^\mu)$, $h = 1, \ldots, N^2$, are the Kaluza-Klein modes of $\lambda^I(x^\mu, X^a)$. In turn, we can assume that the fields on the right hand side of (53) could be considered as one field that takes values in the tensor product Lie algebra Lie $(U(N)) \otimes$ Lie $(U(P))$, which corresponds to the algebra Lie $(U(NP))$. Similarly, the gauge field A_ν can be written as

$$A_\nu(x^\mu, X^a) = A^I_\nu(x^\mu, X^a) \mathcal{T}^I = A^{h,I}_\nu(x^\mu) \mathcal{T}^h \mathcal{T}^I \,, \tag{54}$$

which is interpreted as a gauge field on M^4 that takes values in the Lie $(U(NP))$ algebra. A similar consideration can also be applied in the case of scalar fields.[4]

It is worth noting the enhancement of the gauge symmetry of the 4-dimensional theory compared to the gauge symmetry of the higher-dimensional theory. In other words, we can start with an abelian gauge group in higher dimensions and result with a non-abelian gauge symmetry in the 4-dimensional theory. A defect of this theory is that the scalars are accommodated in the adjoint representation of the 4-dimensional gauge group, which means that they cannot induce the electroweak symmetry breaking. This motivates the realization of non-trivial dimensional reduction schemes, like the one that follows in the next section.

[4] Also, Trtr_G is interpreted as the trace of the $U(NP)$ matrices.

5 Fuzzy CSDR

In order to result with a less defective 4-dimensional gauge theory, we proceed by performing a non-trivial dimensional reduction, that is the fuzzy version of the CSDR.

So, in this section we adopt the CSDR programme in the non-commutative framework, where the extra dimensions are fuzzy coset spaces [34],[5] in order to result with a smaller number of both gauge and scalar fields in the 4-dimensional action (51). In general, the group S acts on the fuzzy coset $(S/R)_F$, and in accordance with the commutative case, CSDR scheme suggests that the fields of the theory must be invariant under an infinitesimal group $S-$transformation, up to an infinitesimal gauge transformation. Specifically, the fuzzy coset in this case is the fuzzy sphere, $(SU(2)/U(1))_F$, so the action of an infinitesimal $SU(2)-$transformation should leave the scalar and gauge fields invariant, up to an infinitesimal gauge transformation

$$\mathcal{L}_b\phi = \delta_{W_b} = W_b\phi \tag{55}$$

$$\mathcal{L}_b A = \delta_{W_b} A = -DW_b\,, \tag{56}$$

where A is the gauge potential expressed as an $1-$form, see (40), and W_b is an anti-hermitian gauge parameter depending only on the coset coordinates X^a. Therefore, W_b is written as

$$W_b = W_b^I \mathcal{T}^I\,, \quad I = 1, 2, \ldots, P^2\,, \tag{57}$$

where $\mathcal{T}^I$ are the hermitian generators of $U(P)$ and $(W_b^I)^\dagger = -W_b^I$, where the † denotes the hermitian conjugation on the X^a coordinates.

Putting into use the covariant coordinates, ϕ_a, (46), and ω_a, defined as

$$\omega_a \equiv X_a - W_a\,, \tag{58}$$

the CSDR constraints, (55) and (56), convert to

$$[\omega_b, A_\mu] = 0 \tag{59}$$

$$C_{bde}\phi^e = [\omega_b, \phi_d]\,. \tag{60}$$

Due to the fact that Lie derivatives respect the $\mathfrak{su}(2)$ commutation relation, (35), one results with the following consistency condition

$$[\omega_a, \omega_b] = C_{ab}^c\omega_c\,, \tag{61}$$

where the transformation of ω_a is given by

$$\omega_a \;\rightarrow\; \omega_a' = g\omega_a g^{-1}\,. \tag{62}$$

[5] See also [45].

In the case of spinor fields, the procedure is quite similar [34].

Let us now consider a higher-dimensional theory with gauge symmetry $G = U(1)$. We are going to perform a fuzzy CSDR, in which the fuzzy sphere is $(S/R)_F = S^2_N$. The $\omega_a = \omega_a(X^b)$ are $N \times N$ antihermitian matrices, therefore they can be considered as elements of Lie$(U(N))$. At the same time, they satisfy the commutation relation of Lie$(SU(2))$, as in the consistency condition, (61). So we have to embed Lie$(SU(2))$ into Lie$(U(N))$. Therefore, if $\mathcal{T}^h$, $h = 1, \ldots, N^2$ are the Lie$(U(N))$ generators, in the fundamental representation, then the convention $h = (a, u)$, $a = 1, 2, 3$, $u = 4, 5, \ldots, N^2$ can be used, obviously with the generators T^a satisfying Lie$(SU(2))$

$$[T^a, T^b] = C^{ab}_c T^c \,. \tag{63}$$

At last, the embedding is defined by the identification

$$\omega_a = T_a \,. \tag{64}$$

So, the constraint (59) implies that the gauge group of the 4-dimensional theory, K, is the centralizer of the image of $SU(2)$ into $U(N)$, that is

$$K = C_{U(N)}(SU(2)) = SU(N-2) \times U(1) \times U(1) \,, \tag{65}$$

where the second $U(1)$ in the right hand side is present due to

$$U(N) \simeq SU(N) \times U(1) \,. \tag{66}$$

Therefore, $A_\mu(x, X)$ are arbitrary functions over x, but they depend on X in a way that take values in Lie(K) instead of Lie$(U(N))$. That means that we result with a 4-dimensional gauge potential which takes values in Lie(K).

Let us now study the next constraint, (60). This one gets satisfied choosing

$$\phi_a = r\phi(x)\omega_a \,, \tag{67}$$

meaning that the degrees of freedom remaining unconstrained are related to the scalar field, $\phi(x)$, which is singlet under the 4-dimensional gauge group, K.

Summing up the results from the above reduction, the consistency condition (61), dictated the embedding of $SU(2)$ into $U(N)$. Although the embedding was realized into the fundamental representation of $U(N)$, we could have used the irreducible N-dimensional representation of $SU(2)$ by identifying $\omega_a = X_a$. If so, the constraint (59) would lead to the $U(1)$ to be the 4-dimensional gauge group, with $A_\mu(x)$ getting values in $U(1)$. The second constraint, (60), implies that, in this case too, $\phi(x)$ is a scalar singlet.

To conclude the whole procedure, one starts with a $U(1)$ higher-dimensional gauge theory on $M^4 \times S^2_N$ and because of the consistency condition, (61), an embedding of $SU(2)$ into $U(N)$ is required.[6] So, the first fuzzy CSDR constraint, (59), gives the 4-dimensional gauge group and from the second one, (60), one obtains the 4-dimensional scalar fields, surviving from the dimensional reduction.

As far as the fermions are concerned, we briefly mention the results of the above dimensional reduction. According to the extended analysis [34], it is proven that the appropriate choice of embedding is

$$S \subset SO(\mathrm{dim}S)\,, \tag{68}$$

which is achieved by $T_a = \frac{1}{2}C_{abc}\Gamma^{bc}$, respecting (63). Therefore, ψ functions as an intertwining operator between the representations of S and $SO(\mathrm{dim}S)$. In accordance to the commutative (non-fuzzy) case, [4], in order to find the surviving fermions in the 4-dimensional theory, one has to decompose the adjoint representation of $U(NP)$ under the product $S_{U(NP)} \times K$, that is

$$U(NP) \supset S_{U(NP)} \times K\,, \tag{69}$$

$$\mathrm{adj}\,U(NP) = \sum_i (s_i, k_i)\,. \tag{70}$$

Also, the decomposition of the spinorial representation σ of $SO(\mathrm{dim}S)$ under S is

$$SO(\mathrm{dim}S) \supset S\,, \tag{71}$$

$$\sigma = \sum_e \sigma_e\,. \tag{72}$$

Thus, if the two irreducible representations s_i, σ_e are identical, the surviving fermions of the 4-dimensional theory (4-dimensional spinors) belong to the k_i representation of gauge group K.

Before we move on, this is a suitable point to compare the higher-dimensional theory $M^4 \times (S/R)$, to its fuzzy extension, $M^4 \times (S/R)_F$. The first similarity has to do with the fact that fuzziness does not affect the isometries, both spaces have the same, $SO(1,3) \times SO(3)$. The second is that the gauge couplings defined on both spaces have the same dimensionality. But, on the other hand, a very striking difference is that of the two, only the non-commutative higher-dimensional theory is renormalizable.[7] In addition, a $U(1)$ initial gauge symmetry on $M^4 \times (S/R)_F$, is enough in order to result with non-abelian structures in four dimensions.[8]

[6]This embedding is achieved non-uniquely, specifically in p_N ways, where p_N is the possible ways one can partition the N into a set of non-increasing, positive integers [46].

[7]The number of counter-terms required to eliminate the divergencies is finite.

[8]Technically, this is possible because $N \times N$ matrices can be decomposed on the $U(N)$ generators.

6 Orbifolds and Fuzzy Extra Dimensions

The introduction of the orbifold structure (similar to the one developed in [47]) in the framework of gauge theories with fuzzy extra dimensions was motivated by the necessity of chiral low energy theories. In order to justify further the renormalizability of the theories constructed so far using fuzzy extra dimensions, we were led to consider the reverse procedure and start from a renormalizable theory in four dimensions and try to reproduce the results of a higher-dimensional theory reduced over fuzzy coset spaces [35–37]. This idea was realized as follows: one starts with a 4-dimensional gauge theory including appropriate scalar fields and a suitable potential leading to vacua that could be interpreted as dynamically generated fuzzy extra dimensions, including a finite Kaluza-Klein tower of massive modes. This reverse procedure gives hope that an initial abelian gauge theory does not have to be higher-dimensional and the non-abelian gauge theory structure could emerge from fluctuations of the coordinates [48]. The whole idea eventually seems to have similarities to the idea of dimensional deconstruction introduced earlier [49].

The inclusion of fermions in such models was desired too, but the best one could achieve for some time contained mirror fermions in bifundamental representations of the low-energy gauge group [36, 37]. Mirror fermions do not exclude the possibility to make contact with phenomenology [50], nevertheless, it is preferred to result with exactly chiral fermions.

Next, the plan that was sketched above is realized. Specifically, we are going to deal with the $\mathbb{Z}_3$ orbifold projection of the $\mathcal{N} = 4$ Supersymmetric Yang Mills (SYM) theory [51], examining the action of the discrete group on the fields of the theory and the superpotential that emerges in the projected theory.

6.1 $\mathcal{N} = 4$ *SYM Field Theory and* $\mathbb{Z}_3$ *Orbifolds*

So, let us begin with an $\mathcal{N} = 4$ supersymmetric $SU(3N)$ gauge theory defined on the Minkowski spacetime. The particle spectrum of the theory (in the $\mathcal{N} = 1$ terminology) consists of an $SU(3N)$ gauge supermultiplet and three adjoint chiral supermultiplets Φ^i , $i = 1, 2, 3$. The component fields of the above supermultiplets are the gauge bosons, A_μ, $\mu = 1, \ldots, 4$, six adjoint real (or three complex) scalars ϕ^a, $a = 1, \ldots, 6$ and four adjoint Weyl fermions ψ^p, $p = 1, \ldots, 4$. The scalars and Weyl fermions transform according to the 6 and 4 representations of the $SU(4)_R$ R-symmetry of the theory, respectively, while the gauge bosons are singlets.

Then, in order to introduce orbifolds, the discrete group $\mathbb{Z}_3$ has to be considered as a subgroup of $SU(4)_R$. The choice of the embedding of $\mathbb{Z}_3$ into $SU(4)_R$ is not unique and the options are not equivalent, since the choice of embedding affects the amount of the remnant supersymmetry [47]:

- Maximal embedding of $\mathbb{Z}_3$ into $SU(4)_R$ is excluded because it would lead to non-supersymmetric models,

- Embedding of $\mathbb{Z}_3$ in an $SU(4)_R$ subgroup:
  - Embedding into an $SU(2)$ subgroup would lead to $\mathcal{N} = 2$ supersymmetric models with $SU(2)_R$ R-symmetry
  - Embedding into an $SU(3)$ subgroup would lead to $\mathcal{N} = 1$ supersymmetric models with $U(1)_R$ R-symmetry.

We focus on the last embedding which is the desired one, since it leads to $\mathcal{N} = 1$ supersymmetric models. Let us consider a generator $g \in \mathbb{Z}_3$, labeled (for convenience) by three integers $\mathbf{a} = (a_1, a_2, a_3)$ [52] satisfying the relation

$$a_1 + a_2 + a_3 = 0 \mod 3\,. \tag{73}$$

The last equation implies that $\mathbb{Z}_3$ is embedded in the $SU(3)$ subgroup, i.e. the remnant supersymmetry is the desired $\mathcal{N} = 1$ [53].

It is expected that since the various fields of the theory transform differently under $SU(4)_R$, $\mathbb{Z}_3$ will act non-trivially on them. Gauge and gaugino fields are singlets under $SU(4)_R$, therefore the geometric action of the $\mathbb{Z}_3$ rotation is trivial. The action of $\mathbb{Z}_3$ on the complex scalar fields is given by the matrix $\gamma(g)_{ij} = \delta_{ij}\omega^{a_i}$, where $\omega = e^{\frac{2\pi}{3}}$ and the action of $\mathbb{Z}_3$ on the fermions ϕ^i is given by $\gamma(g)_{ij} = \delta_{ij}\omega^{b_i}$, where $b_i = -\dfrac{1}{2}(a_{i+1} + a_{i+2} - a_i)$.[9] In the case under study the three integers of the generator g are $(1, 1, -2)$, meaning that $a_i = b_i$.

The matter fields are not invariant under a gauge transformation, therefore $\mathbb{Z}_3$ acts on their gauge indices, too. The action of this rotation is given by the matrix

$$\gamma_3 = \begin{pmatrix} \mathbf{1}_N & 0 & 0 \\ 0 & \omega\mathbf{1}_N & 0 \\ 0 & 0 & \omega^2\mathbf{1}_N \end{pmatrix}. \tag{74}$$

There is no specific reason for these blocks to have the same dimensionality (see e.g. [54–56]). However, since the projected theory must be free of anomalies, the dimension of the three blocks is the same.

After the orbifold projection, the spectrum of the theory consists of the fields that are invariant under the combined action of the discrete group, $\mathbb{Z}_3$, on the "geometric"[10] and gauge indices [52]. As far as the gauge bosons are concerned being singlets, the projection is $A_\mu = \gamma_3 A_\mu \gamma_3^{-1}$. Therefore, taking into consideration (74), the gauge group of the initial theory breaks down to the group $H = SU(N) \times SU(N) \times SU(N)$ in the projected theory.

As we have already stated, the complex scalar fields transform non-trivially under the gauge and $R-$symmetry, so the projection is $\phi^i_{IJ} = \omega^{I-J+a_i}\phi^i_{IJ}$, where I, J are

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[9] Also modulo 3.

[10] In case of ordinary reduction of a 10-dimensional $\mathcal{N} = 1$ SYM theory, one obtains an $\mathcal{N} = 4$ SYM Yang-Mills theory in four dimensions having a global $SU(4)_R$ symmetry which is identified with the tangent space $SO(6)$ of the extra dimensions [16, 17].

gauge indices. Therefore, $J = I + a_i$, meaning that the scalar fields surviving the orbifold projection have the form $\phi_{I,J+a_i}$ and transform under the gauge group H as

$$3 \cdot \left((N, \bar{N}, 1) + (\bar{N}, 1, N) + (1, N, \bar{N})\right) . \tag{75}$$

Similarly, fermions transform non-trivially under the gauge group and $R-$symmetry, too with the projection being $\psi^i_{IJ} = \omega^{I-J+b_i}\psi^i_{IJ}$. Therefore, the fermions surviving the projection have the form $\psi^i_{I,I+b_i}$ accommodated in the same representation of H as the scalars, that is (75), a fact demonstrating the $\mathcal{N} = 1$ remnant supersymmetry. It is worth noting that the representations (75) of the resulting theory are anomaly free.

The fermions, summing up the above results, are accommodated into chiral representations of H and there are three fermionic generations since, as we have mentioned above, the particle spectrum contains three $\mathcal{N} = 1$ chiral supermultiplets.

The interactions of the projected model are given by the superpotential. In order to specify it, one has to begin with the superpotential of the initial $\mathcal{N} = 4$ SYM theory [51]

$$W_{\mathcal{N}=4} = \epsilon_{ijk}\mathrm{Tr}(\Phi^i \Phi^j \Phi^k) , \tag{76}$$

where, Φ^i, Φ^j, Φ^k are the three chiral superfields of the theory. After the projection, the structure of the superpotential remains unchanged, but it encrypts only the interactions among the surviving fields of the $\mathcal{N} = 1$ theory, that is

$$W^{(proj)}_{\mathcal{N}=1} = \sum_I \epsilon_{ijk}\Phi^i_{I,I+a_i}\Phi^j_{I+a_i,I+a_i+a_j}\Phi^k_{I+a_i+a_j,I} . \tag{77}$$

6.2 Dynamical Generation of Twisted Fuzzy Spheres

From the superpotential $W^{proj}_{\mathcal{N}=1}$ that is given in (77), the scalar potential can be extracted:

$$V^{proj}_{\mathcal{N}=1}(\phi) = \frac{1}{4}\mathrm{Tr}\left([\phi^i, \phi^j]^\dagger[\phi^i, \phi^j]\right) , \tag{78}$$

where, ϕ^i are the scalar component fields of the superfield Φ^i. The potential $V^{proj}_{\mathcal{N}=1}(\phi)$ is minimized by vanishing vevs of the fields, so modifications have to be made, in order that solutions interpreted as vacua of a non-commutative geometry to be emerged.

So, in order to result with a minimum of $V^{proj}_{\mathcal{N}=1}(\phi)$, $\mathcal{N} = 1$ soft supersymmetric terms of the form[11]

[11] The SSB terms that will be inserted into $V^{proj}_{\mathcal{N}=1}(\phi)$, are purely scalar. Although this is enough for our purpose, it is obvious that more SSB terms have to be included too, in order to obtain the full SSB sector [57].

$$V_{SSB} = \frac{1}{2}\sum_i m_i^2 \phi^{i\dagger}\phi^i + \frac{1}{2}\sum_{i,j,k} h_{ijk}\phi^i\phi^j\phi^k + h.c. \tag{79}$$

are introduced, where $h_{ijk} = 0$ unless $i + j + k \equiv 0 \bmod 3$. The introduction of these SSB terms should not come as a surprise, since the presence of an SSB sector is necessary anyway for a model with realistic aspirations, see e.g. [57]. The inclusion of the D-terms of the theory is necessary and they are given by

$$V_D = \frac{1}{2}D^2 = \frac{1}{2}D^I D_I\,, \tag{80}$$

where $D^I = \phi_i^\dagger T^I \phi^i$, where T^I are the generators in the representation of the corresponding chiral multiplets.

So, the total potential of the theory is given by

$$V = V_{\mathcal{N}=1}^{proj} + V_{SSB} + V_D\,. \tag{81}$$

A suitable choice for the parameters m_i^2 and h_{ijk} in (79) is $m_i^2 = 1,\ h_{ijk} = \epsilon_{ijk}$. Therefore, the total scalar potential, (81), takes the form

$$V = \frac{1}{4}(F^{ij})^\dagger F^{ij} + V_D\,, \tag{82}$$

where F^{ij} is defined as

$$F^{ij} = [\phi^i, \phi^j] - i\epsilon^{ijk}(\phi^k)^\dagger\,. \tag{83}$$

The first term of the scalar potential, (82), is always positive, therefore, the global minimum of the potential is obtained when

$$[\phi^i, \phi^j] = i\epsilon_{ijk}(\phi^k)^\dagger\,, \quad \phi^i(\phi^i)^\dagger = R^2\,, \tag{84}$$

where $(\phi^i)^\dagger$ denotes the hermitian conjugate of ϕ^i and $[R^2, \phi^i] = 0$. It is clear that the above equations are related to a fuzzy sphere. This becomes more transparent by considering the untwisted fields $\tilde{\phi}^i$, defined by

$$\phi^i = \Omega\tilde{\phi}^i\,, \tag{85}$$

where $\Omega \neq 1$ satisfy the relations

$$\Omega^3 = 1\,, \quad [\Omega, \phi^i] = 0\,, \quad \Omega^\dagger = \Omega^{-1}\,, \quad (\tilde{\phi}^i)^\dagger = \tilde{\phi}^i \ \Leftrightarrow\ (\phi^i)^\dagger = \Omega\phi^i\,. \tag{86}$$

Therefore, (84) reproduces the ordinary fuzzy sphere relations generated by $\tilde{\phi}^i$

$$[\tilde{\phi}^i, \tilde{\phi}^j] = i\epsilon_{ijk}\tilde{\phi}^k\,, \quad \tilde{\phi}^i\tilde{\phi}^i = R^2\,, \tag{87}$$

exhibiting the reason why the non-commutative space generated by ϕ^i is a twisted fuzzy sphere, $\tilde{S}^2_N$.

Next, one can find configurations of the twisted fields ϕ^i, i.e. fields satisfying (84). Such configuration is

$$\phi^i = \Omega(\mathbf{1}_3 \otimes \lambda^i_{(N)})\,, \tag{88}$$

where $\lambda^i_{(N)}$ are the $SU(2)$ generators in the N-dimensional irreducible representation and Ω is the matrix

$$\Omega = \Omega_3 \otimes \mathbf{1}_N\,, \quad \Omega_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}\,, \quad \Omega^3 = \mathbf{1}\,. \tag{89}$$

According to the transformation (85), the "off-diagonal" orbifold sectors (75) convert to the block-diagonal form

$$\phi^i = \begin{pmatrix} 0 & (\lambda^i_{(N)})_{(N,\bar{N},1)} & 0 \\ 0 & 0 & (\lambda^i_{(N)})_{(1,N,\bar{N})} \\ (\lambda^i_{(N)})_{(\bar{N},1,N)} & 0 & 0 \end{pmatrix} = \Omega \begin{pmatrix} \lambda^i_{(N)} & 0 & 0 \\ 0 & \lambda^i_{(N)} & 0 \\ 0 & 0 & \lambda^i_{(N)} \end{pmatrix}\,. \tag{90}$$

Therefore, the untwisted fields generating the ordinary fuzzy sphere, $\tilde{\phi}^i$, are written in a block-diagonal form. Each block can be considered as an ordinary fuzzy sphere, since they separately satisfy the corresponding commutation relations (87). In turn, the above configuration in (90), which corresponds to the vacuum of the theory, has the form of three fuzzy spheres, appearing with relative angles $2\pi/3$. Concluding, the solution ϕ^i can be considered as the twisted equivalent of three fuzzy spheres, conforming with the orbifolding.

Note that the F^{ij} defined in (83), can be interpreted as the field strength of the spontaneously generated fuzzy extra dimensions. The second term of the potential, V_D, induces a change on the radius of the sphere (in a similar way to the case of the ordinary fuzzy sphere [35, 37, 58]).

6.3 Chiral Models After the Fuzzy Orbifold Projection - the $SU(3)_c \times SU(3)_L \times SU(3)_R$ Model

The resulting unification groups after the orbifold projection are various because of the different ways the gauge group $SU(3N)$ is spontaneously broken. The minimal, anomaly free models are $SU(4) \times SU(2) \times SU(2)$, $SU(4)^3$ and $SU(3)^3$.[12]

We focus on the breaking of the latter, which is the trinification group $SU(3)_c \times SU(3)_L \times SU(3)_R$ [60, 61] (see also [62–66] and for a string theory approach see

[12]Similar approaches have been studied in the framework of YM matrix models [59], lacking phenomenological viability.

[49]). At first, the integer N has to be decomposed as $N = n + 3$. Then, for $SU(N)$, the considered embedding is

$$SU(N) \supset SU(n) \times SU(3) \times U(1)\,, \tag{91}$$

from which it follows that the embedding for the gauge group $SU(N)^3$ is

$$SU(N)^3 \supset SU(n) \times SU(3) \times SU(n) \times SU(3) \times SU(n) \times SU(3) \times U(1)^3\,. \tag{92}$$

The three $U(1)$ factors are ignored[13] and the representations are decomposed according to (92), (after reordering the factors) as

$$\begin{aligned}
&SU(n) \times SU(n) \times SU(n) \times SU(3) \times SU(3) \times SU(3)\,,\\
&(n, \bar{n}, 1; 1, 1, 1) + (1, n, \bar{n}; 1, 1, 1) + (\bar{n}, 1, n; 1, 1, 1) + (1, 1, 1; 3, \bar{3}, 1)\\
&+ (1, 1, 1; 1, 3, \bar{3}) + (1, 1, 1; \bar{3}, 1, 3) + (n, 1, 1; 1, \bar{3}, 1) + (1, n, 1; 1, 1, \bar{3})\\
&+ (1, 1, n; \bar{3}, 1, 1) + (\bar{n}, 1, 1; 1, 1, 3) + (1, \bar{n}, 1; 3, 1, 1) + (1, 1, \bar{n}; 1, 3, 1)\,.
\end{aligned} \tag{93}$$

Taking into account the decomposition (91), the gauge group is broken to $SU(3)^3$. Now, under $SU(3)^3$, the surviving fields transform as

$$SU(3) \times SU(3) \times SU(3)\,, \tag{94}$$

$$\big((3, \bar{3}, 1) + (\bar{3}, 1, 3) + (1, 3, \bar{3})\big)\,, \tag{95}$$

which correspond to the desired chiral representations of the trinification group. Under $SU(3)_c \times SU(3)_L \times SU(3)_R$, the quarks and leptons of the first family transform as

$$q = \begin{pmatrix} d & u & h \\ d & u & h \\ d & u & h \end{pmatrix} \sim (3, \bar{3}, 1)\,, \quad q^c = \begin{pmatrix} d^c & d^c & d^c \\ u^c & u^c & u^c \\ h^c & h^c & h^c \end{pmatrix} \sim (\bar{3}, 1, 3)\,, \tag{96}$$

$$\lambda = \begin{pmatrix} N & E^c & \mathrm{v} \\ E & N^c & e \\ \mathrm{v}^c & e^c & S \end{pmatrix} \sim (1, 3, \bar{3})\,,$$

respectively. Similarly, one obtains the matrices for the fermions of the other two families.

It is worth noting that this theory can be upgraded to a two-loop finite theory (for reviews see [62, 67–69]) and give testable predictions [62], too. Additionally, fuzzy orbifolds can be used to break spontaneously the unification gauge group down to MSSM and then to the $SU(3)_c \times U(1)_{em}$.

[13] As anomalous gaining mass by the Green-Schwarz mechanism and therefore they decouple at the low energy sector of the theory [55].

Summarizing this section let us emphasize the general picture of the model that has been constructed. At very high-scale regime, we have an unbroken renormalizable theory. After the spontaneous symmetry breaking, the resulting gauge theory is accompanied by a finite tower of massive Kaluza-Klein modes. Finally, the theory breaks down to an extension of MSSM in the low scale regime. Therefore, we conclude that fuzzy extra dimensions can be used in constructing chiral, renormalizable and phenomenologically viable field-theoretical models.

A natural extension of the above ideas and methods have been reported in ref [70] (see also [71]), realized in the context of Matrix Models (MM). At a fundamental level, the MMs introduced by Banks–Fischler–Shenker–Susskind (BFSS) and Ishibashi–Kawai–Kitazawa–Tsuchiya (IKKT), are supposed to provide a non-perturbative definition of M-theory and type IIB string theory respectively [30, 72]. On the other hand, MMs are also useful laboratories for the study of structures which could be relevant from a low-energy point of view. Indeed, they generate a plethora of interesting solutions, corresponding to strings, D-branes and their interactions [30, 73], as well as to non-commutative/fuzzy spaces, such as fuzzy tori and spheres [74]. Such backgrounds naturally give rise to non-abelian gauge theories. Therefore, it appears natural to pose the question whether it is possible to construct phenomenologically interesting particle physics models in this framework as well. In addition, an orbifold MM was proposed by Aoki-Iso-Suyama (AIS) in [75] as a particular projection of the IKKT model, and it is directly related to the construction described above in which fuzzy extra dimensions arise with trinification gauge theory [38]. By $\mathbb{Z}_3$ - orbifolding, the original symmetry of the IKKT matrix model with matrix size $3N \times 3N$ is generally reduced from $SO(9,1) \times U(3N)$ to $SO(3,1) \times U(N)^3$. This model is chiral and has $D = 4$, $\mathcal{N} = 1$ supersymmetry of Yang-Mills type as well as an inhomogeneous supersymmetry specific to matrix models. The $\mathbb{Z}_3$ - invariant fermion fields transform as bifundamental representations under the unbroken gauge symmetry exactly as in the constructions described above. In the future we plan to extend further the studies initiated in refs [70, 71] in the context of orbifolded IKKT models.

Our current interest is to continue in two directions. Given that the two approaches discussed here led to the $\mathcal{N} = 1$ trinification GUT $SU(3)^3$, one plan is to examine the phenomenological consequences of these models. The models are different in the details but certainly there exist a certain common ground. Among others we plan to determine in both cases the spectrum of the Dirac and Laplace operators in the extra dimensions and use them to study the behaviour of the various couplings, including the contributions of the massive Kaluza-Klein modes. These contributions are infinite or finite in number, depending on whether the extra dimensions are continuous or fuzzy, respectively. We should note that the spectrum of the Dirac operator at least in the case of $SU(3)/U(1) \times U(1)$ is not known.

Another plan is to start with an abelian theory in ten dimensions and with a simple reduction to obtain an $N = (1, 1)$ abelian theory in six dimensions. Finally, reducing the latter theory over a fuzzy sphere, possibly with Chern-Simons terms, to obtain a non-abelian gauge theory in four dimensions provided with soft supersymmetry breaking terms. Recall that the last feature was introduced by hand in the realistic models constructed in the fuzzy extra dimensions framework.

Acknowledgements This research is supported by the Research Funding Program ARISTEIA, Higher Order Calculations and Tools for High Energy Colliders, HOCTools and the ARISTEIA II, Investigation of certain higher derivative term field theories and gravity models (co-financed by the European Union (European Social Fund ESF) and Greek national funds through the Operational Program Education and Lifelong Learning of the National Strategic Reference Framework (NSRF)), by the European Union's ITN programme HIGGSTOOLS, as well as by the Action MP1405 QSPACE from the European Cooperation in Science and Technology (COST). G.Z. would like to thank the organizers and ITP-Heidelberg for the warm and generous hospitality.

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Part II
String Theories and Gravity Theories

Higher Genus Amplitudes in SUSY Double-Well Matrix Model for 2D IIA Superstring

Fumihiko Sugino

Abstract We discuss a simple supersymmetric double-well matrix model which is considered to give a perturbation formulation of two-dimensional type IIA superstring theory on a nontrivial Ramond-Ramond background. Full nonperturbative contributions to the free energy are computed by using the technique of random matrix theory, and the result shows that supersymmetry (SUSY) is spontaneously broken by nonperturbative effects due to instantons. In addition, one-point functions of operators that are not protected by SUSY are obtained to all orders in genus expansion.

1 Introduction

Nonperturbative aspects of noncritical bosonic string theory were vigorously investigated around 1990 by using solvable matrix models (For a review, see [2].), while little has been known for superstring theory, in particular which possesses target-space supersymmetry (SUSY). We here consider a solvable matrix model describing superstring theory with target-space SUSY. We hope our analysis is helpful to understand nonperturbative dynamics of matrix models of super Yang-Mills type for critical superstring theory [1, 3, 10].

2 Supersymmetric Double-Well Matrix Model

We start with a simple matrix model given by the action [13]:

$$S = N\mathrm{tr}\left[\frac{1}{2}B^2 + iB(\phi^2-\mu^2) + \bar{\psi}(\phi\psi+\psi\phi)\right]. \tag{1}$$

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V. Dobrev (ed.), *Lie Theory and Its Applications in Physics*,
Springer Proceedings in Mathematics & Statistics 191,
DOI 10.1007/978-981-10-2636-2_14

B and ϕ are $N \times N$ hermitian matrices, and ψ and $\bar{\psi}$ are $N \times N$ matrices whose components are Grassmann numbers. The action S is invariant under SUSY transformations generated by Q and $\bar{Q}$:

$$Q\phi = \psi, \quad Q\psi = 0, \quad Q\bar{\psi} = -iB, \quad QB = 0, \tag{2}$$

$$\bar{Q}\phi = -\bar{\psi}, \quad \bar{Q}\bar{\psi} = 0, \quad \bar{Q}\psi = -iB, \quad \bar{Q}B = 0, \tag{3}$$

from which one can see the nilpotency: $Q^2 = \bar{Q}^2 = \{Q, \bar{Q}\} = 0$. After integrating out B, we have a scalar potential of a double-well shape: $\frac{1}{2}(\phi^2 - \mu^2)^2$. In case of $\mu^2 > 2$, a large-N saddle point solution for the eigenvalue distribution of the matrix ϕ: $\rho(x) \equiv \frac{1}{N}\mathrm{tr}\,\delta(x - \phi)$ is given by

$$\rho(x) = \begin{cases} \frac{\nu_+}{\pi}\, x \sqrt{(x^2 - a^2)(b^2 - x^2)} & (a < x < b) \\ \frac{\nu_-}{\pi}\, |x| \sqrt{(x^2 - a^2)(b^2 - x^2)} & (-b < x < -a), \end{cases} \tag{4}$$

where $a = \sqrt{\mu^2 - 2}$ and $b = \sqrt{\mu^2 + 2}$. The filling fractions (ν_+, ν_-) satisfying $\nu_+ + \nu_- = 1$ indicate that $\nu_+ N$ $(\nu_- N)$ eigenvalues are around the right (left) minimum of the double-well. The large-N free energy and the expectation values $\left\langle \frac{1}{N}\mathrm{tr}\, B^n \right\rangle$ $(n = 1, 2, \ldots)$ evaluated at the solution turn out to all vanish [13]. This strongly suggests that the solution preserves SUSY. Thus, we conclude that the SUSY minima are infinitely degenerate and parametrized by (ν_+, ν_-) at large N. On the other hand, in case of $\mu^2 < 2$, non SUSY saddle point solution is obtained [14]. Transition between the SUSY phase $(\mu^2 > 2)$ and the SUSY broken phase $(\mu^2 < 2)$ is of the third order.

The partition function after B, ψ and $\bar{\psi}$ are integrated out is expressed as a Gaussian one-matrix model by the Nicolai mapping $H = \phi^2$, where the H-integration is over the *positive definite* hermitian matrices, not over all the hermitian matrices. References [6, 12] discuss that the difference of the integration region has only effects which are nonperturbative in $1/N$, and the model can be regarded as the standard Gaussian matrix model at each order of genus expansion.

The Nicolai mapping changes the operators $\frac{1}{N}\mathrm{tr}\,\phi^{2n}$ $(n = 1, 2, \ldots)$ to regular operators $\frac{1}{N}\mathrm{tr}\, H^n$. Hence, the behavior of their correlators is expected to be described by the Gaussian one-matrix model (the $c = -2$ topological gravity) at least perturbatively in $1/N$. However, the operators $\frac{1}{N}\mathrm{tr}\,\phi^{2n+1}$ $(n = 0, 1, 2, \ldots)$ are mapped to $\pm\frac{1}{N}\mathrm{tr}\, H^{n+1/2}$ that are singular at the origin. They are not observables in the $c = -2$ topological gravity, while they are natural observables as well as $\frac{1}{N}\mathrm{tr}\,\phi^{2n}$ in the original setting (1). Correlation functions among operators

$$\frac{1}{N}\mathrm{tr}\,\phi^{2n+1}, \qquad \frac{1}{N}\mathrm{tr}\,\psi^{2n+1}, \qquad \frac{1}{N}\mathrm{tr}\,\bar{\psi}^{2n+1} \qquad (n = 0, 1, 2, \ldots) \tag{5}$$

at the solution (4) exhibit logarithmic singular behavior of powers of $\ln(\mu^2 - 2)$ [15].

3 2D Type IIA Superstring

The two-dimensional type II superstring theory discussed in Refs. [7, 11, 17, 19] has the target space $(\varphi, x) \in$ (Liouville direction) $\times$ (S^1 with self-dual radius). The holomorphic energy-momentum tensor on the string world-sheet is

$$T = -\frac{1}{2}(\partial x)^2 - \frac{1}{2}\psi_x \partial \psi_x - \frac{1}{2}(\partial\varphi)^2 + \partial^2\varphi - \frac{1}{2}\psi_\ell \partial \psi_\ell \tag{6}$$

excluding ghosts' part. ψ_x and ψ_ℓ are superpartners of x and φ, respectively. Target-space supercurrents in the type IIA theory

$$q_+(z) = e^{-\frac{1}{2}\phi(z) - \frac{i}{2}H(z) - ix(z)}, \qquad \bar{q}_-(\bar{z}) = e^{-\frac{1}{2}\bar{\phi}(\bar{z}) + \frac{i}{2}\bar{H}(\bar{z}) + i\bar{x}(\bar{z})} \tag{7}$$

exist only for the S^1 target space of the self-dual radius. ϕ ($\bar{\phi}$) is the holomorphic (anti-holomorphic) bosonized superconformal ghost, and the fermions are bosonized as $\psi_\ell \pm i\psi_x = \sqrt{2}\, e^{\mp iH}$, $\bar{\psi}_\ell \pm i\bar{\psi}_x = \sqrt{2}\, e^{\mp i\bar{H}}$. In addition, we should care about cocycle factors in order to realize the anticommuting nature between q_+ and $\bar{q}_-$. See [16] for details in the cocycle factors. The supercharges

$$Q_+ = \oint \frac{dz}{2\pi i}\, q_+(z), \qquad \bar{Q}_- = \oint \frac{d\bar{z}}{2\pi i}\, \bar{q}_-(\bar{z}) \tag{8}$$

are nilpotent $Q_+^2 = \bar{Q}_-^2 = \{Q_+, \bar{Q}_-\} = 0$, which indeed matches the property of the supercharges Q and $\bar{Q}$ in the matrix model.

The spectrum except special massive states is represented by the NS vertex operator (in (-1) picture):

$$T_k = e^{-\phi + ikx + p_\ell \varphi}, \qquad \bar{T}_{\bar{k}} = e^{-\bar{\phi} + i\bar{k}\bar{x} + p_\ell \bar{\varphi}}, \tag{9}$$

and by the R vertex operator (in $(-\frac{1}{2})$ picture):

$$V_{k,\epsilon} = e^{-\frac{1}{2}\phi + \frac{i}{2}\epsilon H + ikx + p_\ell \varphi}, \qquad \bar{V}_{\bar{k},\bar{\epsilon}} = e^{-\frac{1}{2}\bar{\phi} + \frac{i}{2}\bar{\epsilon}\bar{H} + i\bar{k}\bar{x} + p_\ell \bar{\varphi}} \tag{10}$$

with $\epsilon, \bar{\epsilon} = \pm 1$. Details in cocycle factors for the vertex operators are also presented in [16]. Locality with the supercurrents, mutual locality, superconformal invariance (including the Dirac equation constraint) and the level matching condition determine physical vertex operators. As discussed in [11], there are two consistent sets of physical vertex operators - "momentum background" and "winding background".

Let us consider the "winding background".[1] The physical spectrum in the "winding background" is given by

$$\begin{array}{lll} (\mathrm{NS},\mathrm{NS}): & T_k\,\bar{T}_{-k} & (k\in\mathbf{Z}+\tfrac{1}{2}),\\ (\mathrm{R}+,\mathrm{R}-): & V_{k,+1}\,\bar{V}_{-k,-1} & (k=\tfrac{1}{2},\,\tfrac{3}{2},\cdots),\\ (\mathrm{R}-,\mathrm{R}+): & V_{-k,-1}\,\bar{V}_{k,+1} & (k=0,1,2,\cdots),\\ (\mathrm{NS},\mathrm{R}-): & T_{-k}\,\bar{V}_{-k,-1} & (k=\tfrac{1}{2},\,\tfrac{3}{2},\cdots),\\ (\mathrm{R}+,\mathrm{NS}): & V_{k,+1}\,\bar{T}_k & (k=\tfrac{1}{2},\,\tfrac{3}{2},\cdots), \end{array} \tag{11}$$

where we take a branch of $p_\ell = 1-|k|$ satisfying the locality bound $p_\ell \le Q/2 = 1$ [22]. We can see that the vertex operators

$$V_{\frac{1}{2},+1}\,\bar{V}_{-\frac{1}{2},-1},\qquad T_{-\frac{1}{2}}\,\bar{V}_{-\frac{1}{2},-1},\qquad V_{\frac{1}{2},+1}\,\bar{T}_{\frac{1}{2}},\qquad T_{-\frac{1}{2}}\,\bar{T}_{\frac{1}{2}} \tag{12}$$

form a quartet under Q_+ and $\bar{Q}_-$ which are isomorphic to (2) and (3), respectively. It leads to correspondence of single-trace operators in the matrix model to integrated vertex operators in the type IIA theory:

$$\begin{aligned} \Phi_1 &= \frac{1}{N}\mathrm{tr}\,\phi \Leftrightarrow \mathcal{V}_\phi(0) \equiv g_s^2\int d^2z\, V_{\frac{1}{2},+1}(z)\,\bar{V}_{-\frac{1}{2},-1}(\bar{z}),\\ \Psi_1 &= \frac{1}{N}\mathrm{tr}\,\psi \Leftrightarrow \mathcal{V}_\psi(0) \equiv g_s^2\int d^2z\, T_{-\frac{1}{2}}(z)\,\bar{V}_{-\frac{1}{2},-1}(\bar{z}),\\ \bar{\Psi}_1 &= \frac{1}{N}\mathrm{tr}\,\bar{\psi} \Leftrightarrow \mathcal{V}_{\bar{\psi}}(0) \equiv g_s^2\int d^2z\, V_{\frac{1}{2},+1}(z)\,\bar{T}_{\frac{1}{2}}(\bar{z}),\\ &\frac{1}{N}\mathrm{tr}\,(-iB) \Leftrightarrow \mathcal{V}_B(0) \equiv g_s^2\int d^2z\, T_{-\frac{1}{2}}(z)\,\bar{T}_{\frac{1}{2}}(\bar{z}), \end{aligned} \tag{13}$$

where the bare string coupling g_s is put in the right hand sides to count the number of external lines of amplitudes in the IIA theory. Furthermore, it can be naturally extended as

$$\begin{aligned} \Phi_{2k+1} &= \frac{1}{N}\mathrm{tr}\,\phi^{2k+1} + (\text{mixing}) \Leftrightarrow \mathcal{V}_\phi(k) \equiv g_s^2\int d^2z\, V_{k+\frac{1}{2},+1}(z)\,\bar{V}_{-k-\frac{1}{2},-1}(\bar{z}),\\ \Psi_{2k+1} &= \frac{1}{N}\mathrm{tr}\,\psi^{2k+1} + (\text{mixing}) \Leftrightarrow \mathcal{V}_\psi(k) \equiv g_s^2\int d^2z\, T_{-k-\frac{1}{2}}(z)\,\bar{V}_{-k-\frac{1}{2},-1}(\bar{z}),\\ \bar{\Psi}_{2k+1} &= \frac{1}{N}\mathrm{tr}\,\bar{\psi}^{2k+1} + (\text{mixing}) \Leftrightarrow \mathcal{V}_{\bar{\psi}}(k) \equiv g_s^2\int d^2z\, V_{k+\frac{1}{2},+1}(z)\,\bar{T}_{k+\frac{1}{2}}(\bar{z}) \end{aligned} \tag{14}$$

for higher $k(=1,2,\cdots)$. "(mixing)" means lower-power operators needed to subtract nonuniversal contributions. In (14), we see that the powers of matrices are interpreted as windings or momenta in the S^1 direction of the type IIA theory.

[1] We can repeat the parallel argument for "momentum background" in the type IIB theory, which is equivalent to the "winding background" in the type IIA theory through T-duality with respect to the S^1 direction.

Note that (R−, R+) operators are singlets under the target-space SUSYs Q_+, $\bar{Q}_-$, and appear to have no counterpart in the matrix model side. Since the expectation value of operators measuring an RR charge Φ_{2k+1} at the solution (4) does not vanish [15], the matrix model is considered to correspond to the type IIA theory on a nontrivial background of the (R−, R+) fields. We may introduce the (R−, R+) background in the form of vertex operators, when the strength of the background $(\nu_+ - \nu_-)$ is small. In this treatment of the background, various correlation functions among the above vertex operators in the type IIA theory are computed in [16], which provides a number of evidence of correspondence between the matrix model and the type IIA theory.

4 Nonperturbative SUSY Breaking in the Matrix Model

In this section, we obtain the full nonperturbative free energy of the matrix model as the Tracy-Widom distribution in random matrix theory in the double scaling limit

$$N \to \infty, \quad \mu^2 \to 2 \quad \text{with } s \equiv N^{2/3}(\mu^2 - 2) \text{ fixed} \tag{15}$$

as discussed in [5, 20]. In its weakly coupled region (s: large), instanton effects can be seen in the matrix model which are nonperturbative in $1/N$. Although such effects are typically of the order e^{-N} and vanish in the simple large-N limit, interestingly we will see that they are nonvanishing in the double scaling limit (15).

The partition function of the matrix model given by the action (1) is expressed as

$$\begin{aligned} Z &= \int d^{N^2}\phi\, e^{-N\frac{1}{2}\mathrm{tr}(\phi^2-\mu^2)^2}\ \det(\phi\otimes\mathbf{1}+\mathbf{1}\otimes\phi) \\ &= \tilde{C}_N \int \Big(\prod_{i=1}^N d\lambda_i\Big)\, \triangle(\lambda)^2 \prod_{i,j=1}^N (\lambda_i+\lambda_j)\, e^{-N\sum_{i=1}^N \frac{1}{2}(\lambda_i^2-\mu^2)^2}, \end{aligned} \tag{16}$$

after integrating out matrices other than ϕ. Here, $\mathbf{1}$ is an $N\times N$ unit matrix, λ_i ($i = 1, \cdots, N$) are eigenvalues of ϕ, and $\triangle(\lambda)$ denotes the Vandermonde determinant $\triangle(\lambda) = \prod_{i>j}(\lambda_i - \lambda_j)$. $\tilde{C}_N$ is an numerical factor depending only on N given by

$$\frac{1}{\tilde{C}_N} = \int \Big(\prod_{i=1}^N d\lambda_i\Big)\, \triangle(\lambda)^2\, e^{-N\sum_{i=1}^N \frac{1}{2}\lambda_i^2} = (2\pi)^{\frac{N}{2}} \frac{\prod_{k=0}^N k!}{N^{\frac{N^2}{2}}}. \tag{17}$$

Contributions to the partition function are divided by sectors labeled by the filling fraction (ν_+, ν_-) as

$$Z = \sum_{\nu_- N=0}^{N} \frac{N!}{(\nu_+ N)!(\nu_- N)!}\, Z_{(\nu_+,\nu_-)} \tag{18}$$

with

$$
Z_{(\nu_+,\nu_-)} \equiv \tilde{C}_N \int_0^\infty \left(\prod_{i=1}^{\nu_+ N} d\lambda_i\right) \int_{-\infty}^0 \left(\prod_{j=\nu_+ N+1}^{N} d\lambda_j\right) \left(\prod_{n=1}^{N} 2\lambda_n\right)
$$

$$
\times \left\{\prod_{n>m} (\lambda_n^2 - \lambda_m^2)^2\right\} e^{-N\sum_{i=1}^N \frac{1}{2}(\lambda_i^2-\mu^2)^2}. \tag{19}
$$

Here, it is easy to see

$$
Z_{(\nu_+,\nu_-)} = (-1)^{\nu_- N} Z_{(1,0)}, \tag{20}
$$

which leads to the vanishing partition function:

$$
Z = (1 + (-1))^N Z_{(1,0)} = 0. \tag{21}
$$

In order for expectation values normalized by the partition function to be well-defined, we regularize the partition function by introducing a factor $e^{-i\alpha\nu_- N}$ with small α in front of $Z_{(\nu_+,\nu_-)}$. The regularized partition function becomes

$$
Z_\alpha \equiv \sum_{\nu_- N=0}^{N} \frac{N!}{(\nu_+ N)!(\nu_- N)!} e^{-i\alpha\nu_- N} Z_{(\nu_+,\nu_-)} = (1 - e^{-i\alpha})^N Z_{(1,0)}. \tag{22}
$$

Notice that calculations in large-N expansion [15] concern the partition function in a single sector ($Z_{(\nu_+,\nu_-)}$), in which this kind of regularization was not needed. On the other hand, since nonperturbative contributions to be computed here possibly communicate among various sectors of filling fractions, we should consider the total partition function (18) and its vanishing value requires the regularization.

The expectation value of $\frac{1}{N}\mathrm{tr}(iB)$ under the regularization (22) is expressed as

$$
\left\langle \frac{1}{N}\mathrm{tr}\,(iB)\right\rangle_\alpha = \frac{1}{N^2}\frac{1}{Z_\alpha}\frac{\partial}{\partial(\mu^2)} Z_\alpha
$$

$$
= \frac{1}{N^2}\frac{1}{Z_{(1,0)}}\frac{\partial}{\partial(\mu^2)} Z_{(1,0)} = \left\langle \frac{1}{N}\mathrm{tr}\,(iB)\right\rangle^{(1,0)} \tag{23}
$$

due to a cancellation of the factor $(1 - e^{-i\alpha})^N$ in (22) between the numerator and the denominator. The regularized expectation value $\left\langle \frac{1}{N}\mathrm{tr}\,(iB)\right\rangle_\alpha$ is independent of α and well-defined in the limit $\alpha \to 0$, and thus serves as an order parameter for spontaneous SUSY breaking.

4.1 Tracy-Widom Distribution

Under the change of variables $x_i = -\lambda_i^2 + \mu^2$ (the Nicolai mapping), the partition function $Z_{(1,0)}$ defined in (19) reduces to Gaussian matrix integrals

$$Z_{(1,0)} = \tilde{C}_N \int_{-\infty}^{\mu^2} \left(\prod_{i=1}^{N} dx_i \right) \triangle(x)^2 \, e^{-N \sum_{i=1}^{N} \frac{1}{2} x_i^2}. \tag{24}$$

It seems almost trivial, but a nontrivial effect arises from the upper bound of the integration region. Techniques in random matrix theory [23] give a closed form for the partition function in the double scaling limit (15):

$$F(s) = -\ln Z_{(1,0)} = \int_s^{\infty} (x-s) q(x)^2 dx, \tag{25}$$

where $q(x)$ satisfies a Painlevé II differential equation

$$q(x)'' = xq(x) + 2q(x)^3 \tag{26}$$

with the boundary condition

$$q(x) \to \mathrm{Ai}(x) \qquad (x \to +\infty). \tag{27}$$

Such a solution is unique and known as the Hastings-McLeod solution [9]. Since Eq. (15) indicates that the string coupling constant $g_s \sim 1/N$ is proportional to $s^{-3/2}$, the region of $s \gg 1$ ($0 < s \ll 1$) describes the weakly (strongly) coupled IIA strings.

4.2 Weak Coupling Expansion

The partition function is given by the Fredholm determinant of the Airy kernel [23]

$$Z_{(1,0)} = \mathrm{Det}(1 - \hat{K}_{\mathrm{Ai}}|_{[s,\infty)}), \tag{28}$$

where the operator $\hat{K}_{\mathrm{Ai}}|_{[s,\infty)}$ can be represented as the integration kernel on the interval $[s, \infty)$:

$$K_{\mathrm{Ai}}(x, y) \equiv \frac{\mathrm{Ai}(x)\mathrm{Ai}'(y) - \mathrm{Ai}'(x)\mathrm{Ai}(y)}{x - y}. \tag{29}$$

From the above fact, it turns out that the weak coupling expansion (large-s expansion) of the free energy is expressed as an instanton sum

$$F = -\ln Z_{(1,0)} = \sum_{k=1}^{\infty} F_{k-\text{inst.}} \tag{30}$$

with

$$\begin{aligned} F_{k-\text{inst.}} &= \frac{1}{k}\int_s^\infty dt_1 \dots dt_k\, K_{\text{Ai}}(t_1, t_2)\, K_{\text{Ai}}(t_2, t_3) \cdots K_{\text{Ai}}(t_k, t_1) \\ &\sim \frac{1}{k}\left(\frac{1}{16\pi s^{3/2}}\, e^{-\frac{4}{3}s^{3/2}}\right)^k \left[1 + a_1^{(k)} s^{-3/2} + a_2^{(k)} s^{-3} + \cdots\right]. \end{aligned} \tag{31}$$

Some of the coefficients are

$$\begin{aligned} &a_1^{(1)} = -\frac{35}{24}, \quad a_2^{(1)} = \frac{3745}{1152}, \quad a_3^{(1)} = -\frac{805805}{82944}, \cdots \\ &a_1^{(2)} = -\frac{35}{12}, \quad a_2^{(2)} = \frac{619}{72}, \quad a_3^{(2)} = -\frac{592117}{20736}, \cdots \\ &a_1^{(3)} = -\frac{35}{8}, \quad a_2^{(3)} = \frac{2059}{128}, \quad a_3^{(3)} = -\frac{184591}{3072}, \cdots \\ &a_1^{(4)} = -\frac{35}{6}, \quad a_2^{(4)} = \frac{3701}{144}, \quad a_3^{(4)} = -\frac{1112077}{10368}, \cdots \\ &\cdots . \end{aligned} \tag{32}$$

The contribution to the free energy has no perturbative part and starts from nonperturbative effects of the instanton action $\frac{4}{3}s^{3/2} \propto N$ and its fluctuations expanded by $s^{-3/2} \propto N^{-1}$. It seems plausible that the nonperturbative contributions are provided by D-brane like objects. The order parameter of the SUSY breaking (with the wave function renormalization factor $N^{4/3}$) $N^{4/3}.\left\langle \frac{1}{N}\text{tr}(iB)\right\rangle^{(1,0)} = -F'(s)$ remains nonzero, implying that the target-space SUSY in the two-dimensional IIA theory is spontaneously broken by D-brane like objects. Corresponding Nambu-Goldstone fermions are identified with $\frac{1}{N}\text{tr}\bar{\psi}$ and $\frac{1}{N}\text{tr}\psi$ associated with the breaking of Q and $\bar{Q}$, respectively [5].

4.3 Strong Coupling Expansion

The Taylor series expansion of (25) around $s = 0$ is

$$\begin{aligned} F(s) = 0.0311059853 - 0.0690913807s + 0.0673670913s^2 \\ -0.0361399144s^3 + \cdots , \end{aligned} \tag{33}$$

which gives strong coupling expansion of the IIA superstring theory. The strong coupling limit is regular and finite. In particular, the expression is smooth around $s = 0$ and there is no obstruction to be continued to the $s < 0$ region (i.e., $\mu^2 < 2$), whereas in Sect. 2 we had mentioned the third order phase transition across the point $\mu^2 = 2$ in the planar limit. Thus, the singularity in the planar limit becomes completely smeared out in the double scaling limit. In the string-theory perspective, singular behavior at the string tree level is smoothed out by quantum effects. Similar phenomenon can be seen in the unitary one-matrix model [21].

5 Higher Genus Amplitudes in the Matrix Model

In this section, we calculate the one-point function of Φ_{2k+1} to all orders in the string perturbation theory. Since the operators are not protected by SUSY, nontrivial large-order behavior is expected here. As discussed in [15], the one-point function at the (ν_+, ν_-) filling fraction is simply related to that at the $(1, 0)$ filling fraction by

$$\left\langle \frac{1}{N} \mathrm{tr}\, \Phi_{2k+1} \right\rangle^{(\nu_+,\nu_-)} = (\nu_+ - \nu_-) \left\langle \frac{1}{N} \mathrm{tr}\, \Phi_{2k+1} \right\rangle^{(1,0)} . \tag{34}$$

So, it is sufficient to consider the sector of the $(1, 0)$ filling fraction alone. The object is recast to the contour integral of the resolvent of ϕ^2 as

$$\left\langle \frac{1}{N} \mathrm{tr}\, \Phi_{2k+1} \right\rangle^{(1,0)} = \oint_{[a,b]} \frac{dz}{2\pi i} z^{2k+1} \cdot 2z \left\langle \frac{1}{N} \mathrm{tr} \frac{1}{z^2 - \phi^2} \right\rangle^{(1,0)} + \cdots , \tag{35}$$

where the integration contour surrounds the support of the eigenvalue distribution $[a, b]$. "$\cdots$" stands for nonuniversal analytic terms in s which we will ignore below. Notice that the resolvent is protected by SUSY because ϕ^2 is essentially equivalent with the auxiliary variable B. The resolvent can be explicitly computed at each order of the $1/N$ expansion by using the result of the Gaussian matrix model [8]. After taking the double scaling limit (15), we end up with the following genus expansion:

$$\begin{aligned} N^{\frac{2}{3}(k+2)} \left\langle \frac{1}{N} \mathrm{tr}\, \Phi^{2k+1} \right\rangle^{(1,0)} &= \\ &= \frac{1}{2\pi^{3/2}} \Gamma\left(k + \frac{3}{2}\right) \sum_{h=0}^{\left[\frac{k+2}{3}\right]} \left(-\frac{1}{12}\right)^h \frac{s^{k-3h+2}}{h!(k-3h+2)!} \ln s \\ &+ \frac{(-1)^{k+1}}{2\pi^{3/2}} \Gamma\left(k + \frac{3}{2}\right) \sum_{h=\left[\frac{k+2}{3}\right]+1}^{\infty} \frac{(3h-k-3)!}{h!} \frac{s^{k+2-3h}}{12^h} . \end{aligned} \tag{36}$$

The infinite series in the third line is divergent and not Borel summable. In fact, the Borel resummation leads to

$$(\text{2nd line}) \simeq \frac{1}{4\pi}\frac{1}{3^{k+5/2}}\frac{s^{k+2}}{(k+\frac{3}{2})(k+\frac{5}{2})}\int_0^\infty dz\,\left(1-\frac{z^2}{z_0^2}\right)^{k+5/2}\, e^{-z}$$
$$+(\text{less singular}) \tag{37}$$

with $z_0 \equiv \frac{4}{3}s^{3/2}$. The integrand has a branch cut singularity at $z = z_0$ which sits on the integration contour. The result of the integral changes depending on avoiding the singularity upwards or downwards. The difference gives the amount of the nonperturbative ambiguity:

$$i(-1)^{k+1}\frac{\Gamma(k+\frac{3}{2})}{2\pi\cdot 3^{k+5/2}}\frac{s^{k+2}}{(k+\frac{3}{2})(k+\frac{5}{2})}\int_{z_0}^\infty dz\left(\frac{z^2}{z_0^2}-1\right)^{k+5/2} e^{-z}, \tag{38}$$

that is of the order $e^{-\frac{4}{3}s^{3/2}}$ coinciding with the leading instanton contribution (31).

Recently, resurgence theory has been discussed in quantum mechanical systems and matrix models, which tells that ambiguity from large-order behavior of perturbation series should cancel with ambiguity from instanton contributions so that the total expression is well-defined (For example, see [4, 18].). It is interesting to compute instanton effects to the one-point function and check whether the resurgence program works in our case.

6 Summary and Discussion

We have discussed a SUSY double-well matrix model and its correspondence to two-dimensional type IIA superstring theory on a nontrivial (R−, R+) background. This is an interesting example of matrix models for superstrings with target-space SUSY, in which various amplitudes are explicitly calculable.

We have seen that nonperturbative effects in the matrix model spontaneously break the SUSY. Since the effects survive in the double scaling limit (15), the result indicates spontaneous SUSY breaking in the type IIA theory by nonperturbative contributions. It is interesting to investigate dynamics of D-branes in the type IIA theory and to reproduce the instanton contributions seen here.

In addition, the one-point function of the non-SUSY operator Φ_{2k+1} has been computed to all orders in genus expansion. The series is divergent and not Borel summable. It is interesting to see that the ambiguity arising from the Borel resummation procedure cancels with that from instanton contributions as the resurgence theory suggests.

Acknowledgements The author would like to thank Michael G. Endres, Tsunehide Kuroki, Shinsuke Nishigaki and Hiroshi Suzuki for collaboration. He is grateful to the organizers of LT-11, especially Professor Vladimir Dobrev, for invitation to the wonderful meeting and for warm hospitality.

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Kruskal–Penrose Formalism for Lightlike Thin-Shell Wormholes

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Abstract The original formulation of the "Einstein–Rosen bridge" in the classic paper of Einstein and Rosen (1935) is historically the first example of a *static* spherically-symmetric wormhole solution. It is *not equivalent* to the concept of the *dynamical* and *non-traversable* Schwarzschild wormhole, also called "Einstein–Rosen bridge" in modern textbooks on general relativity. In previous papers of ours we have provided a mathematically correct treatment of the original "Einstein–Rosen bridge" as a *traversable* wormhole by showing that it requires the presence of a special kind of "exotic matter" located on the wormhole throat – a lightlike brane (the latter was overlooked in the original 1935 paper). In the present note we continue our thorough study of the original "Einstein–Rosen bridge" as a simplest example of a lightlike thin-shell wormhole by explicitly deriving its description in terms of the Kruskal–Penrose formalism for maximal analytic extension of the underlying wormhole spacetime manifold. Further, we generalize the Kruskal–Penrose description to the case of more complicated lightlike thin-shell wormholes with two throats exhibiting a remarkable property of QCD-like charge confinement.

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V. Dobrev (ed.), *Lie Theory and Its Applications in Physics*,
Springer Proceedings in Mathematics & Statistics 191,
DOI 10.1007/978-981-10-2636-2_15

1 Introduction

The principal object of study in the present note is the class of static spherically symmetric *lightlike thin-shell* wormhole solutions in general relativity, i.e., spacetimes with wormhole geometries and "throats" being lightlike ("null") hypersurfaces (for the importance and impact of lightlike hypersurfaces, see Refs. [1–3]). The explicit construction of lightlike thin-shell wormholes based on a self-consistent Lagrangian action formalism for the underlying lightlike branes occupying the wormhole "throats" and serving as material (and electrical charge) sources for the gravity to generate the wormhole spacetime geometry was given in a series of previous papers [4–8].[1]

The celebrated "Einstein–Rosen bridge", originally formulated in the classic paper [10], is historically the first and simplest example of a *static* spherically-symmetric wormhole solution – it is a 4-dimensional spacetime manifold consisting of two identical copies of the exterior Schwarzschild spacetime region matched (glued together) along their common horizon.

Let us immediately emphasize that the original construction in [10] of the "Einstein–Rosen bridge" is *not equivalent* to the notion of the *dynamical* Schwarzschild wormhole, also called "Einstein–Rosen bridge" in several standard textbooks (e.g. Ref. [11]), which employs the formalism of Kruskal–Szekeres maximal analytic extension of Schwarzschild black hole spacetime geometry. Namely, the two regions in Kruskal–Szekeres manifold corresponding to the outer Schwarzschild spacetime region beyond the horizon ($r > 2m$) and labeled (I) and (III) in Ref. [11] are generally *disconnected* and share only a two-sphere (the angular part) as a common border ($U = 0$, $V = 0$ in Kruskal–Szekeres coordinates), whereas in the original Einstein–Rosen "bridge" construction the boundary between the two identical copies of the outer Schwarzschild space-time region ($r > 2m$) is a three-dimensional lightlike hypersurface ($r = 2m$). Physically, the most significant difference is that the "textbook" version of the "Einstein–Rosen bridge" (Schwarzschild wormhole) is *non-traversable*, i.e., there are *no* timelike or lightlike geodesics connecting points belonging to the two separate outer Schwarzschild regions (I) and (III). This is in sharp contrast w.r.t. the original Einstein–Rosen bridge (within its consistent formulation as a lightlike thin-shell wormhole [5]), which is a *traversable wormhole* (see also Sect. 3 below).

However, as explicitly demonstrated in Refs. [5, 6], the originally proposed in [10] Einstein–Rosen "bridge" wormhole solution *does not* satisfy the vacuum Einstein equations at the wormhole "throat". The mathematically consistent formulation of the original Einstein–Rosen "bridge" requires solving Einstein equations of bulk $D = 4$ gravity coupled to a lightlike brane with a well-defined world-volume action [12–15]. The lightlike brane locates itself automatically on the wormhole throat gluing together the two "universes" - two identical copies of the external spacetime region of a Schwarzschild black hole matched at their common horizon, with a special relation

[1] For the general construction of *timelike* thin-shell wormholes, see the book [9].

between the (negative) brane tension and the Schwarzschild mass parameter. This is briefly reviewed in Sect. 2.

Traversability of the correctly formulated Einstein–Rosen bridge as a lightlike thin-shell wormhole is explicitly demonstrated in Sect. 3 in the sense of passing through the wormhole throat from the "left" to the "right" universe within *finite proper time* of a travelling observer.

In Sect. 4 we explicitly construct the Kruskal–Penrose maximal analytic extension of the proper Einstein–Rosen bridge wormhole manifold. In particular, the pertinent Kruskal–Penrose manifold involves a special identification of the future horizon of the "right" universe with the past horizon of the "left" universe, which is the mathematical manifestation of the wormhole traversability.

In Sect. 5 we extend our construction of Kruskal–Penrose maximal analytic extension of the total wormhole manifold to the case of a physically interesting wormhole solution with two "throats" which exhibits a remarkable property of charge and electric flux confinement [16] resembling the quark confinement property of quantum chromodynamics.

Section 6 contains our concluding remarks.

2 Einstein–Rosen Bridge as Lightlike Thin-Shell Wormhole

The Schwarzschild spacetime metric is the simplest static spherically symmetric black hole metric, written in standard coordinates (t, r, θ, φ) (e.g. [11]):

$$ds^2 = -A(r)dt^2 + \frac{1}{A(r)}dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right) \quad , \quad A(r) = 1 - \frac{r_0}{r} \, , \tag{1}$$

where $r_0 \equiv 2m$ (m – black hole mass parameter):

- $r > r_0$ defines the exterior spacetime region; $r < r_0$ is the black hole region;
- r_0 is the horizon radius, where $A(r_0) = 0$ ($r = r_0$ is a non-physical coordinate singularity of the metric (1), unlike the physical spacetime singularity at $r = 0$).

In constructing the maximal analytic extension of the Schwarzschild spacetime geometry – the *Kruskal–Szekeres* coordinate chart – essential intermediate use is made of the so called "tortoise" coordinate r^* (for light rays $t \pm r^* = \text{const}$):

$$\frac{dr^*}{dr} = \frac{1}{A(r)} \quad \longrightarrow \quad r^* = r + r_0 \ln|r - r_0| \, . \tag{2}$$

The Kruskal–Szekeres ("light-cone") coordinates (v, w) are defined as follows (e.g. [11]):

$$v = \pm\frac{1}{\sqrt{2k_h}}e^{k_h\left(t+r^*\right)} \quad , \quad w = \mp\frac{1}{\sqrt{2k_h}}e^{-k_h\left(t-r^*\right)} \tag{3}$$

with all combinations of the overall signs, where $k_h = \frac{1}{2}\partial_r A(r)\,\big|_{r=r_0} = \frac{1}{2r_0}$ is the so called "surface gravity" (related to the Hawking temperature as $\frac{k_h}{2\pi} = k_B T_{\text{hawking}}$). Equation (3) are equivalent to:

$$\mp vw = \frac{1}{k_h} e^{2k_h r^*} \quad , \quad \mp \frac{v}{w} = e^{2k_h t} \ , \tag{4}$$

wherefrom t and r^* are determined as functions of vw.

Depending on the combination of the overall signs Eq. (3) define a doubling the regions of the standard Schwarzschild geometry [11]:

(i) $(+,-)$ – exterior Schwarzschild region $r > r_0$ (region I);
(ii) $(+,+)$ – black hole $r < r_0$ (region II);
(iii) $(-,+)$ – second copy of exterior Schwarzschild region $r > r_0$ (region III);
(iv) $(-,-)$ – "white" hole region $r < r_0$ (region IV).

The metric (1) becomes:

$$ds^2 = \widetilde{A}(vw) dv dw + r^2(vw)\left(d\theta^2 + \sin^2\theta d\varphi^2\right) \quad , \quad \widetilde{A}(vw) \equiv \frac{A\big(r(vw)\big)}{k_h^2 vw} \ , \tag{5}$$

so that now there is no coordinate singularity on the horizon ($v = 0$ or $w = 0$) upon using Eq. (2): $\widetilde{A}(0) = -4$.

In the classic paper [10] Einstein and Rosen introduced in (1) a new radial-like coordinate u via $r = r_0 + u^2$ and let $u \in (-\infty, +\infty)$:

$$ds^2 = -\frac{u^2}{u^2 + r_0} dt^2 + 4(u^2 + r_0) du^2 + (u^2 + r_0)^2 \left(d\theta^2 + \sin^2\theta\, d\varphi^2\right) \ . \tag{6}$$

Thus, (6) describes two identical copies of the exterior Schwarzschild spacetime region ($r > r_0$) for $u > 0$ and $u < 0$, which are formally glued together at the horizon $u = 0$.

Unfortunately, there are serious problems with (6):

- The Einstein–Rosen metric (6) has coordinate singularity at $u = 0$: $\det \|g_{\mu\nu}\|_{u=0} = 0$.
- More seriously, the Einstein equations for (6) acquire an ill-defined non-vanishing "matter" stress-energy tensor term on the r.h.s., which was overlooked in the original 1935 paper!

Indeed, as explained in [5], from Levi-Civita identity $R^0_0 = -\frac{1}{\sqrt{-g_{00}}}\nabla^2_{(3)}\left(\sqrt{-g_{00}}\right)$ we deduce that (6) solves vacuum Einstein equation $R^0_0 = 0$ for all $u \neq 0$. However, since $\sqrt{-g_{00}} \sim |u|$ as $u \to 0$ and since $\frac{\partial^2}{\partial u^2}|u| = 2\delta(u)$, Levi-Civita identity tells us that:

$$R^0_0 \sim \frac{1}{|u|}\delta(u) \sim \delta(u^2) \ , \tag{7}$$

and similarly for the scalar curvature $R \sim \frac{1}{|u|}\delta(u) \sim \delta(u^2)$.

In [5] we proposed a correct reformulation of the original Einstein–Rosen bridge as a mathematically consistent traversable lightlike thin-shell wormhole introducing a different radial-like coordinate $\eta \in (-\infty, +\infty)$, by substituting $r = r_0 + |\eta|$ in (1):

$$ds^2 = -\frac{|\eta|}{|\eta| + r_0} dt^2 + \frac{|\eta| + r_0}{|\eta|} d\eta^2 + (|\eta| + r_0)^2 \left(d\theta^2 + \sin^2\theta\, d\varphi^2\right) . \qquad (8)$$

Equation (8) is the correct spacetime metric for the original Einstein–Rosen bridge:

- Equation (8) describes two "universes" – two identical copies of the exterior Schwarzschild spacetime region for $\eta > 0$ and $\eta < 0$.
- Both "universes" are correctly glued together at their common horizon $\eta = 0$. Namely, the metric (8) solves Einstein equations:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi T^{(brane)}_{\mu\nu} , \qquad (9)$$

  where on the r.h.s. $T^{(brane)}_{\mu\nu} = S_{\mu\nu}\delta(\eta)$ is the energy-momentum tensor of a special kind of *lightlike brane* located on the common horizon $\eta = 0$ – the wormhole "throat".
- The lightlike analogues of W. Israel's junction conditions on the wormhole "throat" are satisfied [5, 6].
- The resulting lightlike thin-shell wormhole is traversable (see Sect. 3 below).

The energy-momentum tensor of lightlike branes $T^{(brane)}_{\mu\nu}$ is self-consistently derived as $T^{(brane)}_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta S_{\rm LL}}{\delta g^{\mu\nu}}$ from the following manifestly reparametrization invariant world-volume Polyakov-type lightlike brane action (written for arbitrary $D = (p + 1) + 1$ embedding spacetime dimension and $(p + 1)$-dimensional brane world-volume):

$$S_{\rm LL} = -\frac{1}{2} \int d^{p+1}\sigma\, T b_0^{\frac{p-1}{2}} \sqrt{-\gamma} \left[\gamma^{ab} \bar{g}_{ab} - b_0(p - 1)\right] , \qquad (10)$$

$$\bar{g}_{ab} \equiv g_{ab} - \frac{1}{T^2} \left(\partial_a u + q\mathcal{A}_a\right) \left(\partial_b u + q\mathcal{A}_b\right) \;, \quad \mathcal{A}_a \equiv \partial_a X^\mu A_\mu \, . \qquad (11)$$

Here and below the following notations are used:

- γ_{ab} is the *intrinsic* Riemannian metric on the world-volume with $\gamma = \det \|\gamma_{ab}\|$; b_0 is a positive constant measuring the world-volume "cosmological constant"; $(\sigma) \equiv (\sigma^a)$ with $a = 0, 1, \ldots, p$; $\partial_a \equiv \frac{\partial}{\partial\sigma^a}$.
- $X^\mu(\sigma)$ are the p-brane embedding coordinates in the bulk D-dimensional spacetime with Riemannian metric $g_{\mu\nu}(x)$ $(\mu, \nu = 0, 1, \ldots, D - 1)$. A_μ is a spacetime electromagnetic field (absent in the present case).

- $g_{ab} \equiv \partial_a X^\mu g_{\mu\nu}(X) \partial_b X^\nu$ is the *induced* metric on the world-volume which becomes *singular* on-shell – manifestation of the lightlike nature of the brane.
- u is auxiliary world-volume scalar field defining the lightlike direction of the induced metric and it is a non-propagating degree of freedom.
- T is *dynamical (variable)* brane tension (also a non-propagating degree of freedom).
- Coupling parameter q is the surface charge density of the LL-brane ($q = 0$ in the present case).

The Einstein Eq. (9) imply the following relation between the lightlike brane parameters and the Einstein–Rosen bridge "mass" ($r_0 = 2m$):

$$-T = \frac{1}{8\pi m} \quad , \quad b_0 = \frac{1}{4} \; , \tag{12}$$

i.e., the lightlike brane dynamical tension T becomes *negative* on-shell – manifestation of "exotic matter" nature.

3 Einstein–Rosen Bridge as Traversable Wormhole

As already noted in [5, 6] traversability of the original Einstein–Rosen bridge is a particular manifestation of the traversability of lightlike "thin-shell" wormholes.[2] Here for completeness we will present the explicit details of the traversability within the proper Einstein–Rosen bridge wormhole coordinate chart (8) which are needed for the construction of the pertinent Kruskal–Penrose diagram in Sect. 4.

The motion of test-particle ("observer") of mass m_0 in a gravitational background is given by the reparametrization-invariant world-line action:

$$S_{\text{particle}} = \frac{1}{2} \int d\lambda \left[\frac{1}{e} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu - e m_0^2 \right] \; , \tag{13}$$

where $\dot{x}^\mu \equiv \frac{dx^\mu}{d\lambda}$, e is the world-line "einbein" and in the present case $(x^\mu) = (t, \eta, \theta, \varphi)$.

For a static spherically symmetric background such as (8) there are conserved Noether "charges" – energy $\mathcal{E}$ and angular momentum $\mathcal{J}$. In what follows we will consider purely "radial" motion ($\mathcal{J} = 0$) so, upon taking into account the "mass-shell" constraint (the equation of motion w.r.t. e) and introducing the world-line proper-time parameter τ ($\frac{d\tau}{d\lambda} = e m_0$), the timelike geodesic equations (world-lines of massive point particles) read:

[2] Subsequently, traversability of the Einstein–Rosen bridge has been studied using Kruskal–Szekeres coordinates for the Schwarzschild black hole [17], or the 1935 Einstein–Rosen coordinate chart (6) [18].

$$\left(\frac{d\eta}{d\tau}\right)^2 = \frac{\mathcal{E}^2}{m_0^2} - A(\eta) \quad , \quad \frac{dt}{d\tau} = \frac{\mathcal{E}}{m_0 A(\eta)} \quad , \quad A(\eta) \equiv \frac{|\eta|}{|\eta| + r_0} \; . \tag{14}$$

where $A(\eta)$ is the "$-g_{00}$" component of the proper Einstein–Rosen bridge metric (8).

For a test-particle starting for $\tau = 0$ at initial position in "our" (right) universe $\eta_0 = \eta(0)$, $t_0 = t(0)$ and *infalling* towards the "throat" the solutions of Eq. (14) read:

$$\frac{\mathcal{E}}{2k_h m_0} \int_{2k_h\eta(\tau)}^{2k_h\eta_0} dy \sqrt{(1+|y|)\Big[(1+\big(1-\frac{m_0^2}{\mathcal{E}^2}\big)|y|\Big]^{-1}} = \tau \; , \tag{15}$$

$$\frac{1}{2k_h} \int_{2k_h\eta(\tau)}^{2k_h\eta_0} dy \frac{1}{|y|} \sqrt{(1+|y|)\Big[(1+\big(1-\frac{m_0^2}{\mathcal{E}^2}\big)|y|\Big]} = t(\tau) - t_0 \; . \tag{16}$$

- Equation (15) shows that the particle will cross the wormhole "throat" ($\eta = 0$) for a finite proper-time $\tau_0 > 0$:

$$\tau_0 = \frac{\mathcal{E}}{2k_h m_0} \int_0^{2k_h\eta_0} dy \sqrt{(1+|y|)\Big[(1+\big(1-\frac{m_0^2}{\mathcal{E}^2}\big)|y|\Big]^{-1}} \; . \tag{17}$$

- It will continue into the second (left) universe and reach any point $\eta_1 = \eta(\tau_1) < 0$ within another **finite** proper-time $\tau_1 > \tau_0$.
- On the other hand, from (16) it follows that $t(\tau_0 - 0) = +\infty$, i.e., from the point of view of a static observer in "our" (right) universe it will take infinite "laboratory" time for the particle to reach the "throat" – the latter appears to the static observer as a future black hole horizon.
- Equation (16) also implies $t(\tau_0 + 0) = -\infty$, which means that from the point of view of a static observer in the second (left) universe, upon crossing the "throat", the particle starts its motion in the second (left) universe from infinite past, so that it will take an infinite amount of "laboratory" time to reach the point $\eta_1 < 0$ – i.e. the "throat" now appears as a past black hole horizon.

In analogy with the usual "tortoise" coordinate r^* for the Schwarzschild black hole geometry (2) let us now introduce Einstein–Rosen bridge "tortoise" coordinate η^* (recall $r_0 = \frac{1}{2k_h}$):

$$\frac{d\eta^*}{d\eta} = \frac{|\eta| + r_0}{|\eta|} \quad \longrightarrow \quad \eta^* = \eta + \text{sign}(\eta) r_0 \ln|\eta| \; . \tag{18}$$

Let us note here an important difference in the behavior of the "tortoise" coordinates r^* (2) and η^* (18) in the vicinity of the horizon. Namely:

$$r^* \to -\infty \quad \text{for} \quad r \to r_0 \pm 0 \; , \tag{19}$$

i.e., when r approaches the horizon either from above or from below, whereas when η approaches the horizon from above or from below:

$$\eta^* \to \mp\infty \quad \text{for} \quad \eta \to \pm 0 \,. \tag{20}$$

For infalling/outgoing massless particles (light rays) Eqs. (15)–(18) imply:

$$t \pm \eta^* = \text{const} \,. \tag{21}$$

For infalling massive particles towards the "throat" ($\eta = 0$) starting at $\eta_0^+ > 0$ in "our" (right) universe and crossing into the second (left) universe, or starting in the second (left) universe at some $\eta_0^- < 0$ and crossing into the "our" (right) universe, we have correspondingly (replacing τ-dependence with functional dependence w.r.t. η using first Eq. (14)):

$$\big[t \pm \eta^*\big](\eta) = \frac{\pm 1}{2k_h} \int_{2k_h \eta}^{2k_h \eta_0^\pm} dy \left(1 + \frac{1}{|y|}\right) \left[\sqrt{(1+|y|)\Big[(1 + \big(1 - \frac{m_0^2}{\mathcal{E}^2}\big)|y|\Big]^{-1}} - 1 \right] \tag{22}$$

4 Kruskal–Penrose Diagram for Einstein–Rosen Bridge

We now define the maximal analytic extension of original Einstein–Rosen wormhole geometry (8) via introducing Kruskal-like coordinates (v, w) as follows:

$$v = \pm \frac{1}{\sqrt{2k_h}} e^{\pm k_h (t+\eta^*)} \quad , \quad w = \mp \frac{1}{\sqrt{2k_h}} e^{\mp k_h (t-\eta^*)} \,, \tag{23}$$

implying:

$$-vw = \frac{1}{2k_h} e^{\pm 2k_h \eta^*} \quad , \quad -\frac{v}{w} = e^{\pm 2k_h t} \,. \tag{24}$$

Here and below η^* is given by (18).

- The upper signs in (23) and (24) correspond to region I ($v > 0$, $w < 0$) describing "our" (right) universe $\eta > 0$.
- The lower signs in (23) and (24) correspond to region II ($v < 0$, $w > 0$) describing the second (left) universe $\eta < 0$.

The metric (8) of Einstein–Rosen bridge in the Kruskal-like coordinates (23) reads:

$$ds^2 = \widetilde{A}(vw)dvdw + \widetilde{r}^2(vw)\left(d\theta^2 + \sin^2\theta d\varphi^2\right) , \tag{25}$$

$$\widetilde{r}(vw) = r_0 + |\eta(vw)| \quad \left(r_0 \equiv \frac{1}{2k_h}\right) ,$$

$$\widetilde{A}(vw) = \frac{A\left(\eta(vw)\right)}{k_h^2 vw} = -\frac{4e^{-2k_h|\eta(vw)|}}{1 + 2k_h|\eta(vw)|} , \tag{26}$$

where $\eta(vw)$ is determined from (24) and (18) as:

$$-vw = \frac{|\eta|}{2k_h}e^{2k_h|\eta|} \quad \longrightarrow \quad |\eta(vw)| = \frac{1}{2k_h}\mathcal{W}(-4k_h^2 vw) , \tag{27}$$

$\mathcal{W}(z)$ being the Lambert (product-logarithm) function ($z = \mathcal{W}(z)e^{\mathcal{W}(z)}$).

Using the explicit expression (18) for η^* in (24) we find for the metric (25) and (26):

- "Throats" (horizons) – at $v = 0$ or $w = 0$;
- In region I the "throat" ($v > 0, w = 0$) is a future horizon ($\eta = 0\,,\ t \to +\infty$), whereas the "throat" ($v = 0, w < 0$) is a past horizon ($\eta = 0\,,\ t \to -\infty$).
- In region II the "throat" ($v = 0, w > 0$) is a future horizon ($\eta = 0\,,\ t \to +\infty$), whereas the "throat" ($v < 0, w = 0$) is a past horizon ($\eta = 0\,,\ t \to -\infty$).

It is customary to replace Kruskal-like coordinates (v, w) (23) with compactified Penrose-like coordinates $(\bar{v}, \bar{w})$:

$$\bar{v} = \arctan(\sqrt{2k_h}\, v) \quad , \quad \bar{w} = \arctan(\sqrt{2k_h}\, w) , \tag{28}$$

mapping the various "throats" (horizons) and infinities to finite lines/points:

- In region I: future horizon ($0 < \bar{v} < \frac{\pi}{2}, \bar{w} = 0$); past horizon ($\bar{v} = 0, -\frac{\pi}{2} < \bar{w} < 0$).
- In region II: future horizon ($\bar{v} = 0, 0 < \bar{w} < \frac{\pi}{2}$); past horizon ($-\frac{\pi}{2} < \bar{v} < 0$, $\bar{w} = 0$).
- i_0 – spacelike infinity ($t =$ fixed, $\eta \to \pm\infty$):
  $i_0 = (\frac{\pi}{2}, -\frac{\pi}{2})$ in region I; $i_0 = (-\frac{\pi}{2}, \frac{\pi}{2})$ in region II.
- $i_\pm$ – future/past timelike infinity ($t \to \pm\infty$, $\eta =$ fixed):
  $i_+ = (\frac{\pi}{2}, 0)$, $i_- = (0, -\frac{\pi}{2})$ in region I; $i_+ = (0, \frac{\pi}{2})$, $i_- = (-\frac{\pi}{2}, 0)$ in region II.
- J_+ – future lightlike infinity ($t \to +\infty, \eta \to \pm\infty,\ t \mp \eta^* =$ fixed):
  $J_+ = (\bar{v} = \frac{\pi}{2}, -\frac{\pi}{2} < \bar{w} < 0)$ in region I;
  $J_+ = (-\frac{\pi}{2} < \bar{v} < 0, \bar{w} = \frac{\pi}{2})$ in region II.
- J_- – past lightlike infinity ($t \to -\infty, \eta \to \pm\infty$), $t \pm \eta^* =$ fixed):
  $J_- = (0 < \bar{v} < \frac{\pi}{2}, \bar{w} = -\frac{\pi}{2})$ in region I:
  $J_- = (\bar{v} = -\frac{\pi}{2}, 0 < \bar{w} < \frac{\pi}{2})$ in region II.

For infalling light rays starting in region I and crossing into region II we have the lightlike geodesic $t + \eta^* = c_1 \equiv$ const. Thus, according to (23) we must identify the

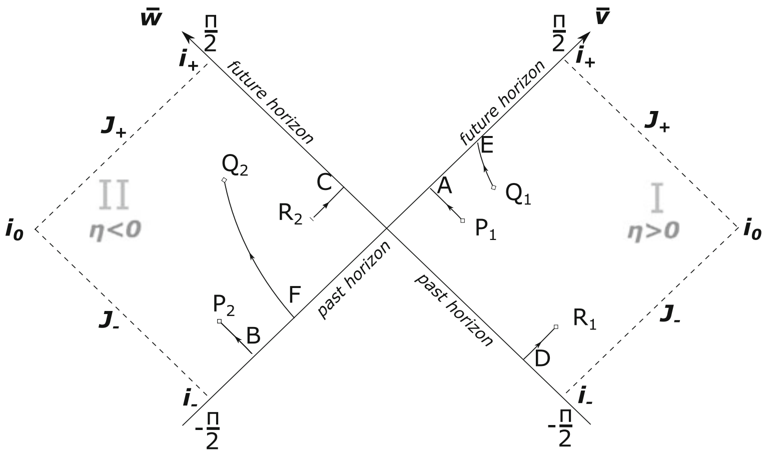

Fig. 1 Kruskal–Penrose diagram of the original Einstein–Rosen bridge

crossing point A on the future horizon of region I having Kruskal-like coordinates $(v = \frac{1}{\sqrt{2k_h}} e^{k_h c_1}, 0)$ with the point B on the past horizon of region II where the light rays enters into region II whose Kruskal-like coordinates are $(v = -\frac{1}{\sqrt{2k_h}} e^{-k_h c_1}, 0)$.

Similarly, for infalling light rays starting in region II and crossing into region I we have $t - \eta^* = c_2 \equiv$ const. Therefore, the crossing point C on the future horizon of region II having Kruskal-like coordinates $(0, w = \frac{1}{\sqrt{2k_h}} e^{k_h c_2})$ must be identified with the exit point D $(0, w = -\frac{1}{\sqrt{2k_h}} e^{-k_h c_2})$ on the past horizon of region I.

Inserting Eqs. (18)–(22) into the definitions of Kruskal-like (23) and Penrose-like (28) coordinates and taking into account the above identifications of horizons, we obtain the following visual representation of the Kruskal–Penrose diagram of the proper Einstein–Rosen bridge geometry (8) as depicted in Fig. 1:

- Future horizon in region I is identified with past horizon in region II as:

$$(\bar{v}, 0) \sim \left(\bar{v} - \frac{\pi}{2}, 0\right) . \tag{29}$$

  Infalling light rays cross from region I into region II via paths $P_1 \rightarrow A \sim B \rightarrow P_2$ – all the way within finite world-line time intervals (the symbol $\sim$ means identification according to (29)). Similarly, infalling massive particles cross from region I into region II via paths $Q_1 \rightarrow E \sim F \rightarrow Q_2$ within finite proper-time interval.
- Future horizon in II is identified with past horizon in I:

$$(0, \bar{w}) \sim \left(0, \bar{w} - \frac{\pi}{2}\right) . \tag{30}$$

Infalling light rays cross from region II into region I via paths $R_2 \to C \sim D \to R_1$ where $C \sim D$ is identified according to (30).

5 Kruskal–Penrose Formalism for Two-Throat Lightlike Thin-Shell Wormhole

Now we will briefly discuss the extension of the construction of Kruskal–Penrose diagram for the proper Einstein–Rosen bridge wormhole to the case of lightlike "thin-shell" wormholes with two throats. To this end we will consider the physically interesting example of the charge-confining two-throat "tube-like" wormhole studied in [16]. It is a solution of gravity interacting with a special non-linear gauge field system and both coupled to a pair of oppositely charged lightlike branes (cf. Eqs. (10) and (11) above).

The full wormhole spacetime consists of three "universes" glued pairwise via the two oppositely charged lightlike branes located on their common horizons:

- Region I: right-most non-compact electrically neutral "universe" – exterior region beyond the Schwarzschild horizon of a Schwarzschild-de Sitter black hole;
- Region II: middle "tube-like" "universe" of Levi-Civita–Bertotti–Robinson type [19–21] with finite radial-like spacial extend and compactified transverse spacial dimensions;
- Region III: left-most non-compact electrically neutral "universe" – exterior region beyond the Schwarzschild horizon of a Schwarzschild-de Sitter black hole, mirror copy of the left-most "universe".
- Most remarkable property is that the whole electric flux generated by the two oppositely charged lightlike branes sitting on the two "throats" is *completely confined* within the finite-spacial-size middle "tube-like" universe – analog of QCD quark confinement!

For a visual representation, see Fig. 2 [16].

Generically, the metric of a spherically symmetric traversable lightlike thin-shell wormhole with two "throats" reads [16] ($-\infty < \eta < \infty$):

$$ds^2 = -A(\eta)dt^2 + \frac{d\eta^2}{A(\eta)} + r^2(\eta)\left(d\theta^2 + \sin^2\theta d\varphi^2\right) , \tag{31}$$

$$A(\eta_1) = 0 \; , \; A(\eta_2) = 0 \; , \; a^{(1)}_{(\pm)} = \pm\frac{\partial}{\partial\eta}A \,\Big|_{\eta_1 \pm 0} > 0 \; , \; a^{(2)}_{(\pm)} = \pm\frac{\partial}{\partial\eta}A \,\Big|_{\eta_2 \pm 0} > 0 \; .$$

Accordingly, for the wormhole "tortoise" coordinate η^* defined as in first Eq. (18) we have in the vicinity of the two horizons $\eta_{1,2}$:

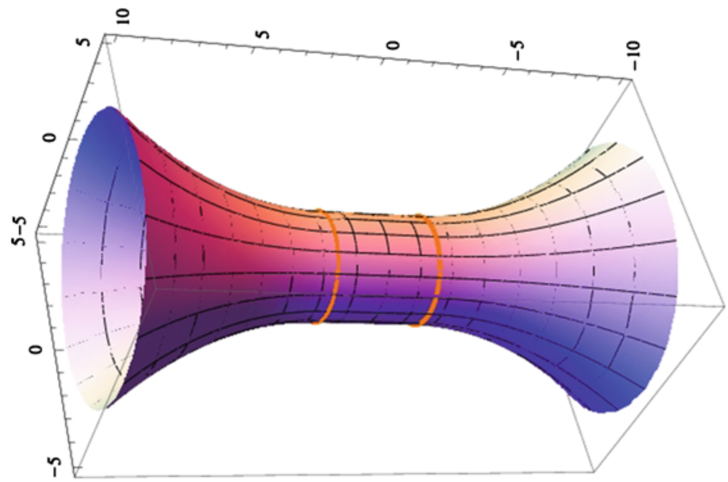

Fig. 2 Shape of $t = const$ and $\theta = \frac{\pi}{2}$ slice of charge-confining wormhole geometry. The whole electric flux is confined within the middle *cylindric* "tube" (region II) connecting the two infinite "funnels" (region I and region III). The *rings* on the edges of the "tube" depict the two oppositely charged lightlike branes

$$\eta^* = \mathrm{sign}(\eta - \eta_1) a^{(1)}_{(\pm)} \ln|\eta - \eta_1| + O\big((\eta - \eta_1)^2\big) , \tag{32}$$

$$\eta^* = \mathrm{sign}(\eta - \eta_2) a^{(2)}_{(\pm)} \ln|\eta - \eta_2| + O\big((\eta - \eta_2)^2\big) . \tag{33}$$

Now we can introduce the Kruskal-like and the compactified Kruskal–Penrose coordinates $(\bar{v}, \bar{w})$ for the maximal analytic extension of the two-throat lightlike thin-shell wormhole generalizing formulas (23) and (28) as follows:

- In region I (right-most universe) – $(+\infty > \eta > \eta_1)$:

$$\bar{v}, \bar{w} = \pm \frac{\pi}{2\sqrt{a^{(1)}_{(-)}}} \pm \frac{1}{\sqrt{a^{(1)}_{(+)}}} \arctan\Big(e^{\frac{1}{2} a^{(1)}_{(+)} (\eta^* \pm t)}\Big) \tag{34}$$

- In region II (middle universe) – $(\eta_1 > \eta > \eta_2)$; here $a^{(1)}_{(-)} = a^{(2)}_{(+)}$ which is satisfied in the case of the charge-confining two-throat "tube" wormhole:

$$\bar{v}, \bar{w} = \pm \frac{1}{\sqrt{a^{(1)}_{(-)}}} \arctan\Big(e^{\frac{1}{2} a^{(1)}_{(-)} (\eta^* \pm t)}\Big) . \tag{35}$$

- In region III (left-most universe) – $(\eta_2 > \eta > -\infty)$:

$$\bar{v}, \bar{w} = \mp \frac{\pi}{2\sqrt{a^{(2)}_{(-)}}} \pm \frac{1}{\sqrt{a^{(2)}_{(-)}}} \arctan\Big(e^{\frac{1}{2} a^{(2)}_{(-)} (\eta^* \pm t)}\Big) . \tag{36}$$

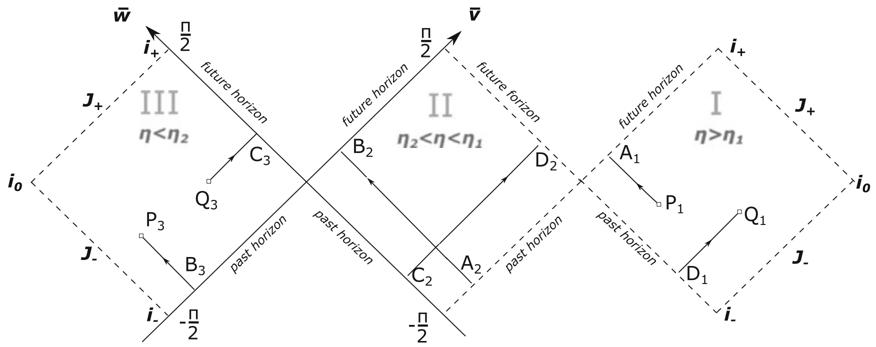

Fig. 3 Kruskal–Penrose diagram of "charge-confining" two-throat wormhole

The resulting Kruskal–Penrose diagram is depicted on Fig. 3.

In particular, infalling light ray starting in region I arrives in region III within finite world-line time interval ("proper-time" in the case of massive particle) on the path $P_1 \to A_1 \sim A_2 \to B_2 \sim B_3 \to P_3$, where the symbol $\sim$ indicates identification of the pertinent future and past horizons of the "glued" together neighboring "universes" analogous to the identification (29), (30) in the simpler case of Einstein–Rosen one-throat wormhole.

And similarly for an infalling light ray starting in region III and arriving in region I within finite world-line time interval on the path $Q_3 \to C_3 \sim C_2 \to D_2 \sim D_1 \to Q_1$.

6 Conclusions

The mathematically correct reformulation [5] of original Einstein–Rosen "bridge" construction, briefly reviewed in Sect. 2 above, shows that it is the simplest example in the class of static spherically symmetric *traversable* lightlike "thin-shell" wormhole solutions in general relativity. The consistency of Einstein–Rosen "bridge" as a traversable wormhole solution is guaranteed by the remarkable special properties of the world-volume dynamics of the lightlike brane, which serves as an "exotic" thin-shell matter (and charge) source of gravity.

In the present note we have explicitly derived the Kruskal-like extension and the associated Kruskal–Penrose diagram representation of the mathematically correctly defined original Einstein–Rosen "bridge" [5] with the following significant differences w.r.t. Kruskal–Penrose extension of the standard Schwarzschild black hole defining the corresponding "textbook" version of Einstein–Rosen "bridge" (the Schwarzschild wormhole) [11]:

- The pertinent Kruskal–Penrose diagram for the proper Einstein–Rosen bridge (Fig. 1) has only two regions corresponding to "our" (right) and the second (left) "universes" unlike the four regions in the standard Schwarzschild black hole case (no black/white hole regions).
- The proper original Einstein–Rosen bridge is a *traversable* static spherically symmetric wormhole unlike the non-traversable non-static "textbook" version. Traversability is equivalent to the pairwise specific identifications of future with past horizons of the neighboring Kruskal regions.

We have also extended the Kruskal–Penrose diagram construction to the case of lightlike "thin-shell" wormholes with two throats.

Acknowledgements E.G., E.N. and S.P. gratefully acknowledge support of our collaboration through the academic exchange agreement between the Ben-Gurion University in Beer-Sheva, Israel, and the Bulgarian Academy of Sciences. S.P. and E.N. have received partial support from European COST actions MP-1210 and MP-1405, respectively. E.N., S.P. and M.S. are also thankful to Bulgarian National Science Fund for support via research grant DFNI-T02/6.

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Metric-Independent Spacetime Volume-Forms and Dark Energy/Dark Matter Unification

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Abstract The method of non-Riemannian (metric-independent) spacetime volume-forms (alternative generally-covariant integration measure densities) is applied to construct a modified model of gravity coupled to a single scalar field providing an explicit unification of dark energy (as a dynamically generated cosmological constant) and dust fluid dark matter flowing along geodesics as an exact sum of two separate terms in the scalar field energy-momentum tensor. The fundamental reason for the dark species unification is the presence of a non-Riemannian volume-form in the scalar field action which both triggers the dynamical generation of the cosmological constant as well as gives rise to a hidden nonlinear Noether symmetry underlying the dust dark matter fluid nature. Upon adding appropriate perturbation breaking the hidden "dust" Noether symmetry we preserve the geodesic flow property of the dark matter while we suggest a way to get growing dark energy in the present universe' epoch free of evolution pathologies. Also, an intrinsic relation between the above modified gravity + single scalar field model and a special quadratic purely kinetic "k-essence" model is established as a weak-versus-strong-coupling duality.

1 Introduction

According to the standard cosmological model (ΛCDM model [1–3]) the energy density of the late time Universe is dominated by two "dark" components - around 70 % made out of "dark energy" [4–6] and around 25 % made out of "dark matter"

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V. Dobrev (ed.), *Lie Theory and Its Applications in Physics*,
Springer Proceedings in Mathematics & Statistics 191,
DOI 10.1007/978-981-10-2636-2_16

[7–9]. Since more than a decade a principal challenge in modern cosmology is to understand theoretically from first principles the nature of both "dark" species of the universe's substance as a manifestation of the dynamics of a single entity of matter. Among the multitude of approaches to this seminal problem proposed so far are the (generalized) "Chaplygin gas" models [10–13], the "purely kinetic k-essence" models [14–17] based on the class of kinetic "quintessence" models [18–21], and more recently – the so called "mimetic" dark matter model [22, 23] and its extensions [24, 25], as well as constant-pressure-ansatz models [26].

Here we will describe a new approach achieving unified description of dark energy and dark matter based on a class of generalized models of gravity interacting with a single scalar field employing the method of non-Riemannian volume-forms on the pertinent spacetime manifold [27–31] (for further developments, see Refs. [32, 33]). Non-Riemannian spacetime volume-forms or, equivalently, alternative generally covariant integration measure densities are defined in terms of auxiliary maximal-rank antisymmetric tensor gauge fields ("measure gauge fields") unlike the standard Riemannian integration measure density given in terms of the square root of the determinant of the spacetime metric. These non-Riemannian-measure-modified gravity-matter models are also called "two-measure gravity theories".

Let us particularly stress that the method of non-Riemannian spacetime volume-forms is a very powerful one having profound impact in any (field theory) models with general coordinate reparametrization invariance, such as general relativity and its extensions [27–39]; strings and (higher-dimensional) membranes [40, 41]; and supergravity [42, 43]. Among its main features we should mention:

- Dynamical generation of cosmological constant as arbitrary integration constant in the solution of the equations of motion for the auxiliary "measure" gauge fields (see also Eq. (6) below).
- Using the canonical Hamiltonian formalism for Dirac-constrained systems we find that the auxiliary "measure" gauge fields are in fact almost pure gauge degrees of freedom except for the above mentioned arbitrary integration constants which are identified with the conserved Dirac-constrained canonical momenta conjugated to the "magnetic" components of the "measure" gauge fields [38, 39].
- Applying the non-Riemannian volume-form formalism to minimal $N = 1$ supergravity the appearance of a dynamically generated cosmological constant triggers spontaneous supersymmetry breaking and mass generation for the gravitino (supersymmetric Brout-Englert-Higgs effect) [42, 43]. Applying the same formalism to anti-de Sitter supergravity allows to produce simultaneously a very large physical gravitino mass and a very small *positive* observable cosmological constant [42, 43] in accordance with modern cosmological scenarios for slowly expanding universe of the present epoch [4–6].
- Employing two independent non-Riemannian volume-forms produces effective scalar potential with two infinitely large flat regions [37, 38] (one for large negative and another one for large positive values of the scalar field φ) with vastly different scales appropriate for a unified description of both the early and late universe' evolution. A remarkable feature is the existence of a stable initial phase of

non-singular universe creation preceding the inflationary phase – stable "emergent universe" without "Big-Bang" [37].

In Sect. 2 below we briefly discuss a non-standard model of gravity interacting with a single scalar field which couples symmetrically to a standard Riemannian as well as to another non-Riemannian volume form (spacetime integration measure density). We show that the auxiliary "measure" gauge field dynamics produces an arbitrary integration constant identified as a dynamically generated cosmological constant giving rise to a the dark energy term in the pertinent energy-momentum tensor. Simultaneously, a hidden strongly nonlinear Noether symmetry of the scalar Lagrangian action is revealed leading to a "dust" fluid representation of the second term in the energy-momentum tensor, which accordingly is identified as a "dust" dark matter flowing along geodesics. Thus, both "dark" species are explicitly unified as an exact sum of two separate contributions to the energy-momentum tensor.

In Sect. 3 some implications for cosmology are briefly considered. Specifically, we briefly study an appropriate perturbation of our modified-measure gravity + scalar-field model which breaks the above crucial hidden Noether symmetry and introduces exchange between the dark energy and dark matter components, while preserving the geodesic flow property of the dark matter fluid. Further, we suggest how to obtain a growing dark energy in the present day universe' epoch without invoking any pathologies of "cosmic doomsday" or future singularities kind [44–46].

In Sect. 4 below we couple the above modified-measure scalar-field model to a quadratic $f(R)$-gravity. We derive the pertinent "Einstein"-frame effective theory which turns out be a very special quadratic purely kinetic "k-essence" gravity-matter model. The main result here is establishing duality (in the standard sense of weak versus strong coupling) between the latter and the original quadratic $f(R)$-gravity plus modified-measure scalar-field model, whose matter part delivers an exact unified description of dynamical dark energy and dust fluid dark matter.

Section 5 contains our concluding remarks.

For further details, in particular, canonical Hamiltonian treatment and Wheeler-DeWitt quantization of the above unified model of dark energy and dark matter, see Refs. [36, 47].

2 Gravity-Matter Theory with a Non-Riemannian Volume-Form in The Scalar Field Action – Hidden Noether Symmetry and Unification of Dark Energy and Dark Matter

Let us consider the following simple particular case of a non-conventional gravity-scalar-field action – a member of the general class of the "two-measure" gravity-matter theories [28–31] (for simplicity we use units with the Newton constant $G_N = 1/16\pi$):

$$S = \int d^4x \sqrt{-g}\, R + \int d^4x \big(\sqrt{-g} + \Phi(B)\big) L(\varphi, X) \; . \tag{1}$$

Here R denotes the standard Riemannian scalar curvature for the pertinent Riemannian metric $g_{\mu\nu}$. The second term in (1) – the scalar field action is constructed in terms of two mutually independent spacetime volume-forms (integration measure densities):

(a) $\sqrt{-g} \equiv \sqrt{-\det \|g_{\mu\nu}\|}$ is the standard Riemannian integration measure density;
(b) $\Phi(B)$ denotes an alternative non-Riemannian generally covariant integration measure density independent of $g_{\mu\nu}$ and defining an alternative non-Riemannian volume-form:

$$\Phi(B) = \frac{1}{3!}\varepsilon^{\mu\nu\kappa\lambda}\partial_\mu B_{\nu\kappa\lambda} \; , \tag{2}$$

where $B_{\mu\nu\lambda}$ is an auxiliary maximal rank antisymmetric tensor gauge field independent of the Riemannian metric, also called "measure gauge field".

$L(\varphi, X)$ is general-coordinate invariant Lagrangian of a single scalar field $\varphi(x)$, the simplest example being:

$$L(\varphi, X) = X - V(\varphi) \quad , \quad X \equiv -\frac{1}{2} g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi \; , \tag{3}$$

As it will become clear below, the final result about the unification of dark energy and dark matter resulting from an underlying hidden Noether symmetry (see (9) below) of the scalar field action (second term in (1)) does *not* depend on the detailed form of $L(\varphi, X)$ which could be of an arbitrary generic "k-essence" form [18–21]:

$$L(\varphi, X) = \sum_{n=1}^{N} A_n(\varphi) X^n - V(\varphi) \; , \tag{4}$$

i.e., a nonlinear (in general) function of the scalar kinetic term X.

Due to general-coordinate invariance we have covariant conservation of the scalar field energy-momentum tensor:

$$T_{\mu\nu} = g_{\mu\nu} L(\varphi, X) + \left(1 + \frac{\Phi(B)}{\sqrt{-g}}\right)\frac{\partial L}{\partial X}\partial_\mu\varphi\,\partial_\nu\varphi \quad , \quad \nabla^\nu T_{\mu\nu} = 0 \qquad . \tag{5}$$

Equivalently, energy-momentum conservation (5) follows from the second-order equation of motion w.r.t. φ. The latter, however, becomes redundant because the modified-measure scalar field action (second term in (1)) exhibits a crucial new property – it yields a *dynamical constraint* on $L(\varphi, X)$ as a result of the equations of motion w.r.t. "measure" gauge field $B_{\mu\nu\lambda}$:

$$\partial_\mu L(\varphi, X) = 0 \quad \longrightarrow \quad L(\varphi, X) = -2M = \text{const} \;, \tag{6}$$

in particular, for (3):

$$X - V(\varphi) = -2M \quad \longrightarrow \quad X = V(\varphi) - 2M \;, \tag{7}$$

where M is arbitrary integration constant. The factor 2 in front of M is for later convenience, moreover, we will take $M > 0$ in view of its interpretation as a dynamically generated cosmological constant.[1] Indeed, taking into account (6), the expression (5) becomes:

$$T_{\mu\nu} = -2M g_{\mu\nu} + \left(1 + \frac{\Phi(B)}{\sqrt{-g}}\right) \frac{\partial L}{\partial X} \partial_\mu\varphi \partial_\nu\varphi \;. \tag{8}$$

As already shown in Ref. [36] the scalar field action in (1) possesses a hidden strongly nonlinear Noether symmetry, namely (1) is invariant (up to a total derivative) under the following nonlinear symmetry transformations:

$$\delta_\epsilon\varphi = \epsilon\sqrt{X} \quad , \quad \delta_\epsilon g_{\mu\nu} = 0 \quad , \quad \delta_\epsilon \mathcal{B}^\mu = -\epsilon \frac{1}{2\sqrt{X}} g^{\mu\nu}\partial_\nu\varphi \big(\Phi(B) + \sqrt{-g}\big) \;, \tag{9}$$

where $\mathcal{B}^\mu \equiv \frac{1}{3!}\varepsilon^{\mu\nu\kappa\lambda} B_{\nu\kappa\lambda}$. Under (9) the action (1) transforms as $\delta_\epsilon S = \int d^4x \partial_\mu \big(L(\varphi, X)\delta_\epsilon \mathcal{B}^\mu\big)$. Then, the standard Noether procedure yields the conserved current:

$$\nabla_\mu J^\mu = 0 \quad , \quad J^\mu \equiv \left(1 + \frac{\Phi(B)}{\sqrt{-g}}\right)\sqrt{2X} g^{\mu\nu}\partial_\nu\varphi \frac{\partial L}{\partial X} \;. \tag{10}$$

$T_{\mu\nu}$ (8) and J^μ (10) can be cast into a relativistic hydrodynamical form:

$$T_{\mu\nu} = -2M g_{\mu\nu} + \rho_0 u_\mu u_\nu \quad , \quad J^\mu = \rho_0 u^\mu \;, \tag{11}$$

where:

$$\rho_0 \equiv \left(1 + \frac{\Phi(B)}{\sqrt{-g}}\right) 2X \frac{\partial L}{\partial X} \quad , \quad u_\mu \equiv \frac{\partial_\mu\varphi}{\sqrt{2X}} \quad , \quad u^\mu u_\mu = -1 \;. \tag{12}$$

For the pressure p and energy density ρ we have accordingly (with ρ_0 as in (12)):

$$p = -2M = \text{const} \quad , \quad \rho = \rho_0 - p = \left(1 + \frac{\Phi(B)}{\sqrt{-g}}\right) 2X \frac{\partial L}{\partial X} + 2M \;, \tag{13}$$

[1]The physical meaning of the "measure" gauge field $B_{\mu\nu\lambda}$ (2) as well as the meaning of the integration constant M are most straightforwardly seen within the canonical Hamiltonian treatment of (1) [36]. For more details about the canonical Hamiltonian treatment of general gravity-matter theories with (several independent) non-Riemannian volume-forms we refer to [38, 39].

where the integration constant M appears as *dynamically generated cosmological constant*.

Thus, $T_{\mu\nu}$ (11) represents an exact sum of two contributions of the two dark species with $p = p_{\rm DE} + p_{\rm DM}$ and $\rho = \rho_{\rm DE} + \rho_{\rm DM}$:

$$p_{\rm DE} = -2M \quad , \quad \rho_{\rm DE} = 2M \quad ; \quad p_{\rm DM} = 0 \quad , \quad \rho_{\rm DM} = \rho_0 \; , \tag{14}$$

i.e., the dark matter component is a dust fluid ($p_{\rm DM} = 0$).

Covariant conservation of $T_{\mu\nu}$ (11) immediately implies *both* (i) the covariant conservation of $J^\mu = \rho_0 u^\mu$ (10) describing dust dark matter "particle number" conservation, and (ii) the geodesic flow equation of the dust dark matter fluid:

$$\nabla_\mu\big(\rho_0 u^\mu\big) = 0 \quad , \quad u_\nu \nabla^\nu u_\mu = 0 \; . \tag{15}$$

3 Some Cosmological Implications

Let us now consider a perturbation of the initial modified-measure gravity + scalar-field action (1) by some additional scalar field Lagrangian $\widehat{L}(\varphi, X)$ independent of the initial scalar Lagrangian $L(\varphi, X)$:

$$\widehat{S} = \int d^4x\sqrt{-g}\, R + \int d^4x\big(\sqrt{-g} + \Phi(B)\big)L(\varphi, X) + \int d^4x\sqrt{-g}\,\widehat{L}(\varphi, X) \; . \tag{16}$$

An important property of the perturbed action (16) is that once again the scalar field φ-dynamics is given by the unperturbed dynamical constraint Eq. (6) of the initial scalar Lagrangian $L(\varphi, X)$, which is completely independent of the perturbing scalar Lagrangian $\widehat{L}(\varphi, X)$.

Henceforth, for simplicity we will take the scalar Lagrangians in the canonical form $L(\varphi, X) = X - V(\varphi)$, $\widehat{L}(\varphi, X) = X - U(\varphi)$, where $U(\varphi)$ is independent of $V(\varphi)$.

The associated scalar field energy-momentum tensor now reads (cf. Eqs. (11)–(13)):

$$\widehat{T}_{\mu\nu} = \widehat{\rho}_0 u_\mu u_\nu + g_{\mu\nu}\big(-4M + V - U\big) \quad , \quad \widehat{\rho}_0 \equiv 2(V - 2M)\left(1 + \frac{\Phi(B)}{\sqrt{-g}}\right) \; , \tag{17}$$

or, equivalently:

$$\widehat{T}_{\mu\nu} = \big(\widehat{\rho} + \widehat{p}\big)u_\mu u_\nu + \widehat{p}\, g_{\mu\nu} \quad , \quad \widehat{p} = -4M + V - U \; , \tag{18}$$

$$\widehat{\rho} = \widehat{\rho}_0 - \widehat{p} = 2(V - 2M)\Big(1 + \frac{\Phi(B)}{\sqrt{-g}}\Big) + 4M + U - V \; , \tag{19}$$

where (7) is used.

The perturbed energy-momentum (17) conservation $\nabla^{\mu}\widehat{T}_{\mu\nu} = 0$ now implies:

- The perturbed action (16) does not any more possess the hidden symmetry (9) and, therefore, the conservation of the dust particle number current $J^{\mu} = \rho_0 u^{\mu}$ (11) is now replaced by:

$$\nabla^{\mu}\left(\widehat{\rho}_0 u_{\mu}\right) + \sqrt{2(V - 2M)}\left(\frac{\partial V}{\partial\varphi} - \frac{\partial U}{\partial\varphi}\right) = 0\,. \tag{20}$$

- *Once again* we obtain the geodesic flow equation for the dark matter "fluid" (second Eq. (15)). Let us stress that this is due to the fact that the perturbed pressure $\widehat{p}$ (second relation in (18)), because of the dynamical constraint (7) triggered by the non-Riemannian volume-form in (16), is a function of φ only but not of X.

Thus, we conclude that the geodesic flow dynamics of the cosmological fluid described by the action (16) persist irrespective of the presence of the perturbation (last term in (16)) as well as of the specific form of the latter.

In the cosmological context, when taking the spacetime metric in the standard Friedmann–Lemaitre–Robertson–Walker (FLRW) form, the scalar field is assumed to be time-dependent only: $\varphi = \varphi(t)$. Thus, in this case the dynamical constraint Eq. (7) and its solution assume the form:

$$\dot{\varphi}^2 = 2\big(V(\varphi) - 2M\big) \quad \longrightarrow \quad \int_{\varphi(0)}^{\varphi(t)} \frac{d\varphi}{\sqrt{2\big(V(\varphi) - 2M\big)}} = \pm t\;. \tag{21}$$

Choosing the $+$ sign in (21) corresponds to $\varphi(t)$ monotonically growing with t irrespective of the detailed form of the potential $V(\varphi)$. The only condition due to consistency of the dynamical constraint (first Eq. (21)) is $V(\varphi) > 2M$ for the whole interval of classically accessible values of φ. Also, note the "strange" looking second-order (in time derivatives) form of the first Eq. (21): $\ddot{\varphi} - \partial V/\partial\varphi = 0$, where we specifically stress on the *opposite* sign in the force term. Thus, it is fully consistent for $\varphi(t)$ to "climb" a growing w.r.t. φ scalar potential.

As already stressed above, the dynamics of the $\varphi(t)$ does not depend at all on the presence of the perturbing scalar potential $U(\varphi)$. Therefore, if we choose the perturbation $U(\varphi)$ in (16) such that the potential difference $U(\varphi) - V(\varphi)$ is a growing function at large φ (e.g., $U(\varphi) - V(\varphi) \sim e^{\alpha\varphi}$, α small positive) then, when $\varphi(t)$ evolves through (21) to large positive values, it (slowly) "climbs" $U(\varphi) - V(\varphi)$ and according to the expression $\widehat{\rho}_{DE} = 4M + U(\varphi) - V(\varphi) = -\widehat{p}$ for the dark energy density (cf. (17) and (18)), the latter will (slowly) grow up! Let us emphasize that in this way we obtain growing dark energy of the "late" universe without any pathologies in the universe' evolution like "cosmic doomsday" or future singularities [44–46].

4 Duality to Purely Kinetic "K-Essence"

Let us now consider a different perturbation of the modified-measure gravity + scalar-field action (1) by replacing the standard Einstein-Hilbert gravity action (the first term in (1)) with a $f(R) = R - \alpha R^2$ extended gravity action in the first-order Palatini formalism:

$$S^{(\alpha)} = \int d^4x \sqrt{-g}\big(R(g,\Gamma) - \alpha R^2(g,\Gamma)\big) + \int d^4x \big(\sqrt{-g} + \Phi(B)\big) L(\varphi, X) , \qquad (22)$$

where $R(g,\Gamma) = g^{\mu\nu} R_{\mu\nu}(\Gamma)$, *i.e.*, with *a priori* independent metric $g_{\mu\nu}$ and affine connection $\Gamma^{\mu}_{\nu\lambda}$.

Since the scalar field action – the second term in (22) – remains the same as in the original action (1), and the hidden nonlinear Noether symmetry (9) does not affect the metric, all results in Sect. 2 remain valid. Namely, the Noether symmetry (9) produces "dust" fluid particle number conserved current (first Eq. (15)) and interpretation of φ as describing simultaneously dark energy (because of the dynamical scalar Lagrangian constraint (6)) and dust dark matter with geodesic dust fluid flow (second Eq. (15)) remains intact.

However, the gravitational equations of motion derived from (22) are not of the standard Einstein form:

$$R_{\mu\nu}(\Gamma) = \frac{1}{2f'_R}\big[T_{\mu\nu} + f(R) g_{\mu\nu}\big] , \qquad (23)$$

where $f(R) = R(g,\Gamma) - \alpha R^2(g,\Gamma)$, $f'_R = 1 - 2\alpha R(g,\Gamma)$ and $T_{\mu\nu}$ is the same as in (8).

The equations of motion w.r.t. independent $\Gamma^{\mu}_{\nu\lambda}$ resulting from (22) yield (for an analogous derivation, see [28]) the following solution for $\Gamma^{\mu}_{\nu\lambda}$ as a Levi-Civita connection:

$$\Gamma^{\mu}_{\nu\lambda} = \Gamma^{\mu}_{\nu\lambda}(\bar{g}) = \frac{1}{2}\bar{g}^{\mu\kappa}\left(\partial_\nu \bar{g}_{\lambda\kappa} + \partial_\lambda \bar{g}_{\nu\kappa} - \partial_\kappa \bar{g}_{\nu\lambda}\right) , \qquad (24)$$

w.r.t. to the Weyl-rescaled metric $\bar{g}_{\mu\nu}$:

$$\bar{g}_{\mu\nu} = f'_R \, g_{\mu\nu} , \qquad (25)$$

so that $\bar{g}_{\mu\nu}$ is called (physical) "Einstein-frame" metric. In passing over to the "Einstein-frame" it is also useful to perform the following φ-field redefinition:

$$\varphi \to \widetilde{\varphi} = \int \frac{d\varphi}{\sqrt{\big(V(\varphi) - 2M\big)}} \quad , \quad X \to \widetilde{X} = -\frac{1}{2}\bar{g}^{\mu\nu}\partial_\mu \widetilde{\varphi} \partial_\nu \widetilde{\varphi} = \frac{1}{f'_R} , \qquad (26)$$

where the last relation follows from the Lagrangian dynamical constraint (7) together with (25).

Derivation of the explicit expressions for the Einstein-frame gravitational equations, *i.e.*, equations w.r.t. Einstein-frame metric (25) and the Einstein-frame scalar field (first Eq. (26)), yields the latter in the standard form of Einstein gravity equations:

$$\overline{R}_{\mu\nu} - \frac{1}{2}\overline{g}_{\mu\nu}\overline{R} = \frac{1}{2}\overline{T}_{\mu\nu} \,. \tag{27}$$

Here the following notations are used:

(i) $\overline{R}_{\mu\nu}$ and $\overline{R}$ are the standard Ricci tensor and scalar curvature of the Einstein-frame metric (25).
(ii) The Einstein-frame energy-momentum tensor:

$$\overline{T}_{\mu\nu} = \overline{g}_{\mu\nu}\overline{L}_{\rm eff} - 2\frac{\partial \overline{L}_{\rm eff}}{\partial \overline{g}^{\mu\nu}} \tag{28}$$

is given in terms of the following effective $\widetilde{\varphi}$-scalar field Lagrangian of a specific quadratic purely kinetic "k-essence" form:

$$\overline{L}_{\rm eff}(\widetilde{X}) = \left(\frac{1}{4\alpha} - 2M\right)\widetilde{X}^2 - \frac{1}{2\alpha}\widetilde{X} + \frac{1}{4\alpha} \,. \tag{29}$$

Thus, the Einstein-frame gravity+scalar-field action reads:

$$S_{\rm k-ess} = \int d^4\sqrt{-\overline{g}}\Big[\overline{R} + \Big(\frac{1}{4\alpha} - 2M\Big)\widetilde{X}^2 - \frac{1}{2\alpha}\widetilde{X} + \frac{1}{4\alpha}\Big] \,. \tag{30}$$

The Einstein-frame effective energy-momentum-tensor (28) in the perfect fluid representation reads (taking into account the explicit form of $\overline{L}_{\rm eff}$ (29)):

$$\bar{T}_{\mu\nu} = \overline{g}_{\mu\nu}\widetilde{p} + \widetilde{u}_\mu\widetilde{u}_\nu(\widetilde{\rho} + \widetilde{p}) \quad , \quad \widetilde{u}_\mu \equiv \frac{\partial_\mu\widetilde{\varphi}}{\sqrt{2\widetilde{X}}} \quad , \quad \overline{g}^{\mu\nu}\widetilde{u}_\mu\widetilde{u}_\nu = -1, \tag{31}$$

$$\widetilde{p} = \left(\frac{1}{4\alpha} - 2M\right)\widetilde{X}^2 - \frac{1}{2\alpha}\widetilde{X} + \frac{1}{4\alpha} \,, \quad \widetilde{\rho} = 3\left(\frac{1}{4\alpha} - 2M\right)\widetilde{X}^2 - \frac{1}{2\alpha}\widetilde{X} - \frac{1}{4\alpha} \tag{32}$$

Let us stress that the quadratic purely kinetic "k-essence" scalar Lagrangian (29) is indeed a very special one:

- The three coupling constants in (29) depend only on two independent parameters (α, M), the second one being a dynamically generated integration constant in the original theory (22).
- The quadratic gravity term $-\alpha R^2$ in (22) is just a small perturbation w.r.t. the initial action (1) when $\alpha \to 0$, whereas the coupling constants in the Einstein-frame effective action (30) diverge as $1/\alpha$, *i.e.*, weak coupling in (22) is equivalent to a strong coupling in (30).

- Due to the apparent Noether symmetry of (29) under constant shift of $\widetilde{\varphi}$ ($\widetilde{\varphi} \to \widetilde{\varphi} +$ const) the corresponding Noether conservation law is identical to the $\widetilde{\varphi}$-equations of motion:

$$\overline{\nabla}_\mu \left(\overline{g}^{\mu\nu} \partial_\nu \widetilde{\varphi} \frac{\partial \widetilde{L}_{\rm eff}}{\partial \widetilde{X}} \right) = 0 \; , \tag{33}$$

where $\overline{\nabla}_\mu$ is covariant derivative w.r.t. the Levi-Civita connection (24) in the $\overline{g}_{\mu\nu}$-(Einstein) frame. Equation (33) is the Einstein-frame counterpart of the "dust" Noether conservation law (10) in the original theory (1) or (22).

Thus, we have found an explicit duality in the usual sense of "weak versus strong coupling" between the original non-standard gravity+scalar-field model providing exact unified description of dynamical dark energy and dust fluid dark matter in the matter sector, on one hand, and a special quadratic purely kinetic "k-essence" gravity-matter model, on the other hand. The latter dual theory arises as the "Einstein-frame" effective theory of its original counterpart.

To make explicit the existence of smooth strong coupling limit $\alpha \to 0$ *on-shell* in the dual "k-essence" energy density $\widetilde{\rho}$ and "k-essence" pressure $\widetilde{p}$ (32) in spite of the divergence of the corresponding constant coefficients, let us consider a reduction of the dual quadratic purely kinetic "k-essence" gravity + scalar-field model (30) for the Friedmann–Lemaitre–Robertson–Walker (FLRW) class of metrics:

$$ds^2 = -N^2(t)dt^2 + a^2(t)\Big[\frac{dr^2}{1-Kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\Big] \; . \tag{34}$$

The FLRW reduction of the $\phi \equiv \widetilde{\varphi}$-equation of motion (33) (using henceforth the gauge $N = 1$) reads:

$$\frac{dp_\phi}{dt} = 0 \quad \longrightarrow \quad p_\phi = a^3\Big[-\frac{1}{2\alpha}\dot{\phi} + \big(\frac{1}{4\alpha} - 2M\big)\dot{\phi}^3\Big] \; , \tag{35}$$

where p_ϕ is the constant conserved canonically conjugated momentum of $\phi \equiv \widetilde{\varphi}$. Thus, the velocity $\dot{\phi} = \dot{\phi}(p_\phi/a^3)$ is a function of the Friedmann scale factor $a(t)$ through the ratio p_ϕ/a^3 and solves the cubic algebraic equation (35) for any α. For small α we get:

$$\dot{\phi}(p_\phi/a^3) \simeq \sqrt{2} + \alpha\Big(4\sqrt{2}M + \frac{p_\phi}{a^3}\Big) + \mathrm{O}(\alpha^2) \; . \tag{36}$$

Then, inserting (36) into the FLRW-reduced $\widetilde{X} = \frac{1}{2}\dot{\phi}^2$ and substituting it into the expressions (32) we obtain for the small-α asymptotics of the "k-essence" energy density and "k-essence" pressure:

$$\widetilde{\rho} = 2M + \sqrt{2}\frac{p_\phi}{a^3} + \alpha\Big[16M^2 + 4\sqrt{2}M\frac{p_\phi}{a^3} + \frac{1}{2}\big(\frac{p_\phi}{a^3}\big)^2\Big] + \mathrm{O}(\alpha^2) \; , \tag{37}$$

$$\widetilde{p} = -2M - \alpha\Big[16M^2 - \frac{1}{2}\big(\frac{p_\phi}{a^3}\big)^2\Big] + \mathrm{O}(\alpha^2) \; . \tag{38}$$

The limiting values $\widetilde{\rho} = 2M + \sqrt{2}\frac{p_\phi}{a^3}$ and $\widetilde{p} = -2M$ precisely coincide with the corresponding values of ρ and p (13) in the FLRW reduced original theory (1) [36].

5 Conclusions

In the present note we have demonstrated the power of the method of non-Riemannian spacetime volume-forms (alternative generally-covariant integration measure densities) by applying it to construct a modified model of gravity coupled to a single scalar field which delivers a unification of dark energy (as a dynamically generated cosmological constant) and dust fluid dark matter flowing along geodesics (due to a hidden nonlinear Noether symmetry). Both "dark" species appear as an exact sum of two separate contributions in the energy-momentum tensor of the single scalar field. Upon perturbation of the scalar field action, which breaks the hidden "dust" Noether symmetry but preserves the geodesic flow property, we show how to obtain a growing dark energy in the late Universe without evolution pathologies. Furthermore, we have established a duality (in the standard sense of weak versus strong coupling) of the above model unifying dark energy and dark matter, on one hand, and a specific quadratic purely kinetic "k-essence" model. This duality elucidates the ability of purely kinetic "k-essence" theories to describe approximately the unification of dark energy and dark matter and explains how the "k-essence" description becomes exact in the strong coupling limit on the "k-essence" side.

Acknowledgements We gratefully acknowledge support of our collaboration through the academic exchange agreement between the Ben-Gurion University in Beer-Sheva, Israel, and the Bulgarian Academy of Sciences. S.P. and E.N. have received partial support from European COST actions MP-1210 and MP-1405, respectively, as well from Bulgarian National Science Fund Grant DFNI-T02/6.

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Large Volume Supersymmetry Breaking Without Decompactification Problem

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Abstract We consider heterotic string backgrounds in four-dimensional Minkowski space, where $\mathcal{N} = 1$ supersymmetry is spontaneously broken at a low scale $m_{3/2}$ by a stringy Scherk-Schwarz mechanism. We review how the effective gauge couplings at 1-loop may evade the "decompactification problem", namely the proportionality of the gauge threshold corrections, with the large volume of the compact space involved in the supersymmetry breaking.

1 Introduction

A sensible physical theory must at least meet two requirements: Be realistic and analytically under control. The first point can be satisfied by considering string theory, which has the advantage to be, at present time, the only setup in which both gravitational and gauge interactions can be described consistently at the quantum level. In this review, we do not consider cosmological issues and thus analyze models defined classically in four-dimensional Minkowski space. The "no-scale models" are particularly interesting since, by definition, they describe in supergravity or string theory classical backgrounds, in which supersymmetry is spontaneously broken at an arbitrary scale $m_{3/2}$ in flat space [1]. In other words, even if supersymmetry is not explicit, the classical vacuum energy vanishes.

The most conservative way to preserve analytical control is to ensure the validity of perturbation theory. In string theory, quantum loops can be evaluated explicitly, when the underlying two-dimensional conformal field theory is itself under control. Clearly, this is the case, when one considers free field on the world sheet, for instance in toroidal orbifold models [2] or fermionic constructions [3]. In these frameworks, the $\mathcal{N} = 1 \to \mathcal{N} = 0$ spontaneous breaking of supersymmetry can be implemented

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V. Dobrev (ed.), *Lie Theory and Its Applications in Physics*,
Springer Proceedings in Mathematics & Statistics 191,
DOI 10.1007/978-981-10-2636-2_17

at tree level *via* a stringy version [4] of the Scherk-Schwarz mechanism [5].[1] In this case, the supersymmetry breaking scale is of order of the inverse volume of the internal directions involved in the breaking. For a single circle of radius R, one has

$$m_{3/2} = \frac{M_{\rm s}}{R}\,, \tag{1}$$

where $M_{\rm s}$ is the string scale, so that having a low $m_{3/2} = \mathcal{O}(10\ \text{TeV})$ imposes the circle to be extremely large, $R = \mathcal{O}(10^{17})$ [6]. Such large directions yield towers of light Kaluza–Klein states and a problem arises from those charged under some gauge group factor G^i. In general, their contributions to the quantum corrections to the inverse squared gauge coupling is proportional to the very large volume and invalidates the use of perturbation theory.

To be specific, let us consider in heterotic string the 1-loop low energy running gauge coupling $g_i(\mu)$, which satisfies [7]

$$\frac{16\pi^2}{g_i^2(\mu)} = k^i\,\frac{16\pi^2}{g_{\rm s}^2} + b^i \ln \frac{M_{\rm s}^2}{\mu^2} + \Delta^i\,. \tag{2}$$

In this expression, $g_{\rm s}$ is the string coupling and k^i is the Kac–Moody level of G^i. The logarithmic contribution, which depends on the energy scale μ, arises from the massless states and is proportional to the β-function coefficient b^i, while the massive modes yield the threshold corrections Δ^i. The main contributions to the latter arise from the light Kaluza–Klein states, which for a single large radius yield

$$\Delta^i = C^i R - b^i \ln R^2 + \mathcal{O}\left(\frac{1}{R}\right), \tag{3}$$

where $C^i = Cb^i - C'k^i$, for some non-negative C and C' that depend on other moduli. When $C^i = \mathcal{O}(1)$, requiring in Eq. (2) the loop correction to be small compared to the tree level contribution imposes $g_{\rm s}^2 R < 1$. In other words, for perturbation theory to be valid, the string coupling must be extremely weak, $g_{\rm s} < \mathcal{O}(10^{-6.5})$. If $C^i > 0$, which implies G^i is not asymptotically free, Eq. (2) imposes the running gauge coupling to be essentially free, $g_i(\mu) = \mathcal{O}(g_{\rm s})$, and G^i describes a hidden gauge group. However, if $C^i < 0$, which is the case if G^i is asymptotically free, the very large tree level contribution proportional to $1/g_{\rm s}^2$ must cancel $C^i R$, up to very high accuracy, for the running gauge coupling to be of order 1 and have a chance to describe realistic gauge interactions. This unnatural fine-tuning is a manifestation of the so-called "decompactification problem", which actually arises generically, when a submanifold of the internal space is large, compared to the string scale, i.e. when

[1] Note that non-perturbative mechanisms based on gaugino condensation could also be considered, but only at the level of the low energy effective supergravity, thus at the price of loosing part of the string predictability.

the internal conformal field theory allows a geometrical interpretation in terms of a compactified space.

To avoid the above described behavior, C^i can be required to vanish. This is trivially the case in the $\mathcal{N}=4$ supersymmetric theories, where actually $b^i=0$ and $\Delta^i=0$. The condition $C^i=0$ remains valid in the theories realizing the $\mathcal{N}=4\to\mathcal{N}=2$ spontaneous breaking, provided $\mathcal{N}=4$ is recovered when the volume is sent to infinity [8]. In this case, the threshold corrections scale logarithmically with the volume and no fine-tuning is required for perturbation theory to be valid. In Sect. 2, we review the construction of models that realize an $\mathcal{N}=1\to\mathcal{N}=0$ spontaneous breaking at a low scale $m_{3/2}$, while avoiding the decompactification problem. The corresponding threshold corrections are computed in Sect. 3 [9, 10].

2 The Non-supersymmetric $\mathbb{Z}_2\times\mathbb{Z}_2$ Models

In the present work, we focus on heterotic string backgrounds in four-dimensional Minkowski space and analyze the gauge coupling threshold corrections. At 1-loop, their formal expression is [7, 11, 12]

$$\Delta^i=\int_{\mathcal{F}}\frac{d^2\tau}{\tau_2}\left(\frac{1}{2}\sum_{a,b}\mathcal{Q}\big[{}^a_b\big](2v)\left(\mathcal{P}_i^2(2\bar{w})-\frac{k^i}{4\pi\tau_2}\right)\tau_2\,Z\big[{}^a_b\big](2v,2\bar{w})-b^i\right)\Bigg|_{v=\bar{w}=0}$$
$$+\,b^i\log\frac{2\,e^{1-\gamma}}{\pi\sqrt{27}}\,, \tag{4}$$

where $\mathcal{F}$ is the fundamental domain of $SL(2,\mathbb{Z})$ and $Z\big[{}^a_b\big](2v,2\bar{w})$ is a refined partition function for given spin structure $(a,b)\in\mathbb{Z}_2\times\mathbb{Z}_2$. $\mathcal{P}_i(2\bar{w})$ acts on the right-moving sector as the squared charge operator of the gauge group factor G^i, while $\mathcal{Q}\big[{}^a_b\big](2v)$ acts on the left-moving sector as the helicity operator,[2]

$$\mathcal{Q}\big[{}^a_b\big](2v)=\frac{1}{16\pi^2}\,\frac{\partial_v^2(\theta\big[{}^a_b\big](2v))}{\theta\big[{}^a_b\big](2v)}-\frac{i}{\pi}\,\partial_\tau\log\eta\equiv\frac{i}{\pi}\,\partial_\tau\left(\log\frac{\theta\big[{}^a_b\big](2v)}{\eta}\right). \tag{5}$$

From now on, we consider $\mathbb{Z}_2\times\mathbb{Z}_2$ orbifold models [2] or fermionic constructions [3] in which the marginal deformations parameterized by the Kähler and complex structures $T_I,U_I,I=1,2,3$, associated to the three internal 2-tori are switched on [9, 14]. In both cases, orbifolds or "moduli-deformed fermionic constructions", $\mathcal{N}=1$ supersymmetry is spontaneously broken by a stringy Scherk-Schwarz mechanism [4]. The associated genus-1 refined partition function is

---

[2]Our conventions for the Jacobi functions $\theta\big[{}^a_b\big](\nu|\tau)$ (or $\theta_\alpha(\nu|\tau)$, $\alpha=1,\ldots,4$) and Dedekind function can be found in [13].

$$
\begin{aligned}
Z(2v, 2\bar{w}) = {} & \frac{1}{\tau_2 (\eta\bar{\eta})^2} \times \\
& \frac{1}{2}\sum_{a,b} \frac{1}{2}\sum_{H_1,G_1} \frac{1}{2}\sum_{H_2,G_2} (-1)^{a+b+ab} \, \frac{\theta[^a_b](2v)}{\eta} \, \frac{\theta[^{a+H_1}_{b+G_1}]}{\eta} \, \frac{\theta[^{a+H_2}_{b+G_2}]}{\eta} \, \frac{\theta[^{a+H_3}_{b+G_3}]}{\eta} \times \\
& \frac{1}{2^N} \sum_{h_I^i, g_I^i} S_L\left[^{a,\, h_I^i,\, H_I}_{b,\, g_I^i,\, G_I}\right] Z_{2,2}\left[^{h_1^i}_{g_1^i}\Big|^{H_1}_{G_1}\right] Z_{2,2}\left[^{h_2^i}_{g_2^i}\Big|^{H_2}_{G_2}\right] Z_{2,2}\left[^{h_3^i}_{g_3^i}\Big|^{H_3}_{G_3}\right] Z_{0,16}\left[^{h_I^i,\, H_I}_{g_I^i,\, G_I}\right](2\bar{w}),
\end{aligned}
\tag{6}
$$

where our notations are as follows:

- The $Z_{2,2}$ conformal blocks arise from the three internal 2-tori. The genus-1 surface having two non-trivial cycles, $(h_I^i, g_I^i) \in \mathbb{Z}_2 \times \mathbb{Z}_2$, $i = 1, 2$, $I = 1, 2, 3$ denote associated shifts of the six coordinates. Similarly, $(H_I, G_I) \in \mathbb{Z}_2 \times \mathbb{Z}_2$ refer to the twists, where we have defined for convenience $(H_3, G_3) \equiv (-H_1 - H_2, -G_1 - G_2)$. Explicitly, we have

$$
Z_{2,2}\left[^{h_I^1,\, h_I^2}_{g_I^1,\, g_I^2}\Big|^{H_I}_{G_I}\right] = \begin{cases} \dfrac{\Gamma_{2,2}\left[^{h_I^1,\, h_I^2}_{g_I^1,\, g_I^2}\right](T_I, U_I)}{(\eta\bar{\eta})^2}\,, \\ \qquad\qquad \text{when } (H_I, G_I) = (0,0) \bmod 2\,, \\ \dfrac{4\eta\bar{\eta}}{\theta\left[^{1-H_I}_{1-G_I}\right]\bar{\theta}\left[^{1-H_I}_{1-G_I}\right]}\, \delta_{\left|^{h_I^1\ H_I}_{g_I^1\ G_I}\right|, 0 \bmod 2}\, \delta_{\left|^{h_I^2\ H_I}_{g_I^2\ G_I}\right|, 0 \bmod 2} \\ \qquad\qquad \text{otherwise}\,, \end{cases}
\tag{7}
$$

  where $\Gamma_{2,2}$ is a shifted lattice that depends on the Kähler and complex structure moduli T_I, U_I of the I^{th} 2-torus. The arguments of the Kronecker symbols are determinants.
- When defining each model, linear constraints on the shifts (h_I^i, g_I^i) and twists (H_I, G_I) may be imposed, leaving effectively N independent shifts.
- $Z_{0,16}$ denotes the contribution of the 32 extra right-moving world sheet fermions. Its dependance on the shifts and twists may generate discrete Wilson lines, which break partially $E_8 \times E_8$ or $SO(32)$.
- The first line contains the contribution of the spacetime light-cone bosons, while the second is that of the left-moving fermions.
- S_L is a conformal block-dependent sign that implements the stringy Scherk-Schwarz mechanism. A choice of S_L that correlates the spin structure (a, b) to some shift (h_I^i, g_I^i) implements the $\mathcal{N} = 1 \to \mathcal{N} = 0$ spontaneous breaking.

The $\mathbb{Z}_2 \times \mathbb{Z}_2$ models contain three $\mathcal{N} = 2$ sectors. For the decompactification problem not to arise, we impose one of them to be realized as a spontaneously broken phase of $\mathcal{N} = 4$. This can be done by demanding the $\mathbb{Z}_2$ action characterized by (H_2, G_2) to be free. The associated generator twists the 2[nd] and 3[rd] 2-tori (i.e. the directions X^6, X^7, X^8, X^9 in bosonic language) and shifts some direction(s) of the 1[st] 2-torus, say X^5 only. To simplify our discussion, we take the generator of the other $\mathbb{Z}_2$, whose action is characterized by (H_1, G_1), to not be free : It twists the 1[st] and 3[rd] 2-tori, and fixes the 2[nd] one. Similarly, we suppose that the product of the two generators, whose action is characterized by (H_3, G_3), twists the 1st and 2nd 2-tori, and fixes the 3rd one. These restrictions impose the moduli T_2, U_2 and T_3, U_3 not to be far from 1, in order to avoid the decompactification problem to occur from the remaining two $\mathcal{N} = 2$ sectors. However, our care in choosing the orbifold action is allowing us to take the volume of the 1st 2-torus to be large.

The above remarks have an important consequence, since the final stringy Scherk-Schwarz mechanism responsible of the $\mathcal{N} = 1 \to \mathcal{N} = 0$ spontaneous breaking must involve the moduli T_1, U_1 only, for the gravitino mass to be light. Thus, this breaking must be implemented *via* a shift along the 1st 2-torus, say X^4, and a non-trivial choice of S_L. Therefore, the sector $(H_1, G_1) = (0, 0)$ realizes the pattern of spontaneous breaking $\mathcal{N} = 4 \to \mathcal{N} = 2 \to \mathcal{N} = 0$, while the other two $\mathcal{N} = 2$ sectors, which have 2nd and 3rd 2-tori respectively fixed, are independent of T_1 and U_1 and thus remain supersymmetric. As a result, we have in the two following independent modular orbits:

$$\begin{aligned} S_L &= (-1)^{ag_1^1 + bh_1^1 + h_1^1 g_1^1}\,, && \text{when} \quad (H_1, G_1) = (0, 0)\,, \\ S_L &= 1\,, && \text{when} \quad (H_1, G_1) \neq (0, 0)\,. \end{aligned} \tag{8}$$

Given the fact that we have imposed $(h_1^2, g_1^2) \equiv (H_2, G_2)$, the 1st 2-torus lattice takes the explicit form

$$\begin{aligned} \Gamma_{2,2}\begin{bmatrix} h_1^1, H_2 \\ g_1^1, G_2 \end{bmatrix}(T_1, U_1) = \sum_{m^i, n^i} (-1)^{m^1 g_1^1 + m^2 G_2}\, e^{2i\pi\bar{\tau}\left[m^1\left(n^1 + \frac{1}{2}h_1^1\right) + m^2\left(n^2 + \frac{1}{2}H_2\right)\right]} \times \\ e^{-\frac{\pi\tau_2}{\mathrm{Im}\,T_1\,\mathrm{Im}\,U_1}\left|T_1\left(n^1 + \frac{1}{2}h_1^1\right) + T_1 U_1\left(n^2 + \frac{1}{2}H_2\right) + U_1 m^1 - m^2\right|^2} \end{aligned} \tag{9}$$

This expression can be used to find the squared scales of spontaneous $\mathcal{N} = 4 \to \mathcal{N} = 2$ and $\mathcal{N} = 2 \to \mathcal{N} = 0$ breaking. For $\mathrm{Re}\,(U_1) \in (-\frac{1}{2}, \frac{1}{2}]$, they are

$$\frac{M_s^2}{\mathrm{Im}\,T_1\,\mathrm{Im}\,U_1}\,, \qquad m_{3/2}^2 = \frac{|U_1|^2 M_s^2}{\mathrm{Im}\,T_1\,\mathrm{Im}\,U_1}\,, \tag{10}$$

where the latter is nothing but the gravitino mass squared of the full $\mathcal{N} = 0$ theory. For these scales to be small compared to M_s, we consider the regime $\mathrm{Im}\,T_1 \gg 1$, $U_1 = \mathcal{O}(i)$.

3 Threshold Corrections

The threshold corrections can be evaluated in each conformal block [9]. Starting with those where $(H_1, G_1) = (0, 0)$, the discussion is facilitated by summing over the spin structures. Focussing on the relevant parts of the refined partition function Z, we have

$$\frac{1}{2}\sum_{a,b}(-1)^{a+b+ab}\quad(-1)^{ag_1^1+bh_1^1+h_1^1g_1^1}\,\theta\big[{}^a_b\big](2v)\,\theta\big[{}^a_b\big]\theta\big[{}^{a+H_2}_{b+G_2}\big]\theta\big[{}^{a-H_2}_{b-G_2}\big]=$$

$$(-1)^{h_1^1g_1^1+G_2(1+h_1^1+H_2)}\,\theta\begin{bmatrix}1-h_1^1\\1-g_1^1\end{bmatrix}^2(v)\,\theta\begin{bmatrix}1-h_1^1+H_2\\1-g_1^1+G_2\end{bmatrix}^2(v)\,, \qquad (11)$$

which shows how many odd $\theta_1(v)\equiv\theta[{}^1_1](v)$ functions (or equivalently how many fermionic zero modes in the path integral) arise for given shift (h_1^1, g_1^1) and twist (H_2, G_2).

Conformal block A : $(h_1^1, g_1^1)=(0,0)$, $(H_2, G_2)=(0,0)$
This block is proportional to $\theta\big[{}^1_1\big]^4(v)=\mathcal{O}(v^4)$. Up to an overall factor $1/2^3$, it is the contribution of the $\mathcal{N}=4$ spectrum of the parent theory, when neither the $\mathbb{Z}_2\times\mathbb{Z}_2$ action nor the stringy Scherk-Schwarz mechanism are implemented. Therefore, it does not contribute to the 1-loop gauge couplings.

Conformal blocks B : $(h_1^1, g_1^1)\neq(0,0)$, $(H_2, G_2)=(0,0)$
They are proportional to $\theta\begin{bmatrix}1-h_1^1\\1-g_1^1\end{bmatrix}^4(v)=\mathcal{O}(1)$. The parity of the winding number along the compact direction X^4 being h_1^1, the blocks with $h_1^1=1$ involve states, which are super massive compared to the pure Kaluza–Klein modes. These blocks are therefore exponentially suppressed, compared to the block $(h_1^1, g_1^1)=(0,1)$. Up to an overall factor $1/2^2$, the latter arises from the spectrum considered in the conformal block A, but in the $\mathcal{N}=4\to\mathcal{N}=0$ spontaneously broken phase, and contributes to the gauge couplings.

Conformal blocks C : $(h_1^1, g_1^1)=(0,0)$, $(H_2, G_2)\neq(0,0)$
They are proportional to $\theta\big[{}^1_1\big](v)^2\theta\big[{}^{1-H_2}_{1-G_2}\big]^2(v)=\mathcal{O}(v^2)$ and do contribute to Δ^i, due to the action of the helicity operator. Reasoning as in the previous case, the parity of the winding number along the compact direction X^5 is H_2, which implies the blocks with $H_2=1$ yield exponentially suppressed contributions, compared to that associated to the block $(H_2, G_2)=(0,1)$. Up to an overall factor $1/2^2$, the latter arises from a spectrum realizing the spontaneous $\mathcal{N}=4\to\mathcal{N}_C=2$ breaking, which contributes to the couplings.

Conformal blocks D : $(h_1^1, g_1^1)=(H_2, G_2)\neq(0,0)$
They are proportional to $\theta\big[{}^{1-H_2}_{1-G_2}\big]^2(v)\theta\big[{}^1_1\big](v)^2=\mathcal{O}(v^2)$. The situation is identical to that of the conformal blocks C, except that the generator of the $\mathbb{Z}_2$ free action responsible of the partial spontaneous breaking of $\mathcal{N}=4$ twists X^6, X^7, X^8, X^9 and shifts X^4, X^5. The dominant contribution to the threshold corrections arises again from the block $(H_2, G_2)=(0,1)$, which describes a spectrum realizing the spontaneous $\mathcal{N}=4\to\mathcal{N}_D=2$ breaking.

Conformal blocks E : $\begin{vmatrix}h_1^1 & H_2\\ g_1^1 & G_2\end{vmatrix}\neq 0$

The remaining conformal blocks have non-trivial determinant $\left|\begin{smallmatrix} h_1^1 & H_2 \\ g_1^1 & G_2 \end{smallmatrix}\right|$, which implies $\theta\left[\begin{smallmatrix} 1-h_1^1 \\ 1-g_1^1 \end{smallmatrix}\right]^2(v)\,\theta\left[\begin{smallmatrix} 1-h_1^1+H_2 \\ 1-g_1^1+G_2 \end{smallmatrix}\right]^2(v) = \mathcal{O}(1)$. However, this condition is also saying that $(h_1^1, H_2) \neq (0,0)$, which means the modes in these blocks have non-trivial winding number(s) along X^4, X^5 or both. Therefore, their contributions to the gauge couplings are non-trivial but exponentially suppressed.

Having analyzed all conformal blocks satisfying $(H_1, G_1) = (0,0)$, we proceed with the study of the modular orbit $(H_1, G_1) \neq (0,0)$, where the sign S_L is trivial. Since the 1st 2-torus is twisted, these blocks are independent of the moduli T_1, U_1 and thus $m_{3/2}$. They can be analyzed as in the case of $\mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathcal{N} = 1$ supersymmetric models. Actually, summing over the spin structures, the relevant terms in the refined partition function Z become

$$\frac{1}{2}\sum_{a,b}(-1)^{a+b+ab}\theta\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right](2v)\,\theta\left[\begin{smallmatrix} a+H_1 \\ b+G_1 \end{smallmatrix}\right]\theta\left[\begin{smallmatrix} a+H_2 \\ b+G_2 \end{smallmatrix}\right]\theta\left[\begin{smallmatrix} a-H_1-H_2 \\ b-G_1-G_2 \end{smallmatrix}\right] =$$
$$(-1)^{(G_1+G_2)(1+H_1+H_2)}\,\theta\left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right](v)\,\theta\left[\begin{smallmatrix} 1-H_1 \\ 1-G_1 \end{smallmatrix}\right](v)\,\theta\left[\begin{smallmatrix} 1-H_2 \\ 1-G_2 \end{smallmatrix}\right](v)\,\theta\left[\begin{smallmatrix} 1+H_1+H_2 \\ 1+G_1+G_2 \end{smallmatrix}\right](v)\,, \quad (12)$$

which invites us to split the discussion in three parts.

$\mathcal{N} = 2$ *conformal blocks, with fixed 2nd 2-torus* : $(H_2, G_2) = (0,0)$
They are proportional to $\theta\left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]^2(v)\theta\left[\begin{smallmatrix} 1-H_1 \\ 1-G_1 \end{smallmatrix}\right]^2(v) = \mathcal{O}(v^2)$. The 2nd internal 2-torus is fixed by the non-free action of the $\mathbb{Z}_2$ characterized by (H_1, G_1). Adding the conformal block A, we obtain an $\mathcal{N} = 2$ sector of the theory, up to an overall factor $1/2$ associated to the second $\mathbb{Z}_2$. This spectrum leads to non-trivial corrections to the gauge couplings.

$\mathcal{N} = 2$ *conformal blocks, with fixed 3rd 2-torus* : $(H_1, G_1) = (H_2, G_2)$
Thy are proportional to $\theta\left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]^2(v)\theta\left[\begin{smallmatrix} 1-H_1 \\ 1-G_1 \end{smallmatrix}\right]^2(v) = \mathcal{O}(v^2)$. Actually, $(H_3, G_3) = (0,0)$, which means that the 3rd 2-torus is fixed by the combined action of the generators of the two $\mathbb{Z}_2$'s. Adding the conformal block A, one obtains the last $\mathcal{N} = 2$ sector of the theory, up to an overall factor $1/2$. Again, this spectrum yields a non-trivial contribution to the gauge couplings.

$\mathcal{N} = 1$ *conformal blocks* : $\left|\begin{smallmatrix} H_1 & H_2 \\ G_1 & G_2 \end{smallmatrix}\right| \neq 0$
The remaining blocks have non-trivial determinant, $\left|\begin{smallmatrix} H_1 & H_2 \\ G_1 & G_2 \end{smallmatrix}\right| \neq 0$, which implies they are proportional to $\theta\left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right](v)\,\theta\left[\begin{smallmatrix} 1-H_1 \\ 1-G_1 \end{smallmatrix}\right](v)\,\theta\left[\begin{smallmatrix} 1-H_2 \\ 1-G_2 \end{smallmatrix}\right](v)\,\theta\left[\begin{smallmatrix} 1+H_1+H_2 \\ 1+G_1+G_2 \end{smallmatrix}\right](v) = \mathcal{O}(v)$. Acting on them with the helicity operator, the result is proportional to

$$\partial_v^2\Big(\theta\left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right](v)\,\theta\left[\begin{smallmatrix} 1-H_1 \\ 1-G_1 \end{smallmatrix}\right](v)\,\theta\left[\begin{smallmatrix} 1-H_2 \\ 1-G_2 \end{smallmatrix}\right](v)\;\;\theta\left[\begin{smallmatrix} 1+H_1+H_2 \\ 1+G_1+G_2 \end{smallmatrix}\right](v)\Big)\Big|_{v=0} \propto$$
$$\partial_v^2\Big(\theta_1(v)\,\theta_2(v)\,\theta_3(v)\,\theta_4(v)\Big)\Big|_{v=0} = 0\,, \quad (13)$$

thanks to the oddness of $\theta_1(v)$ and evenness of $\theta_{2,3,4}(v)$. Thus, these conformal blocks do not contribute to the thresholds.

In the class of models we consider, the effective running gauge coupling associated to some gauge group factor G^i has a universal form at 1-loop [9]. It can be elegantly expressed in terms of three moduli-dependent squared mass scales arising from the corrections associated to the conformal blocks B, C, D,

$$M_B^2 = \frac{M_s^2}{|\theta_2(U_1)|^4 \operatorname{Im} T_1 \operatorname{Im} U_1}, \quad M_C^2 = \frac{M_s^2}{|\theta_4(U_1)|^4 \operatorname{Im} T_1 \operatorname{Im} U_1},$$
$$M_D^2 = \frac{M_s^2}{|\theta_3(U_1)|^4 \operatorname{Im} T_1 \operatorname{Im} U_1}, \tag{14}$$

which are of order $m_{3/2}^2$, and two more scales

$$M_I^2 = \frac{M_s^2}{16\big|\eta(T_I)|^4\,\big|\eta(U_I)|^4 \operatorname{Im} T_I \operatorname{Im} U_I}, \quad I = 2, 3, \tag{15}$$

of order M_s^2 that encode the contributions of the $\mathcal{N} = 2$ sectors associated to the fixed 2[nd] and 3[rd] internal 2-tori. It is also useful to introduce a "renormalized string coupling" [11],

$$\frac{16\,\pi^2}{g^2_{\text{renor}}} = \frac{16\,\pi^2}{g_s^2} - \frac{1}{2}Y(T_2, U_2) - \frac{1}{2}Y(T_3, U_3)\,, \tag{16}$$
$$\text{where} \quad Y(T,U) = \frac{1}{12}\int_{\mathcal{F}} \frac{d^2\tau}{\tau_2}\,\Gamma_{2,2}(T,U)\left[\left(\bar{E}_2 - \frac{3}{\pi\tau_2}\right)\frac{\bar{E}_4\bar{E}_6}{\bar{\eta}^{24}} - \bar{j} + 1008\right],$$

in which $\Gamma_{2,2} = \Gamma_{2,2}\big[{}^{0,\,0}_{0,\,0}\big]$ is the unshifted lattice, while for $q = e^{2i\pi\tau}$, $E_{2,4,6} = 1 + \mathcal{O}(q)$ are holomorphic Eisenstein series of modular weights 2, 4, 6 and $j = 1/q + 744 + \mathcal{O}(q)$ is holomorphic and modular invariant. The inverse squared 1-loop gauge coupling at energy scale $Q^2 = \mu^2\frac{\pi^2}{4}$ is then

$$\frac{16\,\pi^2}{g_i^2(Q)} = k^i\,\frac{16\,\pi^2}{g^2_{\text{renor}}} - \frac{b^i_B}{4}\ln\left(\frac{Q^2}{Q^2+M_B^2}\right) - \frac{b^i_C}{4}\ln\left(\frac{Q^2}{Q^2+M_C^2}\right) - \tag{17}$$
$$- \frac{b^i_D}{4}\ln\left(\frac{Q^2}{Q^2+M_D^2}\right) - \frac{b^i_2}{2}\ln\left(\frac{Q^2}{M_2^2}\right) - \frac{b^i_3}{2}\ln\left(\frac{Q^2}{M_3^2}\right) + \mathcal{O}\left(\frac{m^2_{3/2}}{M_s^2}\right),$$

which depends only on five model-dependent β-function coefficients and the Kac–Moody level. In this final result, we have shifted $M^2_{B,C,D} \to Q^2 + M^2_{B,C,D}$ in order to implement the thresholds at which the sectors B, C or D decouple, i.e. when Q exceeds M_B, M_C or M_D. Thus, this expression is valid as long as Q is lower than the mass of the heavy states we have neglected the exponentially suppressed contributions i.e. the string or GUT scale, depending on the model. Taking Q lower than at least one of the scales M_B, M_C or M_D, the r.h.s. of Eq. (17) scales as $\ln \mathrm{Im}\, T_1$, which is the logarithm of the large 1st 2-torus volume, as expected for the decompactification problem not to arise.

To conclude, we would like to mention two important remarks. First of all, we stress that the $\mathbb{Z}_2 \times \mathbb{Z}_2$ models, where a $\mathbb{Z}_2$ is freely acting and a stringy Scherk-Schwarz mechanism responsible of the final breaking of $\mathcal{N} = 1$ takes place, have non-chiral massless spectra. This is due to the fact that in the $\mathcal{N} = 1, \mathbb{Z}_2 \times \mathbb{Z}_2$ models, chiral families occur from twisted states localized at fixed points. In the models we have considered, fixed points localized on the 2nd and 3rd 2-tori can arise but are independent of the moduli T_1, U_1 i.e. $m_{3/2}$. Thus, taking the large volume limit of the 1st 2-torus, where $\mathcal{N} = 2$ supersymmetry is recovered, one concludes that the twisted states are actually hypermultiplets i.e. couples of families and anti-families.

Second, we point out that in the models analyzed in the present work, the conformal block B is the only non-supersymmetric and non-negligible contribution to the partition function Z, and thus to the 1-loop effective potential. In Refs. [10, 15], it is shown that in some models, the latter is positive semi-definite. The motion of the moduli T_2, U_2 and T_3, U_3 is thus attracted to points [16], where the effective potential vanishes, allowing $m_{3/2}$ to be arbitrary. In other words, the defining properties of the no-scale models, namely arbitrariness of the supersymmetry breaking scale $m_{3/2}$ in flat space, which are valid at tree level, are extended to the 1-loop level. This very fact, characteristic of the so-called "super no-scale models", may have interesting consequences on the smallness of a cosmological constant generated at higher orders. In Ref. [17], other models having 1-loop vanishing cosmological constant are also considered, which however suffer from the decompactification problem.

Acknowledgements I am grateful to A. Faraggi and K. Kounnas, with whom the original work [9] has been done in collaboration. I would also like to thank C. Angelantonj, I. Antoniadis and J. Rizos for fruitful discussions, and the Laboratoire de Physique Théorique of Ecole Normale Supérieure for hospitality.

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Glueball Inflation and Gauge/Gravity Duality

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Abstract We summarize our work on building glueball inflation models with the methods of the gauge/gravity duality. We review the relevant five-dimensional consistent truncation of type IIB supergravity. We consider solutions of this effective theory, whose metric has the form of a dS_4 foliation over a radial direction. By turning on small (in an appropriate sense) time-dependent deformations around these solutions, one can build models of glueball inflation. We discuss a particular deformed solution, describing an ultra-slow roll inflationary regime.

1 Introduction

Composite inflation models [1, 2] provide a possible resolution to the well-known η-problem [3, 4] of inflationary model-building. However, they are quite challenging to study with standard QFT methods, since they involve a strongly-coupled gauge sector. This has motivated interest in developing descriptions of such models via a string-theoretic tool aimed precisely at studying the nonperturbative regime of gauge theories, namely the gauge/gravity duality. Gravitational duals, in which the inflaton arises from the position of a D3-brane probe have been considered in [5–9]. Instead, in [10–12] we studied models, whose inflaton arises from the background fields of the gravitational solution and is thus a glueball in the dual gauge theory.

The backgrounds of interest for us solve the equations of motion of the 5d consistent truncation of type IIB supergravity established in [13]. The latter encompasses a wide variety of prominent gravity duals, like [14–18], as special solutions and thus provides a unifying framework for gauge/gravity duality investigations. The work [10] obtained new non-supersymmetric classes of solutions of this theory, whose metric is of the form of a dS_4 fibration over the fifth direction. These backgrounds provide a useful playground for studying certain strongly-coupled gauge theories in de Sitter space. To have an inflationary model, however, one needs a time-dependent

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V. Dobrev (ed.), *Lie Theory and Its Applications in Physics*,
Springer Proceedings in Mathematics & Statistics 191,
DOI 10.1007/978-981-10-2636-2_18

Hubble parameter. Therefore, in [12] we investigated time-dependent deformations around a solution of [10], in order to search for gravity duals of glueball inflation.

It is worth pointing out that the main cosmological observables of an inflationary model (like the scalar spectral index n_s and the tensor-to-scalar ratio r) are entirely determined by the Hubble parameter and inflaton field as functions of time [19]. Hence, once one has a deformed background in the above set-up, one can immediately compute the desired quantities. This is the sense, in which the time-dependent deformations of the previous paragraph give models of cosmological inflation. In that vein, in [12] we calculated the slow roll parameters for a solution we found there and thus established that it gives a gravity dual of ultra-slow roll glueball inflation. The ultra-slow roll regime may play an important role in understanding the observed low l anomaly in the power spectrum of the CMB. Hence it deserves further study. We also discuss here perspectives for building gravity duals of standard slow roll inflationary models.

2 Effective 5d Theory

In this section we summarize necessary material about the 5d consistent truncation of type IIB supergravity relevant for our considerations. We also recall a particular solution of this theory, whose time-dependent deformations we will investigate in the next subsection.

2.1 Action and Field Equations

Let us briefly review the basic characteristics of the five-dimensional consistent truncation of [13]. Using a particular ansatz for the bosonic fields of type IIB supergravity in terms of certain 5d fields and integrating out five compact dimensions, one reduces the ten-dimensional IIB action to the following five-dimensional one:

$$S = \int d^5x\sqrt{-detg}\left[-\frac{R}{4} + \frac{1}{2}G_{ij}(\Phi)\partial_I\Phi^i\partial^I\Phi^j + V(\Phi)\right] . \tag{1}$$

Here $\{\Phi^i\}$ is a set of 5d scalar fields, that arise from the components of the 10d ones including metric warp factors, $V(\Phi)$ is a rather complicated potential, $G_{ij}(\Phi)$ is a diagonal sigma-model metric and, finally, R is the Ricci scalar of the 5d spacetime metric g_{IJ}. The full expressions for $V(\Phi)$ and $G_{ij}(\Phi)$ can be found in [13]; for a more concise summary, see also [10]. The field equations that the action (1) implies are:

$$\nabla^2 \Phi^i + \mathcal{G}^i{}_{jk}\, g^{IJ} (\partial_I \Phi^j)(\partial_J \Phi^k) - V^i = 0 \quad ,$$
$$-R_{IJ} + 2\, G_{ij}\, (\partial_I \Phi^i)(\partial_J \Phi^j) + \frac{4}{3}\, g_{IJ} V = 0 \quad , \tag{2}$$

where $V^i = G^{ij} V_j$, $V_i = \frac{\partial V}{\partial \Phi^i}$ and $\mathcal{G}^i{}_{jk}$ are the Christoffel symbols of the sigma-model metric G_{ij}.

2.2 A Solution with dS_4 Slicing

We will be interested in time-dependent deformations around a particular solution of the system (2) found in [10]. So let us first recall its form. In the notation of [10] and working within the same subtruncation as there (i.e., with zero NS flux), we have six scalars in the 5d effective theory:

$$\{\Phi^i(x^I)\} = \{p(x^I), x(x^I), g(x^I), \phi(x^I), a(x^I), b(x^I)\} \ . \tag{3}$$

The work [10] found three families of solutions of (2) with a 5d metric of the form

$$ds_5^2 = e^{2A(z)} \left[-dt^2 + s(t)^2 \sum_{m=1}^{3} (dx^m)^2 \right] + dz^2 \ , \tag{4}$$

where $s(t) = e^{\mathcal{H}t}$ with $\mathcal{H} = const$. In all of them, three of the scalars Φ^i vanish identically, namely:

$$g(x^I) = 0 \quad , \quad a(x^I) = 0 \quad , \quad b(x^I) = 0 \quad . \tag{5}$$

Two of those solutions are numerical and one is analytical. For convenience, we will study deformations around the latter. Denoting its metric functions and scalar fields by the subscript 0, we have [10]:

$$A_0(z) = \ln(z + C) + \frac{1}{2}\, \ln\left(\frac{7}{3} \mathcal{H}_0^2 \right) \ ,$$
$$p_0(z) = -\frac{1}{7} \ln(z + C) - \frac{1}{14}\, \ln\left(\frac{7N^2}{9} \right) \ ,$$
$$x_0(z) = -6\, p_0(z) \qquad , \qquad \phi_0 = 0 \quad , \tag{6}$$

where C and N are constants.

Let us mention in passing that the form of (6) is consistent with ALD (asymptotically linear dilaton) behavior at large z. This is not obvious at first sight due to the use of a different coordinate system (in string frame) compared to the conventional one (in Einstein frame), in which the holographic renormalization of ALD backgrounds was developed [20]. This issue was discussed in more detail in [10, 11], where it was also pointed out that the same kind of asymptotics characterizes the walking solutions of [16] as well.

3 Deforming the dS_4 Solution

Now we are ready to turn to the investigation of solutions of the system (2), which are deformations around the zeroth order background (6). Since our aim is to study glueball inflation, we would like to find solutions, whose 5d metric is of the form (4) but with $\dot{\mathcal{H}} \neq 0$. Recall that the Hubble parameter is defined as

$$\mathcal{H} = \frac{\dot{s}}{s} \quad , \tag{7}$$

where for convenience we have denoted $^{\cdot} \equiv \frac{\partial}{\partial t}$. Now, one of the slow roll conditions widely used in inflationary model building[1] is the following [19]:

$$-\frac{\dot{\mathcal{H}}}{\mathcal{H}^2} \ll 1 \ . \tag{8}$$

In view of that, we will look for solutions with time-dependent $\mathcal{H}$ by considering small, in the sense of (8), deformations around an $\mathcal{H} = const$ solution.

For that purpose, let us introduce a small parameter γ, satisfying

$$\gamma \ll 1 \ , \tag{9}$$

and search for solutions that are expansions in powers of this parameter. To do this, we make the following ansatz for the nonvanishing 5d fields:

$$\begin{aligned}
p(t,z) &= p_0(z) \quad , \quad x(t,z) = x_0(z) \ , \\
\phi(t,z) &= \gamma\, \phi_{(1)}(t,z) + \gamma^3 \phi_{(3)}(t,z) + \mathcal{O}(\gamma^5) \ , \\
A(t,z) &= A_0(z) + \gamma^2 A_{(2)}(t,z) + \mathcal{O}(\gamma^4) \ , \\
H(t,z) &= \mathcal{H}_0\, t + \gamma^2 H_{(2)}(t,z) + \mathcal{O}(\gamma^4) \ ,
\end{aligned} \tag{10}$$

where $H(t,z)$ is a warp factor defined via

$$ds_5^2 = e^{2A(t,z)} \left[-dt^2 + e^{2H(t,z)} \sum_{m=1}^{3} (dx^m)^2 \right] + dz^2 \ . \tag{11}$$

[1] One should keep in mind, though, that there are more exotic inflationary regimes, in which one or more of the slow roll conditions can be violated; see [21–23], for instance.

In other words, we keep the scalars $p(x^I)$ and $x(x^I)$ the same as in (6), while allowing small deviations around that zeroth order solution in the scalar ϕ and the metric functions A and H.

It is worth commenting a bit more on the form of the deformation ansatz (10). First of all, in order to obtain solutions with $\dot{\mathcal{H}} \neq 0$, we need to turn on time dependence in at least one scalar. It is convenient to take this scalar to be ϕ since, unlike p and x, it vanishes at zeroth order and, furthermore, it is a flat direction of the potential; see [12]. Therefore, ϕ will play the role of the inflaton in our set-up. Also note that, although we would like to have t-dependent H only, we have allowed z-dependence too, for more generality. And, finally, the different powers of γ in the expansion of ϕ, compared to the expansions of the warp factors, will be of great significance for finding an analytical solution, as will become clear below.

3.1 Equations of Motion

Let us now substitute the ansatz (10) in the system (2) and study the result order by order in γ. Clearly, since we are expanding around a zeroth order solution, there is no contribution at order γ^0.

At order γ, we have the following field equation [12]:

$$\ddot{\phi}_{(1)} + 3\,\mathcal{H}_0\,\dot{\phi}_{(1)} = e^{2A_0}\left(\phi''_{(1)} + 4A'_0\phi'_{(1)}\right) \ , \tag{12}$$

where $' \equiv \frac{\partial}{\partial z}$. To find a solution, let us make the ansatz

$$\phi_{(1)} = \Phi_1(t)\,\Phi_2(z) \tag{13}$$

and solve the eigen-problems

$$\ddot{\Phi}_1 + 3\mathcal{H}_0\dot{\Phi}_1 = \lambda\,\Phi_1 \qquad \text{and} \qquad e^{2A_0}\left(\Phi''_2 + 4A'_0\Phi'_2\right) = \lambda\,\Phi_2 \tag{14}$$

with λ being some constant. One easily obtains that [12]:

$$\Phi_1(t) = C_1\,e^{k_+ t} + C_2\,e^{k_- t} \ , \quad \text{where} \quad k_\pm = -\frac{3\mathcal{H}_0}{2} \pm \frac{\sqrt{9\mathcal{H}_0^2 + 4\lambda}}{2} \tag{15}$$

and $C_{1,2}$ are integration constants, while

$$\Phi_2(z) = C_3(z+C)^{\alpha_+} + C_4(z+C)^{\alpha_-} \ \text{with} \ \ \alpha_\pm = -\frac{3}{2} \pm \frac{3}{2}\sqrt{1 + \frac{4\,\lambda}{21\,\mathcal{H}_0^2}} \tag{16}$$

and $C_{3,4}$ being integration constants.

Note that if $\lambda = 0$, then one is free to add an arbitrary constant to the $\phi_{(1)}$ solution, determined by (14). This will be important in the following.

At order γ^2, we find a coupled system for the warp factor deformations $A_{(2)}$ and $H_{(2)}$, namely [12]:

$$\begin{aligned}
E1:\quad & -\mathcal{H}_0^2\left(\frac{7}{3}(z+C)^2 A''_{(2)} + \frac{56}{3}(z+C)A'_{(2)} + 7(z+C)H'_{(2)} + 6A_{(2)}\right)\\
& + \mathcal{H}_0\big(3\dot{A}_{(2)} + 6\dot{H}_{(2)}\big) + 3\ddot{A}_{(2)} + 3\ddot{H}_{(2)} + \frac{1}{2}\dot{\phi}^2_{(1)} = 0 \quad ,\\
E2:\quad & \mathcal{H}_0^2\left(\frac{7}{3}(z+C)^2\left[A''_{(2)} + H''_{(2)}\right] + \frac{56}{3}(z+C)A'_{(2)} + \frac{49}{3}(z+C)H'_{(2)}\right.\\
& \left. + \, 6A_{(2)}\right) - \mathcal{H}_0\big(5\dot{A}_{(2)} + 6\dot{H}_{(2)}\big) - \ddot{A}_{(2)} - \ddot{H}_{(2)} = 0 \quad ,\\
E3:\quad & 4A''_{(2)} + 3H''_{(2)} + \frac{2}{z+C}\left(4A'_{(2)} + 3H'_{(2)}\right) + \frac{1}{2}\phi'^2_{(1)} = 0 \quad ,\\
E4:\quad & 3\dot{A}'_{(2)} + 3\dot{H}'_{(2)} + 3\,\mathcal{H}_0\, H'_{(2)} + \frac{1}{2}\dot{\phi}_{(1)}\phi'_{(1)} = 0 \quad .
\end{aligned} \tag{17}$$

To solve this rather involved system, let us take for convenience the $\phi_{(1)}$ solution to be:

$$\phi_{(1)} = C_\phi + \tilde{C}e^{kt}(z+C)^\alpha \qquad \text{with} \qquad C_\phi\,,\,\tilde{C} = const \quad , \tag{18}$$

where k is any of $k_\pm$ and α is any of $\alpha_\pm$. Note that the addition of the arbitrary constant C_ϕ in (18) makes no difference for the solutions of (17), since the function $\phi_{(1)}$ enters those equations only through its derivatives. However, the presence of C_ϕ will turn out to be useful later. Plus, it will become clear shortly that it is consistent with (14).

Now, the form of $E3$ in (17), together with (18), suggests looking for a solution with the following ansatz:

$$A_{(2)}(t,z) = e^{2kt}\hat{A}(z) \qquad \text{and} \qquad H_{(2)}(t,z) = \hat{C}_H + e^{2kt}\hat{H}(z)\ , \tag{19}$$

where $\hat{C}_H = const$. Again, we have included an arbitrary constant $\hat{C}_H$, since $H_{(2)}$ enters the system (17) only via its derivatives. Substituting (19) and (18) into (17), one can see that the time-dependence factors out. Thus, one is left with a coupled system of ODEs for the functions $\hat{A}(z)$ and $\hat{H}(z)$. A detailed investigation in [12] showed that this system has a solution only for[2]

$$\alpha = 0\ , \tag{20}$$

[2] To prove this, one also needs to use the fact that (15) and (16) imply the following relation between k and α: $k = -\frac{3}{2}\mathcal{H}_0 \pm \frac{\sqrt{84\alpha^2+252\alpha+81}}{6}\,\mathcal{H}_0$.

in which case both $\hat{A} = const$ and $\hat{H} = const$. Substituting $\alpha = 0$ in (16), we find that $\lambda = 0$ as well. This in turn implies that we are free to add the constant C_ϕ in (18), as commented below (16). Finally, from (15) we now have:

$$k = -3\mathcal{H}_0 \ , \tag{21}$$

where we have taken the value of k_- in order to have time-dependence in the inflaton field ϕ.

3.2 Ultra-Slow Roll Inflation

The solution we described above gives a dual description of an ultra-slow roll glueball inflation model. To see this, let us compute the inflationary slow-roll parameters. They are defined in terms of the inflaton field and Hubble parameter as [19]:

$$\varepsilon = -\frac{\dot{\mathcal{H}}}{\mathcal{H}^2} \qquad \text{and} \qquad \eta = -\frac{\ddot{\phi}}{\mathcal{H}\dot{\phi}} \qquad . \tag{22}$$

From the results of Sect. 3.1, we have that ϕ and $\mathcal{H}$ are given by:

$$\begin{aligned} \phi &= \left(C_\phi + \tilde{C}\, e^{-3\mathcal{H}_0 t}\right)\gamma + \mathcal{O}(\gamma^3) \ , \\ \mathcal{H} &= \mathcal{H}_0 - C_{\mathcal{H}}\, e^{-6\mathcal{H}_0 t}\,\gamma^2 + \mathcal{O}(\gamma^4) \ , \end{aligned} \tag{23}$$

where $C_{\mathcal{H}}$ is some constant; for more details, see [12].[3]

Substituting (23) in (22), we find that the slow roll parameters behave as $\varepsilon = \mathcal{O}(\gamma^2)$ and $\eta = 3 + \mathcal{O}(\gamma^2)$; see [12] for more detailed expressions. In other words, at leading order we have:

$$\varepsilon \ll 1 \qquad \text{and} \qquad \eta = 3 \quad . \tag{24}$$

These are precisely the values of ε and η for the ultra-slow regime, considered in [24, 25]. In fact, our result for the inflaton in (23) also agrees completely with the expression in [25].

It is worth pointing out a similarity between our model and the constant-rate-of-roll solutions of [23]. For that purpose, let us introduce the following series of slow roll parameters:

[3]Note that, since the correction $A_{(2)}$ to the warp factor $A(t, z)$ in (11) also depends on t, as can be seen from (19), one should, in principle, first perform a coordinate transformation $t \to \tau$ that absorbs that dependence, before computing the physical Hubble parameter $\mathcal{H}(\tau)$ and inflaton field $\phi(\tau)$. However, in the present case, this leads to exactly the same expressions as (23) with t substituted by τ, with the only difference being the numerical value of the constant $C_{\mathcal{H}}$. So we will not discuss the details of that transformation here.

$$\varepsilon_1 = -\frac{\dot{\mathcal{H}}}{\mathcal{H}^2} \quad \text{and} \quad \varepsilon_{n+1} = \frac{\dot{\varepsilon}_n}{\mathcal{H}\varepsilon_n} \quad , \tag{25}$$

where obviously $\varepsilon_1 \equiv \varepsilon$. One can easily compute that, at large t, our solution gives [12]:

$$\varepsilon_{2n+1} \to 0 \quad \text{and} \quad \varepsilon_{2n} \to -6 \quad . \tag{26}$$

This is the same asymptotics as in [23]. It would be interesting to investigate whether there is a deeper underlying reason for that.

In conclusion, let us make a few comments regarding other inflationary models in our framework. Although an ultra-slow roll inflationary regime may be desirable to account for the low l anomaly in the CMB power spectrum, it is rather short-lived. So it has to be succeeded by regular slow roll, in order to have enough expansion and thus give a complete inflationary model. To obtain such solutions in our gauge/gravity duality set-up, one may need to study deformations around the numerical solutions of [10], instead of the analytical one (6). It could also be that duals of regular slow roll can be found by modifying the initial ansatz for the deformations around the analytical solution. Finally, it would be interesting to investigate what kind of models can be obtained by going to the next order in γ in the expansions (10), while taking $\phi_{(1)}$, $A_{(2)}$ and $H_{(2)}$ to vanish. This seems to open much wider possibilities for inflationary model building, as the equations of motion for $A_{(4)}$ and $H_{(4)}$ would be independent of $\phi_{(3)}$. Thus, many of the restrictions we encountered here (and as a result of which we ended up with ultra-slow roll) would not occur.

Acknowledgements I have received partial support from the European COST Action MP-1210 and the Bulgarian NSF grant DFNI T02/6.

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Degenerate Metrics and Their Applications to Spacetime

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Abstract The Lie groups preserving degenerate quadratic forms appear in various contexts related to spacetime. The homogeneous Galilei group is the intersection of two such groups. The structure group of sub-Riemannian geometry and of singular semi-Riemannian geometry, as well as of some submanifolds of semi-Riemannian manifolds, is of this kind. Such groups are shown to replace the Lorentz group at a very large class of singularities in general relativity. Also, these groups are shown to be fundamental in Kaluza-Klein theory and in gauge theory, where they provide an explanation why we may not be able to probe extra-dimensional lengths.

1 Introduction

In the following, we will discuss some modern applications of the degenerate metrics in the physics of spacetime. While normally spacetime is considered endowed with a non-degenerate metric, we will see some situations where the spacetime metric is degenerate and what are the implications. These situations include Galilei spacetime, spacetime singularities, and Kaluza theories. But first, let us remind some definitions and fix some notations.

Let (V, g) be a vector space endowed with a symmetric bilinear form g, which in the following will be named *inner product* or *metric*. The signature of g is (r, s, t) if g can be diagonalized to $\text{diag}(-I_t, I_s, O_r)$. We denote by $O(t, s, r)$ the group of transformations of the vector space $\mathbb{R}^n$ which preserve this bilinear form. The metric g is called *degenerate* if $r > 0$. For example, the orthogonal group $O(n) = O(0, n, 0)$ preserves the non-degenerate metric $\text{diag}(1, \ldots, 1)$, the Lorentz group $O(1, 3) = O(1, 3, 0)$ preserves the Lorentz metric $\text{diag}(-1, 1, 1, 1)$, which is non-degenerate, and the general linear group $GL(n) = O(0, 0, n)$ preserves the degenerate metric $g = 0$. In this article, we are interested in $O(t, s, r)$ with $r > 0$.

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V. Dobrev (ed.), *Lie Theory and Its Applications in Physics*,
Springer Proceedings in Mathematics & Statistics 191,
DOI 10.1007/978-981-10-2636-2_19

We define the morphism $\flat : V \to V^*$ by $u \mapsto u^\bullet := \flat(u) = u^\flat = g(u, \_)$. The *radical* $V_\circ := \ker \flat = V^\perp$ is the set of isotropic vectors in V. The *radical annihilator* $V^\bullet := \operatorname{im} \flat \leq V^*$ is the image of $\flat$. It has the property that for any $\eta \in V^\bullet$, $\eta|_{V_\circ} = 0$. The inner product g induces on $V^\bullet$ an inner product defined by $g_\bullet(u_1^\flat, u_1^\flat) := g(u_1, u_2)$, which is the inverse of g iff $\det g \neq 0$. The quotient $V_\bullet := V/V_\circ$ consists in the equivalence classes of the form $u + V_\circ$. On $V_\bullet$, g induces an inner product $g^\bullet(u_1 + V_\circ, u_2 + V_\circ) := g(u_1, u_2)$.

These definitions can be applied to the tangent space of a manifold. Until recently, the state of the art of spacetimes with degenerate metric was the work of D. Kupeli [1, 2], but there were two limitations. The first was that the theory worked only for metrics with constant signature, while in general relativity the signature has to change, if singularities are involved. The second limitation is that the method is not invariant,depending on the choice of a distribution transversal to $\ker g$. But the approach introduced in [3] applies to both constant and changing signature, is invariant, and generalize both Riemann's and Kupeli's results.

For example, degenerate metrics have applications to Galilei's spacetime. The laws of Newtonian mechanics are invariant to Galilei transformations of the space G with coordinates (t, x, y, z). Galilei's transformations are those linear transformations which preserve a degenerate metric on the underlying vector space (also denoted here by G), and one on its dual space G^* [4–6]. The degenerate metric $g_{\text{space}\,\text{ij}}$ on G^* has the rank equal to 3, and its radical is the three-dimensional space $S < G$. It induces a three-dimensional (sub-Riemannian) metric $g_{\text{space}\,\text{ij}}$ on S, which is the Euclidean metric of space. The other degenerate metric is $g_{\text{time}\,\text{ij}} := t \otimes t$, where t is the one-form t defining the time, and which annihilates S.

In the following, we will see some applications of degenerate metrics to spacetime, in the higher-dimensional theories like Kaluza's, and in the problem of singularities in general relativity.

2 Gauge Theory and Kaluza Theory

Let us consider a *fiber bundle* (E, M, π, F), with *total space* E, *fiber* F, and projection $\pi : E \to M$, where the *base space* is a semi-Riemannian manifold with metric g. The pull-back of the metric g is a degenerate metric $\widetilde{g} = \pi^* g$ on the total space E. The *vertical bundle* $V := \ker(\mathrm{d}\pi)$ is a sub-bundle of TE. The vertical tangent space V_p at a point $p \in E$ is the radical of $\widetilde{g}_p$, $\ker(\mathrm{d}\pi)_p = \ker \widetilde{g}_p$.

If there is a free group action of a group G on E, then (E, M, π, F, G) is a *principal G-bundle*, and the dimension of the fiber F is $\dim F = d = \dim G$. In gauge theory, the *gauge connection* $A = A_a^\mu$ defines a horizontal distribution $H < TE$ on the principal bundle. Also, on the vertical bundle is defined a metric $\widetilde{h}_V$. Both of them are gauge invariant, and together they are equivalent to a gauge invariant metric $\widetilde{h}$ on TE which is degenerate on H, given by $\widetilde{h}(X, Y) = \widetilde{h}_V(\pi_V X, \pi_V Y)$, where $\pi_V : TE \to V$ is the projection on V along H, and $H = \ker \widetilde{h}$.

The metric $\widetilde{g}$ induces a non-degenerate metric $\widetilde{g}_H$ on the horizontal distribution H. Because $\widetilde{g}|_V = 0, H, V$ and $\widetilde{g}_H$ allows one to recover $\widetilde{g}$ by $\widetilde{g}(X, Y) = \widetilde{g}_H(\pi_H X, \pi_H Y)$, where $\pi_H : TE \to H$ is the projection on H along V.

The Kaluza metric $\hat{g}_0$ on E is obtained from the two metrics $\widetilde{g}_H$ on H and $\widetilde{h}_V$ on V by $\hat{g}_0(X, Y) = \widetilde{g}(X, Y) + \widetilde{h}(X, Y)$. The components of $\hat{g}_0$ in a frame composed of a horizontal and a vertical frame are denoted by

$$\hat{g}_0 = \begin{pmatrix} g_{ab} & 0 \\ 0 & h_{\alpha\beta} \end{pmatrix}. \tag{1}$$

Locally, one can identify E to the product $E = M \times F$. The metric $\hat{g}_0$ in a frame made of a frame of M and one of F is obtained by a transformation preserving the fibers and projecting the horizontal space H_p onto the space $T_p M$,

$$S = \begin{pmatrix} I_4 & A \\ 0 & I_d \end{pmatrix}. \tag{2}$$

Then, the Kaluza metric (1) takes in an $M \times F$-frame the form

$$\hat{g}_{ij} = S\hat{g}_0 S^T = \begin{pmatrix} g_{ab} + h_{\mu\nu} A^\mu_a A^\nu_b & h_{\mu\beta} A^\mu_a \\ h_{\alpha\nu} A^\nu_b & h_{\alpha\beta} \end{pmatrix}, \tag{3}$$

in terms of the Lorentzian metric g_{ab} on M and the metric $h_{\mu\nu}$ on the fiber. This is the Kaluza metric, generalized to an arbitrary gauge group (see eg Kerner [7]).

In particular, if $G = U(1)$ and $h = 1$, one obtains the original Kaluza theory. The Lagrangian density in the Kaluza theory is the scalar curvature corresponding to $\hat{g}$, leading to the *Einstein-Maxwell equations*, which include the source-free Maxwell equations, and the Einstein equation on M with the electromagnetic stress-energy tensor $T_{ab} = \frac{1}{\mu_0}\left(F_{as}F_b{}^s - \frac{1}{4}F_{st}F^{st}g_{ab}\right)$.

Because the metric $\widetilde{g}$ vanishes on the fiber, any experiments aiming to detect extra dimensions will fail, and the only evidence of extra dimensions are the gauge fields and gauge symmetry.

3 Degenerate Metrics in Singular General Relativity

The two big problems of general relativity are the occurrence of *singularities* [8], and the fact that quantum gravity is perturbatively non-renormalizable. There are two types of singularities: *malign singularities*, which have components $g_{ab} \to \infty$, and *benign singularities*, whose components g_{ab} are smooth and finite, but $\det g \to 0$, so the reciprocal metric g^{ab} is undefined or singular. Both these types of singularities cause problems which prevent us from using the standard geometric tools. The Christoffel symbols $\Gamma^c{}_{ab} = \dfrac{1}{2} g^{cs}(\partial_a g_{bs} + \partial_b g_{sa} - \partial_s g_{ab})$, needed to define covariant derivatives, blow up even in the benign case, because g^{ab} is singular. This makes writing partial differential equations impossible. Because Christoffel's symbols are used in writing the Riemann curvature $R^d{}_{abc} = \Gamma^d{}_{ac,b} - \Gamma^d{}_{ab,c} + \Gamma^d{}_{bs}\Gamma^s{}_{ac} - \Gamma^d{}_{cs}\Gamma^s{}_{ab}$, the Ricci curvature $R_{ab} = R^s{}_{asb}$, and the scalar curvature $R = g^{pq}R_{pq}$, these are also singular, making Einstein's tensor

$$G_{ab} = R_{ab} - \frac{1}{2}Rg_{ab} \tag{4}$$

singular too. Even if g_{ab} are all finite, these equations contain g^{ab}, and $g^{ab} \to \infty$ when $\det g \to 0$.

Fortunately, we can define other geometric objects, which allow us to do what covariant derivative and curvature do outside the singularities, but can also work at singularities. The constructions associated to a metric on a vector space, introduced in Sect. 1, also applies to the tangent bundle TM of a manifold M. In the following, we will work on a manifold M endowed with a metric g which can be degenerate.

The Koszul object is defined as $\mathcal{K} : \mathcal{X}(M)^3 \to \mathbb{R}$, $\mathcal{K}(X, Y, Z) := \frac{1}{2}\{X\langle Y, Z\rangle + Y\langle Z, X\rangle - Z\langle X, Y\rangle - \langle X, [Y, Z]\rangle + \langle Y, [Z, X]\rangle + \langle Z, [X, Y]\rangle\}$. In local coordinates, $\mathcal{K}_{abc} = \mathcal{K}(\partial_a, \partial_b, \partial_c) = \Gamma_{abc}$. When the metric is non-degenerate, the Levi-Civita connection is obtained by $\nabla_X Y = \mathcal{K}(X, Y, \_)^\sharp$. For the degenerate case, when the contraction with g^{ab} is not defined, we can instead successfully work in many cases with the *lower covariant derivative* of a vector field Y in the direction of a vector field X,

$$(\nabla^\flat_X Y)(Z) := \mathcal{K}(X, Y, Z). \tag{5}$$

The *covariant derivative of differential forms* is defined by

$$(\nabla_X \omega)(Y) := X(\omega(Y)) - g_\bullet(\nabla^\flat_X Y, \omega), \tag{6}$$

$$\nabla_X(\omega_1 \otimes \ldots \otimes \omega_s) := \nabla_X(\omega_1) \otimes \ldots \otimes \omega_s + \ldots + \omega_1 \otimes \ldots \otimes \nabla_X(\omega_s), \tag{7}$$

where the 1-forms $\omega, \omega_1, \ldots, \omega_s \in \Gamma(T^\bullet M)$. For non-degenerate metrics it becomes the usual covariant derivative. We denote the space of these sections by $\mathcal{A}^\bullet(M)$. Similarly, we define the covariant derivative of a tensor $T \in \bigotimes^k T^\bullet M$, by $(\nabla_X T)(Y_1, \ldots, Y_k) = X(T(Y_1, \ldots, Y_k)) - \sum_{i=1}^k \mathcal{K}(X, Y_i, {}_\bullet) \quad T(Y_1, \ldots, {}_\bullet, \ldots, Y_k)$, where ${}_\bullet$ stands for contraction with $g_\bullet$.

A *semi-regular semi-Riemannian manifold* is defined by the condition that vector fields admit double covariant derivative, $\nabla_X \nabla^\flat_Y Z \in \mathcal{A}^\bullet(M)$. This is equivalent to $\mathcal{K}(X, Y, {}_\bullet)\mathcal{K}(Z, T, {}_\bullet) \in \mathcal{F}(M)$. We can define the *Riemann curvature* by

$$R(X, Y, Z, T) = (\nabla_X \nabla^\flat_Y Z)(T) - (\nabla_Y \nabla^\flat_X Z)(T) - (\nabla^\flat_{[X,Y]} Z)(T). \tag{8}$$

The Riemann curvature defined above is a smooth radical annihilator tensor field and has the same symmetry properties as the usual Riemann curvature [3].

The Ricci decomposition

$$R_{abcd} = S_{abcd} + E_{abcd} + C_{abcd} \tag{9}$$

where $S_{abcd} = \frac{1}{n(n-1)} R(g \circ g)_{abcd}$, $E_{abcd} = \frac{1}{n-2}(S \circ g)_{abcd}$, $S_{ab} := R_{ab} - \frac{1}{n} R g_{ab}$, and $(h \circ k)_{abcd} := h_{ac}k_{bd} - h_{ad}k_{bc} + h_{bd}k_{ac} - h_{bc}k_{ad}$, holds as well for degenerate metrics [9]. If the Ricci decomposition is such that all of the terms are smooth, then the metric is called *quasi-regular*. Examples include *isotropic singularities* (obtained by conformally scaling a non-degenerate metric), *degenerate warped products* $B \times_f F$ with $\dim B = 1$ and $\dim F = 3$ (in particular, FLRW singularities), Schwarzschild singularities [9]. The *Weyl curvature tensor* $C_{abcd} = R_{abcd} - S_{abcd} - E_{abcd}$ satisfies $C_{abcd} \to 0$ as approaching a quasi-regular singularity [10].

In dimension $n = 4$, if the metric is quasi-regular, we can cast Einstein's equation in the form

$$(G \circ g)_{abcd} + \Lambda(g \circ g)_{abcd} = \kappa(T \circ g)_{abcd}. \tag{10}$$

This is equivalent to Einstein's equation if the metric is non-degenerate, but in addition it also holds smoothly at quasi-regular singularities [9].

We apply now the new methods to the *Friedmann-Lemaître-Robertson-Walker* spacetime, which is the warped product $I \times_a \Sigma$ between a three dimensional Riemannian manifold (Σ, g_Σ) (representing the space) and a one-dimensional Riemannian manifold $(I, -\mathrm{d}t^2)$,

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\mathrm{d}\Sigma^2. \tag{11}$$

In general the warping function $a \in \mathcal{F}(I)$ is taken $a(t) > 0$ for every $t \in I$. Here we allow $a(t) \geq 0$, including possible singularities, which turn out to be quasi-regular [11, 12]. By taking the time component and the trace of Einstein's equation, we get the *Friedmann equation*, the *acceleration equation*, and the *fluid equation* expressing the conservation of mass-energy,

$$\rho = \frac{3}{\kappa}\frac{\dot{a}^2 + k}{a^2}, \ \ \rho + 3p = -\frac{6}{\kappa}\frac{\ddot{a}}{a}, \ \ \dot{\rho} = -3\frac{\dot{a}}{a}(\rho + p). \tag{12}$$

We see that ρ, and in general also p, become *singular* for $a = 0$. This is because they are defined in an *orthonormal frame*, but when $a = 0$, the metric (11) is degenerate, and there is no orthonormal frame.

To obtain the total mass at t, one integrates the 3-*form* $\rho \mathrm{d}_{vol_3}$, where $\mathrm{d}_{vol_3} := \sqrt{g_{\Sigma_t}}\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z = a^3\sqrt{g_\Sigma}\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z = \mathrm{i}_{\partial_t}\mathrm{d}_{vol}$, where d_{vol} is the *volume form*, or *volume element*, $\mathrm{d}_{vol} := \sqrt{-g}\mathrm{d}t \wedge \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z = a^3\sqrt{g_\Sigma}\mathrm{d}t \wedge \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$.

Hence, one should rewrite the equations using 4-forms or scalar densities $\widetilde{\rho} = \rho\sqrt{-g} = \rho a^3\sqrt{g_\Sigma}$ and $\widetilde{p} = p\sqrt{-g} = pa^3\sqrt{g_\Sigma}$, which luckily can be defined in any coordinates/frames. The Friedmann equation and the acceleration equation become

$$\widetilde{\rho} = \frac{3}{\kappa} a \left(\dot{a}^2 + k\right) \sqrt{g_\Sigma}, \;\; \widetilde{\rho} + 3\widetilde{p} = -\frac{6}{\kappa} a^2 \ddot{a} \sqrt{g_\Sigma}. \tag{13}$$

Hence, $\widetilde{\rho}$ and $\widetilde{p}$ are smooth, as it is the densitized stress-energy tensor

$$T_{ab}\sqrt{-g} = (\widetilde{\rho} + \widetilde{p})\, u_a u_b + \widetilde{p} g_{ab}. \tag{14}$$

The equations are smooth in any frame. Consequently, the Einstein tensor density $G_{ab}\sqrt{-g}$ is smooth too, so we can use a densitized version of Einstein's equation, which has all involved quantities finite at the singularity.

The case of black-hole singularities is a bit more difficult, because the singularity $r = 0$ is malign. The Schwarzschild metric is in Schwarzschild coordinates

$$\mathrm{d}s^2 = -\left(1 - \frac{2m}{r}\right)\mathrm{d}t^2 + \left(1 - \frac{2m}{r}\right)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\sigma^2, \tag{15}$$

where $\mathrm{d}\sigma^2 = \mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2$. The singularities $r = 0$ and $r = 2m$ are *malign*, since there g_{ab} has infinite components. The singularity $r = 2m$ can be removed by the Eddington–Finkelstein coordinates [13, 14]. But the singularity $r = 0$ cannot be removed like this. However it can be made semi-regular by the coordinate transformation $\begin{cases} r = \tau^2 \\ t = \xi\tau^4 \end{cases}$ [15]. The four-metric becomes:

$$\mathrm{d}s^2 = -\frac{4\tau^4}{2m - \tau^2}\mathrm{d}\tau^2 + (2m - \tau^2)\tau^4\,(4\xi\mathrm{d}\tau + \tau\mathrm{d}\xi)^2 + \tau^4\mathrm{d}\sigma^2, \tag{16}$$

which is smooth and analytic and quasi-regular at $r = 0$ [9, 15].

The new form of the Schwarzschild metric extends analytically at and beyond $r = 0$. It can be used to construct globally hyperbolic spacetimes with singularities, including evaporating black holes which preserve the spacetime structure [16].

For the Reissner–Nordström metric

$$\mathrm{d}s^2 = -\left(1 - \frac{2m}{r} + \frac{q^2}{r^2}\right)\mathrm{d}t^2 + \left(1 - \frac{2m}{r} + \frac{q^2}{r^2}\right)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\sigma^2, \tag{17}$$

we choose the coordinates ρ and τ, so that $\begin{cases} t = \tau\rho^T \\ r = \rho^S \end{cases}, \begin{cases} S \geq 1 \\ T \geq S + 1 \end{cases}.$

The metric has, in the new coordinates, the following form

$$\mathrm{d}s^2 = -\Delta\rho^{2T-2S-2}\,(\rho\mathrm{d}\tau + T\tau\mathrm{d}\rho)^2 + S^2\Delta^{-1}\rho^{4S-2}\mathrm{d}\rho^2 + \rho^{2S}\mathrm{d}\sigma^2, \tag{18}$$

where $\Delta = \rho^{2S} - 2m\rho^S + q^2$. In the new coordinates $(\tau, \rho, \phi, \theta)$ the metric is analytic at $r = 0$. The electromagnetic potential is $A = -q\rho^{T-S-1}\,(\rho\mathrm{d}\tau + T\tau\mathrm{d}\rho)$, and the electromagnetic field is $F = q(2T - S)\rho^{T-S-1}\mathrm{d}\tau \wedge \mathrm{d}\rho$, and they are analytic everywhere, including at the singularity $\rho = 0$ [17]. Similar results apply to the Kerr–Newman black holes [18].

We have seen thus that the use of degenerate metrics allowed us to remove infinities in important examples of spacetime singularities. Singularities are therefore not as bad as were considered. In addition, it turned out that they may be useful in quantum gravity. In various approaches to quantum gravity, the evidence accumulated so far suggests, or even requires, that in the UV limit there is a *dimensional reduction* to two dimensions [19]. What is under debate is the meaning, the nature, the explicit cause of this spontaneous dimensional reduction. The singularities are naturally accompanied by some of the dimensional reduction effects which were postulated *ad-hoc* in various approaches to quantum gravity. Consequently, if in the perturbative expansions one accounts for the fact that the point-particles in general relativity are spacetime singularities, the dimensional reduction effects appear naturally [20].

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The Heun Functions and Their Applications in Astrophysics

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Abstract The Heun functions are often called the hypergemeotry successors of the 21st century, because of the wide number of their applications. In this proceeding we discuss their application to the problem of perturbations of rotating and non-rotating black holes and highlight some recent results on their late-time ring-down obtained using those functions.

1 The Heun Functions

The Heun functions are gaining popularity due to the vast number of their applications. The Heun project, a site dedicated to gathering scientists working in this area, has already accumulated more than 500 articles on the theory and the applications of those functions. Among the topics are the Schrödinger equation with anharmoic potential, the Teukolsky linear perturbation theory for the Schwarzschild and Kerr metrics, transversable wormholes, quantum Rabi models, confinement of graphene electrons in different potentials, quantum critical systems, crystalline materials, three-dimensional atmospheric and ocean waves, single polymer dynamics, economics, genetics e.t.c (see the bibliography section in [10]).

The general Heun function is defined as the local solution of the following second order Fuchsian ordinary differential equation (ODE) [5, 6]:

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V. Dobrev (ed.), *Lie Theory and Its Applications in Physics*,
Springer Proceedings in Mathematics & Statistics 191,
DOI 10.1007/978-981-10-2636-2_20

$$\frac{d^2}{dz^2}H(z) + \left[\frac{\gamma}{z} + \frac{\delta}{z-1} + \frac{\epsilon}{z-a}\right]\frac{dH(z)}{dz} + \frac{\alpha\beta z - q}{z(z-1)(z-a)}H(z) = 0 \quad (1)$$

normalized to 1 at $z = 0$. Here $\epsilon = \alpha + \beta - \gamma - \delta + 1$. This equation posses 4 regular singularities: $z = 0, 1, a, \infty$ and it generalizes the hypergeometric function, the Lamé function, the Mathieu function, the spheroidal wave functions etc. Its group of symmetries is of order 192.

For comparison, the hypregeometric differential equation has 3 regular singularities $z(z-1)\frac{d^2w(z)}{dz^2} + [c - (a+b+1)z]\frac{dw(z)}{dz} - abw(z) = 0$ with group of symmetries of order 24.

Recalling the definition of irregular singularity:

Definition 1 For an ODE of the form: $P(x)y''(x) + Q(x)y'(x) + R(x)y(x) = 0$, the point x_0 is singular if Q(x)/P(x) or R(x)/P(x) diverge at $x = x_0$. If the limits $\lim_{x\to x_0}\frac{Q(x)}{P(x)}(x - x_0)$ and $\lim_{x\to x_0}\frac{R(x)}{P(x)}(x - x_0)^2$ exist and are finite then the point x_0 is regular singularity, otherwise, it is irregular or essential singularity. The point $x_0 = \infty$ is treated the same way under the change $x = 1/z$.

The general Heun function has 4 regular singularities, from which under the process called confluence of singularities, one obtains 4 different types of confluent Heun functions with fewer singularities but of higher s-rank (See Fig. 1 for illustration).

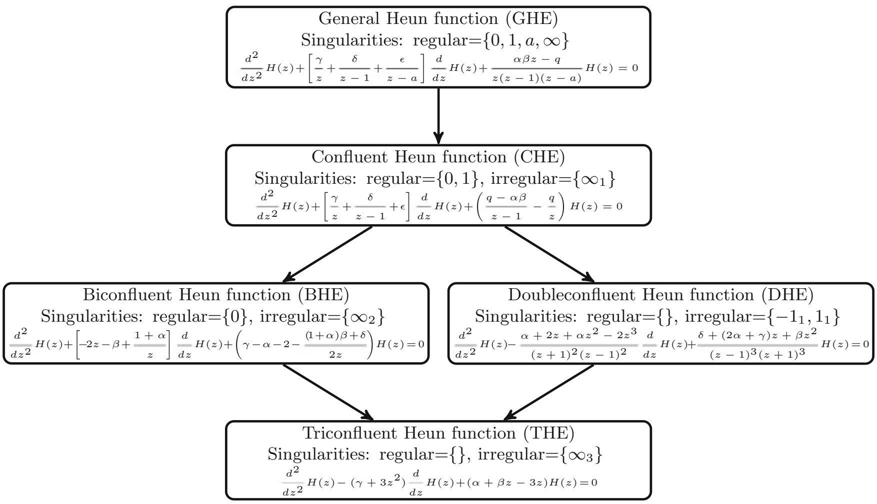

Fig. 1 A scheme of the different confluent ODEs obtainable from the ODE of the general Heun function (in Maple's notations). The subscript next to the irregular singularities is their rank

For the confluent Heun function which we will use below, this process means the redefinition of $\beta = \beta a$, $\epsilon = \epsilon a$, $q = qa$ and taking the limit $a \to \infty$. This gives us the following ODE:

$$\frac{d^2}{dz^2}H(z) - \left(\epsilon - \frac{\delta}{z-1} - \frac{\gamma}{z}\right)\frac{\mathrm{d}}{\mathrm{d}z}H(z) - \left(\frac{\alpha\,\beta - qz}{z-1} + \frac{q}{z}\right)H(z) = 0 \quad (2)$$

In Maple notations, the default form of the solution of this type of ODE is denoted as $HeunC(\alpha, \beta, \gamma, \delta, \eta, z)$ which we adopt. To obtain from Maple's default form Eq. 1, one needs to set $\alpha = -(\epsilon_0^2 - 4q_0)^{1/2}$, $\beta = \gamma_0 - 1$, $\gamma = -1 + \delta_0$, $\delta = -\alpha_0\beta_0 + (1/2)\delta_0\epsilon_0 + (1/2)\epsilon_0\gamma_0$, $\eta = -(1/2)\delta_0\gamma_0 - (1/2)\epsilon_0\gamma_0 + q_0 + 1/2$ and vise versa (the "$_0$" subscript denotes the parameters in Eq. 2.

2 Applications of the Heun Functions in Astrophysics

2.1 Teukolsky Angular Equation and Teukolsky Radial Equation

In the frame of the Teukolsky linear perturbations theory, the late-time ringing of a black hole due to a perturbation of different spin is described by one Master equation. Under the substitution $\Psi(t, r, \theta, \phi) = e^{i(\omega t + m\phi)}S(\theta)R(r)$ (where $m = 0, \pm 1, \pm 2$) this equation splits in two second order ODEs of the confluent Heun type – The Teukolsky Angular Equation (TAE):

$$\frac{d}{du}\left(\left(1-u^2\right)\frac{d}{du}S_{lm}(u)\right) + \left((a\omega u)^2 + 2a\omega s u + E - s^2 - \frac{(m+su)^2}{1-u^2}\right)S_{lm}(u) = 0, \quad (3)$$

and the Teukolsky Radial Equation (TRE):

$$\frac{d^2R_{l,m}(r)}{dr^2} + (1+s)\left(\frac{1}{r-r_+} + \frac{1}{r-r_-}\right)\frac{dR_{l,m}(r)}{dr} + +\left(\frac{K^2}{(r-r_+)(r-r_-)} - is\left(\frac{1}{r-r_+} + \frac{1}{r-r_-}\right)K - \lambda - 4is\omega r\right)\frac{R_{l,m}(r)}{(r-r_+)(r-r_-)} = 0 \quad (4)$$

where $\Delta = r^2 - 2Mr + a^2 = (r - r_-)(r - r_+)$, $K = -\omega(r^2 + a^2) - ma$, $\lambda = E - s(s+1) + a^2\omega^2 + 2am\omega$ and $u = \cos(\theta)$. Here $r_\pm = M \pm \sqrt{M^2 - a^2}$ are the inner and outer horizon of the rotating black hole. Being interested in electromagnetic perturbations we fix the spin to $s = -1$.

In this system, the unknown quantities are the complex frequencies $\omega_{l,m,n}$ giving us the spectrum and the constant of separation $E_{l,m,n}$ which for $a = 0$ is $E = l(l+1)$ (for $s = -1$). The only physical parameters of the system, in agreement with the

No-Hair Theorem, are the rotational parameter a and the mass of the black hole M, which we here fix to $M = 1/2$.

The singularities of the two equations are as follows: for the TRE $r = r_{\pm}$ – regular and $r = \infty$ – irregular. For the TAE, the regular singularities are: $\theta = \pm\pi$ and the irregular is again $\theta = \infty$.

2.2 Boundary Conditions

In order to find the spectrum, we need to solve the central two-point connection problem, imposing appropriate boundary conditions on two of the singular points. Details on the boundary conditions, as well as on the whole approach and the explicit values of the parameters, can be found in [1–4, 7–9]. In brief, we require:

1. On the TAE:
   a. Quasi-normal modes (QNMs): we require angular regularity. This translate into the following determinant

$$W[S_1, S_2] = \frac{\mathrm{HeunC}'(\alpha_1, \beta_1, \gamma_1, \delta_1, \eta_1, (\cos(\pi/6))^2)}{\mathrm{HeunC}(\alpha_1, \beta_1, \gamma_1, \delta_1, \eta_1, (\cos(\pi/6))^2)} + \frac{\mathrm{HeunC}'(\alpha_2, \beta_2, \gamma_2, \delta_2, \eta_2, (\sin(\pi/6))^2)}{\mathrm{HeunC}(\alpha_2, \beta_2, \gamma_2, \delta_2, \eta_2, (\sin(\pi/6))^2)} + p = 0 \tag{5}$$

   where details on the parameters can be seen in [1, 4, 7, 8].
   b. Jet modes: A qualitatively new boundary condition has been used in [8] to obtain the so-called primary jet modes. The condition was that of angular singularity which translates into polynomial condition for the solutions of the TAE, i.e.:

$$\frac{\delta}{\alpha} + \frac{\beta + \gamma}{2} + N + 1 = 0$$
$$\Delta_{N+1}(\mu) = 0$$

   where $\Delta_{N+1}(\mu)$ is tridiagonal determinant [3].

2. On the TRE:
   a. Black hole boundary conditions: For any m, the solution R_2 is valid for frequencies for which $\Re(\omega) \notin (-\frac{ma}{2Mr_+}, 0)$ and also that: $\sin(\arg(\omega) + \arg(r)) < 0$.
   b. Quasi-bound boundary conditions: For any m, the solution R_1 is valid for frequencies for which $\Re(\omega) \notin (-\frac{ma}{2Mr_+}, 0)$ and also that: $\sin(\arg(\omega) + \arg(r)) > 0$.

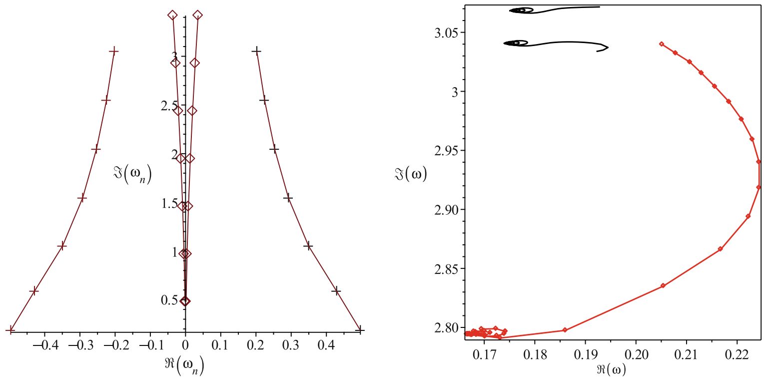

Fig. 2 Examples of the different spectra obtained from the spectral system. **a** Complex plot of the first 7 modes in the QNMs (crosses) and primary jet modes (diamonds) **b** QNMs (point-line) and the non-physical spurious modes (solid lines) for $a = [0, M]$

2.3 Numerical Results

The so described boundary conditions lead to a two-dimensional spectral system on the unknowns ω and E. Because of the complexity of the confluent Heun functions, we use an algorithm developed by the team to find the roots of the system. The numerical results give different spectra of discrete complex frequencies some of which can be seen on Fig. 2. As part of our study, we examined how those spectra change with introduction of rotation ($a \neq 0$), up to the limit $a \rightarrow M$, and we tested the numerical stability of the so-obtained frequencies, in order to ensure they represent physical quantities and not a numerical artifact (an example can be seen on Fig. 2b).

The physically interesting results are the qualitatively different spectra (Fig. 2a), depending on the boundary conditions imposed on the system, which can be used as an independent tool to discover the nature of the physical object emitting electromagnetic or gravitational waves.

3 Conclusion

In this proceeding we discussed the application of the Heun functions to the problem of quasinormal modes of rotating and non-rotating black holes. We presented some of our latest numerical results, key to which is the development of the theory of the Heun functions and their numerical implementation.

Acknowledgements This article was supported by the Foundation "Theoretical and Computational Physics and Astrophysics", by the Bulgarian National Scientific Fund under contracts DO-1-872, DO-1-895, DO-02-136, and Sofia University Scientific Fund, contract 185/26.04.2010, Grants of the Bulgarian Nuclear Regulatory Agency for 2013, 2014 and 2015.

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Part III
Integrable Systems

Boundary Effects on the Supersymmetric Sine-Gordon Model Through Light-Cone Lattice Regularization

Chihiro Matsui

Abstract In this report, we discuss how the boundary condition of the spin-1 XXZ chain affects its low-energy effective field theory. The low-energy effective field theory of the spin-1 XXZ model is known as the supersymmetric (SUSY) sine-Gordon model. As a SUSY model, the theory consists of two subspaces called the Neveu–Schwarz (NS) sector and the Ramond (R) sector. In the Bethe-ansatz contest, the spin chain and its effective field theory are connected via the light-cone lattice regularization in the sense that these two models share the same transfer matrices. Conversely, the effective field theory is obtained in the scaling limit of the spin chain. Using the nonlinear integral equations (NLIEs) for the eigenvalues of the transfer matrices, we derived the scattering matrices of the SUSY sine-Gordon model from the large volume limit analysis of the spin-1 XXZ chain with boundary magnetic fields. At the same time, we derived the conformal dimensions of the SUSY sine-Gordon model in the small volume limit. From these quantities, we found that the different sector of the SUSY sine-Gordon model is realized from the spin-1 XXZ chain depending on the values of boundary magnetic fields.

1 Introduction

Since any real material is a finite-size system, it is important to know boundary effects on physical quantities. To study finite-size systems is important also because they show interesting features such as edge states and boundary critical exponents. Nevertheless, existence of boundaries often destroys good symmetry obtained for infinite systems. This makes it difficult to analyze systems with boundaries.

For the reasons described above, it would be nice to find good symmetries which hold for finite-size systems. One of such examples is the integrable boundary system, whose exact solvability is ensured by the Yang–Baxter equation and the reflection relation [1, 2]. Due to the Yang–Baxter equation and the reflection relation, many-

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V. Dobrev (ed.), *Lie Theory and Its Applications in Physics*,
Springer Proceedings in Mathematics & Statistics 191,
DOI 10.1007/978-981-10-2636-2_21

body scatterings are decomposed into a sequence of two-body scatterings which allows us to find exact scattering and reflection matrices.

An example which possesses these symmetries is the XXZ spin chain associated with $U_q(sl_2)$. The R-matrix is obtained as a solution of the Yang–Baxter equation of the $U_q(sl_2)$-type, while the K-matrices as the diagonal solutions of the reflection relation describe the boundary reflections under boundary magnetic fields. Another example is the sine-Gordon (SG) type model with Dirichlet boundary conditions. The model is obtained through bosonization of the spin chain with boundary magnetic fields. The model serves as the low-energy effective field theory of the spin chain. These two models share the same R-matrix and the K-matrices associated with the $U_q(sl_2)$ [3, 4].

Different methods have been developed for spin chains and quantum field theories, since the former are discrete systems, while the latter are continuum models. The transfer matrix method is often used to solve spin chains by regarding a two-dimensional lattice with time sequence of the transfer matrix. The Bethe-ansatz method is one of the most successful methods to diagonalize a transfer matrix [5]. This method is applied to a system with non-trivial boundaries, as long as the reflection relation holds for the system. For instance, the XXZ model with boundary magnetic fields was solved by the coordinate Bethe-ansatz method [6] and then the method was algebraically reformulated for the diagonal boundary case by introducing the double-row transfer matrix [2].

On the other hand, exact analysis of a continuum theory has been achieved through the bootstrap approach [7]. This method allows us to compute a scattering matrix between any asymptotic states including bound states subsequently from a scattering between asymptotic soliton states. The S-matrix between asymptotic soliton states satisfies the Yang–Baxter equation. The boundary reflection matrix for the asymptotic soliton state is obtained as the solution of the reflection relation. Analogous to the bulk case, the boundary bootstrap principle was developed which subsequently gives reflection matrices for the asymptotic boundary bound states [8].

If one knows the complete correspondence between a spin chain and a quantum field theory, the methods independently developed for discrete and continuum systems can be applied to each other. Therefore, our aim is to derive exact correspondence between a spin chain and a quantum field theory.

In this report, we consider to discretize a quantum field theory, instead of taking the continuum limit of a spin chain. The notion to discretize an integrable quantum field theory was used in the context of the quantum inverse scattering method first in [9]. Among various discretization, we employ the light-cone lattice regularization [10–12]. The light-cone lattice regularization is achieved by setting discrete trajectories of particles. A discretized light-cone then looks like a two-dimensional lattice system. A right-mover runs over a line from left-bottom to right-top, while a left-mover runs over a line from right-bottom to left-top. A scattering occurs only at a vertex. The amplitudes depend on the states of four legs around a vertex, i.e. the presence or absence of a particle. Thus, the scattering amplitudes of the quantum field theory are regarded as the Boltzmann weights of the two-dimensional lattice, i.e. the R-matrix of the spin chain, through the light-cone lattice regularization.

Therefore, through the light-cone lattice regularization, characteristic quantities in quantum field theories, such as S-matrices and conformal dimensions, are derived through the diagonalization of the transfer matrices defined on the light-cone lattice. The diagonalization of the transfer matrices is achieved by two methods. The first one is based on the physical Bethe-ansatz equations, the derivation of which requires the string hypothesis [13–17]. The second is to use the nonlinear integral equations (NLIEs) derived from the analyticity structure of the eigenvalue functions of the transfer matrices [18–21]. In the framework of this method, there is no need to assume the string solutions to the Bethe-ansatz equations. Since Bethe strings deviate at the order of the inverse system size, we use the latter method throughout this report in order to deal with the finite-size system.

Correspondence between the spin chain and the quantum field theory for the spin-$\frac{1}{2}$ case has been closely studied for both the periodic boundary case and the open boundary case with boundary magnetic fields [22–24]. What was found for the periodic spin chain is that the conformal dimension of the sine-Gordon (SG) model admits only an even winding number through the light-cone lattice approach. This is due to the technical problem of the light-cone lattice regularization, which requires the spin chain of even length. Later in [24], it has been suggested that an odd winding number is obtained from the spin-$1/2$ XXZ chain consisting of odd length, although it has not been found yet how to construct the transfer matrix for the odd-length chain on the light-cone lattice. On the other hand, it was found that the spin chain with boundary magnetic fields results in the SG model with Dirichlet boundaries [22, 23]. In this case, the allowed winding number relies on the values of boundary parameters.

Our interest is to find more variation of the exact correspondence between spin chains and quantum field theories. Here we focus on the boundary effects on the supersymmetric sine-Gordon (SSG) model [25]. The SSG model [26] is obtained as the low-energy effective field theory of the spin-1 Zamolodchikov–Fateev (ZF) spin chain [27–29]. Although the SSG model has been discussed through the light-cone approach for both the periodic case [30–32] and the Dirichlet boundary case [28], only the NS sector was obtained, since the authors of [28] limited the range of boundary parameters. Performing analytic continuation, we obtained distinct three regimes of boundary parameters each of which is characterized by a different NLIE. From each set of NLIEs, we derived the scattering matrix and the conformal dimension. We found the Ramond sector in one of three parameter regimes [33].

The plan of this report is as follows. Throughout this report, we analyze the SSG model with Dirichlet boundaries. We focus on the repulsive regime where no breather exists. In Sect. 2, we introduce the SSG model and review known results from both viewpoints of the perturbation of the conformal field theory (CFT) and the integrable quantum field theory. The light-cone lattice regularization is also explained in this section in connection with the spin chain. In Sect. 3, we derive the NLIEs of the spin-1 ZF chain. We show the three distinct regimes of boundary parameters, each of which is characterized by a different NLIE. In the next section, scattering and reflection amplitudes are discussed by taking the infrared (IR) limit. We show that the different NLIEs for three boundary regimes are connected via the boundary bootstrap method.

Then in Sect. 5, the ultraviolet (UV) limit is considered. Conformal dimensions are computed for three distinct regimes and then we show one of them belongs to the R sector. The last section is devoted to concluding remarks and future works.

2 SSG Model with Dirichlet Boundary Conditions

The SSG model is an integrable $(1+1)$-dimensional quantum field theory consisting of a real scalar field Φ and a Majorana fermion Ψ. On a finite system size L, the action of the SSG model is given by

$$\begin{aligned} \mathcal{A}_{\rm SSG} &= \int_{-\infty}^{\infty} dt \int_0^L dx\ \mathcal{L}_{\rm SSG}(x;t), \\ \mathcal{L}_{\rm SSG} &= \frac{1}{2}\partial_\mu \Phi \partial^\mu \Phi + \frac{i}{2}\bar{\Psi}\gamma^\mu \partial_\mu \Psi - \frac{m_0}{2}\cos(\beta\Phi)\bar{\Psi}\Psi + \frac{m_0^2}{2\beta^2}\cos^2(\beta\Phi), \end{aligned} \tag{1}$$

where

$$\Psi = \begin{pmatrix} \psi \\ \bar{\psi} \end{pmatrix}, \quad \gamma^0 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}. \tag{2}$$

A mass parameter m_0 determined in such a way that realizes a proper scaling limit [34] is related to the physical soliton mass via the relation found in [35].

The theory behaves differently depending on a value of the coupling constant β. In the attractive regime $(0 < \beta^2 < \frac{4\pi}{3})$, solitons form bound states called breathers, while the repulsive regime $(\frac{4\pi}{3} < \beta^2 < 4\pi)$ does not admit breathers. Throughout this report, we concentrate on the repulsive regime.

The Dirichlet boundary conditions are given by fixing the value of a scalar field at the boundaries:

$$\Phi(0;t) = \Phi_-, \qquad \Phi(L;t) = \Phi_+. \tag{3}$$

2.1 SSG Model as a Perturbed CFT

From a viewpoint of the renormalization group theory, the SSG model is a perturbation from the $\mathcal{N}=1$ superconformal field theory [30]. The $\mathcal{N}=1$ superconformal field theory consists of free bosons and free fermions compactified on a cylinder with radius $R = \frac{4\sqrt{\pi}}{\beta}$. The third term in the Lagrangian (1) serves as an irrelevant perturbation in the UV limit ($m_0 L \to 0$). In an arbitrary compactification radius, the following two boundary conditions are allowed for a fermion:

$$\Psi(0;t) = \Psi(L;t), \qquad \bar{\Psi}(0;t) = \bar{\Psi}(L;t), \tag{4}$$
$$\Psi(0;t) = -\Psi(L;t), \qquad \bar{\Psi}(0;t) = -\bar{\Psi}(L;t) \tag{5}$$

where the first condition is called the NS boundary condition, while the second is called the R boundary condition.

The highest weight vectors of the current algebra $|m,n\rangle$ are generated from the vacuum $|0,0\rangle$ by using the vertex operator:

$$|m,n\rangle = V_{(m,n)}(z,\bar{z})|0,0\rangle, \tag{6}$$
$$V_{(m,n)} =: e^{i(mR+\frac{n}{R})\phi(z)+i(mR-\frac{n}{R})\bar{\phi}(\bar{z})} : \bar{\Psi}\Psi. \tag{7}$$

Here we used the normal order $: * :$. ϕ and $\bar{\phi}$ are the holomorphic and anti-holomorphic part of a normalized boson defined by

$$\Phi = \frac{1}{4\sqrt{\pi}}(\phi(z) + \bar{\phi}(\bar{z})). \tag{8}$$

Using the vertex operator (7), one finds that the perturbation term in (1) is the primary operator proportional to $V_{(1,0)} + V_{(-1,0)}$.

The winding number m represents the number of windings of a boson which is compactified on a cylinder with radius R (Fig. 1). The momentum number n in (7) must be zero in the presence of boundaries since there is no momentum flux at the boundary.

The energy is given by

$$E(L) = -\frac{\pi}{24L}(c - 24\Delta) + \mathcal{O}(L^{-2}), \tag{9}$$

where the central charge c and the conformal dimension Δ consist of the boson part and fermion part:

$$c = c_{\mathrm{B}} + c_{\mathrm{F}}, \qquad \Delta = \Delta_{\mathrm{B}} + \Delta_{\mathrm{F}}, \tag{10}$$

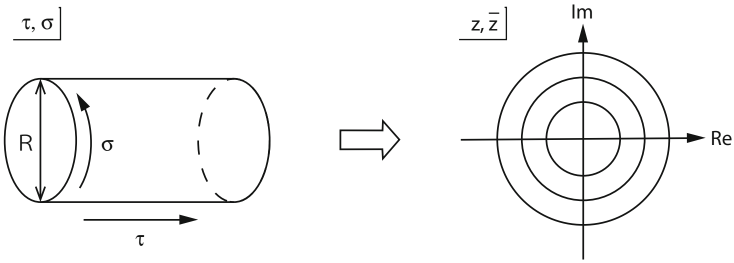


Fig. 1 A conformal map from a cylinder with radius R onto a complex plane. The *circles* in the complex plane represent the contours with respect to time τ

each of which is written by using the winding number and compactification radius:

$$\begin{aligned} &c_{\mathrm{B}} = 1, \qquad c_{\mathrm{F}} = \tfrac{1}{2}, \\ &\varDelta_{\mathrm{B}} = \tfrac{1}{2}\Big(\tfrac{1}{\sqrt{\pi}}(\varPhi_+ - \varPhi_-) + mR\Big)^2, \qquad \varDelta_{\mathrm{F}} = 0, \tfrac{1}{2}, \tfrac{1}{16}, \end{aligned} \tag{11}$$

where $\varDelta_{\mathrm{F}} = 0, \frac{1}{2}$ is for the NS sector and $\varDelta_{\mathrm{F}} = \frac{1}{16}$ is for the R sector.

2.2 *Scattering Theory of the SSG Model*

Supersymmetric solitons are generated by non-commuting operators $A^{\epsilon_j}_{a_j a_{j+1}}$. The superscript denotes a soliton charge $\epsilon_j \in \{\pm\}$, while the subscript denotes an RSOS index $a_j \in \{0, \pm 1\}$. A set of the RSOS indices must satisfies the adjacency condition $|a_j - a_{j+1}| = 1$.

2.2.1 Bulk *S*-Matrix

The soliton creation operator satisfies the following commutation relations:

$$A^{\epsilon_1}_{ab}(\theta_1)A^{\epsilon_2}_{bc}(\theta_2) = \sum_{\epsilon_1', \epsilon_2'} \sum_{d} S^{\epsilon_1\epsilon_2}_{\epsilon_1'\epsilon_2'}\big|^{ac}_{bd}(\theta_1 - \theta_2) A^{\epsilon_2'}_{ad}(\theta_2) A^{\epsilon_1'}_{dc}(\theta_1), \tag{12}$$

which gives the scattering amplitudes between solitons. The parameter θ_j is rapidity of a supersymmetric soliton.

Since the SSG model is an integrable quantum field theory, the *S*-matrix satisfies the Yang–Baxter equation. The *S*-matrix of the SSG model is decomposed into a tensor product of the SG part and the RSOS part [17, 36]:

$$S^{\epsilon_1\epsilon_2}_{\epsilon_1'\epsilon_2'}\big|^{ac}_{bd}(\theta) = S^{\epsilon_1\epsilon_2}_{\epsilon_1'\epsilon_2'}(\theta) \times S^{ac}_{bd}(\theta). \tag{13}$$

As a result, the SG part and the RSOS part independently satisfy the Yang–Baxter equation:

$$\begin{aligned} &S^{\epsilon_1\epsilon_2}_{\epsilon_1'\epsilon_2'}(\theta_1 - \theta_2) S^{\epsilon_1'\epsilon_3}_{\epsilon_1''\epsilon_3'}(\theta_1 - \theta_3) S^{\epsilon_2'\epsilon_3'}_{\epsilon_2''\epsilon_3''}(\theta_2 - \theta_3) \;= \\ &\;=\; S^{\epsilon_2\epsilon_3}_{\epsilon_2'\epsilon_3'}(\theta_2 - \theta_3) S^{\epsilon_1\epsilon_3'}_{\epsilon_1'\epsilon_3''}(\theta_1 - \theta_3) S^{\epsilon_1'\epsilon_2'}_{\epsilon_1''\epsilon_2''}(\theta_1 - \theta_2), \end{aligned} \tag{14}$$

$$\begin{aligned} &S^{ac}_{bg}(\theta_1 - \theta_2) S^{gd}_{ce}(\theta_1 - \theta_3) S^{ac}_{bg}(\theta_2 - \theta_3) \;= \\ &\;=\; S^{bd}_{cg'}(\theta_2 - \theta_3) S^{ag'}_{bf}(\theta_1 - \theta_3) S^{fd}_{g'e}(\theta_1 - \theta_2). \end{aligned} \tag{15}$$

The solution to the SG part has been derived in [7, 37]:

$$S_{\epsilon\epsilon}^{\epsilon\epsilon}(\theta) = S(\theta), \tag{16}$$
$$S_{\epsilon-\epsilon}^{\epsilon-\epsilon}(\theta) = \frac{\sinh \lambda\theta}{\sinh \lambda(i\pi - \theta)} S(\theta), \quad S_{\epsilon-\epsilon}^{-\epsilon\epsilon}(\theta) = i\frac{\sin \pi\lambda}{\sinh \lambda(i\pi - \theta)} S(\theta),$$

where $\epsilon \in \{\pm\}$. The solution is closely related to the R-matrix of the six-vertex model [37]. By setting $u = i\theta$, the overall factor $S(\theta)$ is written by

$$S(\theta) = -\prod_{l=1}^{\infty} \frac{\Gamma(2(l-1)\lambda - \frac{\lambda u}{\pi})\Gamma(2l\lambda + 1 - \frac{\lambda u}{\pi})}{\Gamma((2l-1)\lambda - \frac{\lambda u}{\pi})\Gamma((2l-1)\lambda + 1 - \frac{\lambda u}{\pi})} / (u \to -u) \tag{17}$$
$$= \exp\left[i\int_0^{\infty} \frac{dt}{t} \frac{\sin \frac{\theta t}{\pi} \sinh(\frac{1}{\lambda} - 1)\frac{t}{2}}{\cosh \frac{t}{2} \sinh \frac{t}{2\lambda}}\right]. \tag{18}$$

The parameter λ is determined by a coupling constant β via $\lambda = \frac{2\pi}{\beta^2} - \frac{1}{2}$ [28].

The solution to the RSOS part has been derived in [8, 38–40]:

$$S_{bd}^{ac}(\theta) = X_{bd}^{ac}(\theta)K(\theta), \tag{19}$$

where

$$X_{00}^{\pm\pm}(\theta) = 2^{(i\pi-\theta)/2\pi i}\cos\left(\frac{\theta}{4i} - \frac{\pi}{4}\right), \qquad X_{\pm\pm}^{00}(\theta) = 2^{\theta/2\pi i}\cos\left(\frac{\theta}{4i}\right), \tag{20}$$
$$X_{00}^{\pm\mp}(\theta) = 2^{(i\pi-\theta)/2\pi i}\cos\left(\frac{\theta}{4i} + \frac{\pi}{4}\right), \qquad X_{\pm\mp}^{00}(\theta) = 2^{\theta/2\pi i}\cos\left(\frac{\theta}{4i} - \frac{\pi}{2}\right).$$

The overall factor $K(\theta)$ is written by

$$K(\theta) = \frac{1}{\sqrt{\pi}} \prod_{k-1}^{\infty} \frac{\Gamma(k - \frac{1}{2} + \frac{\theta}{2\pi i})\Gamma(k - \frac{\theta}{2\pi i})}{\Gamma(k + \frac{1}{2} - \frac{\theta}{2\pi i})\Gamma(k + \frac{\theta}{2\pi i})} \tag{21}$$
$$= \frac{-i}{\sqrt{2}\sinh \frac{\theta - i\pi}{4}} \exp\left[i\int_0^{\infty} \frac{dt}{t} \frac{\sin \frac{\theta t}{\pi} \sinh \frac{3t}{2}}{\sinh 2t \cosh \frac{t}{2}}\right]. \tag{22}$$

2.2.2 Boundary Reflection Matrix

In a finite and non-periodic system, a soliton is reflected at a boundary. The soliton creation operator obeys the reflection relation given by

$$A_{ab}^{\epsilon}(\theta)B = \sum_{c}\sum_{\epsilon'} R_{\epsilon'}^{\epsilon}|_{ac}^{b} A_{bc}^{\epsilon'}(-\theta)B. \tag{23}$$

Here we denote the boundary creation operator by B.

As in the case of the bulk S-matrix, the reflection matrix of the SSG model is decomposed into a tensor product of the SG part and the RSOS part [39]:

$$R^{\epsilon}_{\epsilon'}|^{c}_{ab}(\theta) = R^{\epsilon}_{\epsilon'}(\theta) \times R^{c}_{ab}(\theta). \tag{24}$$

Thus, the reflection relation independently holds for the SG part and the RSOS part:

$$\begin{aligned}
&S^{\epsilon_1\epsilon_2}_{\epsilon_2'\epsilon_1'}(\theta_1-\theta_2)R^{\epsilon_2'}_{\epsilon_2''}(\theta_2)S^{\epsilon_2''\epsilon_1'}_{\epsilon_1''\epsilon_2'''}(\theta_1+\theta_2)R^{\epsilon_1''}_{\epsilon_1'''}(\theta_1) \;=\\
&=\; R^{\epsilon_1}_{\epsilon_1'}(\theta_1)S^{\epsilon_1'\epsilon_2}_{\epsilon_2''\epsilon_1''}(-\theta_1-\theta_2)R^{\epsilon_2'}_{\epsilon_2''}(\theta_2)S^{\epsilon_2'\epsilon_1''}_{\epsilon_1'''\epsilon_2'''}(-\theta_1+\theta_2),
\end{aligned} \tag{25}$$

$$\begin{aligned}
&S^{ac}_{bf}(\theta_1-\theta_2)R^{f}_{ag}(\theta_2)S^{gc}_{fd}(\theta_1+\theta_2)R^{d}_{ge}(\theta_1) \;=\\
&=\; R^{b}_{af'}(\theta_1)S^{f'c}_{bg'}(-\theta_1-\theta_2)R^{g'}_{f'e}(\theta_2)S^{ec}_{g'd}(-\theta_1+\theta_2).
\end{aligned} \tag{26}$$

The solution which leads to the Dirichlet boundaries is given by the diagonal matrix [8]:

$$R^{\pm}_{\pm}(\theta) = \cos(\xi \pm \lambda u)R_0(u)\frac{\sigma(\theta,\xi)}{\cos\xi}, \tag{27}$$

where $R_0(u)$ is given by

$$R_0(u) = \prod_{l=1}^{\infty}\left[\frac{\Gamma(4l\lambda-\frac{2\lambda u}{\pi})\Gamma(4\lambda(l-1)+1-\frac{2\lambda u}{\pi})}{\Gamma((4l-3)\lambda-\frac{2\lambda u}{\pi})\Gamma((4l-1)\lambda+1-\frac{2\lambda u}{\pi})}/(u\to -u)\right]. \tag{28}$$

The overall factor $\sigma(\theta,\xi)$ is written by [41, 42]

$$\begin{aligned}
\sigma(\theta,\xi) &= \frac{\cos\xi}{\cos(\xi+\lambda u)}\prod_{l=1}^{\infty}\left\{\left[\frac{\Gamma(\frac{1}{2}+\frac{\xi}{\pi}+(2l-1)\lambda-\frac{\lambda u}{\pi})}{\Gamma(\frac{1}{2}-\frac{\xi}{\pi}+(2l-2)\lambda-\frac{\lambda u}{\pi})}\times\right.\right.\\
&\qquad\left.\left.\times\frac{\Gamma(\frac{1}{2}-\frac{\xi}{\pi}+(2l-1)\lambda-\frac{\lambda u}{\pi})}{\Gamma(\frac{1}{2}+\frac{\xi}{\pi}+2l\lambda-\frac{\lambda u}{\pi})}\right]/(u\to -u)\right\}\\
&= \frac{\cos\xi}{\cos(\xi+\lambda u)}(R_1^{\pm}(\theta)+R_2(\theta)),
\end{aligned} \tag{29}$$

where

$$R_1^{+}(\theta) = \exp\left[i\int_0^{\infty}\frac{dt}{t}\left(\frac{\sinh(1-\frac{2\xi}{\pi\lambda})\frac{t}{2}}{2\sinh\frac{t}{2\lambda}\cosh\frac{t}{2}}+\frac{\sinh(\frac{\xi}{\pi}-\lfloor\frac{\xi}{\pi}-\frac{1}{2}\rfloor-1)\frac{t}{\lambda}}{\sinh\frac{t}{2\lambda}}\right)\sin\frac{\theta t}{\pi}\right] \tag{30}$$

$$R_1^{-}(\theta) = \exp\left[i\int_0^{\infty}\frac{dt}{t}\frac{\sinh(1-\frac{2\xi}{\pi\lambda})\frac{t}{2}}{2\sinh\frac{t}{2\lambda}\cosh\frac{t}{2}}\sin\frac{\theta t}{\pi}\right], \tag{31}$$

$$R_2(\theta) = \exp\left[i \int_0^\infty \frac{dt}{t} \frac{\sinh \frac{3t}{4} \sinh(\frac{1}{\lambda} - 1)\frac{t}{4}}{\sinh t \sinh \frac{t}{4\lambda}} \right]. \tag{32}$$

A boundary parameter ξ is connected to field values at boundaries through $\xi_\pm = \frac{2\pi}{\beta}\Phi_\pm$ [28].

The solution of the RSOS part has been derived in [39, 40]. Different solutions were obtained for two sectors of the superconformal field theory. For the NS sector, the solution is given by

$$R^0_{\sigma\sigma}(\theta;\xi) = P(\theta;\xi), \tag{33}$$

$$R^{\pm 1}_{00}(\theta;\xi) = \left(\cos\frac{\xi}{2} \pm i \sinh\frac{\theta}{2}\right) 2^{i\theta/\pi} K(\theta - i\xi) K(\theta + i\xi) P(\theta;\xi), \tag{34}$$

where

$$P(\theta,\xi) = \frac{\sin\xi - i\sinh\theta}{\sin\xi + i\sinh\theta} P_0(\theta), \tag{35}$$

$$P_0(\theta) = \prod_{k=1}^\infty \left[\frac{\Gamma(k - \frac{\theta}{2\pi i})\Gamma(k - \frac{\theta}{2\pi i})}{\Gamma(k - \frac{1}{4} - \frac{\theta}{2\pi i})\Gamma(k + \frac{1}{4} - \frac{\theta}{2\pi i})} / (\theta \to -\theta) \right] \tag{36}$$

$$= \exp\left(-\frac{\theta}{2\pi} \ln 2 + \frac{1}{8} \int_0^\infty \frac{dt}{t} \frac{\sin \frac{2\theta t}{\pi}}{\cosh^2 t \cosh^2 \frac{t}{2}} \right). \tag{37}$$

Thus only diagonal matrix elements are non-zero in the reflection matrix of the NS sector.

On the other hand, the solution to the R sector is obtained as

$$R^0_{\sigma\sigma}(\theta;\xi) = \cos\frac{\xi}{2} K(\theta - i\xi) K(\theta + i\xi) P(\theta;\xi), \tag{38}$$

$$R^0_{-\sigma\sigma}(\theta;\xi) = -ir^\sigma \sinh\frac{\theta}{2} K(\theta - i\xi) K(\theta + i\xi) P(\theta;\xi), \tag{39}$$

$$R^\sigma_{00}(\theta;\xi) = 2^{i\theta/\pi} P(\theta;\xi). \tag{40}$$

Unlike the NS sector, the reflection matrix of the R sector has non-diagonal elements $R^0_{\pm\mp}(\theta;\xi)$. The matrices (38)–(40) have the block diagonal structure. The subspace which includes the non-diagonal elements is diagonalized with the eigenvalues $\cos\frac{\xi}{2} \pm i \sinh\frac{\theta}{2}$. Thus, we obtain the common eigenvalues for the NS and R sectors from the reflection matrix up to the factor $2^{i\theta/\pi}$, which can be removed by a similarity transformation.

2.3 Light-Cone Lattice Regularization

The light-cone lattice regularization of a quantum field theory is achieved by setting discrete trajectories of particles with a lattice spacing a [10–12]. A discretized light-cone forms a two-dimensional lattice. A right-mover runs over a line from left-bottom to right-top, while a left-mover runs over a line from right-bottom to left-top, (Fig. 2).

Particle scatterings occur only at vertices. The amplitudes depend on the states of four legs around a vertex, i.e. the presence or absence of a particle. Thus, the scattering amplitudes of the quantum field theory are regarded as the Boltzmann weights of the two-dimensional lattice through the light-cone lattice regularization. For an integrable quantum field theory, the two-dimensional lattice obtained as the light-cone lattice is identified with an integrable lattice model. For instance, the light-cone lattice of the SG model is regarded as the time sequence of the transfer matrix of the spin-$\frac{1}{2}$ XXZ model, while that of the SSG model is obtained in the spin-1 ZF model [28, 32]. Thus, particle trajectories in an integrable quantum field theory are described by the transfer matrix of the corresponding integrable spin chain.

Now we focus on the SSG model connected to the spin-1 ZF model. The spin-1 ZF model is defined by

$$\mathcal{H} = \sum_{j=1}^{N-1} \Big[T_j - (T_j)^2 - 2\sin^2\gamma \, (T_j^z + (S_j^z)^2 + (S_{j+1}^z)^2 - (T_j^z)^2) + \\ + 4\sin^2\tfrac{\gamma}{2} \, (T_j^{\perp} T_j^z + T_j^z T_j^{\perp}) \Big] + \mathcal{H}_{\mathrm{B}}, \tag{41}$$

where

$$T_j = \mathbf{S}_j \cdot \mathbf{S}_{j+1}, \qquad T_j^{\perp} = S_j^x S_{j+1}^x + S_j^y S_{j+1}^y, \qquad T_j^z = S_j^z S_{j+1}^z. \tag{42}$$

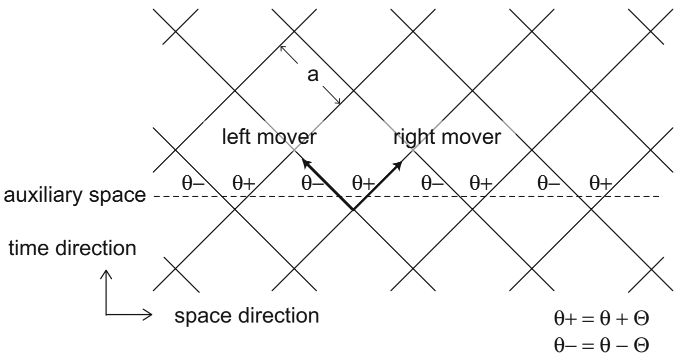


Fig. 2 The light-cone lattice with a lattice spacing a. A *right-mover* runs over a line from *left-bottom* to *right-top*, while a *left-mover* runs over a line from *right-bottom* to *left-top*. The inhomogeneities denoted by $\pm\Theta$ give the rapidities of the *right-movers* and *left-movers*

The operator S_j^α ($\alpha \in \{x, y, z\}$) is the three-dimensional $SU(2)$ generator which nontrivially acts on the jth space of the N-fold tensor product. The parameter γ is an anisotropy parameter which determines a coupling constant of the SSG model via $\beta^2 = 4(\pi - 2\gamma)$. Since β^2 is a real value, γ must be less than $\frac{\pi}{2}$. In a spin chain realm, the condition $\gamma < \frac{\pi}{2}$ indicates that the system is gapless.

Dirichlet boundaries of the SSG model are realized by imposing boundary magnetic fields on the spin-1 ZF model. Since the Dirichlet boundaries do not change the soliton charge, the boundary magnetic fields are imposed in the direction of the z-axis:

$$\mathcal{H}_{\rm B} = h_1(H_-)S_1^z + h_2(H_-)(S_1^z)^2 + h_1(H_+)S_N^z + h_2(H_+)(S_N^z)^2, \tag{43}$$

where two functions h_1 and h_2 share the same parameter H:

$$h_1(H) = \frac{1}{2}\sin 2\gamma \left(\cot \frac{\gamma H}{2} + \cot \frac{\gamma(H+2)}{2}\right), \tag{44}$$

$$h_2(H) = \frac{1}{2}\sin 2\gamma \left(-\cot \frac{\gamma H}{2} + \cot \frac{\gamma(H+2)}{2}\right). \tag{45}$$

The boundary fields are $\frac{2\pi}{\gamma}$-periodic functions with respect to H (Fig. 3). The periodic unit consists of two domains $[-2 + \frac{2\pi n}{\gamma}, \frac{2\pi n}{\gamma}]$ and $[\frac{2\pi(n-1)}{\gamma}, -2 + \frac{2\pi n}{\gamma}]$ ($n \in \mathbb{Z}$), in each of which we expect different physics.

The transfer matrix of the ZF model is obtained from the R-matrix of the 19-vertex model [26]:

$$\begin{aligned} T(\theta) &= R_{0,2N}(\tfrac{\gamma}{\pi}(\theta - \xi_{2N}))R_{0,2N-1}(\tfrac{\gamma}{\pi}(\theta + \xi_{2N-1})) \cdots R_{02}(\tfrac{\gamma}{\pi}(\theta - \xi_2)) \times \\ &\quad \times R_{01}(\tfrac{\gamma}{\pi}(\theta + \xi_1)), \\ \widehat{T}(\theta) &= R_{10}(\tfrac{\gamma}{\pi}(\theta + i\pi + \xi_1))R_{20}(\tfrac{\gamma}{\pi}(\theta + i\pi - \xi_2)) \cdots \\ &\quad \cdots R_{2N-1,0}(\tfrac{\gamma}{\pi}(\theta + i\pi + \xi_{2N-1}))R_{2N,0}(\tfrac{\gamma}{\pi}(\theta + i\pi - \xi_{2N})), \end{aligned} \tag{46}$$

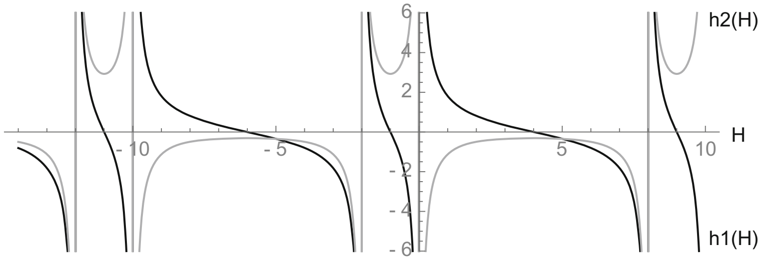


Fig. 3 The boundary magnetic fields as functions of the boundary parameter H. The anisotropy parameter is taken to be $\gamma = \frac{\pi}{5}$. Both functions $h_1(H)$ and $h_2(H)$ possess the $\frac{2\pi}{\gamma}$-periodicity. In each periodic unit, two distinct domains are obtained

where ξ_j is an inhomogeneity parameter which to be taken as 0 for the spin chain. The double-row transfer matrix [4] allows us to deal with boundary reflection:

$$T = \mathrm{tr}_0[K_+(\theta)T(\theta)K_-(\theta)\widehat{T}(\theta)]_{\theta=0}. \tag{47}$$

As we discussed, particle trajectories in the SSG model are described by the same transfer matrix. Corresponding to the rapidity of a right-mover and a left-mover, we choose the inhomogeneity parameters as

$$\xi_{2n-1} = \Theta, \qquad \xi_{2n} = -\Theta \quad (n \in \mathbb{N}). \tag{48}$$

Then the transfer matrices are independently given for the right-mover and the left-mover:

$$\begin{aligned} T_{\mathrm{R}} &= \mathrm{tr}_0[K_+(\theta)T(\theta)K_-(\theta)\widehat{T}(\theta)]_{\theta=\Theta}, \\ T_{\mathrm{L}} &= \mathrm{tr}_0[K_+(\theta)T(\theta)K_-(\theta)\widehat{T}(\theta)]_{\theta=-\Theta}. \end{aligned} \tag{49}$$

The Hamiltonian and total momentum are expressed by using the double-row transfer matrices:

$$\mathcal{H} = \frac{i\gamma}{2\pi a}[\ln T_{\mathrm{R}} + \ln T_{\mathrm{L}}], \qquad \mathcal{P} = \frac{i\gamma}{2\pi a}[\ln T_{\mathrm{R}} - \ln T_{\mathrm{L}}]. \tag{50}$$

3 Nonlinear Integral Equations

3.1 Analyticity Structure of the Transfer Matrix

There are two independent transfer matrices corresponding to the two- and three-dimensional representations of the auxiliary space. The eigenvalues of the two transfer matrices are given by [28]

$$T_1(\theta) = l_1(\theta) + l_2(\theta), \tag{51}$$

$$T_2(\theta) = \lambda_1(\theta) + \lambda_2(\theta) + \lambda_3(\theta), \tag{52}$$

where

$$\begin{aligned} l_1(\theta) &= \sinh \tfrac{\gamma}{\pi}(2\theta + i\pi)B_+(\theta)\phi(\theta + i\pi)\frac{Q(\theta - i\pi)}{Q(\theta)}, \\ l_2(\theta) &= \sinh \tfrac{\gamma}{\pi}(2\theta - i\pi)B_-(\theta)\phi(\theta - i\pi)\frac{Q(\theta + i\pi)}{Q(\theta)} \end{aligned} \tag{53}$$

and

$$
\begin{aligned}
\lambda_1(\theta) &= \sinh \tfrac{\gamma}{\pi}(2\theta - 2i\pi)B_-(\theta - \tfrac{i\pi}{2})B_-(\theta + \tfrac{i\pi}{2}) \ \times \\
&\quad \times\ \phi(\theta - \tfrac{3i\pi}{2})\phi(\theta - \tfrac{i\pi}{2})\frac{Q(\theta + \frac{3i\pi}{2})}{Q(\theta - \frac{i\pi}{2})}, \\
\lambda_2(\theta) &= \sinh \tfrac{\gamma}{\pi}(2\theta)B_+(\theta - \tfrac{i\pi}{2})B_-(\theta + \tfrac{i\pi}{2}) \ \times \\
&\quad \times\ \phi(\theta - \tfrac{i\pi}{2})\phi(\theta + \tfrac{i\pi}{2})\frac{Q(\theta + \frac{3i\pi}{2})Q(\theta - \frac{3i\pi}{2})}{Q(\theta - \frac{i\pi}{2})Q(\theta + \frac{i\pi}{2})}, \\
\lambda_3(\theta) &= \sinh \tfrac{\gamma}{\pi}(2\theta + 2i\pi)B_+(\theta - \tfrac{i\pi}{2})B_+(\theta + \tfrac{i\pi}{2}) \ \times \\
&\quad \times\ \phi(\theta + \tfrac{3i\pi}{2})\phi(\theta + \tfrac{i\pi}{2})\frac{Q(\theta - \frac{3i\pi}{2})}{Q(\theta + \frac{i\pi}{2})}.
\end{aligned} \tag{54}
$$

The function $\phi(\theta)$ gives the phase shift:

$$
\phi(\theta) = \sinh^N \tfrac{\gamma}{\pi}(\theta - \Theta) \sinh^N \tfrac{\gamma}{\pi}(\theta + \Theta) \tag{55}
$$

and the functions $B_\pm(\theta)$ give the boundary effects:

$$
B_\pm(\theta) = \sinh \tfrac{\gamma}{\pi}(\theta \pm \tfrac{i\pi H_+}{2}) \sinh \tfrac{\gamma}{\pi}(\theta \pm \tfrac{i\pi H_-}{2}). \tag{56}
$$

The function $Q(\theta)$ becomes zero at the Bethe roots:

$$
Q(\theta) = \prod_{j=1}^{M} \sinh \tfrac{\gamma}{\pi}(\theta - \theta_j) \sinh \tfrac{\gamma}{\pi}(\theta + \theta_j). \tag{57}
$$

The functions $T_1(\theta)$ and $T_2(\theta)$ are symmetric under $\theta_j \leftrightarrow -\theta_j$, and therefore the Bethe roots symmetrically locate to the origin in a complex plane.

Note that the following relation holds for $T_1(\theta)$ and $T_2(\theta)$:

$$
T_1(\theta - \tfrac{i\pi}{2})T_1(\theta + \tfrac{i\pi}{2}) = l_2(\theta - \tfrac{i\pi}{2})l_1(\theta + \tfrac{i\pi}{2}) + \sinh \tfrac{\gamma}{\pi}(2\theta)T_2(\theta). \tag{58}
$$

This is the fusion relation obtained for the transfer matrices [43] and later algebraically formulated in the context of the thermodynamic Bethe-ansatz [44].

Now we discuss the analyticity structure of the T-functions. The function $T_2(\theta)$ is analytic and nonzero (ANZ) around the real axis except for the origin and the positions of holes. Since the hole indicates an excitation particle, rapidity of a soliton

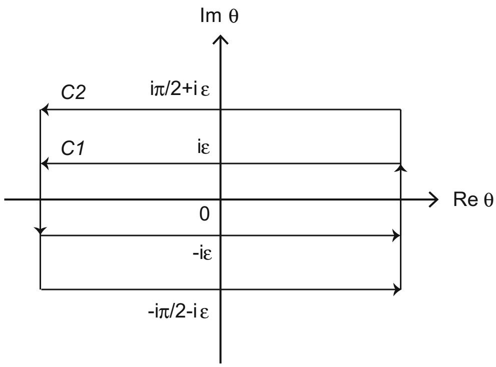


Fig. 4 The contours $\mathcal{C}_1$ and $\mathcal{C}_2$. The *vertical lines* are taken at $\pm\infty$

is characterized by the hole. By writing the positions of holes in the real axis by $h_j \in \mathbb{R}$, we have the following integral equation through the Cauchy theorem:

$$\oint_{\mathcal{C}_1} d\theta\, e^{ik\theta}[\ln T_2(\theta)]'' = \frac{2\pi k}{1-e^{-\pi k}}\Big(1 + \sum_{h_j\in\mathbb{R}} e^{ikh_j}\Big). \tag{59}$$

The contour $\mathcal{C}_1$ is shown in Fig. 4.

For the function $T_1(\theta)$, we need the analyticity structure of $\mathrm{Im}\theta \in [-\frac{\pi}{2}, \frac{\pi}{2})$. By writing the positions of holes by $h_j^{(1)}$, the following integral equation is obtained:

$$\oint_{\mathcal{C}_2} d\theta\, e^{ik\theta}[\ln T_1(\theta)]'' = \frac{2\pi k}{1-e^{-\pi k}}\Big(1 + \sum_{\mathrm{Im}h_j^{(1)}\in[-\frac{\pi}{2},\frac{\pi}{2})} e^{ikh_j^{(1)}}\Big), \tag{60}$$

where the contour $\mathcal{C}_2$ is shown in Fig. 4.

Thus, we obtained two integral equations from the T-functions. In the next subsection, we derive coupled integral equations by rewriting the left-hand sides of equations.

3.2 Nonlinear Integral Equations

We consider the auxiliary functions [45] defined by

$$\begin{aligned} b(\theta) &= \frac{\lambda_1(\theta)+\lambda_2(\theta)}{\lambda_3(\theta)}, \qquad \bar{b}(\theta) = \frac{\lambda_3(\theta)+\lambda_2(\theta)}{\lambda_1(\theta)} = b(-\theta), \\ y(\theta) &= \frac{\sinh\frac{\gamma}{\pi}(2\theta)T_2(\theta)}{l_2(\theta-\frac{i\pi}{2})l_1(\theta+\frac{i\pi}{2})}. \end{aligned} \tag{61}$$

We also define

$$B(\theta) = 1 + b(\theta), \qquad \bar{B}(\theta) = 1 + \bar{b}(\theta),$$
$$Y(\theta) = 1 + y(\theta). \tag{62}$$

The auxiliary functions have different analyticity structure from the T-functions. Besides the origin and the positions of holes, the function $B(\theta)$ has zeros at the positions of roots $\pm\frac{i\pi}{2}$ ($\theta = \theta_k \pm \frac{i\pi}{2}$). Note that these positions become two-string centers in the IR limit [46].

Substituting (61) and (62) into the integral equations (59) and (60), we obtain the nonlinear integral equations (NLIEs) [33]:

$$\begin{aligned}
\ln b(\theta) = & \int_{-\infty}^{\infty} d\theta' \, G(\theta - \theta' - i\epsilon) \ln B(\theta' + i\epsilon) - \\
& - \int_{-\infty}^{\infty} d\theta' \, G(\theta - \theta' + i\epsilon) \ln \bar{B}(\theta' - i\epsilon) \\
& + \int_{-\infty}^{\infty} d\theta' \, G_K(\theta - \theta' - \tfrac{i\pi}{2} + i\epsilon) \ln Y(\theta' - i\epsilon) + \\
& + \, i D_{\text{bulk}}(\theta) + i D_{\text{B}}(\theta) + i D(\theta) + C_b^{(2)}
\end{aligned} \tag{63}$$

$$\begin{aligned}
\ln y(\theta) = & \int_{-\infty}^{\infty} d\theta' \, G_K(\theta - \theta' + \tfrac{i\pi}{2} - i\epsilon) \ln B(\theta' + i\epsilon) + \\
& + \int_{-\infty}^{\infty} d\theta' \, G_K(\theta - \theta' - \tfrac{i\pi}{2} + i\epsilon) \ln \bar{B}(\theta' - i\epsilon) \\
& + i D_{\text{SB}}(\theta) + i D_K(\theta) + C_y.
\end{aligned} \tag{64}$$

The integration constants C_b and C_y are determined from the asymptotic behaviors of the NLIEs. The functions $G(\theta)$ and $G_K(\theta)$ show the phase shift coming from soliton-soliton scatterings:

$$G(\theta) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{e^{-ik\theta} \sinh(\frac{\pi}{\gamma} - 3)\frac{\pi k}{2}}{2\cosh\frac{\pi k}{2} \sinh(\frac{\pi}{\gamma} - 2)\frac{\pi k}{2}}, \qquad G_K(\theta) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{e^{-ik\theta}}{2\cosh\frac{\pi k}{2}}. \tag{65}$$

The effect of the bulk phase shift is contained in

$$D_{\text{bulk}}(\theta) = 2N \arctan \frac{\sinh\theta}{\cosh\Theta}. \tag{66}$$

This is the only term which concerns the scaling limit given by $a \to 0$ by fixing

$$L = Na, \qquad m_0 = \frac{2}{a} e^{-\Theta}. \tag{67}$$

As a result, the scaling limit of $D_{\mathrm{bulk}}(\theta)$ is written by the mass parameter and the system length of the SSG model:

$$2N \arctan \frac{\sinh \theta}{\cosh \Theta} \to 2im_0 L \sinh \theta. \tag{68}$$

The effects of the existence of particles are obtained in

$$\begin{aligned} D(\theta) &= \sum_j c_j \{g_{(j)}(\theta - \tilde{\theta}_j) + g_{(j)}(\theta + \tilde{\theta}_j)\}, \\ g(\theta) &= 2\gamma \int_0^\infty d\theta' \, G(\theta'), \qquad g_K(\theta) = 2\gamma \int_0^\infty d\theta' \, G_K(\theta'), \end{aligned} \tag{69}$$

where $\tilde{\theta}_j - \mathrm{sgn}(\mathrm{Im}\theta_j)\frac{i\pi}{2}$ is an effective Bethe root. The form of the function $g_{(j)}(\theta)$ depends on the types of particles:

$$g_{(j)}(\theta \pm \tilde{\theta}_j) = \begin{cases} g(\theta \pm \tilde{\theta}_j) + g(\theta \pm \tilde{\theta}_j - i\pi \, \mathrm{sgn}(\mathrm{Im}\, \theta)) & \pi < |\mathrm{Im}\tilde{\theta}_j| < \frac{\pi^2}{2\gamma} - \frac{\pi}{2} \\ g(\theta \pm \tilde{\theta}_j + i\epsilon) + g(\theta \pm \tilde{\theta}_j - i\epsilon) & |\mathrm{Im}\tilde{\theta}_j| = \frac{\pi^2}{2\gamma} - \frac{\pi}{2} \\ g_K(\theta \pm \tilde{\theta}_j) & \tilde{\theta}_j = h_j^{(1)} \\ g(\theta \pm \tilde{\theta}_j) & \text{otherwise,} \end{cases} \tag{70}$$

where we choose

$$c_j = \begin{cases} +1 & \text{for holes} \\ -1 & \text{otherwise.} \end{cases} \tag{71}$$

The term $D_K(\theta)$ is interpreted as the effects of the existence of kinks:

$$\begin{aligned} D_K(\theta) &= \lim_{\epsilon \to +0} \widetilde{D}_K(\theta + \tfrac{i\pi}{2} - i\epsilon) \\ \widetilde{D}_K(\theta) &= \sum_j c_j \{g_{(j)}^{(1)}(\theta - \tilde{\theta}_j) + g_{(j)}^{(1)}(\theta + \tilde{\theta}_j)\}, \end{aligned} \tag{72}$$

where $g_{(j)}^{(1)}(\theta)$ is defined by

$$g_{(j)}^{(1)}(\theta \pm \tilde{\theta}_j) = \begin{cases} g_K(\theta \pm \tilde{\theta}_j) + g_K(\theta \pm \tilde{\theta}_j - i\pi \, \mathrm{sgn}(\mathrm{Im}\, \theta)) = 0 \ : \\ \qquad\qquad \pi < |\mathrm{Im}\tilde{\theta}_j| < \frac{\pi^2}{\gamma} - \frac{\pi}{2} \ ; \\ g_K(\theta \pm \tilde{\theta}_j + i\epsilon) + g_K(\theta \pm \tilde{\theta}_j - i\epsilon) \ : \\ \qquad\qquad |\mathrm{Im}\tilde{\theta}_j| = \frac{\pi^2}{\gamma} - \frac{\pi}{2} \ ; \\ g_K(\theta \pm \tilde{\theta}_j) \ : \qquad \text{otherwise.} \end{cases} \tag{73}$$

The boundary terms $D_{\mathrm{B}}(\theta)$ and $D_{\mathrm{SB}}(\theta)$ have different forms depending on the boundary parameters. By writing the boundary terms by

$$D_{\mathrm{B}}(\theta) = F(\theta; H_+) + F(\theta; H_-) + J(\theta) \tag{74}$$

$$D_{\mathrm{SB}}(\theta) = F_y(\theta; H_+) + F_y(\theta; H_-) + J_K(\theta), \tag{75}$$

the last terms do not depend on the boundary parameters:

$$J(\theta) = \int_0^\infty d\theta' \int_{-\infty}^\infty dk\; e^{-ik\theta'} \frac{\cosh\frac{\pi k}{4}\sinh(\frac{\pi}{\gamma}-3)\frac{\pi k}{4}}{\cosh\frac{\pi k}{2}\sinh(\frac{\pi}{\gamma}-2)\frac{\pi k}{2}}, \tag{76}$$

$$J_K(\theta) = 2\widetilde{g}_K(\theta) = \lim_{\epsilon\to+0} 2g_K(\theta + \tfrac{i\pi}{2} - i\epsilon). \tag{77}$$

The functions $F(\theta; H)$ and $F_y(\theta; H)$ include the integrands which cross the branch cut when $H = \pm 1$. Taking into account of the periodicity with respect to H, there are three distinct regimes:

Regime (a): $\mathbf{1 < H_\pm \leq \frac{2\pi}{\gamma} - 1}$

$$F(\theta; H) = \int_0^\infty d\theta' \int_{-\infty}^\infty dk\; e^{-ik\theta'} \frac{\sinh(\frac{\pi}{\gamma} - H)\frac{\pi k}{2}}{2\cosh\frac{\pi k}{2}\sinh(\frac{\pi}{\gamma}-2)\frac{\pi k}{2}}, \tag{78}$$

$$F_y(\theta; H) = 0. \tag{79}$$

Regime (b): $\mathbf{-1 < H_\pm \leq 1}$

$$F(\theta; H) = -\int_0^\infty d\theta' \int_{-\infty}^\infty dk\; e^{-ik\theta'} \frac{\sinh(\frac{\pi}{\gamma} + \pi H - 2)\frac{\pi k}{2}}{2\cosh\frac{\pi k}{2}\sinh(\frac{\pi}{\gamma}-2)\frac{\pi k}{2}}, \tag{80}$$

$$F_y(\theta; H) = \widetilde{g}_K(\theta - \tfrac{i\pi(1-H)}{2}) + \widetilde{g}_K(\theta + \tfrac{i\pi(1-H)}{2}). \tag{81}$$

Regime (c): $\mathbf{-\frac{2\pi}{\gamma} + 1 < H_\pm \leq -1}$

$$F(\theta; H) = -\int_0^\infty d\theta' \int_{-\infty}^\infty \frac{dk}{2\pi}\; e^{-ik\theta'} \frac{\sinh(\frac{\pi}{\gamma} + H)\frac{\pi k}{2}}{2\cosh\frac{\pi k}{2}\sinh(\frac{\pi}{\gamma}-2)\frac{\pi k}{2}}, \tag{82}$$

$$F_y(\theta; H) = 0. \tag{83}$$

3.3 Ground State and Boundary Bound States

In this subsection, we discuss the boundary effects on the ground state from the viewpoint of Bethe roots. Under the presence of strong enough boundary magnetic fields, it has been found that boundary bound states emerge. The existence of boundary bound states was first discussed in [8] as poles in the reflection matrix. Then later, it was discussed by using the q-deformed vertex operator [47] and the Bethe-ansatz method [42, 48]. In a realm of the Bethe-ansatz method, boundary bound states are obtained as imaginary roots of the Bethe-ansatz equations.

The existence of boundary bound states modifies the root density of the bulk in the order of the inverse of the system size. Here we determine the root component of the ground state through calculation of the boundary energy.

The eigenenergy of the SSG model is calculated from the transfer matrix [49]. From (50), the eigenenergy is obtained as

$$\begin{aligned} E &= \frac{1}{4ia}\left(\frac{d}{d\theta}\ln T_2(\theta)\Big|_{\theta=\Theta+\frac{i\pi}{2}} - \frac{d}{d\theta}\ln T_2(\theta)\Big|_{\theta=\Theta-\frac{i\pi}{2}}\right) \\ &= E_{\text{bulk}} + E_{\text{B}} + E_{\text{ex}} + E_{\text{C}}, \end{aligned} \tag{84}$$

where the bulk, particle excitation, and Casimir energy are respectively given by

$$E_{\text{bulk}} = 0, \tag{85}$$

$$E_{\text{ex}} = m_0 \sum_{j=1}^{N_H} \cosh h_j - m_0 \sum_{j=1}^{M_C} \cosh \tilde{c}_j, \tag{86}$$

$$E_C = \frac{m_0}{2\pi} \operatorname{Im} \int_{-\infty-i\epsilon}^{\infty-i\epsilon} d\theta\, e^{-\theta} \ln \bar{B}(\theta). \tag{87}$$

The boundary energy is given by

$$E_{\text{B}} = m_0 + E_{\text{b}}(H_+) + E_{\text{b}}(H_-), \tag{88}$$

$$E_{\text{b}}(H) = \begin{cases} 0 & |H| > 1, \\ m_0 \cos \frac{\pi(1-H)}{2} & |H| < 1. \end{cases} \tag{89}$$

The boundary dependence of the ground-state and first-excitation energy subtracted by E_C is shown in Fig. 5.

Taking into account that the rapidity of a boundary bound state approaches $\frac{i\pi}{2}(1 - H_\pm)$ in the thermodynamic limit [33], we find that the two-string state (depicted by bold dotted line) gives the boundary excitation state for $0 < H < 1$ and $-2 < H < -1$. And, subsequently, this implies that the ground state includes a boundary bound state. This fact is numerically checked in the isotopic case for $(H_+, H_-) = (1.5, 2.2)$, $(1.5, 0.3)$, and $(1.5, -1.8)$. From Fig. 6, it seems that the rapidity of a boundary bound state is fixed at $\theta = \frac{i\pi}{2}(1 - H_-)$.

4 Boundary Effect on the Scattering Theory

Now we discuss the boundary effect of the scattering theory of the SSG model. Since the scattering and reflection matrices of a quantum field theory are defined between

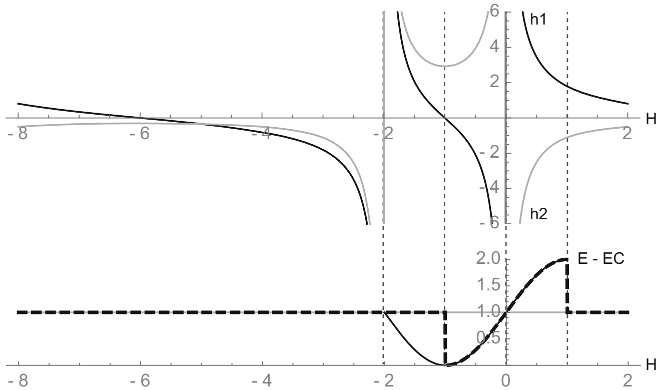
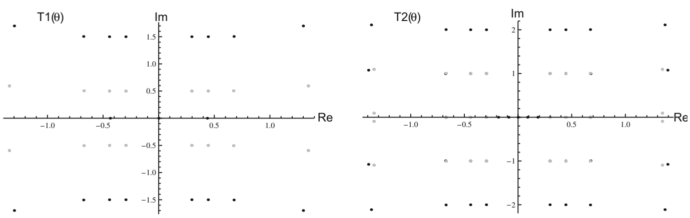
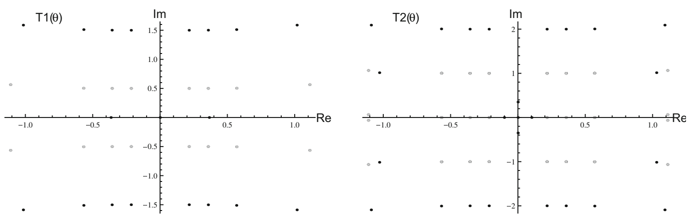


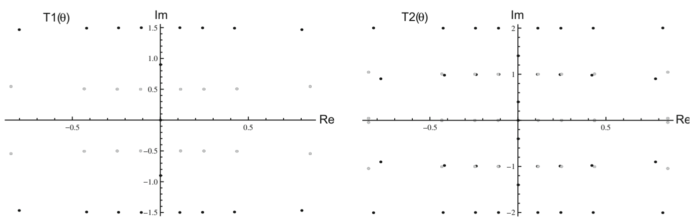
Fig. 5 The ground-state and first-excitation energy subtracted by the Casimir energy is shown together with the behavior of the boundary magnetic fields. The energy of the two-string state is given by the *bold dotted line*. The anisotropy parameter is taken as $\gamma = \frac{\pi}{5}$ and the mass parameter by $m_0 = 1$



(a) $(H_+, H_-) = (1.5, -2.2)$



(b) $(H_+, H_-) = (1.5, 0.3)$



(c) $(H_+, H_-) = (1.5, -1.8)$

Fig. 6 Analyticity structure of $T_1(\theta)$ and $T_2(\theta)$ is plotted for three regimes. The zeros of T-functions are depicted by *black dots*, while the roots by *gray dots*. These are numerically calculated for the 4 two-string roots in the $N = 8$ isotropic chain the asymptotic states obtained long before and after the scattering, we analyze the NLIEs in the IR limit $m_0 L \to \infty$.

In the IR limit, the first and second terms of (63) and (64) become negligibly small [32]. Therefore, we obtain the NLIEs in the IR limit as follows:

$$\begin{aligned}\ln b(\theta) =& \int_{-\infty}^{\infty} d\theta' \; G_K(\theta - \theta' - \tfrac{i\pi}{2} + i\epsilon) \ln Y(\theta' - i\epsilon) + 2im_0 L \sinh\theta \\ &+ iD_{\mathrm{B}}(\theta) + iD(\theta) + i\pi C_b^{(2)}, \end{aligned} \tag{90}$$

$$\ln y(\theta) = iD_{\mathrm{SB}}(\theta) + iD_K(\theta) + i\pi C_y^{(2)}. \tag{91}$$

Since the auxiliary function $B(\theta)$ has zeros at the positions of holes, the function $b(\theta)$ satisfies the quantization condition given by

$$b(h_j) = -1. \tag{92}$$

On the other hand, the quantization condition in a realm of the scattering theory [13, 16] is written in terms of the scattering and reflection amplitudes:

$$e^{2im_0 L \sinh\theta_j} R(\theta_j; \xi_+) \cdot \prod_{\substack{l=1 \\ l \neq j}}^{n} S(\theta_j - \theta_l) S(\theta_j + \theta_l) \cdot R(\theta_j; \xi_-) = 1. \tag{93}$$

By comparing the above two conditions, we obtain the relation between the NLIEs and the scattering theory:

$$\begin{aligned} &\ln R_1^+(\theta) = iF^{(r)}(\theta; H) \qquad (r = a, b, c), \\ &\ln R_2(\theta) = iJ(\theta). \end{aligned} \tag{94}$$

The function $F^{(r)}(\theta; H)$ has different forms depending on the regimes denoted by the superscript r. Remind that the ground state includes a boundary bound state for $-2 < H < -1$ and $0 < H < 1$ (Fig. 5). Thus, change of the function at $H = 1$ is understood as coming from change of the ground-state description. Indeed, we obtain the boundary bootstrap relation between two functions:

$$\begin{aligned} iF^{(b)}(\theta; H) &= iF^{(a)}(\theta; H) + ig(\theta - \tfrac{i\pi}{2}(1-H)) + ig(\theta + \tfrac{i\pi}{2}(1-H), \\ iF_y^{(b)}(\theta; H) &= iF_y^{(a)}(\theta; H) + i\widetilde{g}_K(\theta - \tfrac{i\pi}{2}(1-H)) + i\widetilde{g}_K(\theta + \tfrac{i\pi}{2}(1-H)), \end{aligned} \tag{95}$$

where the rapidity of the boundary bound state is what we obtained from the numerical calculation (Fig. 6). Similarly, the change of the function at $H = -1$ is understood from the emergence of a boundary bound state. The boundary bootstrap relation is obtained as

Table 1 The relation between parameters in the NLIEs and the scattering theory. Three domains require different relations. These domain separation matches that obtained in Fig. 3

$H > 0$	$0 > H > -2$	$-2 > H$
$H = -\frac{2\xi}{\pi\lambda} + \frac{1}{\lambda} + 1$	$H = -\frac{2\xi}{\pi\lambda} + \frac{1}{\lambda} - 1$	$H = -\frac{2\xi}{\pi\lambda} - \frac{1}{\lambda} - 3$

$$\begin{aligned} iF^{(c)}(\theta; H) &= iF^{(b)}(\theta; H) + ig(\theta - \tfrac{i\pi}{2}(1+H)) + ig(\theta + \tfrac{i\pi}{2}(1+H)), \\ iF_y^{(c)}(\theta; H) &= iF_y^{(b)}(\theta; H) + i\widetilde{g}_K(\theta + \tfrac{i\pi}{2}(1+H)) + i\widetilde{g}_K(\theta - \tfrac{i\pi}{2}(1+H)). \end{aligned} \tag{96}$$

Thus, we obtain the relation between parameters in the NLIEs and the scattering theory as Table 1. The parameter relation for the regime (c) is realized from that for the regime (a) through the transformation $H \to -H - \frac{2\pi}{\gamma} - 2$, which gives the soliton-antisoliton exchange transformation:

$$iF^{(a)}(\theta; -H - \tfrac{2\pi}{\gamma} - 2) = iF^{(c)}(\theta; -H - \tfrac{2\pi}{\gamma} - 2) = \ln R_1^-(\theta), \tag{97}$$

while this transformation maps the relation for the regime (b) to itself:

$$iF^{(b)}(\theta; -H-2) - ig_K(\theta - \tfrac{i\pi}{2}(-H-2)) - ig_K(\theta + \tfrac{i\pi}{2}(-H-2)) = \ln R_1^-(\theta). \tag{98}$$

Here we used the $\frac{2\pi}{\gamma}$-periodicity of the reflection amplitude with respect to H. This fact supports our assumption that the two domains obtained in Fig. 3 are described by different physics.

5 Boundary Effect on the UV Behavior

The SSG model shows the conformal invariance in the UV limit $m_0 L \to 0$. The $\mathcal{N} = 1$ super CFT obtained from the UV limit of the SSG model consists of the NS and R sectors, which are realized by the different boundary conditions of fermions. The UV limit of the SSG model has been studied for the periodic case [30–32] and the limited regime of the Dirichlet boundaries [28], and it was found that only the NS sector is realized through the light-cone lattice regularization. We study how the UV behavior is affected by the boundaries in this section.

In the UV limit, some Bethe roots stay finite, while the others go to infinity. Such roots that go to infinity behave as $\theta \sim \hat{\theta} - \ln m_0 L$ in the limit $m_0 L \to 0$ [18]. We introduce the scaling function defined by $f^+(\hat{\theta}) = f(\hat{\theta} - \ln m_0 L)$, which is a step-like function [18, 21, 50]. By focusing on the roots which go to infinity, the NLIEs in the UV limit are written as

$$\ln b^+(\hat{\theta}) = \int_{-\infty}^{\infty} d\hat{\theta}'\; G(\hat{\theta} - \hat{\theta}' - i\epsilon) \ln B^+(\hat{\theta}' + i\epsilon)\; -$$

$$
\begin{aligned}
&- \int_{-\infty}^{\infty} d\hat{\theta}'\, G(\hat{\theta}-\hat{\theta}'+i\epsilon) \ln \bar{B}^{+}(\hat{\theta}'-i\epsilon) \\
&+ \int_{-\infty}^{\infty} d\hat{\theta}'\, G_K(\hat{\theta}-\hat{\theta}'-\tfrac{i\pi}{2}+i\epsilon) \ln Y^{+}(\hat{\theta}'-i\epsilon) + ie^{\hat{\theta}} + \\
&+ i\sum_j c_j g_{(j)}(\hat{\theta}-\hat{\theta}_j) + i\pi \hat{C}_b,
\end{aligned} \tag{99}
$$

$$
\begin{aligned}
\ln y^{+}(\hat{\theta}) = &\int_{-\infty}^{\infty} d\hat{\theta}'\, G_K(\hat{\theta}-\hat{\theta}'+\tfrac{i\pi}{2}-i\epsilon) \ln B^{+}(\hat{\theta}'+i\epsilon) \\
&+ \int_{-\infty}^{\infty} d\hat{\theta}'\, G_K(\hat{\theta}-\hat{\theta}'-\tfrac{i\pi}{2}+i\epsilon) \ln \bar{B}^{+}(\hat{\theta}'-i\epsilon) \; + \\
&+ i\sum_j c_j g^{(1)}_{(j)}(\hat{\theta}-\hat{\theta}_j) + i\pi \hat{C}_y,
\end{aligned} \tag{100}
$$

where $\hat{C}_b$ and $\hat{C}_y$ are determined from the asymptotic behavior of the NLIEs.

The eigenenergy in the UV limit is calculated through the formula (50). Since the central charge and conformal dimensions show up in the particle excitation and Casimir energy, we define:

$$
\begin{aligned}
&E_{\mathrm{CFT}}(L) = E(L) - (E_{\mathrm{bulk}} + E_{\mathrm{B}}) = E_{\mathrm{ex}}(L) + E_C(L), \\
&E_{\mathrm{ex}}(L) = \frac{1}{2L}\sum_{j=1} e^{\hat{h}_j} - \frac{1}{2L}\sum_{\frac{\pi}{2}<|\mathrm{Im}\theta_j|<\frac{3\pi}{2}} e^{\hat{\theta}_j - \frac{i\pi}{2}\mathrm{Im}\theta_j}, \\
&E_C(L) = \frac{1}{2\pi L}\mathrm{Im}\int_{-\infty}^{\infty} d\hat{\theta}\, e^{\hat{\theta}} \ln \bar{B}^{+}(\hat{\theta}).
\end{aligned} \tag{101}
$$

Then $E_{\mathrm{CFT}}(L)$ is expressed by

$$
\begin{aligned}
E_{\mathrm{CFT}}(L) = &\frac{1}{4\pi L}\{L_{+}(b^{+}(\infty)) - L_{+}(b^{+}(-\infty)) + L_{+}(\bar{b}^{+}(\infty)) - L_{+}(\bar{b}^{+}(-\infty)) \\
&\quad + L_{+}(y^{+}(\infty)) - L_{+}(y^{+}(-\infty))\} \\
&+ \frac{i}{8\pi L}\Big[\Big\{e^{\hat{\theta}} + \sum_j c_j g_{(j)}(\hat{\theta}-\hat{\theta}_j) + \pi\hat{C}_b\Big\}(\ln B^{+}(\hat{\theta}) - \ln \bar{B}^{+}(\hat{\theta}))\Big]_{-\infty}^{\infty} \\
&+ \frac{i}{8\pi L}\Big[\Big\{\sum_j c_j g^{(1)}_{(j)}(\hat{\theta}-\hat{\theta}_j) + \pi\hat{C}_y\Big\}\ln Y^{+}(\hat{\theta})\Big]_{-\infty}^{\infty},
\end{aligned} \tag{102}
$$

using the dilogarithm function defined by

$$
L_{+}(x) = \frac{1}{2}\int_0^x dy\, \left(\frac{\ln(1+y)}{y} - \frac{\ln y}{1+y}\right). \tag{103}
$$

From the formula obtained for the dilogarithm function [44, 51–56], we have

$$E_{\mathrm{CFT}}(L) = -\frac{\pi}{L} + \frac{\pi}{2L}\left\{\frac{1}{\sqrt{\pi}}\Big(-\frac{\gamma(H_+ + 1)}{2\sqrt{\pi - 2\gamma}} - \frac{\gamma(H_- + 1)}{2\sqrt{\pi - 2\gamma}}\Big) + m\sqrt{\frac{\pi}{\pi - 2\gamma}}\right\}^2$$
$$+ \frac{1}{16}\Big(\frac{1}{2}(\mathrm{sgn}(1 - H_+) + \mathrm{sgn}(1 - H_-) + \mathrm{sgn}(1 + H_+) + \mathrm{sgn}(1 + H_-))\Big)_{\mathrm{mod}\,2}, \tag{104}$$

where

$$m = -S + \tfrac{1}{4}(\mathrm{sgn}(1 - H_+) + \mathrm{sgn}(1 - H_-) + \mathrm{sgn}(1 + H_+) + \mathrm{sgn}(1 + H_-)). \tag{105}$$

We used the effective soliton charge S defined in [33]. By setting the compactification radius R and the boundary parameters $\Phi_\pm$ as

$$R = \sqrt{\frac{\pi}{\pi - 2\gamma}}, \qquad \Phi_\pm = \mp\frac{\gamma(H_\pm + 1)}{2\sqrt{\pi - 2\gamma}}, \tag{106}$$

the energy (104) is the form of

$$E_{\mathrm{CFT}} = -\frac{\pi}{24L}\Big(c_{\mathrm{B}} + c_{\mathrm{F}} - 24(\Delta_{\mathrm{B}} + \Delta_{\mathrm{F}})\Big), \tag{107}$$

where $c = c_{\mathrm{B}} + c_{\mathrm{F}}$ is the central charge and $\Delta = \Delta_{\mathrm{B}} + \Delta_{\mathrm{F}}$ is the conformal dimension given by

$$c_{\mathrm{B}} = 1, \qquad c_{\mathrm{F}} = \frac{1}{2}, \tag{108}$$

$$\Delta_{\mathrm{B}} = \frac{1}{2}\Big(\frac{\Phi_+ - \Phi_-}{\sqrt{\pi}} + mR\Big)^2, \tag{109}$$

$$\Delta_{\mathrm{F}} = \frac{1}{16}\Big(\frac{1}{2}(\mathrm{sgn}(1 - H_+) + \mathrm{sgn}(1 - H_-) + \mathrm{sgn}(1 + H_+) + \tag{110}$$

$$+\ \mathrm{sgn}(1 + H_-))\Big)_{\mathrm{mod}\,2}. \tag{111}$$

Consequently, m defined in (105) is interpreted as the winding number. Thus, the sector of the SUSY realized through the light-cone lattice regularization depends on the boundary parameters $H_\pm$. In Fig. 7, we showed the boundary dependence of the sectors obtained through the light-cone lattice regularization. This indicates that the R sector is also obtained by properly choosing the boundary parameters, although the sector separation does not match the domain separation obtained in the boundary magnetic fields (Fig. 3).

6 Conclusions

We discussed the boundary effects on the light-cone lattice regularization of the SSG model with Dirichlet boundaries. By regarding the light-cone lattice with the time

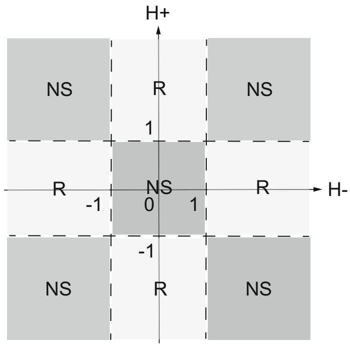


Fig. 7 The boundary dependence of the sectors obtained through the light-cone lattice regularization. A proper choice of boundary parameters admits the R sector

sequence of the transfer matrix of the spin-1 ZF model, we calculated the scattering matrix and the conformal dimensions of the SSG model through the NLIEs.

As a result, we obtained a boundary bound state for certain values of the boundary parameters. The emergence of a boundary bound state was understood from the boundary bootstrap principle of the scattering theory, and subsequently, it explains the three different forms of the NLIEs. The boundary dependence of the NLIEs also leads to the conformal dimension as a function of the boundary parameters. By choosing the boundary parameters properly, we found that both of two sectors of the $\mathcal{N} = 1$ SUSY CFT are obtained through the light-cone lattice regularization.

However, the domain separation obtained in the IR limit does not match the sector separation in the UV limit. Therefore, it is important to understand the physics of the SSG model in an intermediate volume. At the same time, it is an interesting problem to ask how is determined the sector obtained from a certain set of the boundary parameters. Recently, the supercharges were introduced to the integrable spin chain [57, 58]. In order to properly define the super algebra on the spin chain, they used the supercharges which change the length of the integrable spin chain by one. Thus, the supercharges defined in this way connect the even-length chain and odd-length chain, which we expect that give the new insights to the quantum field theories obtained through the light-cone lattice regularization.

Acknowledgements I am grateful to the organizers of the 11th International Workshop "Lie Theory and Its Applications in Physics" for giving me a chance to talk and make fruitful discussions. This work is partially supported by JSPS Grants-in-Aid No. 15K20939.

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Infinite Dimensional Matrix Product States for Long-Range Quantum Spin Models

Roberto Bondesan and Thomas Quella

Abstract We describe a systematic construction of long-range 1D and 2D SU(N) quantum spin models which is based on the algebraic structure of an underlying Wess–Zumino–Witten conformal field theory. The resulting Hamiltonians are put into the context of the Haldane-Shastry model, the paradigmatic example of long-range spin models.

1 Introduction

The analysis of quantum spin models has led to profound insights into the properties of strongly correlated quantum systems. The study of exactly solvable models, the idea of renormalization, the effective field theory approach, including the relevance of topological contributions to the action functional; there are many areas where the conceptual advancement of theoretical physics has gone hand in hand with questions originally posed in spin systems. One of the reasons for the prominent role of spin models is their simplicity. Due to their extremely high degree of symmetry, they have a very clear and concise mathematical description. In spite of this, various natural questions such as those about the existence or absence of gaps, the presence of phase transitions, the breaking of symmetries or the nature of excitations are physically deep and mathematically challenging.

Here we would like to present recent progress in understanding connections between quantum many-body physics and quantum information theory. Our goal is the systematic design of quantum states and systems with definite properties. To be specific, we will use quantum information theoretic methods to impose a certain

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V. Dobrev (ed.), *Lie Theory and Its Applications in Physics*,
Springer Proceedings in Mathematics & Statistics 191,
DOI 10.1007/978-981-10-2636-2_22

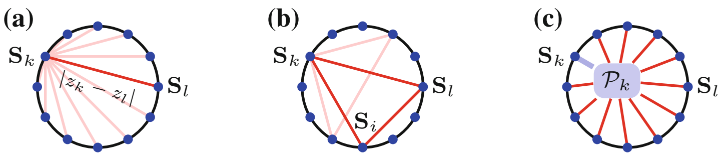

Fig. 1 **a** The spin configuration in the Haldane-Shastry model and the interactions with the spin $\mathbf{S}_k$. **b** The three-spin interactions encountered in the parent Hamiltonian (5). **c** Sketch of the action of the operator $\mathcal{P}_k(\{z_l\})$ (Color figure online)

entanglement structure on the aspired ground state of an SU(N) quantum spin system and we will associate a natural Hamiltonian with it which is then studied in detail. Technically, this goes under the name of *infinite dimensional matrix product states* (∞MPS). Both the construction of the ground state and of the Hamiltonian will make heavy use of the machinery of conformal field theory (CFT) (see [8]).

The physical systems we wish to design are lattice realizations of one-dimensional critical systems or two-dimensional gapped chiral spin liquids, prominent examples of topological phases of matter. Since the latter exhibit massless degrees of freedom at their boundary, these two types of systems are, in fact, intimately related, see the seminal work by Moore and Read for a discussion in the context of continuous fractional quantum Hall samples [17]. While predominantly interested in the realization of 1D critical systems we will also briefly review progress on the 2D side. In this contribution we are aiming at a presentation of the basic philosophy to a non-expert audience while referring to the original articles for most of the more technical aspects.

2 The Haldane-Shastry Model as a Paradigm

Among all known spin chains, the Haldane-Shastry model [11, 24] plays a special role due its analytical tractability and its connections to numerous fields of physics and mathematics. In its original definition for SU(2), it was regarded as a basic model for spin-1/2 degrees of freedom with long-range interactions on a circular 1D lattice with L equidistant sites $z_k = \exp(\frac{2\pi i}{L}k)$ in the complex plane, see Fig. 1a. The interactions take an inverse-square form in the cord distance $|z_k - z_l|^2 = 4\sin^2\frac{\pi}{L}(k-l)$ in the 2D plane. In standard normalization, the Hamiltonian reads

$$H_{\mathrm{HS}} = \left(\frac{2\pi}{L}\right)^2 \sum_{k<l} \frac{\mathbf{S}_k \cdot \mathbf{S}_l}{|z_k - z_l|^2} = \left(\frac{\pi}{L}\right)^2 \sum_{k<l} \frac{\mathbf{S}_k \cdot \mathbf{S}_l}{\sin^2\frac{\pi}{L}(k-l)} \,. \tag{1}$$

This normalization ensures that, for $L \to \infty$ and $|k-l| \ll L$, the leading contribution reduces to the usual Heisenberg Hamiltonian without L-dependent prefactors.

The Haldane-Shastry model exhibits a rather peculiar type of integrability. Indeed, the Hamiltonian may be shown to *commute* with the generators of an infinite-dimensional Yangian $\mathrm{Y}(sl_2)$ [14]. This statement is true for arbitrary chain lengths and it allows for a convenient decomposition of the quantum mechanical Hilbert space into irreducible Yangian multiplets inside of which all states are energetically degenerate. The explicit decomposition and determination of the energy levels can be achieved using the theory of degenerate affine Hecke algebras and symmetric polynomials [20]. Using the exact knowledge of the ground state wave function, one also has access to (dynamical) correlation functions [16].

One of the important physical insights which can be inferred from the analysis of the Haldane-Shastry model is that the elementary excitations are spinons (rather than magnons), i.e. particles which carry half-integer spin and obey fractional exclusion statistics. A configuration of spinons can be described in terms of so-called *motifs* [13] whose combinatorial structure provides an explicit implementation of a generalized form of Pauli's exclusion principle [12]. A detailed discussion of the Haldane-Shastry model, including a more comprehensive guide to the literature, can be found in Haldane's review [13] and in recent work by Greiter [10].

In the thermodynamic limit $L \to \infty$, the Haldane-Shastry model provides a realization of the $\mathrm{SU}(2)_1$ WZW conformal field theory. It is thus not surprising that this CFT also admits an action of the Yangian $\mathrm{Y}(sl_2)$ and that its spectrum may be organized in terms of Yangian multiplets [1, 23]. The investigation of these connections led to remarkable results on finitized (i.e. truncated) characters for representations of affine Lie algebras such as $\widehat{sl}_2$, see [4] and references therein. In contrast to the usual ones in terms of the Weyl–Kac character formula, these expressions have been termed *fermionic* since they are manifestly positive and do not involve alternating sums. This is due to the absence of null states and signals that the spinons indeed provide the correct basis for the description of quasi-particle excitations.

Since this aspect is important for the motivation of our work let us emphasize that some of the main features of the Haldane-Shastry model differ considerably from those of systems which are solvable by standard Bethe ansatz methods. This is even true for the spin-1/2 Heisenberg model which realizes the same $\mathrm{SU}(2)_1$ WZW model as $L \to \infty$. For the Heisenberg model, the Hamiltonian can be constructed from the Yangian $\mathrm{Y}(sl_2)$ but does not commute with it. The Yangian hence plays the role of a spectrum generating symmetry. On a technical level, this is the reason why Bethe ansatz may be used to derive the full spectrum. The Yangian only becomes a true symmetry as $L \to \infty$ [6].[1]

This simple fact has profound physical consequences. First of all, the spinon excitations are interacting in the Heisenberg model but non-interacting in the Haldane-Shastry model. This is reflected in the spinon's scattering matrix which is momentum dependent in the former case while it is a purely statistical phase factor for the latter. Similarly, the Heisenberg model only realizes the $\mathrm{SU}(2)_1$ WZW model up to logarithmic corrections while the latter are absent in the Haldane-Shastry model. In view of the non-interacting nature of the spinons and the absence of logarithmic corrections we are tempted to think of the Haldane-Shastry model as providing an

[1]The quantum affine algebra $\mathcal{U}_q(\widehat{sl}_2)$ of this paper degenerates into the Yangian $\mathrm{Y}(sl_2)$ as $q \to 1$.

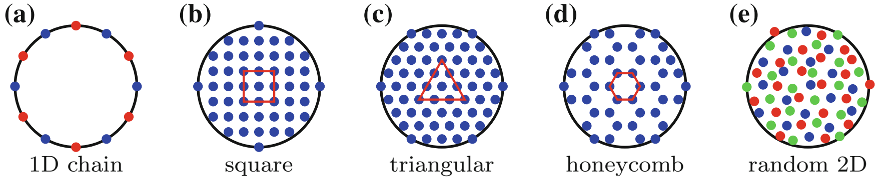

Fig. 2 Examples of 1D and 2D lattices that are of potential interest in physical applications. Different colors denote distinct representations of the physical spin (Color figure online)

"optimal discretization" of the $SU(2)_1$ WZW model. It is currently unclear whether similar "optimal discretizations" also exist for all other WZW theories.

The last statement immediately brings us to the central motivation for our study of ∞MPS. While the Hamiltonian (1) of the Haldane-Shastry model can, of course, immediately be defined for any Lie algebra with an invariant and non-degenerate bilinear form (such that $\mathbf{S}_k \cdot \mathbf{S}_l$ can be defined), the special properties mentioned above only arise for SU(N) and spins transforming in the fundamental (or anti-fundamental) representation. The goal of this note is to come up with natural generalizations of the Haldane-Shastry model which relax its inherent restrictions while trying to preserve its nice properties. In particular, we would like to allow for other symmetry groups, spins transforming in arbitrary representations and general locations on the circle or even in the complex plane. A sketch of possible situations of interest can be found in Fig. 2. As we will see below, all three generalizations can be achieved conveniently in the framework of ∞MPS. The latter provide a convenient tool to imprint the desired entanglement structure on the aspired ground state.

3 Matrix Product States and Their Parent Hamiltonians

The description of a general quantum state in the Hilbert space $\mathcal{H} = \mathcal{V}^{\otimes L}$ of a spin model requires the specification of an exponentially large number of coefficients. However, recent studies suggest that ground states of quantum spin systems have very particular properties and only populate a tiny corner of the full Hilbert space. This statement can be made rather rigorous for gapped 1D systems and it implies that the ground state of all such systems admits a good approximation in terms of a so-called *matrix product state* (MPS)[2]

$$|\psi\rangle = \sum_{\{k_l\}} \operatorname{tr}(A^{k_1} \cdots A^{k_L})\,|k_1 \cdots k_L\rangle \in \mathcal{V}^{\otimes L}\ . \tag{2}$$

The symbols A^k in this formula refer to matrices acting on an auxiliary space $\mathcal{B}$ whose (arbitrary but fixed) dimensionality d depends on the system under consideration

[2]For simplicity we restrict our attention to translation invariant configurations.

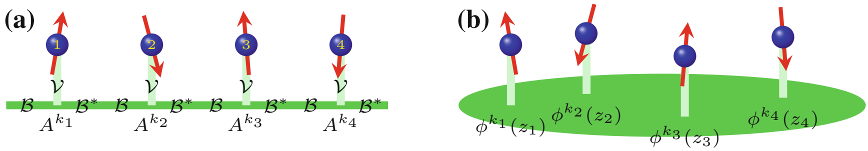

Fig. 3 Sketch of an MPS (*left*) and of an ∞MPS (*right*). In both cases, the states are defined by coupling the physical degrees of freedom (*blue*) to an auxiliary entanglement layer (*green*) which reflects the spatial structure of the system and encodes the state's entanglement (Color figure online)

[21]. It is convenient to view the symbols A as intertwiners $\mathcal{B} \otimes \mathcal{B}^* \to \mathcal{V}$. This has the physical interpretation of attaching two auxiliary spins $\mathcal{B}$ and $\mathcal{B}^*$ to each physical space $\mathcal{V}$. The matrix multiplication and the trace in Eq. (2) then simply correspond to the formation of singlet bonds between auxiliary spins on neighboring physical sites, see Fig. 3 for an illustration.

Let us now try to extract the essential properties of the physical system which gave rise to a specific state $|\psi\rangle$. More precisely, we wish to construct a new prototypical Hamiltonian H which ideally should have $|\psi\rangle$ as its unique zero energy ground state and reflect the basic physical properties of the original system. This so-called *parent Hamiltonian* is constructed as follows. We start with two-site Hamiltonians $h^{(2)}$ which are specific projectors chosen as to annihilate $|\psi\rangle$ (see [21]). If $\mathcal{B} \otimes \mathcal{B}^*$ is a subspace of the two-site Hilbert space $\mathcal{V} \otimes \mathcal{V}$, these operators can simply be chosen to project onto the orthogonal complement $(\mathcal{B} \otimes \mathcal{B}^*)^{\perp}$. By way of construction, the translation invariant sum $H = \sum_k h^{(2)}_{k,k+1}$ satisfies $H \geq 0$ and $H|\psi\rangle = 0$. In other words, the MPS defined in Eq. (2) is a ground state of the Hamiltonian. If the system under study admits the action of a symmetry group and the matrices A have been chosen to be intertwiners, then H will also be invariant. Under certain technical assumptions (and sometimes more generally), the ground state will, moreover, be unique and gapped. Whenever the last two properties fail to be true they can be restored at the cost of block renormalizing the physical spin and the Hamiltonian. Effectively, this introduces interactions beyond neighboring sites.

Let us once more emphasize the change of perspective that we introduced through the back-door. The fundamental object in the previous paragraph was the MPS $|\psi\rangle$ whose properties are fully characterized by the coupling of the physical spins to the auxiliary entanglement layer, i.e. by the matrices A^k. The associated parent Hamiltonian H only entered in a second step and should hence be regarded as a derived concept. The construction is motivated by the hope that the knowledge of the ground state $|\psi\rangle$ will generally be sufficient in order to determine the essential features of the (low energy) physics of the associated quantum system. In the next section we will lift this philosophy to an even more sophisticated level by generalizing MPS to ∞MPS.

4 From CFT to Long-Range Spin Models

In the present note we are mainly interested in the realization of critical 1D systems and 2D gapped chiral topological phases. In both cases, the size of the matrices A^k needs to grow beyond any limit, as can be inferred from the respective scaling laws for the entanglement entropy (see, e.g., [9]). It is then natural to replace the matrices A^k by operators of an auxiliary quantum field theory (QFT) and the trace by a correlation function since this involves a natural prescription of how to deal with products of operators on an infinite dimensional (auxiliary) Hilbert space. The state in the associated quantum spin system then reads

$$|\psi\rangle = \sum\nolimits_{\{k_i\}} \underbrace{\left\langle\phi^{k_1}(z_1)\cdots\phi^{k_L}(z_L)\right\rangle}_{\text{QFT correlator}} |k_1,\ldots,k_L\rangle \in \mathcal{V}^{\otimes L} \; . \tag{3}$$

We note that the coordinates $\{z_l\}$ and hence also the dimensionality of the QFT simply play the role of parameters. In a general QFT, the expression (3) will usually only be defined perturbatively. However, if the underlying QFT is a 2D CFT, the correlator coincides with a chiral conformal block and is mathematically absolutely well-defined. This is the case we will focus on from now on. In the CFT context, a subtle point concerns the fact that conformal blocks are usually not single-valued. Uniqueness (up to phase factors) imposes strong restrictions on the fields which have to be used in Eq. (3). In particular, it is custom to choose the fields $\phi(z)$ to be primary fields with *abelian fusion* in order to avoid an exponential ground state degeneracy.

As we have discussed in Sect. 3, for any MPS of the usual kind there is an associated parent Hamiltonian which is constructed as a sum over projectors. For an ∞MPS, this type of construction turns out to be impossible. When the underlying CFT is a WZW model based on a Lie group G [8], one can nevertheless easily come up with linear operators $\mathcal{P}_k(\{z_l\})$ for each physical site k which annihilate the quantum state $|\psi\rangle$ that has been defined in Eq. (3). This is due to the existence of null fields $\chi(z)$ which are descendants of the primary fields $\phi(z)$, i.e. which can be reached by applying the symmetry generators of the WZW model [8]. A parent Hamiltonian associated with the ∞MPS $|\psi\rangle$ can then be defined as [5]

$$H = \sum_k \mathcal{P}_k\big(\{z_l\}\big)^\dagger \cdot \mathcal{P}_k\big(\{z_l\}\big) \; . \tag{4}$$

As before, the Hamiltonian is manifestly hermitean, positive and invariant under the action of G. Actually it is more appropriate to speak about a whole family of Hamiltonians which are parametrized by the choice of positions $\{z_k\}$. The construction can easily be adapted to also allow for varying representations of the physical spins.

5 Application to SU(N) Spin Models

The procedure just sketched has been suggested in [5] and then further refined in [18]. In these publications it was also established that the parent Hamiltonian associated with the $SU(2)_1$ WZW model, in fact, basically reproduces the Haldane-Shastry model. A generalization of this analysis to $SU(N)_1$ WZW models has been studied in [2, 27]. The new feature that arises here is the possibility to study alternating setups in which part of the physical spins reside in the fundamental representation and others in the anti-fundamental, see Fig. 2a.

For the WZW model at level 1, the relevant null vectors $\chi(z_k)$ for fields $\phi(z_k)$ transforming in the fundamental or the anti-fundamental representation are located at the first energy level. The associated algebraic operators which annihilate the ∞MPS $|\psi\rangle$ can then be chosen to be $\mathcal{P}_k^a\big(\{z_l\}\big) = \sum_{l(\neq k)} w_{kl}\, \mathcal{P}_{k\,b}^{a} S_l^b$ with $w_{kl} = (z_k + z_l)/(z_k - z_l)$ [2, 27]. Here, $\mathcal{P}_{k\,b}^{a}$ is a specific projection matrix which involves the spin operator $\mathbf{S}_k$ at site k and a is an sl_N Lie algebra index. The action of $\mathcal{P}_k^a\big(\{z_l\}\big)$ on the physical spins is sketched in Fig. 1c. Using the concrete form of the projection matrix $\mathcal{P}_{k\,b}^{a}$ and $d_k \in \{0, 1\}$ to distinguish between the two different types of representations, one finally finds the parent Hamiltonian

$$\begin{aligned} H &= \sum_k \mathcal{P}_{k,a}\big(\{z_l\}\big)^{\dagger} \mathcal{P}_k^a\big(\{z_l\}\big) = \sum_k \sum_{i,j(\neq k)} \bar{w}_{ki} w_{kj}\, S_i^a \mathcal{P}_{k,ab} S_j^b \\ &= \sum_k \sum_{i,j(\neq k)} \bar{w}_{ki} w_{kj} \Big\{ -\tfrac{i}{4}\tfrac{N+2}{N+1} f_{abc}\, S_i^a S_j^b S_k^c - \tfrac{N(-1)^{d_k}}{4(N+1)} d_{abc}\, S_i^a S_j^b S_k^c \; + \\ &\quad + \tfrac{N+2}{2(N+1)} \mathbf{S}_i \cdot \mathbf{S}_j \Big\} . \end{aligned} \tag{5}$$

In contrast to the Hamiltonian of the Haldane-Shastry model, this expression involves SU(N)-invariant long-range couplings between *two and three* spins (see Fig. 1b) which are mediated by the structure constants f_{abc} and by the completely symmetric rank-3 tensor d_{abc}. In the general form (5), the Hamiltonian can be used to describe 1D or 2D quantum spin systems with arbitrary positions of the spins.

Let us now specialize to the 1D setup where all spins transform in the fundamental representation and the spin locations are chosen equidistantly on the unit circle. Under these circumstances, the Hamiltonian (5) simplifies considerably and reads

$$H = C_1 \sum_{k\neq l} \frac{\mathbf{S}_k \cdot \mathbf{S}_l}{|z_k - z_l|^2} + \underbrace{C_2\, \mathbf{S}^2 + C_3\, d_{abc} S^a S^b S^c}_{\text{coupling to total spin } \mathbf{S}} + C_4 \,, \tag{6}$$

with L-dependent constants C_i. We have thus succeeded in recovering the SU(N) Haldane-Shastry model from the ∞MPS construction, at least up to terms which couple the total spin to generalized chemical potentials. Of course, these terms do not affect the solvability of the model.

The previous considerations may easily be generalized to 1D configurations in which the spins alternate between the fundamental and the anti-fundamental representation. While the resulting model does not appear to be solvable, numerical evidence using either ground state entanglement scaling [27], the determination of spin-spin correlation functions [27] or exact diagonalization [2] suggests that the theory is critical. For the last type of analysis it is convenient to interpret the system as a loop model at fugacity $N = \dim(\mathcal{V})$. This procedure, which is based on a generalized Schur–Weyl duality of SU(N) with a walled Brauer algebra, allows for a convenient graphical representation of the states and of the Hamiltonian. In this way, computations can be carried out efficiently for arbitrary values of N [2], providing substantial evidence for a critical thermodynamic limit which deviates from the $\mathrm{SU(N)}_1$ WZW model. Nevertheless, the precise identification of the critical theory is an open problem. Since the $\mathrm{SU(N)}_1$ WZW model underlying the ∞MPS construction admits a free field representation in terms of $N-1$ bosons (or N complex fermions modulo a boson), writing down concrete expressions for the CFT correlator featuring in Eq. (3) and hence the ∞MPS $|\psi\rangle$, the unique ground state of H, is straightforward, independent of the particular arrangement of spins [2, 27].

6 Conclusions and Outlook

In this note we have reviewed the general philosophy underlying the construction of ∞MPS states and their associated parent Hamiltonians. As an application we have discussed the specific example of $\mathrm{SU(N)}_1$ WZW models. As we have seen, the resulting family of Hamiltonians provides a natural generalization of the SU(N) Haldane-Shastry model which is recovered for particular types of spins and their positions. A full exposition of our results can be found in the article [2].

The $\mathrm{SU(N)}_1$ WZW theories are rather special in that they admit a realization in terms of free fields. Non-abelian features will only become visible if one considers higher level theories, i.e. $\mathrm{SU(N)}_k$ with $k \geq 2$. To our knowledge, an "optimal discretization" of these theories in the sense of Sect. 2 is currently not available. It is an interesting question whether the ∞MPS construction can remedy this deficiency. Another goal which can be pursued with ∞MPS is the systematic search for parent Hamiltonians which realize 2D non-abelian chiral SU(N) spin liquids. Since these will always be long-ranged, it is also natural to study the effect of truncating the interaction range, see [19] for the corresponding study in the context of SU(2).

Besides tuning the WZW level one can also modify the symmetry group. Here, implementations of the ∞MPS formalism have already been considered for U(1) and SO(N) [25, 26]. It immediately suggests itself to generalize the construction to spins transforming under the supergroup SU(M|N) or close derivatives such as GL(M|N) [3]. Beyond potential applications in string theory, disordered systems and statistical physics this may also provide lattice discretizations of logarithmic CFTs which are naturally associated with supergroup WZW models [22]. It would also be interesting to leave the realm of WZW models and to apply the ∞MPS construction to other types of CFTs such as minimal models.

One of the appealing features of the ∞MPS construction is the exact knowledge of the ground state of the associated Hamiltonian. The ground state can indeed be used to provide a detailed characterization of the system and to verify a number of properties, both in 1D and in 2D. First of all, this includes its criticality or topological non-triviality. The scaling of the ground state entanglement indeed gives direct access to the central charge (in 1D) and to the total quantum dimension of anyonic excitations (in 2D), respectively (see [27] for examples and additional references). Via the calculation of spin-spin correlation functions and ground state overlaps one can also infer data characterizing expected excitations such as conformal dimensions (in 1D) [18, 27] and topological spins of anyons and their modular data (in 2D) [7]. For the latter it is necessary to extend the ideas presented here from the complex plane to higher genus Riemann surfaces such as the torus.

The complete solution of a quantum system, including its thermodynamic properties, also requires knowledge about the excited states. For $SU(2)_1$ it has been shown how low-lying excited states may be constructed systematically in ∞MPS form [15] using CFT correlators involving descendant fields. Whether this still holds true for other symmetries and/or higher levels remains to be investigated.

Let us finally return to the most important aspect of the ∞MPS construction: The systematic design of quantum entanglement in a state through the coupling to an auxiliary entanglement layer. It is tempting to speculate whether this intimate link may allow to lift special structures which are, a priori, only defined in the underlying continuum theory to the associated quantum lattice model. A simple example would be the aforementioned study of excitations (in the spin model) in terms of descendent fields (excitations in the CFT). More interestingly, the continuum theory may exhibit additional structures such as a Yangian symmetry which may also be reflected in the lattice model, at least in disguise. While this is known to be true for the ∞MPS based on $SU(N)_1$ WZW theories, at least as long as the quantum spins all transform in the fundamental representation, the further exploration of these connections may lead to fruitful new insights into the properties of general quantum spin models.

Acknowlegements TQ would like to thank the organizers of the Workshop, specifically Vladimir Dobrev, for setting up such a wonderful scientific program and all its participants for their thought-provoking contributions and lively discussions. TQ is funded by the German Research Foundation (DFG) through Martin Zirnbauer's Leibniz Prize, DFG grant no. ZI 513/2-1. Additional support was received from the DFG through the SFB|TR12 "Symmetries and Universality in Mesoscopic Systems" and the Center of Excellence "Quantum Matter and Materials" (QM^2).

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Group Analysis of a Class of Nonlinear Kolmogorov Equations

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Abstract A class of $(1+2)$-dimensional diffusion-convection equations (nonlinear Kolmogorov equations) with time-dependent coefficients is studied with Lie symmetry point of view. The complete group classification is achieved using a gauging of arbitrary elements (i.e., via reducing the number of variable coefficients) with the application of equivalence transformations. Two possible gaugings are discussed in order to show how equivalence group can serve in making the optimal choice.

1 Introduction

Second-order partial differential equations of the form

$$u_t = Du_{yy} + \nu\,[K(u)]_x\,, \tag{1}$$

where D and ν are nonzero constants, K is a smooth nonlinear function of the dependent variable u, appear in various applications. In particular, they describe diffusion-convection processes [1], model an interaction of particles with two kinds of particles on a lattice [2], arise in mathematical finance, when studying agents' decisions under risk [3, 4]. Equations (1) are called in the literature diffusion-advection equations, nonlinear ultraparabolic equations and nonlinear Kolmogorov equations. They were studied from various points of view. An important study of partial differential equations and especially nonlinear ones is finding Lie groups of point transformations

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V. Dobrev (ed.), *Lie Theory and Its Applications in Physics*,
Springer Proceedings in Mathematics & Statistics 191,
DOI 10.1007/978-981-10-2636-2_23

that leave an equation under study invariant. Such symmetry transformations allow one to apply powerful and what is most important algorithmic method for finding exact solutions of a given nonlinear equation. Moreover, Lie symmetries can serve as a selection criterion of physically important models among possible ones [5]. Lie symmetries of Eq. (1) and the corresponding group invariant solutions were classified by Demetriou et al. [6]. There are also some studies on Lie symmetries of linear Kolmogorov equations [7, 8] and of constant coefficient nonlinear Kolmogorov equations of the form $u_t - u_{yy} - uu_x = f(u)$ [9].

An attempt of group classification of more general than (1) class of diffusion-convection equations in (1+2)-dimensions, namely, the equations with time dependent coefficients

$$u_t = f(t)u_{yy} - g(t)[K(u)]_x\,, \quad fgK_{uu} \neq 0, \tag{2}$$

where f and g are smooth nonvanishing functions of the variable t, K is a smooth nonlinear function of u, was recently made [10]. Nevertheless the classification of Lie symmetries was not achieved therein, in particular, the case $K = u\ln u$ was missed and dimensions of maximal Lie symmetry algebras as well as some of their basis elements for the other cases of extensions were presented incorrectly. Moreover, the important for applications case $K = u^2$ was not studied with Lie symmetry point of view at all.

In this paper we perform the complete group classification of Eq. (2). As class (2) is parameterized by three arbitrary elements, $K(u)$, $f(t)$ and $g(t)$, the group classification problem appears to be too complicated to be solved completely without modern approaches based on the usage of point equivalence transformations. One of such tools is the gauging of arbitrary elements by equivalence transformations (i.e., reducing of a class to its subclass with fewer number of arbitrary elements). To use this technique, we firstly look for the equivalence group of class (2) in Sect. 2. A gauging of arbitrary elements is performed in the same section. In Sect. 3 Lie symmetries of the simplified class are exhaustively classified. In Sect. 4 we discuss how to choose an optimal gauging among possible ones. To illustrate that the chosen gauging is optimal we also adduce results on group classification of class (2) carried out for alternative gauging.

2 Equivalence Transformations

Equivalence transformations are nondegenerate point transformations, that preserve the differential structure of the class under study, change only its arbitrary elements and form a group. There are several kinds of equivalence groups. The *usual equivalence group*, used for solving group classification problems since late 50's, consists of the nondegenerate point transformations of the independent and dependent variables and of the arbitrary elements of the class, where transformations for independent and dependent variables do not involve arbitrary elements of the class [11]. The notion of the *generalized equivalence group*, where transformations of variables of given DEs explicitly depend on arbitrary elements, appeared in works by Meleshko in middle nineties [12, 13]. The *extended equivalence group* is an equivalence group whose transformations include nonlocalities with respect to arbitrary elements [14]. The *generalized extended equivalence group* possesses the properties of both generalized and extended equivalence groups. The group classification problem becomes simpler for solving if one use the widest possible equivalence group, the comparison of usage of usual and generalized extended equivalence groups was presented recently in [15]. Moreover, in some cases the usage of generalized extended equivalence groups is the only way to present the complete group classification, see, e.g., [16].

To derive the equivalence group of the class under consideration we use the direct method [17]. The details of calculations are skipped for brevity and we only present the results.

As it is more convenient for the study of Lie symmetries to consider the equivalent form of the above class,

$$u_t = f(t)u_{yy} - g(t)k(u)u_x, \quad fgk_u \neq 0, \tag{3}$$

where f and g are smooth nonvanishing functions of the variable t, k is an arbitrary smooth nonconstant function of u, we present transformations for both K and $k = K_u$ in the theorems below.

Theorem 1 *The generalized extended equivalence group $\hat{G}^{\sim}$ of class (2) (resp. (3)) is formed by the transformations*

$$\tilde{t} = T(t), \quad \tilde{x} = \delta_1 x + \delta_2 \int g(t)\mathrm{d}t + \delta_3, \quad \tilde{y} = \delta_4 y + \delta_5, \quad \tilde{u} = \delta_6 u + \delta_7,$$
$$\tilde{f}(\tilde{t}) = \frac{{\delta_4}^2}{T_t} f(t), \quad \tilde{g}(\tilde{t}) = \frac{\varepsilon_1}{T_t} g(t),$$
$$\tilde{K}(\tilde{u}) = \frac{\delta_6}{\varepsilon_1}\left(\delta_1 K(u) + \delta_2 u + \varepsilon_2\right), \quad \left(\textit{resp.} \quad \tilde{k}(\tilde{u}) = \frac{1}{\varepsilon_1}(\delta_1 k(u) + \delta_2),\right)$$

where δ_i, $i = 1, \ldots, 7$, ε_1 and ε_2 are arbitrary constants with $\delta_1\delta_4\delta_6\varepsilon_1 \neq 0$, $T(t)$ is an arbitrary smooth function with $T_t \neq 0$.

The usual equivalence group of class (2) (resp. (3)) consists of the above transformations with $\delta_2 = 0$.

It appears that if $K = u^2$ (resp. $k = u$) the class of equations under study admits a wider equivalence group.

Theorem 2 *The generalized extended equivalence group $\hat{G}^{\sim}_1$ of the class*

$$u_t = f(t)u_{yy} - g(t)uu_x, \quad fg \neq 0, \tag{4}$$

comprises the transformations

$$\tilde{t} = T(t), \quad \tilde{x} = X(t)x + \delta_3 \int g(t)X(t)^2 \mathrm{d}t + \delta_4, \quad \tilde{y} = \delta_1 y + \delta_2,$$
$$\tilde{u} = \delta_5 \left(\frac{u}{X(t)} - \delta_6 x + \delta_3 \right), \quad \tilde{g}(\tilde{t}) = \frac{X(t)^2 g(t)}{\delta_5 T_t}, \quad \tilde{f}(\tilde{t}) = \frac{\delta_1^2 f(t)}{\delta_5 T_t},$$

where $X(t) = \left(\delta_6 \int g(t)\mathrm{d}t + \delta_7\right)^{-1}$, δ_i, $i = 1, \ldots, 7$, *are arbitrary constants with* $\delta_1\delta_5(\delta_6^2 + \delta_7^2) \neq 0$, *and* $T(t)$ *is an arbitrary smooth function with* $T_t \neq 0$.

The usual equivalence group of class (4) consists of the above transformations with $\delta_3 = \delta_6 = 0$.

Equivalence transformations generate a subset of a set of admissible transformations [18] which can be interpreted as triples, each of which consists of two fixed equations from a class and a point transformation that links these two equations. The set of admissible transformations considered with the standard operation of composition of transformations is also called the equivalence groupoid [19]. Theorems 1 and 2 give the descriptions of the equivalence groupoids of class (3) with nonlinear k and of class (4), respectively.

As there is one arbitrary function, $T(t)$, in the transformations from the group $\hat{G}^{\sim}$, we can set one of the arbitrary elements f or g of the initial class equals to a nonzero constant value. We choose to perform the gauging $g = 1$ by using the equivalence transformation

$$\tilde{t} = \int g(t)\mathrm{d}t, \quad \tilde{x} = x, \quad \tilde{u} = u. \tag{5}$$

Then, any equation from class (2) (resp. (3)) is mapped to an equation from its subclass that is singled out by the condition $g = 1$. The detailed discussion on optimal choice of gauging is presented in Sect. 4.

Without loss of generality, we can restrict ourselves to the study of the class (2) with $g = 1$ or, what is more convenient, its equivalent form

$$u_t = f(t)u_{yy} - k(u)u_x, \quad f k_u \neq 0, \tag{6}$$

since all results on symmetries, conservation laws, classical solutions and other related objects can be found for Eq. (3) using the similar results derived for Eq. (6).

The equivalence groups of class (6) and its subclass with $k = u$ are presented in the following theorems.

Theorem 3 *The usual equivalence group* $G^{\sim}$ *of class (6) consists of the transformations*

$$\tilde{t} = \varepsilon_1 t + \varepsilon_0, \quad \tilde{x} = \delta_1 x + \delta_2 t + \delta_3, \quad \tilde{y} = \delta_4 y + \delta_5, \quad \tilde{u} = \delta_6 u + \delta_7,$$
$$\tilde{f}(\tilde{t}) = \frac{{\delta_4}^2}{\varepsilon_1} f(t), \quad \tilde{k}(\tilde{u}) = \frac{1}{\varepsilon_1}(\delta_1 k(u) + \delta_2),$$

where δ_i, $i = 1, \ldots, 7$, ε_1 *and* ε_0 *are arbitrary constants with* $\delta_1\delta_4\delta_6\varepsilon_1 \neq 0$.

Theorem 4 *The usual equivalence group $G_1^{\sim}$ of the class*

$$u_t = f(t)u_{yy} - uu_x, \quad f \neq 0, \tag{7}$$

is formed by the transformations

$$\tilde{t} = \frac{\alpha t + \beta}{\gamma t + \delta}, \quad \tilde{x} = \frac{\kappa x + \mu t + \nu}{\gamma t + \delta}, \quad \tilde{y} = \lambda y + \varepsilon,$$
$$\tilde{u} = \frac{1}{\Delta}\left(\kappa(\gamma t + \delta)u - \kappa\gamma x + \delta\mu - \gamma\nu\right), \quad \tilde{f}(\tilde{t}) = \frac{\lambda^2}{\Delta}(\gamma t + \delta)^2 f(t),$$

where α, β, γ, δ, κ, μ, and ν are arbitrary constants defined up to a nonzero multiplier with $\Delta = \alpha\delta - \beta\gamma \neq 0$, $\kappa \neq 0$; λ and ε are arbitrary constants, $\lambda \neq 0$.

Theorem 4 implies that any Eq. (7) with $f = a(t + b)^{-2}$, where $a \neq 0$ and b are constants, is mapped by a point transformation to a constant-coefficient equation from the same class.

We present also the results on equivalence transformations for the subclass of class (3) singled out by the condition $f = 1$, which we will use for the comparison of the cases $f = 1$ and $g = 1$ in Sect. 4.

Theorem 5 *The generalized extended equivalence group $\hat{G}_2^{\sim}$ of the class*

$$u_t = u_{yy} - g(t)k(u)u_x, \quad gk_u \neq 0, \tag{8}$$

comprises the transformations

$$\tilde{t} = \delta_4^2 t + \delta_0, \quad \tilde{x} = \delta_1 x + \delta_2 \int g(t)\mathrm{d}t + \delta_3, \quad \tilde{y} = \delta_4 y + \delta_5, \quad \tilde{u} = \delta_6 u + \delta_7,$$
$$\tilde{g}(\tilde{t}) = \frac{\varepsilon_1}{\delta_4^2} g(t), \quad \tilde{k}(\tilde{u}) = \frac{1}{\varepsilon_1}\left(\delta_1 k(u) + \delta_2\right),$$

where δ_i, $i = 0, 1, \ldots, 7$, and ε_1 are arbitrary constants with $\delta_1\delta_4\delta_6\varepsilon_1 \neq 0$.

Theorem 6 *The generalized extended equivalence group $\hat{G}_3^{\sim}$ of the class*

$$u_t = u_{yy} - g(t)uu_x, \quad g \neq 0, \tag{9}$$

consists of the transformations

$$\tilde{t} = \delta_1^2 t + \delta_2, \quad \tilde{x} = \frac{x + \delta_4}{\gamma_1 \int g(t)\mathrm{d}t + \gamma_2} + \delta_5, \quad \tilde{y} = \delta_1 y + \delta_3,$$
$$\tilde{u} = \delta_6\left(\left(\gamma_1 \int g(t)\mathrm{d}t + \gamma_2\right) u - \gamma_1(x + \delta_4)\right), \quad \tilde{g}(\tilde{t}) = \frac{g(t)}{\delta_1^2\delta_6\left(\gamma_1 \int g(t)\mathrm{d}t + \gamma_2\right)^2},$$

where δ_i, $i = 1, \ldots, 6$, γ_1 and γ_2 are arbitrary constants with $\delta_1\delta_6(\gamma_1^2 + \gamma_2^2) \neq 0$.

3 Lie Symmetries

The group classification problem for class (3) up to $\hat{G}^{\sim}$-equivalence reduces to the similar problem for class (6) up to $G^{\sim}$-equivalence (resp. the group classification problem for class (4) up to $\hat{G}_1^{\sim}$-equivalence reduces to such a problem for class (7) up to $G_1^{\sim}$-equivalence). To solve the group classification problem for class (6) we use the classical approach based on integration of determining equations implied by the infinitesimal invariance criterion [11]. We search for symmetry operators of the form $Q = \tau(t,x,y,u)\partial_t + \xi(t,x,y,u)\partial_x + \eta(t,x,y,u)\partial_y + \theta(t,x,y,u)\partial_u$ generating one-parameter Lie groups of transformations that leave Eq. (6) invariant [11, 20]. It is required that the action of the second prolongation $Q^{(2)}$ of the operator Q on (6) vanishes identically modulo equation (6),

$$Q^{(2)}\{u_t - f(t)u_{yy} + k(u)u_x\}|_{u_t=f(t)u_{yy}-k(u)u_x} = 0. \tag{10}$$

The infinitesimal invariance criterion (10) implies the determining equations, simplest of which result in

$$\tau = \tau(t), \quad \xi = \xi(t,x), \quad \eta = \eta^1(t)y + \eta^0(t), \quad \theta = \varphi(t,x,y)u + \psi(t,x,y),$$

where τ, ξ, η^1, η^0, φ and ψ are arbitrary smooth functions of their variables. Then the rest of the determining equations are

$$\tau f_t = (2\eta^1 - \tau_t)f, \quad 2f\varphi_y = -\eta^1_t y - \eta^0_t, \tag{11}$$

$$(\varphi u + \psi)k_u + (\tau_t - \xi_x)k = \xi_t, \tag{12}$$

$$(\varphi_x u + \psi_x)k + (\varphi_t - f\varphi_{yy})u + \psi_t - f\psi_{yy} = 0. \tag{13}$$

Firstly we integrate equations (12) and (13) for k up to the $G^{\sim}$-equivalence taking into account that $k_u \neq 0$. The method of furcate split [21, 22] is further used. For any operator $Q \in A^{\max}$ Eq. (12) gives some equations on k of the general form

$$(au + b)k_u + ck = d, \tag{14}$$

where a, b, c, and d are constants. The number s of such independent equations is not greater than two, otherwise such equations form incompatible system for k. If $s = 0$, then (14) is not an equation on k but an identity, this corresponds to the case of arbitrary k. If $s = 1$, then the integration of (14) up to the $G^{\sim}$-equivalence gives three different cases: (i) $k = u^n$, $n \neq 0, 1$; (ii) $k = e^u$; (iii) $k = \ln u$. If $s = 2$, then the function k is linear in u, $k = u \bmod G^{\sim}$.

The determining Eq. (13) implies that there exist two essentially different cases of classification: I. $k_{uu} \neq 0$, and II. $k_{uu} = 0$, i.e. $k = u \bmod G^{\sim}$.

Consider firstly the case of arbitrary function k. In this case Eqs. (12) and (13) should be split with respect to k and k_u. The splitting results in the equations $\varphi = \psi = \xi_t = \tau_t - \xi_x = 0$. Therefore $\tau = c_1 t + c_2$, $\xi = c_1 x + c_3$. As $\varphi = 0$, the

Table 1 The group classification of class (6) up to the $G^{\sim}$-equivalence

No.	$f(t)$	Basis of $A^{\max}$
Arbitrary k		
1	$\forall$	∂_x, ∂_y
2	t^ρ	$\partial_x, \partial_y, 2t\partial_t + 2x\partial_x + (\rho+1)y\partial_y$
3	e^t	$\partial_x, \partial_y, 2\partial_t + y\partial_y$
4	1	$\partial_x, \partial_y, \partial_t, 2t\partial_t + 2x\partial_x + y\partial_y$
$k = u^n, n \neq 0, 1$		
5	$\forall$	$\partial_x, \partial_y, nx\partial_x + u\partial_u$
6	t^ρ	$\partial_x, \partial_y, nx\partial_x + u\partial_u, 2t\partial_t + 2x\partial_x + (\rho+1)y\partial_y$
7	e^t	$\partial_x, \partial_y, nx\partial_x + u\partial_u, 2\partial_t + y\partial_y$
8	1	$\partial_x, \partial_y, nx\partial_x + u\partial_u, \partial_t, 2t\partial_t + 2x\partial_x + y\partial_y$
$k = e^u$		
9	$\forall$	$\partial_x, \partial_y, x\partial_x + \partial_u$
10	t^ρ	$\partial_x, \partial_y, x\partial_x + \partial_u, 2t\partial_t + 2x\partial_x + (\rho+1)y\partial_y$
11	e^t	$\partial_x, \partial_y, x\partial_x + \partial_u, 2\partial_t + y\partial_y$
12	1	$\partial_x, \partial_y, x\partial_x + \partial_u, \partial_t, 2t\partial_t + 2x\partial_x + y\partial_y$
$k = \ln u$		
13	$\forall$	$\partial_x, \partial_y, t\partial_x + u\partial_u$
14	t^ρ	$\partial_x, \partial_y, t\partial_x + u\partial_u, 2t\partial_t + 2x\partial_x + (\rho+1)y\partial_y$
15	e^t	$\partial_x, \partial_y, t\partial_x + u\partial_u, 2\partial_t + y\partial_y$
16	1	$\partial_x, \partial_y, t\partial_x + u\partial_u, \partial_t, 2t\partial_t + 2x\partial_x + y\partial_y$

Here n and ρ are arbitrary nonzero constants, and $n \neq 1$

second equation of (11) implies $\eta^1_t = \eta^0_t = 0$, i.e. $\eta^1 = c_4$, and $\eta^0 = c_5$. Here c_i, $i = 1, \dots, 5$, are arbitrary constants. Then the general form of the infinitesimal generator is $Q = (c_1 t + c_2)\partial_t + (c_1 x + c_3)\partial_x + (c_4 y + c_5)\partial_y$ and the first equation of (11) takes the form

$$(c_1 t + c_2) f_t = (2c_4 - c_1) f. \tag{15}$$

This is the classifying equation for f. If f is an arbitrary nonvanishing smooth function, then the latter equation should be split with respect to f and its derivative, which results in $c_1 = c_2 = c_4 = 0$. Therefore, the kernel $A^\cap$ of the maximal Lie invariance algebras of equations from class (6) is $A^\cap = \langle \partial_x,\ \partial_y \rangle$ (Case 1 of Table 1). To perform the further classification we integrate equation (15) up to the $G^{\sim}$-equivalence. All $G^{\sim}$-inequivalent values of f that provide Lie symmetry extensions for equations from class (6) with arbitrary k are exhausted by the following values: $f = t^\rho$, $\rho \neq 0$; $f = e^t$; $f = 1$. The corresponding bases of maximal Lie invariance algebras are presented by Cases 2–4 of Table 1.

If $k = u^n$, $n \neq 0, 1$, then splitting Eqs. (12) and (13) with respect to different powers of u leads to the system $\xi_t = \psi = \varphi_x = 0$, $\varphi_t = f\varphi_{yy}$, $n\varphi + \tau_t - \xi_x = 0$. These equations together with (11) imply $\tau = c_1 t + c_2$, $\xi = (c_1 + nc_6)x + c_3$, $\eta = c_4 y + c_5$, $\varphi = c_6$, where c_i, $i = 1, \dots, 6$, are arbitrary constants. The classifying equation for f takes the form (15). Therefore, the cases of Lie symmetry extensions are given by the same forms of f as in previous case, namely, arbitrary, power, exponential and constant. See Cases 5–8 of Table 1. The dimensions of the respective Lie symmetry algebras increase by one in comparing with the case of arbitrary k. The highest dimension is five, not six as it was stated in the paper by Kumar et al. [10].

The consideration of the cases $k = e^u$ and $k = \ln u$ is rather similar to the case of $k = u^n$ with $n \neq 0, 1$, therefore, we omit the details of calculations. The classification results are presented in Cases 9–16 of Table 1.

Consider the case of linear k, then up to the equivalence we can assume $k = u$. We substitute $k = u$ to Eqs. (12) and (13) and further split them with respect to different powers of u. This leads to the system $\psi = \xi_t$, $\tau_t - \xi_x + \varphi = 0$, $\varphi_x = 0$, $\psi_x + \varphi_t - f\varphi_{yy} = 0$, and $\psi_t - f\psi_{yy} = 0$. We differentiate the first and the second equation of this system with respect to the variable y and get the additional conditions $\varphi_y = \psi_y = 0$. Then also $\psi_t = \psi_{xx} = \varphi_{tt} = 0$ and the second equation of (11) gives $\eta^1_t = \eta^0_t = 0$. The general form of the infinitesimal operator Q is $Q = (c_2t^2 + c_1t + c_0)\partial_t + ((c_2t + c_4)x + c_3t + c_5)\partial_x + (c_6y + c_7)\partial_y + ((c_4 - c_1 - c_2t)u + c_2x + c_3)\partial_u$, where c_i, $i = 0, \ldots, 7$, are arbitrary constants. The classifying equation for f is

$$(c_2t^2 + c_1t + c_0)f_t = (2c_6 - c_1 - 2c_2t)f. \tag{16}$$

If this is not an equation on f but an identity then $c_0 = c_1 = c_2 = c_6 = 0$. Therefore, the constants c_3, c_4, c_5, c_7 appearing in the infinitesimal generator Q are arbitrary and the maximal Lie invariance algebra of the Eq. (7) with arbitrary f is the four-dimensional algebra $\langle \partial_x,\ \partial_y,\ x\partial_x + u\partial_u,\ t\partial_x + \partial_u \rangle$ (Case 1 of Table 2).

The further group classification of Eq. (6) with $k = u$, i.e. Eq. (7), is equivalent to the integration of the equation on f of the form

$$(at^2 + bt + c)f_t = (d - 2at)f, \tag{17}$$

where a, b, c and d are arbitrary constants with $(a, b, c) \neq (0, 0, 0)$. Up to $G^\sim_1$-equivalence the parameter quadruple (a, b, c, d) can be assumed to belong to the set $\{(1, 0, 1, \sigma),\ (0, 1, 0, \rho),\ (0, 0, 1, 1),\ (0, 0, 1, 0)\}$, where σ, ρ are nonzero constants, $\rho \leq -1$. The proof is similar to ones presented in Vaneeva et al. [16, 23]. It is based on the fact that transformations from the equivalence group $G^\sim_1$ can be extended to the coefficients a, b, c and d as follows

$$\begin{aligned} \tilde a &= \mu(a\delta^2 - b\gamma\delta + c\gamma^2), \quad \tilde b = \mu(-2a\beta\delta + b(\alpha\delta + \beta\gamma) - 2c\alpha\gamma),\\ \tilde c &= \mu(a\beta^2 - b\alpha\beta + c\alpha^2), \quad \tilde d = \mu(d\Delta + 2a\beta\delta - 2b\beta\gamma + 2c\alpha\gamma), \end{aligned}$$

where $\Delta = \alpha\delta - \beta\gamma$ and μ is an arbitrary nonzero constant.

Integration of the Eq. (17) for four inequivalent cases of the quadruple (a, b, c, d) gives respectively $f = \frac{e^{\sigma \arctan t}}{t^2+1}$, $f = t^\rho$, $\rho \neq 0$, $f = e^t$ and $f = 1$. We further sub-

Table 2 The group classification of class (7) up to the $G_1^{\sim}$-equivalence

No.	$f(t)$	Basis of $A^{\max}$
1	$\forall$	$\partial_x, \partial_y, x\partial_x + u\partial_u, t\partial_x + \partial_u$
2	$\frac{e^{\sigma \arctan t}}{t^2+1}$	$\partial_x, \partial_y, x\partial_x + u\partial_u, t\partial_x + \partial_u, (t^2+1)\partial_t + tx\partial_x + \frac{1}{2}\sigma y\partial_y$ $+(x - tu)\partial_u$
3	t^ρ	$\partial_x, \partial_y, x\partial_x + u\partial_u, t\partial_x + \partial_u, 2t\partial_t + (\rho+1)y\partial_y - 2u\partial_u$
4	e^t	$\partial_x, \partial_y, x\partial_x + u\partial_u, t\partial_x + \partial_u, 2\partial_t + y\partial_y$
5	1	$\partial_x, \partial_y, x\partial_x + u\partial_u, t\partial_x + \partial_u, \partial_t, 2t\partial_t + y\partial_y - 2u\partial_u$

Here ρ and σ are arbitrary constants with $\rho \neq 0, -2$. Moreover $\rho \leq -1 \bmod G_1^{\sim}$

stitute the obtained inequivalent values of f into Eq. (16) and find the corresponding values of constants c_i and, therefore, the general forms of the infinitesimal generators. The results of the group classification of class (7) are presented in Table 2.

The classification lists presented in Tables 1 and 2 give exhaustive group classification of the class of variable coefficient nonlinear Kolmogorov equations (3).

4 Discussion on the Choice of the Optimal Gauging

Appropriate choice of gauging of the arbitrary elements is a crucial step in solving group classification problems. In our case the gauging $f = 1$ could seem more convenient if one look for the determining equations for finding Lie symmetries. For class (8) they have the form

$$2\eta_y = \tau_t, \quad \eta_{yy} - \eta_t = 2\varphi_y, \quad (\varphi u + \psi)gk_u + [\tau g_t + (\tau_t - \xi_x)g]k = \xi_t,$$
$$(\varphi_x u + \psi_x)gk + (\varphi_t - \varphi_{yy})u + \psi_t - \psi_{yy} = 0.$$

For the case $k \neq u$ the difference in classification is not so crucial (cf. Table 1 with Table 3). Though one can see that for $k = \ln u$ the operator $t\partial_x + u\partial_u$ appearing in Cases 13–16 of Table 1 transforms to various forms in the respective cases of Table 3. For the case $k = u$ the difficulty of group classification of the class (3) with $f = 1$ increases essentially in comparison with the gauging $g = 1$. Solving the determining equations results in the following form of the infinitesimal generator

$$Q = (c_1 t + c_0)\partial_t + [(c_2 x + c_3)\textstyle\int g(t)\mathrm{d}t + c_4 x + c_5]\partial_x +$$
$$(\tfrac{1}{2}c_1 y + c_6)\partial_y + [(c_7 - c_2\textstyle\int g(t)\mathrm{d}t)u + c_2 x + c_3]\partial_u,$$

where c_i, $i = 0, \ldots, 7$, are arbitrary constants. The classifying equation for g is the integro-differential equation $(c_1 t + c_0)g_t + \left(c_1 - c_4 + c_7 - 2c_2\int g(t)\mathrm{d}t\right)g = 0$ (cf. with the classifying Eq. (16) for f that is much simpler). The results of group classification for class (9) are presented in Table 4. Comparing Tables 2 and 4 one can

Table 3 The group classification of class (8) up to the $\hat{G}^{\sim}_{2}$-equivalence

No.	$g(t)$	Basis of $A^{\max}$
Arbitrary k		
1	$\forall$	∂_x, ∂_y
2	t^ρ	$\partial_x, \partial_y, 2t\partial_t + 2(\rho+1)x\partial_x + y\partial_y$
3	e^t	$\partial_x, \partial_y, \partial_t + x\partial_x$
4	1	$\partial_x, \partial_y, \partial_t, 2t\partial_t + 2x\partial_x + y\partial_y$
$k = u^n, n \neq 0, 1$		
5	$\forall$	$\partial_x, \partial_y, nx\partial_x + u\partial_u$
6	t^ρ	$\partial_x, \partial_y, nx\partial_x + u\partial_u, 2t\partial_t + 2(\rho+1)x\partial_x + y\partial_y$
7	e^t	$\partial_x, \partial_y, nx\partial_x + u\partial_u, \partial_t + x\partial_x$
8	1	$\partial_x, \partial_y, nx\partial_x + u\partial_u, \partial_t, 2t\partial_t + 2x\partial_x + y\partial_y$
$k = e^u$		
9	$\forall$	$\partial_x, \partial_y, x\partial_x + \partial_u$
10	t^ρ	$\partial_x, \partial_y, x\partial_x + \partial_u, 2t\partial_t + 2(\rho+1)x\partial_x + y\partial_y$
11	e^t	$\partial_x, \partial_y, x\partial_x + \partial_u, \partial_t + x\partial_x$
12	1	$\partial_x, \partial_y, x\partial_x + \partial_u, \partial_t, 2t\partial_t + 2x\partial_x + y\partial_y$
$k = \ln u$		
13	$\forall$	$\partial_x, \partial_y, \int g(t)\mathrm{d}t\, \partial_x + u\partial_u$
14_a	$t^\rho,\ \rho \neq -1$	$\partial_x, \partial_y, t^{\rho+1}\partial_x + (\rho+1)u\partial_u, 2t\partial_t + 2(\rho+1)x\partial_x + y\partial_y$
14_b	t^{-1}	$\partial_x, \partial_y, \ln t\, \partial_x + u\partial_u, 2t\partial_t + y\partial_y$
15	e^t	$\partial_x, \partial_y, e^t\partial_x + u\partial_u, \partial_t + x\partial_x$
16	1	$\partial_x, \partial_y, t\partial_x + u\partial_u, 2t\partial_t + 2x\partial_x + y\partial_y, \partial_t$

Here n and ρ are arbitrary nonzero constants

Table 4 The group classification of class (9) up to the $\hat{G}^{\sim}_{3}$-equivalence

No.	$g(t)$	Basis of $A^{\max}$
1	$\forall$	$\partial_x, \partial_y, x\partial_x + u\partial_u, \int g(t)\mathrm{d}t\, \partial_x + \partial_u$
2	$\frac{1}{t\cos^2(\nu\ln t)}$	$\partial_x, \partial_y, x\partial_x + u\partial_u, \tan(\nu\ln t)\partial_x + \nu\partial_u,$ $t\partial_t + \nu x\tan(\nu\ln t)\partial_x + \frac{1}{2}y\partial_y + \nu(\nu x - \tan(\nu\ln t)u)\partial_u$
3	$\frac{1}{\cos^2 t}$	$\partial_x, \partial_y, x\partial_x + u\partial_u, \tan t\partial_x + \partial_u, \partial_t + x\tan t\partial_x + (x - u\tan t)\partial_u$
4_a	t^ρ	$\partial_x, \partial_y, x\partial_x + u\partial_u, t^{\rho+1}\partial_x + (\rho+1)\partial_u, 2t\partial_t + 2(\rho+1)x\partial_x + y\partial_y$
4_b	t^{-1}	$\partial_x, \partial_y, x\partial_x + u\partial_u, \ln t\, \partial_x + \partial_u, 2t\partial_t + y\partial_y$
5	e^t	$\partial_x, \partial_y, x\partial_x + u\partial_u, e^t\partial_x + \partial_u, \partial_t + x\partial_x$
6	1	$\partial_x, \partial_y, x\partial_x + u\partial_u, t\partial_x + \partial_u, \partial_t, 2t\partial_t + 2x\partial_x + y\partial_y$

Here ρ and ν are arbitrary nonzero constants. Moreover $\rho < -1 \bmod \hat{G}^{\sim}_{3}$, $\rho \neq -2$

conclude that forms of the basis operators of the maximal Lie invariance algebras are more cumbersome in Table 4.

The links between equations of the form (9) are also more tricky than between equations from class (7). For example, the equation

$$u_t = u_{yy} - \frac{1}{t\cosh^2(\nu \ln t)} u u_x,$$

where the variable coefficient can be rewritten as $\frac{4}{t(t^\nu + t^{-\nu})^2}$, admits the five-dimensional maximal Lie invariance algebra with the basis operators ∂_x, ∂_y, $\tanh(\nu \ln t)\partial_x + \nu\partial_u$, $x\partial_x + u\partial_u$, and $t\partial_t - \nu x \tanh(\nu \ln t)\partial_x + \frac{1}{2}y\partial_y - \nu(\nu x - \tanh(\nu \ln t)u)\partial_u$. The equivalence of this equation and the equation

$$\tilde{u}_{\tilde{t}} = \tilde{u}_{\tilde{y}\tilde{y}} - \tilde{t}^{2\nu-1}\tilde{u}\tilde{u}_{\tilde{x}}$$

from the same class does not seem obvious. Nevertheless, there exists transformation from the equivalence group that establishes a link between these equations, which is

$$\tilde{t} = t, \quad \tilde{x} = \frac{1}{4}x(t^{2\nu} + 1), \quad \tilde{y} = y, \quad \tilde{u} = \frac{u}{t^{2\nu} + 1} + \frac{\nu}{2}x.$$

This shows that the distinguishing inequivalent cases of Lie symmetry extensions for class (9) is also more difficult task than for class (7).

Therefore, the gauging $g = 1$ is without a doubt the right choice to perform a group classification for the class (3) and especially its subclass (4).

So, is there a regular way that can help one to indicate which gauging is preferable among several possible ones? Equivalence group appears to be that indicator showing the right choice of gauging. Indeed, if we compare equivalence groups presented in Theorems 3 and 4 with those adduced in Theorems 5 and 6, we can see that equivalence groups of class (6) and its subclass (7) are of the usual type whereas equivalence groups of class (8) and its subclass (9) remain to be generalized extended as the equivalence group of the initial class. Transformations from the generalized extended groups become point only after fixing arbitrary elements and integrals of g then naturally appear in the forms of Lie symmetry generators and even in the classifying equation. This of course makes the calculations more difficult.

Therefore, the widest possible equivalence group should be necessarily found even before applying Lie invariance criterion to the equations under study in order to choose the optimal gauging and to optimize the process of group classification.

5 Conclusion

The complete group classification of class (2) is performed using the gauging of arbitrary elements by the equivalence transformations. We presented classification lists for the equivalent form of this class, namely, for class (3). The correspondences between k and K are the following: $k = u^n$, $n \neq 0, -1$, $\leftrightarrow K = u^{n+1}$; $k = u^{-1} \leftrightarrow K = \ln u$; $k = e^u \leftrightarrow K = e^u$; $k = \ln u \leftrightarrow K = u \ln u$.

Application of the widest possible (generalized extended) equivalence groups allowed us to write down classification lists in an explicit and concise form. We have also shown that the equivalence group is that indicator which helps one to choose the optimal gauging among several possible ones.

The derived Lie symmetries can be now used to to reduce the nonlinear Kolmogorov equations (2) to ordinary differential equations and, therefore, for finding exact solutions. The reductions can be achieved using two-dimensional subalgebras of the corresponding maximal Lie invariance algebras.

Acknowledgements O.V. would like to thank the Organizing Committee of LT-11 and especially Prof. Vladimir Dobrev for the hospitality and support. O.V. and Yu. K. also acknowledge the support provided by the University of Cyprus. The authors express their gratitude to Roman Popovych for useful discussions.

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Thermoelectric Characteristics of $\mathbb{Z}_k$ Parafermion Coulomb Islands

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Abstract Using the explicit rational conformal field theory partition functions for the $\mathbb{Z}_k$ parafermion quantum Hall states on a disk we compute numerically the thermoelectric power factor for Coulomb-blockaded islands at finite temperature. We demonstrate that the power factor is rather sensitive to the neutral degrees of freedom and could eventually be used to distinguish experimentally between different quantum Hall states having identical electric properties. This might help us to confirm whether non-Abelian quasiparticles, such as the Fibonacci anyons, are indeed present in the experimentally observed quantum Hall states.

1 Introduction: Non-Abelian Anyons and Topological Quantum Computation

We shall start this section with the question of what non-Abelian statistics is. It is well known that when we exchange indistinguishable particles the quantum state acquires a phase $e^{i\pi(\theta/\pi)}$ which is proportional to the statistical angle θ/π. In three-dimensional coordinate space this phase can be either 0 when the particles are bosons, or 1 when the particles are fermions. In two-dimensional space, however, this restriction is not valid and the particles can have any statistical angle between 0 and 1, that's why they are called anyons. For example, the Laughlin anyons corresponding to the fractional quantum Hall (FQH) state with filling factor $1/3$ have $\theta_L/\pi = 1/3$. In addition, while the n-particle quantum states in three dimensions are constructed as representations of the symmetric group $\mathcal{S}_n$, which are symmetric for bosons and antisymmetric for fermions, in two dimensions the n-particle states are build up as representations of the braid group $\mathcal{B}_n$. The *non-Abelian anyons* are such particles in two dimensional space whose n-particle states belong to representations of $\mathcal{B}_n$ whose dimension is bigger than 1. In terms of n-particle states this means that the non-Abelian anyons'

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V. Dobrev (ed.), *Lie Theory and Its Applications in Physics*,
Springer Proceedings in Mathematics & Statistics 191,
DOI 10.1007/978-981-10-2636-2_24

wave functions belong to degenerate multiplets and that the statistical angle θ may be a non-trivial matrix, in which case the statistical phase $\mathrm{e}^{i\theta}$ would be non-Abelian.

The anyonic states of matter are labeled by fusion paths [1] which are defined as concatenation of fusion channels and can be displayed in Bratteli diagrams. The fusion channels are denoted by the index 'c' in the fusion process of two particles of type a and b which is denoted symbolically as $\Psi_a \times \Psi_b = \sum_{c=1}^{g} N_{ab}{}^{c} \Psi_c$, where the fusion coefficients $(N_{ab})^c$ are integers which are symmetric and associative [2]. Put another way, a collection of particles $\{\Psi_a\}$ are non-Abelian anyons if $N_{ab}{}^{c} \neq 0$ for more than one c. As an example we consider the Ising anyons $\Psi_I(z) = \sigma(z) \;:\mathrm{e}^{i\frac{1}{2\sqrt{2}}\phi(z)}:$ realized in a conformal field theory (CFT) with $\widehat{u(1)} \times$ Ising symmetry, where $\phi(z)$ is a normalized $\widehat{u(1)}$ boson and σ is the chiral spin field in the Ising CFT model,

$$\sigma \times \sigma = \mathbb{I} + \psi.$$

Besides the fact that non-Abelian statistics is a new fundamental concept in particle physics it is also important for the so-called topological quantum computation (TQC) [3, 4]. In this context quantum information is encoded into the fusion channels

$$|0\rangle = (\sigma, \sigma)_{\mathbb{I}} \quad \longleftrightarrow \quad \sigma \times \sigma \to \mathbb{I}$$

$$|1\rangle = (\sigma, \sigma)_{\psi} \quad \longleftrightarrow \quad \sigma \times \sigma \to \psi,$$

which is a topological quantity–it is independent of the fusion process details, depending only on the topology of the coordinate space. Fusion channel is also independent of the anyon separation and is preserved when the two particles are separated–if we fuse two particles and then split them again, their fusion channel does not change.

The basic idea of TQC is that quantum information can be encoded into the fusion channels and the quantum gates can be implemented by braiding non-Abelian anyons. As an illustration we can consider 8 Ising anyons, which in the quantum information language can be used to encode 3 topological qubits, and transport adiabatically anyon number 7 along a complete loop around anyon number 6. Then the 8-anyons states are multiplied by a statistical phase $\left(B_6^{(8,+)}\right)^2 = X_3$ which implements the NOT gate $X_3 = \mathbb{I}_2 \otimes \mathbb{I}_2 \otimes X$ on the third qubit [3, 5].

Another promising example of non-Abelian anyons are the Fibonacci anyons [6] realized in the diagonal coset of the $\mathbb{Z}_3$ parafermion FQH states [7, 8] (or, in the three-state Pots model) as the parafermion primary field ϵ corresponding to the nontrivial orbit of the simple-current's action $\mathbb{I} = \{\Lambda_0 + \Lambda_0, \Lambda_1 + \Lambda_1, \Lambda_2 + \Lambda_2\}$ $\epsilon = \{\Lambda_0 + \Lambda_1, \Lambda_1 + \Lambda_2, \Lambda_0 + \Lambda_2\}$ with fusion rules

$$\mathbb{I} \times \mathbb{I} = \mathbb{I}, \quad \mathbb{I} \times \epsilon = \epsilon, \quad \epsilon \times \epsilon = \mathbb{I} + \epsilon.$$

The information encoding for Fibonacci anyons is again in the fusion channels, denoted by the field $\mathbb{I}$ and ϵ of the resulting fusion, however, this time for triples of anyons [6]

$$|0\rangle = ((\epsilon, \epsilon)_{\mathbb{I}}, \epsilon)_\epsilon$$
$$|1\rangle = ((\epsilon, \epsilon)_\epsilon, \epsilon)_\epsilon,$$

and the third state $((\epsilon, \epsilon)_\epsilon, \epsilon)_{\mathbb{I}}$, having a trivial quantum dimension, decouples from the previous two and is called non-computational [6].

Given that non-Abelian statistics is a new concept a natural question arises: how can it be discovered? In the rest of this paper we will discuss, how non-Abelian statistics might be observed in Coulomb-blockade conductance spectrometry and by measuring certain thermoelectric characteristics of Coulomb-blockaded islands.

2 Coulomb Island Spectroscopy

Let us consider a Coulomb-blockaded island, which can be realized as a quantum dot with drain, source and a side gate which is equivalent to single-electron transistor, like in Ref. [9]. This setup is an almost closed quantum system, which still has discrete energy levels and is like a large artificial atom but is highly tunable by Aharonov–Bohm flux and side-gate voltage.

2.1 Coulomb Island's Conductance–CFT Approach

In this section we are going to use the chiral Grand canonical partition function for a disk fractional quantum Hall sample to calculate its thermoelectric properties. In such samples the bulk is inert due to the nonzero mobility gap while the edge is mobile and can be described by a rational unitary CFT [2, 10, 11]. The Grand partition function is

$$Z_{\text{disk}}(\tau, \zeta) = \mathrm{tr}_{\mathcal{H}_{\text{edge}}}\, \mathrm{e}^{-\beta(H-\mu N)} = \mathrm{tr}_{\mathcal{H}_{\text{edge}}}\, \mathrm{e}^{2\pi i\tau(L_0 - c/24)} e^{2\pi i \zeta J_0}, \tag{1}$$

where $H = \hbar \frac{2\pi v_F}{L}\left(L_0 - \frac{c}{24}\right)$ is the edge Hamiltonian expressed in terms of the zero mode L_0 of the Virasoro stress-energy tensor (with a central charge c), $N = -\sqrt{\nu_H}\, J_0$ is the particle number on the edge expressed in terms of the J_0 zero mode of the $\widehat{u(1)}$ current and ν_H is the FQH filling factor. The trace is taken over the edge-states' Hilbert space $\mathcal{H}_{\text{edge}}$ which depends on the number of quasiparticles localized in the bulk. The temperature T and chemical potential μ are related to the modular parameters [2] τ and ζ by

$$\tau = i\pi \frac{T_0}{T}, \quad T_0 = \frac{\hbar v_F}{\pi k_B L}, \quad \zeta = i \frac{\mu}{2\pi k_B T},$$

where v_F is the Fermi velocity at the edge and L is the circumference of the edge. The CFT disk partition function in presence of AB flux $\phi = eB.A/h$, threading the disk, is modified by simply shifting the chemical potential [12]

$$\zeta \to \zeta + \phi\tau, \quad Z^{\phi}_{\mathrm{disk}}(\tau,\zeta) = Z_{\mathrm{disk}}(\tau,\zeta+\phi\tau).$$

It is interesting to note that the side-gate voltage V_g affects the quantum dot (QD) in the same way as the AB flux [13], through the (continuous) externally induced electric charge [14] on the QD $-C_g V_g/e \equiv \nu_H \phi = Q_{\mathrm{ext}}$, where C_g is the capacitance of the gate.

The thermodynamic Grand potential on the edge is defined as usual as $\Omega(T,\mu) = -k_B T \ln Z_{\mathrm{disk}}(\tau,\zeta)$ and the electron number can be computed as [15]

$$\langle N_{\mathrm{el}}(\phi)\rangle_{\beta,\mu_N} = \nu_H \left(\phi + \frac{\mu_N}{\Delta\epsilon}\right) + \frac{1}{2\pi^2}\left(\frac{T}{T_0}\right)\frac{\partial}{\partial\phi}\ln Z_{\phi}(T,\mu_N) \qquad (2)$$

Similarly the edge conductance in the linear-response regime can be computed by [15]

$$G(\phi) = \frac{e^2}{h}\left(\nu_H + \frac{1}{2\pi^2}\left(\frac{T}{T_0}\right)\frac{\partial^2}{\partial\phi^2}\ln Z_{\phi}(T,0)\right). \qquad (3)$$

There are certain difficulties in measuring QD conductance and distinguishing FQH states: experiments are performed in extreme conditions (high B, very low T) with expensive samples. Moreover, there are many doppelgangers [16], i.e., distinct states with the same conductance patterns with differences in the neutral sector where G is not sensitive. Under these conditions the sequential tunneling of electrons one-by-one is dominating the cotunneling, which is a higher-order process associated with almost simultaneous virtual tunneling of pairs of electrons [14], that will not be considered here.

3 Thermopower: A Finer Spectroscopic Tool

The thermopower, or the Seebeck coefficient, is defined [13, 14] as the potential difference V between the leads of the SET when $\Delta T = T_R - T_L \ll T_L$, under the condition that $I = 0$. Usually thermopower is expressed as $S = G_T/G$, where G and G_T are electric and thermal conductances, respectively. However, for the SET configuration $G_T \to 0$ and $G \to 0$ in the Coulomb blockade valleys, so that it is more appropriate to use another expression [14]

$$S \equiv - \lim_{\Delta T \to 0} \frac{V}{\Delta T}\bigg|_{I=0} = -\frac{\langle \varepsilon \rangle}{eT},$$

where $\langle \varepsilon \rangle$ is the average energy of the tunneling electrons. In the CFT approach using the Grand partition function for the FQH edge of the QD we can express the average tunneling energy as the difference between the energy of the QD with $N+1$ electrons and that of the QD with N electrons

$$\langle \varepsilon \rangle^{\phi}_{\beta,\mu_N} = \frac{E^{\beta,\mu_{N+1}}_{\mathrm{QD}}(\phi) - E^{\beta,\mu_N}_{\mathrm{QD}}(\phi)}{\langle N(\phi)\rangle_{\beta,\mu_{N+1}} - \langle N(\phi)\rangle_{\beta,\mu_N}}$$

The total QD energy (in the Grand canonical ensemble) can be written as

$$E^{\beta,\mu_N}_{\mathrm{QD}}(\phi) = \sum_{i=1}^{N_0} E_i + \langle H_{\mathrm{CFT}}(\phi)\rangle_{\beta,\mu_N},$$

where E_i, $i = 1, \ldots, N_0$ are the occupied single-electron states in the bulk of the QD, and $\langle \cdots \rangle_{\beta,\mu}$ is the Grand canonical average of H_{CFT} on the edge at inverse temperature $\beta = (k_B T)^{-1}$ and chemical potential μ. The chemical potentials μ_N and μ_{N+1} of the QD with N and $N+1$ electrons can be chosen as [13]

$$\mu_N = -\frac{1}{2}\Delta\epsilon, \quad \mu_{N+1} = \frac{1}{2}\Delta\epsilon, \quad \Delta\epsilon = \hbar\frac{2\pi v_F}{L}.$$

Another important observable is the thermoelectric power factor [13] $\mathcal{P}_T$ which is defined as the electric power P generated by the temperature difference ΔT

$$P = V^2/R = \mathcal{P}_T(\Delta T)^2, \quad \mathcal{P}_T = S^2 G, \tag{4}$$

where $R = 1/G$ is the electric resistance of the CB island. The average tunneling energy can be expressed in terms of the CFT averages of the Hamiltonian and particle number as follows [13]

$$\langle \varepsilon \rangle^{\phi}_{\beta,\mu_N} = \frac{\langle H_{\mathrm{CFT}}(\phi)\rangle_{\beta,\mu_{N+1}} - \langle H_{\mathrm{CFT}}(\phi)\rangle_{\beta,\mu_N}}{\langle N(\phi)\rangle_{\beta,\mu_{N+1}} - \langle N(\phi)\rangle_{\beta,\mu_N}}. \tag{5}$$

The electron number average can be computed from Eq. (2) and the edge energy average can be obtained from the Grand potential $\Omega_\phi(T,\mu_N) = -k_B T \ln Z_\phi(T,\mu)$ in presence of AB flux ϕ as

$$\langle H_{\mathrm{CFT}}(\phi)\rangle_{\beta,\mu_N} = \Omega_\phi(T,\mu_N) - T\frac{\partial \Omega_\phi(T,\mu_N)}{\partial T} - \mu_N \frac{\partial \Omega_\phi(T,\mu_N)}{\partial \mu}. \tag{6}$$

4 $\mathbb{Z}_k$ Parafermion Quantum Hall Islands

The CFT for the $\mathbb{Z}_k$ parafermion quantum Hall islands (or QDs) contains an electric charge part $\widehat{u(1)}$ and a neutral part which is realized as a parafermion diagonal coset [7]

$$\left(\widehat{u(1)} \otimes \frac{\widehat{su(k)}_1 \oplus \widehat{su(k)}_1}{\widehat{su(k)}_2}\right)^{\mathbb{Z}_k}$$

The total disk partition function for the $\mathbb{Z}_k$ parafermion quantum Hall islands [7] is labeled by two integers l mod $k+2$ and ρ mod k satisfying $l-\rho \leq \rho$ mod k and can be written as follows

$$\chi_{l,\rho}(\tau,\zeta) = \sum_{s=0}^{k-1} K_{l+s(k+2)}(\tau, k\zeta; k(k+2))\mathrm{ch}(\Lambda_{l-\rho+s} + \Lambda_{\rho+s})(\tau), \qquad (7)$$

where $K_l(\tau, k\zeta; k(k+2))$ are the chiral partition functions of the $\widehat{u(1)}$ part while $\mathrm{ch}(\Lambda_\mu + \Lambda_\rho)(\tau)$ are the characters of the neutral part of the CFT. The $\widehat{u(1)}$ part corresponds to Luttinger liquid partition function (with compactification radius $R_c = 1/m$)

$$K_l(\tau,\zeta;m) = \frac{\mathrm{CZ}}{\eta(\tau)} \sum_{n=-\infty}^{\infty} q^{\frac{m}{2}\left(n+\frac{l}{m}\right)^2} \mathrm{e}^{2\pi i\zeta\left(n+\frac{l}{m}\right)},$$

where the modular parameter is related to the temperature

$$q = \mathrm{e}^{-\beta\Delta\varepsilon} = \mathrm{e}^{2\pi i\tau}, \quad \Delta\varepsilon = \hbar\frac{2\pi v_F}{L}$$

and the Dedekind function and Cappelli–Zemba factors [2, 11] are given by

$$\eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n), \quad \mathrm{CZ} = \mathrm{e}^{-\pi\nu_H\frac{(\mathrm{Im}\,\zeta)^2}{\mathrm{Im}\,\tau}}.$$

The neutral partition function are labeled by a level-2 weight $\Lambda_\mu + \Lambda_\rho$ with the condition $0 \leq \mu \leq \rho \leq k-1$ and have the form [7]

$$\mathrm{ch}_{\sigma,Q}(\tau) = q^{\Delta(\sigma)-\frac{c}{24}} \sum_{\substack{m_1,m_2,\ldots,m_{k-1}=0 \\ \sum_{i=1}^{k-1} i\, m_i \equiv Q \mod k}}^{\infty} \frac{q^{\underline{m}.C^{-1}.(\underline{m}-\Lambda_\sigma)}}{(q)_{m_1}\cdots(q)_{m_{k-1}}},$$

$$(q)_n = \prod_{j=1}^{n}(1-q^j), \quad \Delta(\sigma) = \frac{\sigma(k-\sigma)}{2k(k+2)}, \quad c = \frac{2(k-1)}{k+2}$$

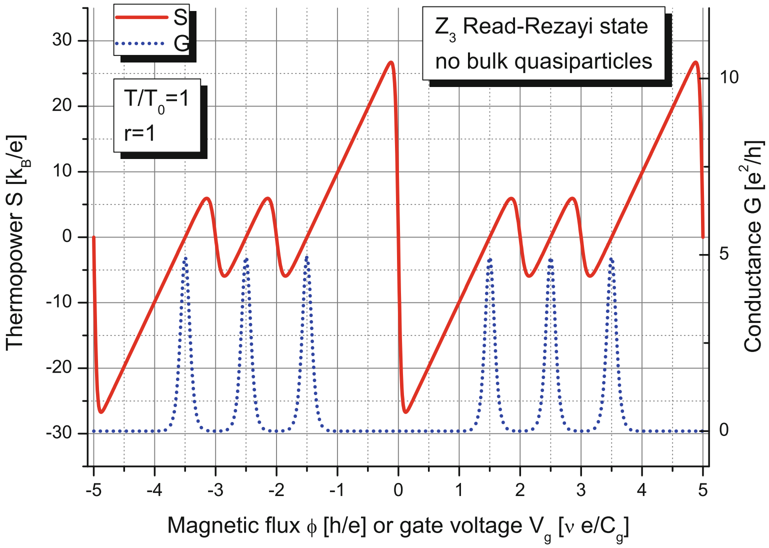

Fig. 1 Conductance peaks (*right Y* scale) and thermopower (*left Y* scale) for the $\mathbb{Z}_3$ parafermion FQH state without bulk quasiparticles

where $\underline{m} = (m_1, \ldots, m_{k-1})$, $0 \le \sigma \le Q \le k-1$ and C^{-1} is the inverse Cartan matrix for $su(k)$. The coset weight labels are related to σ and Q by $\mu = Q - \sigma$, $\rho = Q$. Using the explicit formulas (5) for the thermopower in terms of the average tunneling energy expressed in terms of the Grand canonical averages (2) and (6) and the partition function (7), as well as Eq. (3) for the conductance, we can compute the thermopower for the $\mathbb{Z}_3$ parafermion FQH state. We plot in Fig. 1 the electric conductance and thermopower as functions of the AB flux ϕ, or, equivalently as functions of the side-gate voltage V_g for the $\mathbb{Z}_3$ parafermion FQH state without quasiparticles in the bulk, i.e., for $l = 0$ and $\rho = 0$. Just like the thermopower of metallic quantum dots [14], we see in Fig. 1 that the peaks of the conductance precisely corresponds to the (continuous in the limit $T \to 0$) zeros of the thermopower.

Similarly, we can compute from Eq. (4) the power factor $\mathcal{P}_T$ for the $\mathbb{Z}_3$ parafermion FQH state without quasiparticles in the bulk ($l = 0$ and $\rho = 0$), which is plotted together with the conductance in Fig. 2.

5 Conclusion and Perspectives

The thermoelectric characteristics of Coulomb blockaded QDs, such as the thermopower and especially the thermoelectric power factor, appear to be more sensitive to the neutral modes in the FQH liquid than the tunneling conductance. These could be used as experimental signatures to identify (non-Abelian) Fibonacci anyons [6],

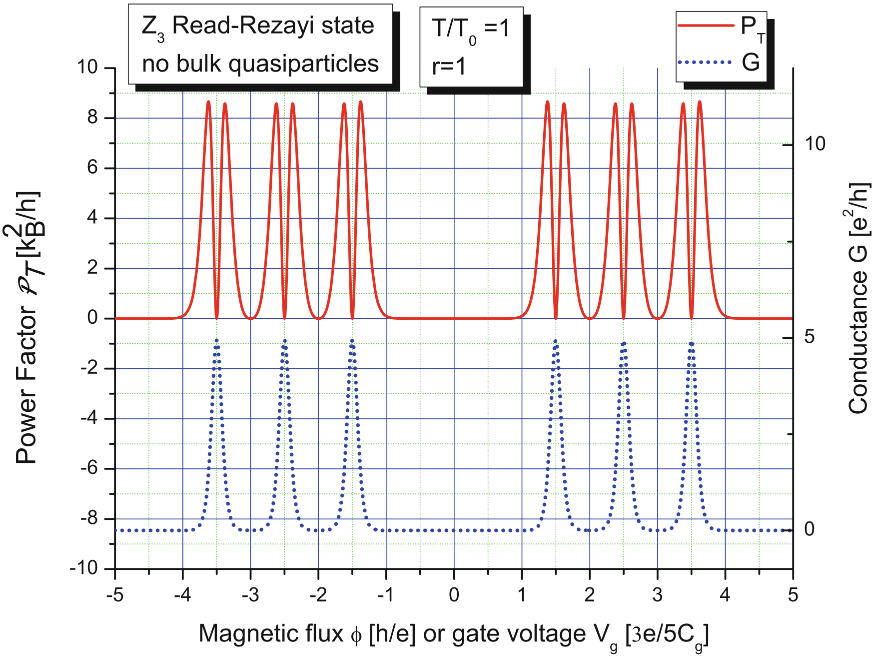

Fig. 2 Power factor and conductance for the $\mathbb{Z}_3$ parafermion FQH state without bulk quasiparticles

which are believed to exist in the $\nu_H = 12/5$ quantum Hall state. This experimentally observed FQH state [17] might be a realization of the particle–hole conjugate of the $\mathbb{Z}_3$ parafermion quantum Hall state in the second Landau level with filling factor $\nu_H = 3 - k/(k+2)$ for $k = 3$.

A recent experiment [18] demonstrated that the power factor of a Coulomb blockaded quantum dot might be directly measurable like the observable plotted in Fig. 3c there. This could allow us to estimate from the experiment the ratio between the Fermi velocities of the charged and neutral edge modes by comparing with the power factor profile computed theoretically from the CFT [8]. Finally, this possibility to distinguish neutral characteristics of FQH states is bringing a new hope that we could eventually decide whether Fibonacci anyons are indeed realized in Nature.

Acknowledgements I thank Andrea Cappelli, Guillermo Zemba and Ady Stern for many helpful discussions. This work has been partially supported by the Alexander von Humboldt Foundation under the Return Fellowship and Equipment Subsidies Programs and by the Bulgarian Science Fund under Contract No. DFNI-E 01/2 and DFNI-T 02/6.

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First Order Hamiltonian Operators of Differential-Geometric Type in 2D

Paolo Lorenzoni and Andrea Savoldi

Abstract We present an alternative approach to the problem of classification of first order Hamiltonian operators of differential-geometric type in 2D.

1 Introduction

Multi-dimensional Hamiltonian operators of hydrodynamic type have been introduced in 1984 by Dubrovin and Novikov [1]. In two dimensions, they are given by

$$P^{ij} = g^{ij}(u)\frac{d}{dx} + b^{ij}_k(u)u^k_x + \tilde{g}^{ij}(u)\frac{d}{dy} + \tilde{b}^{ij}_k(u)u^k_y, \tag{1}$$

where $u = (u^1, \dots, u^n)$. In the non-degenerate case g and $\tilde{g}$ define a pair of compatible flat (pseudo)-metrics [1–3] and satisfy a set of additional constraints coming from the skew-symmetry condition and the Jacobi identity [2].

Hamiltonian operators of this kind have been classified up to $n = 4$ components [4]. A full classification has also been obtained in some special cases: in the semi-simple case (that is, when the affinor $L^i_j = \tilde{g}^{ik}g_{kj}$ has distinct eigenvalues) [3], and in the case of a direct sum of Jordan blocks with distinct eigenvalues [4].

In this paper we are going to present an alternative approach to the classification problem. For simplicity we will consider the case of a direct sum of Jordan blocks with distinct eigenvalues, providing full details in the cases $n = 2, 3$.

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V. Dobrev (ed.), *Lie Theory and Its Applications in Physics*,
Springer Proceedings in Mathematics & Statistics 191,
DOI 10.1007/978-981-10-2636-2_25

1.1 Mokhov's Conditions Rewritten

The set of relations determining when an operator of the form (1) defines a Hamiltonian operator was found by Mokhov (see [2, 3] for further details). Recently, it has been proved that these conditions can be rewritten in the following form [4]:

Theorem 1 *Two flat metrics g and $\tilde{g}$ define a two-dimensional first order Hamiltonian operator of hydrodynamic type if and only if the following conditions are satisfied:*

- *the contravariant (pseudo)-metric $\tilde{g}^{jk}$ is linear in the flat coordinates of g^{ij},*
- *the Nijenhuis torsion of the affinor $L^i_j = \tilde{g}^{il} g_{lj}$ vanishes,*
- *$\tilde{g}$ is a Killing tensor of g: $\nabla^i \tilde{g}^{kj} + \nabla^k \tilde{g}^{ij} + \nabla^j \tilde{g}^{ik} = 0$.*

Moreover, the flatness of g and the above conditions imply the flatness of $\tilde{g}$.

The classification approach presented in [4] is based on the fact that a pair of symmetric (constant) bivectors can be reduced to a normal form called the Segre normal form. The procedure used can be summarized as follows. Working in the flat coordinates of the first metric g, and setting $\tilde{g}^{ij} = a^{ij}_k u^k + \tilde{g}^{ij}_0$ (where a^{ij}_k and $\tilde{g}^{ij}_0$ are constant), we have that g and $\tilde{g}_0$ can be fixed using the Segre classification. The unknowns a^{ij}_k can be found imposing first of all the Killing condition, and then the vanishing of the Nijenhuis torsion.

2 Mokhov's Relations in a Non Holonomic Frame

We can address the problem of classification in a different way. Let us consider the case where L is one Jordan block.

As stated in [5] there exists a moving frame $e_{(1)}, \dots, e_{(n)}$ such that:

$$\begin{aligned} L^i_k e^k_{(p)} &= \lambda e^i_{(p)} + e^i_{(p-1)}, \\ g_{ij} e^i_{(p)} e^j_{(q)} &= \pm \eta_{pq}, \end{aligned}$$

where we set $e^i_{(p-1)} = 0$ if $p = 1$, the metric $\eta_{pq} = \delta_{p,n+1-q}$ is the usual constant anti-diagonal metric and λ is the eigenvalue of L. We point out that the frame is not assumed to be holonomic, which means that

$$[e_{(p)}, e_{(q)}] = c^s_{pq} e_{(s)}. \tag{2}$$

Let us now write the Mokhov's conditions in the non holonomic frame $e_{(i)}$, $i = 1, \dots, n$.

- *Vanishing of the Nijenhuis torsion.*
  According to [5] the vanishing of the Nijenhuis torsion of L implies that

$$e_{(p)}(\lambda) = 0, \qquad \forall p = 1, \dots, n-1. \tag{3}$$

Using this condition, we obtain

$$\begin{aligned}
&[Le_{(i)}, Le_{(j)}] - L[e_{(i)}, Le_{(j)}] - L[Le_{(i)}, e_{(j)}] + L^2[e_{(i)}, e_{(j)}] = \\
&= (L-\lambda I)^2[e_{(i)}, e_{(j)}] - (L-\lambda I)[e_{(i)}, e_{(j-1)}] - (L-\lambda I)[e_{(i-1)}, e_{(j)}] \ + \\
&+ [e_{(i-1)}, e_{(j-1)}] - e_{(i)}(\lambda)e_{(j-1)} + e_{(j)}(\lambda)e_{(i-1)} = \\
&= \sum_{k=1}^{n-2}(c^{k+2}_{ij} - c^{k+1}_{i,j-1} - c^{k+1}_{i-1,j} + c^{k}_{i-1,j-1})e_{(k)} - (c^{n}_{i,j-1} + c^{n}_{i-1,j} \ - \\
&- c^{n-1}_{i-1,j-1})e_{(n-1)} + c^{n}_{i-1,j-1}e_{(n)} - e_{(i)}(\lambda)e_{(j-1)} + e_{(j)}(\lambda)e_{(i-1)} = 0.
\end{aligned}$$

- *Killing condition*
  In a non holonomic frame, the Christoffel symbols are defined as

$$\nabla_{e_{(p)}} e_{(q)} = \Gamma^{s}_{qp} e_{(s)}, \tag{4}$$

and can be written in terms of the coefficients of the commutators $[e_{(p)}, e_{(q)}]$ and of the scalar products $\eta_{pq} = g_{ij} e^{i}_{(p)} e^{j}_{(q)}$ as

$$\Gamma^{s}_{qp} = \frac{1}{2}\eta^{st}(c_{qtp} + c_{ptq} - c_{tqp}), \tag{5}$$

where $c_{tqp} = \eta_{tl} c^{l}_{qp}$. In this context, the Killing condition reads

$$(\nabla_l L^k_m) g_{kn} + (\nabla_n L^k_l) g_{km} + (\nabla_m L^k_n) g_{kl} = 0. \tag{6}$$

Multiplying by $e^{l}_{(p)}, e^{m}_{(q)}, e^{n}_{(r)}$, taking the sum over l, m, n, we get, after some computations

$$\begin{aligned}
&\nabla_{e_{(p)}}(L^k_m) e^{m}_{(q)} g_{kn} e^{n}_{(r)} + (\nabla_{e_{(r)}} L^k_l) e^{l}_{(p)} g_{km} e^{m}_{(q)} + (\nabla_{e_{(q)}} L^k_n) e^{n}_{(r)} g_{kl} e^{l}_{(p)} = \\
&c^{n+1-p}_{r,q-1} + c^{n+1-r}_{p,q-1} + c^{n+1-r}_{q,p-1} + c^{n+1-q}_{r,p-1} + c^{n+1-q}_{p,r-1} + c^{n+1-p}_{q,r-1} \\
&+ e_{(p)}(\lambda)\eta_{qr} + e_{(r)}(\lambda)\eta_{pq} + e_{(q)}(\lambda)\eta_{rp} = 0.
\end{aligned}$$

- *Vanishing of the curvature*:

$$e_{(q)}(\Gamma^{s}_{rp}) - e_{(r)}(\Gamma^{s}_{rq}) + \Gamma^{l}_{rp}\Gamma^{s}_{lq} - \Gamma^{l}_{rq}\Gamma^{s}_{lp} + c^{l}_{pq}\Gamma^{s}_{rl} = 0. \tag{7}$$

Thus, summarizing we have the following conditions

1. $e_{(p)}(\lambda) = 0, \qquad \forall p = 1, \dots, n-1$
2. For $i < j$:

$$\sum_{k=1}^{n-2}(c_{ij}^{k+2} - c_{i,j-1}^{k+1} - c_{i-1,j}^{k+1} + c_{i-1,j-1}^{k})e_{(k)} - (c_{i,j-1}^{n} + c_{i-1,j}^{n} - \\ - c_{i-1,j-1}^{n-1})e_{(n-1)} + c_{i-1,j-1}^{n}e_{(n)} - e_{(i)}(\lambda)e_{(j-1)} + e_{(j)}(\lambda)e_{(i-1)} = 0.$$

3. For $p \leq q \leq r$:

$$c_{r,q-1}^{n+1-p} + c_{p,q-1}^{n+1-r} + c_{q,p-1}^{n+1-r} + c_{r,p-1}^{n+1-q} + c_{p,r-1}^{n+1-q} + c_{q,r-1}^{n+1-p} + \\ e_{(p)}(\lambda)\eta_{qr} + e_{(r)}(\lambda)\eta_{pq} + e_{(q)}(\lambda)\eta_{rp} = 0.$$

4. Vanishing of the curvature (7).

Let us know discuss in detail the cases $n = 2, 3$.

2.1 One 2×2 Jordan Block

Let us consider the case $n = 2$. Applying the previous conditions we obtain

$$c_{12}^{1} = 0, \qquad c_{12}^{2} = e_{(2)}(\lambda).$$

In other words we have the following commutations relation

$$[e_{(1)}, e_{(2)}] = e_{(2)}(\lambda)e_{(2)}.$$

Applying the definition, let us compute now the Christoffel symbols in the non holonomic frame. We have

$$\Gamma_{11}^{1} = -e_{(2)}(\lambda),\ \Gamma_{11}^{2} = 0,\ \Gamma_{12}^{1} = 0,\ \Gamma_{21}^{1} = 0, \\ \Gamma_{22}^{1} = 0,\ \Gamma_{12}^{2} = 0,\ \Gamma_{21}^{2} = e_{(2)}(\lambda),\ \Gamma_{22}^{2} = 0.$$

The vanishing of the curvature (7) implies

$$e_{(1)}(e_{(2)}(\lambda)) = \left(e_{(2)}(\lambda)\right)^2, \tag{8}$$

$$e_{(2)}(e_{(2)}(\lambda)) = 0. \tag{9}$$

It is a straightforward computation to check that the above condition coincides with the condition

$$[\tilde{e}_{(1)}, \tilde{e}_{(2)}] = 0,$$

for the frame

$$(\tilde{e}_{(1)}, \tilde{e}_{(2)}) = (e_{(1)}, e_{(2)})\ J^{-1}, \tag{10}$$

where J is the orthogonal transformation

$$J = \begin{pmatrix} \frac{1}{e_{(2)}(\lambda)} & 0 \\ 0 & e_{(2)}(\lambda) \end{pmatrix}.$$

Therefore, the new frame $\tilde{e}_{(1)}, \tilde{e}_{(2)}$ is holonomic. Notice that after this orthogonal transformation we have

$$JLJ^{-1} = \begin{pmatrix} \lambda & \frac{1}{(e_{(2)}(\lambda))^2} \\ 0 & \lambda \end{pmatrix}$$

In the Mokhov case this is exactly the transformation reducing the affinor to the Mokhov's form.

The system given by (8) and (9) for the unknown function $f = e_{(2)}(\lambda)$, written in the new holonomic frame $\tilde{e}_i = \frac{\partial}{\partial \tilde{u}^i}$, reads

$$\frac{\partial f}{\partial \tilde{u}^1} = f^3, \tag{11}$$

$$\frac{\partial f}{\partial \tilde{u}^2} = 0. \tag{12}$$

The general solution is given by $f(\tilde{u}^1) = \frac{1}{\sqrt{-2\tilde{u}^1 + C_1}}$. Using (10), one can easily see that

$$\tilde{e}_{(1)}(\lambda) = 0, \qquad \tilde{e}_{(2)}(\lambda) = 1,$$

which implies $\lambda = \tilde{u}^2 + C_2$.

Notice that up to shifts of $\tilde{u}^1, \tilde{u}^2$, L coincides with Mokhov's example.

Remark 1 The case $\lambda = const$ is trivial. Indeed, the starting frame is already holonomic, and Mokhov's condition implies $L = const$.

2.2 One 3 × 3 Jordan Block

Let us consider the case $n = 3$. In this case applying conditions 1, 2 and 3 we obtain

$$c^2_{12} = c^3_{12} = c^3_{13} = c^1_{23} = 0,\ c^2_{23} = -c^1_{13},\ c^2_{13} = e_{(3)}(\lambda),$$
$$c^3_{23} = \frac{1}{2} e_{(3)}(\lambda),\ c^1_{12} = -\frac{1}{2} e_{(3)}(\lambda).$$

In other words, we have the following commutations relations

$$
\begin{aligned}
[e_{(1)}, e_{(2)}] &= -\frac{1}{2}e_{(3)}(\lambda)e_{(1)},\\
[e_{(1)}, e_{(3)}] &= c^1_{13}e_{(1)} + e_{(3)}(\lambda)e_{(2)},\\
[e_{(2)}, e_{(3)}] &= -c^1_{13}e_{(2)} + \frac{1}{2}e_{(3)}(\lambda)e_{(3)},
\end{aligned}
$$

where c^1_{13} is an arbitrary function of u^1, u^2, u^3.

Assuming $e_{(3)}(\lambda) \neq 0$ we can reduce L to the Mokhov's form by an orthogonal transformation:

$$
JLJ^{-1} = \begin{pmatrix} \lambda & -\frac{1}{e_{(3)}(\lambda)} & -4\frac{c^1_{13}}{(e_{(3)}(\lambda))^3} \\ 0 & \lambda & -\frac{1}{e_{(3)}(\lambda)} \\ 0 & 0 & \lambda \end{pmatrix},
$$

$$
J = \begin{pmatrix} \frac{1}{e_{(3)}(\lambda)} & -2\frac{c^1_{13}}{(e_{(3)}(\lambda))^2} & -2\frac{(c^1_{13})^2}{(e_{(3)}(\lambda))^3} \\ 0 & -1 & -2\frac{c^1_{13}}{(e_{(3)}(\lambda))} \\ 0 & 0 & e_{(3)}(\lambda) \end{pmatrix}.
$$

Applying the definition, let us compute now the Christoffel symbols in the non holonomic frame. We have

$$
\begin{aligned}
&\Gamma^1_{11} = 0,\ \Gamma^2_{11} = 0,\ \Gamma^3_{11} = 0,\ \Gamma^1_{12} = -\frac{1}{2}e_{(3)}(\lambda),\ \Gamma^2_{12} = 0,\ \Gamma^3_{12} = 0,\\
&\Gamma^1_{13} = -c^1_{13},\ \Gamma^2_{13} = 0,\ \Gamma^3_{13} = 0,\ \Gamma^1_{31} = 0,\ \Gamma^2_{31} = e_{(3)}(\lambda),\ \Gamma^3_{31} = 0,\\
&\Gamma^1_{21} = -e_{(3)}(\lambda),\ \Gamma^2_{21} = 0,\ \Gamma^3_{21} = 0,\ \Gamma^1_{22} = c^1_{13},\ \Gamma^2_{22} = 0,\ \Gamma^3_{22} = 0,\\
&\Gamma^1_{23} = 0,\ \Gamma^2_{23} = 0,\ \Gamma^3_{23} = 0,\ \Gamma^1_{32} = 0,\ \Gamma^2_{32} = -c^1_{13}, \Gamma^3_{32} = \frac{1}{2}e_{(3)}(\lambda),\\
&\Gamma^1_{33} = 0,\ \Gamma^2_{33} = 0,\ \Gamma^3_{33} = c^1_{13}.
\end{aligned}
$$

The vanishing of the curvature (7) implies

$$
e_{(3)}(e_{(3)}(\lambda)) = c^1_{13}e_{(3)}(\lambda), \tag{13}
$$

$$
e_{(1)}(c^1_{13}) = \frac{1}{2}[e_{(3)}(\lambda)]^2, \tag{14}
$$

$$
e_{(2)}(c^1_{13}) = \frac{1}{2}c^1_{13}e_{(3)}(\lambda), \tag{15}
$$

$$
e_{(3)}(c^1_{13}) = 2[c^1_{13}]^2. \tag{16}
$$

It is a straightforward computation that the above condition coincide with the condition

$$[\tilde{e}_{(i)}, \tilde{e}_{(j)}] = 0,$$

for the frame

$$(\tilde{e}_{(1)}, \tilde{e}_{(2)}, \tilde{e}_{(3)}) = (e_{(1)}, e_{(2)}, e_{(3)})\ J^{-1}. \tag{17}$$

Thus, again the new frame is holonomic. Using the commutativity conditions and conditions

$$e_{(1)}(\lambda) = 0, \tag{18}$$

$$e_{(2)}(\lambda) = 0, \tag{19}$$

we can complete the system (13)–(16) for the unknown functions $f = e_{(3)}(\lambda)$ and $c = c^1_{13}$ obtaining

$$e_{(1)}(f) = 0, \qquad e_{(2)}(f) = \frac{1}{2} f^2, \qquad e_{(3)}(f) = cf,$$
$$e_{(1)}(c) = \frac{1}{2} f^2, \qquad e_{(2)}(c) = \frac{1}{2} cf, \qquad e_{(3)}(c) = 2c^2.$$

This system written in the new holonomic frame $\tilde{e}_i = \frac{\partial}{\partial \tilde{u}^i}$ has the general solution

$$c = \frac{4\tilde{u}^1 + C_2}{(2C_1 + \tilde{u}^2)^3}, \qquad f = \frac{2}{2C_1 + \tilde{u}^2}.$$

Finally, using (17)–(19), one can easily see that

$$\tilde{e}_{(1)}(\lambda) = 0, \qquad \tilde{e}_{(2)}(\lambda) = 0, \qquad \tilde{e}_{(3)}(\lambda) = 1,$$

and then $\lambda = u^3 + C_3$. Up to a shift of $\tilde{u}^1$, $\tilde{u}^2$ and $\tilde{u}^3$, this result coincides with the functions providing Mokhov's solutions.

If λ is constant we obtain

$$[e_{(1)}, e_{(2)}] = 0, \qquad [e_{(1)}, e_{(3)}] = ce_{(1)}, \qquad [e_{(2)}, e_{(3)}] = -ce_{(2)},$$

with

$$e_{(1)}(c) = 0, \qquad e_{(2)}(c) = 0, \qquad e_{(3)}(c) = 2c^2.$$

It is easy to check that the frame

$$\tilde{e}_{(1)} = c^{\frac{1}{2}}\, e_{(1)}, \quad \tilde{e}_{(2)} = c^{-\frac{1}{2}}\, e_{(2)}, \quad \tilde{e}_{(3)} = c^{-\frac{3}{2}}\, e_{(3)}$$

is holonomic. Using this fact one can prove that $c = (\tilde{u}^3)^2$ and that

$$\tilde{L} = \begin{pmatrix} \lambda & (\tilde{u}^3)^{-2} & 0 \\ 0 & \lambda & (\tilde{u}^3)^{-2} \\ 0 & 0 & \lambda \end{pmatrix}, \qquad \tilde{\eta} = (\tilde{u}^3)^2 \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

In the new coordinates $\hat{u}^1 = \tilde{u}^1 - \frac{(\tilde{u}^2)^2}{2\tilde{u}^3}$, $\hat{u}^2 = -\frac{\tilde{u}^2}{\tilde{u}^3}$, $\hat{u}^3 = -\frac{1}{\tilde{u}^3}$ we obtain the formulas

$$\hat{L} = \begin{pmatrix} \lambda & \hat{u}^3 & -2\hat{u}^2 \\ 0 & \lambda & \hat{u}^3 \\ 0 & 0 & \lambda \end{pmatrix}, \qquad \hat{\eta} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

that coincide with those obtained in [4].

Acknowledgements We thank Jenya Ferapontov for useful discussions. P.L. is partially supported by the Italian MIUR Research Project *Teorie geometriche e analitiche dei sistemi Hamiltoniani in dimensioni finite e infinite* and by GNFM Progetto Giovani 2014 *Aspetti geometrici e analitici dei sistemi integrabili*.

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Exact Solutions for Generalized KdV Equations with Variable Coefficients Using the Equivalence Method

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Abstract Using an example of variable-coefficient KdV equations we compare effectiveness of the "equivalence method" and the "extended mapping transformation method". It is shown that the "equivalence method" is more efficient. A formula for generation of exact solutions for variable-coefficient KdV equations is derived.

1 Introduction

A number of models for different types of wave processes (including gravity waves and waves in plasma) are reducible to the classical Korteweg–de Vries (KdV) equation or its generalizations. This explains a great interest of researchers in seeking new techniques for finding exact solutions of such equations. Unfortunately the majority of the proposed techniques lead to the equivalent forms of the solutions which are known already. This is because the equivalence of the models and the corresponding solutions is not systematically investigated.

In [1] exact solutions for the "general" KdV equations with variable coefficients of the form

$$u_t - 3M\gamma(t)uu_x + \gamma(t)u_{xxx} + 2\beta(t)u + (\alpha(t) + \beta(t)x)u_x = 0, \tag{1}$$

with $M\gamma \neq 0$ were constructed using the so-called extended mapping transformation method. Here α, β and γ are arbitrary smooth functions of the variable t with $\gamma \neq 0$ and M is a nonzero constant.

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V. Dobrev (ed.), *Lie Theory and Its Applications in Physics*,
Springer Proceedings in Mathematics & Statistics 191,
DOI 10.1007/978-981-10-2636-2_26

In this paper we would like to show that the equivalence transformations are much more efficient tools for finding exact solutions for this model.

2 Derivation of the Solution Formula via the Equivalence Method

It was shown in [2] that any equation from the class (1) is reduced to the standard KdV equation

$$\tilde{u}_{\tilde{t}} - 6\tilde{u}\tilde{u}_{\tilde{x}} + \tilde{u}_{\tilde{x}\tilde{x}\tilde{x}} = 0 \tag{2}$$

by a point transformation (see Example 4 therein). We derive the most general form of this transformation, that is

$$\tilde{t} = \delta_1^3 \int \gamma(t) e^{-3\int \beta(t)dt} dt + \delta_0 \tag{3}$$

$$\tilde{x} = \delta_1 e^{-\int \beta(t)dt} x + \int e^{-\int \beta(t)dt} \Big(\delta_2 \gamma(t) e^{-2\int \beta(t)dt} - \delta_1 \alpha(t) \Big) dt + \delta_3, \tag{4}$$

$$\tilde{u} = \frac{M}{2\delta_1^2} e^{2\int \beta(t)dt} u - \frac{\delta_2}{6\delta_1^3},$$

where δ_j, $j = 0, 1, 2, 3$, are arbitrary constants with $\delta_1 \neq 0$. Then the formula for the generating solutions of Eq. (1) from solutions of the Eq. (2) has the form

$$u = \frac{2\delta_1^2}{M} e^{-2\int \beta(t)dt} \left[\tilde{u}\left(\tilde{t}, \tilde{x}\right) + \frac{\delta_2}{6\delta_1^3} \right], \tag{5}$$

where $\tilde{u}$ is an exact solution of the Eq. (2) and the variables $\tilde{t}$ and $\tilde{x}$ should be replaced by expressions (3) and (4), respectively. See a collection of solutions of the KdV equation (2), for example, in [3].

Using (5) one can construct a number of exact solutions (of different types!) for equations from the class (1). For example, the two-soliton solution of (2) has the form

$$\tilde{u} = -2\frac{\partial^2}{\partial \tilde{x}^2} \ln \left(1 + b_1 e^{a_1\tilde{x} - a_1^3\tilde{t}} + b_2 e^{a_2\tilde{x} - a_2^3\tilde{t}} + A b_1 b_2 e^{(a_1+a_2)\tilde{x} - (a_1^3+a_2^3)\tilde{t}} \right),$$

where a_1, a_2, b_1 and b_2 are arbitrary constants, $A = \left(\frac{a_1-a_2}{a_1+a_2}\right)^2$. This solution leads to the following solution of the Eq. (1):

$$u = \frac{\delta_2}{3M\delta_1} e^{-2\int \beta(t)dt} - \frac{4}{M}\frac{\partial^2}{\partial x^2} \ln\left(1 + b_1 e^{\theta_1} + b_2 e^{\theta_2} + A b_1 b_2 e^{\theta_1+\theta_2}\right),$$

where a_1, a_2, b_1 and b_2 are arbitrary constants, $A = \left(\frac{a_1-a_2}{a_1+a_2}\right)^2$,

$$\theta_i = a_i\delta_1 e^{-\int \beta(t)dt}x + c_i + \\ + a_i \int e^{-\int \beta(t)dt}\left((\delta_2 - a_i^2\delta_1^3)\gamma(t)e^{-2\int \beta(t)dt} - \delta_1\alpha(t)\right)dt,$$

where $c_i = a_i\delta_3 - a_i^3\delta_0$ are constants, $i = 1, 2$. Using the formula (5) one can construct multi-soliton, rational, "one-soliton+one-pole" solutions and, of course, solutions in terms of Jacobi elliptic functions for equations from the class (1) from known solutions of the classical KdV equation.

Concerning the method suggested in [1], it is based mainly on seeking "new general solutions" of first-order ordinary differential equation

$$\phi'^2 = a_0 + a_1\phi + a_2\phi^2 + a_3\phi^3 + a_4\phi^4,$$

where $\phi = \phi(\xi)$ is the unknown function and a_i, $i = 0, \ldots, 4$, are constant parameters. It is well known for a long time that solutions of such equations can be expressed in terms of Jacobi elliptic functions (which are reduced for some values of parameters to trigonometric, hyperbolic or rational functions) [4]. Note that seeking solutions of this equation in the form (4) of [1] does not provide new solutions but only equivalent forms of known solutions. This is a common error in finding exact solutions described in [5]. Consider for simplicity the trigonometric solution found in [1] (see Case 3). It can be checked by direct substitution that in fact this is a solution only for $r = \pm 1$. Thus, consider the particular solution

$$\phi = \frac{1+\sin\xi}{1+\sin\xi \pm \cos\xi} \tag{6}$$

of the equation $\phi'^2 = \frac{1}{4}\left(1 - 2\phi + 2\phi^2\right)^2$. This is the equation appearing in Case 3 of [1] for $r = \pm 1$. Its general solution is

$$\phi = \frac{1}{2}\left(1 \pm \tan\frac{\xi + C}{2}\right) = \frac{1}{2}\left(1 \pm \frac{\sin\frac{\xi+C}{2}\cos\frac{\xi-C}{2}}{\cos\frac{\xi+C}{2}\cos\frac{\xi-C}{2}}\right) \\ = \frac{1}{2}\frac{\cos\xi + \cos C \pm \sin\xi \pm \sin C}{\cos\xi + \cos C},$$

where C is an arbitrary constant. If we set $C = \pi/4$ and perform the shift of the variable ξ on $-\pi/4$ then, taking into account that $\sin(\xi - \pi/4) = \sqrt{2}(\sin\xi - \cos x)/2$, $\cos(\xi - \pi/4) = \sqrt{2}(\sin\xi + \cos\xi)/2$, we get exactly (6) with the positive sign of $\cos\xi$. The solution (6) with the negative sign is equivalent to that with the positive sign up to the reflection $\xi \to -\xi$ and the simultaneous shift on π since $\cos(\pi - x) = -\cos\xi$ and $\sin(\pi - x) = \sin\xi$.

Consider one more equation on ϕ presented in [1], i.e., the equation

$$\phi'^2 = \frac{1}{4}\left(2\phi^2 + (1-m)(1-2\phi)\right)\left(2\phi^2 + (1+m)(1-2\phi)\right).$$

Using the "improved method" particular solutions of this equation were constructed in the form

$$\phi = \frac{\mathrm{cn}(\xi, m)}{\pm 1 \pm \mathrm{sn}(\xi, m) + \mathrm{cn}(\xi, m)}, \quad \phi = \frac{\mathrm{cn}(\xi, m)}{\pm 1 \mp \mathrm{sn}(\xi, m) + \mathrm{cn}(\xi, m)},$$

whereas its general solution can be represented as

$$\phi = \frac{1}{2} + \frac{1}{2\delta}\mathrm{sn}\left(\frac{\delta}{2}\xi + C, \tilde{m}\right),$$

where C is an arbitrary constant, $\delta = m + \sqrt{m^2 - 1}$ and $\tilde{m} = (1 + 8m^2(m^2 - 1) + 4m(2m^2 - 1)\sqrt{m^2 - 1})^{-\frac{1}{2}}$. Formulas for various connections between Jacobi elliptic functions can be found, for example, in [6].

3 Conclusion

In this paper we have shown that each equation from the class (1) is similar to the classical KdV equation (2) with respect to a point transformation. For such equations the equivalence-based approach [2, 7] works much better than other existing methods since it allows one to use the variety of known solutions of the classical KdV equation. Moreover, we have also shown that the "extended mapping deformation method" cannot provide new solutions but only equivalent to known ones. When one deals with variable coefficients KdV or mKdV equations it is necessary to check firstly whether equations under study are reducible to the classical KdV or mKdV equations. The corresponding criteria are given, e.g., in [2] in the course of the study admissible transformations (called also form-preserving or allowed transformations, see definitions in [8–10]) within the classes

$$u_t + f(t)uu_x + g(t)u_{xxx} + h(t)u + (p(t) + q(t)x)u_x + k(t)x + l(t) = 0$$

and

$$u_t + f(t)u^2u_x + g(t)u_{xxx} + h(t)u + (p(t) + q(t)x)u_x + k(t)uu_x + l(t) = 0.$$

Acknowledgements O. Braginets (née Kuriksha) is grateful to the Organizing Committee of LT-11 for the hospitality and support.

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Part IV
Representation Theory

Classifying $A_{\mathfrak{q}}(\lambda)$ Modules by Their Dirac Cohomology

Pavle Pandžić

Abstract This talk is a preliminary report on the joint work with Jing-Song Huang and David Vogan. The main question we address is: to what extent is an $A_{\mathfrak{q}}(\lambda)$ module determined by its Dirac cohomology? The focus of the talk is not so much on explaining this question and its answer, which are mentioned briefly at the end. Rather, the focus is on introducing the whole setting and giving some background material about representation theory, especially the notion of Dirac cohomology.

1 Real Reductive Groups and Their Representations

1.1 Real Reductive Groups

A Lie group G is called reductive if its complexified Lie algebra $\mathfrak{g}$ is reductive, i.e., $\mathfrak{g}$ is the direct sum of its center and simple ideals. We are interested in connected real reductive Lie groups G, with Cartan involution θ, such that $K = G^\theta$ is a (maximal) compact subgroup of G.

The main examples of G, which are sufficient for our purposes, are closed subgroups of $GL(n, \mathbb{C})$, stable under $\theta(g) =^t \bar{g}^{-1}$. For example, G could be $SL(n, \mathbb{R})$, $U(p, q)$, $Sp(2n, \mathbb{R})$, or $O(p, q)_0$. (Here the subscript 0 denotes the connected component of the identity.) The corresponding K are $SO(n) \subset SL(n, \mathbb{R})$; $U(p) \times U(q) \subset U(p, q)$; $U(n) \subset Sp(2n, \mathbb{R})$; $(O(p) \times O(q))_0 \subset O(p, q)_0$.

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V. Dobrev (ed.), *Lie Theory and Its Applications in Physics*,
Springer Proceedings in Mathematics & Statistics 191,
DOI 10.1007/978-981-10-2636-2_27

1.2 Representations

A representation of G is a complex topological vector space V with a continuous G-action by linear operators. More precise definitions would require that the action map

$$G \times V \to V$$

be continuous, or that the map

$$G \to GL(V)$$

be continuous, where $GL(V)$ denotes the groups of invertible continuous linear operators on V equipped with strong topology. These conditions are equivalent under reasonable assumptions on V.

Group representations are the main objects of harmonic analysis and have many applications.

1.3 $(\mathfrak{g}, K)$-*Modules*

To study algebraic properties of representations, it is convenient to introduce their algebraic analogs, $(\mathfrak{g}, K)$-modules. For a representation V of G, let V_K be the space of K-finite vectors in V, i.e., the space of vectors $v \in V$ such that the span of Kv is finite-dimensional. The space V_K has an action of the Lie algebra $\mathfrak{g}_0$ of G. Namely, one can show that each K-finite vector satisfies an elliptic differential equation, so it is in particular smooth. This implies that one can differentiate the G-action to obtain an action of $\mathfrak{g}_0$ on such vectors. Thus the complexified Lie algebra $\mathfrak{g} = (\mathfrak{g}_0)_{\mathbb{C}}$ also acts on V_K.

A $(\mathfrak{g}, K)$-module is a vector space M with a Lie algebra action of $\mathfrak{g}$ and a locally finite action of K, which are compatible, i.e., induce the same action of $\mathfrak{k}_0$, the Lie algebra of K. A typical example is $M = V_K$ as above.

Any $(\mathfrak{g}, K)$-module M can be decomposed under K as

$$M = \bigoplus_{\delta \in \hat{K}} m_\delta E_\delta.$$

Here $\hat{K}$ denotes the set of (isomorphism classes of) irreducible finite-dimensional representations of K, and for each $\delta \in \hat{K}$, E_δ is the space of δ and m_δ is the multiplicity of δ in M. All δ with $m_\delta > 0$ are called the K-types of M. The existence of such a decomposition is one of the basic properties of locally finite representations of compact groups.

M is a Harish-Chandra module if it is finitely generated and all m_δ are finite.

1.4 Example: $G = SU(1,1) \cong SL(2,\mathbb{R})$.

The complexified Lie algebra of G is $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{C})$, consisting of 2x2 matrices of trace 0. $\mathfrak{g}$ has a basis

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

The possible irreducible $(\mathfrak{g}, K)$-modules can be described by the following pictures:

$$\begin{array}{cccc} \bullet & \bullet & \bullet & \dots \\ k & k+2 & k+4 & \dots \end{array} \tag{1}$$

$$\begin{array}{cccc} \dots & \bullet & \bullet & \bullet \\ \dots & -k-4 & -k-2 & -k \end{array} \tag{2}$$

$$\begin{array}{cccc} \bullet & \bullet & \dots & \bullet \\ -n & -n+2 & \dots & n \end{array} \tag{3}$$

$$\begin{array}{ccccc} \dots & \bullet & \bullet & \bullet & \dots \\ \dots & i-2 & i & i+2 & \dots \end{array} \tag{4}$$

where $k > 0$, $n \geq 0$ and i are integers.

Each dot represents a K-type, which is in this case simply a one-dimensional h-eigenspace. The numbers are the h-eigenvalues. e raises the eigenvalue by 2, and f lowers it by 2.

Each of the first three pictures determines a unique irreducible $(\mathfrak{g}, K)$-module. There are however many non-isomorphic modules corresponding to the fourth picture. In order to distinguish between them, one can use the concept of infinitesimal character which we introduce below.

1.5 Infinitesimal Character

Let $U(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$, i.e., the associative algebra with unit, generated by $\mathfrak{g}$, with relations

$$xy - yx = [x, y], \qquad x, y \in \mathfrak{g}.$$

Let $Z(\mathfrak{g})$ be the center of $U(\mathfrak{g})$. By a version of Schur's lemma, all $z \in Z(\mathfrak{g})$ act as scalars on any irreducible $(\mathfrak{g}, K)$-module M. This defines the infinitesimal character of M, $\chi_M : Z(\mathfrak{g}) \to \mathbb{C}$.

Harish-Chandra proved that $Z(\mathfrak{g}) \cong P(\mathfrak{h}^*)^W$, where $P(\mathfrak{h}^*)$ denotes the algebra of polynomial functions on the dual of a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, and W is the Weyl group of $(\mathfrak{g}, \mathfrak{h})$, a certain finite group generated by reflections. In typical matrix examples, $\mathfrak{h}$ can be taken to consist of diagonal matrices in $\mathfrak{g}$, and W acts on $\mathfrak{h}^*$ by operations like permuting, or changing signs, of coordinates.

Since any character of the polynomial algebra $P(\mathfrak{h}^*)$ is given by evaluation at an element of $\mathfrak{h}^*$, it follows that infinitesimal characters correspond to the elements of $\mathfrak{h}^*/W$.

The simplest nontrivial element of $Z(\mathfrak{g})$ is the Casimir element

$$\mathrm{Cas}_{\mathfrak{g}} = \sum b_i d_i,$$

where b_i and d_i are dual bases of $\mathfrak{g}$ with respect to the (slightly modified) Killing form B. For semisimple Lie algebras, the Killing form is defined by

$$B(x, y) = \mathrm{tr}(\mathrm{ad}\, x\, \mathrm{ad}\, y), \qquad x, y \in \mathfrak{g}$$

and if $\mathfrak{g}$ has a center, B can be modified on the center in order to get a nondegenerate form. For matrix groups, another choice is to take

$$B(x, y) = \mathrm{tr}(xy) \qquad x, y \in \mathfrak{g}.$$

For $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$, $Z(\mathfrak{g})$ consists of polynomials in

$$\mathrm{Cas}_{\mathfrak{g}} = \frac{1}{2}h^2 + ef + fe.$$

The infinitesimal character of an irreducible module M can thus be determined from the scalar by which $\mathrm{Cas}_{\mathfrak{g}}$ acts on M. To complete the list of irreducible modules in Sect. 1.4, one shows that for each complex λ which is not an integer of the same parity as $i + 1$, there is a unique module corresponding to the picture (4) on which $\mathrm{Cas}_{\mathfrak{g}}$ acts by λ^2.

2 Dirac Operators and Dirac Cohomology

2.1 The Clifford Algebra for G

Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition. Here $\mathfrak{k}$ and $\mathfrak{p}$ are the ± 1 eigenspaces of the Cartan involution, which was already mentioned as an involution of G, but now it is differentiated and complexified to get an involution of $\mathfrak{g}$. Note that $\mathfrak{k}$ is the complexified Lie algebra of K.

The Clifford algebra $C(\mathfrak{p})$ of $\mathfrak{p}$ with respect to B is the associative algebra with unit, generated by $\mathfrak{p}$, with relations

$$xy + yx = -2B(x, y).$$

2.2 The Dirac Operator for G

Let b_i be any basis of $\mathfrak{p}$ and let d_i be the dual basis with respect to B. The Dirac operator for G is defined by the formula

$$D = \sum_i b_i \otimes d_i \qquad \in U(\mathfrak{g}) \otimes C(\mathfrak{p}).$$

It is easy to see that D is independent of the choice of b_i, and K-invariant for the adjoint action on both factors.

This Dirac operator was introduced by Parthasarathy [21] in order to construct the discrete series representations. He also proved that D^2 is the spin Laplacean:

$$D^2 = -\operatorname{Cas}_{\mathfrak{g}} \otimes 1 + \operatorname{Cas}_{\mathfrak{k}_\Delta} + \text{constant}.$$

Here $\operatorname{Cas}_{\mathfrak{g}}$ respectively $\operatorname{Cas}_{\mathfrak{k}_\Delta}$ are the Casimir elements of $U(\mathfrak{g})$ respectively $U(\mathfrak{k}_\Delta)$, and $\mathfrak{k}_\Delta$ is the diagonal copy of $\mathfrak{k}$ in $U(\mathfrak{g}) \otimes C(\mathfrak{p})$, defined by $\mathfrak{k} \hookrightarrow U(\mathfrak{g})$ and $\mathfrak{k} \to \mathfrak{so}(\mathfrak{p}) \hookrightarrow C(\mathfrak{p})$.

2.3 Dirac Cohomology

Let M be a $(\mathfrak{g}, K)$-module, and let S be a spin module for $C(\mathfrak{p})$. Recall that S is constructed by choosing a pair of dual maximal isotropic subspaces $\mathfrak{p}^+$ and $\mathfrak{p}^-$ and putting

$$S = \bigwedge \mathfrak{p}^+,$$

with $\mathfrak{p}^+$ acting by wedging and $\mathfrak{p}^-$ by contracting. If $\dim \mathfrak{p}$ is even, then $\mathfrak{p} = \mathfrak{p}^+ \oplus \mathfrak{p}^-$, so this completely determines S, and one shows that S is the unique simple $C(\mathfrak{p})$-module. If $\dim \mathfrak{p}$ is odd, there is an element Z of $\mathfrak{p}$ such that $B(Z, Z) = 1$, and such that $\mathfrak{p} = \mathfrak{p}^+ \oplus \mathfrak{p}^- \oplus \mathbb{C}Z$. In this case there are two ways to make Z act on S: either by i on $\bigwedge^{\text{even}} \mathfrak{p}^+$ and by $-i$ on $\bigwedge^{\text{odd}} \mathfrak{p}^+$, or by $-i$ on $\bigwedge^{\text{even}} \mathfrak{p}^+$ and by i on $\bigwedge^{\text{odd}} \mathfrak{p}^+$. This gives two non-isomorphic $C(\mathfrak{p})$-modules, both of them simple, and these two are the only simple $C(\mathfrak{p})$-modules. They are both called spin modules.

Now we consider $M \otimes S$ as a module for the algebra $U(\mathfrak{g}) \otimes C(\mathfrak{p})$ in the obvious way. In particular, the Dirac operator D acts on $M \otimes S$. Following [28], we define the Dirac cohomology of M as

$$H_D(M) = \ker D / \operatorname{Im} D \cap \ker D.$$

Then $H_D(M)$ is a module for the spin double cover $\widetilde{K}$ of K.

Suppose M is unitary, i.e., there is a Hermitian inner product $\langle\,,\rangle$ on M such that K acts by unitary operators and the real Lie algebra $\mathfrak{g}_0$ of G acts by skew-Hermitian operators. Then there is a natural Hermitian inner product on $M \otimes S$ such that D is self adjoint with respect to this inner product. It follows that

$$H_D(M) = \ker D = \ker D^2.$$

Furthermore, $D^2 \geq 0$. The last inequality is called Parthasarathy's Dirac inequality and it was proved in [22]. It is a very useful necessary condition for unitarity, and it was used in several classification results.

2.4 Example: $G = SU(1,1) \cong SL(2,\mathbb{R})$

The modules corresponding to pictures (1)–(3) have $H_D \neq 0$. For each such M, $H_D(M)$ consists of $\widetilde{K}$-types corresponding to the highest weight$+1$ and/or the lowest weight-1.

The modules corresponding to picture (4) all have $H_D = 0$.

2.5 Vogan's Conjecture

Let $\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{a}$ be a fundamental Cartan subalgebra of $\mathfrak{g}$ (i.e., $\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{k}$). View $\mathfrak{t}^* \subset \mathfrak{h}^*$ via extension by 0 over $\mathfrak{a}$.

The following result was conjectured by Vogan [28], and proved by Huang-Pandžić [6].

Theorem 1 *Assume M has infinitesimal character and $H_D(M)$ contains a $\widetilde{K}$-type E_γ of highest weight $\gamma \in \mathfrak{t}^*$.*

Then the infinitesimal character of M is $\gamma + \rho_{\mathfrak{k}}$ up to the Weyl group of $(\mathfrak{g}, \mathfrak{h})$.

2.6 Motivation

Irreducible unitary M with $H_D \neq 0$ are interesting:

- discrete series representations (implicit in [21]);
- many of the $A_q(\lambda)$ modules [10];
- unitary highest weight modules [9, 11];
- some unipotent reps [2, 3];
- also finite-dimensional modules [10, 17].

The study of Dirac cohomology is related to the unitarity problem, for example through Dirac inequality and its improvements. Furthermore, irreducible unitary M with $H_D \neq 0$ should form a nice part of the unitary dual. Dirac cohomology is also related to classical topics like generalized Weyl character formula, generalized Bott–Borel–Weil Theorem, construction of the discrete series representations, and multiplicities of automorphic forms. See [8] for details.

There are connections between Dirac cohomology and $\mathfrak{n}$-cohomology in special cases [9], and to $(\mathfrak{g}, K)$-cohomology [10] (more details below). There are also relations to characters and branching problems [12]. Furthermore, there are several generalizations to other settings:

- quadratic subalgebras [17], with D replaced by a cubic version [5, 16];
- certain Lie superalgebras [7];
- affine Lie algebras [14];
- graded affine Hecke algebras and p-adic groups [4];
- noncommutative equivariant cohomology [1, 18].

It is also possible to construct reps with $H_D \neq 0$ via "algebraic Dirac induction" [20, 23, 24]. Finally, there is a translation principle for the Euler characteristic of H_D, i.e., the Dirac index [19].

2.7 $(\mathfrak{g}, K)$-Cohomology

Let X be a $(\mathfrak{g}, K)$-module with the same infinitesimal character as a finite-dimensional module F. The (twisted) $(\mathfrak{g}, K)$-cohomology of X is the space $H(\mathfrak{g}, K; X) = \mathrm{Ext}^{\cdot}_{(\mathfrak{g},K)}(F, X)$.

If X is unitary, then

$$H(\mathfrak{g}, K; X) = \mathrm{Hom}_{\widetilde{K}}(H_D(F), H_D(X)).$$

(Or twice this if $\dim \mathfrak{p}$ is odd.) See [10] for details and explanations.

3 $A_{\mathfrak{q}}(\lambda)$ Modules

3.1 Definition of $A_{\mathfrak{q}}(\lambda)$ Modules

Let $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ be a θ-stable parabolic subalgebra of $\mathfrak{g}$, i.e., the sum of nonnegative eigenspaces of $\mathrm{ad}(H)$, where H is some fixed element of $i\mathfrak{t}_0$ (recall that $\mathfrak{h}_0 = \mathfrak{t}_0 \oplus \mathfrak{a}_0$ is a fundamental Cartan subalgebra of $\mathfrak{g}_0$). The Levi subalgebra $\mathfrak{l}$ of $\mathfrak{q}$ is the zero eigenspace of $\mathrm{ad}(H)$, while the nilradical $\mathfrak{u}$ of $\mathfrak{q}$ is the sum of positive eigenspaces of $\mathrm{ad}(H)$. We choose positive roots for $(\mathfrak{g}, \mathfrak{h})$ and for $(\mathfrak{l}, \mathfrak{h})$ so that

$$\Delta^+(\mathfrak{g}) = \Delta^+(\mathfrak{l}) \cup \Delta(\mathfrak{u}).$$

As usual, we denote by ρ the half sum of roots in $\Delta^+(\mathfrak{g})$, etc.

The Levi subalgebra $\mathfrak{l}$ of $\mathfrak{q}$ is real, i.e., $\mathfrak{l}$ is the complexification of a subalgebra $\mathfrak{l}_0$ of $\mathfrak{g}_0$. Let L denote the connected subgroup of G corresponding to $\mathfrak{l}_0$. Let $\lambda \in \mathfrak{l}^*$ be admissible, i.e., λ is the complexified differential of a unitary character of L satisfying the following positivity condition:

$$\langle \alpha, \lambda|_{\mathfrak{t}} \rangle \geq 0, \qquad \text{for all} \quad \alpha \in \Delta(\mathfrak{u}).$$

Then λ is orthogonal to all roots of $\mathfrak{l}$, so we can view λ as an element of $\mathfrak{h}^*$.

$A_{\mathfrak{q}}(\lambda)$ modules can be defined by the following result of Vogan and Zuckerman [29].

Theorem 2 ([27, 29]) *Let $\mathfrak{q}$ be a θ-stable parabolic subalgebra of $\mathfrak{g}$ and let $\lambda \in \mathfrak{h}^*$ be admissible as above. Then there is a unique irreducible unitary $(\mathfrak{g}, K)$-module $A_{\mathfrak{q}}(\lambda)$ with the following properties:*

(i) The restriction of $A_{\mathfrak{q}}(\lambda)$ to $\mathfrak{k}$ contains the representation with highest weight $\mu(\mathfrak{q}, \lambda) = \lambda|_{\mathfrak{t}} + 2\rho(\mathfrak{u} \cap \mathfrak{p})$, where $\rho(\mathfrak{u} \cap \mathfrak{p})$ denotes the half sum of (positive) roots in $\mathfrak{u} \cap \mathfrak{p}$;

(ii) $A_{\mathfrak{q}}(\lambda)$ has infinitesimal character $\lambda + \rho$;

(iii) If the representation of $\mathfrak{k}$ occurs in $A_{\mathfrak{q}}(\lambda)$, then its highest weight is of the form

$$\mu(\mathfrak{q}, \lambda) + \sum_{\beta \in \Delta(\mathfrak{u} \cap \mathfrak{p})} n_\beta \beta \tag{5}$$

with n_β non-negative integers. In particular, $\mu(\mathfrak{q}, \lambda)$ is the lowest K-type of $A_{\mathfrak{q}}(\lambda)$ (and its multiplicity is 1).

Vogan and Zuckerman proved in [29] that every irreducible unitary module with nonzero $(\mathfrak{g}, K)$-cohomology is an $A_{\mathfrak{q}}(\lambda)$ module. More generally, it is proved in [25] that any irreducible unitary module with strongly regular infinitesimal character is an $A_{\mathfrak{q}}(\lambda)$ module. In particular, any unitary module with the same infinitesimal character as a finite-dimensional module is an $A_{\mathfrak{q}}(\lambda)$ module.

3.2 Construction of $A_{\mathfrak{q}}(\lambda)$ Modules

The module $A_{\mathfrak{q}}(\lambda)$ is constructed by the so called cohomological induction, starting from the one-dimensional $(\mathfrak{l}, L \cap K)$-module $\mathbb{C}_\lambda$, on which any $x \in \mathfrak{l}$ acts by the scalar $\lambda(x)$. This module is now shifted by the one-dimensional $(\mathfrak{l}, L \cap K)$-module $\bigwedge^{\mathrm{top}} \mathfrak{u}$. From here, there are two versions of the construction. In the first version, we define a $\mathfrak{q}$-action on $Z = \mathbb{C}_\lambda \otimes \bigwedge^{\mathrm{top}} \mathfrak{u}$ by letting $\mathfrak{u}$ act by zero, and consider the produced $(\mathfrak{g}, L \cap K)$-module

$$\mathrm{Hom}_{U(\mathfrak{q})}(U(\mathfrak{g}), Z)_{L \cap K-\text{finite}},$$

with naturally defined actions. One now applies the derived Zuckerman functor in the middle degree to obtain a $(\mathfrak{g}, K)$-module. (The Zuckerman functor roughly speaking extracts the K-finite part of a $(\mathfrak{g}, L \cap K)$ module, but this is very often zero, so one needs to use derived functors.)

The other construction is to first make Z into a $\bar{\mathfrak{q}}$-module by letting $\bar{\mathfrak{u}}$ act by zero, then consider the induced $(\mathfrak{g}, L \cap K)$-module

$$U(\mathfrak{g}) \otimes_{U(\bar{\mathfrak{q}})} Z,$$

and finally apply the derived Bernstein functor in the middle degree. (The Bernstein functor is similar to the Zuckerman functor, it is defined by a dual construction.)

3.3 Dirac Cohomology of $A_{\mathfrak{q}}(\lambda)$ Modules

As shown in [10], $A_{\mathfrak{q}}(\lambda)$ has nonzero Dirac cohomology precisely when $\theta\lambda = \lambda$, and in this case the Dirac cohomology is given by the formula

$$H_D(A_{\mathfrak{q}}(\lambda)) = \bigoplus_{w \in W(\mathfrak{l}, \mathfrak{t})^1} 2^{[\dim \mathfrak{a}/2]} E_{w(\lambda+\rho)-\rho_{\mathfrak{k}}}.$$

Here as before, $\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{a}$ is a fundamental Cartan subalgebra of $\mathfrak{g}$. Positive root systems for $(\mathfrak{g}, \mathfrak{h})$, $(\mathfrak{g}, \mathfrak{t})$, $(\mathfrak{k}, \mathfrak{t})$ and $(\mathfrak{l}, \mathfrak{t})$ are chosen in a compatible way, and ρ and $\rho_{\mathfrak{k}}$ are the half sums of positive roots for $(\mathfrak{g}, \mathfrak{h})$ respectively $(\mathfrak{k}, \mathfrak{t})$. $W(\mathfrak{l}, \mathfrak{t})^1$ consists of the elements of the Weyl group $W(\mathfrak{l}, \mathfrak{t})$ which take the dominant $\mathfrak{l}$-chamber into the dominant $\mathfrak{l} \cap \mathfrak{k}$-chamber. For each integral $\mu \in \mathfrak{t}^*$, E_μ denotes the $\widetilde{K}$-type with highest weight μ.

3.4 Question

Is an $A_{\mathfrak{q}}(\lambda)$ module uniquely determined by its Dirac cohomology? More precisely, suppose that $\mathfrak{q}$ and $\mathfrak{q}'$ are θ-stable parabolic subalgebras of $\mathfrak{g}$ such that the semisimple parts of the real forms of the Levi subalgebras $\mathfrak{l}$ and $\mathfrak{l}'$ have no compact factors. Assuming that $H_D(A_{\mathfrak{q}}(\lambda)) = H_D(A_{\mathfrak{q}'}(\lambda'))$, can we conclude that $\mathfrak{q} = \mathfrak{q}'$ and $\lambda = \lambda'$?

This question arises in the study of elliptic tempered characters (Huang). It is also a natural classification question. Furthermore, it is related to Dirac induction and the issue of reconstructing modules from their Dirac cohomology.

3.5 Answer [HPV]

Yes, if the real forms of $\mathfrak{l}$ and $\mathfrak{l}'$ do not have factors $\mathfrak{so}(2n, 1)$, $\mathfrak{sp}(p, q)$ or the nonsplit $\mathfrak{f}_4$. In particular, the answer is always yes if $\mathfrak{g}$ is of type A, D, E or G. It is also always yes if $(\mathfrak{g}, \mathfrak{k})$ is Hermitian.

The question boils down to the issue whether $W(\mathfrak{l}, \mathfrak{t})^1$ generates $W(\mathfrak{l}, \mathfrak{t})$. The answer involves the study of modifications of Vogan diagrams by simple noncompact reflections.

3.6 Example: $\mathfrak{g}_0 = \mathfrak{so}(2n, 1)$

For each $k = 1, \ldots, n$, there is a θ-stable parabolic subalgebra $\mathfrak{q}_k$ with the semisimple part of the real form of the Levi factor equal to $\mathfrak{so}(2k, 1)$.

The modules $A_{\mathfrak{q}_k}(0)$ are different, but they all have the same Dirac cohomology, consisting of two K-types, with highest weights

$$\left(n - \frac{1}{2}, \ldots, \frac{3}{2}, \frac{1}{2}\right) \quad \text{and} \quad \left(n - \frac{1}{2}, \ldots, \frac{3}{2}, -\frac{1}{2}\right).$$

(We are using the usual coordinates.)

There are also two discrete series representations with infinitesimal character ρ, each with a single K-type in the Dirac cohomology.

Acknowledgements The author was supported by grant no. 4176 of the Croatian Science Foundation and by the Center of Excellence QuantiXLie.

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B–Orbits in Abelian Nilradicals of Types B, C and D: Towards a Conjecture of Panyushev

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Abstract Let B be a Borel subgroup of a semisimple algebraic group G and let $\mathfrak{m}$ be an abelian nilradical in $\mathfrak{b} = \mathrm{Lie}(B)$. Using subsets of strongly orthogonal roots in the subset of positive roots corresponding to $\mathfrak{m}$, D. Panyushev [1] gives in particular classification of $B-$orbits in $\mathfrak{m}$ and $\mathfrak{m}^*$ and states general conjectures on the closure and dimensions of the $B-$orbits in both $\mathfrak{m}$ and $\mathfrak{m}^*$ in terms of involutions of the Weyl group. Using Pyasetskii correspondence between $B-$orbits in $\mathfrak{m}$ and $\mathfrak{m}^*$ he shows the equivalence of these two conjectures. In this Note we prove his conjecture in types B_n, C_n and D_n for adjoint case.

1 Abelian Nilradicals and Panyushev's Conjecture

1.1 Minimal Nilradicals

Let G be a semisimple linear algebraic group over $\mathbb{C}$ and let $\mathfrak{g}$ be its Lie algebra. Let B be its Borel subgroup and $\mathfrak{b} = \mathrm{Lie}(B)$. Let $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{h} \oplus \mathfrak{n}^-$ be its corresponding triangular decomposition, where $\mathfrak{b} = \mathfrak{n} \oplus \mathfrak{h}$. B acts adjointly on $\mathfrak{n}$. For $x \in \mathfrak{n}$ let $B.x$ denote its orbit.

Since the description of $B-$orbits in $\mathfrak{n}$ immediately reduces to simple Lie algebras in what follows we assume that $\mathfrak{g}$ is simple.

Let R be the root system of $\mathfrak{g}$ and W its Weyl group. For $\alpha \in R$ let s_α be the corresponding reflection in W.

Let R^+ (resp. R^-) denote the subset of positive (resp. negative) roots. For $\alpha \in R$ let X_α denote the standard root vector in $\mathfrak{g}$ so that $\mathfrak{n} = \bigoplus\limits_{\alpha\in R^+} \mathbb{C}X_\alpha$. Let $\Delta = \{\alpha_i\}_{i=1}^n \subset R^+$ be a set of simple roots. Let θ be the maximal root in R^+.

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V. Dobrev (ed.), *Lie Theory and Its Applications in Physics*,
Springer Proceedings in Mathematics & Statistics 191,
DOI 10.1007/978-981-10-2636-2_28

Recall that any standard parabolic subgroup P of G is of the form $P = L \ltimes M$ where L is a standard Levy subgroup and M is the unipotent radical of P. If R_L is the root system of $\mathfrak{l} = \mathrm{Lie}(L)$ then $\Delta_L = \Delta \cap R_L$. Let W_P denote Weyl group of $\mathfrak{l}$. Let $\widehat{w}$ be the longest element of W_P.

P is maximal if and only if $\Delta_L = \Delta \setminus \{\alpha_i\}$. We will write $P_{\alpha_i} = P$, $M_{\alpha_i} = M$, $R_{\alpha_i} = R_L, R^+_{\alpha_i} = R^+_L$ and $W_{\alpha_i} = W_P$ in this case. We put $\overline{R}^+_{\alpha_i} := R^+ \setminus R^+_{\alpha_i}$. Put $\mathfrak{m}_{\alpha_i} := \mathrm{Lie}(M_{\alpha_i}) = \bigoplus\limits_{\alpha \in \overline{R}^+_{\alpha_i}} \mathbb{C}X_\alpha$.

A nilradical $\mathfrak{m}$ is abelian if and only if $\mathfrak{m} = \mathfrak{m}_{\alpha_i}$ and in $\theta = \sum\limits_{j=1}^{n} k_j\alpha_j$ one has $k_i = 1$ (cf. [1] for details).

1.2 Strongly Orthogonal Sets and $B-$Orbits in A_n, B_n, C_n, D_n

A set $\mathcal{S} \subset R^+$ is called *strongly orthogonal* if $\alpha \pm \beta \notin R$ for any $\alpha, \beta \in \mathcal{S}$. Given a strongly orthogonal set $\mathcal{S} = \{\beta_i\}_{i=1}^k$ put $\sigma_{\mathcal{S}} := \prod\limits_{i=1}^{k} s_{\beta_i}$. Note that this is an involution. As it is shown in [1] each $B-$orbit in an abelian nilradical $\mathfrak{m}_{\alpha_i}$ has a unique representative of form $\sum\limits_{\alpha \in \mathcal{S}} X_\alpha$ where $\mathcal{S} \subset \overline{R}^+_{\alpha_i}$ is strongly orthogonal.

We choose the following root systems:

- In A_n : $R = \{e_j - e_i\}_{1 \le i \ne j \le n+1}$, $R^+ = \{e_j - e_i\}_{1 \le i < j \le n+1}$, $\Delta = \{e_{i+1} - e_i\}_{i=1}^n$;
- In C_n : $R = \{\pm(e_j \pm e_i)\}_{1 \le i < j \le n} \cup \{\pm 2e_i\}_{i=1}^n$, $R^+ = \{e_j \pm e_i\}_{1 \le i < j \le n} \cup \{2e_i\}_{i=1}^n$, $\Delta = \{2e_1, e_{i+1} - e_i\}_{i=1}^{n-1}$;
- In B_n : $R = \{\pm(e_j \pm e_i)\}_{1 \le i < j \le n} \cup \{\pm e_i\}_{i=1}^n$, $R^+ = \{e_j \pm e_i\}_{1 \le i < j \le n} \cup \{e_i\}_{i=1}^n$, $\Delta = \{e_1, e_{i+1} - e_i\}_{i=1}^{n-1}$;
- In D_n : $R = \{\pm(e_j \pm e_i)\}_{1 \le i < j \le n}$, $R^+ = \{e_j \pm e_i\}_{1 \le i < j \le n}$, $\Delta = \{e_2 + e_1, e_{i+1} - e_i\}_{i=1}^{n-1}$.

We call roots $\alpha = e_j \pm e_i$ or $\alpha = e_i(2e_i)$, $\beta = e_l \pm e_k$ or $\beta = e_k(2e_k)$ *disjoint* if $\{i, j\} \cap \{k, l\} = \emptyset$.

In A_n and C_n the roots α, β are strongly orthogonal iff they are disjoint. In these two cases, (as well as in D_n) root vector X_α is of nilpotency order two. As it is shown in [2, 3] in theses two cases each $B-$orbit of nilpotency order two in $\mathfrak{n}$ has a unique representative of the form $\sum\limits_{i=1}^{k} X_{\beta_i}$ where $\{\beta_i\}_{i=1}^k \subset R^+$ is a strongly orthogonal (i.e. pairwise disjoint) set. On the other hand, each involution of W can be written as a (commutative) product of pairwise disjoint reflections in the unique way, so there is a one-to-one correspondence between the strongly orthogonal sets and involutions of W so that $B-$orbits of nilpotency order 2 are indexed by involutions in these two cases.

As for the cases B_n and D_n there is no bijection between $B-$orbits of nilpotent order 2 in $\mathfrak{n}$ and involutions of W because of two reasons. First of all, a root vector X_{e_i} in B_n and a sum of strongly orthogonal root vectors $X_{e_j-e_i}+X_{e_j+e_i}$ (roots e_j-e_i and e_j+e_i are strongly orthogonal in $\mathfrak{so}_n$) are matrices of nilpotency order 3 both in B_n and D_n. The second obstacle is that different sets of strongly orthogonal roots correspond to the same involution in W, for example, $\sigma_{\{e_j-e_i,e_j+e_i\}}=\sigma_{\{e_i,e_j\}}$ but $X_{e_j-e_i}+X_{e_j+e_i}$ and $X_{e_i}+X_{e_j}$ are representatives of different $B-$orbits (of nilpotency order 3) in B_n. Exactly in the same way $\sigma_{\{e_i,e_j,e_k,e_l\}}$ is connected to 3 different strongly orthogonal sets in D_n namely $\{e_{t_1}-e_{s_1},e_{t_1}+e_{s_1},e_{t_2}-e_{s_2},e_{t_2}+e_{s_2}\}$ where $\{s_1,s_2,t_1,t_2\}=\{i,j,k,l\}$ and $s_r<t_r$ for $r=1,2$ (and additional 7 different strongly orthogonal sets in B_n) and the corresponding sums of roots are representatives of different $B-$orbits (of nilpotency order 3). However when we restrict ourselves to abelian nilradicals there is a bijection between the sets of strongly orthogonal roots in $\overline{R}^+_{\alpha_i}$ and subset of involutions of W so that $B-$orbits are indexed by involutions inside abelian nilradicals in the unique way. Some of these orbits are of nilpotency order 3.

1.3 Abelian Nilradicals in A_n, B_n, C_n, D_n

Abelian nilradicals in A_n, B_n, C_n, D_n are (cf. [1], for example for the details).

(i) In $\mathfrak{sl}_n$ any $\mathfrak{m}_{e_{k+1}-e_k}$ is abelian so that there are $n-1$ abelian nilradicals. They are of the form

$$\mathfrak{m}_{e_{k+1}-e_k}=\bigoplus_{1\le i\le k<j\le n}\mathbb{C}X_{e_j-e_i}$$

One can see at once that in this case $\mathfrak{m}_{e_{k+1}-e_k}$ is a subspace of matrices of nilpotency order 2 and respectively all $B-$orbits there are indexed by sets of pairwise disjoint roots $\{e_{j_s}-e_{i_s}\}_{s=1}^m$ where $i_s\le k$ and $j_s\ge k+1$ for any s : $1\le s\le m$.

(ii) In $\mathfrak{sp}_{2n}$ the abelian nilradical is unique and it is

$$\mathfrak{m}_{2e_1}=\bigoplus_{1\le i<j\le n}\mathbb{C}X_{e_j+e_i}\oplus\bigoplus_{i=1}^{n}\mathbb{C}X_{2e_i}.$$

Again this is a subspace of matrices of nilpotency order 2, so that all the $B-$orbits there are indexed by sets of pairwise disjoint roots $\{2e_{i_s}\}_{s=1}^{l}\cup\{e_{k_t}+e_{j_t}\}_{t=1}^{m}$.

(iii) In $\mathfrak{so}_{2n+1}$ the abelian nilradical is unique and it is

$$\mathfrak{m}_{e_n-e_{n-1}}=\bigoplus_{i=1}^{n-1}\mathbb{C}X_{e_n-e_i}\oplus\bigoplus_{i=1}^{n-1}\mathbb{C}X_{e_n+e_i}\oplus\mathbb{C}X_{e_n}.$$

By [1] $\{X_{e_n}, X_{e_n \pm e_i}, X_{e_n - e_i} + X_{e_n + e_i}\}_{i=1}^{n-1}$ is the set of the (unique) representatives of $B-$orbits in the form of sums of strongly orthogonal root vectors. Note that the corresponding set of involutions $\{s_{e_n}, s_{e_n \pm e_i}, s_{e_n} s_{e_i}\}_{i=1}^{n-1}$ is defined uniquely on this subset.

(iv) In $\mathfrak{so}_{2n}$ there are 3 abelian nilradicals; two of them are isomorphic, namely, $\mathfrak{m}_{e_2 - e_1} \cong \mathfrak{m}_{e_2 + e_1}$. It is enough to consider

$$\mathfrak{m}_{e_2+e_1} = \bigoplus_{1 \le i < j \le n} \mathbb{C} X_{e_j + e_i}$$

This is the subspace of matrices of nilpotency order 2 and a $B-$orbit in it has a unique representative in the form $\sum_{s=1}^{m} X_{e_{j_s} + e_{i_s}}$ where $\{e_{j_s} + e_{i_s}\}_{s=1}^{m}$ is a set of pairwise disjoint roots.
The third nilradical is

$$\mathfrak{m}_{e_n - e_{n-1}} = \bigoplus_{i=1}^{n-1} X_{e_n - e_i} \oplus \bigoplus_{i=1}^{n-1} X_{e_n + e_i}.$$

By [1] $\{X_{e_n \pm e_i},\ X_{e_n - e_i} + X_{e_n + e_i}\}_{i=1}^{n-1}$ is the set of the (unique) representatives of $B-$orbits in the form of sums of strongly orthogonal root vectors. Note that the corresponding set of involutions $\{s_{e_n \pm e_i}, s_{e_n} s_{e_i}\}_{i=1}^{n-1}$ is defined uniquely on this subset.

In particular, as we see, all $B-$orbits in an abelian nilradical for A_n, B_n, C_n and D_n are indexed by strongly orthogonal subsets in $\overline{R}^+_{\alpha_i}$. For a strongly orthogonal set $\mathcal{S} \subset \overline{R}^+_{\alpha_i}$ put $\mathbf{B}_{\mathcal{S}} := B.(\sum\limits_{\alpha \in \mathcal{S}} X_\alpha)$.

1.4 Panyushev's Conjecture

To formulate the conjecture we need the following notation. For $w \in W$ put $\ell(w)$ to be its length, that is $\ell(w) := \#\{\alpha \in R^+ \ : \ w(\alpha) \in R^-\}$. For a strongly orthogonal set $\mathcal{S}$ let $\#(\mathcal{S})$ denote its cardinality. Let $\le$ denote Bruhat order on W.

Respectively, for (coadjoint) $B-$orbits in $\mathfrak{m}^*_\alpha$ Panyushev shows that they are labeled by the same strongly orthogonal sets $\mathcal{S}$ and we denote them by $\mathbf{B}^*_{\mathcal{S}}$.

Conjecture 1 (Panyushev) *Let $\mathfrak{m}_\alpha$ be an abelian nilradical in a simple $\mathfrak{g}$, and W_α be the corresponding Weyl group. Let $\widehat{w}$ denote the longest element of W_α.*

Let $\mathcal{S}, \mathcal{S}' \subset \overline{R}^+_\alpha$ be strongly orthogonal and let $\sigma = \sigma_{\mathcal{S}}$, $\sigma' = \sigma_{\mathcal{S}'}$. Then

(i) $\mathbf{B}_{\mathcal{S}} \subset \overline{\mathbf{B}}_{\mathcal{S}'}$ *if and only if* $\widehat{w}\sigma\widehat{w} \le \widehat{w}\sigma'\widehat{w}$.
(ii) $\dim B_{\mathcal{S}} = \frac{\ell(\widehat{w}\sigma\widehat{w}) + \#(\mathcal{S})}{2}$;

Respectively, for coadjoint orbits one has

(i)* $\mathbf{B}^*_{\mathcal{S}} \subset \overline{\mathbf{B}^*}_{\mathcal{S}'}$ *if and only if* $\sigma \leq \sigma'$.
(ii)* $\dim \mathbf{B}^*_{\mathcal{S}} = \frac{\ell(\sigma)+\#(\mathcal{S})}{2}$;

Panyushev shows, using Pyasetskii correspondence that these two conjectures are equivalent.

Taking into account that by [4] for $B-$orbits $\mathcal{B}, \mathcal{B}'$ one has $\mathcal{B}'$ is in the boundary of $\mathcal{B}$ iff $\mathrm{codim}_{\overline{\mathcal{B}}}\mathcal{B}' = 1$ the part (*ii*) of the conjecture follows straightforwardly from part (*i*).

In cases of A_n and C_n both adjoint and coadjoint $B-$orbits of nilpotency order 2 are indexed by involutions [2, 3, 5, 6] and by [5, 6] for involutions $\sigma, \sigma' \in W$ one has $\mathbf{B}^*_{\sigma} \subset \overline{\mathbf{B}^*}_{\sigma'}$ if and only if $\sigma \leq \sigma'$ so that the conjecture is a private case of a more general phenomenon.

As for B_n and D_n we were informed by M. Ignatyev that general description of inclusions of coadjoint $B-$orbit closures of nilpotent order 2 is not given by restriction of Bruhat order to involutions. We think that this happens because of the same difficulties with bijection between the strongly orthogonal sets and involutions that are described above.

For adjoint orbits in A_n and C_n, in general, the combinatorial order on involutions defined by the inclusion of $B-$orbit closures of nilpotency order 2 is not connected to Bruhat order. However, for $B-$orbits in an abelian nilradical the conjecture is obtained as a straightforward corollary of [2, 3].

In this Note we reprove the conjecture for A_n and C_n and prove it for B_n and D_n for adjoint case. We also provide a simple combinatorial expression for $\ell(\sigma)$ for involutions in S_n, W_{C_n} and W_{D_n}. To do this we introduce link patterns. May be the expression can be obtained from the results of F. Incitti and is known to experts, but we have not found this result in the literature.

2 Link Patterns and $\ell(\sigma)$ for the Weyl Group

Recall that Weyl group of $\mathfrak{sl}_n$ is S_n and its action on roots is obtained by extending linearly $w(e_i) = e_{w(i)}$. Weyl group W_{C_n} of either $\mathfrak{sp}_{2n}$ or $\mathfrak{so}_{2n+1}$ is a group of maps from $\{-n, \dots, -1, 1, \dots, n\}$ onto itself symmetric around zero, namely $i \mapsto j \ \Leftrightarrow\ -i \mapsto -j$ and its action on roots is obtained by extending linearly $w(e_i) = \mathrm{sign}(w(i))e_{|w(i)|}$. Finally, Weyl group W_{D_n} is a subgroup of W_{C_n} of maps sending even number of positive numbers to negative numbers. It acts on roots exactly in the same way as W_{B_n}.

A *link pattern* on n points with k arcs is a graph on n (numbered) vertexes (drawn on a horizontal line) with k disjoint edges $\{(i_s, j_s)\}_{s=1}^k$ (that is, $\{i_s, j_s\} \cap \{i_t, j_t\} = \emptyset$ for $1 \leq s \neq t \leq k$) drawn over the line and called arcs. Vertex $f \notin \{i_s, j_s\}_{s=1}^k$ is called a *fixed point*.

A strongly orthogonal set $\{e_{j_s} - e_{i_s}\}_{s=1}^k$ in $\mathfrak{sl}_n$ (or corresponding involution in S_n) can be drawn as a link pattern on $1, \dots, n$ with edges $\{i_s, j_s\}_{s=1}^k$; respectively a strongly orthogonal set in C_n (or corresponding involution in W_{C_n}) can be drawn as a link pattern symmetric around zero on $-n, \ldots, -1, 1, \ldots, n$ where $2e_i$ corresponds to arc $(-i, i)$ and $e_j \pm e_i$ for $0 < i < j \leq n$ corresponds to two arcs $(\mp i, j)$ and $(\pm i, -j)$. Respectively, for an involution of W_{C_n} to be an element of W_{D_n} we need the even number of cycles of type $(-i, i)$ so that it can be drawn as a link pattern on $-n, \ldots, -1, 1, \ldots, n$ symmetric around zero with even number of arcs over zero.

Given a strongly orthogonal set $\mathcal{S} = \{e_{j_s} - e_{i_s}\}_{s=1}^m$ (resp. $\mathcal{S} = \{e_{j_s} \mp e_{i_s}\}_{s=1}^l \cup \{e_{2k_t}\}_{t=1}^m$) let $P_{\mathcal{S}}$ be the corresponding link pattern. Let $|\mathcal{S}|$ denote the number of arcs in $P_{\mathcal{S}}$. Note that in case of $\mathfrak{sl}_n$ one has $\#(\mathcal{S}) = |\mathcal{S}|$; in case of C_n or D_n one has $\#(\mathcal{S}) \leq |\mathcal{S}| \leq 2\#(\mathcal{S})$ (depending on the roots).

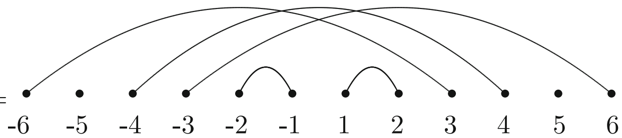
Let $(a_1, b_1) \ldots (a_m, b_m)$, where $m = |\mathcal{S}|$, be the list of arcs of $P_{\mathcal{S}}$ written in such a way that $a_i < b_i$. We also need the following statistics on $P_{\mathcal{S}}$:

(i) set $c(a_s, b_s) := \#\{t \ : \ a_t < a_s < b_t < b_s\}$ to be the number of arcs crossing the given arc (a_s, b_s) on the left and $c(\mathcal{S}) := \sum\limits_{s=1}^m c(a_s, b_s)$ to be the total number of crosses;

(ii) set $r(a_s, b_s) := \#\{t \ : \ a_t > b_s\}$ to be the number of arcs to the right of the given arc (a_s, b_s) and $r(\mathcal{S}) := \sum\limits_{s=1}^m r(a_s, b_s)$ to be the total number of arcs to the right of some arc;

(iii) set $b(a_s, b_s) := \#\{p \ : \ a_s < p < b_s$ and $p \notin \{a_t, b_t\}_{t=1}^m\}$ to be the number of fixed points under the given arc (bridge) (a_s, b_s); and $b(\mathcal{S}) := \sum\limits_{s=1}^m b(a_s, b_s)$ to be the total number of fixed points under the arcs, or in other words the total number of bridges over all fixed points.

For example, let $\mathcal{S} = \{e_2 - e_1, \ e_6 + e_3, \ 2e_4\}$ in C_6, then

$P_{\mathcal{S}} =$ 

and $|\mathcal{S}| = 5$, $c(\mathcal{S}) = 3$, $r(\mathcal{S}) = 1$, $b(\mathcal{S}) = 2$.

Proposition 1 *Let $\mathcal{S}$ be a strongly orthogonal set in either $\mathfrak{sl}_n$ or C_n (D_n) and let $\sigma = \sigma_{\mathcal{S}}$ be an involution in the corresponding Weyl group.*

1. *For $\mathcal{S} = \{e_{j_s} - e_{i_s}\}_{s=1}^k$ in $\mathfrak{sl}_n$ one has*

$$\ell(\sigma) = 2|\mathcal{S}|^2 - |\mathcal{S}| + 2b(\mathcal{S}) - 4r(\mathcal{S}) - 2c(\mathcal{S})$$

2. *For $\mathcal{S} = \{e_{j_s} - e_{i_s}\}_{s=1}^a \cup \{2e_{k_s}\}_{s=1}^d \cup \{e_{m_s} + e_{l_s}\}_{s=1}^f$ in C_n one has for σ in W_{C_n}*

$$\ell(\sigma) = |\mathcal{S}|^2 - a + b(\mathcal{S}) - c(\mathcal{S}) - 2r(\mathcal{S})$$

3. *For* $\mathcal{S} = \{e_{j_s} - e_{i_s}\}_{s=1}^{a} \cup \{e_{k_s}\}_{s=1}^{2d} \cup \{e_{m_s} + e_{l_s}\}_{s=1}^{f}$ *so that* $\sigma \in W_{D_n}$ *one has*

$$\ell(\sigma) = |\mathcal{S}|^2 - |\mathcal{S}| + a + b(\mathcal{S}) - c(\mathcal{S}) - 2r(\mathcal{S})$$

Proof We prove (1) by the induction on $|\sigma|$ and induction on n. It is trivial for $n = 2$. Assume it is true for $\sigma \in S_{n-1}$ and show for $\sigma \in S_n$. Recall that $s_{e_j - e_i} = (i, j)$ in cyclic form so that $|\mathcal{S}| = 1$ iff $\sigma_{\mathcal{S}} = (i, j)$. If $(i, j) \neq (1, n)$ we can regard it as an element of S_{n-1} so that $\ell((i, j))$ is obtained by induction. For $(1, n)$ one has that $(1, n)(e_t - e_s)$ is negative iff $t = n$ or $s = 1$ so that $\ell((1, n)) = 2n - 3$. On the other hand $b((1, n)) = n - 2$ and $c((1, n)) = r((1, n)) = 0$ so that the expression is satisfied.

Now assume this is true for $\sigma_{\mathcal{S}'} \in S_n$ where $|\mathcal{S}'| \leq k - 1$ and show for $\sigma_{\mathcal{S}}$ of $|\mathcal{S}| = k$. Let $\sigma = \sigma'(i, j)$ where $\sigma' = (i_1, j_1) \ldots (i_{k-1}, j_{k-1})$ and $j > j_s$ for any $1 \leq s \leq k - 1$. If $j < n$ we can regard σ as an element of S_{n-1} and the result is obtained by induction on n. If $j = n$ one has

$$\sigma(e_t - e_s) = \sigma'(i, n)(e_t - e_s) = \begin{cases} -(e_n - e_i) \text{ if } (s, t) = (i, n) & (I) \\ e_i - \sigma'(e_s) \text{ if } t = n,\ s \neq i & (II) \\ e_n - \sigma'(e_s) \text{ if } t = i & (III) \\ \sigma'(e_t) - e_n \text{ if } s = i,\ t < n & (IV) \\ \sigma'(e_t - e_s) \text{ otherwise} & (V) \end{cases}$$

Take into account that

- $\sigma'(e_n - e_i) = e_n - e_i$ so that case (I) adds 1 to the length;
- $\sigma'(e_n - e_s) = e_n - \sigma'(e_s) \in R^+$. On the other hand $e_i - \sigma'(e_s) \in R^-$ exactly for $n - 1 - i$ roots since for every $i < s < n$ either $\sigma'(s) = s$ or there exists $r < j$ such that $\sigma'(r) = s$. Thus (II) adds $(n - 1 - i)$ to the length;
- $\sigma(e_i - e_s) = e_n - \sigma'(e_s) \in R^+$ always. On the other hand for any (i_s, j_s) such that $i_s < i < j_s$ one has $\sigma'(e_i - e_{i_s}) = e_i - e_{j_s} \in R^-$ so that in case (III) we have to reduce $c(i, n)$ from the length;
- $\sigma(e_t - e_i) = \sigma'(e_t) - e_n \in R^-$ for all $t\ :\ i < t$ and $\sigma'(e_t - e_i) = \sigma'(e_t) - e_i \in R^-$ iff $t = j_s$ where $i_s < i < j_s < j$. Thus case (IV) adds $n - 1 - i - c(i, n)$ to the length;
- Case (V) does not add anything to the length.

Summarizing, we get $\ell(\sigma) = 2(n - i) - 1 - 2c(i, n) + \ell(\sigma')$.

Put $u(i, n) := \#\{t\ :\ i < i_t, j_t < n\}$ to be the number of arcs under (i, n). One has:
$c(\mathcal{S}) = c(\mathcal{S}') + c(i, n)$;
$b(\mathcal{S}) = b(\mathcal{S}') - c(i, n) + (n - 1 - i) - c(i, n) - 2u(i, n) = b(\mathcal{S}') + (n - 1 - i)$
$- 2c(i, n) - 2u(i, n)$
$r(\mathcal{S}) = r(\mathcal{S}') + (k - 1) - c(i, n) - u(i, n)$ since for any (i_s, j_s) it is either to the left of (i, n) or under (i, n) or crosses it on the left.

Taking all this into account we get straightforwardly $\ell(\sigma) = 2k^2 - k + 2b(\mathcal{S}) - 2c(\mathcal{S}) - 4r(\mathcal{S})$ in accordance with the expression.

(2) Let $\mathcal{S} = \{e_{j_s} - e_{i_s}\}_{s=1}^{a} \cup \{2e_{k_s}\}_{s=1}^{d} \cup \{e_{m_s} + e_{l_s}\}_{s=1}^{f}$ in C_n so that $|\mathcal{S}| = 2a + d + 2f$ and let $\sigma = \sigma_{\mathcal{S}}$. Taking into account that $s_{2e_i} = (-i, i)$, $s_{e_j \pm e_i} = (\mp i, j)(\pm i, -j)$ by (i) its length as an element of S_{2n} is $\ell_{S_{2n}}(\sigma) = |\mathcal{S}|^2 - |\mathcal{S}| + 2b(\mathcal{S}) - 4r(\mathcal{S}) - 2c(\mathcal{S})$. On the other hand, in C_n all the short roots are sums (up to sign) of two roots in $\mathfrak{sl}_{2n}$. Let $x(\sigma)$ be the number of positive long roots $2e_s$ such that $\sigma(2e_s) \in R^-$. Then $\ell(\sigma) = \frac{1}{2}(\ell_{S_{2n}}(\sigma) + x(\sigma))$. Further, note that $s_{e_j - e_i}(2e_s) \in R^+$ always,

$$s_{2e_k}(2e_s) = \begin{cases} -2e_s & \text{if } s = k; \\ 2e_s & \text{otherwise.} \end{cases} \quad \text{and} \quad s_{e_j + e_i}(2e_s) = \begin{cases} -2e_s & \text{if } s = i, j; \\ 2e_s & \text{otherwise.} \end{cases}$$

Thus $x(\sigma) = d + 2f = |\mathcal{S}| - 2a$. Summarizing, we get $\ell(\sigma) = |\mathcal{S}|^2 - a + b(\mathcal{S}) - c(\mathcal{S}) - 2r(\mathcal{S})$.

(3) Finally let $\mathcal{S} = \{e_{j_s} - e_{i_s}\}_{s=1}^{a} \cup \{e_{k_s}\}_{s=1}^{2d} \cup \{e_{m_s} + e_{l_s}\}_{s=1}^{f}$ so that $|\mathcal{S}| = 2a + 2d + 2f$ and $\sigma \in W_{D_n}$. By (2) its length as an element of W_{B_n} is $\ell_{B_n}(\sigma) = |\mathcal{S}|^2 - a + b(\mathcal{S}) - c(\mathcal{S}) - 2r(\mathcal{S})$. Let $x(\sigma)$ be the number of positive short roots e_s such that $\sigma(e_s) \in R^-_{B_n}$. Then $\ell(\sigma) = \ell_{B_n}(\sigma) - x(\sigma)$. As it is shown in (2) $x(\sigma) = |\mathcal{S}| - 2a$. By a straightforward computation we get expression (3) which completes the proof.

3 The Proof of Panyushev's Conjecture

3.1 Case $\mathfrak{sl}_n$

It is known that the conjecture is true for $\mathfrak{sl}_n$ (cf. [1]). The proof is straightforward and we provide it in short here since we use it in what follows.

Let S_n denote a standard symmetric group and $S_{[i,j]}$ a symmetric group on the elements $i, i+1, \ldots, j$. For a strongly orthogonal set $\mathcal{S} \subset R^+$ let $\pi_{i,j}(\mathcal{S}) = \mathcal{S} \cap \{e_l - e_k\}_{i \le k < l \le j}$. By [3], $\mathbf{B}_{\mathcal{S}'} \subset \overline{\mathbf{B}}_{\mathcal{S}}$ for $\mathcal{S}, \mathcal{S}' \subset R^+$ strongly orthogonal sets in $\mathfrak{sl}_n$ iff for any $i, j\ :\ 1 \le i < j \le n$ one has $|\pi_{i,j}(\mathcal{S}')| \le |\pi_{i,j}(\mathcal{S})|$. Moreover these inclusions are generated by elementary moves on link patterns defined as follows:

1. Let $e_j - e_i \in \mathcal{S}$ and let $\mathcal{S}'$ be obtained from $\mathcal{S}$ by exclusion of this root. Then $\mathbf{B}_{\mathcal{S}'} \subset \overline{\mathbf{B}}_{\mathcal{S}}$;
2. Let $e_j - e_i \in \mathcal{S}$ and let $k > j$ be a fixed point of $P_{\mathcal{S}}$. Let $\mathcal{S}'$ be obtained from $\mathcal{S}$ by changing $e_j - e_i$ to $e_k - e_i$, then $\mathbf{B}_{\mathcal{S}'} \subset \overline{\mathbf{B}}_{\mathcal{S}}$;
3. Let $e_j - e_i \in \mathcal{S}$ and let $k < i$ be a fixed point of $P_{\mathcal{S}}$. Let $\mathcal{S}'$ be obtained from $\mathcal{S}$ by changing $e_j - e_i$ to $e_j - e_k$, then $\mathbf{B}_{\mathcal{S}'} \subset \overline{\mathbf{B}}_{\mathcal{S}}$;
4. Let $e_l - e_i, e_k - e_j \in \mathcal{S}$ be such that $i < j < k < l$. Let $\mathcal{S}'$ be obtained from $\mathcal{S}$ by changing $e_l - e_i, e_k - e_j$ to $e_k - e_i, e_l - e_j$, then $\mathbf{B}_{\mathcal{S}'} \subset \overline{\mathbf{B}}_{\mathcal{S}}$.
5. Let $e_j - e_i, e_l - e_k \in \mathcal{S}$ be such that $j < k$. Let $\mathcal{S}'$ be obtained from $\mathcal{S}$ by changing $e_j - e_i, e_l - e_k$ to $e_k - e_i, e_l - e_j$, then $\mathbf{B}_{\mathcal{S}'} \subset \overline{\mathbf{B}}_{\mathcal{S}}$;

For $\mathfrak{m}_{e_{k+1}-e_k}$ one has $W_{e_{k+1}-e_k} = S_k \times S_{[k+1,n]}$ and $\widehat{w} = [k, \dots, 1, n, \dots, k+1]$.

Note that $\mathbf{B}_{\mathcal{S}} \subset \mathfrak{m}_{e_{k+1}-e_k}$ iff $\mathcal{S} = \{e_{j_s} - e_{i_s}\}_{s=1,\ i_s \le k, j_s \ge k+1}^m$, and therefore, $\widehat{w}\sigma_{\mathcal{S}}\widehat{w} = \sigma_{\widehat{\mathcal{S}}}$ where $\widehat{\mathcal{S}} = \{e_{n+1-j_s} - e_{k+1-i_s}\}_{s=1}^m$.

Since on one hand inclusion of $B-$orbit closures is generated by elementary moves on link patterns and on the other hand Bruhat order is generated by products by (i,j) we have only to compare these two actions.

For $\mathcal{S} = \{e_{j_s} - e_{i_s}\}_{s=1}^m$ put $\langle \sigma_{\mathcal{S}} \rangle := \{i_s, j_s\}_{s=1}^m$ to be the list of end points of $P_{\mathcal{S}}$. We have to take into account that the restriction of Bruhat order to involutions is generated by $\sigma < \sigma(i,j)$ only if $\{i,j\} \cap \langle \sigma \rangle = \emptyset$, otherwise we have to compare σ and $(i,j)\sigma(i,j)$.

Let $\sigma = \sigma_{\mathcal{S}}$ where $\mathcal{S} \subset \overline{R}^+_{e_{k+1}-e_k}$. Note that in order for $(i,j)\sigma$ in the first case (resp. $(i,j)\sigma(i,j)$ in the second case) to be $\sigma_{\mathcal{S}'}$ for $\mathcal{S}' \subset \overline{R}^+_{e_{k+1}-e_k}$ one needs to choose $i \le k$ and $j \ge k+1$ (resp. either $i, j \le k$ or $i, j \ge k+1$).

(i) $\sigma \to \sigma(i,j)$: Let $\mathcal{S}' = \{e_{j_s} - e_{i_s}\}_{s=1}^l$ and let $e_j - e_i$ be strongly orthogonal to $\mathcal{S}'$ then $\mathcal{S} = \mathcal{S}' \cup \{e_j - e_i\}$ is strongly orthogonal so that $\mathbf{B}_{\mathcal{S}'} \subset \overline{\mathbf{B}}_{\mathcal{S}}$ by (1) on one hand and on the other hand $\widehat{w}(i,j)\sigma_{\mathcal{S}'}\widehat{w} = (k+1-i, n+1-j)\sigma_{\widehat{\mathcal{S}'}} > \sigma_{\widehat{\mathcal{S}'}}$;

(ii) $\sigma \to (i,j)\sigma(i,j)$ where either $i, j \le k$ or $i, j \ge k+1$ and $|\{i,j\} \cap \langle \sigma \rangle| = 1$: Let $\mathcal{S} = \{e_{j_1} - e_{i_1}\} \cup \mathcal{T}$ where $\mathcal{T} = \{e_{j_s} - e_{i_s}\}_{s=2}^m$. Let $j = i_1$ and $i \notin \{i_s\}_{s=1}^m$ (resp. $i = j_1$ and $j \notin \{j_s\}_{s=1}^m$). Let $\mathcal{S}' = \{e_{j_1} - e_i\} \cup \mathcal{T}$ (resp. $\mathcal{S}' = \{e_j - e_{i_1}\} \cup \mathcal{T}$). Then on one hand by (2) $\mathbf{B}_{\mathcal{S}'} \subset \overline{\mathbf{B}}_{\mathcal{S}}$ iff $i < i_1$ (resp. by (3) iff $j > j_1$). On the other hand $\widehat{w}\sigma_{\mathcal{S}'}\widehat{w} = (k+1-i, n+1-j_1)\sigma_{\widehat{\mathcal{T}}}$ (resp. $\widehat{w}\sigma_{\mathcal{S}'}\widehat{w} = (k+1-i_1, n+1-j)\sigma_{\widehat{\mathcal{T}}}$) and $\widehat{w}\sigma_{\mathcal{S}}\widehat{w} = (k+1-i_1, n+1-j_1)\sigma_{\widehat{\mathcal{T}}}$ so that $\widehat{w}\sigma_{\mathcal{S}'}\widehat{w} < \widehat{w}\sigma_{\mathcal{S}}\widehat{w}$ iff $i < i_1$ (resp. $j > j_1$).

(iii) $\sigma \to (i,j)\sigma(i,j)$ where $\{i,j\} \subset \langle \sigma \rangle$: Let $\mathcal{S} = \{e_{j_1} - e_{i_1}, e_{j_2} - e_{i_2}\} \cup \mathcal{T}$ where $i_1 < i_2 (\le k)$ and $(i,j) = (i_1, i_2)$ (this is equal to action on σ by $(i,j) = (j_1, j_2)$). Then $(i,j)\sigma(i,j) = \sigma_{\mathcal{S}'}$ where $\mathcal{S}' = \{e_{j_1} - e_{i_2}, e_{j_2} - e_{i_1}\} \cup \mathcal{T}$. On one hand by (4) $\mathbf{B}_{\mathcal{S}'} \subset \overline{\mathbf{B}}_{\mathcal{S}}$ iff $j_1 > j_2$, on the other hand $\widehat{w}\sigma_{\mathcal{S}'}\widehat{w} = (k+1-i_1, n+1-j_2)(k+1-i_2, n+1-j_1)\sigma_{\widehat{\mathcal{T}}}$ and $\widehat{w}\sigma_{\mathcal{S}}\widehat{w} = (k+1-i_1, n+1-j_1)(k+1-i_2, n+1-j_2)\sigma_{\widehat{\mathcal{T}}}$ so that $\widehat{w}\sigma_{\mathcal{S}'}\widehat{w} < \widehat{w}\sigma_{\mathcal{S}}\widehat{w}$ iff $j_2 < j_1$.

3.2 Case $\mathfrak{sp}_{2n}$

For $\mathfrak{sp}_{2n}$ the unique abelian nilradical is $\mathfrak{m}_{2e_1}$. In this case $W_{2e_1} = S_n$ and $\widehat{w} = [n, \dots, 1]$. One has $\mathbf{B}_{\mathcal{S}} \subset \mathfrak{m}_{2e_1}$ iff $\mathcal{S} = \{2e_{k_s}\}_{s=1}^d \cup \{e_{m_s} + e_{l_s}\}_{s=1}^f$.

In this case the conjecture is obtained as a straightforward corollary of the result for $\mathfrak{sl}_{2n}$ and the following facts:

1. A set of strongly orthogonal roots $\mathcal{S}$ in C_n can be considered as a set $\widetilde{\mathcal{S}}$ of $|\mathcal{S}|$ strongly orthogonal roots in $\mathfrak{sl}_{2n}$. In these terms for $\mathbf{B}_{\mathcal{S}}, \mathbf{B}_{\mathcal{S}'} \subset R^+$ in $\mathfrak{sp}_{2n}$ one has by [2] $\mathbf{B}_{\mathcal{S}'} \subset \overline{\mathbf{B}}_{\mathcal{S}}$ iff they are restriction to $\mathfrak{sp}_{2n}$ of the orbits $\mathbf{B}'_{\widetilde{\mathcal{S}}}, \mathbf{B}'_{\widetilde{\mathcal{S}'}}$ from $\mathfrak{sl}_{2n}$ such that $\mathbf{B}'_{\widetilde{\mathcal{S}'}} \subset \overline{\mathbf{B}}'_{\widetilde{\mathcal{S}}}$;

2. $\mathfrak{m}_{2e_1}$ of $\mathfrak{sp}_{2n}$ is the restriction to $\mathfrak{sp}_{2n}$ of $\mathfrak{m}_{e_{n+1}-e_n}$ of $\mathfrak{sl}_{2n}$;
3. $\sigma, \sigma' \in W_{C_n}$ are elements of S_{2n} and $\sigma' < \sigma$ in W_{C_n} iff $\sigma' < \sigma$ in S_{2n} – this is shown for example in [7, Sect. 4];
4. $\widehat{w} \in W_{2e_1}$ is identified with the maximal element of $S_n \times S_{[n+1,2n]}$.

3.3 Case $\mathfrak{so}_{2n+1}$

For $\mathfrak{so}_{2n+1}$ the unique abelian nilradical is $\mathfrak{m}_{e_n-e_{n-1}}$. In this case $W_{e_n-e_{n-1}} = W_{B_{n-1}}$ and $\widehat{w} = s_{e_1} \dots s_{e_{n-1}}$.

$\mathbf{B}_{\mathcal{S}} \subset \mathfrak{m}_{e_n-e_{n-1}}$ if either $\mathcal{S} = \{e_n\}$ or $\mathcal{S} = \{e_n \pm e_i\}$, or $\mathcal{S} = \{e_n - e_i, e_n + e_i\}$ for $1 \le i < n$. Note that $\widehat{w} s_{e_n \pm e_i} \widehat{w} = s_{e_n \mp e_i}$, $\widehat{w} s_{e_n} \widehat{w} = s_{e_n}$ and $\widehat{w} \sigma_{\{e_n-e_i, e_n+e_i\}} \widehat{w} = \sigma_{\{e_n-e_i, e_n+e_i\}}$.

The restriction of Bruhat order to our set of involutions is as follows (cf. [7], for example):

$$\begin{array}{ll} s_{e_n+e_{n-1}} > s_{e_n+e_{n-2}} > \cdots > s_{e_n+e_1} > s_{e_n-e_1} > \cdots > s_{e_n-e_{n-1}}\ ; & \\ s_{e_n} > s_{e_n-e_1}\ ; & (i) \\ s_{e_n-e_{n-1}} s_{e_n+e_{n-1}} > \cdots > s_{e_n-e_1} s_{e_n+e_1} > s_{e_n}\ ; & \\ s_{e_n-e_i} s_{e_n+e_i} > s_{e_n+e_i} & (ii) \end{array}$$

Also $s_{e_n+e_i}$ and s_{e_n} are incompatible for any $i < n$. As for inclusions of $B-$orbit closures one has

(i) In order to show $\mathbf{B}_{\{e_n-e_{i-1}\}} \subset \overline{\mathbf{B}}_{\{e_n-e_i\}}$ note that $Exp(aX_{e_i-e_{i-1}}).X_{e_n-e_i} = X_{e_n-e_i} - aX_{e_n-e_{i-1}}$ so that by torus action we get $X_{e_n-e_{i-1}} \in \overline{\mathbf{B}}_{\{e_n-e_i\}}$. This corresponds to $s_{e_n+e_i} > s_{e_n+e_{i-1}}$ for $i\ :\ 2 \le i \le n-1$.
Let us show that $\mathbf{B}_{\{e_n+e_1\}} \subset \overline{\mathbf{B}}_{\{e_n-e_1\}}$. Indeed, $Exp(aX_{e_1}).X_{e_n-e_1} = X_{e_n-e_1} - aX_{e_n} - \frac{a^2}{2}X_{e_n+e_1}$. Further by torus action we get $X_{e_n+e_1} \in \overline{\mathbf{B}}_{\{e_n-e_1\}}$. This corresponds to $s_{e_n+e_1} > s_{e_n-e_1}$.
To show $\mathbf{B}_{\{e_n+e_{i+1}\}} \subset \overline{\mathbf{B}}_{\{e_n+e_i\}}$ note that $Exp(aX_{e_{i+1}-e_i}).X_{e_n+e_i} = X_{e_n+e_i} + aX_{e_n+e_{i+1}}$ so that by torus action we get $X_{e_n+e_{i+1}} \in \overline{\mathbf{B}}_{\{e_n+e_i\}}$. This corresponds to $s_{e_n-e_i} > s_{e_n-e_{i+1}}$ for $i\ :\ 1 \le i \le n-2$.
Exactly in the same way, $Exp(aX_{e_1}).X_{e_n} = X_{e_n} + aX_{e_n+e_1}$ and then by torus action we get $X_{e_n+e_1} \in \overline{\mathbf{B}}_{\{e_n\}}$ which corresponds to $s_{e_n-e_1} < s_{e_n}$.
Obviously $\mathbf{B}_{\{e_n-e_i\}}$ and $\mathbf{B}_{\{e_n\}}$ are incompatible.

(ii) To show $\mathbf{B}_{\{e_n-e_i, e_n+e_i\}} \subset \overline{\mathbf{B}}_{\{e_n-e_j, e_n+e_j\}}$ for $1 \le i < j \le n-1$ we note as before that $Exp(a(X_{e_j-e_i} + X_{e_j+e_i})).(X_{e_n-e_j} + X_{e_n+e_j}) = X_{e_n-e_j} + X_{e_n+e_j} - a(X_{e_n-e_i} + X_{e_n+e_i})$ and then by torus action we get $X_{e_n-e_i} + X_{e_n+e_i} \in \overline{\mathbf{B}}_{\{e_n-e_j, e_n+e_j\}}$ which corresponds to $s_{e_n-e_j} s_{e_n+e_j} > s_{e_n-e_i} s_{e_n+e_i}$ for $1 \le i < j \le n-1$.
To show $\mathbf{B}_{\{e_n\}} \subset \overline{\mathbf{B}}_{\{e_n-e_j, e_n+e_j\}}$ for $1 \le j \le n-1$ note that $Exp(\sqrt{2}X_{e_j}).(X_{e_n-e_j} + X_{e_n+e_j}) = X_{e_n-e_j} - \sqrt{2}X_{e_n}$ and then by torus action we get $X_{e_n} \in \overline{\mathbf{B}}_{\{e_n-e_j, e_n+e_j\}}$ for any $1 \le j \le n-1$ which corresponds to $s_{e_n-e_j} s_{e_n+e_j} > s_{e_n}$ for $1 \le j \le n-1$.

Obviously by torus action we get $X_{e_n-e_i} \in \overline{\mathbf{B}}_{\{e_n-e_i,e_n+e_i\}}$ which provides $s_{e_n-e_i}s_{e_n+e_i} > s_{e_n+e_i}$ for $1 \le i \le n-1$.

3.4 Case $\mathfrak{so}_{2n}$

Recall that there are 3 abelian nilradicals in the case of $\mathfrak{so}_{2n}$ namely $\mathfrak{m}_{e_2-e_1} \cong \mathfrak{m}_{e_2+e_1}$ and $\mathfrak{m}_{e_n-e_{n-1}}$.

Let us start with $\mathfrak{m}_{e_n-e_{n-1}}$ which can be obtained from the previous case. In this case $W_{e_n-e_{n-1}} = W_{D_{n-1}}$ and

$$\widehat{w} = \begin{cases} s_{e_1}\dots s_{e_{n-1}} & \text{if } n = 2k+1; \\ s_{e_2}\dots s_{e_{n-1}} & \text{if } n = 2k; \end{cases}$$

$\mathbf{B}_{\mathcal{S}} \subset \mathfrak{m}_{e_n-e_{n-1}}$ if either $\mathcal{S} = \{e_n \pm e_i\}$ or $\mathcal{S} = \{e_n - e_i, e_n + e_i\}$ for $1 \le i < n$. Note that $\widehat{w}s_{e_n\pm e_i}\widehat{w} = s_{e_n\mp e_i}$ for $i > 1$, $\widehat{w}s_{e_n\pm e_1}\widehat{w} = \begin{cases} s_{e_n\pm e_1} & \text{if } n = 2k; \\ s_{e_n\mp e_1} & \text{if } n = 2k+1; \end{cases}$ and $\widehat{w}s_{e_n-e_i}s_{e_n+e_i}\widehat{w} = s_{e_n-e_i}s_{e_n+e_i}$. The restriction of Bruhat order from W_{B_n} to W_{D_n} provides

$$\begin{aligned} & s_{e_n+e_{n-1}} > s_{e_n+e_{n-2}} > \dots > s_{e_n+e_1}, \quad s_{e_n-e_1} > s_{e_n-e_2} > \\ & > \dots > s_{e_n-e_{n-1}}; && (i) \\ & \quad s_{e_n+e_2} > s_{e_n-e_1}, \; s_{e_n+e_1} > s_{e_n-e_2}; && (ii) \\ & s_{e_n-e_{n-1}}s_{e_n+e_{n-1}} > \dots > s_{e_n-e_1}s_{e_n+e_1}, \; s_{e_n-e_i}s_{e_n+e_i} > \\ & > s_{e_n+e_i}, \; s_{e_n-e_1}s_{e_n+e_1} > s_{e_n\pm e_1} && (iii) \end{aligned}$$

The only differences with W_{B_n} are that $s_{e_n+e_1}$, $s_{e_n-e_1}$ are incompatible (they are of the same length by Proposition 1) and $s_{e_n} \notin W_{D_n}$. As for inclusions of $B-$orbit closures we have to take into account that inclusions of $B-$orbit closures in D_n implies the inclusions of corresponding $B-$orbit closures in B_n so that we have to check only the corresponding cases from Sect. 3.3. We get:

(i)+(ii) Exactly as in $\mathfrak{so}_{2n+1}$ one has $\mathbf{B}_{\{e_n-e_{i-1}\}} \subset \overline{\mathbf{B}}_{\{e_n-e_i\}}$ for $i\,:\, 2 \le i \le n-1$ which corresponds to $s_{e_n+e_{n-1}} > \dots > s_{e_n+e_2}$ and $s_{e_n+e_2} > \begin{cases} s_{e_n+e_1} & \text{if } n = 2k+1; \\ s_{e_n-e_1} & \text{if } n = 2k; \end{cases}$. Further note that $Exp(aX_{e_2+e_1}).X_{e_n-e_2} = X_{e_n-e_2} - aX_{e_n+e_1}$ so that by torus action we get $X_{e_n+e_1} \in \overline{\mathbf{B}}_{e_n-e_2}$ which corresponds to $s_{e_n+e_2} > \begin{cases} s_{e_n-e_1} & \text{if } n = 2k+1; \\ s_{e_n+e_1} & \text{if } n = 2k; \end{cases}$

Exactly as in $\mathfrak{so}_{2n+1}$ one has $\mathbf{B}_{e_n+e_{i+1}} \subset \overline{\mathbf{B}}_{\{e_n+e_i\}}$ for $1 \le i \le n-2$. This corresponds to $s_{e_n-e_i} > s_{e_n-e_{i+1}}$ for $i\,:\, 2 \le i \le n-2$ and $s_{e_n-e_2} < \begin{cases} s_{e_n-e_1} & \text{if } n = 2k+1; \\ s_{e_n+e_1} & \text{if } n = 2k; \end{cases}$.

Let us show that $\mathbf{B}_{\{e_n+e_2\}} \subset \overline{\mathbf{B}}_{\{e_n-e_1\}}$. Indeed, $Exp(aX_{e_2+e_1}).X_{e_n-e_1} = X_{e_n-e_1} + aX_{e_n+e_2}$ so that by torus action we get $X_{e_n+e_2} \in \overline{\mathbf{B}}_{\{e_n-e_1\}}$. This cor-

responds to $s_{e_n-e_2} < \begin{cases} s_{e_n+e_1} & \text{if } n = 2k+1; \\ s_{e_n-e_1} & \text{if } n = 2k; \end{cases}$.
To finish (i) and (ii) we have to check that $\mathbf{B}_{e_n+e_1} \not\subset \overline{\mathbf{B}}_{e_n-e_1}$. This is obtained straightforwardly from the fact $\dim \mathbf{B}_{e_n+e_1} = \dim \mathbf{B}_{e_n-e_1} = n-1$.

(iii) Exactly as in Sect. 3.3 one has $\mathbf{B}_{\{e_n-e_i,\, e_n+e_i\}} \subset \overline{\mathbf{B}}_{\{e_n-e_j, e_n+e_j\}}$ for $1 \le i < j \le n-1$. Obviously by torus action we get $X_{e_n-e_j} \in \overline{\mathbf{B}}_{\{e_n-e_j, e_n+e_j\}}$ and $X_{e_n \pm e_1} \in \overline{\mathbf{B}}_{\{e_n-e_1, e_n+e_1\}}$ so we get all the relations from (iii).

Since $\mathfrak{m}_{e_2+e_1} \cong \mathfrak{m}_{e_2-e_1}$ it is enough to consider $\mathfrak{m}_{e_2+e_1}$. One has

$$\mathfrak{m}_{e_2+e_1} = \bigoplus_{1 \le i < j \le n} \mathbb{C}X_{e_j+e_i}$$

Comparing $\mathfrak{m}_{e_2+e_1}$ with $\mathfrak{m}_{2e_1}$ of $\mathfrak{sp}_n$ one can see at once that root vectors here correspond (up to sign in the sum) to root vectors in $\mathfrak{m}_{2e_1}$ for short roots. In particular, this is a subspace of matrices of nilpotency order 2. The truth of the conjecture for $\mathfrak{m}_{e_2+e_1}$ is obtained from its truth for $\mathfrak{m}_{2e_1}$ by the following facts:

1. The sets of strongly orthogonal roots in $\mathfrak{m}_{e_2+e_1}$ coincide with the sets of strongly orthogonal short roots in $\mathfrak{m}_{2e_1}$.
2. Only for root $\alpha = e_j - e_i$ the action of $Exp(aX_\alpha)$ on roots $X_{e_k+e_s}$ can be non-trivial both in C_n and D_n and this action in both cases coincide up to sign, apart from case $Exp(aX_{e_j-e_i}).X_{e_j+e_i} = \begin{cases} X_{e_j+e_i} + 2aX_{2e_j} & \text{in } C_n; \\ X_{e_j+e_i} & \text{in } D_n; \end{cases}$ which is irrelevant here. Thus, $X_{\mathcal{S}'} \in \overline{\mathbf{B}}_{\mathcal{S}}$ for strongly orthogonal sets $\mathcal{S}'$, $\mathcal{S}$ in $\mathfrak{m}_{e_2+e_1}$ iff $X_{\mathcal{S}'} \in \overline{\mathbf{B}}_{\mathcal{S}}$ in $\mathfrak{m}_{2e_1}$.
3. $\widehat{w}$ of $W_{e_2+e_1}$ (in W_{D_n}) is equal to $\widehat{w}$ of W_{2e_1} (in W_{C_n}).
4. Bruhat order restricted to multiplication of reflections of strongly orthogonal roots of type $e_i + e_j$ coincides for W_{C_n} and W_{D_n}. (cf., for example [7, Sect. 4]).

Acknowledgements We would like to thank Dmitri Panyushev for sharing his paper with us and for discussions during this work. N. Barnea was partially supported by Israel Scientific Foundation grant 797/14.

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Anti de Sitter Holography via Sekiguchi Decomposition

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Abstract In the present paper we start consideration of anti de Sitter holography in the general case of the $(q+1)$-dimensional anti de Sitter bulk with boundary q-dimensional Minkowski space-time. We present the group-theoretic foundations that are necessary in our approach. Comparing what is done for $q = 3$ the new element in the present paper is the presentation of the bulk space as the homogeneous space $G/H = SO(q, 2)/SO(q, 1)$, which homogeneous space was studied by Sekiguchi.

1 Introduction

For the last fifteen years due to the remarkable proposal of [1] the AdS/CFT correspondence is a dominant subject in string theory and conformal field theory. Actually the possible relation of field theory on anti de Sitter space to conformal field theory on boundary Minkowski space-time was studied also before, cf., e.g., [2–7]. The proposal of [1] was further elaborated in [8] and [9]. After that there was an explosion of related research which continues also currently.

Let us recall that the AdS/CFT correspondence has 2 ingredients [1, 8, 9]: 1. the holography principle, which is very old, and means the reconstruction of some objects in the bulk (that may be classical or quantum) from some objects on the boundary; 2. the reconstruction of quantum objects, like 2-point functions on the boundary, from appropriate actions on the bulk.

Our focus is on the first ingredient. We note that until recently the explicit presentation of the holography principle was realized in the Euclidean case, i.e., for the group $SO(q+1, 1)$ relying on Wick rotations of the final results, cf., e.g.,

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V. Dobrev (ed.), *Lie Theory and Its Applications in Physics*,
Springer Proceedings in Mathematics & Statistics 191,
DOI 10.1007/978-981-10-2636-2_29

[9, 10]. Yet it is desirable to show the holography principle by direct construction in Minkowski space-time, i.e., for the conformal group $SO(q, 2)$.

This was done for the case $q = 3$ in detail in [11]. In the present paper we start consideration of the general case of the $(q+1)$-dimensional anti de Sitter bulk with boundary q-dimensional Minkowski space-time. Actually, here we only lay the group-theoretic foundations that are necessary in our approach while the actual construction is postponed to [12]. As historical remark we mention that this approach originated in the construction of the discrete series of unitary representations in [13, 14], which was then applied in [15] for the Euclidean conformal group $SO(4, 1)$. A different approach was applied to the general Euclidean case $SO(N, 1)$ in [10]. Also the nonrelativistic Schrödinger algebra case was considered in [16].

The new element in the present paper is the presentation of the bulk space as the homogeneous space $G/H = SO(q, 2)/SO(q, 1)$. For this we use the Sekiguchi decomposition [17]

$$G \cong |_{\mathrm{loc}}\ \widetilde{N}AH$$

where A is the subgroup of dilatations, $\widetilde{N}$ is isomorphic to the subgroup of translations. The above means that the subgroup $\widetilde{N}AH$ is an open dense set of G, and thus the homogeneous space G/H is locally isomorphic to bulk space $\widetilde{N}A$.

2 Preliminaries

We need some well-known preliminaries to set up our notation and conventions. The Lie algebra $\mathcal{G} = so(q, 2)$ may be defined as the set of $(q+2) \times (q+2)$ matrices X which fulfil the relation:

$$^{t}X\eta + \eta X = 0, \tag{1}$$

where the metric η is given by

$$\eta = (\eta_{AB}) = \mathrm{diag}(-1, 1, \ldots, 1, -1), \quad \mathrm{A, B = 0, 1, \cdots, q+1} \tag{2}$$

Then we can choose a basis $X_{AB} = -X_{BA}$ of $\mathcal{G}$ satisfying the commutation relations

$$[X_{AB}, X_{CD}] = \eta_{AC}X_{BD} + \eta_{BD}X_{AC} - \eta_{AD}X_{BC} - \eta_{BC}X_{AD}. \tag{3}$$

We list the important subalgebras of $\mathcal{G}$:

- $\mathcal{K} = so(q) \oplus so(2)$, generators: $X_{AB}\ :\ (A, B) \in \{1, \ldots, q\}, \{0, q+1\}$, maximal compact subalgebra;
- $\mathcal{Q}$, generators: $X_{AB} : A \in \{1, \ldots, q\},\ B \in \{0, q+1\}$, non-compact completion of $\mathcal{K}$;
- $\mathcal{A} = so(1, 1)$, generator: $D \doteq X_{q,q+1}$, dilatations;

- $\mathcal{M} = so(q-1, 1)$, generators: X_{AB} : $(A, B) \in \{0, \ldots, q-1\}$, Lorentz subalgebra;
- $\mathcal{N}$, generators: $T_\mu = X_{\mu q} + X_{\mu,q+1}$, $\mu = 0, \ldots, q-1$, translations;
- $\widetilde{\mathcal{N}}$, generators: $C_\mu = X_{\mu q} - X_{\mu,q+1}$, $\mu = 0, \ldots, q-1$, special conformal transformations;
- $\mathcal{A}_0 = so(1, 1) \oplus so(1, 1)$, generators: $X_{0,q-1}, X_{q,q+1}$;
- $\mathcal{M}_0 = so(q-2)$, generators: $X_{AB} : (A, B) \in \{1, \ldots, q-2\}$;
- $\mathcal{N}_0$, generators: T_μ, $\mu = 0, \ldots, q-1$, $T'_\mu = X_{\mu 0} + X_{\mu,q-1}$, $\mu = 1, \ldots, q-2$, extended translations;
- $\widetilde{\mathcal{N}}_0$, generators: C_μ, $\mu = 0, \ldots, q-1$, $C'_\mu = X_{\mu 0} - X_{\mu,q-1}$, $\mu = 1, \ldots, q-2$, extended special conformal transformations;
- $\mathcal{H} = so(q, 1)$, generators: $X_{AB} : (A, B) \in \{0, \ldots, q\}$.

The last subalgebra is the analog of the maximal compact subalgebra $so(q+1)$ of the Euclidean conformal algebra $so(q+1, 1)$ of q-dimensional Euclidean space. Thus, it may result from the Wick rotation of the Euclidean conformal algebra $so(q+1, 1)$ to the Minkowskian conformal algebra $so(q, 2)$.

Thus, we have several decompositions:

- $\mathcal{G} = \mathcal{K} \oplus \mathcal{Q}$, Cartan decomposition;
- $\mathcal{G} = \mathcal{K} \oplus \mathcal{A}_0 \oplus \mathcal{N}_0$, (also $\mathcal{N}_0 \to \widetilde{\mathcal{N}}_0$), Iwasawa decomposition;
- $\mathcal{G} = \mathcal{N}_0 \oplus \mathcal{M}_0 \oplus \mathcal{A}_0 \oplus \widetilde{\mathcal{N}}_0$, minimal Bruhat decomposition;
- $\mathcal{G} = \mathcal{N} \oplus \mathcal{M} \oplus \mathcal{A} \oplus \widetilde{\mathcal{N}}$, maximal Bruhat decomposition;
- $\mathcal{G} = \mathcal{H} \oplus \mathcal{A} \oplus \mathcal{N}$, (also $\mathcal{N} \to \widetilde{\mathcal{N}}$), Sekiguchi decomposition [17].

The subalgebra $\mathcal{P}_0 = \mathcal{M}_0 \oplus \mathcal{A}_0 \oplus \widetilde{\mathcal{N}}_0$ is a minimal parabolic subalgebra of $\mathcal{G}$. The subalgebra $\mathcal{P} = \mathcal{M} \oplus \mathcal{A} \oplus \widetilde{\mathcal{N}}$ is a maximal parabolic subalgebra of $\mathcal{G}$.

Finally, we introduce the corresponding Lie groups:
$G = SO_0(q, 2)$ with Lie algebra $\mathcal{G} = so(q, 2)$, $H = SO(q, 1)$ with Lie algebra $\mathcal{H} = so(q, 1)$, $K = SO(q) \times SO(2)$ is the maximal compact subgroup of G, $A_0 = \exp(\mathcal{A}_0) = SO_0(1, 1) \times SO_0(1, 1)$ is abelian simply connected, $N_0 = \exp(\mathcal{N}_0) \cong \widetilde{N}_0 = \exp(\widetilde{\mathcal{N}}_0)$, are abelian simply connected subgroups of G preserved by the action of A_0. The group $M_0 \cong SO_0(q-2)$ (with Lie algebra $\mathcal{M}_0$) commutes with A_0. Further $A = \exp(\mathcal{A}) = SO_0(1, 1)$ is abelian simply connected, $N = \exp(\mathcal{N}) \cong \widetilde{N} = \exp(\widetilde{\mathcal{N}})$, are abelian simply connected subgroups of G preserved by the action of A. The group $M \cong SO_0(q-1, 1)$ (with Lie algebra $\mathcal{M}$) commutes with A.

We mention also some group decompositions:

$$G = KA_0N_0, \text{ (also } N_0 \to \widetilde{N}_0\text{), Iwasawa decomposition;} \tag{4a}$$
$$G \cong |_{\text{loc}}\, \widetilde{N}AMN, \text{ maximal Bruhat decomposition;} \tag{4b}$$
$$G \cong |_{\text{loc}}\, \widetilde{N}AH, \text{ (also } \widetilde{N} \to N\text{), Sekiguchi decomposition} \tag{4c}$$

In (4b, 4c) the groups on the RHS are open dense subsets of G. We should note that in [17] was studied the more general case $SO_0(q, r+1)/SO(q, r)$.

The subgroup $P_0 = M_0A_0N_0$ is a *minimal parabolic subgroup* of G. The subgroup $P = MAN$ is a *maximal parabolic subgroup* of G. Parabolic subgroups are important because the representations induced from them generate all admissible irreducible representations of semisimple groups [18, 19]. The group (algebra) $SO_0(q, 2)$ ($so(q, 2)$) has one more maximal (cuspidal) parabolic subgroup (subalgebra) which we do not give here for the lack of space, cf., e.g., [20, 21] for $q = 4$.

3 Elementary Representations

We use the approach of [22] which we adapt in a condensed form here. We work with so-called *elementary representations* (ERs). They are induced from representations of the parabolic subgroups. Here we work with the maximal parabolic $P = MAN$, where we use (non-unitary) finite-dimensional representations λ of $M = SO(q-1, 1)$ in the space V_λ, (non-unitary) characters of A represented by the conformal weight Δ, and the factor N is represented trivially. For further use we give explicit parametrization of λ:

$$\lambda = (\lambda_1, \dots, \lambda_{\hat{q}}) , \quad \hat{q} \doteq [\tfrac{1}{2}q] \tag{5}$$

where $[w]$ is the largest integer not greater than w. The numbers λ_i are all integer or all half-integer and they fulfill the following conditions:

$$\begin{aligned} &|\lambda_1| \le \lambda_2 \le \cdots \le \lambda_{q/2} , \quad \text{for } q \text{ even} \\ &0 \le \lambda_1 \le \lambda_2 \le \cdots \le \lambda_{(q-1)/2} , \quad \text{for } q \text{ odd} \end{aligned} \tag{6}$$

The data λ, Δ is enough to determine a weight $\chi \in \mathcal{H}^*_{\mathcal{G}}$, where $\mathcal{H}_{\mathcal{G}}$ is the Cartan subalgebra of $\mathcal{G}$, cf. [22]. Thus, we shall denote the ERs by C^χ. Sometimes we shall write: $\chi = [\lambda, \Delta]$. The representation spaces are C^∞ functions on G/P, or equivalently, on the locally isomorphic group N with appropriate asymptotic conditions (which we do not need explicitly, cf., e.g., [21, 22]). We recall that $\widetilde{N}$ is isomorphic to q-dimensional Minkowski space-time $\mathfrak{M}$ whose elements will be denoted by $x = (x_0, \dots, x_{q-1})$, while the corresponding elements of $\widetilde{N}$ will be denoted by n_x. The Lorentzian inner product in $\mathfrak{M}$ is defined as usual:

$$\langle x, x' \rangle \doteq x_0 x'_0 - \cdots - x_{q-1} x'_{q-1} , \tag{7}$$

and we use the notation $\boldsymbol{x}^2 = \langle x, x \rangle$.

The representation action is given as follows:

$$(T^\chi(g)\varphi)(x) = y^{-\Delta} D^\lambda(m)\, \varphi(x') \tag{8}$$

the various factors being defined from the local Bruhat decomposition (4b) $G \cong_{\text{loc}} \widetilde{N}AMN$:

$$g^{-1}\, \widetilde{n}_x = \widetilde{n}_{x'}\, a_y^{-1} m^{-1} n^{-1} , \tag{9}$$

where $y \in \mathbb{R}_+$ parametrizes the elements $a \in A$, $m \in M$, $D^\lambda(m)$ denotes the representation action of M on the space V_λ, $n \in N$.

On these functions the infinitesimal action of our representations looks as follows:

$$\begin{aligned}
T_\mu &= \partial_\mu, \quad \partial_\mu \doteq \frac{\partial}{\partial x_\mu}, \quad \mu = 0, \dots, q-1, \\
D &= -\sum_{\mu=0}^{q-1} x_\mu \partial_\mu - \Delta, \\
X_{0a} &= x_0 \partial_a + x_a \partial_0 + \mathfrak{s}_{0a}, \quad a = 1, \dots, q-1, \\
X_{ab} &= -x_a \partial_b + x_b \partial_a + \mathfrak{s}_{ab}, \quad 1 \leq a < b \leq q-1, \\
C_\mu &= -2\eta_{\mu\mu}\, x_\mu D + \boldsymbol{x}^2 \partial_\mu - 2 \sum_{\nu=0}^{q-1} x^\nu \mathfrak{s}_{\mu\nu},
\end{aligned} \tag{10}$$

where $\mathfrak{s}_{\mu\nu}$ are the infinitesimal generators of $D^\lambda(m)$.

We recall several facts about elementary representations [15, 22]:

- The Casimir operators $\mathcal{C}_i$ of $\mathcal{G}$ have constant values on the ERs:

$$\mathcal{C}_i(\{X\})\, \varphi(x) = \chi_i(\lambda, \Delta)\, \varphi(x), \qquad i = 1, \dots, \operatorname{rank} G = \left[\tfrac{1}{2}q\right] + 1, \tag{11}$$

  where $\{X\}$ denotes symbolically the generators of the Lie algebra $\mathcal{G}$ of G, the action of which is given in (10).
- On the ERs are defined the integral Knapp–Stein G_χ operators which intertwine the representation χ with the representation $\tilde{\chi} \doteq [\tilde{\lambda}, q - \Delta]$, where $\tilde{\lambda}$ is the mirror image of λ. We recall that the mirror image $\tilde{\lambda}$ is equivalent to λ when q is odd, while for q even and λ parametrized as in (5): $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_{q/2})$ we have $\tilde{\lambda} = (-\lambda_1, \lambda_2, \dots, \lambda_{q/2})$.
- The representations χ and $\tilde{\chi}$ are called *partially equivalent* due to the existence of the intertwining operator G_χ between them. The representations are called *equivalent* if the intertwining operator G_χ is onto and invertible.
- We also recall that the Casimirs χ_i have the same values on the partially equivalent ERs:

$$\chi_i(\lambda, \Delta) = \chi_i(\tilde{\lambda}, q - \Delta) \tag{12}$$

In the above general definition $\varphi(x)$ are considered as elements of the finite-dimensional representation space V^λ in which act the operators $D^\lambda(m)$. The representation space C^χ can be thought of as the space of smooth sections of the homogeneous vector bundle (called also vector G-bundle) with base space G/P and fibre V_λ, (which is an associated bundle to the principal P-bundle with total space G). Actually, we do not need this description, but following [22] we replace the above homogeneous vector bundle with a line bundle again with base space G/P. The resulting functions $\hat{\varphi}$ can be thought of as smooth sections of this line bundle.

In the case when the representation λ is of symmetric traceless tensors of rank ℓ, i.e., $\lambda = (0, \ldots, 0, \ell)$, we can be more explicit following [15]. Namely, the functions $\hat{\varphi}$ are scalar functions over an extended space $\mathfrak{M} \times \mathfrak{M}_0$, where $\mathfrak{M}_0$ is a cone parametrized by the variable $\zeta = (\zeta_0, \ldots, \zeta_{q-1})$ subject to the condition:

$$\zeta^2 = \langle \zeta, \zeta \rangle = \zeta_0^2 - \cdots - \zeta_{q-1}^2 = 0. \tag{13}$$

The functions on the extended space will be denoted as $\hat{\varphi}(x, \zeta)$. The internal variable ζ will carry the representation D^λ. Thus, on the functions $\hat{\varphi}$ the infinitesimal generators $\mathfrak{s}_{\mu\nu}$ from (10) are given as follows:

$$\mathfrak{s}_{0a} = \zeta_0 \frac{\partial}{\partial \zeta_a} + \zeta_a \frac{\partial}{\partial \zeta_0}, \qquad \mathfrak{s}_{ab} = -\zeta_a \frac{\partial}{\partial \zeta_b} + \zeta_b \frac{\partial}{\partial \zeta_a}, \tag{14}$$

4 Bulk Representations

It is well known that the group $SO(q, 2)$ is called also anti de Sitter group, as it is the group of isometry of (q + 1)-dimensional anti de Sitter space:

$$\xi^A \xi^B \eta_{AB} = 1 \,, \quad A, B = 0, \ldots, q+1 \,. \tag{15}$$

There are several ways to parametrize anti de Sitter space. For $q = 3$ in the paper [11] was utilized the same local Bruhat decomposition (4b) that we used in the previous section. In the present paper we shall use the Sekiguchi decomposition (4c), i.e., the factor-space $G/H \cong \widetilde{N}A$. In fact, we use isomorphic (w.r.t. [11]) coordinates $(x, y) = (x_0, \ldots, x_{q-1}, y)$, $y \in \mathbb{R}_+$. In this setting anti de Sitter space is called *bulk* space, while q-dimensional Minkowski space-time is called *boundary* space, as it is identified with the bulk boundary value $y = 0$.

It is natural to discuss representations on anti de Sitter space $\widetilde{N}A$ which are induced from the subgroup $H = SO(q, 1)$. Namely, we consider the representation space:

$$\hat{C}_\tau = \{\phi \in C^\infty(\mathbb{R}^q \times \mathbb{R}_{>0} \,, \hat{V}_\tau)\} \tag{16}$$

where τ is an arbitrary finite-dimensional irrespective of H, $\hat{V}_\tau$ is the finite-dimensional representation space of τ, with representation action:

$$(\hat{T}^\tau(g)\phi)(x, y) = \tilde{D}^\tau(h)\, \phi(x', y') \tag{17}$$

where the Sekiguchi decomposition is used:

$$g^{-1}\widetilde{n}_x a_y = \widetilde{n}_{x'} a_{y'} h^{-1}, \quad g \in G,\ h \in H,\ n_x, n_{x'} \in N,\ a_y, a_{y'} \in A \tag{18}$$

and $\tilde{D}^{\tau}(k)$ is the representation matrix of τ in $\hat{V}_{\tau}$. For later use we give the parametrization of the relevant subgroups:

$$H = \left\{ h = \begin{bmatrix} h' & 0 \\ 0 & \pm 1 \end{bmatrix} | h \in SO_0(q,2), \quad h' \in SO_0(q,1), \right\} \cong SO(q,1) \tag{19}$$

$$A = \left\{ a_y = \begin{pmatrix} \mathbb{I}_q & 0 & 0 \\ 0 & \cosh(s) & \sinh(s) \\ 0 & \sinh(s) & \cosh(s) \end{pmatrix} | y = e^s, s \in \mathbb{R} \right\} \tag{20}$$

$$\tilde{N} = \left\{ \tilde{n}_x = \begin{bmatrix} \mathbb{I}_1 & 0 & t & t \\ 0 & \mathbb{I}_{q-1} & s^{\dagger} & s^{\dagger} \\ t & -s & 1 + \frac{t^2}{2} - \frac{s^2}{2} & \frac{t^2}{2} - \frac{s^2}{2} \\ -t & s & \frac{s^2}{2} - \frac{t^2}{2} & 1 + \frac{s^2}{2} - \frac{t^2}{2} \end{bmatrix} | \ (t,s) = \frac{x}{\sqrt{2}} \in \mathbb{R}^q \right\} \tag{21}$$

The infinitesimal generators of (17) are given as follows:

$$\begin{aligned} \hat{T}_{\mu} &= \partial_{\mu}, \quad \mu = 0, \dots, q-1 \\ \hat{D} &= -\sum_{\mu=0}^{q-1} x_{\mu}\partial_{\mu} - y\partial_y, \\ \hat{X}_{0a} &= x_0\partial_a + x_a\partial_0 + \mathfrak{s}_{0a}, \quad a = 1, \dots, q-1 \\ \hat{X}_{ab} &= -x_a\partial_b + x_b\partial_a + \mathfrak{s}_{ab}, \quad 1 \le a < b \le q-1 \\ \hat{C}_{\mu} &= -2\eta_{\mu\mu}x_{\mu}D + (\boldsymbol{x}^2 + y^2)\partial_{\mu} - 2\sum_{\nu=0}^{q-1} x^{\nu}\mathfrak{s}_{\mu\nu} - 2y\Gamma_{\mu}, \end{aligned} \tag{22}$$

where $\mathfrak{s}_{\mu\nu}$, Γ_{μ} are infinitesimal generators of $\tilde{D}^{\tau}(h)$, such that (due to the compatibility of λ and τ) $\mathfrak{s}_{\mu\nu} = \frac{1}{4}[\Gamma_{\mu}, \Gamma_{\nu}]$, $[\mathfrak{s}_{\mu\nu}, \Gamma_{\rho}] = \eta_{\nu\rho}\Gamma_{\mu} - \eta_{\mu\rho}\Gamma_{\nu}$.

Note that the realization of $so(q,2)$ on the boundary given in (10) may be obtained from (22) by replacing $y\partial_y \to \Delta$ and then taking the limit $y \to 0$.

What is important is that, unlike the ERs, the representations (17) are highly reducible. Our aim is to extract from $\hat{C}_{\tau}$ representations that may be equivalent to C_{χ} , $\chi = [\lambda, \Delta]$. The first condition for this is that the M-representation λ is contained in the restriction of the H-representation τ to M, i.e., $\lambda \in \tau|_M$. Another condition is that the two representations would have the same Casimir values $\lambda_i(\lambda, \Delta)$.

This procedure is actually well understood and used in the construction of the discrete series of unitary representations, cf. [13, 14], (also [11, 15] for $q = 4$). The method utilizes the fact that in the bulk the Casimir operators are not fixed numerically. Thus, when a vector-field realization of the anti de Sitter algebra $so(q,2)$ (e.g., (22)) is substituted in the bulk Casimirs the latter turn into differential operators. In contrast, the boundary Casimir operators are fixed by the quantum numbers of the fields under consideration. Then the bulk/boundary correspondence forces eigenvalue equations involving the Casimir differential operators. Actually the 2nd order Casimir is enough for this purpose. That corresponding eigenvalue 2nd order differential equation is used to find the two-point Green function in the bulk which is then used to construct the boundary-to-bulk integral intertwining operator. This operator maps a boundary field to a bulk field. For our setting this will be given in detail in [12].

Having in mind the degeneracy of Casimir values for partially equivalent representations (e.g., (12)) we add also the appropriate asymptotic condition. Furthermore, from now on we shall suppose that Δ is real.

Thus, the representation (partially) equivalent to the ER χ is defined as:

$$\hat{C}^{\tau}_{\chi} = \{ \phi \in \hat{C}_{\tau} : \ \mathcal{C}_i(\{\hat{X}\})\, \phi(x,y) = \lambda_i(\lambda, \Delta)\, \phi(x,y), \quad \forall i, \quad \lambda \in \tau|_M , \\ \phi(x,y) \sim y^{\Delta}\, \varphi(x) \text{ for } y \to 0 \} \tag{23}$$

where $\hat{X}$ denotes the action (22) of $\mathcal{G}$ on the bulk fields.

In the case of symmetric traceless tensors of rank ℓ for both M and H we can extend the functions on the bulk extended also with the cone $\mathfrak{M}_0$. These extended functions will be denoted by $\phi(x, y, \zeta)$. On these functions we have the infinitesimal action given by (22) with $\mathfrak{s}_{\mu\nu}$ are given by (14), while Γ_μ are certain finite-dimensional matrices which we shall give in [12].

5 Two Parametrizations of Bulk Space

As we mentioned in [11] we used as parametrization of the bulk space the coset $G/MN =|_{\mathrm{loc}} \widetilde{N}A$. The local coordinates of this coset come from the Bruhat decomposition:

$$g = \{g_{AB}\} = \widetilde{n}_x a_y m n \tag{24}$$

which exists for $g \in G$ forming a dense subset of G. The local coordinates of the above bulk are:

$$y = \tfrac{1}{2}(g_{qq} + g_{q,q+1} + g_{q+1,q} + g_{q+1,q+1}) \ , \tag{25}$$
$$x_\mu = \frac{g_{\mu,q} + g_{\mu,q+1}}{g_{qq} + g_{q,q+1} + g_{q+1,q} + g_{q+1,q+1}} \ , \ \mu = 0, \dots, q-1$$

The parametrization used in the present paper for the bulk space is the coset $G/H =|_{\mathrm{loc}} \widetilde{N}A$. Certainly, it is isomorphic to the bulk above, however, the local coordinates are different, namely, the latter. come from the Sekiguchi decomposition:

$$g = \{g_{AB}\} = \widetilde{n}_x a_y h \tag{26}$$

Explicitly, they are given as follows:

$$y = |g_{q+1,q} + g_{q+1,q+1}| \tag{27}$$
$$x_\mu = \frac{g_{\mu,q+1}}{g_{q+1,q} + g_{q+1,q+1}} \quad \mu = 0, \ldots, q-1$$

Comparing the two parametrizations (25) and (27) we see that the latter is simpler and thus easier to implement. Thus, in the follow-up paper [12] we shall use the Sekiguchi decomposition (26).

Acknowledgements The first author has received partial support from COST actions MP-1210 and MP-1405, and from Bulgarian NSF Grant DFNI T02/6.

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Localization and the Canonical Commutation Relations

Patrick Moylan

Abstract Let $\mathbf{W}_n(\mathbb{R})$ be the Weyl algebra of index n. We have shown that by using extension and localization, it is possible to construct homomorphisms of $\mathbf{W}_n(\mathbb{R})$ onto its image in a localization, or a quotient thereof, of $\mathfrak{U}(\mathfrak{so}(2,q))$, the universal enveloping algebra of $\mathfrak{so}(2,q)$, for n depending upon q [1]. Here we treat the $\mathfrak{so}(2,1)$ case in complete detail. We establish an isomorphism of skew fields, specifically, $\widetilde{\mathfrak{D}(\mathfrak{so}(2,1))} \simeq \mathfrak{D}_{(1,1)}(\mathbb{R})$ where $\mathfrak{D}_{1,1}(\mathbb{R})$ is the fraction field of $\mathbf{W}_{1,1}(\mathbb{R}) \simeq \mathbf{W}_1(\mathbb{R}) \otimes \mathbb{R}(y)$ with $\mathbb{R}(y)$ being the ring of polynomials in the indeterminate y and $\widetilde{\mathfrak{D}(\mathfrak{so}(2,1))}$ is a certain extension of the skew field of fractions of $\mathfrak{U}(\mathfrak{so}(2,1))$, which is described below. We give applications of this result to representations. In particular we are able to construct representations of $\mathbf{W}_1(\mathbb{R})$ out of representations of $\mathfrak{so}(2,1)$. Thus, we are able, for this lowest dimensional case, to obtain the canonical commutation relations and representations of them out of $\mathfrak{so}(2,1)$ symmetry. Using similar results in higher dimensions [1] we are able to construct representations of $\mathbf{W}_n(\mathbb{R})$ out of representations of $\mathfrak{so}(2,q)$.

1 Introduction

Localization, or formation of quotients, is a powerful tool in mathematics with many known applications. It is used to relate different algebraic structures which share some common similarities. An important example is the Gelfand-Kirillov conjecture [2] which is very much related to what we do here. Another example, extremely interesting from the physical viewpoint, is an isomorphism between Lie field extensions of $SO_0(1,4)$ and the Poincaré group which was first demonstrated in [3].

In this paper we study a physically interesting problem similar in nature to the ones just mentioned, namely, given a real Lie algebra, L, is it possible to obtain an embedding of $\mathbf{W}_n(\mathbb{R})$ (for certain n depending upon L) into a commutative algebraic

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V. Dobrev (ed.), *Lie Theory and Its Applications in Physics*, Springer Proceedings in Mathematics & Statistics 191,
DOI 10.1007/978-981-10-2636-2_30

extension of a localization of $\mathfrak{U}(L)$ or a suitable quotient thereof? We have studied in some detail the case of $L = \mathfrak{so}(2, q)$ and have reported some of our results in [1]. Our method of approach rests upon the conformal realization of $\mathfrak{so}(2, q)$ as vector-valued differential operators on a q dimensional pseudo-Euclidean space of signature $(1, q-1)$ and it essentially amounts to obtaining of the canonical commutation relations out of conformal symmetry. Usually the canonical commutation relations are derived from some form of translational symmetry (homogeneity of space) with additional assumptions, e.g. a system of imprimitivity [4, 5]. The arguments presented herein rest solely upon conformal symmetry and extension and localization. For us, the noncommutativity of the momentum and position operators comes from the noncommutativity of pseudorotations in different directions of conformal space. Our arguments provide a different and perhaps more convincing derivation than those presented in the just mentioned references.

2 Facts About Enveloping Algebras, Weyl Algebras and $SO_0(2, 1)$

$\mathfrak{U}(L)$ denotes the universal enveloping algebra of a Lie algebra L over $\mathbb{R}$. It is a ring with identity which we denote by 1. $\mathfrak{U}(L)$ is a filtered algebra i.e. $\mathfrak{U}_r(L) = \{p_r(X_1, X_2, ...X_n)|\deg(p_r) \leq i; X_i \in L\}$ with $\mathfrak{U}_0(L) = \mathbb{R}$, $\mathfrak{U}_1(L) = \mathbb{R} \cdot 1 + L$ and $\mathfrak{U}_r(L)\, \mathfrak{U}_s(L) \subset \mathfrak{U}_{r+s}(L)$. A basis for $\mathfrak{U}_r(L)$ is $X_1^{q_1} X_2^{q_2} \ldots X_n^{q_n}$ with $q_1 + q_2 + \ldots q_n \leq i$. The graded algebra associated with $\mathfrak{U}(L)$ is $G = G^0 \oplus G^1 \oplus \ldots \oplus G^i \oplus \ldots$ with $G^i = \mathfrak{U}_i(L)/U_{i-1}(L)$ $(U_{-1}(L) = 0)$. $\mathfrak{U}(L)$ admits a fraction field which we denote by $\mathfrak{D}(L)$ [6]. Denote the centers of $\mathfrak{U}(L)$ and $\mathfrak{D}(L)$ by $\mathfrak{Z}(L)$ and $\mathfrak{c}(L)$, respectively.

The Weyl algebra $\mathbf{W}_n(\mathbb{R})$ is determined by the $2n$ generators $p_1, \ldots, p_n, q_1, \ldots, q_n$ with relations

$$[p_i, p_j] = [q_i, q_j] = 0 \tag{1}$$

$$[p_i, q_j] = \delta_{ij} \tag{2}$$

for all $i, j \leq n$. Given a collection of free variables $y_1, \ldots, y_s$ we define

$$\mathbf{W}_{n,s}(\mathbb{R}) := \mathbf{W}_n(\mathbb{R}) \otimes \mathbb{R}[y_1, \ldots, y_s]. \tag{3}$$

Being a Noetherian domain [7] the algebra $\mathbf{W}_{n,s}(\mathbb{R})$ also admits a field of fractions denoted $\mathfrak{D}_{n,s}(\mathbb{R})$. In $\mathbf{W}_{n,s}(\mathbb{R})$ we have the filtration $\mathbf{W}_{n,s}(\mathbb{R})_0 \subset \mathbf{W}_{n,s}(\mathbb{R})_1 \subset \ldots$ where $\mathbf{W}_{n,s}(\mathbb{R})_i$ is set of all polynomials of degree $\leq i$ in $q_1, \ldots, q_n, p_1, \ldots, p_n$ with coefficients in $\mathbb{R}[y_1, \ldots, y_s]$. The associated graded ring $\mathbf{G}_{n,s}(\mathbb{R})$ is isomorphic to the polynomial ring $\mathbb{R}[p_1, \ldots, p_s; q_1, \ldots, q_s; y_1, \ldots, y_s]$.

Let $\mathbb{R}[x_i, \ldots x_j]$ be the ring of polynomial functions in n variables $x_1, x_2, ..., x_n$ with coefficients in $\mathbb{R}$. We have a homomorphism ρ from $\mathbf{W}_n(\mathbb{R})$ into End

($\mathbb{R}[x_i, \ldots x_j]$) determined by $\rho(p_i) = \partial/\partial x_i (= \partial_i)$ and $\rho(x_j) =$ "multiplication by" x_j. Since $\mathbf{W}_n(\mathbb{R})$ is simple [6], it is easy to see that ρ is injective. Thus this representation of $\mathbf{W}_n(\mathbb{R})$ is isomorphic to $\mathbf{W}_n(\mathbb{R})$. From an algebraic standpoint, we can and shall use this representation of $\mathbf{W}_n(\mathbb{R})$ interchangeably with $\mathbf{W}_n(\mathbb{R})$.

$SO(2,1) = \{g \in SL(3,\mathbb{R}) | g^{\dagger}\beta_0 g = \beta_0\}$ $(\beta_0 = \mathrm{diag}(1,\ 1, -1))$. $SO_0(2,1)$ is the connected component and $\mathfrak{so}(2,1) = \{X \in \mathfrak{sl}(3,\mathbb{R}) | X^{\dagger}\beta_0 + \beta_0 X = 0\}$ is the Lie algebra of $SO_0(2,1)$. A basis of $\mathfrak{g} = \mathfrak{so}(2,1)$ is $\mathbf{L}_{ij}$ $(i,\ j = -1, 0, 1,\ i < j)$. We let $\mathbf{L}_{ij} = -\mathbf{L}_{ji}$ for $i > j$, and the $\mathbf{L}_{ij}$ satisfy the following commutation relations:

$$[\mathbf{L}_{ab},\ \mathbf{L}_{bc}] = -e_b\, \mathbf{L}_{ac} \tag{4}$$

with $e_{-1} = e_0 = -e_1 = 1$. All other commutators vanish. The relation of the $\mathbf{L}_{ij}$ to a basis for $(2,\mathbb{R})$ is $H = -2i\mathbf{L}_{-10}$, $X^{\pm} = \mathbf{L}_{-11} \mp i\mathbf{L}_{01}$ where H and $X^{\pm}$ are the basic generators of $(2,\mathbb{R})$. The $*$ structure on the complexification $(2,\mathbb{C})$ of $\mathfrak{so}(2,1)$ compatible with physical requirements of skew-symmetry of the $\mathfrak{so}(2,1)$ generators is: $\mathbf{L}_{ij}^{\dagger} = -\mathbf{L}_{ij} \Longleftrightarrow H^{\dagger} = H, X^{\pm\dagger} = -X^{\mp}$. The Casimir operator of $SO_0(2,1)$ is:

$$\Delta = \mathbf{L}_{-10}^2 - \mathbf{L}_{01}^2 - \mathbf{L}_{-11}^2 = X^+X^- + \frac{1}{4}H^2 - \frac{1}{2}H - \frac{1}{4}. \tag{5}$$

It generates the center $\mathfrak{z}(\mathfrak{so}(2,1))$ of the enveloping algebra $\mathfrak{U}(\mathfrak{so}(2,1))$.

Some subgroups of $SO_0(2,1)$ which we shall need are:

$$K = \left\{ k(u) = \begin{bmatrix} u_1 & u_2 & 0 \\ -u_2 & u_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \middle|\ u\ =\ \begin{pmatrix} u_1 & u_2 \\ -u_2 & u_1 \end{pmatrix} \in SO(2) \right\} \tag{6}$$

$$A = \left\{ a(\tau) = \begin{bmatrix} \mathrm{ch}(\emptyset) & 0 & \mathrm{sh}(\tau) \\ 0 & 1 & 0 \\ \mathrm{sh}(\tau) & 0 & \mathrm{ch}(\emptyset) \end{bmatrix} \middle| \tau \in \mathbb{R} \right\} \tag{7}$$

$$H = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & \hat{p}_0 & \hat{p} \\ 0 & \hat{p} & \hat{p}_0 \end{bmatrix} \middle| \begin{pmatrix} \hat{p}_0 & \hat{p} \\ \hat{p} & \hat{p}_0 \end{pmatrix} \in SO_0(1,1) \quad (\hat{p}_0^2 - \hat{p}^2 = 1, p_0 > 0) \right\} \tag{8}$$

$$N = \left\{ n(a) = \begin{bmatrix} 1 - \frac{a^2}{2} & a & \frac{a^2}{2} \\ -a & 1 & a \\ -\frac{a^2}{2} & a & 1 + \frac{a^2}{2} \end{bmatrix} \middle| a \in \mathbb{R} \right\} \tag{9}$$

$$\tilde{N} = \left\{ \tilde{n}(x) = \begin{bmatrix} 1 - \frac{x^2}{2} & -x & -\frac{x^2}{2} \\ x & 1 & x \\ \frac{x^2}{2} & x & 1 + \frac{x^2}{2} \end{bmatrix} \middle| x \in \mathbb{R} \right\}. \tag{10}$$

For $SO_0(2,1)$ the centralizer of A in K, which is denoted by M, turns out to be the trivial subgroup i.e. $M = C_A(K) = \{I\}$ where I is the identity on $\mathbb{R}^3$. Well known decompositions of $SO_0(2,1)$ are the Iwasawa decomposition: $SO_0(2,1) \simeq KAN$; and the Bruhat decomposition: $SO_0(2,1) \simeq \tilde{N}MAN$. The Langlands decomposition of the parabolic subgroup is $P = MAN$.

3 Parabolic Induction: Elementary Representations

A character of A is a homomorphism $\chi : A \to \mathbb{C}$. Let A^* be the space of all characters. For $\sigma \in \mathbb{C}$ let $\chi_\sigma \in A^* \ni \chi_\sigma(a(\tau)) = e^{-(\sigma+1/2)\tau}$. Let ρ be an irreducible representation of M. Since $M = \{I\}$, $\rho(I) = (+1)$ and $\rho = 1$, i.e. the identity representation. Consider $1 \otimes \chi_\sigma : MA \to \mathbb{C}$ and extend this mapping to a mapping from the parabolic subgroup $P = MAN$ to $\mathbb{C}$ by requiring that it act trivially on N. Thus $P \ni p = ma(\tau)n \longrightarrow (1 \otimes \chi_\sigma \otimes 1)(p) = \chi_\sigma(a(\tau)) \in \mathbb{C}$. The elementary representations (or generalized principal series) are thus defined as [8]:

$$\mathrm{Ind}_P^G(1 \otimes \chi_\sigma \otimes 1) = \{\mathcal{F} \in C^\infty(G) | \mathcal{F}(gp) = (1 \otimes \chi_\sigma \otimes 1)(p)\mathcal{F}(g), g \in G, p \in P\}$$

$$= \{\mathcal{F} \in C^\infty(G) |\, \mathcal{F}(gp) = e^{-(\sigma+1/2)\tau}\mathcal{F}(g), g \in G, p = ma(\tau)n \in P\}. \qquad (11)$$

G (= $SO_0(2,1)$) acts by left translation: $\mathcal{F}(g) \overset{g_1}{\longrightarrow} \mathcal{F}(g_1 g)$.

Parallelizations of $\mathrm{Ind}_P^G(1 \otimes \chi_\sigma \otimes 1)$ associated with the Iwasawa and Bruhat decompositions are defined as follows. Let $L^{(\sigma)}(G/P)$ is the line bundle over $u \in G/P$ whose elements are equivalence classes of ordered pairs $(u, s) \in G \times \mathbb{C}$ with respect to the equivalence relation ($p = ma(\tau)n \in P$):

$$(u, s) \sim (up, \rho(m)\chi_\sigma(a(\tau))s). \qquad (12)$$

Let $C^\infty(G/P, L^{(\sigma)})$ be the space of all C^∞ sections of $L^{(\sigma)}(G/P)$. The Iwasawa decomposition gives us the isomorphism

$$C^\infty(G/P, L^{(\sigma)}) \ni \psi \longrightarrow \phi \in C^\infty(K/M). \qquad (13)$$

This defines a parallelization of the bundle $L^{(\sigma)}(G/P)$, which we call the "spherical parallelization." There is a similar description for the Bruhat decomposition which parallelization we call the "flat parallelization." It associates sections of $L^{(\sigma)}(G/P)$ with functions on $C^\infty(\tilde{N})$ with appropriate asymptotic conditions which we do not specify [9].

Table 1 Presentations of the infinitesimal generators in the spherical and flat parallelizations

Generator	Spherical parallelization	Flat parallelization[a]
$\mathbf{L}_{-1,1}$	$\sin\theta\partial_\theta + w\cos\theta$	$S + w$
$\mathbf{L}_{0,1}$	$\cos\theta\partial_\theta - w\sin\theta$	$(1 - \frac{x_0^2}{4})\partial_0 + \frac{1}{2}x_0(S+w)$
$\mathbf{L}_{-1,0}$	$-\partial_\theta$	$(1 + \frac{x_0^2}{4})\partial_0 - \frac{1}{2}x_0(S+w)$

[a] $x_0 = 2x$, $\partial_0 = \partial_{x_0}$, $S = x_0\partial_0$, and $w = \sigma + \frac{1}{2}$

The action of $\mathrm{Ind}_P^G(1 \otimes \chi_\sigma \otimes 1)$ on $C^\infty(K/M)$ (spherical parallelization) is:

$$\pi^\sigma(g)\phi(u) = e^{-(\sigma+1/2)\tau}\phi(g^{-1}u) = e^{-(\sigma+1/2)\tau}\phi(g^{-1}u) \tag{14}$$

where $u \in K/M \simeq K$, and $g^{-1}k(u) = k(u')a(\tau)n$ with $g^{-1}u := u'$. The action of $\mathrm{Ind}_P^G(1 \otimes \chi_\sigma \otimes 1)$ on $C^\infty(\tilde{N})$ (flat parallelization) is related to the action in the curved parallelization by

$$\psi(x) = \left(\frac{2}{1+u_1}\right)^{-(\sigma+1/2)}\phi(u) \tag{15}$$

where $K^*/M \ni u \to x = \pi(u) \in \tilde{N}$ is that of stereographic projection from $K^*/M \simeq S^1\backslash\{P\}$ onto $\tilde{N} \simeq \mathbb{R}$ with P the excluded point of the projection and $K^* \simeq K\backslash\{P\}$.

Let $L_{ij}f = d\pi^\sigma(\mathbf{L}_{ij})f := \frac{d\pi^\sigma(e^{t\mathbf{L}_{ij}})}{dt}\Big|_{t=0} f$ for f a C^∞ function on either K/M (spherical parallelization) or $\tilde{N}$ (flat parallelization). Using this we obtain the values given in Table 1 for the infinitesimal action of the generators $\mathbf{L}_{ij}$ of $\mathfrak{g} = \mathfrak{so}(2,1)$ in the representation $\mathrm{Ind}_P^G(1 \otimes \chi_\sigma \otimes 1)$.

4 An Embedding of $\mathfrak{D}(\mathfrak{so}(2,1))$ into $\mathfrak{D}_{1,1}(\mathbb{R})$ and the Isomorphism $\mathfrak{D}(\widetilde{\mathfrak{so}(2,1)}) \simeq \mathfrak{D}_{1,1}(\mathbb{R})$

Recall that $\mathbf{W}_{1,1}(\mathbb{R}) = \mathbf{W}_1(\mathbb{R}) \otimes \mathbb{R}[Y]$ with associated skew field $\mathfrak{D}_{1,1}(\mathbb{R})$. Based on the results in Table 1 we define $\tilde{\tau} : \mathfrak{so}(2,1) \to \mathbf{W}_{1,1}(\mathbb{R})$ by

$$\tilde{\tau}(\mathbf{L}_{-11}) = S + Y\cdot 1 = x_0\partial_0 + Y\cdot 1 \tag{16}$$

$$\tilde{\tau}(\mathbf{L}_{01}) = -\partial_0 + \frac{1}{4}x_0 S + \frac{1}{2}Y x_0 \tag{17}$$

$$\tilde{\tau}(\mathbf{L}_{-10}) = -\partial_0 - \frac{1}{4}x_0 S - \frac{1}{2}Y x_0. \tag{18}$$

$\tilde{\tau}$ extends to a homomorphism $\tilde{\tau} : \mathfrak{U}(\mathfrak{so}(2,1)) \to \mathbf{W}_{1,1}(\mathbb{R})$ with $\tilde{\tau}(1) = 1$. Since $\mathfrak{so}(2,1)$ is simple we can show that $\ker(\tilde{\tau}) = 0$, so that $\tilde{\tau}$ extends to an injective homomorphism of $\mathfrak{D}(\mathfrak{so}(2,1))$ into $\mathfrak{D}_{1,1}(\mathbb{R})$. Calculations show that

$$\tilde{\tau}(\Delta) = \tilde{\tau}(\mathbf{L}_{-10}^2 - \mathbf{L}_{01}^2 - \mathbf{L}_{-11}^2) = Y^2 - Y. \tag{19}$$

The polynomial equation $\Delta - Z^2 + Z = 0$ has no solution in $\mathfrak{D}(\mathfrak{so}(2,1)$, so $Y \notin \tilde{\tau}(\mathfrak{U}(\mathfrak{so}(2,1)))$. Thus we let $\tilde{\tau}(Z) = Y$ and consider the (commutative) extension field

$$\widetilde{\mathfrak{D}(\mathfrak{so}(2,1))} = \{a + Zb \mid a, b \in \mathfrak{D}(\mathfrak{so}(2,1)) \text{ with } [Z, a] = 0 \ \forall\, a\}\,. \tag{20}$$

Theorem 1 *$\tilde{\tau}$: $\widetilde{\mathfrak{D}(\mathfrak{so}(2,1))} \to \mathfrak{D}_{1,1}(\mathbb{R})$ is an isomorphism.*

Proof The only nontrivial part is to prove surjectivity. For this we define

$$\mathbf{P}_0 := -\frac{1}{2}\,(\mathbf{L}_{-10} + \mathbf{L}_{01}) \quad \text{and} \quad \mathbf{Q}_0 := (\mathbf{L}_{-11} - Z)\mathbf{P}_0^{-1} \tag{21}$$

and show that

$$\tilde{\tau}(\mathbf{P}_0) = \partial_0, \quad \text{and} \quad \tilde{\tau}\,(\mathbf{Q}_0) = x_0. \ \square \tag{22}$$

A physically acceptable $*$ structure on $\mathfrak{D}_{1,1}(\mathbb{R})$ is: $x_0^\dagger = x_0$ and $\partial_0^\dagger = -\partial_0$ i.e. x_0 is symmetric and ∂_0 is skew-symmetric. The physically acceptable $*$ structure associated with $\widetilde{\mathfrak{D}(\mathfrak{so}(2,1))}$ is: $\mathbf{L}_{ij}^\dagger = -\mathbf{L}_{ij}$. We want $\tilde{\tau}$ to be compatible with these $*$ structures i.e. $\tilde{\tau}$ should preserve these $*$ structures. Define $Z^\dagger$ such that $Z + Z^\dagger = 1$ and $\tilde{\tau}(Z^\dagger) = Y^\dagger$. Then

$$\Delta^\dagger = \Delta \Longleftrightarrow (Y^2)^\dagger = (Y^\dagger)^2. \tag{23}$$

Theorem 2 *For $Z^\dagger$ such that $Z + Z^\dagger = 1$, $\tilde{\tau}$ is compatible with the $*$ structures on $\widetilde{\mathfrak{D}(\mathfrak{so}(2,1))}$ and $\mathfrak{D}_{1,1}(\mathbb{R})$, i.e.*

$$\tilde{\tau}(\mathbf{Q}_0^\dagger) = x_0^\dagger \ \textit{and} \ \tilde{\tau}(\mathbf{P}_0^\dagger) = -\partial_0^\dagger. \tag{24}$$

Proof Straightforward calculation.

5 Applications: Representations of $\mathbf{W_1}(\mathbb{R})$ from Representations of $\mathfrak{so}(2,1)$

Let $\tau = \tilde{\tau}^{-1}$ then $\tau : \mathbf{W}_1(\mathbb{R}) \longrightarrow \widetilde{\mathfrak{D}(\mathfrak{so}(2,1))}$ is an isomorphism onto its image in $\widetilde{\mathfrak{D}(\mathfrak{so}(2,1))}$ such that $\tau(\partial_0) = \mathbf{P}_0$ and $\tau(x_0) = \mathbf{Q}_0$. In order to construct representations of $\mathbf{W}_1(\mathbb{R})$ out of representations of $\mathfrak{U}(\mathfrak{so}(2,1))$ by using τ, we need the following Lemma which is proved in [6]:

Lemma 1 *Suppose $f : R \longrightarrow R_1$ is a ring homomorphism and Q is a left (resp. right) quotient ring of R with respect to S. If $f(s)$ is a unit in R_1 for every $s \in S$, then there exists a (unique) ring homomorphism $g : Q \longrightarrow R_1$ which extends f.*

This criterion implies for our case that the action of $\mathbf{P}_0$ in a given representation $d\pi$ of $\mathfrak{U}(\mathfrak{so}(2,1))$ must be invertible. For Hilbert space representations this means that zero should lie in the resolvent set of $d\pi(\mathbf{P}_0)$. Likely candidates for representations of $\mathfrak{so}(2,1)$ for which $d\pi(\mathbf{P}_0)$ is invertible are highest or lowest weight representations. The reason for this is the following [10]. Let $d\pi(\mathfrak{so}(2,1))$ be a lowest (highest) weight representation of $\mathfrak{so}(2,1)$. If $d\pi$ comes from a unitary representation of $SO_0(2,1)$, then we say that it is an infinitesimally unitarizable *lowest (highest) weight representation*. A linear operator A on a Hilbert space $\mathcal{H}$ is *positive* if

$$(\psi, A\psi) \geq 0 \quad \text{for all} \quad \psi \in \mathcal{H}$$

(similarly for *strictly positive*). $d\pi(\mathbf{P}_0)$ is a positive, self adjoint operator if and only if $d\pi(\mathfrak{so}(2,1))$ is an (infinitesimally) unitarizable lowest weight representation of $\mathfrak{so}(2,1)$.

The unitary irreducible representations of $SO_0(2,1)$ are contained as a subset in a famous work of V. Bargmann [11]. They are: i) trivial representation; ii) principal series: $\mathrm{Ind}_P^G(1 \otimes \chi_{\sigma=i\rho} \otimes 1)$ $(0 < \rho < \infty)$; iii) D_0^+ and D_0^-:$V_K = V^+ \oplus V^- \subset \mathrm{Ind}_P^G(1 \otimes \chi_{\sigma=-\frac{1}{2}} \otimes 1)$ (V^+ positive & V^- negative energy). $\mathfrak{U}(\mathfrak{g})$ acts on the quotient module V_K/Y_0 via the quotient action of $\mathfrak{g}$ on V_K/Y_0 and $D_0^+ = V^+/Y_0$, $D_0^- = V^-/Y_0$. Furthermore $0 \to Y_0 \xrightarrow{\iota} V_K \xrightarrow{\pi} V_K/Y_0 \to 0$ is short exact sequence of $\mathfrak{g}$ equivariant $\mathfrak{U}(\mathfrak{g})$ module homomorphisms; iv) discrete series: D_ℓ^+ and D_ℓ^- $(\ell = 1, 2, 3, ...)$: (same description of quotient structure); v) complementary series: $\mathrm{Ind}_P^G(1 \otimes \chi_{\sigma=-c} \otimes 1)$ $(0 < c \leq \frac{1}{2})$. None of these representations satisfy both the conditions of Theorem 2 and of the Lemma. In order to find a representation which satisfies all conditions of the Theorem 2 and also of the Lemma, we found it necessary to go to a covering group of $SO_0(2,1)$.

Theorem 3 (Ørsted, Segal (IES)) [12] *Let $\bar{\bar{G}}$ denote the four-fold cover of $SO_0(2,1)$ and consider the representation of $\bar{\bar{G}}$ induced from $\bar{P}$ of weight $w = \frac{1}{2}$ $(\sigma = 0)$. This representation is equivalent to the direct sum of two unitary positive and negative energy irreducible representations of $\bar{\bar{G}}$.*

Proposition 1 *Let $\mathcal{H}_{1/2}^+$ and $\mathcal{H}_{1/2}^-$ respectively denote the positive and negative energy subspaces of the Hilbert space associated with the representation in Theorem 3. Then on either $\mathcal{H}_{1/2}^+$ or $\mathcal{H}_{1/2}^-$, $\mathbf{P}_0$ acts as a skew-symmetric operator and $\mathbf{Q}_0$ acts as a well-defined symmetric operator. Thus we have on $\mathcal{H}_{1/2}^+$ or $\mathcal{H}_{1/2}^-$ representations of $\mathbf{W}_1(\mathbb{R})$ in which x_0 and ∂_0 act, respectively, as symmetric and skew-symmetric operators.*

We also have the result that $\widetilde{\mathfrak{D}(\mathfrak{so}(2,1))} \simeq \widetilde{\mathfrak{D}(\mathfrak{iso}(1,1))}$ where $\mathfrak{iso}(1,1) \simeq H \rtimes N$ [13]. (This result is a lower dimensional and hence much simpler case of the isomorphism established in ref. [3].) Combining this with $\widetilde{\mathfrak{D}(\mathfrak{so}}(2,1)) \simeq \mathfrak{D}_{1,1}(\mathbb{R})$ we get

$$\widetilde{\mathfrak{D}(\mathfrak{so}}(2,1)) \simeq \mathfrak{D}_{1,1}(\mathbb{R}) \simeq \widetilde{\mathfrak{D}(\mathfrak{iso}}(1,1)) \tag{25}$$

From this it is clear that $\mathfrak{D}_{1,1}(\mathbb{R})$ is stable under the Lie algebra contraction $\mathfrak{so}(2,1) \to \mathfrak{iso}(1,1)$. In other words, Planck's constant remains unchanged as the contraction parameter goes to zero, i.e. as the conformal invariant radius of S^1 becomes infinite [14]. We have obtained generalizations to $SO(2,q)$ groups like $SO(2,3)$ and $SO(2,4)$, results of which are partially described in [1].

Acknowledgements Sincere thanks to Alexey Petukhov for several useful discussions resulting in many improvements from the original version presented in Varna at LT-11. I would also like to thank Eyal Subag for pointing out that Eq. (25) implies that $\mathfrak{D}_{1,1}(\mathbb{R})$ is stable under the Lie algebra contraction $\mathfrak{so}(2,1) \to \mathfrak{iso}(1,1)$. I am also grateful to Joseph A. Wolf for clearing up some confusion about the definition of generalized principal series representations.

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Permutation-Symmetric Three-Body O(6) Hyperspherical Harmonics in Three Spatial Dimensions

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Abstract We have constructed the three-body permutation symmetric O(6) hyperspherical harmonics which can be used to solve the non-relativistic three-body Schrödinger equation in three spatial dimensions. We label the states with eigenvalues of the $U(1) \otimes SO(3)_{rot} \subset U(3) \subset O(6)$ chain of algebras and we present the corresponding $K \leq 4$ harmonics. Concrete transformation properties of the harmonics are discussed in some detail.

1 Introduction

Hyperspherical harmonics are an important tool for dealing with quantum-mechanical three-body problem, being of a particular importance in the context of bound states [1–6]. However, before our recent progress [7], a systematical construction of permutation-symmetric three-body hyperspherical harmonics was, to our knowledge, lacking (with only some particular cases being worked out – e.g. those with total orbital angular momentum $L = 0$, see Refs. [5, 8]).

In this note, we report the construction of permutation-symmetric three-body O(6) hyperspherical harmonics using the $U(1) \otimes SO(3)_{rot} \subset U(3) \subset O(6)$ chain of algebras, where $U(1)$ is the "democracy transformation", or "kinematic rotation" group for three particles, $SO(3)_{rot}$ is the 3D rotation group, and $U(3)$, $O(6)$ are the usual Lie groups. This particular chain of algebras is mathematically very natural, since the $U(1)$ group of "democracy transformations" is the only nontrivial (Lie) subgroup of full hyperspherical $SO(6)$ symmetry (the symmetry of nonrelativistic kinetic energy) that commutes with spatial rotations. Historically, this chain was also suggested in the recent review of the Russian school's work, Ref. [9], and indicated

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V. Dobrev (ed.), *Lie Theory and Its Applications in Physics*, Springer Proceedings in Mathematics & Statistics 191,
DOI 10.1007/978-981-10-2636-2_31

by the previous discovery of the dynamical $O(2)$ symmetry of the Y-string potential, Ref. [10]. The name "democracy transformations" comes from the close relation of these transformations with permutations: (cyclic) particle permutations form a discrete subgroup of this $U(1)$ group.

2 Three-Body Hyperspherical Coordinates

A natural set of coordinates for parametrization of three-body wave function $\Psi(\boldsymbol{\rho}, \boldsymbol{\lambda})$ (in the center-of-mass frame of reference) is given by the Euclidean relative position Jacobi vectors $\boldsymbol{\rho} = \frac{1}{\sqrt{2}}(\mathbf{x}_1 - \mathbf{x}_2)$, $\boldsymbol{\lambda} = \frac{1}{\sqrt{6}}(\mathbf{x}_1 + \mathbf{x}_2 - 2\mathbf{x}_3)$. The overall six components of the two vectors can be seen as specifying a position in a six-dimensional configuration space $x_\mu = (\boldsymbol{\lambda}, \boldsymbol{\rho})$, which, in turn, can be parameterized by hyperspherical coordinates as $\Psi(R, \Omega_5)$. Here $R = \sqrt{\boldsymbol{\rho}^2 + \boldsymbol{\lambda}^2}$ is the hyper-radius, and five angles Ω_5 parametrize a hyper-sphere in the six-dimensional Euclidean space. Three (Φ_i; $i = 1, 2, 3$) of these five angles (Ω_5) are just the Euler angles associated with the orientation in a three-dimensional space of a spatial reference frame defined by the (plane of) three bodies; the remaining two hyper-angles describe the shape of the triangle subtended by three bodies; they are functions of three independent scalar three-body variables, e.g. $\boldsymbol{\rho} \cdot \boldsymbol{\lambda}$, $\boldsymbol{\rho}^2$, and $\boldsymbol{\lambda}^2$. Due to the connection $R = \sqrt{\boldsymbol{\rho}^2 + \boldsymbol{\lambda}^2}$, this shape-space is two-dimensional, and topologically equivalent to the surface of a three-dimensional sphere. A spherical coordinate system can be further introduced in this shape space. Among various (in principle infinitely many) ways that this can be accomplished, the one due to Iwai [6] stands out as the one that fully observes the permutation symmetry of the problem. Namely, of the two Iwai (hyper)spherical angles (α, ϕ): $(\sin\alpha)^2 = 1 - \left(\frac{2\boldsymbol{\rho}\times\boldsymbol{\lambda}}{R^2}\right)^2$, $\tan\phi = \left(\frac{2\boldsymbol{\rho}\cdot\boldsymbol{\lambda}}{\boldsymbol{\rho}^2-\boldsymbol{\lambda}^2}\right)$, the angle α does not change under permutations, so that all permutation properties are encoded in the ϕ-dependence of the wave functions.

Nevertheless, in the construction of hyperspherical harmonics, we will, unlike the most of the previous attempts in this context, refrain from use of any explicit set of angles, and express harmonics as functions of Cartesian Jacobi coordinates.

3 O(6) Symmetry of the Hyperspherical Approach

The motivation for hyperspherical approach to the three-body problem comes from the fact that the equal-mass three-body kinetic energy T is O(6) invariant and can be written as

$$T = \frac{m}{2}\dot{R}^2 + \frac{K^2_{\mu\nu}}{2mR^2}. \tag{1}$$

Here, $K_{\mu\nu}, (\mu, \nu = 1, 2, \ldots, 6)$ denotes the $SO(6)$ "grand angular" momentum tensor

$$\begin{aligned} K_{\mu\nu} &= m\left(\mathbf{x}_\mu \dot{\mathbf{x}}_\nu - \mathbf{x}_\nu \dot{\mathbf{x}}_\mu\right) \\ &= \left(\mathbf{x}_\mu \mathbf{p}_\nu - \mathbf{x}_\nu \mathbf{p}_\mu\right). \end{aligned} \tag{2}$$

$K_{\mu\nu}$ has 15 linearly independent components, that contain, among themselves three components of the "ordinary" orbital angular momentum: $\mathbf{L} = \mathbf{l}_\rho + \mathbf{l}_\lambda = m\left(\boldsymbol{\rho} \times \dot{\boldsymbol{\rho}} + \boldsymbol{\lambda} \times \dot{\boldsymbol{\lambda}}\right)$.

It is due to this symmetry of the kinetic energy that the decomposition of the wave function and potential energy into $SO(6)$ hyperspherical harmonics becomes a natural way to tackle the three-body quantum problem.

In this particular physical context, the six dimensional hyperspherical harmonics need to have some desirable properties. Quite generally, apart from the hyperangular momentum K, which labels the O(6) irreducible representation, all hyperspherical harmonics must carry additional labels specifying the transformation properties of the harmonic with respect to (w.r.t.) certain subgroups of the orthogonal group. The symmetries of most three-body potentials, including the three-quark confinement ones, are: parity, rotations and permutations (spatial exchange of particles).

Therefore, the goal is to find three-body hyperspherical harmonics with well defined transformation properties with respect to thee symmetries. Parity is directly related to K value: $P = (-1)^{\mathrm{K}}$, the rotation symmetry implies that the hyperspherical harmonics must carry usual quantum numbers L and m corresponding to $SO(3)_{rot} \supset SO(2)$ subgroups and permutation properties turn out to be related with a continuous $U(1)$ subgroup of "democracy transformations", as will be discussed below.

4 Labels od Permutation-Symmetric Three-Body Hyperspherical Harmonics

We introduce the complex coordinates:

$$X_i^{\pm} = \lambda_i \pm i\rho_i, \quad i = 1, 2, 3. \tag{3}$$

Nine of 15 hermitian $SO(6)$ generators $K_{\mu\nu}$ in these new coordinates become

$$iL_{ij} \equiv X_i^+ \frac{\partial}{\partial X_j^+} + X_i^- \frac{\partial}{\partial X_j^-} - X_j^+ \frac{\partial}{\partial X_i^+} - X_j^- \frac{\partial}{\partial X_i^-}, \tag{4}$$

$$2Q_{ij} \equiv X_i^+ \frac{\partial}{\partial X_j^+} - X_i^- \frac{\partial}{\partial X_j^-} + X_j^+ \frac{\partial}{\partial X_i^+} - X_j^- \frac{\partial}{\partial X_i^-}. \tag{5}$$

Here L_{ij} have physical interpretation of components of angular momentum vector $\mathbf{L}$. The symmetric tensor Q_{ij} decomposes as (5) + (1) w.r.t. rotations, while the trace:

$$Q \equiv Q_{ii} = \sum_{i=1}^{3} X_i^+ \frac{\partial}{\partial X_i^+} - \sum_{i=1}^{3} X_i^- \frac{\partial}{\partial X_i^-} \tag{6}$$

is the only scalar under rotations, among all of the $SO(6)$ generators. Therefore, the only mathematically justified choice is to take eigenvalues of this operator for an additional label of the hyperspherical harmonics. Besides, this trace Q is the generator of the forementioned democracy transformations, a special case of which are the cyclic permutations – which in addition makes this choice particularly convenient on an route to construction of permutation-symmetric hyperspherical harmonics. The remaining five components of the symmetric tensor Q_{ij}, together with three antisymmetric tensors L_{ij} generate the $SU(3)$ Lie algebra, which together with the single scalar Q form an $U(3)$ algebra, Ref. [9].

Overall, labelling of the O(6) hyperspherical harmonics with labels K, Q, L and m corresponds to the subgroup chain $U(1) \otimes SO(3)_{rot} \subset U(3) \subset SO(6)$. Yet, these four quantum numbers are in general insufficient to uniquely specify an $SO(6)$ hyperspherical harmonic and an additional quantum number must be introduced to account for the remaining multiplicity. This is the multiplicity that necessarily occurs when $SU(3)$ unitary irreducible representations are labelled w.r.t. the chain $SO(2) \subset SO(3) \subset SU(3)$ (where $SO(3)$ is "matrix embedded" into $SU(3)$), and thus is well documented in the literature. In this context the operator:

$$\mathcal{V}_{LQL} \equiv \sum_{ij} L_i Q_{ij} L_j \tag{7}$$

(where $L_i = \frac{1}{2}\varepsilon_{ijk} L_{jk}$ and Q_{ij} is given by Eq. (5)) has often been used to label the multiplicity of $SU(3)$ states. This operator commutes both with the angular momentum L_i, and with the "democracy rotation" generator Q:

$$\left[\mathcal{V}_{LQL}, L_i\right] = 0; \quad \left[\mathcal{V}_{LQL}, Q\right] = 0$$

Therefore we demand that the hyperspherical harmonics be eigenstates of this operator:

$$\mathcal{V}_{LQL} \mathcal{Y}_{L,m}^{KQ\nu} = \nu \mathcal{Y}_{L,m}^{KQ\nu};$$

Thus, ν will be the fifth label of the hyperspherical harmonics, beside the (K, Q, L, m).

5 Tables of Hyperspherical Harmonics of Given K, Q, L, M and ν

Below we explicitly list all hyperspherical harmonics for K ≤ 4, labelled by the quantum numbers (K, Q, L, m, ν) (we will not delve here into lengthy details of the derivation of the expressions). We list only the harmonics with $m = L$ and $Q \geq 0$, as the rest can be easily obtained by acting on them with standard lowering operators and by using the permutation symmetry properties of hyperspherical harmonics: $\mathcal{Y}^{\mathrm{K}Q\nu}_{L,m}(\lambda, \rho) = (-1)^{\mathrm{K}-L}\mathcal{Y}^{\mathrm{K}-Q-\nu}_{L,m}(\lambda, -\rho)$. We use the (more compact) spherical complex coordinates: $X^{\pm}_0 \equiv \lambda_3 \pm i\rho_3$, $X^{\pm}_{(\pm)} \equiv \lambda_1 \pm i\rho_1 + (\pm)(\lambda_2 \pm i\rho_2)$, $|X^{\pm}|^2 = X^{\pm}_+ X^{\pm}_- + (X^{\pm}_0)^2$, while we are also explicitly writing out the K ≤ 3 harmonics in terms of Jacobi coordinates.

$$\mathcal{Y}^{0,0,0}_{0,0}(X) = \frac{1}{\pi^{3/2}}$$

$$\mathcal{Y}^{1,1,-1}_{1,1}(X) = \frac{\sqrt{\frac{3}{2}}X^+_+}{\pi^{3/2}R} = \frac{\sqrt{\frac{3}{2}}\left(\lambda_1 + i\left(\lambda_2 + \rho_1 + i\rho_2\right)\right)}{\pi^{3/2}\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \rho_1^2 + \rho_2^2 + \rho_3^2}}$$

$$\mathcal{Y}^{2,0,0}_{1,1}(X) = \frac{\sqrt{3}\left(X^-_+ X^+_0 - X^+_+ X^-_0\right)}{\pi^{3/2}R^2} = \frac{2\sqrt{3}\left(\lambda_3\left(\rho_2 - i\rho_1\right) + i\left(\lambda_1 + i\lambda_2\right)\rho_3\right)}{\pi^{3/2}\left(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \rho_1^2 + \rho_2^2 + \rho_3^2\right)}$$

$$\mathcal{Y}^{2,0,0}_{2,2}(X) = \frac{\sqrt{3}X^+_+ X^-_+}{\pi^{3/2}R^2} = \frac{\sqrt{3}\left(\lambda_1 + i\left(\lambda_2 + \rho_1 + i\rho_2\right)\right)\left(\lambda_1 + i\lambda_2 - i\rho_1 + \rho_2\right)}{\pi^{3/2}\left(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \rho_1^2 + \rho_2^2 + \rho_3^2\right)}$$

$$\mathcal{Y}^{2,2,0}_{0,0}(X) = \frac{\sqrt{2}|X^+|^2}{\pi^{3/2}R^2}$$
$$= \frac{\sqrt{2}\left(2i\lambda_1\rho_1 + 2i\lambda_2\rho_2 + 2i\lambda_3\rho_3 + \lambda_1^2 + \lambda_2^2 + \lambda_3^2 - \rho_1^2 - \rho_2^2 - \rho_3^2\right)}{\pi^{3/2}\left(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \rho_1^2 + \rho_2^2 + \rho_3^2\right)}$$

$$\mathcal{Y}^{2,2,-3}_{2,2}(X) = \frac{\sqrt{\frac{3}{2}}\left(X^+_+\right)^2}{\pi^{3/2}R^2} = \frac{\sqrt{\frac{3}{2}}\left(\lambda_1 + i\left(\lambda_2 + \rho_1 + i\rho_2\right)\right)^2}{\pi^{3/2}\left(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \rho_1^2 + \rho_2^2 + \rho_3^2\right)}$$

$$\mathcal{Y}^{3,1,3}_{1,1}(X) = \frac{\sqrt{6}\left(X^-_+|X^+|^2 - \frac{1}{2}R^2 X^+_+\right)}{\pi^{3/2}R^3}$$
$$= \frac{\sqrt{6}\left(\left(\lambda_1 + i\lambda_2 - i\rho_1 + \rho_2\right)\left(\left(\lambda_1 + i\rho_1\right)^2 + \left(\lambda_2 + i\rho_2\right)^2 + \left(\lambda_3 + i\rho_3\right)^2\right)\right)}{\pi^{3/2}\left(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \rho_1^2 + \rho_2^2 + \rho_3^2\right)} -$$
$$\frac{\frac{\sqrt{6}}{2}\left(\lambda_1 + i\left(\lambda_2 + \rho_1 + i\rho_2\right)\right)}{\pi^{3/2}\left(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \rho_1^2 + \rho_2^2 + \rho_3^2\right)}$$

$$\mathcal{Y}_{2,2}^{3,1,-5}(X) = \frac{\sqrt{5}X_{+}^{+}\left(X_{+}^{-}X_{0}^{+} - X_{+}^{+}X_{0}^{-}\right)}{\pi^{3/2}R^{3}}$$
$$= \frac{2\sqrt{5}\left(\lambda_1 + i\left(\lambda_2 + \rho_1 + i\rho_2\right)\right)\left(\lambda_3\left(\rho_2 - i\rho_1\right) + i\left(\lambda_1 + i\lambda_2\right)\rho_3\right)}{\pi^{3/2}\left(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \rho_1^2 + \rho_2^2 + \rho_3^2\right)^{3/2}}$$

$$\mathcal{Y}_{3,3}^{3,1,-2}(X) = \frac{\sqrt{15}\left(X_{+}^{+}\right)^2 X_{+}^{-}}{2\pi^{3/2}R^{3}}$$
$$= \frac{\sqrt{15}\left(\lambda_1 + i\left(\lambda_2 + \rho_1 + i\rho_2\right)\right)^2\left(\lambda_1 + i\lambda_2 - i\rho_1 + \rho_2\right)}{2\pi^{3/2}\left(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \rho_1^2 + \rho_2^2 + \rho_3^2\right)^{3/2}}$$

$$\mathcal{Y}_{1,1}^{3,3,-1}(X) = \frac{\sqrt{3}X_{+}^{+}\left|X^{+}\right|^2}{\pi^{3/2}R^{3}}$$
$$= \frac{\sqrt{3}(\lambda_1+i(\lambda_2+\rho_1+i\rho_2))\left(2i\lambda_1\rho_1+2i\lambda_2\rho_2+2i\lambda_3\rho_3+\lambda_1^2+\lambda_2^2+\lambda_3^2-\rho_1^2-\rho_2^2-\rho_3^2\right)}{\pi^{3/2}\left(\lambda_1^2+\lambda_2^2+\lambda_3^2+\rho_1^2+\rho_2^2+\rho_3^2\right)^{3/2}}$$

$$\mathcal{Y}_{3,3}^{3,3,-6}(X) = \frac{\sqrt{5}\left(X_{+}^{+}\right)^3}{2\pi^{3/2}R^{3}} = \frac{\sqrt{5}\left(\lambda_1 + i\left(\lambda_2 + \rho_1 + i\rho_2\right)\right)^3}{2\pi^{3/2}\left(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \rho_1^2 + \rho_2^2 + \rho_3^2\right)^{3/2}}$$

$$\mathcal{Y}_{0,0}^{4,0,0}(X) = -\frac{\sqrt{3}\left(R^4 - 2\left|X^{-}\right|^2\left|X^{+}\right|^2\right)}{\pi^{3/2}R^4}$$

$$\mathcal{Y}_{2,2}^{4,0,-\sqrt{105}}(X) = \frac{-12\sqrt{14}R^2X_{+}^{+}X_{+}^{-}+\sqrt{105\left(11-\sqrt{105}\right)}\left(X_{+}^{-}\right)^2\left|X^{+}\right|^2+\sqrt{105\left(11+\sqrt{105}\right)}\left(X_{+}^{+}\right)^2\left|X^{-}\right|^2}{14\pi^{3/2}R^4}$$

$$\mathcal{Y}_{2,2}^{4,0,\sqrt{105}}(X) = \frac{-12\sqrt{14}R^2X_{+}^{+}X_{+}^{-}+\sqrt{105\left(11+\sqrt{105}\right)}\left(X_{+}^{-}\right)^2\left|X^{+}\right|^2+\sqrt{105\left(11-\sqrt{105}\right)}\left(X_{+}^{+}\right)^2\left|X^{-}\right|^2}{14\pi^{3/2}R^4}$$

$$\mathcal{Y}_{3,3}^{4,0,0}(X) = \frac{3\sqrt{5}X_{+}^{+}X_{+}^{-}\left(X_{+}^{-}X_{0}^{+} - X_{+}^{+}X_{0}^{-}\right)}{2\pi^{3/2}R^4}$$

$$\mathcal{Y}_{4,4}^{4,0,0}(X) = \frac{3\sqrt{\frac{5}{2}}\left(X_{+}^{+}\right)^2\left(X_{+}^{-}\right)^2}{2\pi^{3/2}R^4}$$

$$\mathcal{Y}_{1,1}^{4,2,2}(X) = \frac{3\left(X_{+}^{-}X_{0}^{+} - X_{+}^{+}X_{0}^{-}\right)\left|X^{+}\right|^2}{\pi^{3/2}R^4}$$

$$\mathcal{Y}_{2,2}^{4,2,2}(X) = \frac{\sqrt{\frac{3}{7}}X_+^+\left(5X_+^-\left|X^+\right|^2 - 2R^2X_+^+\right)}{\pi^{3/2}R^4}$$

$$\mathcal{Y}_{3,3}^{4,2,-13}(X) = \frac{3\sqrt{\frac{5}{2}}\left(X_+^+\right)^2\left(X_+^-X_0^+ - X_+^+X_0^-\right)}{2\pi^{3/2}R^4}$$

$$\mathcal{Y}_{4,4}^{4,2,-5}(X) = \frac{\sqrt{15}\left(X_+^+\right)^3 X_+^-}{2\pi^{3/2}R^4}$$

$$\mathcal{Y}_{0,0}^{4,4,0}(X) = \frac{\sqrt{3}\left|X^+\right|^4}{\pi^{3/2}R^4}$$

$$\mathcal{Y}_{2,2}^{4,4,-3}(X) = \frac{3\sqrt{\frac{5}{14}}\left(X_+^+\right)^2\left|X^+\right|^2}{\pi^{3/2}R^4}$$

$$\mathcal{Y}_{4,4}^{4,4,-10}(X) = \frac{\sqrt{15}\left(X_+^+\right)^4}{4\pi^{3/2}R^4}$$

6 Permutation Symmetric Hyperspherical Harmonics

There is a small step remaining from obtaining the hyperspherical harmonics labelled by quantum numbers (K, Q, L, m, ν) to achieving our goal, which is to construct hyperspherical functions with well-defined values of parity $P = (-1)^{\mathrm{K}}$, rotational group quantum numbers (L, m), and permutation symmetry M (mixed), S (symmetric), and A (antisymmetric).[1] In this section we clarify how to obtain the latter as linear combinations of the former.

Properties under particle permutations of the functions $\mathcal{Y}_{J,m}^{KQ\nu}(\boldsymbol{\lambda}, \boldsymbol{\rho})$ are inferred from the transformation properties of the coordinates $X_i^{\pm}$: under the transpositions (two-body permutations) $\{\mathcal{T}_{12}, \mathcal{T}_{23}, \mathcal{T}_{31}\}$ of pairs of particles (1,2), (2,3) and (3,1), the Jacobi coordinates transform as:

[1]The mixed symmetry representation of the S_3 permutation group being two-dimensional, there are two different state vectors (hyperspherical harmonics) in each mixed permutation symmetry multiplet, usually denoted by M_ρ and M_λ.

$$
\begin{aligned}
\mathcal{T}_{12}:\ & \boldsymbol{\lambda} \to \boldsymbol{\lambda}, \quad \boldsymbol{\rho} \to -\boldsymbol{\rho}, \\
\mathcal{T}_{23}:\ & \boldsymbol{\lambda} \to -\frac{1}{2}\boldsymbol{\lambda} + \frac{\sqrt{3}}{2}\boldsymbol{\rho}, \quad \boldsymbol{\rho} \to \frac{1}{2}\boldsymbol{\rho} + \frac{\sqrt{3}}{2}\boldsymbol{\lambda}, \\
\mathcal{T}_{31}:\ & \boldsymbol{\lambda} \to -\frac{1}{2}\boldsymbol{\lambda} - \frac{\sqrt{3}}{2}\boldsymbol{\rho}, \quad \boldsymbol{\rho} \to \frac{1}{2}\boldsymbol{\rho} - \frac{\sqrt{3}}{2}\boldsymbol{\lambda}.
\end{aligned} \tag{8}
$$

That induces the following transformations of complex coordinates $X_i^{\pm}$:

$$
\begin{aligned}
\mathcal{T}_{12}:\ & X_i^{\pm} \to X_i^{\mp}, \\
\mathcal{T}_{23}:\ & X_i^{\pm} \to e^{\pm\frac{2i\pi}{3}} X_i^{\mp}, \\
\mathcal{T}_{31}:\ & X_i^{\pm} \to e^{\mp\frac{2i\pi}{3}} X_i^{\mp}.
\end{aligned} \tag{9}
$$

None of the quantum numbers K, L and m change under permutations of particles, whereas the values of the "democracy label" Q and multiplicity label ν are inverted under all transpositions: $Q \to -Q$, $\nu \to -\nu$.

Apart from the changes in labels, transpositions of two particles generally also result in the appearance of an additional phase factor multiplying the hyperspherical harmonic. For values of K, Q, L and m with no multiplicity, we readily derive (Ref. [7]) the following transformation properties of h.s. harmonics under (two-particle) particle transpositions:

$$
\begin{aligned}
\mathcal{T}_{12}:\ & \mathcal{Y}_{L,m}^{KQ\nu} \to (-1)^{K-J} \mathcal{Y}_{L,m}^{K,-Q,-\nu}, \\
\mathcal{T}_{23}:\ & \mathcal{Y}_{L,m}^{KQ\nu} \to (-1)^{K-L} e^{\frac{2Qi\pi}{3}} \mathcal{Y}_{L,m}^{K,-Q,-\nu}, \\
\mathcal{T}_{31}:\ & \mathcal{Y}_{L,m}^{KQ\nu} \to (-1)^{K-L} e^{-\frac{2Qi\pi}{3}} \mathcal{Y}_{L,m}^{K,-Q,-\nu}.
\end{aligned} \tag{10}
$$

There are three distinct irreducible representations of the S_3 permutation group - two one-dimensional (the symmetric S and the antisymmetric A ones) and a two-dimensional (the mixed M one). In order to determine to which representation of the permutation group any particular h.s. harmonic $\mathcal{Y}_{L,m}^{KQ\nu}$ belongs, one has to consider various cases, with and without multiplicity, see Ref. [7]; here we simply state the results of the analysis conducted therein. The following linear combinations of the h.s. harmonics,

$$
\mathcal{Y}_{L,m,\pm}^{K|Q|\nu} \equiv \frac{1}{\sqrt{2}} \left(\mathcal{Y}_{L,m}^{K|Q|\nu} \pm (-1)^{K-L} \mathcal{Y}_{L,m}^{K,-|Q|,-\nu} \right). \tag{11}
$$

are no longer eigenfunctions of Q operator but are (pure sign) eigenfunctions of the transposition $\mathcal{T}_{12}$ instead:

$$
\mathcal{T}_{12}: \mathcal{Y}_{L,m,\pm}^{K|Q|\nu} \to \pm \mathcal{Y}_{L,m,\pm}^{K|Q|\nu}.
$$

They are the appropriate h.s. harmonics with well-defined permutation properties:

1. for $Q \not\equiv 0$ (mod 3), the harmonics $\mathcal{Y}^{K|Q|\nu}_{L,m,\pm}$ belong to the mixed representation M, where the $\pm$ sign determines the M_ρ, M_λ component,
2. for $Q \equiv 0$ (mod 3), the harmonic $\mathcal{Y}^{K|Q|\nu}_{L,m,+}$ belongs to the symmetric representation S and $\mathcal{Y}^{K|Q|\nu}_{L,m,-}$ belongs to the antisymmetric representation A.

The above rules define the representation of S_3 for any given h.s. harmonic.

7 Summary

In this paper we have reported on our recent construction of permutation symmetric three-body $SO(6)$ hyperspherical harmonics. In the Sect. 5 we have displayed explicit forms the harmonic functions labelled by quantum numbers K, Q, L, m and ν, postponing explanation of their derivation to [7]. In Sect. 6 we demonstrated that simple linear combinations $\mathcal{Y}^{K|Q|\nu}_{L,m,\pm}$ of these functions have well defined permutation properties. To our knowledge, this is the first time that such hyperspherical harmonics are constructed in full generality.

Acknowledgements This work was financed by the Serbian Ministry of Science and Technological Development under grant numbers OI 171031, OI 171037 and III 41011.

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Quantum Plactic and Pseudo-Plactic Algebras

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Abstract We review the Robinson–Schensted–Knuth correspondence in the light of the quantum Schur–Weyl duality. The *quantum plactic algebra* is defined to be a Schur functor mapping a tower of left modules of Hecke algebras into a tower of $U_q\mathfrak{gl}$-modules. The functions on the quantum group carry a $U_q\mathfrak{gl}$-bimodule structure whose combinatorial spirit emerges in the RSK algorithm. The bimodule structure on the algebra of biletter words is used for a functorial formulation of the *quantum pseudo-plactic algebra*. The latter algebra has been proposed by Daniel Krob and Jean-Yves Thibon as a higher noncommutative analogue of the quantum torus.

1 Schur–Weyl Duality

Let us denote by $\mathfrak{S}$ the tower of the symmetric groups $\mathfrak{S}_r$, $\mathfrak{S} = \bigsqcup_{r\geq 0} \mathfrak{S}_r$. The group algebra $\mathbb{C}[\mathfrak{S}_r]$ is the space of functions on $\mathfrak{S}_r$. The algebra $\mathbb{C}[\mathfrak{S}_r]$ is a left and right module over itself. Similarly, the space of functions $\mathbb{C}[\mathfrak{S}]$ is a left and right module on itself, provided that each $\mathbb{C}[\mathfrak{S}_r]$ acts nontrivially only on its own level r.

The Hecke algebra $\mathcal{H}_r(q)$ is a deformation of the group algebra $\mathbb{C}[\mathfrak{S}_r]$

$$\begin{aligned} T_iT_{i+1}T_i &= T_{i+1}T_iT_{i+1} && i = 1, \ldots, r-1 \\ T_iT_j &= T_jT_i && |i-j| \geq 2 \\ T_i^2 &= 1 + (q - q^{-1})T_i && i = 1, \ldots, r-1. \end{aligned} \tag{1}$$

The specialization of the formal parameter q to $q = 1$ yields the Coxeter relations of the symmetric group $\mathfrak{S}_r$generated by the elementary transpositions $s_i = (i\, i+1)$. For generic values of q one has an isomorphism of algebras $\mathcal{H}_r(q) \cong \mathbb{C}[\mathfrak{S}_r]$.

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V. Dobrev (ed.), *Lie Theory and Its Applications in Physics*,
Springer Proceedings in Mathematics & Statistics 191,
DOI 10.1007/978-981-10-2636-2_32

In full parallel with $\mathbb{C}[\mathfrak{S}]$ we define the tower of algebras $\mathcal{H}(q) = \bigoplus_{r\geq 0} \mathcal{H}_r(q)$. The algebra $\mathcal{H}(q)$ is a left and right module over itself. For q in generic position the $\mathcal{H}(q)$-bimodule $\mathcal{H}(q)$ is isomorphic to the $\mathbb{C}[\mathfrak{S}]$-bimodule $\mathbb{C}[\mathfrak{S}]$.

The Drinfeld-Jimbo R-matrix $\hat{R}_q \in \mathrm{End}(V^{\otimes 2})$ in the basis $V = \bigoplus_{i\in I} \mathbb{C}(q)e_i$ is

$$\hat{R}_q = \sum_{i,j\in I} q^{\delta_{ij}} e^i_j \otimes e^j_i + (q - q^{-1}) \sum_{i,j\in I: i<j} e^i_j \otimes e^i_j \qquad e^i_j \in \mathfrak{gl}(V). \tag{2}$$

$\hat{R}_q$ satisfies the Hecke relation $\hat{R}_q^2 = 1 + (q - q^{-1})\hat{R}_q$ and provides a representation of the Hecke algebra $\pi(\mathcal{H}_r(q)) \in \mathrm{End}(V^{\otimes r})$ with generators $\pi(T_i) = \hat{R}_{q_{ii+1}}$.

The coordinate ring of $GL(V)$ is the algebra of the regular functions $\mathbb{C}[GL(V)]$ generated by the matrix elements $(x^i_j)_{i,j\in I}$ on $GL(V)$. The algebra $\mathbb{C}[GL(n)]$ is a Hopf algebra with a coproduct $\Delta x^i_j = \sum_s x^i_s \otimes x^s_j$.

The dual Hopf algebra of $\mathbb{C}[GL(V)]$ is the Universal Enveloping Algebra (UEA) $U\mathfrak{gl}(V) \cong \mathbb{C}[GL(V)]^*$, that is, the algebra of the vector fields on $GL(V)$.

The coordinate ring of the quantum group $GL_q(V)$ is the (Hopf algebra) deformation of the *commutative* algebra $\mathbb{C}[GL(V)]$ to a noncommutative algebra $\mathbb{C}_{\hat{R}_q}[GL(V)]$ with Faddeev-Reshetikhin-Takhtadjan (FRT) relations

$$\sum_{a,b} (\hat{R}_q)^{ij}_{ab} x^a_k x^b_l = \sum_{a,b} x^i_a x^j_b (\hat{R}_q)^{ab}_{kl}. \tag{3}$$

The Hopf algebra dual to $\mathbb{C}_{\hat{R}_q}[GL(V)]$ is the Drinfeld quantum UEA $U_q\mathfrak{gl}(V)$.

The FRT relations live in the centralizer of the representation ρ of the Hecke algebra $\mathrm{End}_{\mathcal{H}_2(q)}(V^{\otimes 2})$, similarly to the relations of $\mathbb{C}[GL(V)]$, $x^a_k x^b_l = x^b_l x^a_k$ which live in $\mathrm{End}_{\mathfrak{S}_2}(V^{\otimes 2})$, the centralizer of the (place permutation) action of $\mathfrak{S}_2$.

The *classical* Schur–Weyl duality is the statement that on the tensor power $V^{\otimes r}$ the (place permutation) action of $\mathbb{C}[\mathfrak{S}_r]$ and the diagonal action by matrix multiplication of the group $GL(V)$ (which is dense in $\mathfrak{gl}(V)$), are commutant of each other,

$$\mathbb{C}[\mathfrak{S}_r] = \mathrm{End}_{GL(V)}(V^{\otimes r}) \qquad \mathbb{C}[GL(V)]_r = \mathrm{End}_{\mathfrak{S}_r}(V^{\otimes r}).$$

The dual formulation of Schur–Weyl duality is the double centralizing property between the UEA $U\mathfrak{gl}(V)$ and $\mathbb{C}[\mathfrak{S}_r]$ in $\mathrm{End}(V^{\otimes r})$.

The *quantum* Schur–Weyl duality is the double centralizing property in $\mathrm{End}(V^{\otimes r})$ between the actions of $\mathcal{H}_r(q)$ and the quantum group $GL_q(V)$ (through the coaction of $\mathbb{C}_{\hat{R}_q}[GL(V)]$). Its dual formulation is between the actions of the Drinfeld quantum UEA $U_q\mathfrak{gl}(V)$ and the Hecke algebra $\mathcal{H}_r(q)$.

The quantum Schur–Weyl duality implies that the $(U_q\mathfrak{gl}(V), \mathcal{H}_r(q))$-bimodule $V^{\otimes r}$ splits into irreducible left $U_q\mathfrak{gl}(V)$-modules V_λ (*Schur modules*) and right $\mathcal{H}_r(q)$-modules S^λ (*Specht modules*), $V^{\otimes r} \cong \bigoplus_{\lambda\vdash r} V_\lambda \otimes S^\lambda$. The left Schur module V_λ is the image of the Schur functor $S_\lambda(\_\_)$ determined by the left Specht module S_λ as

$$V_\lambda = S_\lambda(V) := V^{\otimes r} \otimes_{\mathcal{H}_r(q)} S_\lambda.$$

In the classical limit $q = 1$ we get a $(U\mathfrak{gl}(V), \mathbb{C}[\mathfrak{S}_r])$-bimodule $V^{\otimes r}$.

2 RSK Correspondence

The celebrated Robinson–Schensted–Knuth (RSK) algorithm sets bijection between words written with letters of an alphabet I (an ordered set) and pairs of Young Tableaux [4]. The set of words becomes a free monoid I^* when one takes as multiplication the juxtaposition of words. Let the alphabet be the index set I of an ordered basis $\{e_i\}_{i\in I}$ of a vector space $V = \oplus_{i\in I}\mathbb{C}e_i$. The linear combinations of words $w = w_1w_2 \dots w_r \in I^*$ with coefficients in $\mathbb{C}$ span the free associative algebra generated in V, i.e., the tensor algebra $T(V) = \bigoplus_{r\geq 0} V^{\otimes r}$, graded by the word's length.

There are three layers in the Robinson–Schensted–Knuth correspondence [4] stemming from three levels in the representation theory;

(i) The Robinson bijection stems from the decomposition of $\mathbb{C}[\mathfrak{S}]$ into irreducible Specht left $\mathbb{C}[\mathfrak{S}_r]$-modules S_λ and right $\mathbb{C}[\mathfrak{S}_r]$-modules $(S^\lambda = \mathrm{Hom}_{\mathbb{C}[\mathfrak{S}_r]}(S_\cdot, \mathbb{C}))$

$$\mathbb{C}[\mathfrak{S}] \cong \bigoplus_{r\geq 0}\left(\bigoplus_{\lambda\vdash r} S_\lambda \otimes S^\lambda\right) \tag{4}$$

The basis vectors in the Specht modules S_λ and S^λ are in bijection with the Standard Young Tableaux $STab(\lambda)$ of shape λ.

The basis in $\mathbb{C}[\mathfrak{S}_r]$ is labelled by permutations $\sigma \in \mathfrak{S}_r$. The Robinson correspondence is between a word without repetition (identified with a permutation $\sigma \in \mathfrak{S}_r$) and a pair of standard Young tableaux with r boxes reflecting the decomposition (4)

$$\begin{array}{ccc} \mathfrak{S}_r & \longleftrightarrow & \bigsqcup_{\lambda\vdash r} STab(\lambda) \times STab(\lambda) \\ \sigma & \longleftrightarrow & (P(\sigma), Q(\sigma)) \end{array}.$$

(ii) The Robinson–Schensted bijection parallels the decomposition of the tensor algebra $T(V)$ as a $(U\mathfrak{gl}(V), \mathbb{C}[\mathfrak{S}])$-bimodule

$$T(V) \cong \bigoplus_{r\geq 0}\left(\bigoplus_{\lambda\vdash r} S_\lambda(V) \otimes S^\lambda\right).$$

The basis of the irreducible (left) Schur module $V_\lambda = S_\lambda(V)$ is indexed by the semistandard Young tableaux $Tab(\lambda, I)$ of shape λ, while the basis of the right Specht module S_λ is indexed by $STab(\lambda)$.

A word $w = w_1 w_2 \dots w_r \in I_r^*$ is identified with a tensor algebra element in $V^{\otimes r}$. Robinson–Schensted algorithm maps a word $w \in I^*$ to a pair of Young tableaux: a (left) semistandard Young tableau $P(w)$ with entries in I and a (right) standard Young tableau $Q(w)$

$$\begin{array}{ccc} I_r^* & \longleftrightarrow & \bigsqcup_{\lambda \vdash r} Tab(\lambda, I) \times STab(\lambda) \\ w & \longleftrightarrow & (P(w), Q(w)) \end{array}.$$

The pair $(P(w), Q(w))$ in the Robinson–Schensted bijection is asymmetric, the left tableau $P(w)$ is semistandard and the right one $Q(w)$ is standard. The symmetry is restored by extending the correspondence to biwords.

(iii) The Robinson–Schensted–Knuth bijection is the combinatorial counterpart of the Peter–Weyl theorem. The polynomial functions $\mathbb{C}[GL(V)]$ are dense in the space $\mathbb{C}[V^* \otimes V]$. The decomposition of the $(U\mathfrak{gl}(V), U\mathfrak{gl}(V))$-bimodule $\mathbb{C}[V^* \otimes V]$ into irreducible left and right Schur modules yields

$$\mathbb{C}[GL(V)] \cong \bigoplus_{r \geq 0} \mathbb{C}[V \otimes V^*]_r \cong \bigoplus_{r \geq 0} \left(\bigoplus_{\lambda \vdash r} S_\lambda(V^*) \otimes S^\lambda(V) \right). \tag{5}$$

The basis of the product of left and right Schur modules is labelled by pairs of left and right semistandard Young tableaux as in the RSK bijection.

A biword $\begin{bmatrix} u \\ v \end{bmatrix}$ is a word written in biletters $\begin{pmatrix} u_i \\ v_i \end{pmatrix} \in I \times I$, $\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} u_1 & \dots & u_r \\ v_1 & \dots & v_r \end{bmatrix}$. An element in $\mathbb{C}[V \otimes V^*]_r$ is encoded into the commutative biword, i.e., the monomial written in commutative biletters.[1]

We conclude that the RSK correspondence between (the commutative class of) a biword $\begin{bmatrix} u \\ v \end{bmatrix}$ and a pair of semistandard Young tableaux mirrors decomposition (5)

$$\begin{array}{ccc} I_r^* \times I_r^* & \longleftrightarrow & \bigsqcup_{\lambda \vdash r} Tab(\lambda, I) \times Tab(\lambda, I) \\ \begin{bmatrix} u \\ v \end{bmatrix} & \longleftrightarrow & (P(u), Q(v)) \end{array}.$$

Although the parallel driven between RSK bijection and representation theory in terms of the classical Schur–Weyl duality is suggestive, one can be puzzled by the apparent absence of symmetry on the side of the combinatorial objects. It turns out that the symmetries $U\mathfrak{gl}(V)$ and $\mathbb{C}[\mathfrak{S}_r]$ are present but in a sense are "frozen" on the combinatorial side. To really understand the symmetry behind the RSK algorithm one has to consider first the quantum Schur–Weyl duality, i.e., to consider the representations of $U_q\mathfrak{gl}(V)$ and $\mathcal{H}(q)$ and then to take in these representations the singular crystal limit $q = 0$. This idea appeared for the first time in the seminal

[1] It is worth noting that a noncommutative word $w = w_1 w_2 \dots w_r$ can be written as a monomial in commutative biletters $\begin{pmatrix} w_i \\ i \end{pmatrix}$, e.g., $\begin{bmatrix} w_1 & w_2 & \dots & w_r \\ 1 & 2 & \dots & r \end{bmatrix}$.

paper of Etsuro Date, Michio Jimbo and Tetuji Miwa [1]. It seems that the paper [1] become a source of inspiration for Masaki Kashiwara and led to his ground-breaking theory of the crystal bases.

The three RSK levels are to be related to the double centralizing property in the following modules

(R) the Hecke algebra $\mathcal{H}(q)$ as a $(\mathcal{H}(q), \mathcal{H}(q))$-bimodule,

(RS) the tensor algebra $T(V)$ over $\mathbb{C}(q)$ as a $(U_q\mathfrak{gl}(V), \mathcal{H}(q))$-bimodule,

(RSK) the coordinate ring $\mathbb{C}_{\hat{R}_q}[GL(V)]$ as a $(U_q\mathfrak{gl}(V), U_q\mathfrak{gl}(V))$-bimodule.

Loosely speaking each layer of the RSK bijection has its own plactic-type algebra.

3 Plactic Monoid and Robinson–Schensted Bijection

Let us now concentrate on the Robinson–Schensted bijection. The correspondence $w \to P(w)$ becomes many-to-one if we forget the standard $Q(w)$ tableau. The words having the same P tableau are called P-equivalent. By construction the P-equivalence classes of words $w \in I^*$ are mapped bijectively to the Semistandard Young Tableaux.

The P-equivalence on words of length 3 implies the so called the Knuth relations

$$\begin{aligned} xzy &\equiv zxy \quad x \le y < z \\ yxz &\equiv yzx \quad x < y \le z \end{aligned} \tag{6}$$

The Knuth relations generate equivalence relation $\equiv$ between the words in the free monoid I^*.

Definition 1 The plactic monoid $\mathfrak{P}(I)$ on an alphabet I is the quotient of the free monoid I^* by the equivalence relation $\equiv$

$$\mathfrak{P}(I) = I^*/\equiv .$$

Let V be a vector space, such that $V = \oplus_{i\in I}\mathbb{C}e_i$. The plactic algebra $\mathfrak{P}(V)$ is the algebra of the plactic monoid $\mathfrak{P}(I)$, it is the factor algebra of the tensor algebra $T(V)$ by the two-sided ideal $(\mathfrak{Knuth}(V))$ generated by the Knuth relations (6)

$$\mathfrak{P}(V) = T(V)/(\mathfrak{Knuth}(V)).$$

Donald Knuth has proven that the plactic classes of the $\equiv$-equivalence are the same as the classes of the P-equivalence. Therefore the classes in $\mathfrak{P}(V)$ are in bijection with the Semistandard Young Tableaux.

4 Parastatistics Algebra $U\mathfrak{n}(V)$

Let $\mathfrak{n}(V)$ be the free 2-nilpotent graded Lie algebra $\mathfrak{n}(V) = V \oplus \wedge^2 V$ generated by the vector space V. In other words the Lie bracket of $\mathfrak{n}$ is

$$[x, y] := \begin{cases} x \wedge y & x, y \in V \\ 0 & \text{otherwise} \end{cases}. \tag{7}$$

We will deal with the UEA $U\mathfrak{n}(V)$ which is a functor on $\mathfrak{n}(V)$

$$U\mathfrak{n}(V) = U(V \oplus \wedge^2 V) = T(V)/([[V, V], V]) =: PS(V)$$

The algebra $U\mathfrak{n}(V)$ is referred to as the *parastatistics algebra* and denoted also as $PS(V)$ (see e.g. [6]) because it appears as a Fock-like space for the algebra of the parastatistics creation and annihilation operators [2].

The following theorem shows that there is a bijection between the parastatistics algebra $U\mathfrak{n}(V)$ and the plactic algebra $\mathfrak{P}(V)$.

Theorem 1 *In the decomposition of the left $U\mathfrak{gl}(V)$-module $U\mathfrak{n}(V)$ into irreducibles every Schur module appears once and exactly once*[2]

$$U\mathfrak{n}(V) \cong \bigoplus_{\lambda} S_{\lambda}(V).$$

The basis of $U\mathfrak{n}(V)$ is labelled by Semistandard Young tableaux $Tab(I) = \oplus_{\lambda} Tab(\lambda, I)$.

Proof The Poincaré–Birkhoff–Witt theorem implies $U(V \oplus \wedge^2 V) \cong S(V) \otimes S(\wedge^2 V)$. The character of the $GL(V)$-module $U(V \oplus \wedge^2 V)$ is the LHS of the Cauchy identity

$$\prod_i \frac{1}{1 - x_i} \prod_{i<j} \frac{1}{1 - x_i x_j} = \sum_{\lambda} s_{\lambda}(x)$$

whereas on the RHS appear the Schur polynomials $s_{\lambda}(x) = chS_{\lambda}(V)$, these are the characters of the Schur modules $S_{\lambda}(V)$ which implies the statement. □

Although the algebras $\mathfrak{P}(V)$ and $U\mathfrak{n}(V)$ have the same amount of elements these algebras enjoy very different symmetries. The Lie type relations of the parastatistics algebra $U\mathfrak{n}(V)$ span an irreducible $U\mathfrak{gl}(V)$-module $[[V, V], V] \cong S_{(2,1)}(V) \subset V^{\otimes 3}$. Therefore the quotient algebra $T(V)/([[V, V], V])$ is a $U\mathfrak{gl}(V)$-module. In contrast the plactic algebra $\mathfrak{P}(V)$ has no apparent symmetry. We are going to show that the plactic algebra $\mathfrak{P}(V)$ arises as a tropicalization of the parastatistics algebra $U\mathfrak{n}(V)$.

[2] Although some of the Schur modules $V_{\lambda} = S_{\lambda}(V)$ are vanishing, those with height $ht(\lambda) > \dim V$.

5 Quantum Plactic Algebra

Let us denote by $\mathfrak{Lie}(V) = \bigoplus_{n\geq 1} \mathfrak{Lie}^n(V)$ the free Lie algebra generated in V. Jean-Louis Loday constructed in his work [5] a system of *Eulerian* idempotents $\{e_n\}_{n\geq 1}$ such that $e_n \in \mathbb{C}[\mathfrak{S}_n]$ projects to the degree n primitive elements in $T(V)$, i.e., $V^{\otimes n}e_n = \mathfrak{Lie}^n(V)$. There exists a system of quantum Eulerian idempotents $\{e_n(q)\}_{n\geq 1}$ projecting to the quantum free Lie algebra $\mathfrak{Lie}_q(V)$ of quasi-primitive elements of $T(V) = U_q\mathfrak{L}ie(V)$. Although the problem of finding the quantum Eulerian idempotents $e_r(q) \in \mathcal{H}_r(q)$ is a difficult one and it is not solved in general the cubic idempotent $e_3(q)$ is known [6, 7]

$$\begin{aligned} e_3(q) := \frac{1}{[3]}\left(T_{123} - \frac{1}{2}(T_{231} + T_{213} + T_{132} + T_{312}) + T_{321}\right) \\ + \frac{q-q^{-1}}{2[3]}\,(T_{213} - T_{312} - T_{231} + T_{132})\,. \end{aligned} \tag{8}$$

An orthogonal $\widetilde{e}_3(q)$ is obtained from $e_3(q)$ by the involution $\widetilde{T}_\sigma \to (-1)^\sigma T_\sigma$, $q \to q^{-1}$.

With the knowledge of the quantum Eulerian idempotent $e_3(q) \in \mathcal{H}_3(q)$ we construct the two-dimensional left $\mathcal{H}_3(q)$-module as a left ideal $\mathfrak{Lie}^3_q = \mathcal{H}_3(q)e_3(q)$

$$\begin{aligned} \mathfrak{Lie}^3_q\left[\begin{smallmatrix}12\\3\end{smallmatrix}\right] &:= q(T_{213} - T_{231}) + T_{123} - T_{132} - T_{231} + T_{321} + q^{-1}(T_{312} - T_{132})\,, \\ \mathfrak{Lie}^3_q\left[\begin{smallmatrix}13\\2\end{smallmatrix}\right] &= q(T_{132} - T_{312}) + T_{123} - T_{213} - T_{312} + T_{321} + q^{-1}(T_{231} - T_{213}). \end{aligned}$$

By quantum Schur–Weyl duality the $\mathcal{H}_3(q)$-module $\mathfrak{Lie}^3_q$ defines a Schur functor as

$$\mathfrak{Lie}^3_q(V) := V^{\otimes 3} \otimes_{\mathcal{H}_3(q)} \mathfrak{Lie}^3_q = V^{\otimes 3}e_3(q).$$

We are now ready to introduce the quantum plactic algebra $\mathfrak{P}_q(V)$ which is a $U_q\mathfrak{gl}(V)$-module. It will interpolate between the plactic algebra $\mathfrak{P}(V)$ and the parastatistics algebra $U\mathfrak{n}(V)$.

Definition 2 The quantum plactic algebra $\mathfrak{P}_q(V)$ is the tensor algebra $T(V)$ quotient by the ideal generated by the quantum Lie cubic relations $\mathfrak{Lie}^3_q(V)$

$$\mathfrak{P}_q(V) = T(V)/(\mathfrak{Lie}^3_q(V)) \tag{9}$$

The quantum Lie cubic relations $\mathfrak{Lie}^3_q(V)$ span a left $U_q\mathfrak{gl}(V)$-module with basis

$$\begin{aligned} \mathfrak{Lie}^3_q\left[\begin{smallmatrix}i_1,i_2\\i_3\end{smallmatrix}\right] &:= [[e_{i_1}, e_{i_3}], e_{i_2}]_{q^2} + q[[e_{i_1}, e_{i_2}], e_{i_3}] && \text{with } i_1 < i_2 < i_3 \\ \mathfrak{Lie}^3_q\left[\begin{smallmatrix}i_1,i_1\\i_2\end{smallmatrix}\right] &:= [[e_{i_1}, e_{i_2}], e_{i_1}]_q && \text{with } i_1 < i_2 \\ \mathfrak{Lie}^3_q\left[\begin{smallmatrix}i_1,i_3\\i_2\end{smallmatrix}\right] &:= [e_{i_2}, [e_{i_1}, e_{i_3}]]_{q^2} + q[e_{i_1}, [e_{i_2}, e_{i_3}]] && \text{with } i_1 < i_2 < i_3 \\ \mathfrak{Lie}^3_q\left[\begin{smallmatrix}i_1,i_2\\i_2\end{smallmatrix}\right] &:= [e_{i_2}, [e_{i_1}, e_{i_2}]]_q && \text{with } i_1 < i_2 \end{aligned}$$

Here the deformed commutator is defined as $[x, y]_Q := xy - Qyx$.

The $U_q\mathfrak{gl}(V)$-module $\mathfrak{Lie}^3_q(V)$ is isomorphic to the Schur module $V_{(2,1)}(q)$ hence its elements are indexed by Semistandard Young Tableaux of shape $\yng(2,1)$.

$\mathfrak{P}_q(V)$ was first designed as deformed parastatistics algebra[3] $PS_q(V) := U_q\mathfrak{n}(V)$ [6].

We have the following commutative diagram between left $U_q\mathfrak{gl}(V)$-modules

$$\begin{tikzcd}
([[V^*, V^*], V^*]) \arrow[r, "\cong"] \arrow[d, hook] & (\mathfrak{Lie}^3(V)) \arrow[d, hook] & (\mathfrak{Lie}^3_q(V)) \arrow[l, "q=1"'] \arrow[r, "q=0"] \arrow[d, hook] & (\mathfrak{Lie}^3_0(V)) \arrow[d, hook] \\
T(V^*) \arrow[r, "\cong"] & T(V) \arrow[d, two heads] & T(V) \arrow[l, "q=1"'] \arrow[r, "q=0"] \arrow[d, two heads] & T(V) \arrow[d, two heads] \\
\mathbb{C}[V, [V, V]] \arrow[r, "\cong"] \arrow[u, hook] & U\mathfrak{n}(V) & \mathfrak{P}_q(V) \arrow[l, "q=1"'] \arrow[r, "q=0"] & \mathfrak{P}(V)
\end{tikzcd} \tag{10}$$

$U_q\mathfrak{gl}(V)$-module $\mathfrak{Lie}^3_q(V)$ is a quantum deformation of the space of cubic Lie elements where the value $q = 1$ is the "classical limit" $\mathfrak{Lie}^3(V)$. On the other hand the Kashiwara crystal limit $q = 0$ of the module $\mathfrak{Lie}^3_q(V)$ span the Knuth relations

$$[[V, V], V] \xleftarrow{q=1} \mathfrak{Lie}^3_q(V) \xrightarrow{q=0} \mathfrak{Knuth}(V) \ ,$$

so the plactic algebra $\mathfrak{P}(V)$ is a tropicalization of the polynomial algebra $\mathbb{C}[V, [V, V]]$.

Let us stress that $U_q\mathfrak{gl}(V)$-modules in the diagram (10) are Schur functors. Dropping the argument V of the left $U_q\mathfrak{gl}(V)$-modules in the diagram (10) we recover a (Schur–Weyl dual) diagram between the left Specht $\mathcal{H}(q)$-modules.

Dropping V in the Schur functor $T(V) = \bigoplus_{r\geq 0} V^{\otimes r} \otimes_{\mathcal{H}_r(q)} \mathcal{H}_r(q)$ we end up with the left $\mathcal{H}(q)$-module $\mathcal{H}(q) = \bigoplus_{r\geq 0} \mathcal{H}_r(q)$. For generic q the $\mathcal{H}(q)$-module $\mathcal{H}(q)$ is isomorphic to the $\mathbb{C}[\mathfrak{S}]$-module $\mathbb{C}[\mathfrak{S}]$. It turns out that the Malvenuto-Reutenauer Hopf algebra $\mathfrak{MR} := (\mathbb{C}[\mathfrak{S}], *, \Delta)$[8] can be promoted to a Hopf algebra q-$\mathfrak{MR} = (\mathcal{H}(q), *, \Delta)$ with similar definitions of $*$ and Δ(see [7]).

Lemma 1 *Standard quantum plactic algebra $\mathfrak{P}_q$ is the factor of the left $\mathcal{H}(q)$-module $\mathcal{H}(q)$ by the Hopf ideal generated by the left $\mathcal{H}_3(q)$-module $\mathfrak{Lie}^3_q$*

$$\mathfrak{P}_q = q\text{-}\mathfrak{MR}/(\mathfrak{Lie}^3_q).$$

The Hopf algebra $(\mathfrak{P}_q, *, \Delta)$ is a Hopf factor algebra of q-$\mathfrak{MR}$ algebra.

[3] Another way to get $\mathfrak{Lie}^3_q(V)$ is through the parabolic subalgebra $\mathfrak{gl}(V) \ltimes \mathfrak{n}(V) \subset \mathfrak{so}_{1+2\dim V}$. Its radical $\mathfrak{n}(V)$ defines $U_q\mathfrak{n}(V)$ uniquely from the quantum Serre relations of $U_q\mathfrak{so}_{2n+1}$.

Indeed on checks that $\mathfrak{Lie}_q^3 \subset \mathcal{H}_3(q)$ is primitive $\Delta \mathfrak{Lie}_q^3 = \mathfrak{Lie}_q^3 \otimes \varnothing + \varnothing \otimes \mathfrak{Lie}_q^3$ hence the ideal $(\mathfrak{Lie}_q^3) = \bigoplus_{r \geq 0} (\mathfrak{Lie}_q^3)(r) = \sum_{i+j=r-3} \mathcal{H}_i(q) * \mathfrak{Lie}_q^3(3) * \mathcal{H}_j(q)$ is a coideal.

The crystal limit $q = 0$ of the standard quantum plactic algebra $(\mathfrak{P}_q, *, \Delta)$ turns out to be the Poirier-Reutenauer Hopf algebra [9]. We conclude that $(\mathfrak{P}_q, *, \Delta)$ is a family of Hopf structures(parametrized by q) on the set of Standard Young Tableaux with (see [7] for more details).

6 Quantum Pseudo-Plactic Algebra and Quantum Torus

Our motivation for exploring the RSK algorithm as a quantum group bimodule correspondence is to shed a new light of the *quantum pseudo-plactic algebra* introduced with relation to noncommutative characters and Schur functions [3].

The coordinate ring $\mathbb{C}_{\hat{R}_q}[GL(V)]$ is a $(U_q\mathfrak{gl}(V), U_q\mathfrak{gl}(V))$-bimodule

$$\mathbb{C}_{\hat{R}_q}[GL(V)] \cong T(W)/(\hat{R}W \otimes W - W \otimes W\hat{R})$$

generated by the bimodule $W := V^* \otimes V = \bigoplus_{i,j \in I} \mathbb{C}(q) x_j^i$. The FRT relations (3) read

$$\begin{array}{lll} x_k^j x_k^i = q x_k^i x_k^j & x_j^k x_i^k = q x_i^k x_j^k & j > i \\ x_l^j x_k^i = x_k^i x_l^j + (q - q^{-1}) x_l^i x_k^j & x_k^j x_l^i = x_l^i x_k^j & j > i \quad l > k \end{array}$$

When $q = 1$ the restriction of the commutative ring $\mathbb{C}[GL(V)]$ to the subring of the diagonal matrix elements x_i^i yields $\mathbb{C}[\mathbb{T}]$, the *commutative* functions on the torus $\mathbb{T}$. Under deformation when $q \neq \pm 1$ we have $x_j^j x_i^i = x_i^i x_j^j + (q - q^{-1}) x_j^i x_i^j$ for $j > i$ thus the diagonal matrix elements $x_i^i \in \mathbb{T}^*$ do not close a quadratic algebra anymore.

Daniel Krob and Jean-Yves Thibon conjectured in [3] that the commutative ring $\mathbb{C}[\mathbb{T}]$ is deformed to a diagonal cubic subalgebra of $\mathbb{C}_{\hat{R}_q}[GL(V)]$, the so called *quantum pseudo-plactic algebra* $\mathfrak{PP}_q(\mathbb{T})$ (rescaled but equivalent to the one in [3]).

Definition 3 Quantum pseudo-plactic algebra $\mathfrak{PP}_q(\mathbb{T}^*)$ generated in $\mathbb{T}^* = \bigoplus_{i \in I} \mathbb{C} x_i^i$

$$\mathfrak{PP}_q(\mathbb{T}^*) = T(\mathbb{T}^*)/(\mathfrak{L}_q(\mathbb{T}^*))$$

is the factor of the tensor algebra $T(\mathbb{T}^*)$ over $\mathbb{C}(q)$ by the ideal spanned by the subspace (defining the pseudo-Knuth relations)

$$\begin{array}{lll} \mathfrak{L}_{q\,i_3}^{\ i_1,i_2} := [[x_{i_1}^{i_1}, x_{i_3}^{i_3}], x_{i_2}^{i_2}] & \textit{with} & i_1 < i_2 < i_3 \\ \mathfrak{L}_{q\,i_2}^{\ i_1,i_1} := [[x_{i_1}^{i_1}, x_{i_2}^{i_2}], x_{i_1}^{i_1}]_{q^2} & \textit{with} & i_1 < i_2 \\ \mathfrak{L}_{q\,i_2}^{\ i_1,i_2} := [x_{i_2}^{i_2}, [x_{i_1}^{i_1}, x_{i_2}^{i_2}]]_{q^2} & \textit{with} & i_1 < i_2 \end{array} \quad . \tag{11}$$

Remark $\mathfrak{PP}_q(\mathbb{T})$ has the quantum torus coordinate ring $x_j^j x_i^i = q^2 x_i^i x_j^j$ as a factor.

Lemma 2 *The isotypic decomposition of the identity* $id_{\mathcal{H}_3(q)} = \oplus_{\lambda\vdash 3} E_\lambda$ *has a finer splitting* $E_{(2,1)} = e_3(q) + \widetilde{e}_3(q)$ *where* $\widetilde{e}_3(q)$ *is the orthogonal of the q-Eulerian idempotent* $e_3(q)$. *The pseudo-Knuth relations* $\mathfrak{L}_q(\mathbb{T}^*) = 0$ *are obtain by restriction to* $\mathbb{T}^*$

$$\mathfrak{L}_q(\mathbb{T}^*) = \mathfrak{L}_q(W)|_{\mathbb{T}^*} \qquad \mathfrak{L}_q(W) = \widetilde{e}_3(q) W \otimes W \otimes W e_3(q) \qquad e_3(q)\widetilde{e}_3(q) = 0$$

with $e_3(q), \widetilde{e}_3(q) \in \mathrm{End}(V^{\otimes 3})$ *evaluated in the Hecke algebra representation* $\pi_{\hat{R}_q}$.

Sketch of the proof. The vanishing of $\mathfrak{L}_q(W)$ is implied from the FRT relations (2). The upper and lower indices coincide as multisets allowing the restriction to $\mathbb{T}^*$. □

We can summarize the hierarchy of algebras in the commutative diagram

$$\begin{tikzcd}
([\mathbb{T}^*, \mathbb{T}^*]) \arrow[r, hook] \arrow[d, hook] & ([W, W]) \arrow[d, hook] & (\hat{R}WW - WW\hat{R}) \arrow[l, "q=1"'] \arrow[d, hook] & (\mathfrak{L}_q(\mathbb{T})) \arrow[l, hook'] \arrow[d, hook] \\
T(\mathbb{T}^*) \arrow[r, hook] \arrow[d] & T(W) \arrow[d] & T(W) \arrow[l, "q=1"'] \arrow[d] & T(\mathbb{T}^*) \arrow[l, hook'] \arrow[d] \\
\mathbb{C}[\mathbb{T}] \arrow[r, hook] & \mathbb{C}[GL(V)] & \mathbb{C}_{\hat{R}_q}[GL(V)] \arrow[l, "q=1"'] & \mathfrak{PP}_q(\mathbb{T}^*) \arrow[l, hook']
\end{tikzcd} \tag{12}$$

The functions $\mathbb{C}_{\hat{R}_q}[GL(V)]_r \cong (W^{\otimes r})^{\mathcal{H}_r(q)}$ carry left and right $U_q\mathfrak{gl}(V)$)-representation

$$(V^{*\otimes r})^A \otimes_{\mathcal{H}_r(q)} (V^{\otimes r})_B \cong \bigoplus_{\Lambda\vdash r} S^A_\Lambda(V^*) \otimes_{\mathcal{H}_r(q)} S^\Lambda_B(V) \qquad A, B \in I^r.$$

$\mathbb{C}_{\hat{R}_q}[GL(V)]_r$ is the Schur functor image of the $(\mathcal{H}_r(q), \mathcal{H}_r(q))$-bimodule $\mathcal{H}_r(q)$. The elements $T^\alpha_\beta := T_{\alpha^{-1}} \otimes T_\beta \in \mathcal{H}_r(q) \otimes \mathcal{H}_r(q)$ with indices $\alpha, \beta \in \mathfrak{S}_r$ project to

$$\mathcal{H}_r(q)^\alpha_\beta \cong \oplus_{\Lambda\vdash r} S^\alpha_\Lambda \otimes_{\mathcal{H}_r(q)} S^\Lambda_\beta \quad \ni \quad \bar{T}^\alpha_\beta := T_{\alpha^{-1}} \otimes_{\mathcal{H}_r(q)} T_\beta = (T^\alpha_\beta)^{\mathcal{H}_r(q)}.$$

The diagonal restriction $(\mathcal{H}(q)\otimes\mathcal{H}(q))^{sym} := \bigoplus_{r\geq 0} \left(\bigoplus_{\alpha\in\mathfrak{S}_r} \mathbb{C}T^\alpha_\alpha\right)$ of the bimodule $\mathcal{H}(q)^{\otimes 2}$ is a $\mathcal{H}(q)$-module with a left action

$$T_\rho \cdot T^\sigma_\sigma := T_{\rho^{-1}} T_{\sigma^{-1}} \otimes T_\sigma T_\rho \qquad \text{(similarly for } \bar{T}^\sigma_\sigma \in \mathcal{H}^{sym}(q)).$$

The pseudo-Knuth bimodule $\mathfrak{L}_q(W)$ is a submodule of the $(U_q\mathfrak{gl}(V), U_q\mathfrak{gl}(V))$-bimodule $W^{\otimes 3}$ and it defines a Schur functor through a $(\mathcal{H}_r(q), \mathcal{H}_r(q))$-bimodule

$$\mathfrak{L}_q := \mathcal{H}_3(q) e_3(q) \otimes \widetilde{e}_3(q) \mathcal{H}_3(q) \qquad \text{such that} \qquad \mathfrak{L}_q^{\mathcal{H}_r(q)} = 0.$$

Definition 4 The standard quantum pseudo-plactic algebra $\mathfrak{P}\mathfrak{P}_q$ is the quotient of the $\mathcal{H}(q)$-module $(\mathcal{H}(q)\otimes\mathcal{H}(q))^{sym}$ by the submodule generated in $\mathfrak{L}_q^{sym}$

$$\mathfrak{P}\mathfrak{P}_q = (\mathcal{H}(q)\otimes\mathcal{H}(q))^{sym}/(\mathfrak{L}_q^{sym}) \,, \qquad\qquad \mathfrak{L}_q^{sym} = T_{213}^{213} - T_{231}^{231} - T_{132}^{132} + T_{312}^{312}.$$

The conjecture of Krob and Thibon about the quantum pseudo-plactic algebra [3] is a corollary of the statement $\mathcal{H}^{sym}(q) \cong \mathfrak{P}\mathfrak{P}_q$. We will give the details elsewhere.

Acknowledgements It is my pleasure to thank Peter Dalakov, Michel Dubois-Violette, Gérard Duchamp and Milen Yakimov for many enlightening discussions. This work has been supported by Grant DFNI T02/6 of the Bulgarian National Science Foundation and a Koç University grant.

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Conformal Invariance of the $1D$ Collisionless Boltzmann Equation

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Abstract Dynamical symmetries of the collisionless Boltzmann transport equation, with an external driving force, are derived in $d = 1$ spatial dimensions. Both positions and velocities are considered as independent variables. The Lie algebra of dynamical symmetries is isomorphic to the $2D$ projective conformal algebra, but we find new non-standard representations. Several examples with explicit external forces are presented.

1 Physical Background and Motivation

Consider a system of classical particles, described by an effective single-particle (Wigner) distribution function $f = f(t, \mathbf{r}, \mathbf{p})$, such that

$$\mathrm{d}N = f(t, \mathbf{r}, \mathbf{p})\mathrm{d}\mathbf{r}\,\mathrm{d}\mathbf{p} \tag{1}$$

is the number of particles in a cell in phase space, of volume $\mathrm{d}\mathbf{r}\,\mathrm{d}\mathbf{p}$, and centred at position $\mathbf{r}$ and $\mathbf{p}$. The classical *Boltzmann transport equation* (BTE)

$$\frac{\partial f}{\partial t} + \frac{\mathbf{p}}{m} \cdot \frac{\partial f}{\partial \mathbf{r}} + \mathbf{F} \cdot \frac{\partial f}{\partial \mathbf{p}} = \left(\frac{\partial f}{\partial t}\right)_{\mathrm{coll}} \tag{2}$$

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V. Dobrev (ed.), *Lie Theory and Its Applications in Physics*,
Springer Proceedings in Mathematics & Statistics 191,
DOI 10.1007/978-981-10-2636-2_33

describes the effects of particle transport under the influence of an external force $\mathbf{F} = \mathbf{F}(t, \mathbf{r})$ [1, 5, 6, 11]. The right-hand-side of the BTE describes the effect of collisions between particles, for instance in a diluted gas. Obtaining an explicit form requires knowledge of the statistics the particles obey, for example the Maxwell-Boltzmann, Fermi-Dirac or Bose-Einstein distributions.

A different situation arises in the case of long-ranged interactions between the particles, for instance from gravitational interactions between stars in galactical dynamics or from electromagnetic interactions between the charge carriers in a plasma. Then, the evolution cannot be reduced to a succession of isolated binary encounters which become overwhelmed by the long-range effects of the acting forces. As pointed out by Jeans in 1915 for galactical dynamics and by Vlasov in 1938 for plasmas, in such cases the collision term in (2) should be left out, for an overview see [9] and references therein. One commonly refers to Eq. (2) with a vanishing right-hand-side as the *collisionless Boltzmann equation* (CBE) [9]. In plasma physics, the forces are determined self-consistently by using for $\mathbf{F}$ the Laplace-Lorentz force, where the electric and magnetic fields are determined from the Maxwell-equations. Together with the CBE, this leads to the *Vlasov-Maxwell system* [16]. In the non-relativistic limit, where the magnetic field also vanishes, the remaining electric field is found from the Poisson equation. Then the problem becomes equivalent to the *Jeans-Poisson system* [10].

The CBE continues as a theme of intensive recent research, see [3, 12, 13] and references therein.[1] The classic *Jeans' theorem* states that the stationary distribution function of the CBE only depends on the integrals of motion. Relationship with Landau damping and physicists' derivations of the CBE can be found e.g. in [2–4, 15]. Here, we shall study a class of dynamical symmetries of the CBE.[2]

Throughout this work, we shall restrict to $d = 1$ space dimension.

The symmetries of the CBE will be found through an analogy with a non-standard representation of the conformal algebra, in one time and one space dimension. The Lie algebra is spanned by the generators $\langle X_n, Y_n \rangle_{n\in\mathbb{Z}}$, which obey [7, 8]

$$
\begin{gathered}
[X_n, X_m] = (n-m)X_{n+m}, \quad [X_n, Y_m] = (n-m)Y_{n+m}, \\
[Y_n, Y_m] = \mu(n-m)Y_{n+m}
\end{gathered} \tag{3}
$$

where μ^{-1} is a constant universal velocity ('speed of sound/light'). Explicitly

$$
\begin{aligned}
X_n &= -t^{n+1}\partial_t - \mu^{-1}[(t+\mu r)^{n+1} - t^{n+1}]\partial_r - (n+1)\frac{\gamma}{\mu}[(t+\mu r)^n - t^n] - \\
&\quad - (n+1)xt^n \\
Y_n &= -(t+\mu r)^{n+1}\partial_r - (n+1)\gamma(t+\mu r)^n
\end{aligned} \tag{4}
$$

[1] In plasma physics, the CBE is often called the *Vlasov equation* [16], although its application to galactical dynamics by Jeans occurred more than 20 years earlier [10].

[2] This paper contains the main results of our original work [14], presented by the first author at the LT-11 conference.

where x, γ are constants. Writing $X_n = \ell_n + \bar{\ell}_n$ and $Y_n = \mu\bar{\ell}_n$, the generators $\langle \ell_n, \bar{\ell}_n \rangle_{n\in\mathbb{Z}}$ satisfy $[\ell_n, \ell_m] = (n-m)\ell_{n+m}$, $[\bar{\ell}_n, \bar{\ell}_m] = (n-m)\bar{\ell}_{n+m}$, $[\ell_n, \bar{\ell}_m] = 0$. Provided $\mu \neq 0$, the Lie algebra (3) is isomorphic to a pair of Virasoro algebras $\mathfrak{vect}(S^1) \oplus \mathfrak{vect}(S^1)$ with a vanishing central charge [7, 14]. However, this isomorphism does not imply that physical systems described by two different representations of the conformal algebra (3) were trivially related. For example, it is well-known that if one uses (i) the generators of the standard representation of conformal invariance or else (ii) the non-standard representation (4) in order to find co-variant two-point functions, the resulting scaling forms are different [7, 8].

The representation (4) acts as a dynamical symmetry on the equation of motion

$$\hat{S}\phi(t,r) = (-\mu\partial_t + \partial_r)\phi(t,r) = 0. \tag{5}$$

Indeed, since (with $n \in \mathbb{Z}$)

$$[\hat{S}, X_n] = -(n+1)t^n\hat{S} + n(n+1)\mu\left(x - \frac{\gamma}{\mu}\right)t^{n-1}, \quad [\hat{S}, Y_n] = 0 \tag{6}$$

a solution ϕ of with scaling dimension $x_\phi = x = \gamma/\mu$ is mapped onto another solution of (5). Hence the space of solutions of the equation (5) is invariant under (4). This is the analogue of the conformal invariance of the $2D$ Laplace equation.

In what follows, we shall consider the analogues of the maximal finite-dimensional sub-algebra $\langle X_{\pm1,0}, Y_{\pm1,0}\rangle \cong \mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R})$, for $\mu \neq 0$. Specifically, the generators X_{-1}, Y_{-1} describe time- and space-translations, Y_0 is a (conformal) Galilei transformation, X_0 gives the dynamical scaling $t \mapsto \lambda t$ of $r \mapsto \lambda r$ (with $\lambda \in \mathbb{R}$) such that the so-called 'dynamical exponent' $z = 1$ since both time and space are re-scaled in the same way and finally X_{+1}, Y_{+1} give 'special' conformal transformations. In the context of statistical mechanics of conformally invariant phase transitions, one characterises co-variant quasi-primary scaling operators through the invariant parameters $(x, \gamma/\mu)$, where x is the scaling dimension.

Returning to the Boltzmann equation, we consider Eq. (5) in the form

$$\hat{B}f = (\mu\partial_t + v\partial_r)f(t,r,v) = 0 \tag{7}$$

with the distribution function $f = f(t,r,v)$ and where we consider the 'velocity' v as an additional variable. Equation (7) is a simple collisionless Boltzmann equation, yet without an external force, and in one space dimension. From (6), its solution space is conformally invariant. In Sect. 2, we shall generalise the above representation of the conformal algebra to the situation with v as a further variable. Furthermore, we shall distinguish the representations relevant when (i) no external force term is included in the CBE and (ii) when an external force $F = F(t,r,v)$, possibly depending on time, spatial position and velocity, is included. Can one identify situations of potential physical interest with a non-trivial conformal symmetry ? We conclude in Sect. 3.

2 Symmetries of Collisionless Boltzmann Equations

As a preparation for the construction of dynamical symmetries of the $1D$ CBE, we point out that some of the symmetries of Eq. (7), with $v = 1$, can be generalised by replacing X_1, $Y_{0,1}$ as follows, where k is a constant

$$\begin{aligned} X_1 &= -\left(t^2 + kr^2\right)\partial_t - \left(2tr + \frac{k-\mu^2}{\mu}r^2\right)\partial_r - 2xt + 2\mu xr, \\ Y_0 &= -kr\partial_t - \left(t + \frac{k-\mu^2}{\mu}r\right)\partial_r + \mu x \\ Y_1 &= -\left(2ktr + \frac{k(k-\mu^2)}{\mu}r^2\right)\partial_t \\ &\quad - \left(t^2 + 2\frac{k-\mu^2}{\mu}tr + \frac{k(k-\mu^2)+\mu^4}{\mu^2}r^2\right)\partial_r + 2\mu x(t - \mu r). \end{aligned} \tag{8}$$

For $n, n', m, m' \in \{0, \pm 1\}$ they satisfy the following commutation relations

$$\begin{aligned} &[X_n, X_{n'}] = (n - n')X_{n+n'}, \quad [X_n, Y_m] = (n - m)Y_{n+m} \\ &[Y_m, Y_{m'}] = (m - m')\left(kX_{m+m'} + \frac{k-\mu^2}{\mu}Y_{m+m'}\right). \end{aligned} \tag{9}$$

This Lie algebra, spanned by the modified generators (together with usual ones X_{-1}, X_0, Y_{-1}), is isomorphic to $\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R})$ [14, Proposition 1], and to (3). Because of this isomorphism, the Lie algebra (9) will also be called a *conformal algebra*.

Explicitly, we shall look for representations of (9), also including the velocity ν as additional variable and acting as dynamical symmetries of the CBE

$$\hat{B} f(t,r,v) = (\mu\partial_t + v\partial_r + F(t,r,v)\partial_v)\, f(t,r,v) = 0 \tag{10}$$

where the force $F(t,r,v)$ is taken to be an arbitrary function, with its shape to be determined from the dynamical symmetries. Dynamical symmetries of the CBE Eq. (10) are obtained along the lines of the construction of local scale-invariance in time-dependent critical phenomena [7, 8]. In particular, we require invariance under time-translation and dynamical scaling:

1. Invariance under the time-translations X_{-1} implies $\partial_t F = 0$, that is $F = F(r, v)$.
2. The generator of scale-transformations (dilatations) reads [14]

$$X_0 = -t\partial_t - \frac{r}{z}\partial_r - \frac{1-z}{z}v\partial_v - x/z. \tag{11}$$

If the *dynamical exponent* $z \neq 1$, then the generators will contain an explicit v-dependence. The invariance requirement $[\hat{B}, X_0] = -\hat{B}$ leads to a further reduction of the force term:

$$(r\partial_r + (1-z)v\partial_v - (1-2z))F(r,v) = 0 \quad \Longrightarrow \quad F(r,v) = r^{1-2z}\varphi(r^{z-1}v), \tag{12}$$

where $\varphi(u)$ is an arbitrary function, of the scaling variable $u := r^{z-1}v$. It turns out to be convenient to make a change of independent variables $(t,r,v) \mapsto (t,r,u)$.

In order to generate a representation of conformal algebra (9), the generator Y_{-1} of space-translations must be fixed. We use the ansatz[3]

$$Y_{-1} = -r^{1-z}u\partial_r - r^{-z}\Phi(u)\partial_u, \quad \Phi(u) = (z-1)u^2 + \varphi(u). \tag{13}$$

Clearly, $[\hat{B}, Y_{-1}] = 0$, without further conditions on $\Phi(u)$. The dilatation generator $X_0 = -t\partial_t - \frac{r}{z}\partial_r - \frac{x}{z}$ is independent of u. The CBE becomes

$$\hat{B}f(t,r,u) = \left(\mu\partial_t + r^{1-z}u\partial_r + r^{-z}\Phi(u)\partial_u\right) f(t,r,u) = 0. \tag{14}$$

Having fixed the generators X_{-1}, X_0, Y_{-1}, one must now find representations of the conformal algebra (9). However, the determination of the yet undetermined function $\Phi(u)$ in the CBE Eq. (14) and in the form (13) of Y_{-1} requires to use the commutators of the entire conformal algebra (9), as well as the symmetry conditions (by construction $\lambda_{X_{-1}} = \lambda_{Y_{-1}} = 0$ and $\lambda_{X_0} = -1$)

$$[\hat{B}, X_1] = \lambda_{X_1}(t,r,v)\hat{B}, \quad [\hat{B}, Y_0] = \lambda_{Y_0}(t,r,v)\hat{B}, \quad [\hat{B}, Y_1] = \lambda_{Y_1}(t,r,v)\hat{B}. \tag{15}$$

In fact, writing the yet unknown generators X_1, Y_0, Y_1 as vector fields in t, r, u with unknown coefficients, application of these criteria allows to specify the t- and r-dependence of these functions

$$\begin{aligned}
Y_0 &= -r^z a_0(u)\partial_t - (r^{1-z}u + rb_0(u))\partial_r - (r^{-z}\Phi(u)t + c_0(u))\partial_u - d_0(u) \\
X_1 &= -(t^2 + r^{2z}a_{12}(u))\partial_t - ((2/z)tr + r^{z+1}b_{12}(u))\partial_r \\
&\quad -r^z c_{12}(u)\partial_u - (2/z)xt - r^z d_{12}(u) \\
Y_1 &= -\left(2tr^z a_0(u) + r^{2z}A(u)\right)\partial_t - \left(t^2 r^{1-z}u + 2trb_0(u) + r^{z+1}B(u)\right)\partial_r \\
&\quad -\left(t^2 r^{-z}\Phi(u) + 2tc_0(u) + r^z C(u)\right)\partial_u + (2/z)\mu xt - r^z D(u),
\end{aligned} \tag{16}$$

with the four functions $A = A(u)$, $B = B(u)$, $C = C(u)$, $D = D(u)$

[3] The usual form of space translations does not work [14]. Y_{-1} is found (i) as a symmetry of the CBE and (ii) it forms a closed Lie algebra with the other basic generators $X_{-1,0}$. The ansatz (13) is a particular solution the differential equation following from this. It leads to a Boltzmann operator $\hat{B} = -\mu X_{-1} - Y_{-1}$ linear in the generators. We believe this to be a natural auxiliary hypothesis.

$$A = 2zb_0a_{12} + c_0a'_{12} - za_0b_{12} - a'_0c_{12}, \quad C = zb_0c_{12} + c_0c'_{12} - c'_0c_{12} - a_{12}\Phi$$
$$B = \frac{2}{z}a_0 + zb_0b_{12} + c_0b'_{12} - ua'_{12} - b'_0c_{12}, \quad D = \frac{2}{z}xa_0 + zb_0d_{12} + c_0d'_{12} \quad (17)$$

Further application of Eq. (9), (15) gives $q = (k - \mu^2)$, $\lambda_{Y_0} = -k/\mu = -(\mu + q)$ and

$$\lambda_{X_1}(t, r, u) = -2t - 2r^z a_0(u)/\mu, \quad \lambda_{Y_1}(t, r, u) = -\frac{2k}{\mu}t - \frac{2k}{\mu^2}r^z a_0(u) \quad (18)$$

for the eigenvalues, along with a system of 24 coupled non-linear differential equations. They are listed in [14]. From these, the functions $a_0(u)$, $b_0(u)$, $c_0(u)$, $d_0(u)$, $a_{12}(u)$, $b_{12}(u)$, $c_{12}(u)$, $d_{12}(u)$ can be found, but a complete classification of their solutions does not yet exist. We shall illustrate their content through examples.

2.1 CBEs Without External Force

In the ansatz (13), consider the case $\Phi(u) = (z - 1)u^2$, that is $\varphi(u) = 0$. The resulting generators are re-expressed in the original variables (t, r, v). Then

$$X_{-1} = -\partial_t, \quad X_0 = -t\partial_t - \frac{r}{z}\partial_r - \frac{1-z}{z}v\partial_v - \frac{x}{z}, \quad Y_{-1} = -v\partial_r. \quad (19)$$

The CBE (14) coincides with Eq. (7). Several cases can be distinguished [14].

Case A: $k = 0$, $a_0(u) = 0$, z arbitrary.

Theorem 1 *For an arbitrary dynamical exponent z, the generators (19) and*

$$\begin{aligned} X_1^A &= -t^2\partial_t - \left(\frac{2}{z}tr + \mu\frac{z-2}{z}\frac{r^2}{v}\right)\partial_r - \frac{2(1-z)}{z}(vt - \mu r)\partial_v - \frac{2}{z}xt + \frac{2}{z}\mu x\frac{r}{v} \\ Y_0^A &= -\left(tv - \frac{\mu}{z}r\right)\partial_r - \frac{z-1}{z}\mu v\partial_v + \mu\frac{x}{z} \\ Y_1^A &= -\left(t^2v - \frac{2}{z}\mu tr - \mu^2\frac{z-2}{z}\frac{r^2}{v}\right)\partial_r - \frac{2}{z}(z-1)\mu(vt - \mu r)\partial_v + \\ &\quad + \frac{2}{z}\mu x\left(t - \mu\frac{r}{v}\right) \end{aligned} \quad (20)$$

span a six-dimensional representation of the conformal algebra (9). They are dynamical symmetries of the CBE (7), with $\lambda_{X_1^A} = -2t$, $\lambda_{Y_0^A} = \lambda_{Y_1^A} = 0$, see Eq. (15).

Case B1: $k = 0$, $a_0(u) = A_{110}u^{z/(1-z)}$, $z \neq 1$.

Theorem 2 *Let $z \neq 1$ and A_{110} be arbitrary constants. The generators (19) and*

$$
\begin{aligned}
X_1^{B1} &= -\left(t^2 + A_{110} r v^{(2z-1)/(1-z)} + \frac{A_{110}^2}{4\mu^2} v^{2z/(1-z)}\right)\partial_t \\
&\quad -\left(\frac{2}{z}tr + \mu\frac{z-2}{z}\frac{r^2}{v} + \frac{A_{110}}{\mu} r v^{z/(1-z)} + \frac{A_{110}^2}{4\mu^3} v^{(z+1)/(1-z)}\right)\partial_r \\
&\quad -\frac{2(1-z)}{z}(vt - \mu r)\partial_v - \frac{2}{z}xt + \frac{2}{z}\mu x\frac{r}{v} \\
Y_0^{B1} &= -\frac{A_{110}}{2} v^{z/(1-z)}\partial_t - \left(tv - \frac{\mu}{z}r + \frac{A_{110}}{2\mu} v^{1/(1-z)}\right)\partial_r - \frac{z-1}{z}\mu v\partial_v + \mu\frac{x}{z} \\
Y_1^{B1} &= -A_{110}(t v^{z/(1-z)} - \mu r v^{(2z-1)/(1-z)})\partial_t \\
&\quad -\left(t^2 v - \frac{2}{z}\mu t r - \mu^2\frac{z-2}{z}\frac{r^2}{v} + \frac{A_{110}}{\mu}(t v^{1/(1-z)} - \mu r v^{z/(1-z)})\right)\partial_r \\
&\quad -\frac{2}{z}(z-1)\mu(vt - \mu r)\partial_v + \frac{2}{z}\mu x t - \mu^2 x\frac{2}{z}\frac{r}{v}
\end{aligned}
\tag{21}
$$

span a representation of the conformal algebra (9). They act as symmetries of the CBE (7), with $\lambda_{X_1^{B1}} = -(2t + \frac{A_{110}}{\mu}v^{z/(1-z)})$, and $\lambda_{Y_0^{B1}} = \lambda_{Y_1^{B1}} = 0$.

The representation (20) is recovered by setting first $A_{110} = 0$. Afterwards, one may let $z \to 1$. When also $\nu = 1$, one finally recovers (9) with $k = 0$.

Case B2: $k \neq 0$, $a_0(u) = A_{12}u^{-1}$, z arbitrary. Consistency is achieved if $A_{12} = k = \mu$.

Theorem 3 *For an arbitrary constant z, the generators (19) and*

$$
\begin{aligned}
X_1^{B2} &= -\left(t^2 + \mu r^2 v^{-2}\right)\partial_t - \left(\frac{2}{z}tr + \frac{z + \mu(z-2)}{z}\frac{r^2}{v}\right)\partial_r \\
&\quad -\frac{2(1-z)}{z}(vt - \mu r)\partial_v - \frac{2}{z}xt + \frac{2\mu x}{z}\frac{r}{v} \\
Y_0^{B2} &= -\mu r v^{-1}\partial_t - \left(tv - \left(\frac{\mu}{z} - 1\right)r\right)\partial_r - \frac{z-1}{z}\mu v\partial_v + \mu\frac{x}{z} \\
Y_1^{B2} &= -\mu\left(\frac{2tr}{v} + (1-\mu)\frac{r^2}{v^2}\right)\partial_t - \left(t^2 v - \frac{2}{z}(z-\mu)tr + \right. \\
&\quad \left. + \frac{z(1-\mu) - (z-2)\mu^2}{z}\frac{r^2}{v}\right)\partial_r \\
&\quad -\frac{2}{z}(z-1)\mu(vt - \mu r)\partial_v + \frac{2}{z}\mu x t - \frac{2\mu^2 x}{z}\frac{r}{v}
\end{aligned}
\tag{22}
$$

span a representation of the Lie algebra (9) with $k = \mu = 1 - q$. They are symmetries of (7), with $\lambda_{X_1^{B2}} = -2(t + \frac{r}{z}v^{-1})$, $\lambda_{Y_0^{B2}} = -1$, $\lambda_{Y_1^{B2}} = -2\left(t + \frac{1}{z\mu}rv^{-1}\right)$.

Remark One might wonder if the representations (20), (21), (22), acting on functions $f = f(t, r, v)$, and having a dynamical exponent $z \neq 1$, could be extended to representations of an infinite-dimensional algebra, isomorphic to, say, $\mathfrak{vect}(S^1) \oplus \mathfrak{vect}(S^1)$. However, the answer turns out to be negative [14].

2.2 CBEs with External Forces

We now look at situations where in the ansatz (13) the function $\varphi(u) \neq 0$. An important open problem is the classification of solutions $\varphi(u)$. Here, we merely give some simple examples. The proofs are obtained through straightforward computation of the required commutators [14].

Example 1: $\Phi(u) = 0$. In this simple-looking case, we find

$$\begin{aligned}
a_0(u) &= \frac{k}{z}\frac{1}{u}, \quad b_0 = \frac{k}{z\mu} - \frac{\mu}{z}, \quad c_0(u) = 0, \quad d_0 = -\frac{\mu x}{z}, \\
a_{12}(u) &= \frac{k}{z^2}\frac{1}{u^2}, \quad b_{12}(u) = \frac{k-\mu^2}{\mu z^2}\frac{1}{u}, \quad c_{12}(u) = 0, \quad d_{12}(u) = -\frac{2\mu x}{z^2}\frac{1}{u}
\end{aligned} \tag{23}$$

and inserting into (17), we finally arrive at the generators

$$\begin{aligned}
X_{-1} &= -\partial_t \quad , \qquad X_0 = -t\partial_t - \frac{r}{z}\partial_r - \frac{1-z}{z}v\partial_v - \frac{x}{z} \\
X_1 &= -\left(t^2 + \frac{k}{z^2}\frac{r^2}{v^2}\right)\partial_t - \left(\frac{2}{z}tr + \frac{k-\mu^2}{z^2\mu}\frac{r^2}{v}\right)\partial_r \\
&\quad -(1-z)\left(\frac{2}{z}tv + \frac{k-\mu^2}{z^2\mu}r\right)\partial_v - \frac{2}{z}xt + \frac{2\mu x}{z^2}\frac{r}{v}, \\
Y_{-1} &= -v\partial_r - (1-z)r^{-1}v^2\partial_v, \\
Y_0 &= -\frac{k}{z}rv^{-1}\partial_t - \left(tv + \frac{k-\mu^2}{z\mu}r\right)\partial_r - (1-z)\left(tr^{-1}v^2 + \frac{k-\mu^2}{z\mu}v\right)\partial_v \\
&\quad + \frac{\mu x}{z}, \\
Y_1 &= -\left(\frac{2k}{z}trv^{-1} + \frac{k(k-\mu^2)}{z^2\mu}\frac{r^2}{v^2}\right)\partial_t - \left(t^2v + 2\frac{k-\mu^2}{z\mu}tr\right. \\
&\quad \left. + \frac{k(k-\mu^2)+\mu^4}{z^2\mu^2}\frac{r^2}{v}\right)\partial_r - (1-z)\left(t^2r^{-1}v^2 + 2\frac{k-\mu^2}{z\mu}tv\right. \\
&\quad \left. + \frac{k(k-\mu^2)+\mu^4}{z^2\mu^2}r\right)\partial_v + \frac{2}{z}\mu xt - \frac{2\mu^2 x}{z^2}\frac{r}{v}.
\end{aligned} \tag{24}$$

Theorem 4 *For an arbitrary constant z, the generators (24) span a representation of the Lie algebra (9), and act as dynamical symmetries of the following CBE*

$$\hat{B}f(t,r,v) = (\mu\partial_t + v\partial_r + (1-z)r^{-1}v^2\partial_v)f(t,r,v) = 0 \tag{25}$$

with $\lambda_{X_1} = -2\left(t + \frac{k}{z\mu}\frac{r}{v}\right)$, $\lambda_{Y_0} = -k/\mu$, $\lambda_{Y_1} = -2\left(\frac{k}{\mu}t + \frac{k}{z\mu^2}\frac{r}{v}\right)$.

Example 2: $k = 0$. Here, $\Phi(u)$ left arbitrary. This leads first to $a_0 = c_0 = 0$ and $b_0 = -\mu/z$, $d_0 = -\mu x/z$. From Eq. (17), taking into account $q = -\mu$, we obtain $a_{12}(u) = 0$ and

$$A(u) = 0, \quad B(u) = -\mu b_{12}(u), \quad C(u) = -\mu c_{12}(u), \quad D(u) = -\mu d_{12}. \tag{26}$$

Theorem 5 *Let* $\Phi(u) = (z-1)u^2 + \varphi(u)$ *and z be arbitrary. Form the generators*

$$\begin{aligned}
Y_{-1} &= -v\partial_r - (1-z)\left(r^{-1}v^2 + \frac{r^{1-2z}}{1-z}\Phi(u)\right)\partial_v = -v\partial_r - r^{1-2z}\varphi(u)\partial_v,\\
Y_0 &= -\left(tv - \frac{\mu}{z}r\right)\partial_r - (1-z)\left(\frac{r^{1-2z}}{1-z}\varphi(u)t - \frac{\mu}{z}v\right)\partial_v + \frac{\mu x}{z},\\
Y_1 &= -\left(t^2v - 2\frac{\mu}{z}tr - \mu r^{z+1}b_{12}(u)\right)\partial_r + \frac{2}{z}\mu xt + \mu r^z d_{12}(u)\\
&\quad -(1-z)\left(t^2\frac{r^{1-2z}}{1-z}\varphi(u) - \frac{2}{z}\mu tv - n\mu r^z v b_{12}(u) - \mu\frac{r^{1-2z}}{1-z}c_{12}(u)\right)\partial_v,\\
X_1 &= -t^2\partial_t - \left(\frac{2}{z}tr + r^{z+1}b_{12}(u)\right)\partial_r\\
&\quad -(1-z)\left(\frac{2}{z}tv + r^z v b_{12}(u) + \frac{r^{1-2z}}{1-z}c_{12}(u)\right)\partial_v - \frac{2}{z}xt - r^z d_{12}(u),\\
X_{-1} &= -\partial_t \quad , \qquad X_0 = -t\partial_t - \frac{r}{z}\partial_r - \frac{1-z}{z}v\partial_v - \frac{x}{z}
\end{aligned} \tag{27}$$

where the functions $\varphi(u)$, $b_{12}(u)$ *and* $d_{12}(u)$ *satisfy*

$$\begin{aligned}
&[(z-1)u^2 + \varphi(u)]^2 b_{12}''(u) + 3zu[(z-1)u^2 + \varphi(u)]b_{12}'(u)\\
&+z[(z+1)u^2 - 2u\varphi'(u) + 3\varphi(u)]b_{12}(u) + [(2-z)u - \varphi'(u)]2\mu/z = 0
\end{aligned} \tag{28}$$

$$zud_{12}(u) + [(z-1)u^2 + \varphi(u)]d_{12}'(u) + 2\mu x/z = 0 \tag{29}$$

and $c_{12}(u) = 2zub_{12}(u) + ((z-1)u^2 + \varphi(u))b_{12}'(u) + 2\mu/z$. *For any triplet of solutions of the system (28), (29), the generators (27) span a representation of the conformal algebra (9) and act as dynamical symmetries of the following CBE*

$$\hat{B}f(t,r,v) = (\mu\partial_t + v\partial_r + r^{1-2z}\varphi(u)\partial_v)f(t,r,v) = 0 \tag{30}$$

where $\lambda_{X_1} = -2t$ *and* $\lambda_{Y_0} = \lambda_{Y_1} = 0$*, for an arbitrary dynamical exponent* z*.*

A reasonable-looking "physical requirement" might be that the force term should only depend on the position r, that is $\varphi(u) = \varphi_0$ should be a constant. Then the representation (27) can be worked out. To do so, one must solve the system

$$
\begin{aligned}
& [(z-1)u^2 + \varphi_0]^2 b''_{12}(u) + 3zu[(z-1)u^2 + \varphi_0]b'_{12}(u) \\
& \qquad + z[(z+1)u^2 + 3\varphi_0]b_{12}(u) + 2\mu\frac{2-z}{z}u = 0 \qquad (31) \\
& zud_{12}(u) + [(z-1)u^2 + \varphi_0]d'_{12}(u) + 2\mu x/z = 0. \qquad (32)
\end{aligned}
$$

For any value of z, the solution of Eq. (32) is

$$
d_{12}(u) = -\delta_0[(z-1)u^2 + \varphi_0]^{\frac{z}{2(1-z)}} \int_R \mathrm{d}u\, [(z-1)u^2 + \varphi_0]^{\frac{z-2}{2(1-z)}}, \qquad (33)
$$

where δ_0 is a constant. The solution of the equation (31) for z arbitrary can be expressed in terms of hyper-geometric functions [14]. An elementary solution of the system (31), (32) exists for $z = 2$ and reads (b_{120}, b_{121} are constants)

$$
b_{12}(u) = b_{120}\frac{u}{(u^2+\varphi_0)^2} + b_{121}\frac{u^2-\varphi_0}{(u^2+\varphi_0)^2}, \quad d_{12}(u) = -\mu x\frac{u}{u^2+\varphi_0}. \qquad (34)
$$

Inserting this into Eq. (27), for $z = 2$, gives a finite-dimensional representation of the dynamical conformal symmetry algebra (9) of the collisionless Boltzmann equation

$$
\hat{B} f(t, r, v) = (\mu\partial_t + v\partial_r + \varphi_0 r^{-3}\partial_v) f(t, r, v) = 0 \qquad (35)
$$

with an external long-ranged force $F(r) = \varphi_0 r^{-3}$.

3 Conclusions

The collisionless Boltzmann equation (CBE), either without or with an external force, admits in $d = 1$ space dimension dynamical symmetries, whose Lie algebras are isomorphic to the $2D$ conformal Lie algebras (3), (9). Several new non-standard representations of the conformal Lie algebra, with generic values of the dynamical exponent z, were found: either (20), (21), (22) for the CBE (7) without an external force, or else (24), (27) for the CBEs (25), (30) with external forces, respectively. In the latter case, the auxiliary conditions (28), (29) on the force field $\varphi(u)$ lead for velocity-independent forces, and for $z = 2$, to the simple CBE (35). Physical consequences of these new symmetries remain to be studied.

Acknowledgements Most of this work was done during the visits of S.S. at Université de Lorraine Nancy and of M.H. at XI$^{\text{th}}$ International workshop *"Lie theories and its applications in physics"*. These visits was supported by PHC Rila. M.H. was partly supported by the Collège Doctoral Nancy-Leipzig-Coventry (Systèmes complexes à l'équilibre et hors équilibre) of UFA-DFH. S.S. has received partial support from the Bulgarian National Science Fund Grant DFNI-T02/6.

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On Reducibility Criterions for Scalar Generalized Verma Modules Associated to Maximal Parabolic Subalgebras

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Abstract In this short article we discuss reducibility criterions for scalar generalized Verma modules for maximal parabolic subalgebras of simply-laced Lie algebras.

Keywords Generalized Verma modules · Reducible points · Jantzen's criterion

2010 MSC: Primary17B10; Secondary22E46

Introduction

Generalized Verma modules (also known as parabolic Verma modules) are one of the central objects in representation theory of Lie algebras. The aim of this short paper is to introduce a simplification trick to determine the reducibility of so-called scalar generalized Verma modules, associated to maximal parabolic subalgebras for simply-laced Lie algebras.

Reducibility of generalized Verma modules is studied in a number of different settings in the literature such as the unitarity of cohomologically induced representations [6, 27, 29] and the roots of b-functions [7, 14, 26, 28]. It is also related to the study of homomorphisms between generalized Verma modules, intertwining differential operators between principal series representations, and branching laws [2, 4, 5, 10, 12, 15–22]. Recently reducibility of scalar generalized Verma modules associated to maximal parabolic subalgebras $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ is determined in [8] for $\mathfrak{u}$ abelian and in [9] for $\mathfrak{u}$ general. For more details on the reducibility of generalized Verma modules, consult, for instance, [9, Introduction].

A criterion due to Jantzen [13] is a very powerful tool to determine the reducibility of generalized Verma modules. Nonetheless, it is not easy to apply in general. In [6] and [9], the authors consider some algorithms and strategies that reduce the possibilities to apply Jantzen's criterion directly. In the strategies there is a step that requires tedious computations to carry out by hand for the Lie algebras of type E. The main result of this article is to introduce a trick to simplify such computations,

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V. Dobrev (ed.), *Lie Theory and Its Applications in Physics*,
Springer Proceedings in Mathematics & Statistics 191,
DOI 10.1007/978-981-10-2636-2_34

so that one can determine reducibility for type E Lie algebras by hand easily. This is done in Proposition 14. We note that in a number of cases the simplification trick even simplifies the strategy itself. (See Sect. 2.1.)

Now we briefly describe the rest of this paper. This paper consists of two sections. In Sect. 1, we quickly review generalized Verma modules and Jantzen's criterion. In this section a strategy to determine the reducibility is also recalled from [9] (Sect. 1.2). The simplification trick is discussed in Sect. 2. We also demonstrate how to apply the trick by a simple example in the section (Sect. 2.1).

1 Generalized Verma Modules and Jantzen's Criterion

The aim of this section is to briefly review generalized Verma modules and Jantzen's criterion. To do so we start by introducing the notation.

Let $\mathfrak{g}$ be a finite dimensional complex simple Lie algebra with rank greater than one. Fix a Cartan subalgebra $\mathfrak{h}$ and write $\Delta \equiv \Delta(\mathfrak{g}, \mathfrak{h})$ for the set of roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$. Let $\mathfrak{g}_\alpha$ denote the root space for $\alpha \in \Delta$. We choose a positive system Δ^+ of Δ and set $\mathfrak{b} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_\alpha$, a Borel subalgebra containing $\mathfrak{h}$. We denote by Π the set of simple roots of Δ^+.

Let $\langle \cdot, \cdot \rangle$ be the inner product on the weight space $\mathfrak{h}^*$ induced by the Killing form of $\mathfrak{g}$. For $\alpha \in \Delta$, we write $\alpha^\vee = 2\alpha/\langle \alpha, \alpha \rangle$, the coroot of α, and write s_α for the root reflection with respect to α. Let ρ denote half the sum of the positive roots of $\mathfrak{g}$.

For any ad($\mathfrak{h}$)-invariant proper subspace $V \subset \mathfrak{g}$, we denote by $\Delta(V)$ the set of roots α with $\mathfrak{g}_\alpha \subset V$. We set $\Delta^+(V) := \Delta^+ \cap \Delta(V)$ and $\Pi(V) := \Pi \cap \Delta(V)$. We write $\rho(V)$ for half the sum of the positive roots in $\Delta^+(V)$. For any Lie algebra $\mathfrak{s}$, we denote by $\mathcal{U}(\mathfrak{s})$ the universal enveloping algebra of $\mathfrak{s}$.

Fix a maximal parabolic subalgebra $\mathfrak{q}$ containing $\mathfrak{b}$ and write $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ for the Levi decomposition with $\mathfrak{l}$ the Levi factor and $\mathfrak{u}$ the nilpotent radical. For $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$, let $W(\mathfrak{l})$ denote the subgroup of the Weyl group W of $\mathfrak{g}$ generated by $\{s_\alpha \ : \ \alpha \in \Pi(\mathfrak{l})\}$.

It is well-known that there is a one-to-one correspondence between standard maximal parabolic subalgebras and simple roots. Let α_0 be the simple root that corresponds to $\mathfrak{q}$. If λ_0 is the fundamental weight for α_0, then $\Delta(\mathfrak{l})$ and $\Delta(\mathfrak{u})$ may be given by

$$\Delta(\mathfrak{l}) = \{\beta \in \Delta \ : \ \langle \lambda_0, \beta \rangle = 0\} \quad \text{and} \quad \Delta(\mathfrak{u}) = \{\beta \in \Delta \ : \ \langle \lambda_0, \beta \rangle > 0\}. \tag{1}$$

Now we set

$$\mathbf{P}^+_{\mathfrak{l}} := \{\nu \in \mathfrak{h}^* \ : \ \langle \nu, \alpha^\vee \rangle \in 1 + \mathbb{Z}_{\geq 0} \text{ for all } \alpha \in \Pi(\mathfrak{l})\}.$$

For $\nu \in \mathbf{P}^+_{\mathfrak{l}}$, let $E(\nu - \rho)$ be the finite dimensional simple $\mathcal{U}(\mathfrak{l})$-module with highest weight $\nu - \rho$. Extend $E(\nu - \rho)$ to be a $\mathcal{U}(\mathfrak{q})$-module by letting $\mathfrak{u}$ act trivially. Then define the *generalized Verma module* $M_\mathfrak{q}(\nu)$ with highest weight $\nu - \rho$ [24] by means of

$$M_{\mathfrak{q}}(\nu) := \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{q})} E(\nu - \rho).$$

Define

$$\mathbf{P}^{+}_{\mathfrak{l}}(1) = \{\nu \in \mathfrak{h}^* \ : \ \langle \nu, \alpha^\vee \rangle = 1 \text{ for all } \alpha \in \Pi(\mathfrak{l})\}. \tag{2}$$

It is easy to see that $\dim_{\mathbb{C}}(E(\nu - \rho)) = 1$ if and only if $\nu \in \mathbf{P}^{+}_{\mathfrak{l}}(1)$. In this case the generalized Verma module $M_{\mathfrak{q}}(\nu)$ is called a *scalar* generalized Verma module [1].

Observe that, as $\mathfrak{g}$ has rank greater than one and $\mathfrak{q}$ is a maximal parabolic subalgebra determined by α_0, the set $\mathbf{P}^{+}_{\mathfrak{l}}(1)$ is given by

$$\mathbf{P}^{+}_{\mathfrak{l}}(1) = \{t\lambda_0 + \rho(\mathfrak{l}) \ : \ t \in \mathbb{C}\}.$$

We set

$$\Theta_t := t\lambda_0 + \rho(\mathfrak{l}) \ \text{ with } t \in \mathbb{C}.$$

Then any scalar generalized Verma modules of $\mathfrak{q}$ may be parametrized by $t \in \mathbb{C}$ as

$$M_{\mathfrak{q}}[t] \equiv M_{\mathfrak{q}}(\Theta_t) = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{q})} \mathbb{C}_{\Theta_t - \rho} \tag{3}$$

with infinitesimal character Θ_t. Moreover, since $\rho(\mathfrak{u}) = \rho - \rho(\mathfrak{l})$, we have $\rho(\mathfrak{u}) \in \mathfrak{z}(\mathfrak{l})^*$ with $\mathfrak{z}(\mathfrak{l})$ the center of $\mathfrak{l}$ and $\mathfrak{z}(\mathfrak{l})^*$ the dual of $\mathfrak{z}(\mathfrak{l})$; thus, $\rho(\mathfrak{u}) = c_0\lambda_0$ for some $c_0 \in \mathbb{C}$. Therefore the scalar generalized Verma module $M_{\mathfrak{q}}[t]$ may be expressed as

$$M_{\mathfrak{q}}[t] = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{q})} \mathbb{C}_{(t-c_0)\lambda_0}$$

with infinitesimal character

$$\Theta_t = (t - c_0)\lambda_0 + \rho.$$

Observe that since the weight $2\rho(\mathfrak{u})$ is integral and $\langle \rho(\mathfrak{u}), \alpha_0^\vee \rangle \geq 1$, we have $c_0 \in \frac{1}{2}\mathbb{Z}_{>0} = (\frac{1}{2} + \mathbb{Z}_{\geq 0}) \cup (1 + \mathbb{Z}_{\geq 0})$.

1.1 Jantzen's Criterion

In [13], Jantzen introduced a very powerful criterion that determines whether or not a given generalized Verma module is irreducible. Although the criterion works for any generalized Verma modules, we only state here the specialization of the criterion to the present situation. For the general statement of Jantzen's criterion see, for instance, [13, Satz 3] or [11, Theorem 9.13].

In order to state the Jantzen's criterion it is important to introduce the following notation. For $\lambda \in \mathfrak{h}^*$, define

$$Y(\lambda) := D^{-1} \sum_{w \in W(\mathfrak{l})} (-1)^{\ell(w)} e^{w\lambda}, \tag{4}$$

where $\ell(w)$ denotes the length of $w \in W(\mathfrak{l})$, e^{μ} is a function on $\mathfrak{h}^*$ which takes values 1 at μ and 0 elsewhere, and $D = e^{\rho} \prod_{\alpha \in \Delta^+}(1 - e^{-\alpha})$ is the Weyl denominator.

Proposition 1 below shows some properties of $Y(\lambda)$.

Proposition 1 *([23, Corollary A.1.5] and [25, Corollary 2.2.10]) We have the following two properties:*

(1) If $\lambda \in \mathfrak{h}^*$ satisfies $\langle \lambda, \alpha \rangle = 0$ for some $\alpha \in \Delta(\mathfrak{l})$, then $Y(\lambda) = 0$. Conversely, if $\lambda \in \mathfrak{h}^*$ satisfies $\langle \lambda, \alpha^\vee \rangle \in \mathbb{Z} \setminus \{0\}$ for all $\alpha \in \Delta(\mathfrak{l})$, then $Y(\lambda) \neq 0$.
(2) For $\lambda \in \mathfrak{h}^*$ and $w \in W(\mathfrak{l})$, we have $Y(w\lambda) = (-1)^{\ell(w)} Y(\lambda)$.

If

$$S_t := \{\beta \in \Delta(\mathfrak{u}) \ : \ \langle \Theta_t, \beta^\vee \rangle \in 1 + \mathbb{Z}_{\geq 0}\}, \tag{5}$$

$$R_t := \left\{\beta \in S_t \ : \ Y(s_\beta(\Theta_t)) \neq 0\right\}, \tag{6}$$

then Jantzen's criterion for scalar generalized Verma modules associated to a maximal parabolic subalgebra $\mathfrak{q}$ reads as follows. This specialization of the criterion is motivated by [25, Theorem 2.2.11].

Theorem 2 (Jantzen's criterion) ([13, Satz 3]) Let $\mathfrak{q}$ be a maximal parabolic subalgebra. Then the scalar generalized Verma module $M_{\mathfrak{q}}(\Theta_t)$ is irreducible if and only if $R_t = \emptyset$ or

$$\sum_{\beta \in R_t} Y(s_\beta(\Theta_t)) = 0.$$

To use Jantzen's criterion we need to determine whether or not $\sum_{\beta \in R_t} Y(s_\beta(\Theta_t))$ is zero. Then it is useful to know when terms $Y(s_\beta(\Theta_t))$ cancel out in $\sum_{\beta \in R_t} Y(s_\beta(\Theta_t))$. Proposition 3 below deals with this issue. For the proof, see, for instance, [23, Proposition A.2.4].

Proposition 3 *Suppose that $R_t \neq \emptyset$. Then the sum $\sum_{\beta \in R_t} Y(s_\beta \Theta_t)$ is zero if and only if $R_t = \{\beta_1, \beta_2, \ldots, \beta_k\} \cup \{\delta_1, \delta_2, \ldots, \delta_k\}$ with odd $w_1, w_2, \ldots w_k \in W(\mathfrak{l})$ so that $s_{\beta_i}\Theta_t = w_i s_{\delta_i}\Theta_t$ for all i.*

Corollary 4 ([9, Corollary 1.11]) *If there exists $\beta \in R_t$ so that no $\delta \in R_t \setminus \{\beta\}$ satisfies $\langle \Theta_t, \delta \rangle \langle \lambda_0, \delta^\vee \rangle = \langle \Theta_t, \beta \rangle \langle \lambda_0, \beta^\vee \rangle$, then $M_{\mathfrak{q}}[t]$ is reducible.*

Remark 5 In the type E cases the converse of Corollary 4 turns out to hold (but only after doing all the computations). See [9, Sect. 3].

1.2 Strategy

Although Jantzen's criterion is very powerful, it is in general not easy to determine whether or not $\sum_{\beta \in R_t} Y(s_\beta(\Theta_t))$ is zero. The purpose of this section is to recall from [9] a useful strategy that reduces the number of parameters $t \in \mathbb{C}$ for $M_\mathfrak{q}[t] \equiv M_\mathfrak{q}(\Theta_t)$ that need to be checked by Jantzen's criterion directly.

We start by recalling the following two facts.

Lemma 6 ([9, Lemma 1.3]) *If* $t \leq 0$, *then* $M_\mathfrak{q}[t]$ *is irreducible.*

Fact 7 ([9, Sect. 1.3]) There are a finite number of families of the form $q + \mathbb{Z}_{\geq 0}$, $q \in \mathbb{Q}$, containing all reducible points.

Now we give the strategy as follows; the strategy consists of three steps.

Step 1: Determine the families

$$q + \mathbb{Z}_{\geq 0} \text{ for which } \langle \Theta_t, \beta^\vee \rangle \in \mathbb{Z}, \text{ for some } \beta \in \Delta(\mathfrak{u}),\ t \in q + \mathbb{Z}_{\geq 0}. \tag{7}$$

By Lemma 6 and Fact 7, we assume that $q \in \mathbb{Q} \cap (0, 1]$. Lemma 8 below shows that we only need to check reducibility of $M_\mathfrak{q}[t]$ for a finite number of values of t.

Lemma 8 ([9, Lemma 3.2]) If $t \geq c_0$ and $t \in q + \mathbb{Z}_{\geq 0}$ (q as in (7)), then $M_\mathfrak{q}[t]$ is reducible.

Step 2: For each $t = q + m,\ m \in \mathbb{Z}_{\geq 0}$ (as in (7)) with $t < c_0$, one can determine if Corollary 4 applies.
Step 3: For the remaining values of t, we apply the following algorithm.

(i) Determine whether or not $R_t = \emptyset$. If $R_t = \emptyset$, then $M_\mathfrak{q}[t]$ is irreducible.
(ii) If $R_t \neq \emptyset$, then set $R_{t,0} := R_t$. Take $\beta \in R_{t,0}$ and look for $\delta \in R_{t,0} \setminus \{\beta\}$ so that $ws_\delta(\Theta_t) = s_\beta(\Theta_t)$ for some $w \in W(\mathfrak{l})$ of odd length. If there is such δ, then set $R_{t,1} := R_{t,0} \setminus \{\beta, \delta\}$. If $R_{t,1} = \emptyset$, then $M_\mathfrak{q}[t]$ is irreducible. Otherwise, do the above process for $R_{t,1}$. If this process can be repeated until $R_{t,k} = \emptyset$ for some $k \in \mathbb{Z}_{\geq 0}$, then $M_\mathfrak{q}[t]$ is irreducible. Otherwise it is reducible.

2 Simplification Tricks

Applying Proposition 3 (or (ii) in Step 3 in the strategy) requires some tedious computations for the Lie algebras of type E. In this section we discuss some tricks that simplify such computations. We resume the notation from the previous sections, but $\mathfrak{g}$ is further assumed to be a simply-laced Lie algebra, unless otherwise specified.

For $\beta \in \Delta$, let $\mathrm{ht}(\beta)$ be the height of β, namely, if $\beta = \sum_{\alpha \in \Pi} m_\alpha \alpha$, then $\mathrm{ht}(\beta) = \sum_{\alpha \in \Pi} m_\alpha$. Since Δ is simply-laced, we have $\langle \rho, \beta^\vee \rangle = \mathrm{ht}(\beta)$.

Observe from (1) that $\Delta(\mathfrak{u})$ may be given as $\Delta(\mathfrak{u}) = \bigcup_{j\geq 1} \Delta(j)$, where $\Delta(j) := \{\beta \in \Delta : \langle \lambda_0, \beta^\vee \rangle = j\}$. We define γ_j to be the root in $\Delta(j)$ so that, for all $\beta \in \Delta(j)$, we have $\mathrm{ht}(\gamma_j) \geq \mathrm{ht}(\beta)$, the root in $\Delta(j)$ of greatest height. Note that such γ_j is unique for each $j > 0$, as $\mathfrak{q}$ is maximal.

For each $j \geq 1$, we set $S_t(j) := S_t \cap \Delta(j)$. Since $\Theta_t = (t-c)\lambda_0 + \rho$, it follows from (5) that

$$S_t(j) = \{\beta \in \Delta(j) : (t-c)j + \mathrm{ht}(\beta) \in \mathbb{Z}_{>0}\}. \tag{8}$$

By (8), it is clear that $S_t(j) \neq \emptyset$ if and only if $\gamma_j \in S_t(j)$.

Proposition 9 *Given $t_0 \in \mathbb{C}$, suppose that $S_{t_0} = \bigcup_{k=1}^{r} S_{t_0}(j_k)$ with $j_1 < \cdots < j_r$ and $S_{t_0}(j_k) \neq \emptyset$ for all k. If*

(a) $Y(s_{\gamma_{j_r}}(\Theta_{t_0})) \neq 0$ and
(b) $\langle \Theta_{t_0}, \gamma_{j_r} \rangle \geq \langle \Theta_{t_0}, \gamma_{j_k} \rangle$ for all $k = 1, \ldots, r-1$,

then $M_{\mathfrak{q}}[t_0]$ is reducible.

Proof By Corollary 4, it suffices to check that there does not exists $\delta \in R_t \setminus \{\gamma_{j_r}\}$ so that $\langle \lambda_0, \delta^\vee \rangle \langle \Theta_t, \delta \rangle = \langle \lambda_0, \gamma_{j_k}^\vee \rangle \langle \Theta_t, \gamma_{j_r} \rangle$ under the hypothesis (b). Suppose that such $\delta \in S_t$ does exist. Then $\delta \in S_t(j_k)$ for some j_k. Therefore,

$$\begin{aligned}\langle \lambda_0, \delta^\vee \rangle \langle \Theta_t, \delta \rangle &= j_k \langle \Theta_t, \delta \rangle \\ &\leq j_k \langle \Theta_t, \gamma_{j_k} \rangle \text{ since } \gamma_{j_k} \text{ has maximal height in } \Delta(j_k) \\ &\leq \langle \lambda_0, \gamma_{j_r}^\vee \rangle \langle \Theta_t, \gamma_{j_r} \rangle \text{ by hypothesis } (b).\end{aligned}$$

The inequality is strict unless $j_k = j_r$, in which case we would have $\mathrm{ht}(\delta) = \mathrm{ht}(\gamma_{j_r})$, so $\delta = \gamma_{j_r}$ as γ_{j_r} is a unique root with height $\mathrm{ht}(\gamma_{j_r})$. This contradicts the choice of δ.

Remark 10 If $S_t = S_t(j_r) \neq \emptyset$, then there is nothing to check for Condition (b); the identity $\langle \lambda_0, \delta^\vee \rangle \langle \Theta_t, \delta \rangle = \langle \lambda_0, \gamma_{j_k}^\vee \rangle \langle \Theta_t, \gamma_{j_r} \rangle$ never holds for $\delta \in S_t(j_r) \setminus \{\gamma_{j_r}\}$.

By using the simply-lacedness, one may give a sufficient condition for Condition (a) of Proposition 9. To show this, we first give the following well-known fact. See, for instance, [11, Theorem 9.12].

Proposition 11 *Suppose that $\mathfrak{g}$ is a complex simple Lie algebra (not necessarily simply-laced). Let $\lambda - \rho$ be dominant integral for $\Delta^+(\mathfrak{l})$. If $\langle \lambda, \beta^\vee \rangle \notin \mathbb{Z}_{>0}$ for all $\beta \in \Delta(\mathfrak{u})$, then $M_{\mathfrak{q}}(\lambda)$ is irreducible. The converse also holds if λ is regular.*

Lemma 12 [23, Proposition A.4.4] Suppose that $\mathfrak{g}$ is simply-laced and let $\alpha \in \Delta(\mathfrak{l})$ and $\beta \in S_t$. Then $\langle s_\beta(\Theta_t), \alpha^\vee \rangle = 0$ if and only if $\beta - \alpha \in \Delta$ and $\mathrm{ht}(\alpha) = \langle \Theta_t, \beta^\vee \rangle$.

Observe that if $t \in t_0 + \mathbb{Z}_{\geq 0}$, then $\langle \Theta_t, \beta^\vee \rangle \geq \langle \Theta_{t_0}, \beta^\vee \rangle$. Indeed, we have

$$\langle \Theta_t, \beta^\vee \rangle = (t-c) + \mathrm{ht}(\beta) \geq (t_0 - c) + \mathrm{ht}(\beta) = \langle \Theta_{t_0}, \beta^\vee \rangle.$$

Now we set

$$\mathrm{ht}(\mathfrak{l}) := \max\{\mathrm{ht}(\alpha) : \alpha \in \Delta^+(\mathfrak{l})\}.$$

Lemma 13 *Suppose that* $\beta \in S_{t_0}$. *If* $\langle \Theta_{t_0}, \beta^\vee \rangle > \mathrm{ht}(\mathfrak{l})$*, then* $Y(s_\beta(\Theta_t)) \neq 0$ *for all* $t \in t_0 + \mathbb{Z}_{\geq 0}$.

Proof Let $t \in t_0 + \mathbb{Z}_{>0}$. Since $\langle \Theta_{t_0}, \beta^\vee \rangle > \mathrm{ht}(\mathfrak{l})$, by the observation above, we have $\langle \Theta_t, \beta^\vee \rangle > \mathrm{ht}(\mathfrak{l})$. In particular, $\langle \Theta_t, \beta^\vee \rangle > \mathrm{ht}(\alpha)$ for all $\alpha \in \Delta(\mathfrak{l})$. It then follows from Lemma 12 that $\langle s_\beta(\Theta_t), \alpha^\vee \rangle \neq 0$ for all $\alpha \in \Delta(\mathfrak{l})$. Now Proposition 1 concludes the lemma.

We now state a condition for reducibility for $M_\mathfrak{q}[t] \equiv M_\mathfrak{q}(\Theta_t)$ that applies in the simply-laced cases and is easy to apply.

Proposition 14 *Given* $t_0 \in \mathbb{C}$*, suppose that* $S_{t_0} = \bigcup_{k=1}^r S_{t_0}(j_k)$ *with* $j_1 < \cdots < j_r$ *and* $S_{t_0}(j_k) \neq \emptyset$ *for all* k. *If*

(a) $\langle \Theta_{t_0}, \gamma_{j_r}^\vee \rangle > \mathrm{ht}(\mathfrak{l})$ *and*
(b) $\langle \Theta_{t_0}, \gamma_{j_r} \rangle \geq \langle \Theta_{t_0}, \gamma_{j_k} \rangle$ *for all* $k = 1, \ldots, r-1$,

then $M_\mathfrak{q}[t]$ *is reducible for all* $t \in t_0 + \mathbb{Z}_{\geq 0}$.

Proof Since the condition $t \in t_0 + \mathbb{Z}_{\geq 0}$ implies that $\langle \Theta_t, \beta^\vee \rangle \geq \langle \Theta_{t_0}, \beta^\vee \rangle$, this is an immediate consequence of Proposition 9 and Lemma 13.

2.1 Example

We now demonstrate how to apply the simplification tricks in a simple example. Let $\mathfrak{g}$ be of type E_7 and $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ be the maximal parabolic subalgebra corresponding to the simple root α_3. (We use the Bourbaki convention [3] for the numbering of the simple roots.) In this case $\Delta(\mathfrak{u}) = \bigcup_{j=1}^3 \Delta(j)$ and c_3 for $\rho(\mathfrak{u}) = c_3\lambda_3$ is $c_3 = 11/2$. By inspection we have $\mathrm{ht}(\gamma_3) = 17$, $\mathrm{ht}(\gamma_2) = 15$, $\mathrm{ht}(\gamma_1) = 10$, and $\mathrm{ht}(\mathfrak{l}) = 5$.

A simple observation shows that if $M_\mathfrak{q}[t]$ is reducible, then there exists $\beta_0 \in \Delta(\mathfrak{u})$ and $k \in 1 + \mathbb{Z}_{\geq 0}$ so that

$$t = \frac{k - \langle \rho, \beta_0^\vee \rangle}{\langle \lambda_3, \beta_0^\vee \rangle} + c_3$$

(see Sect. 1.3 of [9]); in particular, $t \in \frac{1}{6}\mathbb{Z}_{>0}$. It thus suffices to consider $t \in \frac{1}{6}\mathbb{Z}_{>0}$.

One can easily check that $S_t \neq \emptyset$ if and only if $t \in (\frac{1}{6} + \mathbb{Z}_{\geq 0}) \cup (\frac{1}{2} + \mathbb{Z}_{\geq 0}) \cup (\frac{5}{6} + \mathbb{Z}_{\geq 0}) \cup (1 + \mathbb{Z}_{\geq 0})$ with S_t as follows:

- $t \in \frac{1}{6} + \mathbb{Z}_{\geq 0}$: $S_t = S_t(3)$;
- $t \in \frac{1}{2} + \mathbb{Z}_{\geq 0}$: $S_t = \bigcup_{k=1}^3 S_t(k)$;
- $t \in \frac{5}{6} + \mathbb{Z}_{\geq 0}$: $S_t = S_t(3)$;

– $t \in 1 + \mathbb{Z}_{\geq 0}$: $S_t = S_t(2)$.

For the sake of simplicity we exhibit the case of $t \in \frac{1}{6} + \mathbb{Z}_{\geq 0}$ only; one can proceed in the other cases similarly. Note that in this case, as $S_t = S_t(3)$, there is nothing to check in the hypothesis (b) in Proposition 14.

Claim: For $t \in \frac{1}{6} + \mathbb{Z}_{\geq 0}$, $M_{\mathfrak{q}}[t]$ is reducible if and only if $t = \frac{7}{6} + \mathbb{Z}_{\geq 0}$.

Write $t = \frac{1}{6} + m$ with $m \in \mathbb{Z}_{\geq 0}$. Observe that if $\langle \Theta_t, \gamma_2^\vee \rangle > \mathrm{ht}(\mathfrak{l})$, then $m \geq 2$. Thus, by Proposition 14, $M_{\mathfrak{q}}[t]$ is reducible for all $t \in \frac{1}{6} + 2 + \mathbb{Z}_{\geq 0}$. We then need to check $t = \frac{7}{6}$ and $t = \frac{1}{6}$.

(1) $t = \frac{7}{6}$: We claim that $\Theta_{\frac{7}{6}}$ is regular. Indeed, for $\beta \in \Delta(k) \subset \Delta(\mathfrak{u})$ for $k = 1, 2, 3$, we have $\langle \Theta_{\frac{7}{6}}, \beta^\vee \rangle = -\frac{13}{3}k + \mathrm{ht}(\beta)$. Thus, if $\langle \Theta_{\frac{7}{6}}, \beta^\vee \rangle = 0$, then $k = 3$ and $\mathrm{ht}(\beta) = 13$. However, one can easily check that $\mathrm{ht}(\beta) \geq 16$ for all $\beta \in \Delta(3)$. Thus, $\langle \Theta_{\frac{7}{6}}, \beta^\vee \rangle \neq 0$ for all $\beta \in \Delta(\mathfrak{u})$. Since $\langle \Theta_{\frac{7}{6}}, \alpha^\vee \rangle = \mathrm{ht}(\alpha)$ for all $\alpha \in \Delta(\mathfrak{l})$, this shows that $\Theta_{\frac{7}{6}}$ is regular. Now Proposition 11 concludes that $M_{\mathfrak{q}}[\frac{7}{6}]$ is reducible.

(2) $t = \frac{1}{6}$: We claim that $R_{\frac{1}{6}} = \emptyset$. First it is easy to check that $S_{\frac{1}{6}} = \{\gamma_3\}$. Since $\gamma_3 - \alpha_1 \in \Delta$ and $\langle \Theta_{\frac{1}{6}}, \gamma_3^\vee \rangle = \mathrm{ht}(\alpha_1) = 1$, by Lemma 12, we have $\langle s_{\gamma_3}(\Theta_{\frac{1}{6}}), \alpha_1^\vee \rangle = 0$. It follows from Proposition 1 that $Y(s_{\gamma_3}(\Theta_{\frac{1}{6}})) = 0$; in particular, $R_{\frac{1}{6}} = \emptyset$. Now Jantzen's criterion concludes that $M_{\mathfrak{q}}[\frac{1}{6}]$ is irreducible.

By similar arguments one can show that $M_{\mathfrak{q}}[t]$ is reducible if and only if $t \in (\frac{7}{6} + \mathbb{Z}_{\geq 0}) \cup (\frac{3}{2} + \mathbb{Z}_{\geq 0}) \cup (\frac{5}{6} + \mathbb{Z}_{\geq 0}) \cup (1 + \mathbb{Z}_{\geq 0})$.

Acknowledgements The author is thankful to Roger Zierau for his invaluable comments on this manuscript.

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Part V
Supersymmetry and Quantum Groups

On Finite W-Algebras for Lie Superalgebras in Non-Regular Case

Elena Poletaeva

Abstract We study finite W-algebras associated to non-regular nilpotent elements for the queer Lie superalgebra $Q(n)$. We give an explicit presentation of the finite W-algebra for $Q(3)$

1 Introduction

A finite W-algebra is a certain associative algebra attached to a pair $(\mathfrak{g}, e)$, where $\mathfrak{g}$ is a complex semi-simple Lie algebra and $e \in \mathfrak{g}$ is a nilpotent element. It is a generalization of the universal enveloping algebra $U(\mathfrak{g})$.

Finite W-algebras for semi-simple Lie algebras were introduced by A. Premet [17] (see also [7]). In the case of Lie superalgebras, finite W-algebras have been extensively studied by mathematicians and physicists in [1, 2, 12–16, 20–22]. E. Ragoucy and P. Sorba first observed that in the case when $\mathfrak{g}$ is the general linear Lie algebra and e consists of n Jordan blocks each of the same size l, the finite W-algebra for $\mathfrak{g}$ is isomorphic to the truncated Yangian of level l associated to $\mathfrak{gl}(n)$, which is a certain quotient of the Yangian Y_n for $\mathfrak{gl}(n)$ [19]. J. Brundan and A. Kleshchev generalized this result to an arbitrary nilpotent e, and obtained a realization of the finite W-algebra for the general linear Lie algebra as a quotient of a so-called shifted Yangian [3] (see also [4]).

For the general linear Lie superalgebra $\mathfrak{g} = \mathfrak{gl}(m|n)$, a connection between finite W-algebras for $\mathfrak{g}$ and super-Yangians was firstly observed by C. Briot and E. Ragoucy [1]. In a more recent article, J. Brown, J. Brundan and S. Goodwin described principal finite W-algebras for $\mathfrak{gl}(m|n)$ associated to regular nilpotent e as truncations of shifted super-Yangians of $\mathfrak{gl}(1|1)$ [2]. After that, Y. Peng described the finite W-algebra for $\mathfrak{g} = \mathfrak{gl}(m|n)$ associated to an e in the case when the Jordan type of e satisfies the

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V. Dobrev (ed.), *Lie Theory and Its Applications in Physics*,
Springer Proceedings in Mathematics & Statistics 191,
DOI 10.1007/978-981-10-2636-2_35

following condition: $e = e_m \oplus e_n$, where e_m is principal nilpotent in $\mathfrak{gl}(m|0)$ and the sizes of the Jordan blocks of e_n are all greater or equal to m [12].

In [13] we obtained the precise description of the principal finite W-algebras for classical Lie superalgebras of Type I and defect one. In [14] we described principal finite W-algebras for certain orthosymplectic Lie superalgebras and obtained partial results for the exceptional Lie superalgebra $F(4)$. In [15] we described principal finite W-algebras for the family of simple exceptional Lie superalgebras $D(2, 1; \alpha)$ and for the universal central extension of $\mathfrak{psl}(2|2)$. In [16] we studied in detail the queer Lie superalgebra $Q(n)$ in the regular case. In particular, we proved that the principal finite W-algebra for $Q(n)$ is isomorphic to a quotient of the super-Yangian of $Q(1)$.

An interesting problem is to extend the result of J. Brundan and A. Kleshchev to $Q(n)$ associated to an arbitrary nilpotent element. An attempt should be made in the case when the Jordan blocks of the nilpotent are of the same size. We formulate a conjecture about this case in Sect. 5. In Sect. 3.1 we consider the finite W-algebra associated to a nilpotent with a different type of Jordan blocks. In Sect. 4.1 we explicitly describe the finite W-algebra for $Q(3)$.

2 Preliminaries

Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be a Lie superalgebra with reductive even part $\mathfrak{g}_{\bar{0}}$. Let $\chi \in \mathfrak{g}_{\bar{0}}^* \subset \mathfrak{g}^*$ be an even nilpotent element in the coadjoint representation, i.e., the closure of the $G_{\bar{0}}$-orbit of χ in $\mathfrak{g}_{\bar{0}}^*$ (where $G_{\bar{0}}$ is the algebraic reductive group of $\mathfrak{g}_{\bar{0}}$) contains zero.

Definition 1 *The annihilator of χ in $\mathfrak{g}$ is*

$$\mathfrak{g}^\chi = \{x \in \mathfrak{g} \mid \chi([x, \mathfrak{g}]) = 0\}.$$

Definition 2 *A good $\mathbb{Z}$-grading for χ is a $\mathbb{Z}$-grading $\mathfrak{g} = \oplus_{j\in\mathbb{Z}}\mathfrak{g}_j$* satisfying the following two conditions:

(1) $\chi(\mathfrak{g}_j) = 0$ if $j \neq -2$;
(2) $\mathfrak{g}^\chi$ belongs to $\bigoplus_{j\geq 0} \mathfrak{g}_j$.

Note that $\chi([\cdot, \cdot])$ defines a non-degenerate skew-symmetric even bilinear form on $\mathfrak{g}_{-1}$. Let $\mathfrak{l}$ be a maximal isotropic subspace with respect to this form. We consider a nilpotent subalgebra $\mathfrak{m} = (\oplus_{j\leq -2}\mathfrak{g}_j) \bigoplus \mathfrak{l}$ of $\mathfrak{g}$. The restriction of χ to $\mathfrak{m}$, $\chi : \mathfrak{m} \longrightarrow \mathbb{C}$, defines a one-dimensional representation $\mathbb{C}_\chi =< v >$ of $\mathfrak{m}$. Let I_χ be the left ideal of $U(\mathfrak{g})$ generated by $a - \chi(a)$ for all $a \in \mathfrak{m}$.

Definition 3 The induced $\mathfrak{g}$-module

$$Q_\chi := U(\mathfrak{g}) \otimes_{U(\mathfrak{m})} \mathbb{C}_\chi \cong U(\mathfrak{g})/I_\chi$$

is called *the generalized Whittaker module*.

Definition 4 *The finite W-algebra* associated to the nilpotent element χ is

$$W_\chi := \mathrm{End}_{U(\mathfrak{g})}(Q_\chi)^{op}.$$

As in the Lie algebra case, the superalgebras W_χ are all isomorphic for different choices of good $\mathbb{Z}$-gradings and maximal isotropic subspaces $\mathfrak{l}$ (see [22]). If $\mathfrak{g}$ admits an even non-degenerate $\mathfrak{g}$-invariant supersymmetric bilinear form, then $\mathfrak{g} \simeq \mathfrak{g}^*$ and $\chi(x) = (e|x)$ for some nilpotent $e \in \mathfrak{g}_{\bar{0}}$ (i.e. ad$e$ is a nilpotent endomorphism of $\mathfrak{g}$). By the Jacobson–Morozov theorem e can be included in $\mathfrak{sl}(2) =< e, h, f >$. As in the Lie algebra case, the linear operator ad$h$ defines a Dynkin $\mathbb{Z}$-grading $\mathfrak{g} = \bigoplus_{j\in\mathbb{Z}} \mathfrak{g}_j$, where

$$\mathfrak{g}_j = \{x \in \mathfrak{g} \mid \mathrm{ad}h(x) = jx\}.$$

As follows from the representation theory of $\mathfrak{sl}(2)$, the Dynkin $\mathbb{Z}$-grading is good for χ. Let $\mathfrak{g}^e := \mathrm{Ker}(\mathrm{ad}e)$. Clearly, $\mathfrak{g}^e = \mathfrak{g}^\chi$. Note that as in the Lie algebra case, $\dim \mathfrak{g}^e = \dim \mathfrak{g}_0 + \dim \mathfrak{g}_1$.

Note that by Frobenius reciprocity

$$\mathrm{End}_{U(\mathfrak{g})}(Q_\chi) = \mathrm{Hom}_{U(\mathfrak{m})}(\mathbb{C}_\chi, Q_\chi).$$

That defines an identification of W_χ with the subspace

$$Q_\chi^{\mathfrak{m}} = \{u \in Q_\chi \mid au = \chi(a)u \text{ for all } a \in \mathfrak{m}\}.$$

In what follows we denote by $\pi : U(\mathfrak{g}) \to U(\mathfrak{g})/I_\chi$ the natural projection. By above

$$W_\chi = \{\pi(y) \in U(\mathfrak{g})/I_\chi \mid (a - \chi(a))y \in I_\chi \text{ for all } a \in \mathfrak{m}\},$$

or, equivalently,

$$W_\chi = \{\pi(y) \in U(\mathfrak{g})/I_\chi \mid \mathrm{ad}(a)y \in I_\chi \text{ for all } a \in \mathfrak{m}\}. \tag{2.1}$$

The algebra structure on W_χ is given by

$$\pi(y_1)\pi(y_2) = \pi(y_1 y_2)$$

for $y_i \in U(\mathfrak{g})$ such that $\mathrm{ad}(a)y_i \in I_\chi$ for all $a \in \mathfrak{m}$ and $i = 1, 2$.

Definition 5 A nilpotent $\chi \in \mathfrak{g}_{\bar{0}}^*$ is called *regular* nilpotent if $G_{\bar{0}}$-orbit of χ has maximal dimension, i.e. the dimension of $\mathfrak{g}_{\bar{0}}^\chi$ is minimal. (Equivalently, a nilpotent $e \in \mathfrak{g}_{\bar{0}}$ is *regular* nilpotent, if $\mathfrak{g}_{\bar{0}}^e$ attains the minimal dimension, which is equal to rank$\mathfrak{g}_{\bar{0}}$.)

Theorem 1 (B. Kostant, [6]) *For a reductive Lie algebra $\mathfrak{g}$ and a regular nilpotent element $e \in \mathfrak{g}$, the finite W-algebra W_χ is isomorphic to the center of $U(\mathfrak{g})$.*
This theorem does not hold for Lie superalgebras, since W_χ must have a non-trivial odd part, and the center of $U(\mathfrak{g})$ is even.

Definition 6 Define a $\mathbb{Z}$-grading on $T(\mathfrak{g})$ by setting the degree of $g \in \mathfrak{g}_j$ to be $j+2$. This induces a filtration on $U(\mathfrak{g})$ and therefore on $U(\mathfrak{g})/I_\chi$ which is called the *Kazhdan filtration*. We will denote by Gr_K the corresponding graded algebras. Since by (2.1) $W_\chi \subset U(\mathfrak{g})/I_\chi$, we have an induced filtration on W_χ.

Theorem 2 (A. Premet, [17]) *Let $\mathfrak{g}$ be a semi-simple Lie algebra. Then the associated graded algebra $Gr_K W_\chi$ is isomorphic to $S(\mathfrak{g}^\chi)$.*
To generalize this result to the super case, we assume that $\mathfrak{l}'$ is some subspace in $\mathfrak{g}_{-1}$ satisfying the following two properties:

(1) $\mathfrak{g}_{-1} = \mathfrak{l} \oplus \mathfrak{l}'$,

(2) $\mathfrak{l}'$ contains a maximal isotropic subspace with respect to the form defined by $\chi([\cdot,\cdot])$ on $\mathfrak{g}_{-1}$.

If $\dim(\mathfrak{g}_{-1})_{\bar{1}}$ is even, then $\mathfrak{l}'$ is a maximal isotropic subspace. If $\dim(\mathfrak{g}_{-1})_{\bar{1}}$ is odd, then $\mathfrak{l}^\perp \cap \mathfrak{l}'$ is one-dimensional and we fix $\theta \in \mathfrak{l}^\perp \cap \mathfrak{l}'$ such that $\chi([\theta,\theta]) = 2$. It is clear that $\pi(\theta) \in W_\chi$ and $\pi(\theta)^2 = 1$.

Let $\mathfrak{p} = \bigoplus_{j\geq 0} \mathfrak{g}_j$. By the PBW theorem, $U(\mathfrak{g})/I_\chi \simeq S(\mathfrak{p} \oplus \mathfrak{l}')$ as a vector space. Therefore $Gr_K(U(\mathfrak{g})/I_\chi)$ is isomorphic to $S(\mathfrak{p} \oplus \mathfrak{l}')$ as a vector space. The good $\mathbb{Z}$-grading of $\mathfrak{g}$ induces the grading on $S(\mathfrak{p} \oplus \mathfrak{l}')$. For any $X \in U(\mathfrak{g})/I_\chi$ let $Gr_K(X)$ denote the corresponding element in $Gr_K(U(\mathfrak{g})/I_\chi)$, and $P(X)$ denote the highest weight component of $Gr_K(X)$ in this $\mathbb{Z}$-grading.

Theorem 3 ([16], Proposition 2.7) *Let $y_1, \dots, y_p$ be a basis in $\mathfrak{g}^\chi$ homogeneous in the good $\mathbb{Z}$-grading. Assume that there exist $Y_1, \dots, Y_p \in W_\chi$ such that $P(Y_i) = y_i$ for all $i = 1, \dots, p$. Then*
(a) if $\dim(\mathfrak{g}_{-1})_{\bar{1}}$ is even, then $Y_1, \dots, Y_p$ generate W_χ, and if $\dim(\mathfrak{g}_{-1})_{\bar{1}}$ is odd, then $Y_1, \dots, Y_p$ and $\pi(\theta)$ generate W_χ;
(b) if $\dim(\mathfrak{g}_{-1})_{\bar{1}}$ is even, then $Gr_K W_\chi \simeq S(\mathfrak{g}^\chi)$, and if $\dim(\mathfrak{g}_{-1})_{\bar{1}}$ is odd, then $Gr_K W_\chi \simeq S(\mathfrak{g}^\chi) \otimes \mathbb{C}[\xi]$, where $\mathbb{C}[\xi]$ is the exterior algebra generated by one element ξ.

As an example, we will give a presentation of the finite W-algebra for $\mathfrak{gl}(3)$ in non-regular case. A. Premet described finite W-algebras for semi-simple Lie algebras and all minimal nilpotent orbits (like considered here) in [18].

Example 1 Let $\mathfrak{g} = \mathfrak{gl}(3)$.

$$e = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Note that e is a non-regular nilpotent element, $\mathfrak{sl}(2) =< e, h, f >$ defines a Dynkin $\mathbb{Z}$-grading on $\mathfrak{gl}(3)$, whose degrees on the elementary matrices e_{ij} are

$$\begin{pmatrix} 0 & 1 & 2 \\ -1 & 0 & 1 \\ -2 & -1 & 0 \end{pmatrix}. \tag{2.2}$$

Recall that $\mathfrak{g}$ admits a non-degenerate $\mathfrak{g}$-invariant symmetric form $(x|y) = \mathrm{tr}(xy)$. Let $\chi \in \mathfrak{g}^*$ be defined by $\chi(x) := (e|x)$. Then $\chi(e_{3,1}) = 1$, and χ is zero on other basis elements. We have that $\mathfrak{g}^\chi = \mathfrak{g}^e$, where

$$\begin{aligned} \dim \mathfrak{g}^e &= \dim \mathfrak{g}_0 + \dim \mathfrak{g}_1 = 5, \\ \mathfrak{g}^e &=< e_{11} + e_{33}, e_{22}, e_{12}, e_{23}, e_{13} > . \end{aligned} \tag{2.3}$$

Thus χ is non-regular, and the $\mathbb{Z}$-grading given in (2.2) is good for χ. We set

$$\mathfrak{m} = \mathfrak{g}_{-2} \oplus \mathfrak{l}, \quad \mathfrak{l} =< e_{2,1} >,$$

and follow the definition of W_χ. Note that $\mathfrak{m}$ is spanned by $e_{2,1}$ and $e_{3,1}$.

Proposition 4 *(a) W_χ is generated by the elements $E_{1,1}, E_{2,2}, E_{1,2}, E_{2,3}, E_{1,3}$, where*

$$\begin{aligned} E_{1,1} &= \pi(e_{1,1} + e_{3,3}), \\ E_{2,2} &= \pi(e_{2,2}), \\ E_{1,2} &= \pi(e_{1,2} + (e_{2,2} - e_{1,1})e_{3,2} + 2e_{3,2}), \\ E_{2,3} &= \pi(e_{2,3}), \\ E_{1,3} &= \pi(e_{1,3} + e_{2,3}e_{3,2} + (e_{1,1} - e_{2,2})(e_{2,2} - e_{3,3}) + 2(e_{3,3} - e_{2,2})). \end{aligned}$$

(b) The nonzero commutation relations between these generators are as follows:

$$\begin{aligned} &[E_{1,1}, E_{1,2}] = E_{1,2}, \quad [E_{1,1}, E_{2,3}] = -E_{2,3}, \\ &[E_{2,2}, E_{1,2}] = -E_{1,2}, \quad [E_{2,2}, E_{2,3}] = E_{2,3}, \\ &[E_{1,2}, E_{2,3}] = E_{1,3}, \quad [E_{1,2}, E_{1,3}] = -6E_{1,2} + 2E_{1,1}E_{1,2} - 4E_{2,2}E_{1,2}, \\ &[E_{2,3}, E_{1,3}] = 4E_{2,2}E_{2,3} - 2E_{1,1}E_{2,3}. \end{aligned}$$

Proof Note that

$$\begin{aligned} &P(E_{1,1}) = e_{1,1} + e_{3,3}, \quad P(E_{2,2}) = e_{2,2}, \\ &P(E_{1,2}) = e_{1,2}, \quad P(E_{2,3}) = e_{2,3}, \quad P(E_{1,3}) = e_{1,3}. \end{aligned}$$

The result follows from Theorem 3 and (2.3). $\square$

3 The Queer Lie Superalgebra $Q(n)$

Recall that the *queer* Lie superalgebra is defined as follows

$$Q(n) := \{\left(\begin{array}{c|c} A & B \\ \hline B & A \end{array}\right) \mid A, B \text{ are } n \times n \text{ matrices}\}.$$

Let $\operatorname{otr}\left(\begin{array}{c|c} A & B \\ \hline B & A \end{array}\right) = \operatorname{tr} B$.

Remark 1 $Q(n)$ has one-dimensional center $< z >$, where $z = 1_{2n}$. Let

$$SQ(n) = \{X \in Q(n) \mid \operatorname{otr} X = 0\}.$$

The Lie superalgebra $\tilde{Q}(n) := SQ(n)/ < z >$ is simple for $n \geq 3$, see [5].

Note that $\mathfrak{g} = Q(n)$ admits an *odd* non-degenerate $\mathfrak{g}$-invariant supersymmetric bilinear form

$$(x|y) := \operatorname{otr}(xy) \text{ for } x, y \in \mathfrak{g}.$$

Therefore, we identify the coadjoint module $\mathfrak{g}^*$ with $\Pi(\mathfrak{g})$, where Π is the change of parity functor.
Let $e_{i,j}$ and $f_{i,j}$ be standard bases in $\mathfrak{g}_{\bar{0}}$ and $\mathfrak{g}_{\bar{1}}$ respectively:

$$e_{i,j} = \left(\begin{array}{c|c} E_{ij} & 0 \\ \hline 0 & E_{ij} \end{array}\right), \quad f_{i,j} = \left(\begin{array}{c|c} 0 & E_{ij} \\ \hline E_{ij} & 0 \end{array}\right),$$

where E_{ij} are elementary $n \times n$ matrices.

3.1 The Finite W-Algebra for $Q(n)$

Let $n \geq 3$. Consider $\mathfrak{sl}(2) =< e, h, f >$, where

$$e = e_{1,n}, \quad h = e_{1,1} - e_{n,n}, \quad f = e_{n,1}.$$

Note that e is a non-regular nilpotent element, h defines a Dynkin $\mathbb{Z}$-grading $\mathfrak{g} = \oplus_{i=-2}^{2} \mathfrak{g}_i$, where

$$\begin{aligned}
&\mathfrak{g}_2 =< e_{1,n} \mid f_{1,n} >, \quad \mathfrak{g}_{-2} =< e_{n,1} \mid f_{n,1} >, \\
&\mathfrak{g}_{-1} =< e_{i,1}, e_{n,i} \mid f_{i,1}, f_{n,i} > \text{ for } i = 2, \ldots, n-1, \\
&\mathfrak{g}_1 =< e_{1,i}, e_{i,n} \mid f_{1,i}, f_{i,n} > \text{ for } i = 2, \ldots, n-1, \\
&\mathfrak{g}_0 =< e_{1,1}, e_{n,n}, e_{i,j} \mid f_{1,1}, f_{n,n}, f_{i,j} > \text{ for } i, j = 2, \ldots, n-1.
\end{aligned}$$

Let $E = f_{1,n}$. Since we have an isomorphism $\mathfrak{g}^* \simeq \Pi(\mathfrak{g})$, an even non-regular nilpotent $\chi \in \mathfrak{g}^*$ can be defined by $\chi(x) := (x|E)$ for $x \in \mathfrak{g}$. Then $\chi \in \mathfrak{g}^*_{\bar{0}}$, and $\chi(e_{n,1}) = 1$, and χ is zero on other basis elements. We have that

$$\mathfrak{g}^\chi = \mathfrak{g}^E = \mathfrak{g}^\chi_{\bar{0}} \oplus \mathfrak{g}^\chi_{\bar{1}},$$

where

$$\mathfrak{g}^\chi_{\bar{0}} =< e_{1,1} + e_{n,n}, e_{i,j} \mid i = 1, \dots, n-1, j = 2, \dots, n >, \tag{3.1}$$

$$\mathfrak{g}^\chi_{\bar{1}} =< f_{1,1} - f_{n,n}, f_{i,j} \mid i = 1, \dots, n-1, j = 2, \dots, n > . \tag{3.2}$$

Thus the $\mathbb{Z}$-grading is good for χ. We set

$$\mathfrak{m} = \mathfrak{g}_{-2} \oplus \mathfrak{l}, \quad \text{where } \mathfrak{l} =< e_{i,1} \mid f_{i,1} >, i = 2, \dots, n-1,$$

and follow the definition of W_χ given in Sect. 2.

Theorem 5 *(a) W_χ is generated by even elements $E_{i,j}$, where*

$$\begin{aligned}
&E_{i,j} = \pi(e_{i,j}) \textit{ for } i = 2, \dots, n-1, j = 2, \dots, n,\\
&E_{1,1} = \pi(e_{1,1} + e_{n,n}),\\
&E_{1,k} = \pi(e_{1,k} + \textstyle\sum_{i=2, i\neq k}^{n-1} e_{i,k}e_{n,i} + (e_{k,k} - e_{1,1})e_{n,k} + \sum_{i=2, i\neq k}^{n-1} f_{i,k} f_{n,i} +\\
&\quad +(f_{k,k} - f_{1,1}) f_{n,k}) \qquad \textit{for } k = 2, \dots, n-1 \textit{ and } n \geq 4,\\
&E_{1,2} = \pi(e_{1,2} + (e_{2,2} - e_{1,1})e_{3,2} + (f_{2,2} - f_{1,1}) f_{3,2}) \textit{ for } n = 3,\\
&E_{1,n} = \pi(e_{1,n} + \textstyle\sum_{i=2}^{n-1} e_{i,n}e_{n,i} - \sum_{i=3}^{n-1} e_{i,2}e_{2,i} + (e_{1,1} - e_{2,2})(e_{2,2} - e_{n,n}) +\\
&\qquad \textstyle\sum_{i=2}^{n-1} f_{i,n} f_{n,i} - \sum_{i=3}^{n-1} f_{i,2} f_{2,i} + (f_{1,1} - f_{2,2})(f_{2,2} - f_{n,n})) \textit{ for } n \geq 4,\\
&E_{1,3} = \pi(e_{1,3} + e_{2,3}e_{3,2} + f_{2,3} f_{3,2} + (e_{1,1} - e_{2,2})(e_{2,2} - e_{3,3}) +\\
&\quad +(f_{1,1} - f_{2,2})(f_{2,2} - f_{3,3})) \textit{ for } n = 3.
\end{aligned}$$

and odd elements $F_{i,j}$, where

$$\begin{aligned}
&F_{i,j} = \pi(f_{i,j}) \quad \textit{for } i = 2, \dots, n-1, j = 2, \dots, n,\\
&F_{1,1} = \pi(f_{1,1} - f_{n,n}),\\
&F_{1,k} = \pi(f_{1,k} + \textstyle\sum_{i=2, i\neq k}^{n-1} f_{i,k}e_{n,i} - \sum_{i=2, i\neq k}^{n-1} e_{i,k} f_{n,i} - (e_{1,1} + e_{k,k}) f_{n,k} +\\
&\quad + (f_{k,k} - f_{1,1})e_{n,k}) \quad \textit{for } k = 2, \dots, n-1 \textit{ and } n \geq 4,\\
&F_{1,2} = \pi(f_{1,2} - (e_{1,1} + e_{2,2}) f_{3,2} + (f_{2,2} - f_{1,1})e_{3,2}) \quad \textit{for } n = 3,\\
&F_{1,n} = \pi(f_{1,n} + \textstyle\sum_{i=2}^{n-1} f_{i,n}e_{n,i} - \sum_{i=2}^{n-1} e_{i,n} f_{n,i} + \sum_{i=3}^{n-1} f_{i,2}e_{2,i} -\\
&\quad - \textstyle\sum_{i=3}^{n-1} e_{i,2} f_{2,i} + (e_{1,1} - e_{2,2})(f_{2,2} - f_{n,n}) +\\
&\quad + (f_{2,2} - f_{1,1})(e_{2,2} + e_{n,n})) \quad \textit{for } n \geq 4,\\
&F_{1,3} = \pi(f_{1,3} + f_{2,3}e_{3,2} - e_{2,3} f_{3,2} + (e_{1,1} - e_{2,2})(f_{2,2} - f_{3,3}) +\\
&\quad + (f_{2,2} - f_{1,1})(e_{2,2} + e_{3,3})) \textit{ for } n = 3.
\end{aligned}$$

(b) $Gr_K W_\chi \simeq S(\mathfrak{g}^\chi)$.

Proof Note that

$$P(E_{i,j}) = e_{i,j}, \quad P(F_{i,j}) = f_{i,j} \text{ for } i = 1, \ldots, n-1, \, j = 2, \ldots, n,$$
$$E_{1,1} = e_{1,1} + e_{n,n}, \quad F_{1,1} = f_{1,1} - f_{n,n}.$$

The result follows from Theorem 3 and (3.1), (3.2). □

4 The Harish-Chandra Homomorphism

In this section we recall the notion of the Harish-Chandra homomorphism for Lie superalgebras (see [16], Sect. 3.1). We essentially use the injectivity Theorem (Theorem 6). In the case of semi-simple Lie algebras the injectivity Theorem appeared in [8, 17]. It is also well-known to physicists since 1990s.

We can assume that $\mathfrak{g}$ is a basic Lie superalgebra or $Q(n)$. Let $\mathfrak{p} \subset \mathfrak{g}$ be a parabolic subalgebra such that $\mathfrak{n}^- \subset \mathfrak{m} \subset \mathfrak{p}^-$, where $\mathfrak{n}^-$ denotes the nilradical of the opposite parabolic $\mathfrak{p}^-$. Let $\mathfrak{s}$ be the Levi subalgebra of $\mathfrak{p}$, $\mathfrak{n}$ be its nilradical and $\mathfrak{m}^{\mathfrak{s}} = \mathfrak{m} \cap \mathfrak{s}$. Note that $\mathfrak{m} = \mathfrak{n}^- \oplus \mathfrak{m}^{\mathfrak{s}}$. We denote by $Q_\chi^{\mathfrak{s}}$ the induced module $U(\mathfrak{s}) \otimes_{U(\mathfrak{m}^{\mathfrak{s}})} C_\chi$, where by χ we understand the restriction of χ on $\mathfrak{s}$. Let

$$\bar{W}_\chi^{\mathfrak{s}} = \mathrm{End}_{U(\mathfrak{s})}(Q_\chi^{\mathfrak{s}})^{op} = (Q_\chi^{\mathfrak{s}})^{\mathfrak{m}^{\mathfrak{s}}}.$$

Let J_χ (respectively $J_\chi^{\mathfrak{s}}$) be the left ideal in $U(\mathfrak{p})$ (respectively in $U(\mathfrak{s})$) generated by $a - \chi(a)$ for all $a \in \mathfrak{m}^{\mathfrak{s}}$. Finally, let $\bar{\vartheta} : U(\mathfrak{p}) \to U(\mathfrak{s})$ denote the projection with the kernel $\mathfrak{n}U(\mathfrak{p})$. Note that $\bar{\vartheta}(J_\chi) = J_\chi^{\mathfrak{s}}$. Thus, the projection $\vartheta' : U(\mathfrak{p})/J_\chi \to U(\mathfrak{s})/J_\chi^{\mathfrak{s}}$ is well defined.

Note that we have an isomorphism of vector spaces $Q_\chi \simeq U(\mathfrak{p})/J_\chi$, hence W_χ can be identified with a subspace in $(U(\mathfrak{p})/J_\chi)^{\mathfrak{m}^{\mathfrak{s}}}$. On the other hand, $\bar{W}_\chi^{\mathfrak{s}}$ can be identified with the subspace $(U(\mathfrak{s})/J_\chi^{\mathfrak{s}})^{\mathfrak{m}^{\mathfrak{s}}}$. Consider a map $\vartheta : W_\chi \to U(\mathfrak{s})/J_\chi^{\mathfrak{s}}$ obtained by the restriction of ϑ' to W_χ. Since $\mathrm{ad}\mathfrak{m}^{\mathfrak{s}}(\mathfrak{n}) \subset \mathfrak{n}$, ϑ maps $\mathrm{ad}\mathfrak{m}^{\mathfrak{s}}$-invariants to $\mathrm{ad}\mathfrak{m}^{\mathfrak{s}}$-invariants. In other words, $\vartheta(W_\chi) \subset \bar{W}_\chi^{\mathfrak{s}}$. Furthermore, one can easily see that $\vartheta : W_\chi \to \bar{W}_\chi^{\mathfrak{s}}$ is a homomorphism of algebras.

Theorem 6 ([16], Theorem 3.1)
The homomorphism $\vartheta : W_\chi \to \bar{W}_\chi^{\mathfrak{s}}$ is injective.

Corollary 1 *In the case when $\mathfrak{m}^{\mathfrak{s}} = 0$, we have that ϑ is a homomorphism $W_\chi \to U(\mathfrak{s})$.*

4.1 The Harish-Chandra Homomorphism for $\mathfrak{g} = Q(3)$

In this section we explicitly describe the Harish-Chandra homomorphism in the case when $\mathfrak{g} = Q(3)$ and obtain defining relations in W_χ.

Recall that $\mathfrak{sl}(2) =< e, h, f >$, where

$$e = e_{1,3}, \quad h = e_{1,1} - e_{3,3}, \quad f = e_{3,1}.$$

Note that e is a non-regular nilpotent element, h defines a Dynkin $\mathbb{Z}$-grading of $\mathfrak{g}$ whose degrees on the elementary matrices are

$$\left(\begin{array}{ccc|ccc} 0 & 1 & 2 & 0 & 1 & 2 \\ -1 & 0 & 1 & -1 & 0 & 1 \\ -2 & -1 & 0 & -2 & -1 & 0 \\ \hline 0 & 1 & 2 & 0 & 1 & 2 \\ -1 & 0 & 1 & -1 & 0 & 1 \\ -2 & -1 & 0 & -2 & -1 & 0 \end{array}\right). \tag{4.1}$$

Note that $\mathfrak{p} = \mathfrak{s} \oplus \mathfrak{n}$, $\mathfrak{p}^- = \mathfrak{s} \oplus \mathfrak{n}^-$, where

$$\begin{aligned}
&\mathfrak{s} =< e_{1,1}, e_{2,2}, e_{3,3}, e_{2,3}, e_{3,2} | f_{1,1}, f_{2,2}, f_{3,3}, f_{2,3}, f_{3,2} >,\\
&\mathfrak{n}^- =< e_{3,1}, e_{2,1} | f_{3,1}, f_{2,1} >,\\
&\mathfrak{n} =< e_{1,2}, e_{1,3} | f_{1,2}, f_{1,3} >,\\
&\mathfrak{m} = \mathfrak{n}^-, \quad \mathfrak{m}^{\mathfrak{s}} = 0.
\end{aligned}$$

Lemma 1 *The images of the generators of W_χ under the homomorphism ϑ are the following elements of $U(\mathfrak{s})$:*

$$\begin{aligned}
\vartheta(E_{1,1}) &= e_{1,1} + e_{3,3},\\
\vartheta(E_{2,2}) &= e_{2,2},\\
\vartheta(E_{1,2}) &= (e_{2,2} - e_{1,1})e_{3,2} + (f_{2,2} - f_{1,1})f_{3,2},\\
\vartheta(E_{2,3}) &= e_{2,3},\\
\vartheta(E_{1,3}) &= e_{2,3}e_{3,2} + f_{2,3}f_{3,2} + (e_{1,1} - e_{2,2})(e_{2,2} - e_{3,3}) +\\
&\quad + (f_{1,1} - f_{2,2})(f_{2,2} - f_{3,3}),\\
\vartheta(F_{1,1}) &= f_{1,1} - f_{3,3},\\
\vartheta(F_{2,2}) &= f_{2,2},\\
\vartheta(F_{1,2}) &= -(e_{1,1} + e_{2,2})f_{3,2} + (f_{2,2} - f_{1,1})e_{3,2},\\
\vartheta(F_{2,3}) &= f_{2,3},\\
\vartheta(F_{1,3}) &= f_{2,3}e_{3,2} - e_{2,3}f_{3,2} + (e_{1,1} - e_{2,2})(f_{2,2} - f_{3,3}) +\\
&\quad + (f_{2,2} - f_{1,1})(e_{2,2} + e_{3,3}).
\end{aligned}$$

Theorem 7 *W_χ is generated by even elements $E_{1,1}$, $E_{2,2}$, $E_{1,2}$, $E_{2,3}$, $E_{1,3}$ and odd elements $F_{1,1}$, $F_{2,2}$, $F_{1,2}$, $F_{2,3}$, $F_{1,3}$. The defining relations are as follows between even generators:*

$$
\begin{aligned}
&[E_{1,1}, E_{2,2}] = 0, \quad [E_{1,1}, E_{1,2}] = E_{1,2}, \quad [E_{1,1}, E_{2,3}] = -E_{2,3},\\
&[E_{2,2}, E_{1,2}] = -E_{1,2}, \quad [E_{2,2}, E_{2,3}] = E_{2,3}, \quad [E_{1,2}, E_{2,3}] = E_{1,3},\\
&[E_{1,2}, E_{1,3}] = 2E_{1,1}E_{1,2} - 4E_{2,2}E_{1,2} + 2F_{1,1}F_{1,2} - 4E_{1,2},\\
&[E_{2,3}, E_{1,3}] = 4E_{2,3}E_{2,2} - 2E_{2,3}E_{1,1} + 2F_{2,3}F_{1,1} + 4E_{2,3};
\end{aligned}
$$

between odd generators:

$$
\begin{aligned}
&[F_{1,1}, F_{2,2}] = [F_{1,2}, F_{1,2}] = [F_{2,3}, F_{2,3}] = 0,\\
&[F_{1,1}, F_{1,1}] = 2E_{1,1}, \quad [F_{1,1}, F_{1,2}] = E_{1,2}, \quad [F_{1,1}, F_{2,3}] = -E_{2,3},\\
&[F_{2,2}, F_{2,2}] = 2E_{2,2}, \quad [F_{2,2}, F_{1,2}] = E_{1,2}, \quad [F_{2,2}, F_{2,3}] = E_{2,3},\\
&[F_{1,2}, F_{2,3}] = E_{1,3} - 2E_{1,1}E_{2,2};
\end{aligned}
$$

between even and odd generators:

$$
\begin{aligned}
&[E_{1,1}, F_{1,1}] = [E_{1,1}, F_{2,2}] = [E_{2,2}, F_{1,1}] = [E_{2,2}, F_{2,2}] = 0,\\
&[E_{1,1}, F_{1,2}] = F_{1,2}, \quad [E_{1,1}, F_{2,3}] = -F_{2,3},\\
&[E_{2,2}, F_{1,2}] = -F_{1,2}, \quad [E_{2,2}, F_{2,3}] = F_{2,3},\\
&[E_{1,2}, F_{1,1}] = -F_{1,2}, \quad [E_{1,2}, F_{2,2}] = F_{1,2}, \quad [E_{1,2}, F_{1,2}] = 0,\\
&[E_{1,2}, F_{1,3}] = 2E_{1,1}F_{1,2} - 2F_{1,1}E_{1,2},\\
&[E_{1,2}, F_{2,3}] = F_{1,3}, \quad [E_{2,3}, F_{1,1}] = -F_{2,3}, \quad [E_{2,3}, F_{2,2}] = -F_{2,3},\\
&[E_{2,3}, F_{1,2}] = -F_{1,3} - 2E_{2,2}F_{1,1}, \quad [E_{2,3}, F_{2,3}] = 0,\\
&[E_{2,3}, F_{1,3}] = -2F_{2,3}E_{1,1} + 2F_{2,3}E_{2,2} + 2F_{2,3}.
\end{aligned}
$$

Proof Follows from Lemma 1. □

5 Connection with Super-Yangians

Super-Yangian $Y(Q(n))$ was introduced by M. Nazarov in [9]. Recall that $Y(Q(n))$ is the associative unital superalgebra over $\mathbb{C}$ with the countable set of generators

$$T_{i,j}^{(m)} \text{ where } m = 1, 2, \ldots \text{ and } i, j = \pm 1, \pm 2, \ldots, \pm n.$$

The $\mathbb{Z}_2$-grading of the algebra $Y(Q(n))$ is defined as follows:

$$p(T_{i,j}^{(m)}) = p(i) + p(j), \text{ where } p(i) = 0 \text{ if } i > 0, \text{ and } p(i) = 1 \text{ if } i < 0.$$

The defining relations are as follows

$$
\begin{aligned}
&([T_{i,j}^{(m+1)}, T_{k,l}^{(r-1)}] - [T_{i,j}^{(m-1)}, T_{k,l}^{(r+1)}]) \cdot (-1)^{p(i)p(k)+p(i)p(l)+p(k)p(l)} =\\
&T_{k,j}^{(m)}T_{i,l}^{(r-1)} + T_{k,j}^{(m-1)}T_{i,l}^{(r)} - T_{k,j}^{(r-1)}T_{i,l}^{(m)} - T_{k,j}^{(r)}T_{i,l}^{(m-1)}\\
&+(-1)^{p(k)+p(l)}(-T_{-k,j}^{(m)}T_{-i,l}^{(r-1)} + T_{-k,j}^{(m-1)}T_{-i,l}^{(r)} + T_{k,-j}^{(r-1)}T_{i,-l}^{(m)} - T_{k,-j}^{(r)}T_{i,-l}^{(m-1)}),
\end{aligned}
$$

$$T^{(m)}_{-i,-j} = (-1)^m T^{(m)}_{i,j},$$

where $m, r = 1, \ldots$ and $T^{(0)}_{i,j} = \delta_{ij}$.

Recall that $Y(Q(n))$ is a Hopf superalgebra (see [10]) with comultiplication given by the formula

$$\Delta(T^{(r)}_{i,j}) = \sum_{s=0}^{r}\sum_{k}(-1)^{(p(i)+p(k))(p(j)+p(k))}T^{(s)}_{i,k}\otimes T^{(r-s)}_{k,j}.$$

The evaluation homomorphism $\mathrm{ev} : Y(Q(n)) \to U(Q(n))$ is defined as follows

$$T^{(1)}_{i,j} \mapsto -e_{j,i},\quad T^{(1)}_{-i,j} \mapsto -f_{j,i} \text{ for } i, j > 0,\quad T^{(0)}_{i,j} \mapsto \delta_{ij},\quad T^{(r)}_{i,j} \mapsto 0 \text{ for } r > 1,$$

see [10]. Observe that the map

$$\Delta_l : Y(Q(n)) \longrightarrow Y(Q(n))^{\otimes l},$$

where

$$\Delta_l := \Delta_{l-1,l} \circ \cdots \circ \Delta_{2,3} \circ \Delta$$

is a homomorphism of associative algebras.

Conjecture. Let e be an even nilpotent element in $Q(n)$ whose Jordan blocks are all of the same size l, and let $k = \frac{n}{l}$. Then the finite W-algebra for $Q(n)$ is isomorphic to the image of $Y(Q(k))$ under the homomorphism

$$\mathrm{ev}^{\otimes l} \circ \Delta_l : Y(Q(k)) \longrightarrow (U(Q(k)))^{\otimes l}.$$

Remark 2 In [16] we considered the case when e is regular ($l = n$). We used the opposite comultiplication

$$\Delta^{op}(T^{(r)}_{i,j}) = \sum_{s=0}^{r}\sum_{k} T^{(r-s)}_{k,j}\otimes T^{(s)}_{i,k},$$

and instead of the evaluation homomorphism we used the homomorphism
$U : Y(Q(1)) \to U(Q(1))$ defined in [11], which is given by

$$T^{(r)}_{1,1} \mapsto (-1)^r e^{(r)}_{1,1},\quad T^{(r)}_{-1,1} \mapsto (-1)^r f^{(r)}_{1,1} \quad \text{for } r > 0,\quad T^{(0)}_{i,j} \mapsto \delta_{ij}.$$

We proved that the finite W-algebra for $Q(n)$ is isomorphic to the image of $Y(Q(1))$ under the homomorphism $U^{\otimes n} \circ \Delta^{op}_n$, where $\Delta^{op}_n := \Delta^{op}_{n-1,n} \circ \cdots \circ \Delta^{op}_{2,3} \circ \Delta^{op}$, and we can now prove that

$$(U^{\otimes n} \circ \Delta^{op}_n)(Y(Q(1))) = (\mathrm{ev}^{\otimes n} \circ \Delta_n)(Y(Q(1))).$$

Acknowledgements This work was supported by a grant from the Simons Foundation (#354847, Elena Poletaeva). I thank the organizers of the 11-th International Workshop "Lie Theory and Its Applications in Physics" (LT-11), 15-21 June 2015, Varna, Bulgaria, for the very interesting conference and hospitality. I am grateful to V. Serganova for very helpful discussions.

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The Joseph Ideal for $\mathfrak{sl}(m|n)$

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Abstract Using deformation theory, Braverman and Joseph obtained an alternative characterisation of the Joseph ideal for simple Lie algebras, which included even type A. In this note we extend that characterisation to define a remarkable quadratic ideal for $\mathfrak{sl}(m|n)$. When $m - n > 2$, we prove that the ideal is primitive and can also be characterised similarly to the construction of the Joseph ideal by Garfinkle.

1 Preliminaries

We use the notation $\mathfrak{g} = \mathfrak{sl}(m|n)$. See [4] for the definition and more information on $\mathfrak{sl}(m|n)$ and Lie superalgebras. We take the Borel subalgebra $\mathfrak{b}$ to be the space of upper triangular matrices and the Cartan subalgebra $\mathfrak{h}$ diagonal matrices, both with zero supertrace. With slight abuse of notation we will write elements of $\mathfrak{h}^*$ as elements of $\mathbb{C}^{m+n}$, using bases $\{\epsilon_j, i = 1, \ldots, m\}$ of $\mathbb{C}^m$ and $\{\delta_j, i = 1, \ldots, n\}$ of $\mathbb{C}^n$, with the restriction that the coefficients add up to zero. With this choice and convention, the system of positive roots is given by $\Delta^+ = \Delta_0^+ \cup \Delta_1^+$, where

$$\Delta_0^+ = \{\epsilon_i - \epsilon_j | 1 \leq i < j \leq m\} \cup \{\delta_i - \delta_j | 1 \leq i < j \leq n\},$$

$$\Delta_1^+ = \{\epsilon_i - \delta_j | 1 \leq i \leq m, 1 \leq j \leq n\}.$$

The Borel subalgebra leads to a triangular decomposition of $\mathfrak{g}$ given by $\mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ where $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^+$. A highest weight vector v_λ of a weight module M satisfies

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V. Dobrev (ed.), *Lie Theory and Its Applications in Physics*,
Springer Proceedings in Mathematics & Statistics 191,
DOI 10.1007/978-981-10-2636-2_36

$\mathfrak{n}^{+} \cdot v_\lambda = 0$ and $\mathfrak{h} \cdot v_\lambda = \lambda(\mathfrak{h}) \cdot v_\lambda$. The corresponding weight $\lambda \in \mathfrak{h}^*$ will be called a highest weight. We use the notation $L(\lambda)$ for the simple module with highest weight $\lambda \in \mathfrak{h}^*$. We also set $\rho_0 = \frac{1}{2}\sum_{\alpha\in\Delta_0^+}\alpha$ and $\rho = \rho_0 - \frac{1}{2}\sum_{\gamma\in\Delta_1^+}\gamma$, so concretely

$$\rho = \frac{1}{2}\sum_{i=1}^{m}(m-n-2i+1)\epsilon_i + \frac{1}{2}\sum_{j=1}^{n}(n+m-2j+1)\delta_j. \tag{1}$$

We choose the form $(\cdot,\cdot)$ on $\mathbb{C}^{m+n}$, and on $\mathfrak{h}^*$ by restriction, by setting $(\epsilon_i, \epsilon_j) = \delta_{ij}$, $(\delta_j, \delta_k) = -\delta_{jk}$ and $(\epsilon_i, \delta_j) = 0$.

From now on we consider only weights λ which are *integral*, that is $(\lambda+\rho, \alpha^\vee) \in \mathbb{Z}$ for all $\alpha \in \Delta_0$, with $\alpha^\vee := 2\alpha/(\alpha,\alpha)$. We denote this subset of $\mathfrak{h}^*$ by P_0. If $(\lambda+\rho, \alpha^\vee) > 0$, for all $\alpha \in \Delta_0^+$, we say that $\lambda \in P_0$ is *dominant regular*.

Denote by C the quadratic *Casimir operator*. It is an element of the center of $U(\mathfrak{g})$ and it acts on a highest weight vector of weight λ by the scalar

$$C \cdot v_\lambda = (\lambda + 2\rho, \lambda)v_\lambda. \tag{2}$$

We denote by $M^\vee$ the dual module of M in category $\mathcal{O}$, see e.g. [8, Chap. 3]. The functor $\vee$ is exact and contravariant, we have that $L(\lambda)^\vee \cong L(\lambda)$ and for finite dimensional modules $(M \otimes N)^\vee \cong M^\vee \otimes N^\vee$.

We set $V = \mathbb{C}^{m|n}$ the natural representation of $\mathfrak{g}$. We will use the notation $A^i{}_j$ for an element in $V \otimes V^*$ and we have the identification $V \otimes V^* \cong V^* \otimes V$ given by $A^i{}_j \cong (-1)^{|i||j|}A_j{}^i$, where $|\cdot|$ is the parity function, i.e., $|i| = 0$ for $i \in \{1,\dots,m\}$ and $|i| = 1$ for $i \in \{m+1,\dots,m+n\}$. We define the *supertrace* str as the $\mathfrak{g}$-morphism

$$\mathrm{str}\colon V \otimes V^* \to \mathbb{C} \quad A^i{}_j \mapsto \sum_i (-1)^{|i|}A^i{}_i.$$

If $m \neq n$ the supertrace gives a decomposition of $V \otimes V^*$ in a traceless and a pure trace part. The Lie superalgebra $\mathfrak{g}$ consist exactly of the traceless elements in $V \otimes V^*$. We will use the identification $V \otimes V^* \cong V^* \otimes V$ for taking the supertrace of higher order tensor powers. For example, if $A \in V \otimes V^* \otimes V \otimes V^*$, then the supertrace over the first and last component is given by

$$\mathrm{str}_{1,4}\colon V \otimes V^* \otimes V \otimes V^* \to V^* \otimes V; \quad A^i{}_j{}^k{}_l \mapsto \sum_i (-1)^{|i|+|i|(|k|+|j|)}A^i{}_j{}^k{}_i.$$

With these conventions str always corresponds to a $\mathfrak{g}$-module morphism.

We will also use the *Killing form*

$$\langle\cdot,\cdot\rangle\colon \mathfrak{g}\times\mathfrak{g} \to \mathbb{C} \quad \langle A, B\rangle = 2(m-n)\sum_{i,j}(-1)^{|i|}A^i{}_j B^j{}_i,$$

which satisfies $\langle A, B\rangle = \mathrm{str}_{\mathfrak{g}}(\mathrm{ad}_{\mathfrak{g}}(A)\mathrm{ad}_{\mathfrak{g}}(B))$. This is an invariant, even, supersymmetric form. If $m-n\neq 0$ it is non-degenerate. We introduce the corresponding $\mathfrak{g}$-module morphism $\mathcal{K} = 2(m-n)\,\mathrm{str}_{1,4}\circ \mathrm{str}_{2,3}$:

$$\mathcal{K} : V\otimes V^*\otimes V\otimes V^* \to \mathbb{C};\quad A^i{}_j{}^k{}_l \mapsto 2(m-n)\sum_{i,j}(-1)^{|i|}A^i{}_j{}^j{}_i.$$

In particular, for A, B in $\mathfrak{g}$ we have $\mathcal{K}(A\otimes B) = \langle A, B\rangle$.

2 Second Tensor Power of the Adjoint Representation for $\mathfrak{sl}(m|n)$

In this section we will always set $\mathfrak{g} = \mathfrak{sl}(m|n)$ with $m\neq n$. We will also always assume $m\neq 0\neq n$. In case $m=1$ one needs to replace all ϵ_2 occurring in formulae by δ_1 and for $n=1$ one replaces δ_{n-1} by ϵ_m. Furthermore V will be the natural $\mathfrak{sl}(m|n)$ module and we identify $\mathfrak{sl}(m|n)$ with the corresponding tensors in $V\otimes V^*$.

Theorem 1 *For $\mathfrak{g} = \mathfrak{sl}(m|n)$ with $|m-n|>2$, the second tensor power of the adjoint representation $\mathfrak{g}\otimes\mathfrak{g}\cong \mathfrak{g}\odot\mathfrak{g}\oplus\mathfrak{g}\wedge\mathfrak{g}$ decomposes as*

$$\begin{aligned}\mathfrak{g}\odot\mathfrak{g} &\cong L_{2\epsilon_1-\delta_{n-1}-\delta_n}\oplus L_{\epsilon_1+\epsilon_2-2\delta_n}\oplus L_{\epsilon_1-\delta_n}\oplus L_0,\\ \mathfrak{g}\wedge\mathfrak{g} &\cong L_{2\epsilon_1-2\delta_n}\oplus L_{\epsilon_1+\epsilon_2-\delta_{n-1}-\delta_n}\oplus L_{\epsilon_1-\delta_n}.\end{aligned}$$

We define the *Cartan product* $\mathfrak{g}\circledcirc\mathfrak{g}$ as the direct summand of $\mathfrak{g}\otimes\mathfrak{g}$ isomorphic to $L_{2\epsilon_1-\delta_{n-1}-\delta_n}$.

To give an explicit expression for the decomposition of the symmetric part we will use a projection operator $\chi\colon \mathfrak{g}\odot\mathfrak{g}\to\mathfrak{g}\odot\mathfrak{g}$ given by $\chi := \phi\circ\mathrm{str}_{2,3}$, where ϕ is the $\mathfrak{g}$-module morphism $\phi\colon V\otimes V^*\to V\otimes V^*\otimes V\otimes V^*$ defined in Lemma 2.

Theorem 2 *According to the decomposition of $\mathfrak{g}\odot\mathfrak{g}$ in Theorem 1, respecting that order, a tensor $A\in\mathfrak{g}\odot\mathfrak{g}$ decomposes as $A = B+C+D+E$, where*

- $B^i{}_j{}^k{}_l = \frac{1}{2}(A^i{}_j{}^k{}_l - \chi(A)^i{}_j{}^k{}_l) + \frac{1}{2}(-1)^{|i||j|+|i||k|+|j||k|}(A^k{}_j{}^i{}_l - \chi(A)^k{}_j{}^i{}_l)$,
  i.e., B is the super symmetrisation in the upper indices of $A-\chi(A)$;
- $C = A-\chi(A)-B$,
  i.e., C is the super antisymmetrisation in the upper indices of $A-\chi(A)$;
- $E = (2(m-n)^2)^{-1}\mathcal{K}(A)\phi(\delta)$;
- $D = \chi(A)-E$.

By construction $\mathrm{str}_{2,3}(B) = 0 = \mathrm{str}_{2,3}(C)$ *and* $\mathcal{K}(D)=0$.

The explicit formula for $\phi(\delta)$, where $\delta = \delta^i{}_j$ is the Kronecker delta, is given by

$$(\phi(\delta))^i{}_l{}^k{}_j = ((m-n)^2-1)^{-1}\left((-1)^{|k|}(m-n)\delta^i{}_l\delta^k{}_j - \delta^i{}_j\delta^k{}_l\right).$$

The remainder of this section is devoted to the proof of these theorems.

Lemma 1 *The possible highest weights of the* $\mathfrak{g}$*-module* $\mathfrak{g} \otimes \mathfrak{g}$ *are*

$$2\epsilon_1 - 2\delta_n,\ 2\epsilon_1 - \delta_{n-1} - \delta_n,\ \epsilon_1 + \epsilon_2 - 2\delta_n,\ \epsilon_1 + \epsilon_2 - \delta_{n-1} - \delta_n,\ \epsilon_1 - \delta_n,\ 0.$$

The space of highest weight vectors for $\epsilon_1 - \delta_n$ *has at most dimension 2 and for the other weights at most 1.*

Proof A highest weight vector v_λ in $\mathfrak{g} \otimes \mathfrak{g}$ is of the form

$$v_\lambda = X_{\epsilon_1 - \delta_n} \otimes A + \cdots, \text{ where } A \in \mathfrak{g}.$$

Thus the highest weight λ is of the form $\lambda = \epsilon_1 - \delta_n + \mu$ with $\mu \in \Delta \cup \{0\}$. Since it also has to be regular dominant we have the following possibilities for λ :

$$2\epsilon_1 - 2\delta_n,\ 2\epsilon_1 - \delta_{n-1} - \delta_n,\ \epsilon_1 + \epsilon_2 - 2\delta_n,\ \epsilon_1 + \epsilon_2 - \delta_{n-1} - \delta_n,\ \epsilon_1 - \delta_n,\ 0, \text{ and}$$

$$\begin{aligned} 2\epsilon_1 - \epsilon_m - \delta_n,\ \epsilon_1 + \epsilon_2 - \epsilon_m - \delta_n,\ \epsilon_1 - \epsilon_m,\ \delta_1 - \delta_n, \\ \epsilon_1 + \delta_1 - 2\delta_n,\ \epsilon_1 + \delta_1 - \delta_{n-1} - \delta_n,\ \epsilon_1 - \epsilon_m + \delta_1 - \delta_n. \end{aligned} \tag{3}$$

A corresponding highest weight vector v_λ has to satisfy $[X, v_\lambda] = 0$ for all $X \in \mathfrak{n}^+$. Writing out this condition for all positive simple roots vectors, we deduce that there are no highest weight vectors corresponding to the weights in (3) and that the dimension of the space of highest weight vectors for $\epsilon_1 - \delta_n$ is at most 2. The fact that for the other possibilities the dimension is at most 1 follows from the dimension of the corresponding root space in $\mathfrak{g}$, which is always 1. □

We want to construct a $\mathfrak{g}$-module morphism $\phi\colon V \otimes V^* \to V \otimes V^* \otimes V \otimes V^*$, such that its image is in $\mathfrak{g} \odot \mathfrak{g}$ and $\mathrm{str}_{2,3} \circ \phi = \mathrm{id}$. Thus this morphism has to satisfy the following properties for all $B \in V \otimes V^*$

1. $\mathrm{str}_{1,2}\, \phi(B) = 0$
2. $\phi(B)^i{}_j{}^k{}_l = (-1)^{(|i|+|j|)(|k|+|l|)} \phi(B)^k{}_l{}^i{}_j$
3. $\mathrm{str}_{2,3}\, \phi(B) = B$.

Lemma 2 *Consider the map* $\phi\colon V \otimes V^* \to V \otimes V^* \otimes V \otimes V^*$ *given by*

$$\begin{aligned} \phi(B)^i{}_j{}^k{}_l = a \Big(&(-1)^{|k|} B^i{}_l \delta^k{}_j + (-1)^{(|i|+|j|)(|k|+|l|)+|i|} B^k{}_j \delta^i{}_l + \frac{-2}{m-n} B^i{}_j \delta^k{}_l \\ &+ \frac{-2}{m-n} (-1)^{(|i|+|j|)(|k|+|l|)} B^k{}_l \delta^i{}_j + c_1 (-1)^{|k|} \mathrm{str}(B) \delta^i{}_l \delta^k{}_j + c_2\, \mathrm{str}(B) \delta^i{}_j \delta^k{}_l \Big). \end{aligned}$$

For the constants $a = \frac{m-n}{(m-n)^2-4}$, $c_1 = \frac{(m-n)^2+2}{(m-n)(1-(m-n)^2)}$ *and* $c_2 = \frac{3}{(m-n)^2-1}$, *the map* ϕ *is a* $\mathfrak{g}$*-module morphism satisfying conditions 1, 2, 3. above.*

Proof One can easily see that $\phi(B)$ is supersymmetric for the indices (i, j) and (k, l), hence it satisfies the second condition. The first condition leads to

$$c_1 + (m-n)c_2 - 2(m-n)^{-1} = 0,$$

while the third condition gives us the following two equations:

$$a((m-n) - 4(m-n)^{-1}) = 1 \qquad \text{and} \qquad 1 + (m-n)c_1 + c_2 = 0.$$

This system of equations has as solution the constants given in the lemma. One can also check directly that ϕ is indeed a $\mathfrak{g}$-module morphism. □

Proof of Theorem 2. Define the $\mathfrak{g}$-module morphism $\chi\colon \mathfrak{g} \odot \mathfrak{g} \to \mathfrak{g} \odot \mathfrak{g}$ by $\chi = \phi \circ \mathrm{str}_{2,3}$. Since $\mathrm{str}_{2,3} \circ \phi = \mathrm{id}$, we have $\chi^2 = \chi$. This implies that the representation splits up into $\ker \chi = \mathrm{im}(1-\chi)$ and $\mathrm{im}\,\chi = \ker(1-\chi)$. Hence

$$\mathfrak{g} \odot \mathfrak{g} = \ker \chi \oplus \mathrm{im}\,\chi.$$

We have $\mathrm{im}\,\chi = \mathrm{im}\,\phi \cong V \otimes V^*$, since ϕ is injective. From Sect. 1 we know that $V \otimes V^* \cong L_{\epsilon_1-\delta_n} \oplus L_0$, where this decomposition is based on the supertrace.

Let $q \in \mathrm{End}_{\mathfrak{g}}(\ker \chi)$ denote the super symmetrisation in the upper indices, so $q^2 = q$ and hence $\ker \chi = \ker q \oplus \mathrm{im}\, q$.

In the proof of Theorem 1 we will show that $\mathfrak{g} \wedge \mathfrak{g}$ has three direct summands. From Lemma 1 we know that $\mathfrak{g} \otimes \mathfrak{g}$ contains at most seven highest weight vectors, of which thus three are already contained in $\mathfrak{g} \wedge \mathfrak{g}$. Therefore $\ker q$ and $\mathrm{im}\, q$ each contain exactly one highest weight vector. Since $\ker q \oplus \mathrm{im}\, q$ is self-dual in category $\mathcal{O}$ this implies that they are both simple modules. Therefore $\mathfrak{g} \odot \mathfrak{g} = \ker q \oplus \mathrm{im}\, q \oplus L_{\epsilon_1-\delta_n} \oplus L_0$ is a decomposition in simple modules. One can verify, by tracking the highest weights of the respective subspaces, that $\ker q = L_{\epsilon_1+\epsilon_2-2\delta_n}$ and that $\mathrm{im}\, q = L_{2\epsilon_1-\delta_{n-1}-\delta_n}$.

By construction, the expressions for projections on simple summands follow. □

Proof of Theorem 1. We have already dealt with the symmetric part in the proof of Theorem 2. For the antisymmetric part we remark that $\mathrm{str}_{1,4}\, \mathrm{str}_{2,3}(A) = 0$ for all $A \in \mathfrak{g} \wedge \mathfrak{g}$. Thus $\mathrm{str}_{2,3}$ is a $\mathfrak{g}$-module morphism from $\mathfrak{g} \wedge \mathfrak{g}$ to $\mathfrak{g} \cong L_{\epsilon_1-\delta_n}$. Consider the $\mathfrak{g}$-module morphism $\psi\colon \mathfrak{g} \to \mathfrak{g} \wedge \mathfrak{g}$ given by

$$B \mapsto (m-n)^{-1} \left((-1)^{|k|} B^i{}_l \delta^k{}_j - (-1)^{(|i|+|j|)(|k|+|l|)+|i|} B^k{}_j \delta^i{}_l \right).$$

For this morphism it holds that $\mathrm{str}_{2,3} \circ \psi = \mathrm{id}$. Denote by q again the symmetrisation in the upper indices. Then we find in the same way as for the symmetric part

$$\mathfrak{g} \wedge \mathfrak{g} = \ker q \oplus \mathrm{im}\, q \oplus \mathrm{im}\, \psi,$$

and $\ker q \cong L_{\epsilon_1+\epsilon_2-\delta_{n-1}-\delta_n}$, $\mathrm{im}\, q \cong L_{2\epsilon_1-2\delta_n}$ and $\mathrm{im}\, \psi \cong L_{\epsilon_1-\delta_n}$. □

3 The Joseph Ideal for $\mathfrak{sl}(m|n)$

In this section we define and characterise the Joseph ideal for $\mathfrak{g} = \mathfrak{sl}(m|n)$, where from now on we always assume $|m-n| > 2$. Similar results for $\mathfrak{osp}(m|2n)$ have been obtained in [5].

We define a one-parameter family $\{\mathcal{J}_\lambda \mid \lambda \in \mathbb{C}\}$ of quadratic two-sided ideals in the tensor algebra $T(\mathfrak{g}) = \oplus_{j\geq 0} \otimes^j \mathfrak{g}$, where $\mathcal{J}_\lambda$ is generated by

$$\{X \otimes Y - X \circledcirc Y - \frac{1}{2}[X, Y] - \lambda\langle X, Y\rangle \mid X, Y \in \mathfrak{g}\} \\ \subset \mathfrak{g} \otimes \mathfrak{g} \oplus \mathfrak{g} \oplus \mathbb{C} \subset T(\mathfrak{g}). \tag{4}$$

By construction there is a unique ideal J_λ in the universal enveloping algebra $U(\mathfrak{g})$, which satisfies $T(\mathfrak{g})/\mathcal{J}_\lambda \cong U(\mathfrak{g})/J_\lambda$. Now we define $\lambda^c := -1/(8(m-n+1))$.

Theorem 3 *(i) For $\lambda \neq \lambda^c$, the ideal J_λ has finite codimension, more precisely $J_\lambda = U(\mathfrak{g})$ for $\lambda \notin \{0, \lambda^c\}$ and $J_\lambda = \mathfrak{g}U(\mathfrak{g})$ for $\lambda = 0$.*

(ii) For $\lambda = \lambda^c$, the ideal J_λ has infinite codimension.

From now on we call the ideal J_{λ^c} the *Joseph ideal*. If $m - n > 2$, we give another characterisation of the Joseph ideal, which generalises the characterisation in [7] to type A (super and classical). The classical case, $n = 0$, was already obtained through different methods in the proof of Proposition 3.1 in [1]. For this we need the *canonical antiautomorphism* τ of $U(\mathfrak{g})$, defined by $\tau(X) = -X$ for $X \in \mathfrak{g}$.

Theorem 4 *Let $\mathfrak{g} = \mathfrak{sl}(m|n)$ with $m - n > 2$. Any two-sided ideal $\mathfrak{K}$ in $U(\mathfrak{g})$ of infinite codimension, with $\tau(\mathfrak{K}) = \mathfrak{K}$, such that the graded ideal $gr(\mathfrak{K})$ in $\odot\mathfrak{g}$ satisfies*

$$(gr(\mathfrak{K}) \cap \odot^2\mathfrak{g}) \oplus \mathfrak{g} \circledcirc \mathfrak{g} = \odot^2\mathfrak{g},$$

is equal to the Joseph ideal J_{λ^c}.

In the remainder of this section we will prove both theorems.

Proof of Theorem 3 Similarly to the proof of Theorem 2.1 in [6] for $\mathfrak{gl}(m)$, to which we refer for more details, we construct a special tensor S in $\otimes^3\mathfrak{g}$, which we will reduce inside $T(\mathfrak{g})/\mathcal{J}_\lambda$ in two different ways. This will show that for λ different from λ^c, the ideal $\mathcal{J}_\lambda$ contains $\mathfrak{g}$. Note that the existence of the tensor S in the setting of [6] was already non-constructively proved in [3].

Consider $T \in \mathfrak{g}$ and define the tensor S as

$$\begin{aligned} S^a{}_b{}^c{}_d{}^e{}_f = (-1)^{|d|}\delta^e{}_d\delta^c{}_f T^a{}_b - \frac{1}{m-n}\delta^c{}_d\delta^e{}_f T^a{}_b \\ - (-1)^{|b|+(|a|+|b|)(|c|+|d|)}\delta^e{}_b\delta^a{}_f T^c{}_d + \frac{1}{m-n}\delta^a{}_b\delta^e{}_f T^c{}_d \\ + (-1)^{|b|+(|a|+|b|)|c|+|d||e|}\delta^a{}_d\delta^e{}_b T^c{}_f \end{aligned}$$

$$
\begin{aligned}
&- \frac{1}{m-n}(-1)^{|d|+(|a|+|b|)(|c|+|d|)}\delta^a{}_d\delta^e{}_f T^c{}_b \\
&- (-1)^{(|c|+|d|)|b|+|d|+|b||e|}\delta^c{}_b\delta^e{}_d T^a{}_f + \frac{1}{m-n}(-1)^{|b|}\delta^c{}_b\delta^e{}_f T^a{}_d.
\end{aligned}
$$

One can calculate that $\mathrm{str}_{1,2}\, S = \mathrm{str}_{3,4}\, S = \mathrm{str}_{5,6}\, S = 0$, hence $S \in \otimes^3\mathfrak{g}$. Remark that we also defined S so that it is antisymmetric in the indices (a, b) and (c, d), hence $S \in \mathfrak{g} \wedge \mathfrak{g} \otimes \mathfrak{g}$. Since the Cartan product lies in $\mathfrak{g} \odot \mathfrak{g}$, the Cartan part with respect to the first four indices a, b, c, d vanishes. Now we consider (for each a, b), the tensor in $\otimes^2\mathfrak{g}$ corresponding to the indices c, d, e, f. First we symmetrise, to find a tensor in $\odot^2\mathfrak{g}$. When we apply $1 - \chi$ to that tensor and then symmetrise in the upper indices, we obtain zero. Theorem 2 thus shows that S also has no part lying in $\mathfrak{g} \otimes \mathfrak{g} \circledcirc \mathfrak{g}$.

Now, on the one hand, we can reduce S using the fact that the Cartan part vanishes with respect to the first four indices a, b, c, d. Then we find

$$
S \simeq -\frac{1}{2}(m-n)(m-n-2)T \qquad \mathrm{mod}\ \mathcal{J}_\lambda.
$$

If, on the other hand, we reduce S using the fact that the Cartan part vanishes with respect to the last four indices c, d, e, f, we find

$$
S \simeq (m-n)(m-n-2)(2\lambda(m-n+1) - \frac{1}{4})T \quad \mathrm{mod}\ \mathcal{J}_\lambda.
$$

Therefore, if $\lambda \neq \lambda^c$, then T is an element of $\mathcal{J}_\lambda$. Hence, we have proven that $\mathfrak{g} \subset \mathcal{J}_\lambda$ for $\lambda \neq \lambda^c$. This also implies for $\lambda \neq 0$, by Eq. (4), that $\mathbb{C} \subset \mathcal{J}_\lambda$. Hence $\mathcal{J}_\lambda = T(\mathfrak{g})$ for $\lambda \notin \{0, \lambda^c\}$ and $\mathcal{J}_0 = \oplus_{k>0} \otimes^k \mathfrak{g}$. This proves part (i) of Theorem 3. Part (ii) will follow from the construction in Sect. 4. □

To prove Theorem 4, we will need two lemmata. First we define I_2 as the complement representation of $\mathfrak{g} \circledcirc \mathfrak{g}$ in $\mathfrak{g} \otimes \mathfrak{g}$ and recursively

$$
I_k = I_{k-1} \otimes \mathfrak{g} + \mathfrak{g} \otimes I_{k-1} \ \text{ for } k > 2. \tag{5}
$$

Denote by λ^k the highest weight occurring in $\odot^k\mathfrak{g}$, then

$$
\lambda^k = \begin{cases} k\epsilon_1 - \delta_{n-k+1} - \delta_{n-k+2} - \cdots - \delta_{n-1} - \delta_n & \text{for } k \leq n, \\ k\epsilon_1 - (k-n)\epsilon_m - \delta_1 - \delta_2 - \cdots - \delta_{n-1} - \delta_n & \text{for } k \geq n. \end{cases}
$$

Lemma 3 *Let* $\mathfrak{g} = \mathfrak{sl}(m|n)$ *with* $m - n > 2$. *Then* $\otimes^k\mathfrak{g} \cong L(\lambda^k) \oplus I_k$.

Proof Set $\beta_2 = \mathfrak{g} \circledcirc \mathfrak{g}$ and define the submodule β_k of $\odot^k\mathfrak{g}$ by

$$
\beta_k := \beta_{k-1} \otimes \mathfrak{g} \cap \mathfrak{g} \otimes \beta_{k-1}, \qquad \text{for } k > 2.
$$

We will show by induction that $\beta_k = L(\lambda^k)$ and that this is a direct summand in $\otimes^k \mathfrak{g}$. This holds for $k = 2$ by definition. Now we assume that it holds for k and start by proving that all highest weight vectors in β_{k+1} are in the 1 dimensional subspace of $\odot^k \mathfrak{g}$ of the vectors with the highest occurring weight λ^{k+1}.

Let v_μ be a highest weight vector in β_{k+1}. Then

$$v_\mu = X \otimes v_{\lambda^k} + \cdots ,$$

where v_{λ^k} is a highest weight vector in $\beta_k = L(\lambda^k)$ and $X \in \mathfrak{g}$ is a Cartan element or a root vector. It follows that $\mu = \alpha + \lambda^k$ for $\alpha \in \Delta$ or $\mu = \lambda^k$.

First assume $\mu = \lambda^k$. Equation (2) implies that $Cv_\mu = (\lambda^k, \lambda^k + 2\rho)v_\mu$, for C the Casimir operator. Similarly to Lemma 4.5 in [5] it follows that C acts on β_{k+1} through $(\lambda^{k+1}, \lambda^{k+1} + 2\rho)$. A highest weight vector v_μ in β_{k+1} hence implies

$$(\lambda^k, \lambda^k + 2\rho) = (\lambda^{k+1}, \lambda^{k+1} + 2\rho).$$

Using (1) it follows that $(\lambda^k, \lambda^k + 2\rho) = 2k(k + m - n - 1)$, so the displayed condition is equivalent to $2k = -m + n$. As this contradicts $m - n > 2$, we conclude that there is no highest weight vector in β_{k+1} with weight $\mu = \lambda^k$.

Now assume $\mu = \lambda^k + \alpha$ for $\alpha \in \Delta$. We will consider the case $k \geq n$, the case $k < n$ being similar. Since μ has to be dominant regular, the possibilities for α are

$$\begin{gathered}\epsilon_1 - \epsilon_m, \epsilon_1 - \epsilon_{m-1}, \epsilon_2 - \epsilon_m, \epsilon_1 - \delta_n, \epsilon_2 - \delta_n, \\ \epsilon_2 - \epsilon_{m-1}, \epsilon_m - \delta_n, \delta_1 - \delta_n, -\epsilon_1 + \epsilon_2, -\epsilon_1 + \epsilon_m, \delta_1 - \epsilon_1, \delta_1 - \epsilon_m, \delta_1 - \epsilon_{m-1}.\end{gathered} \tag{6}$$

Observe that for example $\epsilon_2 - \epsilon_{m-1}$ can not occur since applying $X_{\epsilon_1 - \epsilon_2}$ to the highest weight vector should be zero, but the result would contain a term with the factor $X_{\epsilon_1 - \epsilon_{m-1}}$ which can not be compensated for. By choosing the appropriate simple root vector, we can eliminate all the possibilities in (6).

For the root $\epsilon_1 - \delta_n$ the Casimir operator acts on v_μ by

$$(\lambda^k + \epsilon_1 - \delta_n, \lambda^k + \epsilon_1 - \delta_n + 2\rho) = 2k(k + m - n) + 2(m - n - 1).$$

Since this is different from $(\lambda^{k+1}, \lambda^{k+1} + 2\rho) = 2(k + 1)(k + m - n)$, this excludes $\epsilon_1 - \delta_n$. Similarly for $\epsilon_2 - \delta_n$, $\epsilon_1 - \epsilon_{m-1}$ and $\epsilon_2 - \epsilon_m$ we get

$$\begin{aligned}(\lambda^k + \epsilon_2 - \delta_n, \lambda^k + \epsilon_2 - \delta_n + 2\rho) &= 2k(k + m - n - 1) + 2(m - n - 2), \\ (\lambda^k + \epsilon_1 - \epsilon_{m-1}, \lambda^k + \epsilon_1 - \epsilon_{m-1} + 2\rho) &= 2k(k + m - n) + 2(m - 1), \\ (\lambda^k + \epsilon_2 - \epsilon_m, \lambda^k + \epsilon_2 - \epsilon_m + 2\rho) &= 2k(k + m - n) + 2(m - n - 1).\end{aligned}$$

Because $k \geq n$, these expressions are different from $(\lambda^{k+1}, \lambda^{k+1} + 2\rho)$. Hence there exists no v_μ in β_{k+1} for these roots.

We conclude that the only possibility is $\epsilon_1 - \epsilon_m$ for $k \geq n$. For $k < n$ we find similarly that only $\epsilon_1 - \delta_{n-k}$ is possible. Therefore β_{k+1} contains only one highest weight vector, up to multiplicative constant, namely $v_{\lambda^{k+1}}$. The submodule of β_{k+1} (which is also a submodule of $\otimes^{k+1}\mathfrak{g}$) generated by such a highest weight vector must therefore be isomorphic to $L(\lambda^{k+1})$. Since $\otimes^{k+1}\mathfrak{g}$ is self-dual for $\vee$, $L(\lambda^{k+1})$ must also appear as a quotient of $\otimes^{k+1}\mathfrak{g}$. However, as the weight λ^{k+1} appears with multiplicity one in $\otimes^{k+1}\mathfrak{g}$, we find $[\otimes^{k+1}\mathfrak{g} : L(\lambda^{k+1})] = 1$ and $L(\lambda^{k+1})$ must be a direct summand.

In particular $L(\lambda^{k+1})$ has a complement inside β_{k+1}. By the above, the latter complement is a finite dimensional weight module which has no highest weight vectors, implying that it must be zero, so $\beta_{k+1} \cong L(\lambda^{k+1})$. Hence we find indeed that for all $k \geq 2$ we have $\beta_k \cong L(\lambda^k)$ and that this is a direct summand in $\otimes^k\mathfrak{g}$.

We have a non-degenerate form on $\otimes^k\mathfrak{g}$ such that $\beta_k^\perp = I_k$ (see Sect. 4 in [5].) Hence $\dim \otimes^k\mathfrak{g} = \dim \beta_k + \dim I_k$. Since $I_k \cap L(\lambda^k) = 0$ we conclude $\otimes^k\mathfrak{g} = L(\lambda^k) \oplus I_k$, which finishes the proof of the lemma. $\square$

Any two sided ideal $\mathcal{L}$ in $T(\mathfrak{g})$ is a submodule for the adjoint representation. Set $T_{\leq k}(\mathfrak{g}) = \oplus_{j \leq k} \otimes^j \mathfrak{g}$ and define the modules $\mathcal{L}_k \subseteq \otimes^k\mathfrak{g}$ by

$$\mathcal{L}_k = \left((\mathcal{L} + T_{\leq k-1}(\mathfrak{g})) \cap T_{\leq k}(\mathfrak{g})\right) / T_{\leq k-1}(\mathfrak{g}).$$

One can easily prove that if there is a strict inclusion $\mathcal{L}^1 \subsetneq \mathcal{L}^2$, then there must be some k for which $\mathcal{L}^1_k \subsetneq \mathcal{L}^2_k$, see e.g. the proof of Theorem 5.4 in [5].

Lemma 4 *Let $\mathfrak{g} = sl(m|n)$ with $m - n > 2$. Consider a two-sided ideal $\mathfrak{K}$ in $U(\mathfrak{g})$. If $\mathfrak{K}$ contains J_{λ^c} and has infinite codimension, then $\mathfrak{K} = J_{\lambda^c}$.*

Proof Let $\mathcal{J}_\lambda$ be as defined in (4) and denote by $\mathcal{K}$ the kernel of the composition. $T(\mathfrak{g}) \twoheadrightarrow U(\mathfrak{g}) \twoheadrightarrow U(\mathfrak{g})/\mathfrak{K}$. We have that $(\mathcal{J}_\lambda)_k = I_k$ with I_k as defined in (5). Since $J_{\lambda^c} \subset \mathfrak{K}$, also $(\mathcal{J}_{\lambda^c})_k \subset \mathcal{K}_k$ holds. If $\mathcal{K}$ would be strictly bigger than $\mathcal{J}_{\lambda^c}$, then for some k, $\mathcal{K}_k$ would be bigger than $(\mathcal{J}_{\lambda^c})_k = I_k$. Lemma 3 would then imply that $\mathcal{K}_k = \otimes^k\mathfrak{g}$ and thus also $\mathcal{K}_l = \otimes^l\mathfrak{g}$ for all $l \geq k$, since $\mathcal{K}$ is a two-sided ideal. This is a contradiction with the infinite codimension of $\mathfrak{K}$. Therefore we conclude that $\mathcal{K} = \mathcal{J}_{\lambda^c}$ and thus $\mathfrak{K} = J_{\lambda^c}$. $\square$

Proof of Theorem 4 From the assumed property of $gr(\mathfrak{K})$ follows that for each $X, Y \in \mathfrak{g}$, we have

$$XY + (-1)^{|X||Y|}YX - 2X \circledcirc Y + Z(X, Y) + c(X, Y) \in \mathfrak{K}, \tag{7}$$

where $Z(X, Y) \in \mathfrak{g}$ and $c(X, Y) \in \mathbb{C}$. Since $\mathfrak{K}$ is a two-sided ideal, we can interpret Z and c as $\mathfrak{g}$-module morphism from $\mathfrak{g} \odot \mathfrak{g}$ to $\mathfrak{g}$ and to $\mathbb{C}$ respectively. Furthermore we assumed $\mathfrak{K}$ to be invariant under the canonical automorphism τ. So applying τ to (7) and subtracting we get that $2Z(X, Y)$ is in $\mathfrak{K}$. If Z would be a morphism different from zero, then it follows from the simplicity of $\mathfrak{g}$ under the adjoint operation that Z is surjective. Hence $\mathfrak{g} \subset \mathfrak{K}$, a contradiction with the infinite codimension of $\mathfrak{K}$. From

Theorem 1 it also follows that $c(X, Y) = \lambda\langle X, Y\rangle$ for some constant λ. This implies that $J_\lambda \subset \mathfrak{K}$. Since $\mathfrak{K}$ has infinite codimension, Theorem 3 and Lemma 4 imply that $\lambda = \lambda^c = -\frac{1}{8(m-n+1)}$ and $\mathfrak{K} = J_{\lambda^c}$. □

4 A Minimal Realisation and Primitivity of the Joseph Ideal

In [2] the authors construct polynomial realisations for $\mathbb{Z}$-graded Lie algebras. Consider the 3-grading on $\mathfrak{gl}(m|n)$ by the eigenspaces of $\mathrm{ad} H_{\epsilon_1}$. We consider the corresponding 3-grading $\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_+$ inherited by the subalgebra $\mathfrak{g} = \mathfrak{sl}(m|n)$.

The procedure in [2, Sect. 3] then gives realisations of $\mathfrak{g}$ as (complex) polynomial differential operators on a real flat supermanifold with same dimensions as $\mathfrak{g}_-$, so on $\mathbb{R}^{m-1|n}$. We choose coordinates x_i with corresponding partial differential operators ∂_i, for $2 \le i \le m+n$, both are even for $i \le m$ and odd otherwise.

As $\mathfrak{g}_0 \cong \mathfrak{gl}(m-1|n)$, the space of characters $\mathfrak{g}_0 \to \mathbb{C}$ is in bijection with $\mathbb{C}$. If we apply the construction in [2, Sect. 3] to the character corresponding to $\mu \in \mathbb{C}$, we find a realisation π_μ satisfying

$$\pi_\mu(X_{\epsilon_j-\epsilon_1}) = x_j \quad \text{and} \quad \pi_\mu(X_{\epsilon_1-\epsilon_j}) = (\mu - \mathbb{E})\partial_j \quad \text{for} \quad 2 \le j \le m+n, \tag{8}$$

with $\mathbb{E} = \sum_{i=2}^{m+n} x_i\partial_i$. The other expressions for π_μ follow from the above and the fact that, since π_μ is a realisation, we have for all X, Y in $\mathfrak{g}$

$$\pi_\mu(X)\pi_\mu(Y) - (-1)^{|X||Y|}\pi_\mu(Y)\pi_\mu(X) = \pi_\mu([X, Y]).$$

Furthermore for $A \in \mathfrak{g} \odot \mathfrak{g}$, let $A = B + C + D + E$ be the decomposition given in Theorem 2. If we choose $\mu = \frac{n-m}{2}$, then we can calculate

$$\pi_{\frac{n-m}{2}}(C) = 0 = \pi_{\frac{n-m}{2}}(D) \quad \text{and} \quad \pi_{\frac{n-m}{2}}(E) = \lambda^c\mathcal{K}(A),$$

$$\text{with} \;\; \lambda^c = -\frac{1}{8(m-n+1)}.$$

Therefore we conclude

$$\left(\pi_{\frac{n-m}{2}}(X \otimes Y) - \pi_{\frac{n-m}{2}}(X \circledcirc Y) - \frac{1}{2}\pi_{\frac{n-m}{2}}([X, Y]) - \lambda^c\pi_{\frac{n-m}{2}}(\langle X, Y\rangle)\right) = 0. \tag{9}$$

Now we interpret π_μ as a representation of $\mathfrak{g}$ on the space of polynomials, i.e., on $S(\mathfrak{g}_-)$. Equation (9) then implies that the annihilator ideal of the representation $\pi_{\frac{n-m}{2}}$ contains the Joseph ideal J_{λ^c}. Since the representation is infinite dimensional, the Joseph ideal must have infinite codimension, which proves part (ii) of Theorem 3. For $m - n > 2$ it follows from Lemma 4 that the Joseph ideal is even equal to the annihilator ideal. Furthermore in this case, it follows clearly from Eq. (8) that the representation is simple.

In conclusion, we find that for $m - n > 2$, the Joseph ideal J_{λ^c} is primitive.

Acknowledgments SB is a PhD Fellow of the Research Foundation - Flanders (FWO). KC is supported by the Research Foundation - Flanders (FWO) and by Australian Research Council Discover-Project Grant DP140103239. The authors thank Jean-Philippe Michel for raising the question which led to the study in Theorem 4.

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“Spread” Restricted Young Diagrams from a 2D WZNW Dynamical Quantum Group

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Abstract The Fock representation of the Q-operator algebra for the diagonal 2D $\widehat{su}(n)_k$ WZNW model where $Q = (Q^i_j)$, $Q^i_j = a^i_\alpha \otimes \bar{a}^\alpha_j$, and $a^i_\alpha, \bar{a}^\beta_j$ are the chiral WZNW “zero modes”, has a natural basis labeled by $su(n)$ Young diagrams $Y_{\mathbf{m}}$ subject to the “spread” restriction

$$\boxed{\mathrm{spr}\,(Y_{\mathbf{m}}) := \#(\text{columns}) + \#(\text{rows}) \leq k + n =: h.}$$

1 Introduction

This work contains a brief exposition of new results based on ideas and techniques, some parts of which have been already made public in [20, 22, 23]. The latter relied, in turn, on the notion of quantum matrix algebras generated by the chiral zero modes of the $SU(n)_k$ Wess–Zumino–Novikov–Witten (WZNW) model introduced in [21] (see also [16, 18, 25]). The relation of such algebraic objects with quantum groups [7, 26] has been anticipated in [1, 12]. For generic values of the deformation parameter q the Fock representation of the chiral zero modes' algebra is a model space of $U_q(s\ell(n))$ [4, 16, 20]. In the most interesting applications the deformation parameter q is a root of unity (in our case we take $q = e^{-i\frac{\pi}{h}},\ h = k + n$). It has been shown, in particular, in [19] that the Fock representation of the chiral zero modes' algebra for $n = 2$ carries a representation of the *restricted* (finite dimensional) quantum group $\overline{U}_q(s\ell(2))$ of [13, 14] containing, as submodules or quotient modules, all irreducible representations of the latter.

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V. Dobrev (ed.), *Lie Theory and Its Applications in Physics*, Springer Proceedings in Mathematics & Statistics 191,
DOI 10.1007/978-981-10-2636-2_37

Combining the left and right chiral zero modes' algebras, one obtains a particular ($2D$ zero modes') *dynamical* quantum group [9]. Its role in the description of the internal (sector) structure of the $n = 2$ WZNW model has been studied in [8, 17] where it has been shown that it provides in a natural way a finite extension of the unitary model. The setting is reminiscent to the axiomatic (cohomological) approach to gauge theories, the quantum group playing the role of generalized gauge symmetry.

The main statement of the present paper is that the Fock representation of the $2D$ zero modes' algebra has a basis that is in a one-to-one correspondence with the finite set of $su(n)$ Young diagrams [15] restricted by a specific "spread" condition. These fit into a rectangle of size $(n-1) \times (h-1)$ which is thus wider than the $(n-1) \times k$ rectangle containing the unitary $\widehat{su}(n)_k$ fusion sectors [5]. (The ring structure of the latter, i.e., the Verlinde algebra, is conveniently described by a suitable representation of the phase model hopping operator algebra or, in other terms, of the affine local plactic algebra [27, 30].) Note that the spread restriction is more stringent than just the fitting into the $(n-1) \times (h-1)$ rectangle requirement.

It would be interesting to find out if the affine algebra representations (some of which non-integrable) corresponding to the finite set of "spread restricted" $su(n)$ diagrams constitute a sensible extension of the unitary WZNW model. We hope that, on the long run, the present approach could help to better understand the adequate "gauge" symmetry (the $2D$ counterpart of the Doplicher–Roberts [6] compact group) governing the "addition of non-abelian charges", i.e., the fusion rules, in RCFT (cf. [2, 3, 10, 11, 24, 28, 29]).

2 Definitions: $SU(n)_k$ WZNW Zero Modes

We will recall here the basic assumptions about the chiral and $2D$ WZNW "zero modes" and their Fock representation, justified by the consistent application of the principles of canonical quantization (see e.g. [20, 23]).

The mutually commuting left and right $SU(n)_k$ WZNW chiral zero modes' algebras $\mathcal{M}_q$, $\bar{\mathcal{M}}_q$ are generated by operators $\{q^{p_j}, a^i_\alpha\}$ and $\{q^{\bar{p}_j}, \bar{a}^\alpha_i\}$, respectively, satisfying identical exchange relations:

$$
\begin{gathered}
q^{p_i} q^{p_j} = q^{p_j} q^{p_i} \ , \quad \prod_{j=1}^{n} q^{p_j} = 1 \ , \quad q^{p_{j\ell}} a^i_\alpha = a^i_\alpha \, q^{p_{j\ell} + \delta^i_j - \delta^i_\ell} \quad (p_{j\ell} := p_j - p_\ell) \ , \\
q^{\bar{p}_i} q^{\bar{p}_j} = q^{\bar{p}_j} q^{\bar{p}_i} \ , \quad \prod_{j=1}^{n} q^{\bar{p}_j} = 1 \ , \quad q^{\bar{p}_{j\ell}} \, \bar{a}^\alpha_i = \bar{a}^\alpha_i \, q^{\bar{p}_{j\ell} + \delta_{ij} - \delta_{i\ell}} \quad (\bar{p}_{j\ell} := \bar{p}_j - \bar{p}_\ell)
\end{gathered}
\tag{1}
$$

all indices run from 1 to n. Bilinear combinations of chiral zero modes intertwine dynamical and constant R-matrices; the left and right sector quadratic exchange relations following from

$$\hat{R}_{12}(p)\, a_1\, a_2 = a_1\, a_2\, \hat{R}_{12}\ , \quad \hat{R}_{12}\, \bar{a}_1\, \bar{a}_2 \ = \ \bar{a}_1\, \bar{a}_2\, \hat{\hat{R}}_{12}(\bar{p}) \quad (\hat{\hat{R}}_{12}(\bar{p}) = (\hat{R}_{12}(\bar{p}))^t)\ , \tag{2}$$

respectively, coincide as well when written in components:

$$\begin{aligned} & a^j_\beta a^i_\alpha\, [p_{ij} - 1] = a^i_\alpha a^j_\beta\, [p_{ij}] - \ a^i_\beta\, a^j_\alpha\, q^{\epsilon_{\alpha\beta} p_{ij}} \quad (\text{for} \quad i \neq j \quad \text{and} \quad \alpha \neq \beta)\ , \\ & [a^j_\alpha, a^i_\alpha] = 0\ , \qquad a^i_\alpha a^i_\beta = q^{\epsilon_{\alpha\beta}}\, a^i_\beta a^i_\alpha\ , \qquad [p] := \frac{q^p - q^{-p}}{q - q^{-1}}\ ; \end{aligned} \tag{3}$$

$$\begin{aligned} & \bar{a}^\beta_j \bar{a}^\alpha_i\, [\bar{p}_{ij} - 1] = \bar{a}^\alpha_i \bar{a}^\beta_j\, [\bar{p}_{ij}] - \ \bar{a}^\beta_i\, \bar{a}^\alpha_j\, q^{\epsilon_{\alpha\beta} \bar{p}_{ij}} \quad (\text{for} \quad i \neq j \quad \text{and} \quad \alpha \neq \beta)\ , \\ & [\bar{a}^\alpha_j, \bar{a}^\alpha_i] = 0\ , \qquad \bar{a}^\alpha_i \bar{a}^\beta_i = q^{\epsilon_{\alpha\beta}}\, \bar{a}^\beta_i \bar{a}^\alpha_i \end{aligned} \tag{4}$$

(the antisymmetric symbol $\epsilon_{\alpha\beta} = \pm 1$ for $\alpha \gtrless \beta$ and vanishes for $\alpha = \beta$). They are supplemented by appropriate n-linear determinant conditions,

$$\det(a) = \mathcal{D}_q(p)\ , \qquad \det(\bar{a}) = \mathcal{D}_q(\bar{p})\ , \tag{5}$$

where $\mathcal{D}_q(p) := \prod_{i<j} [p_{ij}]$ and

$$\begin{aligned} \det(a) &:= \frac{1}{[n]!}\, \epsilon_{i_1 \dots i_n}\, a^{i_1}_{\alpha_1} \dots a^{i_n}_{\alpha_n}\, \varepsilon^{\alpha_1 \dots \alpha_n}\ , \\ \det(\bar{a}) &:= \frac{1}{[n]!}\, \varepsilon_{\alpha_1 \dots \alpha_n}\, \bar{a}^{i_1}_{\alpha_1} \dots \bar{a}^{i_n}_{\alpha_n}\, \epsilon^{i_1 \dots i_n}\ , \end{aligned} \tag{6}$$

$\epsilon_{i_1 \dots i_n} = \epsilon^{i_1 \dots i_n}$ and $\varepsilon_{\alpha_1 \dots \alpha_n} = \varepsilon^{\alpha_1 \dots \alpha_n}$ being the "ordinary" and "quantum" fully (q-) antisymmetric n-tensors, respectively. Finally, for $q^h = -1$ the chiral zero modes' algebras $\mathcal{M}_q$, $\bar{\mathcal{M}}_q$ possess non-trivial two-sided ideals such that the corresponding factor algebras $\mathcal{M}^{(h)}_q$ and $\bar{\mathcal{M}}^{(h)}_q$ are characterized by the additional relations

$$(a^i_\alpha)^h = 0\ , \quad q^{2h\, p_{j\ell}} = 1\ , \qquad (\bar{a}^\alpha_i)^h = 0\ , \quad q^{2h\, \bar{p}_{j\ell}} = 1\ . \tag{7}$$

(Strictly speaking, the two algebras are identified with the corresponding *non-commutative polynomial* rings in a^i_α and $\bar{a}^\alpha_i$ over the fields of rational functions of q^{p_j} and $q^{\bar{p}_j}$, respectively.) We will be interested in the Fock space representation $\mathcal{F}^{(h)} \otimes \bar{\mathcal{F}}^{(h)}$ of $\mathcal{M}^{(h)}_q \otimes \bar{\mathcal{M}}^{(h)}_q$, where

$$\mathcal{F}^{(h)} = \mathcal{M}^{(h)}_q\, |0\rangle\ , \qquad \bar{\mathcal{F}}^{(h)} = \bar{\mathcal{M}}^{(h)}_q\, |0\rangle\ . \tag{8}$$

The action of the generating elements on the vacuum vector is subject to

$$\begin{aligned} & q^{p_{j\ell}}\, |0\rangle = q^{\ell - j}\, |0\rangle = q^{\bar{p}_{j\ell}}\, |0\rangle\ , \qquad j, \ell = 1, \dots, n\ , \\ & a^i_\alpha\, |0\rangle = 0 = \bar{a}^\alpha_i\, |0\rangle\ , \qquad i = 2, \dots, n\ . \end{aligned} \tag{9}$$

It follows that monomials in a^i_α, $\bar{a}^\alpha_i$ generate eigenvectors of q^{p_j} and $q^{\bar{p}_j}$, respectively, with eigenvalues of $p_{j\ell}$, $\bar{p}_{j\ell}$ corresponding to *shifted* $su(n)$ weights (e.g. $p_{jj+1} = \lambda_j + 1$, the vacuum quantum numbers being given by the components of the Weyl vector).

Defining $Q^i_j = a^i_\alpha \otimes \bar{a}^\alpha_j \in \mathcal{M}^{(h)}_q \otimes \bar{\mathcal{M}}^{(h)}_q$, we will call the corresponding operator algebra "the Q-algebra" (of the $SU(n)_k$ WZNW model). Our task will be to describe the structure of its vacuum representation as a subspace of the "extended" (carrier) space $\mathcal{F}^{(h)} \otimes \bar{\mathcal{F}}^{(h)}$.

This has been done in a completely satisfactory way for $n = 2$ (in [20, 23]; see also [17]) and the emerging picture is easy to describe. It turns out that in this case the diagonal elements of the matrix $Q = (Q^i_j)$ commute with the off-diagonal ones and both generate two copies of the (finite dimensional) *restricted quantum group* $\overline{U}_q(s\ell(2))$ of [13, 14, 19]. The corresponding Fock space representations are however quite different: while the one generated by the off-diagonal elements of Q is one dimensional, the diagonal Q-operators span a subspace $\mathcal{F}^{diag}$ of dimension $h = k + 2$ in the h^4-dimensional $\mathcal{F}^{(h)} \otimes \bar{\mathcal{F}}^{(h)}$. Furthermore, there is a natural scalar product on $\mathcal{F}^{diag}$ which is positive semidefinite, the subspace of zero-norm vectors $\mathcal{F}''$ being one-dimensional, $\mathcal{F}'' = \mathbb{C}\,(Q^1_1)^{h-1}\,|0\rangle$. One obtains in effect a finite dimensional toy generalization of axiomatic gauge theory, the role of the pre-physical subspace $\mathcal{F}'$ being played by $\mathcal{F}^{diag}$ and such that the physical subquotient

$$\mathcal{F}^{phys} = \mathcal{F}'/\mathcal{F}'' \simeq \oplus_{p=1}^{h-1} \mathcal{F}^{phys}_p\,, \qquad \mathcal{F}^{phys}_p := \mathbb{C}\,(Q^1_1)^{p-1}\,|0\rangle \tag{10}$$

contains exactly the fusion sectors $\mathcal{F}^{phys}_p$ $(p = 2I + 1)$ of the unitary model.

It is this picture that we would like to generalize to $n \geq 3$ when $q = e^{-i\frac{\pi}{h}}$, $h = k + n$.

3 The Q-Algebra for $n \geq 3$ and the Space $\mathcal{F}'$

For the lack of space we will only sketch in this section the derivation of the results for $n \geq 3$ postponing most of the interesting details to a forthcoming publication. First of all, it is easy to see that (3), (4) and (7) imply

$$(Q^i_j)^h = 0\,. \tag{11}$$

Combining further the quadratic exchange relations for the left and right sector zero modes (2), we obtain those for the Q-operators in a *dynamical quantum group* form [9, 25]:

$$\hat{R}_{12}(p)\, Q_1\, Q_2 = Q_1\, Q_2\, \hat{\bar{R}}_{12}(\bar{p})\,. \tag{12}$$

(As in the case of chiral exchange relations (3), (4), we will actually postulate relations obtained after getting rid of the denominators in the entries of the two dynamical

R-matrices.) A straightforward computation shows that, as a result, any two entries of the matrix Q belonging to the same row or column commute.

All this has been already proved in [23] where the problem of commutativity of diagonal and off-diagonal elements has been also addressed. The novelty we would like to announce here answers this question and also describes the $n \geq 3$ counterpart of the pre-physical space $\mathcal{F}'$ of Sect. 2, providing a basis in it labeled by (a certain finite set of) $su(n)$ Young diagrams.

To this end we first introduce again the space $\mathcal{F}^{diag}$ generated from the vacuum by diagonal Q-operators. Due to (9), the vacuum is annihilated by all Q^i_j except for $i = j = 1$. Depicting the action of each diagonal operator Q^j_j by adding a box to the j-th row of a table with n rows, we can make correspond to any vector in $\mathcal{F}^{diag}$ generated by a monomial a unique tableau with boxes numbered in the order of appearance in the product (counted from the right) of the specific operator.

It is not clear from the outset even if $\mathcal{F}^{diag}$ is finite dimensional. However, one immediately realizes that any *single row table* containing more than $h-1$ boxes should vanish, due to (11) implying $v_h^{(1)} := (Q^1_1)^h\,|0\rangle = 0$. Consider next the vector $v_h^{(2)} := Q^2_2\,(Q^1_1)^{h-1}\,|0\rangle$. Noting that on $\mathcal{F}^{diag}$ the eigenvalues of $p_{j\ell}$ and $\bar{p}_{j\ell}$ coincide and that (12) implies, for any $v \in \mathcal{F}^{diag}$,

$$Q^i_j\, v = 0 \quad \text{or} \quad Q^j_i\, v = 0 \quad \Rightarrow \quad [p_{ij}+1]\, Q^i_i Q^j_j\, v = [p_{ij}-1]\, Q^j_j Q^i_i\, v \tag{13}$$

(see [23]), we deduce from (1) and (9) that (since $[h-2] = [2] \neq 0$)

$$\begin{aligned} &[p_{21}+1]\, Q^2_2\,(Q^1_1)^{h-1}\,|0\rangle = [p_{21}-1]\, Q^1_1\, Q^2_2\,(Q^1_1)^{h-2}\,|0\rangle\,, \quad \text{i.e.,} \\ &-[h-2]\, v_h^{(2)} = -[h]\, w_h^{(2)} \equiv 0 \quad \Rightarrow \quad v_h^{(2)} = 0\,, \end{aligned} \tag{14}$$

where $w_h^{(2)} := Q^1_1\, Q^2_2\,(Q^1_1)^{h-2}\,|0\rangle$. Similar observations suggest to introduce the space $\mathcal{F}' \subseteq \mathcal{F}^{diag}$ as the linear span of vectors of the type

$$v_{\mathbf{m}} := (Q^i_i)^{m_i} \dots (Q^2_2)^{m_2} (Q^1_1)^{m_1}\,|0\rangle\,, \quad \mathbf{m} = (m_1, m_2, \dots, m_i, 0, \dots, 0) \tag{15}$$

with

$$1 \leq i \leq n-1\,, \quad m_i \leq m_{i-1} \leq \dots \leq m_1\,, \quad m_1 + i \leq h\,. \tag{16}$$

It turns out that it has remarkable properties.

- $\mathcal{F}'$ is finite dimensional
- it is annihilated by any off-diagonal Q-operator: $Q^i_j\,\mathcal{F}' = 0$ for $i \neq j$
- the basis (15) can be labeled by admissible $su(n)$ *Young diagrams* $Y_{\mathbf{m}}$ of *maximal hook length* not exceeding $h-1$ or, which is the same, of *spread*

$$\text{spr}\,(Y_{\mathbf{m}}) := i + m_1 \leq h\,. \tag{17}$$

The last assertion needs some clarification. We first verify, by using (13), that reordering the factors in the Q-monomial (15) reproduces one and the same vector, up to a non-zero coefficient, as far the "standard $su(n)$ rule" $m_j \leq m_{j-1}$ is respected at any step (i.e., also in subdiagrams obtained by removing an arbitrary number of Q-operators from the left); so under this condition the numeration of boxes is irrelevant. What we call "maximal hook length" (of a non-trivial Young diagram) is the hook length of the box in the upper left (NW) corner; we have used the term "spread" for the sum of the numbers of columns and rows.

To show that $\mathcal{F}' = \mathcal{F}^{diag}$, we have to prove that any action violating the conditions (16), or equivalently – in the "diagrammatic language" – that

- adding a box to the n-th row
- adding an extra box to the j-th row when $m_j = m_{j-1}$ or, finally,
- adding an extra box to the first column or row of a diagram saturating the spread inequality (17), i.e., for which spr $(Y_{\mathbf{m}}) = h$

all lead to a zero vector (or to another vector in $\mathcal{F}'$). The proof relies essentially on the careful application of (13) (and, for the first point, also on the determinant conditions (5)). The only difficulty arises when one violates the spread inequality by adding an extra box to the first row of the diagram (i.e., $m_1 \to m_1 + 1$ for $i + m_1 = h$).

In this last case the problem can be reduced to hook shaped diagrams (all boxes in a diagram saturating (17) become irrelevant except those in the first row and the first column which form its "backbone"). In effect, for $i > 2$ we just have to generalize (14), introducing

$$v_h^{(i)} = Q_i^i Q_1^1 \, v \ , \quad w_h^{(i)} = Q_1^1 Q_i^i \, v \ , \quad v := Q_{i-1}^{i-1} \dots Q_2^2 \, (Q_1^1)^{h-i} \, |0\rangle \ . \tag{18}$$

The argument why $v_h^{(i)} = 0$ is the same as in (14); we only have to evaluate $[p_{i1} \pm 1] \, v = [1 - h \pm 1] \, v$. The problem is to show that $w_h^{(i)} = 0$ too. To tackle it, we need a new technique which is introduced in the next section.

Before going to it we will make the following important remark. It is obvious that the $su(n)$ Young diagrams of spread restricted by h fit into a rectangle of size $(n-1) \times (h-1)$. (The spread condition imposes a stronger restriction, except for $n = 2$). In the WZNW setting the zero modes are coupled to elementary chiral vertex operators (CVO) with similar intertwining properties [20]; having this in mind, we note that the *integrable* representations of the affine algebra $\widehat{su}(n)_k$ (or the fusion sectors of the unitary model) are labeled by *all* $su(n)$ Young diagrams that fit into the "narrower" rectangle of size $(n-1) \times k$. Extending the analogy with the $n = 2$ case, cf. (10), we would expect, in particular, the vectors $v_{\mathbf{m}}$ (15) to have zero norm if they correspond to diagrams outside the "unitary" rectangle. (All such diagrams have thus also boxes in the additional $(n-1) \times (n-1)$ square) (Figs. 1, 2, 3, 4, 5, 6, 7, 8).

The following figures illustrate the above ideas and notions.

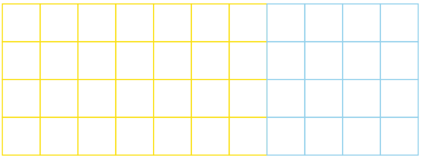

Fig. 1 The $(n-1) \times (h-1)$ *rectangle* for $n = 5$, $k = 7$

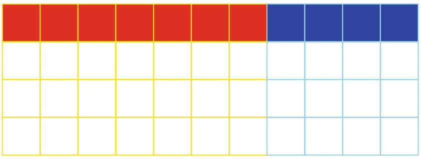

Fig. 2 *Single row* diagram Y of maximal spread $(\mathrm{spr}(Y) = 12)$

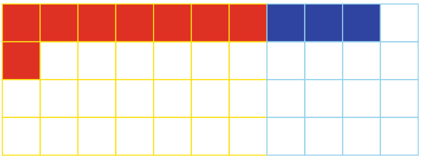

Fig. 3 *Hook-shaped* diagram of maximal spread

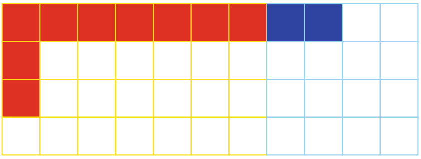

Fig. 4 *Hook-shaped* diagram of maximal spread

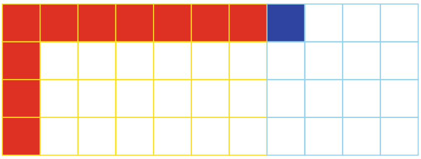

Fig. 5 *Hook-shaped* diagram of maximal spread

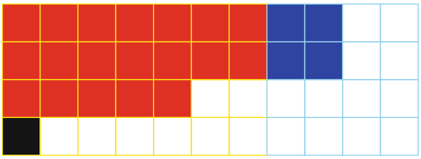

Fig. 6 A “*forbidden*” diagram (the *black box* is extra)

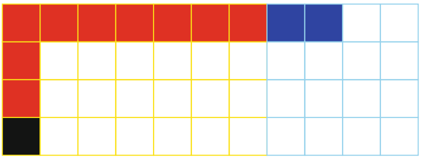

Fig. 7 A “*forbidden*” *hook-shaped* diagram (the “backbone” of Fig. 6)

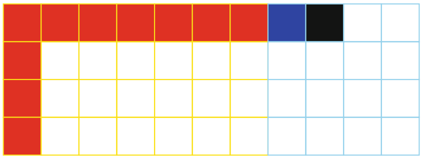

Fig. 8 The same ("*forbidden*"?) *hook-shaped* diagram as in Fig. 7

4 Chiral q-Symmetric and q-Antisymmetric Bilinears

It turns out that the cumbersome bilinear exchange relations (3), (4) assume an amazingly simple form when written in terms of the corresponding q-antisymmetric and q-symmetric bilinear combinations

$$a^i_\alpha a^j_\beta = A^{ij}_{\alpha\beta} + S^{ij}_{\alpha\beta}\,, \quad A^{ij}_{\alpha\beta} = -q^{-\epsilon_{\alpha\beta}} A^{ij}_{\beta\alpha}\,, \quad S^{ij}_{\alpha\beta} = q^{\epsilon_{\alpha\beta}} S^{ij}_{\beta\alpha} \tag{19}$$

defined by

$$[2]\, A^{ij}_{\alpha\beta} := a^i_{\alpha'} a^j_{\beta'} A^{\alpha'\beta'}_{\alpha\beta} = \begin{cases} q^{-\epsilon_{\alpha\beta}} a^i_\alpha a^j_\beta - a^i_\beta a^j_\alpha\,, & \alpha \neq \beta \\ 0\,, \quad \alpha = \beta \end{cases} \tag{20}$$

and

$$[2]\, S^{ij}_{\alpha\beta} := a^i_{\alpha'} a^j_{\beta'} S^{\alpha'\beta'}_{\alpha\beta} = \begin{cases} q^{\epsilon_{\alpha\beta}} a^i_\alpha a^j_\beta + a^i_\beta a^j_\alpha\,, & \alpha \neq \beta \\ [2]\, a^i_\alpha a^j_\alpha \;\; (\equiv [2]\, a^j_\alpha a^i_\alpha)\,, & \alpha = \beta \end{cases}\,, \tag{21}$$

respectively, and their bar analogs

$$[2]\, \bar{A}^{\alpha\beta}_{ij} := A^{\alpha\beta}_{\alpha'\beta'}\, \bar{a}^{\alpha'}_i \bar{a}^{\beta'}_j\,, \qquad [2]\, \bar{S}^{\alpha\beta}_{ij} := S^{\alpha\beta}_{\alpha'\beta'}\, \bar{a}^{\alpha'}_i \bar{a}^{\beta'}_j\,. \tag{22}$$

A simple calculation [20] shows that relations (3), (4) for $i \neq j$ and $\alpha \neq \beta$ are equivalent to

$$\begin{aligned} [p_{ij}+1]\, A^{ij}_{\alpha\beta} &= -[p_{ij}-1]\, A^{ji}_{\alpha\beta}\,, \qquad & S^{ij}_{\alpha\beta} &= S^{ji}_{\alpha\beta}\,, \\ [\bar{p}_{ij}+1]\, \bar{A}^{\alpha\beta}_{ij} &= -[\bar{p}_{ij}-1]\, \bar{A}^{\alpha\beta}_{ji}\,, \qquad & \bar{S}^{\alpha\beta}_{ij} &= \bar{S}^{\alpha\beta}_{ji}\,. \end{aligned} \tag{23}$$

The remaining relations (3), (4) look equally simple in these terms:

$$S^{ij}_{\alpha\alpha} = S^{ji}_{\alpha\alpha}\,, \quad A^{ii}_{\alpha\beta} = 0\,, \qquad \bar{S}^{\alpha\alpha}_{ij} = \bar{S}^{\alpha\alpha}_{ji}\,, \quad \bar{A}^{\alpha\beta}_{ii} = 0\,. \tag{24}$$

Note that

$$S^{ij}_{\alpha\beta} \otimes \bar{A}^{\alpha\beta}_{\ell m} = 0 = A^{ij}_{\alpha\beta} \otimes \bar{S}^{\alpha\beta}_{\ell m} \tag{25}$$

(where summation over α and β is assumed) since e.g.

$$S^{ij}_{\alpha\beta} \otimes \bar{A}^{\alpha\beta}_{\ell m} = (q^{\epsilon_{\alpha\beta}} S^{ij}_{\beta\alpha}) \otimes (-q^{-\epsilon_{\alpha\beta}} \bar{A}^{\beta\alpha}_{\ell m}) = - S^{ij}_{\beta\alpha} \otimes \bar{A}^{\beta\alpha}_{\ell m} \ , \tag{26}$$

and hence

$$Q^i_\ell \, Q^j_m = (S^{ij}_{\alpha\beta} + A^{ij}_{\alpha\beta}) \otimes (\bar{S}^{\alpha\beta}_{\ell m} + \bar{A}^{\alpha\beta}_{\ell m}) = S^{ij}_{\alpha\beta} \otimes \bar{S}^{\alpha\beta}_{\ell m} + A^{ij}_{\alpha\beta} \otimes \bar{A}^{\alpha\beta}_{\ell m} \ . \tag{27}$$

These relations turn out to be crucial as they shed light on the quantum group "internal structure" of monomials in the Q-operators. (Their importance can be anticipated in the derivation of (11), see [23], which is actually based on the q-symmetry underlying the chiral expansion of $Q^i_{\ j} Q^i_{\ j}$.) To illustrate the idea, we will demonstrate that, presented as in (18), the vector $w^{(i)}_h$ only contains the q-antisymmetric part of $Q^1_{\ 1} Q^i_{\ i}$. Indeed, it follows from $[p_{i1} - 1]\, v = 0$ and (23) that

$$[p_{i1} + 1] A^{i1}_{\alpha\beta}\, v = -[p_{i1} - 1] A^{1i}_{\alpha\beta}\, v = 0 \quad \Rightarrow \quad v^{(i)}_h = S^{i1}_{\alpha\beta} \otimes \bar{S}^{\alpha\beta}_{i1}\, v \ . \tag{28}$$

Now taking into account that $v^{(i)}_h = 0$ and $S^{i1}_{\alpha\beta} = S^{1i}_{\alpha\beta}$, $\bar{S}^{\alpha\beta}_{i1} = \bar{S}^{\alpha\beta}_{1i}$ we infer

$$0 = S^{i1}_{\alpha\beta} \otimes \bar{S}^{\alpha\beta}_{i1}\, v = S^{1i}_{\alpha\beta} \otimes \bar{S}^{\alpha\beta}_{1i}\, v \quad \Rightarrow \quad w^{(i)}_h = A^{1i}_{\alpha\beta} \otimes \bar{A}^{\alpha\beta}_{1i}\, v \ . \tag{29}$$

Using this property, we have been able to show "by brute force", in the case $n = 3$, that $w^{(2)}_h = 0$ for small values of the level k. Finding the appropriate combinatorial arguments in the general case (of arbitrary n, i and h) remains a challenge.

Acknowledgements The authors thank Peter Dalakov, Joris Van der Jeugt, Ivan Penkov, Todor Popov, Ivan Todorov, Fernando Rodriguez Villegas and Paul Zinn-Justin for their interest and for sharing valuable information on the subject with them. LH would like to express his gratitude to Prof. V. K. Dobrev and the members of the local Organizing Committee of the 11th International Workshop "Lie Theory and Its Applications in Physics" (Varna, Bulgaria, 15–21 June 2015). The work of LH has been supported in part by Grant DFNI T02/6 of the Bulgarian National Science Foundation.

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Part VI
Vertex Algebras and Lie Algebra Structure Theory

Vertex Operator Algebras Associated with $\mathbf{Z}/k\mathbf{Z}$-Codes

Tomoyuki Arakawa, Hiromichi Yamada and Hiroshi Yamauchi

Abstract We construct a vertex operator algebra associated with a $\mathbf{Z}/k\mathbf{Z}$-code of length n for an integer $k \geq 2$. We realize it inside a lattice vertex operator algebra as the commutant of a certain subalgebra. The vertex operator algebra is isomorphic to a known one in the cases $k = 2, 3$.

1 Introduction

Let $L_{\widehat{sl}_2}(k, 0)$ be an integrable highest weight module for an affine Kac–Moody Lie algebra $\widehat{sl}_2$ at level k. Then $L_{\widehat{sl}_2}(k, 0)$ is a simple vertex operator algebra and it contains a Heisenberg vertex operator algebra generated by a Cartan subalgebra of sl_2. The commutant $K(sl_2, k)$ of the Heisenberg vertex operator algebra in $L_{\widehat{sl}_2}(k, 0)$ is called the parafermion vertex operator algebra of type sl_2. Its central charge is $2(k-1)/(k+2)$. The properties of $K(sl_2, k)$ and its irreducible modules have been studied in [1, 3, 4], etc.

In fact, irreducible modules $M^{i,j}$, $0 \leq i \leq k$, $j \in \mathbf{Z}/k\mathbf{Z}$ with $M^{i,j} \cong M^{k-i,k-i+j}$ were constructed in [3]. In [1], it was shown that the $k(k+1)/2$ irreducible modules $M^{i,j}$, $0 \leq j < i \leq k$ form a complete set of representatives of equivalence classes of irreducible modules. There is a $\mathbf{Z}/k\mathbf{Z}$-symmetry among $M^j = M^{k,j}$, $j \in \mathbf{Z}/k\mathbf{Z}$. In

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V. Dobrev (ed.), *Lie Theory and Its Applications in Physics*,
Springer Proceedings in Mathematics & Statistics 191,
DOI 10.1007/978-981-10-2636-2_38

this notation M^0 is the parafermion vertex operator algebra $K(sl_2, k)$. The top level of M^j is one dimensional with weight $j(k-j)/k$.

A $\mathbf{Z}/k\mathbf{Z}$-code D of length n means an additive subgroup of $(\mathbf{Z}/k\mathbf{Z})^n$. In this paper we construct a vertex operator algebra M_D associated with D. Actually, M_D is a direct sum of M_ξ, $\xi \in D$, where $M_\xi = M^{i_1} \otimes \cdots \otimes M^{i_n}$, $\xi = (i_1, \ldots, i_n)$ is an irreducible module for a tensor product of n copies of M^0. The vertex operator algebra M_D is defined to be the commutant of a certain subalgebra in a lattice vertex operator algebra V_{Γ_D}. The lattice Γ_D is a positive definite even lattice obtained by using the code D. We also discuss the case Γ_D is an odd lattice and M_D is a vertex operator superalgebra when k is even.

The vertex operator algebra M_D is isomorphic to a known one [7, 10] in the cases $k = 2, 3$. We generalize the construction for an arbitrary k.

2 Irreducible M^0-Modules $M^{(j)}$, $0 \le j \le k-1$

In this section we shall construct irreducible modules $M^{(j)}$, $0 \le j \le k-1$ for M^0 inside certain irreducible modules for a lattice vertex operator algebra. Those irreducible modules for M^0 will be used as building blocks of $\mathbf{Z}/k\mathbf{Z}$-code vertex operator algebras in Sect. 3.

Let $L = \mathbf{Z}\alpha_1 + \cdots + \mathbf{Z}\alpha_k$ with $\langle \alpha_i, \alpha_j \rangle = 2\delta_{ij}$ and $\gamma = \alpha_1 + \cdots + \alpha_k$. Let

$$N = \{\alpha \in L \mid \langle \alpha, \gamma \rangle = 0\}.$$

Then $N = \sum_{p=1}^{k-1} \mathbf{Z}(\alpha_p - \alpha_{p+1})$. Set $R = N \oplus \mathbf{Z}\gamma$, which is a sublattice of L with the same rank. The dual lattice $\{\alpha \in \mathbf{Q} \otimes_{\mathbf{Z}} X \mid \langle \alpha, X \rangle \subset \mathbf{Z}\}$ of an integral lattice $(X, \langle \cdot\,, \cdot \rangle)$ is denoted by X°. Then $L^\circ = \frac{1}{2}L$ and $(\mathbf{Z}\gamma)^\circ = \mathbf{Z}\frac{1}{2k}\gamma$.

Lemma 1 (1) *$R \subset L \subset L^\circ \subset R^\circ$ with $R^\circ = N^\circ \oplus (\mathbf{Z}\gamma)^\circ$.*
(2) *$L = \cup_{i=0}^{k-1}(R + i\alpha_k)$; disjoint.*
(3) *$L + \mathbf{Z}\frac{1}{2k}\gamma = \cup_{i=0}^{k-1} \cup_{j=0}^{2k-1} (R + i\alpha_k + \frac{j}{2k}\gamma)$; disjoint.*

Let $\beta_p = \alpha_p - \alpha_{p+1}$, $1 \le p \le k-1$ and

$$\lambda = \frac{1}{2k}(\beta_1 + 2\beta_2 + \cdots + (k-1)\beta_{k-1}) = \frac{1}{2k}\gamma - \frac{1}{2}\alpha_k.$$

Then $\{\frac{1}{2}\beta_2, \ldots, \frac{1}{2}\beta_{k-1}, \lambda\}$ is a $\mathbf{Z}$-basis of N° and $N^\circ/N \cong (\mathbf{Z}_2)^{k-2} \times \mathbf{Z}_{2k}$.

We define $V_{R^\circ} = M(1) \otimes \mathbf{C}\{R^\circ\}$ and $Y(v, z) \in (\mathrm{End}\, V_{R^\circ})\{z\}$ for $v \in V_{R^\circ}$ as in [2, 5, 9] so that (V_{R°, Y) is a generalized vertex algebra [2, Theorem 9.8].

The lattice R° is a nondegenerate rational lattice of rank k and R is an even sublattice of R° with the same rank. We choose

$$\left\{\frac{1}{2}\beta_2, \ldots, \frac{1}{2}\beta_{k-1}, \lambda, \frac{1}{2k}\gamma\right\} \tag{1}$$

as a $\mathbf{Z}$-basis of R°. Let $s = 4k$, so that $\frac{s}{2}\langle\alpha,\beta\rangle \in \mathbf{Z}$ for $\alpha, \beta \in R^\circ$. Define an alternating $\mathbf{Z}$-bilinear map $c_0 : R^\circ \times R^\circ \to \mathbf{Z}/s\mathbf{Z}$ as in [9, Remark 6.4.12] and a $\mathbf{Z}$-bilinear map $\varepsilon_0 : R^\circ \times R^\circ \to \mathbf{Z}/s\mathbf{Z}$ as in [5, Proposition 5.2.3] with respect to the $\mathbf{Z}$-basis (1). Then

$$\varepsilon_0(\alpha, \beta + p\lambda) = 0 \tag{2}$$

for $\alpha, \beta \in N$, $p \in \mathbf{Z}$ and

$$\varepsilon_0(\alpha, \gamma/2k) = \varepsilon_0(\gamma/2k, \alpha) = 0 \tag{3}$$

for $\alpha \in R^\circ$.

Let $\widehat{R^\circ}$ be a central extension of R° by a cyclic group $\langle\kappa_s\rangle$ of order s with section $e_\alpha \in \widehat{R^\circ}$, $\alpha \in R^\circ$ and 2-cocycle ε_0. The multiplication in $\widehat{R^\circ}$ is given by

$$e_\alpha e_\beta = e_{\alpha+\beta}\kappa_s^{\varepsilon_0(\alpha,\beta)}$$

for $\alpha, \beta \in R^\circ$.

Set $\mathbf{C}\{R^\circ\} = \mathbf{C}[\widehat{R^\circ}]/(\kappa_s - \omega_s)\mathbf{C}[\widehat{R^\circ}]$, where ω_s is a primitive s-th root of unity. For simplicity of notation we denote both $e_\alpha \in \widehat{R^\circ}$ and $\iota(e_\alpha) \in \mathbf{C}\{R^\circ\}$ in the notation of [2, 5, 9] by e^α. Then $\{e^\alpha \mid \alpha \in R^\circ\}$ is a basis of $\mathbf{C}\{R^\circ\}$ and the multiplication in $\mathbf{C}\{R^\circ\}$ is

$$e^\alpha e^\beta = \omega_s^{\varepsilon_0(\alpha,\beta)} e^{\alpha+\beta}$$

for $\alpha, \beta \in R^\circ$. By (3), the following lemma holds.

Lemma 2 $e^\alpha e^{p\gamma/2k} = e^{p\gamma/2k} e^\alpha = e^{\alpha+p\gamma/2k}$ *for* $\alpha \in R^\circ$, $p \in \mathbf{Z}$.

For any subset X of R°, let $\mathbf{C}\{X\} = \mathrm{span}\{e^\alpha \mid \alpha \in X\} \subset \mathbf{C}\{R^\circ\}$ as in [2, 5, 9]. We write $M_{\mathbf{C}\otimes_{\mathbf{Z}}X}(1)$ for the Heisenberg vertex operator algebra $M(1)$ generated by $\mathbf{C} \otimes_{\mathbf{Z}} X$ when we want to clarify the generators. Let $V_X = M_{\mathbf{C}\otimes_{\mathbf{Z}}X}(1) \otimes \mathbf{C}\{X\}$.

In this notation, $V_{R^\circ} = M_{\mathbf{C}\otimes_{\mathbf{Z}}R}(1) \otimes \mathbf{C}\{R^\circ\}$. Since R° is an orthogonal sum of N° and $(\mathbf{Z}\gamma)^\circ$, we have $\mathbf{C}\{R^\circ\} = \mathbf{C}\{N^\circ\} \otimes \mathbf{C}\{(\mathbf{Z}\gamma)^\circ\}$ as associative algebras by (3) and $M_{\mathbf{C}\otimes_{\mathbf{Z}}R}(1) = M_{\mathbf{C}\otimes_{\mathbf{Z}}N}(1) \otimes M_{\mathbf{C}\otimes_{\mathbf{Z}}\mathbf{Z}\gamma}(1)$ as vertex operator algebras. Hence

$$V_{R^\circ} = V_{N^\circ} \otimes V_{(\mathbf{Z}\gamma)^\circ},$$

where $V_{N^\circ} = M_{\mathbf{C}\otimes_{\mathbf{Z}}N}(1) \otimes \mathbf{C}\{N^\circ\}$ and $V_{(\mathbf{Z}\gamma)^\circ} = M_{\mathbf{C}\otimes_{\mathbf{Z}}\mathbf{Z}\gamma}(1) \otimes \mathbf{C}\{(\mathbf{Z}\gamma)^\circ\}$ are subalgebras of the generalized vertex algebra V_{R°.

By the definition of the vertex operator $Y(v,z) \in (\mathrm{End}\, V_{R^\circ})\{z\}$ for $v \in V_{R^\circ}$, the component operator $v_{(n)} \in \mathrm{End}\, V_{R^\circ}$ of $Y(v,z) = \sum_{n\in\mathbf{Q}} v_{(n)} z^{-n-1}$ has the property that

$$v_{(n)} w \in V_{R+\alpha+\beta}$$

for $v \in V_{R+\alpha}$, $w \in V_{R+\beta}$, $\alpha, \beta \in R^\circ$.

Note that V_X for $X = R, N, \mathbf{Z}\gamma$ and L are vertex operator algebras, V_{L° is a V_L-module, and V_{R° is a V_R-module. We have $V_R = V_N \otimes V_{\mathbf{Z}\gamma}$.

We consider a linear isomorphism $\psi_p : V_{R^\circ} \to V_{R^\circ}$ defined by

$$\psi_p : u \otimes e^\alpha \mapsto u \otimes (e^\alpha e^{p\gamma/2k}) = u \otimes e^{\alpha+p\gamma/2k}$$

for $u \in M_{\mathbf{C}\otimes_{\mathbf{Z}} R}(1), \alpha \in R^\circ, p \in \mathbf{Z}$. Then $\psi_p(V_{R+\mu}) = V_{R+\mu+p\gamma/2k}$ for $\mu \in R^\circ$. Moreover, the following lemma holds.

Lemma 3 (1) *$\psi_p : V_{R^\circ} \to V_{R^\circ}$ is an isomorphism of V_N-modules.*

(2) *$\gamma(0)\psi_p(v) = \psi_p(\gamma(0)v) + p\psi_p(v)$ and $\gamma(m)\psi_p(v) = \psi_p(\gamma(m)v)$ if $m \neq 0$ for $v \in V_{R^\circ}$, $m \in \mathbf{Z}$.*

Let

$$H = \gamma(-1)\mathbf{1}, \qquad E = e^{\alpha_1} + \cdots + e^{\alpha_k}, \qquad F = e^{-\alpha_1} + \cdots + e^{-\alpha_k}$$

be as in [3]. Let V^{aff} be the subalgebra of the vertex operator algebra V_L generated by H, E and F. Then $V^{\text{aff}} \cong L_{\widehat{sl}_2}(k, 0)$ and $V^{\text{aff}} \supset V_{\mathbf{Z}\gamma}$. We denote $M^{0,j}$ of [3, Lemma 4.2] by M^j for simplicity of notation. Thus

$$V^{\text{aff}} \cong \bigoplus_{j=0}^{k-1} M^j \otimes V_{\mathbf{Z}\gamma - j\gamma/k} \tag{4}$$

as $M^0 \otimes V_{\mathbf{Z}\gamma}$-modules, where

$$M^j = \{v \in V^{\text{aff}} \mid \gamma(m)v = -2j\delta_{m,0}v \text{ for } m \geq 0\}, \tag{5}$$

and in particular $M^0 = K(sl_2, k)$. Note that M^0 is also the commutant of $V_{\mathbf{Z}\gamma}$ in V^{aff}. Since the coset $\mathbf{Z}\gamma - j\gamma/k$ is determined by j modulo k, the index j can be considered as an element of $\mathbf{Z}/k\mathbf{Z}$ in (4).

The commutant of $V_{\mathbf{Z}\gamma}$ in V_L is V_N and so $M^0 \subset V_N$. Let T be the commutant of V^{aff} in V_L. The vertex operator algebra T was studied in [6]. Note that T is also the commutant of M^0 in V_N and the commutant of T in V_N is M^0. Moreover, we have

$$V^{\text{aff}} = \{v \in V_L \mid (\omega_T)_{(1)}v = 0\}, \tag{6}$$

where

$$\omega_T = \frac{1}{2(k+2)} \sum_{1 \leq p < q \leq k} \left(\frac{1}{2}\big((\alpha_p - \alpha_q)(-1)\big)^2 \mathbf{1} - 2\big(e^{\alpha_p - \alpha_q} + e^{\alpha_q - \alpha_p}\big) \right)$$

is the conformal vector of the vertex operator algebra T.

The vertex operator algebra V_L decomposes into a direct sum of irreducible V_R-modules $V_L = \bigoplus_{i=0}^{k-1} V_{R+i\alpha_k}$ by Lemma 1. Let $\sigma = \exp(2\pi\sqrt{-1}\gamma(0)/2k)$ be a linear isomorphism induced by the action of $\gamma(0)$ on V_{R°. Then

$$\{v \in V_L \mid \sigma v = \exp(2\pi i\sqrt{-1}/k)v\} = V_{R+i\alpha_k}.$$

Hence (4) implies that $V^{\mathrm{aff}} \cap V_{R-j\alpha_k} \cong M^j \otimes V_{\mathbf{Z}\gamma - j\gamma/k}$ as $M^0 \otimes V_{\mathbf{Z}\gamma}$-modules. Thus,

$$\{v \in V_{R-j\alpha_k} \mid (\omega_T)_{(1)}v = 0\} \cong M^j \otimes V_{\mathbf{Z}\gamma - j\gamma/k} \tag{7}$$

as $M^0 \otimes V_{\mathbf{Z}\gamma}$-modules by (6).

Since $\omega_T \in V_N$ and $M^0 \subset V_N$, the map ψ_{2j} commutes with $(\omega_T)_{(1)}$ and it is also an isomorphism of M^0-modules by Lemma 3. Therefore, taking the images of both sides of (7) under the map ψ_{2j} we have

$$\{v \in V_{R-j\alpha_k+j\gamma/k} \mid (\omega_T)_{(1)}v = 0\} \cong M^j \otimes V_{\mathbf{Z}\gamma}. \tag{8}$$

Recall that $\lambda = \gamma/2k - \alpha_k/2 \in N^\circ$. Thus $R - j\alpha_k + j\gamma/k = R + 2j\lambda$. Let

$$N^j = N + 2j\lambda.$$

Then $R + 2j\lambda = \{(x, y) \mid x \in N^j, y \in \mathbf{Z}\gamma\}$. Hence $V_{R+2j\lambda} = V_{N^j} \otimes V_{\mathbf{Z}\gamma}$ and

$$V_{N^j} = \{v \in V_{R+2j\lambda} \mid (\omega_\gamma)_{(1)}v = 0\},$$

where $\omega_\gamma = \frac{1}{4k}\gamma(0)^2\mathbf{1}$ is the conformal vector of $V_{\mathbf{Z}\gamma}$. We set

$$M^{(j)} = \{v \in V_{N^j} \mid (\omega_T)_{(1)}v = 0\}. \tag{9}$$

In particular, $M^{(0)} = M^0$. The coset N^j is determined by j modulo k. Thus the index j of $M^{(j)}$ can be considered as an element of $\mathbf{Z}/k\mathbf{Z}$. Since $(\omega_T)_{(1)}$ and $(\omega_\gamma)_{(1)}$ commute, (8) implies that $M^{(j)} \cong M^j$ as M^0-modules. We also have $\psi_{2j}(M^j) = M^{(j)}$.

Next, we shall describe the top level of $\psi_{2j}(M^j)$. Recall the element

$$(F_{-1})^j\mathbf{1} = j! \sum_{I \subset \{1,2,\ldots,k\}, |I|=j} e^{-\alpha_I}$$

of the vertex operator algebra V_L for $0 \le j \le k$, where $\alpha_I = \sum_{p\in I}\alpha_p$ for a subset I of $\{1, 2, \ldots, k\}$ [3, Sect. 4]. The element $(F_{-1})^j\mathbf{1}$ is contained in V^{aff}. Moreover, $\gamma(m)(F_{-1})^j\mathbf{1} = -2j\delta_{m,0}(F_{-1})^j\mathbf{1}$ for $m \ge 0$. Thus $(F_{-1})^j\mathbf{1} \in M^j$ by (5) and

$$\psi_{2j}((F_{-1})^j\mathbf{1}) = j! \sum_{I \subset \{1,2,\ldots,k\}, |I|=j} e^{-\alpha_I + j\gamma/k} \in \psi_{2j}(M^j). \tag{10}$$

Note that (10) is valid for $0 \le j \le k$.

Let $\omega_L, \omega_{\mathrm{aff}}$ and ω be the conformal vectors of the vertex operator algebras V_L, V^{aff} and M^0, respectively. Then $\omega_{\mathrm{aff}} = \omega + \omega_\gamma$ and $\omega_L = \omega_T + \omega_{\mathrm{aff}}$. Moreover, $(F_{-1})^j\mathbf{1}$

is an eigenvector for the operators $(\omega_L)_{(1)}$, $(\omega_T)_{(1)}$ and $(\omega_\gamma)_{(1)}$ with eigenvalues j, 0 and j^2/k, respectively. Thus it is also an eigenvector for $\omega_{(1)}$ with eigenvalue $j(k-j)/k$. Since the top level of the irreducible M^0-module M^j is one dimensional and of weight $j(k-j)/k$ by Theorem 4.4 and Proposition 4.5 of [3], this implies that $\mathbf{C}(F_{-1})^j\mathbf{1}$ is the top level of M^j for $0 \le j \le k-1$. Thus the top level of the irreducible M^0-module $\psi_{2j}(M^j)$ is $\mathbf{C}\psi_{2j}((F_{-1})^j\mathbf{1})$ for $0 \le j \le k-1$, since ψ_{2j} is an isomorphism of M^0-modules by Lemma 3.

Note that $\alpha_I \in N + j\alpha_k$ for any subset I of $\{1, 2, \ldots, k\}$ with $|I| = j$ and $N^j = N - j\alpha_k + j\gamma/k \subset N^\circ$ but $N - j\alpha_k \not\subset N^\circ$ if $1 \le j \le k-1$. Note also that $\psi_{2j}((F_{-1})^j\mathbf{1})$ is an eigenvector for the operators $(\omega_L)_{(1)}$ and $(\omega_\gamma)_{(1)}$ with eigenvalues $j(k-j)/k$ and 0, respectively.

By the above argument, we have the following theorem.

Theorem 1 (1) *$M^{(j)} = \psi_{2j}(M^j) \cong M^j$ as M^0-modules.*

(2) *The top level of the irreducible M^0-module $M^{(j)}$ is $\mathbf{C}\psi_{2j}((F_{-1})^j\mathbf{1})$.*

(3) *$u_{(m)}v \in M^{(i+j)}$ for $u \in M^{(i)}$, $v \in M^{(j)}$, $m \in \mathbf{Z}$.*

3 Lattice Γ_D and VOA or VOSA M_D

In this section we fix a positive integer n. Define a standard scalar product on $(\mathbf{Z}/k\mathbf{Z})^n$ by

$$(\xi|\eta) = i_1 j_1 + \cdots + i_n j_n \in \mathbf{Z}/k\mathbf{Z}$$

for $\xi = (i_1, \ldots, i_n)$, $\eta = (j_1, \ldots, j_n) \in (\mathbf{Z}/k\mathbf{Z})^n$. Let

$$N_\xi = \{(x_1, \ldots, x_n) \mid x_r \in N^{i_r}, 1 \le r \le n\} \subset (N^\circ)^{\oplus n} \tag{11}$$

for $\xi = (i_1, \ldots, i_n) \in (\mathbf{Z}/k\mathbf{Z})^n$, where $(N^\circ)^{\oplus n}$ is an orthogonal sum of n copies of the dual lattice N° of N. For $\alpha \in N_\xi$ and $\beta \in N_\eta$, we have

$$\langle \alpha, \beta \rangle \in -\frac{2}{k}(\xi|\eta) + 2\mathbf{Z}. \tag{12}$$

Let D be an additive subgroup of $(\mathbf{Z}/k\mathbf{Z})^n$. We consider two cases.

Case A. $(\xi|\xi) = 0$ for all $\xi \in D$.

Case B. k is even, $(\xi|\eta) \in \{0, k/2\}$ for all $\xi, \eta \in D$, and $(\xi|\xi) = k/2$ for some $\xi \in D$.

Remark 1 If k is odd, Case A occurs if and only if D is self-orthogonal. However, the condition that $(\xi|\xi) = 0$ for all $\xi \in D$ does not imply self-orthogonality of D if k is even.

Let

$$\Gamma_D = \bigcup_{\xi \in D} N_\xi \subset (N^\circ)^{\oplus n}, \tag{13}$$

which is a sublattice of $(N^\circ)^{\oplus n}$. We see from (12) that Γ_D is an integral lattice if and only if $(\xi|\eta) \in \{0, k/2\}$ for all $\xi, \eta \in D$. More precisely, the following lemma holds.

Lemma 4 (1) *Γ_D is a positive definite even lattice if and only if D is in Case A.*

(2) *Γ_D is a positive definite odd lattice if and only if If k is even and D is in Case B.*

If k is even and D is in Case B, we set $\Gamma_D^{\bar{p}} = \{\alpha \in \Gamma_D \mid \langle \alpha, \alpha \rangle \in p + 2\mathbf{Z}\}$ for $p = 0, 1$.

Let $V_{\Gamma_D} = M_{\mathbf{C}\otimes_{\mathbf{Z}}\Gamma_D}(1) \otimes \mathbf{C}\{\Gamma_D\}$. If D is in Case A, then V_{Γ_D} is a vertex operator algebra. If k is even and D is in Case B, then $V_{\Gamma_D} = V_{\Gamma_D^{\bar{0}}} \oplus V_{\Gamma_D^{\bar{1}}}$ is a vertex operator superalgebra.

By the definition (11) of N_ξ, we have

$$V_{N_\xi} = V_{N^{i_1}} \otimes \cdots \otimes V_{N^{i_n}} \subset (V_{N^\circ})^{\otimes n},$$

which is an irreducible $(V_N)^{\otimes n}$-submodule of $(V_{N^\circ})^{\otimes n}$. It follows from (13) that $V_{\Gamma_D} = \bigoplus_{\xi \in D} V_{N_\xi}$.

We set

$$M_\xi = \{v \in V_{N_\xi} \mid (\omega_{T^{\otimes n}})_{(1)} v = 0\},$$

where $\omega_{T^{\otimes n}}$ is the conformal vector of the vertex operator subalgebra $T^{\otimes n}$ of $(V_N)^{\otimes n}$. Then

$$M_\xi = M^{(i_1)} \otimes \cdots \otimes M^{(i_n)}$$

for $\xi = (i_1, \ldots, i_n) \in (\mathbf{Z}/k\mathbf{Z})^n$ by (9), which is an irreducible module for $M_{\mathbf{0}} = (M^{(0)})^{\otimes n}$ with $\mathbf{0} = (0, \ldots, 0)$ the zero codeword. We have $u_{(m)} v \in M_{\xi+\eta}$ for $u \in M_\xi$, $v \in M_\eta$, $m \in \mathbf{Z}$.

The top level of M_ξ is one dimensional with weight $\left(\sum_{p=1}^n i_p\right) - \frac{(\xi|\xi)}{k}$, where i_p and $(\xi|\xi)$ are considered to be nonnegative integers.

Let M_D be the commutant of $T^{\otimes n}$ in V_{Γ_D}. Then

$$M_D = \{v \in V_{\Gamma_D} \mid (\omega_{T^{\otimes n}})_{(1)} v = 0\} = \bigoplus_{\xi \in D} M_\xi.$$

Theorem 2 (1) *If D is in Case A, then M_D is a simple vertex operator algebra of CFT-type with central charge $2n(k-1)/(k+2)$.*

(2) *If k is even and D is in Case B, then $M_D = M_D^{\bar{0}} \oplus M_D^{\bar{1}}$ is a simple vertex operator superalgebra, where the even part $M_D^{\bar{0}}$ and the odd part $M_D^{\bar{1}}$ are given by*

$$M_D^{\bar{0}} = \bigoplus_{\xi \in D, (\xi|\xi)=0} M_\xi, \qquad M_D^{\bar{1}} = \bigoplus_{\xi \in D, (\xi|\xi)=k/2} M_\xi.$$

4 Examples

The vertex operator algebra M_D is already known for some k and n. The cases $k = 2, 3$ were studied in [7, 10]. The cases $k = 5$, $n = 2$, $D = \{(00), (12), (24), (31), (43)\}$ and $k = 9$, $n = 1$, $D = \{(0), (3), (6)\}$ appeared in [8]. The following example looks new.

Let $k = 6$, $n = 1$ and $D = \{(0), (3)\}$. Then

$$M_D = M^{(0)} \oplus M^{(3)} \cong L_{NS}(5/4, 0) \oplus L_{NS}(5/4, 3),$$

where $L_{NS}(5/4, 0)$ is a simple Neveu–Schwarz algebra of central charge $5/4$ and $L_{NS}(5/4, 3)$ is its irreducible highest weight module with highest weight 3.

Indeed, the top level of $M^{(3)}$ is one dimensional with weight $3/2$. Let v be an element of the top level of $M^{(3)}$ such that $v_{(2)}v = (5/6)\mathbf{1}$. Then $v_{(m)}v = 0$ for $m \geq 3$, $v_{(1)}v = 0$ and $v_{(0)}v = 2\omega$, where ω is the conformal vector of $M^{(0)}$. Hence

$$L_n = \omega_{(n+1)}, \quad G_{n-1/2} = v_{(n)} \quad (n \in \mathbf{Z})$$

satisfy the relations for the Neveu–Schwarz algebra of central charge $5/4$. Thus the subalgebra generated by ω and v in the lattice vertex operator superalgebra V_{Γ_D} is isomorphic to $L_{NS}(5/4, 0)$. The parafermion vertex operator algebra M^0 has a weight 3 primary vector W^3. The vector W^3 is a highest weight vector for $L_{NS}(5/4, 0)$ and it generates $L_{NS}(5/4, 3)$.

Acknowledgements The authors would like to thank Ching Hung Lam for valuable discussions. Tomoyuki Arakawa was partially supported by JSPS Grant-in-Aid for Scientific Research No. 25287004, Hiromichi Yamada was partially supported by JSPS Grant-in-Aid for Scientific Research No. 26400040, Hiroshi Yamauchi was partially supported by JSPS Grant-in-Aid for Scientific Research No. 24740027.

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Vertex Algebras in Higher Dimensions Are Homotopy Equivalent to Vertex Algebras in Two Dimensions

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Abstract There is a differential graded operad associated to quadratic configuration spaces, whose class of algebras naturally contains the class of all vertex algebras. We have found that under certain shift of the degree in the cohomology these operads are isomorphic in cohomology for any even spatial dimension.

1 Real Configuration Spaces and Related Operads

Configuration spaces have been studied long ago in mathematics (see [3]). By definition, the real n-th configuration space over $\mathbb{R}^D$ is the set of all configurations of n points in $\mathbb{R}^D$, which are distinct. It is shortly denoted by $\mathbf{F}_{\mathbb{R},n}$,

$$\mathbf{F}_{\mathbb{R},n}\,(\equiv\,\mathbf{F}_{\mathbb{R},n}^{(D)})\;:=\;\{(\mathrm{x}_1,\dots,\mathrm{x}_n)\,|\,\mathrm{x}_1,\dots,\mathrm{x}_n\;\in\;\mathbb{R}^D,\;\mathrm{x}_j\neq\mathrm{x}_k\;(1\leqslant j<k\leqslant n)\}\,. \tag{1}$$

These spaces obey a very rich structure. In particular, there are several operads that are associated to the sequence of all configuration spaces (over $\mathbb{R}^D$). One of the most simple operads is the so called *little balls/cubes operad* (see [8, Sect. 2.2]).

Before explaining the latter operad let us remind that the operads provide a generalization of the notion of a "type of algebra". They consists of a sequence of spaces $\mathcal{M}(n)$ equipped with several structure maps. If we think of $\mathcal{M}(n)$ as a space of "n-ary operations" (i.e., operations with n inputs and one output) then there are structure maps that axiomatize the composition,

$$\mathcal{M}(n)\times\mathcal{M}(j_1)\times\cdots\times\mathcal{M}(j_n)\;\ni\;(\mu,\mu_1,\dots,\mu_n)\;\longmapsto\;\mu\circ(\mu_1,\dots,\mu_n)\;\in\;\mathcal{M}(\ell)\,, \tag{2}$$

of an n-ary operation $\mu_n\in\mathcal{M}(n)$ with n other operations $\mu_1\in\mathcal{M}(j_1)$, ..., $\mu_n\in\mathcal{M}(j_n)$, and it gives a result that belongs to the space $\mathcal{M}(\ell)$ of operation with

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V. Dobrev (ed.), *Lie Theory and Its Applications in Physics*,
Springer Proceedings in Mathematics & Statistics 191,
DOI 10.1007/978-981-10-2636-2_39

$$\ell = j_1 + \cdots + j_n \tag{3}$$

inputs. In addition, the permutation group $\mathcal{S}_n$ is supposed to act on $\mathcal{M}(n)$ for every n axiomatizing the exchange of inputs of an n-ary operation. There are natural conditions of associativity for the operadic compositions and equivariance (compatibility) for the compositions with respect to the permutation actions. The reader can find further information in [7].

In the case of the little balls operad, the space $\mathcal{M}(n)$ consists of all closed balls $\overline{B}_1, \dots, \overline{B}_n$ in $\mathbb{R}^D$, which do not intersect each other and are contained in an open ball B_0,

$$\mathcal{M}(n) = \big\{(B_0; B_1, \dots, B_n) \,\big|\, \overline{B}_1, \dots, \overline{B}_n \subset B_0\,,\ \overline{B}_j \cap \overline{B}_k = \emptyset\ (1 \leqslant j < k \leqslant n)\big\} \tag{4}$$

The operadic composition

$$\begin{aligned}&(B_0; B_1, \dots, B_n) \circ \big((B_{1,0}; B_{1,1}, \dots, B_{1,j_1}), \dots, (B_{n,0}; B_{n,1}, \dots, B_{n,j_n})\big)\\ &= (B_0, B'_{1,1}, \dots, B'_{1,j_1}, \dots, B'_{n,1}, \dots, B'_{n,j_n})\end{aligned} \tag{5}$$

is then obtained by transforming each configuration $(B_{k,0}, B_{k,1}, \dots, B_{k,j_k})$ with translations and dilations in such a way that we can plug $B_{k,0}$ into B_k, i.e.,

$$(B_{k,0}, B_{k,1}, \dots, B_{k,j_k}) \overset{\substack{\text{translations}\\ \text{\& dilations}}}{\longmapsto} (B'_{k,0}, B'_{k,1}, \dots, B'_{k,j_k})\,, \text{ so that } B'_{k,0} = B_k \tag{6}$$

for every $k = 1, \dots, n$. Note that $\mathcal{M}(n)$ is homotopy equivalent to the configuration space $\mathbf{F}_{\mathbb{R},n}$ and hence, the above opearadic compositions induce maps between the homology spaces (with rational coefficients),

$$H_\bullet\big(\mathcal{M}(n), \mathbb{Q}\big) = H_\bullet(\mathbf{F}_{\mathbb{R},n}, \mathbb{Q})\,. \tag{7}$$

In this way, the sequence of spaces $H_\bullet(\mathbf{F}_{\mathbb{R},n}, \mathbb{Q})$ becomes an *algebraic operad*, i.e., an operad whose operadic spaces are vector spaces and the operadic compositions are multilinear maps.

There is a straight forward generalization of the little balls operad. Let

$$r \subseteqq \mathbb{R}^D \times \mathbb{R}^D \tag{8}$$

be a homogeneous, closed, binary relation and denote

$$\mathbf{F}_{r\,;\,n} := \{(\mathrm{x}_1, \dots, \mathrm{x}_n) \,|\, \mathrm{x}_1, \dots, \mathrm{x}_n \in \mathbb{R}^D,\ (\mathrm{x}_j, \mathrm{x}_k) \notin r\ (1 \leqslant j < k \leqslant n)\} \tag{9}$$

$$\begin{aligned}\mathcal{M}^{(r)}(n) = \big\{(B_0; B_1, \dots, B_n) \,\big|\, &\overline{B}_1, \dots, \overline{B}_n \subset B_0 \subset \mathbb{R}^D\,,\\ &(\overline{B}_j \times \overline{B}_k) \cap r = \emptyset\ (1 \leqslant j < k \leqslant n)\big\}\,,\end{aligned} \tag{10}$$

with operadic composition given by (2). Then we obtain again an operad and the sequence

$$H_\bullet\big(\mathcal{M}^{(r)}(n), \mathbb{Q}\big) \cong H_\bullet(\mathbf{F}_{r\,;\,n}, \mathbb{Q}) \tag{11}$$

is an algebraic operad.

2 Quadratic Configuration Spaces and Related Operads

As a particular example of the operad $\mathcal{M}^{(r)}(n)$ (10) let us consider the complex vector space $\mathbb{C}^D$ ($\cong\mathbb{R}^{2D}$ as a real vector space) equipped with a quadratic homogeneous relation

$$r \subset \mathbb{C}^D \times \mathbb{C}^D\,, \quad r := \{(\mathrm{x}, \mathrm{y}) \in \mathbb{C}^D \,|\, (\mathrm{x}-\mathrm{y})^2 = 0\}\,, \tag{12}$$

$$\mathrm{x}^2 \equiv \mathrm{x}\cdot\mathrm{x} := (x^1)^2 + \cdots + (x^D)^2 \quad \text{for} \quad \mathrm{x} := (x^1, \ldots, x^D) \in \mathbb{C}^D\,. \tag{13}$$

Then, following [9, 10] we call $\mathbf{F}_{r\,;\,n}$ (9) a *quadratic configuration space* and denote it by $\mathbf{F}_{\mathbb{C},n}$

$$\mathbf{F}_{\mathbb{C},n} := \big\{(\mathrm{x}_1, \ldots, \mathrm{x}_n) \in (\mathbb{C}^D)^{\times n} \,\big|\, (\mathrm{x}_j - \mathrm{x}_k)^2 \neq 0 \; (1 \leqslant j < k \leqslant n)\big\}\,. \tag{14}$$

Note in particular, that

$$\mathbf{F}_{\mathbb{C},n} \cap (\mathbb{R}^D)^{\times n} = \mathbf{F}_{\mathbb{R},n}\,. \tag{15}$$

We observe also that $\mathbf{F}_{\mathbb{C},n}$ are *complex affine varieties* and the ring of regular functions on $\mathbf{F}_{\mathbb{C},n}$ coincides with the algebra of rational functions with quadratic singularities,

$$\widetilde{\mathscr{O}}_n := \mathscr{O}\big(\mathbf{F}_{\mathbb{C},n}\big) = \mathbb{C}\big[\mathrm{x}_1, \ldots, \mathrm{x}_n\big]\bigg[\Big(\prod_{1\, j < k \leqslant n} (\mathrm{x}_j - \mathrm{x}_k)^2\Big)^{-1}\bigg]. \tag{16}$$

In physics terminology, one can say that the elements of $\widetilde{\mathscr{O}}_n$ are the rational n-point functions with *light-cone singularities*. One can divide the configuration spaces by the action of the translations and pass to the reduced configuration spaces $\mathbf{F}_n\big/\mathbb{C}^D$, whose algebra of regular functions $\mathscr{O}_n$ consists of the translation invariant functions belonging to $\widetilde{\mathscr{O}}_n$

$$\mathscr{O}_n := \mathscr{O}\big(\mathbf{F}_{\mathbb{C},n}\big/\mathbb{C}^D\big) = \mathbb{C}\big[\mathrm{x}_1 - \mathrm{x}_n, \ldots, \mathrm{x}_{n-1} - \mathrm{x}_n\big]\bigg[\Big(\prod_{1 \leqslant j < k \leqslant n} (\mathrm{x}_j - \mathrm{x}_k)^2\Big)^{-1}\bigg] \tag{17}$$

As the quotient by the translations do not change topology up to a homotopy equivalence we obtain the same operad structure on the homology spaces.

Let us remind the result of Grothendieck [4], which identifies the algebraic de Rham cohomologies of $\widetilde{\mathscr{O}}_n$ (resp., $\mathscr{O}_n$) with the Betti cohomology of the corresponding complex affine variety $\mathbf{F}_{\mathbb{C},n}$ (resp., $\mathbf{F}_{\mathbb{C},n}/\mathbb{C}^D$). In the case of $\widetilde{\mathscr{O}}_n$ (as well as, $\mathscr{O}_n$) the algebraic de Rham complex has a simple construction,

$$\begin{aligned}\Omega^k\big(\mathscr{O}_{\mathbb{C},n}\big) \; := \; \textstyle\bigwedge^k_{\mathscr{O}_{\mathbb{C},n}} \Omega^1\big(\mathscr{O}_{\mathbb{C},n}\big) := \mathrm{Span}_{\mathbb{C}}\Big\{ f \, dx_{j_1}^{\mu_1} \wedge \cdots \wedge dx_{j_k}^{\mu_k} \;\Big| \\ f \in \widetilde{\mathscr{O}}_{\mathbb{C},n} \,, \quad \mu_1, \dots, \mu_k = 1, \dots, D \,, \; j_1, \dots, j_k = 1, \dots, n \Big\} .\end{aligned} \tag{18}$$

Then

$$H^k\big(\widetilde{\mathscr{O}}_{\mathbb{C},n}\big) \; := \; \frac{\mathrm{Ker}\Big(\Omega^k\big(\widetilde{\mathscr{O}}_{\mathbb{C},n}\big) \xrightarrow{d} \Omega^{k+1}\big(\mathscr{O}_{C,n}\big)\Big)}{\mathrm{Image}\Big(\Omega^{k-1}\big(\mathscr{O}_{\mathbb{C},n}\big) \xrightarrow{d} \Omega^k\big(\mathscr{O}_{\mathbb{C},n}\big)\Big)} \,, \tag{19}$$

with respect to the de Rham differential:

$$\begin{aligned} & d\big(f dx_{j_1}^{\mu_1} \wedge \cdots \wedge dx_{j_k}^{\mu_k}\big) \\ & := \sum_{j=1}^{n} \sum_{\mu=1}^{D} \frac{\partial f}{\partial x_j^{\mu}} \, dx_j^{\mu} \wedge dx_{j_1}^{\mu_1} \wedge \cdots \wedge dx_{j_k}^{\mu_k} \,, \end{aligned} \tag{20}$$

where $f(\mathrm{x}_1, \dots, \mathrm{x}_n) \in \mathscr{O}_{\mathbb{C},n}$. Now, the Grothendieck's theorem implies that

$$H^k\big(\widetilde{\mathscr{O}}_n\big) \; \cong \; H^k\big(\mathbf{F}_{\mathbb{C},n}; \mathbb{C}\big) \,. \tag{21}$$

In fact, $H^k\big(\widetilde{\mathscr{O}}_{\mathbb{Q},n}\big) \otimes_{\mathbb{Q}} \mathbb{C} \cong \big(H_k\big(\mathbf{F}_{\mathbb{C},n}; \mathbb{Z}\big) \otimes_{\mathbb{Z}} \mathbb{C}\big)^*$, where $\widetilde{\mathscr{O}}_{\mathbb{Q},n}$ is the algebra $\widetilde{\mathscr{O}}_n$ (16) with coefficients in $\mathbb{Q}$ (instead of $\mathbb{C}$), and the natural $\mathbb{Z}$-bilinear paring $H^k\big(\mathbf{F}_{\mathbb{Q},n}\big) \times H_k\big(\mathbf{F}_{\mathbb{C},n}; \mathbb{Z}\big) \to \mathbb{C}$ gives rise to the space ($\mathbb{Z}$-module) of periods related to the quadratic configuration spaces $\mathbf{F}_{\mathbb{C},n}$, which play a very important role in renormalization theory as residues of Feynman amplitudes in massless Quantum Field Theories (see [11, 13]). Furthermore, there is a differential graded operad associated to the sequence of algebras $\mathscr{O}_n$, whose cohomologies coincide with the operad $\big(H_{\bullet}(\mathbf{F}_{\mathbb{C},n}, \mathbb{Q})\big)_{n \geqslant 2}$ and it has an application to both: the theory of vertex algebras and the renormalization [10, 12].

Remark 1 For the operadic point of view on vertex algebras we would like to mention also the papers [5, 6], where certain *partial* operads are proposed for this purpose. However, the operad suggested in [10] is not a *partial* operad but an "ordinary" symmetric operad (as defined for example in [7]). The price for this simplification is perhaps that the latter operad has more algebras than the vertex algebras. Nevertheless, there is a simple criterion for separating the class of vertex algebras among all others (more details will be published in [12]).

3 Cohomologies of Quadratic Configuration Spaces up to Three Points and Their Application in the Theory of Vertex Algebras

The main new result in the present work is the computation of the cohomology spaces of $\mathbf{F}_{\mathbb{C},n}$ for $n = 2, 3$. In general, the problem of finding all cohomology spaces of $\mathbf{F}_{\mathbb{C},n}$ for all $n = 2, 3, \ldots$ is very difficult.

A standard approach for studying configuration spaces is via the sequence of maps

$$q_{n+1} : \mathbf{F}_{\mathbb{C},n+1} \longrightarrow \mathbf{F}_{\mathbb{C},n} \,:\, (\mathrm{x}_1, \ldots, \mathrm{x}_{n+1}) \longmapsto (\mathrm{x}_1, \ldots, \mathrm{x}_n) \tag{22}$$

that forget about the last point (for $n = 2, 3, \ldots$). The fiber of q_{n+1} at the point $(\mathrm{x}_1, \ldots, \mathrm{x}_n) \in \mathbf{F}_{\mathbb{C},n}$ is

$$M_{\mathrm{x}_1,\ldots,\mathrm{x}_n} := \left\{\mathrm{z} \in \mathbb{C}^D \,\middle|\, (\mathrm{z} - \mathrm{x}_j)^2 \neq 0 \text{ for all } j = 1, \ldots, n\right\} = \mathbb{C}^D \setminus \bigcup_{j=1}^{n} Q_{\mathrm{x}_j}\,, \tag{23}$$

i.e., it is the complement of union of quadrics of a type

$$Q_{\mathrm{x}} := \left\{\mathrm{z} \in \mathbb{C}^D \,\middle|\, (\mathrm{z} - \mathrm{x})^2 = 0\right\}. \tag{24}$$

In case $\mathbb{C} \mapsto \mathbb{R}$ the fibers are

$$M^{\mathbb{R}}_{\mathrm{x}_1,\ldots,\mathrm{x}_n} = \left\{\mathrm{z} \in \mathbb{R}^D \,\middle|\, \mathrm{z} \neq \mathrm{x}_j \text{ for all } j = 1, \ldots, n\right\},$$

their homeomorphism type does not depend on $(\mathrm{x}_1, \ldots, \mathrm{x}_n)$, and each of them is homotopy equivalent to a bouquet of $(D-1)$-spheres. In particular the projections q_n are fibrations and one may use iterated Leray–Serre spectral sequences or the Leray–Hirsch theorem in order to obtain the Betti cohomology of $\mathbf{F}_{\mathbb{R},n}$, see [1–3].

Let us point out that for both cases, $\mathbb{C}$ and $\mathbb{R}$, the maps (22), $q_{n+1} : \mathbf{F}_{\mathbb{C},n+1} \longrightarrow \mathbf{F}_{\mathbb{C},n}$ and $q_{n+1} : \mathbf{F}_{\mathbb{R},n+1} \longrightarrow \mathbf{F}_{\mathbb{R},n}$ (respectively) are fiber bundles for $n = 1, 2$. This is due to the fact that in these cases the bases $\mathbf{F}_{\mathbb{C},n}$ are homogeneous spaces of the group of Euclidean motions with dilations on $\mathbb{C}^D$. In the real case, the maps $q_{n+1} : \mathbf{F}_{\mathbb{R},n+1} \longrightarrow \mathbf{F}_{\mathbb{R},n}$ remain fiber bundles for any n (the fibers being homotopy equivalent to a bouquet of spheres, as we have pointed out). However, over $\mathbb{C}$ and $n > 2$ the fibers $M_{\mathrm{x}_1,\ldots,\mathrm{x}_n}$ (23) are in general non-isomorphic.

The case $n = 2$ is relatively simple. We have an isomorphism

$$\mathbf{F}_{\mathbb{C},2} \cong M_0 \times \mathbb{C}^D \,:\, (\mathrm{x}_1, \mathrm{x}_2) \mapsto (\mathrm{x}_1 - \mathrm{x}_2, \mathrm{x}_2)\,. \tag{25}$$

For $M_0 = \mathbb{C}^D \backslash Q_0$ then we use the projection

$$M_0 \longrightarrow \mathbb{C}\backslash\{0\} \,:\, \mathrm{x}_1 - \mathrm{x}_2 \longmapsto (\mathrm{x}_1 - \mathrm{x}_2)^2\,, \tag{26}$$

which is a bundle with fibers isomorphic to the complex $(D-1)$-sphere $\mathbb{S}_{\mathbb{C}}^{D-1}$. For even $D > 2$ we then derive by the Leray–Hirsch theorem that

$$H^k(\mathbf{F}_{\mathbb{C},2}) \,=\, 0 \quad \text{for} \quad k \neq 0,\ 1,\ D-1,\ D\,, \tag{27}$$

$$H^1(\mathbf{F}_{\mathbb{C},2}) \,=\, \mathbb{C}\left[\sum_{\mu=1}^{D} \frac{z^\mu dz^\mu}{\mathrm{z}^2}\right], \tag{28}$$

$$H^{D-1}(\mathbf{F}_{\mathbb{C},2}) \,=\, \mathbb{C}\left[\sum_{\mu=1}^{D} \frac{(-1)^{\mu+1}\, z^\mu\, dz^1 \wedge \cdots \wedge \widehat{dz^\mu} \wedge \cdots \wedge dz^D}{(\mathrm{z}^2)^{\frac{D}{2}}}\right], \tag{29}$$

$$H^D(\mathbf{F}_{\mathbb{C},2}) \,=\, \mathbb{C}\left[\frac{dz^1 \wedge \cdots \wedge dz^D}{(\mathrm{z}^2)^{\frac{D}{2}}}\right], \tag{30}$$

where $\mathrm{z} = \mathrm{x}_1 - \mathrm{x}_2$. The role of the fact that D is restricted to be even is that only then are the representatives of the cohomology classes in (29) and (30) rational functions.

Let us introduce

$$\begin{aligned}
\omega_{j,k}^{(1)} &:= \sum_{\mu=1}^{D} \frac{(x_j^\mu - x_k^\mu)\, d(x_j^\mu - x_k^\mu)}{(\mathrm{x}_j - \mathrm{x}_k)^2}\,, \\
\omega_{j,k}^{(D-1)} &:= \sum_{\mu=1}^{D} \frac{(-1)^{\mu+1}\, d(x_j^\mu - x_k^\mu)}{\big(\mathrm{x}_j - \mathrm{x}_k^2\big)^{\frac{D}{2}}}\, d(x_j^1 - x_k^1) \wedge \cdots \\
&\qquad \wedge \widehat{d(x_j^\mu - x_k^\mu)} \wedge \cdots \wedge d(x_j^D - x_k^D)\,,
\end{aligned} \tag{31}$$

for $j, k = 1, \ldots, n$ and $j \neq k$, while for $j = k$ we set for convenience

$$\omega_{(k,k)}^{(m)} \,:=\, 0\,.$$

We also have

$$\omega_{j,k}^{(m)} \,=\, \omega_{k,j}^{(m)}\,.$$

Then we have found the following basis for the cohomologies of $\mathscr{O}_{\mathbb{C},3}$ (ordered by the form degree):

$$
\begin{aligned}
\text{deg. } 0 \ &: \ [1]\,,\\
\text{deg. } 1 \ &: \ [\omega_{1,2}^{(1)}]\,,\ [\omega_{1,3}^{(1)}]\,,\ [\omega_{2,3}^{(1)}]\,,\\
\text{deg. } 2 \ &: \ [\omega_{1,2}^{(1)}\,\omega_{1,3}^{(1)}]\,,\ [\omega_{1,2}^{(1)}\,\omega_{2,3}^{(1)}]\,,\ [\omega_{1,3}^{(1)}\,\omega_{2,3}^{(1)}]\,,\\
\text{deg. } 3 \ &: \ [\omega_{1,2}^{(1)}\,\omega_{1,3}^{(1)}\,\omega_{2,3}^{(1)}]\,,\\
\text{deg. } D-1 \ &: \ [\omega_{1,2}^{(D-1)}]\,,\ [\omega_{1,3}^{(D-1)}]\,,\ [\omega_{2,3}^{(D-1)}]\,,\\
\text{deg. } D \ &: \ [\omega_{1,2}^{(1)}\,\omega_{1,2}^{(D-1)}]\,,\ [\omega_{1,2}^{(D-1)}\,\omega_{1,3}^{(1)}]\,,\ [\omega_{1,2}^{(D-1)}\,\omega_{2,3}^{(1)}]\,,\\
&\quad\ [\omega_{1,2}^{(1)}\,\omega_{1,3}^{(D-1)}]\,,\ [\omega_{1,2}^{(1)}\,\omega_{2,3}^{(D-1)}]\,,\ [\omega_{1,3}^{(1)}\,\omega_{1,3}^{(D-1)}]\,,\\
&\quad\ [\omega_{2,3}^{(1)}\,\omega_{2,3}^{(D-1)}]\,,\ [\omega_{1,3}^{(1)}\,\omega_{2,3}^{(D-1)}]\,,\\
\text{deg. } D+1 \ &: \ [\omega_{1,2}^{(1)}\,\omega_{1,2}^{(D-1)}\,\omega_{1,3}^{(1)}]\,,\ [\omega_{1,2}^{(1)}\,\omega_{1,2}^{(D-1)}\,\omega_{2,3}^{(1)}]\,,\\
&\quad\ [\omega_{1,2}^{(D-1)}\,\omega_{1,3}^{(1)}\,\omega_{2,3}^{(1)}]\,,\ [\omega_{1,2}^{(1)}\,\omega_{1,3}^{(1)}\,\omega_{1,3}^{(D-1)}]\,,\\
&\quad\ [\omega_{1,2}^{(1)}\,\omega_{2,3}^{(1)}\,\omega_{2,3}^{(D-1)}]\,,\ [\omega_{1,2}^{(1)}\,\omega_{1,3}^{(1)}\,\omega_{2,3}^{(D-1)}]\,,\\
\text{deg. } D+2 \ &: \ [\omega_{1,2}^{(1)}]\,[\omega_{1,2}^{(D-1)}]\,[\omega_{1,3}^{(1)}]\,[\omega_{2,3}^{(1)}]\,,\\
\text{deg. } 2D-2 \ &: \ [\omega_{1,2}^{(D-1)}\,\omega_{1,3}^{(D-1)}]\,,\ [\omega_{1,2}^{(D-1)}\,\omega_{2,3}^{(D-1)}]\,,\\
\text{deg. } 2D-1 \ &: \ [\omega_{1,2}^{(1)}\,\omega_{1,2}^{(D-1)}\,\omega_{1,3}^{(D-1)}]\,,\ [\omega_{1,2}^{(1)}\,\omega_{1,2}^{(D-1)}\,\omega_{2,3}^{(D-1)}]\,,\\
&\quad\ [\omega_{1,2}^{(D-1)}\,\omega_{1,3}^{(1)}\,\omega_{1,3}^{(D-1)}]\,,\ [\omega_{1,2}^{(D-1)}\,\omega_{2,3}^{(1)}\,\omega_{2,3}^{(D-1)}]\,,\\
&\quad\ [\omega_{1,2}^{(D-1)}\,\omega_{1,3}^{(1)}\,\omega_{2,3}^{(D-1)}]\,,\\
\text{deg. } 2D \ &: \ [\omega_{1,2}^{(1)}\,\omega_{1,2}^{(D-1)}\,\omega_{1,3}^{(1)}\,\omega_{1,3}^{(D-1)}]\,,\\
&\quad\ [\omega_{1,2}^{(1)}\,\omega_{1,2}^{(D-1)}\,\omega_{2,3}^{(1)}\,\omega_{2,3}^{(D-1)}]\,,\\
&\quad\ [\omega_{1,2}^{(1)}\,\omega_{1,2}^{(D-1)}\,\omega_{1,3}^{(1)}\,\omega_{2,3}^{(D-1)}]\,.
\end{aligned}
$$

In particular, with the shift $D \mapsto 2$ we obtain an isomorphism in cohomology for al even spatial dimensions.

For the application of the above result to the theory of vertex algebras it is important that the vertex algebras can be viewed as algebras over an operad built only by the first two quadratic configuration spaces $\mathbf{F}_{\mathbb{C},n}$ for $n = 2, 3$ [12]. The key argument for this is that the axiomatic conditions on vertex algebras are formulated only for Operator Product Expansions of two fields and hence, they use only two and three point functions on the spatial variables. This indicates a new kind of “homotopy equivalence” of the theories on operadic level for any even spatial dimension D.

Acknowledgements The author thanks for the useful discussions with Spencer Bloch, Francis Brown, Pierre Cartier and Maxim Kontsevich on various topics related to this work during his visits at Institut des Hautes Études Scientifiques (IHÉS, Bures-sur-Yvette, France). The author is grateful to the referee for the careful reading of the manuscript and for the suggestions of several corrections. This work has been partially supported by Grant DFNI T02/6 of the Bulgarian National Science Foundation.

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Automorphisms of Multiloop Lie Algebras

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Abstract Multiloop Lie algebras are twisted forms of classical (Chevalley) simple Lie algebras over a ring of Laurent polynomials in several variables $k[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$. These algebras occur as centreless cores of extended affine Lie algebras (EALA's) which are higher nullity generalizations of affine Kac-Moody Lie algebras. Such a multiloop Lie algebra $\mathcal{L}$, also called a Lie torus, is naturally graded by a finite root system Δ, and thus possess a significant supply of nilpotent elements. We compute the difference between the full automorphism group of L and its subgroup generated by exponents of nilpotent elements. The answer is given in terms of Whitehead groups, also called non-stable K_1-functors, of simple algebraic groups over the field of iterated Laurent power series $k((x_1))\ldots((x_n))$. As a corollary, we simplify one step in the proof of conjugacy of Cartan subalgebras in EALA's due to Chernousov, Neher, Pianzola and Yahorau, under the assumption $\mathrm{rank}(\Delta) \geq 2$.

1 Multiloop Lie Algebras, Lie Tori, and EALA's

Let k be an algebraically closed field of characteristic 0. We fix a compatible set of primitive m-th roots of unity $\xi_m \in k$, $m \geq 1$. Let G be an adjoint simple algebraic group over k (a Chevalley group), and $L = \mathrm{Lie}(G)$ be the corresponding simple Lie algebra over k. It is well-known that

$$\mathrm{Aut}_k(L) \cong \mathrm{Aut}_k(G) \cong G \rtimes N, \tag{1}$$

where N is the finite group of automorphisms of the Dynkin diagram of the root system of L and G.

Fix two integers $n \geq 0, m \geq 1$ and let $\sigma = (\sigma_1, \ldots, \sigma_n)$ be an n-tuple of pairwise commuting elements of period m in $\mathrm{Aut}_k(L)$. Such an n-tuple determines a $\mathbb{Z}^n$-grading on L with

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V. Dobrev (ed.), *Lie Theory and Its Applications in Physics*,
Springer Proceedings in Mathematics & Statistics 191,
DOI 10.1007/978-981-10-2636-2_40

$$L_{i_1\dots i_n} = \{x \in L \mid \sigma_j(x) = \xi_m^{i_j} x,\ 1 \le j \le n\}. \tag{2}$$

Set $R = k[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$, and let $\tilde{R} = k[x_1^{\pm \frac{1}{m}}, \dots, x_n^{\pm \frac{1}{m}}]$, $m \ge 1$, be another copy of R, considered as an R-algebra via the natural embedding $R \subseteq \tilde{R}$. Then $\tilde{R}/R$ is a Galois ring extension [11, p. 8] with the Galois group

$$\mathrm{Gal}(\tilde{R}/R) \cong (\mathbb{Z}/m\mathbb{Z})^n. \tag{3}$$

Definition 1 The multiloop Lie algebra $\mathcal{L}(L, \sigma)$ is the $\mathbb{Z}^n$-graded k-Lie subalgebra

$$\mathcal{L}(L, \sigma) = \bigoplus_{(i_1,\dots,i_n)\in\mathbb{Z}^n} L_{i_1\dots i_n} \otimes x_1^{\frac{i_1}{m}} \dots x_n^{\frac{i_n}{m}} \tag{4}$$

of the k-Lie algebra $L \otimes_k \tilde{R}$.

Note that, considered as an R-Lie algebra, the algebra $\mathcal{L}(L, \sigma)$ is an $\tilde{R}/R$-twisted form of the R-Lie algebra $L \otimes_k R$, i.e.

$$\mathcal{L}(L, \sigma) \otimes_R \tilde{R} \cong (L \otimes_k R) \otimes_R \tilde{R}. \tag{5}$$

Let Δ be a finite root system in the sense of [4] together with the 0-vector, which we include following the tradition in the theory of extended affine Lie algebras. We set $\Delta^\times = \Delta \setminus \{0\}$, $Q = \mathbb{Z}\Delta$, and

$$\Delta^\times_{ind} = \{\alpha \in \Delta^\times \mid \tfrac{1}{2}\alpha \notin \Delta\}. \tag{6}$$

The importance of multiloop Lie algebras stems from the fact that they provide explicit realizations for a class of infinite-dimensional Lie algebras over k called Lie tori. This was shown by B. Allison, S. Berman, J. Faulkner and A. Pianzola in [2].

Definition 2 [2, Definition 1.1.6] A *Lie Λ-torus of type Δ* is a $Q \times \Lambda$-graded Lie algebra $\mathcal{L} = \bigoplus_{(\alpha,\lambda)\in Q\times\Lambda} \mathcal{L}_\alpha^\lambda$ over k satisfying

1. $\mathcal{L}_\alpha^\lambda = 0$ for all $\alpha \in Q \setminus \Delta$ and all $\lambda \in \Lambda$.
2. $\mathcal{L}_\alpha^0 \ne 0$ for all $\alpha \in \Delta^\times_{ind}$.
3. Λ is generated by the set of all $\lambda \in \Lambda$ such that $\mathcal{L}_\alpha^\lambda \ne 0$ for some $\alpha \in \Delta$.
4. For all $(\alpha, \lambda) \in \Delta^\times \times \Lambda$ such that $\mathcal{L}_\alpha^\lambda \ne 0$, there exist elements $e_\alpha^\lambda \in \mathcal{L}_\alpha^\lambda$ and $f_\alpha^\lambda \in \mathcal{L}_{-\alpha}^{-\lambda}$ satisfying

$$\mathcal{L}_\alpha^\lambda = k e_\alpha^\lambda, \quad \mathcal{L}_{-\alpha}^{-\lambda} = k f_\alpha^\lambda, \quad \text{and} \quad [[e_\alpha^\lambda, f_\alpha^\lambda], x] = \langle \beta, \alpha^\vee \rangle x \tag{7}$$

for all $x \in \mathcal{L}_\beta^\mu$, $(\beta, \mu) \in \Delta \times \Lambda$.

5. $\mathcal{L}$ is generated as a k-Lie algebra by the subspaces $\mathcal{L}_\alpha^\lambda$, $(\alpha, \lambda) \in \Delta^\times \times \Lambda$.

If $\Lambda = \mathbb{Z}^n$, then n is called the *nullity* of $\mathcal{L}$.

In what follows we will always assume that $\Lambda = \mathbb{Z}^n$. By [2, Lemma 1.3.5 and Proposition 1.4.2], if a centreless Lie torus $\mathcal{L}$ with $\Lambda \cong \mathbb{Z}^n$ is finitely generated over its centroid (fgc), then the centroid is isomorphic as a k-algebra to

$$k[\mathbb{Z}^n] \cong k[x_1^{\pm 1}, \ldots, x_n^{\pm 1}] = R. \tag{8}$$

Note that, according to an announced result of E. Neher [14, Theorem 7(b)], all Lie tori are fgc, except for just one class of Lie tori of type A_n called quantum tori; see [2, Remark 1.4.3].

If a centreless Lie torus $\mathcal{L}$ is fgc, the Realization theorem [2, Theorem 3.3.1] asserts that $\mathcal{L}$ as a Lie algebra over its centroid R is $\mathbb{Z}^n$-graded isomorphic to a multiloop algebra $\mathcal{L}(L, \sigma)$.

An *extended affine Lie algebra* is a pair (E, H) consisting of a Lie algebra E over k and subalgebra H satisfying the following axioms (EA1)–(EA6).

(EA1) E has an invariant nondegenerate symmetric bilinear form $(\cdot|\cdot)$.
(EA2) H is a non-trivial finite-dimensional toral and self-centralizing subalgebra of E.

Such an H induces a decomposition of E via the adjoint representation:

$$\begin{aligned} E &= \textstyle\bigoplus_{\alpha \in H^*} E_\alpha, \\ E_\alpha &= \{e \in E : [h, e] = \alpha(h)e \text{ for all } h \in H\}. \end{aligned} \tag{9}$$

One defines

$$\begin{aligned} \Psi &= \{\alpha \in H^* : E_\alpha \neq 0\} \quad (\textit{set of roots of } (E, H)), \\ \Psi^0 &= \{\alpha \in \Psi : (\alpha \mid \alpha) = 0\} \quad (\textit{null roots}), \\ \Psi^{\mathrm{an}} &= \{\alpha \in \Psi : (\alpha \mid \alpha) \neq 0\} \quad (\textit{anisotropic roots}). \end{aligned} \tag{10}$$

Next one defines the *core of* (E, H) as the subalgebra E_c of E generated by all anisotropic root spaces E_α, $\alpha \in \Psi^{\mathrm{an}}$. We can now state the remaining four axioms.

(EA3) For any $\alpha \in \Psi^{\mathrm{an}}$ and $x_\alpha \in E_\alpha$, the operator $\mathrm{ad} x_\alpha$ is locally nilpotent on E.
(EA4) Ψ^{an} is connected in the sense that for any decomposition $\Psi^{\mathrm{an}} = \Psi_1 \cup \Psi_2$ with $(\Psi_1 \mid \Psi_2) = 0$ we have $\Psi_1 = \emptyset$ or $\Psi_2 = \emptyset$.
(EA5) $\{e \in E : [e, E_c] = 0\} \subset E_c$.
(EA6) The subgroup $\mathbb{Z}\Psi^0 \subset H^*$ is isomorphic to $\mathbb{Z}^n$ for some $n \geq 0$.

The relationship between Lie tori and EALA's is described by the following theorem.

Theorem 1 [15, Theorem 6] *Let (E, H) be an EALA, and let $\mathcal{L} = E_c/Z(E_c)$ be its centreless core. Then $\mathcal{L}$ is a centreless $\mathbb{Z}^n$-Lie torus over k. Conversely, for any centreless Lie torus $\mathcal{L}$ there is a (non-unique) EALA (E, H) such that $\mathcal{L} \cong E_c/Z(E_c)$.*

2 Automorphisms of Multiloop Lie Algebras and EALA's

Let $\mathcal{L}$ be a fgc centreless Lie torus over k with the centroid $R \cong k[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$. As explained above, the Lie torus $\mathcal{L}$ is a $\tilde{R}/R$-twisted form of a split simple Lie algebra $L \otimes_k R$. Consequently, the group scheme of R-equivariant automorphisms $\mathrm{Aut}_R(\mathcal{L})$ is a twisted form of the group scheme $\mathrm{Aut}_R(L \otimes_k R)$, and $G = \mathrm{Aut}_R(\mathcal{L})^\circ$ is an adjoint simple reductive group scheme over R. Moreover,

$$\mathrm{Lie}(\mathrm{Aut}_R(\mathcal{L})^\circ) \cong \mathcal{L} \tag{11}$$

as Lie algebras over R, e.g. [10, Proposition 4.10].

One can show that there is a short exact sequence of group homomorphisms

$$1 \to \mathrm{Aut}_R(\mathcal{L}) \to \mathrm{Aut}_k(\mathcal{L}) \to \mathrm{Aut}_k(R), \tag{12}$$

where the first two arrows are the natural ones, and the third arrow sends every $f \in \mathrm{Aut}_k(\mathcal{L})$ to the automorphism of the centroid R mapping $\chi \in R$ to $f\chi f^{-1}$. In the sequence (12), one has $\mathrm{Aut}_k(R) \cong (k^\times)^n \rtimes \mathrm{GL}_n(\mathbb{Z})$, and the group $\mathrm{Aut}_R(\mathcal{L})$ fits into the short exact sequence

$$1 \to G \to \mathrm{Aut}_R(\mathcal{L}) \to \mathrm{Out}_R(\mathcal{L}) \to 1, \tag{13}$$

where $\mathrm{Out}_R(\mathcal{L}) = 1$ if L has type B_l, C_l, E_7, E_8, F_4, G_2, and is an $\tilde{R}/R$-twisted form of $\mathbb{Z}/d\mathbb{Z}$ or $(\mathbb{Z}/2\mathbb{Z})^2$ in other cases. The aim of the present text is to study G.

Definition 3 [20] Let H be a simple algebraic group, K an arbitrary field of characteristic 0. The *Whitehead group of H* is the quotient group

$$W(K, H) = H(K)/H(K)^+, \tag{14}$$

where $H(K)^+ = \langle g \in H(K) \ : \ g \text{ is unipotent} \rangle$.

It follows from the main result of [12] that the subgroup $H(K)^+$ of Definition 3 also equals the (normal) subgroup of $H(K)$ generated by all $g \in H(K)$ such that there is a morphism of algebraic K-varieties $\phi_g : \mathbb{A}^1_K \to H$ satisfying $\phi_g(0) = e$, $\phi_g(1) = g$.

Theorem 2 [19, Theorem 1.3] *Let k be an algebraically closed field of characteristic 0, Δ be a finite root system of rank ≥ 2, and $\Lambda = \mathbb{Z}^n$, $n \geq 1$. Let $\mathcal{L}$ be a* *centreless Lie Λ-torus of type Δ over k that is finitely generated over its centroid $R \cong k[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$. Set $G = \mathrm{Aut}_R(\mathcal{L})^\circ$, and*

$$E_{exp}(\mathcal{L}) = \langle \exp(\mathrm{ad}_x) \ : \ x \in \mathcal{L}_\alpha^\lambda, \ (\alpha, \lambda) \in \Delta \times \Lambda, \ \alpha \neq 0 \rangle. \tag{15}$$

Then the natural inclusion $G(R) \leq G\big(k((x_1)) \dots ((x_n))\big)$ induces an isomorphism of groups

$$G(R)/E_{exp}(\mathcal{L}) \cong W\big(k((x_1)) \dots ((x_n)), G\big). \tag{16}$$

Using this theorem, we slightly shorten the proof of [7, Theorem 0.1] under the assumption that the grading root system of its centreless core has rank ≥ 2.

Theorem 3 *Let (E, H) be an extended affine Lie algebra over k such that its centreless core $\mathcal{L}$ is a fgc $\mathbb{Z}^n$-Lie torus of type Δ, where $\mathrm{rank}(\Delta) \geq 2$. Assume that E admits the second structure (E, H') of an extended affine Lie algebra. Then there exists an automorphism f of the Lie algebra E such that $f(H) = H'$.*

Proof By [7, Corollary 3.2] the core corresponding to the second structure (E, H') on E coincides with $\mathcal{L}$. The two canonical images H_{cc} and H'_{cc} of H and H' in $\mathcal{L}$ are Borel–Mostow MADs of $\mathcal{L}$ in the sense of [6, §13] by [1, Corollary 5.5]. The assumption $\mathrm{rank}(\Psi) - \mathrm{rank}(\Psi^0) \geq 2$ implies that $\mathrm{rank}(\Delta) \geq 2$. Let $R \cong k[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ be the centroid of $\mathcal{L}$. By [6, Theorem 12.1 and Proposition 13.1] any two Borel–Mostow MADs of $\mathcal{L}$ are conjugate by an element $g \in \mathrm{Aut}_R(\mathcal{L})^\circ(R) = G(R)$. This follows from the fact that any Borel–Mostow MAD is the unique maximal ad-diagonalizable k-subalgebra of the Lie algebra $\mathrm{Lie}(S)$, where S is a maximal split R-subtorus of G such that $\mathrm{Cent}_G(S)$ is loop reductive. Let S and S' be the R-tori corresponding to H_{cc} and H'_{cc} respectively. The tori S and S' by [19, Lemma 2.8] may be provided with a pair of parabolic subgroups P and P' such that $gPg^{-1} = P'$ and $L = \mathrm{Cent}_G(S)$ (resp. $L' = \mathrm{Cent}_G(S')$) is a Levi subgroup of P (resp. P'). Set $F = k((x_1)) \dots ((x_n))$. Then the pairs (L_F, P_F) and $(L'_F, P_{F'})$ are conjugate in G_F by an element $h \in G(F)^+$ by [3, Proposition 6.11(i)]. Then $h^{-1}g \in L(F)$. By [6, Theorem 10.2] (see also [19, Theorem 1.1]) one has $L(F) = L(R) \cdot (G(F)^+ \cap L(F))$. Therefore, adjusting h, we can assume that $h^{-1}g \in L(R)$, and hence $h \in G(R) \cap G(F)^+$. Then by Theorem 2 we have $h \in E_{exp}(\mathcal{L})$. Therefore, h is a product of $\exp(\mathrm{ad}_x)$ for some $x \in \mathcal{L}$ such that $x \in \mathcal{L}_\alpha^\lambda$, $(\alpha, \lambda) \in \Delta \times \Lambda$, $\alpha \neq 0$. By definition of the centreless core any such element x lifts to an element $\tilde{x}$ in E_α for some $\alpha \in \Psi^{\mathrm{an}}$, see [16, Sect 6.3]. Then $\mathrm{ad}_{\tilde{x}}$ is locally nilpotent, and $\exp(\mathrm{ad}_{\tilde{x}})$ is a well-defined automorphism of E. Thus, h lifts to an automorphism f of E. This implies that it is enough to prove the theorem in case where $H_{cc} = H'_{cc}$. Then the proof is finished by [7, Theorem 7.1].

3 How to Compute $W\big(k((x_1))\dots((x_n)), G\big)$?

Set $F = k((x_1))\dots((x_n))$. Note that F is of characteristic 0. As in the previous section, we denote by G the R-group scheme $\mathrm{Aut}_R(\mathcal{L})^\circ$, where $R \cong k[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$, and $\mathcal{L}$ is a centreless fgc $\mathbb{Z}^n$-Lie torus of type Δ, and a twisted form of the Lie algebra $L \otimes_k R$. Note that the root system type of the split simple Lie algebra L is the same as the "absolute" root system type of G over the algebraic closure $\bar{F}$ of F.

Let G^{sc} be the simply connected cover of G, and set $C = \mathrm{Cent}(G^{sc})$. We have an exact sequence of pointed sets

$$1 \to C(F) \to G^{sc}(F) \to G(F) \to H^1(F, C(\bar{F})) \to H^1(F, G^{sc}(\bar{F})), \qquad (17)$$

where $H^1(F, -)$ is the Galois, or étale cohomology of F with values in the corresponding groups. In this sequence, the maps between groups are group homomorphisms [18]. Clearly, $G^{sc}(F)^+$ is mapped into $G(F)^+$. Furthermore, the group $G(F)^+$ is perfect (e.g. by the main result of [20]), and therefore has trivial image in the abelian group $H^1(F, C(\bar{F}))$. Summing up, one obtains the following exact sequence of group homomorphisms:

$$1 \to C(F) \to W(F, G^{sc}) \to W(F, G) \to H^1(F, C(\bar{F})) \to H^1(F, G^{sc}(\bar{F})). \qquad (18)$$

The **Kneser–Tits problem** asks to compute $W(K, G^{sc})$ for any field K and any simply connected simple algebraic K-group G^{sc}. See [9] for a survey of available results. It turns out that $W(F, G^{sc})$ is trivial in many cases, but not always. We cite some particular cases below. We keep the notation introduced in the beginning of the present section.

Theorem 4 [5, 9, 21] *One has* $W(F, G^{sc}) = 1$ *whenever*

- L *is of type* A_l $(l \geq 1)$*, and* $l+1$ *is square-free or* $\mathrm{rank}(\Delta) \geq \lfloor \frac{l}{2} \rfloor$*;*
- L *is of type* B_l, C_l $(l \geq 2)$*,* F_4*,* G_2*;*
- L *is of type* D_l $(l \geq 4)$*, and* $\mathrm{rank}(\Delta) \geq \lfloor \frac{l}{2} \rfloor$ *or* Δ *is of type* B_m $(m \geq 2)$*;*
- L *is of type* E_6*, and* $\mathrm{rank}(\Delta) \geq 2$*;*
- L *is of type* E_7 *or* E_8*, and* $\mathrm{rank}(\Delta) \geq 3$.

Note that Theorem 4 implies that $W(F, G^{sc}) = 1$ whenever $\mathrm{rank}(\Delta) \geq \lfloor \frac{\mathrm{rank}(\Phi)}{2} \rfloor$, where Φ is the root system of L of arbitrary type.

Theorem 5 [9, Theorem 8.6] *One has* $W(F, G^{sc}) = 1$ *whenever* $n \leq 2$*, except possibly in the case where* $n = 2$*,* L *is of type* E_7*, and* $\mathrm{rank}(\Delta) = 1$.

Another important member of the sequence (18) is the group $H^1(F, C(\bar{F}))$. This group is computed in all cases, see [13]. We list below the easiest part of the answer.

- If L is of type E_8, F_4, G_2, then $C(\bar{F}) = 1$ and $H^1(F, C(\bar{F})) = 1$.
- If L is of type B_l, C_l, E_7, then $H^1(F, C(\bar{F})) \cong (\mathbb{Z}/2\mathbb{Z})^n$.
- If L if of type D_{2l}, $l \geq 2$, then $H^1(F, C(\bar{F})) \cong (\mathbb{Z}/2\mathbb{Z})^{2n}$.

- If L is of inner type A_l, $l \geq 2$, then $H^1(F, C(\bar{F})) \cong (\mathbb{Z}/l\mathbb{Z})^n$.
- If L is of inner type D_{2l+1}, $l \geq 2$, then $H^1(F, C(\bar{F})) \cong (\mathbb{Z}/4\mathbb{Z})^n$.
- If L is of inner type E_6, then $H^1(F, C(\bar{F})) \cong (\mathbb{Z}/3\mathbb{Z})^n$.

Finally, the group $H^1(F, G^{sc}(\bar{F}))$ of (18) is

- trivial if $n \leq 2$ (a known case of Serre's conjecture II, see e.g. [11, Sect 9.2.1]);
- tricky in general; there are some case-by-case computations via "cohomological invariants", see [8, 11, 17] and references therein.

Combining the above-mentioned results, we conclude the following.

Theorem 6 *Let k be an algebraically closed field of characteristic* 0. *Set $R = k[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$, and $F = k((x_1))\ldots((x_n))$. Let L be a split simple Lie algebra over k, and let $\mathcal{L}$ be a twisted form of the Lie algebra $L \otimes_k R$ which is a centreless fgc $\mathbb{Z}^n$-Lie torus of type Δ with the centroid R. Let G be the R-group scheme* $\mathrm{Aut}_R(\mathcal{L})^\circ$.

1. *If L has type F_4, G_2, or L has type E_8 and* $\mathrm{rank}(\Delta) \geq 3$, *then* $W(F, G) = 1$.
2. *If $n = 1$, or $n = 2$ and $\big(\text{type of } L,\ \mathrm{rank}(\Delta)\big) \neq (E_7, 1)$, then*

$$W(F, G) \cong H^1(F, C(\bar{F}))$$

is computed explicitly in all cases [13]. In particular, if L is of type B_l, C_l, E_7, or inner type D_{2l+1}, A_l, E_6, then $W(F, G) \cong (\mathbb{Z}/d\mathbb{Z})^n$ for a suitable $d \geq 2$.

Acknowledgements The author was supported at different stages of her work by the postdoctoral grant 6.50.22.2014 "Structure theory, representation theory and geometry of algebraic groups" and the travel grant 6.41.667.2015 at St. Petersburg State University, by the J.E. Marsden postdoctoral fellowship of the Fields Institute, and by the RFBR grants 14-01-31515-mol_a, 13-01-00429-a, 16-01-00750-a.

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Contraction Admissible Pairs of Complex Six-Dimensional Nilpotent Lie Algebras

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Abstract All possible pairs of complex six-dimensional nilpotent Lie algebras are considered and necessary contraction conditions are verified. The complete set of the Lie algebra couples that do not admit contraction is obtained.

1 Introduction

Notion of contraction originates from the work of I. Segal [1] and later in the work [2] of E. Inonu and E. Wigner it was shown that different physical theories are connected by the contractions of their underlying symmetry algebras. There exist two parallel branches of scientists studying contractions in modern science: the first branch is "algebraical", they mainly study the varieties of Lie algebras by means of deformations and orbit closures (degenerations); another one is "physical", it deals with the limit processes between different theories and with the applications of contractions and deformations to physics, e.g. quantization, uncoupling of the coupled systems, etc.

Both problems of description of all possible contractions of a fixed Lie algebra or description of all contractions of Lie algebras of a fixed dimension are rather complicated (e.g., up to now the complete classification of contractions is known only for dimensions not greater then four [3], or for some subsets). Nevertheless, some sets of Lie algebras are closed with respect to contractions and can be studied independently. One of the closed sets is the set of nilpotent Lie algebras of a fixed dimension. In this paper we will deal with complex six-dimensional nilpotent Lie algebras. Note, that complex six-dimensional nilpotent Lie algebras have been already considered

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V. Dobrev (ed.), *Lie Theory and Its Applications in Physics*,
Springer Proceedings in Mathematics & Statistics 191,
DOI 10.1007/978-981-10-2636-2_41

by Seeley [4], but this paper contains a number of misprints and mistakes, in particular, for the algebras 12346_C and 1246 from classification by Seeley the Jacobi identity is not valid, and there is no contraction from the algebra 246_C to the algebra $13 + 13$ that is indicated in [4]. So we would like to correct and enhance the result as far as it forms a base for the proof of the statement that all contractions of nilpotent Lie algebras of dimensions up to six are equivalent to the generalized Inonu–Wigner contractions.

The effective method that allows one to study all inequivalent contractions of some closed family of Lie algebras consists of two main steps: (i) to exclude all the pairs of Lie algebras that do not admit any contraction; (ii) to construct explicitly the contraction matrix for the rest of the pairs.

We will focus on the step (i), that forms a necessary prerequisite for the complete investigation of all possible contractions of six-dimensional Lie algebras.

2 Contractions and Necessary Conditions in Case of Nilpotent Lie Algebras

At the beginning of this section let us fix the notations and basic definitions. Let V be an n-dimensional vector space over the field of complex numbers and $\mathcal{L}_n$ is the set of all possible Lie brackets on V. We identify $\mu \in \mathcal{L}_n$ with the Lie algebra $g = (V, \mu)$. $\mathcal{L}_n$ is an algebraic subset of the variety $V^* \otimes V^* \otimes V$ of bilinear maps from $V \times V$ to V. The group $GL(V)$ acts on $\mathcal{L}_n$ in the following way:

$$(U \cdot \mu)(x, y) = U\big(\mu(U^{-1}x, U^{-1}y)\big) \quad \text{for all } U \in GL(V),\ \mu \in \mathcal{L}_n,\ x, y \in V.$$

Denote the orbit of $\mu \in \mathcal{L}_n$ under the action of $GL(V)$ by $\mathcal{O}(\mu)$ and the closure of it with respect to the Zariski topology on $\mathcal{L}_n$ by $\overline{\mathcal{O}(\mu)}$.

Definition 1 The Lie algebra $g_0 = (V, \mu_0)$ is called a *contraction* of the Lie algebra $g = (V, \mu)$ if $\mu_0 \in \overline{\mathcal{O}(\mu)}$.

For the explicit calculations and application of contractions we fix a basis $\{e_1, \dots, e_n\}$ of V and use the one-to-one correspondence between Lie bracket $\mu \in \mathcal{L}_n$ and a structure constant tensor (c_{ij}^k). In this case the notion of a contraction matrix U_ε ($U\colon (0, 1] \to GL(V)$) and a parameterized family of new Lie brackets $[x, y]_\varepsilon$ is determined by

$$[x, y]_\varepsilon = U_\varepsilon{}^{-1}[U_\varepsilon x, U_\varepsilon y] \quad \text{for all } \varepsilon \in (0, 1],\ x, y \in V,$$

and the contraction is defined as follows:

Definition 2 If the limit $\lim\limits_{\varepsilon\to+0}[x, y]_\varepsilon = \lim\limits_{\varepsilon\to+0} U_\varepsilon{}^{-1}[U_\varepsilon x, U_\varepsilon y] =: [x, y]_0$ exists for any $x, y \in V$, then $[\cdot, \cdot]_0$ is a well-defined Lie bracket and $g_0 = (V, [\cdot, \cdot]_0)$ is called a *contraction* of the Lie algebra g.

To prove the non-existence of contraction from a fixed Lie algebra g to a Lie algebra g_0 we use the necessary contraction conditions which are algebraic quantities uniquely calculated and semiinvariant under contractions (the semiinvariance means an existence of inequalities between the corresponding quantities of the initial and contracted algebras). The sets of invariant and semiinvariant quantities are rather large (see [3–6], etc.), but some of them do not work for nilpotent Lie algebras (the rank of Cartan subalgebra, unimodularity, nilradical dimension, adjoint representation trace, etc.). Below we present the subset of necessary conditions that is sufficient for determining the existence of contractions between complex six-dimensional nilpotent Lie algebras.

Theorem 1 *If the Lie algebra g_0 is a contraction of the Lie algebra g, then the following set of conditions holds true:*

(1) $\dim \mathcal{O}(g_0) < \dim \mathcal{O}(g)$, *where $\mathcal{O}(g)$ is the orbit under action of $GL(V)$ in the variety $\mathcal{L}_n$;*
(2) $n_{\mathrm{Ai}}(g_0) \geq n_{\mathrm{Ai}}(g)$, *where $n_{\mathrm{Ai}}(g)$ is the maximal dimension of Abelian ideals;*
(3) $\dim g_0^l \leq \dim g^l$, $l = 1, 2, 3$, *where $g^1 = [g, g]$, $g^2 = [g^1, g]$, $g^3 = [g^2, g]$;*
(4) $\dim Z(g_0) \geq \dim Z(g)$, *where $Z(g)$ denotes the center of g;*
(5) $\dim C_g(g') \leq \dim C_g(g_0{}')$, *where $C_g(g')$ is the centralizer of the derived algebra;*
(6) if $s(g)$ is a subalgebra of g, then there exists a subalgebra $s(g_0)$ of g_0 of the same dimension, such that $s(g_0)$ is the contraction of $s(g)$;
(7) let g be represented by the structure s, which lies in a Borel-stable closed subset S, then g_0 must also be represented by a structure in S.

If this set of conditions is satisfied by a pair of complex nilpotent six-dimensional Lie algebras then this pair certainly admits a contraction.

Proof Proof of the conditions (1)–(4) can be found in [3], necessary condition (7) is presented in [4]. Condition (6) is a part of more general semi-invariant chain of structures due to [7].

To prove the necessary condition (5) we consider the notion of sequential contraction, which is equivalent to Definition 2 and apply the technique presented on p. 12 of [3].

Proof of the second part of the theorem is given by Tables 1 and 2, to construct the necessary contraction matrices it is enough to consider generalized Inönü–Wigner contractions and repeated contractions (i.e., composition of contractions). Description of the generalized Inönü–Wigner contractions can be reduced to the study of the coinciding diagonal differentiations of the initial and resulting Lie algebras.

Note that we do not consider non-proper contractions that send a Lie algebra to itself and are equivalent to isomorphism transformations.

3 Complex Six-Dimensional Nilpotent Lie Algebras and Their Invariant Quantities

Below we list all inequivalent six-dimensional nilpotent Lie algebras over the complex field according to the Magnin classification [8]. Each Lie algebra is represented by non-zero commutation relation in a basis $\{e_1,\ e_2,\ e_3,\ e_4,\ e_5,\ e_6\}$. Note that we skip Abelian algebra from our consideration as far as any Lie algebra contracts to it.

$g_{6.20}\colon [e_1,e_2]=e_3,\ [e_1,e_3]=e_4,\ [e_1,e_4]=e_5,\ [e_2,e_3]=e_5,\ [e_2,e_5]=e_6,$
$[e_3,e_4]=-e_6;$

$g_{6.19}\colon [e_1,e_2]=e_3,\ [e_1,e_3]=e_4,\ [e_1,e_4]=e_5,\ [e_1,e_5]=e_6,\ [e_2,e_3]=e_5,$
$[e_2,e_4]=e_6;$

$g_{6.18}\colon [e_1,e_2]=e_3,\ [e_1,e_3]=e_4,\ [e_1,e_4]=e_5,\ [e_2,e_5]=e_6,\ [e_3,e_4]=-e_6;$

$g_{6.17}\colon [e_1,e_2]=e_3,\ [e_1,e_3]=e_4,\ [e_1,e_4]=e_5,\ [e_1,e_5]=e_6,\ [e_2,e_3]=e_6;$

$g_{6.16}\colon [e_1,e_2]=e_3,\ [e_1,e_3]=e_4,\ [e_1,e_4]=e_5,\ [e_1,e_5]=e_6;$

$g_{6.15}\colon [e_1,e_2]=e_3,\ [e_1,e_3]=e_4,\ [e_1,e_5]=e_6,\ [e_2,e_3]=e_5,\ [e_2,e_4]=e_6;$

$g_{6.14}\colon [e_1,e_2]=e_3,\ [e_1,e_3]=e_4,\ [e_1,e_4]=e_5,\ [e_2,e_3]=e_6;$

$g_{6.13}\colon [e_1,e_2]=e_4,\ [e_1,e_4]=e_5,\ [e_1,e_5]=e_6,\ [e_2,e_3]=e_5,\ [e_3,e_4]=-e_6;$

$g_{6.12}\colon [e_1,e_2]=e_4,\ [e_1,e_4]=e_5,\ [e_1,e_5]=e_6,\ [e_2,e_3]=e_6,\ [e_2,e_4]=e_6;$

$g_{6.11}\colon [e_1,e_2]=e_4,\ [e_1,e_4]=e_5,\ [e_1,e_5]=e_6,\ [e_2,e_3]=e_6;$

$g_{6.10}\colon [e_1,e_2]=e_4,\ [e_1,e_3]=e_5,\ [e_1,e_4]=e_6,\ [e_3,e_5]=e_6;$

$g_{6.9}\colon [e_1,e_2]=e_4,\ [e_1,e_3]=e_5,\ [e_2,e_5]=e_6,\ [e_3,e_4]=e_6;$

$g_{6.8}\colon [e_1,e_2]=e_4,\ [e_1,e_4]=e_5,\ [e_2,e_3]=e_5,\ [e_2,e_4]=e_6;$

$g_{6.7}\colon [e_1,e_2]=e_4,\ [e_1,e_3]=e_5,\ [e_1,e_4]=e_6,\ [e_2,e_3]=-e_6;$

$g_{6.6}\colon [e_1,e_2]=e_4,\ [e_2,e_3]=e_6,\ [e_2,e_4]=e_5;$

$g_{6.5}\colon [e_1,e_2]=e_4,\ [e_1,e_4]=e_5,\ [e_2,e_3]=e_6,\ [e_2,e_4]=e_6;$

$g_{6.4}\colon [e_1,e_2]=e_4,\ [e_1,e_3]=e_6,\ [e_2,e_4]=e_5;$

$g_{6.3}\colon [e_1,e_2]=e_4,\ [e_1,e_3]=e_5,\ [e_2,e_3]=e_6;$

$g_{6.2}\colon [e_1,e_2]=e_5,\ [e_1,e_5]=e_6,\ [e_3,e_4]=e_6;$

$g_{6.1}\colon [e_1,e_2]=e_5,\ [e_1,e_4]=e_6,\ [e_2,e_3]=e_6;$

$g_{5.6}\oplus g_1\colon [e_1,e_2]=e_3,\ [e_1,e_3]=e_4,\ [e_1,e_4]=e_5,\ [e_2,e_3]=e_5;$

$g_{5.5}\oplus g_1\colon [e_1,e_2]=e_3,\ [e_1,e_3]=e_4,\ [e_1,e_4]=e_5;$

$g_{5.4}\oplus g_1\colon [e_1,e_2]=e_3,\ [e_1,e_3]=e_4,\ [e_2,e_3]=e_5;$

$g_{5.3}\oplus g_1\colon [e_1,e_2]=e_3,\ [e_1,e_4]=e_5,\ [e_2,e_3]=e_5;$

$g_{5.2}\oplus g_1\colon [e_1,e_2]=e_4,\ [e_1,e_3]=e_5;$

$g_{5.1}\oplus g_1\colon [e_1,e_2]=e_5,\ [e_3,e_4]=e_5;$

$g_4\oplus 2g_1\colon [e_1,e_2]=e_3,\ [e_1,e_3]=e_4;$

$g_3 \oplus g_3\colon [e_1, e_2] = e_3,\ [e_4, e_5] = e_6;$
$g_3 \oplus 3g_1\colon [e_1, e_2] = e_3;$
$6g_1.$

The most powerful necessary condition of Theorem 1 is the first one (orbit dimension), since it proves the impossibility of contraction for the more then half of the pairs. Indeed, the orbit dimension gives us the following order on the variety of complex nilpotent six-dimensional Lie algebras:

$$
\begin{aligned}
&28 = \dim \mathcal{O}(g_{6.20});\\
&27 = \dim \mathcal{O}(g_{6.19}) = \dim \mathcal{O}(g_{6.18});\\
&26 = \dim \mathcal{O}(g_{6.17}) = \dim \mathcal{O}(g_{6.15}) = \dim \mathcal{O}(g_{6.13});\\
&25 = \dim \mathcal{O}(g_{6.16}) = \dim \mathcal{O}(g_{6.14}) = \dim \mathcal{O}(g_{6.12}) = \dim \mathcal{O}(g_{6.9});\\
&24 = \dim \mathcal{O}(g_{6.11}) = \dim \mathcal{O}(g_{6.10}) = \dim \mathcal{O}(g_{6.5}) = \dim \mathcal{O}(g_{5.6} \oplus g_1);\\
&23 = \dim \mathcal{O}(g_{6.8}) = \dim \mathcal{O}(g_{6.4}) = \dim \mathcal{O}(g_{5.5} \oplus g_1);\\
&22 = \dim \mathcal{O}(g_{6.7}) = \dim \mathcal{O}(g_{6.2});\\
&21 = \dim \mathcal{O}(g_{6.6}) = \dim \mathcal{O}(g_{5.4} \oplus g_1) = \dim \mathcal{O}(g_{5.3} \oplus g_1);\\
&20 = \dim \mathcal{O}(g_3 \oplus g_3);\\
&19 = \dim \mathcal{O}(g_{6.1}) = \dim \mathcal{O}(g_4 \oplus 2g_1);\\
&18 = \dim \mathcal{O}(g_{6.3});\\
&17 = \dim \mathcal{O}(g_{5.2} \oplus g_1);\\
&15 = \dim \mathcal{O}(g_{5.1} \oplus g_1);\\
&12 = \dim \mathcal{O}(g_3 \oplus 3g_1).
\end{aligned}
$$

Let us consider dimensions of maximal Abelian ideals for the Lie algebras that still should be investigated after the first necessary condition. We obtain
$n_{\mathrm{Ai}} = 4$ for the algebras $g_{6.19}, g_{6.17}, g_{6.15}, g_{6.14}, g_{6.13}, g_{6.12}, g_{6.11}, g_{6.10}, g_{6.9}, g_{6.8}, g_{6.7}, g_{6.5}, g_{6.4}, g_{6.3}, g_{6.2}, g_{6.1}, g_{5.6} \oplus g_1, g_{5.4} \oplus g_1, g_{5.3} \oplus g_1, g_{5.1} \oplus g_1, g_3 \oplus g_3$;
$n_{\mathrm{Ai}} = 5$ for the algebras $g_{6.16}, g_{6.6}, g_{5.5} \oplus g_1, g_{5.2} \oplus g_1, g_4 \oplus 2g_1, g_3 \oplus 3g_1$.

Next we consider the first three dimensions of the lower central series for the desired algebras. We have
$\dim g^1 = 4$ for the algebras $g_{6.16}$ and $g_{6.14}$;
$\dim g^1 = 3$ for the algebras $g_{6.13}$, $g_{6.6}$ and $g_{6.3}$;
$\dim g^1 = 2$ for the algebras $g_{6.2}$, $g_{6.1}$, $g_{5.3} \oplus g_1$ and $g_3 \oplus g_3$.
$\dim g^2 = 2$ for the algebras $g_{6.11}$, $g_{6.8}$, $g_{6.5}$, $g_{5.6} \oplus g_1$, $g_{5.5} \oplus g_1$ and $g_{5.4} \oplus g_1$;
$\dim g^2 = 1$ for the algebras $g_{6.10}$, $g_{6.9}$ and $g_4 \oplus 2g_1$;
$\dim g_3 \oplus {g_3}^2 = 0$;
$\dim {g_{6.15}}^3 = 1$ and $\dim {g_{6.16}}^3 = 2$;
$\dim {g_{6.5}}^3 = 0$ and $\dim {g_{g_{5.5} \oplus g_1}}^3 = 1$.

Other refinement we obtain by utilizing the dimension of the center. We have
dim $Z = 1$ for the algebras $g_{6.11}$, $g_{6.10}$ and $g_{6.2}$;
dim $Z = 2$ for the algebras $g_{6.14}$, $g_{6.8}$, $g_{6.5}$, $g_{6.4}$, $g_{6.1}$, $g_{5.6} \oplus g_1$, $g_{5.3} \oplus g_1$, $g_{5.1} \oplus g_1$ and $g_3 \oplus g_3$;
dim $Z = 3$ for the algebras $g_{6.3}$, $g_{5.4} \oplus g_1$ and $g_{5.2} \oplus g_1$.

Next we make use of the dimensions of the centralizers of the derived algebras. We obtain
dim $C_g(g') = 5$ for the algebras $g_{6.11}$, $g_{6.7}$, $g_{6.4}$ and $g_{6.2}$;
dim $C_g(g') = 4$ for the algebras $g_{6.8}$ and $g_{5.4} \oplus g_1$.

The subalgebra criterion is applied to the three pairs of algebras $(g_{6.18}, g_{6.15})$, $(g_{6.18}, g_{6.13})$ and $(g_{6.17}, g_{6.9})$. We prove non-existence of the contraction using the five-dimensional subalgebra $\langle e_2, e_3, e_4, e_5, e_6\rangle$ of $g_{6.18}$. This subalgebra is isomorphic to $g_{5.1}$ and contracts to the Abelian one or to the algebra $g_3 \oplus 2g_1$, but none of these three five-dimensional subalgebras are subalgebras of $g_{6.15}$ or $g_{6.13}$. The same situation takes place for the subalgebra $\langle e_2, e_3, e_4, e_5, e_6\rangle$ of $g_{6.17}$, which is isomorphic to $g_3 \oplus 2g_1$ and contracts to Abelian one, but $g_{6.9}$ does not have subalgebra $g_3 \oplus 2g_1$ or $5g_1$.

The criterion based on Borel-stable structures is applied to the pairs: $(g_{6.17}, g_{6.12})$, $(g_{5.6} \oplus g_1, g_{6.4})$, $(g_{5.6} \oplus g_1, g_3 \oplus g_3)$, $(g_{6.8}, g_3 \oplus g_3)$, $(g_{6.7}, g_3 \oplus g_3)$, $(g_{5.3} \oplus g_1, g_3 \oplus g_3)$. Note that in the case of the group GL_n the Borel subgroups are formed by lower or upper triangular matrices. Bellow we present all necessary Borel-stable structures that contradict the existence of contraction inside the considered pairs of algebras.

For $g_{6.17}$ we consider the structure $[\langle e_2, e_3, e_4, e_5, e_6\rangle, \langle e_4, e_5, e_6\rangle] \subseteq \langle 0\rangle$, for $g_{6.12}$ we obtain $[\langle e_2, e_3, e_4, e_5, e_6\rangle, \langle e_4, e_5, e_6\rangle] \subseteq \langle e_6\rangle \neq \langle 0\rangle$.

Absolutely the same structures fit for the cases $(g_{5.3} \oplus g_1, g_3 \oplus g_3)$ and $(g_{5.6} \oplus g_1, g_{6.4})$.

For the pairs $(g_{5.6} \oplus g_1, g_3 \oplus g_3)$ and $(g_{6.7}, g_3 \oplus g_3)$ the first structure is the same, but the second one for $g_3 \oplus g_3$ is $[\langle e_2, e_3, e_4, e_5, e_6\rangle, \langle e_3, e_5, e_6\rangle] \subseteq \langle e_6\rangle \neq \langle 0\rangle$.

Finally, the pair $(g_{6.8}, g_3 \oplus g_3)$ is eliminated by $[\langle e_1, e_3, e_4, e_5, e_6\rangle, \langle e_3, e_5, e_6\rangle] \subseteq \langle 0\rangle$ for $g_{6.8}$.

The main result of the paper is summed up in a form of tables placed at the end of the paper. The numbers in tables code the necessary condition from the Theorem 1 that contradicts the contraction existence; the names of the columns denote the initial algebra and the names of the rows are the contraction results. The letter "c" means that a contraction exists and the letter "i" indicates the pair of the identical (same) Lie algebras.

From the Tables 1 and 2 we can see that the Lie algebra $g_{6.20}$ is a generic for the variety of six-dimensional complex Lie algebras in the sense that all the rest of nilpotent Lie algebras are obtained from it by means of contractions. Therefore the $g_{6.20}$-invariant theories consolidate all the theories with the 6D nilpotent invariance.

Table 1 Admissible contractions of 6D nilpotent Lie algebras 1. The numbers in the tables code the necessary condition from the Theorem 1 that contradicts the contraction existence, the names of the columns denote the initial algebra and the names of the rows are the contraction results. The letter "c" means that a contraction exists and the letter "i" indicates the pair of the identical (same) Lie algebras

$\mathfrak{g}_0\backslash\mathfrak{g}$	$\mathfrak{g}_{6.20}$	$\mathfrak{g}_{6.19}$	$\mathfrak{g}_{6.18}$	$\mathfrak{g}_{6.17}$	$\mathfrak{g}_{6.16}$	$\mathfrak{g}_{6.15}$	$\mathfrak{g}_{6.14}$	$\mathfrak{g}_{6.13}$	$\mathfrak{g}_{6.12}$	$\mathfrak{g}_{6.11}$	$\mathfrak{g}_{6.10}$	$\mathfrak{g}_{6.9}$	$\mathfrak{g}_{6.8}$	$\mathfrak{g}_{6.7}$	$\mathfrak{g}_{6.6}$	$\mathfrak{g}_{6.5}$
$\mathfrak{g}_{6.20}$	i	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$\mathfrak{g}_{6.19}$	c	i	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$\mathfrak{g}_{6.18}$	c	1	i	1	1	1	1	1	1	1	1	1	1	1	1	1
$\mathfrak{g}_{6.17}$	c	c	c	i	1	1	1	1	1	1	1	1	1	1	1	1
$\mathfrak{g}_{6.16}$	c	c	c	c	i	3	1	3	1	1	1	1	1	1	1	1
$\mathfrak{g}_{6.15}$	c	c	6	1	1	i	1	1	1	1	1	1	1	1	1	1
$\mathfrak{g}_{6.14}$	c	c	c	c	1	c	i	3	1	1	1	1	1	1	1	1
$\mathfrak{g}_{6.13}$	c	c	6	1	1	1	1	i	1	1	1	1	1	1	1	1
$\mathfrak{g}_{6.12}$	c	c	c	7	1	c	1	c	i	1	1	1	1	1	1	1
$\mathfrak{g}_{6.11}$	c	c	c	c	2	c	4	c	c	i	1	3	1	1	1	1
$\mathfrak{g}_{6.10}$	c	c	c	c	2	c	4	c	c	1	i	c	1	1	1	1
$\mathfrak{g}_{6.9}$	c	c	c	6	1	c	1	c	1	1	1	i	1	1	1	1
$\mathfrak{g}_{6.8}$	c	c	c	c	2	c	c	c	c	5	3	3	i	1	1	c
$\mathfrak{g}_{6.7}$	c	c	c	c	2	c	c	c	c	c	c	c	c	i	1	c
$\mathfrak{g}_{6.6}$	c	c	c	c	c	c	c	c	c	c	c	c	c	c	i	c
$\mathfrak{g}_{6.5}$	c	c	c	c	2	c	c	c	c	1	1	3	1	1	1	i
$\mathfrak{g}_{6.4}$	c	c	c	c	2	c	c	c	c	c	c	c	1	1	1	c
$\mathfrak{g}_{6.3}$	c	c	c	c	2	c	c	c	c	c	c	c	c	c	2	c
$\mathfrak{g}_{6.2}$	c	c	c	c	2	c	c	c	c	c	c	c	4	1	1	4
$\mathfrak{g}_{6.1}$	c	c	c	c	2	c	c	c	c	c	c	c	c	c	2	c

(continued)

Table 1 (continued)

$\mathfrak{g}_0 \backslash \mathfrak{g}$	$\mathfrak{g}_{6.20}$	$\mathfrak{g}_{6.19}$	$\mathfrak{g}_{6.18}$	$\mathfrak{g}_{6.17}$	$\mathfrak{g}_{6.16}$	$\mathfrak{g}_{6.15}$	$\mathfrak{g}_{6.14}$	$\mathfrak{g}_{6.13}$	$\mathfrak{g}_{6.12}$	$\mathfrak{g}_{6.11}$	$\mathfrak{g}_{6.10}$	$\mathfrak{g}_{6.9}$	$\mathfrak{g}_{6.8}$	$\mathfrak{g}_{6.7}$	$\mathfrak{g}_{6.6}$	$\mathfrak{g}_{6.5}$
$\mathfrak{g}_{5.6} \oplus \mathfrak{g}_1$	c	c	c	c	2	c	c	c	c	1	1	3	1	1	1	1
$\mathfrak{g}_{5.5} \oplus \mathfrak{g}_1$	c	c	c	c	c	c	c	c	c	c	3	3	1	1	1	3
$\mathfrak{g}_{5.4} \oplus \mathfrak{g}_1$	c	c	c	c	2	c	c	c	c	5	3	3	c	5	2	c
$\mathfrak{g}_{5.3} \oplus \mathfrak{g}_1$	c	c	c	c	2	c	c	c	c	c	c	c	c	c	2	c
$\mathfrak{g}_{5.2} \oplus \mathfrak{g}_1$	c	c	c	c	c	c	c	c	c	c	c	c	c	c	c	c
$\mathfrak{g}_{5.1} \oplus \mathfrak{g}_1$	c	c	c	c	2	c	c	c	c	c	c	c	c	c	2	c
$\mathfrak{g}_4 \oplus 2\mathfrak{g}_1$	c	c	c	c	c	c	c	c	c	c	c	c	c	c	c	c
$\mathfrak{g}_3 \oplus \mathfrak{g}_3$	c	c	c	c	2	c	c	c	c	c	c	c	7	7	2	c
$\mathfrak{g}_3 \oplus 3\mathfrak{g}_1$	c	c	c	c	c	c	c	c	c	c	c	c	c	c	c	c

Table 2 Admissible contractions of 6D nilpotent Lie algebras 2. The numbers in tables code the necessary condition from the Theorem 1 that contradicts the contraction existence, the names of the columns denote the initial algebra and the names of the rows are the contraction results. The letter "c" means that a contraction exists and the letter "i" indicates the pair of the identical (same) Lie algebras

$\mathfrak{g}_0\backslash\mathfrak{g}$	$\mathfrak{g}_{6.4}$	$\mathfrak{g}_{6.3}$	$\mathfrak{g}_{6.2}$	$\mathfrak{g}_{6.1}$	$\mathfrak{g}_{5.6}\oplus\mathfrak{g}_1$	$\mathfrak{g}_{5.5}\oplus\mathfrak{g}_1$	$\mathfrak{g}_{5.4}\oplus\mathfrak{g}_1$	$\mathfrak{g}_{5.3}\oplus\mathfrak{g}_1$	$\mathfrak{g}_{5.2}\oplus\mathfrak{g}_1$	$\mathfrak{g}_{5.1}\oplus\mathfrak{g}_1$	$\mathfrak{g}_4\oplus 2\mathfrak{g}_1$	$2\mathfrak{g}_3$	$\mathfrak{g}_3\oplus 3\mathfrak{g}_1$
$\mathfrak{g}_{6.20}$	1	1	1	1	1	1	1	1	1	1	1	1	1
$\mathfrak{g}_{6.19}$	1	1	1	1	1	1	1	1	1	1	1	1	1
$\mathfrak{g}_{6.18}$	1	1	1	1	1	1	1	1	1	1	1	1	1
$\mathfrak{g}_{6.17}$	1	1	1	1	1	1	1	1	1	1	1	1	1
$\mathfrak{g}_{6.16}$	1	1	1	1	1	1	1	1	1	1	1	1	1
$\mathfrak{g}_{6.15}$	1	1	1	1	1	1	1	1	1	1	1	1	1
$\mathfrak{g}_{6.14}$	1	1	1	1	1	1	1	1	1	1	1	1	1
$\mathfrak{g}_{6.13}$	1	1	1	1	1	1	1	1	1	1	1	1	1
$\mathfrak{g}_{6.12}$	1	1	1	1	1	1	1	1	1	1	1	1	1
$\mathfrak{g}_{6.11}$	1	1	1	1	1	1	1	1	1	1	1	1	1
$\mathfrak{g}_{6.10}$	1	1	1	1	1	1	1	1	1	1	1	1	1
$\mathfrak{g}_{6.9}$	1	1	1	1	1	1	1	1	1	1	1	1	1
$\mathfrak{g}_{6.8}$	1	1	1	1	c	1	1	1	1	1	1	1	1
$\mathfrak{g}_{6.7}$	c	1	1	1	c	1	1	1	1	1	1	1	1
$\mathfrak{g}_{6.6}$	c	1	3	1	c	c	1	1	1	1	1	1	1
$\mathfrak{g}_{6.5}$	1	1	1	1	1	1	1	1	1	1	1	1	1
$\mathfrak{g}_{6.4}$	i	1	1	1	7	1	1	1	1	1	1	1	1
$\mathfrak{g}_{6.3}$	c	i	3	3	c	2	c	3	1	1	2	3	1
$\mathfrak{g}_{6.2}$	4	1	i	1	4	2	1	1	1	1	1	1	1
$\mathfrak{g}_{6.1}$	c	1	c	i	c	2	4	c	1	1	1	c	1

(continued)

Table 2 (continued)

$g_0 \backslash g$	$g_{6.4}$	$g_{6.3}$	$g_{6.2}$	$g_{6.1}$	$g_{5.6} \oplus g_1$	$g_{5.5} \oplus g_1$	$g_{5.4} \oplus g_1$	$g_{5.3} \oplus g_1$	$g_{5.2} \oplus g_1$	$g_{5.1} \oplus g_1$	$g_4 \oplus 2g_1$	$2g_3$	$g_3 \oplus 3g_1$
$g_{5.6} \oplus g_1$	1	1	1	1	i	1	1	1	1	1	1	1	1
$g_{5.5} \oplus g_1$	1	1	1	1	c	i	1	1	1	1	1	1	1
$g_{5.4} \oplus g_1$	5	1	5	1	c	2	i	1	1	1	1	1	1
$g_{5.3} \oplus g_1$	c	1	c	1	c	2	4	i	1	1	1	1	1
$g_{5.2} \oplus g_1$	c	c	c	c	c	c	c	c	i	1	c	c	1
$g_{5.1} \oplus g_1$	c	4	c	c	c	2	4	c	4	i	2	c	1
$g_4 \oplus 2g_1$	c	1	c	1	c	c	c	c	1	1	i	3	1
$g_3 \oplus g_3$	c	1	c	1	7	2	4	7	1	1	1	i	1
$g_3 \oplus 3g_1$	c	c	c	c	c	c	c	c	c	c	c	c	i

Acknowledgements SP acknowledges the support of SGS15/215/OHK4/3T/14, project of the Czech Technical University in Prague. MN is grateful for the hospitality extended to her at the Department of mathematics, FNSPE, Czech Technical University in Prague.

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About Filiform Lie Algebras of Order 3

R.M. Navarro

Abstract The aim of this work is to review recent advances in generalizing filiform Lie (super)algebras into the theory of Lie algebras of order F. Recall that the latter type of algebras constitutes the underlying algebraic structure of fractional supersymmetry. In this context filiform Lie algebras of order F emerged in [16], and a further study can be found in [17].

1 Introduction

Filiform Lie algebras were introduced by Vergne [21] verifying an important property, i.e. all of them can be obtained by using a deformation of the model filiform Lie algebra. Moreover, this result has been extended for Lie superalgebras [1, 6, 9, 10].

On the other hand, some generalizations of lie (super)algebras that have been studied due to their physics applications are color Lie superalgebras [11–13, 18] and Lie algebras of order F [7, 19, 20]. In particular, Lie algebras of order F were considered to implement non-trivial extensions of the Poincaré symmetry different from the usual supersymmetric extension. Therefore, Lie algebras of order F can be regarded as the algebraic structure associated to fractional supersymmetry [4, 5, 14, 15].

Consequently, we center our present work in reviewing recent advances in "filiform Lie algebras of order F". These algebras were introduced in [16] as a generalization of filiform Lie (super)algebras into the theory of Lie algebras of order F. A further study can be found in [17], in which families of filiform Lie algebras of order 3 have been obtained by using infinitesimal deformations.

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V. Dobrev (ed.), *Lie Theory and Its Applications in Physics*,
Springer Proceedings in Mathematics & Statistics 191,
DOI 10.1007/978-981-10-2636-2_42

2 Preliminaries

Now, we recall some basic results of Lie algebras of order F introduced in [3, 7, 19, 20].

Definition 1 Let $F \in \mathbb{N}^*$. A $\mathbb{Z}_F$-graded $\mathbb{C}$-vector space $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 \cdots \oplus \mathfrak{g}_{F-1}$ is called a complex **Lie algebra of order** F if the following hold:

- $\mathfrak{g}_0$ is a complex Lie algebra.
- For all $i = 1, \dots, F-1$, $\mathfrak{g}_i$ is a representation of $\mathfrak{g}_0$. If $X \in \mathfrak{g}_0$, $Y \in \mathfrak{g}_i$, then $[X, Y]$ denotes the action of $X \in \mathfrak{g}_0$ on $Y \in \mathfrak{g}_i$ for all $i = 1, \dots, F-1$.
- For all $i = 1, \dots, F-1$, there exists an F-Linear, $\mathfrak{g}_0$-equivariant map, $\{\cdots\}$: $S^F(\mathfrak{g}_i) \longrightarrow \mathfrak{g}_0$, where $S^F(\mathfrak{g}_i)$ denotes the F-fold symmetric product of $\mathfrak{g}_i$.
- For all $X_i \in \mathfrak{g}_0$ and $Y_j \in \mathfrak{g}_k$, the following "Jacobi identities" hold:

$$[[X_1, X_2], X_3] + [[X_2, X_3], X_1] + [[X_3, X_1], X_2] = 0. \tag{1}$$

$$[[X_1, X_2], Y_3] + [[X_2, Y_3], X_1] + [[Y_3, X_1], X_2] = 0. \tag{2}$$

$$[X, \{Y_1, \dots, Y_F\}] = \{[X, Y_1], \dots, Y_F\} + \cdots + \{Y_1, \dots, [X, Y_F]\}. \tag{3}$$

$$\sum_{j=1}^{F+1} [Y_j, \{Y_1, \dots, Y_{j-1}, Y_{j+1}, \dots, Y_{F+1}\}] = 0. \tag{4}$$

As a Lie algebra of order 1 is exactly a Lie algebra and a Lie algebra of order 2 is a Lie superalgebra, then we can consider that the Lie algebras of order 3 are generalization of the Lie (super)algebras.

Proposition 1 *[3] Let* $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_{F-1}$ *be a Lie algebra of order* F*, with* $F > 1$. *For any* $i = 1, \dots, F-1$, *the subspaces* $\mathfrak{g}_0 \oplus \mathfrak{g}_i$ *inherits the structure of a Lie algebra of order* F. *We call these type of algebras* **elementary Lie algebras of order** F.

We will consider in our study elementary Lie algebras of order 3, $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$, it can be found examples of elementary Lie algebras of order 3 in [7].

3 Filiform Lie Algebras of Order 3

Prior to give the definition of "*filiform Lie algebras of order* 3", it is necessary to know the concept "*filiform module*".

Definition 2 [16] Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_{F-1}$ be a Lie algebra of order F. $\mathfrak{g}_i$ is called a $\mathfrak{g}_0$-**filiform module** if there exists a decreasing subsequence of vector subspaces in its underlying vectorial space V, $V = V_m \supset \cdots \supset V_1 \supset V_0$, with dimensions $m, m-1, \dots 0$, respectively, $m > 0$, and such that $[\mathfrak{g}_0, V_{i+1}] = V_i$.

Definition 3 [16] Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_{F-1}$ be a Lie algebra of order F. Then $\mathfrak{g}$ is a **filiform Lie algebra of order** F if the following conditions hold:

(1) $\mathfrak{g}_0$ is a filiform Lie algebra.
(2) $\mathfrak{g}_i$ has structure of $\mathfrak{g}_0$-filiform module, for all i, $1 \leq i \leq F-1$

If we consider an homogeneous basis of a Lie algebra of order 3, then it will be completely determined by its structure constants. These structure constants verify the polynomial equations that come from the Jacobi identity and these equations endow to the Lie algebras of order 3 of structure of algebraic variety, called $\mathcal{L}_{n,m,p}$. The subset composed of all filiform Lie algebras of order 3 will be denoted by $\mathcal{F}_{n,m,p}$.

Prior to continuing it is convenient to find a suitable basis or so called adapted basis. This is an open problem in general, but it has been solved in some cases [11, 16].

Theorem 1 [16] *Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$ be a Lie algebra of order 3. If $\mathfrak{g}$ is a filiform Lie algebra of order 3, then there exists an adapted basis of $\mathfrak{g}$, namely $\{X_0, \dots, X_n, Y_1, \dots, Y_m, Z_1, \dots, Z_p\}$ with $\{X_0, X_1, \dots, X_n\}$ a basis of $\mathfrak{g}_0$, $\{Y_1, \dots, Y_m\}$ a basis of $\mathfrak{g}_1$ and $\{Z_1, \dots, Z_p\}$ a basis of $\mathfrak{g}_2$, such that:*

$$\begin{cases} [X_0, X_i] = X_{i+1}, & 1 \leq i \leq n-1, \\ [X_0, Y_j] = Y_{j+1}, & 1 \leq j \leq m-1, \\ [X_0, Z_k] = Z_{k+1}, & 1 \leq k \leq p-1, \\ [X_i, X_j] = \sum_{k=0}^{n} C_{ij}^k X_k, & 1 \leq i < j \leq n, \\ [X_i, Y_j] = \sum_{k=1}^{m} D_{ij}^k Y_k, & 1 \leq i \leq n, 1 \leq j \leq m, \\ [X_i, Z_j] = \sum_{k=1}^{p} E_{ij}^k Z_k, & 1 \leq i \leq n, 1 \leq j \leq p, \\ \{Y_i, Y_j, Y_l\} = \sum_{k=0}^{n} F_{ijl}^k X_k, & 1 \leq i \leq j \leq l \leq m, \\ \{Z_i, Z_j, Z_l\} = \sum_{k=0}^{n} G_{ijl}^k X_k, & 1 \leq i \leq j \leq l \leq p, \end{cases}$$

X_0 will be called the characteristic vector.

It can be noted, that the simplest filiform Lie algebra of order 3 will be given, exactly, by the only non-null bracket products that follow

$$\begin{cases} [X_0, X_i] = X_{i+1}, \ 1 \le i \le n-1 \\ [X_0, Y_j] = Y_{j+1}, \ 1 \le j \le m-1 \\ [X_0, Z_k] = Z_{k+1} \ \ 1 \le k \le p-1 \end{cases}$$

We will call to this algebra as the **model filiform Lie algebra of order** 3 or μ_0. This algebra of order 3 will play the same role as the model filiform algebra, i.e. by using infinitesimal deformations of it, families of filiform Lie algebras of order 3 are obtained. Thus, next we are going to introduce the concepts of pre-infinitesimal and infinitesimal deformations.

Definition 4 [16] Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be an elementary Lie algebra of order 3 and let $A = (\mathfrak{g}_0 \wedge \mathfrak{g}_0) \oplus (\mathfrak{g}_0 \wedge \mathfrak{g}_1) \oplus S^3(\mathfrak{g}_1)$. The linear map $\psi : A \longrightarrow \mathfrak{g}$ is called a **pre-infinitesimal deformation** of $\mathfrak{g}$ if it satisfies

$$\mu \circ \psi + \psi \circ \mu = 0$$

with μ representing the law of $\mathfrak{g}$.

Definition 5 [7] Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be an elementary Lie algebra of order 3 and let $A = (\mathfrak{g}_0 \wedge \mathfrak{g}_0) \oplus (\mathfrak{g}_0 \wedge \mathfrak{g}_1) \oplus S^3(\mathfrak{g}_1)$. The linear map $\psi : A \longrightarrow \mathfrak{g}$ is called an **infinitesimal deformation** of $\mathfrak{g}$ if it satisfies $\mu \circ \psi + \psi \circ \mu = 0$ and $\psi \circ \psi = 0$, with μ representing the law of $\mathfrak{g}$.

Remark 1 [16] If ψ is a pre-infinitesimal deformation of a model filiform elementary Lie algebra of order 3 law μ_0 with $\psi(X_0, X) = 0$ for all $X \in \mu_0$, then the law $\mu_0 + \psi$ is a filiform Lie algebra of order 3 law iff ψ is an infinitesimal deformation. Thus, by using infinitesimal deformations of the associated model elementary Lie algebra it can be obtained families of filiform elementary Lie algebras of order 3.

Let denote by $Z(\mu_0)$ the vector space composed by all the pre-infinitesimal deformations ψ of the model filiform elementary Lie algebra of order 3, μ_0, verifying that $\psi(X_0, X) = 0, \ \forall \ X \in \mu_0$. Then, we have the following decomposition (see [16]) of $Z(\mu_0) =$

$Z(\mu_0) \cap Hom(\mathfrak{g}_0 \wedge \mathfrak{g}_0, \mathfrak{g}_0) \oplus Z(\mu_0) \cap Hom(\mathfrak{g}_0 \wedge \mathfrak{g}_1, \mathfrak{g}_1) \oplus Z(\mu_0) \cap Hom (S^3(\mathfrak{g}_1), \mathfrak{g}_0)$

$:= A \oplus B \oplus C$,

with $\mathfrak{g}_0 =< X_0, X_1, \ldots, X_n >$ and $\mathfrak{g}_1 =< Y_1, \ldots, Y_m >$.

It can be observed that of the three subspaces of pre-infinitesimal deformations A, B and C, the most important for our goal is C because any ψ belonging to C will be an infinitesimal deformation, i.e. $\psi \circ \psi = 0$. Then, if $\psi \in C$ we always have that $\mu_0 + \psi$ is a filiform elementary Lie algebra of order 3.

Remark 2 [16] We note that if ψ is a pre-infinitesimal deformation of the model filiform elementary Lie algebra of order 3, $\psi = \psi_1 + \psi_2 + \psi_3 \in Z(\mu_0)$, then $Im\psi_3 \subset \langle X_1, \ldots X_n \rangle$, that is $\psi_3 : S^3(\mathfrak{g}_1) \longrightarrow \mathfrak{g}_0/\mathbb{C}X_0$ such that

$$[X_0, \psi_3(Y_i, Y_j, Y_k)] - \psi_3([X_0, Y_i], Y_j, Y_k) - \psi_3(Y_i, [X_0, Y_j], Y_k) -$$

$$- \psi_3(Y_i, Y_j, [X_0, Y_k]) = 0$$

with $1 \leq i \leq j \leq k \leq m$.

3.1 $\mathfrak{sl}(2, \mathbb{C})$-*Module Method*

Throughout this section we are going to explain briefly the $\mathfrak{sl}(2, \mathbb{C})$-module method to compute the dimensions of C, for more details see [16].

It is noted $\mathfrak{sl}(2, \mathbb{C})$ as $< X_-, H, X_+ >$ with the relations: $[X_+, X_-] = H$, $[H, X_+] = 2X_+$, $[H, X_-] = -2X_-$. Then if we have a n-dimensional $\mathfrak{sl}(2, \mathbb{C})$-module $V =< e_1, \ldots, e_n >$, there exists in V a unique structure (up to isomorphism) of an irreducible $\mathfrak{sl}(2, \mathbb{C})$-module [2, 8], i.e.

$$\begin{cases} X_+ \cdot e_i = e_{i+1}, & 1 \leq i \leq n-1, \\ X_+ \cdot e_n = 0, & \\ H \cdot e_i = (-n + 2i - 1)e_i, & 1 \leq i \leq n. \end{cases}$$

e_n is called the maximal vector of V and its weight, called the highest weight of V, is equal to $n - 1$.

The space $\mathrm{Hom}(\otimes_{i=1}^{k} V_i, V_0)$, with V_i $\mathfrak{sl}(2, \mathbb{C})$-modules, can be endowed also with structure of $\mathfrak{sl}(2, \mathbb{C})$-module:

$$(\xi \cdot \varphi)(x_1, \ldots, x_k) = \xi \cdot \varphi(x_1, \ldots, x_k) - \sum_{i=1}^{i=k} \varphi(x_1, \ldots, \xi \cdot x_i, x_{i+1}, \ldots, x_n)$$

with $\xi \in \mathfrak{sl}(2, \mathbb{C})$ and $\varphi \in \mathrm{Hom}(\otimes_{i=1}^{k} V_i, V_0)$. In particular, an element $\varphi \in \mathrm{Hom}(V_1 \otimes V_1 \otimes V_1, V_0)$ is invariant if $X_+ \cdot \varphi = 0$, i.e.

$$X_+ \cdot \varphi(x_1, x_2, x_3) - \varphi(X_+ \cdot x_1, x_2, x_3) - \varphi(x_1, X_+ \cdot x_2, x_3) - \varphi(x_1, x_2, X_+ \cdot x_3) = 0, \quad (5)$$

$\forall x_1, x_2, x_3 \in V_1$. Thus, φ is invariant if and only if φ is a maximal vector.

We consider φ, on the other hand, as a pre-infinitesimal deformation that belongs to C, so according to Remark 2 we have: $\varphi : S^3(\mathfrak{g}_1) \longrightarrow \mathfrak{g}_0/\mathbb{C}X_0$ with

$$[X_0, \varphi(Y_i, Y_j, Y_k)] - \varphi([X_0, Y_i], Y_j, Y_k) - \varphi(Y_i, [X_0, Y_j], Y_k) - \quad (6)$$

$$- \varphi(Y_i, Y_j, [X_0, Y_k]) = 0$$

and $1 \leq i \leq j \leq k \leq m$. Then, if we consider the structure of irreducible $\mathfrak{sl}(2, \mathbb{C})$-module in $V_0 = \langle X_1, \ldots, X_n \rangle = \mathfrak{g}_0/\mathbb{C}X_0$ and in $V_1 = \langle Y_1, \ldots, Y_m \rangle = \mathfrak{g}_1$:

$$\begin{cases} X_+ \cdot X_i = X_{i+1}, \ 1 \le i \le n-1, \ X_+ \cdot X_n = 0, \\ X_+ \cdot Y_j = Y_{j+1}, \ 1 \le j \le m-1, \ X_+ \cdot Y_m = 0. \end{cases}$$

We can identify the multiplications $X_+ \cdot X_i$ and $X_+ \cdot Y_j$ with the brackets $[X_0, X_i]$ and $[X_0, Y_j]$ respectively. Thus, (5) and (6) are equivalent.

Proposition 2 [16] *Any symmetric multi-linear map φ, $\varphi : S^3 V_1 \longrightarrow V_0$ will be an element of C if and only if φ is a maximal vector of the $\mathfrak{sl}(2, \mathbb{C})$-module* $\mathrm{Hom}(S^3 V_1, V_0)$, *with $V_0 = \langle X_1, \ldots, X_n \rangle$ and $V_1 = \langle Y_1, \ldots, Y_m \rangle$.*

As each irreducible $\mathfrak{sl}(2, \mathbb{C})$-module has (up to nonzero scalar multiples) a unique maximal vector, then the dimension of C is equal to the number of summands of any decomposition of $\mathrm{Hom}(S^3 V_1, V_0)$ into the direct sum of irreducible $\mathfrak{sl}(2, \mathbb{C})$-modules. Thanks to the symmetric structure of the weights we have that the dimension of C is equal to the dimension of the subspace of Hom $(S^3 V_1, V_0)$ spanned by the vectors of weight 0 or 1. Therefore, in that way are calculated the dimensions of the following section.

3.2 Dimension and Basis

Throughout this section we are going to list the main results obtained in [16] and [17].

Theorem 2 [16, 17] *Let be $C = Z(\mu_0) \cap \mathrm{Hom}(S^3 \mathfrak{g}_1, \mathfrak{g}_0)$, then*

If *m=3* and *n* odd ,

$$dim\ C = \begin{cases} 2 & \text{if } n = 1 \\ 6 & \text{if } n = 3 \\ 8 & \text{if } n = 5 \\ 10 & \text{if } n = 2k+1, \ k \ge 3 \end{cases}$$

If *m=3* and *n* even ,

$$dim\ C = \begin{cases} 4 & \text{if } n = 2 \\ 7 & \text{if } n = 4 \\ 9 & \text{if } n = 6 \\ 10 & \text{if } n = 2k, \ k \ge 4 \end{cases}$$

If *m=4* and *n* even ,

$$dim\ C = \begin{cases} 6 & \text{if } n = 2 \\ 12 & \text{if } n = 4 \\ 16 & \text{if } n = 6 \\ 18 & \text{if } n = 8 \\ 20 & \text{if } n = 2k, \ k \ge 5 \end{cases}$$

Theorem 3 [16] *If $C = Z(\mu_0) \cap \mathrm{Hom}(S^3 \mathfrak{g}_1, \mathfrak{g}_0)$ and $m = 3$ and n is odd, then we have the following vector basis of C*

- $\{\varphi_{1,1}, \varphi_{3,1}\}$ *if* $n = 1$
- $\{\varphi_{1,3}, \varphi_{1,2}, \varphi_{1,1}, \varphi_{3,3}, \varphi_{3,2}, \varphi_{3,1}\}$ *if* $n = 3$
- $\{\varphi_{1,5}, \varphi_{1,4}, \varphi_{1,3}, \varphi_{1,2}, \varphi_{1,1}, \varphi_{3,5}, \varphi_{3,4}, \varphi_{3,3}\}$ *if* $n = 5$

- $\{\varphi_{1,n}, \varphi_{1,n-1}, \varphi_{1,n-2}, \varphi_{1,n-3}, \varphi_{1,n-4}, \varphi_{3,n}, \varphi_{3,n-1}, \varphi_{3,n-2}, \varphi_{1,3,n-5}, \varphi_{1,3,n-6}\}$ *if* $n \geq 7$

  with

$$\varphi_{1,s} = \begin{cases} \varphi_{1,s}(Y_1, Y_1, Y_1) = X_s \\ \varphi_{1,s}(Y_1, Y_1, Y_2) = \frac{1}{3}X_{s+1} \\ \varphi_{1,s}(Y_1, Y_2, Y_2) = \frac{1}{6}X_{s+2} \\ \varphi_{1,s}(Y_2, Y_2, Y_2) = \frac{1}{6}X_{s+3} \\ \varphi_{1,s}(Y_2, Y_2, Y_3) = \frac{1}{18}X_{s+4} \\ \varphi_{1,s}(Y_1, Y_3, Y_3) = -\frac{1}{18}X_{s+4} \\ \varphi_{1,s}(Y_2, Y_3, Y_3) = \frac{1}{36}X_{s+5} \\ \varphi_{1,s}(Y_3, Y_3, Y_3) = \frac{1}{36}X_{s+6} \end{cases} \quad \varphi_{3,s} = \begin{cases} \varphi_{3,s}(Y_1, Y_1, Y_3) = X_s \\ \varphi_{3,s}(Y_1, Y_2, Y_2) = -\frac{1}{2}X_s \\ \varphi_{3,s}(Y_1, Y_2, Y_3) = \frac{1}{2}X_{s+1} \\ \varphi_{3,s}(Y_2, Y_2, Y_2) = -\frac{3}{2}X_{s+1} \\ \varphi_{3,s}(Y_2, Y_2, Y_3) = -\frac{1}{2}X_{s+2} \\ \varphi_{3,s}(Y_1, Y_3, Y_3) = X_{s+2} \\ \varphi_{3,s}(Y_2, Y_3, Y_3) = -\frac{1}{4}X_{s+3} \\ \varphi_{3,s}(Y_3, Y_3, Y_3) = -\frac{1}{4}X_{s+4} \end{cases}$$

$$\varphi_{1,3,s} = \begin{cases} \varphi_{1,3,s}(Y_1, Y_1, Y_1) = X_s, & \varphi_{1,3,s}(Y_1, Y_1, Y_3) = \frac{1}{15}X_{s+2} \\ \varphi_{1,3,s}(Y_1, Y_1, Y_2) = \frac{1}{3}X_{s+1}, & \varphi_{1,3,s}(Y_1, Y_2, Y_2) = \frac{2}{15}X_{s+2} \\ \varphi_{1,3,s}(Y_2, Y_2, Y_2) = \frac{1}{15}X_{s+3}, & \varphi_{1,3,s}(Y_1, Y_2, Y_3) = \frac{1}{30}X_{s+3} \\ \varphi_{1,3,s}(Y_2, Y_2, Y_3) = \frac{1}{45}X_{s+4}, & \varphi_{1,3,s}(Y_1, Y_3, Y_3) = \frac{1}{90}X_{s+4} \\ \varphi_{1,3,s}(Y_2, Y_3, Y_3) = \frac{1}{90}X_{s+5}, & \varphi_{1,3,s}(Y_3, Y_3, Y_3) = \frac{1}{90}X_{s+6} \end{cases}$$

It can be consulted [17] for the corresponding basis in the cases $m = 3$ and n even and $m = 4$ and $n = 2k$ with $k \geq 5$.

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Algebraic Structures Related to Racah Doubles

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Abstract In Oste and Van der Jeugt, SIGMA, 12 (2016), [13], we classified all pairs of recurrence relations connecting two sets of Hahn, dual Hahn or Racah polynomials of the same type but with different parameters. We examine the algebraic relations underlying the Racah doubles and find that for a special case of Racah doubles with specific parameters this is given by the so-called Racah algebra.

1 Introduction

In [13], we classified all pairs of recurrence relations connecting two sets of Hahn, dual Hahn or Racah polynomials (classical finite and discrete hypergeometric orthogonal polynomials) of the same type but with different (shifted) parameters. This coupling of two sets of polynomials was dubbed ‘doubling’ (yielding Hahn doubles, dual Hahn doubles and Racah doubles respectively) and it was seen to fit in the context of obtaining a “new” system of orthogonal polynomials, following the technique due to Chihara [4, 5], from a set of orthogonal polynomials and its kernel partner related by a Christoffel-Geronimus transform pair [11, 15]. When applied to polynomials satisfying a discrete orthogonality relation, the coefficients of the three-term recurrence relation of the new system are stored in the tridiagonal Jacobi matrix, which by construction has zero diagonal. In the following, we will use the term ‘two-diagonal’ to refer to tridiagonal matrices with zero diagonal.

These Jacobi matrices could be interpreted as representation matrices of an algebra. For instance, the simplest case is that of the symmetric Krawtchouk polynomials. Here, the recurrence relation [10, (9.11.3)] for normalized Krawtchouk functions reads (for $n = 0, 1, \ldots, N$)

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V. Dobrev (ed.), *Lie Theory and Its Applications in Physics*,
Springer Proceedings in Mathematics & Statistics 191,
DOI 10.1007/978-981-10-2636-2_43

$$\sqrt{n(N-n+1)}\tilde{K}_{n-1}(x) + \sqrt{(n+1)(N-n)}\tilde{K}_{n+1}(x) = (N-2x)\tilde{K}_n(x), \quad (1)$$

where $\tilde{K}_n(x)$ stands for the normalized Krawtchouk functions $\tilde{K}_n(x) \sim K_n(x; \frac{1}{2}, N)$ which are scaled Krawtchouk polynomials $K_n(x; p, N)$ [10, 12] with $p = 1/2$. As the Krawtchouk polynomials are self-dual, the same relation holds when interchanging n and x. This bispectrality can then be encoded in an algebraic structure. Writing down (1) for $n = 0, 1, \ldots, N$, and putting this in matrix form, the coefficient matrix of the left hand side of (1) is just the two-diagonal $(N+1) \times (N+1)$ matrix

$$M = \begin{pmatrix} 0 & M_0 & & & \\ M_0 & 0 & M_1 & & \\ & M_1 & 0 & M_2 & \\ & & \ddots & \ddots & \ddots \end{pmatrix}, \quad (2)$$

with $M_n = \sqrt{(n+1)(N-n)}$. Here we recognise the action of J_+ and J_- appearing in the context of unitary representations of the Lie algebra $\mathfrak{su}(2)$. The simple two-diagonal structure of M makes it particularly interesting as a model for a finite quantum oscillator, namely the $\mathfrak{su}(2)$ oscillator model [1, 2].

Similarly, the bispectrality of the Racah polynomials is encoded in the algebraic structure known as the Racah algebra [6, 7]. The Racah algebra is a unital associative algebra over $\mathbb{C}$ with three generators K_1, K_2 and K_3 which obey the following relations in the generic presentation,

$$\begin{aligned} [K_1, K_2] &= K_3, \\ [K_2, K_3] &= a_2 K_2^2 + a_1\{K_1, K_2\} + c_1 K_1 + d K_2 + e_1, \\ [K_3, K_1] &= a_1 K_1^2 + a_2\{K_1, K_2\} + c_2 K_2 + d K_1 + e_2. \end{aligned} \quad (3)$$

For the realization on the space of Racah polynomials, the matrix representation of the element K_2 is the tridiagonal Jacobi matrix of the Racah recurrence relation and K_1 is the diagonal matrix containing the quadratic expression $n(n+\alpha+\beta+1)$ for $n = 0, \ldots, N$ (which corresponds to the right hand side of the Racah difference equation). In this case the coefficients in (3) are functions of the Racah parameters $\alpha, \beta, \gamma, \delta$, given explicitly in [6]. This algebra also appears in the context of the Racah problem of $\mathfrak{su}(2)$ to derive the symmetry group of the $6j$-symbols [7].

We are now interested in the algebraic structures underlying the two-diagonal Jacobi matrices obtained through the doubling process. From [13], we observe that the dual Hahn and Racah doubles reduce to symmetric Krawtchouk polynomials when setting all parameters to trivial values and leaving no parameter unspecified. For general parameters, however, it is less trivial. In [13], the algebraic structures behind the matrices of the three cases of dual Hahn doubles were examined explicitly and were found to be extensions of $\mathfrak{su}(2)$. Such extensions consist of the addition of a parity operator P, while the standard $\mathfrak{su}(2)$ relations are altered to

$$
\begin{aligned}
&P^2 = 1, \qquad PJ_0 = J_0P, \qquad PJ_\pm = -J_\pm P, \\
&[J_0, J_\pm] = \pm J_\pm, \\
&[J_+, J_-] = 2J_0 + 2(\gamma + \delta + 1)J_0P - (2j+1)(\gamma - \delta)P + (\gamma - \delta)I,
\end{aligned} \tag{4}
$$

for the case **dual Hahn I**. The appearing parameters γ, δ here are the ones occurring also in the dual Hahn doubles, while the factor $(2j+1)$ can be interpreted as the addition of a central element. When $\gamma = \delta = -1/2$, the equations coincide with the $\mathfrak{su}(2)$ relations. These parameter values correspond to a reduction to the symmetric Krawtchouk case where the recurrence relations reduce to (1). The algebraic relations for the other dual Hahn cases are similar to (4) and can be found in [13, Sect. 7]. While the general algebras have two parameters, special cases with only one parameter are of importance for the construction of finite oscillator models [8, 9].

The remaining question is of course whether the two-diagonal matrices arising from the Hahn and Racah doubles can also be interpreted as representation matrices of an algebra, and whether this is related to (an extension of) a known (Lie) algebra. In the current work, we shall consider a special case of a Racah double and show that for this special case the underlying algebraic structure is related to the Racah algebra. Hereto, in Sect. 2 we will briefly summarize the definition and properties of the Racah polynomials.

2 Racah Polynomials

Racah polynomials $R_n(\lambda(x); \alpha, \beta, \gamma, \delta)$ of degree n ($n = 0, 1, \ldots, N$) in the variable $\lambda(x) = x(x + \gamma + \delta + 1)$ are defined by [10, 12]

$$
R_n(\lambda(x); \alpha, \beta, \gamma, \delta) = {}_4F_3\left(\begin{matrix} -n, n+\alpha+\beta+1, -x, x+\gamma+\delta+1 \\ \alpha+1, \beta+\delta+1, \gamma+1 \end{matrix} ; 1 \right),
$$

where one of the denominator parameters should be $-N$. Herein, the function ${}_4F_3$ is the generalized hypergeometric series [3].

For the (discrete) orthogonality relation (depending on the choice of which parameter relates to $-N$) we refer to [10, (9.2.2)]. Under certain restrictions for the parameters, such that the weight function w and the squared norm h_n of the orthogonality relation are positive, we can define orthonormal Racah functions as follows

$$
\tilde{R}_n(\lambda(x); \alpha, \beta, \gamma, \delta) \equiv \sqrt{w(x; \alpha, \beta, \gamma, \delta)}\, R_n(\lambda(x); \alpha, \beta, \gamma, \delta) \Big/ \sqrt{h_n(\alpha, \beta, \gamma, \delta)}.
$$

We now turn to a result obtained in [13, Appendix] corresponding to the case Racah I:

Proposition 1 *Let $\alpha + 1 = -N$, and suppose that $\gamma, \delta > -1$ and $\beta > N + \gamma$ or $\beta < -N - \delta - 1$. Consider two $(2N+2) \times (2N+2)$ matrices U and M, defined as follows. U has elements ($n, x \in \{0, 1, \ldots, N\}$):*

$$U_{2n,N-x} = U_{2n,N+x+1} = \frac{(-1)^n}{\sqrt{2}} \tilde{R}_n(\lambda(x); \alpha, \beta, \gamma, \delta + 1),$$
$$U_{2n+1,N-x} = -U_{2n+1,N+x+1} = -\frac{(-1)^n}{\sqrt{2}} \tilde{R}_n(\lambda(x); \alpha, \beta + 1, \gamma + 1, \delta);$$

M is the two-diagonal $(2N+2) \times (2N+2)$-matrix of the form (2) with

$$M_{2k} = \sqrt{\frac{(N - \beta - k)(\gamma + 1 + k)(N + \delta + 1 - k)(k + \beta + 1)}{(N - \beta - 2k)(2k - N + 1 + \beta)}},$$
$$M_{2k+1} = \sqrt{\frac{(\gamma + N - \beta - k)(k + 1)(N - k)(k + \delta + \beta + 2)}{(N - \beta - 2k - 2)(2k - N + 1 + \beta)}}. \tag{5}$$

Then U is orthogonal, and the columns of U are the eigenvectors of M, i.e. $MU = UD$, where D is a diagonal matrix containing the eigenvalues of M:

$$D = \mathrm{diag}(-\epsilon_N, \ldots, -\epsilon_1, -\epsilon_0, \epsilon_0, \epsilon_1, \ldots, \epsilon_N), \quad \epsilon_k = \sqrt{(k + \gamma + 1)(k + \delta + 1)}. \tag{6}$$

In short, the pair of polynomials $R_n(\lambda(x); \alpha, \beta, \gamma, \delta + 1)$ and $R_n(\lambda(x); \alpha, \beta + 1, \gamma + 1, \delta)$ form a "Racah double", and the relation $MU = UD$ governs the corresponding recurrence relations with M taking the role of a Jacobi matrix [13].

In the context of Chihara's method, the Racah polynomials have already occurred in relation to the Bannai-Ito polynomials, which were shown to have an explicit expression in terms of the Wilson polynomials [14]. Examining the hypergeometric functions expressions given by equations (5.18) and (5.19) from [14], one sees that they are of type "Racah III" in the terminology of [13], whereas the ones appearing in Proposition 1 are of type "Racah I".

3 Algebraic Structure

We start by taking M to be the Jacobi matrix of the first Racah double, i.e. the $(2N+2) \times (2N+2)$ matrix of the form (2) with entries given by (5). We now propose that M can be interpreted as the representation matrix of an algebra, and set out to determine the relations governing this algebra. A natural candidate for this algebraic structure is the Racah algebra (3). As such, we define $K_2 = M$. Inspired by the realization on the space of the ordinary Racah polynomials, we take the generator K_1 to be a diagonal matrix on the Racah double, containing a general quadratic expression in the degree n, say $K_1 = \mathrm{diag}[(pn^2 + qn + r); n = 0, \ldots, 2N + 1]$. We set out to determine the coefficients p, q, r for which the commutation relations are of the form (3).

A direct computation shows that defining $K_3 = [K_1, K_2]$ we have the following commutation relation

$$[K_3, K_1] = -2p\{K_1, K_2\} + (p^2 - q^2 + 4pr)K_2,$$

which is a quadratic relation of the same form as the Racah algebra relations (3). This relation actually holds for general matrices K_2 of the form (2) and K_1 as above, relying on the symmetric two-diagonal structure of K_2.

Next, we examine whether the remaining commutator $[K_2, K_3]$ can be cast in the same form as (3). However, in general this does not seem to be the case for K_1 of the above form. By direct computation we find that is possible, but only for specific values of the Racah parameters γ and δ, namely when both are equal to either $-1/2$ or $-N - 3/2$. Only for one of these parameter values, the choice $q = 2p(\beta - N)$ yields the following relation

$$[K_2, K_3] = -2pK_2^2 - K_1 + (2N + 1)(\beta + 1)p + r\,.$$

There are two arbitrary constants left, namely p and r. If one chooses the value $p = -1/2$, we arrive at the following commutation relations for K_1, K_2 and K_3

$$\begin{aligned}
[K_1, K_2] &= K_3 \\
[K_2, K_3] &= K_2^2 - K_1 - \frac{1}{2}(2N + 1)(\beta + 1) + r \\
[K_3, K_1] &= \{K_1, K_2\} + \Big(\frac{1}{4} - (\beta - N)^2 - 2r\Big)K_2.
\end{aligned}$$

Note that by means of the constant r, corresponding just to an affine transformation of K_1, one can fix the coefficient of K_2 in the third relation.

For the $(2N + 1) \times (2N + 1)$ matrices of another Racah double, namely Racah III [13, Appendix] similar results hold, only yielding the Racah algebra for either $\gamma = \delta = -1/2$ or $\gamma = \beta - 1/2$ and $\delta = -\beta - 1/2$. For the special parameter values $\gamma = \delta = -1/2$ the Jacobi matrices of those two Racah doubles can actually be unified in a single expression, matching the appropriate Racah double for even and for odd dimensions. For this special case, the spectrum of these matrices also reduces to equidistant integers or half-integers, as seen from (6), making them interesting candidates for finite quantum oscillator models. We will pursue the in-depth study of this oscillator model and the unification of the two Racah doubles in a separate paper.

We believe that for general parameter values of the Racah doubles the algebraic structure is also related to the Racah algebra. The matrix representation of K_1, however, will be more complicated and should follow from investigating the bispectrality of these polynomials. Only for the specific parameter values obtained here, does K_1 reduce to the same form as for the ordinary Racah polynomials. We intend to analyse this in further work.

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A Note on Strongly Graded Lie Algebras

Antonio J. Calderón Martín and Diouf Mame Cheikh

Abstract We show that if a strongly graded Lie algebra with a symmetric support $(\mathfrak{L}, [\cdot, \cdot])$ is centerless, then $\mathfrak{L}$ is the direct sum of the family of its minimal graded-ideals, each one being a graded-simple strongly graded Lie algebra.

1 Introduction and Previous Definitions

We begin by noting that all of the Lie algebras are considered of arbitrary dimension and over an arbitrary base field $\mathbb{K}$.

A *graded Lie algebra* is a Lie algebra $\mathfrak{L}$ that can be expressed as the direct sum of linear subspaces indexed by the elements of an abelian group $(G, +)$, that is,

$$\mathfrak{L} = \bigoplus_{g \in G} \mathfrak{L}_g$$

with any $\mathfrak{L}_g$ a linear subspace of $\mathfrak{L}$ such that $[\mathfrak{L}_g, \mathfrak{L}_h] \subset \mathfrak{L}_{g+h}$ for any $h \in G$. We call the *support* of the grading to the set $\Sigma := \{g \in G \setminus \{0\} : \mathfrak{L}_g \neq 0\}$. The set Σ is called *symmetric* if $\Sigma = -\Sigma$. Finally, we recall that a graded Lie algebra $\mathfrak{L} = \bigoplus_{g \in G} \mathfrak{L}_g$ is called a *strongly graded Lie algebra* if the condition $[\mathfrak{L}_g, \mathfrak{L}_h] = \mathfrak{L}_{g+h}$ holds for any $g, h \in G$ such that $g + h \in \Sigma$, $\mathfrak{L}_0 = \sum_{g \in \Sigma} [\mathfrak{L}_g, \mathfrak{L}_{-g}]$, and for any $x \in \mathfrak{L}_0$ and $g \in \Sigma$ we have that either $\mathrm{ad}(x)(\mathfrak{L}_g) = 0$ or $\mathrm{ad}(x)(\mathfrak{L}_g) = \mathfrak{L}_g$, where $\mathrm{ad}(x)(y) := [x, y]$ denotes de adjoint operator, (see [3, 4]).

As examples of strongly graded Lie algebras we can consider the finite dimensional semisimple Lie algebras, the graded algebras associated to L^*-algebras and

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V. Dobrev (ed.), *Lie Theory and Its Applications in Physics*,
Springer Proceedings in Mathematics & Statistics 191,
DOI 10.1007/978-981-10-2636-2_44

to semisimple locally finite split Lie algebras with the group-gradings induced by their split decompositions; and the split Lie algebras considered in [1, Sect. 2] among other classes of Lie algebras (see [5–7]).

The regularity conditions will be understood in graded sense. That is, a linear space I of a strongly graded Lie algebra $\mathfrak{L} = \bigoplus_{g \in G} \mathfrak{L}_g$ is called a *graded-ideal* if $[I, \mathfrak{L}] \subset I$ and $I = \bigoplus_{g \in G} (I \cap \mathfrak{L}_g)$ with $I \cap \mathfrak{L}_0 \neq 0$. A strongly graded Lie algebra $\mathfrak{L}$ will be called *graded-simple* if $[\mathfrak{L}, \mathfrak{L}] \neq 0$ and its only graded-ideals are $\{0\}$ and $\mathfrak{L}$.

2 Main Results

In the following, $\mathfrak{L}$ denotes a strongly graded Lie algebra of maximal length and

$$\mathfrak{L} = \bigoplus_{g \in G} \mathfrak{L}_g = \mathfrak{L}_0 \oplus \left(\bigoplus_{g \in \Sigma} \mathfrak{L}_g \right)$$

the corresponding grading. We will consider the set $-\Sigma = \{-g : g \in \Sigma\} \subset G$.

Definition 1 Let $g, h \in \Sigma$. We say that g is *connected* to h if there exists a family $g_1, g_2, \ldots, g_n \in \Sigma$ satisfying the following conditions:

1. $g_1 = g$.
2. $\{g_1 + g_2, g_1 + g_2 + g_3, \ldots, g_1 + \cdots + g_{n-1}\} \subset \pm\Sigma$.
3. $g_1 + g_2 + \cdots + g_n = \epsilon\beta$ for some $\epsilon \in \{\pm 1\}$.

We also say that $\{g_1, \ldots, g_n\}$ is a *connection* from g to h.

It is straightforward to verify that the relation connection is an equivalence connection, see [1] or [2]. So we can consider the quotient set

$$\Sigma/ \sim = \{[g] : g \in \Sigma\}.$$

Now, for any $[g] \in \Sigma/ \sim$ we are going to introduce the linear subspace

$$\mathfrak{L}_{[g]} := W_{[g]} \oplus V_{[g]}$$

where

$$W_{[g]} := \sum_{h \in [g]} [\mathfrak{L}_h, \mathfrak{L}_{-h}] \subset \mathfrak{L}_0$$

and

$$V_{[g]} := \bigoplus_{h \in [g]} \mathfrak{L}_h.$$

We recall that *the center* of $\mathfrak{L}$ is the set $\mathcal{Z}(\mathfrak{L}) = \{x \in \mathfrak{L} : [x, \mathfrak{L}] = 0\}$.

Proposition 1 *For any* $[g] \in \Sigma/\sim$ *we have that any* $\mathfrak{L}_{[g]}$ *is a graded-ideal of* $\mathfrak{L}$. *If furthermore* $\mathcal{Z}(\mathfrak{L}) = 0$ *and the support is symmetric then* $\mathfrak{L}_{[g]}$ *is graded-simple.*

Proof First, let us show that

$$[\mathfrak{L}_{[g]}, \mathfrak{L}] = \left[W_{[g]} \oplus V_{[g]}, \mathfrak{L}_0 \oplus \left(\bigoplus_{h \in [g]} \mathfrak{L}_h \right) \oplus \left(\bigoplus_{k \notin [g]} \mathfrak{L}_k \right) \right] \subset \mathfrak{L}_{[g]}. \tag{1}$$

By the grading we have

$$\left[W_{[g]}, \bigoplus_{h \in [g]} \mathfrak{L}_h \right] + [V_{[g]}, \mathfrak{L}_0] \subset \left[\mathfrak{L}_0, \bigoplus_{h \in [g]} \mathfrak{L}_h \right] \subset V_{[g]}.$$

Jacobi identity also shows $[W_{[g]}, \mathfrak{L}_0] = \sum_{h \in [g]} [[\mathfrak{L}_h, \mathfrak{L}_{-h}], \mathfrak{L}_0] \subset W_{[g]}$.

Since in case $[\mathfrak{L}_u, \mathfrak{L}_v] \neq 0$ for some $u, v \in \Sigma$ with $u + v \neq 0$, the connections $\{u, v\}$ and $\{u, v, -u\}$ imply $[u] = [u + v] = [v]$, we get

$$\left[V_{[g]}, \bigoplus_{h \in [g]} \mathfrak{L}_h \right] \subset \mathfrak{L}_{[g]}$$

and

$$\left[V_{[g]}, \bigoplus_{k \notin [g]} \mathfrak{L}_k \right] = 0. \tag{2}$$

Taking now into account the fact $W_{[g]} := \sum_{h \in [g]} [\mathfrak{L}_h, \mathfrak{L}_{-h}]$, Jacobi identity together with Eq. (2) finally give us

$$\left[W_{[g]}, \bigoplus_{k \notin [g]} \mathfrak{L}_k \right] = 0 \tag{3}$$

and so Eq. (1) holds.

Since Eq. (3) shows $[W_{[h]}, V_{[g]}] = 0$ for any $[g] \neq [h]$ we get $W_{[g]} \neq 0$. From here, we can also assert that $\mathfrak{L}_{[g]}$ is a strongly graded ideal of $\mathfrak{L}$ admitting the grading

$$\mathfrak{L}_{[g]} = W_{[g]} \oplus \left(\bigoplus_{h \in [g]} \mathfrak{L}_h \right). \tag{4}$$

Suppose now $(\mathfrak{L}, [\cdot, \cdot])$ is centerless and Σ symmetric, and let us show $\mathfrak{L}_{[g]}$ is graded-simple. Consider a graded-ideal I of $\mathfrak{L}_{[g]}$. By Eq. (4) we can write

$$I = (I \cap W_{[g]}) \oplus \left(\bigoplus_{h \in [g]} (I \cap \mathfrak{L}_h) \right)$$

with $I \cap W_{[g]} \neq 0$. For any $0 \neq x \in I \cap W_{[g]}$, the fact $(\mathfrak{L}, [\cdot, \cdot])$ is centerless together with Eq. (3) give us that there exists $h \in [g]$ such that $[x, \mathfrak{L}_h] \neq 0$. From here we get $[I \cap W_{[g]}, \mathfrak{L}_h] = \mathfrak{L}_h$ and so $0 \neq \mathfrak{L}_h \subset I$.

Given now any $u \in [g] \setminus \{\pm h\}$, the fact that h and u are connected allows us to take a connection $\{g_1, g_2, \dots, g_n\}$ from h to u. Since $g_1, g_2, g_1 + g_2 \in \Sigma$ we have

$$[\mathfrak{L}_{g_1}, \mathfrak{L}_{g_2}] = \mathfrak{L}_{g_1+g_2} \subset I$$

as consequence of $\mathfrak{L}_{g_1} = \mathfrak{L}_h \subset I$. In a similar way

$$[\mathfrak{L}_{g_1+g_2}, \mathfrak{L}_{g_3}] = \mathfrak{L}_{g_1+g_2+g_3} \subset I$$

and we finally get by following this process that

$$\mathfrak{L}_{g_1+g_2+g_3+\cdots+g_n} = \mathfrak{L}_{\epsilon u} \subset I$$

for some $\epsilon \in \pm 1$. From here we have $W_{[g]} \subset I$ and taking also into account that Eq. (3) allows us to assert $[W_{[g]}, \mathfrak{L}_u] = \mathfrak{L}_u$ for any $u \in [g]$, we also get that $V_{[g]} \subset I$. We have showed $I = \mathfrak{L}_{[g]}$ and so $\mathfrak{L}_{[g]}$ is graded-simple.

Theorem 1 *Any centerless strongly graded Lie algebra $\mathfrak{L}$ with a symmetric support is the direct sum of the family of its minimal graded-ideals, each one being a graded-simple strongly graded Lie algebra.*

Proof Since we can write the disjoint union $\Sigma = \bigcup_{[g] \in \Sigma \setminus \sim} [g]$ we have $\mathfrak{L} = \sum_{[g] \in \Sigma \setminus \sim} \mathfrak{L}_{[g]}$. Let us now verify the direct character of the sum: given $x \in \mathfrak{L}_{[g]} \cap \sum_{\substack{[h] \in \Sigma / \sim \\ h \nsim g}} \mathfrak{L}_{[h]}$, since by Eqs. (2) and (3) we have $[\mathfrak{L}_{[g]}, \mathfrak{L}_{[h]}] = 0$ for $[g] \neq [h]$, we obtain

$$\left[x, \mathfrak{L}_{[g]}\right] + \left[x, \sum_{\substack{[h] \in \Sigma / \sim \\ h \nsim g}} \mathfrak{L}_{[h]} \right] = 0.$$

From here $[x, \mathfrak{L}] = 0$ and so $x = 0$, as desired. Consequently we can write

$$\mathfrak{L} = \bigoplus_{[g]\in\Sigma\backslash\sim} \mathfrak{L}_{[g]}.$$

Finally, Proposition 1 completes the proof.

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Part VII
Various Mathematical Results

Toeplitz Operators with Discontinuous Symbols on the Sphere

Tatyana Barron and David Itkin

Abstract We obtain asymptotics of norms for Toeplitz operators with specific discontinuous symbols on S^2.

1 Introduction

This note follows the presentation in the talk by the first author at the "Lie theory and its applications in physics" workshop in Varna, Bulgaria, in June 2015. This is an expanded and more detailed version of the talk.

(Berezin–)Toeplitz operators are linear operators $T_f^{(k)}$ that act on spaces of holomorphic sections of $\mathcal{L}^{\otimes k}$, where $\mathcal{L}$ is a holomorphic hermitian line bundle on a Kähler manifold X and k is a positive integer. The symbol f is a function on X. Most results in literature on Berezin–Toeplitz quantization are obtained under the assumption that f is C^∞. In a recent paper [2] asymptotics of Toeplitz operators with C^m symbols are obtained. For the case of C^∞ symbols explicit constructions of Toeplitz operators on $X = \mathbb{P}^1(\cong S^2)$, with the Fubini-Study form, and on the 2-dimensional torus, were worked out in [3]. See also [5].

After giving the necessary background and definitions in Sect. 2, we make explicit calculations for Toeplitz operators with specific discontinuous symbols on S^2 in Sect. 3. The outcome is summarized in Theorem 4. For these specific discontinuous symbols the semiclassical behaviour of the norm of a Toeplitz operator is akin to that for C^2 symbols.

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V. Dobrev (ed.), *Lie Theory and Its Applications in Physics*, Springer Proceedings in Mathematics & Statistics 191,
DOI 10.1007/978-981-10-2636-2_45

2 Preliminaries

Let μ be the normalized Lebesgue measure on $S^1 = \{z \in \mathbb{C}|\ |z| = 1\}$. Functions $e_n = e_n(z) = z^n, z \in S^1, n \in \mathbb{Z}$ form an orthonormal basis in $L^2 = L^2(S^1, d\mu)$. Call $f \in L^2$ *analytic* if $\int_{S^1} f\bar{e}_n d\mu = 0$ for all $n < 0$. Denote by H^2 the space of all analytic functions in L^2 and by $P : L^2 \to H^2$ the orthogonal projector.

Definition 1 (*see e.g.* [7]) Let φ be a bounded measurable function on S^1. A *Toeplitz operator* for φ is $T_\varphi = PM_\varphi : H^2 \to H^2$ (or, to write this differently, $T_\varphi g = P(\varphi g)$ for g in H^2).

This construction has been generalized from S^1 to disk, spheres, balls, $\mathbb{C}^n$, domains in $\mathbb{C}^n$ and extensively studied over the years, for various function spaces. Common ingredients are measure and orthogonal projection $P : L^2 \to H^2$, and a *Toeplitz operator* is $T_\varphi = PM_\varphi : H^2 \to H^2$.

Boutet de Monvel and Guillemin (see, in particular, [6]) greatly generalized this concept. Their work, and work by Berezin, applied in the context of quantization of Kähler manifolds, led to the following definition of a (Berezin–)Toeplitz operator.

Let (X, ω) be a compact connected Kähler manifold of complex dimension n. Assume that the Kähler form $\frac{\omega}{2\pi}$ is integral. Then there is a holomorphic hermitian line bundle $\mathcal{L} \to X$ such that the curvature of the hermitian connection is equal to $-i\omega$. For a positive integer k let $V^{(k)} = H^0(X, \mathcal{L}^{\otimes k})$ (the space of holomorphic sections of $\mathcal{L}^{\otimes k}$). It is a finite-dimensional complex vector space, dim $V^{(k)} \sim$ const k^n + l.o.t. as $k \to \infty$. The inner product on $V^{(k)}$ is obtained from the hermitian metric on $\mathcal{L}$.

Definition 2 ([4]) For a bounded measurable function f on X a *(Berezin)–Toeplitz operator* for f is

$$T_f^{(k)} = \Pi^{(k)} \circ M_f^{(k)} \in End(V^{(k)})$$

where

$$M_f^{(k)} : V^{(k)} \to L^2(X, \mathcal{L}^{\otimes k})$$

$$s \mapsto fs,$$

and $\Pi^{(k)} : L^2(X, \mathcal{L}^{\otimes k}) \to V^{(k)}$ is the orthogonal projector. The function f is said to be the *symbol* of $T_f^{(k)}$. The operator $T_f = \bigoplus_{k=0}^{\infty} T_f^{(k)}$ is also referred to as the *(Berezin)–Toeplitz operator* for *f*.

Remark 1 For $\alpha, \beta \in \mathbb{C}$ and bounded measurable functions f, g $T^{(k)}_{\alpha f+\beta g} = \alpha T_f^{(k)} + \beta T_g^{(k)}$.

Composition of Berezin–Toeplitz operators is a Berezin–Toeplitz operator.

Remark 2 Quantization, mentioned earlier, is a concept from physics, that broadly means passing from classical mechanics to quantum mechanics. In mathematics "quantization" can have various meanings, e.g. a map between vector spaces with prescribed properties, or a deformation of an algebra. Often by quantization people mean a way to associate a linear operator on a Hilbert space to a function on the classical phase space so that a version of Dirac's conditions holds. In the context of this paper, that corresponds to Berezin–Toeplitz quantization. X, regarded as a symplectic manifold, is the classical phase space, $V^{(k)}$ is the Hilbert space of quantum-mechanical wave functions, the positive integer k is formally interpreted as $1/\hbar$, where $\hbar$ is the Planck constant, the limit $k \to \infty$ is called *the semi-classical limit*, the quantum observables are (Berezin–)Toeplitz operators $T_f^{(k)}$, and Berezin–Toeplitz quantization is the map $C^\infty(X) \to End(V^{(k)})$, $f \mapsto T_f^{(k)}$. See, for example, [12] for a survey on Berezin–Toeplitz quantization.

Useful asymptotic properties of Toeplitz operators are summarized in the following well-known theorem.

Theorem 1 ([4]) *For $f, f_1, \ldots, f_p \in C^\infty(X)$, as $k \to \infty$*

$$||T_{f_1}^{(k)} \ldots T_{f_p}^{(k)} - T_{f_1 \ldots f_p}^{(k)}|| = O(\frac{1}{k})$$

$$tr(T_{f_1}^{(k)} \ldots T_{f_p}^{(k)}) = k^n(\int_X f_1 \ldots f_p \frac{\omega^n}{n!} + O(\frac{1}{k}))$$

$$||ik[T_{f_1}^{(k)}, T_{f_2}^{(k)}] - T_{\{f_1, f_2\}}^{(k)}|| = O(\frac{1}{k})$$

There is $C > 0$ s.t.

$$|f|_\infty - \frac{C}{k} \leq ||T_f^{(k)}|| \leq |f|_\infty \tag{1}$$

Here $\{., .\}$ denotes the Poisson bracket for ω and $|.|_\infty$ is the **sup**-norm.

Remark 3 Under the assumptions of the theorem we immediately get, as a corollary: as $k \to \infty$

$$||[T_{f_1}^{(k)}, T_{f_2}^{(k)}]|| = O(\frac{1}{k}).$$

Remark 4 See [8] for another proof of (1). See [9, 10] for Theorem 1 in a more general setting.

$f \in C^\infty(X)$ has been a quite standard assumption. In [2] we address the case when f is C^m (not necessarily C^∞) and prove, in particular, the following:

Theorem 2 ([2]) *Let $f_1, \ldots, f_p \in C^m(X)$. As $k \to \infty$*

$$||T_{f_1}^{(k)} \ldots T_{f_p}^{(k)} - T_{f_1 \ldots f_p}^{(k)}|| = \begin{cases} O(k^{-1}) & if\, m = 2 \\ o(k^{-1/2}) & if\, m = 1 \\ o(1) & if\, m = 0 \end{cases}$$

$$\frac{1}{k^n} tr(T^{(k)}_{f_1} \dots T^{(k)}_{f_p}) = \int_X f_1 \dots f_p \frac{\omega^n}{n!} + \begin{cases} O(k^{-1}) & if\, m = 2 \\ O(k^{-1/2}) & if\, m = 1 \\ o(1) & if\, m = 0 \end{cases}$$

$$||ik[T^{(k)}_{f_1}, T^{(k)}_{f_2}] - T^{(k)}_{\{f_1, f_2\}}|| = \begin{cases} o(1) & if\, m = 2 \\ o(k^{-1/2}) & if\, m = 3 \\ O(k^{-1}) & if\, m = 4 \end{cases}$$

Theorem 3 ([2]) *Suppose $f \in L^\infty(X)$ and there is $x_0 \in X$ such that f is continuous at x_0 and $|f(x_0)| = |f|_\infty$. Then*

$$\lim_{k \to \infty} ||T^{(k)}_f|| = |f|_\infty.$$

If $f \in C^1(X)$ then $\exists\, C > 0$ s.t.

$$|f|_\infty - \frac{C}{\sqrt{k}} \le ||T^{(k)}_f|| \le |f|_\infty.$$

If $f \in C^2(X)$ then $\exists\, C > 0$ s.t.

$$|f|_\infty - \frac{C}{k} \le ||T^{(k)}_f|| \le |f|_\infty.$$

3 Analysis on the Sphere

Let $X = S^2 \cong \mathbb{P}^1$, with the Fubini-Study form, and let $\mathcal{L}$ be the hyperplane bundle. It is a standard fact ([11] 4.1.1. or [3] 3.1) that $V^{(k)}$ can be identified with the space of polynomials in z of degree $\le k$, with the inner product

$$\langle \phi, \psi \rangle = \frac{i}{2\pi} \int_{\mathbb{C}} \frac{\phi(z)\overline{\psi(z)}}{(1 + |z|^2)^{k+2}} dz d\bar{z}.$$

The space of holomorphic sections of $\mathcal{L}^{\otimes k}$ is usually described as the space of homogeneous degree k polynomials in ζ_0, ζ_1, where ζ_0, ζ_1 are homogeneous coordinates on $\mathbb{P}^1$. On the affine chart $\{[\zeta_0 : \zeta_1] \in \mathbb{P}^1 | \zeta_1 \neq 0\}$, with $z = \zeta_0/\zeta_1$, we get the description above.

Remark 5 For representation theorists we make a note of the fact that X is a coadjoint orbit of $SU(2)$ and $V^{(k)}$ is an irreducible representation of $SU(2)$. An orthonormal basis in $V^{(k)}$ is $\varphi^{(k)}_j = \sqrt{\frac{(k+1)!}{j!(k-j)!}} z^j,\ j = 0, \dots, k$ (this is easily verified by a calculation in polar coordinates on $\mathbb{R}^2$). Realize X as $\{(\xi, \eta, \zeta) \in \mathbb{R}^3 \mid \xi^2 + \eta^2 + (\zeta - \frac{1}{2})^2 = \frac{1}{4}\}$. Let $p_0 = (0, 0, 1)$. A standard stereographic projection [1] is $\sigma : X - \{p_0\} \to$

$\mathbb{C} \cong \mathbb{R}^2$, $(\xi, \eta, \zeta) \mapsto z = x + iy$, where $x = \frac{\xi}{1-\zeta}$, $y = \frac{\eta}{1-\zeta}$ and we have:

$$\xi = \frac{x}{x^2 + y^2 + 1},\ \eta = \frac{y}{x^2 + y^2 + 1},\ \zeta = \frac{x^2 + y^2}{x^2 + y^2 + 1}.$$

Define $f, g, h : X \to \mathbb{R}$ by

$$f(\xi, \eta, \zeta) = \begin{cases} 1 \text{ if } \zeta < 1/2 \\ 0 \text{ if } \zeta \geq 1/2 \end{cases}$$

$$g(\xi, \eta, \zeta) = \begin{cases} 1 \text{ if } \zeta < 1/5 \\ 0 \text{ if } \zeta \geq 1/5 \end{cases}$$

$$h(\xi, \eta, \zeta) = \begin{cases} 0 \text{ if } \zeta < 1/2 \\ \zeta \text{ if } \zeta \geq 1/2 \end{cases}$$

for $(\xi, \eta, \zeta) \in X \subset \mathbb{R}^3$. Stereographic projection gives the following functions on the xy-plane:

$$\hat{f}(x, y) = \begin{cases} 1 \text{ if } x^2 + y^2 < 1 \\ 0 \text{ if } x^2 + y^2 \geq 1 \end{cases}$$

$$\hat{g}(x, y) = \begin{cases} 1 \text{ if } x^2 + y^2 < 1/4 \\ 0 \text{ if } x^2 + y^2 \geq 1/4 \end{cases}$$

$$\hat{h}(x, y) = \begin{cases} 0 & \text{if } x^2 + y^2 < 1 \\ \frac{x^2+y^2}{x^2+y^2+1} & \text{if } x^2 + y^2 \geq 1 \end{cases}$$

defined by

$$f\Big|_{X-\{p_0\}} = \hat{f} \circ \sigma,\ g\Big|_{X-\{p_0\}} = \hat{g} \circ \sigma,\ h\Big|_{X-\{p_0\}} = \hat{h} \circ \sigma.$$

Let ρ be a bounded measurable function on X (measurable with respect to the Lebesgue measure). Write $T_\rho^{(k)}$ as a matrix, in the basis $(\varphi_j^{(k)})$. The lj-th entry of this matrix is

$$(T_\rho^{(k)})_{lj} = \langle T_\rho^{(k)} \varphi_j^{(k)}, \varphi_l^{(k)} \rangle = \langle \Pi^{(k)} M_\rho^{(k)} \varphi_j^{(k)}, \varphi_l^{(k)} \rangle = \langle \rho \varphi_j^{(k)}, \varphi_l^{(k)} \rangle =$$

$$= \frac{i}{2\pi} \frac{(k+1)!}{\sqrt{j!l!(k-j)!(k-l)!}} \int_{\mathbb{C}} \hat{\rho}(z, \bar{z}) z^j \bar{z}^l \frac{dz d\bar{z}}{(1 + |z|^2)^{k+2}}.$$

Theorem 4 *For sufficiently large k*

(i) $||T_f^{(k)}|| = 1 - \frac{1}{2^{k+1}}$,

(ii) $||T_g^{(k)}|| = 1 - (\frac{4}{5})^{k+1}$,

(iii) $1 - \frac{2}{k} \leq ||T_h^{(k)}|| \leq 1$.

Proof Switching to the polar coordinates (r, Θ) on $\mathbb{R}^2$, we see that if $\rho(\xi, \eta, \zeta)$ depends only on ζ, then the corresponding function $\hat{\rho}$ on $\mathbb{R}^2$, determined by $\rho\Big|_{X-\{p_0\}} = \hat{\rho} \circ \sigma$, depends only on r, and (assuming that ρ is bounded and measurable) $(T_\rho^{(k)})_{lj} = 0$ for $l \neq j$.

Since $\varphi_j^{(k)}$, $0 \leq j \leq k$, form an orthonormal basis in $V^{(k)}$, the inner product of two vectors $u = \sum_{m=0}^{k} u_m \varphi_m^{(k)}$ and $v = \sum_{m=0}^{k} v_m \varphi_m^{(k)}$ is $\langle u, v\rangle = \sum_{m=0}^{k} u_m \bar{v}_m$. If a transformation $A \in End(V^{(k)})$ is represented by a diagonal matrix $\begin{pmatrix} \lambda_0 & & & \\ & \lambda_1 & & \\ & & \dots & \\ & & & \lambda_k \end{pmatrix}$, then the operator norm $||A|| = \max_{0 \leq j \leq k} |\lambda_j|$ (proof: assume $\lambda_{i_0} = \max |\lambda_j| \neq 0$, then

$$\max_{||v||=1} ||Av|| = \max_{||v||=1} \sqrt{\sum_{j=0}^{k} |\lambda_j v_j|^2} = |\lambda_{i_0}| \max_{||v||=1} \sqrt{\sum_{j=0}^{k} |\frac{\lambda_j}{\lambda_{i_0}} v_j|^2} \leq$$

$$\leq |\lambda_{i_0}| \max_{||v||=1} \sqrt{\sum_{j=0}^{k} |v_j|^2} = |\lambda_{i_0}|$$

therefore $||A|| \leq |\lambda_{i_0}|$, and it is equality because $||A\varphi_{i_0}^{(k)}|| = |\lambda_{i_0}|$).

In the calculations the following identity will be useful: for $0 < a \leq 1, 0 \leq j \leq k$

$$\int_0^a \frac{r^{2j+1}}{(1+r^2)^{k+2}} dr = \frac{1}{2}\Big[\frac{j!(k-j)!}{(k+1)!} - \frac{1}{k+1} \sum_{m=0}^{j} \frac{(k-m)!j!(a^2)^{j-m}}{k!(j-m)!(1+a^2)^{k+1-m}}\Big]$$

(it is proved by substitution $u = r^2$ and repeated integration by parts).
Proof of *(i)*:

$$||T_f^{(k)}|| = \max_{0 \leq j \leq k} |(T_f^{(k)})_{jj}| = \max_{0 \leq j \leq k} \Big| \frac{i}{2\pi} \frac{(k+1)!}{j!(k-j)!} \int_{\mathbb{C}} \hat{f}(z, \bar{z}) |z|^{2j} \frac{dz d\bar{z}}{(1+|z|^2)^{k+2}} \Big| =$$

$$= \max_{0 \leq j \leq k} \frac{(k+1)!}{j!(k-j)!} 2 \int_0^1 \frac{r^{2j+1}}{(1+r^2)^{k+2}} dr = \max_{0 \leq j \leq k} \Big[1 - \frac{k!}{j!(k-j)!)} \Big(\frac{1}{2^{k+1}} + \frac{j}{k2^k} +$$

$$+ \frac{j(j-1)}{k(k-1)2^{k-1}} + \dots + \frac{j!(k-j+1)!}{k!2^{k-j+2}} + \frac{j!(k-j)!}{k!2^{k-j+1}} \Big) \Big] = 1 - \frac{1}{2^{k+1}}.$$

The last equality holds because maximum, for each k, is achieved for $j = 0$, since $\frac{k!}{j!(k-j)!} \geq 1$.

Proof of *(ii)*:

$$||T_g^{(k)}|| = \max_{0 \leq j \leq k} |(T_g^{(k)})_{jj}| = \max_{0 \leq j \leq k} \left| \frac{i}{2\pi} \frac{(k+1)!}{j!(k-j)!} \int_{\mathbb{C}} \hat{g}(z, \bar{z}) |z|^{2j} \frac{dz d\bar{z}}{(1+|z|^2)^{k+2}} \right| =$$

$$\max_{0 \leq j \leq k} \frac{(k+1)!}{j!(k-j)!} 2 \int_0^{1/2} \frac{r^{2j+1}}{(1+r^2)^{k+2}} dr = \max_{0 \leq j \leq k} \left[1 - \frac{k!}{j!(k-j)!} \left(\frac{(\frac{1}{4})^j}{(\frac{5}{4})^{k+1}} + \frac{j(\frac{1}{4})^{j-1}}{k(\frac{5}{4})^k} + \right.\right.$$

$$\left.\left. \frac{j(j-1)(\frac{1}{4})^{j-2}}{k(k-1)(\frac{5}{4})^{k-1}} + \cdots + \frac{j!(k-j+1)!\frac{1}{4}}{k!(\frac{5}{4})^{k-j+2}} + \frac{j!(k-j)!}{k!(\frac{5}{4})^{k-j+1}} \right) \right] = 1 - \left(\frac{4}{5} \right)^{k+1}.$$

The last equality holds because maximum, for each k, is achieved for $j = 0$, since for $j \geq 1$

$$1 - \frac{k!}{j!(k-j)!} \left(\frac{(\frac{1}{4})^j}{(\frac{5}{4})^{k+1}} + \frac{j(\frac{1}{4})^{j-1}}{k(\frac{5}{4})^k} + \cdots + \frac{j!(k-j+1)!\frac{1}{4}}{k!(\frac{5}{4})^{k-j+2}} + \frac{j!(k-j)!}{k!(\frac{5}{4})^{k-j+1}} \right) \leq$$

$$1 - \frac{k!}{j!(k-j)!} \frac{j!(k-j)!}{k!(\frac{5}{4})^{k-j+1}} = 1 - \frac{1}{(\frac{5}{4})^{k-j+1}} \leq 1 - \frac{1}{(\frac{5}{4})^{k+1}}.$$

Proof of *(iii)*:

$$||T_h^{(k)}|| = \max_{0 \leq j \leq k} |(T_h^{(k)})_{jj}| = \max_{0 \leq j \leq k} \left| \frac{i}{2\pi} \frac{(k+1)!}{j!(k-j)!} \int_{\mathbb{C}} \hat{h}(z, \bar{z}) |z|^{2j} \frac{dz d\bar{z}}{(1+|z|^2)^{k+2}} \right| =$$

$$\max_{0 \leq j \leq k} \frac{(k+1)!}{j!(k-j)!} 2 \int_1^{\infty} \frac{r^{2j+3}}{(1+r^2)^{k+3}} dr.$$

Making a substitution $u = r^2$ and by repeated integration by parts we get:

$$\int_1^{\infty} \frac{r^{2j+3}}{(1+r^2)^{k+3}} dr = \frac{1}{2} \left(\frac{1}{(k+2)2^{k+2}} + \frac{j+1}{(k+2)(k+1)} \sum_{m=0}^{j} \frac{(k-m)!j!}{k!(j-m)!2^{k+1-m}} \right),$$

therefore

$$||T_h^{(k)}|| = \max_{0 \leq j \leq k} \frac{(k+1)!}{j!(k-j)!} \left(\frac{1}{(k+2)2^{k+2}} + \frac{j+1}{(k+2)(k+1)2^{k+1}} + \right.$$

$$+\frac{(j+1)j}{(k+2)(k+1)k2^k}+\cdots+\frac{(j+1)!}{(k+2)(k+1)k\ldots(k-j+1)2^{k-j+1}}\Bigg)\geq$$

$$\geq\frac{k+1}{k+2}\Big(1-\frac{1}{2^{k+2}}\Big)\geq 1-\frac{2}{k}.$$

The second inequality holds for sufficiently large k, and the first inequality is obtained by setting $j=k$ and observing that for $j=k$

$$\frac{(k+1)!}{j!(k-j)!}\Bigg(\frac{1}{(k+2)2^{k+2}}+\frac{j+1}{(k+2)(k+1)2^{k+1}}+\frac{(j+1)j}{(k+2)(k+1)k2^k}+\ldots$$

$$+\frac{(j+1)!}{(k+2)(k+1)k\ldots(k-j+1)2^{k-j+1}}\Bigg)=\frac{k+1}{k+2}\Bigg(\frac{1}{2^{k+2}}+\frac{1}{2^{k+1}}+\frac{1}{2^k}+\cdots+\frac{1}{2}\Bigg)=$$

$$\frac{k+1}{2(k+2)}\frac{1-(\frac{1}{2})^{k+2}}{1-\frac{1}{2}}=\frac{k+1}{k+2}\Big(1-\frac{1}{2^{k+2}}\Big).$$

$||T_h^{(k)}||\leq 1$, since

$$||T_h^{(k)}||\leq\max_{0\leq j\leq k}\frac{(k+1)!}{j!(k-j)!}2\int_0^\infty\frac{r^{2j+3}}{(1+r^2)^{k+3}}dr,$$

and by substitution $u=r^2$ and repeated integration by parts we find that

$$\int_0^\infty\frac{r^{2j+3}}{(1+r^2)^{k+3}}dr=\frac{1}{2}\frac{(j+1)!(k-j)!}{(k+2)!},$$

hence

$$||T_h^{(k)}||\leq\max_{0\leq j\leq k}\frac{j+1}{k+2}=\frac{k+1}{k+2}<1.$$

Acknowledgements The authors are thankful to the referee for the suggestions. Research of the first author is supported in part by the Natural Sciences and Engineering Research Council of Canada.

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Multiplication of Distributions in Mathematical Physics

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Abstract We expose a mathematical method that permits to treat calculations in form of multiplications of distributions that arise in various areas of mathematical physics, starting with an analysis of the famous Schwartz impossibility result (1954), then a construction of products of distributions, with examples and references of use in various domains of physics: classical and quantum mechanics, stochastic analysis and general relativity.

1 Introduction

In 1954 L. Schwartz published a celebrated note "Impossibility of the multiplication of distributions" [27], which had a strong impact on the subsequent development of physics (axiomatic field theory). Later in 1983 L. Schwartz presented (to the academy) a note "A general multiplication of distributions" by one of the authors [8]. We analyze this apparent contradiction in a very simple way and we observe that the impossibility proof is no more than the loss of a relatively minor property. To multiply distributions it suffices to construct a differential calculus in which the idealization that transforms the "irregular functions that represent physical quantities" into mathematical generalized functions is less crude than in distribution theory,

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V. Dobrev (ed.), *Lie Theory and Its Applications in Physics*,
Springer Proceedings in Mathematics & Statistics 191,
DOI 10.1007/978-981-10-2636-2_46

in other words mathematics closer to physics than distribution theory. This will be explained throughout the paper. We sketch how this permits to give a mathematical sense to calculations in physics and to state equations of physics in a more precise way which can resolve ambiguities usually connected with the appearance of products of distributions which are not defined within distribution theory.

2 An Analysis of the Schwartz Impossibility Result

To prove his claim L. Schwartz stated a list of properties to be satisfied by any hypothetical differential algebra $\mathcal{A}(\mathbb{R})$ containing at least some distributions (here we assume $\mathcal{D}'(\mathbb{R}) \subset \mathcal{A}(\mathbb{R})$ for simplicity and we abbreviate these spaces by $\mathcal{D}'$ and $\mathcal{A}$ respectively), and he put in evidence a contradiction in this set of properties [13, 19, 24, 27], starting calculations with the continuous functions $x(ln|x|-1)$ and $x^2(ln|x|-1)$ because he stated the properties with continuous functions. As a consequence his proof does not put (3) in evidence, as it stems here from the extreme simplicity of (1) and (2). For clarity we start here with the Heaviside function. To understand the whole situation we compare the two formulas (1) and (2) below where H denotes the Heaviside function ($H(x) = 0$ if $x < 0$, $H(x) = 1$ if $x > 0$, $H(0)$ undefined). These formulas are

$$\int_{\mathbb{R}} (H^2(x) - H(x))\phi(x)dx = 0 \, \forall \phi \in \mathcal{C}_c^\infty(\mathbb{R}), \tag{1}$$

and

$$\int_{\mathbb{R}} (H^2(x) - H(x))H'(x)(x)dx = \left[\frac{H^3}{3} - \frac{H^2}{2}\right]_{-\infty}^{+\infty} = \frac{1}{3} - \frac{1}{2} \neq 0. \tag{2}$$

These two formulas are clear if one assumes that the Heaviside function is an idealization of a smooth function with a jump from the value 0 to the value 1 on a very small region around $x = 0$. Formula (1) shows that $H^2 = H$ in $\mathcal{D}' \subset \mathcal{A}$ and formula (2) shows that $H^2 \neq H$ in $\mathcal{A}$, hence a contradiction which proves the impossibility of the multiplication of distributions. But there is a subtle mistake hidden in this reasoning! In (2) H^2 is the square of H in $\mathcal{A}$ since (2) does not make sense in $\mathcal{D}'$. To compare (2) with (1) the H^2 in (1) should be the same as in (2). Therefore, for comparison, the quantities $(H^2 - H)$ in (1) and (2) are both the same and are an element of $\mathcal{A}$; nothing tells it is an element of $\mathcal{D}'$. Therefore (1, 2) prove that in $\mathcal{A}$

$$\int F(x)\phi(x)dx = 0 \, \forall \phi \in \mathcal{C}_c^\infty(\mathbb{R}) \nRightarrow F = 0. \tag{3}$$

Indeed choose $F = H^2 - H$ above. The lack of validity of the familiar implication that fails from (3) does not prove the impossibility of the multiplication of distributions and does not prohibit the existence of a suitable algebra $\mathcal{A}$.

In $\mathcal{D}'$ one has $H^2 = H$ from (1); in $\mathcal{A}$ one has $H^2 \neq H$ from (2). Again this looks very much like an absurdity!. The explanation is that the square of H is not the same in $\mathcal{D}'$ and $\mathcal{A}$. Is this an incoherence, i.e. are these two objects, both denoted H^2, really different? Contrarily to all appearance the answer is no! Look at (H^2 in $\mathcal{A}$). We want to observe that it is (H^2 in $\mathcal{D}'$), i.e. H; to this end we observe of course this object (H^2 in $\mathcal{A}$) in the way the objects of $\mathcal{D}'$ are defined i.e. we consider

$$\int_{\mathbb{R}} (H^2\ in\ \mathcal{A})(x)\phi(x)dx$$

and from (1) we observe nothing other than (H in $\mathcal{D}'$). In conclusion (H^2 in $\mathcal{A}$) is different from (H in $\mathcal{A}$) but when (H^2 in $\mathcal{A}$) is observed according to the definition of distributions to compare with the classical objects in $\mathcal{D}'$ it appears to be H as this should be for coherence. The above systematically holds for all operations in the algebra $\mathcal{G}(\Omega)$ considered in the next section. Therefore in this algebra there is a perfect coherence between all new and all classical calculations.

3 A Differential Algebra Containing the Distributions

One can construct a differential algebra $\mathcal{G}(\Omega)$ containing a copy isomorphic to the vector space $\mathcal{D}'(\Omega)$, $\Omega \subset \mathbb{R}^n$ open, in the situation

$$\mathcal{C}^\infty(\Omega) \subset \mathcal{C}^0(\Omega) \subset \mathcal{D}'(\Omega) \subset \mathcal{G}(\Omega). \tag{4}$$

The partial derivatives in $\mathcal{G}(\Omega)$ induce on $\mathcal{D}'(\Omega)$ the partial derivatives in the sense of distributions; the multiplication in $\mathcal{G}(\Omega)$ induces the classical multiplication of $\mathcal{C}^\infty$ functions: $\mathcal{C}^\infty(\Omega)$ is a faithful subalgebra of $\mathcal{G}(\Omega)$. The Schwartz impossibility result implies that the algebra $\mathcal{C}^0(\Omega)$ is not a subalgebra of $\mathcal{G}(\Omega)$, but if f, g are two continuous functions on Ω and if $f \bullet g \in \mathcal{G}(\Omega)$ denotes their (new) product in $\mathcal{G}(\Omega)$, then we have the coherence

$$\int_\Omega (f \bullet g)(x)\phi(x)dx = \int_\Omega f(x)g(x)\phi(x)dx\ \forall\phi \in \mathcal{C}_c^\infty(\Omega) \tag{5}$$

for a natural integration in $\mathcal{G}(\Omega)$. The basic idea is that the elements of $\mathcal{G}(\Omega)$ are mathematical idealizations (that can represent physical quantities) that remain closer to physics than distributions: they are equivalence classes of families (f_ϵ) of $\mathcal{C}^\infty$ functions for a rather strict equivalence relation such that the property $(lim_{\epsilon\to 0} \int_\Omega f_\epsilon(x)\phi(x)dx = 0\ \forall\phi \in \mathcal{C}_c^\infty(\Omega))$ does not imply that the family (f_ϵ) is null in the quotient defining $\mathcal{G}(\Omega)$, as this is the case in distribution theory.

L. Nachbin and L. Schwartz supported fast publication in book form to speed divulgation [11, 12], that were soon complemented by [5, 13, 23, 24]. This theory is presented in form of a differential calculus dealing with infinitesimal quantities and infinitely large quantities in [3, 4], as well as in various expository texts [10, 14], … and has been extended to manifolds in view of its use in general relativity [18–22, 28–31], …

The problem that served as a first application in 1986 was the one of calculating jump conditions for a system used in industry (design of armor) to model very strong collisions [9, 13]. The system showed multiplications of distributions of the form $H \times \delta$ where δ is the Dirac distribution. We observed the existence of different possible jump conditions, all of them stable [17].

We recall that in $\mathcal{G}(\Omega)$ one has two concepts that can play the role of the equality of functions: of course the equality in $\mathcal{G}(\Omega)$ which is coherent with all operations (on particular the multiplication and the derivation) and the concept in left hand side of the non-implication (3) that we state as "association" since it is not really a weak equality (since different elements of $\mathcal{G}(\Omega)$ can be associated) and denote by the symbol $\approx$:

$$F \approx G \Leftrightarrow \int_\Omega (F - G)(x)\phi(x)dx = 0 \;\forall \phi \in \mathcal{C}_c^\infty(\Omega). \tag{6}$$

The association is coherent with the derivation but not with the multiplication. We recall from (1, 2) that $H^2 \neq H$ and $H^2 \approx H$.

A solution was obtained as follows: state with the equality in $\mathcal{G}(\Omega)$ the laws of physics which are considered true at a very small scale (may be 10^{-7} meters) and state with the association the laws or properties valid only at a far larger scale (may be 10^{-4} meters). This is explained in detail in [13] p. 69, with calculations of shock waves for nonconservative systems and references. The results that followed from this statement were in perfect agreement with observations and experiments. In various interesting cases one obtains the remarkable result that the jumps of different physical variables are represented by the same Heaviside function in $\mathcal{G}(\mathbb{R})$, [13] p. 72. Same explanations for another problem are given in [2].

4 Calculations of the Hamiltonian Formalism of Interacting Fields

The canonical Hamiltonian formalism (exposed in detail in [15, 16]) consists in a formal solution of the interacting field equations

$$\left(-\partial_t^2 + \sum_{i=1}^{3} \partial_{x_i}^2 - m^2\right)\Phi(x,t) = g\Phi(x,t)^N, \;\; \Phi(x,\tau) = \Phi_0(x,\tau) \tag{7}$$

where $m, g \in \mathbb{R}$ and $\Phi_0(x,t)$ is the free field operator (explicitly known: it is a distribution valued in a space of unbounded operators on a Hilbert space). The Hamiltonian formalism constructs a solution of (7) according to a formula

$$\Phi(x,t) = exp(-i(t-\tau)H_0(\tau))\Phi_0(x,\tau)exp(i(t-\tau)H_0(\tau)) \tag{8}$$

where $H_0(\tau)$ is obtained by plugging formally the free field into the formula of the total Hamiltonian corresponding to (7). Further calculations give the related formula

$$\Phi(x,t) = (S_\tau(t))^{-1}\Phi_0(x,t)(S_\tau(t)) \tag{9}$$

which gives the interacting field as a function of the free field at the same time. The formal operator $S = S_{-\infty}(+\infty)$ is called the scattering operator. Note that it depends on the real parameter g called coupling constant. If Φ_1, Φ_2 are two normalized orthogonal states then the formula $| < \Phi_1, S\Phi_2 > |$ represents the probability that the state Φ_2 would become Φ_1 after interaction.

What can be done with the context of Sect. 2 as mathematical tool? First one remarks the basic point that in these formal calculations, see for instance [15, 16], two basic mathematical difficulties are intimately mixed: multiplications of distributions, treated as $\mathcal{C}^\infty$ functions, and unbounded operators, treated as bounded operators. Indeed the free field Φ_0 is a distribution in x, not a function. The context of Sect. 2 is adapted to multiplication of distributions but brings nothing concerning unbounded operators, therefore it does not elucidate completely nicely these calculations. Anyway all calculations finally make sense mathematically [16] and one obtains a scattering operator $S = S(g)$ and transition probabilities $| < \Phi_1, S\Phi_2 > |$. What are these mathematical objects (which in the context of Sect. 3 make sense mathematically)?

The exponentials in (8) make sense from a proof that $H_0(\tau)$ admits a self-adjoint extension and one obtains a scattering operator $S = S(g)$ [16]. What is $| < \Phi_1, S(g)\Phi_2 > |$?: it depends on a parameter ϵ that tends to 0 and for each value of ϵ it is in between 0 and 1. We believe as quasi certain it has no limit (in the usual sense) when $\epsilon \to 0$ and therefore it oscillates endlessly inside the real interval [0, 1] when $\epsilon \to 0$. As an obvious example of such an oscillating object consider $|cos(\frac{g}{\epsilon})|$. Because of the periodicity of the function cosine, to this objects one can associate a well defined real number, here $\frac{2}{\pi}$, to be checked at once from numerical calculations by computing an average for a large number of very small values of ϵ chosen at random. Such average values exist for all quasi periodic functions, see [15]. The presence of complex exponentials and the self-adjointness property of $H_0(\tau)$ suggest that $| < \Phi_1, S(g)\Phi_2 > |$ is a quasi periodic function in the variable $\frac{1}{\epsilon}$ and therefore this oscillating function of ϵ would have a mean value as $\epsilon \to 0$ (the variable ϵ is of course not intrinsic but it plays only an auxiliary role and does not influence the final result, which appears very robust). In short the infinite quantities in the formal perturbation series are replaced by oscillations to be treated by computer calculations of an average value. To test this method one should compute the numerical value so obtained in a case for which one has an experimental result. The computer calculations look difficult and this has not been done after the premature death of A. Gsponer in 2009.

5 Stochastic Analysis

Stochastic differential equations (SDEs) serve to model many important phenomena in mathematical physics. An important class of SDEs in $\mathbb{R}^d$ is of the following form

$$\partial_t U(t,x) = \mathcal{L}U(t,x) + \eta(t,x), \quad U(0,x) = F(x) \tag{10}$$

where $\mathcal{L}$ is a differential operator and $\eta(t,x)$ is a space-time noise. The solutions of (10) are necessarily in a space of generalized functions because of the non-differentiability of the process driven by the equation. Therefore the meaning of the nonlinear part of $\mathcal{L}$ is not obvious. One way to sort out this problem is to consider the solutions of (10) as generalized stochastic processes, that is, processes whose paths are generalized functions. More precisely, by analogy with the association (6), we say that a family of smooth martingales $(U_\epsilon)_{\epsilon>0}$ is a weak solution of the equation (10) in the sense (6) if both 1 and 2 below hold:

$$\forall \phi \in \mathcal{C}_c^\infty(]0,T[\times\mathbb{R}^d,\mathbb{R}^d)\ \ lim_{\epsilon\to 0} < \mathcal{L}U_\epsilon, \phi >= \int_{[0,T]\times\mathbb{R}^d} \phi(t,x)d\eta(t,x) \tag{11}$$

and

$$U_\epsilon(0,x) = F(x)\ \forall\epsilon > 0. \tag{12}$$

One notices that

1. S. Albeverio, M. Oberguggenberger and F. Russo, among others, proposed already in the nineties to solve nonlinear SDEs in the framework of $\mathcal{G}(\Omega)$, [1, 25, 26].
2. One observes that if η_ϵ is a regularisation of the noise η and if U_ϵ is a solution of (10) driven by the noise η_ϵ, then under very general conditions $(U_\epsilon)_\epsilon > 0$ is a weak solution of the Eq. (10) in the sense (6).
3. Choosing the Burgers operator $\mathcal{L}U = \partial_t U - \Delta_x U - \nabla_x \|U\|^2$ and $\eta = \nabla_x\partial_t W(t,x)$, where $W(t,x)$ is a space-time white noise, we obtain the stochastic Burgers equation. The Cole-Hopf family is a weak solution of the Burgers equation in the sense (10), see [7].
4. In the case $d = 1$ the Hopf-Cole family is associated to a distribution, see [6].

6 Conclusion

After an analysis of the Schwartz impossibility result we have presented a context of multiplication of distributions having all natural requested properties. Then we have presented selected applications in continuum mechanics, quantum mechanics and in stochastic PDEs. For general relativity we refer to [19–22, 28–31].

Acknowledgements P. Catuogno has been partly supported by the grant CNPq 460713/2014-0. J.F. Colombeau has been supported by the grant FAPESP 2012 18940-7. Ch Olivera has been supported by the grants FAPESP 2015/04723-2 and CNPq 460713/2014-0.

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About Arbitrage and Holonomy

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Abstract I make a brief survey of the realization that arbitrage in finance is holonomy of a gauge connection (I am neither an economist nor an econophysicist. Nevertheless I would like to advertise in this mathematical physics workshop a very simple but in my view beautiful and important observation. I will assume the potential reader is acquainted with elementary notions from gauge theory.).

1 Introduction

Physics deals with simpler systems and their mathematical modeling is easier, more developed, and spectacularly successful. No wonder that it is the most advanced of all sciences. The social sciences deal with systems that are much more complex to an extent that some believe mathematical modeling is unapplicable there. The successes of physics have induced a constant trend of transfer of ideas, methods, and models from physics to the social sciences. Neoclassical economics is the foundation of much of present day economic theory. In close analogy with classical physics the mathematical method of neoclassical economics amounts to constraint optimization – the maximization of utility. Utility sometimes is explained as "happiness" - whatever that means. The high point of neoclassical economics is the Arrow–Debreu theorem of the existence of general equilibrium when no one can increase their utility without decreasing the utility of others. The mathematical engine making the Arrow–Debreu theorem possible is the Brouwer fixed point theorem. Though a beautiful mathematical result the Arrow–Debreu theorem cannot be the end of the story. Indeed Brower's theorem is an existence result so there could be many equilibria. What is worse the general equilibrium theory does not answer key questions about stability of fixed points and how fast the economy approaches a fixed point. The origin for these shortcomings is that general equilibrium theory has no dynamics.

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V. Dobrev (ed.), *Lie Theory and Its Applications in Physics*,
Springer Proceedings in Mathematics & Statistics 191,
DOI 10.1007/978-981-10-2636-2_47

There is a much simpler subdomain of Economics and that is Financial Economics. For one thing the mysterious "happiness" or utility in the case of economics is simply money. Another aspect of finance that renders it similar to physics is the availability of huge amounts of data. The volumes traded in financial derivatives are enormous and since everything goes automatically that data is recorded and is generally available. Thus it should not be surprising that the most successful[1] mathematical model in the social sciences is in Finance, i.e., the Black-Scholes-Merton theory of option[2] pricing. The BSM model is build on two basic assumptions: the assumption of no-arbitrage and the assumption that the stochastic process describing the logarithm of the stock price is a Brownian motion.

The macroscopic manifestation of random walk/Brownian motion is diffusion. Turning things around the microscopic process underlying diffusion is random walk or its continuous limit Brownian motion. The equation governing diffusion is known to physicists for 200 years – this is the heat or Fokker–Planck equation. It should not come as a surprise that the Black–Scholes equation is the heat equation. On the other hand the understanding of Brownian motion came a century later in the work of Louis Bechelier[3] [1], but five years earlier than the similar paper of Einstein. In its simplest form a random walk is obtained by adding independent random steps, i.e., if ξ_t is the position of the random walker at time t then $\xi_{t+1} = \xi_t + X_{t+1}$ where $(X_t)_t$ is (an affine transform of) a Bernoulli process. i.e., the X_t are independent identically distributed binary random variables. The fact that the steps X_t are independent is crucial because we can apply the Central Limit Theorem and from a random walk obtain a Brownian motion. The rational is that the price of a financial asset undergoes continuously trading and at each trade its price is pushed up or down independently of the previous trade. The independence assumption makes sense in a market close to equilibrium. In a market in turmoil herding effects are typical and one cannot assume independence.

The other key assumption in the BSM model is the principle of no-arbitrage. In two words arbitrage is riskless profit. For more details and the use of no-arbitrage in mathematical finance see for example [2]. We will discuss this crucial economic notion further in the text. We will see that arbitrage is naturally associated to nontrivial holonomy of a gauge theory of prices and no-arbitrage is the same as the flatness of the gauge connection. No-arbitrage is again a property that one would expect from a market in equilibrium. In a market close to equilibrium if an arbitrage opportunity appears traders will flock to the opportunity and the prices will change in directions to eliminate the arbitrage. The principle of no-arbitrage implies the "law of one price" – if two financial securities produce the same outcomes they should have the same price.

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[1] Creating a multi-trillion dollar industry is a success by most standards.

[2] A "call option" is a contract giving the owner the right but not the obligation to buy at a future moment of time an underlying financial security, e.g., a stock, for a fixed price. The fundamental question of mathematical finance is what is the present day price of an option.

[3] On March 29, 1900 Bechelier defended his Ph.D. thesis under the supervision of Poincaré. That date is fair to view as the birthday of mathematical finance.

Armed with the law of one price one can solve the BSM problem of pricing a call option C based on an underlying stock S. In a nutshell we can form a portfolio of the stock and cash that replicates the option. This is not surprising because the fluctuations of the option are due to the fluctuations of the stock. Then by the law of one price the option and the replicating portfolio should have the same price.

Despite the success of the BSM model it is only a first approximation. Its two main assumption random walk of prices and no-arbitrage are also the source of its vices. These two assumptions are true for equilibrium but how to get away from equilibrium? For a beautiful discussion and a call for a disequilibrium dynamical economics from the point of view of a mathematical physicist see the paper of Smolin [9].

2 Arbitrage

The nontechnical definition of arbitrage is riskless profit. Let us illustrate it with two examples.

First consider three currencies A, B, and C with exchange rates $A/B = 1.2$, $B/C = 1.1$, and $C/A = 0.9$. By $A/B = 1.2$ we mean that 1 unit of A is exchanged for 1.2 units of B. A standard assumption is that there are no transaction costs[4] and buy-sell spreads, i.e., the exchange rate of A for B is the multiplicative inverse of the exchange rate of B for A. The product of the three numbers above is 1.188. If such a situation exists we have a "money pump", we can borrow[5] one unit of A, exchange it for B, the result exchange for C, the result exchange for A, returning the one unit of A that we borrowed we are left with 0.188 units of A. We can go around this loop as long as it exists and starting with nothing we end up with positive wealth. Going around the loop in the other direction we will acquire a sure loss.

For the second example assume that we have two banks B_1 and B_2. Assume the interest rate of B_i is r_i for a certain period of time T. Again assume there are no transaction costs and the interest rate for loans and deposits are equal. One unit of cash deposited in B_i today will amount to $R_i = (1 + r_i/n)^{nT}$ units if interest is compounded n times during the period of time T and $R_i = e^{r_i T}$ units of cash if the interest is compounded continuously. For our example take $r_1 = 0.1$, $r_2 = 0.2$, and $T = 1$. If interest is compounded continuously we will have $R_1 = 1.105$ and $R_1 = 1.221$. To make a sure profit we borrow one unit of cash from the first bank and deposit it in the second bank. At the end of the time interval we withdraw our deposit in the second bank and cover our loan from the first bank being left with

[4] Transaction costs are like friction in physics so it is natural as a first approximation to ignore it. Moreover in finance transaction costs in machine trading could be very low.

[5] Another standard assumption is that we can borrow freely without any cost any asset. In financial jargon we have gone "short" in that asset. Thus the first step in the above transactions can be stated: "we sell short 1 unit of A and buy 1.2 units of B ...".

0.116 units of cash. Going around the loop in the opposite direction we will acquire a sure loss of the same amount. Again we have a "money pump".

In real life we can see a situation similar to the above but with a crucial modification – the bank giving the higher interest rate is as a rule of thumb the riskier, so if we are lucky we can make money from a loop as the above but it is also probable that we can loose all our money in the riskier bank. Hence the need for the word "riskless" in the definition of arbitrage.

3 Gauge Theory of Trade Networks

Consider the simplest possible trade network – two agents A and B, the first has a tradable asset[6] a and the second has asset b. To trade the agents need a means to compare a and b or an exchange rate: if β unit of b are exchanged for α units of a, set $U_{ba} = \beta/\alpha$. Here U_{ba} is a positive real number. Set $U_{ba}U_{ab} = 1$. We can picture this by a graph with two nodes and a directed edge from a to b decorated by the exchange rate U_{ab}. The important observation is that the units in which each asset is measured are a matter of convention. So we have a "gauge" freedom to rescale at every node or in a more fancy way a local action of $GL_+(1, \mathbb{R})$. The exchange rate U_{ab} is the connection on our graph allowing for the "parallel transport" or the exchange of one asset for another. Under local gauge transformations $g_a, g_b \in GL_+(1, \mathbb{R})$ the connection transforms as $U_{ab} \mapsto g_a U_{ab} g_b^{-1}$. In the case of three assets we will have a triangular graph with the exchange rates or connections U_{ab}, U_{bc}, U_{ca} on the edges and the freedom to rescale at every vertex independently. Transporting around a loop, in this case the triangle, we obtain the holonomy[7] $U_{ab}U_{bc}U_{ca}$. When the holonomy is not equal to one we have arbitrage, or equivalently, no-arbitrage corresponds to a flat connection.

If we have a complex network of financial securities in different countries the network will look something like a backbone consisting of nodes corresponding to the major currencies connected by edges labeled with the foreign exchange rates. Each currency node is the center of a star made of edges connecting the currency with the financial securities tradable in this currency with the current prices of the securities labeling the corresponding edges.[8] This network describes "space" at a certain moment of time.

To evolve our network in time we have to evolve every node. To compare the value of an asset at time t and $t + 1$ we need a connection in the time direction. This is exactly the discounting factor with which a future value is discounted to compare

[6]To avoid going into financial definitions probably it is better to call it a "tradable thing", anything that can be traded, goods, services, financial instruments.

[7]Because our gauge theory is on a graph it is better to use the word holonomy instead of curvature, since curvature is infinitesimal holonomy and on a graph this does not make sense.

[8]The price of any tradable thing is the connection on the edge between this thing and the corresponding currency.

to present value. When a bank account at interest rate r compounded continuously is "transported" from t to $t+1$ the amount is changed by e^r. Hence discount factors are the gauge connections in the time direction. Starting with our simplest graph with just one edge evolving in time for one time period we obtain a quadrilateral with vertices (a,t), $(a,t+1)$, (b,t), and $(b,t+1)$ and "space-like" edges labeled by exchange rates $U_{ab;t}$ and $U_{ab;t+1}$ and "time-like" edges labeled by discounting factors $U_{a;t,t+1}$ and $U_{b;t,t+1}$. The product $U_{ab;t}U_{b;t,t+1}U_{ba;t+1}U_{a;t+1,t}$ is the holonomy around this quadrilateral loop. If it is not one we have arbitrage.

We have described a trade network as a graph equipped with a principle $GL_+(1,\mathbb{R})$ bundle (the ability to rescale the unit of measure at every node) and a connection living on the edges allowing us to "parallel transport", i.e., to exchange or trade and evolve in time.

The identification of prices, exchange rates, and discount factors with gauge connections probably has occurred to several people but it is clearly stated probably for the first time in the works of Ilinski and collaborators, see for example [5, 6]. See also [15]. The propagation of gauge theory ideas and methods from physics to finance can be seen as a manifestation of Mack's "pushing Einstein's principles to the extreme" [7]. It is an amusing turn of fate that the proposal of Weyl [14], marking the birth of gauge theory, while unapplicable to electrodynamics finds a new incarnation in finance.

4 Gauge Theory of Welfare Economics

Here I want to take a brief look at the use of gauge connections to define consumer price indices (CPI). The Malaney–Weinstein theory [8, 13] is probably the first use of gauge theory in economics. One considers a basket (of goods and services) and how its value changes with time. Consider the consumption of N different goods and services. A basket is a vector $\mathbf{q}$ in $\mathbb{R}^N$ where the i-th component q_i is the quantity of the i-th good or service in the basket. A basket is the counterpart of a portfolio of financial securities. Prices live in the dual vector space, i.e., they are given by a price covector $\mathbf{p} \in \mathbb{R}^N$. The value of the basket is the pairing of the price covector and the basket vector. In a simpler language the basket can be represented as a column vector, the prices as a row vector and the pairing as the product $\mathbf{pq} \in \mathbb{R}$. The big (political and economic) question is to calculate the cost of living adjustment (COLA) which is the basis for government support to maintain a minimum standard of living. The problem is that both baskets and prices change with time. Examples of CPI are the Laspeyres index $(\mathbf{p}^1\mathbf{q}^0)/(\mathbf{p}^0\mathbf{q}^0)$ or the Paasche index $(\mathbf{p}^1\mathbf{q}^1)/(\mathbf{p}^0\mathbf{q}^1)$. Here the superscript indicates the moment of time. From the viewpoint of gauge theory one sees immediately that something is wrong with these indices – we are doing operations on variables at different times. Such variables live in different fibers and in order to do anything meaningful with them we have to "parallel transport" them to the same fiber for which we need a connection. The connection of Malaney–Weinstein has the crucial property to take into account that tastes change. Over long periods of time baskets are in practice incomparable – now automobiles with internal combustion engines are everywhere and horse carriages are only a tourist attraction but two hundred years ago automobiles were nonexistent. The key observation is that baskets don't change abruptly. It took time for cars to take over carriages. At a given moment of time we have a fixed price covector. Its kernel is a codimension one subspace in the space of baskets. This is the space of barter baskets. The Malaney–Weinstein connection $A = (\mathbf{q}d\mathbf{p})/(\mathbf{q}\mathbf{p})$ is such that the covariant derivative of the value of a basket is zero for baskets that are modified by changes in the barter direction.

The Malaney–Weinstein connection has been employed in finance by Farinelli and Vazquez [12].

5 Nonequilibrium Dynamics

How to introduce dynamics into economics and move away from equilibrium points is certainly a research program for generations ahead but the view point of gauge theory could help. We have learned the most important lesson that in a gauge theory the observables are gauge invariant quantities. An example of a gauge invariant quantity is holonomy. If equilibrium is related to no-arbitrage then we could attempt to mimic the quantum dynamics of gauge theory in physics, i.e., QED. Very promising simulations along the proposal for lattice QED of finance of Ilinski have been performed in [3, 4]. The model of Ilinski is not without critiques [10, 11]. The use of gauge invariant quantities to describe the dynamics is undisputable. On the other hand the use of a Boltzmann–Gibbs type of partition function given by a path integral $Z = \int DU \exp(-\beta S(U))$ with Boltzmann probability density $e^{-\beta S}$ is rather adhoc in the financial context.

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On Some Exact Solutions of Heat and Mass Transfer Equations with Variable Transport Coefficients

Ilya I. Ryzhkov and Irina V. Stepanova

Abstract Solutions of stationary and non-stationary heat and mass transfer equations describing thermal diffusion in a binary mixture are investigated. The dependence of physical properties on temperature and concentration is taken into account. The resulting differential equations are non–linear and require nontrivial approaches to their study and construction of exact solutions.

1 Introduction

The equation describing heat conduction or solute diffusion is widely investigated under the assumption of constant transport coefficients. The existence and uniqueness theorems for initial and boundary value problems are proved, and exact solutions are constructed for different types of imposed conditions. Efficient numerical methods for solving diffusion problems are also developed. In case of variable transport coefficients, the study of diffusion equation becomes more complex and requires application of powerful analytical or numerical methods.

In this paper, equations describing thermal diffusion process are investigated. Thermal diffusion refers to the separation of gas or liquid mixture driven by the temperature gradient. This process is described by heat and mass transfer equations, where the thermal conductivity as well as diffusion and thermal diffusion coefficients are in general functions of temperature and concentration. Investigation of these equations can be performed with the help of Lie symmetry analysis [9]. A review of exact solutions for non–linear diffusion–convection equation is given in [4] without physical interpretation.

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V. Dobrev (ed.), *Lie Theory and Its Applications in Physics*,
Springer Proceedings in Mathematics & Statistics 191,
DOI 10.1007/978-981-10-2636-2_48

In this work, we construct exact solutions for stationary and non–stationary one–dimensional thermal diffusion equations. We obtain solutions describing separation process in aqueous solution of sodium chloride with physical properties depending on temperature.

2 Problem Statement

Let us consider a binary mixture filling the space between impermeable parallel plates maintained at different constant temperatures. This simple configuration corresponds to an experimental setup, which is used for measuring transport coefficients in different multicomponent mixtures [5]. In this case, the one-dimensional thermal diffusion equations have the form

$$\frac{\partial T}{\partial t} = \frac{\partial}{\partial z}\left(\kappa(T, C)\frac{\partial T}{\partial z}\right), \tag{1}$$

$$\frac{\partial C}{\partial t} = \frac{\partial}{\partial z}\left(D(T, C)\frac{\partial C}{\partial z} + C(1 - C)D_T(T, C)\frac{\partial T}{\partial z}\right). \tag{2}$$

Here t is the time and z is the space coordinate, T is the temperature, C is the concentration of selected mixture component, κ is the thermal conductivity, and D and D_T are diffusion and thermal diffusion coefficients, respectively. We impose constant temperatures on the parallel plates and assume that the diffusion flux through these plates vanishes. In this case, the boundary conditions for Eqs. (1) and (2) are written as follows

$$z = 0: \quad T = T_0 - \frac{\triangle T}{2} \equiv T_1, \qquad z = L: \quad T = T_0 + \frac{\triangle T}{2} \equiv T_2, \tag{3}$$

$$z = 0 \quad \text{and} \quad z = L: \quad D(T, C)\frac{\partial C}{\partial z} + C(1 - C)D_T(T, C)\frac{\partial T}{\partial z} = 0, \tag{4}$$

where $\triangle T$ is the temperature difference between the plates at $z = 0$ and $z = L$. Relations (3) and (4) correspond to typical experimental conditions. At the initial moment of time $t = 0$, we impose constant temperature $T = T_0$ and constant concentration $C = C_0$ for $0 < z < L$.

The following sections are devoted to constructing stationary and non–stationary solutions of the above–described problem.

3 Solution of Steady–State Problem

In the stationary state, the functions T and C depend only on the space coordinate z and do not depend on time t. Thus, the terms in the left–hand side of Eqs. (1) and (2) must be dropped, and the initial conditions are out of consideration. However, the conservation of mass for the selected component requires

$$\frac{1}{L}\int_0^L C dz = C_0. \tag{5}$$

This relation is obtained by integrating Eq. (2) with respect to z from $z = 0$ to $z = L$ and then with respect to t taking into account the initial condition $C = C_0$ at $t = 0$.

Stationary Eqs. (1) and (2) can be integrated once using boundary conditions (4). Furthermore, if $D(T, C) \neq 0$, then these equations reduce to the following form after integration:

$$\kappa(T, C)\frac{dT}{dz} = \kappa_0, \tag{6}$$

$$\frac{dC}{dz} + C(1 - C)S_T(T, C)\frac{dT}{dz} = 0, \tag{7}$$

where κ_0 is the constant and $S_T = D_T/D$ is the Soret coefficient. Construction of solution for this problem was studied in our paper [8] in details. Here we present a brief description of the method. We assume that the variable T is independent and z is dependent one, i.e. $z = z(T)$, $C = C(T)$. Using the technique suggested, e.g. in [6], we rewrite Eqs. (6) and (7) as

$$\frac{dz}{dT} = \frac{\kappa(T, C)}{\kappa_0}, \tag{8}$$

$$\frac{dC}{dT} = -S'_T(T, C), \tag{9}$$

where $S'_T = C(1 - C)S_T(T, C)$. Boundary conditions (3) become

$$z(T_1) = 0, \qquad z(T_2) = L\,. \tag{10}$$

The solution of (8) and (10) is given by

$$z = \frac{1}{\kappa_0}\int_{T_1}^{T} \kappa\big(\tau, C(\tau)\big)d\tau, \qquad \kappa_0 = \frac{1}{L}\int_{T_1}^{T_2} \kappa\big(T, C(T)\big)dT. \tag{11}$$

Additional condition (5) transforms into

$$\int_{T_1}^{T_2} \kappa(T, C(T))(C(T) - C_0)dT = 0. \tag{12}$$

If the coefficients $\kappa(T, C)$ and $S_T(C, T)$ are known, then we can integrate Eq. (9) analytically or numerically with condition (12). Then one should use solution (11) to invert the function $z = z(T)$, and reconstruct the temperature and concentration profiles $T(z)$ and $C(z)$. Note that for numerical calculation of $C(T)$, we should impose the boundary condition $C(T_1) = C_1$, where C_1 is found in order to satisfy condition (12). For arbitrary form of coefficients $\kappa(T,C)$ and $S_T(T,C)$, we can completely solve this problem by numerical method only. The shooting procedure can be employed to find the value C_1. For more details, see paper [8].

3.1 Example of Exact Solution

Here we give example of an exact solution, which describes thermodiffusion separation in binary mixture. At first, we should mention that for constant physical properties, the solution represents a linear dependence of functions T and C on the space coordinate z. When variable physical properties are taken into account, the solution becomes non–linear.

Let us consider aqueous solution of sodium chloride as a working binary mixture. The thermal conductivity and Soret coefficient are assumed to be functions of temperature only. We find the functional dependencies of these coefficients with the help of measured data from paper [2] by least squares method at mean concentration $C_0 = 0.0285$. They are as follows

$$\kappa(T) = k_0 + k_1 T, \qquad S_T(T) = \frac{a_0 + a_1 T + a_2 T^2}{b + T}, \tag{13}$$

where

$$k_0 = 0.57,\ k_1 = 0.0016,$$

$$a_0 = -2.181 \cdot 10^{-2},\ a_1 = 1.03 \cdot 10^{-3},\ a_2 = 6.53 \cdot 10^{-5},\ b = 17.218.$$

The exact solution for functions T and C under the assumption $C(1-C) \sim C_0(1-C_0)$ is presented by the formulas

$$T = -\frac{a_0}{a_1} + \sqrt{T_0^2 + 2\triangle T\left(\frac{1}{2} - \frac{z}{L}\right)\left(T_0 + \frac{a_0}{a_1}\right) + \frac{\triangle T^2}{4} + \frac{a_0}{a_1}\left(2T_0 + \frac{a_0}{a_1}\right)} \tag{14}$$

$$\begin{aligned} C = C_0 + C_0(1-C_0)\Big[&\frac{a_2}{2}T^2 + (a_1 - ba_2)T + (b^2 a_2 - ba_1 + a_0)\Big(1 + \\ &+ \ln(2T + 2b) + \frac{1}{2\triangle T}\big((2T_0 + 2b - \triangle T)\ln(2T_0 + 2b - \triangle T) - \\ &-(2T_0 + 2b + \triangle T)\ln(2T_0 + 2b + \triangle T)\big)\Big) + a_2\Big(bT_0 - \frac{\triangle T^2}{24} - \frac{T_0^2}{2}\Big) - a_1 T_0\Big]. \end{aligned} \tag{15}$$

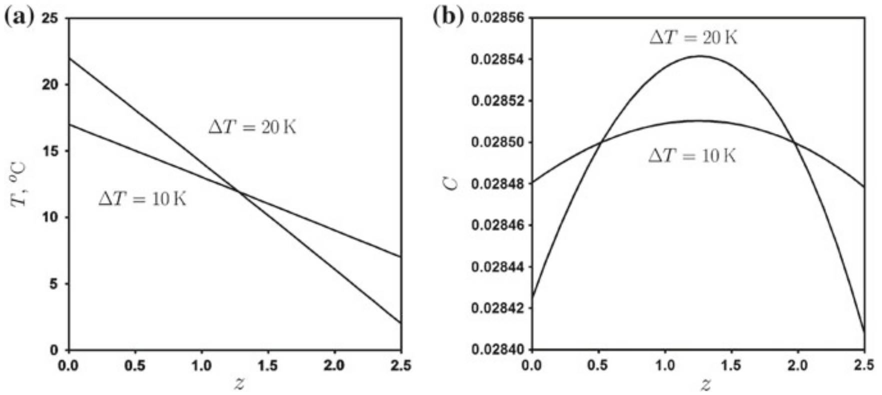

Fig. 1 Temperature (**a**) and concentration (**b**) profiles in aqueous solution of sodium chloride

To obtain the profile $C = C(z)$ in explicit form, one should substitute T from (14) into (15). The profiles of temperature T and concentration C are shown in Fig. 1 for various temperature differences ΔT between the plates and mean temperature $T_0 = 12\,°\mathrm{C}$. It is easy to see that temperature distribution is close to linear despite of the fact that formula (14) demonstrates a nonlinear dependence of T on z. The concentration profiles show that the separation effect becomes stronger with increasing $\triangle T$ from 10 K up to 20 K. The concentration reaches maximum at the point where the Soret coefficient is equal to zero and then it decreases. It should be emphasized that the function C is essentially nonlinear when the Soret coefficient is not constant.

4 Solution of Non–stationary Problem

In this section, we consider non–stationary problem for concentration (see Eq. (2)) assuming that the temperature distribution described by Eq. (1) is stationary. It often occurs in practical situations that the temperature profile is established much more rapidly than the concentration profile due to large difference in characteristic times. It is supposed that the coefficient κ =const. Then from Eq. (1) and boundary conditions (3) it follows that

$$T = T_0 + \triangle T\left(\frac{1}{2} - \frac{z}{L}\right).$$

As in Sect. 3.1, we take aqueous solution of sodium chloride as a working binary mixture. Coefficients D and D_T are linear and quadratic functions in z, respectively, due to linear dependence of temperature on coordinate z. We substitute these functions into Eq. (2) assuming $C(1-C) \sim C_0(1-C_0)$, where $C_0 = 0.0285$ as before. After some calculations, we obtain the equation for concentration C in the form

$$\frac{\partial C}{\partial t} = \frac{\partial}{\partial z}\left(p(z)\frac{\partial C}{\partial z}\right) + \Phi(z), \tag{16}$$

where $p(z) = m_1 z + m_2$, $\Phi(z) = m_3 z + m_4$. Constants m_i, $i = 1, \ldots, 4$, can be found from coefficients in formulas (13). The initial and boundary conditions for Eq. (16) have the form

$$C(t=0) = C_0, \qquad \left.\frac{\partial C}{\partial z}\right|_{z=0} = C_1, \qquad \left.\frac{\partial C}{\partial z}\right|_{z=L} = C_2, \tag{17}$$

where $C_1 = -D_{T0}\triangle T/(D_0 L)$, $C_2 = -D_{T1}\triangle T/(D_1 L)$ are constants. Here D_0, D_{T0} are values of coefficients D and D_T at $z = 0$, and D_1, D_{T1} are values of those coefficients at $z = L$.

The solution of problem (16) and (17) can be found with a help of Green function according to handbook [7]. This solution is constructed by means of series with respect to Bessel functions. These series have very slow convergence. It is necessary to take about one thousand terms for obtaining the solution with good accuracy. Here we describe another way of solving this problem using the Laplace transform [1]. Application of this transform leads to the boundary value problem for ordinary differential equation for the Laplace image $\tilde{C}$ of the function C. Solution of this problem is obtained in the form

$$\begin{aligned}
\tilde{C} &= (M_1 + iM_2) I_0\left(\frac{2\sqrt{p(m_1 z + m_2)}}{m_1}\right) - \frac{2M_2}{\pi} K_0\left(\frac{2\sqrt{p(m_1 z + m_2)}}{m_1}\right) + \\
&\quad + \frac{m_3 z + m_4 + pC_0}{p^2} + \frac{m_1 m_3}{p^3}, \\
M_1 &= \frac{\pi}{2p^{3/2}(K_1^0 I_1^L - I_1^0 K_1^L)}\Big[\sqrt{m_2}(m3/p - C_1)(iI_1^L + 2K_1^L/\pi) - \\
&\quad - \sqrt{m_1 L + m_2}(m_3/p - C_2)(iI_1^0 + 2K_1^0/\pi)\Big], \\
M_2 &= \frac{\pi[\sqrt{m_1 L + m_2}(m_3/p - C_2)I_1^0 - \sqrt{m_2}(m3/p - C_1)I_1^L]}{2p^{3/2}(K_1^0 I_1^L - I_1^0 K_1^L)}.
\end{aligned}$$

Here I_0, K_0, I_1 and K_1 are the modified Bessel functions of zero and first order. In the expressions for constants M_1 and M_2, indices 0 and L mean the value of corresponding functions at $z = 0$ and $z = L$, respectively, i is the imaginary unit, p is the complex parameter of the Laplace transform. Inversion of the function $\tilde{C}$ is performed with a help of numerical procedure suggested in [3]. As a result, solution of non–stationary problem is obtained. Evolution of concentration profiles with respect to time t is presented in Fig. 2. Maximum of concentration corresponds to the point where Soret coefficient is equal to zero. There is no separation of mixture at this point. Thereby, more salty and more dense component is located in central part of the layer. With increasing time, the unsteady concentration profile tends to the stationary solution (see red line in Fig. 2).

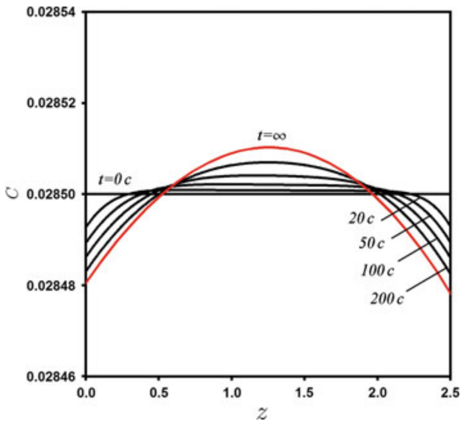

Fig. 2 Concentration profile at different times; $t = \infty$ corresponds to the stationary profile

5 Conclusion

We have studied solutions of stationary and non-stationary heat and mass transfer equations describing thermal diffusion in a binary mixture. The dependence of physical properties on temperature and concentration is taken into account. Examples of solutions, which describe the separation of sodium chloride aqueous solution between parallel plates with different temperatures, are constructed and analyzed.

Acknowledgements The work is supported by Programm III.22.4 Mathematical analysis of models of inhomogeneous media dynamics, project 'Mathematical and numerical modelling of conjugate problems of continuum mechanics'.

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A Star Product for the Volume Form Nambu-Poisson Structure on a Kähler Manifold

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Abstract Every symplectic (1-plectic) manifold (M, ω) of dimension $2n$ may be regarded as a $(2n-1)$-plectic manifold by wedging together n copies of the symplectic 2-form to get the Liouville volume form. This volume form defines a Nambu-Poisson structure $\{., \ldots, .\}_{NP}$ of order $2n$ on $C^{\infty}(M)$. When the manifold is Kähler, the Kähler structure can be used to define a star product (known as the Berezin-Toeplitz star product) for the Poisson algebra $C^{\infty}(M)$. For a Kähler manifold (M, ω) we define a higher order analogue of the Berezin-Toeplitz star product on the Nambu-Poisson algebra $(C^{\infty}(M), \{., \ldots, .\}_{NP})$.

1 Introduction

In the paper [2], Bordemann, Meinrenken and Schlichenmaier define an operator quantization that works for any compact Kähler manifold (M, ω) with integral Kähler form. The Kähler form ω is called integral when the cohomology class of the rescaled Kähler form $\frac{\omega}{2\pi}$ is an integral cohomology class. By an operator quantization we mean a mapping from the algebra of smooth functions on M (which is a commutative algebra under the operation of pointwise multiplication of functions, as well as a Poisson algebra with the bracket defined by $\{f, g\} := \omega(X_f, X_g)$, where X_f is the Hamiltonian vector field of $f \in C^{\infty}(M)$) to the algebra of operators on some Hilbert space $\mathscr{H}$ (which is a noncommutative algebra under the operation of composition of operators as well as a Lie algebra with the commutator of operators as the Lie bracket) $C^{\infty}(M) \xrightarrow{Q} Op(\mathscr{H}), f \mapsto \hat{f}$. Here and throughout the text $C^{\infty}(M)$ denotes smooth complex valued functions on M. The quantization mapping is required to

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V. Dobrev (ed.), *Lie Theory and Its Applications in Physics*,
Springer Proceedings in Mathematics & Statistics 191,
DOI 10.1007/978-981-10-2636-2_49

satisfy certain properties[1] [3] which include $\mathbb{C}$-linearity and a version of Dirac's quantization conditions[2]:

$$1 \mapsto const(\hbar)I,$$

$$\{f, g\} \mapsto const(\hbar)[\hat{f}, \hat{g}].$$

In [2] the authors solve the problem of quantizing a compact Kähler manifold (M, ω) with integral Kähler form by constructing a sequence of Hilbert spaces $\mathscr{H}^k$ ($k \in \{1, 2, \dots\}$) along with a sequence of mappings $C^\infty(M) \xrightarrow{T^{(k)}} End(\mathscr{H}^k)$. The authors go on to prove that their mappings satisfy the axioms of quantization asymptotically, they establish the following theorem:

Theorem 1.1 *([2] Thm. 4.1, 4.2, [8] Thm. 3.3) For $f, g \in C^\infty(M)$, as $k \to \infty$,*

(i)

$$\left|\left| ik\left[T_f^{(k)}, T_g^{(k)}\right] - T_{\{f,g\}}^{(k)} \right|\right| = O\left(\frac{1}{k}\right),$$

(ii) there is a constant $C = C(f) > 0$ such that

$$|f|_\infty - \frac{C}{k} \leq \left|\left| T_f^{(k)} \right|\right| \leq |f|_\infty .$$

Theorem 1.1 establishes the second of Dirac's quantization conditions up to a term of order $\frac{1}{k}$, which is the best one can do in this setting [6]. This construction is known as Berezin-Toeplitz operator[3] quantization.

The term quantization is borrowed by mathematicians from physicists. In the context of classical mechanics $C^\infty(M)$ is known as the classical algebra of observables. For mathematicians every $f \in C^\infty(M)$ is called an observable, although from the physical point of view only real valued functions have any meaning. In this context quantization leads to a quantum mechanical version of the classical system that is being quantized. Quantization in the mathematical sense is a much broader concept than it is from the physical point of view. For mathematicians the term quantization can refer to any number of mappings $C^\infty(M) \xrightarrow{Q} Op(\mathscr{H})$ satisfying similar but not identical axioms and that work for manifolds with different levels of structure.

The focus of the present article will be on deformation quantization rather than on operator quantization. In deformation quantization rather than assigning operators to functions (which is the method of operator quantization) one starts with the Poisson algebra $(C^\infty(M), \{\cdot, \cdot\})$ and deforms it into a noncommutative Lie algebra like $(\mathscr{H}, [\cdot, \cdot])$. Foundational work on deformations of algebras was done by

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[1] These properties are known as the axioms of quantization, the exact axioms that are chosen depend somewhat on the mathematical setting. For a discussion of the reasons behind this and of the axioms that are used in the Kähler setting see [3].

[2] $const(\hbar)$ denotes a constant that depends on $\hbar$.

[3] Because of the role played by the operators on the Hilbert space $\mathscr{H}$.

Gerstenhaber [5], the Lie algebra case in particular was done by Nijenhuis and Richardson [7].

Definition 1.2 A deformation of a Lie algebra $\mathfrak{g}$ with Lie bracket $\{f, g\}$ is a family of Lie brackets depending on a parameter t, defined by

$$\{f, g\}_t := \{f, g\} + \sum_{n=1}^{\infty} t^n C_n(f, g),$$

where the C_n are bilinear, skew symmetric $\mathfrak{g}$-valued maps, $C_n : \wedge^2 \mathfrak{g} \to \mathfrak{g}$. The brackets $\{f, g\}_t$ take values in $\mathfrak{g}[[t]]$, so we must view the Lie algebra $\mathfrak{g}$ as a subalgebra of the algebra of formal power series $\mathfrak{g}[[t]]$ in order for this definition to make sense.[4] Furthermore we require that the deformed bracket $\{f, g\}_t$ satisfies the Jacobi identity for every t.

The following definition captures the notion of deformation that is appropriate for quantization. Let $A = C^\infty(M)[[t]]$, the space of formal power series with coefficients in $C^\infty(M)$. A product $\star$ on A is called a **(formal) star product** if it is an associative $\mathbb{C}$-linear product such that

(i) $A/tA \cong C^\infty(M)$, in particular $f \star g \mod t = fg$, for $f, g \in C^\infty(M) \subset C^\infty(M)[[t]]$.
(ii) $\frac{1}{t}(f \star g - g \star f) \mod t = -i\{f, g\}$,

where $f, g \in C^\infty(M)$. We can also write

$$f \star g = \sum_{j=0}^{\infty} C_j(f, g) t^j, \tag{1}$$

with $C_j(f, g) \in C^\infty(M)$. The C_j should be $\mathbb{C}$-bilinear. The conditions (i) and (ii) can be reformulated as

$$C_0(f, g) = fg, \tag{2}$$

and

$$C_1(f, g) - C_1(g, f) = -i\{f, g\} \tag{3}$$

The bilinearity of the C_j guarantees that the star product will be bilinear. The condition (i) says that the star product is in fact a deformation[5] of the associative algebra $(C^\infty(M), \bullet)$, where $f \bullet g$ is the usual pointwise multiplication of functions. Every star product defines a skew symmetric bracket of functions by the formula

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[4]If the bracket $\{f, g\}_t$ converges for all t and for all $f, g \in C^\infty(M)$ we would be in the most ideal situation, of course this will depend on the details of the definition of the C_i.

[5]in the sense of [5].

$$[f, g]_Q := \frac{1}{t}(f \star g - g \star f). \tag{4}$$

Condition (*ii*) is equivalent to the correspondence principle[6] (of quantum mechanics) for the quantum bracket defined by (4). That is, (*ii*) says that $[f, g]_Q$ is a deformation of the Poisson algebra $(C^\infty(M), \{,\})$ [5].

In the article [9], Schlichenmaier applies the Berezin-Toeplitz operator quantization (Theorem 1.1) to give a proof of the following theorem.

Theorem 1.3 *([9] Thm 2.2) There exists a unique (formal) star product on $C^\infty(M)$*

$$f \star g \equiv \sum_{j=0}^{\infty} \nu^j C_j(f, g), \tag{5}$$

in such a way that for $f, g \in C^\infty(M)$ and for every $N \in \mathbb{N}$ we have with suitable constants $K_N(f, g)$ such that for all m

$$||T_f^{(m)} T_g^{(m)} - \sum_{0 \le j < N} \left(\frac{1}{m}\right)^j T^{(m)}_{C_j(f,g)}|| \le K_N(f, g) \left(\frac{1}{m}\right)^N \tag{6}$$

2 A Deformation of the Nambu Bracket

In [1] we extend the Berezin-Toeplitz quantization to the multisymplectic manifold $(M, \frac{\omega^n}{n!})$ with the Nambu-Poisson structure $(C^\infty(M), \{., \ldots, .\}_{NP})$ and we prove an analog of Theorem 1.1. As has been explained already, Berezin-Toeplitz quantization leads to a deformation quantization and the Berezin-Toeplitz star product [9]. In this section we perform this step in the multisymplectic setting [1]. We propose to define a higher order analogue of the Berezin-Toeplitz star product.

Consider the $(2n - 1)$-plectic manifold (M, Ω) obtained from the symplectic manifold (M, ω), where $\Omega = \frac{1}{n!}\omega^n$. Define a star product (the terminology is justified by Proposition 2.1) of $2j$ ($j \le n$) functions in $C^\infty(M)$ by the formula

$$\star(f_1, f_2, \ldots, f_{2j-1}, f_{2j}) = \sum_{i=0}^{\infty} t^i D_i(f_1, f_2, \ldots, f_{2j-1}, f_{2j}) \tag{7}$$

where

$$D_i(f_1, f_2, \ldots, f_{2j-1}, f_{2j}) := C_i(f_1, f_2) \ldots C_i(f_{2j-1}, f_{2j}) \tag{8}$$

[6] This principal says that the classical theory should be recovered in the limit $t \to 0$.

and the C_i are the coefficients of some star product (such as the Berezin-Toeplitz star product, theorem 1.3). The following formula defines a Nambu-Poisson bracket $\{., ..., .\} : \bigwedge^{2n} C^\infty(M) \to C^\infty(M)$

$$df_1 \wedge ... \wedge df_{2n} = \{f_1, ..., f_{2n}\}\Omega \tag{9}$$

In order to prove our main result (Proposition 2.1) we will use a formula for the $2n$-ary bracket defined by (9), in terms of the usual Poisson bracket and a $(2n-2)$-ary bracket. One may define a $2j$-ary bracket ($j \leq n$) for any Poisson manifold $(M, \{., .\})$ of dimension $2n$ by the formula,

$$\{f_1, \dots, f_{2j}\} = \frac{1}{2^j j!} \sum_{\sigma \in S_{2j}} \varepsilon(\sigma) \prod_{i=1}^{j} \{f_{\sigma(2i-1)}, f_{\sigma(2i)}\} \tag{10}$$

In [10] (Chap. 2) we demonstrated that the $(2n)$-ary bracket defined by (9) agrees with the one defined by (10) when $j = n$, so that (10) may be regarded as a generalization of the Nambu-Poisson bracket (9), [9] contains the same result. It should be noted that for $j < n$ the bracket defined by the formula (10) is not a Nambu-Poisson bracket because it does not satisfy the fundamental identity [4]. Those identities are not needed to establish the relation (11) that we will need for the proof of Proposition 2.1.

$$\begin{aligned}\{f_1, \dots, f_{2j}\} = {} & \{f_1, f_2\}\{f_3, \dots, f_{2j}\} + \\ & + \sum_{i=3}^{2j-1} (-1)^i \{f_1, f_i\}\{f_2, \dots, f_{i-1}, f_{i+1}, \dots, f_{2j}\} + \\ & + \{f_1, f_{2j}\}\{f_2, \dots, f_{2j-1}\}\end{aligned} \tag{11}$$

For $j = 3$ the formula reads

$$\begin{aligned}\{f_1, f_2, f_3, f_4, f_5, f_6\} = {} & \{f_1, f_2\}\{f_3, f_4, f_5, f_6\} - \{f_1, f_3\}\{f_2, f_4, f_5, f_6\} + \\ & + \{f_1, f_4\}\{f_2, f_3, f_5, f_6\} - \{f_1, f_5\}\{f_2, f_3, f_4, f_6\} + \\ & + \{f_1, f_6\}\{f_2, f_3, f_4, f_5\}\end{aligned} \tag{12}$$

Formula (11) can be established by simply substituting the definition (10) for all of the brackets on both sides of the relation (11).

Proposition 2.1 *The $2j$-ary product defined by (7) satisfies the following properties:*

1. $D_0(f_1, f_2, \dots, f_{2j-1}, f_{2j}) = f_1 f_2 \dots f_{2j-1} f_{2j}$
2. $\sum_{\sigma \in S_{2j}} \varepsilon(\sigma) D_1(f_{\sigma(1)}, f_{\sigma(2)}, \dots, f_{\sigma(2j-1)}, f_{\sigma(2j)}) = j!(-i)^j \{f_1, f_2, \dots, f_{2j-1}, f_{2j}\}$

When $j = n$ the product defined by (7) can be viewed as a generalization of the Berezin-Toeplitz star product (1.3) where the Nambu-Poisson bracket (9) plays the role of the Poisson bracket.

Proof Proof of Proposition 2.1

The proof is an induction with base case $j = 1$ provided by (3) and involving (11) in the induction step. Throughout the proof $\varepsilon(\sigma)$ stands in for the sign of the permutation σ.

$$
\begin{aligned}
&\sum_{\sigma \in S_{\{1,\ldots,2j\}}} \varepsilon(\sigma) C_1(f_{\sigma(1)}, f_{\sigma(2)}) \ldots C_1(f_{\sigma(2j-1)}, f_{\sigma(2j)}) = \\
&= j(C_1(f_1, f_2) - C_1(f_2, f_1))(\sum_{\gamma \in S_{\{1,\ldots,2j\} \setminus \{1,2\}}} \varepsilon(\gamma) C_1(f_{\gamma(3)}, f_{\gamma(4)}) \ldots \\
&\ldots C_1(f_{\gamma(2j-1)}, f_{\gamma(2j)})) \\
&+ \sum_{i=3}^{2j-1} j(C_1(f_1, f_i) - C_1(f_i, f_1))(\sum_{\gamma \in S_{\{1,\ldots,2j\} \setminus \{1,i\}}} \varepsilon(\gamma) C_1(f_{\gamma(2)}, f_{\gamma(3)}) \ldots \\
&\ldots C_1(f_{\gamma(i-1)}, f_{\gamma(i+1)}) \ldots C_1(f_{\gamma(2j-1)}, f_{\gamma(2j)})) + \\
&+ j(C_1(f_1, f_{2j}) - C_1(f_{2j}, f_1))(\sum_{\gamma \in S_{\{1,\ldots,2j\} \setminus \{1,2j\}}} \varepsilon(\gamma) C_1(f_{\gamma(3)}, f_{\gamma(4)}) \ldots \\
&\ldots C_1(f_{\gamma(2j-2)}, f_{\gamma(2j-1)})) = \\
&= j!(-i)^j (\{f_1, f_2\}\{f_3, \ldots, f_{2j}\} + \sum_{i=3}^{2j-1} (-1)^i \{f_1, f_i\}\{f_2, \ldots, f_{i-1}, f_{i+1}, \ldots, f_{2j}\} + \\
&+ \{f_1, f_{2j}\}\{f_2, \ldots, f_{2j-1}\}) = j!(-i)^j \{f_1, \ldots, f_{2j}\}.
\end{aligned}
$$

This proves 2., 1. follows from (8) and (2).

Remark 2.2 The requirement

$$\sum_{\sigma \in S_{2j}} \varepsilon(\sigma) D_1(f_{\sigma(1)}, f_{\sigma(2)}, \ldots, f_{\sigma(2j-1)}, f_{\sigma(2j)}) = j!(-i)^j \{f_1, f_2, \ldots, f_{2j-1}, f_{2j}\}$$

is a generalization of the requirement $C_1(f, g) - C_1(g, f) = -i\{f, g\}$ which we make for binary star products. As we will see by the next definition, this constraint ensures that our star products (which are deformations, of the associative product of two or more functions, vis-a-vis condition 1.) lead to infinitesimal deformations of the $2j$-ary bracket (9) (we will denote this family of deformations by $\star[f_1, \ldots, f_{2j}]_t$).

$$\star[f_1, \ldots, f_{2j}]_t := \frac{1}{t} \left(\sum_{\sigma \in S_{2j}} \varepsilon(\sigma) \star (f_{\sigma(1)}, \ldots, f_{\sigma(2j)}) \right)$$

$$= \sum_{\sigma \in S_{2j}} \varepsilon(\sigma) D_1(f_{\sigma(1)}, \ldots, f_{\sigma 2j}) + t \sum_{\sigma \in S_{2j}} \varepsilon(\sigma) D_2(f_{\sigma(1)} \ldots, f_{\sigma(2j)}) + O(t^2)$$

$$= j! i^j \{f_1, \ldots, f_{2j}\} + t \sum_{\sigma \in S_{2j}} \varepsilon(\sigma) D_2(f_{\sigma(1)}, \ldots, f_{\sigma(2j)}) + O(t^2)$$

$$= j! i^j \{f_1, \ldots, f_{2j}\} + t\alpha(f_1, \ldots, f_{2j}) + O(t^2)$$

where

$$\alpha : \wedge^{2j} C^\infty(M) \to C^\infty(M)$$

Remark 2.3 The star product $\star(., \ldots, .)$ satisfies some additional properties that are worth mentioning.

1. $\star(1, f_2, \ldots, f_{2j}) = \star(f_2, 1, \ldots, f_{2j}) = \cdots = \star(f_2, f_3, \ldots, f_{2j-1}, 1) =$ $= f_2 f_3 \ldots f_{2j-1} f_{2j}$. This means, in other words, that multiplication by a constant c in classical mechanics corresponds throughout the deformation to multiplication by the constant power series c. This is one of the usual axioms of quantization. This property is equivalent to $D_k(1, f_2, \ldots, f_{2j}) = D_k(f_2, 1, \ldots, f_{2j}) = \cdots = D_k(f_2, f_3, \ldots, f_{2j-1}, 1) = 0$ for $k \geq 1$, it follows from the corresponding property of 3.5 and the definition of the D_k.
2. A star product of $2j$ functions should be called **local** if for all $f, g \in C^\infty(M)$, the support $suppD_j(f_1, \ldots, f_{2j})$ is contained in $suppf_1 \bigcap \cdots \bigcap suppf_{2j}$ for all $j \in \mathbb{N}_0$. This is the obvious generalization of locality of a star product for $j = 1$. The locality of our star product depends on the locality of the star product defined by Schlichenmaier, see [9] and the comments therein.

Acknowledgements This paper follows the talk given in June 2015 at the "Lie theory and its applications in physics" workshop in Varna. It explains results from one section of my Ph.D. thesis defended at the University of Western Ontario in August 2015. I am thankful to my Ph.D. advisors T. Barron and M. Pinsonnault for discussions. I would also like to thank the anonymous referees for helpful comments regarding the exposition.

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