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Gualtiero Badin
Fulvio Crisciani

Variational Formulation of Fluid and Geophysical Fluid Dynamics

Mechanics, Symmetries and Conservation Laws

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In loving memory of Prof. Giuseppe Furlan
(1935–2016)

Foreword

In science, as in other walks of life, we are often tempted to do something that will have an immediate impact that seems original and that will garner more funding and get us promoted but that may have little true benefit in the long term. And so we do it, and thereby make a Faustian bargain, not really thinking about the longer term. But that long term might be better served if we could make more of a Proustian bargain in which we remember the accomplishments of the past, search for the meaning in the science, build on secure foundations and so make true advances, even if slowly and intermittently. To proceed this way, we need a proper exposition of those foundations and how they relate to the more applied concerns that we deal with on a daily basis, and it is this noble task that the authors of this book have set themselves. They have returned to the very fundamentals of Geophysical Fluid Dynamics and given us a compelling account of how Hamilton's principle and variational methods provide a secure footing to the subject and give an underlying meaning to its results.

Hamilton's principle provides one of the most fundamental and elegant ways of looking at mechanics. The laws of motion—whether they may be Newton's laws in classical mechanics or the equations of quantum mechanics—emerge naturally by way of a systematic variational treatment from clear axioms. The connection of the conservation properties of the system to the underlying symmetries is made transparent, approximations may be made consistently, and the formulation provides a solid basis for practical applications. In this book, the authors apply this methodology to Geophysical Fluid Dynamics, starting with a derivation of the equations of motion themselves and progressing systematically to approximate equation sets for use with the rapidly rotating and stratified flows that we encounter in meteorology and oceanography. Along the way, we encounter such things as Noether's Theorem, Lagrangian and Eulerian viewpoints, the relabelling symmetry that gives rise to potential vorticity conservation, semi-geostrophic dynamics and the conservation of wave activity. The method also has great practical benefit, for it is only by use of approximate equation sets that we are able to compute the future state of the weather—the lack of the proper use of approximate or filtered equations can be thought of as the cause of the failure of Richardson's heroic effort to numerically predict the weather in 1922, and the proper use of an approximate set was vital for the success of the effort some 30 years later by Charney, Fjortoft and von Neumann, and still today we use approximate equation sets in climate and weather models.

The treatment in the book is unavoidably mathematical, but it is not “advanced”, for it makes use only of fairly standard methods in variational calculus and a little bit of group theory. The book should be accessible to anyone who has such a background although it is not, reusing one of Clifford Truesdell’s many memorable remarks, a mountain railway that will take the reader on a scenic tour of all the famous results with no effort on the reader’s part. But with just a little work, the book will benefit meteorologists and oceanographers who wish to learn about variational methods, and it will benefit physicists and applied mathematicians who wish to learn about Geophysical Fluid Dynamics. And the book reminds us once again that Geophysical Fluid Dynamics is a branch of theoretical physics, as it has always been but as we sometimes forget.

April 2017

Geoffrey K. Vallis
University of Exeter, Exeter, England

Preface

The motion of fluids from the smaller to the large scales is described by a complex interplay between the momentum equations and the equations describing the thermodynamics of the system under consideration. The emerging motion comprises several scales, ranging from microscales, to planetary scales, often linked by non-trivial self-similar scalings. At the same time, the motion of classic fluids is described by a specific branch of continuum mechanics. It comes thus natural that one would like to describe the rich phenomenology of the fluid and geophysical fluid motion in a systematic way from first principles, derived by continuum mechanics. One of these first principles is given by Hamilton's principle, which allows to obtain the equations of motion through a variational treatment of the system.

A famous call for the need of a systematic derivation of the equations of Fluid and Geophysical Fluid Dynamics lies in the memorandum sent by the mathematician John von Neumann to Oswald Veblen, written in 1945 and here reported in the Introduction to Chap. 3. The quote reads: "*The great virtue of the variational treatment […] is that it permits efficient use, in the process of calculation, of any experimental or intuitive insight […]. It is important to realize that it is not possible, or possible to a much smaller extent, if one performs the calculation by using the original form of the equations of motion—the partial differential equations. […] Symmetry, stationarity, similitude properties […] applying such methods to hydrodynamics would be of the greatest importance since in many hydrodynamical problems we have very good general evidence of the above-mentioned sort about the approximate aspect of the solution, and the refining of this to a solution of the desired precision is what presents disproportionate computational difficulties […]*" (see reference to von Neumann (1963) of Chap. 3). While sadly von Neumann intended to make practical use of such a treatment to study the aftershocks created by nuclear explosions, the quote still summarizes some of the most important features of the variational method: "*Symmetry, stationarity, similitude properties*". With these properties, von Neumann clearly had in mind the self-similar structure of fluid flows ("*similitude*"), which is indeed the feature that allows us to study different scales of motion through a proper rescaling of the system; he probably had in mind also the study of the stability of the system under consideration ("*stationarity*"); but he mentions also one of the most important results from field-theory that is the study of what he calls with the word "*symmetry*". Continuous symmetries in mechanical systems have in fact the property to be related to conserved quantities, as it is well known by probably the most beautiful theorem in mathematical physics, the celebrated "*Noether's Theorem*". In the specific case of fluid dynamics, the continuum hypothesis is associated to a specific symmetry that is the particle relabelling symmetry. Application of Noether's Theorem results in the fundamental conservation of vorticity in fluids, which is itself linked to the conservation of circulation and of potential vorticity, all quantities that have primary importance in a huge number of applications, ranging from fluids, geophysical fluids, plasmas and astrophysical fluids. It is from the particle relabelling symmetry and Noether's Theorem that one sees that the conservation of vorticity is a fundamental property of the system and does not emerge just from skilful manipulation of the partial differential equations describing the dynamics.

The aim of this book is to go through the development of these concepts.

In Chap. 1, we give a résumé of the aspects of Fluid and Geophysical Fluid Dynamics, starting from the continuum hypothesis and then presenting the governing equations and the conservation of potential vorticity as well as energy and enstrophy, in various approximations.

In Chap. 2, we review the Lagrangian formulation of dynamics starting from Hamilton's Principle of First Action. In the second part of the chapter, Noether's Theorem is presented both for material particles and for continuous systems such as fluids.

In this way, Chap. 1 will serve as an introduction to Fluid and Geophysical Fluid Dynamics to students and researchers of subjects such as physics and mathematics. Chapter 2 will instead serve as an introduction to analytical mechanics to students of applied subjects, such as engineering, climatology, meteorology and oceanography.

In Chap. 3, we first introduce the Lagrangian density for the ideal fluid. The equations of motion are rederived using Hamilton's principle first in the Lagrangian and then in the Eulerian frameworks. The relationship between the two frameworks is thus revealed from the use of canonical transformations. Noether's Theorem is then applied to derive the conservation laws corresponding to the continuous symmetries of the Lagrangian density. Particular attention will be given to the particle relabelling symmetry, and the associated conservation of vorticity.

In Chap. 4, we extend the use of Hamilton's principle to continuously stratified fluids and to uniformly rotating flows. Different sets of approximated equations, which constitute different commonly used approximation in Geophysical Fluid Dynamics, are considered, as well as the form taken by the conservation of potential vorticity in each of them. Finally, the variational methods are applied to study some selected topics of wave dynamics.

Technical derivations of equations that might interrupt the flow of the reading are reported in a number of appendices.

This book should be considered as an elementary introduction. Bibliographical notes at the end of each chapter will guide the reader to more advanced treatments of the subject.

Hamburg, Germany Gualtiero Badin
Trieste, Italy Fulvio Crisciani
April 2017

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Gualtiero Badin

Deep gratitude to my wife Isabella who, in the course of our happy marriage, always had to face the unpleasant aspects of daily life alone, so that I was free to devote myself to physics—thank you.

Fulvio Crisciani

Contents

Chapter 1
Fundamental Equations of Fluid and Geophysical Fluid Dynamics

Abstract The motion of fluids from the smaller to the large scales, i.e. until the oceans and atmospheric currents, is described by a complex interplay of the momentum equations and the equations describing the thermodynamics of the specific system. The resulting set of equations constitutes the branch of physics and applied mathematics called Fluid and Geophysical Fluid Dynamics. The continuum hypothesis and the governing equations of Fluid and Geophysical Fluid Dynamics in their inviscid form are here synthetically reviewed. Emphasis is given to the conservation of energy, enstrophy and potential vorticity, which are written in various approximations. The obtained relationships constitute the basis for the development of the following chapters. Chapter 1 aims thus to give only a résumé of the aspects of Fluid and Geophysical Fluid Dynamics which will be considered from the Lagrangian and Hamiltonian point of view in the other chapters. For this reason, several steps in deriving the governing equations are omitted and only the outlines are mostly reported.

Keywords Fluid dynamics · Geophysical fluid dynamics · Ideal fluid · Conservation laws · Rotating flows · Stratified flows · Potential vorticity · Ertel's theorem · Circulation · Shallow water equations · Quasi-geostrophic equations

1.1 Introduction

Fluid dynamics deals with a wide range of scales of motions, ranging from the micro till the planetary scales, and linked by self-similar laws. Once the laws governing the velocity $\mathbf{u}$, the pressure p and the density ρ of these fluids are established, and the main goal is to understand, in terms of mathematical models suitably idealized, the rich physical phenomenology exhibited by the fluids. The governing equations are based on the continuous distribution of the fields under consideration. At the larger scales, Geophysical Fluid Dynamics deals with large-scale motions of fluids in the oceans (marine currents) and in the atmosphere (winds), as viewed by a terrestrial observer, i.e. by an observer whose frame of reference is fixed with the Earth. On this subject, Joseph Pedlosky [12] said *"One of the key features of*

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Geophysical Fluid Dynamics is the need to combine approximate forms of the basic fluid-dynamical equations of motion with careful and precise analysis. The approximations are required to make any progress possible, while precision is demanded to make the progress meaningful". In this chapter, this continuum hypothesis and the governing equations are synthetically reviewed in the standard (i.e. nonvariational) approach. While the oceans and the atmosphere are made of viscous, and thus dissipative, fluids, we will here concentrate in the nondissipative equations, in order to allow for a Hamiltonian formulation of the dynamics in the following chapters. Notice that not all the approximations derived in this chapter will be rederived from variational principles in the following chapters and vice versa. Some attention will however be dedicated here to additional approximations, as the nonvariational derivation highlights some physical processes that might be of interest especially for the more applied readers.

1.2 The Continuum Hypothesis

Fluid matter is slippery not only from a practical point of view, but also in the attempt to establish principles and laws of classical physics which govern its evolution. For instance, while the dynamics of a pointlike massive body is, basically, that of Newton's second law, the application of the same equation to a fluid according to the *Lagrangian description* looks problematic as far as an operative definition of a "pointlike fluid body" is not established. In order to attribute the velocity and the acceleration to individual bodies of fluid consistently with Newton's second law, the concept of parcel is introduced and defined as a volume of fluid whose amount is the same at any time. Obviously, a parcel is not pointlike but, rather, it has a finite volume which, in general, changes its shape at each time. Hence, one should pose the question: "How large is a parcel?". The answer goes beyond the simple definition of parcel (in the sense that, a priori, its volume and the related mass could be arbitrary) and relies on the possibility to perform measurements, at the macroscopic scale and in the framework of classical physics, on the fluid, i.e. on its parcels. In a measure process, a probe is put into the fluid and the output of the instrument is a number (usually referred to SI units) which comes from the interaction of the probe with the fluid; this number quantifies a physical property of the fluid. The process can be repeated for each point and time. The volume of fluid to which the instrument responds is much larger than the volume in which variations due to molecular fluctuations take place; in this way, an undesired random variability of the output is avoided. Thus, the volume of a parcel, and of the parcels that interact with the probe, is far larger than the typical distance among the molecules of the fluid. This is a lower bound for the volume of a parcel. On the other hand, the uniqueness of the number provided by the instrument in a single measurement means that, in the interaction with the parcel, the probe feels a uniformly distributed property of the fluid. Thus, *the property of the fluid* of the parcel that interacts with the probe, and hence *of every parcel, is spread uniformly over the volume of the parcel*. This is the *continuum hypothesis*, which ultimately relies on the unicity of the response of a process of measurement within the framework of classical physics. By varying the position of the probe, different outputs are expected and usually obtained. This fact poses an upper bound to the volume of the parcels; in fact, the possibility to detect the variations is associated with spatial distribution of physical quantities whenever the probe interacts with different parcels of the same fluid demands that the volume of the parcels be far smaller of the total volume involved in the measure process.

The same measure process described above can be interpreted from a different point of view, by associating with each position of the probe the related numerical output of the measurement and by assuming the possibility to extend ideally this mapping to the whole volume of fluid under investigation. In this way, named *Eulerian description*, the physical property of the fluid is referred, point by point, to the volume that includes it in terms of a field. Although the Eulerian description is not fit for Newton's second law, it is related to the Lagrangian description by a kinematic constraint according to which the field property at a given location and time must equal the property possessed by the parcel occupying that position on that instant.

In the *Lagrangian description*, instead, the fluid looks like a continuous variety of parcels, so, in principle, each parcel can be identified by a fixed tern of labels and by time. As a parcel does not change the labels in the course of motion, the coordinates of the parcel are a function of the labels and of time; in other words, in the *Lagrangian description*, labels and time are independent variables while the coordinates are variables dependent on the labels and on time. Hence, if a quantity is ascribed to a parcel following the motion, the rate of change of the quantity is simply its differentiation with respect to time. Label coordinates refer to a certain label space, while the space coordinates refer to a certain location space, and the relationship between these two spaces is represented by a nonsingular mapping (in each point of the location space, there is a "labelled" parcel, and in each "labelled" point of the label space, definite space coordinates can be attributed to the parcel occupying that point) whose time evolution describes fluid motion. No criterion to assign the labels is established a priori, provided that each parcel keeps the same labels for all time.

1.3 Derivation of the Equations of Motion

1.3.1 Conservation of Mass

By definition, the mass of any parcel is conserved in time. Thus, if $V(t)$ is the material volume of a certain parcel, then

$$\frac{d}{dt}\int_{V(t)} \rho(\mathbf{r}, t)dV' = 0 \,. \tag{1.1}$$

Notice that in the following, the independent variables will sometimes be omitted.

Now, for any scalar θ included in the parcel of volume $V(t)$, the equation

$$\frac{d}{dt}\int_{V(t)} \theta dV' = \int_{V(t)} \left(\frac{D\theta}{Dt} + \theta \; div \; \mathbf{u}\right) dV' \tag{1.2}$$

allows to transfer the time derivative of a space integral inside the space integral itself. A derivation of (1.2) is reported in Appendix A. The velocity $\mathbf{u}$ appearing in the Lagrangian derivative $D/Dt = \partial/\partial t + \mathbf{u} \cdot \nabla$ of (1.2) is the velocity of the parcel. By using (1.2), Eq. (1.1) becomes

$$\int_{V(t)} \left(\frac{D\rho}{Dt} + \rho \; div \; \mathbf{u}\right) dV' = 0 \; . \tag{1.3}$$

Because Eq. (1.3) holds for every parcel, the governing equation of the density field

$$\frac{D\rho}{Dt} + \rho \; div \; \mathbf{u} = 0 \tag{1.4}$$

or, equivalently,

$$\frac{\partial\rho}{\partial t} + div(\rho\mathbf{u}) = 0 \tag{1.5}$$

immediately follows.

1.3.2 Incompressibility and Density Conservation

A fluid is said to be incompressible when the density of parcels is not affected by changes in the pressure. Thus, the rate of change of ρ following the motion is zero

$$\frac{D\rho}{Dt} = 0 \; . \tag{1.6}$$

In other words, parcels move on trajectories along which the density field takes a constant value. If Eq. (1.6) holds true, then (1.4) implies

$$div \; \mathbf{u} = 0 \; , \tag{1.7}$$

which means that the current is solenoidal. In turn, Eq. (1.7) means that every stream tube must be either close, or end on the boundary of the fluid, or extend in a unbounded way in some direction.

1.3.3 Momentum Equation in an Inertial Frame of Reference

Given that the parcel is an individual portion of fluid, in analogy with the Newton's second law for a pointlike massive object, the time derivative of the linear momentum of a parcel is assumed to be equal to the sum of the forces applied to it. The linear momentum of a parcel is defined by $\int_{V(t)} \rho \mathbf{u} dV'$, so its acceleration is given by

$$\frac{d}{dt}\int_{V(t)} \rho \mathbf{u} dV' \,. \tag{1.8}$$

Unlike a pointlike mass moving without interacting with the surrounding matter, the forces applied to a parcel are not only body forces, such as gravity, but also surface forces due to the interaction of each parcel with those surrounding it. The body forces can be represented by the quantity

$$\int_{V(t)} \rho \mathbf{F} dV' \,, \tag{1.9}$$

where $\mathbf{F}$ is a force per unit mass that includes gravity acceleration $-g\hat{\mathbf{k}}$, where we have used the standard notation in which $\left(\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}\right)$ indicate the orthogonal unit vectors for the (x, y, z) tern. In the following, we will also indicate with $\left(\hat{\mathbf{n}}, \hat{\mathbf{t}}\right)$ the normal and the tangent unit vectors at a certain point of a material surface, respectively. The fundamental surface force mainly comes from the pressure reciprocally exerted at the boundary of the parcels in contact. The parcel included into $V(t)$ experiences the force

$$-\int_{V(t)} \nabla p dV' \,, \tag{1.10}$$

where $p = p(\mathbf{r}, t)$ is the pressure field. By using (1.8)–(1.10), Newton's second law results in the equation

$$\frac{d}{dt}\int_{V(t)} \rho \mathbf{u} dV' = \int_{V(t)} (\rho \mathbf{F} - \nabla p)\, dV' \,. \tag{1.11}$$

The l.h.s of (1.11) can be rearranged using (1.2) according to the chain of equalities

$$\begin{aligned}
\frac{d}{dt}\int_{V(t)} \rho \mathbf{u} dV' &= \int_{V(t)} \left[\frac{D}{Dt}(\rho \mathbf{u}) + \rho \mathbf{u}\, div\, \mathbf{u}\right] dV' \\
&= \int_{V(t)} \left[\mathbf{u}\frac{D\rho}{Dt} + \rho \frac{D\mathbf{u}}{Dt} + \rho \mathbf{u}\, div\, \mathbf{u}\right] dV' \\
&= \int_{V(t)} \left[\rho \frac{D\mathbf{u}}{Dt} + \mathbf{u}\left(\frac{D\rho}{Dt} + \rho\, div\, \mathbf{u}\right)\right] dV' \,.
\end{aligned} \tag{1.12}$$

Recalling (1.4)–(1.5), Eq. (1.12) simplifies into

$$\frac{d}{dt}\int_{V(t)} \rho \mathbf{u} dV' = \int_{V(t)} \rho \frac{D\mathbf{u}}{Dt} dV' \tag{1.13}$$

so, with the aid of (1.13), Eq. (1.11) becomes

$$\int_{V(t)} \left(\rho \frac{D\mathbf{u}}{Dt} - \rho \mathbf{F} + \nabla p \right) dV' = 0 \, . \tag{1.14}$$

After a trivial rearrangement and the use of position $\mathbf{F} = -g^* \hat{\mathbf{k}}$, Eq. (1.14) yields the so-called *Euler's equation* in the form

$$\frac{D\mathbf{u}}{Dt} = -\frac{\nabla p}{\rho} - g^* \hat{\mathbf{k}} \, . \tag{1.15}$$

Equation (1.15) looks fit for flows of a massive ($-g^* \hat{\mathbf{k}} \neq \mathbf{0}$) but nonrotating Earth or, more realistically, for flows that do not feel Earth's rotation. This point will be clarified at the end of Sect. 1.5.

1.4 Elementary Symmetries of the Euler's Equation

Consider the Euler's equation (1.15) in absence of the body force $\mathbf{F}$,

$$\frac{D\mathbf{u}}{Dt} = -\frac{\nabla p}{\rho} \, . \tag{1.16}$$

Equation (1.16) is invariant under the symmetry transformation

$$\mathbf{r} \to \mathbf{r}' = g_r(\mathbf{r}) \, , \tag{1.17a}$$

$$t \to t' = g_t(t) \, , \tag{1.17b}$$

$$p \to p' = g_p(p) \, , \tag{1.17c}$$

if it satisfies

$$\frac{D\mathbf{u}}{Dt} = -\frac{\nabla p}{\rho} \Rightarrow \frac{D\mathbf{u}'}{Dt'} = -\frac{\nabla' p'}{\rho} \, , \tag{1.18}$$

i.e. if the equations of motion do not change under the transformation (1.17a)–(1.17c). In (1.17a)–(1.17c), $g_r \in G_r$, $g_t \in G_t$, $g_p \in G_p$ are symmetry transformations that belong to the one-parameter groups G_r , G_t, G_p. Notice that the transformation of the pressure field is

$$g_p(p) = p + \bar{p} \quad \text{or} \quad g_p(p) = Cp \; , \tag{1.19}$$

and it is thus determined through the identification of either $\bar{p}$ or the nondimensional constant C so that the transformed pressure field depends on the specific case under consideration and satisfies the invariance of the original equation. In certain cases, g_p will be a function of the space and time coordinates.

For the following symmetries, the theses and proofs will proceed through the statement of the symmetric transformation on the independent variables. The corresponding transformations of the velocity field and on the time and space derivatives are thus determined directly.

1.4.1 Continuous Symmetries

Equation (1.16) satisfies the following continuous symmetries:

1.4.1.1 Gauge Invariance for the Pressure Field

$$g_r(\mathbf{r}) = \mathbf{r} \; , \tag{1.20a}$$

$$g_t(t) = t \; , \tag{1.20b}$$

$$g_p(p) = p + F(t) \; , \tag{1.20c}$$

where $F(t)$ is an arbitrary function of time. Observing that the transformation does not act on the independent variables t, $\mathbf{r}$ and on the dependent variable $\mathbf{u}$, the invariance is trivially proved upon substitution of (1.20a)–(1.20c) in (1.16).

1.4.1.2 Space Translations

$$g_r(\mathbf{r}) = \mathbf{r} + \mathbf{c} \; , \tag{1.21a}$$

$$g_t(t) = t \; , \tag{1.21b}$$

$$g_p(p) = p \; , \tag{1.21c}$$

where $\mathbf{c} \in \mathbb{R}^3$ is a constant vector.

Proof Time differentiation of (1.21a) shows that the transformation does not act on the velocity field, so that $\mathbf{u}' = \mathbf{u}$. Consider, for simplicity and without loss of generality, the space translation in the x direction $x' = x + c$. Because c is a constant,

$$\frac{\partial}{\partial x'} = \frac{\partial}{\partial x} \; . \tag{1.22}$$

And the invariance of (1.16) under (1.21a)–(1.21c) is proved upon substitution and setting $\bar{p} = 0$, or $C = 1$.

Importantly, it should be noted that (1.16) is invariant also under nonuniform translations

$$g_r(\mathbf{r}) = \mathbf{r} + \mathbf{c}(t) \ , \tag{1.23a}$$
$$g_t(t) = t \ , \tag{1.23b}$$
$$g_p(p) = p - \rho \mathbf{r} \frac{d^2\mathbf{c}}{dt^2} \ . \tag{1.23c}$$

Proof Start by noting that (1.23a) yields

$$\mathbf{u}' = \mathbf{u} + \frac{d\mathbf{c}}{dt} \ . \tag{1.24}$$

Using index notation, (1.23a), (1.23b) can be written as

$$g_x(x_i) = x_i + c_i(t) \ , \tag{1.25a}$$
$$g_t(t) = t \ . \tag{1.25b}$$

Equation (1.16) and the incompressibility Eq. (1.7), using Einstein's summation over repeated indices, take the form

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho} \frac{\partial p}{\partial x_i} \ , \tag{1.26}$$
$$\frac{\partial u_i}{\partial x_i} = 0 \ . \tag{1.27}$$

Time differentiation of the first equation of (1.25a) yields

$$u_i' = u_i + \frac{dc_i}{dt} \ . \tag{1.28}$$

Under (1.25a), (1.25b), the space derivatives transform as

$$\begin{aligned}\frac{\partial}{\partial x_i} &= \frac{\partial x_j'}{\partial x_i} \frac{\partial}{\partial x_j'} \\ &= \frac{\partial \left(x_j + c_j\right)}{\partial x_i} \frac{\partial}{\partial x_j'} \\ &= \delta_{ij} \frac{\partial}{\partial x_j'} \\ &= \frac{\partial}{\partial x_i'} \ ,\end{aligned} \tag{1.29}$$

where δ_{ij} is the Kronecker delta symbol, while the time derivative transforms as

$$\begin{aligned}\frac{\partial}{\partial t} &= \frac{\partial t'}{\partial t}\frac{\partial}{\partial t'} + \frac{\partial x_i'}{\partial t}\frac{\partial}{\partial x_i'} \\ &= \frac{\partial}{\partial t'} + \frac{\partial\left(x_i + c_i\right)}{\partial t}\frac{\partial}{\partial x_i'} \\ &= \frac{\partial}{\partial t'} + \frac{dc_i}{dt}\frac{\partial}{\partial x_i'} \; . \end{aligned} \tag{1.30}$$

Substitution in (1.16) gives

$$\left(\frac{\partial}{\partial t'} + \frac{dc_i}{dt}\frac{\partial}{\partial x_i'}\right)\left(u_i' - \frac{dc_i}{dt}\right) + \left(u_j' - \frac{dc_j}{dt}\right)\frac{\partial}{\partial x_j'}\left(u_i' - \frac{dc_i}{dt}\right) = -\frac{1}{\rho}\frac{\partial}{\partial x_j'}\left(p' + \bar{p}\right) \; . \tag{1.31}$$

After multiplications, (1.31) yields

$$\frac{\partial u_i'}{\partial t'} + u_j'\frac{\partial u_i'}{\partial x_j'} + \frac{1}{\rho}\frac{\partial p'}{\partial x_j'} = -\frac{d^2 c_i}{dt^2} - \frac{1}{\rho}\frac{\partial \bar{p}}{\partial x_j'} \; . \tag{1.32}$$

Using (1.26), the l.h.s. of (1.32) is zero, so that (1.32) reduces to (1.16) if

$$\bar{p} = -\rho x_j \frac{d^2 c_j}{dt^2} \; , \tag{1.33}$$

or, returning to vector notation,

$$\bar{p} = -\rho \mathbf{r} \cdot \frac{d^2 \mathbf{c}}{dt^2} \; , \tag{1.34}$$

proving the invariance of (1.16) under (1.23a), (1.23b) and (1.24). Notice that in this case, the correction term to the pressure field, $\bar{p}$, depends on $\mathbf{r}$.

1.4.1.3 Time Translations

$$g_r(\mathbf{r}) = \mathbf{r} \; , \tag{1.35a}$$

$$g_t(t) = t + \tau \; , \tag{1.35b}$$

$$g_p(p) = p \; , \tag{1.35c}$$

where $\tau \in [0, +\infty)$ is a constant.

Proof As for the space translation invariance, time differentiation of (1.35a) shows that the transformation does not act on the velocity field, so that $\mathbf{u}' = \mathbf{u}$. The invariance is thus trivially proved to hold for $\bar{p} = 0$ or $C = 1$ upon substitution of (1.35a)–(1.35c) in (1.16) and noting that both the spatial and the time derivatives remain unchanged

$$\nabla' = \nabla \ , \tag{1.36a}$$

$$\frac{\partial}{\partial t'} = \frac{\partial}{\partial t} \ . \tag{1.36b}$$

1.4.1.4 Invariance Under Galilean Transformations

$$g_r(\mathbf{r}) = \mathbf{r} + \mathbf{U}t \ , \tag{1.37a}$$

$$g_t(t) = t \ , \tag{1.37b}$$

$$g_p(p) = p \ , \tag{1.37c}$$

where $\mathbf{U} \in \mathbb{R}^3$ is a constant velocity.

Proof Time differentiation of (1.37a) yields

$$\mathbf{u}' = \mathbf{u} + \mathbf{U} \ . \tag{1.38}$$

To prove the invariance of (1.16), consider, for simplicity and without loss of generality, translations along the x direction, so that $x' = x + Ut$. The time derivative transforms as

$$\begin{aligned} \frac{\partial}{\partial t'} &= \frac{\partial t}{\partial t'}\frac{\partial}{\partial t} + \frac{\partial x}{\partial t}\frac{\partial}{\partial x} \\ &= \frac{\partial}{\partial t} + \frac{\partial (x' - Ut)}{\partial t}\frac{\partial}{\partial x} \\ &= \frac{\partial}{\partial t} - U\frac{\partial}{\partial x} \ . \end{aligned} \tag{1.39}$$

The time derivative for the general case transforms thus as

$$\frac{\partial}{\partial t'} = \frac{\partial}{\partial t} - \mathbf{U} \cdot \nabla \ , \tag{1.40}$$

while the advective term in the Lagrangian derivative transforms as

$$\left(\mathbf{u}' \cdot \nabla'\right) \mathbf{u}' = (\mathbf{u} \cdot \nabla)\, \mathbf{u} + (\mathbf{U} \cdot \nabla)\, \mathbf{u} \ . \tag{1.41}$$

Due to the mutual cancellation of last terms on the r.h.s. of (1.40) and (1.41), the Galilean covariance of (1.16) is proved.

1.4.1.5 Rotations

$$g_r(\mathbf{r}) = \mathbf{Rr} \,, \tag{1.42a}$$
$$g_t(t) = t \,, \tag{1.42b}$$
$$g_p(p) = p \,, \tag{1.42c}$$

where $\mathbf{R} \in SO(3)$ is the three-dimensional orthogonal rotation matrix, with property

$$det\ \mathbf{R} = 1 \,. \tag{1.43}$$

Proof Time differentiation of (1.42a) yields

$$\mathbf{u}' = \mathbf{Ru} \,. \tag{1.44}$$

Because the pressure field is a scalar, the pressure term remains unaltered

$$p' = p \,, \tag{1.45}$$

i.e. $\bar{p} = 0$ or $C = 1$. Using again index notation and Einstein's summation over repeated indices,

$$x_i = R_{ji} x'_j \,, \tag{1.46}$$
$$u_i = R_{ji} u'_j \,, \tag{1.47}$$

and, because of (1.43),

$$x'_i = R_{ij} x_j \,, \tag{1.48}$$
$$u'_i = R_{ij} u_j \,. \tag{1.49}$$

Then, the space derivatives transform as

$$\begin{aligned}
\frac{\partial}{\partial x_i} &= \frac{\partial x'_j}{\partial x_i} \frac{\partial}{\partial x'_j} \\
&= \frac{\partial \left(R_{jk} x_k \right)}{\partial x_i} \frac{\partial}{\partial x'_j} \\
&= R_{jk} \frac{\partial x_k}{\partial x_i} \frac{\partial}{\partial x'_j} \\
&= R_{jk} \delta_{ki} \frac{\partial}{\partial x'_j} \\
&= R_{ji} \frac{\partial}{\partial x'_j} \,,
\end{aligned} \tag{1.50}$$

Equations (1.42a)–(1.42c) and (1.50) thus yield, from (1.16)

$$\begin{aligned}\frac{\partial u_i}{\partial t}+u_j\frac{\partial u_i}{\partial x_j}+\frac{1}{\rho}\frac{\partial p}{\partial x_i}&=\frac{\partial}{\partial t'}\left(R_{ki}u'_k\right)+\left(R_{mj}u'_m\right)R_{nj}\frac{\partial}{\partial x'_n}\left(R_{ki}u'_k\right)+\frac{1}{\rho}R_{ki}\frac{\partial p'}{\partial x'_k}\\&=R_{ki}\left[\frac{\partial u'_k}{\partial t}+\left(R_{mj}R_{nj}\right)u'_m\frac{\partial u'_k}{\partial x'_n}+\frac{1}{\rho}\frac{\partial p'}{\partial x'_k}\right]\\&=0\,.\end{aligned}\tag{1.51}$$

Equation (1.51) yields

$$\frac{\partial u'_k}{\partial t}+\left(R_{mj}R_{nj}\right)u'_m\frac{\partial u'_k}{\partial x'_n}+\frac{1}{\rho}\frac{\partial p'}{\partial x'_k}=0\,.\tag{1.52}$$

Noting that

$$\left[R_{mj}R_{nj}\right]_{mn}=\delta_{mn}\,,\tag{1.53}$$

Equation (1.52) reduces to (1.16).

1.4.2 Discrete Symmetries

The continuous symmetries discussed in the previous section are satisfied also in the case of discrete transformations. Further, we consider here some additional discrete symmetries, namely the time-reversal symmetry, the symmetry by parity reflections and the symmetry by scaling invariance.

1.4.2.1 Time Reversal

$$g_r(\mathbf{r})=\mathbf{r}\,,\tag{1.54a}$$
$$g_t(t)=-t\,,\tag{1.54b}$$
$$g_p(p)=p\,.\tag{1.54c}$$

Proof Direct application of (1.54b) implies

$$\frac{\partial}{\partial t'}=-\frac{\partial}{\partial t}\,.\tag{1.55}$$

In turn, the time differentiation of (1.54a) yields

$$\mathbf{u}'=-\mathbf{u}\,.\tag{1.56}$$

The invariance of (1.16) under the symmetry transformation (1.54a)–(1.54c) is thus proved upon substitution, with $\bar{p} = 0$ or $C = 1$.

1.4.2.2 Parity

$$g_r(\mathbf{r}) = -\mathbf{r} \ , \tag{1.57a}$$

$$g_t(t) = t \ , \tag{1.57b}$$

$$g_p(p) = p \ . \tag{1.57c}$$

Proof The time differentiation of (1.57a) yields

$$\mathbf{u}' = -\mathbf{u} \ . \tag{1.58}$$

Noting that (1.57a) implies the transformation of the gradient operator as

$$\nabla' = -\nabla \ , \tag{1.59}$$

substitution in (1.16) proves the symmetry of the system for $\bar{p} = 0$ or $C = 1$.

1.4.2.3 Scaling Invariance

$$g_r(\mathbf{r}) = a\mathbf{r} \ , \tag{1.60a}$$

$$g_t(t) = bt \ , \tag{1.60b}$$

$$g_p(p) = \left(\frac{a}{b}\right)^2 p \ , \tag{1.60c}$$

where $a, b \in \mathbb{R},\ a, b \neq 0$.

Proof Under (1.60a)–(1.60b), the partial derivatives transform as

$$\nabla \to \nabla' = \frac{1}{a}\nabla \ , \tag{1.61}$$

$$\frac{\partial}{\partial t} \to \frac{\partial}{\partial t'} = \frac{1}{b}\frac{\partial}{\partial t} \ , \tag{1.62}$$

Further, time differentiation of (1.60a) yields

$$\mathbf{u}' = \frac{a}{b}\mathbf{u} \ . \tag{1.63}$$

With the position $p' = Cp$, substitution into (1.16) yields

$$\frac{b^2}{a}\frac{D\mathbf{u}'}{Dt'} = -\frac{a}{C}\frac{\nabla' p'}{\rho} , \tag{1.64}$$

which implies that (1.16) is invariant under (1.60a)–(1.60c) for

$$C = \left(\frac{a}{b}\right)^2 . \tag{1.65}$$

1.4.3 Role of Gravity in Breaking the Symmetries of the Euler's Equation

What happens when the gravitational force $\mathbf{F} = -g\hat{\mathbf{k}}$ is reinserted in the Euler's equation? The additional term introduces a nonhomogeneity in the vertical component of the equation of motion. In the horizontal directions, (1.15) retains thus the symmetries of (1.16). In the vertical direction, however, for the continuous symmetries, the nonhomogeneous term acts to break the rotational symmetry. Indicating with the subscript 3, the vertical component of the velocity with the addition of the gravity acceleration (1.51) can be written as

$$\begin{aligned}\frac{\partial u_3}{\partial t} + u_j\frac{\partial u_3}{\partial x_j} + \frac{1}{\rho}\frac{\partial p}{\partial x_3} + g &= \frac{\partial}{\partial t'}\left(R_{k3}u'_k\right) + \left(R_{mj}u'_m\right)R_{nj}\frac{\partial}{\partial x'_n}\left(R_{k3}u'_k\right) + \frac{1}{\rho}R_{k3}\frac{\partial p'}{\partial x'_k} + g \\ &= R_{k3}\left[\frac{\partial u'_k}{\partial t'} + \left(R_{mj}R_{nj}\right)u'_m\frac{\partial u'_k}{\partial x'_n} + \frac{1}{\rho}\frac{\partial p'}{\partial x'_k}\right] + g \\ &= 0 .\end{aligned} \tag{1.66}$$

It is thus visible that it is not possible to factorize the term R_{k3} from the gravity term and thus to satisfy the symmetry under rotations of the Euler's equation.

The nonhomogeneous term acts to break also all the discrete symmetries in the vertical direction. To prove this statement for the time-reversal and parity symmetries, it is necessary to note only that the gravity acceleration term does not change sign under the transformations (1.54a)–(1.54c) and (1.57a)–(1.57c), respectively. To prove instead the lack of scaling invariance, it is simply possible to notice that (1.60a)–(1.60c) does not act on the gravity acceleration.

1.5 Momentum Equation in a Uniformly Rotating Frame of Reference

An observer fixed with the Earth rotates uniformly at the rate Ω around the constant vector $\mathbf{\Omega}$. This reference is openly not inertial, so Eqs. (1.4) and (1.15) must be reformulated in a uniformly rotating frame. Consider an inertial frame located, only for mathematical simplicity, in the centre of the Earth and a parcel $P\left(\mathbf{r}(t)\right)$ where

$\mathbf{r}$ is the position vector of the parcel (embedded in the ocean or in the atmosphere) with respect to Earth's centre. Let $\mathbf{u}_I(t)$ and $\mathbf{u}_R(t)$ be the velocity of the same parcel P detected in the inertial system (subscript I) and in a system fixed with the Earth (subscript R), thus uniformly rotating according to $\boldsymbol{\Omega}$. The system fixed with the Earth is a right-handed Cartesian one, with the z-axis antiparallel to the local gravity and the plane (x, y) tangent to the terrestrial sphere in a given point, of latitude ϕ_0. In the Northern Hemisphere, the y-axis points northward (and therefore the x-axis points eastward). In the Southern Hemisphere, the y-axis points southward to form again a right-handed Cartesian tern. The vectors $\mathbf{u}_I(t)$ and $\mathbf{u}_R(t)$ turn out to be linked by the relationship

$$\mathbf{u}_I(t) = \mathbf{u}_R(t) + \boldsymbol{\Omega} \times \mathbf{r}(t) \; . \tag{1.67}$$

In particular, $\mathbf{u}_R = (u_R, v_R, w_R) = (Dx/Dt,\ Dy/Dt,\ Dz/Dt)$. Likewise, the acceleration of P in the inertial system, say $(D\mathbf{u}_I/Dt)_I$, is linked to the acceleration in a system fixed with the Earth, say $(D\mathbf{u}_R/Dt)_R$, by the relationship

$$\left(\frac{D\mathbf{u}_I}{Dt}\right)_I = \left(\frac{D\mathbf{u}_R}{Dt}\right)_R + 2\boldsymbol{\Omega} \times \mathbf{u}_R + \boldsymbol{\Omega} \times [\boldsymbol{\Omega} \times \mathbf{r}(t)] \; . \tag{1.68}$$

In the transformation law for the acceleration (1.68), the fundamental new term is Coriolis' acceleration $2\boldsymbol{\Omega} \times \mathbf{u}_R$ which takes place whenever $\mathbf{u}_R \neq 0$. The related force per unit mass is the further body force, besides gravity, which is typical of Geophysical Fluid Dynamics. The importance of Coriolis' acceleration is explained by evaluating the ratio $|D\mathbf{u}_R \,/\, Dt| \,/\, |2\boldsymbol{\Omega} \times \mathbf{u}_R|$ whose typical value, named Rossby number (ε), is mostly lesser than unit on geophysical scales. This result shows that Coriolis' acceleration may be prevalent on the local acceleration, as actually happens in the evolution of the geostrophic current $\mathbf{u}_G = \hat{\mathbf{k}} \times \nabla p \,/\, 2\Omega\rho$.

Unlike the momentum equation, both the Lagrangian derivative and the gradient operator are covariant in passing from an inertial to a uniformly rotating frame of reference, that is

$$\left(\frac{D}{Dt}\right)_I = \left(\frac{D}{Dt}\right)_R \quad \text{and} \quad \nabla_I = \nabla_R \; . \tag{1.69}$$

By using Eqs. (1.67)–(1.69), the continuity equation turns out to retain its form (1.5) also in a uniformly rotating system

$$\frac{D\rho}{Dt} + \rho \; div \, \mathbf{u}_R = 0 \tag{1.70}$$

or, equivalently,

$$\frac{D\rho}{Dt} + div \, (\rho \mathbf{u}_R) = 0 \; . \tag{1.71}$$

By using again (1.67)–(1.69), the momentum Eq. (1.15) becomes

$$\frac{\partial \mathbf{u}_R}{\partial t} + (\mathbf{u}_R \cdot \nabla)\, \mathbf{u}_R + 2\boldsymbol{\Omega} \times \mathbf{u}_R + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}) = -\frac{\nabla p}{\rho} - g^* \hat{\mathbf{k}}\,. \tag{1.72}$$

Because in Geophysical Fluid Dynamics all the fields are referred only to systems fixed with the Earth, the subscript R can be dropped without ambiguity from (1.70)–(1.72).

In the absence of rotation, the gravitational acceleration $\mathbf{g}^*$ alone would tend to form a spherical planet, while, in a rotating one, the combined gravitational ($\mathbf{g}^*$) and centrifugal ($-\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r})$) accelerations cause a planet with a flattened ellipsoidal shape. Things adjust themselves in such a way that the resulting acceleration has the direction of the local vertical, say $\hat{\mathbf{k}}$, so that $g\hat{\mathbf{k}} = \mathbf{g}^* + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r})$. As a consequence, the surface of the planet is a geopotential (Φ) surface: every particle at rest on a surface of equation $\Phi = const$ will remain at rest unless it undergoes additional forces. Now, the direction of the geopotential is used to define the local vertical direction $\hat{\mathbf{k}}$ and to regard the surfaces of equation $\Phi = const$ as they were true spheres. So a local Cartesian reference can be established in the usual way and the f-plane or β-plane approximations can be introduced [15]. Thus, the governing equations are written as

$$\frac{D\rho}{Dt} + \rho\, div\, \mathbf{u} = 0\,, \tag{1.73}$$

$$\frac{D\mathbf{u}}{Dt} + 2\boldsymbol{\Omega} \times \mathbf{u} = -\frac{\nabla p}{\rho} - g\hat{\mathbf{k}}\,. \tag{1.74}$$

In systems, such as streams and tornadoes, where the Rossby number is far larger than unity, the horizontal pressure gradient equilibrates the local acceleration rather than Coriolis acceleration. In this case, the dominant dynamic balance is that of Euler's equation (1.15) in spite of Earth's rotation.

1.5.1 Vorticity Equation

Equations (1.72) and (1.74) are the basis to establish the governing equation of a scalar which plays a very important role in Geophysical Fluid Dynamics, that is potential vorticity, and in finding the conditions, expressed by Ertel's theorem, for the conservation of this scalar. Some preliminaries are in order.

The relative vorticity $\boldsymbol{\omega}$ is, by definition, the curl of the velocity

$$\boldsymbol{\omega} = rot\ \mathbf{u}, \tag{1.75}$$

while the absolute vorticity $\boldsymbol{\omega}_a$ is the sum of the relative vorticity (1.75) plus the planetary vorticity $2\boldsymbol{\Omega}$, that is

$$\boldsymbol{\omega}_a = \boldsymbol{\omega} + 2\boldsymbol{\Omega}\,. \tag{1.76}$$

According to position (1.76), the total vorticity takes into account both the contribution of relative vorticity due to the shear of the current, or of the wind, and the contribution arising from the rotation of the fluid Earth, as a whole, around $\mathbf{\Omega}$. For future purposes, it is useful to recall the component of absolute vorticity on the z-axis, that is $\boldsymbol{\omega}_a \cdot \hat{\mathbf{k}}$, which is given by

$$\boldsymbol{\omega}_a \cdot \hat{\mathbf{k}} = \boldsymbol{\omega} \cdot \hat{\mathbf{k}} + 2\mathbf{\Omega} \cdot \hat{\mathbf{k}} \approx \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} + f_0 + \beta_0 y \, . \tag{1.77}$$

In (1.77), $(\partial v/\partial x) - (\partial u/\partial y)$ is the component of $rot\ \mathbf{u}$ on the z-axis. To explain the quantity $f_0 + \beta_0 y$ of the same equation, start from the equation $2\mathbf{\Omega} \cdot \hat{\mathbf{k}} = 2\Omega \sin(\phi)$, where ϕ is the latitude of the parcel which (1.76) refers to. If ϕ is not too far from the latitude ϕ_0 of the contact point between the surface of the Earth and the Cartesian plane, then $\sin(\phi) \approx \sin(\phi_0) + (\phi - \phi_0)\cos(\phi_0)$. By introducing the Coriolis parameter $f_0 = 2\Omega \sin(\phi_0)$, the planetary vorticity gradient $\beta_0 = 2\Omega \cos(\phi_0) / R$, where R is the mean radius of the Earth, and the approximation $\phi - \phi_0 \approx y / R$, one obtains

$$\begin{aligned} 2\mathbf{\Omega} \cdot \hat{\mathbf{k}} &= 2\Omega \sin(\phi) \\ &\approx 2\Omega \sin(\phi_0) + (\phi - \phi_0)\, 2\Omega \cos(\phi_0) \\ &\approx f_0 + \frac{2\Omega}{R} y \cos(\phi_0) \\ &= f_0 + \beta_0 y \, . \end{aligned} \tag{1.78}$$

Equation (1.78) shows that

$$2\mathbf{\Omega} \cdot \hat{\mathbf{k}} \approx f_0 + \beta_0 y \, , \tag{1.79}$$

in accordance with (1.77). The approximation

$$2\mathbf{\Omega} \cdot \hat{\mathbf{k}} \approx f_0 \tag{1.80}$$

defines the dynamics in the so-called f-plane. The approximation (1.79) defines instead the dynamics in the so-called β-plane.

After these preliminaries, consider the application of the operator *rot* to (1.74). The result is

$$\frac{D\boldsymbol{\omega}_a}{Dt} = (\boldsymbol{\omega}_a \cdot \nabla)\, \mathbf{u} - \boldsymbol{\omega}_a\, div\, \mathbf{u} - \frac{\nabla p \times \nabla \rho}{\rho^2} \, . \tag{1.81}$$

1.5.2 Planar Flows with Constant Density

A special case of Eq. (1.81) is that in which the fluid density is a constant and the flow is planar. Under these hypotheses, each term at the r.h.s of (1.81) identically vanishes so this equation becomes

$$\frac{D}{Dt}\left(\boldsymbol{\omega} + 2\boldsymbol{\Omega}\right) = 0 \; . \tag{1.82}$$

Assume the motion on the (x, y) plane. Then, fluid incompressibility $\partial u/\partial x + \partial v/\partial y = 0$ is satisfied by setting

$$\mathbf{u} = \hat{\mathbf{k}} \times \nabla\psi \; , \tag{1.83}$$

where ψ is a certain differentiable *stream function*. From (1.81) and (1.83), one obtains

$$\boldsymbol{\omega} = \left(\nabla^2\psi\right)\hat{\mathbf{k}} \; . \tag{1.84}$$

The use of (1.84) together with (1.79) into (1.82) results in the equation

$$\frac{D}{Dt}\left(\nabla^2\psi + \beta_0 y\right) = 0 \; . \tag{1.85}$$

The latter equation yields the evolution law of the stream function and hence of the flow $\mathbf{u}$. Noting that (1.83) implies $\mathbf{u}\cdot\nabla = -\partial\psi/\partial y\ \partial/\partial x + \partial\psi/\partial x\ \partial/\partial y$, Eq. (1.85) can be finally restated, in terms of the *Jacobian determinant* $J(\cdot,\cdot)$, according to the more usual form

$$\frac{\partial}{\partial t}\nabla^2\psi + J\left(\psi, \nabla^2\psi + \beta_0 y\right) = 0 \; . \tag{1.86}$$

For a list of basic properties of the Jacobian determinant, see for instance the Appendix A of [2].

Equation (1.86) is dimensional. For reasons of homogeneity with the rest of the book, the derivation of the nondimensional version of (1.86) is in order. This is achieved by introducing the relationships

$$(x, y) = L(x', y'),\; t = (L/U)t',\; \psi = UL\psi',\; \nabla^2 = \nabla'^2/L^2,\; J = J'/L^2 \; , \tag{1.87}$$

where $L,\ U$ are representative of the scale of the motion taken into account, while the quantities with apex are nondimensional. In terms of the nondimensional variables, after some trivial rearrangements Eq. (1.86) becomes

$$\frac{\partial}{\partial t'}\nabla'^2\psi' + J'\left(\psi', \nabla'^2\psi' + \frac{\beta_0 L^2}{U}y'\right) = 0 \; . \tag{1.88}$$

Finally, by resorting to the nondimensional parameter

$$\beta = \frac{\beta_0 L^2}{U} \; , \tag{1.89}$$

the nondimensional version of (1.86) is obtained from (1.88) in the form

$$\frac{\partial}{\partial t}\nabla^2\psi + J\left(\psi, \nabla^2\psi + \beta y\right) = 0\,. \tag{1.90}$$

once the apices have been dropped. Thus, the only formal difference between (1.86) and (1.90) lies in the use of (1.89) in the latter equation in place of in the former one.

1.6 Elementary Symmetries of the Vorticity Equation

In order to see how the symmetries of the Euler's equation are modified in a rotating framework, consider now the two-dimensional vorticity Eq. (1.90) in the f-plane,

$$\frac{\partial}{\partial t}\nabla^2\psi + J\left(\psi, \nabla^2\psi\right) = 0\,, \tag{1.91}$$

with (1.83), i.e.

$$u = -\frac{\partial\psi}{\partial y}\,, \tag{1.92a}$$

$$v = \frac{\partial\psi}{\partial x}\,. \tag{1.92b}$$

In analogy with the proofs of the invariance of the Euler's equation, Eq. (1.91) is invariant under the symmetry transformations $g_t \in G_t$, $g_r(\mathbf{r})$, $g_p \in G_p$

$$\mathbf{r} \to \mathbf{r}' = g_r(\mathbf{r})\,, \tag{1.93a}$$

$$t \to t' = g_t(t)\,, \tag{1.93b}$$

$$\psi \to \psi' = g_p(\psi)\,, \tag{1.93c}$$

if it satisfies the formal invariance

$$\frac{\partial}{\partial t}\nabla^2\psi + J\left(\psi, \nabla^2\psi\right) = 0 \;\Rightarrow\; \frac{\partial}{\partial t'}\nabla'^2\psi' + J'\left(\psi', \nabla'^2\psi'\right) = 0\,. \tag{1.94}$$

As for the Euler's equations, the transformation of the stream function is determined through the identification of the field $\bar{\psi}$ so that the transformed stream function $\psi' = \psi + \bar{\psi}$ or $\psi' = C\psi$, depending on the specific case under consideration, satisfies the invariance of the original equation.

Once again, for the following symmetries, the theses and proofs will proceed though the statement of the symmetric transformation on the independent variables.

1.6.1 Continuous Symmetries

1.6.1.1 Gauge Symmetry for the Stream Function

$$g_r(\mathbf{r}) = \mathbf{r} \ , \tag{1.95a}$$

$$g_t(t) = t \ , \tag{1.95b}$$

$$g_p(\psi) = \psi + F(t) \ , \tag{1.95c}$$

where $F(t)$ is an arbitrary function of time. The invariance is trivially proved noting that the stream function enters the vorticity equation only within space derivatives.

1.6.1.2 Space Translations

$$g_r(\mathbf{r}) = \mathbf{r} + \mathbf{c} \ , \tag{1.96a}$$

$$g_t(t) = t \ , \tag{1.96b}$$

$$g_p(\psi) = \psi \ , \tag{1.96c}$$

where $\mathbf{c} \in \mathbb{R}^2$ is a constant vector. Direct substitution of (1.96a)–(1.96c) in (1.91) shows that the symmetry is proved for $\bar{\psi} = 0$ or $C = 1$.

1.6.1.3 Time Translations

$$g_r(\mathbf{r}) = \mathbf{r} \ , \tag{1.97a}$$

$$g_t(t) = t + \tau \ , \tag{1.97b}$$

$$g_p(\psi) = \psi \ , \tag{1.97c}$$

where $\tau \in [0, +\infty)$ is a constant. Also in this case, direct substitution shows that the symmetry is proved for $\bar{\psi} = 0$ or $C = 1$.

1.6.1.4 Invariance Under Galilean Transformations

$$g_x(x) = x + F(t) \ , \tag{1.98a}$$

$$g_y(y) = y \ , \tag{1.98b}$$

$$g_t(t) = t \ , \tag{1.98c}$$

$$g_p(\psi) = \psi - \frac{dF(t)}{dt} y \ , \tag{1.98d}$$

and

$$g_x(x) = x \ , \tag{1.99a}$$

$$g_y(y) = y + F(t) \ , \tag{1.99b}$$

$$g_t(t) = t \ , \tag{1.99c}$$

$$g_p(\psi) = \psi + \frac{dF(t)}{dt}x \ , \tag{1.99d}$$

where $F(t)$ is an arbitrary function of time.

Proof Consider (1.98a)–(1.98d). The space derivatives are invariant under Galilean transformation

$$\frac{\partial}{\partial x} \rightarrow \frac{\partial}{\partial x'} = \frac{\partial x}{\partial x'}\frac{\partial}{\partial x} = \frac{\partial\left(x' - F(t)\right)}{\partial x'}\frac{\partial}{\partial x} = \frac{\partial}{\partial x} \ , \tag{1.100}$$

$$\frac{\partial}{\partial y} \rightarrow \frac{\partial}{\partial y'} = \frac{\partial}{\partial y} \ . \tag{1.101}$$

Setting $\psi' = \psi + \bar{\psi}$, the zonal velocity is

$$u' = -\frac{\partial\psi'}{\partial y'} = -\frac{\partial\psi}{\partial y'} - \frac{\partial\bar{\psi}}{\partial y'} = -\frac{\partial\psi}{\partial y} - \frac{\partial\bar{\psi}}{\partial y} = u - \frac{\partial\bar{\psi}}{\partial y'} \ . \tag{1.102}$$

At the same time, the time derivative of (1.98a) yields

$$\frac{dx'}{dt} = \frac{dx}{dt} + \frac{dF(t)}{dt} \ . \tag{1.103}$$

Equating (1.102) and (1.103) yields

$$\bar{\psi} = -\frac{dF(t)}{dt}y \ , \tag{1.104}$$

so that

$$\psi' = \psi - \frac{dF(t)}{dt}y \ . \tag{1.105}$$

Notice that, with the use of (1.101) and (1.105), the relative vorticity results invariant under (1.98a)–(1.98d)

$$\nabla'^2\psi' = \nabla^2\psi \ . \tag{1.106}$$

Considering that the partial derivative respect to time transforms as

$$\begin{aligned}\frac{\partial}{\partial t'} &= \frac{\partial t}{\partial t'}\frac{\partial}{\partial t} + \frac{\partial x}{\partial t}\frac{\partial}{\partial x}\\ &= \frac{\partial}{\partial t} + \frac{\partial(x' - F(t))}{\partial t}\frac{\partial}{\partial x}\\ &= \frac{\partial}{\partial t} - \frac{dF(t)}{dt}\frac{\partial}{\partial x}\ ,\end{aligned} \tag{1.107}$$

direct substitution of (1.105)–(1.107) in (1.91), and mutual cancellation of the additional terms arising from the time derivative and the velocity advection, shows the Galilean covariance of the vorticity equation.

Analogously, (1.99d) yields

$$\psi' = \psi + \frac{dF(t)}{dt}x\ . \tag{1.108}$$

In the commonly analysed case in which $g(t) = \mathbf{U}t$, where $\mathbf{U} \in \mathbb{R}^3$ is a constant velocity, (1.98a)–(1.98d) and (1.99a)–(1.99d) yield, respectively,

$$g_x(x) = x + Ut\ , \tag{1.109a}$$
$$g_y(y) = y\ , \tag{1.109b}$$
$$g_t(t) = t\ , \tag{1.109c}$$
$$g_p(\psi) = \psi - Uy\ , \tag{1.109d}$$

and

$$g_x(x) = x\ , \tag{1.110a}$$
$$g_y(y) = y + Ut\ , \tag{1.110b}$$
$$g_t(t) = t\ , \tag{1.110c}$$
$$g_p(\psi) = \psi + Ux\ . \tag{1.110d}$$

It is interesting to consider the covariance of the momentum equation, in the presence of rotation, under a Galilean transformation. Consider the separate components of the two-dimensional momentum equations in the f-plane

$$\frac{\partial u}{\partial t} + (\mathbf{u}\cdot\nabla)\,u - fv = -\frac{1}{\rho}\frac{\partial p}{\partial x}\ , \tag{1.111a}$$
$$\frac{\partial v}{\partial t} + (\mathbf{u}\cdot\nabla)\,v + fu = -\frac{1}{\rho}\frac{\partial p}{\partial y}\ , \tag{1.111b}$$

and consider case (1.98a)–(1.98d). Using (1.107) and setting $p' = p + \bar{p}$, after a simple cancellation (1.111a)–(1.111b), yield

$$\frac{\partial \bar{p}}{\partial x'} = 0 \,, \tag{1.112a}$$

$$f\frac{dF(t')}{dt'} = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial y'} \tag{1.112b}$$

that gives

$$\bar{p} = -f\rho\frac{dF(t')}{dt'}y' \,. \tag{1.113}$$

Analogously, (1.99a)–(1.99d) yield

$$\bar{p} = f\rho\frac{dF(t')}{dt'}x' \,. \tag{1.114}$$

It should be noted that the correction term to the pressure field, $\bar{p}$, depends on the space coordinate, as for the symmetry under nonuniform translations for the Euler's equation in a nonrotating framework.

1.6.1.5 Rotations

$$g_x(x) = x\cos\theta - y\sin\theta \,, \tag{1.115a}$$

$$g_y(y) = x\sin\theta + y\cos\theta \,, \tag{1.115b}$$

$$g_t(t) = t \,, \tag{1.115c}$$

$$g_p(\psi) = \psi \,, \tag{1.115d}$$

where $\theta \in [0, 2\pi]$ increasing anticlockwise. Equation (1.91) is invariant under (1.115a)–(1.115d) for $\bar{\psi} = 0$ or $C = 1$. Notice that in this case, the two-dimensional orthogonal rotation matrix $\mathbf{R} \in SO(2)$ is

$$\mathbf{R} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \,. \tag{1.116}$$

More in general, it can be proved that (1.91) is invariant under time-dependent rotations

$$g_x(x) = x\cos(\Theta t) - y\sin(\Theta t) \,, \tag{1.117a}$$

$$g_y(y) = x\sin(\Theta t) + y\cos(\Theta t) \,, \tag{1.117b}$$

$$g_t(t) = t \,, \tag{1.117c}$$

$$g_p(\psi) = \psi \,, \tag{1.117d}$$

where Θ is a constant with units of the inverse of a time. The invariance of (1.91) under (1.115a)–(1.115d) and (1.117a)–(1.117d) can be proved through direct substitution of the transformations in the vorticity equation.

A special case of (1.115a)–(1.115d) is represented by the reflection through the origin

$$g_r(\mathbf{r}) = -\mathbf{r} \ , \tag{1.118a}$$

$$g_t(t) = t \ , \tag{1.118b}$$

$$g_p(\psi) = \psi \ , \tag{1.118c}$$

corresponding to a rotation of $\theta = \pi$ and that is an invariant transformation thanks to the mutual product of the partial derivatives in the Jacobian. The transformation of the velocity field can be obtained from direct time differentiation of (1.115a)–(1.115d) or (1.117a)–(1.117d). In particular, the case of reflection through the origin (1.118a)–(1.118c) implies the reversal of the velocity field

$$u \to u' = -\frac{\partial \psi'}{\partial y'} = \frac{\partial \psi}{\partial y} \ \Rightarrow \ u' = -u \ , \tag{1.119}$$

$$v \to v' = \frac{\partial \psi'}{\partial x'} = -\frac{\partial \psi}{\partial x} \ \Rightarrow \ v' = -v \ . \tag{1.120}$$

1.6.2 Discrete Symmetries

1.6.2.1 Time Reversal

$$g_r(\mathbf{r}) = \mathbf{r} \ , \tag{1.121a}$$

$$g_t(t) = -t \ , \tag{1.121b}$$

$$g_p(\psi) = -\psi \ . \tag{1.121c}$$

Proof The proof proceeds noting from time derivative of (1.121a) and (1.121b) that the time-reversal transformation reverses the velocities and the time derivative,

$$\mathbf{u} \to \mathbf{u}' = -\mathbf{u} \ , \tag{1.122}$$

$$\frac{\partial}{\partial t} \to \frac{\partial}{\partial t'} = -\frac{\partial}{\partial t} \ , \tag{1.123}$$

Equation (1.121c) is instead justified by

$$u \to u' = -\frac{\partial \psi'}{\partial y'} = -\frac{\partial \psi'}{\partial y} = -u \ \Rightarrow \ \psi' = -\psi \ , \tag{1.124}$$

or analogously for v, so that

$$\nabla'^2\psi' = -\nabla^2\psi \; . \tag{1.125}$$

This shows automatically the invariance of the two-dimensional vorticity equation under the time-reversal symmetry (1.121a)–(1.121c).

It is instructive to see how the two-dimensional momentum equations in the f-plane are invariant under time-reversal transformations. Consider the momentum equations

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\,\mathbf{u} + 2\boldsymbol{\Omega}\times\mathbf{u} = -\frac{1}{\rho}\nabla p \; . \tag{1.126}$$

Setting $p' = p + \bar{p}$, it is visible that (1.126) is invariant under the time-reversal transformation (1.121a)–(1.121c) for $\bar{p} = 0$. In particular, it is important to recognize that the time-reversal transformation acts on reversing the rotation frequency, so that

$$\boldsymbol{\Omega}' = -\boldsymbol{\Omega} \; , \tag{1.127}$$

and thus

$$f' = -f \; . \tag{1.128}$$

1.6.2.2 Scaling Invariance

$$g_r(\mathbf{r}) = a\mathbf{r} \; , \tag{1.129a}$$

$$g_t(t) = bt \; , \tag{1.129b}$$

$$g_p(\psi) = \frac{b}{a^2}\psi \; , \tag{1.129c}$$

where $a, b \in \mathbb{R},\ a, b \neq 0$.

Proof Notice that (1.129c) implies

$$\nabla \to \nabla' = \frac{1}{a}\nabla \; , \tag{1.130}$$

$$\nabla^2 \to \nabla'^2 = \frac{1}{a^2}\nabla^2 \; , \tag{1.131}$$

$$\frac{\partial}{\partial t} \to \frac{\partial}{\partial t'} = \frac{1}{b}\frac{\partial}{\partial t} \; . \tag{1.132}$$

Rewriting (1.91) as

$$\frac{\partial}{\partial t}\nabla^2\psi + \hat{\mathbf{k}}\cdot\nabla\psi\times\nabla\left(\nabla^2\psi\right) = 0 \; , \tag{1.133}$$

and setting $\psi' = C\psi$, one has

$$a^2 bC \frac{\partial}{\partial t'} \nabla'^2 \psi' + a^4 C^2 \hat{\mathbf{k}} \cdot \nabla' \psi' \times \nabla' \left(\nabla'^2 \psi' \right) = 0 \,. \tag{1.134}$$

Simplification shows that (1.91) is invariant if

$$C = \frac{b}{a^2} \,. \tag{1.135}$$

1.6.3 *Breaking of Symmetries of the Vorticity Equation in the β Plane*

It is interesting to consider the case in which the two-dimensional vorticity equation is in the β-plane, as expressed in (1.90)

$$\frac{\partial}{\partial t} \nabla^2 \psi + J \left(\psi, \nabla^2 \psi + \beta y \right) = 0 \,. \tag{1.136}$$

In this case, the Galilean covariance for translations along the y direction (1.99a)–(1.99d) and the symmetry under rotations (1.115a)–(1.115d) and (1.117a)–(1.117d) are no longer satisfied.

To prove this statement, consider the two-dimensional vorticity equation in the β-plane (1.90) written as

$$\frac{\partial}{\partial t} \nabla^2 \psi + J \left(\psi, \nabla^2 \psi \right) + \beta \frac{\partial \psi}{\partial x} = 0 \,. \tag{1.137}$$

Direct substitution of the symmetry transformations shows that (1.137) is invariant under gauge transformations for the stream function, space and time translations, time reversal, parity and scaling. In particular, one should notice that the time-reversal symmetry is not broken by the β term due to the request that $\beta' = -\beta$.

The lack of symmetry under Galilean transformation along the y direction is simply proven noticing that, under (1.99a)–(1.99d), (1.137) transforms as

$$\frac{\partial}{\partial t'} \nabla'^2 \psi' + J \left(\psi', \nabla'^2 \psi' \right) + \beta \frac{\partial \psi'}{\partial x'} = -\beta \frac{dg(t)}{dt} \,, \tag{1.138}$$

which differs from (1.137) due to the additional term on the r.h.s. of the equation.

Similarly, the lack of rotational symmetry can be proved through direct application of the rotational transformation (1.115a)–(1.115d) and (1.117a)–(1.137).

It is thus visible that the symmetries that are not satisfied are the ones relying on the isotropic properties of the vorticity equation, which are indeed broken by the insertion of the β term.

1.7 Energy and Enstrophy Conservation

The inviscid and unforced nature of (1.90) suggests that energy is conserved by any flow governed by the equation above. To verify this, each term of equation (1.90) is multiplied by ψ and the products are integrated over a certain domain D with the use of suitable boundary conditions. This leads in a straight way to the conservation of the integrated kinetic energy of the flow. Because of the identity

$$\psi\left[\frac{\partial}{\partial t}\nabla^2\psi + J\left(\psi, \nabla^2\psi + \beta y\right)\right] = div\left(\psi\frac{\partial}{\partial t}\nabla^2\psi\right) - \frac{1}{2}\frac{\partial}{\partial t}|\nabla\psi|^2 + \frac{1}{2}\, div\left[\psi^2\nabla\left(\nabla^2\psi + \beta y\right)\times\hat{\mathbf{k}}\right], \tag{1.139}$$

Equation (1.90) implies that the r.h.s of (1.139) is zero, and hence, by using also the divergence theorem, integration over the fluid domain gives

$$\oint_{\partial D}\psi\frac{\partial}{\partial t}\nabla\psi\cdot\hat{\mathbf{n}}ds - \frac{1}{2}\frac{d}{dt}\int_D|\nabla\psi|^2\,dxdy + \frac{1}{2}\oint_{\partial D}\psi^2\nabla\left(\nabla^2\psi + \beta y\right)\times\hat{\mathbf{k}}\cdot\hat{\mathbf{n}}\,ds = 0\,. \tag{1.140}$$

The first integral of (1.140) is equivalent to

$$\bar{\psi}\oint_{\partial D}\frac{\partial}{\partial t}\mathbf{u}\cdot\hat{\mathbf{t}}\,ds\,, \tag{1.141}$$

where $\bar{\psi}$ is the spatially constant value taken by the stream function at the boundary. On the other hand, $\oint_{\partial D}(\partial/\partial t)\ \mathbf{u}\cdot\hat{\mathbf{t}}ds = 0$ as a consequence of the vanishing of the normal velocity on a closed boundary. Moreover, the third integral of (1.140) is equivalent to

$$\frac{\bar{\psi}^2}{2}\oint_{\partial D}\nabla\left(\nabla^2\psi + \beta y\right)\cdot\hat{\mathbf{t}}\,ds\,, \tag{1.142}$$

but Stoke's theorem implies $\oint_{\partial D}\nabla\left(\nabla^2\psi + \beta y\right)\cdot\hat{\mathbf{t}}ds = 0$. In conclusion, Eqs. (1.140), (1.141) and (1.142) yield the conservation of the integrated kinetic energy K, that is

$$\frac{dK}{dt} = 0\,, \tag{1.143}$$

where

$$K = \int_D\frac{1}{2}|\nabla\psi|^2\,dxdy\,. \tag{1.144}$$

An evolution equation for the integrated potential enstrophy

$$Z = \frac{1}{2} \int_D \left(\nabla^2 \psi\right)^2 dxdy \,, \tag{1.145}$$

can be inferred from (1.90) in analogy with the integrated kinetic energy. In this case, each term of equation (1.90) is multiplied by $\nabla^2 \psi$ and the products are integrated over the domain D by means of suitable boundary conditions. In this way, sufficient conditions for the conservation of (1.145) can be inferred. To this purpose, consider the identity

$$\nabla^2 \psi \left[\frac{\partial}{\partial t} \nabla^2 \psi + J\left(\psi, \nabla^2 \psi + \beta y\right) \right] = \frac{1}{2} \frac{\partial}{\partial t} \left(\nabla^2 \psi\right)^2 + \frac{1}{2} J\left(\psi, \left(\nabla^2 \psi\right)^2\right) + \beta \nabla^2 \psi \frac{\partial \psi}{\partial x} \,. \tag{1.146}$$

Because of (1.90), integration of the r.h.s. of (1.146) over D yields

$$\frac{dZ}{dt} + \frac{\bar{\psi}}{2} \oint_{\partial D} \nabla \left(\nabla^2 \psi\right)^2 \cdot \hat{\mathbf{t}} \, ds + \beta \int_D \nabla^2 \psi \frac{\partial \psi}{\partial x} dxdy = 0 \,. \tag{1.147}$$

The second term of (1.147) is zero because of Stoke's theorem, so the latter equation simplifies into

$$\frac{dZ}{dt} = -\beta \int_D \nabla^2 \psi \frac{\partial \psi}{\partial x} dxdy \,. \tag{1.148}$$

Equation (1.148) shows immediately that a sufficient condition for the conservation of potential enstrophy is $\beta = 0$, that is to say that a flow confined in a region of the f-plane, and governed by (1.90), conserves V. Another condition, independent of β, presupposes a fluid domain of the kind,

$$D = [0 \leq x \leq X] \times [0 \leq y \leq Y] \,, \tag{1.149}$$

and a doubly periodic stream function such that

$$\psi(x, y) = \psi(x + X, y), \ \forall y \in [0, Y] \,, \tag{1.150a}$$

$$\psi(x, y) = \psi(x, y + Y), \ \forall x \in [0, X] \,. \tag{1.150b}$$

In fact, Eq. (1.148) is equivalent to

$$\frac{dZ}{dt} = -\beta \int_D \left[\frac{\partial \psi}{\partial x} \left(\frac{\partial \psi}{\partial x}\right)^2 + \frac{\partial \psi}{\partial y} \left(\frac{\partial \psi}{\partial x} \frac{\partial \psi}{\partial x}\right) - \frac{1}{2} \frac{\partial}{\partial x} |\nabla \psi|^2 \right] dxdy \,, \tag{1.151}$$

but, because of (1.149) and (1.150a), (1.150b), each of the three integrals of (1.151) is identically zero, so the conservation of potential enstrophy,

$$\frac{dZ}{dt} = 0\,, \tag{1.152}$$

follows. A further condition for the conservation of (1.145) demands that ∂D recedes to infinity and ψ becomes spatially constant at infinity. In fact, Eq. (1.148) can be restated, with the aid of the divergence theorem, as

$$\frac{dZ}{dt} = -\beta \oint_{\partial D} \frac{\partial \psi}{\partial x} \nabla\psi \cdot \hat{\mathbf{n}}\, ds + \frac{\beta}{2} \oint_{\partial D} |\nabla\psi|^2\, dy\,. \tag{1.153}$$

If, at infinity, $\psi = \bar{\psi},\ \forall(x, y) \in \partial D$ and hence $\partial\bar{\psi}/\partial x = \nabla\bar{\psi} = 0$, then Eq. (1.152) is again recovered.

1.8 Conservation Laws

1.8.1 Kelvin's Circulation Theorem and Conservation of Circulation

The Euler's equation (1.90) has a further conservation law, namely the conservation of circulation, which is stated in Kelvin's Circulation Theorem

Theorem 1.1 *Consider a closed contour $\partial C(t)$ bounding an area C. For inviscid, barotropic flows under the influence of only conservative body forces, the circulation*

$$\Gamma = \oint_{\partial C(t)} \mathbf{u} \cdot d\mathbf{r} \tag{1.154}$$

is conserved, i.e.

$$\frac{d\Gamma}{dt} = 0\,, \tag{1.155}$$

where $d\mathbf{r}$ is an infinitesimal element of ∂C.

Proof By Stokes' theorem, (1.154) corresponds to

$$\Gamma = \iint_C \boldsymbol{\omega} \cdot \hat{\mathbf{n}}\, dC\,, \tag{1.156}$$

which means the circulation corresponds to the flux of vorticity through the surface C. Assuming that $\mathbf{u}$ is smooth enough to allow for time differentiation under the integral and using (1.2), the time derivative of (1.154) yields

$$
\begin{aligned}
\frac{d\Gamma}{dt} &= \oint_{\partial C(t)} \frac{D\mathbf{u}}{Dt} \cdot d\mathbf{r} + \oint_{\partial C(t)} \mathbf{u}\frac{d\mathbf{r}}{dt} \\
&= \oint_{\partial C(t)} \frac{D\mathbf{u}}{Dt} \cdot d\mathbf{r} + \oint_{\partial C(t)} \mathbf{u}d\mathbf{u} \\
&= \oint_{\partial C(t)} \frac{D\mathbf{u}}{Dt} \cdot d\mathbf{r} + \frac{1}{2}\oint_{\partial C(t)} d|u|^2 \,.
\end{aligned}
\tag{1.157}
$$

The last integral on the right-hand side of (1.157) disappears as it is the integral of a perfect differential along a closed contour.

Using the two-dimensional Euler's equation and the continuity equation, with ρ assumed to be constant, (1.157) yields

$$
\frac{d\Gamma}{dt} = -\oint_{\partial C(t)} \nabla\left(\frac{p}{\rho}\right) \cdot d\mathbf{r} \,. \tag{1.158}
$$

It should be noted that (1.158) could be written without loss of generality for the absolute circulation $\Gamma_a = \Gamma + 2\Omega C$.

The integral on the right-hand side of (1.158) can be written as

$$
\begin{aligned}
-\oint_{\partial C(t)} \nabla\left(\frac{p}{\rho}\right) \cdot d\mathbf{r} &= -\iint_C rot\left[\nabla\left(\frac{p}{\rho}\right)\right] \cdot \hat{\mathbf{n}} dC \\
&= \iint_C \frac{1}{\rho^2}\left[\nabla\rho \times \nabla p\right] \cdot \hat{\mathbf{n}} dC \,.
\end{aligned}
\tag{1.159}
$$

The term within square brackets in (1.159) is the baroclinic vector already seen in the equation for the absolute vorticity (1.81). If

$$
\nabla\rho \times \nabla p = 0 \,, \tag{1.160}
$$

which means if

$$
\rho = \rho(p) \,, \tag{1.161}
$$

situation that corresponds to the case in which the surfaces of constant ρ and p coincide, the combination of (1.158) and (1.159) yields

$$
\frac{d\Gamma}{dt} = 0 \,. \tag{1.162}
$$

1.8.2 *Potential Vorticity and Ertel's Theorem*

By expressing $div\,\mathbf{u}$ with the aid of (1.73), Eq. (1.81) takes the form

$$\frac{D}{Dt}\frac{\boldsymbol{\omega}_a}{\rho} = \left(\frac{\boldsymbol{\omega}_a}{\rho}\cdot\nabla\right)\mathbf{u} - \frac{\nabla p \times \nabla \rho}{\rho^3} \,. \tag{1.163}$$

Quite independently of the equations above, suppose that q is a certain scalar which is conserved in the course of the motion

$$\frac{Dq}{Dt} = 0\,. \tag{1.164}$$

Equation (1.164) implies the identity

$$\frac{\boldsymbol{\omega}_a}{\rho}\cdot\frac{D}{Dt}\nabla q = -\nabla q \cdot \left[\left(\frac{\boldsymbol{\omega}_a}{\rho}\cdot\nabla\right)\mathbf{u}\right] \tag{1.165}$$

while the dot product of ∇q with (1.163) yields

$$\nabla q \cdot \frac{D}{Dt}\frac{\boldsymbol{\omega}_a}{\rho} = \nabla q \cdot \left[\left(\frac{\boldsymbol{\omega}_a}{\rho}\cdot\nabla\right)\mathbf{u}\right] - \nabla q \cdot \frac{\nabla p \times \nabla \rho}{\rho^3} \,. \tag{1.166}$$

Addition of (1.165) with (1.166) results in

$$\frac{D}{Dt}\left(\frac{\boldsymbol{\omega}_a}{\rho}\cdot\nabla q\right) = -\nabla q \cdot \frac{\nabla p \times \nabla \rho}{\rho^3} \tag{1.167}$$

which governs the evolution of potential vorticity Π defined by

$$\Pi = \frac{\boldsymbol{\omega}_a}{\rho}\cdot\nabla q\,. \tag{1.168}$$

Based on (1.167) Ertel's theorem claims that:

Theorem 1.2 *If* (1.160) *or, as alternative,* (1.161) *hold, then the potential vorticity* (1.168) *of an inviscid flow is conserved following the motion.*

In the framework of Geophysical Fluid Dynamics, the gradient of the most of geophysical scalars, say q, can be approximated by

$$\nabla q \approx \frac{\partial q}{\partial z}\hat{\mathbf{k}}\,. \tag{1.169}$$

Approximation (1.169) is based on the very small value of the aspect ratio $\delta = H\,/\,L$ in the oceans and in the atmosphere, which implies $(\partial/\partial x)\,/\,(\partial/\partial z) \approx (\partial/\partial y)\,/\,(\partial/\partial z) = O\,(H/L) << 1$ and which in turn implies a stronger variability in the vertical direction rather than in the horizontal for most of the geophysical scalars. Hence, if potential vorticity is conserved and approximation (1.169) together with (1.79) is used into (1.168), the conserved scalar is

$$\Pi = \frac{1}{\rho}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} + f_0 + \beta_0 y\right)\frac{\partial q}{\partial z} \; . \tag{1.170}$$

Usually, the short notation $\varsigma = (\partial v \,/\, \partial x) - (\partial u \,/\, \partial y)$ is used, so that (1.170) can be written as

$$\Pi = \frac{\varsigma + f_0 + \beta_0 y}{\rho}\frac{\partial q}{\partial z} \; . \tag{1.171}$$

What is q? This quantity depends on the type of fluid under consideration. In the following, different approximations of the governing equations of Geophysical Fluid Dynamics will be considered and the corresponding form for the conservation of potential vorticity will be reported.

It should be noted that the derivation of Ertel's potential vorticity can be derived from Kelvin's Circulation Theorem. For a proof, see Appendix B.

1.9 Conservation of Potential Vorticity and Models of Geophysical Flows

1.9.1 Shallow-Water Model with Primitive Equations

The shallow-water model deals with a constant-density, single-layer fluid in frictionless motion under the effect of Earth's rotation and, in case, of a bathymetric modulation. The fluid is included between the bottom, say in $z = h(x, y)$, and the free surface, say in $z = H + \eta(x, y, t)$. The constant H is a vertical length scale which is representative of the full thickness of the layer and coincides with the depth of the motion. Furthermore, a typical horizontal length L is necessary as well to estimate, for instance, the gradient of the free surface elevation η. The fundamental assumption of the shallow-water model is that L is much greater than H or, in other terms, that the aspect ratio

$$\delta = H \,/\, L << 1 \; . \tag{1.172}$$

A constant density fluid implies

$$div\, \mathbf{u} = 0 \; , \tag{1.173}$$

while (1.74) can be written as

$$\frac{D\mathbf{u}}{Dt} + 2\boldsymbol{\Omega} \times \mathbf{u} = -\frac{\nabla \tilde{p}}{\rho} \tag{1.174}$$

where $\tilde{p}$ is the perturbation pressure. Because only the local normal component of the Earth's rotation is dynamically significant, then

$$\boldsymbol{\Omega} \approx \left(\boldsymbol{\Omega} \cdot \hat{\mathbf{k}}\right) \hat{\mathbf{k}} = f(y)\hat{\mathbf{k}} \tag{1.175}$$

where $f(y) = f_0 + \beta_0 y$ according to (1.79). Substitution of (1.175) into (1.174) produces the following set of scalar equations:

$$\frac{Du}{Dt} - f(y)v = -\frac{1}{\rho}\frac{\partial \tilde{p}}{\partial x} , \tag{1.176a}$$

$$\frac{Dv}{Dt} + f(y)u = -\frac{1}{\rho}\frac{\partial \tilde{p}}{\partial y} , \tag{1.176b}$$

$$\frac{Dw}{Dt} = -\frac{1}{\rho}\frac{\partial \tilde{p}}{\partial z} . \tag{1.176c}$$

The subsequent development of (1.173), (1.176a)–(1.176c) requires a scaling analysis of these equations. To achieve this, consider the nondimensional quantities (with the apex) $(x', y') = (x/L,\ y/L),\ z' = z/H,\ (u', v') = (u/U,\ v/U),\ w' = w/W,\ \ p' = \bar{p}/P$. In terms of them, Eq. (1.173) takes the form

$$\frac{\partial u'}{\partial x'} + \frac{\partial v'}{\partial y'} + \frac{WL}{UH}\frac{\partial w'}{\partial z'} = 0. \tag{1.177}$$

The three-dimensional nature of the flow demands that all the space derivatives are comparable among them, and therefore, $WL\ /\ UH = 1$, that is to say

$$W = \delta U. \tag{1.178}$$

Note that, because of (1.172), Eq. (1.178) implies $W \ << \ U$. Consider now Eqs. (1.176a) and (1.176b). Because

$$\frac{Du}{Dt} \approx \frac{Dv}{Dt} \approx \frac{U^2}{L} \ ,\ \ f(y)u \approx f(y)v \approx f_0 U \ ,\ \ \frac{\partial \tilde{p}}{\partial x} \approx \frac{\partial \tilde{p}}{\partial y} \approx \frac{P}{L} \ ,$$

the order of magnitude P of $\tilde{p}$, estimated from (1.176a) and (1.176b), turns out to be

$$P = \rho L \ max \left\{U^2/L,\ \ f_0 U\right\} = \rho L U \ max \left\{U/L,\ \ f_0\right\} \ . \tag{1.179}$$

On the other hand, the order of magnitude of the l.h.s of (1.176c) is

$$\frac{Dw}{Dt} = \frac{\delta U^2}{L} \tag{1.180}$$

while the r.h.s of the same equation can be estimated by using (1.179), whence

$$\frac{1}{\rho}\frac{\partial \tilde{p}}{\partial z} \approx \frac{LU \ max \left\{U/L,\ \ f_0\right\}}{H} \approx \frac{U \ max \left\{U/L,\ \ f_0\right\}}{\delta} \ . \tag{1.181}$$

From (1.180), (1.181) and assuming $U \ / \ f_0 L = \varepsilon \leq O(1)$, one obtains

$$\left|\frac{Dw}{Dt}\right| / \left|\frac{1}{\rho}\frac{\partial \bar{p}}{\partial z}\right| \approx \frac{\delta^2 U/L}{max\ \{U/L,\ f_0\}} \leq \delta^2 \ . \tag{1.182}$$

Inequality (1.182) shows that the r.h.s. of (1.176c) is much greater that the l.h.s. so the former cannot be balanced by the latter and thus (1.176c) at the leading order gives

$$\frac{\partial \tilde{p}}{\partial z} = 0 \ . \tag{1.183}$$

Equation (1.183) shows that the perturbation pressure is depth independent, so the total pressure p can be written as

$$p = \rho g \left[\eta(x, y, t) - z\right] + p\,(z = \eta) \tag{1.184}$$

Thus, total pressure is hydrostatic and, in this framework, the term $p\,(z = \eta)$ is usually constant. In turn, (1.184) yields

$$\frac{1}{\rho}\frac{\partial p}{\partial x} = g\frac{\partial \eta}{\partial x} \ , \tag{1.185a}$$

$$\frac{1}{\rho}\frac{\partial p}{\partial y} = g\frac{\partial \eta}{\partial y} \ , \tag{1.185b}$$

so the r.h.s of (1.176a) and (1.176b) is depth independent. Therefore, the horizontal components of the current appearing at the l.h.s. of the same equations can be consistently assumed depth independent as well, i.e.

$$\frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = 0 \ . \tag{1.186}$$

Based on (1.185a)–(1.185b) and (1.186), Eqs. (1.176a) and (1.176b) take the final form

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - f(y)v = -g\frac{\partial \eta}{\partial x} \ , \tag{1.187a}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + f(y)u = -g\frac{\partial \eta}{\partial y} \ , \tag{1.187b}$$

or, in vector notation,

$$\frac{D\mathbf{u}}{Dt} + f(y)\hat{\mathbf{k}} \times \mathbf{u} = -g\nabla\eta \ . \tag{1.188}$$

Moreover, because of (1.186), the vertical integration of (1.173) in the intervals $(-H + h, \eta)$ and $(-H + h, z)$ gives

$$(H + \eta - h)\,\nabla_H \cdot \mathbf{u} + \frac{D}{Dt}\,(H + \eta - h) = 0\,, \tag{1.189}$$

and

$$(z + H - h)\,\nabla_H \cdot \mathbf{u} + \frac{D}{Dt}\,(z + H - h) = 0\,. \tag{1.190}$$

Then, the elimination of the horizontal divergence between (1.189) and (1.190) results in the Eq. (1.164), where

$$q = \frac{z - h}{H + \eta - h}\,. \tag{1.191}$$

Hence, the potential vorticity (1.171) takes the form

$$\Pi = \frac{\varsigma + f_0 + \beta_0 y}{H + \eta - h}\,. \tag{1.192}$$

Equations (1.187a), (1.187b) and the conservation of (1.192) govern the shallow-water model. Once the horizontal boundary conditions and the initial conditions are fixed, equations above can be integrated to produce the horizontal current field and the free surface elevation field.

1.9.2 Quasi-geostrophic Shallow-Water Model

Large-scale circulation of geophysical flows is characterized by a horizontal length scale L and a typical horizontal current U such that the Rossby number

$$\varepsilon = \frac{U}{f_0 L} << 1\,. \tag{1.193}$$

Equation (1.193) states that the Coriolis acceleration is far greater than local acceleration. In this case, the dominant dynamic balance involves the Coriolis acceleration and the pressure gradient to produce the geostrophic current

$$\mathbf{u}_G = (g/f_0)\,\hat{\mathbf{k}} \times \nabla\eta\,. \tag{1.194}$$

Equation (1.194) allows us to evaluate the scale of the perturbation pressure. In fact, if U is the intensity of the geostrophic current and E is the amplitude of the free surface elevation, then

$$E = f_0 U L\,/\,g\,. \tag{1.195}$$

Estimate (1.195) is used, in turn, to derive the nondimensional version of (1.188), that is

$$\varepsilon \frac{D\mathbf{u}'}{Dt} + \left(1 + \varepsilon \beta y'\right) \hat{\mathbf{k}} \times \mathbf{u}' = -\nabla' \eta' , \tag{1.196}$$

where $\beta = \beta_0 L^2 / U$, with, in this context, $\beta = O(1)$. *In the limit of a vanishingly small Rossby number*, Eq. (1.196) takes the form $\hat{\mathbf{k}} \times \mathbf{u}' = -\nabla' \eta'$ whence the nondimensional geostrophic current

$$\mathbf{u}'_0 = \hat{\mathbf{k}} \times \nabla' \eta'_0 \tag{1.197}$$

follows. The subscript *0* is reminiscent of the position $\varepsilon = 0$ made in (1.196) to infer (1.197). Based on (1.197), the nondimensional relative vorticity at the geostrophic level of approximation, say ς'_0, is determined as a function of η'_0

$$\varsigma'_0 = \hat{\mathbf{k}} \cdot \nabla' \times \mathbf{u}'_0 = \hat{\mathbf{k}} \cdot \nabla' \times \left(\hat{\mathbf{k}} \times \nabla' \eta'_0\right) = \nabla'^2 \eta'_0 \, . \tag{1.198}$$

Equation (1.197) implies $\nabla'_H \cdot \mathbf{u}'_0 = 0$ so $\partial w'_0 / \partial z' = 0$. Thus, w'_0 is depth independent. In particular, the vertical velocity at the level of the free surface is given, in dimensional variables, by $w(z = \eta) = D\eta / Dt$. At the geostrophic level of approximation, the latter equation yields

$$w'_0 = \varepsilon F D \frac{D\eta'_0}{Dt} \tag{1.199}$$

where the Froude number $F = (f_0 L)^2 / gH \le O(1)$. But the geostrophic level of approximation presupposes a vanishingly small Rossby number, and therefore, (1.199) implies

$$w'_0 = 0 \, . \tag{1.200}$$

The bathymetric modulation $h(x, y)$ must be consistent with the geostrophic current (1.194). The point is that a too marked bottom modulation could force the current out of balance (1.194); however, this situation does not take place provided that

$$h \le \varepsilon H h' \tag{1.201}$$

where $h' = O(1)$ is the nondimensional bathymetric profile.

After these preliminaries, consider the nondimensional version of the conservation of potential vorticity in the version fit for the shallow-water model, that is the equation for the conservation of (1.192). The nondimensional version of the conservation of (1.192) can be written as

$$\left(\frac{\partial}{\partial t'} + \mathbf{u}' \cdot \nabla'\right) \frac{1 + \varepsilon \left(\varsigma' + \beta y'\right)}{1 + \varepsilon \left(F\eta' - h'\right)} = 0 \tag{1.202}$$

where $\beta = O(1)$, $F = O(1)$, $\varepsilon << 1$, and all the variables with apex are of $O(1)$. Because Eq. (1.202) includes the (small) parameter ε, the fields $\mathbf{u}'$, ς', η' appearing into the same equation are expected to depend parametrically on ε as well and *to converge to their geostrophic limit for* $\varepsilon \to 0$. Thus, according to (1.197)–(1.200),

$$\left(\frac{\partial}{\partial t'} - \frac{\partial \eta_0'}{\partial y'}\frac{\partial}{\partial x'} + \frac{\partial \eta_0'}{\partial x'}\frac{\partial}{\partial y'} + O(\varepsilon)\right)\frac{1+\varepsilon\left\{\nabla'^2\eta_0' + O(\varepsilon) + \beta y'\right\}}{1+\varepsilon\left\{F\left[\eta_0' + O(\varepsilon)\right] - h'\right\}} = 0\,. \tag{1.203}$$

By using the truncated expansion

$$\frac{1+\varepsilon\left\{\nabla'^2\eta_0' + O(\varepsilon) + \beta y'\right\}}{1+\varepsilon\left\{F\left[\eta_0' + O(\varepsilon)\right] - h'\right\}} = 1 + \varepsilon\left(\nabla'^2\eta_0' - F\eta_0' + h' + \beta y'\right) + O\left(\varepsilon^2\right)$$

up to the first order in ε, Eq. (1.203) becomes

$$\left(\frac{\partial}{\partial t'} - \frac{\partial \eta_0'}{\partial y'}\frac{\partial}{\partial x'} + \frac{\partial \eta_0'}{\partial x'}\frac{\partial}{\partial y'}\right)\left(\nabla'^2\eta_0' - F\eta_0' + h' + \beta y'\right) = 0\,. \tag{1.204}$$

Equation (1.204) is the evolution equation for the free surface elevation at the geostrophic level of approximation η_0' in which the inputs are the bathymetric profile (if any) *h'(x',y')*, the Froude number F and the nondimensional planetary vorticity gradient β. Once $\eta_0'(x', y')$ is determined by (1.204) together with suitable boundary and initial conditions, the time-dependent geostrophic current is given by (1.197) and the time-dependent relative vorticity at the geostrophic level of approximation by (1.198). Conventionally, the notation $\eta_0' = \psi$ is adopted and (1.204) is written, after dropping the apices, in the equivalent form

$$\frac{\partial}{\partial t}\left(\nabla^2\psi - F\psi\right) + J\left(\psi, \nabla^2\psi + h + \beta y\right) = 0\,. \tag{1.205}$$

Further forms, equivalent to (1.205), are admissible. Moreover, if $F = O(\varepsilon)$, then the term $F\psi$ disappears from (1.205). On the other hand, in the *f*-plane, Eq. (1.205) does not include the term βy.

1.9.3 Energy and Enstrophy Conservation for the Quasi-geostrophic Shallow Water Model

Equation (1.205) is a generalization of equation (1.90) which includes fluctuations of the free surface of the fluid, by means of the term $-F\partial\psi/\partial t$, and a bottom modulation, by means of the term $J(\psi, h)$. One can start from the identity

$$\psi\left[\frac{\partial}{\partial t}\left(\nabla^2\psi-F\psi\right)+J\left(\psi,\nabla^2\psi+h+\beta y\right)\right]=div\,\left(\psi\frac{\partial}{\partial t}\nabla\psi\right)$$
$$-\frac{1}{2}\frac{\partial}{\partial t}\left|\nabla\psi\right|^2-\frac{F}{2}\frac{\partial}{\partial t}\psi^2+\frac{1}{2}div\,\left[\psi^2\nabla\left(\nabla^2\psi+h+\beta y\right)\times\hat{\mathbf{k}}\right], \qquad (1.206)$$

where the r.h.s. is zero because of (1.90). Integration of the r.h.s. of (1.206) over the while fluid domain D yields, after little algebra, the equation

$$\frac{d}{dt}\int_D\frac{1}{2}\left(|\nabla\psi|^2+F\psi^2\right)dxdy=\bar{\psi}\oint_{\partial D}\frac{\partial}{\partial t}\nabla\psi\cdot\hat{\mathbf{n}}$$
$$+\frac{\bar{\psi}^2}{2}\oint_{\partial D}\nabla\left(\nabla^2\psi+h+\beta y\right)\cdot\hat{\mathbf{t}}\,ds\,. \qquad (1.207)$$

Both the circuit integrals on the r.h.s. of (1.207) vanish separately, so the conservation of the integrated mechanical energy

$$E=\int_D\frac{1}{2}\left(|\nabla\psi|^2+F\psi^2\right)dxdy\,, \qquad (1.208)$$

that is

$$\frac{dE}{dt}=0\,, \qquad (1.209)$$

immediately follows from (1.207). The basic difference between (1.209) and (1.90) lies in the presence, in the former equation, of the available potential energy

$$APE=\frac{F}{2}\int_D\psi^2dxdy\,, \qquad (1.210)$$

which comes from the possibility of the fluid parcels to gain, or to loose, an amount of potential energy owing to their vertical motion associated with the free surface fluctuations. Note that bottom topography plays no role in the derivation of (1.209) because only *its gradient* appears in the second circuit integral of (1.207).

Consider now Eq. (1.205) with flat bottom, that is

$$\frac{\partial}{\partial t}\left(\nabla^2\psi-F\psi\right)+J\left(\psi,\nabla^2\psi+\beta y\right)=0\,. \qquad (1.211)$$

In this context, the integrated potential enstrophy is defined as

$$Z=\frac{1}{2}\int_D\left(\nabla^2\psi-F\psi\right)^2dxdy\,, \qquad (1.212)$$

and the determination of criteria for the conservation of V is based on methods very similar to the case (1.145). Thus, consider the identity

$$\left(\nabla^2\psi - F\psi\right)\left[\frac{\partial}{\partial t}\left(\nabla^2\psi - F\psi\right) + J\left(\psi, \nabla^2\psi + \beta y\right)\right] = \frac{1}{2}\frac{\partial}{\partial t}\left(\nabla^2\psi - F\psi\right)^2$$
$$+ \left(\nabla^2\psi - F\psi\right) J\left(\psi, \nabla^2\psi\right) + \beta\left(\nabla^2\psi - F\psi\right)\frac{\partial\psi}{\partial x}\,. \qquad (1.213)$$

Because (1.148), integration of the r.h.s. of (1.213) over D yields

$$\frac{dZ}{dt} = -\beta\int_D\left(\nabla^2\psi\frac{\partial\psi}{\partial x} - \frac{F}{2}\frac{\partial}{\partial x}\psi^2\right)dxdy\,. \qquad (1.214)$$

Equation (1.214) is an obvious generalization of (1.148). Quite analogously, Eq. (1.214) shows that a sufficient condition for the conservation of potential enstrophy (1.212) is $\beta = 0$. Another condition presupposes again boundary condition (1.149) and the doubly periodic stream function (1.150b). In fact, Eq. (1.213) is equivalent to

$$\frac{dZ}{dt} = -\beta\int_D\left[\frac{\partial}{\partial x}\left(\frac{\partial\psi}{\partial x}\right)^2 + \frac{\partial}{\partial y}\left(\frac{\partial\psi}{\partial x}\frac{\partial\psi}{\partial x}\right)\right.$$
$$\left. -\frac{1}{2}\frac{\partial}{\partial x}\left(|\nabla\psi|^2 + F\psi^2\right)\right]dxdy\,, \qquad (1.215)$$

and each of the three integrals at the r.h.s. of (1.215) are separately zero because of (1.149) and (1.150b). Hence

$$\frac{dZ}{dt} = 0\,, \qquad (1.216)$$

and the potential enstrophy (1.212) is conserved.

A further condition for the conservation of (1.212) demands that ∂D recedes to infinity and ψ becomes spatially constant at infinity. In fact, Eq. (1.214) can be restated, with the aid of the divergence theorem, as

$$\frac{dZ}{dt} = -\beta\oint_{\partial D}\frac{\partial\psi}{\partial x}\nabla\psi\cdot\hat{\mathbf{n}}\,ds + \frac{\beta}{2}\oint_{\partial D}|\nabla\psi|^2\,dy + \frac{\beta F}{2}\oint_{\partial D}\psi^2 dy\,. \qquad (1.217)$$

If, at infinity, $\psi = \bar{\psi}\ \forall(x, y) \in \partial D$ and hence $\partial\bar{\psi}/\partial x = \nabla\bar{\psi} = 0$, then Eq. (1.217) becomes

$$\frac{dZ}{dt} = \frac{\beta F}{2}\bar{\psi}^2\oint_{\partial D}dy\,. \qquad (1.218)$$

In turn, $dy = \cos\theta(s)ds$, where $\theta(s)$ is the angle between the unit vectors $\hat{\mathbf{j}}$ (fixed) and $\hat{\mathbf{t}}$ (varying along ∂D). Therefore, $\oint_{\partial D} dy = \oint_{\partial D}\cos\theta(s)ds = 0$ and Eq. (1.216) again holds true. In conclusion, conservation of potential enstrophy is not affected by free surface fluctuations.

1.9.4 Quasi-geostrophic Model of a Density Conserving Ocean

In general, a stratified flow can be conceived as resulting from a disturbance of a hypothetical rest state, in which the current $\mathbf{u} = \mathbf{0}$, the density has a "standard" profile $\rho_s = \rho_s(z)$, and the pressure $p_S(z)$ is in hydrostatic equilibrium with ρ_s. The disturbance that generates the motion ($\mathbf{u} \neq \mathbf{0}$) comes from the superposition of a density anomaly $\tilde{\rho}(\mathbf{r}, t)$ to the standard density and of a perturbation pressure $\tilde{p}(\mathbf{r}, t)$ to the standard pressure so that the horizontal gradient of $\tilde{p}(\mathbf{r}, t)$ acts on the fluid parcels and forces them to move. In the ocean case, far from coastal areas and with the exception of the fluid layer immediately below the free surface, large-scale circulation is characterized by a phenomenology which is summarized as follows:

- The current is almost exactly in geostrophic balance with the pressure gradient:

$$\mathbf{u} \approx \mathbf{u}_g \quad \text{where} \quad \rho_s f_0 \mathbf{u}_g = \hat{\mathbf{k}} \times \nabla \tilde{p} \tag{1.219}$$

- The fluid body is in hydrostatic equilibrium:

$$\frac{\partial p}{\partial z} + g\rho = 0 \tag{1.220}$$

  and therefore

$$\frac{\partial \tilde{p}}{\partial z} + g\tilde{\rho} = 0. \tag{1.221}$$

- The density of each fluid parcel is conserved following the motion.

To prove the conservation of density, one should resort to the equation of state

$$\alpha = \alpha\,(p, \vartheta, S)\,, \tag{1.222}$$

where $\alpha = 1\,/\,\rho$ is the specific volume, p is the pressure, ϑ is the absolute temperature, and S is the salinity, and the thermodynamic equation

$$\delta e = \vartheta\delta\eta - p\delta\alpha + \mu\delta S \tag{1.223}$$

representing a small reversible change of internal energy e, entropy η and salinity of the sea water, in which

$$p = -\frac{\partial}{\partial \alpha} e\,(\alpha, \eta, S)\;\;, \tag{1.224}$$

$$\vartheta = \frac{\partial}{\partial \eta} e\,(\alpha, \eta, S)\;\;, \tag{1.225}$$

and

$$\mu = \frac{\partial}{\partial S} e(\alpha, \eta, S) \ , \tag{1.226}$$

is the chemical potential of salt in sea water.

With the aid of (1.222) and of (1.225), the specific volume can be written as an implicit function of pressure, entropy and salinity as $\alpha = \alpha(p, \partial e(\alpha, \eta, S)/\partial\eta, S)$, whence

$$\alpha = \alpha(p, \eta, S) \ . \tag{1.227}$$

In turn, Eq. (1.227) implies

$$\frac{D\alpha}{Dt} = \frac{\partial\alpha}{\partial p}\frac{Dp}{Dt} + \frac{\partial\alpha}{\partial\eta}\frac{D\eta}{Dt} + \frac{\partial\alpha}{\partial S}\frac{DS}{Dt} \ . \tag{1.228}$$

In a isentropic and salinity-conserving ocean, by definition

$$\frac{D\eta}{Dt} = 0 \, ,$$

and

$$\frac{DS}{Dt} = 0 \, ,$$

so Eq. (1.228) simplifies into

$$\frac{D\alpha}{Dt} = \left(\frac{\partial\alpha}{\partial p}\right)\left(\frac{Dp}{Dt}\right) \ .$$

Recalling moreover that $\alpha = 1/\rho$ and

$$\frac{\partial\rho}{\partial p} = \frac{1}{c^2} \ ,$$

where c is the speed of sound in sea water, Eq. (1.228) eventually becomes

$$\frac{D\rho}{Dt} = \frac{1}{c^2}\frac{Dp}{Dt} \ . \tag{1.229}$$

In the perspective of Geophysical Fluid Dynamics, the speed of sound is far larger than the speed of every fluid parcel and looks practically “infinite”. Hence, a reasonable approximation of (1.229) is

$$\frac{D\rho}{Dt} = 0 \, . \tag{1.230}$$

Thus, the potential vorticity conserved in an isentropic and salinity-conserving ocean is

$$\Pi = \frac{\varsigma + f_0 + \beta_0 y}{\rho} \frac{\partial \rho}{\partial z} . \tag{1.231}$$

The conservation of potential vorticity (1.231) can be cast in a quasi-geostrophic form, to give the evolution of the geostrophic current and of the related geostrophic fields. The aim of this section is to derive this equation.

For mathematical simplicity, both the free surface, in $z = 0$, and the bottom, in $z = -H$, are assumed flat and therefore

$$w(z = 0) = 0 \text{ and } w(z = -H) = 0 . \tag{1.232}$$

The magnitude P of the perturbation pressure can be inferred from (1.219) by substituting positions $\mathbf{u}_g = U\mathbf{u}'$ and $\tilde{p} = Pp'$ into it ($\mathbf{u}'$ and p' are $O(1)$ nondimensional quantities) to obtain

$$P = f_0 \rho_s U L . \tag{1.233}$$

Estimate (1.233) allows to scale the density anomaly $\tilde{\rho}$ by using (1.221). In fact, from (1.221) and (1.233) one has $\tilde{\rho} = O\,(f_0 \rho_s U L \;/\; gH)$, that is

$$\tilde{\rho} = \varepsilon F \rho_s \rho' . \tag{1.234}$$

Hence, the nondimensional version of (1.221) is

$$\frac{\partial p'}{\partial z'} + \rho' = 0 . \tag{1.235}$$

The total density ρ is given by $\rho = \rho_s + \tilde{\rho}$ so, owing to (1.234),

$$\rho = \rho_s \left(1 + \varepsilon F \rho'\right) . \tag{1.236}$$

Equations (1.233) and (1.236) are now used to express the momentum equation

$$\frac{D\mathbf{u}}{Dt} + (f_0 + \beta_0 y)\, \hat{\mathbf{k}} \times \mathbf{u} = -\frac{\nabla \tilde{p}}{\rho} , \tag{1.237}$$

in the nondimensional version consistently with (2.35) and (2.36). The resulting nondimensional equation turns out to be

$$\varepsilon \frac{D\mathbf{u}'}{Dt'} + \left(1 + \beta y'\right) \hat{\mathbf{k}} \times \mathbf{u}' = -\frac{\nabla' p'}{1 + \varepsilon F \rho'} . \tag{1.238}$$

As the geostrophic level of approximation presupposes a vanishingly small Rossby number ε, at such level ($\varepsilon << 1$) Eq. (1.238) yields $\hat{\mathbf{k}} \times \mathbf{u}'_0 = -\nabla' p'_0$, whence the nondimensional geostrophic current follows in the form

$$\mathbf{u}_0' = \hat{\mathbf{k}} \times \nabla' p_0' \, . \tag{1.239}$$

Because of (1.239), the nondimensional relative vorticity at the geostrophic level of approximation ς_0' is expressed as a function of p_0', that is

$$\varsigma_0' = \nabla'^2 p_0' \, . \tag{1.240}$$

The derivation of (1.240) goes along the same line as (1.198). Moreover, (1.239) also implies $\partial w_0' \, / \, \partial z' = 0$. The latter equation together with (1.232) yields

$$w_0' = 0 \, . \tag{1.241}$$

Equations (1.235) and (1.236) are now used to evaluate the factor $\rho^{-1} \left(\partial \rho / \partial z \right)$ appearing in the potential vorticity (1.231). To achieve this, consider the chain of equalities

$$\begin{aligned} \frac{1}{\rho}\frac{\partial \rho}{\partial z} &= \frac{\partial}{\partial z} \ln(\rho) \\ &= \frac{\partial}{\partial z} \ln\left[\rho_s \left(1 + \varepsilon F \rho'\right)\right] \\ &\approx \frac{\partial}{\partial z} \ln(\rho_s) + \frac{\varepsilon F}{H}\frac{\partial \rho'}{\partial z'} \\ &= \frac{1}{\rho_s}\frac{d\rho_s}{dz} - \frac{\varepsilon F}{H}\frac{\partial^2 p'}{\partial z'^2} \\ &= -\frac{F}{H}\left(S + \varepsilon \frac{\partial^2 p'}{\partial z'^2}\right) \, . \end{aligned} \tag{1.242}$$

In (1.242), Burger's number

$$S = \left(\frac{H N_s}{f_0 L}\right)^2 \tag{1.243}$$

also known as the stratification parameter, has been introduced. Indeed, (1.243) is a function of depth only, through the depth-dependent buoyancy frequency squared

$$N_s^2 = -\frac{g}{\rho_s}\frac{d\rho_s}{dz} \, . \tag{1.244}$$

With the aid of (1.242), the nondimensional version of the potential vorticity conservation

$$\frac{D}{Dt}\left(\frac{\varsigma + f_0 + \beta_0 y}{\rho}\frac{\partial \rho}{\partial z}\right) = 0 \tag{1.245}$$

takes the form, analogous to (1.202),

$$\left(\frac{\partial}{\partial t'} + \mathbf{u}' \cdot \nabla'\right)\left[\left(1 + \varepsilon\varsigma' + \varepsilon\beta y'\right)\left(S + \varepsilon\frac{\partial^2 p'}{\partial z'^2}\right)\right] = 0\,. \tag{1.246}$$

In (1.246), both β and S are $O(1)$ quantities, while $\varepsilon << 1$. As Eq. (1.246) depends parametrically on ε, the fields $\mathbf{u}'$, ς', p' are expected to depend parametrically on ε as well and to converge to their geostrophic limits for $\varepsilon \to 0$. The situation is quite similar to the quasi-geostrophic, shallow-water model, however with an exception that will be soon clear. The leading orders terms of potential vorticity appearing in (1.246) constitute the quantity

$$S + \varepsilon S\left(\varsigma' + \beta y'\right) + \varepsilon\frac{\partial^2 p'}{\partial z'^2} \tag{1.247}$$

so, recalling (1.239), (1.240) and (1.241) and using the expansions

$$p' = p'_0 + O(\varepsilon)\,, \tag{1.248a}$$

$$u' = -\frac{\partial p'_0}{\partial y'} + \varepsilon u'_1 + O\left(\varepsilon^2\right)\,, \tag{1.248b}$$

$$v' = \frac{\partial p'_0}{\partial x'} + \varepsilon v'_1 + O\left(\varepsilon^2\right)\,, \tag{1.248c}$$

$$w' = w'_1 + O\left(\varepsilon^2\right)\,, \tag{1.248d}$$

$$\varsigma'_0 = \nabla'^2 p'_0 + O\left(\varepsilon\right)\,, \tag{1.248e}$$

Equation (1.246) can be written as

$$\left(\frac{\partial}{\partial t'} - \frac{\partial p'_0}{\partial y'}\frac{\partial}{\partial x'} + \frac{\partial p'_0}{\partial x'}\frac{\partial}{\partial y'} + \mathbf{u}'_1 \cdot \nabla' + O\left(\varepsilon^2\right)\right)$$
$$\times\left\{S + \varepsilon S\left[\nabla'^2 p'_0 + \beta y'\right] + \varepsilon\frac{\partial^2 p'_0}{\partial z'^2} + O\left(\varepsilon^2\right)\right\} = 0\,. \tag{1.249}$$

Unlike the shallow-water model, the presence of the $O(1)$ stratification parameter (1.243) in the potential vorticity demands the explicit expansion of the Lagrangian derivative up to the first order in ε (see the term $\varepsilon\mathbf{u}'_1 \cdot \nabla'$) to establish a consistent conservation equation at the higher level of approximation. Setting

$$\frac{D_0}{Dt} = \frac{\partial}{\partial t'} - \frac{\partial p'_0}{\partial y'}\frac{\partial}{\partial x'} + \frac{\partial p'_0}{\partial x'}\frac{\partial}{\partial y'}\,,$$

that is

$$\frac{D}{Dt'} = \frac{D_0}{Dt'} + O\left(\varepsilon\right)\,, \tag{1.250}$$

Equation (1.249) yields, at the first order in ε,

$$\frac{D_0}{Dt'}\left(\nabla'^2 p'_0 + \beta y'\right) + \frac{1}{S}\frac{D_0}{Dt'}\frac{\partial^2 p'_0}{\partial z'^2} + \frac{1}{S}w'_1\frac{\partial S}{\partial z'} = 0\,. \tag{1.251}$$

The last step is to express w'_1, appearing in the third term at the l.h.s. of (1.251), as a function of p'_0. To this purpose, Eq. (1.230) is, first, reconsidered in the nondimensional form

$$w' = \frac{\varepsilon}{S}\frac{D\rho'}{Dt'}\,. \tag{1.252}$$

Then, substitution of (1.235) into (1.252) results in the equation

$$w' = -\frac{\varepsilon}{S}\frac{D}{Dt'}\frac{\partial p'}{\partial z'}\,. \tag{1.253}$$

where $w' = \varepsilon w'_1 + O\left(\varepsilon^2\right),\ p' = p'_0 + O(\varepsilon)$, and the Lagrangian derivative is given by (1.250). Finally, up to the first order in ε, Eq. (1.253) takes the form

$$w' = -\frac{1}{S}\frac{D_0}{Dt'}\frac{\partial p'_0}{\partial z'}\,. \tag{1.254}$$

The vertical velocity w' can be eliminated from (1.251) and (1.254) to obtain, after little algebra, the equation

$$\frac{D_0}{Dt'}\left[\nabla'^2 p'_0 + \beta y' + \frac{\partial}{\partial z'}\left(\frac{1}{S}\frac{\partial p'_0}{\partial z'}\right)\right] = 0\,. \tag{1.255}$$

Equation (1.255) is just the quasi-geostrophic version of (1.245). Analogously to the shallow-water model, the notation $p'_0 = \psi$ is here adopted so Eq. (1.255) can be written, after dropping the apices, as

$$\frac{\partial}{\partial t}\left[\nabla^2\psi + \frac{\partial}{\partial z}\left(\frac{1}{S}\frac{\partial\psi}{\partial z}\right)\right] + J\left[\psi, \nabla^2\psi + \frac{\partial}{\partial z}\left(\frac{1}{S}\frac{\partial\psi}{\partial z}\right) + \beta y\right] = 0\,. \tag{1.256}$$

The quantity

$$\frac{\partial}{\partial z}\left(\frac{1}{S}\frac{\partial\psi}{\partial z}\right) \tag{1.257}$$

is named "thermal vorticity" and it is the component of (1.256) arising from the isentropic condition (1.230). In the f-plane approximation, Eqs. (1.255) and (1.256) do not include the terms $\beta y'$ and βy, respectively.

At the geostrophic level of approximation, vertical boundary conditions (1.232), to be applied to the integrals of (1.256), follow from (1.254). In fact, written in terms of ψ, Eq. (1.254) takes the form

$$\frac{\partial}{\partial t}\frac{\partial\psi}{\partial z} + J\left(\psi, \frac{\partial\psi}{\partial z}\right) = 0 \text{ in } z' = -1 \text{ and } z' = 0\,. \tag{1.258}$$

In full analogy with Eqs. (1.139) and (1.206), energy conservation of the baroclinic flow govened by (1.256) follows from space integration of the product of the latter equation times the stream function. However, space integration is now 3D and, in the vertical integration, say from $z = -1$ to $z = 0$, also boundary conditions (1.232) in the form (1.258) are invoked to include the flow in this interval. Thus, consider

$$\int_{-1}^{0} dz \left\{ \int_D \psi \frac{\partial}{\partial t}\left[\nabla^2\psi + \frac{\partial}{\partial z}\left(\frac{1}{S}\frac{\partial \psi}{\partial z}\right)\right] dxdy \right\}$$
$$+ \int_{-1}^{0} dz \left\{ \int_D \psi J\left(\psi, \nabla^2\psi + \frac{\partial}{\partial z}\left(\frac{1}{S}\frac{\partial \psi}{\partial z}\right) + \beta y\right) dxdy \right\} = 0\,. \qquad (1.259)$$

Owing to the constant value taken by the stream function on ∂D, the 2D integral involving the Jacobian in the second term of (1.259) is identically zero. Hence, (1.259) can be developed further on as

$$-\frac{1}{2}\frac{d}{dt}\int_{-1}^{0} dz \int_D |\nabla\psi|^2\, dxdydz + \int_{-1}^{0}\int_D \psi \frac{\partial}{\partial z}\frac{\partial}{\partial t}\left(\frac{1}{S}\frac{\partial \psi}{\partial z}\right) dxdydz = 0\,. \qquad (1.260)$$

Integration by parts of the second term of (1.260) yields, with the aid of (1.258),

$$\int_{-1}^{0}\int_D \psi \frac{\partial}{\partial z}\frac{\partial}{\partial t}\left(\frac{1}{S}\frac{\partial \psi}{\partial z}\right) dxdydz$$
$$= \int_D \left[\psi \frac{1}{S} J\left(\psi, \frac{\partial \psi}{\partial z}\right)\right]_{z=-1}^{z=0} dxdy$$
$$-\frac{1}{2}\int_{-1}^{0}\int_D \frac{1}{S}\frac{\partial}{\partial t}\left(\frac{\partial \psi}{\partial z}\right)^2 dxdydz\,. \qquad (1.261)$$

In turn, Stokes' theorem allows to write the first integral at the r.h.s. of (1.261) as

$$\int_D \left[\frac{1}{S}\psi J\left(\psi, \frac{\partial \psi}{\partial z}\right)\right]_{z=-1}^{z=0} dxdy = -\left[\frac{1}{2S}\int_D J\left(\psi^2, \frac{\partial \psi}{\partial z}\right) dxdy\right]_{z=-1}^{z=0}$$
$$= -\left[\frac{\bar{\psi}^2}{2S}\oint_{\partial D} \nabla \frac{\partial \psi}{\partial z}\cdot \hat{\mathbf{t}} ds\right]_{z=-1}^{z=0} = 0\,. \qquad (1.262)$$

On the whole, substitution of (1.261) into (1.260) gives the conservation of total mechanical energy

$$\frac{dE}{dt} = 0\,, \qquad (1.263)$$

where

$$E = \frac{1}{2}\int_{-1}^{0}\int_D \left[|\nabla\psi|^2 + \frac{1}{S}\left(\frac{\partial \psi}{\partial z}\right)^2\right] dxdydz\,. \qquad (1.264)$$

In baroclinic flows, the deformation, at each depth, of the isopycnals from the flat configuration is the source of the available potential energy

$$APE = \frac{1}{2}\int_{-1}^{0}\frac{1}{S}\int_{D}\left(\frac{\partial\psi}{\partial z}\right)^2 dxdydz\ , \tag{1.265}$$

appearing in (1.264).

1.9.5 Quasi-geostrophic Model of a Potential Temperature-Conserving Atmosphere

In this section, we consider an isentropic atmosphere, without dissipation near the (flat) ground and in the absence of radiative forcing. The equation of state is, formally, the same as that of a perfect gas, i.e.

$$p = \rho RT\ . \tag{1.266}$$

A change of internal energy means

$$\delta e = c_V\delta T\ , \tag{1.267}$$

where c_V is the specific heat at constant volume, and Eq. (1.223) takes the form

$$c_V\delta T = T\delta\eta - p\delta\alpha\ . \tag{1.268}$$

By using (1.266) and (1.267), Eq. (1.268) yields

$$\delta\eta = c_V\frac{\delta T}{T} - R\frac{T}{p}\delta\left(\frac{p}{T}\right)\ ,$$

that is to say

$$\delta\eta = c_V\delta\left(\ln T\right) - R\delta\left(\ln\frac{p}{T}\right)\ .$$

Recalling Carnot's Law

$$c_P - c_V = R\ , \tag{1.269}$$

where c_P is the specific heat at constant pressure, one eventually obtains

$$\delta\eta = c_P\delta\left(\ln T\right) - R\delta\left(\ln p\right)\ . \tag{1.270}$$

Because the atmosphere is assumed isentropic, $\delta\eta = 0$, and Eq. (1.270) gives

$$\frac{D}{Dt}\ln\left[\left(\frac{T}{T_0}\right)^{c_P}\left(\frac{p_0}{p}\right)^{R}\right]=0\,, \tag{1.271}$$

where T_0 and p_0 are constants which make nondimensional the argument of the logarithm. In terms of the specific heat ratio $\gamma = c_P \,/\, c_V$, Eq. (1.271) takes the equivalent form

$$\frac{D}{Dt}\left[\frac{T}{T_0}\left(\frac{p_0}{p}\right)^{\frac{\gamma-1}{\gamma}}\right]=0 \tag{1.272}$$

that states the conservation of the potential temperature $\theta = T\,(p_0\,/\,p)^{\frac{\gamma-1}{\gamma}}$. Thus, the potential vorticity conserved in an isentropic atmosphere is

$$\Pi = \frac{\varsigma + f_0 + \beta_0 y}{\rho}\frac{\partial\theta}{\partial z}\,. \tag{1.273}$$

In analogy with (1.236), the nondimensional potential temperature is written as

$$\theta = \theta_s\left(1+\varepsilon F\theta'\right)\,. \tag{1.274}$$

where $\theta_s(z)$ is the standard part of θ and $\theta'(x', y', z', t')$ is its nondimensional anomaly. A part from higher order terms, the relationship

$$\theta' = \frac{\partial p'}{\partial z'}\,, \tag{1.275}$$

allows to express the factor $\rho^{-1}\partial\theta \,/\, \partial z \approx \rho^{-1}\partial\theta_s \,/\, \partial z$ appearing in (1.273) as follows

$$\frac{1}{\rho}\frac{\partial\theta}{\partial z} = \frac{1}{H}\frac{d\theta_s}{dz'} + \frac{\varepsilon F}{H}\frac{\partial}{\partial z'}\left(\theta_s\frac{\partial p'}{\partial z'}\right)\,. \tag{1.276}$$

With the aid of (1.276) and in analogy with (1.246), the nondimensional version of the potential vorticity conservation equation for the atmosphere

$$\frac{D}{Dt}\frac{\varsigma + f_0 + \beta_0 y}{\rho}\frac{\partial\theta}{\partial z} = 0\,,$$

takes the form

$$\left(\frac{\partial}{\partial t'} + \mathbf{u}'\cdot\nabla'\right)\left\{\frac{1+\varepsilon\left(\varsigma' + \beta y'\right)}{\rho_s}\left[\frac{d\theta_s}{dz'} + \varepsilon F\frac{\partial}{\partial z'}\left(\theta_s\frac{\partial p'}{\partial z'}\right)\right]\right\} = 0\,. \tag{1.277}$$

By resorting to the well-known formal expansions of $\mathbf{u}'$, ς', p' in powers of ε applied to (1.277), the latter equation yields, up to the leading order,

$$w_1' \rho_s \frac{\partial}{\partial z'} \left(\frac{1}{\rho_s} \frac{d\theta_s}{dz'} \right) + \frac{d\theta_s}{dz'} \frac{D_0}{Dt'} \left(\nabla'^2 p_0' + \beta y' \right) + F \frac{D_0}{Dt'} \frac{\partial}{\partial z'} \left(\theta_s \frac{\partial p_0'}{\partial z'} \right) = 0 \,. \quad (1.278)$$

The first-order vertical velocity w_1' is the same as (1.254), so elimination of w_1' from (1.254) and (1.278) results in the conservation equation

$$\frac{D_0}{Dt'} \left[\nabla'^2 p_0' + \beta y' - \frac{\rho_s}{S^2} \frac{\partial}{\partial z'} \frac{S}{\rho_s} + \frac{1}{S} \frac{\partial^2 p_0'}{\partial z'^2} \right] = 0 \,, \quad (1.279)$$

which is analogous to (1.255) for the ocean. Finally, in terms of the stream function $\psi = p_0'$, resorting to the Jacobian determinant and after dropping the apices, Eq. (1.279) takes the usual form

$$\frac{\partial}{\partial t} \left[\nabla^2 \psi + \frac{1}{\rho_s} \frac{\partial}{\partial z} \left(\frac{\rho_s}{S} \frac{\partial \psi}{\partial z} \right) \right] + J \left[\psi, \nabla^2 \psi + \frac{1}{\rho_s} \frac{\partial}{\partial z} \left(\frac{\rho_s}{S} \frac{\partial \psi}{\partial z} \right) + \beta y \right] = 0 \,. \quad (1.280)$$

The boundary condition at the flat ground, say in $z' = 0$, is the same as (1.258) which expresses the vanishing of the first-order vertical velocity. The boundary condition at large heights, say for $z' \to \infty$, is based on the principle that the energy density $E'(\psi)$ of every finite portion of fluid is finite, so ψ is requested to satisfy the relationship

$$\sup_{0 \le z' \le \infty} E'(\psi) < \infty \,. \quad (1.281)$$

In the case of an isothermal, undisturbed atmosphere, the standard density $\rho_s(z')$ is determined by the perfect gas law together with the hydrostatic equilibrium, whence

$$\rho_s(z') = \rho_{s0} \exp\left(-Hz'/H_\rho \right) \,, \quad (1.282)$$

follows. In (1.282), H/H_ρ is the ratio between the vertical scale of the motion H and the density height scale H_ρ. Under assumption (1.282), relationship (1.281) takes the form

$$\sup_{0 \le z' < \infty} \left\{ \exp\left(-Hz'/H_\rho \right) \left[\left| \nabla_H' \psi \right|^2 + \frac{1}{S} \left(\frac{\partial \psi}{\partial z'} \right)^2 \right] \right\} < \infty \,. \quad (1.283)$$

The main physical reason of the difformity between (1.256), i.e. the ocean model, and (1.283), i.e. the atmosphere model, lies in the compressibility of air in place of the incompressibility of sea water. As a consequence, in the ocean, H/H_ρ is vanishingly small while in the troposphere $H/H_\rho = O(1)$. This explains the different forms of thermal vorticity in the two systems.

The inference of energy conservation of atmospheric flows is similar to the ocean case, the differences being due to the vertical extension of the domain, that is $(0 \le z < \infty)$, to the (approximate) exponential decay to zero of air density with height and to the (assumed) periodic behaviour with respect to longitude of the motion on the beta-plane. Thus, a 3D integral here means $\int_0^\infty \int_D dx dy dz$. Once (1.280) is multiplied by

$\rho_s \psi / \rho_{s0}$ and the product is integrated, the following equation, analogous to (1.259), is found

$$-\frac{d}{dt}\int_0^\infty \int_D \frac{\rho_s}{2\rho_{s0}} |\nabla \psi|^2 \, dxdydz + \int_0^\infty \int_D \frac{1}{\rho_{s0}} \psi \frac{\partial}{\partial z}\frac{\partial}{\partial t}\left(\frac{\rho_s}{S}\frac{\partial \psi}{\partial z}\right) dxdydz = 0 \,. \tag{1.284}$$

By means of the equation

$$w_I = -\frac{1}{S}\left[\frac{\partial}{\partial t}\frac{\partial \psi}{\partial z} + J\left(\psi, \frac{\partial \psi}{\partial z}\right)\right] , \tag{1.285}$$

and an integration by parts with respect to z in the second term of (1.284), one obtains

$$-\frac{d}{dt}\int_0^\infty \int_D \frac{\rho_s}{2\rho_{s0}}\left[|\nabla \psi|^2 + \frac{1}{S}\left(\frac{\partial \psi}{\partial z}\right)^2\right] dxdydz + \int_0^\infty \int_D \frac{1}{\rho_{s0}}\frac{\partial}{\partial z}\left[\rho_s \psi w_I - \frac{\rho_s}{2S} J\left(\psi^2, \frac{\partial \psi}{\partial z}\right)\right] dxdydz \,. \tag{1.286}$$

Equation (1.286) is equivalent to

$$-\frac{d}{dt}\int_0^\infty \int_D \frac{\rho_s}{2\rho_{s0}}\left[|\nabla \psi|^2 + \frac{1}{S}\left(\frac{\partial \psi}{\partial z}\right)^2\right] dxdydz = \left[\int_D \frac{\rho_s}{\rho_{s0}} \psi w_I dxdy\right]_{z=0}^{z=\infty} \,. \tag{1.287}$$

In the absence of pressure work at the top of the atmosphere, the integrand of the r.h.s. of (1.287) is zero and the total mechanical energy E of the flow is conserved

$$\frac{dE}{dt} = 0 \,, \tag{1.288}$$

where

$$E = \frac{1}{2}\int_0^\infty \frac{\rho_s}{\rho_{s0}} \int_D \left[|\nabla \psi|^2 + \frac{1}{S}\left(\frac{\partial \psi}{\partial z}\right)^2\right] dxdydz \,. \tag{1.289}$$

1.9.6 Conservation of Pseudo-Enstrophy in a Baroclinic Quasi-geostrophic Model

While the previous sections pointed out the conservation of energy in the baroclinic, quasi-geostrophic models of the ocean and atmosphere in the β-plane, a particular caution has to be used for the derivation of the correspondent of the enstrophy.

Consider again the conservation of the quasi-geostrophic potential vorticity q for a stratified fluid in the β-plane and bounded in the vertical domain $z^- < z < z^+$. In nondimensional form,

$$\frac{D_0}{Dt} q = -\beta \frac{\partial \psi}{\partial x} , \tag{1.290}$$

where

$$\frac{D_0}{Dt} = \frac{\partial}{\partial t} + J(\psi, \cdot) , \tag{1.291a}$$

$$q = \nabla_H^2 \psi + \frac{\partial \theta}{\partial z} , \tag{1.291b}$$

$$\theta = \frac{1}{S} \frac{\partial \psi}{\partial z} , \tag{1.291c}$$

$$\frac{\partial \psi}{\partial x} = v . \tag{1.291d}$$

The vertical boundary conditions $w_1(z = z^-) = w_1(z = z^+) = 0$ imply

$$\frac{D_0 w_1}{Dt} = 0 , \text{ in } z = z^- , \; z = z^+ , \tag{1.292}$$

that means

$$\frac{D_0 \theta}{Dt} = 0 , \text{ in } z = z^- , \; z = z^+ . \tag{1.293}$$

In the horizontal directions, assume double periodic boundary conditions, or assume that the velocity goes to zero at infinite. If V is the volume occupied by the fluid,

$$V = S \times \left[z^-, z^+ \right] , \tag{1.294}$$

where S is the section of the β plane occupied by the fluid. Multiplication of (1.290) by q yields

$$\frac{D_0}{Dt} q^2 = -\beta q \frac{\partial \psi}{\partial x} . \tag{1.295}$$

However, it is possible to see that

$$\begin{aligned} q \frac{\partial \psi}{\partial x} &= \nabla_H^2 \psi \frac{\partial \psi}{\partial x} + \frac{\partial \theta}{\partial z} \frac{\partial \psi}{\partial x} \\ &= \nabla_H \cdot \left(\frac{\partial \psi}{\partial x} \nabla_H \psi \right) - \nabla_H \psi \cdot \frac{\partial}{\partial x} \nabla_H \psi + \frac{\partial}{\partial z} (\theta v) - \theta \frac{\partial v}{\partial z} \\ &= \nabla_H \cdot \left(\frac{\partial \psi}{\partial x} \nabla_H \psi \right) - \frac{1}{2} \frac{\partial}{\partial x} \left| \nabla_H \psi \right|^2 + \frac{\partial}{\partial z} (\theta v) - \frac{1}{2S} \frac{\partial}{\partial x} \left(\frac{\partial \psi}{\partial z} \right)^2 \\ &= \nabla_H \cdot \left(\frac{\partial \psi}{\partial x} \nabla_H \psi \right) - \frac{1}{2} \frac{\partial}{\partial x} \left[\left| \nabla_H \psi \right|^2 + \frac{1}{S} \left(\frac{\partial \psi}{\partial z} \right)^2 \right] + \frac{\partial}{\partial z} (\theta v) . \end{aligned} \tag{1.296}$$

With the use of (1.296) and taking into account the boundary conditions, the integration of (1.295) over V yields

$$\frac{1}{2}\frac{d}{dt}\int_V q^2 dV = -\beta \int_S [\theta v]_{z^-}^{z^+} dS \,. \tag{1.297}$$

The multiplication of (1.290) by θ yields instead

$$\theta \frac{D_0}{Dt} q = -\beta\theta v \,, \tag{1.298}$$

which is equivalent to

$$\frac{D_0}{Dt}(\theta q) - q\frac{D_0}{Dt}\theta = -\beta\theta v \,, \tag{1.299}$$

Using (1.293), the integration of (1.299) along $z = z^-$ yields

$$\frac{d}{dt}\int_S [\theta q]_{z^-} dS = -\beta \int_S [\theta v]_{z^-} dS \,. \tag{1.300}$$

Analogously, the integration of (1.299) along $z = z^+$ yields

$$\frac{d}{dt}\int_S [\theta q]_{z^+} dS = -\beta \int_S [\theta v]_{z^+} dS \,. \tag{1.301}$$

The subtraction of (1.300) from (1.301) provides

$$\frac{d}{dt}\int_S [\theta q]_{z^-}^{z^+} dS = -\beta \int_S [\theta v]_{z^-}^{z^+} dS \,. \tag{1.302}$$

Finally, the subtraction of (1.302) from (1.297) gives

$$\frac{d}{dt}\left\{\int_V \frac{1}{2}q^2 dV - \int_S [\theta q]_{z^-}^{z^+} dS\right\} = 0 \,, \tag{1.303}$$

which expresses the conservation of the pseudo-enstrophy

$$Z = \int_V \frac{1}{2}q^2 dV - \int_S [\theta q]_{z^-}^{z^+} dS \,. \tag{1.304}$$

From (1.304), it is possible to make two important observations:

- Differently from the barotropic case, the pseudo-enstrophy Z is *not* positive defined;
- From the multiplication of (1.293) with θ and the following integration on S, it is possible to derive

$$\frac{d}{dt}\int_S \left[\theta^2\right]_z^- dS = 0\,, \tag{1.305a}$$

$$\frac{d}{dt}\int_S \left[\theta^2\right]_z^+ dS = 0\,, \tag{1.305b}$$

which express the conservation of the positive defined quantities

$$\int_S \left[\theta^2\right]_{z^-} dS\,,\quad \int_S \left[\theta^2\right]_{z^+} dS\,. \tag{1.306}$$

1.9.7 Surface Quasi-geostrophic Dynamics

The conserved quantities (1.306) suggest the possibility to formulate a quasi-geostrophic model based on the material conservation of the potential temperature θ at the boundaries. To obtain such a model consider, without loss of generality, the motion to take place on the f-plane, and set the potential vorticity as zero

$$q = \nabla_H^2\psi + \frac{\partial\theta}{\partial z} = 0\,. \tag{1.307}$$

The dynamics are thus restricted to the advection of potential temperature by the geostrophic flow at the boundaries

$$\frac{D_0\theta}{Dt} = 0\,,\ \text{in}\ z = z^-\,,\ z = z^+\,, \tag{1.308}$$

where, as previously stated,

$$\theta = \frac{1}{S}\frac{\partial\psi}{\partial z}\,. \tag{1.309}$$

The resulting dynamics that take the name of surface quasi-geostrophic approximation reduce the three-dimensional quasi-geostrophic problem to a two-dimensional problem. The approximation so obtained conserves the quantities

$$\int_S \left[\theta^2\right]_{z^-} dS\,,\quad \int_S \left[\theta^2\right]_{z^+} dS\,. \tag{1.310}$$

Further, in analogy with the two-dimensional Euler's equation, the multiplication of (1.308) with ψ and the integration over S, with the use of double periodic boundary conditions or with the requirement that the flow goes to zero at infinity, yields

$$\frac{d}{dt}\int_S \psi\theta dS = 0\,,\ \text{in}\ z = z^-\,,\ z = z^+\,, \tag{1.311}$$

i.e. to the conservation of the quantities

$$\int_S [\psi\theta]_{z^-}\, dS \,, \int_S [\psi\theta]_{z^+}\, dS \,. \tag{1.312}$$

Curiously, (1.310) and (1.312) have the same form of the conservation of enstrophy and kinetic energy, respectively, for the two-dimensional Euler's equation, where the relative vorticity $\nabla^2\psi$ replaces the potential temperature θ. However, in the surface quasi geostrophic equation, (1.310) takes the place of the energy conservation, as it expresses the conservation of kinetic energy, while (1.312) yields instead the conservation of the helicity of the flow.

1.10 Bibliographical Note

In this bibliographical note and in the one at the end of the next chapter, we will focus on review books, while in the chapters dedicated to the variational derivation of the Fluid and Geophysical Fluid Dynamics equations, we will report also a selection of research articles.

A number of textbooks are available on the derivation of the equations of Fluid and Geophysical Fluid Dynamics. The thermodynamical equations, as well as the equations for fluid dynamics, are carefully derived in the book by Batchelor [1]. A treatment of the thermodynamical equations for the special case of the ocean is reported in the book by Salmon [13]. The fundamental equations of Geophysical Fluid Dynamics are rigorously derived in the book by Pedlosky [12]. The quasi-geostrophic equations are the focus of the book by Cavallini and Crisciani [2]. Finally, a number of applications to the phenomenology of the atmosphere and the ocean are, for example, reported in the book by Vallis [15]. For other books, see for example [3–11, 14, 16].

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Chapter 2
Mechanics, Symmetries and Noether's Theorem

Abstract The Lagrangian description of mechanics allows to derive the equations of motion from a variational principle based on conserved quantities of the system. In the first part of this chapter, the Lagrangian formulation of dynamics and the properties of the Lagrangian operator are synthetically reviewed starting from Hamilton's Principle of First Action. In the second part of the chapter, the important link between continuous symmetries of the Lagrangian operator and conserved quantities of the system is introduced through Noether's Theorem. The proof of the Theorem is reported both for material particles and for continuous systems such as fluids.

Keywords Classical mechanics · Particle mechanics · Continuum mechanics · Variational principle · Symmetry · Noether's Theorem · Lagrangian dynamics · Hamiltonian dynamics · Canonical transformations

2.1 Introduction

In this chapter, we will make a short review of the Lagrangian formalism of classical mechanics, both for systems with a finite number of degrees of freedom, i.e. for systems of point particles, and for systems with an infinite number of degrees of freedom, i.e. for continuous systems. The description for point particles can be applied, for example, to study the advection of a passive tracer in a fluid flow, while the description for continuous systems provides a framework to derive the equations of motion for fluids.

The Lagrangian formalism is based on a variational principle and it introduces a number of useful advantages to the study of the dynamics. In particular, the Lagrangian formulation allows to establish an important link between the symmetries of the resulting Lagrangian operator and the conservation laws of the system through the formulation of Noether's Theorem. As it will be seen in the following chapters, this link provides an important and powerful concept and instrument to analyse the system, giving a physical base for the conservation of energy through the invariance under time translations and for the conservation of vorticity through the

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particle relabelling symmetry, which is a unique feature for fluids. This fact holds even for a system resulting from approximate equations, if the conservation laws concerning the original system are preserved. The reader familiar with Geophysical Fluid Dynamics will recognize the immediate usefulness of this property, given that Geophysical Fluid Dynamics relies on approximations, some of which were introduced in Chap. 1.

2.2 Hamilton's Principle of Least Action

Consider a system of N point particles with masses $(m_1, \ldots, m_N)$ and positions $(r_1(t), \ldots, r_N(t))$ at time t. In the following, the shorter notation m_i and $\mathbf{r}_i(t)$, with $i = 1, \ldots, N$, will be often used. The point particles move in a potential V. Unless specified, we will often take the potential as function of position only, say $V(\mathbf{r}_i(t))$. Each particle moves following Newton's second law. While this system is generally described by $3N$ degrees of freedom, the presence of constraints in the system can act to make the coordinates not independent: if the constraints are expressed as $f(\mathbf{r}_1, \ldots, \mathbf{r}_k, t) = 0$, which take also the name of *holonomic* constraints, then only $n = 3N - k$ independent coordinates can be determined in terms of *generalized coordinates* $q_i(t)$ that form a parametric representation of the nonindependent positions $\mathbf{r}_i(t)$. Notice that the $q_i(t)$ must not be orthogonal coordinates. The space defined by the generalized coordinates is also known as *configuration space*. The description provided by Newton's second law is based on a differential principle. The motion of the point particles in the configuration space can be equivalently described using Hamilton's Principle of Least Action, which is based on the minimization of an action functional and which provides an integral description of the evolution of the system, i.e. a description depending on the entire path of the system in configuration space. Consider the motion of the system in the time interval $[t_1, t_2]$, and assume that the position of the system at the extremes t_1 and t_2 is fixed. Let T be the kinetic energy of the system. Hamilton's Principle of Least Action thus states (see, e.g. [4]):

Theorem 2.1 *Out of all possible paths by which the system point could travel from its position at time t_1 to its position at time t_2, it actually travels along that path for which the integral*

$$I = \int_{t_1}^{t_2} (T - V)\, dt \tag{2.1}$$

is an extremum, whether a minimum or maximum.

The function $L = T - V$ takes the name of *Lagrange function*, or simply *Lagrangian*.

Given $T = T(\dot{q}_1, \ldots, \dot{q}_n)$ and $V = V(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n, t)$, where n is the total number of generalized coordinates $q_i(t)$, the Lagrangian is a function of the kind

$$L = L(q_i, \dot{q}_i, t), \ (i = 1, \ldots, n) . \tag{2.2}$$

Thus, according to the Principle of Least Action, the task is to find an extremum of the functional

$$I = \int_{t_1}^{t_2} L(q_i, \dot{q}_i, t) dt \ , \tag{2.3}$$

in the space $C^2_{[t_1, t_2]}$ of twice differentiable functions $q_i(t)$ such that

$$q_i(t_1) = q_i(t_2) = 0 . \tag{2.4}$$

Functions $q_i(t)$ can be conceived as being labelled by a parameter l and given by

$$q_i(t, l) = q_i(t, 0) + l\eta_i(t) \ , \tag{2.5}$$

where $\eta_i(t) \in C^2_{[t_1, t_2]}$ are arbitrary functions that satisfy

$$\eta_i(t_1) = \eta_i(t_2) = 0 . \tag{2.6}$$

Substitution of (2.5) into (2.3) yields

$$I(l) = \int_{t_1}^{t_2} L[q_i(t, 0) + l\eta_i(t), \dot{q}_i(t, 0) + l\dot{\eta}_i(t), t] dt . \tag{2.7}$$

Equation (2.7) shows that (2.3) is actually a function of the free parameter l, so the variation δI of $I(l)$ is given by

$$\delta I = \frac{\partial I}{\partial l} dl = \int_{t_1}^{t_2} \sum_{i=1}^{n} \left(\frac{\partial L}{\partial q_i} \frac{\partial q_i}{\partial l} dl + \frac{\partial L}{\partial \dot{q}_i} \frac{\partial \dot{q}_i}{\partial l} dl \right) dt . \tag{2.8}$$

Because of (2.5), $\partial q_i / \partial l = \eta_i$, $\partial \dot{q}_i / \partial l = \dot{\eta}_i$, and therefore, Eq. (2.8) take the form

$$\begin{aligned}
\delta I &= \sum_{i=1}^{n} \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q_i} \eta_i + \frac{\partial L}{\partial \dot{q}_i} \dot{\eta}_i \right) dt dl \\
&= \sum_{i=1}^{n} \int_{t_1}^{t_2} \left[\frac{\partial L}{\partial q_i} \eta_i + \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \eta_i \right) - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) \eta_i \right] dt dl \\
&= \sum_{i=1}^{n} \left\{ \int_{t_1}^{t_2} \left[\frac{\partial L}{\partial q_i} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) \right] \eta_i dt + \left[\frac{\partial L}{\partial \dot{q}_i} \eta_i \right]_{t_1}^{t_2} \eta_i \right\} dl
\end{aligned}$$

$$= \sum_{i=1}^{n} \int_{t_1}^{t_2} \left[\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) \right] \eta_i dt dl \; , \tag{2.9}$$

where the identity,

$$\frac{\partial L}{\partial \dot{q}_i}\dot{\eta}_i = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\eta_i\right) - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\eta_i \tag{2.10}$$

has been used. The arbitrariness of functions $\eta_i(t)$ leads us to conclude that

$$\delta I = 0 \iff \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0, \; (i = 1, \ldots, n) \; .$$

In other words, Hamilton's Principle of Least Action implies the *Euler–Lagrange equations*

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0, \; (i = 1, \ldots, n) \; , \tag{2.11}$$

and vice versa.

In particular, if $T = \sum_{i=1}^{n} m_i \dot{q}_i^2/2$ and $V = V(q_1, \ldots, q_n)$, the Euler–Lagrange equations take the form of Newton's second law, that is

$$m_i \ddot{q}_i^2 + \frac{\partial V}{\partial q_i} = 0, \; (i = 1, \ldots, n) \; , \tag{2.12}$$

so Newton's second law is an extremal for the action

$$I = \int_{t_1}^{t_2} \left[\left(\sum_{i=1}^{n} \frac{1}{2} m_i \dot{q}_i^2 \right) + V(q_1, \ldots, q_n) \right] dt \; . \tag{2.13}$$

In (2.10),

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \sum_{i=1}^{n} \left(\dot{q}_i \frac{\partial}{\partial q_i} + \ddot{q}_i \frac{\partial}{\partial \dot{q}_i} \right) \; , \tag{2.14}$$

and the presence of $\ddot{q}_i(t)$ in (2.14) explains why the space of twice differentiable functions $q_i(t)$ is the framework in which the variational problem $\delta I = 0$ is posed. Because (2.11) is thus a set of n second-order equations, the problem needs the specification of $2n$ initial conditions for q_i and $\dot{q}_i$ to be specified either at t_1 or t_2.

The Euler–Lagrange equations and the action principle show one of the benefits of the Lagrangian formulation of dynamics, that is that it is possible to derive the equations of motion from the knowledge of two scalars, T and V, rather than from all forces acting on the system.

2.3 Lagrangian Function, Euler–Lagrange Equations and D'Alembert's Principle

The Lagrangian function and the Euler–Lagrange equations can be derived also in a different way than present in the previous section, making use of two fundamental principles, namely the *virtual work's* and *D'Alembert's* principles. The second of them, of dynamical nature, is just an extension of the first, which is instead of statistical nature.

The virtual work's principle is applied to a system of point particles in an equilibrium, so that the sum of the forces applied to the point particle i

$$\mathbf{F}_i = 0 \tag{2.15}$$

is zero. Further, consider the *virtual displacement* $\delta\mathbf{r}_i$ of the point particle i as the displacement given by the infinitesimal change of the configuration of the entire system following the forces and the constraints associated with the system itself, occurring while time is held constant. The virtual displacement differs from real displacements in the way that real displacements imply a temporal evolution of the forces and constraints, which instead does not take place in the virtual case.

Equation (2.15) implies $\mathbf{F}_i \cdot \mathbf{r}_i = 0$, and thus,

$$\sum_i \mathbf{F}_i \cdot \delta\mathbf{r}_i = 0 \,, \tag{2.16}$$

where the sum is intended as over all the point particles of the system. The sum of the forces acting on point particle i can be separated in two classes: the applied forces $\mathbf{F}_i^{(a)}$ and the forces of constraint $\mathbf{f}_i$. Under this partition, (2.16) yields

$$\sum_i \mathbf{F}_i^{(a)} \cdot \delta\mathbf{r}_i + \sum_i \mathbf{f}_i \cdot \delta\mathbf{r}_i = 0 \,. \tag{2.17}$$

In the framework of systems in which the net virtual work of the forces of constraint is zero, i.e. $\mathbf{f}_i \cdot \delta\mathbf{r}_i = 0$, (2.17) can be written as

$$\sum_i \mathbf{F}_i^{(a)} \cdot \delta\mathbf{r}_i = 0 \,. \tag{2.18}$$

Equation (2.18) expresses the *virtual work's* principle. It should be noted that (2.18) does not apply, for example, in the presence of dissipative forces. While (2.18) refers here to a steady system, it could be easily applied to unsteady systems substituting (2.15) with Newton's law $\mathbf{F}_i - \dot{\mathbf{p}}_i = 0$. Under this assumption, (2.16) yields

$$\sum_i (\mathbf{F}_i - \mathbf{p}_i) \cdot \delta\mathbf{r}_i = 0 \,, \tag{2.19}$$

and, once again under the assumption that the net virtual work of the forces of constraint is zero, (2.19) yields

$$\sum_i \left(\mathbf{F}_i^{(a)} - \mathbf{p}_i\right) \cdot \delta\mathbf{r}_i = 0\,. \tag{2.20}$$

Equation (2.20) is *D'Alembert's principle*, which states that every state of the motion can be considered as a state in mechanical equilibrium.

The introduction of the Lagrangian function follows from a formal development of (2.20) making use of the generalized coordinates q_j in place of the vectors $\mathbf{r}_i$, following the transformation equations

$$r_i = r_i\left(q_1, \ldots, q_{3N-k}, t\right),\ i = 1, \ldots, N\,, \tag{2.21}$$

where N is the number of point particles and k is the number of holonomic constraints. From (2.21), one gets

$$\delta\mathbf{r}_i = \sum_j \frac{\partial \mathbf{r}_i}{\partial q_j}\delta q_j\,, \tag{2.22}$$

$$\mathbf{v}_i = \mathbf{r}_i = \sum_j \frac{\partial \mathbf{r}_i}{\partial q_j}\dot{q}_j + \frac{\partial \mathbf{r}_i}{\partial t}\,. \tag{2.23}$$

Using (2.22), the first term of (2.20) yields, omitting the superscript,

$$\sum_i \mathbf{F}_i \cdot \delta\mathbf{r}_i = \sum_i \sum_j \mathbf{F}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j}\delta q_j\,. \tag{2.24}$$

If $Q_j = \sum_i \mathbf{F}_i \cdot \partial\mathbf{r}_i/\partial q_j$ is the j component of a generalized force, (2.24) becomes

$$\sum_i \mathbf{F}_i \cdot \delta\mathbf{r}_i = \sum_j Q_j \delta q_j\,. \tag{2.25}$$

If the forces derive from a scalar potential V, so that $F_i = -\nabla_i V(q_j)$, the j component of the generalized force for conservative systems can be written as

$$Q_j = -\sum_i \nabla_i V \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} = -\frac{\partial V}{\partial q_j}\,. \tag{2.26}$$

Using (2.22), the second term of (2.20) yields instead

$$\sum_i \dot{\mathbf{p}}_i \cdot \delta\mathbf{r}_i = \sum_i m_i \ddot{\mathbf{r}}_i \cdot \delta\mathbf{r}_i = \sum_j \left(\sum_i m_i \ddot{\mathbf{r}}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j}\right)\delta q_j\,. \tag{2.27}$$

Because the generalized coordinates are independent, (2.20), (2.25) and (2.27) yield

$$\sum_i m_i \ddot{\mathbf{r}}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} - Q_j = 0\,, \tag{2.28}$$

which is equivalent to

$$\sum_i \left[m_i \dot{\mathbf{r}}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} - m_i \dot{\mathbf{r}}_i \cdot \frac{d}{dt}\frac{\partial \mathbf{r}_i}{\partial q_j} \right] - Q_j = 0\,. \tag{2.29}$$

Remembering that $\dot{\mathbf{r}}_i = \mathbf{v}_i$ and using

$$\frac{\partial \mathbf{r}_i}{\partial q_j} = \frac{\partial \mathbf{v}_i}{\partial \dot{q}_j}\,,$$

and

$$\frac{d}{dt}\frac{\partial \mathbf{r}_i}{\partial q_j} = \frac{\partial}{\partial q_j}\left(\sum_k \frac{\partial \mathbf{r}_i}{\partial q_k}\dot{q}_k + \frac{\partial \mathbf{r}_i}{\partial t} \right) = \frac{\partial \mathbf{v}_i}{\partial q_j}\,,$$

which come from (2.23), Eq. (2.29) can be written as

$$\sum_i \left[\frac{d}{dt}\left(m_i \mathbf{v}_i \cdot \frac{\partial \mathbf{v}_i}{\partial \dot{q}_j} \right) - m_i \mathbf{v}_i \cdot \frac{\partial \mathbf{v}_i}{\partial q_j} \right] - Q_j = 0\,,$$

or

$$\sum_i \left[\frac{d}{dt}\frac{\partial}{\partial \dot{q}_j}\frac{m_i \mathbf{v}_i^2}{2} - \frac{\partial}{\partial q_j}\frac{m_i \mathbf{v}_i^2}{2} \right] - Q_j = 0\,, \tag{2.30}$$

where $T = \sum_i m_i \mathbf{v}_i^2/2$ is the total kinetic energy of the system. With (2.26), (2.30) yields

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j} - \frac{\partial}{\partial q_j}(T - V) = 0\,. \tag{2.31}$$

Because $\partial V/\partial \dot{q}_j = 0$, and introducing the Lagrangian function

$$L = T - V\,, \tag{2.32}$$

Equation (2.31) takes the final form

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0\,,\ i = 1, \ldots, 3N - k\,, \tag{2.33}$$

which are the Euler–Lagrange equations (2.11).

It should be noted that (2.33) is valid only if dissipative forces are not present and if the potential is independent from the velocities. For notable exception, such as the Lorentz force and Rayleigh's dissipation, see e.g. [4]. Finally, it should be also noted that (2.33) involves energy terms as it derives from D'Alembert's principle, which, in dimensional form, expresses an energy balance.

2.4 Covariance of the Lagrangian with Respect to Generalized Coordinates

As mentioned in the Introduction, one of the advantages of the action principle is that it is covariant with respect to a change of generalized coordinates.

To see this, reconsider Eq. (2.11) and assume that $\{Q_k\}_{k=1,\ldots,n}$ is another set of generalized coordinates. Then, $q_i = f_i(Q_1, \ldots, Q_n)$, that is, in short,

$$q_i = f_i(Q_k) \ , \ \frac{\partial f_i}{\partial Q_k} \neq 0 \, \forall i, \ \forall k \ . \tag{2.34}$$

Hence,

$$\dot{q}_i = \sum_{k=1}^{n} \frac{\partial f_i}{\partial Q_k} \dot{Q}_k \ . \tag{2.35}$$

Conversely, if $\hat{f}$ is the inverse coordinate transformation, i.e. $Q_i = \hat{f}_i(q_1, \ldots, q_n)$, one has

$$\dot{Q}_i = \sum_{k=1}^{n} \frac{\partial \hat{f}_i}{\partial q_k} \dot{q}_k \ . \tag{2.36}$$

Quantities that transform under change of coordinate as (2.36) are called *covariant vectors*. It should be noted that traditionally covariant (and their correspondent contravariant) vectors are indicated with the use of subscripts and superscripts. In this book, we will, however, not employ this notation.

Based on (2.34) and (2.35), the Lagrangian (2.2) transforms as

$$L(q, \dot{q}, t) = L\left[f(Q), \nabla_Q f \cdot \dot{\mathbf{Q}}, t\right] = \tilde{L}(Q, \dot{Q}, t) \ , \tag{2.37}$$

where

$$q = q_1 \ldots q_n \ , \tag{2.38a}$$

$$Q = Q_1 \ldots Q_n \ , \tag{2.38b}$$

$$\nabla_Q f \cdot \dot{\mathbf{Q}} = \sum_{k=1}^{n} \frac{\partial f}{\partial Q_k} \dot{Q}_k \ , \tag{2.38c}$$

In the following, we will assume the position $i = 1, \ldots, n$. Starting from (2.37), the quantities $\partial\tilde{L}/\partial Q_k,\ \partial\tilde{L}/\partial\dot{Q}_k,\ d/dt(\partial\tilde{L}/\partial\dot{Q}_k)$ can be evaluated as follows: the first two quantities are

$$\begin{aligned}
\frac{\partial\tilde{L}}{\partial Q_k} &= \frac{\partial}{\partial Q_k} L(q_i, \dot{q}_i, t) \\
&= \frac{\partial L}{\partial q_i}\frac{\partial q_i}{\partial Q_k} + \frac{\partial L}{\partial \dot{q}_i}\frac{\partial \dot{q}_i}{\partial Q_k} \\
&= \frac{\partial L}{\partial q_i}\frac{\partial f_i}{\partial Q_k} + \frac{\partial L}{\partial \dot{q}_i}\frac{\partial}{\partial Q_k}\sum_j \frac{\partial f_i}{\partial Q_j}\dot{Q}_j \\
&= \frac{\partial L}{\partial q_i}\frac{\partial f_i}{\partial Q_k} + \frac{\partial L}{\partial \dot{q}_i}\sum_j \frac{\partial^2 f_i}{\partial Q_k \partial Q_j}\dot{Q}_j\,,
\end{aligned} \tag{2.39}$$

and

$$\frac{\partial\tilde{L}}{\partial \dot{Q}_k} = \frac{\partial L}{\partial \dot{q}_i}\frac{\partial \dot{q}_i}{\partial \dot{Q}_k} = \frac{\partial L}{\partial \dot{q}_i}\frac{\partial}{\partial \dot{Q}_k}\sum_j \frac{\partial f_i}{\partial Q_j}\dot{Q}_j = \frac{\partial L}{\partial \dot{q}_i}\frac{\partial f_i}{\partial Q_k}\,. \tag{2.40}$$

The time derivative of (2.40) gives

$$\begin{aligned}
\frac{d}{dt}\frac{\partial\tilde{L}}{\partial \dot{Q}_k} &= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\frac{\partial f_i}{\partial Q_k} + \frac{d}{dt}\left(\frac{\partial f_i}{\partial Q_k}\right)\frac{\partial L}{\partial \dot{q}_i} \\
&= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\frac{\partial f_i}{\partial Q_k} + \frac{\partial L}{\partial \dot{q}_i}\sum_j \frac{\partial^2 f_i}{\partial Q_k \partial Q_j}\dot{Q}_j\,.
\end{aligned} \tag{2.41}$$

From (2.39), one has

$$\frac{\partial L}{\partial \dot{q}_i}\sum_j \frac{\partial^2 f_i}{\partial Q_k \partial Q_j}\dot{Q}_j = \frac{\partial\tilde{L}}{\partial Q_k} - \frac{\partial L}{\partial q_i}\frac{\partial f_i}{\partial Q_k}\,, \tag{2.42}$$

and substitution of (2.42) into (2.41) implies

$$\frac{d}{dt}\frac{\partial\tilde{L}}{\partial \dot{Q}_k} - \frac{\partial\tilde{L}}{\partial \dot{Q}_k} = \frac{\partial f_i}{\partial Q_k}\left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i}\right)\,. \tag{2.43}$$

Finally, because of the second equation of (2.34) and (2.11), Eq. (2.43) yields

$$\frac{d}{dt}\frac{\partial\tilde{L}}{\partial \dot{Q}_k} - \frac{\partial\tilde{L}}{\partial \dot{Q}_k} = 0\,. \tag{2.44}$$

Equation (2.44) retains openly the same form as (2.11), so the covariance of (2.11) under (2.34) is proved.

2.5 Role of Constraints

As stated in Sect. 2.2, the presence of constraints in the system can introduce mutual dependencies between the generalized coordinates. To see how the equations of motion can be derived from Hamilton's principle even in the presence of constraints, consider first the general problem of finding the extrema of a function $\phi(x_1, \dots, x_n)$ that is not subject to constraints. The extrema can thus be located at the points where $\nabla\phi = 0$. In the presence of constraints determined by m equations $f_\alpha(x_1, \dots, x_n) = 0$, $\alpha = 1, \dots, m$, the problem can be solved finding the extrema of the auxiliary function

$$F(x_1, \dots, x_n, \lambda_1, \dots, \lambda_m) = \phi(x_1, \dots, x_n) + \sum_{\alpha=1}^{m} [\lambda_\alpha f_\alpha(x_1, \dots, x_n)] \ , \tag{2.45}$$

where

$$\lambda_\alpha, \ \alpha = 1, \dots, m \ , \tag{2.46}$$

are the *indeterminate Lagrange multipliers* of the system. The problem of finding the extrema of the function $\phi(x_1, \dots, x_n)$ subject to constraints is thus turned into the problem of finding the extrema of the auxiliary function $F(x_1, \dots, x_n, \lambda_1, \dots, \lambda_m)$ in the absence of constraints.

Consider now a system which is described by the Lagrangian $L(q_1, \dots, q_n, \dot{q}_1, \dots, \dot{q}_n, t)$ subject to m constraints that we assume can be expressed in the form

$$f_\alpha(q_1, \dots, q_n, \dot{q}_1, \dots, \dot{q}_n, t) = 0, \ (\alpha = 1, \dots, m) \ . \tag{2.47}$$

In analogy with the previous example, the derivation of the equations of motion can thus be obtained from Hamilton's Principle of Least Action as

$$\delta \int_{t_1}^{t_2} L_c dt = 0 \ , \tag{2.48}$$

where

$$L_c = L + \sum_{\alpha=1}^{m} \lambda_\alpha f_\alpha \ . \tag{2.49}$$

Notice that

$$\lambda_\alpha = \frac{\partial L_c}{\partial f_\alpha} \ . \tag{2.50}$$

Hamilton's principle (2.48) thus leads to

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = F_i, \ (i = 1, \dots, n) \ , \tag{2.51}$$

where F_i, $i = 1, \ldots, n$, are the *generalized forces*

$$F_i = \sum_{\alpha=1}^{m} \left\{ \lambda_\alpha \left[\frac{\partial f_\alpha}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial f_\alpha}{\partial \dot{q}_i} \right) \right] \right\}, \quad (i = 1, \ldots, n) \ . \tag{2.52}$$

Equations (2.47) and (2.51) constitute thus a system of $n + m$ equations in $n + m$ variables describing a system under the action of the generalized forces (2.52) exerted by constraints.

2.6 Canonical Variables and Hamiltonian Function

Consider the Euler–Lagrange equations (2.11), and assume that the potential V is a function only of the position of the point particles and not of the generalized velocities. Then,

$$\frac{\partial L}{\partial \dot{q}_i} = \frac{\partial T}{\partial \dot{q}_i} - \frac{\partial V}{\partial \dot{q}_i} = \frac{\partial T}{\partial \dot{q}_i} = \frac{\partial}{\partial \dot{q}_i} \sum_{i=1}^{n} \frac{1}{2} m_i \dot{q}_i^2 = m_i \dot{q}_i = p_i \ . \tag{2.53}$$

Equation (2.53) defines thus the *generalized* or *conjugate momentum*

$$p_i = \frac{\partial L}{\partial \dot{q}_i} \ . \tag{2.54}$$

The insertion of (2.54) in (2.11) gives thus the equation for the time evolution of p_i

$$\dot{p}_i = \frac{\partial L}{\partial q_i} \ . \tag{2.55}$$

It is important to notice that as the generalized coordinates q_i are not Cartesian, the conjugate momentum p_i does not necessarily correspond to the linear momentum. The pair of generalized variables (q_i, p_i) takes also the name of *canonical variables*. Notice that (2.54) and (2.55) allow to write the differential of the Lagrangian

$$dL = \sum_{i=1}^{n} \left(\frac{\partial L}{\partial q_i} dq_i + \frac{\partial L}{\partial \dot{q}_i} d\dot{q}_i \right) + \frac{\partial L}{\partial t} dt \ , \tag{2.56}$$

in the form

$$dL = \sum_{i=1}^{n} (\dot{p}_i dq_i + p_i d\dot{q}_i) + \frac{\partial L}{\partial t} dt \ , \tag{2.57}$$

which will be useful for the derivation of the canonical form of the equations of motion.

Consider now the case in which the Lagrangian does not depend on a given generalized coordinate q_j. In that case, the coordinate is called *cyclic* and (2.11) yields

$$\frac{\partial L}{\partial q_j} = 0 \Rightarrow \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} = 0 \,. \tag{2.58}$$

With (2.54), the r.h.s of (2.58) implies

$$\frac{dp_j}{dt} = 0 \,, \tag{2.59}$$

which shows that in the case in which the generalized coordinate q_j is cyclic, the corresponding conjugate momentum p_j is a constant of the motion. Conversely, if the conjugate momentum p_j is a conserved quantity of the system, the Lagrangian L does not depend on the corresponding generalized coordinate q_j.

The Lagrangian formulation of mechanics here developed depends on the set of coordinates $(q_i, \dot{q}_i, t)$. It is possible, however, to build a different formulation aiming at describing the equations of motion in terms of first-order equations in function of the canonical coordinates (q_i, p_i, t). The new formulation can be built through a Legendre transform of $L(q_i, \dot{q}_i, t)$ that defines the function

$$H(q, p, t) = \sum_i \dot{q}_i p_i - L(q_i, \dot{q}_i, t) \,, \tag{2.60}$$

which takes the name of *Hamiltonian function*, or simply *Hamiltonian*. A definition and some of the mathematical properties of the Legendre transform are reported in Appendix C. One should notice the important difference in the dependent variables between the Hamiltonian function $H(q, p, t) = H(q_1, \ldots, q_n, p_1, \ldots, p_n, t)$ and the Lagrangian $L(q, \dot{q}, t) = L(q_1, \ldots, q_n, \dot{q}_1, \ldots, q_n, t)$.

Notice that (2.60) yields, equivalently,

$$L(q_i, \dot{q}_i, t) = \sum_i \dot{q}_i p_i - H(q, p, t) \,, \tag{2.61}$$

which will sometimes be used.

2.7 Hamilton's Equations

The differential of (2.60) is

$$dH = \sum_i (\dot{q}_i dp_i + p_i d\dot{q}_i) - dL \,, \tag{2.62}$$

so that, using (2.57),

$$dH = \sum_i (\dot{q}_i dp_i - \dot{p}_i dq_i) - \frac{\partial L}{\partial t} dt \,. \tag{2.63}$$

Equation (2.63) can be compared with the differential of the Hamiltonian obtained from the chain rule

$$dH = \sum_i \left(\frac{\partial H}{\partial q_i} dq_i + \frac{\partial H}{\partial p_i} dp_i \right) + \frac{\partial H}{\partial t} dt \,. \tag{2.64}$$

Direct comparison of (2.63) and (2.64) gives the *canonical equations*

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \,, \tag{2.65a}$$

$$-\dot{p}_i = \frac{\partial H}{\partial q_i} \,, \tag{2.65b}$$

$$-\frac{\partial L}{\partial t} = \frac{\partial H}{\partial t} \,, \tag{2.65c}$$

the first two of which, i.e.

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \,, \tag{2.66a}$$

$$\dot{p}_i = -\frac{\partial H}{\partial q_i} \,, \tag{2.66b}$$

take the name of *Hamilton's equations*.

Remark 2.1 In comparison with the Euler–Lagrange equations (2.11), which leads to a set of n second-order equations, Hamilton's equations are $2n$ first-order ordinary differential equations for $q_i(t)$ and $p_i(t)$. The initial value problem can be solved specifying the initial conditions $q_i(0)$ and $p_i(0)$. For geometric reasons, one should notice that Hamiltonian dynamics take place in even dimensional spaces.

Hamilton's equations (2.66a), (2.66b) can be derived, in full analogy with the method of Sect. 2.2, from the Least Action Principle by finding an extremum of the functional

$$I(l) = \int_{t_1}^{t_2} \left[\sum_{i=1}^{n} \dot{q}_i p_i - H(q, p, t) \right] dt \,, \tag{2.67}$$

where

$$q_i = q_i(t, 0) + l\eta_i(t),\;\; p_i = p_i(t, 0) + l\varphi_i(t)\;, \tag{2.68}$$

and

$$\eta_i(t_1) = \eta_i(t_2) = \varphi_i(t_1) = \varphi_i(t_2) = 0\;. \tag{2.69}$$

In fact, according to (2.67),

$$\begin{aligned}
\delta I &= \frac{\partial I}{\partial l} dl \\
&= \int_{t_1}^{t_2} \left[\sum_{i=1}^{n} \left(\frac{\partial \dot{q}_i}{\partial l} p_i + \dot{q}_i \frac{\partial p_i}{\partial l} - \frac{\partial H}{\partial q_i}\frac{\partial q_i}{\partial l} - \frac{\partial H}{\partial p_i}\frac{\partial p_i}{\partial l} \right) dl \right] dt \\
&= \int_{t_1}^{t_2} \left[\sum_{i=1}^{n} \left(\dot{\eta}_i p_i + \dot{q}_i \varphi_i - \frac{\partial H}{\partial q_i}\eta_i - \frac{\partial H}{\partial p_i}\varphi_i \right) dl \right] dt \\
&= \int_{t_1}^{t_2} \left\{ \sum_{i=1}^{n} \left[\left(\dot{q}_i - \frac{\partial H}{\partial p_i} \right) \varphi_i + \dot{\eta}_i p_i - \frac{\partial H}{\partial q_i}\eta_i \right] dl \right\} dt \\
&= \int_{t_1}^{t_2} \left\{ \sum_{i=1}^{n} \left[\left(\dot{q}_i - \frac{\partial H}{\partial p_i} \right) \varphi_i - \left(\dot{p}_i + \frac{\partial H}{\partial q_i} \right) \eta_i \right] dl \right\} dt + [p_i \eta_i]_{t_1}^{t_2}\, dl\;.
\end{aligned} \tag{2.70}$$

Because of the arbitrariness of functions η_i, φ_i and (2.69), from (2.70) one concludes that

$$\delta I = 0 \iff \left(\dot{q}_i = \frac{\partial H}{\partial p_i} \text{ and } \dot{p}_i = -\frac{\partial H}{\partial q_i} \right) \forall i = 1, \ldots, n\;. \tag{2.71}$$

Finally, from the last equality of (2.53) and the definition of Lagrangian

$$H(q, p, t) = \sum_{i=1}^{n} \dot{q}_i p_i - (T - V)\;, \tag{2.72}$$

so that, if the potential is a function only of the generalized coordinates, from (2.53),

$$H = T + V = E\;, \tag{2.73}$$

so that the Hamiltonian is the total energy of the system.

In matrix form, Hamilton's equations (2.66a), (2.66b) can be written as

$$\begin{pmatrix} \dot{q}_i \\ \dot{p}_i \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial H}{\partial q_i} \\ \frac{\partial H}{\partial p_i} \end{pmatrix}\;. \tag{2.74}$$

The matrix

$$\mathbf{J} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} . \tag{2.75}$$

takes the name of *symplectic*, or sometimes *co-symplectic*, matrix. The term "symplectic" was introduced by the mathematician Hermann Weyl; it has origins from ancient Greek and it means "intertwined", as it clearly combines the variables q_i and p_i. It should be noted that $\mathbf{J}$ is antisymmetric that means that its transpose is its negative, i.e.

$$\mathbf{J}^T = -\mathbf{J} . \tag{2.76}$$

Further, it can be easily seen that the inverse of $\mathbf{J}$ is its transpose, so that

$$\mathbf{J}^T = \mathbf{J}^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -\mathbf{J} . \tag{2.77}$$

The properties

$$\mathbf{J}\mathbf{J}^T = \mathbf{J}^T\mathbf{J} = 1 , \tag{2.78a}$$

$$\mathbf{J}^2 = 1 , \tag{2.78b}$$

$$\det \mathbf{J} = 1 , \tag{2.78c}$$

follow directly from (2.77) and (2.76).

The relation (2.75) offers a trivial geometrical interpretation of Hamilton's equations in terms of the symplectic matrix. Consider in fact the rotation of coordinates in the plane, which can be written as

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = R \begin{pmatrix} x \\ y \end{pmatrix} , \tag{2.79}$$

where $\mathbf{R}$ is the two-dimensional orthogonal rotation matrix, $\mathbf{R} \in SO(2)$, described in 1.116 and reported here again for convenience

$$\mathbf{R} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} . \tag{2.80}$$

It is visible thus that $\mathbf{J}$ is the matrix corresponding to a rotation through $\pi/2$ in the clockwise direction and, following (2.74), the flow in phase space equals the gradient of H rotated by the same angle. Further, one can see that a point in the phase space moves with the speed $(\dot{q}_i^2 + \dot{p}_i^2)^{1/2} = |\nabla H|$.

The previous discussion can be generalized observing that the canonical equations (2.66a), (2.66b) are not symmetric, due to the negative sign in the equation for $\dot{p}_i$ that is missing in the equation for $\dot{q}_i$. The two equations can be written in symmetric form introducing the symplectic notation defining the vector $\boldsymbol{z}$ so that

$$z_i = q_i \ , \tag{2.81a}$$

$$z_{i+n} = p_i \ , \tag{2.81b}$$

with $i = 1, \ldots, n$. In matrix form, $\boldsymbol{z}$ can be written as

$$\boldsymbol{z} = \begin{pmatrix} z_1 \\ \vdots \\ z_n \\ z_{n+1} \\ \vdots \\ z_{2n} \end{pmatrix} = \begin{pmatrix} q_1 \\ \vdots \\ q_n \\ p_1 \\ \vdots \\ p_n \end{pmatrix} . \tag{2.82}$$

In the same way, define the vectors

$$\left(\frac{\partial H}{\partial \boldsymbol{z}}\right)_i = \frac{\partial H}{\partial q_i} \ , \tag{2.83}$$

$$\left(\frac{\partial H}{\partial \boldsymbol{z}}\right)_{i+n} = \frac{\partial H}{\partial p_i} \ , \tag{2.84}$$

with $i = 1, \ldots, n$. Once again, in matrix form,

$$\frac{\partial H}{\partial \boldsymbol{z}} = \begin{pmatrix} \frac{\partial H}{\partial q_1} \\ \vdots \\ \frac{\partial H}{\partial q_n} \\ \frac{\partial H}{\partial p_1} \\ \vdots \\ \frac{\partial H}{\partial p_n} \end{pmatrix} . \tag{2.85}$$

The canonical equations (2.65c) can thus be rewritten using (2.82) and (2.85) as

$$\dot{\boldsymbol{z}} = \mathbf{J} \frac{\partial H}{\partial \boldsymbol{z}} \ , \tag{2.86}$$

where $\mathbf{J}$ is a $2n \times 2n$ squared matrix made by the four blocks composed by two $n \times n$ null matrices and two identity matrices $\mathbf{I}$ composed as

$$\mathbf{J} = \begin{pmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{I} & \mathbf{0} \end{pmatrix} . \tag{2.87}$$

If $n = 2$, $\mathbf{J}$ reduces to (2.75). At the next order,

$$\mathbf{J} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} , \tag{2.88}$$

and so forth.

Remark 2.2 The symplectic form of Hamilton's equations allows for a geometric formulation of mechanics, which has many important features that are subject of current research. While the description of the dynamics on the symplectic manifold is an essential part of the study of classical dynamics and of mathematical physics, a description of the motion in local coordinates is to be preferred when explicit quantitative results are wanted. In this book, we will follow this second route. For a description of classical dynamics on the manifold, with attention also to infinite dimensional systems and to fluid dynamics, the reader is referred, for example, to [2, 9, 13].

2.8 Canonical Transformations and Generating Functions

In this section, we want to define what are the conditions to transform a set of canonical coordinates into a new set of canonical coordinates. To do so, consider the transformations of the kind

$$Q_i = Q_i(q, p, t) \ , \ i = 1, \ldots, n \ , \tag{2.89a}$$

$$P_i = P_i(q, p, t) \ , \ i = 1, \ldots, n \ , \tag{2.89b}$$

where $q = q_1, \ldots, q_n$ and $p = p_1, \ldots, p_n$ are canonical coordinates. The dependent variables Q_i and P_i are also canonical coordinates *provided that* there exists some function $K(Q, P, t)$ such that

$$\dot{P}_i = -\frac{\partial K}{\partial Q_i} \ , \ \dot{Q}_i = \frac{\partial K}{\partial P_i} \ . \tag{2.90}$$

The relationship between (q_i, p_i) and (Q_i, P_i) is based on the variational principles $\delta \int_{t_1}^{t_2} L(q, \dot{q}, t) dt = 0$ and $\delta \int_{t_1}^{t_2} L(Q, \dot{Q}, t) dt = 0$ which, according to (2.61), take the form

$$\delta \int_{t_1}^{t_2} [\dot{q}_i p_i - H(q, p, t)] \, dt = 0 \ , \tag{2.91}$$

and

$$\delta \int_{t_1}^{t_2} \left[\dot{Q}_i P_i - H(Q, P, t)\right] dt = 0 \ , \tag{2.92}$$

respectively. We stress that if (Q_i, P_i) are canonical coordinates, then the simultaneous validity of (2.91) and (2.92) holds true. In turn, this request is realized if the integrands of (2.91) and (2.92) differ, at most, by the total derivative of an arbitrary function of both the old and the new canonical coordinates, say $F = F(q, p, Q, P, t)$. In fact, the difference between the integrals in (2.91) and in (2.92) is given by

$$\int_{t_1}^{t_2} \frac{dF}{dt} dt = F(q, p, Q, P, t_2) - F(q, p, Q, P, t_1) = 0 ,$$

as the phase space coordinates have zero variations at the end points. Apart from the time variable, the number of independent variables of $F = F(q, p, Q, P, t)$ is not $4n$; in fact, relationships (2.89a), (2.89b) reduce those independent to $2n$, so that only the following possibilities are allowed:

$$F = F_1(q, Q, t) \; , \; F = F_2(q, P, t) \; , \; F = F_3(p, Q, t) \; , \; F = F_4(p, P, t) \; , \tag{2.93}$$

where, for instance, $F_1(q, Q, t) = F_1(q_1, \ldots, q_n, Q_1, \ldots, Q_n, t)$, and so on. We now proceed to evaluate

$$\dot{q}_i p_i - H = \dot{Q}_i P_i - K + \frac{dF_1}{dt} , \tag{2.94}$$

keeping in mind that, in this particular case (i.e. $F = F_1$), q and Q are independent. Equation (2.94) yields

$$\dot{q}_i p_i - H = \dot{Q}_i P_i - K + \frac{\partial F_1}{\partial t} + \frac{\partial F_1}{\partial q_i}\dot{q}_i + \frac{\partial F_1}{\partial Q_i}\dot{Q}_i , \tag{2.95}$$

and, owing to the independence between q and Q, we have

$$p_i = \frac{\partial F_1}{\partial q_i} , \tag{2.96a}$$

$$P_i = -\frac{\partial F_1}{\partial Q_i} , \tag{2.96b}$$

$$K = H + \frac{\partial F_1}{\partial t} . \tag{2.96c}$$

Equation (2.96a) can be solved to give

$$Q_i = Q_i(q, p, t) , \tag{2.97}$$

that is (2.89a). Once (2.97) is known, Eq. (2.96b) can be used to single out (2.89b). Moreover, Eq. (2.96c) connects the new and the old Hamiltonian functions. The method involving the generating functions F_2, F_3 and F_4 is analogous, with the following choices of F_2, F_3 and F_4 allowing to carry out the computations explicitly in the same way:

$$F_2(q, P, t) = F_1(q, Q, t) - P_i Q_i \ , \tag{2.98}$$

with $\partial F_2/\partial Q_i = 0$ because of (2.96b),

$$F_3(q, P, t) = F_1(q, Q, t) - p_i q_i \ , \tag{2.99}$$

with $\partial F_3/\partial q_i = 0$ because of (2.96a), and

$$F_4(q, P, t) = F_1(q, Q, t) + P_i Q_i - p_i q_i \ , \tag{2.100}$$

with $\partial F_4/\partial q = 0$ because of (2.96a) and $\partial F_4/\partial Q = 0$ because of (2.96b). Substitution of (2.98) into (2.94) yields

$$p_i = \frac{\partial F_2}{\partial q_i} \ , \tag{2.101a}$$

$$Q_i = \frac{\partial F_2}{\partial P_i} \ , \tag{2.101b}$$

$$K = H + \frac{\partial F_2}{\partial t} \ . \tag{2.101c}$$

Equation (2.101a) can be solved to give

$$P_i = P_i(q, p, t) \ , \tag{2.102}$$

that is to say (2.89b). Once (2.102) is known, Eq. (2.101b) can be used to single out (2.89a). The connection between the Hamiltonians is expressed by (2.101c). Substitution of (2.99) into (2.94) yields

$$q_i = -\frac{\partial F_3}{\partial p_i} \ , \tag{2.103a}$$

$$P_i = -\frac{\partial F_3}{\partial Q_i} \ , \tag{2.103b}$$

$$K = H + \frac{\partial F_3}{\partial t} \ . \tag{2.103c}$$

Equation (2.103a) can be solved to give

$$Q_i = Q_i(q, p, t) \ , \tag{2.104}$$

that is to say (2.89a). Once (2.104) is known, Eq. (2.103b) can be used to single out (2.89b). The connection between the Hamiltonians is here expressed by (2.103c). Substitution of (2.100) into (2.94) yields

$$q_i = -\frac{\partial F_4}{\partial p_i} \,, \tag{2.105a}$$

$$Q_i = \frac{\partial F_4}{\partial P_i} \,, \tag{2.105b}$$

$$K = H + \frac{\partial F_4}{\partial t} \,. \tag{2.105c}$$

Equation (2.105a) can be solved to give

$$P_i = P_i(q, p, t) \,, \tag{2.106}$$

that is to say (2.89b). Once (2.106) is known, Eq. (2.105b) can be used to single out (2.89a). The connection between the Hamiltonians is now expressed by (2.105c).

2.8.1 Phase Space Volume as Canonical Invariant: Liouville's Theorem and Poisson Brackets

The results from the previous section can be easily extended to the symplectic formalism. Consider a system described by a set of canonical coordinates under (2.86). Recalling the definition of covariant vectors from Sect. 2.4, in this section we will show that the transformation (2.89a), (2.89b) is covariant.

Defining

$$\zeta_i = Q_i(z_j) \,, \tag{2.107a}$$

$$\zeta_{i+n} = P_i(z_j) \,, \tag{2.107b}$$

$(i, j = 1, \ldots, n)$, if K is the Hamiltonian in the new coordinates, satisfying

$$H(\boldsymbol{z}) = K(\boldsymbol{\zeta}) \,, \tag{2.108}$$

the equation of motions must satisfy

$$\dot{\boldsymbol{\zeta}} = \mathbf{J}\frac{\partial K}{\partial \boldsymbol{\zeta}} \,. \tag{2.109}$$

At the same time,

$$\dot{\zeta}_i = \frac{\partial \zeta_i}{\partial z_j}\dot{z}_j, \;\; (i, j = 1, \ldots, n) \,, \tag{2.110}$$

which shows that $\boldsymbol{z}$ transforms covariantly. Equation (2.110) thus yields

$$\begin{aligned}\dot{\zeta}_i &= \frac{\partial \zeta_i}{\partial z_j} J_{jk} \frac{\partial H}{\partial z_k} \\ &= \frac{\partial \zeta_i}{\partial z_j} J_{jk} \frac{\partial K}{\partial \zeta_l} \frac{\partial \zeta_l}{\partial z_k} \\ &= \mathbf{M}_{il} \frac{\partial K}{\partial \zeta_l} \,, \end{aligned} \tag{2.111}$$

$(i, j, k, l = 1, \ldots, n)$, where

$$\mathbf{M}_{il} = \frac{\partial \zeta_i}{\partial z_j} J_{jk} \frac{\partial \zeta_l}{\partial z_k} \,, \tag{2.112}$$

is the transformed symplectic operator. These last relationships show that $\mathbf{J}$ transforms as a *contravariant* (hence the name co-symplectic) tensor with rank 2. Comparison between (2.111), (2.112) and (2.110) states that the transformation of coordinates is canonical, i.e., it preserves the form of Hamilton's equations, if

$$\left(\frac{\partial \boldsymbol{\zeta}}{\partial \mathbf{z}}\right) \mathbf{J} \left(\frac{\partial \boldsymbol{\zeta}}{\partial \mathbf{z}}\right)^T = \mathbf{J} \,, \tag{2.113}$$

or, equivalently, if

$$\left(\frac{\partial \boldsymbol{\zeta}}{\partial \mathbf{z}}\right)^T \mathbf{J} \left(\frac{\partial \boldsymbol{\zeta}}{\partial \mathbf{z}}\right) = \mathbf{J} \,. \tag{2.114}$$

An important consequence of the these results is that the phase space defined by the canonical coordinates (q_i, p_i) has the property that the volume

$$V = \int dq_1 \ldots dq_n dp_1 \ldots dp_n \tag{2.115}$$

is a canonical invariant, i.e., it is invariant under a canonical transformation of coordinates. To demonstrate this property, indicate the infinitesimal volume in the original set of coordinates with

$$dq_1 \ldots dq_n dp_1 \ldots dp_n \,, \tag{2.116}$$

and with

$$dQ_1 \ldots dQ_n dP_1 \ldots dP_n \,, \tag{2.117}$$

the volume in a set of transformed coordinates (2.89a), (2.89b), under a canonical transformation. As well known from multivariate calculus, the two infinitesimal volumes are connected by the determinant of the Jacobian matrix, i.e., the determinant of a symplectic matrix that is equal to 1. This is an important result that states that in conservative systems, element of volumes in the phase space is conserved. The consequences of this result are wide and have applications in the different fields of mathematics and physics, ranging from chaotic dynamics to statistical mechanics. One of them that has consequences that have parallels with fluid dynamics is given by Liouville's theorem. If $\rho(q_1, \dots, q_n, p_1, \dots, p_n) dq_1 \dots dq_n dp_1 \dots dp_n$ is the probability that a trajectory of the system in phase space is in the infinitesimal volume $dq_1 \dots dq_n dp_1 \dots dp_n$, where $\rho(q_1, \dots, q_n, p_1, \dots, p_n)$ is the probability density function, the invariance of the volume yields

$$\begin{aligned} \frac{d\rho}{dt} &= \frac{\partial \rho}{\partial t} + \sum_{i=1}^{n} \left(\frac{\partial \rho}{\partial q_i} \dot{q}_i + \frac{\partial \rho}{\partial p_i} \dot{p}_i \right) \\ &= \frac{\partial \rho}{\partial t} + \sum_{i=1}^{n} \left(\frac{\partial \rho}{\partial q_i} \frac{\partial H}{\partial p_i} - \frac{\partial \rho}{\partial p_i} \frac{\partial H}{\partial q_i} \right) \\ &= 0 \, . \end{aligned} \tag{2.118}$$

Equation (2.118) has the same form of fluid dynamics equation for the conservation of density (1.6), or, in general, for the evolution of a passive tracer stirred by a flow characterized by a stream function that in (2.118) corresponds to the Hamiltonian.

In (2.118), the object

$$\{\rho, H\} = \sum_{i=1}^{n} \left(\frac{\partial \rho}{\partial q_i} \frac{\partial H}{\partial p_i} - \frac{\partial \rho}{\partial p_i} \frac{\partial H}{\partial q_i} \right) \tag{2.119}$$

is called a *canonical Poisson bracket*. For a system in equilibrium, $\partial \rho / \partial t = 0$, and (2.118) yields $\{\rho, H\} = 0$, which is satisfied if $\rho = \rho(H)$, i.e. if ρ is a function of the energy of the system.

In general, given two functions $F(q_1, \dots, q_n, p_1, \dots, p_n)$ and $G(q_1, \dots, q_n, p_1, \dots, p_n)$, a canonical Poisson bracket is defined as

$$\{F, G\} = \sum_{i=1}^{n} \left(\frac{\partial F}{\partial q_i} \frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i} \frac{\partial G}{\partial q_i} \right) \, . \tag{2.120}$$

By the chain rule of differentiation, the time evolution of one of the functions, say F, can be written as

$$\frac{dF}{dt} = \sum_{i=1}^{n} \left(\frac{\partial F}{\partial q_i} \frac{dq_i}{dt} + \frac{\partial F}{\partial p_i} \frac{dp_i}{dt} \right) \, . \tag{2.121}$$

Using Hamilton's equations (2.66a), (2.66b), Eq. (2.121) gives

$$\frac{dF}{dt} = \sum_{i=1}^{n} \left(\frac{\partial F}{\partial q_i} \frac{\partial H}{\partial p_i} - \frac{\partial F}{\partial p_i} \frac{\partial H}{\partial q_i} \right) \, , \tag{2.122}$$

so that, using (2.120), (2.122) yields

$$\frac{dF}{dt} = \{F, H\} \tag{2.123}$$

that shows that the evolution of a generic function can be determined from the Poisson bracket of the function with the Hamiltonian of the system. Notice that (2.123) is in agreement with the definition of the canonical equations (2.66a), (2.66b),

$$\frac{dq_i}{dt} = \{q_i, H\} = \frac{\partial H}{\partial p_i}\,, \tag{2.124}$$

$$\frac{dp_i}{dt} = \{p_i, H\} = -\frac{\partial H}{\partial q_i}\,. \tag{2.125}$$

Relation (2.123) shows also that

$$\{F, H\} = 0 \iff \frac{dF}{dt} = 0\,, \tag{2.126}$$

i.e. if a function F commutes with the Hamiltonian H, it is an invariant of the motion of the system. In particular, the invariance of the Hamiltonian is trivially proved, as

$$\{H, H\} = 0\,, \tag{2.127}$$

in agreement with the energy conservation of the system. The Poisson bracket is an important geometric object in mechanics, and it is worth showing its algebraic properties:

Theorem 2.2 *Given the functions f, g, h, the Poisson bracket satisfies the following properties (the proofs are reported in Appendix I):*

1. *Self-commutation*

$$\{f, f\} = 0\,. \tag{2.128}$$

2. *Skew-symmetry*

$$\{f, g\} = -\{g, f\}\,. \tag{2.129}$$

3. *Distributive property*

$$\{\alpha f + \beta g, h\} = \alpha\{f, h\} + \beta\{g, h\}\,. \tag{2.130}$$

   where $\alpha, \beta \in \mathbb{R}$.

4. *Associative property*

$$\{fg, h\} = f\{g, h\} + \{f, h\}h\,. \tag{2.131}$$

5. *The Jacobi identity*

$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0 \,. \tag{2.132}$$

These properties define a nonassociative Lie algebra.

In a similar way as was done at the beginning of this section, it is possible to express the Poisson bracket in the new set of coordinates. Given two functions $f(q_1, \ldots, q_n, p_1, \ldots, p_n)$ and $g(q_1, \ldots, q_n, p_1, \ldots, p_n)$ and indicating with $F(Q_1, \ldots, Q_n, P_1, \ldots, P_n)$ and $G(Q_1, \ldots, Q_n, P_1, \ldots, P_n)$ the corresponding functions in the transformed coordinates, the Poisson bracket is

$$\begin{aligned}\{f, g\} &= \left(\frac{\partial f}{\partial \mathbf{z}}\right)^T \mathbf{J} \left(\frac{\partial g}{\partial \mathbf{z}}\right) \\ &= \left(\frac{\partial f}{\partial \boldsymbol{\zeta}} \frac{\partial \boldsymbol{\zeta}}{\partial \mathbf{z}}\right)^T \mathbf{J} \left(\frac{\partial g}{\partial \boldsymbol{\zeta}} \frac{\partial \boldsymbol{\zeta}}{\partial \mathbf{z}}\right) \\ &= \left(\frac{\partial f}{\partial \boldsymbol{\zeta}}\right)^T \mathbf{J} \left(\frac{\partial g}{\partial \boldsymbol{\zeta}}\right) \,,\end{aligned} \tag{2.133}$$

where, in the last passage, we have used property (2.114). The relation (2.133) shows that the Poisson bracket is a canonical invariant of the system.

2.8.2 Casimir Invariants and Invertible Systems

A special case of (2.126) is given by a function C that commutes with every other functions F,

$$\{F, C\} = 0, \ \forall F \,. \tag{2.134}$$

In this case, the function C takes the name of *Casimir invariant*, as (2.123) with $F = H$ trivially implies

$$\{C, H\} = 0, \ \Rightarrow \ \frac{dC}{dt} = 0 \,. \tag{2.135}$$

Writing (2.134) in symplectic form and considering that F is an arbitrary function yield

$$\mathbf{J}\frac{\partial C}{\partial \mathbf{z}} = \mathbf{0} \,. \tag{2.136}$$

Relation (2.136) implies that $\partial C/\partial \mathbf{z}$ belongs to the kernel of $\mathbf{J}$ and has important consequences linked to the invertibility of the dynamics of the system. In fact, if $\mathbf{J}^{-1}$ exists, i.e., the Hamiltonian formulation is invertible, (2.136) implies

$$\frac{\partial C}{\partial \mathbf{z}} = \mathbf{0}, \ \Rightarrow \ C = \text{const} \,. \tag{2.137}$$

In this case, the Casmir function is said to be *trivial*. If instead $\mathbf{J}^{-1}$ does not exists, i.e. the Hamiltonian formulation is not invertible, there exists at least one Casimir that is nontrivial. In this case, the dynamics is invariant under the summation of a Casmir to the Hamiltonian; in fact,

$$\mathbf{J}\frac{\partial(H+C)}{\partial \mathbf{z}} = \mathbf{J}\frac{\partial H}{\partial \mathbf{z}} + \mathbf{J}\frac{\partial C}{\partial \mathbf{z}} = \mathbf{J}\frac{\partial H}{\partial \mathbf{z}} = \dot{\mathbf{z}} \,. \tag{2.138}$$

The geometric interpretation of (2.138) relies in the idea that the motion takes place on hypersurfaces where $\partial C/\partial \mathbf{z} = \mathbf{0}$, which are also called *symplectic leaves*. Each of these surfaces is a regular Hamiltonian phase space where the dynamics are regulated by H.

Remark 2.3 As remarked at the end of Sect. 2.7, in the remaining of this book we will not make use of the Poisson bracket. It is, however, difficult to overestimate the importance that this object covers in modern mechanics, making it thus deserving of this very short introduction.

2.9 Noether's Theorem for Point Particles

In this section, we will study the important link between continuous symmetries and conserved quantities, which is explicated by Noether's Theorem, using the symmetries of the Lagrangian operator. The theorem will be first exposed in its formulation for point particles and will then be extended to continuous systems, which include the case of fluids.

Consider a system of point particles, Noether's Theorem thus states that

Theorem 2.3 *If the Lagrangian function is invariant under a continuous and infinitesimal transformation of its spatial and temporal variables, the transformation defines a scalar quantity that is a constant of motion of the system.*

The proof of the theorem will be divided into a mathematical preliminary and it will then be linked to the physics of the system.

2.9.1 Mathematical Preliminary

We start by defining the functional

$$I = \int_{t_1}^{t_2} L\left(q_i, \dot{q}_i, t\right) dt \ , \tag{2.139}$$

and the transformations

$$t' = t + \delta t \, , \tag{2.140a}$$

$$q_i'(t') = q_i(t) + \delta q_i(t) \, , \tag{2.140b}$$

$$\dot{q}_i'(t') = \dot{q}_i(t) + \delta \dot{q}_i(t) \, . \tag{2.140c}$$

The quantities δt, $\delta q_i(t)$, $\delta \dot{q}_i(t)$ are arbitrary differentiable functions of time whose higher order amplitudes are negligible. Notice that these arbitrary differentiable functions can eventually be a constant. In the examples that will follow the proof, different forms of $\delta t, \; \delta q_i(t)$ and $\delta \dot{q}_i(t)$ will be introduced.

The transformations in (2.140a)–(2.140c) generate, through (2.139), a functional variation δI defined as

$$\delta I = \int_{R'} L\left(t', q_i', \dot{q}_i'\right) dt' - \int_R L\left(t, q_i, \dot{q}_i\right) dt \, , \tag{2.141}$$

where $R = [t_1, t_2], \; R' = \left[t_1', t_2'\right]$ and $\int_R dt = \int_{R'} dt'$. Equation (2.141) shows that the functional variation δI is the difference between (2.139) calculated after and before the transformations (2.140a)–(2.140c). As it is shown in Appendix D, Eq. (2.141) yields, at first order, the functional variation in the form

$$\delta I = \int_R \left\{ \frac{D}{Dt} \left[\left(L - \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i \right) \delta t + \frac{\partial L}{\partial \dot{q}_i} \delta q_i \right] + \left[\frac{\partial L}{\partial q_i} - \frac{D}{Dt} \frac{\partial L}{\partial \dot{q}_i} \right] (\delta q_i - \dot{q}_i \delta t) \right\} dt \, , \tag{2.142}$$

where

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \dot{q}_i \frac{\partial}{\partial q_i} + \ddot{q}_i \frac{\partial}{\partial \dot{q}_i} \, . \tag{2.143}$$

If the integrand in (2.139) is the Lagrangian $L = T - V$, and using the initial and final conditions $\delta t(t_1) = \delta t(t_2) = 0 \; , \; \delta q_i(t_1) = \delta q_i(t_2) = 0$, due to Hamilton's principle, the equations of motion (see Sect. 2.2)

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0 \tag{2.144}$$

hold. In this (physically relevant) case, (2.142) yields

$$\delta I = \int_R \frac{D}{Dt} \left[\left(L - \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i \right) \delta t + \frac{\partial L}{\partial \dot{q}_i} \delta q_i \right] dt \, . \tag{2.145}$$

2.9.2 Symmetry Transformations and Proof of the Theorem

Among the transformations (2.140a)–(2.140c), of particular interest are the transformations denoted as

$$t' = t + \delta_S t \; , \tag{2.146a}$$
$$q_i'(t') = q_i(t) + \delta_S q_i(t) \; , \tag{2.146b}$$
$$\dot{q}_i'(t') = \dot{q}_i(t) + \delta_S \dot{q}_i(t) \; , \tag{2.146c}$$

for which the Lagrangian function transforms under two conditions, namely

$$L'\left(t', q_i', \dot{q}_i'\right) dt' = L\left(t, q_i, \dot{q}_i\right) dt \; , \tag{2.147a}$$
$$L'\left(t', q_i', \dot{q}_i'\right) = L\left(t', q_i', \dot{q}_i'\right) + \frac{D}{Dt'}\left(\delta_S \Omega\right) \; . \tag{2.147b}$$

Equation (2.147a) is valid because the functional (2.139) is a scalar and, under the action of the transformations (2.140a)–(2.140c) (and (2.146a)–(2.146c) as a special case) must transform as a scalar. Equation (2.147b), instead, expresses the covariance of Lagrangian function specifically under the *symmetry transformations* (2.146a)–(2.146c). This means that the functional dependence of the Lagrangian on the space–time coordinates remains unaltered under transformations (2.146a)–(2.146c), apart from the term $D/Dt'\,(\delta_S \Omega)$. The presence of this additional term can be explained observing that the equations of motion (2.144) are invariant under the divergence transformation

$$L \to L + \frac{D}{Dt}\left[\delta_S \Omega(t, q_i)\right] \; , \tag{2.148}$$

where $\partial \Omega / \partial \dot{q}_i = 0$, as demonstrated in Appendix E. The strategy of the proof will now be to substitute the transformations (2.146a)–(2.146c) in (2.147a) and (2.147b) and to compare the results in order to find what kind of constraints (2.146a)–(2.146c) pose on the equations of motion (2.144).

Substitution of (2.146a)–(2.146c) in (2.147a) yields

$$L'\left(t + \delta_S t, q_i + \delta_S q_i, \dot{q}_i + \delta_S \dot{q}_i\right) dt' = L\left(t, q_i, \dot{q}_i\right) dt \; , \tag{2.149}$$

while (2.147b) yields

$$\begin{aligned} &L'\left(t + \delta_S t, q_i + \delta_S q_i, \dot{q}_i + \delta_S \dot{q}_i\right) dt' \\ &= L\left(t + \delta_S t, q_i + \delta_S q_i, \dot{q}_i + \delta_S \dot{q}_i\right) dt' + \frac{D\left(\delta_S \Omega\right)}{Dt} dt' \; . \end{aligned} \tag{2.150}$$

Equating (2.149) and (2.150) yields

$$L\left(t + \delta_S t, q_i + \delta_S q_i, \dot{q}_i + \delta_S \dot{q}_i\right) dt' - L\left(t, q_i, \dot{q}_i\right) dt + \frac{D\left(\delta_S \Omega\right)}{Dt} dt' = 0 \; . \tag{2.151}$$

Because

$$dt' = dt\left[1 + \frac{d}{dt}\left(\delta_S t\right)\right] \; , \tag{2.152}$$

and because $\delta_S \ll 1$, ignoring terms of higher order, one gets

$$\frac{D}{Dt'}\left(\delta_S \Omega\right) dt' = \frac{D}{Dt}\left(\delta_S \Omega\right) dt \ . \tag{2.153}$$

Starting from (2.153), we thus pose two goals:

- To derive a test function that allows to prove the invariance of the Lagrangian function upon a continue and infinitesimal symmetry transformation in the class represented by (2.146a)–(2.146c);
- To obtain the correspondent conserved quantity.

In reference to the first goal, Eq. (2.151) can be rewritten using (2.152) and (2.153), so that

$$L\left(t + \delta_S t, q_i + \delta_S q_i, \dot{q}_i + \delta_S \dot{q}_i\right)\left[1 + \frac{d}{dt}\left(\delta_S t\right)\right] - L\left(t, q_i, \dot{q}_i\right) dt = -\frac{D}{Dt}\left(\delta_S \Omega\right) \ . \tag{2.154}$$

At first order, (2.154) yields

$$\left[\delta_S t \frac{\partial}{\partial t} + \delta_S q_i \frac{\partial}{\partial q_i} + \delta_S \dot{q}_i \frac{\partial}{\partial \dot{q}_i} + \frac{d\left(\delta_S t\right)}{dt}\right] L\left(t, q_i, \dot{q}_i\right) = -\frac{D}{Dt}\left(\delta_S \Omega\right) \ . \tag{2.155}$$

The test consists thus in applying the operator between square brackets on the l.h.s. of (2.155) to the assigned Lagrangian, in order to verify if it produces the term on the r.h.s.. Notice that the function $\delta_S \Omega(t, q_i)$ is arbitrary and may be null.

In reference to the second goal, consider the time integration of (2.151), i.e.

$$\int_{R'} L\left(t + \delta_S t, q_i + \delta_S q_i, \dot{q}_i + \delta_S \dot{q}_i\right) dt' - \int_R L\left(t, q_i, \dot{q}_i\right) dt + \int_R \frac{D}{Dt}\left(\delta_S \Omega\right) dt = 0 \, . \tag{2.156}$$

The first two terms in (2.156) are the functional variation (2.141) that, because of Hamilton's principle, must have the form (2.145), yielding

$$\int_R \frac{D}{Dt}\left[\left(L - \frac{\partial L}{\partial \dot{q}_i}\dot{q}_i\right)\delta_S t + \frac{\partial L}{\partial \dot{q}_i}\delta_S q_i + \delta_S \Omega\right] dt = 0 \, . \tag{2.157}$$

Because (2.157) is independent on the integration interval, from the same equation it is possible to derive the conservation law

$$\frac{D}{Dt}\left[\left(L - \frac{\partial L}{\partial \dot{q}_i}\dot{q}_i\right)\delta_S t + \frac{\partial L}{\partial \dot{q}_i}\delta_S q_i + \delta_S \Omega\right] = 0 \, , \tag{2.158}$$

or

$$\left(L - \frac{\partial L}{\partial \dot{q}_i}\dot{q}_i\right)\delta_S t + \frac{\partial L}{\partial \dot{q}_i}\delta_S q_i + \delta_S \Omega = const \ . \tag{2.159}$$

To summarize: every continuous and differential transformation (2.146a)–(2.146c) that turns (2.155) into an identity produces a conserved quantity in the form of (2.159).

2.9.3 Some Examples

As a preamble notice that in classical mechanics for point particles, the Lagrangian has usually the shape

$$L = \sum_i \frac{m_i}{2}\left[(\dot{x})^2 + (\dot{y})^2 + (\dot{z})^2\right] - V\left(\sum_{i \neq j} r_{ij}\right) , \tag{2.160}$$

where, using Cartesian coordinates, $r_{ij} = \left[\left(x_i - x_j\right)^2 + \left(y_i - y_j\right)^2 + \left(z_i - z_j\right)^2\right]^{1/2}$.

2.9.3.1 Invariance for Translations of Amplitude l Along the x-axis

In terms of equation (2.146a)–(2.146c), this symmetry transformation is

$$t' = t , \tag{2.161a}$$

$$x_i' = x_i + l, \quad y_i' = y_i, \quad z_i' = z_i , \tag{2.161b}$$

$$\dot{x}_i' = \dot{x}_i, \quad \dot{y}_i' = \dot{y}_i, \quad \dot{z}_i' = \dot{z}_i , \tag{2.161c}$$

where $\delta_S q_i = l$ if $q_i = x_i$ and $\delta_S q_i = 0$ if $q_i \neq x_i$. It is visible that, because $x_i - x_j = (x_i + l) - (x_j + l)$, the distance r_{ij} between two particles does not change for translations, so that the Lagrangian (2.160) does not change as well. It is, however, possible to proceed formally, recurring to (2.155), that for this case yields

$$\sum_k \frac{\partial L}{\partial x_k} = -\frac{D\Omega}{Dt} . \tag{2.162}$$

The sum in (2.162) is applied to the space derivatives with respect to x for all the particles of the system, so that

$$\sum_k \frac{\partial L}{\partial x_k} = -\sum_i \frac{\partial V}{\partial x_i} - \sum_j \frac{\partial V}{\partial x_j} = -\sum_{i \neq j}\left(\sum_i \frac{\partial V}{\partial x_i} + \sum_j \frac{\partial V}{\partial x_j}\right) = 0 . \tag{2.163}$$

Equation (2.163) shows that (2.155) is satisfied for $\Omega = const$. According to (2.159), the corresponding conserved quantity is

$$\sum_k \frac{\partial L}{\partial \dot{x}_k} = \sum_k m_k \dot{x}_k \; . \tag{2.164}$$

The term on the r.h.s. of (2.164), i.e. the conserved quantity, is the total (i.e. of the entire system) linear momentum along the x-axis.

2.9.3.2 Invariance for Time Translations of Amplitude τ

In terms of equation (2.146a)–(2.146c), the symmetry transformation is

$$t' = t + \tau \; , \tag{2.165a}$$

$$x'_i = x_i, \quad y'_i = y_i, \quad z'_i = z_i \; , \tag{2.165b}$$

$$\dot{x}'_i = \dot{x}_i, \quad \dot{y}'_i = \dot{y}_i, \quad \dot{z}'_i = \dot{z}_i \; , \tag{2.165c}$$

where $\delta_S t = \tau$. Equation (2.160) shows that the Lagrangian does not depend explicitly on time and thus (2.155) follows immediately from the invariance $\partial L/\partial t = 0$ with $\Omega = const$. As a consequence, (2.159) yields

$$L - m_i(\dot{x}_i)^2 = const \; . \tag{2.166}$$

Because $m_i(\dot{x}_i)^2 = 2T$, (2.166) is the conservation of the total (kinetic plus potential) energy of the system,

$$T + V = const \; . \tag{2.167}$$

2.9.3.3 Invariance for Rotations of Amplitude α Around the z-axis

In terms of equation (2.146a)–(2.146c), the transformation is

$$t' = t \; , \tag{2.168a}$$

$$x'_i = x_i + \alpha y_i, \quad y'_i = y_i - \alpha x_i, \quad z'_i = z_i \; , \tag{2.168b}$$

$$\dot{x}'_i = \dot{x}_i + \alpha \dot{y}_i, \quad \dot{y}'_i = \dot{y}_i - \alpha \dot{x}_i, \quad \dot{z}'_i = \dot{z}_i \; , \tag{2.168c}$$

so that

$$\delta_S q_i = \alpha y_i \quad \text{if} \quad q_i = x_i \; , \tag{2.169a}$$

$$\delta_S q_i = -\alpha x_i \quad \text{if} \quad q_i = y_i \; , \tag{2.169b}$$

$$\delta_S q_i = 0 \quad \text{if} \quad q_i = z_i \; . \tag{2.169c}$$

It should be noted that in this case the quantities $\delta_S q_i$ are not constants, but they are the product of an infinitesimal parameter α by a coordinate that is a function of time. The same considerations are valid for the $\delta_S \dot{q}_i$. Notice also that the infinitesimal parameter α derives from the finite rotation

$$x' = \cos\alpha x + \sin\alpha y \ , \tag{2.170a}$$

$$y' = -\sin\alpha x + \cos\alpha y \ , \tag{2.170b}$$

after the truncation at first order of the expansion in α of the trigonometrical functions, following the well-known approximations $\cos\alpha \approx 1$ and $\sin\alpha \approx \alpha$. Using (2.169a)–(2.169c), Eq. (2.155) yields

$$\left[\alpha\left(y_i\frac{\partial}{\partial x_i} - x_i\frac{\partial}{\partial y_i}\right) + \alpha\left(\dot{y}_i\frac{\partial}{\partial \dot{x}_i} - \dot{x}_i\frac{\partial}{\partial \dot{y}_i}\right)\right](T - V) = -\frac{D}{Dt}(\delta_S \Omega) \ . \tag{2.171}$$

Because

$$\left(\dot{y}_i\frac{\partial}{\partial \dot{x}_i} - \dot{x}_i\frac{\partial}{\partial \dot{y}_i}\right)T = m_i\left(\dot{y}_i\dot{x}_i - \dot{x}_i\dot{y}_i\right) = 0 \ ,$$

and

$$\left(y_i\frac{\partial}{\partial x_i} - x_i\frac{\partial}{\partial y_i}\right)V = 0 \ ,$$

as it is verified from the identity

$$\frac{\partial}{\partial x_i}(x_m - x_n) = \frac{\partial}{\partial y_i}(y_m - y_n) = \delta_{im} - \delta_{in} = 0 \ , \tag{2.172}$$

where δ_{kl} is a Kronecker delta, then (2.171) is immediately verified for $\Omega = const$, for all values of α. As a consequence, (2.159) takes the form

$$\delta_S q_i \frac{\partial L}{\partial q_i} = const \ , \tag{2.173}$$

or, using (2.169a)–(2.169c),

$$\frac{\partial}{\partial x_i}(x_m - x_n) = \frac{\partial}{\partial y_i}(y_m - y_n) = \delta_{im} - \delta_{in} = 0 \ , \tag{2.174}$$

that is

$$m_i\left(y_i\dot{x}_i - x_i\dot{y}_i\right) = const \ . \tag{2.175}$$

Equation (2.175) expresses the conservation of the total angular momentum about the z-axis.

2.9.3.4 Invariance for a Translation with Constant Velocity Along the x-axis

In terms of (2.146a)–(2.146c), the transformation is

$$t' = t \ , \tag{2.176a}$$

$$x_i' = x_i - t\delta v, \quad y_i' = y_i, \quad z_i' = z_i \ , \tag{2.176b}$$

$$\dot{x}_i' = \dot{x}_i - \delta v, \quad \dot{y}_i' = \dot{y}_i, \quad \dot{z}_i' = \dot{z}_i \ , \tag{2.176c}$$

so that $\delta_S q_i = -t\delta v$ if $q_i = x_i$ else $\delta_S q_i = 0$; $\delta_S \dot{q}_i = -\delta v$ if $q_i = x_i$ else $\delta_S \dot{q}_i = 0$. With this form for the transformation, (2.155) yields

$$\left[\sum_i \left(-t\delta v \frac{\partial}{\partial x_i} - \delta v \frac{\partial}{\partial \dot{x}_i}\right)\right] L = -\frac{D}{Dt}\left(\delta_S \Omega\right) \ . \tag{2.177}$$

Because $\sum_i \partial V/\partial x_i = 0$, (2.177) simplifies to

$$\sum_i \frac{\partial T}{\partial \dot{x}_i} = \frac{D\Omega}{Dt} \ , \tag{2.178}$$

that is

$$\sum_i m_i x_i = \frac{D\Omega}{Dt} \ . \tag{2.179}$$

Thus, if

$$\Omega = \sum_i m_i x_i \ , \tag{2.180}$$

Equation (2.177) becomes an identity. Given (2.180), Eq. (2.159) determines the conserved quantity

$$\sum_i \left(-t \frac{\partial T}{\partial \dot{x}_i} + m_i x_i\right) = const \ , \tag{2.181}$$

that is

$$\sum_i m_i x_i - t \sum_i m_i \dot{x}_i = const \ . \tag{2.182}$$

If

$$x_c = \frac{\sum_i m_i x_i}{\sum_i m_i} \ , \tag{2.183}$$

is the coordinate of the centre of mass of the system about the x-axis, and using (2.164), then (2.182) describes the motion of the centre of mass, that, as it is well known, is uniform in the absence of forces that are external to the system.

2.9.3.5 Damped Oscillator

The following example is drawn from [15] and reconsidered here in full details. It is noticeable because, although the single-particle system taken into account does not conserve neither energy nor momentum, Noether's Theorem allows to derive a conservation law which cannot emerge from the Euler–Lagrange equation alone. The starting point is the following Lagrangian governing a damped oscillator

$$L = \frac{1}{2}\left[m\dot{x}^2 - kx^2\right]\exp\left(\frac{bt}{m}\right) . \tag{2.184}$$

Hence, the evolution is given by the Euler–Lagrange equation (2.11) which yields the ODE

$$m\ddot{x} + b\dot{x} + kx = 0 . \tag{2.185}$$

In Eq. (2.185), $b\dot{x}$ ($b > 0$) is the damping term while kx ($k > 0$) is the restoring force. The first of them obviously prevents energy conservation; in fact, (2.185) implies

$$m\dot{x}\ddot{x} + b\dot{x}\dot{x} + kx = 0 .$$

that is to say, after time integration on (t_1, t_2)

$$E(t_2) - E(t_1) + b\int_{t_1}^{t_2}(\dot{x})^2 dt = 0 , \tag{2.186}$$

where $E = (m/2)(\dot{x})^2 + (k/2)x^2$ is total energy of the oscillating particle. Equation (2.186) openly shows that kinetic energy is not conserved because of the energy sink $-b\int_{t_1}^{t_2}(\dot{x})^2 dt$. The restoring force prevents linear momentum conservation. In fact, time integration of (2.185) with $x(t_1) = x(t_2) = 0$ immediately gives

$$p(t_2) - p(t_1) + k\int_{t_1}^{t_2} x dt = 0 , \tag{2.187}$$

where $p = m\dot{x}$ is the linear momentum of the particle. According to Eq. (2.187), the linear momentum is not conserved because of the sink $-k\int_{t_1}^{t_2} x dt$. After these preliminaries, consider again (2.184). The Lagrangian conserves its form under the space–time infinitesimal transformation

$$t' = t + \varepsilon , \tag{2.188a}$$

$$x' = x - \varepsilon\left(\frac{bx}{2m}\right) , \tag{2.188b}$$

where positive powers of ε are hereafter neglected with respect to the first one. Thus, (2.188a), (2.188b) can be reverted to give

$$t = t' - \varepsilon \ , \tag{2.189a}$$

$$x = x' + \varepsilon \left(\frac{bx'}{2m} \right) \ , \tag{2.189b}$$

Substitution of (2.189a), (2.189b) into (2.184) proves invariance. In fact,

$$\begin{aligned} L &= \frac{1}{2} \left[m\dot{x}'^2 - kx'^2 \right] \left(1 + \varepsilon \frac{b}{2m} \right)^2 \exp\left(\frac{b}{m}(t' - \varepsilon) \right) \\ &= \frac{1}{2} \left[m\dot{x}'^2 - kx'^2 \right] \left(1 + \varepsilon \frac{b}{m} \right) \exp\left(\frac{bt'}{m} \right) \exp\left(-\varepsilon \frac{b}{m} \right) \\ &= \frac{1}{2} \left[m\dot{x}'^2 - kx'^2 \right] \left(1 + \varepsilon \frac{b}{m} \right) \left(1 - \varepsilon \frac{b}{m} \right) \exp\left(\frac{bt'}{m} \right) \\ &= \frac{1}{2} \left[m\dot{x}'^2 - kx'^2 \right] \exp\left(\frac{bt'}{m} \right) \ , \end{aligned}$$

that is to say

$$L(t, x, \dot{x}) = L(t', x', \dot{x}') \ . \tag{2.190}$$

Invariance (2.190) leads to a conserved quantity according to Noether's Theorem. Transformations (2.188a), (2.188b) allow us to identify

$$\delta_s t = \varepsilon \ , \ \delta_s q = \delta_s x = \varepsilon \left(\frac{b}{2m} \right) x \ , \tag{2.191}$$

so the conserved quantity of the general theory (2.159) takes here the form

$$L - \frac{\partial L}{\partial \dot{x}} \dot{x} - \left(\frac{b}{2m} x \right) \frac{\partial L}{\partial \dot{x}} = const \ . \tag{2.192}$$

Finally, substitution of (2.184) into (2.192) produces the explicit version of the conserved quantity, that is to say

$$\left[m(\dot{x})^2 + kx^2 + bx\dot{x} \right] \exp\left(\frac{bt}{m} \right) = const \ . \tag{2.193}$$

In turn, the constant at the r.h.s. of (2.193) can be singled out by substituting the general integral of the ODE (2.185) into the l.h.s of the same equation. The exponential damping of the oscillator compensates exactly the exponential factor at the l.h.s. of (2.193).

2.10 Lagrangian Formulation for Fields: Lagrangian Depending on a Scalar Function

The Lagrangian formulation for point particles reviewed in the previous sections can be easily expanded for systems with an infinite number of degrees of freedom, i.e. for continuous systems such as fluids. In this section, the formulation of Lagrangian dynamics will be specifically introduced making use of a scalar function, i.e. a stream function, ψ. Later in the chapter, the formulation will be presented for Lagrangian functions depending on vector functions.

In a continuous system, the independent variables are the time t and the space coordinates $x,\ y,\ z$, while the dependent variables that will be considered here are the current function ψ and its first derivatives, e.g. $\partial\psi/\partial t$, etc. The notation

$$(t, x, y, z) = (q_0, q_1, q_2, q_3) \tag{2.194}$$

and its abbreviation

$$(q_0, q_1, q_2, q_3) = q \tag{2.195}$$

will be used when convenient; for example, we will use the notation $\int_R d(q) = \int_R \prod_{k=0}^{3} dq_k$, where $R \subset \mathbb{R}^4$ is the integration domain. Using this notation yields

$$\psi\,(t, x, y, z) = \psi\,(q_0, q_1, q_2, q_3) = \psi\,(q)\ , \tag{2.196}$$

while the partial derivatives are expressed with the index that identifies the variable with respect to which the derivative is taken, e.g.

$$\frac{\partial\psi}{\partial q_k} = \psi_k\ ,\ \ \frac{\partial^2\psi}{\partial q_k \partial q_l} = \psi_{kl}\ . \tag{2.197}$$

Using this notation, the Lagrangian L is a function of all the independent and dependent variables and it will be indicated as

$$L\,(q_k, \psi, \psi_i)\ , \tag{2.198}$$

where $k = 0, 1, 2, 3$ and $i = 0, 1, 2, 3$.

The symbol D/Dq_k generalizes the Lagrangian derivative with time as the only independent variable, and it is defined as

$$\frac{D}{Dq_k} = \frac{\partial}{\partial q_k} + \psi_k \frac{\partial}{\partial\psi} + \psi_{kl} \frac{\partial}{\partial\psi_l}\ . \tag{2.199}$$

An important observation comes from looking at the functional

$$I[\psi] = \int_R L\,(q_k, \psi, \psi_i)\, d(q)\ . \tag{2.200}$$

From (2.200), it is visible that the Lagrangian $L\,(q_k, \psi, \psi_i)$ does not have the same dimensions as the Lagrangian function used in the discrete formulation. The Lagrangian function used in continuous systems takes also the name of *Lagrangian density*. In fluid dynamics, the Lagrangian density is analogous to the Lagrangian function defined for point particles, i.e., it is defined as $L = T - V$ where now T and V represent the kinetic and potential energy densities. As it will be seen in the next chapter, in fluid dynamics the kinetic energy density is expressed as the kinetic energy of the parcels that fill up the continuum that constitutes the fluid. The potential energy density is instead given by the contribution of the internal energy, linked to the thermodynamics property of the fluid, and the external potential.

Analogously to the discrete case, the request

$$\delta I = 0 \tag{2.201}$$

allows to derive the equations of motion for the system. To do so, start by labelling the functions $\psi(q),\ \psi_i(q)$ by the label l

$$\psi(q, l) = \psi(q, 0) + l\phi(q)\ , \tag{2.202a}$$

$$\psi_i(q, l) = \psi_i(q, 0) + l\phi_i(q)\ , \tag{2.202b}$$

where $\phi(q),\ \phi_i(q) \in C^2_R$ are arbitrary functions satisfying $\phi = 0,\ \phi_i = 0$ on the boundary of the domain ∂R. Then, up to leading order in l, (2.201) corresponds to

$$\begin{aligned}
\delta I &= \int_R \left[L\,(q_k, \psi + l\phi, \psi_i + l\phi_i) - L\,(q_k, \psi, \psi_i)\right] d(q) \\
&= \int_R \left[L\,(q_k, \psi, \psi_i) + \frac{\partial L}{\partial \psi} l\phi + \frac{\partial L}{\partial \psi_i} l\phi_i - L\,(q_k, \psi, \psi_i)\right] d(q) \\
&= l \int_R \left(\frac{\partial L}{\partial \psi}\phi + \frac{\partial L}{\partial \psi_i}\phi_i\right) d(q)\ ,
\end{aligned} \tag{2.203}$$

where Einstein's summation over repeated indices has been used.

Equation (2.203) yields

$$\delta I = 0 \iff \int_R \left(\frac{\partial L}{\partial \psi}\phi + \frac{\partial L}{\partial \psi_i}\phi_i\right) d(q) = 0\ . \tag{2.204}$$

Because the functions ϕ depend only on q,

$$\frac{\partial \phi}{\partial q_i} = \frac{D\phi}{Dq_i}\ , \tag{2.205}$$

holds, so that (2.204) yields

$$\delta I = 0 \iff \int_R \left(\frac{\partial L}{\partial \psi} \phi + \frac{\partial L}{\partial \psi_i} \frac{D\phi}{Dq_i} \right) d(q) = 0 \, . \tag{2.206}$$

Splitting the integrals in (2.206), it is possible to see that the second integral can be rewritten as

$$\int_R \frac{\partial L}{\partial \psi_i} \frac{D\phi}{Dq_i} d(q) = \int_R \left[\frac{D}{Dq_i} \left(\frac{\partial L}{\partial \psi_i} \phi \right) - \phi \frac{D}{Dq_i} \frac{\partial L}{\partial \psi_i} \right] d(q) \, , \tag{2.207}$$

so that (2.206) yields

$$\delta I = 0 \iff \int_R \left[\phi \left(\frac{\partial L}{\partial \psi} - \frac{D}{Dq_i} \frac{\partial L}{\partial \psi_i} \right) + \frac{D}{Dq_i} \left(\frac{\partial L}{\partial \psi_i} \phi \right) \right] d(q) = 0 \, . \tag{2.208}$$

Consider now the integral of the second term in the square brackets of (2.208),

$$\int_R \frac{D}{Dq_i} \left(\frac{\partial L}{\partial \psi_i} \phi \right) d(q) \, . \tag{2.209}$$

To simplify the notation, let

$$P_i = \frac{\partial L}{\partial \psi_i} \phi \, , \tag{2.210}$$

with boundary conditions

$$P_i(\partial R) = 0 \, . \tag{2.211}$$

In the simplified notation, (2.209) is written as

$$\int_R \frac{DP_i}{Dq_i} d(q) = \int_R \frac{DP_i}{Dq_i} dq_i \prod_{k \neq i} dq_k \, . \tag{2.212}$$

Considering the integral

$$\int_R \frac{DP_i}{Dq_i} dq_i \, , \tag{2.213}$$

it is possible to see that

$$\frac{DP_i}{Dq_i} dq_i = \frac{\partial P_i}{\partial q_i} dq_i + \frac{\partial P_i}{\partial \psi} \frac{\partial \psi}{\partial q_i} dq_i + \frac{\partial P_i}{\partial \psi_l} \frac{\partial \psi_l}{\partial q_i} dq_i = dP_i \, , \tag{2.214}$$

so that (2.213) yields

$$\int_R \frac{DP_i}{Dq_i} dq_i = \int_R dP_i = 0 \, . \tag{2.215}$$

Using (2.215), (2.208) reduces to

$$\delta I = 0 \iff \int_R \left[\phi\left(\frac{\partial L}{\partial \psi} - \frac{D}{Dq_i}\frac{\partial L}{\partial \psi_i}\right)\right] d(q) = 0\,. \tag{2.216}$$

Because the functions ϕ are arbitrary, (2.216) yields

$$\frac{D}{Dq_i}\frac{\partial L}{\partial \psi_i} - \frac{\partial L}{\partial \psi} = 0\,, \tag{2.217}$$

which are the Euler–Lagrange equations for continuous systems.

In (2.217), it is possible to define

$$\frac{D}{Dq_i}\frac{\partial L}{\partial \psi_i} - \frac{\partial L}{\partial \psi} = -\frac{\delta \mathscr{L}}{\delta \psi}\,, \tag{2.218}$$

where

$$\mathscr{L} = \int_{R_V} L dV\,, \tag{2.219}$$

where $dV = dq_1 dq_2 dq_3$ and $R_V \subset \mathbb{R}^3$. The r.h.s. of (2.218) defines the *functional derivative* of the functional $\mathscr{L}$. For a definition of functional derivatives, see Appendix F.

2.10.1 Hamiltonian for Scalar Fields

Analogously to the point-particle formulation for dynamics, from the Lagrangian density for scalar fields it is possible to define a Hamiltonian density from a Legendre transform of the Lagrangian density, i.e., in analogy with (2.61)

$$H = \frac{\partial L}{\partial \dot{\psi}}\dot{\psi} - L\,. \tag{2.220}$$

From (2.217), we define the canonical momentum density as

$$\pi = \frac{\partial L}{\partial \dot{\psi}}\,, \tag{2.221}$$

where we have underlined the role of the time derivative using the dot symbol. From (2.220), the Hamiltonian density of the system can thus be defined as

$$H = \pi\dot{\psi} - L\,. \tag{2.222}$$

It should be noted that, from the definition of the canonical momentum density (2.221), the Hamiltonian density (2.222) singles out the time variable from the spatial variables. This is different from the Lagrangian formulation in which the space and time variables are instead treated in the same way.

Defining the Hamiltonian as the integral of the Hamiltonian density over the spatial volume,

$$\mathscr{H} = \int_{R_V} H dV \ . \tag{2.223}$$

Hamilton's equations are defined as

$$\dot{\psi} = \frac{\delta \mathscr{H}}{\delta \pi} \ , \ \dot{\pi} = -\frac{\delta \mathscr{H}}{\delta \psi} \ . \tag{2.224}$$

The comparison between (2.224) and (2.66a), (2.66b) shows that in the case of fields the partial derivative is replaced by a functional derivative, as a straight consequence of the fact that $\mathscr{H}$ is indeed a functional.

Also for the case of scalar fields, one can introduce the concept of canonical transformations and generating functionals. Because the definition of these is essentially the same for scalar and vector fields, a complete exposition will be delayed to Sect. 2.12.2.

2.11 Noether's Theorem for Fields with the Lagrangian Depending on a Scalar Function

2.11.1 Mathematical Preliminary

As for the proof for point particles, we start the proof of Noether's Theorem for fields with the Lagrangian depending on a scalar function, with a mathematical preliminary. Given the functional (2.200) and the infinitesimal transformations of the independent variables

$$q_k' = q_k + \delta q_k \ , \tag{2.225}$$

where δq_k have the same meaning of the $\delta q_k(t)$ in the discrete system. The transformations (2.225) change the domain of integration from R into a different integration domain R'. Using (2.225), it is possible to define the dependent variables

$$\psi'(q') = \psi(q) + \delta\psi(q) \ , \tag{2.226a}$$

$$\psi_k'(q') = \psi_k(q) + \delta\psi_k(q) \ , \tag{2.226b}$$

where $q' = \left(q_o', q_1', q_2', q_3'\right)$ indicates the set of independent variables defined by (2.225). Using (2.225) and (2.226a), (2.226b), it is possible to introduce the

Lagrangian density $L\left(q'_k, \psi', \psi'_i\right)$ and the variation

$$\delta I = \int_{R'} L\left(q'_k, \psi', \psi'_i\right) d(q') - \int_R L\left(q_k, \psi, \psi_i\right) d(q) \,. \tag{2.227}$$

Taking into account the infinitesimal behaviour of δq_k in (2.225), the change of variables implies

$$d(q') = \left[1 + \frac{\partial\left(\delta q_k\right)}{\partial q_k}\right] d(q) \,, \tag{2.228a}$$

$$d(q) = \left[1 - \frac{\partial\left(\delta q_k\right)}{\partial q_k}\right] d(q') \,, \tag{2.228b}$$

where the quantities in the square brackets indicate the Jacobian of the transformation (2.225) and its inverse. As shown in Appendix G, from the expansion of (2.227) it is possible to derive, at first order,

$$\delta I = \int_R \left[\frac{D}{Dq_k}\left(L\delta q_k - \frac{\partial L}{\partial \psi_k}\psi_l \delta q_l + \frac{\partial L}{\partial \psi_k}\delta\psi\right) + \left(\frac{\partial L}{\partial \psi} - \frac{D}{Dq_k}\frac{\partial L}{\partial \psi_k}\right)\left(\delta\psi - \psi_l \delta q_l\right)\right] d(q) \,. \tag{2.229}$$

In the hypothesis that along the boundary of R hold the conditions $\delta q_k = 0,\ \delta\psi = 0$, the request of stationarity of (2.200) yields the validity of (2.217), so that (2.229) simplifies into

$$\delta I = \int_R \frac{D}{Dq_k}\left(L\delta q_k - \frac{\partial L}{\partial \psi_k}\psi_l \delta q_l + \frac{\partial L}{\partial \psi_k}\delta\psi\right) d(q) \,. \tag{2.230}$$

2.11.2 *Linking Back to the Physics*

Consider a particular case of the transformations (2.225) and (2.226a), (2.226b), represented by

$$q'_k = q_k + \delta_S q_k \,, \tag{2.231a}$$

$$\psi'(q') = \psi(q) + \delta_S \psi(q) \,, \tag{2.231b}$$

$$\psi'_k(q') = \psi_k(q) + \delta_S \psi_k(q) \,. \tag{2.231c}$$

Equations (2.231a), (2.231b) transform the Lagrangian density as

$$L'\left(q', \psi', \psi'_k\right) d(q') = L\left(q, \psi, \psi_k\right) d(q) \,, \tag{2.232a}$$

$$L'\left(q', \psi', \psi'_k\right) = L\left(q', \psi', \psi'_k\right) + \frac{D}{Dq'_k}\left(\delta_S \Omega_k\right) \,. \tag{2.232b}$$

Equation (2.232a) shows that (2.200) is a scalar that is invariant under (2.225) and (2.231a), (2.231b), while (2.232b) expresses the covariance of the Lagrangian density under the transformations (2.231a), (2.231b). The divergence transformation that appears in (2.232b) is allowed because, as it is shown in Appendix H, the equations of motion (2.217) are invariant under the substitution

$$L \to L + \frac{D}{D_k}\left(\delta_S \Omega_K\right) \,, \tag{2.233}$$

under the hypothesis $\partial\Omega_k/\partial\psi_k = 0$. Using (2.231a), (2.231b), (2.232a) takes the form

$$L'\left(q + \delta_S q, \psi + \delta_S \psi, \psi_k + \delta_S \psi_k\right) d(q') = L\left(q, \psi, \psi_k\right) d(q) \,, \tag{2.234}$$

and, analogously, from (2.232b) follows that

$$\begin{aligned} &L'\left(q + \delta_S q, \psi + \delta_S \psi, \psi_k + \delta_S \psi_k\right) d(q') = \\ &L\left(q + \delta_S q, \psi + \delta_S \psi, \psi_k + \delta_S \psi_k\right) d(q') + \frac{D}{Dq'_k}\left(\delta_S \Omega_k\right) d(q') \,. \end{aligned} \tag{2.235}$$

Because of (2.228a), (2.228b), and considering infinitesimal transformations of the kind (2.231a), (2.231b), at order zero the approximation

$$\frac{D}{Dq'_k}\left(\delta_S \Omega_k\right) d(q') = \frac{D}{Dq_k}\left(\delta_S \Omega_k\right) d(q)$$

holds and (2.235) yields

$$\begin{aligned} &L'\left(q + \delta_S q, \psi + \delta_S \psi, \psi_k + \delta_S \psi_k\right) d(q') = \\ &L\left(q + \delta_S q, \psi + \delta_S \psi, \psi_k + \delta_S \psi_k\right) d(q') + \frac{D}{Dq_k}\left(\delta_S \Omega_k\right) d(q) \,. \end{aligned} \tag{2.236}$$

From the comparison between (2.234) and (2.236), it follows that

$$\begin{aligned} &L\left(q, \psi, \psi_k\right) d(q) = \\ &L\left(q + \delta_S q, \psi + \delta_S \psi, \psi_k + \delta_S \psi_k\right) d(q') + \frac{D}{Dq_k}\left(\delta_S \Omega_k\right) d(q) \,, \end{aligned} \tag{2.237}$$

where, using (2.228a), (2.228b),

$$L\left(q+\delta_S q, \psi+\delta_S\psi, \psi_k+\delta_S\psi_k\right)= \left[L\left(q,\psi,\psi_k\right)-\frac{D}{Dq_k}\left(\delta_S\Omega_k\right)\right]\left[1-\frac{\partial\left(\delta_S q_k\right)}{\partial q_k}\right]. \tag{2.238}$$

Equation (2.238) can be further developed using a Taylor expansion truncated at first order, and eliminating all the terms of order higher that the first, yielding

$$\left[\delta_S q\frac{\partial}{\partial q}+\delta_S\psi\frac{\partial}{\partial\psi}+\delta_S\psi_k\frac{\partial}{\partial\psi_k}+\frac{\partial\left(\delta_S q_k\right)}{\partial q_k}\right]L\left(q,\psi,\psi_k\right)=-\frac{D}{Dq_k}\left(\delta_S\Omega_k\right) . \tag{2.239}$$

Equation (2.239) allows to verify the invariance of a certain Lagrangian density under a transformation of the kind (2.231a), (2.231b) and with an appropriate choice for $\Omega_i(q,\psi)$. To find the corresponding conserved quantity, (2.237) is integrated with respect to $d(q)$ over the domain R (or over R' with respect to $d(q')$), yielding

$$\int_{R'} L\left(q',\psi',\psi'_k\right)d(q')-\int_R L\left(q,\psi,\psi_k\right)d(q)+\int_R\frac{D}{Dq_k}\left(\delta_S\Omega_k\right)d(q)=0 . \tag{2.240}$$

The first two terms in (2.240) are just Eq. (2.227), with δ_S in place of a generic δ. Assuming the validity of (2.217), it is possible to substitute (2.230) in (2.240), with δ_S instead of δ, yielding

$$\int_R\frac{D}{Dq_k}\left(L\delta_S q_k-\frac{\partial L}{\partial\psi_k}\psi_l\delta_S q_l+\frac{\partial L}{\partial\psi_k}\delta_S\psi+\delta_S\Omega_k\right)d(q)=0 . \tag{2.241}$$

Due to the arbitrariness of the integration interval, (2.241) implies the conservation law

$$\frac{D}{Dq_k}\left[\left(L\delta_{kl}q_k-\frac{\partial L}{\partial\psi_k}\psi_l\right)\delta_S q_l+\frac{\partial L}{\partial\psi_k}\delta_S\psi+\delta_S\Omega_k\right]=0 , \tag{2.242}$$

where δ_{kl} is the Kronecker delta.

Summarizing: every continuous and infinitesimal transformation (2.231a), (2.231b) that change (2.239) to an identity creates a conserved quantity with form (2.242).

2.12 Lagrangian Formulation for Fields: Lagrangian Density Dependent on Vector Functions

The previous exposition of the Lagrangian formulation for continuous systems and for the corresponding version of Noether's Theorem was based on a scalar function, i.e. the current function ψ. In this section, we will summarize the Lagrangian formalism for continuous systems, such as fluids, in which the Lagrangian is expressed as a function of a vector function. The results will be shortly presented without formal proofs, which instead follow closely, apart from technical details, the proofs for the Lagrangian formulation that makes use of the scalar function.

Introduce the notation

$$(t, x, y, z) = (x_0, x_1, x_2, x_3) \ , \tag{2.243}$$

and its abbreviation

$$(x_0, x_1, x_2, x_3) = x \ , \tag{2.244}$$

where, now, $\int_R d(x) = \int_R \prod_{n=0}^{3} dx_n$, where $R \subset \mathbb{R}^4$ is the domain of integration. The dependent variables are the components

$$q^1, \ q^2, \ q^3 \ , \tag{2.245}$$

of the position vector $\mathbf{q}$. The derivatives of each component of $\mathbf{q}$ with respect to each of the independent variable are

$$\frac{\partial q^\mu}{\partial x_n}, \ \mu = 1, 2, 3, \ n = 0, 1, 2, 3 \ . \tag{2.246}$$

The following abbreviations will also be used:

$$\frac{\partial q^\mu}{\partial x_n} = q_n^\mu, \ \frac{\partial^2 q^\nu}{\partial x_n \partial x_l} = q_{nl}^\nu \ . \tag{2.247}$$

Using this notation, the Lagrangian density is a function of the independent variables, the dependent variables and their first derivatives, and it is indicated as

$$L\left(x_n, q^\mu, q_n^\nu\right) \ . \tag{2.248}$$

The Lagrangian derivative is instead expressed as

$$\begin{aligned} \frac{D}{Dx_k} &= \frac{\partial}{\partial x_k} + \frac{\partial q^\mu}{\partial x_k}\frac{\partial}{\partial q^\mu} + \frac{\partial}{\partial x_k}\left(\frac{\partial q^\mu}{\partial x_l}\right)\frac{\partial}{\partial\left(\partial q^\mu/\partial x_l\right)} \\ &= \frac{\partial}{\partial x_k} + q_k^\mu \frac{\partial}{\partial q^\mu} + q_{kl}^\mu \frac{\partial}{\partial q_l^\mu} \ . \end{aligned} \tag{2.249}$$

It is important to notice that the sub- and superscripts should not be confused with covariant and contravariant vectors, as commonly used in physics.

As for the previous section, define the functional

$$I = \int_R L\left(x_n, q^\mu, q_n^\mu\right) d(x) \ . \tag{2.250}$$

The equations of motion can be derived from the request

$$\delta I = 0\,. \tag{2.251}$$

To do so, label the functions $q^\mu,\ q_n^\mu$ by the label l

$$q^\mu(x,l) = q^\mu(x,0) + l\,Q^\mu(x)\,, \tag{2.252a}$$
$$q_n^\mu(x,l) = q_n^\mu(x,0) + l\,Q_n^\mu(x)\,, \tag{2.252b}$$

where $Q^\mu(x),\ Q_n^\mu(x) \in C_R^2$ are arbitrary functions satisfying $q_n = 0,\ q_n^\mu = 0$ on the boundary of the domain ∂R. Then, up to leading order in l, (2.251) corresponds to

$$\begin{aligned}\delta I &= \int_R \left[L\left(x_n, q^\mu + l\,Q^\mu, q_n^\mu + l\,Q_n^\mu\right) - L\left(x_n, q^\mu, q_n^\mu\right)\right] d(x)\\ &= \int_R \left[L\left(x_n, q^\mu, q_n^\mu\right) + \frac{\partial L}{\partial q^\mu} l\,Q^\mu + \frac{\partial L}{\partial q_n^\mu} l\,Q_n^\mu - L\left(x_n, q^\mu, q_n^\mu\right)\right] d(x)\\ &= l\int_R \left(\frac{\partial L}{\partial q^\mu} Q^\mu + \frac{\partial L}{\partial q_n^\mu} Q_n^\mu\right) d(x)\,,\end{aligned} \tag{2.253}$$

where Einstein's summation over repeated indices has been used. Equation (2.253) thus yields

$$\delta I = 0 \iff \int_R \left(\frac{\partial L}{\partial q^\mu} Q^\mu + \frac{\partial L}{\partial q_n^\mu} Q_n^\mu\right) d(x) = 0\,. \tag{2.254}$$

As in the analysis reported in the previous section, because the functions Q^μ depend only on x, it is possible to set

$$\frac{\partial Q^\mu}{\partial x_n} = \frac{D Q^\mu}{D x_n}\,, \tag{2.255}$$

so that (2.254) yields

$$\int_R \left(\frac{\partial L}{\partial q^\mu} Q^\mu + \frac{\partial L}{\partial q_n^\mu}\frac{D Q^\mu}{D x_n}\right) d(x) = 0\,. \tag{2.256}$$

The second integral in (2.256) can be rewritten as

$$\int_R \frac{\partial L}{\partial q_n^\mu}\frac{D Q^\mu}{D x_n} d(x) = \int_R \left[\frac{D}{D x_n}\left(\frac{\partial L}{\partial q_n^\mu} Q^\mu\right) - Q^\mu \frac{D}{D x_n}\left(\frac{\partial L}{\partial q_n^\mu}\right)\right] d(x)\,, \tag{2.257}$$

so that (2.256) yields

$$\int_R \left[Q^\mu\left(\frac{\partial L}{\partial q^\mu} - \frac{D}{D x_n}\frac{\partial L}{\partial q_n^\mu}\right) + \frac{D}{D x_n}\left(\frac{\partial L}{\partial q_n^\mu} Q^\mu\right)\right] d(x) = 0\,. \tag{2.258}$$

Considering the integral of the second term in the square brackets of (2.258),

$$\int_R \frac{D}{Dx_n}\left(\frac{\partial L}{\partial q_n^\mu} Q^\mu\right) d(x) \; , \tag{2.259}$$

and setting

$$P_n = \frac{\partial L}{\partial q_n^\mu} Q^\mu \; , \tag{2.260}$$

with boundary conditions

$$P_n(\partial R) = 0 \; , \tag{2.261}$$

Equation (2.259) yields

$$\int_R \frac{DP_i}{Dx_n} d(x) = \int_R \frac{DP_n}{Dx_n} dx_n \prod_{k \neq n} dx_k \; . \tag{2.262}$$

Consider now the integral

$$\int_R \frac{DP_n}{Dx_n} dx_n \; , \tag{2.263}$$

because

$$\frac{DP_n}{Dx_n} dx_n = \frac{\partial P_x}{\partial x_n} dx_n + \frac{\partial P_n}{\partial q^\mu} \frac{\partial q^\mu}{\partial x_n} dx_n + \frac{\partial P_n}{\partial q_l^\mu} \frac{\partial q_l^\mu}{\partial x_n} dx_n = dP_n \; , \tag{2.264}$$

Equation (2.263) yields

$$\int_R \frac{DP_n}{Dx_n} dx_n = \int_R dP_n = 0 \; , \tag{2.265}$$

so that (2.258) reduces to

$$\int_R \left[Q^\mu \left(\frac{\partial L}{\partial q^\mu} - \frac{D}{Dx_n} \frac{\partial L}{\partial q_n^\mu} \right) \right] d(x) = 0 \; . \tag{2.266}$$

Because the functions ϕ are arbitrary, (2.266) yields the equations of motion, i.e. the Euler–Lagrange equations

$$\frac{D}{Dx_k} \frac{\partial L}{\partial q_k^\mu} - \frac{\partial L}{\partial q^\mu} = 0 \; . \tag{2.267}$$

2.12.1 Hamilton's Equations for Vector Fields

Consider the **q** field defined in the previous section. In full analogy with point-particle dynamics and for the formulation of dynamics for scalar fields, from (2.267), we define the canonical momentum densities as

$$\pi^{\mu} = \frac{\partial L}{\partial \dot{q}^{\mu}} \; . \tag{2.268}$$

The quantities $q^{\mu}(x),\; p^{\mu}(x)$ define an infinite dimensional phase space of the field and its development in time. As was already noted in the derivation of the Hamiltonian density for scalar fields, the time variable is here separated from the spatial variables. In the following, we will thus split the domain $R = R_T \times R_V$, where $R_T \subset \mathbb{R}$ and $R_V \subset \mathbb{R}^3$ are the time and space integration domains. We will also use the contracted form $dV = dx_1 dx_2 dx_3$.

In the same way as it was done for point particles, if the Lagrangian density does not contain the field quantity q^{μ} explicitly, the Euler–Lagrange equation (2.267) yields the conservation law

$$\frac{D}{Dx_k}\frac{\partial L}{\partial q^{\mu}_k} = 0 \; , \tag{2.269}$$

i.e.

$$\frac{\partial \pi^{\mu}}{\partial t} + \frac{\partial}{\partial x_i}\frac{\partial L}{\partial q^{\mu}_i} = 0 \; . \tag{2.270}$$

Integration over R_V and use of the divergence theorem yield thus the conservation of the quantity

$$\int_{R_V} \pi^{\mu} dV \; . \tag{2.271}$$

Given (2.269), the Hamiltonian density can be written as

$$H = \pi^{\mu}\dot{q}^{\mu} - L \; . \tag{2.272}$$

Finally, denoting the Hamiltonian of the system with (2.223), it is possible to take the variations of the latter with respect to q^{μ} and π^{μ}, obtaining thus Hamilton's equations

$$\dot{q}^{\mu} = \frac{\delta \mathscr{H}}{\delta \pi^{\mu}} \; , \; \dot{\pi}^{\mu} = -\frac{\delta \mathscr{H}}{\delta q^{\mu}} \; . \tag{2.273}$$

2.12.2 Canonical Transformations and Generating Functionals for Vector Fields

From the previous definitions, it is possible to introduce the formalism for canonical transformations for fields depending on vector functions. Given the field functions

q^{μ} and the momentum densities π^{μ}, we can consider integral transformations into new field functions $\overline{q}^{\mu}$, $\overline{\pi}^{\mu}$. The old and new fields are related to the transformation functionals

$$\int_{R_V} \overline{q}^{\mu} dV = G^{(1,\mu)}[t, q^{\mu}, \pi^{\mu}] , \tag{2.274a}$$

$$\int_{R_V} \overline{\pi}^{\mu} dV = G^{(2,\mu)}[t, q^{\mu}, \pi^{\mu}] , \tag{2.274b}$$

where now $dV = dx_1 dx_2 dx_3$. Notice, once again, the splitting of the domain R into the time and spatial subdomains. Assuming that the integrals can be solved, we have thus

$$\int_{R_V} q^{\mu} dV = F^{(1,\mu)}[t, \overline{q}^{\mu}, \overline{\pi}^{\mu}] , \tag{2.275a}$$

$$\int_{R_V} \pi^{\mu} dV = F^{(2,\mu)}[t, \overline{q}^{\mu}, \overline{\pi}^{\mu}] . \tag{2.275b}$$

If there exists a functional

$$\overline{\mathscr{H}}[t, \overline{q}^{\mu}, \overline{\pi}^{\mu}] = \int_{R_V} \overline{H} dV , \tag{2.276}$$

preserving the form of Hamilton's equations (2.273), i.e. yielding

$$\dot{\overline{q}}^{\mu} = \frac{\delta \overline{\mathscr{H}}}{\delta \overline{\pi}^{\mu}} , \quad \dot{\overline{\pi}}^{\mu} = -\frac{\delta \overline{\mathscr{H}}}{\delta \overline{q}^{\mu}} , \tag{2.277}$$

then the transformations (2.274a), (2.274b), or equivalently (2.275a), (2.275b), are said to be canonical.

The dynamics in the original and transformed coordinates are given, respectively, by the variational principles

$$\delta \int_{R} (\pi^{\mu} \dot{q}^{\mu} - H)\, d(x) = 0 , \tag{2.278a}$$

$$\delta \int_{R} \left(\overline{\pi}^{\mu} \dot{\overline{q}}^{\mu} - \overline{H} \right) d(x) = 0 . \tag{2.278b}$$

From (2.278a) consider the functional

$$\Phi = \mathscr{L} dt = \delta \int_{R_V} (\pi^{\mu} dq^{\mu} - H dt)\, dV , \tag{2.279}$$

which takes also the name of Pfaffian functional. In (2.279), we have used the notation (2.219). It is possible to prove (see, e.g. [14]) that two Pfaffian functionals differing by a differential with respect to a parameter are equivalent. In our case, said parameter is clearly the time. This allows to set the condition for the transformation of coordinates to be canonical if

$$\int_{R_V} (\pi^\mu dq^\mu - H dt)\, dV - \int_{R_V} \left(\overline{\pi}^\mu d\overline{q}^\mu - \overline{H} dt\right) dV = d\mathscr{F} \,, \tag{2.280}$$

where

$$\mathscr{F}[t, q^\mu, \pi^\mu, \overline{q}^\mu, \overline{\pi}^\mu] = \int_{R_V} F dV \,. \tag{2.281}$$

In analogy with the case of point particles, one can define the different generating functionals

$$\mathscr{F}_1 = \mathscr{F}_1[t, q^\mu, \overline{q}^\mu] \,, \tag{2.282a}$$

$$\mathscr{F}_2 = \mathscr{F}_2[t, q^\mu, \overline{\pi}^\mu] \,, \tag{2.282b}$$

$$\mathscr{F}_3 = \mathscr{F}_3[t, \pi^\mu, \overline{q}^\mu] \,, \tag{2.282c}$$

$$\mathscr{F}_4 = \mathscr{F}_4[t, \pi^\mu, \overline{\pi}^\mu] \,, \tag{2.282d}$$

from which the transformation rules for the transformation of variables to be canonical can be derived. For example, for the particular case of the generating functional $\mathscr{F}_1$, one has

$$\pi^\mu = \frac{\delta \mathscr{F}_1}{\delta q^\mu} \,, \tag{2.283a}$$

$$\overline{\pi}^\mu = -\frac{\delta \mathscr{F}_1}{\delta \overline{q}^\mu} \,, \tag{2.283b}$$

$$\overline{H} = H + \frac{\delta \mathscr{F}_1}{\delta t} \,, \tag{2.283c}$$

which has the same form of (2.96a)–(2.96c), apart from the use of the functional derivatives instead of the partial derivatives due to the fact that $\mathscr{F}_1$ is now a functional. Analogously, the transformation rules set by the generating functionals $\mathscr{F}_2$, $\mathscr{F}_3$, $\mathscr{F}_4$ take the same form as the ones set by the generating functions introduced in Sect. 2.8.

2.13 Noether's Theorem for Fields: Lagrangian Density Dependent on Vector Functions

Having introduces the dynamics for vector fields, we can now derive Noether's Theorem for this particular case. The derivation will be similar to the one for fields depending on a scalar function.

Defining the symmetry transformations

$$x'_n = x_n + \delta_S x_n \ , \tag{2.284a}$$
$$q'^{\mu}(x') = q^{\mu}(x) + \delta_S q^{\mu}(x) \ , \tag{2.284b}$$
$$q'^{\mu}_n(x') = q^{\mu}_n(x) + \delta_S q^{\mu}_n(x) \ , \tag{2.284c}$$

the Lagrangian density transforms as

$$L'\left(x'_n, q'^{\mu}, q'^{\mu}_n\right) d(x') = L\left(x_n, q^{\mu}, q^{\mu}_n\right) d(x) \ , \tag{2.285a}$$
$$L'\left(x'_n, q'^{\mu}, q'^{\mu}_n\right) = L\left(x'_n, q'^{\mu}, q'^{\mu}_n\right) + \frac{D}{Dx'_k}\left(\delta_S \Omega_k\right) \ . \tag{2.285b}$$

Equation (2.285a) shows that the functional (2.250) is a scalar invariant under the transformations (2.284a)–(2.284c), while (2.285b) shows the invariance of the form of the Lagrangian density with respect to (2.284a)–(2.284c). The divergence transformation $D\left(\delta_S \Omega_k\right)/Dx'_k$ in (2.285b)) takes into account the invariance of (2.267) upon the substitution $L \rightarrow L + D\left(\delta_S \Omega_k\right)/Dx_k$.

Following the same way of reasoning of the previous sections, after some algebra, (2.285a) and (2.285b) yield

$$\left[\delta_S x_n \frac{\partial}{\partial x_n} + \delta_S q^{\mu} \frac{\partial}{\partial q^{\mu}_n} + \frac{\partial\left(\delta_S x_n\right)}{\partial x_n}\right] L\left(x_n, q^{\mu}, q^{\mu}_n\right) = -\frac{D}{Dx_k}\left(\delta_S \Omega_k\right) \ . \tag{2.286}$$

If (2.286) is satisfied for a certain symmetry transformation (2.284a)–(2.284c) and for a quantity $\Omega_k(x, q)$, the Lagrangian is invariant under that transformation. The corresponding conserved quantity is defined from the equation

$$\frac{D}{Dx_k}\left[\left(L\delta_{kl} - \frac{\partial L}{\partial q^{\mu}_k} q^{\mu}_l\right)\delta_S x_l + \frac{\partial L}{\partial q^{\mu}_k}\delta_S q^{\mu} + \delta_S \Omega_k\right] = 0 \ . \tag{2.287}$$

2.14 Bibliographical Note

Numerous excellent books exist on classical mechanics, such as [4]. The proofs of Noether's Theorem here exposed follow instead [7]. For a historical, critical analysis of [7], see, e.g. [10]. For a self-contained exposition of Noether's Theorem and its implications, see [15]. The same book reports an interesting discussion on why the Lagrangian is made up by the kinetic energy *minus* the potential energy based on the equipartition of energy. The original formulation of Noether's Theorem was formulated by Emmy Noether herself as *"a combination of the methods of the formal calculus of variations and of Lie's theory of groups"* [16]. A didactical proof and discussion of Noether's Theorem that makes use of Lie groups can be found for example in the book by Olver [17]. For other books see, for example [1, 3, 5, 6, 8, 9, 11, 12, 18–20].

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Chapter 3
Variational Principles in Fluid Dynamics, Symmetries and Conservation Laws

Abstract In this chapter, the Lagrangian density associated with fluid dynamics will be introduced. The equations of motion will be rederived from the Lagrangian density using Hamilton's principle. In particular, Hamilton's principle will be applied mainly in the Lagrangian framework, where the analogy to a system of point particles will simplify the calculations. The same principle will, however, be applied also in the Eulerian framework, and the relationship between the two frameworks will be revealed from the use of canonical transformations. Noether's Theorem will be applied to derive the conservation laws corresponding to the continuous symmetries of the Lagrangian density for the ideal fluid. Particular attention will be given to the particle relabelling symmetry and the associated conservation of vorticity.

Keywords Fluid dynamics · Geophysical fluid dynamics · Ideal fluid · Variational principle · Conservation laws · Circulation · Lagrangian labels · Lagrangian for ideal fluid · Hamilton's principle for fluids · Relabelling symmetry · Lin constraints

3.1 Introduction: Lagrangian Coordinates and Labels

In this chapter, we will introduce and analyse the Lagrangian form of the equations for the ideal fluid. The dynamical equations will be derived applying Hamilton's principle, thus making use of variational principles. It is interesting to report a quote from a memorandum sent by the mathematician John von Neumann to Oswald Veblen, in 1945, on the necessity for further development of the study of variational calculations applied to fluid dynamics: *"The great virtue of the variational treatment [...] is that it permits efficient use, in the process of calculation, of any experimental or intuitive insight [...]. It is important to realize that it is not possible, or possible to a much smaller extent, if one performs the calculation by using the original form of the equations of motion - the partial differential equations. [...] Symmetry, stationarity, similitude properties [...] applying such methods to hydrodynamics would be of the greatest importance since in many hydrodynamical problems we have very good general evidence of the above-mentioned sort about the approximate aspect of the solution, and the refining of this to a solution of the desired precision is what presents disproportionate computational difficulties [...]"* ([23], p. 357).

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In order to proceed to introduce Hamilton's principle for fluids, the concept of parcels and labels, already briefly introduced in Sect. 1.2, will be studied more in depth. Here, we thus point out preliminarily (H_1) the basic kinematic constraint which links the Eulerian with the Lagrangian descriptions, and (H_2) the conservation principle satisfied by a parcel, that is to say:

(H_1) The field property at a given location and time must equal the property possessed by the parcel occupying that position at that instant;
(H_2) Because the portion of fluid included in the volume of a parcel is always the same in the course of motion, the mass of a parcel in conserved in time.

In order to draw consequences from (H_1), let $q(\mathbf{x}, t)$ be the value of a certain field q in the point $\mathbf{x}$ at the time t. A suitable Cartesian reference (x, y, z) is understood in the Euclidean space R, hereafter named location space. On the other hand, based on the assumption that a parcel is a identifiable piece of matter, the identification can be realized by associating to each parcel a label, say $\mathbf{a} = (a_1, a_2, a_3)$, varying in the so-called labelling space R_a. For instance, if $\mathbf{x}_0 = (x_0, y_0, z_0)$ is the position of the centre of mass at the "initial" time $t = 0$, the identification $(a_1, a_2, a_3) = (x_0, y_0, z_0)$ allows to single out the considered parcel in the label space at every time after that initial. A one-to-one mapping between R and R_a is postulated. In order to simplify calculations, it is sometimes useful to choose the label $\mathbf{a}$ such that the initial density is uniform in label space. Unless stated otherwise, and without loss of generality, in this book we will make use of this choice starting from Sect. 3.2.

Remark 3.1 The continuum hypothesis allows us to neglect any granular aspect of the fluid, so infinitesimal material volumes of fluid are conceivable. Thus, each parcel is characterized, time by time, only by the location of its centre of mass. In particular, the coordinates of the centre of mass at the "initial" time coincide with the labels of the parcel. For this reason, a one-to-one mapping between the location space and the labelling space can be assumed.

In the Lagrangian framework, the independent variables are $a_1,\ a_2,\ a_3,\ t$, so, in the labelling space

$$x = x(a_1, a_2, a_3, t)\ , \tag{3.1a}$$

$$y = y(a_1, a_2, a_3, t)\ , \tag{3.1b}$$

$$z = z(a_1, a_2, a_3, t)\ , \tag{3.1c}$$

or, concisely,

$$\mathbf{x} = \mathbf{x}(\mathbf{a}, t)\ . \tag{3.2}$$

It should be noted that the time t is assumed to be the same in both frameworks. After these preliminaries, the kinematic constraint (H_1) takes the form

$$q(\mathbf{x}, t) = q(\mathbf{x}(\mathbf{a}, t), t) = q_a(\mathbf{a}, t)\ , \tag{3.3}$$

where q_a is the value of the quantity q taken by the parcel labelled by $a_1,\ a_2,\ a_3$. The r.h.s of (3.3) describes the evolution of q by means of the alternation of different parcels in the course of time according to the trajectories (3.1a)–(3.1c) or (3.2). In other words, supposing that the parcel $\mathbf{a}$, having the value q_a, crosses point $\mathbf{x}$ at the time t, in the same point $\mathbf{x}$ and in the same time t the field $q(\mathbf{x}, t)$ takes the value q_a.

With reference to a given parcel, the tern of labels $\mathbf{a}$ is held fixed at every time, so the time growth rate of $q_a(\mathbf{a}, t)$ is simply $\partial q_a/\partial t$. For instance, in Lagrangian coordinates, owing to (3.1a)–(3.1c), the velocity vector $\mathbf{u}$ is expressed by

$$\mathbf{u}(\mathbf{a}, t) = \frac{\partial x(\mathbf{a}, t)}{\partial t}\hat{\mathbf{i}} + \frac{\partial y(\mathbf{a}, t)}{\partial t}\hat{\mathbf{j}} + \frac{\partial z(\mathbf{a}, t)}{\partial t}\hat{\mathbf{k}}\ . \tag{3.4}$$

Consider now (H_2) and set, with reference to (3.3), $q = \rho$ where ρ is density. In the location space, the mass dm of a parcel enclosed into the volume $dV = dx\ dy\ dz$ is $dm = \rho(x, y, z, t)\ dx\ dy\ dz$, while in the label space, $dm = \rho_a(a_1, a_2, a_3, t)da_1\ da_2\ da_3$. Hence,

$$\rho(x, y, z, t)\ dx\ dy\ dz = \rho_a(a_1, a_2, a_3, t)da_1\ da_2\ da_3\ . \tag{3.5}$$

The assumed one-to-one mapping between R and R_a implies

$$dx\ dy\ dz = \frac{\partial(\mathbf{x})}{\partial(\mathbf{a})}da_1\ da_2\ da_3\ . \tag{3.6}$$

where

$$\frac{\partial(\mathbf{x})}{\partial(\mathbf{a})} = |det\ \mathbf{J}| \tag{3.7}$$

is the determinant of the Jacobi matrix

$$\mathbf{J} = \begin{pmatrix} \frac{\partial x}{\partial a_1} & \frac{\partial x}{\partial a_2} & \frac{\partial x}{\partial a_3} \\ \frac{\partial y}{\partial a_1} & \frac{\partial y}{\partial a_2} & \frac{\partial y}{\partial a_3} \\ \frac{\partial z}{\partial a_1} & \frac{\partial z}{\partial a_2} & \frac{\partial z}{\partial a_3} \end{pmatrix}, \tag{3.8}$$

i.e. the determinant of the mapping above. Because of (3.6), Eq. (3.5) takes the form

$$\rho(x, y, z, t)\frac{\partial(\mathbf{x})}{\partial(\mathbf{a})} = \rho_a(a_1, a_2, a_3, t)\ . \tag{3.9}$$

In terms of the specific volume $\alpha = 1/\rho$, Eq. (3.9) is

$$\alpha = \frac{\partial(\mathbf{x})}{\partial(\mathbf{a})}\alpha_a\ . \tag{3.10}$$

Notice that (3.10) implies that $\partial(\mathbf{x})/\partial(\mathbf{a}) = \alpha/\alpha_a > 0$, which ensures that the existence of the inverse mapping between R and R_a. Equation (3.9) implies

$$\frac{D}{Dt}\left[\rho(\mathbf{x},t)\frac{\partial(\mathbf{x})}{\partial(\mathbf{a})}\right]=\frac{\partial\rho_a}{\partial t}\ ,$$

i.e.

$$\frac{\partial(\mathbf{x})}{\partial(\mathbf{a})}\frac{D\rho}{Dt}+\rho\frac{D}{Dt}\frac{\partial(\mathbf{x})}{\partial(\mathbf{a})}=\frac{\partial\rho_a}{\partial t}\ . \tag{3.11}$$

Because of the Euler's relation (see Appendix J)

$$\frac{D}{Dt}\frac{\partial(\mathbf{x})}{\partial(\mathbf{a})}=\frac{\alpha}{\alpha_a}\,div\,\mathbf{u}\ ,$$

Equation (3.11) yields

$$\frac{1}{\alpha_a\rho}\left(\frac{D\rho}{Dt}+\rho\,div\,\mathbf{u}\right)=\frac{\partial\rho_a}{\partial t}\ . \tag{3.12}$$

Relationship (3.12) shows that the counterpart of mass conservation in the location space, expressed by

$$\frac{D\rho}{Dt}+\rho\,div\,\mathbf{u}=0\ , \tag{3.13}$$

is given in the labelling space by

$$\frac{\partial\rho_a}{\partial t}=0\ , \tag{3.14}$$

so $\rho_a=\rho_a(\mathbf{a})$. After this preamble, we are now ready to introduce the Lagrangian density for fluids, which will allow to derive the equations for fluid dynamics making use of Hamilton's principle. The derivation of the Lagrangian will first be made in the Lagrangian description. In this case, Hamilton's principle will require the passage from the sum over a number of material particles to the integral over a parcel of fluid. Apart from this difference, the formulation of Hamilton's principle in the Lagrangian description presents strong analogies with the same principle formulated for material particles.

3.2 Lagrangian Density in Labelling Space

As anticipated in the Introduction of this chapter, in order to simplify the notation, in the rest of the chapter we will consider without loss of generality the initial density in label space to be uniform. This allows to define

$$\tilde{a}_1 = \rho_a^{1/3} a_1 \ , \tag{3.15a}$$
$$\tilde{a}_2 = \rho_a^{1/3} a_2 \ , \tag{3.15b}$$
$$\tilde{a}_3 = \rho_a^{1/3} a_3 \ . \tag{3.15c}$$

With these, (3.5) becomes

$$\rho \, dx \, dy \, dz = d\tilde{a}_1 \, d\tilde{a}_2 \, d\tilde{a}_3 \ , \tag{3.16}$$

and thus, dropping the tildes,

$$\frac{1}{\rho} = \frac{\partial(\mathbf{x})}{\partial(\mathbf{a})} \ , \tag{3.17}$$

or, equivalently,

$$\alpha = \frac{\partial(\mathbf{x})}{\partial(\mathbf{a})} \ . \tag{3.18}$$

We recall from Chap. 2 that the Lagrangian density has the form

$$L = T - V \ , \tag{3.19}$$

where T is the kinetic energy density and V is the potential energy density. Owing to (3.4) and (3.5), the kinetic energy density in the labelling space is

$$T = \frac{1}{2} \left| \frac{\partial \mathbf{x}(\mathbf{a}, t)}{\partial t} \right|^2 \ . \tag{3.20}$$

The potential energy density takes into account both the intrinsic, i.e. internal, energy of the parcel caused by mechanical compression and heating and the energy arising from its embedding into a force field induced by an external potential.

If $e(\mathbf{a}, t)$ is the internal energy for unit mass and $\dot{Q}$ is the rate of heating for unit mass, the first principle of thermodynamics states that

$$\frac{\partial e}{\partial t} = -p \frac{\partial \alpha}{\partial t} + \dot{Q} \ . \tag{3.21}$$

Moreover, if $\eta(\mathbf{a}, t)$ is the fluid entropy for unit mass, then

$$\vartheta \frac{\partial \eta}{\partial t} = \dot{Q} \ , \tag{3.22}$$

where ϑ is the absolute temperature. Substitution of (3.22) into (3.21) trivially yields

$$\frac{\partial e}{\partial t} = -p \frac{\partial \alpha}{\partial t} + \vartheta \frac{\partial \eta}{\partial t} \ , \tag{3.23}$$

and, in turn, one easily verifies that Eq. (3.23) is satisfied by a function

$$e = e(\alpha, \eta) , \tag{3.24}$$

where

$$\frac{\partial e}{\partial \alpha} = -p , \quad \frac{\partial e}{\partial \eta} = \vartheta , \tag{3.25}$$

i.e. (1.224) and (1.225), respectively.

If no heat is added to the fluid or transferred between adjacent parcels, then $\dot{Q} = 0$ so, because of (3.22),

$$\frac{\partial \eta}{\partial t} = 0 . \tag{3.26}$$

Now, recalling (3.18) and (3.24), the internal energy takes the form

$$e = e\left(\frac{\partial(\mathbf{x})}{\partial(\mathbf{a})}, \eta(\mathbf{a})\right) , \tag{3.27}$$

If the external potential is represented by the field

$$\phi = \phi(\mathbf{x}(\mathbf{a}), t) , \tag{3.28}$$

the total potential energy per unit mass is the sum of (3.27) with (3.28), and hence,

$$V = e\left(\frac{\partial(\mathbf{x})}{\partial(\mathbf{a})}, \eta(\mathbf{a})\right) + \phi(\mathbf{x}(\mathbf{a}), t) . \tag{3.29}$$

Owing to (3.20) and (3.29), the Lagrangian density is

$$L = \frac{1}{2}\left|\frac{\partial \mathbf{x}(\mathbf{a}, t)}{\partial t}\right|^2 - e\left(\frac{\partial(\mathbf{x})}{\partial(\mathbf{a})}, \eta(\mathbf{a})\right) - \phi(\mathbf{x}(\mathbf{a}), t) , \tag{3.30}$$

and, consequently, the action I is given by

$$I = \int_{t_1}^{t_2} \int_{R_a} \left[\frac{1}{2}\left|\frac{\partial \mathbf{x}(\mathbf{a}, t)}{\partial t}\right|^2 - e\left(\frac{\partial(\mathbf{x})}{\partial(\mathbf{a})}, \eta(\mathbf{a})\right) - \phi\left(\mathbf{x}(\mathbf{a}), t\right)\right] d(\mathbf{a}) dt . \tag{3.31}$$

3.2.1 Hamilton's Equations

Notice that, under the same reasoning that lead to (2.61), the Lagrangian density (3.30) can be rewritten as

$$L = \mathbf{u} \cdot \frac{\partial \mathbf{x}}{\partial t} - H \ , \tag{3.32}$$

where

$$\mathbf{u} = \frac{\delta L}{\delta \dot{\mathbf{x}}} \ , \tag{3.33}$$

and

$$H = T + V = \frac{1}{2} \left| \frac{\partial \mathbf{x}}{\partial t} \right|^2 + e + \phi \ , \tag{3.34}$$

is the Hamiltonian density. These relations lead to Hamilton's equations

$$\dot{x}_i = \frac{\delta \mathscr{H}}{\delta u_i} \ , \ \dot{u}_i = -\frac{\delta \mathscr{H}}{\delta x_i} \ , \tag{3.35}$$

where

$$\mathscr{H} = \int_{R_a} H d(\mathbf{a}) \ . \tag{3.36}$$

More generally, writing explicitly the contribution by ρ_a, the Lagrangian density is

$$L = \rho_a \mathbf{u} \cdot \frac{\partial \mathbf{x}}{\partial t} - H \ , \tag{3.37}$$

with

$$H = \rho_a \left\{ \frac{1}{2} \left| \frac{\partial \mathbf{x}}{\partial t} \right|^2 + e + \phi \right\} \ , \tag{3.38}$$

and Lagrangian functional

$$\mathscr{L} = \int_{R_a} L d(\mathbf{a}) \ . \tag{3.39}$$

One can thus define the conjugate momentum density

$$\boldsymbol{\pi}_a = \frac{\delta \mathscr{L}}{\delta \dot{\mathbf{x}}} = \rho_a \mathbf{u} \ , \tag{3.40}$$

and the correspondent Hamilton's equations

$$\dot{x}_i = \frac{\delta \mathscr{H}}{\delta \pi_{a,i}} \ , \ \dot{\pi}_{a,i} = -\frac{\delta \mathscr{H}}{\delta x_i} \ . \tag{3.41}$$

It should be noted that the conjugate momentum density corresponds to a mass flux.

Completely analogously, in location space, the Lagrangian density is

$$L = \boldsymbol{\pi} \cdot \frac{\partial \mathbf{x}}{\partial t} - H \ , \tag{3.42}$$

where the conjugate momentum density is

$$\boldsymbol{\pi} = \frac{\delta \mathscr{L}}{\delta \dot{\mathbf{x}}} = \rho \mathbf{u} \ . \tag{3.43}$$

In (3.42), the Hamiltonian density is

$$H = \rho \left\{ \frac{1}{2} u^2 + e + \phi \right\} \ , \tag{3.44}$$

which yields Hamilton's equations

$$\dot{x}_i = \frac{\delta \mathscr{H}}{\delta \pi_i} \ , \ \dot{\pi}_i = -\frac{\delta \mathscr{H}}{\delta x_i} \ . \tag{3.45}$$

3.3 Hamilton's Principle for Fluids

The action (3.31) allows the derivation of the equations of motion for fluids through the application of Hamilton's principle. The calculations will follow the derivation of the Euler–Lagrange equations for fields reported in Sect. 2.13 and based on (2.251). However, unlike Sect. 2.13, particle labels play the role of Lagrangian coordinates and are here the independent variables. We set

$$\mathbf{x}(\mathbf{a}, t, l) = \mathbf{x}(\mathbf{a}, t, 0) + l\mathbf{Q}(\mathbf{a}, t) \ , \tag{3.46}$$

which are considered in analogy with (2.252a)–(2.252b) and in which the functions $\mathbf{Q}(\mathbf{a}, t) \in C_R^2$ in R_a satisfy suitable initial and boundary conditions depending on the problem. In particular, recalling Chap. 1, we will take into account fluids in either *(i)* a closed domain, with no flow across ∂R_a, satisfying

$$\mathbf{Q}(\mathbf{a}, t_1) = \mathbf{Q}(\mathbf{a}, t_2) = 0 \ , \ \mathbf{Q}(\mathbf{a}, t) = 0 \ \forall \mathbf{a} \in \partial R_a \ , \tag{3.47}$$

(ii) in a periodic domain or *(iii)* in a unbounded domain. In this last case, $\mathbf{Q}$ and their derivatives are requested to become zero at infinity. In the following, we will focus on case *(i)*, without any loss of generality for the derivations. Looking first at the kinetic energy term, using (3.46), the variation of the Lagrangian (3.30) yields, at first order in l

$$L_T(\mathbf{x} + l\mathbf{Q}) \approx \frac{1}{2} \left(\left| \frac{\partial \mathbf{x}}{\partial t} \right|^2 + 2 \frac{\partial \mathbf{x}}{\partial t} \cdot \frac{\partial \, (l\mathbf{Q})}{\partial t} \right) ,$$

so that

$$L_T(\mathbf{x}+l\mathbf{Q}) - L_T(\mathbf{x}) \approx \frac{\partial \mathbf{x}}{\partial t} \cdot \frac{\partial\,(l\mathbf{Q})}{\partial t}\;, \tag{3.48}$$

and hence,

$$\delta I_T = \int_{t_1}^{t_2} dt \int_{R_a} d(\mathbf{a}) \frac{\partial \mathbf{x}}{\partial t} \cdot \frac{\partial\,(l\mathbf{Q})}{\partial t}\;. \tag{3.49}$$

The identity

$$\frac{\partial \mathbf{x}}{\partial t} \cdot \frac{\partial\,(l\mathbf{Q})}{\partial t} = \frac{\partial}{\partial t}\left(\frac{\partial \mathbf{x}}{\partial t} \cdot l\mathbf{Q}\right) - \frac{\partial^2 \mathbf{x}}{\partial t^2} \cdot l\mathbf{Q}\;, \tag{3.50}$$

where

$$\int_{t_1}^{t_2} dt \frac{\partial}{\partial t}\left(\frac{\partial \mathbf{x}}{\partial t} \cdot l\mathbf{Q}\right) = 0\;, \tag{3.51}$$

because of the first equation of (3.47), allows to write (3.49) as

$$\delta I_T = -\int_{t_1}^{t_2} dt \int_{R_a} d(\mathbf{a}) \frac{\partial^2 \mathbf{x}}{\partial t^2} \cdot l\mathbf{Q}\;. \tag{3.52}$$

The potential energy terms yield instead the contribution to the variation of the action, after linearization in e and ϕ,

$$\begin{aligned} \delta I_V &= \int_{t_1}^{t_2} dt \int_{R_a} d(\mathbf{a}) \left[-\frac{\partial e}{\partial \alpha}\delta\alpha - \nabla\phi \cdot l\mathbf{Q}\right] \\ &= \int_{t_1}^{t_2} dt \int_{R_a} d(\mathbf{a}) \left[-\frac{\partial e}{\partial \alpha}\delta\frac{\partial(\mathbf{x})}{\partial(\mathbf{a})} - \nabla\phi \cdot l\mathbf{Q}\right]\;. \end{aligned} \tag{3.53}$$

Recalling that $\partial e/\partial\alpha = -p$, the space integral of the first term of the integrand can be written as

$$\int_{R_a} d(\mathbf{a}) p\delta \frac{\partial(\mathbf{x})}{\partial(\mathbf{a})} = \int_R d(\mathbf{x}) p\rho\delta \frac{\partial(\mathbf{x})}{\partial(\mathbf{a})} = \int_R d(\mathbf{x}) p\; div\;\;(l\mathbf{Q})\;\;, \tag{3.54}$$

where, in the last step, we have used the relation (J.9) from Appendix J. Using the divergence theorem, (3.54) finally yields

$$\begin{aligned} &\int_R d(\mathbf{x}) p\; div\;\;(l\mathbf{Q}) \\ &= \oint_{\partial R} pl\mathbf{Q} \cdot \hat{\mathbf{n}} ds - \int_R d(\mathbf{x}) \nabla p \cdot l\mathbf{Q} \\ &= -\int_R d(\mathbf{x}) \nabla p \cdot l\mathbf{Q}\;, \end{aligned} \tag{3.55}$$

where, once again, we have made use of the second equation of (3.47) in the last step.

From (3.52) and (3.55), the equation

$$\delta I = \int_{t_1}^{t_2} dt \int_R d(\mathbf{x}) \left\{ \rho(\mathbf{x}, t) \left[-\frac{\partial^2 \mathbf{x}}{\partial t^2} - \nabla\phi \right] - \nabla p \right\} \cdot l\mathbf{Q} \,, \tag{3.56}$$

follows. In turn, Eq. (3.56) implies

$$\delta I = 0 \Leftrightarrow \rho \frac{\partial^2 \mathbf{x}}{\partial t^2} + \rho\nabla\phi + \nabla p = 0 \,,$$

that is to say

$$\frac{\partial^2 \mathbf{x}}{\partial t^2} = -\frac{1}{\rho}\nabla p - \nabla\phi \,. \tag{3.57}$$

To summarize, using Hamilton's principle (2.251), Eq. (3.57) yields the equation of motion governing the fluids in the Lagrangian framework.

3.4 Hamilton's Principle in the Eulerian Framework

The description of the Lagrangian density and the corresponding equations of motion for fluids in Lagrangian coordinates can be extended to the Eulerian framework. Consider the Eulerian velocity $\mathbf{u}(\mathbf{x}, t)$ of a density and entropy conserving fluid. The action corresponding to (3.31) is

$$I = \int_{t_1}^{t_2} \int_R \rho \left[\frac{1}{2} u^2(\mathbf{x}, t) - e\,(\alpha, \eta) - \phi\,(\mathbf{x}, t) \right] d(\mathbf{x}) dt \,, \tag{3.58}$$

or, introducing the mass flux defined as (3.43),

$$I = \int_{t_1}^{t_2} \int_R \left\{ \frac{1}{2} \frac{\pi^2}{\rho} + \rho\,[-e\,(\alpha, \eta) - \phi\,(\mathbf{x}, t)] \right\} d(\mathbf{x}) dt \,. \tag{3.59}$$

As shown by [17], the action (3.58) can be constrained by the conservation of the particle labels

$$\rho \frac{\partial \mathbf{a}}{\partial t} + (\boldsymbol{\pi} \cdot \nabla)\mathbf{a} = 0 \,. \tag{3.60}$$

The constraints associated with these conservation laws enter thus in the action that can thus be written as

$$I = \int_{t_1}^{t_2} \int_R \left\{ \frac{1}{2}\frac{\pi^2}{\rho} - \rho(e+\phi) + \mathbf{\Lambda} \cdot \left(\rho \frac{\partial \mathbf{a}}{\partial t} + (\boldsymbol{\pi} \cdot \nabla)\mathbf{a} \right) \right\} d(\mathbf{x})dt \ , \tag{3.61}$$

where $\mathbf{\Lambda} = (\lambda_1(\mathbf{x}, t), \lambda_2(\mathbf{x}, t), \lambda_3(\mathbf{x}, t))$ are Lagrange multipliers. The number of unknowns in the action functional can be reduced with the following observation: the conservation of density and entropy in Eulerian framework introduces constraints to the physical systems, due to the presence of the velocity field in the total derivative of the conserved quantities. Using the mass flux, the conservation of density and entropy can be rewritten as

$$\frac{\partial \rho}{\partial t} + \ div\, \boldsymbol{\pi} = 0 \ , \tag{3.62}$$

and

$$\rho \frac{\partial \eta}{\partial t} + (\boldsymbol{\pi} \cdot \nabla)\eta = 0 \ , \tag{3.63}$$

respectively. It is thus possible to make the following assumptions:

(i) since the entropy is materially conserved, it is possible to identify one of the parcel labels with it, for example $a_3 = \eta$;
(ii) at the same time, the conservation of density suggests that it is possible to identify the density with one of the remaining parcel labels, for example a_2.

Making use of (i) and (ii), and renaming $a_1 = a$, the action becomes

$$\begin{aligned} I = \int_{t_1}^{t_2} \int_R \Bigg\{ & \frac{1}{2}\frac{\pi^2}{\rho} - \rho(e+\phi) \\ & + \lambda_1 \left(\rho \frac{\partial a}{\partial t} + (\boldsymbol{\pi} \cdot \nabla)a \right) \\ & + \lambda_2 \left(\frac{\partial \rho}{\partial t} + \ div\, \boldsymbol{\pi} \right) \\ & + \lambda_3 \left(\rho \frac{\partial \eta}{\partial t} + (\boldsymbol{\pi} \cdot \nabla)\eta \right) \Bigg\} d(\mathbf{x})dt \ , \end{aligned} \tag{3.64}$$

where the density constraint has been rewritten combining (3.60) and (3.62). Hamilton's principle requires (3.64) to be stationary upon variations in the variables depending on $(\mathbf{x}, t)$, namely

$$\boldsymbol{\pi}(\mathbf{x}, t), \ \rho(\mathbf{x}, t), \ \eta(\mathbf{x}, t), \ a(\mathbf{x}, t), \ \lambda_1(\mathbf{x}, t), \ \lambda_2(\mathbf{x}, t), \ \lambda_3(\mathbf{x}, t) \ .$$

The variation of the action (3.64) in the mass flux $\boldsymbol{\pi}(\mathbf{x}, t) + l\mathbf{Q}(\mathbf{x}, t)$ involves the terms $\pi^2\, \rho$, $\lambda_1(\boldsymbol{\pi} \cdot \nabla)a$, $\lambda_2\, div\, \boldsymbol{\pi}$, $\lambda_3(\boldsymbol{\pi} \cdot \nabla)\eta$, thus yielding, up to the first order in l,

$$\delta I_p = \int_{t_1}^{t_2} dt \int_R d(\mathbf{x}) \left\{ \left[\frac{1}{\rho}\boldsymbol{\pi} + \lambda_1 \nabla a - \nabla \lambda_2 + \lambda_3 \nabla \eta \right] \cdot (l\mathbf{Q}) \right\} \ , \tag{3.65}$$

whence yields

$$\mathbf{u} = \boldsymbol{\pi}/\rho = -\lambda_1 \nabla a + \nabla \lambda_2 - \lambda_3 \nabla \eta \ . \tag{3.66}$$

Equation (3.66) allows to calculate the relative vorticity of the flow as

$$\boldsymbol{\omega} = \ rot\ \mathbf{u} = \nabla a \times \nabla \lambda_1 + \nabla \eta \times \nabla \lambda_3 \ . \tag{3.67}$$

The variation of the action (3.64) in the Eulerian density $\rho(\mathbf{x}, t) + l\chi(\mathbf{x}, t)$ involves the terms $\pi^2 \rho$, $-\rho(e(\mathbf{x}, t) + \phi(\mathbf{x}, t))$, $\lambda_1 \rho \partial a/\partial t$, $\lambda_2 \partial \rho/\partial t$ and $\lambda_3 \rho \partial \eta/\partial t$. Up to first order in l, the first term yields

$$\int_{t_1}^{t_2} dt \int_R d(\mathbf{x}) \left[-\frac{u^2}{2} \right] l\chi \ . \tag{3.68}$$

The second term, instead, yields

$$\begin{aligned} &\int_{t_1}^{t_2} dt \int_R d(\mathbf{x}) \left\{ -(\rho + l\chi) \left[e(\rho + l\chi, \eta) + \phi \right] + \rho[e(\rho, \eta) + \phi] \right\} \\ &= \int_{t_1}^{t_2} dt \int_R d(\mathbf{x}) \left\{ -l\chi\phi - (\rho + l\chi) e(\rho + l\chi, \eta) + \rho e(\rho, \eta) \right\} \ . \end{aligned} \tag{3.69}$$

In particular, up to the first order in l,

$$\begin{aligned} &-(\rho + l\chi) e(\rho + l\chi, \eta) + \rho e(\rho, \eta) \\ &= -(\rho + l\chi) \left[e(\rho, \eta) + \frac{\partial e}{\partial \rho} l\chi \right] + \rho e(\rho, \eta) \\ &= -l\chi \left[e(\rho, \eta) + \rho \frac{\partial e}{\partial \rho} \right] , \end{aligned}$$

so, the above integral takes the form

$$\int_{t_1}^{t_2} dt \int_R d(\mathbf{x}) \left[-l\chi \left(e + \rho \frac{\partial e}{\partial \rho} + \phi \right) \right] .$$

Recalling (1.224), one finds

$$p = -\frac{\partial e}{\partial \alpha} \ \Rightarrow \ \rho \frac{\partial e}{\partial \rho} = \frac{p}{\rho} \ \Rightarrow \ e + \rho \frac{\partial e}{\partial \rho} = e + \frac{p}{\rho} = h \ ,$$

where

$$h = e + \frac{p}{\rho} \tag{3.70}$$

is the enthalpy. Thus, the integral takes the form

$$\int_{t_1}^{t_2} dt \int_R d(\mathbf{x}) \left[-l\chi \, (h+\phi)\right] \, . \tag{3.71}$$

Trivially, the remaining integrals yield

$$\int_{t_1}^{t_2} dt \int_R d(\mathbf{x}) \left[l\chi \left(\lambda_1 \frac{\partial a}{\partial t} - \frac{\partial \lambda_2}{\partial t} + \lambda_3 \frac{\partial \eta}{\partial t} \right) \right] \, , \tag{3.72}$$

where, in the second term, the fact that $\chi(\mathbf{x}, t_1) = \chi(\mathbf{x}, t_2) = 0$ has been used, so, on the whole,

$$\delta I_\rho = \int_{t_1}^{t_2} dt \int_R d(\mathbf{x}) \left\{ \left[\frac{1}{2}u^2 + (h+\phi) \right] - \lambda_1 \frac{\partial a}{\partial t} + \frac{\partial \lambda_2}{\partial t} - \lambda_3 \frac{\partial \eta}{\partial t} \right\} (-l\chi) \, , \tag{3.73}$$

whence

$$\left[\frac{1}{2}u^2 + (h+\phi) \right] - \lambda_1 \frac{\partial a}{\partial t} + \frac{\partial \lambda_2}{\partial t} - \lambda_3 \frac{\partial \eta}{\partial t} = 0 \, . \tag{3.74}$$

The quantity on the l.h.s. of (3.74) assumes the name of *generalized Bernoulli function*.

The variation of the action (3.64) in the Eulerian density entropy $\eta(\mathbf{x}, t) + lE(\mathbf{x}, t)$ involves the terms $-\rho e(\alpha, \eta)$ and $\lambda_3 \rho D\eta/Dt$. The related variation is

$$\delta I_\eta = \int_{t_1}^{t_2} dt \int_R d(\mathbf{x}) \left\{ -\rho e(\alpha, \eta + lE) + \lambda_3 \rho \frac{D}{Dt}(\eta + lE) + \rho e(\alpha, \eta) - \lambda_3 \rho \frac{D\eta}{Dt} \right\} \, ,$$

that is to say

$$\delta I_\eta = \int_{t_1}^{t_2} dt \int_R d(\mathbf{x}) \left\{ -\rho \frac{\partial e}{\partial \eta} lE + \lambda_3 \rho \frac{D}{Dt}(lE) \right\} \, . \tag{3.75}$$

Because $E(\mathbf{x}, t) = 0 \; \forall \mathbf{x} \in \partial R$ and $E(\mathbf{x}, t_1) = E(\mathbf{x}, t_2) = 0$, (3.75) is equivalent to

$$\delta I_\eta = -\int_{t_1}^{t_2} dt \int_R d(\mathbf{x}) \rho \left[\frac{\partial e}{\partial \eta} + \frac{D\lambda_3}{Dt} \right] lE \, ,$$

whence, using (1.225),

$$\rho \frac{\partial \lambda_3}{\partial t} + \boldsymbol{\pi} \cdot \nabla \lambda_3 = -\rho \vartheta \, . \tag{3.76}$$

The variations with respect to $a, \; \lambda_1, \; \lambda_2, \; \lambda_3$ can be obtained along the same lines as above and result in the following equations

$$\frac{D\lambda_1}{Dt} = 0 \, , \tag{3.77}$$

$$\frac{Da}{Dt} = 0 , \tag{3.78}$$

$$\frac{D\rho}{Dt} = 0 , \tag{3.79}$$

$$\frac{D\eta}{Dt} = 0 , \tag{3.80}$$

respectively. The equation for the velocity (or, equivalently, for the mass flux) (3.66), together with the dynamic equations for the terms appearing in the same equation, i.e. (3.74), (3.76)–(3.80), defines the *Clebsch–Lin representation* of the equations of motion in the Eulerian framework. For a set of examples of the equations of motion in the Clebsch–Lin representation in different approximations used in fluid dynamics, see [32].

From this set of equations, it is possible to derive the equations of motion in a formulation that is independent on the Lagrange multipliers of the system. To eliminate λ_2, sum the time derivative of (3.66) with the gradient of (3.74), obtaining

$$\begin{aligned} &\frac{\partial \mathbf{u}}{\partial t} + \nabla\left(\frac{1}{2}u^2 + h + \phi\right) \\ &+ \left[\frac{\partial}{\partial t}(\lambda_1 \nabla a) - \nabla\left(\lambda_1 \frac{\partial a}{\partial t}\right)\right] \\ &+ \left[\frac{\partial}{\partial t}(\lambda_3 \nabla \eta) - \nabla\left(\lambda_3 \frac{\partial \eta}{\partial t}\right)\right] = 0 . \end{aligned} \tag{3.81}$$

With the use of (3.77) and (3.78), the term within the first square bracket yields

$$\frac{\partial}{\partial t}(\lambda_1 \nabla a) - \nabla\left(\lambda_1 \frac{\partial a}{\partial t}\right) = [(\mathbf{u} \cdot \nabla)a]\, \nabla\lambda_1 - [(\mathbf{u} \cdot \nabla)\lambda_1]\, \nabla a . \tag{3.82}$$

The term within the second square bracket can instead be evaluated with the use of (3.76) and (3.80), yielding

$$\frac{\partial}{\partial t}(\lambda_3 \nabla \eta) - \nabla\left(\lambda_3 \frac{\partial \eta}{\partial t}\right) = [(\mathbf{u} \cdot \nabla)\eta]\, \nabla\lambda_3 - [(\mathbf{u} \cdot \nabla)\lambda_3]\, \nabla\eta - \vartheta\, \nabla\eta . \tag{3.83}$$

With (3.82) and (3.83), (3.81) becomes

$$\begin{aligned} &\frac{\partial \mathbf{u}}{\partial t} + \nabla\left(\frac{1}{2}u^2 + h + \phi\right) \\ &+ \{[(\mathbf{u} \cdot \nabla)a]\, \nabla\lambda_1 - [(\mathbf{u} \cdot \nabla)\lambda_1]\, \nabla a\} \\ &+ \{[(\mathbf{u} \cdot \nabla)\eta]\, \nabla\lambda_3 - [(\mathbf{u} \cdot \nabla)\lambda_3]\, \nabla\eta - \vartheta\, \nabla\eta\} = 0 . \end{aligned} \tag{3.84}$$

Using the vector identity

$$\mathbf{A} \times (\nabla b \times \nabla c) = (\mathbf{A} \cdot \nabla c)\nabla b - (\mathbf{A} \cdot \nabla b)\nabla c \ ,$$

it is possible to rewrite (3.84) as

$$\begin{aligned} &\frac{\partial \mathbf{u}}{\partial t} + \nabla\left(\frac{1}{2}u^2 + h + \phi\right) - \vartheta \nabla \eta \\ &+ \mathbf{u} \times (\nabla\lambda_1 \times \nabla a + \nabla\lambda_3 \times \nabla \eta) = 0 \ . \end{aligned} \tag{3.85}$$

It is visible that the term within the last brackets is the negative of the relative vorticity (3.67), so that (3.85) can be written as

$$\frac{\partial \mathbf{u}}{\partial t} + \nabla\left(\frac{1}{2}u^2 + h + \phi\right) - \vartheta \nabla \eta - \mathbf{u} \times \boldsymbol{\omega} = 0 \ , \tag{3.86}$$

or, using the identity

$$(\mathbf{u} \cdot \nabla)\mathbf{u} = \frac{1}{2}\nabla u^2 - \mathbf{u} \times \boldsymbol{\omega} \ ,$$

Equation (3.86) yields

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\nabla\phi - (\nabla h - \vartheta \nabla \eta) \ . \tag{3.87}$$

The first term on the r.h.s. of (3.87) represents the forcing created by the external potential ϕ. Making use of (1.225) and (3.70), we have

$$\nabla h - \vartheta \nabla \eta = -\frac{1}{\rho}\nabla p \ ,$$

so that (3.87) yields

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\frac{1}{\rho}\nabla p - \nabla\phi \ . \tag{3.88}$$

Equations (3.88), together with (3.79) and (3.80), form the equations of motion for the ideal fluid in the Eulerian framework.

While a derivation of the equation of motion in an Eulerian framework is possible, as was here exposed, the analogy between the fluid motion in the Lagrangian framework and the well-established formulation of classical mechanics for point particles suggests that it is often more convenient to perform the analysis of the system in this framework. Unless otherwise specified, the remaining of the chapter will thus follow the formulation in the Lagrangian framework.

3.4.1 Equivalence of the Lagrangian and Eulerian Forms of Hamilton's Principle

The derivation of the equations of motion from Hamilton's principle in the Lagrangian and Eulerian framework suggests that there must be a link between the two formulations. Throughout this section, we will show explicitly the factor $\rho_a(\mathbf{a}, t)$, due to the fact that this enters in the variational derivatives.

Anticipating the result, we will show that Hamilton's principle applied to the Lagrangian density (3.37) can be transformed by means of a canonical transformation into a variational principle for the fields $\mathbf{u}(\mathbf{x}, t)$, $\rho(\mathbf{x}, t)$ and $\eta(\mathbf{x}, t)$. To do this, we will demonstrate that it is possible to find a canonical transformation that, at each time step, acts on the label variables $\mathbf{a}(\mathbf{x}, t)$ and the conjugate momentum density $\boldsymbol{\pi}_a(\mathbf{x}, t)$, into the new generalized coordinates $(\mathbf{Q}(\mathbf{x}, t),\ \mathbf{P}_a(\mathbf{x}, t))$. At each time t, we will thus require

$$(\mathbf{a}(\mathbf{x}, t), \boldsymbol{\pi}_a(\mathbf{x}, t)) \rightarrow (\mathbf{Q}(\mathbf{x}, t), \mathbf{P}_a(\mathbf{x}, t)) \ . \tag{3.89}$$

Under this transformation, the Lagrangian density (3.37) will take the form (3.42) upon the identification of the new generalized momenta $\mathbf{P}_a(\mathbf{x}, t)$ with the same variables used in the Lin constraints (3.60), (3.62) and (3.63), i.e. $a = a_1$, ρ, and η. This choice for the new generalized momenta should not be surprising, as we have seen in the previous section that these quantities were chosen as constraints for Hamilton's principle in the Eulerian framework as they are associated with conservation laws.

Recalling the introduction of canonical transformations for vector fields, following [28] at each time t, we introduce a generating functional $\mathscr{F}_1$ so that (2.283a)–(2.283b) yield

$$\pi_{a,i} = \frac{\delta \mathscr{F}_1}{\delta a_i} \ , \tag{3.90a}$$

$$P_{a,i} = -\frac{\delta \mathscr{F}_1}{\delta Q_i} \ . \tag{3.90b}$$

Because $\mathscr{F}_1$ does not depend on time, (2.283c) yields instead the time invariance of the Hamiltonian. Introducing the *ad hoc* form for the generating functional

$$\mathscr{F}_1 = -\int_{R_a} d(\mathbf{a})\rho_a(\mathbf{a}, t)\,[aQ_1(\mathbf{x}, t) + Q_2(\mathbf{x}, t) + \eta(\mathbf{a}, t)Q_3(\mathbf{x}, t)] \ , \tag{3.91}$$

from (3.90b) one has

$$P_{a,1}(\mathbf{x},t) = -\frac{\delta \mathscr{F}_1}{\delta Q_1(\mathbf{x},t)} = \rho(\mathbf{x},t)a(\mathbf{x},t)\ , \tag{3.92a}$$

$$P_{a,2}(\mathbf{x},t) = -\frac{\delta \mathscr{F}_1}{\delta Q_2(\mathbf{x},t)} = \rho(\mathbf{x},t)\ , \tag{3.92b}$$

$$P_{a,3}(\mathbf{x},t) = -\frac{\delta \mathscr{F}_1}{\delta Q_3(\mathbf{x},t)} = \rho(\mathbf{x},t)\eta(\mathbf{x},t)\ , \tag{3.92c}$$

which show that, as anticipated, the new generalized momenta are a, ρ and η.

Analogously, the evaluation of (3.90a), making use of (3.43), gives

$$\mathbf{u}(\mathbf{x},t) = -\left[a\nabla Q_1(\mathbf{x},t) + \nabla Q_2(\mathbf{x},t) + \eta\nabla Q_3(\mathbf{x},t)\right]\ , \tag{3.93}$$

which is equivalent to the Clebsch–Lin formulation of the velocity field (3.66) upon the identification

$$Q_i \to \lambda_i\ . \tag{3.94}$$

Differently from the previous section, however, no constraints have been imposed to the new generalized coordinates Q_i.

Using the new generalized coordinate Q_i, and the new generalized momenta (3.92a)–(3.92c), the Lagrangian density (3.37) transforms as

$$\begin{aligned} L &= \sum_{i=1}^{3} \frac{DQ_i}{Dt} P_{a,i} - H \\ &= \sum_{i=1}^{3} \frac{D\lambda_i}{Dt} P_{a,i} - H \\ &= \rho\left(a\frac{D\lambda_1}{Dt} + \frac{D\lambda_2}{Dt} + \eta\frac{D\lambda_3}{Dt}\right) - H\ , \end{aligned} \tag{3.95}$$

which, after integration by parts, is completely equivalent to the Lagrangian density in (3.64).

3.5 Symmetries and Conservation Laws

The general concept of invariance of the Lagrangian density under symmetry transformations was introduced in Chap. 2. Before to explore the symmetries inherent to fluid dynamics, we would like to report the concept of symmetry as expressed at the beginning of [9]: *"The word $\sigma\upsilon\mu\mu\epsilon\tau\rho\iota\alpha$ was in Greek times synonymous with something well proportioned, well balanced and was therefore related to the ancient concept of beauty. Later, it also acquired another, more restricted, meaning,* *which is just the one that is very important in science: A figure or structure is said to possess a symmetry under a mapping of space upon itself if it is carried into itself by that mapping - that is, it is invariant under that mapping"*. In the context of our investigation, the space will be both the Euclidean and the label spaces; the related transformations will be applied to the Lagrangian density whose resulting invariance in form (i.e. covariance) implies, through Noether's Theorem, the conservation of fundamental quantities possessed by fluid flows.

The final aim will be to find the underlying cause for the material conservation of potential vorticity, i.e. to link it to some fundamental symmetry in fluid dynamics.

The exposition will follow that of [22, 30, 31].

3.5.1 Preliminaries and Notation

We will now reconsider the formulation of Noether's Theorem, derived in Sect. 2.13, with a slight simplification of the notation, that will allow for a clear separation between the time t and labels a_j, $j = 1, 2, 3$, which is clearly valid in classical (i.e. nonrelativistic) mechanics. Reconsider the conserved quantity (2.287), i.e.

$$\frac{D}{Dx_k}\left[\left(L\delta_{kl} - \frac{\partial L}{\partial q_k^\mu} q_l^\mu\right)\delta_S x_l + \frac{\partial L}{\partial q_k^\mu}\delta_S q^\mu + \delta_S \Omega_k\right] = 0\ , \tag{3.96}$$

where, again, sub- and superscripts should not be confused with covariant and contravariant vectors and where sum over repeated indices applies. Equation (3.96) can be simplified neglecting the subscripts in the quantities δ_S and rearranging terms within the square brackets, obtaining

$$\frac{D}{Dx_k}\left[L\delta x_k + \frac{\partial L}{\partial q_k^\mu}\left(\delta q^\mu - q_l^\mu \delta x_l\right) + \delta\Omega_k\right] = 0\ . \tag{3.97}$$

It is thus possible to rename variables to adapt them to the more common notation used in Hamiltonian fluid dynamics as

$$q^\mu \rightarrow x^\mu \rightarrow x_i\ ,\ \ (i = 1, 2, 3)\ , \tag{3.98a}$$

$$x_l \rightarrow a_l\ ,\ \ (a_0 = t\ ,\ a_j = a_1, a_2, a_3)\ , \tag{3.98b}$$

$$q_k^\mu \rightarrow \frac{\partial x_i}{\partial a_k}\ ,\ \ \left(q_0^\mu \rightarrow \frac{\partial x_i}{\partial t}\right)\ . \tag{3.98c}$$

With (3.98a)–(3.98c), the quantity within the round brackets in (3.97) can be rewritten as

$$\delta q^\mu - q_l^\mu \delta x_l = \delta x_i - \frac{\partial x_i}{\partial a_l}\delta a_l = \delta x_i - \frac{\partial x_i}{\partial t}\delta t - \frac{\partial x_i}{\partial a_j}\delta a_j = \Delta x_i\ , \tag{3.99}$$

so that (3.97) can be rewritten as

$$\frac{D}{Dt}\left[L\delta t + \frac{\partial L}{\partial(\partial x_i/\partial t)}\Delta x_i + \delta\Omega_0\right] + \frac{D}{Da_j}\left[L\delta a_j + \frac{\partial L}{\partial(\partial x_i/\partial a_j)}\Delta x_i + \delta\Omega_j\right] = 0\,. \tag{3.100}$$

After this change of notation, we can proceed with the application of Noether's Theorem.

With (3.17), the Lagrangian density in labelling space (3.30) is

$$L = \sum_{i=1}^{3}\frac{1}{2}\left(\frac{\partial x_i}{\partial t}\right)^2 - e\left(\frac{\partial(\mathbf{x})}{\partial(\mathbf{a})}, \eta(\mathbf{a})\right)\,. \tag{3.101}$$

This Lagrangian density possesses a number of symmetries. We will, however, focus here on two important symmetries and their corresponding conservation laws: the *time translations* symmetry

$$t' = t + \delta t\,, \tag{3.102}$$

where δt is a constant, and the *particle relabelling symmetry*

$$a'_j = a_j + \delta a_j\,, \tag{3.103}$$

where the δa_j are quantities to be determined. The invariance of (3.101) under (3.102) and (3.103) leads to conservation laws that are particular cases of the equation

$$\frac{d}{dt}\int_{R_a}\left[L\delta t + \frac{\partial L}{\partial\left(\partial x_i/\partial t\right)}\left(\delta x_i - \frac{\partial x_i}{\partial t}\delta t - \frac{\partial x_i}{\partial a_j}\delta a_j\right)\right] d(\mathbf{a}) = 0\,, \tag{3.104}$$

that is the integral over the label domain of (3.100). The comparison between (3.104) and (3.96) shows that $\delta_S\Omega_k = 0$, which will be the case for most of the applications here considered. In the following, we will analyse (3.102) and (3.103) separately.

3.5.2 Time Translations Symmetry

Considering (3.102), only the kinetic term of (3.101) is involved in the transformation. Because $\partial/\partial t' = \partial/\partial t$, the Lagrangian density is manifestly invariant and, as a consequence, the integrand of (3.104) yields

$$\frac{1}{2}\sum_{i=1}^{3}\left(\frac{\partial x_i}{\partial t}\right)^2 - e - \frac{1}{2}\sum_{i=1}^{3}\left[2\left(\frac{\partial x_i}{\partial t}\right)^2\right] = -\left[\frac{1}{2}\sum_{i=1}^{3}\left(\frac{\partial x_i}{\partial t}\right)^2 + e\right], \tag{3.105}$$

aside for a constant multiplicative factor δt. With (3.105), (3.104) yields

$$\frac{d}{dt}\int_{R_a}\left[\frac{1}{2}\sum_{i=1}^{3}\left(\frac{\partial x_i}{\partial t}\right)^2+e\right]d(\mathbf{a})=0\,, \tag{3.106}$$

that is

$$\frac{d}{dt}\int_{R}\rho\left[\frac{1}{2}\sum_{i=1}^{3}\left(\frac{\partial x_i}{\partial t}\right)^2+e\right]d(\mathbf{x})=0\,. \tag{3.107}$$

Equation (3.107) shows the conservation of the total mechanical energy that emerges as a consequence of the invariance of the Lagrangian density under time translations.

3.5.3 Particle Relabelling Symmetry

Considering (3.103), only the internal energy is affected by the transformation. In the following, we will consider first homentropic and then nonhomentropic flows.

3.5.3.1 Homentropic Flows

Recall that for homentropic flows

$$e=e(\alpha)\,. \tag{3.108}$$

We are thus interested in preliminarily finding the quantities δa_j such that, considering $\delta e=(\partial e/\partial\alpha)\delta\alpha$, yield

$$\delta\alpha=0\,, \tag{3.109}$$

upon the insertion of (3.103) into (3.108). In compact notation, (3.109) can be written as

$$\frac{\partial(\mathbf{a}+\delta\mathbf{a})}{\partial(\mathbf{x})}=\frac{\partial(\mathbf{a})}{\partial(\mathbf{x})}\,, \tag{3.110}$$

where

$$\frac{\partial(\mathbf{a}+\delta\mathbf{a})}{\partial(\mathbf{x})}=\frac{\partial(\mathbf{a})}{\partial(\mathbf{x})}+\delta\frac{\partial(\mathbf{a})}{\partial(\mathbf{x})}\,. \tag{3.111}$$

The determination of the quantities δa_j that satisfy (3.109) translates thus in the determination of the tern $\delta\mathbf{a}=(\delta a_1,\delta a_2,\delta a_3)$ that satisfies

$$\delta\frac{\partial(\mathbf{a})}{\partial(\mathbf{x})}=0\,. \tag{3.112}$$

This equation can be written explicitly as

$$\frac{\partial(\delta a_1, a_2, a_3)}{\partial(\mathbf{x})} + \frac{\partial(a_1, \delta a_2, a_3)}{\partial(\mathbf{x})} + \frac{\partial(a_1, a_2, \delta a_3)}{\partial(\mathbf{x})} = 0 \,,$$

that is

$$\frac{\partial(\mathbf{a})}{\partial(\mathbf{x})}\left[\frac{\partial(\delta a_1, a_2, a_3)}{\partial(\mathbf{a})} + \frac{\partial(a_1, \delta a_2, a_3)}{\partial(\mathbf{a})} + \frac{\partial(a_1, a_2, \delta a_3)}{\partial(\mathbf{a})}\right] = 0 \,. \tag{3.113}$$

Setting the quantity between square brackets in (3.113) to zero corresponds to set

$$\begin{vmatrix} \frac{\partial}{\partial a_1}\delta a_1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 0 & 0 \\ 0 & \frac{\partial}{\partial a_2}\delta a_2 & 0 \\ 0 & 0 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{\partial}{\partial a_3}\delta a_3 \end{vmatrix} = 0 \,, \tag{3.114}$$

that is

$$div_a\, \delta\mathbf{a} = 0 \,. \tag{3.115}$$

This equation is satisfied if

$$\delta\mathbf{a} = rot_a\, \delta\mathbf{A}(\mathbf{a}) \,, \tag{3.116}$$

for the vector function $\mathbf{a} \to \delta\mathbf{A}(\mathbf{a})$. Summarizing: so far we have proven that the internal energy e is invariant under the relabelling transformation (3.103) if (3.116) is satisfied.

Given (3.116), the corresponding conserved quantity is determined by (3.104) in the form

$$\frac{d}{dt}\int_{R_a} \frac{\partial L}{\partial(\partial x_i/\partial t)} \frac{\partial x_i}{\partial a_j} \delta a_j d(\mathbf{a}) = 0 \,. \tag{3.117}$$

Introducing the vector $\mathbf{B}$ with components

$$B_j = \frac{\partial L}{\partial(\partial x_i/\partial t)} \frac{\partial x_i}{\partial a_j} \,, \tag{3.118}$$

Equation (3.117) takes the form

$$\frac{d}{dt}\int_{R_a} \mathbf{B} \cdot rot_a\, \delta\mathbf{A}(\mathbf{a}) d(\mathbf{a}) = 0 \,. \tag{3.119}$$

Using the identity $\mathbf{B} \cdot rot_a\, \delta\mathbf{A}(\mathbf{a}) = div_a\, [\delta\mathbf{A}(\mathbf{a}) \times \mathbf{B}] + \delta\mathbf{A}(\mathbf{a}) \cdot rot_a\, \mathbf{B}$ and the boundary conditions for $\mathbf{B}$ on ∂R_a, Eq. (3.119) is equivalent to

$$\frac{d}{dt}\int_{R_a} rot_a\, \mathbf{B} \cdot \delta\mathbf{A}(\mathbf{a}) d(\mathbf{a}) = 0 \,. \tag{3.120}$$

Given the arbitrariness of the function $\delta\mathbf{A}(\mathbf{a})$, it is possible to set

$$\delta\mathbf{A}(\mathbf{a}) = \delta(a_1 - \bar{a}_1)\,\delta(a_2 - \bar{a}_2)\,\delta(a_2 - \bar{a}_2)\ , \tag{3.121}$$

where the product of the three Dirac delta functions and the following eliminations of the overbars. In this way, (3.120) can be written as a local conservation property rather than global, that is

$$\frac{\partial}{\partial t}\left[rot_a\,\mathbf{B}(t,\mathbf{a})\right] = 0\ , \tag{3.122}$$

which implies the conservation of the quantity

$$\boldsymbol{\omega}_a(t,\mathbf{a}) = rot_a\,\mathbf{B}(t,\mathbf{a})\ . \tag{3.123}$$

To determine the vector

$$\mathbf{B} = \sum_{j=1}^{3} B_j\hat{\mathbf{e}}_j\ , \tag{3.124}$$

we develop its components using (3.118)

$$\begin{aligned} B_j &= \frac{\partial L}{\partial u}\frac{\partial x}{\partial a_j} + \frac{\partial L}{\partial v}\frac{\partial y}{\partial a_j} + \frac{\partial L}{\partial w}\frac{\partial z}{\partial a_j} \\ &= u\frac{\partial x}{\partial a_j} + v\frac{\partial y}{\partial a_j} + w\frac{\partial z}{\partial a_j}\ . \end{aligned} \tag{3.125}$$

Substitution of (3.125) into (3.124) yields

$$\begin{aligned} \mathbf{B} &= \sum_{j=1}^{3}\left(u\frac{\partial x}{\partial a_j}\hat{\mathbf{e}}_j + v\frac{\partial y}{\partial a_j}\hat{\mathbf{e}}_j + w\frac{\partial z}{\partial a_j}\hat{\mathbf{e}}_j\right) \\ &= u\nabla_a x + v\nabla_a y + w\nabla_a z\ , \end{aligned} \tag{3.126}$$

i.e., by components,

$$B_j = u_i\frac{\partial x_i}{\partial a_j}\ . \tag{3.127}$$

The inversion of (3.126) yields the vector

$$\mathbf{u} = B_1\nabla a_1 + B_2\nabla a_2 + B_3\nabla a_3\ , \tag{3.128}$$

with components

$$u_j = B_i\frac{\partial a_i}{\partial x_j}\ . \tag{3.129}$$

Equation (3.128) defines $\mathbf{B}$ as a pseudo-momentum.

Taking any scalar θ, which is conserved on fluid parcels, leads to the conservation of

$$\Pi_a = (rot_a\, \mathbf{B}) \cdot \nabla_a \theta \ . \tag{3.130}$$

In particular, the conserved scalar can be identified with the i^{th} label of the fluid, a_i, and as shown in Appendix K, one has the conservation of the quantity

$$\Pi_i = \frac{1}{\rho} \left(\boldsymbol{\omega} \cdot \nabla a_i \right) \ , \tag{3.131}$$

which can be recognised as the conservation of potential vorticity introduced in Chap. 1. In (3.131), the subscript i indicates the i^{th} label of the fluid, a_i.

Remark 3.2 The derivation of (3.131) is a fundamental result for fluid dynamics, which shows that the conservation of potential vorticity does not simply arises from skilful manipulation of the partial differential equations of the system, but that it is associated through Noether's Theorem to a particular symmetry, the particle relabelling symmetry, which is due to the continuum nature of fluids.

3.5.3.2 Kelvin's Circulation Theorem

The fact that the vector $\mathbf{B}$ is a pseudo-momentum leads naturally also to the conservation of circulation. In fact, the integration of (3.122) over the domain R_a yields

$$\frac{\partial}{\partial t} \int_{R_a} \left[rot_a\, \mathbf{B}(t, \mathbf{a}) \right] d(\mathbf{a}) = \frac{\partial}{\partial t} \int_{\partial R_a} \mathbf{B}(t, \mathbf{a}) \cdot d\mathbf{a} = 0 \ , \tag{3.132}$$

where in the second step Stokes' Theorem has been applied. But because of (3.126),

$$\mathbf{B} \cdot d\mathbf{a} = \mathbf{u} \cdot d\mathbf{x} \ , \tag{3.133}$$

and (3.132) yields

$$\frac{\partial}{\partial t} \int_{\partial R} \mathbf{u} \cdot d\mathbf{x} = 0 \ , \tag{3.134}$$

which corresponds to Kelvin's Circulation Theorem, also introduced in Chap. 1, which is related to the conservation of potential vorticity, as shown in Appendix B.

3.5.3.3 Nonhomentropic Flows

For nonhomentropic flows, the internal energy is

$$e = e\left(\frac{\partial(\mathbf{x})}{\partial(\mathbf{a})}, \eta(\mathbf{a})\right) = e(\alpha, \eta) \; . \tag{3.135}$$

We are now interested in determining the quantities δa_j so that, given $\delta e = (\partial e/\partial\alpha)_\eta \delta\alpha + (\partial e/\partial\eta)_\alpha \delta\eta$, one has

$$\delta\alpha = 0 \; , \; \delta\eta = 0 \; . \tag{3.136}$$

This case differs from the homentropic case; in fact, now the vector function $\mathbf{a} \to \delta\mathbf{A}(\mathbf{a})$ satisfying (3.116) can now be determined by the second equation in (3.136). Because

$$\delta\eta(\mathbf{a}) = 0 \Rightarrow \eta(\mathbf{a} + \delta\mathbf{a}) - \eta(\mathbf{a}) = 0 \Rightarrow \nabla_a \eta \cdot \delta\mathbf{a} = 0 \; ,$$

and because of (3.116), $\delta\mathbf{A}$ is constrained by the equation

$$\nabla_a \eta \cdot rot_a \; \delta\mathbf{A} = 0 \; . \tag{3.137}$$

This equation is identically satisfied by

$$\delta\mathbf{A} = \delta s(\mathbf{a}) \nabla_a \eta \; , \tag{3.138}$$

for any scalar function $\delta s(\mathbf{a})$; in fact,

$$\nabla_a \eta \cdot rot_a \; (\delta s(\mathbf{a}) \nabla_a \eta) = \nabla_a \eta \cdot (\nabla_a \delta s \times \nabla_a \eta + \delta s \; rot_a \; \nabla_a \eta) = 0 \; .$$

Summarizing: the internal energy for a nonhomentropic flow is invariant under (3.103) if

$$\delta\mathbf{a} = rot_a \; [\delta s(\mathbf{a}) \nabla_a \eta] \; . \tag{3.139}$$

Given Eq. (3.139), we can apply Noether's Theorem: for nonhomentropic flows, (3.119) takes the form

$$\frac{d}{dt} \int_{R_a} \mathbf{B} \cdot rot_a \; [\delta s(\mathbf{a}) \nabla_a \eta(\mathbf{a})] \, d(\mathbf{a}) = 0 \; , \tag{3.140}$$

where $\mathbf{B}$ is the pseudo-momentum vector with components defined by (3.118). Using the identity $\mathbf{B} \cdot rot_a \; [\delta s \nabla_a \eta] = div_a \; [\delta s \nabla_a \eta \times \mathbf{B}] + \delta s \nabla_a \eta \cdot rot_a \, \mathbf{B}$, and the boundary conditions for $\mathbf{B}$ on ∂R_a, (3.140) is equivalent to

$$\frac{d}{dt} \int_{R_a} rot_a \, \mathbf{B} \cdot \nabla_a \eta \delta s(\mathbf{a}) d(\mathbf{a}) = 0 \; . \tag{3.141}$$

Again, given the arbitrariness of $\delta s(\mathbf{a})$, it is possible to set

$$\delta s(\mathbf{a}) = \delta\left(a_1 - \bar{a}_1\right) \delta\left(a_2 - \bar{a}_2\right) \delta\left(a_2 - \bar{a}_2\right) \; , \tag{3.142}$$

where the product of the three Dirac delta functions and the following eliminations of the overbars allows to write (3.141) as a local conservation law instead of a global conservation law, that is

$$\frac{\partial}{\partial t}\left[rot_a\, \mathbf{B}(t, \mathbf{a}) \cdot \nabla_a \eta(\mathbf{a})\right] = 0\ . \tag{3.143}$$

Following the same steps leading to (3.131), (3.143) can be expressed in physical space as

$$\frac{D}{Dt}\left[\frac{1}{\rho}(rot\ \mathbf{u}(t, \mathbf{x})) \cdot \nabla \eta(\mathbf{x})\right] = 0 \tag{3.144}$$

that corresponds to the conservation of the potential vorticity in the presence of continuous stratification

$$\Pi = \frac{1}{\rho}(rot\ \mathbf{u}) \cdot \nabla \eta\ . \tag{3.145}$$

Remark 3.3 The particle relabelling symmetry and, hence the conservation of (potential) vorticity, is present also in magneto-hydrodynamics (see, e.g. [27]). This symmetry is, however, absent from the other continuum theory of classical physics, that is the theory of elasticity. The reason for that lies in the mathematical structure of the Lagrangian density for fluid dynamics, in which the derivatives $\partial x_i/\partial a_j$ enter the Lagrangian only through the Jacobian $\partial(\mathbf{x})/\partial(\mathbf{a})$. *This immediately shows that the particle relabelling symmetry corresponds to the request that the relabelling of fluid particles does not affect the distribution of mass*. This is not true in elasticity theory, where the derivatives enter in the internal energy e separately.

3.6 Bibliographical Note

On the derivation of the equations of motion for the ideal fluid using a variational approach, see, among others, [2, 3, 8, 14]. For early work on the variational derivation of the compressible equations for fluid, see instead [7, 33]. These lists are, however, surely incomplete. In general, the review article by Salmon [30] and the seminal book by the same author [31] contain a large number of historically interesting references. [20, 21] offer also great reviews on the subject. Regarding the derivation of the equations of motion using the Clebsch representation, see also [10–12]. A group theory approach to the Clebsch representation was formulated by [1, 19]. Particularly interesting is the analysis made by [4] both on the significance of the Lin constraints and on the fact that if the initial conditions are such that the vortex lines are knotted, the Clebsch representation presents singularities. For further discussion on the canonical transformation connecting the Lagrangian and Eulerian forms of Hamilton's principle, see, e.g. [5, 6, 13]. For the study of symmetries and in particular of the particle relabelling symmetry in fluid dynamics, see, e.g. [15, 16, 18, 25–27, 29, 30, 34]. As reported by [27], the same reference, the first reference on the particle relabelling symmetry was in [24], where the symmetry was called "exchange invariance".

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Chapter 4
Variational Principles in Geophysical Fluid Dynamics and Approximated Equations

Abstract In this chapter, the variational principle of Hamilton is applied to different examples from Geophysical Fluid Dynamics. Hamilton's principle is extended to uniformly rotating flows and to incompressible flows. After an example in finite dimensions consisting of the motion of point vortices, a set of approximated equations is considered, that is, rotating shallow water equations, rotating Green–Naghdi equations and semi-geostrophic equations. Equations of the first and second kind conserve potential vorticity as a consequence of the invariance of the related action functional under relabelling symmetry. Equation of the third kind takes into account also an ageostrophic part of the flow and conserves the so-called transformed potential vorticity which is based on a special Legendre transformation on the coordinates. The case of continuously stratified fluids is then analysed. Finally, the variational approach is applied to wave dynamics, where it can be used to both derive the equations of motion and to obtain the dispersion relation for nonlinear problems as well as the conservation of the wave activity of the system.

Keywords Fluid dynamics · Geophysical fluid dynamics · Ideal fluid · Variational principle · Conservation laws · Rotating flows · Stratified flows · Potential vorticity · Ertel's theorem · Circulation · Shallow water equations · Quasi-geostrophic equations · Lagrangian labels · Relabelling symmetry · Point vortices · Approximated equations · Semi-geostrophy · Green–Naghdi equations · Wave dynamics · Surface waves · Luke's variational principle · Whitham's averaged variational principle · Wave Activity · Klein–Gordon equation · Korteweg–deVries (KdV) equation

4.1 Introduction

In this chapter, some relevant variations on the action functional (3.31), that is,

$$I = \int_{t_1}^{t_2} \int_{R_a} \left[\frac{1}{2} \left| \frac{\partial \mathbf{x}(\mathbf{a}, t)}{\partial t} \right|^2 - e\left(\frac{\partial(\mathbf{x})}{\partial(\mathbf{a})}, \eta(\mathbf{a}) \right) - \phi\left(\mathbf{x}(\mathbf{a}), t\right) \right] d(\mathbf{a}) dt \ ,$$

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are expounded having in view the geophysical context and in particular exploring the effects of rotation and stratification and of different possibilities to scale the Lagrangian density. These variations will be here only anticipated, serving thus as a summary, while their derivations are postponed in the following subsections.

In a uniformly rotating frame of reference with a constant rotation vector $\mathbf{\Omega}$, the action functional (3.31) becomes

$$I = \int_{t_1}^{t_2} \int_{R_a} \left[\frac{1}{2} |\dot{\mathbf{x}}|^2 + \dot{\mathbf{x}} \cdot (\mathbf{\Omega} \times \mathbf{x}) - e - \phi \right] d(\mathbf{a}) dt \ .$$

As a basic consequence, the conserved potential vorticity Π takes the form

$$\Pi = \frac{1}{\rho} (rot\ \mathbf{u} + 2\mathbf{\Omega}) \cdot \nabla \eta \ .$$

Incompressibility of an ideal fluid changes functional (3.31) into

$$I = \int_{t_1}^{t_2} dt \int_{R_a} d(\mathbf{a}) \left[\frac{1}{2} \left| \frac{\partial \mathbf{x}}{\partial t} \right|^2 - \phi + \lambda \left(\frac{\partial (\mathbf{x})}{\partial (\mathbf{a})} - \alpha \right) \right] ,$$

Before to introduce different approximations to the Lagrangian density, we consider the motion of point vortices on the plane, which constitutes a finite dimensional example with relevance for geophysical flows and which is based on the assumption that, for a vortex in r_0,

$$\omega = \Gamma \delta (\mathbf{r} - \mathbf{r}_0) ,$$

where Γ is the circulation and δ is the Dirac delta function. For N interacting point vortices, this system possesses the Lagrangian function

$$L = -\frac{1}{4\pi} \sum_{i \neq j,\ i,j=1}^{N} \Gamma_i \Gamma_j \log \left| \mathbf{r}_i - \mathbf{r}_j \right| + \sum_{i=1}^{N} \Gamma_i \left(x_i^2 + y_i^2 \right) ,$$

where the application of Noether's Theorem shows that the first term on the r.h.s. corresponds to the Hamiltonian and the second term corresponds to the conserved angular momentum.

After that, the rotating shallow water equations are obtained by means of

$$I = \int_{t_1}^{t_2} \int_{R_h} \frac{1}{2} \rho \left[|\dot{\mathbf{x}}_h|^2 + \dot{\mathbf{x}}_h \cdot (2\mathbf{\Omega} \times \mathbf{x}_h) - g\eta \right] d(\mathbf{x}_h) dt ,$$

where $\mathbf{x}_h = \mathbf{x}_h (a_1, a_2, t)$, $z = z (a_1, a_2, a_3, t)$ and $a_3 = 0$ at the free surface in $z = \eta$. In the shallow water equations, the particle relabelling symmetry leads to the conservation of potential vorticity

$$\Pi = \frac{1}{H+\eta}\varsigma \ ,$$

in the inertial case and

$$\Pi = \frac{\varsigma + f}{H+\eta}$$

in the uniformly rotating case, respectively, where we recall that $\varsigma = \hat{\mathbf{k}} \cdot rot\ \mathbf{u}$. The shallow water equations can be extended to the two-layer case, where the action functional comprises an interaction term between the two layers. In this case, the potential vorticity results to be conserved layer-wise.

If the assumption of a small vertical velocity is released, the rotating shallow water equations turn into the Green–Naghdi equations whose action functional in the physical space is

$$I = \int_{t_1}^{t_2} \int_R \rho \left[\frac{1}{2} |\dot{\mathbf{x}}_h|^2 + \frac{1}{2} |\dot{z}|^2 + \dot{\mathbf{x}}_h \cdot (2\boldsymbol{\Omega} \times \mathbf{x}_h) - gz \right] d(\mathbf{x}) dt \ .$$

Following, the geostrophic momentum approximation and the semi-geostrophic dynamics on the f-plane are introduced. The geostrophic momentum approximation takes into account a suitably defined ageostrophic velocity which enters both in the momentum and in the continuity equations to yield a more accurate description of the flow dynamics with respect to the shallow water approximation. The semi-geostrophic dynamics is a further development of this subject in which the so-called geostrophic coordinates (x_s, y_s) are introduced together with the free surface elevation

$$\eta_s = \frac{\partial(a_1, a_2)}{\partial(x_s, y_s)} - H \ ,$$

as a function of them. Hence, potential vorticity is conserved in the form

$$\left[\frac{\partial}{\partial t} + \dot{x} \frac{\partial}{\partial x_s} + \dot{y} \frac{\partial}{\partial y_s} \right] \frac{f_0}{\eta_s + H} = 0 \ .$$

As a further development from shallow water dynamics, a continuously stratified fluid demands, within the Boussinesq approximation, a functional of the kind

$$I = \int_{t_1}^{t_2} dt \int_{R_a} d(\mathbf{a}) \left[\frac{1}{2} \rho_b(a_3) \left| \frac{\partial \mathbf{x}}{\partial t} \right|^2 - \rho_b(a_3)\phi + p \left(\frac{\partial(\mathbf{x})}{\partial(\mathbf{a})} - \alpha \right) \right] ,$$

where $\rho_b(a_3)$ is a background density field.

Chapter 4 ends with an analysis of selected topics in wave dynamics. It is shown that in the physical space the equations for surface waves are derived by the action functional

$$I = \int_{t_1}^{t_2} \int_R \rho \left[\frac{\partial \varphi}{\partial t} + \frac{1}{2} (\nabla \varphi)^2 + gz \right] d(\mathbf{x}) dt \ ,$$

where φ is a velocity potential. Next, Whitham's variational principle is introduced, showing a powerful tool to obtain the dispersion relation for nonlinear problems in wave dynamics as well as the conservation of the wave activity of the system.

4.2 Hamilton's Principle, Rotation and Incompressibility

4.2.1 Lagrangian Density in a Rotating Frame of Reference

The Lagrangian formulation of the dynamics for the ideal fluid can be easily extended to a uniformly rotating frame of reference. To do so, we recall from Sect. 1.5 that the relation between the velocity in an inertial frame of reference and a frame of reference rotating with uniform rate $\boldsymbol{\Omega}$ is given by (1.67), that is,

$$\dot{\mathbf{x}}_I = \dot{\mathbf{x}}_R + \boldsymbol{\Omega} \times \mathbf{x}_R(t) \ . \tag{4.1}$$

Using (4.1) and assuming that both the internal energy and the external potential are unaffected by rotation, the action (3.31) can be rewritten as

$$I = \int_{t_1}^{t_2} \int_{R_a} \left[\frac{1}{2} |\dot{\mathbf{x}} + \boldsymbol{\Omega} \times \mathbf{x}|^2 - e - \phi \right] d(\mathbf{a}) dt \ . \tag{4.2}$$

Consider the variation δI obtained applying (3.46) with boundary conditions (3.47). The variation of the kinetic energy term can be written as

$$\begin{aligned} \delta T =& \frac{1}{2} \left[\dot{\mathbf{x}} + l\dot{\mathbf{Q}} + \boldsymbol{\Omega} \times (\mathbf{x} + l\mathbf{Q}) \right] \cdot \left[\dot{\mathbf{x}} + l\dot{\mathbf{Q}} + \boldsymbol{\Omega} \times (\mathbf{x} + l\mathbf{Q}) \right] \\ &- \frac{1}{2} (\dot{\mathbf{x}} + \boldsymbol{\Omega} \times \mathbf{x}) \cdot (\dot{\mathbf{x}} + \boldsymbol{\Omega} \times \mathbf{x}) \\ =& (\dot{\mathbf{x}} + \boldsymbol{\Omega} \times \mathbf{x}) \cdot \left(l\dot{\mathbf{Q}} + \boldsymbol{\Omega} \times l\mathbf{Q} \right) + \frac{1}{2} \left(l\dot{\mathbf{Q}} + \boldsymbol{\Omega} \times l\mathbf{Q} \right) \cdot \left(l\dot{\mathbf{Q}} + \boldsymbol{\Omega} \times l\mathbf{Q} \right) \\ =& \dot{\mathbf{x}} \cdot l\dot{\mathbf{Q}} + \dot{\mathbf{x}} \cdot (\boldsymbol{\Omega} \times l\mathbf{Q}) + (\boldsymbol{\Omega} \times \mathbf{x}) \cdot l\dot{\mathbf{Q}} + (\boldsymbol{\Omega} \times \mathbf{x}) \cdot (\boldsymbol{\Omega} \times l\mathbf{Q}) \ , \end{aligned} \tag{4.3}$$

where the term $\left(l\dot{\mathbf{Q}} + \boldsymbol{\Omega} \times l\mathbf{Q} \right) \cdot \left(l\dot{\mathbf{Q}} + \boldsymbol{\Omega} \times l\mathbf{Q} \right)$ has been set to zero at first order in $l\mathbf{Q}$. The last line can be reordered factorizing the terms proportional to $l\mathbf{Q}$ and $l\dot{\mathbf{Q}}$, obtaining

$$\delta T = (\dot{\mathbf{x}} + \boldsymbol{\Omega} \times \mathbf{x}) \cdot l\dot{\mathbf{Q}} + (\dot{\mathbf{x}} \times \boldsymbol{\Omega}) \cdot l\mathbf{Q} + (\boldsymbol{\Omega} \cdot \boldsymbol{\Omega}) (\mathbf{x} \cdot l\mathbf{Q}) - (\boldsymbol{\Omega} \cdot l\mathbf{Q}) (\mathbf{x} \cdot \boldsymbol{\Omega}) \ . \tag{4.4}$$

It is now possible to make use of the identities

$$(\dot{\mathbf{x}} + \boldsymbol{\Omega} \times \mathbf{x}) \cdot l\dot{\mathbf{Q}} = \frac{\partial}{\partial t}\left[(\dot{\mathbf{x}} + \boldsymbol{\Omega} \times \mathbf{x}) \cdot l\mathbf{Q}\right] - (\ddot{\mathbf{x}} + \boldsymbol{\Omega} \times \dot{\mathbf{x}}) \cdot l\mathbf{Q}\,, \tag{4.5}$$

and

$$\begin{aligned}(\boldsymbol{\Omega} \cdot \boldsymbol{\Omega})\,(\mathbf{x} \cdot l\mathbf{Q}) - (\boldsymbol{\Omega} \cdot l\mathbf{Q})\,(\mathbf{x} \cdot \boldsymbol{\Omega}) &= \left[(\boldsymbol{\Omega} \cdot \boldsymbol{\Omega})\,\mathbf{x} - (\mathbf{x} \cdot \boldsymbol{\Omega})\,\boldsymbol{\Omega}\right] \cdot l\mathbf{Q} \\ &= \left[-\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{x})\right] \cdot l\mathbf{Q}\,.\end{aligned} \tag{4.6}$$

Applying (4.5) and (4.6)–(4.4) yields

$$\begin{aligned}\delta T =& \{-(\ddot{\mathbf{x}} + \boldsymbol{\Omega} \times \dot{\mathbf{x}}) - \boldsymbol{\Omega} \times \dot{\mathbf{x}} - [\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{x})]\} \cdot l\mathbf{Q} \\ &+ \frac{\partial}{\partial t}\left[(\dot{\mathbf{x}} + \boldsymbol{\Omega} \times \mathbf{x}) \cdot l\mathbf{Q}\right]\,.\end{aligned} \tag{4.7}$$

Putting together (4.7) and the terms that do not depend on the angular frequency, i.e. (3.56), yields

$$\delta I = \int_{t_1}^{t_2} dt \int_R d(\mathbf{x})\,\{\rho\,[-\ddot{\mathbf{x}} - [\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{x})] - (2\boldsymbol{\Omega} \times \dot{\mathbf{x}}) - \nabla\phi] - \nabla p\} \cdot l\mathbf{Q}\,, \tag{4.8}$$

where the integration of last term of (4.7) cancels using the boundary conditions. Once again, Hamilton's principle requires that $\delta I = 0$, so that

$$\ddot{\mathbf{x}} + (2\boldsymbol{\Omega} \times \dot{\mathbf{x}}) + [\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{x})] = -\frac{1}{\rho}\nabla p - \nabla\phi\,. \tag{4.9}$$

In the particular case, relevant for Geophysical Fluid Dynamics, in which ϕ is the gravitational potential, $\phi = gz$ and $\nabla\phi = g\hat{\mathbf{k}}$ and (4.9) corresponds to (1.72), with the second term on the r.h.s. representing the Coriolis force and the third term representing the centrifugal acceleration. As discussed in Sect. 1.5, for the flows here considered the centrifugal acceleration can be included in the gravitational term. In the Lagrangian density, this can be accomplished by modifying the term corresponding to the external potential in (4.2) as

$$\phi(\mathbf{x}) \to \phi(\mathbf{x}) + \frac{1}{2}\,|\boldsymbol{\Omega} \times \mathbf{x}|^2\,, \tag{4.10}$$

which yields

$$\begin{aligned}I &= \int_{t_1}^{t_2}\int_{R_a}\left[\frac{1}{2}\,|\dot{\mathbf{x}} + \boldsymbol{\Omega} \times \mathbf{x}|^2 - \frac{1}{2}\,|\boldsymbol{\Omega} \times \mathbf{x}|^2 - e - \phi\right] d(\mathbf{a})dt \\ &= \int_{t_1}^{t_2}\int_{R_a}\left[\frac{1}{2}\,|\dot{\mathbf{x}}|^2 + \dot{\mathbf{x}} \cdot (\boldsymbol{\Omega} \times \mathbf{x}) - e - \phi\right] d(\mathbf{a})dt\,,\end{aligned} \tag{4.11}$$

with Lagrangian density

$$L = \frac{1}{2}\,|\dot{\mathbf{x}}|^2 + \dot{\mathbf{x}}\cdot(\boldsymbol{\Omega}\times\mathbf{x}) - e - \phi\;, \tag{4.12}$$

and equations of motion

$$\ddot{\mathbf{x}} + (2\boldsymbol{\Omega}\times\dot{\mathbf{x}}) = -\frac{1}{\rho}\nabla p - \nabla\phi\;. \tag{4.13}$$

It should be noted that, in the presence of rotation, the conjugate momentum density is

$$\boldsymbol{\pi}_a = \frac{\delta\mathscr{L}}{\delta\dot{\mathbf{x}}} = \rho_a\,[\mathbf{u} + (\boldsymbol{\Omega}\times\mathbf{x})]\;, \tag{4.14}$$

with analogous definition for $\boldsymbol{\pi}$.

Remark 4.1 The Lagrangian density (4.12) introduces the rotation term $\dot{\mathbf{x}}\cdot(\boldsymbol{\Omega}\times\mathbf{x})$, for which the identity

$$2\boldsymbol{\Omega} = rot\;(\boldsymbol{\Omega}\times\mathbf{x})\;, \tag{4.15}$$

holds. Interestingly, the rotation term shows a strong analogy with the term arising in the Lagrangian density for a particle in a magnetic field. In the latter case, the kinetic energy density includes the term $q\dot{\mathbf{x}}\cdot\mathbf{A}$, which represents the interaction of a particle with charge q in a magnetic vector potential $\mathbf{A}$. The analogy becomes apparent upon the substitution $\mathbf{A} = \boldsymbol{\Omega}\times\mathbf{x}$. The magnetic vector potential is linked to the magnetic field by $\mathbf{B} = rot\mathbf{A} = 2\boldsymbol{\Omega}$. The Coriolis force $\dot{\mathbf{x}}\times 2\boldsymbol{\Omega}$ becomes thus analogous to the Lorentz force $q\dot{\mathbf{x}}\times\mathbf{B}$.

Remark 4.2 The insertion of rotation corresponds to a gauge symmetry of the system. Using (4.15), the terms in (4.8) that correspond to the rotation are

$$\delta I_\Omega = \int_{t_1}^{t_2} dt \int_R d(\mathbf{x})\rho\,[\ddot{\mathbf{x}} + \;rot\;\;(\boldsymbol{\Omega}\times\mathbf{x})\times\mathbf{x}]\cdot l\mathbf{Q}\;. \tag{4.16}$$

Equation (4.16) is invariant under the gauge transformation symmetry

$$\boldsymbol{\Omega}\times\mathbf{x} \to \boldsymbol{\Omega}\times\mathbf{x} + \nabla\varphi\;, \tag{4.17}$$

where φ is an arbitrary differentiable scalar.

4.2.2 Relabelling Symmetry in a Rotating Framework

The derivation of the potential vorticity conservation from the relabelling symmetry can be extended in a straightforward way to the case of a fluid in a rotating framework, with the vorticity $\nabla\times\mathbf{u}$ replaced by the absolute vorticity. This is visible using

Noether's Theorem. In fact, using the Lagrangian density in a rotating framework (4.12), making use of (3.118), the pseudo-vector **B** can be written as

$$\tilde{\mathbf{B}} = \mathbf{B} - \boldsymbol{\Omega} \times \mathbf{x} \,. \tag{4.18}$$

With (4.18), the conservation law (3.143) becomes

$$\frac{\partial}{\partial t}\left[rot_a\, \tilde{\mathbf{B}}(t, \mathbf{a}) \cdot \nabla_a \eta(\mathbf{a})\right] = 0 \,. \tag{4.19}$$

Following again the same steps leading to (3.131), Eq. (4.19) yields the conservation of the potential vorticity in a rotating framework

$$\Pi = \frac{1}{\rho}(rot\ \mathbf{u} + 2\boldsymbol{\Omega}) \cdot \nabla \eta \,. \tag{4.20}$$

4.2.3 Role of Incompressibility

The derivation of the equations of motion (3.57) from the Lagrangian density (3.30) does not make any assumption on the incompressibility of the ideal fluid under consideration, and it is thus valid for compressible fluids. For incompressible fluids, such as, at first approximation, the ocean, the Lagrangian density is determined by replacing the internal energy term with a constraint derived from (3.18)

$$L = \frac{1}{2}\left|\frac{\partial \mathbf{x}}{\partial t}\right|^2 - \phi + \lambda \left(\frac{\partial(\mathbf{x})}{\partial(\mathbf{a})} - \alpha\right) , \tag{4.21}$$

where λ is a Lagrange multiplier, introduced in Sect. 2.5. In the Lagrangian density (4.21), we have thus replaced the thermodynamical term of (3.30) with a dynamical constraint.

Applying the variation (3.46) with boundary conditions (3.47) to the last term on the r.h.s. of the action functional (3.31), now written as

$$I = \int_{t_1}^{t_2} dt \int_{R_a} d(\mathbf{a}) \left[\frac{1}{2}\left|\frac{\partial \mathbf{x}}{\partial t}\right|^2 - \phi + \lambda \left(\frac{\partial(\mathbf{x})}{\partial(\mathbf{a})} - \alpha\right)\right] , \tag{4.22}$$

yields

$$\delta I_\lambda = \int_{t_1}^{t_2} dt \int_{R_a} d(\mathbf{a}) \lambda \delta \frac{\partial(\mathbf{x})}{\partial(\mathbf{a})} \,. \tag{4.23}$$

It is visible that the spatial integrand in (4.23) matches with the first line in (3.54) upon the identification of the Lagrange multiplier with the pressure, i.e.

$$\lambda = p \; . \tag{4.24}$$

In this case, the application of Hamilton's principle to the Lagrangian density (3.30) yields the same equations of motion (3.13) and (3.57) obtained from the Lagrangian density for the compressible ideal fluid (3.31).

4.3 A Finite Dimensional Example: Dynamics of Point Vortices

Before to start the treatment of infinite dimensional example of approximated equations using variational techniques, it is worth giving a look at a classical finite dimensional example which is relevant to Geophysical Fluid Dynamics, which is given by the inviscid point vortex model. Jules Charney, one of the fathers of the quasi-geostrophic dynamics described in Chap. 1, wrote [13]: *"[…] the continuous vorticity distribution in two-dimensional flow may be approximated by a finite set of parallel rectilinear vortex filaments of infinitesimal cross-section and finite strength, whose motion is governed by a set of ordinary differential equations. This is analogous to replacing a continuous mass distribution by a set of gravitating mass points. It has the virtue that mass, energy, linear and angular momentum continue to be conserved, and that the motions represented are those of conceivable, though idealized, physical systems"*. In this section, we will present the simplest, inviscid point vortex model for the Euler equation.

Consider a two-dimensional incompressible, inviscid flow in the (x, y) Cartesian plane. The point vortex model consists in dividing the fluid into a number of separated regions, with the area of each region arbitrarily small. Each vortex approaches thus a single point with infinite vorticity and finite circulation. This can be seen in the following way. Consider a closed contour $\partial C(t)$ containing the vortex and moving with the fluid. Then, as seen in Chap. 1, Kelvin's Circulation Theorem states that, under appropriate conditions, the circulation Γ, given by (1.54), is a conserved quantity. The point vortex approximation thus requests that, for a vortex in r_0,

$$\omega = \Gamma \delta(\mathbf{r} - \mathbf{r}_0) \; , \tag{4.25}$$

where δ is the Dirac delta function, so that $\delta(\mathbf{r}-\mathbf{r}_0) = \infty$ if $\mathbf{r} = \mathbf{r}_0$ and $\delta(\mathbf{r}-\mathbf{r}_0) = 0$ if $\mathbf{r} \neq \mathbf{r}_0$. With the introduction of the stream function ψ, (4.25) becomes

$$\nabla^2 \psi = \Gamma \delta(\mathbf{r} - \mathbf{r}_0) \; . \tag{4.26}$$

The solution of (4.26) can thus be calculated using the Green's function G,

$$\psi(\mathbf{r}) = \iint_C G(\mathbf{r} - \mathbf{r}_0)\omega(\mathbf{r}_0)dxdy \; , \tag{4.27}$$

where G satisfies $|\nabla G| \to 0$ as $|\mathbf{r} - \mathbf{r}_0| \to \infty$. To determine G, it is useful to pass to polar coordinates, with radial coordinate r and with $\mathbf{r}_0 = \mathbf{0}$. Consider C to be the circle with radius R, then the integration of the Poisson equation (4.26) yields

$$\int_{r\leq R} \delta(r) dA = 1 = \int_{r\leq R} div\,(\nabla G) dA\ . \tag{4.28}$$

Using the divergence theorem,

$$\begin{aligned} 1 &= \oint_{r=R} \nabla G \cdot \hat{\mathbf{n}} dl \\ &= \oint_{r=R} G'(r) dl \\ &= 2\pi R G'(R)\ , \end{aligned} \tag{4.29}$$

which in turn yields

$$G'(r) = \frac{1}{2\pi} r\ , \tag{4.30}$$

and thus

$$G(r) = \frac{1}{2\pi} \log\left(|r| + const\right)\ . \tag{4.31}$$

Notice that the constant acts to make the argument of the logarithm nondimensional. Returning to Cartesian coordinates, using $\mathbf{r}_0 \neq 0$ and using (4.31) and (4.27), thus yields

$$\psi(\mathbf{r}) = \frac{\Gamma}{2\pi} \log|\mathbf{r} - \mathbf{r}_0|\ . \tag{4.32}$$

Equation (4.32) yields the velocity at $\mathbf{r}$ induced by an inviscid point vortex at $\mathbf{r}_0$,

$$u(\mathbf{r}) = \frac{dx}{dt} = -\frac{\partial\psi}{\partial y} = -\frac{\Gamma}{2\pi}\frac{y - y_0}{|\mathbf{r} - \mathbf{r}_0|^2}\ , \tag{4.33a}$$

$$v(\mathbf{r}) = \frac{dy}{dt} = \frac{\partial\psi}{\partial x} = \frac{\Gamma}{2\pi}\frac{x - x_0}{|\mathbf{r} - \mathbf{r}_0|^2}\ . \tag{4.33b}$$

Equations (4.26), (4.32), (4.33a) and (4.33b) have a singularity at $\mathbf{r} = \mathbf{r}_0$. The stream function (4.32) has an additional singularity for $\mathbf{r} \to \infty$. Equations (4.25) and (4.32) can be generalized to calculate the vorticity and the associated stream function induced by N inviscid point vortices as

$$\omega(\mathbf{r}) = \sum_{i=1}^{N} \Gamma_i \delta(\mathbf{r} - \mathbf{r}_i) \ , \tag{4.34a}$$

$$\psi(\mathbf{r}) = \frac{1}{2\pi} \sum_{i=1}^{N} \Gamma_i \log |\mathbf{r} - \mathbf{r}_i| \ , \tag{4.34b}$$

for $i = 1, \ldots, N$. In particular, (4.33a) and (4.33b) can be generalized to calculate the velocity of the vortex i, with components $(dx_i/dt, dy_i/dt)$, induced by $N - 1$ point vortices $j \neq i$, as

$$u(\mathbf{r}_i) = \frac{dx_i}{dt} = -\frac{\partial \psi}{\partial y_i} = -\frac{1}{2\pi} \sum_{j \neq i, i=1}^{N} \Gamma_j \frac{y_i - y_j}{|\mathbf{r}_i - \mathbf{r}_j|^2} \ , \tag{4.35a}$$

$$v(\mathbf{r}_i) = \frac{dy_i}{dt} = \frac{\partial \psi}{\partial x_i} = \frac{1}{2\pi} \sum_{j \neq i, i=1}^{N} \Gamma_j \frac{x_i - x_j}{|\mathbf{r}_i - \mathbf{r}_j|^2} \ . \tag{4.35b}$$

The system (4.35a), (4.35b) shows that the trajectories of the vortices are embedded in a $2N$-dimensional phase space.

The system possesses the Lagrangian function

$$L = -\frac{1}{4\pi} \sum_{i \neq j,\ i,j=1}^{N} \Gamma_i \Gamma_j \log \left| \mathbf{r}_i - \mathbf{r}_j \right| + \sum_{i=1}^{N} \Gamma_i \left(x_i^2 + y_i^2 \right) \ . \tag{4.36}$$

The equations of motion are given by the Euler–Lagrange equations involving this Lagrangian function.

The application of Noether's Theorem to (4.36) shows that this system yields a number of conserved quantities. From (2.159), the time translation symmetry, corresponding to (2.165a)–(2.165c), applied to (4.36), yields the conservation of

$$H = -\frac{1}{4\pi} \sum_{i \neq j,\ i,j=1}^{N} \Gamma_i \Gamma_j \log \left| \mathbf{r}_i - \mathbf{r}_j \right| \ . \tag{4.37}$$

The notation used to indicate this quantity is not accidental, as this quantity is the energy of the system and H is indeed the Hamiltonian function.

The invariance for translations along the x-axis, corresponding to (2.161a)–(2.161c), applied to (4.36), yields the conservation of the linear momentum

$$M_x = \frac{1}{\Gamma} \sum_{i=1}^{N} \Gamma_i x_i \ , \tag{4.38}$$

where $\Gamma = \sum_{i=1}^{N} \Gamma_i$ is also a conserved quantity of the system due to Kelvin's circulation theorem. Analogously, the translation symmetry along the y-axis yields the conservation of

$$M_y = \frac{1}{\Gamma} \sum_{i=1}^{N} \Gamma_i y_i \ . \tag{4.39}$$

Finally, the invariance for rotations around the z axis (2.168a)–(2.168c) applied to (4.36) yields the conservation of the quantity

$$I = \sum_{i=1}^{N} \Gamma_i \left(x_i^2 + y_i^2 \right) \ , \tag{4.40}$$

which can be recognized as the total angular momentum of the system.

The system has also a scaling symmetry for

$$(x, y) \to \lambda(x', y') \ , \tag{4.41a}$$

$$t \to \lambda^2 t' \ , \tag{4.41b}$$

with $\lambda \in \mathbb{R}$.

Finally, the system has also the following discrete symmetries

$$t \to -t \ , \ \Gamma \to -\Gamma \ , \tag{4.42a}$$

$$(x, y) \to (-x, -y) \ , \tag{4.42b}$$

and a discrete symmetry under cyclic permutation of indices. For more details on the symmetries of the point vortex system, see, e.g. [12].

Remark 4.3 With (4.37) and (4.40), it is possible to see that the Lagrangian function (4.36) is

$$L = H + I \ , \tag{4.43}$$

which shows that the Lagrangian is not given by the kinetic energy minus the potential energy (the energy due to the self-interaction of the vortices is trivially zero) but by the sum of the kinetic energy and total angular momentum. From the considerations reported in Chap. 2, it should not surprise that the Lagrangian function has this form. For example, for two-dimensional fluid flows, if the domain is not simply connected, the Hamiltonian is not given by the energy of the system but it includes also a term coming from the circulation. For more details, see, e.g. [69].

The Hamiltonian function (4.37) allows to introduce Hamilton's equations for the point vortex i as

$$\Gamma_i \frac{dx_i}{dt} = \frac{\partial H}{\partial y_i} \,, \tag{4.44a}$$

$$\Gamma_i \frac{dy_i}{dt} = -\frac{\partial H}{\partial x_i} \,. \tag{4.44b}$$

One should notice that this is an example in which the conjugate momentum does not correspond to the linear momentum. Equations (4.44a), (4.44b) can be rewritten in symplectic form. Defining $\mathbf{z} = (x_1, \ldots, x_N, y_1, \ldots, y_N)$, one has

$$\Gamma \frac{d\mathbf{z}}{dt} = \mathbf{J} \frac{\partial H}{\partial \mathbf{z}} \,, \tag{4.45}$$

where $\mathbf{J}$ is given by (2.75).

Considered two functions $f(x_1, \ldots, x_N, y_1, \ldots, y_N)$ and $g(x_1, \ldots, x_N, y_1, \ldots, y_N)$, the Poisson bracket associated with the symplectic form (4.45) is

$$\{f, g\} = \sum_{i=1}^{N} \frac{1}{\Gamma_i} \left(\frac{\partial f}{\partial x_i} \frac{\partial g}{\partial y_i} - \frac{\partial f}{\partial y_i} \frac{\partial g}{\partial x_i} \right) \,, \tag{4.46}$$

which satisfies

$$\{x_i, \Gamma_i y_i\} = 1 \,, \tag{4.47}$$

and

$$\{x_i, y_j\} = 0 \iff i \neq j \,, \; (i, j = 1, \ldots, N) \,. \tag{4.48}$$

This is a beautiful property of the point vortex equations that shows that for this system the motion in the Cartesian plane corresponds to the motion in the phase plane. The use of the Poisson bracket (4.46) gives an other proof of the invariance of the Hamiltonian (4.37), due to the fact $\dot{H} = \{H, H\} = 0$. The invariance of M_x, M_y and I can also be inferred from (4.37) and (4.46), and in fact, direct calculation gives

$$\{M_x, H\} = \{M_y, H\} = \{I, H\} = 0 \,. \tag{4.49}$$

Generally, the presence of K invariants reduces the dimension of the phase space from $2N$ to $2(N - K)$, and the system is said to have $N - K$ degrees of freedom. If $N = K$, the system is said to be completely integrable. For the point vortex model, the system is completely integrable for $N \leq 3$, which results in regular or quasi-periodic trajectories. For $N > 3$, the system is generally (but not always, as it could be in the case of systems which possess additional conserved quantities) nonintegrable, which can result in chaotic trajectories.

4.4 Approximated Equations

The variational derivation for the equations of motion of the ideal fluid presented in the previous sections can be extended to an entire gallery of different classes of dynamics and approximations. We will now proceed to derive some of the most important approximations used for fluids and geophysical fluids. The derivation of the different Lagrangian densities is important not only for the inference of the equations of motion but also for the analysis of their symmetries and conservation laws. In fact, one of the advantages of the Lagrangian and Hamiltonian formulation comes from the fact that if the symmetries of a system are preserved by the approximation applied to the original equations, the approximated system will retain the same conservation laws of the original system.

4.4.1 Rotating Shallow Water Equations

An important case of the fluid motion, which is relevant also for Geophysical Fluid Dynamics, is given by the rotating shallow water equations, already presented in Sect. 1.9.1. Let's recall the setting: consider a fluid in a single layer with constant density and under the effect of the Earth's rotation. The fluid is included between a bottom boundary $z = -H + h'(x, y)$, where H is the characteristic depth of the layer of the fluid and the dependence on the horizontal coordinates in $h'(x, y)$ represents the bathymetric modulations, and a free surface at $z = \eta(x, y, t)$. Without loss of generality, we can consider, for simplicity, the case of a flat rigid bottom boundary, i.e. $h'(x, y) = 0$. Using the request of uniform density, (3.10) can be rewritten as

$$\frac{\partial(\mathbf{x})}{\partial(\mathbf{a})} = 1 \ . \tag{4.50}$$

As the perturbation pressure only depends on the modulation of the free surface, it is independent of the depth. This request results in the relationship

$$\frac{\partial \mathbf{u}_h}{\partial z} = 0 \ , \tag{4.51}$$

where $\mathbf{u}_h = (u, v)$, which means that the fluid is constrained to move in columns. Under this hypothesis, the relationship (3.1a)–(3.1c) between the particle positions and labels is modified in

$$x = x(a_1, a_2, t) \ , \tag{4.52a}$$

$$y = y(a_1, a_2, t) \ , \tag{4.52b}$$

$$z = z(a_1, a_2, a_3, t) \ . \tag{4.52c}$$

The relations (4.52a)–(4.52c) show that each column of fluid is a material filament which is parallel to the local direction of gravity. For this filament, the pair (a_1, a_2) identifies, for example, the barycentre, while a_3 identifies a parcel of the filament itself. In particular, $a_3 = 0$ identifies the parcel located at the free surface of the fluid, so that

$$a_3 = 0 \; at \; z = \eta \; . \tag{4.53}$$

Using (4.52a)–(4.52c) and the definition (3.7), the relation (4.50) yields

$$\frac{\partial(\mathbf{x})}{\partial(\mathbf{a})} = \begin{vmatrix} \frac{\partial x}{\partial a_1} & \frac{\partial x}{\partial a_2} & 0 \\ \frac{\partial y}{\partial a_1} & \frac{\partial y}{\partial a_2} & 0 \\ \frac{\partial z}{\partial a_1} & \frac{\partial z}{\partial a_2} & \frac{\partial z}{\partial a_3} \end{vmatrix} = 1 \; , \tag{4.54}$$

that is

$$\frac{\partial(x, y)}{\partial(a_1, a_2)} \frac{\partial z}{\partial a_3} = 1 \; ,$$

or

$$\frac{\partial a_3}{\partial z} = \frac{\partial(x, y)}{\partial(a_1, a_2)} \; . \tag{4.55}$$

The first-order ordinary differential equation (4.55) can be integrated making use of (4.53), yielding

$$a_3 = \frac{\partial(x, y)}{\partial(a_1, a_2)}(z - \eta) \; , \tag{4.56}$$

or

$$\frac{\partial(x, y)}{\partial(a_1, a_2)} = \frac{a_3}{z - \eta} \; . \tag{4.57}$$

It should be noted that only the r.h.s of (4.57) depends on the vertical variables. Equation (4.57) can thus be referred to any depth without changing its l.h.s. It is convenient to consider the situation at the bottom, that is, in $z = -H$, and express the vertical component a_3 of the label of the parcel at the bottom as a function of the position of the filament to which the parcel belongs, say $a_3 = -H_a(a_1, a_2)$. Because any parcel at the boundary never leaves it,

$$\frac{\partial H_a}{\partial t} = 0 \; , \tag{4.58}$$

which implies that H_a is a constant. Without loss of generality, we can rescale the particle labels a_1 and a_2 by H_a, so that (4.57) yields

$$\frac{\partial(x, y)}{\partial(a_1, a_2)} = \frac{1}{H + \eta} \; , \tag{4.59}$$

whence

$$\eta = \frac{\partial(a_1, a_2)}{\partial(x, y)} - H \ , \tag{4.60}$$

and, from (4.57) and (4.60),

$$z = \eta + a_3(\eta + H) \ . \tag{4.61}$$

From the time derivative of (4.60), it is possible to derive easily the continuity equation, in fact, using (4.58),

$$\begin{aligned}
\frac{\partial \eta}{\partial t} &= \frac{\partial}{\partial t}\left[\frac{\partial(a_1, a_2)}{\partial(x, y)}\right] = \frac{\partial}{\partial t}\left[\frac{\partial(x, y)}{\partial(a_1, a_2)}\right]^{-1} \\
&= -\left[\frac{\partial(\dot{x}, y)}{\partial(a_1, a_2)} + \frac{\partial(x, \dot{y})}{\partial(a_1, a_2)}\right]\left[\frac{\partial(x, y)}{\partial(a_1, a_2)}\right]^{-2} \\
&= -\left[\frac{\partial(\dot{x}, y)}{\partial(x, y)} + \frac{\partial(x, \dot{y})}{\partial(x, y)}\right]\left[\frac{\partial(x, y)}{\partial(a_1, a_2)}\right]^{-1} \\
&= -\frac{\partial(a_1, a_2)}{\partial(x, y)}\, div\, \mathbf{u} \\
&= -(\eta + H)\, div\, \mathbf{u} \ ,
\end{aligned} \tag{4.62}$$

where in the last step (4.59) has been used. Equation (4.62) thus yields the continuity equation

$$\frac{\partial \eta}{\partial t} + (\eta + H)\, div\, \mathbf{u} = 0 \ . \tag{4.63}$$

After this preamble, it is possible to derive the Lagrangian density and thus the action functional, for the rotating shallow water model. We make the following assumptions:

(i) We neglect the terms in the kinetic energy involving the vertical velocity. This choice is motivated by the fact that, as shown in the scaling arguments of Sect. 1.9.1 and in particular in Eq. (1.178), the vertical velocity is typically much smaller than the horizontal velocities;
(ii) We omit the internal energy term $e(\alpha, \eta)$ (notice that in this position, η indicates the entropy and not the free surface elevation). This choice is motivated by the fact that we are considering an incompressible, adiabatic flow. Further, the internal energy is unaffected by variations in the positions;
(iii) We retain, however, the role of the external potential that we identify with the gravitational potential.

The reader should compare these assumptions with the less restrictive ones introduced in Sect. 4.4.5. Under these assumptions, the action (4.11) can be rewritten as

$$I = \int_{t_1}^{t_2} \int_R \rho \left[\frac{1}{2} |\dot{\mathbf{x}}_h|^2 + \dot{\mathbf{x}}_h \cdot (\boldsymbol{\Omega} \times \mathbf{x}_h) - gz \right] d(\mathbf{x}) dt \; , \tag{4.64}$$

where $\mathbf{x}_h = (x, y)$. Using (4.61), the vertical integration of the gravitational term in (4.64) yields

$$\int_{-H}^{\eta} \rho g z dz = \int_{-1}^{0} g \left[\eta + a_3(\eta + H) \right] da_3 = \frac{1}{2} g \eta + \; const \; , \tag{4.65}$$

so that (4.64) can be rewritten as

$$I = \int_{t_1}^{t_2} \int_{R_h} \frac{1}{2} \rho \left[|\dot{\mathbf{x}}_h|^2 + \dot{\mathbf{x}}_h \cdot (2\boldsymbol{\Omega} \times \mathbf{x}_h) - g\eta \right] d(\mathbf{x}_h) dt \; , \tag{4.66}$$

where the spatial domain is now restricted to the horizontal coordinates and where the constant in (4.65) has been neglected as it acts to change the minimum of the potential but not its shape or location. Clearly, the action (4.66) corresponds to the Lagrangian density

$$L = \frac{1}{2} \rho \left[|\dot{\mathbf{x}}_h|^2 + \dot{\mathbf{x}}_h \cdot (2\boldsymbol{\Omega} \times \mathbf{x}_h) - g\eta \right] \; . \tag{4.67}$$

Setting $\delta I = 0$ and using the results from (4.9) and

$$\delta \eta = \eta(\mathbf{x}_h + l\mathbf{Q}) - \eta(\mathbf{x}_h) \approx \nabla_h \eta \cdot l\mathbf{Q} \; , \tag{4.68}$$

Hamilton's principle yields the rotating shallow water equations

$$\ddot{\mathbf{x}}_h + (2\boldsymbol{\Omega} \times \dot{\mathbf{x}}_h) = -g \nabla_h \eta \; , \tag{4.69}$$

which should be coupled to the continuity equation (4.63).

With definition (4.14), the Lagrangian density (4.67) yields the conjugate momentum density

$$\boldsymbol{\pi} = \frac{\delta \mathscr{L}}{\delta \dot{\mathbf{x}}} = \rho \left[\mathbf{u}_h + (\boldsymbol{\Omega} \times \mathbf{x}_h) \right] \; , \tag{4.70}$$

or, assuming $\boldsymbol{\Omega} = \Omega \hat{\mathbf{k}}$, where Ω is a constant, by components

$$\pi_1 = \rho \left(u - \Omega y \right) \; , \; \pi_2 = \rho \left(v + \Omega x \right) \; . \tag{4.71}$$

The Lagrangian density can thus be written as

$$L = \rho \left[(u - \Omega y) \, \dot{x} + (v + \Omega x) \, \dot{y} \right] - H \; , \tag{4.72}$$

where the Hamiltonian density is

$$H = \frac{1}{2}\rho\left[u^2 + v^2 + g\eta\right] . \tag{4.73}$$

4.4.1.1 The Particle Relabelling Symmetry and the Shallow Water Equations

The application of the particle relabelling symmetry for the rotating shallow water equations follows directly the derivation made in Sect. 3.5.3. In the following, we will consider, for simplicity, the nonrotating case. As seen in Sect. 4.2.2, the extension to the rotating case is straightforward and will be mentioned at the end of the section.

Consider the Lagrangian density (4.67) in labelling coordinates. The request (3.112) now yields

$$\delta\frac{\partial(\mathbf{a})}{\partial(\mathbf{x})} = 0 \Rightarrow \frac{\partial}{\partial a_1}\delta a_1 + \frac{\partial}{\partial a_2}\delta a_2 = 0 , \tag{4.74}$$

which is satisfied upon introducing a scalar function $\delta\psi$ so that

$$\delta a_1 = -\frac{\partial}{\partial a_2}\delta\psi \ , \ \delta a_2 = \frac{\partial}{\partial a_1}\delta\psi \ . \tag{4.75}$$

With (4.75), the corresponding conserved quantity can be determined from Noether's Theorem. Written in components and using the definition of the pseudo-momentum $\mathbf{B}$, (3.117) yields

$$\begin{aligned} &\frac{d}{dt}\int_{R_a}\left[-B_i\frac{\partial x_i}{\partial a_1}\left(\frac{\partial}{\partial a_2}\delta\psi\right) + B_i\frac{\partial x_i}{\partial a_2}\left(\frac{\partial}{\partial a_1}\delta\psi\right)\right]d(\mathbf{a}) \\ &= -\frac{d}{dt}\int_{R_a}\left[\frac{\partial}{\partial a_1}\left(B_i\frac{\partial x_i}{\partial a_2}\right) - \frac{\partial x_i}{\partial a_2}\left(B_i\frac{\partial x_i}{\partial a_1}\right)\right]\delta\psi d(\mathbf{a}) = 0 , \end{aligned} \tag{4.76}$$

which corresponds to the conservation of the quantity

$$\begin{aligned} \Pi_a &= \frac{\partial}{\partial a_1}\left(B_i\frac{\partial x_i}{\partial a_2}\right) - \frac{\partial x_i}{\partial a_2}\left(B_i\frac{\partial x_i}{\partial a_1}\right) \\ &= \frac{\partial B_i}{\partial a_1}\frac{\partial x_i}{\partial a_2} - \frac{\partial B_i}{\partial a_2}\frac{\partial x_i}{\partial a_1} \\ &= \frac{\partial(B_2, y)}{\partial(a_1, a_2)} + \frac{\partial(B_1, x)}{\partial(a_1, a_2)} \\ &= \frac{\partial(x, y)}{\partial(a_1, a_2)}\left(\frac{\partial(B_2, y)}{\partial(x, y)} + \frac{\partial(B_1, x)}{\partial(x, y)}\right) . \end{aligned} \tag{4.77}$$

Using (4.59), (4.77) can be written as

$$\begin{aligned}
\Pi_a &= \frac{1}{H+\eta}\left(\frac{\partial(B_2, y)}{\partial(x, y)} + \frac{\partial(B_1, x)}{\partial(x, y)}\right) \\
&= \frac{1}{H+\eta}\left(\frac{\partial B_2}{\partial x} - \frac{\partial B_1}{\partial y}\right) \\
&= \frac{1}{H+\eta} rot_a\, \mathbf{B}\,. \qquad (4.78)
\end{aligned}$$

Finally, using the steps leading to (3.131), we have the conservation of the shallow water potential vorticity

$$\begin{aligned}
\Pi &= \frac{1}{H+\eta}\left(\frac{\partial(B_2, y)}{\partial(x, y)} + \frac{\partial(B_1, x)}{\partial(x, y)}\right) \\
&= \frac{1}{H+\eta}\left(\frac{\partial B_2}{\partial x} - \frac{\partial B_1}{\partial y}\right) \\
&= \frac{1}{H+\eta}\hat{\mathbf{k}} \cdot rot\, \mathbf{u} \\
&= \frac{1}{H+\eta}\varsigma\,, \qquad (4.79)
\end{aligned}$$

which can be recognized as the conservation of potential vorticity for the shallow water equations introduced in Chap. 1.

Remark 4.4 The analysis of the shallow water equations in Lagrangian form shows one of the main advantages of this formulation, that is, as stated in the introduction to this section, if the symmetries of a system are preserved by the approximation applied to the original equations, the approximated system, i.e. the shallow water system, will retain the same conservation laws of the original system. In this case, the approximations leading to the shallow water equations do not break the particle relabelling symmetry and thus allow for the derivation of the conservation of potential vorticity.

In the presence of rotation, following Sect. 4.2.2 the conservation of potential vorticity is modified as

$$\Pi = \frac{\varsigma + 2\boldsymbol{\Omega}\cdot\hat{\mathbf{k}}}{H+\eta} = \frac{\varsigma + f}{H+\eta}\,. \qquad (4.80)$$

4.4.2 Two-Layer Shallow Water Equations

4.4.2.1 Particle Labels in a Multilayer System

The analysis reported in the previous section can easily be extended to a layered rotating shallow water model. Following [66], consider an incompressible fluid in the shallow water approximation consisting of N vertical layers with density ρ_i, $i = 1, \ldots, N$. We set that each layer of fluid is bounded by an upper surface $z = \eta_i(x, y, t)$ and a lower surface $z = \eta_{i+1}(x, y, t)$. The lowermost layer is bounded by the topography that will be here considered as flat. Each layer is thus characterized by the height

$$h_i = \eta_i - \eta_{i+1} \ . \tag{4.81}$$

This configuration needs boundary conditions in the vertical. We assume for the upper surface of the uppermost layer to be stress-free and that the pressure is continuous across each internal surface, which yields

$$p_1 = 0 \ , \ \text{on } z = \eta_1 \ , \tag{4.82a}$$

$$p_i = p_{i+1} \ , \ \text{on } z = \eta_{i+1} \ . \tag{4.82b}$$

With these definitions in mind, we define with the subscript i, the layer under consideration. In each layer, the fluid parcels are indicated by the labels

$$\mathbf{a}_i = (a_i, b_i, c_i) \ , \tag{4.83}$$

where the notation $(a_1, a_2, a_3) \to (a, b, c)$ has been used to avoid confusion between the subscripts. In each layer, the position of each parcel is thus

$$\mathbf{x}_i = \mathbf{x}_i(\mathbf{a}_i, t) \ , \tag{4.84}$$

which is a generalization of (3.2) for multilayer dynamics. We choose the labels $\mathbf{a}_i$ to satisfy the conservation of mass in the form

$$\rho_i dx_i dy_i dz_i = da_i db_i dc_i \ , \tag{4.85}$$

i.e. we have assumed the density in label space to be uniform in each layer.

The one-to-one mapping between R and R_a implies that within each layer the density is related to the Jacobian determinant of the map by

$$\frac{\partial(\mathbf{x}_i)}{\partial(\mathbf{a}_i)} = \frac{1}{\rho_i} \ . \tag{4.86}$$

In analogy with the previous section, also for the multilayer case the perturbation pressure only depends on the modulation of the free surfaces, and it is thus independent of the depth. This request results in the relationship

$$\frac{\partial \mathbf{u}_{h,i}}{\partial z} = 0 \ , \tag{4.87}$$

where $\mathbf{u}_{h,i} = (u_i, v_i)$, which means that the fluid is constrained to move in columns. Under this hypothesis, the relationship (4.84) between the particle positions and labels is modified in

$$x_i = x_i(a_i, b_i, t) \ , \tag{4.88a}$$

$$y_i = y_i(a_i, b_i, t) \ , \tag{4.88b}$$

$$z_i = z_i(a_i, b_i, c_i, t) \ . \tag{4.88c}$$

With this assumption, (4.86) becomes

$$\frac{\partial(x_i, y_i)}{\partial(a_i, b_i)} \frac{\partial z_i}{\partial c_i} = \frac{1}{\rho_i} \ . \tag{4.89}$$

Equation (4.89) can be integrated in the vertical with boundary condition $c_i = 0$ at the base of each layer, so that

$$z_i = \frac{1}{\rho_i} \frac{\partial(a_i, b_i)}{\partial(x_i, y_i)} c_i + \eta_{i+1} \ . \tag{4.90}$$

In order to satisfy (4.81), we must have that at the top of the layer $c_i = 1$ and

$$h_i = \frac{1}{\rho_i} \frac{\partial(a_i, b_i)}{\partial(x_i, y_i)} \ , \tag{4.91}$$

so that

$$z_i = h_i c_i + \eta_{i+1} \ . \tag{4.92}$$

By induction, and assuming flat bottom, (4.92) can be written as

$$z_i = h_i c_i + \sum_{j=i+1}^{N} h_j = h_i c_i + \sum_{j=i+1}^{N} \frac{1}{\rho_j} \frac{\partial(a_j, b_j)}{\partial(x_j, y_j)} \ . \tag{4.93}$$

The vertical coordinate in each layer z_i thus depends on the labels in all the layers below, allowing in this way a dynamical coupling between layers.

4.4.2.2 Lagrangian Density for a Two-Layer System

Once that we have introduced the particle labels and reference system for a multilayer system, we can proceed to define the Lagrangian density for the two-layer system.

The action functional of the two-layer shallow water can be written as

$$I = \int_{t_1}^{t_2} \left[\left(\sum_{i=1}^{2} \int_{R_{\mathbf{a}_i,h}} L_i \, da_i db_i \right) + \left(\int_{R_{a_1,h}} \int_{R_{a_2,h}} L_{12} \, da_1 db_1 da_2 db_2 \right) \right] dt \ . \tag{4.94}$$

The Lagrangian densities $L_i,\ i = 1, 2$ are

$$L_i = \frac{1}{2}\rho_i \left(\left| \frac{\partial \mathbf{x}_{h,i}}{\partial t} \right|^2 + \dot{\mathbf{x}}_{h,i} \cdot (2\boldsymbol{\Omega} \times \mathbf{x}_{h,i}) - g \frac{1}{\rho_i} \frac{\partial(a_i, b_i)}{\partial(x_i, y_i)} \right)$$
$$= \frac{1}{2}\rho_i \left(\left| \frac{\partial \mathbf{x}_i}{\partial t} \right|^2 + \dot{\mathbf{x}}_{h,i} \cdot (2\boldsymbol{\Omega} \times \mathbf{x}_{h,i}) - g h_i \right) . \tag{4.95}$$

The Lagrangian density within the second bracket in (4.94), L_{12}, represents the interaction between the particles in the two layers and can be written as [52]

$$L_{12} = -g\rho_1 \delta(\mathbf{x}_{h,1} - \mathbf{x}_{h,2}) \ , \tag{4.96}$$

which implies that the fluid parcels in the upper and lower layers interact upon "touching" each other, i.e. if they are aligned in the vertical.

Dropping the h subscripts and using

$$da_1 db_1 = h_1 (\mathbf{x}_1, t)\, dx_1 dy_1 \ , \tag{4.97}$$

the (labelling) space integral of the interaction term in (4.94) yields

$$\int_{R_{a_1}} \int_{R_{a_2}} L_{12}\, da_1 db_1 da_2 db_2 = -g \int_{R_1} \left[\rho_1 \int_{R_{a_2}} \delta(\mathbf{x}_1 - \mathbf{x}_2)\, da_2 db_2 \right] h_1(\mathbf{x}_1, t)\, dx_1 dy_1$$
$$= -g\rho_2 \int_{R_{a_2}} \frac{\rho_1}{\rho_2} h_1(\mathbf{x}_2, t)\, da_2 db_2 \ . \tag{4.98}$$

The action functional (4.94) is thus

$$I = \int_{t_1}^{t_2} \left\{ \rho_1 \int_{R_{a_1}} \left[\frac{1}{2} |\dot{\mathbf{x}}_1|^2 + \dot{\mathbf{x}}_1 \cdot (2\boldsymbol{\Omega} \times \mathbf{x}_1) - \frac{1}{2} g h_1 \right] da_1 db_1 \right.$$
$$\left. + \rho_2 \int_{R_{a_2}} \left[\frac{1}{2} |\dot{\mathbf{x}}_2|^2 + \dot{\mathbf{x}}_2 \cdot (2\boldsymbol{\Omega} \times \mathbf{x}_2) - \frac{1}{2} g h_2 - \frac{\rho_1}{\rho_2} g h_1(\mathbf{x}_2, t) \right] da_2 db_2 \right\} dt. \tag{4.99}$$

Variations with respect to $\mathbf{x}_1$ yield the equations of motion for the upper layer, i.e. layer 1

$$\frac{\partial \mathbf{u}_1}{\partial t} + f\hat{\mathbf{k}} \times \mathbf{u}_1 = -g\nabla(h_1 + h_2) \ , \tag{4.100}$$

while variations with respect to $\mathbf{x}_2$ yield the equations of motion for the lower layer, i.e. layer 2

$$\frac{\partial \mathbf{u}_2}{\partial t} + f\hat{\mathbf{k}} \times \mathbf{u}_2 = -g \left(\nabla h_2 + \frac{\rho_1}{\rho_2} \nabla h_1 \right) . \tag{4.101}$$

The application of Noether's Theorem to the particle relabelling symmetry yields instead immediately the conservation of the layer-wise potential vorticity

$$\Pi_i = \frac{\varsigma_i + f}{h_i} \ . \tag{4.102}$$

where

$$\varsigma_i = \hat{\mathbf{k}} \cdot rot\ \mathbf{u}_i \ . \tag{4.103}$$

The analysis reported in this section can be easily extended to a multilayer rotating shallow water model [66].

4.4.3 Rotating Green–Naghdi Equations

In the derivation of the rotating shallow water system, it has been assumed that the vertical velocities are much smaller than their horizontal counterpart (see assumption *(i)* of Sect. 4.4.1). It is, however, possible to relax this assumption, adding the terms proportional to $\dot{z}$ in the kinetic energy component of the action (4.64)

$$I = \int_{t_1}^{t_2} \int_R \rho \left[\frac{1}{2} |\dot{\mathbf{x}}_h|^2 + \frac{1}{2} |\dot{z}|^2 + \dot{\mathbf{x}}_h \cdot (2\boldsymbol{\Omega} \times \mathbf{x}_h) - gz \right] d(\mathbf{x}) dt \ . \tag{4.104}$$

Using (4.61),

$$\dot{z} = \dot{\eta}\,(1 + a_3) \ , \tag{4.105}$$

so that the additional term in (4.104) yields

$$\int_{-1}^{0} \frac{1}{2} \left[\dot{\eta}\,(1 + a_3)\right]^2 da_3 = -\frac{1}{6}\dot{\eta}^2 \ . \tag{4.106}$$

Using (4.106), the action functional (4.66) can thus be rewritten as

$$I = \int_{t_1}^{t_2} \int_{R_h} \frac{1}{2}\rho \left[|\dot{\mathbf{x}}_h|^2 - \frac{1}{3}\,(\dot{\eta})^2 + \dot{\mathbf{x}}_h \cdot (2\boldsymbol{\Omega} \times \mathbf{x}_h) - g\eta \right] d(\mathbf{x}_h) dt \ . \tag{4.107}$$

To obtain the additional terms arising in the equations of motion from the new term in (4.104), one needs to consider only the variation of

$$I_\eta = -\int_{t_1}^{t_2} \int_{R_h} \frac{1}{2}\rho \left(\frac{1}{3}\,(\dot{\eta})^2 \right) d(\mathbf{x}_h) dt \ . \tag{4.108}$$

The variation yields

$$
\begin{aligned}
\delta I_\eta &= -\int_{t_1}^{t_2}\int_{R_h}\frac{1}{2}\rho\frac{1}{3}\left\{\left[\frac{\partial}{\partial t}\eta(t+\delta t)\right]^2-\left(\frac{\partial\eta}{\partial t}\right)^2\right\}d(\mathbf{x}_h)dt\\
&= -\int_{t_1}^{t_2}\int_{R_h}\frac{1}{2}\rho\frac{1}{3}\left[\left(\frac{\partial\eta}{\partial t}+\frac{\partial}{\partial t}\delta\eta\right)^2-\left(\frac{\partial\eta}{\partial t}\right)^2\right\}d(\mathbf{x}_h)dt\\
&= -\int_{t_1}^{t_2}\int_{R_h}\frac{1}{2}\rho\frac{1}{3}\left(2\frac{\partial\eta}{\partial t}\frac{\partial}{\partial t}\delta\eta\right)d(\mathbf{x}_h)dt\\
&= -\int_{t_1}^{t_2}\int_{R_h}\frac{1}{2}\rho\frac{1}{3}\left[2\frac{\partial}{\partial t}\left(\dot\eta\delta\eta\right)-2\ddot\eta\delta\eta\right]d(\mathbf{x}_h)dt\\
&= \frac{1}{3}\int_{t_1}^{t_2}\int_{R_h}\rho\ddot\eta\delta\eta d(\mathbf{x}_h)dt\ ,
\end{aligned}
\tag{4.109}
$$

where

$$
\delta\eta=\frac{\partial\eta}{\partial t}\delta t\ .
\tag{4.110}
$$

Using the transformation

$$
\int_{R_h}d(\mathbf{x}_h)=\int_{R_{a,h}}\frac{\partial(x,y)}{\partial(a_1,a_2)}d(\mathbf{a}_h)\ ,
\tag{4.111}
$$

Equation (4.109) can be further developed, in fact

$$
\begin{aligned}
\int_{R_{a,h}}da_1\,da_2\left\{\frac{1}{3}\ddot\eta\delta\eta\right\} &= -\int_{R_{a,h}}da_1\,da_2\left\{\frac{1}{3}\ddot\eta\eta^2\delta\left(\frac{1}{\eta}\right)\right\}\\
&\approx -\int_{R_{a,h}}da_1\,da_2\left\{\frac{1}{3}\ddot\eta\eta^2\delta\left(\frac{\partial(x,y)}{\partial(a_1,a_2)}\right)\right\}\\
&= -\int_{R_{a,h}}da_1\,da_2\left\{\frac{1}{3}\ddot\eta\eta^2\left(\frac{\partial(lQ_1,y)}{\partial(a_1,a_2)}+\frac{\partial(x,lQ_2)}{\partial(a_1,a_2)}\right)\right\}\\
&= \int_{R_{a,h}}da_1\,da_2\left\{\left(\frac{1}{3}\frac{\partial(x,y)}{\partial(a_1,a_2)}\right)\ddot\eta\eta^2\left(\frac{\partial(lQ_1,y)}{\partial(x,y)}+\frac{\partial(x,lQ_2)}{\partial(x,y)}\right)\right\}\\
&= \int_{R_{a,h}}da_1\,da_2\left\{\left(\frac{1}{3}\frac{\partial(x,y)}{\partial(a_1,a_2)}\right)\left[lQ_1\frac{\partial(\ddot\eta\eta^2,y)}{\partial(x,y)}+lQ_2\frac{\partial(x,\ddot\eta\eta^2)}{\partial(x,y)}\right]\right\}\\
&= \int_{R_{a,h}}da_1\,da_2\left\{\left(\frac{1}{3}\frac{\partial(x,y)}{\partial(a_1,a_2)}\right)\left[lQ_1\frac{\partial}{\partial x}(\ddot\eta\eta^2)+lQ_2\frac{\partial}{\partial y}(\ddot\eta\eta^2)\right]\right\}\\
&\approx \int_{R_{a,h}}da_1\,da_2\left\{\frac{1}{3\eta}\nabla\left(\ddot\eta\eta^2\right)\cdot l\mathbf{Q}\right\}\ ,
\end{aligned}
\tag{4.112}
$$

where in the third and in the last step, the constant terms proportional to H of (4.60) have been neglected. On the fourth step, the Jacobi identity

$$
A\frac{\partial(B,C)}{\partial(x,y)}+B\frac{\partial(C,A)}{\partial(x,y)}+C\frac{\partial(A,B)}{\partial(x,y)}=0
\tag{4.113}
$$

has been used with $A = \ddot{\eta}\eta^2$ and the two different cases: *(i)* $B = \delta x = l Q_1$ and $C = y$; *(ii)* $B = x$ and $C = \delta y = l Q_2$.

Combining (4.112), (4.109) and the variation of the remaining terms of (4.107), i.e. (4.69), yields the *rotating Green–Naghdi equations* [22]

$$\ddot{\mathbf{x}}_h + (2\mathbf{\Omega} \times \dot{\mathbf{x}}_h) = -g\nabla_h \eta - \frac{1}{3\eta}\nabla_h \left(\ddot{\eta}\eta^2\right) . \qquad (4.114)$$

These equations do not only take into account the presence of vertical accelerations that might be important at small scales, but, through the additional nonlinear, dispersive term they rectify a known problem of the rotating shallow water equations, namely the fact that in the shallow water equations the short-wavelength gravity waves have unbounded phase speeds, which is problematic at the small scales [78].

Finally, as the additional terms in the Lagrangian density for the Green–Naghdi equations do not violate the relabelling symmetry, these equations must satisfy the conservation of a form of the potential vorticity. For the nonrotating case (to allow for comparison with (4.79)), the additional term on the r.h.s. of (4.114) gives rise to the term contributing to the potential vorticity

$$\frac{1}{3}\frac{\partial\left(\dot{\eta}, \eta\right)}{\partial(a_1, a_2)} .$$

With this term, the potential vorticity in physical space takes the form

$$\Pi = \frac{1}{H + \eta}\left[\varsigma + \frac{1}{3}\frac{\partial\left(D\eta/Dt, \eta\right)}{\partial(x, y)}\right] , \qquad (4.115)$$

which can be seen to differ from (4.79) for the presence of the additional nonlinear term in the square brackets, deriving from the last term on the r.h.s. of (4.114). Further details can be found in [34].

4.4.4 Shallow Water Semi-geostrophic Dynamics

In this section, we will develop an approximation of the shallow water equations which are nearly geostrophic but that include a first-order correction to the flow.

The resulting dynamics, first developed in variational form by [53, 54], is called *semi-geostrophic* [28] and was first derived for the shallow water case by [27]. As the extension of the semi-geostrophic dynamics to the beta-plane still retains some difficulties (see, e.g. [17]), the following treatment will be done in the f-plane.

4.4.4.1 The Geostrophic Momentum Approximation

From Sect. 4.4.1, start by considering the Lagrangian density in labelling space for the rotating shallow water equations as

$$L = (u - \Omega y)\,\dot{x} + (v + \Omega x)\,\dot{y} - H\ , \tag{4.116}$$

with

$$H = \frac{1}{2}\left[u^2 + v^2 + g\eta\right]\ , \tag{4.117}$$

and where u and v are the components of the conjugate momentum density. Through variations of the action functional corresponding to (4.116), with the usual boundary conditions (3.47), one recovers the rotating shallow water equations, e.g. variations in x and y yield the two separate components of (4.69), while variations in u and v yield, respectively, the relations $u = \dot{x}$ and $v = \dot{y}$. The terms proportional to u, v in (4.116) are responsible for the acceleration terms appearing in the rotating shallow water equations, and under the request of the flow to be nearly geostrophic it is natural to first pose the constraint

$$\mathbf{u} = 0\ . \tag{4.118}$$

In this case, the Lagrangian density (4.116) takes the form

$$L = -\Omega y\dot{x} + \Omega x\dot{y} - \frac{1}{2}g\eta\ . \tag{4.119}$$

Variations of the action functional associated to (4.118) with boundary conditions (3.47) yield the geostrophic balance

$$(2\boldsymbol{\Omega} \times \dot{\mathbf{x}}_h) = -g\nabla_h\eta\ . \tag{4.120}$$

An asymptotic derivation of (4.120) from the nondimensionalization of the Lagrangian density is reported in Appendix L.

Neglecting the h subscript, at the next level of approximation, the rather restrictive constraint (4.118) is replaced by the position

$$\mathbf{u} = \mathbf{u}_0\ , \tag{4.121}$$

where

$$\mathbf{u}_0 = \frac{g}{f_0}\hat{\mathbf{k}} \times \nabla h\ , \tag{4.122}$$

i.e. the u, v terms in (4.116) are replaced by the geostrophic components of the velocity. In (4.122), following Chap. 1 we have set $2\Omega = f_0$. The approximation from the imposition of (4.121) in (4.116) takes also the name of *geostrophic momentum* *approximation* [20, 21]. With this constraint, (4.116) takes the form

$$L = (u_0 - \Omega y)\,\dot{x} + (v_0 + \Omega x)\,\dot{y} - H\ , \tag{4.123}$$

with

$$H = \frac{1}{2}\left[u_0^2 + v_0^2 + g\eta\right]\ . \tag{4.124}$$

Using Hamilton's principle, it is possible to derive the equations of motion corresponding to (4.123). Setting the variation $\mathbf{x} = \mathbf{x} + l\mathbf{Q}$, at first order in l, (4.123) yields

$$\begin{aligned}\delta L = &(u_0 - \Omega y)\, l\dot{Q}_1 + (v_0 + \Omega x)\, l\dot{Q}_2 - \Omega l Q_2 \dot{x} + \Omega l Q_1 \dot{y}\\ &+ (\dot{\mathbf{x}} - \mathbf{u}_0)\cdot\delta\mathbf{u}_0 - \frac{1}{2} g\delta\eta\ .\end{aligned} \tag{4.125}$$

Defining the ageostrophic velocity

$$\mathbf{u}_1 = \dot{\mathbf{x}} - \mathbf{u}_0\ , \tag{4.126}$$

and using (3.50) and (3.51) for the first two terms on the r.h.s, (4.125) can be written as

$$\delta L = (-\dot{u}_0 + f_0\dot{y})\, l Q_1 + (-\dot{v}_0 - f_0\dot{x})\, l Q_2 - \frac{1}{2} g\delta\eta + \mathbf{u}_1\cdot\delta\mathbf{u}_0\ . \tag{4.127}$$

The integration of the last term on the r.h.s of (4.127) gives (see, e.g. [53, 54])

$$\begin{aligned}&\int_{t_1}^{t_2}\int_{R_a}(\mathbf{u}_1\cdot\delta\mathbf{u}_0)\, d(\mathbf{a})dt\\ &= \int_{t_1}^{t_2}\int_{R_a}\left\{\left[-(\mathbf{u}_0\cdot\nabla)\mathbf{u}_1 - \frac{g}{f_0\eta}\nabla\left(\eta^2\hat{\mathbf{k}}\cdot\nabla\times\mathbf{u}_1\right)\right]\cdot l\mathbf{Q}\right\} d(\mathbf{a})dt\\ &= \int_{t_1}^{t_2}\int_{R_a}\left\{\left[-(\mathbf{u}_0\cdot\nabla)\mathbf{u}_1 - \frac{g}{f_0\eta}\nabla\left(\eta^2\varsigma_1\right)\right]\cdot l\mathbf{Q}\right\} d(\mathbf{a})dt\ ,\end{aligned} \tag{4.128}$$

where R_a is the part of the domain restricted to (a_1, a_2), $d(\mathbf{a}) = da_1 da_2$, and where we have defined

$$\varsigma_1 = \hat{\mathbf{k}}\cdot\nabla\times\mathbf{u}_1\ . \tag{4.129}$$

Using (4.128) into the application of Hamilton's principle to the action functional corresponding to (4.127) yields the equations of motion

$$\frac{\partial}{\partial t}\mathbf{u}_0 + \mathbf{u}_0 \cdot \nabla \mathbf{u}_0 + [\mathbf{u}_0 \cdot \nabla \mathbf{u}_1 + \mathbf{u}_1 \cdot \nabla \mathbf{u}_0] + f_0 \hat{\mathbf{k}} \times [\mathbf{u}_0 + \mathbf{u}_1]$$
$$= -g\nabla\eta - \frac{g}{f_0 \eta}\nabla\left[\eta^2 \varsigma_1\right] , \tag{4.130}$$

and

$$\frac{\partial \eta}{\partial t} + \nabla \cdot [(\mathbf{u}_0 + \mathbf{u}_1)(\eta + H)] = 0 , \tag{4.131}$$

which are called *geostrophic momentum equations*. The new equations of motion (4.130) comprise *(i)* a geostrophic contribution $\partial \mathbf{u}_0/\partial t + \mathbf{u}_0 \cdot \nabla \mathbf{u}_0$. This term is analogous to the Lagrangian derivative of the momentum term in the quasi-geostrophic approximation introduced in Chap. 1, in which only the geostrophic velocity advects the momentum, and *(ii)* a new advective term $\mathbf{u}_0 \cdot \nabla \mathbf{u}_1 + \mathbf{u}_1 \cdot \nabla \mathbf{u}_0$, which, summed to the geostrophic contribution, yields an approximation to the total acceleration

$$\frac{D\mathbf{u}}{Dt} = \left[\frac{\partial}{\partial t} + (\mathbf{u}_0 + \mathbf{u}_1) \cdot \nabla\right](\mathbf{u}_0 + \mathbf{u}_1) \ . \tag{4.132}$$

The approximation of (4.132) in (4.130) does not comprise the terms $\partial \mathbf{u}_1/\partial t$ and $\mathbf{u}_1 \cdot \nabla \mathbf{u}_1$. This makes the geostrophic momentum equations a set of balanced equations, which filter out gravity waves solutions. As further differences, notice that both the Coriolis term in (4.130) and the advective velocity in (4.131) include the contribution of the total, $\mathbf{u}_0 + \mathbf{u}_1$, velocity. The relative vorticity associated with the ageostrophic velocity also contributes to a new additional term, here written as the last term on the r.h.s. of (4.130). This set of equations is clearly more complicated than the rotating shallow water system, and very few physical remarks can be made on the new equations just by eye inspection. To get more insight on the system, we will now go back to the Lagrangian density (4.123).

4.4.4.2 Geostrophic Coordinates and Semi-geostrophic Dynamics

The treatment of (4.123) is particularly difficult, due to the functional dependence of the geostrophic velocity on the particle locations, i.e. $\mathbf{u}_0 = \mathbf{u}_0[\mathbf{x}(\mathbf{a}, t)]$, which makes the system noncanonical. To overcome this difficulty, it is possible to introduce a Legendre transformation on the coordinates of the form

$$x_s = x + \frac{v_0}{f} , \tag{4.133a}$$

$$y_s = y - \frac{u_0}{f} , \tag{4.133b}$$

or, using (4.122),

$$x_s = x + \frac{g}{f^2}\frac{\partial \eta}{\partial x} \,, \tag{4.134a}$$

$$y_s = y + \frac{g}{f^2}\frac{\partial \eta}{\partial y} \,. \tag{4.134b}$$

The new coordinates (x_s, y_s) are also called *geostrophic coordinates*, and they have the following properties: *(i)* they are Lagrangian coordinates, in the sense that they evolve following the geostrophic flow; *(ii)* their name is justified by the observation that

$$\dot{\mathbf{x}}_s = \dot{\mathbf{x}} + \dot{\mathbf{u}}_0 = \dot{\mathbf{x}} = \mathbf{u}_0 \,; \tag{4.135}$$

(iii) multiplication of (4.133a), (4.133b) by f shows that the geostrophic coordinates are related to the absolute momentum $f\mathbf{x}_s = f\mathbf{x} + \mathbf{u}_0$. Obviously, the fact that the new coordinates are Lagrangian implies that one has to be careful with the boundary conditions, i.e. one would need that the flow is either periodic in the horizontal directions or it goes to zero at infinity. Without loss of generality in the following, we will assume the second possibility.

With (4.133a), (4.133b), the Lagrangian density (4.123) takes a form similar to (4.119), that is

$$L_s = (-\Omega y_s)\,\dot{x} + (\Omega x_s)\,\dot{y} - H \,. \tag{4.136}$$

From (4.60), we also define

$$\eta_s + H = \frac{\partial(a_1, a_2)}{\partial(x_s, y_s)} \,. \tag{4.137}$$

The time derivation of (4.137) yields the conservation of mass in transformed coordinates

$$\frac{\partial \eta_s}{\partial t} + \frac{\partial}{\partial x_s}\left[(\eta_s + H)\,\dot{x}\right] + \frac{\partial}{\partial y_s}\left[(\eta_s + H)\,\dot{y}\right] = 0 \,. \tag{4.138}$$

At the same time, the analogy between the Lagrangian densities (4.136) and (4.119) shows that in the transformed coordinates, the system satisfies the conservation of the transformed potential vorticity

$$\Pi_s = \frac{f_0}{\eta_s + H} \,, \tag{4.139}$$

in the form

$$\left[\frac{\partial}{\partial t} + \dot{x}\frac{\partial}{\partial x_s} + \dot{y}\frac{\partial}{\partial y_s}\right]\Pi_s = 0 \,. \tag{4.140}$$

Notice that, using (4.137), the potential vorticity (4.139) is

$$\begin{aligned}\Pi_s &= \frac{f_0}{\eta_s + H} = f_0 \frac{\partial(x_s, y_s)}{\partial(a_1, a_2)} \\ &= f_0 \frac{\partial(x_s, y_s)}{\partial(x, y)} \frac{\partial(x, y)}{\partial(a_1, a_2)} \\ &= \frac{f_0}{\eta + H} \frac{\partial(x_s, y_s)}{\partial(x, y)} \\ &= \frac{f_0}{\eta + H} \left\{ 1 + \frac{1}{f_0} \left[\frac{\partial v_0}{\partial x} - \frac{\partial u_0}{\partial y} \right] + \frac{1}{f_0^2} \frac{\partial(u_0, v_0)}{\partial(x, y)} + \text{h.o.t.} \right\} ,\end{aligned} \tag{4.141}$$

where $h.o.t.$ indicates higher-order terms.

Scaling shows that the terms within the curl brackets are respectively at zeroth, first and second order of the Rossby number. At first order in the Rossby number, (4.141) reduces thus to

$$\Pi_s = \frac{f_0 + \varsigma_0}{\eta + H} , \tag{4.142}$$

which shows that, neglecting the higher-order terms on the r.h.s, Π_s is an approximation to the potential vorticity of the full rotating shallow water equations (4.80). The comparison between (4.138) and (4.140) shows that the two are compatible only if there exists a function Φ_s that satisfies

$$f_0 \dot{y} = \frac{\partial \Phi_s}{\partial x_s} , \tag{4.143a}$$

$$f_0 \dot{x} = -\frac{\partial \Phi_s}{\partial y_s} . \tag{4.143b}$$

Meanwhile, the application of Hamilton's principle to the action functional associated with (4.136) for variations $\mathbf{x}_s = \mathbf{x}_s + l\mathbf{Q}_s$ and with the previously specified boundary conditions yields the equations of motion

$$f_0 \dot{y} = \frac{\delta \mathscr{H}}{\delta x_s} , \tag{4.144a}$$

$$f_0 \dot{x} = -\frac{\delta \mathscr{H}}{\delta y_s} , \tag{4.144b}$$

where, following the notation used in Chap. 2, $\mathscr{H}$ indicates the Hamiltonian functional found through the integration of the Hamiltonian density H over R_a. Direct comparison of (4.144a), (4.144b) with (4.143a), (4.143b) shows that

$$\frac{\delta \mathscr{H}}{\delta x_s} = \frac{\partial \Phi_s}{\partial x_s} = f_0 \dot{y} , \tag{4.145}$$

and

$$\frac{\delta \mathscr{H}}{\delta y_s} = \frac{\partial \Phi_s}{\partial y_s} = -f_0 \dot{x} \ , \tag{4.146}$$

which are satisfied for

$$\Phi_s = \frac{1}{2}\left(u_0^2 + v_0^2\right) + g\eta \ , \tag{4.147}$$

which shows that Φ is a Bernoulli function. The equalities in (4.145) and (4.146) can be determined also through the evaluation of the functional derivatives of $\mathscr{H}$. In fact, we have

$$\begin{aligned}\delta \mathscr{H} &= \int_{R_a} \left(u_0 \delta u_0 + v_0 \delta v_0 + \frac{1}{2} g \delta \eta \right) d(\mathbf{a}) \\ &= \int_{R_a} \left(u_0 \delta u_0 + v_0 \delta v_0 + g \frac{\partial \eta}{\partial x} \delta x + g \frac{\partial \eta}{\partial y} \delta y \right) d(\mathbf{a}) \ .\end{aligned} \tag{4.148}$$

Transforming the δx and δy terms using the definition of the transformed coordinates (4.134a), (4.134b), and using the geostrophic balance to express the horizontal derivatives of the dynamic height η greatly simplifies (4.148) as

$$\begin{aligned}\delta \mathscr{H} &= \int_{R_a} \left[u_0 \delta u_0 + v_0 \delta v_0 + f_0 v_0 \delta \left(x_s - \frac{v_0}{f_0} \right) - f_0 u_0 \delta \left(y_s + \frac{u_0}{f_0} \right) \right] d(\mathbf{a}) \\ &= f_0 \int_{R_a} \left(v_0 \delta x_s - u_0 \delta y_s \right) d(\mathbf{a}) \ ,\end{aligned} \tag{4.149}$$

so that

$$\frac{\delta \mathscr{H}}{\delta x_s} = f_0 v_0 \ , \tag{4.150a}$$

$$\frac{\delta \mathscr{H}}{\delta y_s} = -f_0 u_0 \ . \tag{4.150b}$$

The system made by (4.134a), (4.134b), (4.141), (4.145)–(4.147) is the shallow water semi-geostrophic model. As already stated in the introduction, this is a first-order correction to the quasi-geostrophic equations that retains, however, its balanced structure. Notably, this system has the capability to create fronts in finite time. This can be seen mathematically through a manipulation of the potential vorticity equation (4.141), which gives rise to a Monge-Ampère type equation, which has finite-time singularities. For more details, see, e.g. [17].

4.4.5 Continuously Stratified Fluid

The considerations carried over in the previous sections for the fluid with uniform density can be extended to the case with continuous stratification. We will do so following [39]. Consider the Lagrangian density (4.21). Without loss of generality, we will consider the case of a nonrotating framework. Differently from the previous sections, we will now request that the labels correspond to a background reference state, i.e. to an equilibrium position **a**. Under this assumption and (4.24), the action functional (4.22) is written as

$$I = \int_{t_1}^{t_2} dt \int_{R_a} d(\mathbf{a}) \left[\frac{1}{2}\rho_a(\mathbf{a}) \left| \frac{\partial \mathbf{x}}{\partial t} \right|^2 - \rho_a(\mathbf{a})\phi + p \left(\frac{\partial(\mathbf{x})}{\partial(\mathbf{a})} - \alpha \right) \right] . \tag{4.151}$$

Notice that, due to the request on the labels, the density in label space is no longer equal to one. We will now set the stratification making use of the Boussinesq approximation introduced in Chap. 1. The density in the kinetic energy term is replaced by a constant density $\rho_a(\mathbf{a}) = \rho_0$. The density in the remaining terms is instead written assuming

$$\rho_a(\mathbf{a}) = \rho_b(a_3) , \tag{4.152}$$

where $\rho_b(a_3)$ is a background density field depending on a_3 due to the identification of this label with z_0 reported at the beginning of the previous chapter. We now identify the external potential with the gravitational potential, so that

$$\phi = gz , \tag{4.153}$$

from which we write

$$\rho_a(\mathbf{a})\phi = \rho_b(a_3)gz = \rho_b(a_3)g(z - a_3) + \rho_b(a_3)ga_3 . \tag{4.154}$$

Notice that the last term is a constant that does not change the shape of the potential and will thus be ignored. In a similar way, the pressure term can be written as a background term depending only on the z coordinate and a perturbation term, so that

$$\begin{aligned} p\left(\frac{\partial(\mathbf{x})}{\partial(\mathbf{a})} - \alpha\right) &= (p_b(z) + \delta p)\left(\frac{\partial(\mathbf{x})}{\partial(\mathbf{a})} - \alpha\right) \\ &= -\alpha\left(p_b(z) - p_b(a_3)\right) + \delta p\left(\frac{\partial(\mathbf{x})}{\partial(\mathbf{a})} - \alpha\right) + p_b(z)\frac{\partial(\mathbf{x})}{\partial(\mathbf{a})} - \alpha p_b(a_3) , \end{aligned} \tag{4.155}$$

where, once again, the last two terms are constants and can thus be ignored in the following calculations. With these manipulations, the action integral (4.151) yields

$$I = \int_{t_1}^{t_2} dt \int_{R_a} d(\mathbf{a}) \left[\frac{1}{2}\rho_0 \left| \frac{\partial \mathbf{x}}{\partial t} \right|^2 + \delta p \left(\frac{\partial(\mathbf{x})}{\partial(\mathbf{a})} - \alpha \right) - \mathscr{V} \right] , \tag{4.156}$$

where

$$\mathscr{V} = \rho_b(a_3)g(z - a_3) - [\alpha\left(p_b(z) - p_b(a_3)\right)] . \tag{4.157}$$

With a little algebra, and assuming the hydrostatic balance to be valid, (4.157) can be further written as

$$\mathscr{V} = g \int_{a_3}^{z} ds \, [\rho_b(a_3) - \rho_b(s)] \ . \tag{4.158}$$

Applying Hamilton's principle with variation (3.46)–(4.156), with boundary conditions (3.47), yields the momentum equations for the continuously stratified, incompressible fluid

$$\frac{\partial^2 \mathbf{x}}{\partial t^2} = -\frac{1}{\rho_0} \nabla \delta p - \frac{g}{\rho_0} [\rho_b(a_3) - \rho_b(z)] \, \mathbf{k} \ , \tag{4.159}$$

which should be compared, upon replacement $\delta p \rightarrow p$, with Eq. (3.57). The additional term on the r.h.s. of (4.159) corresponds to a vertical acceleration imparted to the fluid parcel by the buoyancy force when the parcel moves from its equilibrium position a_3 to z.

Variations of (4.156) on the pressure yield instead the incompressibility constraint $\partial(\mathbf{x})/\partial(\mathbf{a}) = \alpha$.

4.5 Selected Topics in Wave Dynamics

In this final section, the variational methods will be applied to some selected problems in wave dynamics. In particular, we will first introduce Luke's variational principle to obtain the equations for surface water waves. Later, we will introduce Whitham's variational principle, which is a powerful tool that can be used to obtain the wave dispersion relation for nonlinear problems as well as the conservation of the wave activity of the system.

4.5.1 Potential Flows and Surface Water Waves

One of the simplest examples of wave dynamics in geophysical flows comes from the treatment of surface water waves. Consider a fluid flow governed by the Euler equation in a gravitational field (1.74). As for the shallow water equations, the fluid is comprised between a flat bottom boundary at $z = -H$ and an upper boundary set at $z = \eta(x, y, t)$.

Without loss of generality, we can consider the horizontal boundary conditions as double periodic. For the study of these system, we introduce two approximations:

- we assume that the waves evolve in time at scales much shorter than the Earth's rotation, allowing thus to neglect the effects of Earth's rotation, and the Euler's equation is reduced to (1.15);

- at the spatial scales under consideration, we consider the flow to be unsteady and irrotational.

The second hypothesis corresponds to taking the relative vorticity (1.75) as zero

$$\boldsymbol{\omega} = rot\,\mathbf{u} = 0\,. \tag{4.160}$$

As the vorticity is a solution of Euler equation, (4.160) corresponds to take the vorticity as zero at initial time and then integrates it at all times. Equation (4.160) is satisfied by the existence of a single-valued velocity potential φ satisfying

$$\mathbf{u} = \nabla\varphi\,. \tag{4.161}$$

With (4.161), the continuity equation (1.7) yields *Laplace equation*

$$\begin{aligned} div\,\mathbf{u} &= div\,\nabla\varphi \\ &= \nabla^2\varphi \\ &= 0\,. \end{aligned} \tag{4.162}$$

We start by setting kinematic boundary conditions to our problem. At $z = \eta$, the rate of change of η following a fluid parcel must be given by the vertical component of (4.161), so that

$$\frac{D\eta}{Dt} - \frac{\partial\varphi}{\partial z} = 0\,,\;\; z = \eta\,, \tag{4.163}$$

or, equivalently,

$$\frac{\partial\eta}{\partial t} + \frac{\partial\varphi}{\partial x}\frac{\partial\eta}{\partial x} + \frac{\partial\varphi}{\partial y}\frac{\partial\eta}{\partial y} - \frac{\partial\varphi}{\partial z} = 0\,,\;\; z = \eta\,. \tag{4.164}$$

Under the assumption of flat bottom boundary, there the boundary condition is instead

$$\frac{\partial\varphi}{\partial z} = 0\,,\;\; z = -H\,. \tag{4.165}$$

Returning to the dynamical equations, using (4.161) Euler equation can be written as

$$\nabla\left[\frac{\partial\varphi}{\partial t} + \frac{1}{2}(\nabla\varphi)^2 + \frac{p}{\rho} + gz\right] = 0\,. \tag{4.166}$$

At the interface between water and air, the term p/ρ is adsorbed in the $\partial\varphi/\partial t$ term so that, upon integration in space, (4.166) yields

$$\frac{\partial \varphi}{\partial t} + \frac{1}{2}(\nabla \varphi)^2 + g\eta = C(t)\ ,\ z = \eta\ , \tag{4.167}$$

where $C(t)$ is an arbitrary function of time determined by the pressure imposed at the boundaries. As the flow is determined only by the pressure gradient and not by the pressure itself, without loss of generality we can set $C(t) = 0$ and (4.167) becomes

$$\frac{\partial \varphi}{\partial t} + \frac{1}{2}(\nabla \varphi)^2 + g\eta = 0\ ,\ z = \eta\ , \tag{4.168}$$

which is the *Bernoulli equation* at the boundary $z = \eta$. For $-H < z < \eta$, the Bernoulli equation is

$$\frac{\partial \varphi}{\partial t} + \frac{1}{2}(\nabla \varphi)^2 + \frac{p}{\rho} + gz = 0\ . \tag{4.169}$$

Since the surface given by $z = \eta$ is a material surface, from (4.168) one must have

$$\frac{D}{Dt}\left[\frac{\partial \varphi}{\partial t} + \frac{1}{2}(\nabla \varphi)^2 + g\eta\right] = 0\ ,\ z = \eta\ , \tag{4.170}$$

that is

$$\frac{\partial^2 \varphi}{\partial t^2} + 2\nabla\varphi \cdot \nabla\left(\frac{\partial \varphi}{\partial t}\right) + \frac{1}{2}\nabla\varphi \cdot \nabla(\nabla\varphi)^2 + g\frac{\partial \varphi}{\partial z} = 0\ ,\ z = \eta\ , \tag{4.171}$$

Equations (4.162), (4.164), (4.165) and (4.171) can be solved for φ and η. In the following, in the spirit of this book, this set of equations will be rederived from variational principles.

4.5.2 Luke's Variational Principle

The variational derivation will follow [32] and takes the name of Luke's variational principle. Following the previous arguments, the action functional for the system under consideration is

$$I = \int_{t_1}^{t_2}\int_R \rho\left[\frac{\partial \varphi}{\partial t} + \frac{1}{2}(\nabla \varphi)^2 + gz\right] d(\mathbf{x})dt\ . \tag{4.172}$$

The action (4.172) corresponds to the Lagrangian density

$$L = \rho\left[\frac{\partial \varphi}{\partial t} + \frac{1}{2}(\nabla \varphi)^2 + gz\right]\ . \tag{4.173}$$

Application of Hamilton's principle to (4.172) for variations in $\varphi(\mathbf{x}, t) + l\Phi(\mathbf{x}, t)$ yields

$$\delta I = \int_{t_1}^{t_2} \int_R \rho \left[\delta \left(\frac{\partial \varphi}{\partial t} \right) + \frac{\partial \varphi}{\partial x} \delta \left(\frac{\partial \varphi}{\partial x} \right) + \frac{\partial \varphi}{\partial y} \delta \left(\frac{\partial \varphi}{\partial y} \right) + \frac{\partial \varphi}{\partial z} \delta \left(\frac{\partial \varphi}{\partial z} \right) \right] d(\mathbf{x}) dt$$
$$+ \int_{t_1}^{t_2} \int_{R_h} \rho \left[\frac{\partial \varphi}{\partial t} + \frac{1}{2} (\nabla \varphi)^2 + gz \right]_{z=\eta} \delta \left(\frac{\partial \varphi}{\partial z} \right)_{z=\eta} d(\mathbf{x}_h) dt \ , \tag{4.174}$$

that is

$$\delta I = \int_{t_1}^{t_2} \int_{R_h} \rho \left\{ \left[\frac{\partial}{\partial t} \int_{-H}^{\eta} \Phi dz + \frac{\partial}{\partial x} \int_{-H}^{\eta} \frac{\partial \varphi}{\partial x} \Phi dz + \frac{\partial}{\partial y} \int_{-H}^{\eta} \frac{\partial \varphi}{\partial y} \Phi dz \right] \right.$$
$$- \int_{-H}^{\eta} \left(\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 \varphi}{\partial z^2} \right) \Phi dz$$
$$- \left[\left(\frac{\partial \eta}{\partial t} + \frac{\partial \varphi}{\partial x} \frac{\partial \eta}{\partial x} + \frac{\partial \varphi}{\partial y} \frac{\partial \eta}{\partial y} - \frac{\partial \varphi}{\partial z} \right) \Phi \right]_{z=\eta}$$
$$+ \left[\frac{\partial \varphi}{\partial z} \Phi \right]_{z=-H}$$
$$\left. + \int_{t_1}^{t_2} \int_{R_h} \left[\frac{\partial \varphi}{\partial t} + \frac{1}{2} (\nabla \varphi)^2 + gz \right]_{z=\eta} \delta \left(\frac{\partial \varphi}{\partial z} \right)_{z=\eta} \right\} d(\mathbf{x}_h) dt \ . \tag{4.175}$$

Setting Φ to be zero on the domain boundary ∂R_h, the first term of (4.175) vanishes. The second term is the only term that is not defined on one of the vertical boundaries. Away from those, (4.175) is thus satisfied by

$$\nabla^2 \varphi = 0 \ , \ z \neq -H \ , \ z \neq \eta \ , \tag{4.176}$$

which corresponds to (4.162). On the vertical boundaries, since (4.175) must be satisfied by arbitrary values of Φ, one has

$$\left(\frac{\partial \eta}{\partial t} + \frac{\partial \varphi}{\partial x} \frac{\partial \eta}{\partial x} + \frac{\partial \varphi}{\partial y} \frac{\partial \eta}{\partial y} - \frac{\partial \varphi}{\partial z} \right) = 0 \ , \ z = \eta \ , \tag{4.177a}$$

$$\frac{\partial \varphi}{\partial z} = 0 \ , \ z = -H \ , \tag{4.177b}$$

$$\frac{\partial \varphi}{\partial t} + \frac{1}{2} (\nabla \varphi)^2 + g\eta = 0 \ , \ z = \eta \ , \tag{4.177c}$$

which correspond, respectively, to (4.164), (4.165) and (4.168).

4.5.3 Whitham's Averaged Variational Principle and Conservation of Wave Activity

The set of equations derived in the previous section can generally be treated just in special cases due to their nonlinear form. In this section, we will follow [78] to introduce a variational way to treat the case of dispersive waves.

Consider an action functional with general form

$$I = \int_{t_1}^{t_2} L\left(\frac{\partial\varphi}{\partial t}, \frac{\partial\varphi}{\partial x_i}, \varphi\right) d(\mathbf{x})dt \ , \tag{4.178}$$

where the subscripts indicate partial derivatives. Further, assume that (4.178) allows for wave solutions in the form

$$\varphi = \mathbb{R}\left\{Ae^{i\theta}\right\} = a\cos(\theta + \alpha) \ , \tag{4.179}$$

where $a = |A|$ (which should not be confused with the notation for the particle label), $\alpha = arg\{A\}$ and

$$\theta = \mathbf{k}\cdot\mathbf{x} - \omega t \ . \tag{4.180}$$

In (4.180), $\mathbf{k} = (k_1, k_2, k_3)$ is the wavenumber vector and ω is the wave frequency, defined as

$$k_i = \frac{\partial\theta}{\partial x_i} \ , \tag{4.181a}$$

$$\omega = -\frac{\partial\theta}{\partial t} \ . \tag{4.181b}$$

For linear problems, ω and $\mathbf{k}$ are constants, and thus, from (4.180), θ is a linear function of x_i and t. For nonlinear problems, $\omega = \omega(\mathbf{x}, t)$, $\mathbf{k} = \mathbf{k}(\mathbf{x}, t)$ and $A = A(\mathbf{x}, t)$, and thus, θ is a nonlinear function of x_i and t. In this case, the definitions (4.181a), (4.181b) take *local* meaning. Notice also that (4.181a), (4.181b) satisfy

$$\frac{\partial k_i}{\partial t} + \frac{\partial\theta}{\partial x_i} = 0 \ , \tag{4.182}$$

and

$$rot\ \mathbf{k} = \mathbf{0} \ . \tag{4.183}$$

With position (4.179), the arguments of the Lagrangian density in (4.178) are simply

$$\frac{\partial \varphi}{\partial t} = \omega a \sin(\theta + \alpha) \ , \tag{4.184a}$$

$$\frac{\partial \varphi}{\partial x_i} = -k_i a \sin(\theta + \alpha) \ . \tag{4.184b}$$

The Euler–Lagrange equation corresponding to the application of Hamilton's principle to (4.178) can be written as

$$\frac{\partial L_1}{\partial t} + \frac{\partial L_2}{\partial x_i} - L_3 = 0 \ , \tag{4.185}$$

where

$$L_1 = \frac{\partial L}{\partial \varphi_t} \ , \tag{4.186a}$$

$$L_2 = \frac{\partial L}{\partial \varphi_{x_i}} \ , \tag{4.186b}$$

$$L_3 = \frac{\partial L}{\partial \varphi} \ . \tag{4.186c}$$

It should be noted that the second term in (4.185) and Eq. (4.186b) imply summation over repeated indices. Equation (4.185) is of second order, justifying the two constants of integration a and α. Neglecting the phase α, the quantities ω, $\mathbf{k}$ and a are connected by the dispersion relation

$$G(\omega, \mathbf{k}, a) = 0 \ . \tag{4.187}$$

For linear problems, the dispersion relation (4.187) does not depend on A.

Whitham [78] proposed that, after the replacement of the arguments of L by (4.184a), (4.184b), which changes the arguments of L into ω, $\mathbf{k}$ and a, the Lagrangian density can be averaged over one period of the phase

$$L_{av}(\omega, \mathbf{k}, a) = \frac{1}{2\pi} \int_0^{2\pi} L d\theta \ . \tag{4.188}$$

The original motivation for (4.188) was the assumption, based on intuitive grounds, that the energy must be balanced overall. Notice that L_{av} could be also a function of $\mathbf{x}$ and t, due to nonlinearities and nonuniformity of the medium. After this definition, it is possible to introduce the averaged action

$$I_{av} = \int_{t_1}^{t_2} L_{av} d(\mathbf{x}) dt \ . \tag{4.189}$$

The averaged equations can thus be studied setting

$$\delta I_{av} = 0 \,, \tag{4.190}$$

which also takes the name of *averaged variational principle*.

As L_{av} does not depend on derivatives of a, variation of (4.189) in the amplitude yields

$$\frac{\partial L_{av}}{\partial a}(\omega, \mathbf{k}, a) = 0 \,. \tag{4.191}$$

This equation is a functional relationship between ω, $\mathbf{k}$ and a, and is thus the dispersion relation. As a simple example, it can be noted that because of linear problems L must be quadratic in φ and its derivatives, in that case L_{av} takes the form

$$L_{av} = G(\omega, \mathbf{k})a^2 \,. \tag{4.192}$$

In this case, (4.191) yields immediately

$$G(\omega, \mathbf{k}) = 0 \,. \tag{4.193}$$

Notice that in (4.193), G does not depend on a.

Variations of (4.189) in θ yield instead

$$\frac{\partial}{\partial t}\frac{\partial L_{av}}{\partial \omega} - \frac{\partial}{\partial x_i}\frac{\partial L_{av}}{\partial k_i} = 0 \,. \tag{4.194}$$

Defining the *wave action*

$$\mathscr{A} = \frac{\partial L_{av}}{\partial \omega} \,, \tag{4.195}$$

and its flux

$$\mathscr{B}_i = -\frac{\partial L_{av}}{\partial k_i} \,, \tag{4.196}$$

Equation (4.194) can be written as the hyperbolic equation

$$\frac{\partial}{\partial t}\mathscr{A} + \frac{\partial}{\partial x_i}\mathscr{B}_i = 0 \,, \tag{4.197}$$

which is thus an equation for the conservation of the wave action.

Using (4.181a), (4.181b), it is important to notice that L_{av} only depends on the derivatives of θ and not on the level set of the same variable. Equation (4.197) is also invariant for time translations, so that, by Noether's Theorem, one has the conservation of the quantity

$$\mathscr{E} = \omega\mathscr{A} \,, \tag{4.198}$$

that is the average energy density of the system. The wave activity $\mathscr{A} = \mathscr{E}/\omega$ is a known adiabatic invariant for slow modulations of linear vibrating systems.

Notice that in a system with wave–mean flow interactions, it is the total energy given by the energy of the mean flow and of the disturbances, to be conserved, and not the wave energy.

From the previous relationships, the energy equation can be written as

$$\frac{\partial}{\partial t}\left(\omega\frac{\partial L_{av}}{\partial\omega} - L_{av}\right) + \frac{\partial}{\partial x_i}\left(-\omega\frac{\partial L_{av}}{\partial k_i} - L_{av}\right) = 0\,, \tag{4.199}$$

Interchanging the roles of t and x_i yields the averaged momentum equation

$$\frac{\partial}{\partial t}\left(k_i\frac{\partial L_{av}}{\partial\omega}\right) + \frac{\partial}{\partial x_i}\left(-k_j\frac{\partial L_{av}}{\partial k_i} - L_{av}\delta_{ij}\right) = 0\,. \tag{4.200}$$

The quantity

$$k_i\frac{\partial L_{av}}{\partial\omega} = \frac{k_i}{\omega}\mathscr{E}\,, \tag{4.201}$$

is the momentum density, and it is a vector in the direction $\mathbf{k}$ and with magnitude $\mathscr{E}/c$, where c is the phase speed of the wave.

4.5.4 Example 1: The Linear Klein–Gordon Equation

As an example, we consider the linear Klein–Gordon equation

$$\frac{\partial^2\varphi}{\partial t^2} + \beta^2\varphi = \alpha^2\nabla^2\varphi\,. \tag{4.202}$$

This is a hyperbolic equation representing, in classical terms, the vibrations for a displacement φ with a restoring force proportional to φ and a dispersive term. Notably, this is an important equation also in quantum mechanics where it represents a relativistic wave equation.

The Lagrangian density corresponding to (4.202) is

$$L = \frac{1}{2}\left[\left(\frac{\partial\varphi}{\partial t}\right)^2 - \alpha^2\left(\frac{\partial\varphi}{\partial x_i}\right)^2\varphi - \beta^2\varphi^2\right]. \tag{4.203}$$

It should be noted that the Lagrangian density is a quadratic in φ.

The averaged Lagrangian (4.188) for (4.203) is

$$L_{av} = \frac{1}{4}\left(\omega^2 - \alpha^2k^2 - \beta^2\right)a^2\,. \tag{4.204}$$

Hence, direct application of (4.191) yields the dispersion relation

$$\omega = \left(\alpha^2 k^2 + \beta^2\right)^{1/2} \ , \tag{4.205}$$

while the application of (4.195) and (4.198) yields the averaged wave activity

$$\mathscr{A} = \frac{1}{2}\omega a^2 \ , \tag{4.206}$$

and averaged energy density

$$\mathscr{E} = \frac{1}{2}\omega^2 a^2 = \left(\alpha^2 k^2 + \beta^2\right) a^2 \ . \tag{4.207}$$

4.5.5 Example 2: The Nonlinear Klein–Gordon Equation

The Klein–Gordon equation can be generalized to its nonlinear form, here written for simplicity in 1D

$$\frac{\partial^2 \varphi}{\partial t^2} + \beta^2 \varphi + r\varphi^3 = \alpha^2 \frac{\partial^2 \varphi}{\partial x^2} \ . \tag{4.208}$$

The corresponding Lagrangian density is

$$L = \frac{1}{2}\left[\left(\frac{\partial \varphi}{\partial t}\right)^2 - \alpha^2 \left(\frac{\partial \varphi}{\partial x}\right)^2 \varphi - \beta^2 \varphi^2 - \frac{1}{2} r \varphi^4\right] . \tag{4.209}$$

As a result, the averaged Lagrangian (4.188) for (4.209) is

$$L_{av} = \frac{1}{4}\left(\omega^2 - \alpha^2 k^2 - \beta^2\right) a^2 - \frac{3}{32} r a^4 \ . \tag{4.210}$$

It can be noted that the nonlinear term is responsible for the term of $O\left(a^4\right)$. Direct application of (4.191) yields the dispersion relation

$$\omega = \left(\alpha^2 k^2 + \beta^2 + \frac{3}{4} r a^2\right)^{1/2} \ . \tag{4.211}$$

One should notice that the dispersion relation depends now on a. The application of (4.195) and (4.198) shows that the wave activity assumes the same shape (4.206). The energy density takes, however, the form

$$\mathscr{E} = \frac{1}{2}\left(\alpha^2 k^2 + \beta^2 + \frac{3}{4} r a^2\right) a^2 \ . \tag{4.212}$$

4.5.6 Example 3: The Korteweg–DeVries (KdV) Equation

As a last example, consider the nonlinear 1D Korteweg–deVries (KdV) equation

$$\frac{\partial \eta}{\partial t} + 6\eta \frac{\partial \eta}{\partial x} + \frac{\partial^3 \eta}{\partial x^3} = 0 \,. \tag{4.213}$$

In (4.213), the factor 6 was chosen for normalization purposes. This equation is particularly important, as its solutions can take the form of solitary waves as well as trains of them. Its treatment is, however, much more difficult.

Setting

$$\eta = \frac{\partial \varphi}{\partial x} \,, \tag{4.214}$$

the KdV equation (4.213) becomes

$$\frac{\partial^2 \varphi}{\partial t \partial x} + 6 \frac{\partial \varphi}{\partial x} \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^4 \varphi}{\partial x^4} = 0 \,, \tag{4.215}$$

with Lagrangian density

$$L = -\frac{1}{2} \frac{\partial \varphi}{\partial t} \frac{\partial \varphi}{\partial x} - \left(\frac{\partial \varphi}{\partial x} \right)^3 + \frac{1}{2} \left(\frac{\partial^2 \varphi}{\partial x^2} \right)^2 \,. \tag{4.216}$$

The problem can be treated analytically with some smart changes of variables. In particular, it is possible to take

$$\varphi = \beta x - \gamma t + \Phi(\theta) \,, \tag{4.217}$$

where Φ is a periodic function of θ. In (4.217), it is possible to define the pseudo-phase

$$\psi = \beta x - \gamma t \,. \tag{4.218}$$

One should notice the analogy between (4.218) and (4.180), so that

$$\beta = \frac{\partial \psi}{\partial x} \,, \tag{4.219a}$$

$$\gamma = -\frac{\partial \psi}{\partial t} \,. \tag{4.219b}$$

and, in analogy with (4.182),

$$\frac{\partial \beta}{\partial t} + \frac{\partial \gamma}{\partial x} = 0 \,, \tag{4.220}$$

hold.

With (4.217), (4.218), using (4.180) and (4.214) one has

$$\eta = \beta + k\frac{\partial \Phi}{\partial \theta} \ . \tag{4.221}$$

It should be noted that, due to the periodicity of Φ, β refers to the mean value of η, observation that will be used for the averaged Lagrangian. In terms of (4.221), the KdV equation (4.213) can be written as

$$-U\frac{\partial \eta}{\partial \theta} + 6\eta\frac{\partial \eta}{\partial \theta} + k^2\frac{\partial^3 \eta}{\partial \theta^3} = 0 \ , \tag{4.222}$$

where we have indicated with

$$U = \frac{\omega}{k} \ , \tag{4.223}$$

the nonlinear phase velocity. Equation (4.222) has integral

$$-U\eta + 3\eta^2 + k^2\frac{\partial^2 \eta}{\partial \theta^2} + B = 0 \ , \tag{4.224}$$

where A is an integration constant. Further integration yields

$$-U\eta^2 + 2\eta^3 + k^2\left(\frac{\partial \eta}{\partial \theta}\right)^2 + 2B\eta - 2A = 0 \ , \tag{4.225}$$

where B is a second integration constant. Using (4.225), the Lagrangian density (4.216) becomes

$$L = k^2\left(\frac{\partial \eta}{\partial \theta}\right)^2 + \left\{B + \frac{1}{2}(\gamma - U)\beta\right\}\eta - A \ . \tag{4.226}$$

To calculate the averaged Lagrangian, the phase integral of the first term of (4.226) yields

$$\frac{1}{2\pi}\int_0^{2\pi} k^2\left(\frac{\partial \eta}{\partial \theta}\right)^2 d\theta = \frac{1}{2\pi}\oint k^2\frac{\partial \eta}{\partial \theta}d\eta = kW \ , \tag{4.227}$$

where

$$W = \frac{1}{2\pi}\oint\left[2A - 2B\eta + U\eta^2 - 2\eta^3\right]^{1/2} d\eta \ . \tag{4.228}$$

Notice that $W = W(A, B, U)$. Meanwhile, because β refers to the mean value of η, as seen from (4.221), the phase integral of the second term of (4.226) yields

$$\frac{1}{2\pi}\int_0^{2\pi} \eta d\theta = \beta \ . \tag{4.229}$$

With (4.227) and (4.229), the averaged Lagrangian is finally

$$L_{av} = kW + \beta B + \frac{1}{2}\beta\gamma - \frac{1}{2}U\beta^2 - A \; . \tag{4.230}$$

The averaged Lagrangian is thus a functional $L_{av} = L_{av}(\omega, k, A; \gamma, \beta, B)$. It is thus possible to take the following variations:

- with respect to θ, which gives

$$\frac{\partial}{\partial t}\frac{\partial L_{av}}{\partial \omega} - \frac{\partial}{\partial x}\frac{\partial L_{av}}{\partial k} = 0 \; ; \tag{4.231}$$

- with respect to ψ, which gives

$$\frac{\partial}{\partial t}\frac{\partial L_{av}}{\partial \gamma} - \frac{\partial}{\partial x}\frac{\partial L_{av}}{\partial \beta} = 0 \; ; \tag{4.232}$$

- with respect to A, which gives

$$k\frac{\partial W}{\partial A} = 1 \; ; \tag{4.233}$$

- and with respect to B, which gives

$$\beta = -k\frac{\partial W}{\partial B} \; . \tag{4.234}$$

Equation (4.232) with (4.230) results in

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\beta\right) + \frac{\partial}{\partial x}\left(U\beta - \frac{1}{2}\gamma - B\right) = 0 \; . \tag{4.235}$$

Comparison between (4.235) and (4.220) shows that

$$\gamma = -kU\frac{\partial W}{\partial B} - B \; , \tag{4.236}$$

so that, using also (4.234), Eq. (4.235) yields

$$\frac{\partial}{\partial t}\left(k\frac{\partial W}{\partial B}\right) + \frac{\partial}{\partial x}\left(kU\frac{\partial W}{\partial B} + B\right) = 0 \; . \tag{4.237}$$

As the averaged Lagrangian (4.230) does not depend on ω and k, Eq. (4.231) is automatically satisfied and does not give information on the dynamics. It is instead possible to rewrite it in the form of an averaged momentum Eq. (4.200) so that

$$\frac{\partial}{\partial t}\left(k\frac{\partial L_{av}}{\partial \omega} + \beta\frac{\partial L_{av}}{\partial \gamma}\right) + \frac{\partial}{\partial x}\left(L_{av} - k\frac{\partial L_{av}}{\partial k} - \beta\frac{\partial L_{av}}{\partial \beta}\right) = 0 \; . \tag{4.238}$$

Rewriting (4.182) as

$$\frac{\partial k}{\partial t} + \frac{\partial}{\partial x}(kU) = 0\,, \tag{4.239}$$

where we have used (4.223), Eq. (4.238) can be rewritten, making use of (4.233) and (4.239), as

$$\frac{\partial}{\partial t}\left(k\frac{\partial W}{\partial U}\right) + \frac{\partial}{\partial x}\left(kU\frac{\partial W}{\partial U} - A\right) = 0\,. \tag{4.240}$$

Equations (4.237), (4.239) and (4.240) may be viewed as a set of three hyperbolic equations for A, B and U, with k given by (4.233). These equations can be written in symmetric form as

$$\frac{\partial W_B}{\partial t} + U\frac{\partial W_B}{\partial x} + W_A\frac{\partial B}{\partial x} = 0\,, \tag{4.241a}$$

$$\frac{\partial W_U}{\partial t} + U\frac{\partial W_U}{\partial x} - W_A\frac{\partial A}{\partial x} = 0\,, \tag{4.241b}$$

$$\frac{\partial W_A}{\partial t} + U\frac{\partial W_A}{\partial x} - W_A\frac{\partial U}{\partial x} = 0\,, \tag{4.241c}$$

where the subscript indicates partial derivatives. Once (4.241a)–(4.241c) are solved, the wave frequency ω can be determined as

$$\omega = \frac{U}{W_A}\,, \tag{4.242}$$

where we made use of (4.233), and the mean value β can be determined as

$$\beta = -\frac{W_B}{W_A}\,. \tag{4.243}$$

For a solution making use of the method of characteristics, see [78].

4.6 Bibliographical Note and Suggestions for Further Reading

For research articles on the derivation of approximated equations of fluid and geophysical flows using Hamilton's principle, see, for example, [5, 10, 11, 14, 15, 18, 19, 24, 26, 32, 34, 42, 47, 52, 56, 64–67, 70–72, 74–77]. This is obviously an incomplete, selected list. For alternative derivations of the equations of motion for the ideal fluid and for the quasi-geostrophic equations using Hamilton's principle, see [23, 73]. For the particular case of the semi-geostrophic equations, see [42–45, 50, 51, 53, 54, 57, 58, 63]. For recent developments on semi-geostrophic dynamics, see, e.g. [4, 46]. For a review on the consequences of the relabelling symmetry for the conservation of potential vorticity in ocean dynamics, see the already cited [39]. Despite the fact that this book does not cover the derivation of the equations of motion for the ideal fluid and its approximations in Hamiltonian form, and in particular their associated Poisson brackets, for example, in the excellent reviews by [35, 36, 55, 59–61, 69]. The Hamiltonian formulation is associated with particularly powerful criteria for the nonlinear stability, originally formulated in [2, 3] and with important advancements by, amongst others, [1, 9, 16, 29, 30, 33, 38, 48, 49, 68]. See, e.g. the reviews quoted above, including [25, 62]. Further, the Hamiltonian form allows for the formulation of numerical schemes that well preserve the conserved quantities, see, e.g. [31]. For the representation of dissipation in Hamiltonian fluid flows, see e.g. [8, 37] and references therein. Finally, alternatives to the Hamiltonian form for the equations for fluids are available. One of them, given by Nambu [40], proposes a form of the equations that is solely based on Liouville's Theorem. The Nambu form of the equations of fluid and geophysical fluid dynamics is particularly promising. For examples of its use, see, e.g. [6, 7, 41] and references therein. While the lists reported in this section are necessarily incomplete, they are representative of a good sample of different works on these topics and may provide thus as a starting point for further reading.

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Appendix A
Derivation of Equation (1.2)

Equation (1.2) is derived in many textbooks of fluid dynamics. Here, it is explained in its simplest, i.e. one-dimensional, form. Then, the generalization to the three-dimensional form trivially follows. Define

$$\Theta(t) = \int_{x_0(t)}^{x_1(t)} \theta(x,t)dx \ . \tag{A.1}$$

In (A.1), $\theta(x,t)$ is a certain fluid field and $[x_0(t), x_1(t)]$ is a material interval on $\mathbb{R}$ which, in general, moves along $\mathbb{R}$. In the course of motion, the interval can expand or shrink, but its content of fluid is left unchanged in time. According to Leibniz formula,

$$\frac{d\Theta}{dt} = \int_{x_0(t)}^{x_1(t)} \frac{\partial\theta}{\partial t}dx + \theta\,(x_1,t)\,\frac{dx_1}{dt} - \theta\,(x_0,t)\,\frac{dx_0}{dt} \ . \tag{A.2}$$

From the physical point of view, dx_0/dt is the velocity $u\,(x_0,t)$ of the lower extreme of the material interval, while $dx_0/dt = u\,(x_1,t)$ is the velocity of the upper extreme of the same interval. Thus, (A.2) can be written as

$$\frac{d\Theta}{dt} = \int_{x_0(t)}^{x_1(t)} \frac{\partial\theta}{\partial t}dx + \theta\,(x_1,t)\,u\,(x_1,t) - \theta\,(x_0,t)\,u\,(x_0,t) \ . \tag{A.3}$$

or also as

$$\frac{d\Theta}{dt} = \int_{x_0(t)}^{x_1(t)} \left[\frac{\partial\theta}{\partial t} + \frac{\partial}{\partial x}\,(\theta u)\right]dx = \int_{x_0(t)}^{x_1(t)} \left[\frac{\partial\theta}{\partial t} + u\frac{\partial\theta}{\partial x} + \theta\frac{\partial u}{\partial x}\right]dx \ . \tag{A.4}$$

Equation (A.4) shows that variations in time of (A.1) can be ascribed to an explicit variation in time of θ inside the considered material interval, or to the translation of the material interval over a part of $\mathbb{R}$ where θ is not constant or, finally, to the

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deformation of the material interval due to different velocities of its extremes. By means of the introduction of the Lagrangian derivative

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} \ , \tag{A.5}$$

Equation (A.4) takes the form

$$\frac{d\Theta}{dt} = \int_{x_0(t)}^{x_1(t)} \left(\frac{D\theta}{Dt} + \theta \frac{\partial u}{\partial x} \right) dx \ . \tag{A.6}$$

The three-dimensional generalization of (A.5) is expressed by

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{u}\frac{\partial}{\partial x} \ , \tag{A.7}$$

where $\mathbf{u} = (u, v, w)$ and, in turn, the three-dimensional generalization of (A.6) is

$$\frac{d\Theta}{dt} = \int_{V(t)} \left(\frac{D\theta}{Dt} + \theta \ div \ \mathbf{u} \right) dV' \ . \tag{A.8}$$

where D/Dt is given by (A.7). In (A.8), $V(t)$ is the material volume which is the three-dimensional equivalent of $[x_0(t), x_1(t)]$.

Appendix B
Derivation of the Conservation of Potential Vorticity from Kelvin's Circulation Theorem

Consider a system in an uniformly rotating reference frame. The following arguments can be easily reduced to an inertial framework simply by setting $\Omega \to 0$. Following Chap. 1, define the absolute circulation

$$\Gamma_a = \oint_{\partial C} (\mathbf{u} + \boldsymbol{\Omega} \times \mathbf{r}) \cdot d\mathbf{r} \,, \tag{B.1}$$

where C is a material surface. Using Euler's equation in a rotating framework (1.74) and following the same steps leading to (1.158), the time derivative of (B.1) yields

$$\frac{d\Gamma_a}{dt} = -\oint_{\partial C} \frac{\nabla p}{\rho} \cdot d\mathbf{r} \,, \tag{B.2}$$

that is

$$\frac{d\Gamma_a}{dt} = -\oint_{\partial C} \frac{dp}{\rho} \,. \tag{B.3}$$

If ρ is constant, the conservation of the circulation (B.1) follows directly from (B.3).

Using Stokes' theorem, the term on the l.h.s. of (B.3) can be written as

$$\begin{aligned} \frac{d\Gamma_a}{dt} &= \frac{d}{dt} \oint_{\partial C} (\mathbf{u} + \boldsymbol{\Omega} \times \mathbf{r}) \cdot d\mathbf{r} \\ &= \frac{d}{dt} \int_C \hat{\mathbf{k}} \cdot rot \; (\mathbf{u} + \boldsymbol{\Omega} \times \mathbf{r}) \, dC \,. \end{aligned} \tag{B.4}$$

Using the identity $rot \; (\boldsymbol{\Omega} \times \mathbf{r}) = 2\boldsymbol{\Omega}$, (B.4) yields

$$\frac{d\Gamma_a}{dt} = \frac{d}{dt} \int_C \hat{\mathbf{k}} \cdot (\boldsymbol{\omega} + 2\boldsymbol{\Omega}) \, dC \,, \tag{B.5}$$

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where $\boldsymbol{\omega}$ is the relative vorticity (1.75). If C is an infinitesimal surface, the term on the r.h.s. of (B.5) can be written as

$$\frac{D}{Dt}\left[(\boldsymbol{\omega}+2\boldsymbol{\Omega})\cdot\hat{\mathbf{k}}\delta C\right], \tag{B.6}$$

where δC is the area of the integration surface. Hence, if the term on the r.h.s. of (B.3) is zero, i.e. if the quantity dp/ρ is an exact differential on the isobaric surface ∂C, then the quantity

$$\Pi = (\boldsymbol{\omega}+2\boldsymbol{\Omega})\cdot\hat{\mathbf{k}}\delta C \tag{B.7}$$

is materially conserved.

This is proved trivially if ρ is a constant. Under this hypothesis, it is possible to introduce a material volume enclosed between two surfaces of equal pressure, with base δC and height H. The resulting volume is thus

$$\delta V = H\delta C\,. \tag{B.8}$$

Because of the hypothesis of constant density, (B.8) is materially conserved, so that (B.6) yields

$$\frac{D}{Dt}\left[\frac{(\boldsymbol{\omega}+2\boldsymbol{\Omega})}{H}\cdot\hat{\mathbf{k}}\right]=0\,. \tag{B.9}$$

Equation (B.9) is the conservation of potential vorticity for fluids with internal energy that does not depend on the entropy. In fact, in that case

$$\frac{\partial e}{\partial \eta}=0 \;\Rightarrow \alpha=\alpha(p)\,, \tag{B.10}$$

where e is the internal energy, η the entropy and $\alpha = 1/\rho$ is the specific volume of the fluid.

The hypothesis of constant density can be relaxed in favour of a more general condition on the thermodynamics of the fluid under consideration. Denoting with ϑ the absolute temperature of the fluid, one has

$$\vartheta d\eta = de + pd\alpha\,, \tag{B.11}$$

from which is it possible to derive

$$\frac{dp}{\rho} = d\left(e+\frac{p}{\rho}\right)-\vartheta d\eta\,. \tag{B.12}$$

The quantity

$$h = e + \frac{p}{\rho} \tag{B.13}$$

is the enthalpy density of the fluid. Equation (B.12) yields directly

$$\oint_{\partial C} \frac{dp}{\rho} = \oint_{\partial C} dh - \oint_{\partial C} \vartheta d\eta \, . \tag{B.14}$$

For an ideal fluid, the first term on the r.h.s. of (B.14) is clearly zero. Assuming entropy conservation, i.e.

$$\frac{D\eta}{Dt} = 0 \, , \tag{B.15}$$

from Eq. (B.14) one gets that the material path ∂C lies on an isentrope, so that $d\eta = 0$, which results in the term on the r.h.s. of (B.3) being zero, yielding

$$\frac{D}{Dt}\left[(\boldsymbol{\omega} + 2\boldsymbol{\Omega}) \cdot \hat{\mathbf{k}} \delta C\right] = 0 \, . \tag{B.16}$$

The vector $\hat{\mathbf{k}}\delta C$ can be further developed taking into account the conservation of entropy on C. In fact, in this case the unit vector $\hat{\mathbf{k}}$ is orthogonal to the isosurface C and thus

$$\hat{\mathbf{k}} = \frac{\nabla\eta}{|\nabla\eta|} \, . \tag{B.17}$$

Further, the surface element δC can be thought as the base of a cylinder with volume δV, lying between two isosurfaces (i.e. isentropes) set at the δH distance, so that

$$\delta V = \delta H \delta C \, . \tag{B.18}$$

The change of entropy $\delta\eta$ between the two isentropes is

$$\delta\eta = \delta H \, |\nabla\eta| \, . \tag{B.19}$$

Using (B.18), (B.19) yields

$$\delta C = \frac{\delta V \, |\nabla\eta|}{\delta\eta} \, , \tag{B.20}$$

and thus,

$$\Pi = (\boldsymbol{\omega} + 2\boldsymbol{\Omega}) \cdot \frac{\nabla\eta}{\delta\eta} \frac{\rho\delta V}{\rho} \, . \tag{B.21}$$

Because both the entropy $\delta\eta$ and the mass $\rho\delta V$ are conserved, (B.21) yields the conservation of potential vorticity

$$\frac{D}{Dt}\left[(\boldsymbol{\omega} + 2\boldsymbol{\Omega}) \cdot \frac{\nabla\eta}{\rho}\right] = 0 \,, \tag{B.22}$$

for fluids that satisfy $\partial e/\partial\eta \neq 0$ and $\partial\alpha/\partial\vartheta \neq 0$.

Appendix C
Some Simple Mathematical Properties of the Legendre Transformation

Consider a function of one variable $y = f(x),\ x \in \mathbb{R}$ that is strictly convex, i.e. $f''(x) > 0$, where the prime indicates derivative respect to x. The Legendre transform allows the transformation of the function $f(x)$ into the new function $g(p),\ p \in \mathbb{R}$. The transform is defined geometrically through the function

$$F(p, x) = px - f(x)\ . \tag{C.1}$$

which describes the distance between $f(x)$ and the straight line $y = px$. The function F has a maximum at the point $x = x(p)$ at which $f(x)$ is at its farthest from the straight line $y = px$. The point $x(p)$ is thus defined by the condition $\partial F/\partial x = 0$, i.e.

$$f'(x) = p\ . \tag{C.2}$$

Because f is convex, if $x(p)$ exists, it must be unique. The Legendre transform of $f(x)$ is thus the function

$$g(p) = max_x[px - f(x)] = F(p, x(p))\ . \tag{C.3}$$

We list here some mathematical properties of the Legendre transformation.

1. *Multivariable Legendre transform* If $f(q_1, \ldots, q_N)$ is a convex function of the vector $\mathbf{q} = (q_1, \ldots, q_N)$, i.e. f is twice differentiable and its Hessian matrix $\partial^2 f/\partial \mathbf{q}^2$ is positive semi-definite in the entire domain, then it is possible to define the function

$$F(\mathbf{p}, \mathbf{q}) = \mathbf{pq} - f(\mathbf{q})\ , \tag{C.4}$$

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for

$$\mathbf{p} = (p_1, \ldots, p_N) = \partial f / \partial \mathbf{q} \ , \tag{C.5}$$

so that the Legendre transform is

$$g(\mathbf{p}) = max_{\mathbf{q}}[\mathbf{pq} - f(\mathbf{q})] = F(\mathbf{p}, \mathbf{q}(\mathbf{p})) \ . \tag{C.6}$$

As an example, consider the function $f = f(x, y)$. Its differential is

$$df = u\,dx + v\,dy \ , \tag{C.7a}$$

$$u = \frac{\partial f}{\partial x} \ , \tag{C.7b}$$

$$v = \frac{\partial f}{\partial y} \ , \tag{C.7c}$$

so that one can define the Legendre transformation as

$$g(u, y) = f(x, y) - ux \ . \tag{C.8}$$

The differential of g yields

$$\begin{aligned} dg &= df - x\,du - u\,dx \\ &= u\,dx + v\,dy - x\,du - u\,dx \\ &= v\,dy - x\,du \ , \end{aligned} \tag{C.9}$$

Comparison with the form

$$dg = \frac{\partial g}{\partial u} du + \frac{\partial g}{\partial y} dy \tag{C.10}$$

gives the relations

$$v = \frac{\partial g}{\partial y} \ , \tag{C.11a}$$

$$x = -\frac{\partial g}{\partial u} \ . \tag{C.11b}$$

All the properties listed below hold also for the multivariable case.

2. *The Legendre transform preserves convexity* The differential of $g(p)$ can be written as

$$dg = p\,dx + x\,dp - f'(x)dx \ , \tag{C.12}$$

where $f'(x) = df/dx$. Using $f'(x) = p$, (C.2) becomes

$$dg = x\,dp\,, \tag{C.13}$$

or

$$g' = \frac{dg}{dp} = x\,. \tag{C.14}$$

The second derivative yields

$$g'' = \frac{dx}{dp} = \frac{1}{dp/dx} = \frac{1}{f''(x)} > 0 \tag{C.15}$$

that demonstrates that Legendre transform of $f(x)$ is a convex function itself.

3. *The Legendre transform is an involution* To demonstrate this statement, use (C.2) to rewrite (C.1) as

$$f(x) = x\,f'(x) - F(p) = F'(p)\,p - F(p)\,. \tag{C.16}$$

For fixed x, (C.16) is equivalent to

$$f(x) = max_p[xp - g(p)]\,, \tag{C.17}$$

which yields $x = g'(p)$ for the location of the maximum in p. The comparison between (C.17) and (C.3) shows that the Legendre transform is an involution, i.e., it is its own inverse.

4. *Young's inequality* Consider two functions, $f(x)$ and $g(p)$, $x, p \in \mathbb{R}$, which are Legendre transforms of one another. Then, (C.3) and (C.17) imply

$$F(p, x) = px - f(x) \leq g(p)\,. \tag{C.18}$$

f and g are called *dual in the sense of Young*, and (C.18) is called *Young's inequality*.

Appendix D
Derivation of Equation (2.142)

In order to derive (2.142), we start substituting (2.140a)–(2.140c) in (2.141), which yields

$$\delta I = \int_{R'} L\left(t + \delta t, q_i + \delta_L q_i, \dot{q}_i + \delta_L \dot{q}_i\right) dt' - \int_R L\left(t, q_i, \dot{q}_i\right) dt\ . \tag{D.1}$$

Ignoring terms of higher order, (D.1) yields

$$\begin{aligned} & L\left(t + \delta t, q_i + \delta_L q_i, \dot{q}_i + \delta_L \dot{q}_i\right) \\ & = L\left(t, q_i, \dot{q}_i\right) + \frac{\partial L}{\partial t}\delta t + \frac{\partial L}{\partial q_i}\delta_L q_i + \frac{\partial L}{\partial \dot{q}_i}\delta_L \dot{q}_i\ . \end{aligned} \tag{D.2}$$

The insertion of (D.2) in (D.1) gives

$$\delta I = \int_{R'} \left[L\left(t, q_i, \dot{q}_i\right) + \frac{\partial L}{\partial t}\delta t + \frac{\partial L}{\partial q_i}\delta_L q_i + \frac{\partial L}{\partial \dot{q}_i}\delta_L \dot{q}_i \right] dt' - \int_R L\left(t, q_i, \dot{q}_i\right) dt\ . \tag{D.3}$$

Equation (D.3) can be further developed expressing the two integrals over the same time domain. To do so, the first integral can be modified with the substitution

$$\int_{R'} dt' \to \int_R \left(1 + \frac{\partial \delta t}{\partial t}\right) dt\ ,$$

so that, ignoring higher order terms

$$\delta I = \int_R \left[\frac{\partial}{\partial t}(L\delta t) + \frac{\partial L}{\partial q_i}\delta_L q_i + \frac{\partial L}{\partial \dot{q}_i}\delta_L \dot{q}_i \right] dt\ . \tag{D.4}$$

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Because of (2.140a)–(2.140c), $\delta q_i = q_i'(t') - q_i(t)$, $\delta \dot{q}_i = \dot{q}_i'(t') - \dot{q}_i(t)$, so that δq_i and $\delta \dot{q}_i$ do not depend only on the independent variable t but also on t'. In order to obviate this problem, it is possible to introduce a new, auxiliary, variation, called *total variation* and denoted with δ_T, so that

$$q_i'(t') = q_i(t') + \delta_T q_i(t') \; and \tag{D.5a}$$

$$\dot{q}_i'(t') = \dot{q}_i(t') + \delta_T \dot{q}_i(t') \tag{D.5b}$$

that differ from (2.140a)–(2.140c) as (D.5a), (D.5b) refers to coordinates calculated at the same time. Using (2.140a)–(2.140c) and (D.5a), (D.5b), it is possible to calculate the following relationships

$$\begin{aligned}
\delta q_i(t) &= q_i'(t') - q_i(t) \\
&\approx q_i'(t) + \frac{\partial q_i'}{\partial t}\delta t - q_i(t) \\
&= \delta_T q_i(t) + \frac{\partial q_i'}{\partial t}\delta t - q_i(t) \\
&= \delta_T q_i(t) + \frac{\partial}{\partial t}\left[q_i(t) + \delta q_i(t)\right]\delta t \\
&\approx \delta_T q_i(t) + \dot{q}_i \delta t \; ,
\end{aligned} \tag{D.6}$$

$$\begin{aligned}
\delta \dot{q}_i(t) &= \dot{q}_i'(t') - \dot{q}_i(t) \\
&\approx \dot{q}_i'(t) + \frac{\partial \dot{q}_i'}{\partial t}\delta t - \dot{q}_i(t) \\
&= \delta_T \dot{q}_i(t) + \frac{\partial \dot{q}_i'}{\partial t}\delta t - \dot{q}_i(t) \\
&= \delta_T \dot{q}_i(t) + \frac{\partial}{\partial t}\left[\dot{q}_i(t) + \delta \dot{q}_i(t)\right]\delta t \\
&\approx \delta_T \dot{q}_i(t) + \ddot{q}_i \delta t \; .
\end{aligned} \tag{D.7}$$

Equations (D.6) and (D.7) allow to make the substitutions

$$\delta q_i(t) \to \delta_T q_i(t) + \dot{q}_i(t)\delta t \; ,$$
$$\delta \dot{q}_i(t) \to \delta_T \dot{q}_i(t) + \ddot{q}_i(t)\delta t \; ,$$

into (D.4), which yields

$$\begin{aligned}
\delta I &= \int_R \left[\frac{\partial}{\partial t}(L\delta t) + \frac{\partial L}{\partial q_i}\left(\delta_T q_i + \dot{q}_i \delta t\right) + \frac{\partial L}{\partial \dot{q}_i}\left(\delta_T \dot{q}_i + \ddot{q}_i \delta t\right)\right] dt \\
&= \int_R \left[\frac{\partial}{\partial t}(L\delta t) + \frac{\partial L}{\partial q_i}\dot{q}_i \delta t + \frac{\partial L}{\partial \dot{q}_i}\ddot{q}_i \delta t + \frac{\partial L}{\partial q_i}\delta_T q_i + \frac{\partial L}{\partial \dot{q}_i}\delta_T \dot{q}_i\right] dt \; .
\end{aligned} \tag{D.8}$$

Because δt a function only of time,

$$\frac{\partial L}{\partial q_i}\delta t = \frac{\partial}{\partial q_i}(L\delta t) ,$$
$$\frac{\partial L}{\partial \dot{q}_i}\delta t = \frac{\partial}{\partial \dot{q}_i}(L\delta t) ,$$

and thus, the first three terms on the last integrand of (D.8) reduce to $D(L\delta t)/Dt$, and (D.8) yields

$$\delta I = \int_R \left[\frac{D}{Dt}(L\delta t) + \frac{\partial L}{\partial q_i}\delta_T q_i + \frac{\partial L}{\partial \dot{q}_i}\delta_T \dot{q}_i \right] dt . \tag{D.9}$$

Using the identity

$$\frac{\partial L}{\partial \dot{q}_i}\delta_T \dot{q}_i = \frac{D}{Dt}\left(\frac{\partial L}{\partial \dot{q}_i}\delta_T q_i \right) - \delta_T q_i \frac{D}{Dt}\frac{\partial L}{\partial \dot{q}_i} , \tag{D.10}$$

Equation (D.9) yields

$$\delta I = \int_R \left\{ \frac{D}{Dt}\left(L\delta t + \frac{\partial L}{\partial \dot{q}_i}\delta_T q_i \right) + \left[\frac{\partial L}{\partial q_i} - \frac{D}{Dt}\frac{\partial L}{\partial \dot{q}_i} \right] \delta_T q_i \right\} dt . \tag{D.11}$$

At this point, the total variation δ_T can be eliminated using $\delta_T q_i(t) = \delta q_i(t) - \dot{q}_i(t)\delta t$, obtaining thus (2.142).

Appendix E
Invariance of the Equations of Motion (2.144) Under a Divergence Transformation

The proof of the invariance of the equations of motion under a divergence transformation lies in applying the transformation

$$L \to L + \frac{D}{Dt}\left[\Omega(t, q_i)\right] \ , \tag{E.1}$$

to (2.144) and, knowing that $\partial\Omega/\partial\dot{q}_i = 0$, in verifying that

$$\frac{D}{Dt}\frac{\partial}{\partial\dot{q}_i}\left(L + \frac{D\Omega}{Dt}\right) - \frac{\partial}{\partial q_i}\left(L + \frac{D\Omega}{Dt}\right) = 0 \ . \tag{E.2}$$

In fact,

$$\begin{aligned}
&\frac{D}{Dt}\left(\frac{\partial L}{\partial\dot{q}_i} + \frac{\partial}{\partial\dot{q}_i}\frac{D\Omega}{Dt}\right) - \frac{\partial}{\partial q_i}\left(L + \frac{D\Omega}{Dt}\right)\\
&= \frac{D}{Dt}\left(\frac{\partial}{\partial\dot{q}_i}\frac{D\Omega}{Dt}\right) - \frac{\partial}{\partial q_i}\left(\frac{D\Omega}{Dt}\right)\\
&= \frac{D}{Dt}\left[\frac{\partial}{\partial\dot{q}_i}\left(\frac{\partial\Omega}{\partial t} + \dot{q}_i\frac{\partial\Omega}{\partial q_i}\right)\right] - \frac{\partial}{\partial q_i}\left(\frac{\partial\Omega}{\partial t} + \dot{q}_i\frac{\partial\Omega}{\partial q_i}\right)\\
&= \left(\frac{\partial}{\partial t} + \dot{q}_i\frac{\partial}{\partial q_i}\right)\frac{\partial\Omega}{\partial q_i} - \frac{\partial^2\Omega}{\partial t\partial q_i} + \dot{q}_i\frac{\partial^2\Omega}{\partial q_i^2} = 0 \ .
\end{aligned} \tag{E.3}$$

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Appendix F
Functional Derivatives

In this Appendix, we will give a brief definition of functional derivatives. Consider a generic functional F defined as

$$F[q] = \int_R \mathscr{F}\left(x, q, q_x, q_{xx}, \ldots\right) dx\ , \tag{F.1}$$

where the subscripts indicate partial derivatives. Given the variation $q + lQ$, with Q being an arbitrary function vanishing at the integration boundaries x_i and x_f, one has

$$\delta F = \int_{x_i}^{x_f} \left[\frac{\partial \mathscr{F}}{\partial q} lQ + \frac{\partial \mathscr{F}}{\partial q_x} lQ_x + \frac{\partial \mathscr{F}}{\partial q_{xx}} lQ_{xx} + \ldots \right] dx\ . \tag{F.2}$$

Upon integration by parts, (F.2) yields

$$\delta F = \int_{x_i}^{x_f} lQ \left(\frac{\partial \mathscr{F}}{\partial q} - \frac{d}{dx}\frac{\partial \mathscr{F}}{\partial q_x} + \frac{d^2}{dx^2}\frac{\partial \mathscr{F}}{\partial q_{xx}} - \cdots \right) dx + \left[\frac{\partial \mathscr{F}}{\partial q_x} lQ + \cdots \right]_{x_i}^{x_f}\ . \tag{F.3}$$

Under the request that Q vanishes at the integration boundaries, (F.3) can be written as

$$\delta F = \left\langle \frac{\delta F}{\delta q}, lQ \right\rangle\ , \tag{F.4}$$

where we have defined the functional derivative of F with respect to q as

$$\frac{\delta F}{\delta q} = \frac{\partial \mathscr{F}}{\partial q} - \frac{d}{dx}\frac{\partial \mathscr{F}}{\partial q_x} + \frac{d^2}{dx^2}\frac{\partial \mathscr{F}}{\partial q_{xx}} - \ldots\ , \tag{F.5}$$

and where the angle brackets indicate a inner product.

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Appendix G
Derivation of Equation (2.229)

To derive (2.229), substitute (2.225) and (2.226a)–(2.226b) in (2.227), so that

$$\begin{aligned}\delta I &= \int_R^{\prime} L\left(q_k+\delta q_k, \psi+\delta\psi, \psi_k+\delta\psi_k\right) d(q') - \int_R L\left(q_k, \psi, \psi_k\right) d(q) \\ &= \int_R \left\{ L\left(q_k+\delta q_k, \psi+\delta\psi, \psi_k+\delta\psi_k\right)\left[1+\frac{\partial\left(\delta q_k\right)}{\partial q_k}\right] - L\left(q_k, \psi, \psi_k\right)\right\} d(q) \, .\end{aligned} \tag{G.1}$$

At first order, (G.1) yields

$$\delta I = \int_R \left[L\frac{\partial\left(\delta q_k\right)}{\partial q_k} + \frac{\partial L}{\partial q_k}\delta q_k + \frac{\partial L}{\partial \psi}\delta\psi + \frac{\partial L}{\partial \psi_k}\delta\psi_k \right] d(q) \, , \tag{G.2}$$

where, following (2.226a)–(2.226b),

$$\delta\psi(q) = \psi'(q') - \psi(q) \, , \tag{G.3a}$$
$$\delta\psi_k(q) = \psi_k'(q') - \psi_k(q) \, . \tag{G.3b}$$

Equations (G.3a)–(G.3b) show that the variations of the dependent variables are functions also of $q_0' \ldots q_3'$, while the integration in (G.2) is performed with respect to variables $q_0 \ldots q_3$. This difficulty is solved introducing the auxiliary total variations

$$\delta_T \psi(q') = \psi'(q') - \psi(q') \, , \tag{G.4a}$$
$$\delta_T \psi_k(q') = \psi_k'(q') - \psi_k(q') \, . \tag{G.4b}$$

Using (G.4a)–(G.4b), the variations of the dependent variables become function of the variables $q_0 \ldots q_3$ only. In fact,

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$$\begin{aligned}\delta\psi(q) &= \psi'(q') - \psi(q)\\ &\approx \psi'(q) + \psi'_k(q')\delta q_k - \psi(q)\\ &= \delta_T\psi(q) + [\psi_k(q) + \delta\psi_k(q)]\,\delta q_k \approx \delta_T\psi(q) + \psi_k(q)\delta q_k\ ,\end{aligned} \tag{G.5a}$$

$$\begin{aligned}\delta\psi_k(q) &= \psi'_k(q') - \psi_k(q)\\ &\approx \psi'_k(q) + \psi'_{kl}(q')\delta q_l - \psi_k(q)\\ &= \delta_T\psi_k(q) + [\psi_{kl}(q) + \delta\psi_{kl}(q)]\,\delta q_l \approx \delta_T\psi_k(q) + \psi_{kl}(q)\delta q_l\ .\end{aligned} \tag{G.5b}$$

Equations (G.5a) and (G.5b) allow to make, respectively, the substitutions

$$\delta\psi(q) \to \delta_T\psi(q) + \psi_k(q)\delta q_k\ ,\ \delta\psi_k(q) \to \delta_T\psi_k(q) + \psi_{kl}(q)\delta q_l\ , \tag{G.6}$$

into (G.2), that yield

$$\begin{aligned}\delta I &= \int_R \left[L\frac{\partial\left(\delta q_k\right)}{\partial q_k} + \frac{\partial L}{\partial q_k}\delta q_k + \frac{\partial L}{\partial\psi}\left(\delta_T\psi + \psi_k\delta q_k\right) + \frac{\partial L}{\partial\psi_k}\left(\delta_T\psi_k + \psi_{kl}\delta q_l\right)\right]d(q)\\ &= \int_R \left[L\frac{\partial\left(\delta q_k\right)}{\partial q_k} + \frac{\partial L}{\partial q_k}\delta q_k + \frac{\partial L}{\partial\psi}\delta_T\psi + \frac{\partial L}{\partial\psi}\psi_k\delta q_k + \frac{\partial L}{\partial\psi_k}\delta_T\psi_k + \frac{\partial L}{\partial\psi_k}\psi_{kl}\delta q_l\right]d(q)\ .\end{aligned} \tag{G.7}$$

Because

$$\begin{aligned}&L\frac{\partial\left(\delta q_k\right)}{\partial q_k} + \frac{\partial L}{\partial q_k}\delta q_k + \frac{\partial L}{\partial\psi}\psi_k\delta q_k + \frac{\partial L}{\partial\psi_l}\psi_{lk}\delta q_k\\ &= \frac{\partial}{\partial q_k}\left(L\delta q_k\right) + \psi_k\frac{\partial}{\partial\psi}\left(L\delta q_k\right) + \psi_{kl}\frac{\partial}{\partial\psi_l}\left(L\delta q_k\right)\\ &= \frac{D}{Dq_k}\left(L\delta q_k\right)\ ,\end{aligned} \tag{G.8}$$

and

$$\delta_T\psi_k = \frac{\partial}{\partial q_k}\left(\delta_T\psi\right)\ , \tag{G.9}$$

substitution of (G.8) and (G.9) in (G.7) yields the equation

$$\delta I = \int_R \left[\frac{D}{Dq_k}\left(L\delta q_k\right) + \frac{\partial L}{\partial\psi}\delta_T\psi + \frac{\partial L}{\partial\psi_k}\frac{\partial}{\partial q_k}\delta_T\psi\right]d(q)\ . \tag{G.10}$$

Using the identity

$$\frac{\partial L}{\partial\psi_k}\frac{\partial}{\partial q_k}\left(\delta_T\psi\right) = \frac{\partial L}{\partial\psi_k}\frac{D}{Dq_k}\left(\frac{\partial L}{\partial\psi_k}\delta_T\psi\right) - \frac{D}{Dq_k}\left(\frac{\partial L}{\partial\psi_k}\right)\delta_T\psi\ , \tag{G.11}$$

(G.10) yields

$$\delta I = \int_R \left[\frac{D}{Dq_k} \left(L\delta q_k + \frac{\partial L}{\partial \psi_k} \delta_T \psi \right) + \left(\frac{\partial L}{\partial \psi} - \frac{D}{Dq_k} \frac{\partial L}{\partial \psi_k} \right) \delta_T \psi \right] d(q) \ . \quad \text{(G.12)}$$

It is now possible to go back to the original variations, substituting $\delta_T \psi = \delta\psi - \psi_l \delta q_l$ in (G.12), so that

$$\delta I = \int_R \left[\frac{D}{Dq_k} \left(L\delta q_k - \frac{\partial L}{\partial \psi_k} \psi_l \delta q_l + \frac{\partial L}{\partial \psi_k} \delta\psi \right) + \left(\frac{\partial L}{\partial \psi} - \frac{D}{Dq_k} \frac{\partial L}{\partial \psi_k} \right) (\delta\psi - \psi_l \delta q_l) \right] d(q) \quad \text{(G.13)}$$

that corresponds to (2.229).

Appendix H
Invariance of the Equations of Motion (2.217) Under a Divergence Transformation

The invariance can be proved applying the transformation

$$L \to L + \frac{D}{Dq_k}\left(\delta_S \Omega_k\right)$$

to (2.217) and verifying that

$$\left(\frac{D}{Dq_k}\frac{\partial}{\partial \psi_k} - \frac{\partial}{\partial \psi}\right)\frac{D}{Dq_k}\left(\delta_S \Omega_k\right) = 0\,, \tag{H.1}$$

under the hypothesis

$$\frac{\partial}{\partial \psi_k}\left(\delta_S \Omega_k\right) = 0\,. \tag{H.2}$$

From (H.1) and (H.2), it follows that

$$\begin{aligned}
&\left(\frac{D}{Dq_k}\frac{\partial}{\partial \psi_k} - \frac{\partial}{\partial \psi}\right)\left[\frac{\partial}{\partial q_k}\left(\delta_S \Omega_k\right) + \psi_k \frac{\partial}{\partial \psi}\left(\delta_S \Omega_k\right)\right] \\
&= \frac{D}{Dq_k}\frac{\partial}{\partial \psi_k}\left[\psi_k \frac{\partial}{\partial \psi}\left(\delta_S \Omega_k\right)\right] - \frac{\partial}{\partial \psi}\frac{\partial}{\partial q_k}\left(\delta_S \Omega_k\right) - \frac{\partial}{\partial \psi}\left[\psi_k \frac{\partial}{\partial \psi}\left(\delta_S \Omega_k\right)\right] \\
&= \frac{D}{Dq_k}\left[\frac{\partial}{\partial \psi}\left(\delta_S \Omega_k\right)\right] - \frac{\partial}{\partial \psi}\frac{\partial}{\partial q_k}\left(\delta_S \Omega_k\right) - \psi_k \frac{\partial^2}{\partial \psi^2}\left(\delta_S \Omega_k\right) \\
&= \frac{\partial}{\partial q_k}\frac{\partial}{\partial \psi}\left(\delta_S \Omega_k\right) + \psi_k \frac{\partial}{\partial \psi}\frac{\partial}{\partial \psi}\left(\delta_S \Omega_k\right) - \frac{\partial}{\partial \psi}\frac{\partial}{\partial q_k}\left(\delta_S \Omega_k\right) - \psi_k \frac{\partial^2}{\partial \psi^2}\left(\delta_S \Omega_k\right) = 0\,.
\end{aligned} \tag{H.3}$$

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Appendix I
Proofs of the Algebraic Properties of the Poisson Bracket

Let $f(\boldsymbol{\eta})$, $g(\boldsymbol{\eta})$ and $h(\boldsymbol{\eta})$ be three functions of $\boldsymbol{\eta}$, and let $\alpha,\ \beta\ \in \mathbb{R}$ be two constants. Using the symplectic notation, the Poisson bracket can be written as the inner product

$$\{f, g\} = \left\langle \frac{\partial f}{\partial \boldsymbol{\eta}}, \mathbf{J}\frac{\partial g}{\partial \boldsymbol{\eta}} \right\rangle , \tag{I.1}$$

which satisfy commutativity with its arguments. The Poisson bracket thus satisfies the following properties:

1. *Skew-symmetry and self-commutativity*

$$\{f, g\} = -\{g, f\} . \tag{I.2}$$

Proof Because $\mathbf{J}$ is skew-symmetric,

$$\{f, g\} = -\left\langle \mathbf{J}\frac{\partial f}{\partial \boldsymbol{\eta}}, \frac{\partial g}{\partial \boldsymbol{\eta}} \right\rangle . \tag{I.3}$$

But because of the commutativity of the inner product

$$-\left\langle \mathbf{J}\frac{\partial f}{\partial \boldsymbol{\eta}}, \frac{\partial g}{\partial \boldsymbol{\eta}} \right\rangle = -\left\langle \frac{\partial g}{\partial \boldsymbol{\eta}}, \mathbf{J}\frac{\partial f}{\partial \boldsymbol{\eta}} \right\rangle = -\{g, f\} \tag{I.4}$$

that demonstrates (I.2). The property of self-commutation

$$\{f, f\} = 0 \tag{I.5}$$

follows directly, as $\{f, f\} = -\{f, f\} \Rightarrow \{f, f\} = 0$.

2. *Distributive Property*

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$$\{\alpha f + \beta g, h\} = \alpha\{f, h\} + \beta\{g, h\} \,. \tag{I.6}$$

Proof

$$\begin{aligned}\{\alpha f + \beta g, h\} &= \left\langle \frac{\partial\,(\alpha f + \beta g)}{\partial \boldsymbol{\eta}}, \mathbf{J}\frac{\partial h}{\partial \boldsymbol{\eta}} \right\rangle \\ &= \left\langle \alpha\frac{\partial f}{\partial \boldsymbol{\eta}} + \beta\frac{\partial g}{\partial \boldsymbol{\eta}}, \mathbf{J}\frac{\partial h}{\partial \boldsymbol{\eta}} \right\rangle \\ &= \alpha\left\langle \frac{\partial f}{\partial \boldsymbol{\eta}}, \mathbf{J}\frac{\partial h}{\partial \boldsymbol{\eta}} \right\rangle + \beta\left\langle \frac{\partial g}{\partial \boldsymbol{\eta}}, \mathbf{J}\frac{\partial h}{\partial \boldsymbol{\eta}} \right\rangle \\ &= \alpha\{f, h\} + \beta\{g, h\} \,. \end{aligned} \tag{I.7}$$

3. *Associative property*

$$\{fg, h\} = f\{g, h\} + \{f, h\}g \,. \tag{I.8}$$

Proof

$$\begin{aligned}\{fg, h\} &= \left\langle \frac{\partial (fg)}{\partial \boldsymbol{\eta}}, \mathbf{J}\frac{\partial h}{\partial \boldsymbol{\eta}} \right\rangle \\ &= \left\langle \frac{\partial f}{\partial \boldsymbol{\eta}}g + f\frac{\partial g}{\partial \boldsymbol{\eta}}, \mathbf{J}\frac{\partial h}{\partial \boldsymbol{\eta}} \right\rangle \\ &= f\left\langle \frac{\partial g}{\partial \boldsymbol{\eta}}, \mathbf{J}\frac{\partial h}{\partial \boldsymbol{\eta}} \right\rangle + \left\langle \frac{\partial f}{\partial \boldsymbol{\eta}}, \mathbf{J}\frac{\partial h}{\partial \boldsymbol{\eta}} \right\rangle g \\ &= f\{g, h\} + \{f, h\}g \,. \end{aligned} \tag{I.9}$$

4. *Jacobi identity*

$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0 \,. \tag{I.10}$$

Proof If the last term of (I.10) is expanded using (2.120),

$$\begin{aligned}\{h, \{f, g\}\} &= \left\{ h, \sum_{i=1}^{n}\left(\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} \right) \right\} \\ &= \sum_{i=1}^{n}\left(\left\{ h, \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} \right\} - \left\{ h, \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} \right\} \right) \\ &= \sum_{i=1}^{n}\left(\frac{\partial f}{\partial q_i}\left\{ h, \frac{\partial g}{\partial p_i} \right\} + \frac{\partial g}{\partial p_i}\left\{ h, \frac{\partial f}{\partial q_i} \right\} \right. \\ &\quad \left. - \frac{\partial f}{\partial p_i}\left\{ h, \frac{\partial g}{\partial q_i} \right\} - \frac{\partial g}{\partial q_i}\left\{ h, \frac{\partial f}{\partial p_i} \right\} \right) \,. \end{aligned} \tag{I.11}$$

Using the identity

$$\left\{f, \frac{\partial g}{\partial a}\right\} = \frac{\partial}{\partial a}\{f, g\} - \left\{\frac{\partial f}{\partial a}, g\right\} \tag{I.12}$$

in (I.11) gives

$$\begin{aligned}
\{h, \{f, g\}\} = \sum_{i=1}^{n} \Bigg[& \frac{\partial f}{\partial q_i}\left(\frac{\partial}{\partial p_i}\{h, g\} - \left\{\frac{\partial h}{\partial p_i}, g\right\}\right) + \frac{\partial g}{\partial p_i}\left(\frac{\partial}{\partial q_i}\{h, f\} - \left\{\frac{\partial h}{\partial q_i}, f\right\}\right) \\
& - \frac{\partial f}{\partial p_i}\left(\frac{\partial}{\partial q_i}\{h, g\} - \left\{\frac{\partial h}{\partial q_i}, g\right\}\right) - \frac{\partial g}{\partial q_i}\left(\frac{\partial}{\partial p_i}\{h, f\} - \left\{\frac{\partial h}{\partial p_i}, f\right\}\right)\Bigg] \\
= & -\{f, \{g, h\}\} - \{g, \{h, f\}\} \\
& + \sum_{i=1}^{n}\left[-\frac{\partial f}{\partial q_i}\left\{\frac{\partial h}{\partial p_i}, g\right\} - \frac{\partial g}{\partial p_i}\left\{\frac{\partial h}{\partial q_i}, f\right\}\right. \\
& \left. + \frac{\partial f}{\partial p_i}\left\{\frac{\partial h}{\partial q_i}, g\right\} + \frac{\partial g}{\partial q_i}\left\{\frac{\partial h}{\partial p_i}, f\right\}\right].
\end{aligned} \tag{I.13}$$

Using (I.13) in (I.10) and expanding gives

$$\begin{aligned}
& \{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = \\
& = \sum_{i=1}^{n}\left[-\frac{\partial f}{\partial q_i}\left\{\frac{\partial h}{\partial p_i}, g\right\} - \frac{\partial g}{\partial p_i}\left\{\frac{\partial h}{\partial q_i}, f\right\} + \frac{\partial f}{\partial p_i}\left\{\frac{\partial h}{\partial q_i}, g\right\} + \frac{\partial g}{\partial q_i}\left\{\frac{\partial h}{\partial p_i}, f\right\}\right] \\
& = -\sum_{i,j=1}^{n} \frac{\partial f}{\partial q_i}\frac{\partial^2 h}{\partial p_i \partial q_j}\frac{\partial g}{\partial p_j} + \sum_{i,j=1}^{n} \frac{\partial f}{\partial q_i}\frac{\partial^2 h}{\partial p_i \partial p_j}\frac{\partial g}{\partial q_j} \\
& - \sum_{i,j=1}^{n} \frac{\partial g}{\partial p_i}\frac{\partial^2 h}{\partial q_i \partial q_j}\frac{\partial f}{\partial p_j} + \sum_{i,j=1}^{n} \frac{\partial g}{\partial p_i}\frac{\partial^2 h}{\partial q_i \partial p_j}\frac{\partial f}{\partial q_j} \\
& + \sum_{i,j=1}^{n} \frac{\partial f}{\partial p_i}\frac{\partial^2 h}{\partial q_i \partial q_j}\frac{\partial g}{\partial p_j} - \sum_{i,j=1}^{n} \frac{\partial f}{\partial p_j}\frac{\partial^2 h}{\partial q_i \partial p_j}\frac{\partial g}{\partial q_j} \\
& + \sum_{i,j=1}^{n} \frac{\partial g}{\partial q_i}\frac{\partial^2 h}{\partial p_i \partial q_j}\frac{\partial f}{\partial p_j} - \sum_{i,j=1}^{n} \frac{\partial g}{\partial q_i}\frac{\partial^2 h}{\partial p_i \partial p_j}\frac{\partial f}{\partial q_j}
\end{aligned} \tag{I.14}$$

where, due to the symmetry in the summation pairs i, j, the sum is zero.

Appendix J
Some Identities Concerning the Jacobi Determinant

The 3D Jacobi determinant has been defined in (3.7). By using the same notation, one can prove that

$$\frac{\partial\,(\mathbf{a})}{\partial\,(\mathbf{b})}\frac{\partial\,(\mathbf{b})}{\partial\,(\mathbf{c})} = \frac{\partial\,(\mathbf{a})}{\partial\,(\mathbf{c})}\,. \tag{J.1}$$

In particular, (J.1) implies

$$\frac{\partial\,(\mathbf{a})}{\partial\,(\mathbf{b})}\frac{\partial\,(\mathbf{b})}{\partial\,(\mathbf{a})} = 1\,. \tag{J.2}$$

If $dx/dt = u,\ dy/dt = v,\ dz/dt = w$, then the differentiation rule of a functional determinant yields

$$\frac{\partial}{\partial t}\frac{\partial\,(\mathbf{x})}{\partial\,(\mathbf{a})} = \frac{\partial\,(u, y, z)}{\partial\,(a_1, a_2, a_3)} + \frac{\partial\,(x, v, z)}{\partial\,(a_1, a_2, a_3)} + \frac{\partial\,(x, y, w)}{\partial\,(a_1, a_2, a_3)}\,. \tag{J.3}$$

Moreover, a straightforward computation shows that

$$\frac{\partial}{\partial x}\frac{\partial\,(\mathbf{x})}{\partial\,(\mathbf{a})} = \frac{\partial\,(1, y, z)}{\partial\,(a_1, a_2, a_3)} + \frac{\partial\,(x, 0, z)}{\partial\,(a_1, a_2, a_3)} + \frac{\partial\,(x, y, 0)}{\partial\,(a_1, a_2, a_3)} = 0\,. \tag{J.4}$$

and analogously

$$\frac{\partial}{\partial y}\frac{\partial\,(\mathbf{x})}{\partial\,(\mathbf{a})} = 0\,,\quad \frac{\partial}{\partial z}\frac{\partial\,(\mathbf{x})}{\partial\,(\mathbf{a})} = 0\,. \tag{J.5}$$

From (J.4) and (J.5), one concludes

$$\nabla\frac{\partial\,(\mathbf{x})}{\partial\,(\mathbf{a})} = 0\,. \tag{J.6}$$

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Consider

$$\frac{D}{Dt}\frac{\partial\,(\mathbf{x})}{\partial\,(\mathbf{a})} = \frac{\partial}{\partial t}\frac{\partial\,(\mathbf{x})}{\partial\,(\mathbf{a})} + \mathbf{u}\cdot\nabla\frac{\partial\,(\mathbf{x})}{\partial\,(\mathbf{a})}\;. \tag{J.7}$$

Because of (J.3) and (J.6), using also (3.10), equation (J.7) becomes

$$\begin{aligned}\frac{D}{Dt}\frac{\partial\,(\mathbf{x})}{\partial\,(\mathbf{a})} &= \frac{\partial\,(u,y,z)}{\partial\,(a_1,a_2,a_3)} + \frac{\partial\,(x,v,z)}{\partial\,(a_1,a_2,a_3)} + \frac{\partial\,(x,y,w)}{\partial\,(a_1,a_2,a_3)} \\ &= \frac{\partial\,(\mathbf{x})}{\partial\,(\mathbf{a})}\left[\frac{\partial\,(u,y,z)}{\partial\,(x,y,z)} + \frac{\partial\,(x,v,z)}{\partial\,(x,y,z)} + \frac{\partial\,(x,y,w)}{\partial\,(x,y,z)}\right] \\ &= \frac{\alpha}{\alpha_a}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)\;.\end{aligned}$$

that is to say

$$\frac{D}{Dt}\frac{\partial\,(\mathbf{x})}{\partial\,(\mathbf{a})} = \frac{\alpha}{\alpha_a}\,div\,\mathbf{u}\;, \tag{J.8}$$

which is also called *Euler's relation*.

In a similar way, it is useful also to calculate the variation of the determinant of the Jacobian. To do so, consider the variation (3.46) with boundary conditions (3.47). We thus have

$$\begin{aligned}\delta\frac{\partial(\mathbf{x})}{\partial(\mathbf{a})} &= \frac{\partial(lQ_x,y,z)}{\partial(a_1,a_2,a_3)} + \frac{\partial(x,lQ_y,z)}{\partial(a_1,a_2,a_3)} + \frac{\partial(x,y,lQ_z)}{\partial(a_1,a_2,a_3)} \\ &= \frac{\partial(\mathbf{x})}{\partial(\mathbf{a})}\left[\frac{\partial(lQ_x,y,z)}{\partial(x,y,z)} + \frac{\partial(x,lQ_y,z)}{\partial(x,y,z)} + \frac{\partial(x,y,lQ_z)}{\partial(x,y,z)}\right] \\ &= \frac{\alpha}{\alpha_a}\,div\;(l\mathbf{Q})\;\;.\end{aligned} \tag{J.9}$$

Appendix K
Derivation of (3.131)

Throughout this appendix the sum over repeated indices will be used. Consider the i component of $rot_a\,\mathbf{B}$,

$$[rot_a\,\mathbf{B}]_i = \varepsilon_{ijk}\frac{\partial B_k}{\partial a_j}\ , \tag{K.1}$$

where the Levi-Civita tensor, defined in three dimensions as

$$\varepsilon_{ijk} = \begin{cases} 1 \quad if\ (i,j,k)\ is\ (1,2,3)\ ,\ (2,3,1)\ or\ (3,1,2)\ , \\ -1 \quad if\ (i,j,k)\ is\ (1,3,2)\ ,\ (2,1,3)\ or\ (3,2,1) \\ 0 \qquad\qquad if\ i=j\ or\ i=k\ or\ j=k\ , \end{cases}\ , \tag{K.2}$$

has been used. Notice that the Levi-Civita tensor is antisymmetric. One can use the property

$$\varepsilon_{ijk} = \delta_{im}\delta_{km}\varepsilon_{mjn}\ , \tag{K.3}$$

that, together with

$$\delta_{im} = \frac{\partial a_i}{\partial a_m}\ , \tag{K.4}$$

allows to write (K.1) as

$$\begin{aligned} [rot_a\,\mathbf{B}]_i &= \varepsilon_{ijk}\frac{\partial B_k}{\partial a_j} \\ &= \varepsilon_{mjn}\frac{\partial a_i}{\partial a_m}\frac{\partial B_k}{\partial a_j}\frac{\partial a_k}{\partial a_n} \\ &= \frac{\partial(a_i, B_k, a_k)}{\partial(a_1, a_2, a_3)}\ . \end{aligned} \tag{K.5}$$

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The conservation of potential vorticity follows directly from (K.5). In fact, taking any scalar θ, which is conserved on fluid parcels, (3.122) leads to the conservation of the quantity

$$\Pi_a = (rot_a\, \mathbf{B}) \cdot \nabla_a \theta \; . \tag{K.6}$$

Through the identification of the conserved scalar with the ith label of the fluid, using (K.5) one thus has the conservation of the quantity

$$\begin{aligned} \Pi_i &= \frac{1}{\rho} \frac{\partial(\mathbf{a})}{\partial(\mathbf{x})} \frac{\partial(a_i, B_k, a_k)}{\partial(a_1, a_2, a_3)} \\ &= \frac{1}{\rho} \frac{\partial(a_i, B_k, a_k)}{\partial(x, y, z)} \; . \end{aligned} \tag{K.7}$$

Using again the Levi-Civita tensor, (K.7) can be written as

$$\Pi_i = \frac{1}{\rho} \epsilon_{lmn} \frac{\partial a_i}{\partial x_l} \frac{\partial B_k}{\partial x_m} \frac{\partial a_k}{\partial x_n} \; . \tag{K.8}$$

Using (3.127), the $\partial B_k / \partial x_m$ term can be written as

$$\begin{aligned} \frac{\partial B_k}{\partial x_m} &= \frac{\partial}{\partial x_m} \left[u_o \frac{\partial x_o}{\partial a_k} \right] \\ &= \frac{\partial u_o}{\partial x_m} \frac{\partial x_o}{\partial a_k} + u_o \frac{\partial^2 x_o}{\partial x_m \partial a_k} \; . \end{aligned} \tag{K.9}$$

Using (K.9), (K.8) yields

$$\begin{aligned} \Pi_i &= \frac{1}{\rho} \epsilon_{lmn} \frac{\partial a_i}{\partial x_l} \left[\frac{\partial u_o}{\partial x_m} \frac{\partial x_o}{\partial a_k} \right] \frac{\partial a_k}{\partial x_n} \\ &+ \frac{1}{\rho} \epsilon_{lmn} \frac{\partial a_i}{\partial x_l} \left[u_o \frac{\partial^2 x_o}{\partial x_m \partial a_k} \right] \frac{\partial a_k}{\partial x_n} \; . \end{aligned} \tag{K.10}$$

Due to the antisymmetry of the Levi-Civita tensor, the second term on the r.h.s. of (K.10) is zero, leaving

$$\Pi_i = \frac{1}{\rho} \epsilon_{lmn} \frac{\partial u_n}{\partial x_m} \frac{\partial a_i}{\partial x_l} \; . \tag{K.11}$$

Using the identity

$$\omega_i = [rot\, \mathbf{u}]_i = \epsilon_{ijk} \frac{\partial u_k}{\partial x_j} \; , \tag{K.12}$$

equation (K.11) yields

$$\Pi_i = \frac{1}{\rho}\left(\boldsymbol{\omega} \cdot \nabla a_i\right) \ , \tag{K.13}$$

which, upon the identification $\theta = a_i$, can be recognized as the potential vorticity (K.6).

Appendix L
Scaling the Rotating Shallow Water Lagrangian Density

In order to derive a quasi-geostrophic shallow water model, it is useful to consider how the Lagrangian density (4.67), i.e.

$$L = \frac{1}{2}\rho\left[|\dot{\mathbf{x}}_h|^2 + \dot{\mathbf{x}}_h \cdot (2\boldsymbol{\Omega} \times \mathbf{x}_h) - g\eta\right] , \tag{L.1}$$

can be written in nondimensional form. Recall the nondimensionalization introduced in Sect. 1.9.2, i.e.

$$(x', y') = (x/L,\ y/L) ,\ z' = z/H , \tag{L.2a}$$

$$(u', v') = (u/U,\ v/U) ,\ w' = w/W , \tag{L.2b}$$

$$\eta' = \eta/E , \tag{L.2c}$$

$$2\boldsymbol{\Omega} = f_0\boldsymbol{\Omega}' , \tag{L.2d}$$

where the apex indicates the nondimensional variables and with

$$E = \frac{f_0 U L}{g} , \tag{L.3}$$

because of geostrophy, and where, without loss of generality, the motion has been restricted to the f-plane. Introducing once again the nondimensional Rossby number

$$\varepsilon = \frac{U}{f_0 L} ,$$

the nondimensional form of (L.1) is thus

$$L' = \varepsilon\left|\dot{\mathbf{x}}'_h\right|^2 + \dot{\mathbf{x}}'_h \cdot (\boldsymbol{\Omega}' \times \mathbf{x}'_h) - \eta' , \tag{L.4}$$

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where $L = \mathscr{L} L'$ and $\mathscr{L} = H_0 U L f_0/2$ have been used. Hamilton's principle applied to (L.4) yields thus the nondimensional rotating shallow water equations

$$\varepsilon \ddot{\mathbf{x}}_h' + (2\boldsymbol{\Omega}' \times \dot{\mathbf{x}}_h') = -\nabla_h' \eta' \, , \tag{L.5}$$

which should be compared with (1.196), here written in the f-plane. The nondimensional form of the continuity equation (4.63) is

$$F \frac{\partial \eta'}{\partial t'} + (1 + \varepsilon F \eta') \, div' \, \mathbf{u}' = 0 \, , \tag{L.6}$$

where, once again,

$$F = \frac{(f_0 L)^2}{g H} \tag{L.7}$$

is the Froude number, introduced in Sect. 1.9.2.

From (L.5), it is possible to derive the shallow water quasi-geostrophic equations in the same way as it was done in Chap. 1, i.e. omitting primes and the h subscript, assuming $\varepsilon \ll 1$ it is possible to make an asymptotic expansion of the terms appearing in (L.4)

$$L = L_0 + \varepsilon L_1 + \cdots \, , \tag{L.8a}$$

$$\mathbf{x} = \mathbf{x}_0 + \varepsilon \mathbf{x}_1 + \cdots \, , \tag{L.8b}$$

$$\eta = \eta_0 + \varepsilon \eta_1 + \cdots \, . \tag{L.8c}$$

Notice that, by (4.60), one has

$$\eta_0 = \frac{\partial(a_1, a_2)}{\partial(x_0, y_0)} - H \, . \tag{L.9}$$

At $O(1)$ in ε, i.e. in the limit of vanishingly small Rossby number (L.4) yields

$$L_0 = \dot{\mathbf{x}}_0 \cdot (\boldsymbol{\Omega} \times \mathbf{x}_0) - \eta_0 \, . \tag{L.10}$$

The application of Hamilton's principle to (L.10) yields the geostrophic balance (1.197),

$$2\boldsymbol{\Omega} \times \dot{\mathbf{x}}_0 = -\nabla \eta_0 \, . \tag{L.11}$$

In the same way, expansion to higher orders in ε can be performed.

Remark L.1 Notice that if instead of the f-plane one would have used the β-plane approximation, i.e. if (1.79) and (1.175) would have been used, the same line of reasoning would have held the so-called *planetary geostrophic* equations.