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Franco Strocchi

Gauge Invariance and Weyl-polymer Quantization

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Introduction

The standard formulation of Quantum Mechanics (QM) is based on *Heisenberg-Dirac canonical quantization* of the Heisenberg canonical variables q, p. The *Weyl quantization*, i.e. the canonical commutation relations formulated in terms of the unitary exponentials of the canonical variables (Weyl operators), is usually ignored in the textbooks of QM addressed to physicists. Its use in the mathematically minded presentations of QM is usually motivated by the better behavior and mathematical control of the unitary Weyl operators with respect to the Heisenberg canonical variables, which are necessarily represented by unbounded operators.

The connection between the two quantizations is provided by the Stone-von Neumann theorem, which states their equivalence under general regularity conditions (Chap. 1). This may explain why, from a practical point of view, one may feel satisfied with the Heisenberg-Dirac quantization. The point is that the Dirac-Heisenberg quantization implicitly assumes that all the canonical variables described observables and therefore the regularity of their exponentials (Weyl operators) is required by their existence as (unbounded) operators in the Hilbert representation space.

The regularity condition at the basis of the Stone-von Neumann theorem is standard in the mathematical analysis and classification of Lie group representations and in the quantum mechanical case it amounts to consider the strongly (equivalently weakly) continuous (unitary) representations of the Heisenberg group.

However, for a class of physically interesting systems, especially in connection with quantum gauge theories, it has become apparent that the Dirac-Heisenberg quantization is not compatible with a gauge invariant ground state, only the Weyl quantization being allowed.

In these cases, the inequivalence of the two quantization methods arises by the lack of regularity of the one-parameter groups generated by the Weyl operators, so that the corresponding generators, i.e. the Heisenberg canonical variables, cannot be defined as (self-adjoint) operators in the Hilbert space of states.

The physical reason at the basis of such a *lack of regularity* is that the QM description of a class of physical systems involves canonical variables, not all of which correspond to observable quantities; some of them are introduced for the description of the states, namely of the representations of the algebra of observables, for which purpose only their exponentials are needed to exist as well-defined operators in the Hilbert space of states. Actually, the embedding of the observable algebra $\mathcal{A}$ (of canonical variables) into a larger canonical algebra $\mathcal{F}$, which contains the intertwiners between the inequivalent representations of $\mathcal{A}$, qualifies as a general strategy for a complete description of the system.

Thus, what might at first sight look as an uninteresting singular, if not pathological, case turns out to be crucial for the quantum description of physically interesting systems.

For example, for a particle on a circle, or more generally in a periodic structure, only the periodic functions of the position are observable and there is no compelling physical reason for the existence of the position operator, which would require the regularity of the corresponding Weyl operators, whose role is to act as intertwiners between inequivalent representations of the observables algebra (starting from the ground/vacuum state representation).

In general, this lack of regularity may be related to the existence of a gauge group. Typically, one has the Weyl algebra $\mathcal{A}_W$ generated by the exponentials of the full set of canonical variables needed for the description of the states of the system, but only a subalgebra $\mathcal{A}$ describes observables. Generically, $\mathcal{A}$ has a non-trivial center $\mathcal{Z}$, which generates transformations having the meaning of gauge symmetries (*gauge transformations*). Thus, the algebra $\mathcal{A}_W$ of canonical variables contains both gauge dependent and gauge invariant (i.e. observable) variables.

Clearly, the regularity condition must be satisfied by the exponentials of the observable variables, otherwise the representation is not physical, but there is no physical reason for the regularity condition of the gauge dependent Weyl operators.

Actually, as it shall be discussed in these notes (Chap. 2), the representation of the Weyl algebra by a gauge invariant ground state in general requires the non-regularity of the gauge dependent Weyl operators and implies the impossibility of defining the corresponding generators as well-defined operators in the corresponding Hilbert space (*non-regular representation of the Heisenberg group or of the Weyl algebra*).

Relevant quantum mechanical examples of such a structure are the electron in a periodic potential (*Bloch electron*), the *Quantum Hall electron*, the *particle on a circle*, where the gauge transformations are, respectively, the lattice translations, the magnetic translations and the rotations of 2π.

The general mathematical structure of non-regular realizations of the canonical commutation relations in Weyl form provides also a mathematically consistent solution of the problem of the equivalence of the procedures which interchange quantization and imposition of the constraints on the physical states, e.g. the *Gauss' law* (i.e. the gauge invariance) constraint in *Gauge Quantum Field Theories* (GQFT).

A cheap widespread solution of the quantization problem, compatibly with the existence of gauge invariant states, is to admit non-normalizable states. Non-normalizable state vectors are often used in quantum mechanical calculations as pioneered by Dirac, but their mathematical oddness is harmless, since they are used as simplifying limiting extrapolations of well-defined normalizable vectors (e.g. the plane waves as the limit of narrower and narrower wave packets of momentum). Much more problematic and actually mathematically inconsistent is to consider quantizations built on a *cyclic* non-normalizable vector, typically the ground state, because then *all* the so-obtained vectors are non-normalizable and *all* transition amplitudes are divergent. As discussed in these notes (Chap. 3), a much more satisfactory solution of this problem is Weyl quantization with non-regular representations.

The non-regular Weyl representations for the quantization of systems with a gauge symmetry exhibit the following characteristic structures, which play an important role in the analysis of the vacuum structure in Quantum Chromodynamics (QCD) and do not have a counterpart in the Dirac-Heisenberg canonical quantization:

i) a *gauge invariance constraint* (typically a Gauss' law constraint) in operator form compatible with canonical Weyl quantization
ii) *superselected charges* defined by the center of the observable algebra
iii) *gauge invariant ground states*, defining inequivalent representations of the observable algebra, labeled by the spectrum of the superselected charges (the strict analog of the so-called θ sectors of QCD)
iv) *absence of "Goldstone states"* associated to the spontaneous breaking of symmetries conjugated to the gauge transformations (like the chiral symmetry in QCD).

Such features are not peculiar of QCD and also appear in all (finite dimensional) QM models with a gauge symmetry generated by the center of the observable algebra. In our opinion, the realization of such general structures and their very clear and simple realization in (finite dimensional) QM mechanical models, fully under control, discussed in Chap. 3, sheds light on the more difficult infinite dimensional Gauge Quantum Field Theory models.

In particular, the occurrence and relevance of non-regular representations of field algebras is exemplified by the massless scalar field in two spacetime dimensions and by the (positive) realization of the temporal gauge in Quantum Electrodynamics (QED) (Chap. 4).

Non-regular Weyl quantization also provides a strategy for a derivation of the vacuum structure and chiral symmetry breaking in QCD in a more acceptable and convincing mathematical setting, as discussed in Chap. 4, Sect. 4. In this respect, these notes may be regarded as a supplement and a mathematical glossary to the standard somewhat heuristic arguments about the vacuum structure in QCD and the $U(1)$ axial symmetry breaking.

Non-regular representations arise also in quantizations of *diffeomorphism covariant theories* with a diffeomorphism invariant ground state, as discussed in Chap. 5. They play a crucial role in string quantization and in Loop Quantum Gravity, where they have been advocated under the name of *polymer quantizations*. Their features (like the occurrence of non-separable Hilbert spaces, the impossibility of defining gauge dependent fields as Hilbert space operators, etc.) should not be regarded as serious mathematical difficulties or physical oddnesses, the quantum mechanical models (some of which very familiar) providing well-sounded prototypes of them.

The physical relevance of non-regular representations of the Heisenberg group raises the problem of their classification, i.e. a generalization of the classical Stone-von-Neumann theorem, which characterizes the regular ones.

Such a generalization may be obtained by exploiting the simple form of the Gelfand spectrum of the maximal abelian subalgebra $\mathcal{A}_Z$ of the Weyl algebra generated by the pairs $U_i(-2\pi/l)$, $V_i(\lambda)$, $i = 1, \ldots\ d$ (d the space dimensions), formally corresponding to the exponentials $\exp -i(2\pi/\lambda)q_i$, $\exp i\lambda\, p_i$. Such an algebra is called the *Zak algebra* and its Gelfand spectrum Σ is given by d copies of the two-dimensional torus $\Sigma = (\mathbf{T}^2)^d$.

Then, one may prove that all the representations of the Weyl algebra which are *spectrally multiplicity free* as representations of its Zak algebra (a condition which generalizes irreducibility) and are *strongly measurable* (a condition which replaces regularity in non-separable spaces) are unitarily equivalent to a representation of the Weyl algebra of the same form of the standard Schrödinger representation on $L^2(\Sigma, d\mu)$, with $d\mu$ a (positive) translationally invariant Borel measure, which reduces to the Lebesgue measure iff the regularity condition is satisfied (Chap. 6).

The conditions which yield such a classification are satisfied by all the non-regular representations of physical interest discussed in Chaps. 2–5 below; thus, they all have the above form, with a corresponding Borel measure on $(\mathbf{T}^2)^d$.

Technical parts, which may be skipped in a first reading, are marked with a *.

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Chapter 1
Heisenberg Quantization and Weyl Quantization

1 Heisenberg and Weyl Quantizations

The standard formulation of quantum mechanics relies on the so-called canonical quantization prescriptions at the basis of Dirac formulation.[1] The starting point is the identification of the canonical variables q, p, which in the classical case describe the configurations of the system; then the quantization procedure amounts to replacing the classical canonical Poisson brackets by commutators (in units in which $\hbar = 1$)

$$[q_i, p_j] = i\delta_{ij}, \quad [q_i, q_j] = 0 = [p_i, p_j], \quad i, j = 1, \ldots, s, \tag{1.1.1}$$

hereafter called *canonical commutation relations (CCR) in Heisenberg form*.

The next step in the formulation of quantum mechanics is the choice of a representation of the canonical variables q, p as self-adjoint operators in the Hilbert space $\mathcal{H}$ of states and the standard choice is the *Schrödinger representation*, according to which $\mathcal{H}$ is given by the square integrable (wave) functions, $\mathcal{H} = L^2(\mathbf{R}^s)$, with q the multiplication operator and p the generator of translations:

$$(q\psi)(x) = x\psi(x), \qquad (p\psi)(x) = -i\partial_x \psi(x),$$

for all ψ in a suitable dense domain D (see below).[2]

The relevant physical and mathematical question is whether such a representation covers all the physically interesting cases and more generally what is the status of other possibly existing representations.

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[1]P.A.M. Dirac, *The principles of Quantum mechanics*, Oxford University Press 1986, Chap. IV.

[2]The unitary equivalent so-called Heisenberg representation is obtained by (the unitary) Fourier transform: $(q\tilde{\psi})(k) = i\partial_k \tilde{\psi}(k)$, $(p\tilde{\psi})(k) = k\tilde{\psi}(k)$.

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The implicit physical assumption underlying Heisenberg quantization is that the canonical variables q, p, promoted to Hilbert space operators, have a direct physical interpretation, i.e. they describe observable quantities. This implies that they must be represented by (necessarily unbounded) self-adjoint operators. In general, one cannot require that they have the same domain of self-adjointness and in order to give a meaning to the commutation relations Eq. (1.1.1), q and p should at least have a common dense domain D of essential self-adjointness, such that

$$q_j D \subset D, \; p_j D \subset D, \quad q_i p_j - p_j q_i = i\delta_{ij}\mathbf{1}, \quad \text{on} \quad D. \tag{1.1.2}$$

Without further conditions the commutation relations in the form of Eq. (1.1.2) admit a plenty of representations which are not unitarily equivalent to the Schrödinger representation. Thus, one is facing the mathematical and physical problem of focusing the conditions which select the standard Schrödinger choice; this will be the object of the present section.

To this purpose we recall the following basic concepts.

The (real) *Heisenberg Lie algebra*,, henceforth denoted by $\mathcal{L}^s_H$ is the Lie algebra with generators $\tilde{Q}_i, \tilde{P}_i, \tilde{Z}$, $i = 1, \ldots, s$, and Lie brackets $[\tilde{Q}_i, \tilde{P}_j] = \delta_{ij}\tilde{Z}$, all other Lie brackets vanishing. The relation with the canonical variables q_i, p_i is given by $q_j \to -i\tilde{Q}_j, p_j \to -i\tilde{P}_j, \mathbf{1} \to -i\tilde{Z}$.

The non-compact *Heisenberg group* $\mathbf{H}_s$ is obtained by the exponential map $(\alpha, \beta, \gamma) = e^{\alpha\tilde{Q}+\beta\tilde{P}+\gamma\tilde{Z}}$, $\alpha, \beta \in \mathbf{R}^s$, $\gamma \in \mathbf{R}$, so that the group law is

$$(\alpha, \beta, \gamma)(\alpha', \beta', \gamma') = (\alpha + \alpha', \beta + \beta', \gamma + \gamma' + \tfrac{1}{2}(\alpha\beta' - \alpha'\beta)).$$

The so defined Heisenberg group has a non-trivial center, generated by $(0, 0, z)$, so that its irreducible representations are not faithful. Hence, it is convenient to consider the reduced Heisenberg group, obtained by restricting z to be a real number modulo 2π; in the following for simplicity, the reduced group shall still be referred to as the Heisenberg group.[3] Clearly, it is generated by the elements $U(\alpha) \equiv (\alpha, 0, 0)$, $V(\beta) \equiv (0, \beta, 0)$, $e^{iz}\mathbf{1} \equiv (0, 0, z)$, z being a real number modulo 2π.

The *Weyl algebra*, also briefly called *CCR algebra* is the algebra generated by the elements $U(\alpha), V(\beta), \alpha, \beta \in \mathbf{R}^s$, with product rules provided by the group laws of the Heisenberg group

$$U(\alpha)\, V(\beta) = V(\beta)\, U(\alpha) e^{-i\alpha\beta}, \tag{1.1.3}$$

$$U(\alpha)U(\alpha') = U(\alpha + \alpha'), \quad V(\beta)V(\beta') = V(\beta + \beta'), \tag{1.1.4}$$

[3]For more details on the Heisenberg group, see G.B. Folland, *Harmonic Analysis in Phase Space*, Princeton University Press 1989, esp. Chap. 1.

also called *canonical commutation relations in Weyl form* or briefly the *Weyl relations*. Clearly, by the Weyl relations, all monomials of U's and V's reduce to products of the form $UVe^{iz}\mathbf{1}$.

One may define a * operation by $U(\alpha)^* = U(-\alpha)$, $V(\beta)^* = V(-\beta)$, and there is a unique norm $|| \ ||$, with $||U(\alpha)|| = ||V(\beta)|| = 1$ and satisfying $||A^* A|| = ||A||^2$, $\forall A$.[4] The norm closure of the Weyl algebra with such a norm defines the *Weyl C^*-algebra $\mathcal{A}_W$*.[5]

Definition 1.1 *A* **representation** *of a (real) Lie algebra L in a Hilbert space $\mathcal{H}$ is a homomorphism π of L into a set of linear operators in $\mathcal{H}$ having a common invariant dense domain D. A representation is therefore identified by the pair $(\pi, D \subseteq \mathcal{H})$.*

The representation is said to be **self-adjoint** *if $\forall X \in L$, $i\pi(X)$ is essentially self-adjoint on D.*

A representation π of a Lie group G in a Hilbert space $\mathcal{H}$ is a homomorphism of G into a set of bounded operators in $\mathcal{H}$, i.e. $\pi(g_1 g_2) = \pi(g_1)\,\pi(g_2)$, $\pi(e) = \mathbf{1}$, $\forall g_1, g_2 \in G$, e denoting the identity.

The representation of G is called **unitary** *if $\forall g \in G$, $\pi(g)$ is a unitary operator.*

A unitary representation of G is called **regular** *if the representatives of the one-parameter subgroups of G are continuous in the group parameters with respect to the strong Hilbert space topology.*

A self-adjoint representation of the Heisenberg Lie algebra defines what shall be called a **Heisenberg quantization**. *A representation of the Weyl algebra (equivalently a unitary representation of the Heisenberg group) defines a* **Weyl quantization**, *which is called* **regular** *if so is the defining representation.*[6]

A regular Weyl quantization defines a Heisenberg quantization, but the converse is not true, without further (mathematical) conditions discussed in the next Section.

2 * From Heisenberg to Weyl Quantization

The quantization advocated by Weyl in his pioneering work on quantum mechanics and adopted in the mathematical physics literature on canonical quantization, uses the commutation relations in Weyl form and considers the unitary representations of

[4] See J. Slawny, Comm. Math. Phys. **24**, 151 (1971); J. Manuceau, M Sirugue, D. Testard and A. Verbeure, Comm. Math. Phys. **32**, 231 (1973).

[5] We recall that a C^*-algebra $\mathcal{A}$ is a complex Banach algebra with an involution * such that the norm satisfies $||A^* A|| = ||A||^2$, $\forall A \in \mathcal{A}$.

[6] This means that for any pair of Hilbert space vectors $\Psi, \Phi \in \mathcal{H}_\pi$, the matrix elements $F_{\Psi\Phi}(\alpha, \beta) \equiv (\Psi, \pi(U(\alpha)\,V(\beta))\,\Phi)$, of the representatives of the group elements, are continuous functions of α, β.

the Heisenberg group.[7] Clearly, Weyl strategy can be regarded as a regularized (and actually more general) version of Heisenberg quantization, since it is formulated in terms of the (bounded) Weyl variables U, V. Moreover, by Stone-Von-Neumann (SvN) theorem, the requirement of regularity of the unitary representations of the commutation relations in Weyl form, uniquely leads to Schrödinger quantum mechanics (see below).

Furthermore one has

Theorem 2.1 *Let $U(\alpha), V(\beta)$, $\alpha, \beta \in \mathbf{R}^n$ be weakly continuous unitary groups satisfying the Weyl relations (1.1.3), (1.1.4), in a separable Hilbert space $\mathcal{H}$, then there is a dense domain D of essential self-adjointness for the generators q, p satisfying Eqs. (1.1.2)*

The proof of Theorem 1.2 follows from Stone-von Neumann uniqueness theorem on the unitary equivalence of all weakly continuous irreducible unitary representations of the Weyl relations (see below). A direct proof exploits the construction and properties of the dense Gårding domain D, defined as the linear space spanned by the vectors

$$W(f)\psi \equiv \int d\alpha\, d\beta\, f(\alpha,\beta) U(\alpha)\, V(\beta)\, \psi$$

with f of the form $f(\alpha,\beta) = \alpha^n\, \beta^m\, e^{-a(\alpha^2+\beta^2)}$, $m, n \in \mathbf{N}$, $a > 0$. The argument is essentially the same as for the proof of Stone's theorem on weakly continuous groups of unitary operators, by which weak continuity is equivalent to the condition that the generators are represented by self-adjoint operators.[8]

Thus, strongly continuous unitary representations of the Heisenberg group define self-adjoint representations of the Heisenberg Lie algebra. However, not all the representations of the Heisenberg Lie algebra are obtained in this way. In fact, even if the Heisenberg canonical variables q, p are represented by essentially self-adjoint operators on a common invariant dense domain D, on which they satisfy the canonical commutation relations (1.1.2), they need not to exponentiate to the Heisenberg group, nor be equivalent to the Schrödinger representation.

An interesting physical question is to clarify the physical requirements which lead from Heisenberg quantization to *Schrödinger quantum mechanics*. For this purpose, as we shall see, two properties are at issue: the exponentiation of the Heisenberg variables to unitary groups satisfying the Weyl relations and the weak continuity of such unitary groups.

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[7] H. Weyl, *The Theory of Groups and Quantum Mechanics*, Dover 1931; W. Thirring, *A Course in Mathematical Physics, Vol. 3, Quantum Mechanics of Atoms and Molecules*, Springer 1981. For an extensive and excellent analysis of Weyl quantization see D.A. Dubin, M.A. Hennings and T.B. Smith, *Mathematical aspects of Weyl quantization and phase*, World Scientific 2000.

[8] See M. Reed and B. Simon, *Methods of Modern Mathematical Physics, Vol. I,*, Academic Press 1972, Sect. VIII.4, and *Vol. II*, Problem 30, p. 341.

From a mathematical point of view the property that the Heisenberg variables q, p exponentiate to one-parameter weakly continuous unitary groups is equivalent to their self-adjointness (by Stone's theorem); therefore, it is implied by the physical requirement that q, p describe (unbounded) observables and are therefore described by self-adjoint operators in the Hilbert space of states.

The strong continuity is a standard condition in the theory of representations of Lie groups and by von Neumann theorem, for representations π in separable Hilbert spaces, it is equivalent to the weak measurability of $\pi(U(\alpha))$, $\pi(V(\beta))$.[9] Therefore, it is hard to think of a weaker condition, if the generators of the Heisenberg group are required to be observable variables.

Given for granted that the Heisenberg variables must at least satisfy Eqs. (1.1.2), the Weyl relations for the corresponding unitary groups may be derived by the condition that on the common dense domain D also the quadratic Nelson operator $q^2 + p^2$ associated to the Heisenberg Lie algebra is essentially self-adjoint. By Nelson-Stinespring theorems this implies that on D all the polynomials of q or of p are essentially self-adjoint on D, i.e. all such polynomials are uniquely defined as self–adjoint operators.[10] A physical motivation for such a condition is the selection of those representations of the Heisenberg canonical variables which guarantee the construction of at least the polynomial functions separately of q and of p, as self-adjoint operators.[11] Under such conditions the Weyl relations follow from Eqs. (1.1.2) by the Rellich-Dixmier theorem.

Theorem 2.2 *If q and p satisfy the Heisenberg relations in the form (1.1.2) and also $q^2 + p^2$ is essentially self-adjoint of D, then the unitary groups defined by q and p satisfy the Weyl relations.*

Proof Under the same hypotheses Dixmier proved that the action of q and p is unitarily equivalent to that of a direct sum of the Schrödinger representations and for all of them the Weyl relations hold.[12]

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[9] See e.g. M. Reed and B. Simon, *Methods of Modern Mathematical Physics*, Vol. I, Academic Press 1979, Chap. VIII, Sect. 4, Theorem VIII.9.

[10] For the Nelson-Stinespring theorems see, e.g., A.O. Barut and R. Raczka, *Theory of Group Representations and Applications*, World Scientific 1986, Chap. 12, Sect. 2.

[11] One cannot require that all the elements of the enveloping algebra of the Heisenberg Lie algebra, namely the polynomial algebra generated by the Heisenberg canonical variables q, p, are essentially self-adjoint on D.

[12] For the proof of Dixmier theorem and more generally for the discussion of mathematical conditions on the generators of the Heisenberg Lie algebra, which ensure their exponentiation to the Heisenberg group, see C.R. Putnam, *Commutation Properties of Hilbert Space Operators and Related Topics*, Springer 1967, esp. Chap. IV; P.E.T Jorgensen and R.T. Moore, *Operator Commutation Relations*, Reidel 1984.

3 Stone-von Neumann Theorem and Schrödinger Quantum Mechanics

Weyl quantization allows for a simple formulation of the condition which leads to Schrödinger quantum mechanics.

Definition 3.1 *A unitary representation π of a Lie group G in a Hilbert space $\mathcal{H}$ is said to be* **irreducible** *if the only closed invariant subspaces are $\{0\}$ and $\mathcal{H}$.*

Proposition 3.2 *Given a unitary irreducible representation π of a Lie group G in a Hilbert space $\mathcal{H}$, let $\mathcal{A}_G$ denote the algebra generated by (complex) linear combinations and products of the operators $\pi(g)$, $g \in G$. Then any vector $\Psi \in \mathcal{H}$ is cyclic for $\mathcal{A}_G$, i.e. $\overline{\mathcal{A}_G \Psi} = \mathcal{H}$.*

Proof Clearly, denoting by $*$ the Hilbert space adjoint one has $\mathcal{A}_G^* = \mathcal{A}_G$ and $\mathcal{H}_1 \equiv \overline{\mathcal{A}_G \Psi}$ is a closed invariant subspace.

Theorem 3.3 (Stone-von Neumann Uniqueness Theorem) *All unitary regular irreducible representations of the canonical commutation relations in Weyl form are unitarily equivalent to the Schrödinger representation:*

$$(U(\alpha)\,\psi)(x) = e^{i\alpha x}\psi(x), \quad (V(\beta)\psi)(x) = \psi(x+\beta), \quad \psi \in L^2(\mathbf{R}^n). \tag{1.3.1}$$

Proof Since, by the Weyl relations the monomials of elements of the form $W(\alpha, \beta) = e^{i\alpha\beta}\, U(\alpha)\, V(\beta)$ reduce to an element of the same form, a unitary irreducible representation is completely determined, up to isometries, by the expectations of the $W(\alpha, \beta)$'s on a (cyclic) vector Ψ. In fact, if π, π' are two unitary irreducible representations in $\mathcal{H}, \mathcal{H}'$, respectively, and for a pair of (cyclic) vectors $\Psi \in \mathcal{H}$, $\Psi' \in \mathcal{H}'$, one has $(\Psi, \pi(A_G)\Psi) = (\Psi', \mathcal{A}_G\, \Psi')$, then the mapping $U : \pi(A)\Psi \to \pi'(A)\Psi'$, $\forall A \in \mathcal{A}_G$, and its inverse are densely defined and preserve the scalar products. Then, U defines an isometry between the two representations.

Now, for any unitary regular representation π, $\pi(W(\alpha, \beta))$ is a continuous bounded operator-valued function, so that the integral

$$P \equiv (1/2\pi) \int d\alpha\, d\beta\, e^{-(\alpha^2+\beta^2)/4}\, \pi(W(\alpha, \beta)) = P^*, \tag{1.3.2}$$

exists and defines a bounded operator. Furthermore, since

$$e^{-(\alpha^2+\beta^2)/4}\pi(W(\alpha, \beta)) = \int d\gamma d\delta\, e^{-i(\alpha\delta+\beta\gamma)}\, \pi(W(-\gamma, \delta))P\pi(W(\gamma, -\delta)),$$

P cannot vanish. Finally, by Gaussian integrations one proves that

$$P\, \pi(W(\alpha, \beta))\, P = e^{-(\alpha^2+\beta^2)/4}\, P. \tag{1.3.3}$$

Therefore, P is a non-trivial projection and there exists at least a vector $\Psi \in P\mathcal{H}$ such that $(\Psi, W(\alpha, \beta)\,\Psi) = e^{-(\alpha^2+\beta^2)/4}$. Hence, all unitary irreducible regular representation are isomorphic and it is not difficult to show that the Schrödinger representation (1.3.1) is unitary, regular and irreducible.[13]

Stone-von Neumann theorem plays a very important role for the foundations of quantum mechanics, since it clarifies the relation between Heisenberg quantization and Schrödinger wave mechanics.

The regularity conditions at the basis of Stone-von Neumann theorem were regarded so natural and harmless that the problem of a possible inequivalence between Heisenberg and Schrödinger quantization is not even mentioned in most textbooks on quantum mechanics. In fact, the problem of existence of inequivalent representations of the canonical commutation relations has since then been regarded to arise only in the case of infinite degrees of freedom.

Summarizing, the following relations emerge:

1. **Heisenberg quantization** at the basis of the standard presentation of quantum mechanics requires the addition of some technical, yet very reasonable, condition in order to lead to Schrödinger quantum mechanics (see e.g. Theorem 2.1).

   From a physical point of view, Heisenberg quantization applies to systems described by canonical variables which have the interpretation of observable quantities. As we shall see in the following Chaps. 2, 3, this innocent looking condition is not satisfied by a class of interesting physical systems, typically those described by canonical variables with a gauge symmetry; in these cases, the requirement of a gauge invariant ground state excludes Heisenberg quantization.
2. **Weyl quantization** is more general than Heisenberg quantization and, in fact, applies also to canonical quantum systems with a gauge symmetry and/or with strong delocalization (see the following Chapters). The regularity condition provides the strict link between Weyl quantization and Schrödinger quantum mechanics; such regularity may fail for the Weyl exponentials of gauge dependent canonical variables, as it happens in the case of representations defined by a gauge invariant state.
3. **Gauge dependent canonical variables**. The above structure suggests that an important point for the quantization problem is the distinction between canonical variables which have the meaning of observable quantities and those which are instrumental for the description of the states, but whose corresponding operators do not belong to the algebra of observables. This phenomenon is typically associated to the presence of a gauge symmetry and the non-observable canonical variables are not gauge invariant.

For example, the Weyl C^*-algebra generated by the canonical variables may be taken as the C^*-algebra of observables in the case of N distinguishable particles, but

[13] See, e.g., F. Strocchi, *An Introduction to the Mathematical Structure of Quantum Mechanics*, 2nd expanded edition, World Scientific 2010, p. 63; hereafter this book will sometimes be referred to as Strocchi (2010).

this is no longer the case for N identical particles, characterized by presence of the gauge group of permutations.

These considerations suggest that the basic ingredient for description of quantum systems is the **C^*-algebra of observables** and the quantization problem, i.e. the identification of the quantum states of the system, reduces to the analysis of the representations of its C^*-algebra of observables.[14]

Such a structure is already present, actually in a stronger form, in one of the Dirac-von Neumann axioms, where it is assumed that the observables are described by *the* set of self-adjoint operators in the Hilbert space $\mathcal{H}$ of vector states, so that they generate the C^*-algebra of *all* bounded operators in $\mathcal{H}$.

As discussed in the references of footnote 14, the C^*-algebraic structure of the observables is enough for deriving the other Dirac-von Neumann axioms: (i) the description of the states by Hilbert space vectors follows from the Gelfand-Naimark-Segal representation theorem, (ii) the representation of the observables by Hilbert space operators follows from Gelfand-Naimark theorem on the characterization of abstract C^*-algebras, (iii) finally, by Stone-von Neumann theorem, Schrödinger quantum mechanics follows from Heisenberg uncertainty relations codified by the canonical commutation relations which define the non-abelian C^*-algebraic structure of $\mathcal{A}_W$.

The formulation of quantum mechanics in terms of the algebra of observables allows for a simple description of the so-called *superselection rules*, which in the conventional formulation correspond to the existence of quantum numbers commuting with all observables. In the C^*-algebraic formulation, superselection rules are defined by the existence of inequivalent physically acceptable representations of the observable algebra.

Therefore, in the cases in which the algebra of observables is given by the Weyl algebra, so that the regularity of the representations of the Weyl algebra is physically motivated and may be taken as a criterion of physical acceptability, Stone-von Neumann uniqueness theorem implies that there are no superselection rules; this is the case of quantum systems of N distinguishable particles.

In view of the relevance of the C^*-algebraic structure of the observables and in order to provide a simple dictionary for the discussion of the following Chapters we recall a few basic facts[15] about representations of C^*-algebras, which in the following shall always be assumed to be *unital*, i.e. to have an identity $\mathbf{1}$.

A *state* ω of a C^* algebra $\mathcal{A}$ is a positive linear functional on $\mathcal{A}$, i.e. $\forall A,\, B \in \mathcal{A}$, $\omega(A) \in \mathbf{C}$, $\omega(A + B) = \omega(A) + \omega(B)$, $\omega(A^* A) \geq 0$. A positive functional is necessarily continuous: $|\omega(A)| \leq ||A||\, \omega(\mathbf{1})$. Without loss of generality, one can assume that the states are normalized, i.e. $\omega(\mathbf{1}) = 1$.

[14]For a presentation of quantum mechanics based on the physically motivated C^*-algebraic structure of the observables, see F. Strocchi, *An Introduction to the Mathematical Structure of Quantum mechanics*, 2nd expanded ed. World Scientific 2010; Eur. Phys. J. Plus (2012), **127**:12.

[15]For a more extended account see e.g. Strocchi (2010).

A state is *pure* if it cannot be written as a convex linear combination of two other states.

A *representation* π of a C^*-algebra $\mathcal{A}$ is a homomorphisms of $\mathcal{A}$ into a C^*-subalgebra of the algebra $\mathcal{B}(\mathcal{H})$ of bounded linear operators on a Hilbert space $\mathcal{H}$, i.e. $\pi(A+B) = \pi(A) + \pi(B)$, $\pi(AB) = \pi(A)\,\pi(B)$, $\pi(\mathcal{A})^* = \pi(A^*)$. It follows that $||\pi(A)|| \leq ||A||$.

A representation π is faithful if it is an isomorphism.

Theorem 3.4 (GNS) *Given a state ω on a C^*-algebra $\mathcal{A}$, there exists a representation π_ω on a Hilbert space $\mathcal{H}_\omega$ with the property that $\mathcal{H}_\omega$ contains a cyclic vector Ψ_ω, such that $\forall A \in \mathcal{A}$,*

$$\omega(A) = (\Psi_\omega, \pi_\omega(A)\,\Psi_\omega). \tag{1.3.4}$$

Any other representation, π'_ω in a Hilbert space $\mathcal{H}'_\omega$ with a cyclic vector Ψ'_ω satisfying Eq. (1.3.4), i.e. $(\Psi'_\omega, \pi'_\omega(A)\,\Psi'_\omega) = \omega(A)$, is unitarily equivalent to π_ω.

Theorem 3.5 *The GNS representation defined by a state ω is irreducible if and only if ω is pure.*

Theorem 3.6 (Gelfand-Naimark) *A C^*-algebra $\mathcal{A}$, with identity, is isomorphic to a C^*-algebra of (bounded) operators in a Hilbert space.*

A *multiplicative linear functional* m on an abelian (unital) C^*-algebra $\mathcal{A}$, is a homomorphism of $\mathcal{A}$ into the set **C** (of complex numbers), i.e. a mapping which preserves all the algebraic relations:

$$m(AB) = m(A)\,m(B), \quad m(A+B) = m(A) + m(B).$$

Clearly, if m is not the trivial homomorphism, $m(\mathbf{1}) = 1$.

The *Gelfand spectrum* of an abelian (unital) C^*-algebra $\mathcal{A}$ is the set $\Sigma(\mathcal{A})$ of its multiplicative linear functionals.

Theorem 3.7 (Gelfand) *An abelian (unital) C^*-algebra $\mathcal{A}$ is isomorphic to the C^*-algebra of continuous functions on a compact Hausdorff topological space, which is its Gelfand spectrum $\Sigma(\mathcal{A})$, with the topology induced by the weak* topology.*

For a handy account of these notions see Strocchi (2010).

Chapter 2
Delocalization, Gauge Invariance and Non-regular Representations

1 Delocalization and Gauge Invariance

As discussed in the previous chapter, the physical criterium of regularity of the unitary representations of the Heisenberg group corresponds to the interpretation of the generators q, p as (unbounded) observable variables, so that they must be described by self-adjoint operators. This implies that the states of the system can be described in terms of L^2 wave functions on the spectrum of the position q (or, equivalently, of the momentum p). This amounts to assume that the states of the system have good *localization* properties. The aim of this section is to provide evidence that this innocent looking assumption cannot be adopted for the quantum description of some interesting physical systems.

The physical reason for the lack of regularity is that some generators of the Heisenberg group may not correspond to observables and therefore they need not be described by self-adjoint operators.

A typical example is when the configurations of the quantum systems cannot have L^2 localization. For example, for a quantum particle (typically an electron) in a periodic potential (*Bloch electron*), by Bloch theorem the energy eigenstates, in particular the ground state, are described by quasi periodic wave function, which cannot belong to L^2.

This means that in the representations defined by the Bloch states the "position" variable q does not exist as a (densely defined) self-adjoint operator and therefore the corresponding representation of the Heisenberg group is not regular.

Another class of examples arises if one ask for a quantization of a system with a ground state characterized by the property of being the eigenstate of a canonical observable with continuous spectrum. Then, the unitary one-parameter group generated by the conjugated canonical variable cannot be weakly continuous

F. Strocchi, *Gauge Invariance and Weyl-polymer Quantization*,
Lecture Notes in Physics 904, DOI 10.1007/978-3-319-17695-6_2

and one has to consider non-regular representations (physically interesting models shall be discussed below).[1]

Quite generally, in order to describe the set of physically relevant states of some physical systems, it is convenient to introduce a canonical algebra $\mathcal{A}_W$ larger than the observable algebra $\mathcal{A}$. Correspondingly, the Heisenberg group G_H generated by the larger algebra $\mathcal{A}_W$ contains as a proper subgroup the group G_{obs}, called the *Heisenberg observable subgroup*. generated by $\mathcal{A}$.

Such a structure is particularly interesting if G_{obs} has a non-trivial center $\mathcal{Z}$. The physical origin and interpretation of such a structure is that the center $\mathcal{Z}$ of G_{obs} defines *gauge transformations*, and the generators of $\mathcal{Z}$ have the meaning of *superselected charges*, since they commute with the observable algebra.

The physical consequence is that the observable algebra has inequivalent representations labeled by the spectrum of $\mathcal{Z}$, called superselected sectors and the variables conjugated to the elements of $\mathcal{Z}$ describe *charge raising/lowering* operators.

The physically motivated condition that one is interested in gauge invariant states implies that they must be invariant under $\mathcal{Z}$, i.e. be eigenstates of the generators of $\mathcal{Z}$. Hence, in such representations the one-parameter groups generated by the variables conjugated to $\mathcal{Z}$ cannot be weakly continuous.

A very simple example is the *two body problem*, if one declares that one does not observe the position of the center of mass. Then, the Heisenberg group generated by the canonical variables q_1, q_2, p_1, p_2 contains as observable subgroup, G_{obs}, the group generated by

$$q = q_1 - q_2, \quad p = (m_2 p_1 - m_1 p_2)/(m_1 + m_2), \quad P = p_1 + p_2.$$

The gauge invariant states are the eigenstates of P; in particular, the ground state of the Hamiltonian

$$H = P^2/2M + p^2/2\mu + V(q_1 - q_2), \quad M \equiv m_1 + m_2, \quad \mu \equiv m_1\, m_2/M,$$

corresponds to the zero eigenvalue of P. Then, the GNS representation of the (twelve dimensional) Heisenberg group defined by such ground state is not regular.

As discussed in the Introduction, the cheap widespread solution of admitting non-normalizable states is mathematically unacceptable, since a non-normalizable cyclic ground state implies that *all* transition amplitudes are divergent.

Actually, for the gauge invariant quantization of gauge models, as the examples mentioned above, rather than considering the fake escape of non-normalizable state

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[1]For a mathematical analysis of non-regular representations of the Weyl algebra see the pioneering paper by R. Baume, J. Manuceau, A. Pellet and M. Sirugue, Comm. Math Phys. **38**, 29 (1974); a systematic analysis from the point of view of gauge quantum models is given by F. Acerbi, G. Morchio and F. Strocchi, Jour, Math. Phys. **34**, 899 (1992); Lett. Math. Phys. **27**, 1 (1993); Lett. Math. Phys. **26**, 13 (1992).

vectors, one should better consider non-regular representations of the Heisenberg group, or equivalently of the Weyl algebra. This leads to quantizations in which only the exponentials of some canonical variables (hereafter called singular variables, typically those which are not gauge invariant) can be represented, but not the variables of which they are the formal exponentials.

Rather than working with mathematically inconsistent non-normalizable state vectors one should use non-regular representations built on normalizable cyclic vectors, which yield well defined expectations for the regular canonical variables (typically gauge invariant variables) and (only) for the exponentials of the non-regular ones.

The general lesson from the examples mentioned above is that the standard canonical quantization in terms of the Heisenberg canonical variables q, p, may not always be possible, since the canonical variables do not always have a direct physical interpretation and their spectrum may not be observable. A safer and more general quantization is that done in terms of the Weyl operators, whose existence is related to the much milder requirement of existence of translations and boosts.

2 The Representation Defined by a Translationally Invariant State

We consider for simplicity the one-dimensional case. The group of space translations α_β, $\beta \in \mathbf{R}$, is described by the one-parameter group $V(\beta)$, and a state ω_0 on $\mathcal{A}_W$ is translationally invariant if

$$\omega_0(\alpha_\beta(A)) = \omega_0(V(\beta)AV(\beta)^*) = \omega_0(A), \;\; \forall A \in \mathcal{A}_W. \tag{2.2.1}$$

An interesting problem is to characterize the GNS representation π_0 defined by ω_0. As we shall see below, such a representation occurs in the quantization of several physical systems and it is also of interest for analogies between quantum mechanical and gauge field theory models. It is also the prototype of non- regular representations of the Heisenberg group.

Proposition 2.1 *The GNS representation π_0 defined by a pure translationally invariant state ω_0 is unitarily equivalent to the following representation*

$$\omega_0(U(\alpha)\, V(\beta)) = 0, \;\; \textit{if} \;\; \alpha \neq 0, \quad \omega_0(V(\beta)) = e^{i\beta\bar{p}}, \;\; \bar{p} \in \mathbf{R}. \tag{2.2.2}$$

Thus, the one-parameter group $U(\alpha)$ is non-regularly represented. The GNS representation space $\mathcal{H}_0$ contains as representative of ω_0 a cyclic vector Ψ_0 such that (denoting by the same symbols the elements of the Weyl algebra and their representatives)

$$V(\beta)\Psi_0 = e^{i\beta\bar{p}}\, \Psi_0, \quad (U(\alpha)\, \Psi_0, \, U(\alpha')\, \Psi_0) = 0, \;\; \textit{if} \;\; \alpha \neq \alpha'. \tag{2.2.3}$$

The linear span D of the vectors $U(\alpha)\Psi_0$, $\alpha \in \mathbf{R}$ *is dense in* $\mathcal{H}_0$, *which is* **non-separable**.

The generator of the one-parameter group $U(\alpha)$ *does not exist, but nevertheless a generic vector of D*

$$\Psi_A = A\Psi_0, \quad A = \sum_{n \in \mathbf{Z}} a_n U(\alpha_n), \quad \{a_n\} \in l^2,$$

may be represented by a wave function $\psi_A(x) = \sum_{n \in \mathbf{Z}} a_n \, e^{i\alpha_n x}$, *with scalar product given by the ergodic mean*

$$(\psi_A, \psi_A) = \sum_{n \in \mathbf{Z}} |a_n|^2 = \lim_{L \to \infty} (2L)^{-1} \int_{-L}^{L} dx \, \bar{\psi}_A(x) \, \psi_A(x). \tag{2.2.4}$$

The spectrum of $V(\beta)$ *is a pure point spectrum.*

Proof Quite generally, by Eq. (2.2.1) and the Weyl commutation relations one has, $\forall \beta \in \mathbf{R}$,

$$\omega_0(U(\alpha)\, V(\gamma))) = \omega_0(V(\beta) U(\alpha)\, V(\gamma) V(\beta)^*) = \omega_0(U(\alpha)\, V(\gamma)) e^{i\alpha\beta}. \tag{2.2.5}$$

Thus one has

$$\omega_0(U(\alpha) V(\gamma)) = 0, \ \text{ unless } \ \alpha = 0, \quad \omega_0(V(\gamma)) \neq 0. \tag{2.2.6}$$

Hence, the representative of the unitary group $U(\alpha)$, $\alpha \in \mathbf{R}$, is not weakly continuous and the representation is not regular.

In order to completely characterize the representation, we note that the translational invariance of ω_0 implies that the operator $T(\beta)$ defined by

$$T(\beta)\, A\, \Psi_0 = \alpha_\beta(A)\, \Psi_0, \quad T(\beta)\, \Psi_0 = \Psi_0, \quad \forall A \in \mathcal{A}_W,$$

is a unitary operator and satisfies $T(\beta) A T(\beta)^* = \alpha_\beta(A)$. In fact, the mapping $T(\beta)$ (together with its inverse) defined on the dense set $\mathcal{A}_W \Psi_0$ by

$$T(\beta)\Psi_0 = \Psi_0, \quad T(\beta)\, A\, \Psi_0 = \alpha_\beta(A)\Psi_0, \quad \forall A \in \mathcal{A}_W,$$

preserves the scalar product as a consequence of the invariance of ω_0,

$$(T(\beta) A\, \Psi_0, T(\beta)\, B\Psi_0) = (\Psi_0, \alpha_\beta(A^*\, B)\, \Psi_0) = \omega_0(\alpha_\beta(A^* B)) =$$
$$= \omega_0(A^* B) = (A\Psi_0,\, B\Psi_0).$$

Hence, $T(\beta)$ is unitary. Furthermore, by construction $T(\beta)^*\, V(\beta)$ commutes with $\mathcal{A}_W$ (since both $T(\beta)$ and $V(\beta)$ generate the same automorphism), and by the irreducibility of the representation (following from ω_0 being pure) $T(\beta)^* V(\beta) = e^{i\theta(\beta)}\mathbf{1}$. The group law requires $\theta(\beta) = \bar{p}\beta, \bar{p} \in \mathbf{R}$, and therefore $V(\beta)\Psi_0 = e^{i\bar{p}\beta}\,\Psi_0$. Then, Eq. (2.2.6) implies Eqs. (2.2.2), (2.2.3).

The Weyl operators are represented in the following way:

$$(U(\alpha)\psi)(x) = e^{i\alpha x}\,\psi(x), \quad (V(\beta)\psi)(x) = \psi(x+\beta).$$

The rest of the Proposition follows from a direct check that the cyclic vector defined by the wave function $\psi_{\mathbf{1}} = 1$ yields the same expectations as ω_0; therefore the corresponding GNS representations are unitarily equivalent. Moreover, one has

$$V(\beta)\,U(\alpha)\Psi_0 = e^{i(\bar{p}+\alpha)\beta}\,U(\alpha)\,\Psi_0.$$

3 Bloch Electron and Non-regular Quantization

It is a very good approximation to describe an electron in a periodic crystal by a Schrödinger equation with a periodic potential; for simplicity we consider the one-dimensional case.[2] In this case the Hamiltonian is $H = -d^2/dx^2 + W(x)$, with the potential satisfying the periodicity condition $W(x+a) = W(x)$, for a suitable a.

The spectrum of H is purely continuous, so that the improper eigenvectors, and in particular the improper ground state, are not described by square integrable functions and therefore are not normalizable. On the other hand, much of the wisdom on periodic structures in solids makes extensive use of such improper states and in order to bypass the difficulties of non-normalizability it has becomes standard to restrict the wave functions to an elementary unit cell. At the basis of this prescription is the so called Floquet-Bloch theorem,[3] according to which the energy improper eigen-functions can be chosen of the form

$$\psi_k^n(x) = e^{ikx}v_k^n(x), \quad v_k^n(x+a) = v_k^n(x), \quad k \in [0, 2\pi/a), \quad n \in \mathbf{N}, \tag{2.3.1}$$

(*Bloch electrons*). Such improper states do not play the mere role of limiting extrapolations of well defined vectors (like the plane waves in the free case), since all the physically relevant states used in the treatment of periodic structures in solids belong to the cyclic representation defined by the ground state and the

[2]For an excellent treatment of the Schrödinger operators with periodic potentials, see M. Reed and B. Simon, *Methods of Modern Mathematical Physics, Vol. IV*, Academic Press 1978, Sect. XIII.16; in particular pp. 287–301 for the one-dimensional case. For an expository presentation of the one-dimensional case, see e.g. A.A. Cottey, Am. J. Phys. **39**, 1235 (1971). For the discussion of the related physical problem see e.g. J.M. Ziman, *Principles of the Theory of Solids*, Cambridge Univ. Press 1964, Chap. 1; N.W. Ashcroft, *Solid State Physics*, Saunders College Publ. 1976, p. 132–141.

[3]G. Floquet, Ann. École Norm. Sup. **12**, 47 (1883); W. Magnus and S. Winkler, *Hill's Equation*, Wiley 1966; F. Bloch, Z. Physik **52**, 555 (1928).

latter corresponds to the improper wave function $\psi_0^0(x) = v_0^0(x)$ (Eq. (2.3.1) with $k = 0, n = 0$). This justifies to look for a mathematical control of the status of such representations also in order to obtain well defined rules for computing transition amplitudes etc.

Another motivation for such an investigation is to clarify the analogy drawn between the θ vacua representations in quantum chromodynamics (QCD) and that of the ground state of the Bloch electron.[4]

Proposition 3.1 [5] *Let $W(x)$ be a bounded measurable periodic potential, $W(x) = W(x + a)$, then there exists one and only one irreducible representation $(\pi, \mathcal{K})$ of the CCR algebra $\mathcal{A}_W$ in which the Hamiltonian*

$$H = p^2/2 + W(x)$$

is well defined, as a strong limit of elements of $\mathcal{A}_W$ (on a dense domain), and has a ground state $\Psi_0 \in \mathcal{K}$.

Moreover, such a representation is independent of W, in the class mentioned above, and it is the unique non-regular representation π_0 in which the subgroup $V(\beta)$, $\beta \in \mathbf{R}$ is regularly represented; its generator p has a discrete spectrum.

The Hilbert space $\mathcal{K}$ of π_0 consists of the formal sums

$$\psi(x) = \sum_{n \in \mathbf{Z}} c_n \, e^{i\alpha_n x}, \quad \{c_n\} \in l^2(\mathbf{C}), \quad x \in \mathbf{R}, \quad \alpha_n \in \mathbf{R}, \tag{2.3.2}$$

with scalar product given by the ergodic mean

$$(\psi, \psi) = \sum_{n \in \mathbf{Z}} |c_n|^2 = \lim_{L \to \infty} (2L)^{-1} \int_L^L dx \, \bar{\psi}(x) \, \psi(x). \tag{2.3.3}$$

The Weyl operators are represented by

$$(\pi_o(U(\alpha))\psi)(x) = e^{i\alpha x}\psi(x), \quad (\pi_0(V(\beta))\psi)(x) = \psi(x + \beta). \tag{2.3.4}$$

The (orthogonal) decomposition of $\mathcal{K}$ over the spectrum of $V(a)$ is

$$\mathcal{K} = \sum_{\theta \in [0, 2\pi)} \oplus \, \mathcal{H}_\theta, \quad V(a) \, \mathcal{H}_\theta = e^{i\theta} \, \mathcal{H}_\theta, \quad \theta \in [0, 2\pi). \tag{2.3.5}$$

[4] R. Jackiw, Rev. Mod. Phys. **49**, 681 (1977), Sect. III. G.

[5] J. Löffeholz, G. Morchio and F. Strocchi, Lett. Math. Phys. **35**, 251 (1995).

The spectrum of p in $\mathcal{H}_\theta$ is $\sigma(p)|_{\mathcal{H}_\theta} = \{2\pi n/a + \theta/a,\ \ n \in \mathbf{Z}\}$ and the wave functions $\psi_\theta \in \mathcal{H}_\theta$ are quasi periodic of the form

$$\psi_\theta(x) = e^{i\theta x/a} \sum c_n e^{i2\pi n x/a}. \tag{2.3.6}$$

*(**Bloch waves**). The unique ground state is a vector of $\mathcal{H}_{\theta=0}$.*

Proof

1) *The representation is non-regular for $U(\alpha)$.*
   A bounded measurable potential belongs to the strong closure of $\mathcal{A}_W$ and therefore if H is a well defined operator as a strong limit of elements of $\mathcal{A}_W$, so is $H_0 = p^2/2m$. Hence, p^2 is well defined and so is its square root p, which is the generator of $V(\beta)$. Therefore, by Stone's theorem $V(\beta)$ is weakly continuous, i.e. regularly represented in H.
   The Weyl commutation relations and the weak continuity of $V(\beta)$ imply $U(\alpha)\, p\, U(-\alpha) = p - \alpha$, i.e. the spectrum $\sigma(p)$ of p is homogeneous. From the irreducibility of π_0 it follows that three cases are possible:

   i) $\sigma(p)$ is absolutely continuous; hence, by irreducibility $\sigma(p)$ has no multiplicity, so that $U(\alpha)$ act as translations on $\sigma(p)$ and the absolute continuity of the spectral measures of p with respect to the Lebesgue measure implies that $U(\alpha)$ is weakly continuous. Then, by the SvN theorem the representation is equivalent to the Schrödinger representation and there is no ground state;
   ii) $\sigma(p)$ is a pure point spectrum; then, if Ω_λ is a state corresponding to an eigenvector of p with eigenvalue λ, one has, $\forall A \in \mathcal{A}_W$, $\Omega_\lambda(A\, V(\beta)) = e^{i\lambda\beta}\, \Omega_\lambda(A)$, and

   $$\Omega_\lambda(U(\alpha)V(\beta)) = \Omega_\lambda(V(\gamma)U(\alpha)\, V(\beta)V(-\gamma)) = e^{i\alpha\gamma}\,\Omega_\lambda(U(\alpha)\, V(\beta)),$$

   i.e.

   $$\Omega_\lambda(U(\alpha)V(\beta)) = 0, \quad \forall \alpha \neq 0, \quad \Omega_\lambda(V(\beta)) = e^{i\lambda\beta}.$$

   In conclusion, the representation is unitarily equivalent to the *non-regular* representation of a translationally invariant state, Proposition 2.1. Then, Eqs. (2.3.2), (2.3.3) hold;
   iii) $\sigma(p)$ is purely singular, to be discussed later.

2) *The spectrum of $V(a)$ and of the Hamiltonian.*
   In such a representation the potential can be written in the following form $W(x) = \sum_n v_n\, U(2\pi n/a)$, where v_n are the coefficients of the Fourier expansion of the periodic function $W(x)$ in $[0, a]$ and the series is strongly convergent on D. For the analysis of the spectrum of H, since $V(a) = e^{ipa}$ commutes with H, it is convenient to decompose $\mathcal{K}$ according to the spectrum of $V(a)$, which is purely discrete and coincides with the whole circle, (Eq. (2.3.5)).

Then,

$$\psi(x) = \sum_n c_n U(\alpha_n)\psi_{\mathbf{1}} = \sum_n c_n \exp(i\alpha_n x) \in \mathcal{H}_\theta,$$

implies $V(a)\psi(x) = e^{i\theta}\,\psi(x)$ and on the other hand, by the Weyl commutation relations,

$$V(a)\sum_n c_n U(\alpha_n)\,\psi_{\mathbf{1}} = \sum_n c_n e^{i\alpha_n(x+a)}.$$

Hence, one must have $\alpha_n = \theta/a + 2\pi n/a$, $n \in \mathbf{Z}$ and $\psi(x)$ can be written as in Eq. (2.3.6), i.e. $\psi(x)$ is a quasi periodic function of the form of Eq. (2.3.1), with $k = \theta/a$.

3) *The unique ground state belongs to* $\mathcal{H}_{\theta=0}$.

Since $V(a)$ commutes with H, the subspaces $\mathcal{H}_\theta$ reduce $\mathcal{K}$ and in $\mathcal{H}_\theta$ the Hamiltonian reduces to $H_\theta = H_{0,\theta} + W(x)$, $H_{0,\theta} = p_\theta^2/2m$. Since $W(x)$ is bounded, it is a bounded operator in each $\mathcal{H}_\theta$ and therefore it is infinitesimally smaller than $H_{0,\theta}$ in the sense of Kato, (i.e. $||W\psi_\theta|| \leq a||H_{0,\theta}\psi_\theta|| + b||\psi_\theta||$, with $\inf a = 0$). Since the spectrum of $H_{0,\theta}$ is discrete, so is the spectrum of H_θ; this implies that ground states exist.

Moreover, the boundedness of $W(x)$ in $\mathcal{H}_\theta$ implies that e^{-H_θ} has a strictly positive kernel, i.e. $e^{-H_\theta}\,\psi_\theta(x) > 0$, $\forall \psi_\theta \geq 0$. Now, if $\Psi_0 \in \mathcal{H}$ is the ground state, it must have a non-vanishing projection $\psi_{0,\theta}$ on at least one $\mathcal{H}_\theta$, corresponding to $\inf\,\sigma(H_\theta)$. By a generalized Perron-Frobenius theorem the corresponding wave function may be chosen strictly positive

$$\mathcal{H}_\theta \ni \psi_{0,\theta}(x) = e^{i\varphi}|\psi_{0,\theta}(x)|,\ \varphi \in \mathbf{R},\ \ |\psi_{0,\theta}(x)| \neq 0,\ \ a.e. \tag{2.3.7}$$

Since $|\psi_{0,\theta}(x)|$ is a periodic function, it belongs to $\mathcal{H}_{\theta=0}$ and Eq. (2.3.7) is consistent only for $\theta = 0$. The ground state is unique because any other ground state would be described by a positive wave function and therefore could not be orthogonal to Ψ_0.[6]

iii) The possibility of a representation characterized by a purely singular $\sigma(p)$ is excluded by the following argument. As in the previous case, one may decompose the Hilbert space $\mathcal{K}$ of such a representation according to the spectrum of $V(a)$, $\mathcal{K} = \int d\nu(\theta)\,\mathcal{K}_\theta$, with a measure $d\nu(\theta)$ now singular with respect to the Lebesgue measure and translationally invariant because

[6]For the permanence of a discrete spectrum under a bounded perturbation see M. Reed and B. Simon, *Methods of Modern Mathematical Physics, Vol. IV*, Academic Press 1972, Theors. XII.11, XII.13. For the generalized Perron-Frobenius theorem see J. Glimm and A. Jaffe, *Quantum Physics. A Functional Integral Point of View*, Springer 1987, Sect. 3.3; for a simple account see F. Strocchi, *An Introduction to the Mathematical Structure of Quantum Mechanics*, 2nd ed., World Scientific 2008, Sect. 6.4.

$\tilde{U}(\alpha) \equiv U(\alpha/a)$ maps $\mathcal{K}_\theta$ into $\mathcal{K}_{\theta+\alpha}$. In each space $\mathcal{K}_\theta$ corresponding to such a decomposition, the spectrum of p is discrete and coincides there with the spectrum in the subspace $\mathcal{H}_\theta$ of the preceding case. Then, the representation of the subalgebra generated by $V(\beta)$, $\beta \in \mathbf{R}$ and by $U(2\pi n/a)$, $n \in \mathbf{Z}$, is quasi equivalent to that in $\mathcal{H}_\theta$. The Hamiltonian H is therefore defined in $\mathcal{K}_\theta$ and has the same spectrum as in $\mathcal{H}_\theta$; by the same arguments if inf $\sigma(H)$ is an eigenvalue the corresponding eigenvector must belong to $\mathcal{K}_{\theta=0}$, which must therefore appear as a discrete component of $\mathcal{K}$. Since the spectrum of p is purely discrete in $\mathcal{K}_{\theta=0}$, by irreducibility it is purely discrete in $\mathcal{K}$ and has no singular component there.

The above theorem allows to recover in a simple (mathematically rigorous) way the basic features of the analysis of the energy spectrum in the case of periodic potentials.[7]

a) **Band structure**. The energy spectrum $\{E_n(\theta)\}$ is characterized by bands, classified by the quantum number $n \in \mathbf{Z}$; within each band the energy levels are functions of the parameter $\theta \in [0, 2\pi/a)$.
b) **Description in terms of the elementary cell**. Equation (2.3.6) defines an isomorphism between $\mathcal{H}_\theta$ and $L^2([0,a), dx/a)$, so that the scalar product in $\mathcal{H}_\theta$ reduces to an L^2 product with integration over the *elementary cell* [0, a]. In this identification, p is represented by the self adjoint extension of the differential operator $-id/dx$, corresponding to the boundary conditions $\psi(a) = e^{i\theta}\,\psi(0)$. The generic function $\psi \in \mathcal{K}$ can be expressed as a denumerable superposition of ψ_{θ_k}

$$\psi(x) = \sum_{\theta_k} c(\theta_k)\psi_{\theta_k}(x)$$

and since $(\psi_\theta, \varphi_{\theta'}) = 0$ if $\theta \neq \theta'$, the product (2.3.3) reduces to a sum of products in $L^2([0,a], dx/a)$, i.e. in the elementary cell.

The energy is a continuous function of θ, since $\tilde{U}(\theta) : \mathcal{H}_{\theta=0} \to \mathcal{H}_\theta$, so that

$$(\psi_\theta, H\,\psi_\theta) = (\psi_0, (H + p\theta/m + \theta^2/2m)\,\psi_0).$$

[7] M. Reed and B. Simon, *Methods of Modern Mathematical Physics, Vol. IV*, Academic Press, Sect. XIII.16.

4 Gauge Invariance and Non-regular Canonical Quantization

4.1 Gauge Invariance and Superselection Rules

A general case leading to non-regular representations is when (i) a quantum system is described by canonical variables, generating a Heisenberg group G_H, but only a subset of them, and consequently only a subgroup $G_{obs} \subset G_H$, describes observable quantities, (ii) G_{obs} is generated by a Heisenberg subgroup and by an abelian subgroup $\mathcal{G}$ which commutes with G_{obs}. Then, $\mathcal{G}$ generates a group of transformations α_g, $g \in \mathcal{G}$, which leave the observables pointwise invariant

$$\alpha_g(A) = A, \quad \forall A \in G_{obs}, \quad \forall g \in \mathcal{G}, \tag{2.4.1}$$

i.e. $\mathcal{G}$ has the meaning of a **gauge group**,.

The elements of G_H generate a C^*-algebra $\mathcal{F}_W$, called *field algebra*, and the elements of G_{obs} generate a *C^*-algebra $\mathcal{A}$ of observables*, characterized by gauge invariance, Eq. (2.4.1) (as discussed in Chap. 1, Sect. 3). $\mathcal{A}$ has a non-trivial *center* $\mathcal{Z}$ generated by the elements of $\mathcal{G}$. A representation of $\mathcal{F}_W$ is *physical* if G_{obs} is regularly represented.

In the irreducible representations of $\mathcal{A}$, $\mathcal{Z}$ is represented by multiples of the identity. The generators of $\mathcal{G}$ have the meaning of **superselected charges** and the points θ of the spectrum $\sigma(\mathcal{Z})$ of $\mathcal{Z}$ label inequivalent representations $(\mathcal{H}_\theta, \pi_\theta)$ of $\mathcal{A}$, called **θ sectors**. Stone-von Neumann uniqueness theorem does not apply and this can be traced back to the fact that, contrary to the Weyl C^*-algebra, $\mathcal{A}$ is not simple.

By definition a *gauge invariant state ω* on $\mathcal{F}_W$ satisfies $\omega(\alpha_g(F)) = \omega(F)$, $\forall F \in \mathcal{F}_W$ and therefore, in the GNS representation π_ω of $\mathcal{F}_W$ defined by ω, the gauge transformations are implemented by unitary operators $U(g)$ defined by (Ψ_ω denotes the vector which represents ω)

$$U(g)\Psi_\omega = \Psi_\omega, \quad U(g)\pi_\omega(F)\Psi_\omega = \pi_\omega(\alpha_g(F))\,\Psi_\omega, \quad \forall F \in \mathcal{F}_W.$$

Let $V(g)$ denote the element of $\mathcal{G}$ which defines α_g: $\alpha_g(F) = V(g)\,F\,V(g)^{-1}$, $\forall F \in \mathcal{F}_W$; then, $\pi_\omega(V(g))U(g)^*$ commutes with $\mathcal{F}_W$ and, in each irreducible representation of $\mathcal{F}_W$, $\pi_\omega(V(g))\,U(g)^* = e^{i\theta(g)}\mathbf{1}$. Hence, Ψ_ω is an eigenvector of $\pi_\omega(V(g))$, with eigenvalue $e^{i\theta(g)}$,

$$\pi_\omega(V(g))\Psi_\omega = \pi_\omega(V(g))U(g)^*\Psi_\omega = e^{i\theta(g)}\Psi_\omega. \tag{2.4.2}$$

Thus, the analysis of Sect. 2, applies, with the result that the GNS representation π_ω of $\mathcal{F}_W$, equivalently of G_H, defined by a gauge invariant state ω is non-regular,

$$\mathcal{H}_{\pi_\omega} = \sum_{\theta \in \sigma(\mathcal{Z})} \oplus \, \mathcal{H}_\theta,$$

and the subspaces $\mathcal{H}_\theta$ carrying disjoint irreducible representations of $\mathcal{A}$ are proper subspaces of the non-separable space $\mathcal{H}_{\pi_\omega}$.

Proposition 4.1 *Let G_H be the Heisenberg group defined by the set of canonical variables $\{q_i, p_i\}$, $\mathcal{F}_W$ the corresponding canonical C^*-algebra, $\mathcal{A} \subset \mathcal{F}_W$ the C^*-subalgebra of observables and $\mathcal{G}$ the commutative group of gauge transformations, defined by a subgroup $\mathcal{G} \subset G_H$.*

Then, the GNS representation of $\mathcal{F}_W$ defined by a gauge invariant state is a non-regular representation of $\mathcal{F}_W$, as well as of the Heisenberg group G_H, and the elements of $\mathcal{G}$ define **superselection rules**.

Relevant examples of such a structure are provided by quantum mechanical models, in particular those exhibiting strong analogies with gauge quantum field theories (see in particular the following chapter).

4.2 Gauge Invariance in the Two-Body Problem

The description of the quantum two-body problem is provided by the Weyl field algebra $\mathcal{F}_W$ generated by the exponentials $u(\alpha)$, $v(\beta)$ of the center of mass canonical variables Q, P, and by the exponentials $U(\alpha)$, $V(\beta)$ of the relative canonical variables q, p. The Hamiltonian has the form

$$H = P^2/2M + p^2/2\mu + V(q) \tag{2.4.3}$$

and, for the purpose of discussing the bound state spectrum and in particular the lowest energy level, the position of the center of mass is irrelevant. It is therefore natural to consider as observable C^*-algebra $\mathcal{A}$ the algebra generated by the canonical variable q, p, P. Since the center of mass position is not observed, the translations $v(\beta)$ of the center of mass have the meaning of *gauge transformations*.

Therefore, the representations of the canonical field algebra $\mathcal{F}_W$ defined by a gauge invariant state ω are characterized by the property that the vector Ψ_ω, which represents ω, is an eigenvector of $v(\beta)$, equivalently of P. In particular, the lowest energy state ω_0 must satisfy $\omega_0(P^2) = 0$, so that the corresponding vector Ψ_0 satisfies

$$0 = (\Psi_0, P^2\, \Psi_0) = ||P\, \Psi_0||^2, \quad \text{i.e.} \quad P\, \Psi_0 = 0. \tag{2.4.4}$$

As discussed in Sect. 2, the representations of the canonical algebra $\mathcal{F}_W$ defined by gauge invariant states are non-regular; in fact, Eq. (2.4.4) is incompatible with the canonical commutation relations in the Heisenberg form. It has been suggested to bypass such incompatibility by allowing Ψ_0 to be non-normalizable.[8] In our opinion, such a choice would have catastrophic consequence on the GNS representation defined by such a ground state; by the cyclicity of Ψ_0 all vectors of such a representation would be non-normalizable, all matrix elements (including the ground state expectations of gauge invariant operators) would be divergent and one could not extract finite results in a consistent mathematical way.

However, Eq. (2.4.4) is compatible with the CCR in Weyl form. Thus, a canonical quantization is not forbidden, provided it is done in terms of the Weyl algebra, rather than of the Heisenberg algebra; actually, it is uniquely determined and coincides with the non-regular representation discussed in Sect. 2. The vector states are not represented by square integrable functions on the spectrum of Q, but one can still describe them by wave functions of the center of mass position by using a non-L^2 scalar product (see Eq. (2.2.4)).

In our opinion, from a mathematical point of view, the non-regularity of the representation is a much better price to pay, rather than living with non-normalizable state vectors. The advantages of such a quantization is that the states are described by *normalizable* vectors of a Hilbert space, the basic quantum mechanical rules are not violated, the observable subalgebra $\mathcal{A}$ is regularly represented in the standard way, the canonical variables which are not gauge invariant are non-regularly represented, only their exponentials being well defined.

The quantization discussed above sheds light on the quantization of gauge field theories, in particular on the quantization of the temporal gauge.[9]

4.3 Non-regular Representations and Symmetry Breaking

We briefly recall that, given a C^*-algebra $\mathcal{A}$, an algebraic *symmetry* is an automorphism β of $\mathcal{A}$; given a state ω, the symmetry is *unbroken* in the corresponding representation space if β is implemented by a unitary operator $T(\beta)$ there, i.e.

$$\pi_\omega(\beta(A)) = T(\beta)\, \pi_\omega(A)\, T(\beta)^*, \quad \forall A \in \mathcal{A}. \tag{2.4.5}$$

This means that the representation defined by the state ω_β, $\omega_\beta(A) \equiv \omega(\beta(A))$ is unitary equivalent to π_ω:

$$\pi_{\omega_\beta}(A) = T(\beta)\, \pi_\omega(A)\, T^*(\beta).$$

[8] R. Jackiw, Topological Investigations of Quantized Gauge Theories, in S.B. Treiman, R. Jackiw, B. Zumino and E. Witten, *Current Algebra and Anomalies*, World Scientific 1985.

[9] J. Löffelholz, G. Morchio and F. Strocchi, J. Math. Phys. **44**, 5095 (2003).

In this case, β gives rise to a Wigner symmetry in $\mathcal{H}_\omega$, i.e. all transition amplitudes are invariant. Otherwise, if there is no unitary operator which implements β in $\mathcal{H}_\omega$, by Wigner theorem on symmetries at least one transition amplitude is not invariant and the symmetry β is said to be *broken* in $\mathcal{H}_\omega$. An algebraic symmetry is said to be *regular* if it maps regular representations into regular ones.[10]

In the case of quantum systems described by the canonical Weyl algebra, any regular algebraic symmetry is unbroken in any regular irreducible representation, since, by Stone-von Neumann theorem, all such representations are unitarily equivalent. Thus, the important phenomenon of *symmetry breaking*, in the strong sense of a loss of symmetry as defined above, (not merely as the non-invariance of the ground state) cannot appear in the case of Heisenberg quantization, more generally in the case of regular Weyl quantization.

The situation drastically changes in the case of non-regular Weyl quantization. A distinguished case is when one has the structure discussed in Sect. 4.1, namely a canonical algebra $\mathcal{F}_W$ and an observable (gauge invariant) subalgebra $\mathcal{A}$, with a non-trivial center $\mathcal{Z} \subset \mathcal{A}$.

Clearly, any symmetry β of $\mathcal{A}$, defined by an element of $\mathcal{F}_W$, is implemented by a unitary operator $T(\beta)$ in the non-regular representation π of $\mathcal{F}_W$, defined by a gauge invariant state ω_θ, $\theta \in \sigma(\mathcal{Z})$.

However, if β does not commute with the gauge group $\mathcal{G}$, β is broken in *each* irreducible representation $\mathcal{H}_\theta$ of the observable subalgebra $\mathcal{A}$, i.e. β fails to define a Wigner symmetry of the gauge invariant states of $\mathcal{H}_\theta = \overline{\mathcal{A}\,\Psi_{\omega_\theta}}$, because $T(\beta)$ does not leave $\mathcal{H}_\theta$ invariant.

In the regular irreducible representation, π_r of $\mathcal{F}_W$, the symmetry β is unbroken but the elements of $\mathcal{Z}$ have a continuous spectrum in $\mathcal{H}_{\pi_r}$ and there is no gauge invariant (proper) state vector in $\mathcal{H}_{\pi_r}$.

Proposition 4.2 *Let $\mathcal{F}_W$ denote the canonical field C^*-algebra defined by a Heisenberg group G_H, $\mathcal{A}$ the observable C^*-subalgebra, $\mathcal{Z}$ the non-trivial center of $\mathcal{A}$ generated by the commutative subgroup $\mathcal{G} \subset \mathcal{G}_H$ (gauge group), then*

i) any algebraic symmetry β of $\mathcal{A}$, defined by an element of G_H which does not commute with $\mathcal{G}$, is spontaneously broken in each irreducible representation of $\mathcal{A}$ (θ sector);

ii) in any representation of $\mathcal{F}_W$ defined by a gauge invariant state ω, the one-parameter subgroups which do not commute with $\mathcal{G}$ are non-regularly represented, so that the corresponding generators cannot be defined as operators in $\mathcal{H}_\omega$, only their exponentials exist.

For representations of $\mathcal{A}$ defined by a ground state ω_0, (more generally by a state ω invariant under time translations), the non-invariance of ω_0,

$$< A > \equiv \omega_0(A) \neq \omega_0(\beta(A)), \quad \text{for some} \quad A \in \mathcal{A}, \tag{2.4.6}$$

[10]For a discussion of the meaning and the mechanism of spontaneous symmetry breaking see: F. Strocchi, *Symmetry Breaking*, 2nd ed., Springer 2008.

is still compatible with β giving rise to a Wigner symmetry in the GNS representation space $\mathcal{H}_{\omega_0}$. In this case, if β commutes with the dynamics, Eq. (2.4.6) implies degeneracy of the ground state. This is what happens if (2.4.6) holds for β defined by an element of the field algebra $\mathcal{F}_W$ which commutes also with the gauge group.

A one-parameter group β^λ, $\lambda \in \mathbf{R}$, of symmetries shall be called a *continuous symmetry*. A symmetry is called *internal* if it commutes with the one-parameter group α_t, $t \in \mathbf{R}$, of the time translations. In the following, the breaking of an internal symmetry shall be called *spontaneous symmetry breaking*.

4.4 Goldstone Theorem and Non-regular Representations

The spontaneous breaking of a continuous symmetry in the quantum theory of infinitely extended systems is usually accompanied by a strong constraint on the energy spectrum; in fact, if the symmetry commutes with the dynamics (i.e. if the Hamiltonian is symmetric) the Goldstone theorem predicts the absence of an energy gap with respect to the ground state, in the channels related to the ground state by the broken generators.[11]

Here, we investigate a possible *quantum mechanical version of the Goldstone theorem* which mimics as closely as possible the formulation and proof for infinitely extended quantum systems.

For this purpose, given a C^*-algebra $\mathcal{A}$, a one-parameter group β^λ, $\lambda \in \mathbf{R}$ of automorphisms of $\mathcal{A}$ and a representation π of $\mathcal{A}$ defined by a ground state ω_0, we consider :

i) the infinitesimal variation of a generic element $F = \pi(A), A \in \mathcal{A}$,

$$\delta F = \delta(\pi(A)) = \frac{d\,\pi(\beta^\lambda(A))}{d\lambda}|_{\lambda=0},$$

ii) the generation of the continuous symmetry β^λ by elements of the strong closure $\pi(\mathcal{A})''$ of $\pi(\mathcal{A})$, in the sense that there is a sequence $Q_n = Q_n^* \in \pi(\mathcal{A})''$, $n = 1, ..,$ such that

$$\delta F = i \lim_{n\to\infty} [\,Q_n,\, F\,].$$

If Q_n converges weakly to a self adjoint operator Q, then β^λ is implementable by the unitary operator $e^{i\lambda Q}$, the symmetry is not broken and $< \delta F > \equiv \omega_0(F) \neq 0$ implies that ω_0 is not invariant, i.e. $Q\Psi_0 \neq 0$. Furthermore, if β^λ commutes with the time translations α_t, $\Psi_\lambda \equiv e^{i\lambda Q}\Psi_0$ is a family of *degenerate ground*

[11] For a review and critical discussion of the Goldstone theorem see F. Strocchi, *Symmetry Breaking*, 2nd ed., Springer 2008, Chap. 15.

states. In this case, one gets a picture close to the standard heuristic formulation of spontaneous breaking of a continuous symmetry, based on the following oversimplified assumptions: (i) the continuous symmetry is generated by a charge Q, in the sense that $\delta F = i[Q, F]$, (ii) the Hamiltonian is symmetric, i.e. $[Q, H] = 0$, (iii) $< \delta F > \neq 0$; the conclusion being that $Q\Psi_0 \neq 0$ has zero energy.

Closer to the infinite-dimensional case is the case in which there is no sequence Q_n, with the above property, which converges weakly to a self-adjoint operator Q; then, if $< \delta F > \neq 0$, the symmetry is broken in the strong sense of loss of symmetry and $< \delta F >$ is the strict analog of a *symmetry breaking order parameter*, which characterizes symmetry breaking in quantum field theory or in many body theory. Similarly, $< \delta F >= i \lim_{n\to\infty} < [Q_n, F] >$ plays the role of the *symmetry breaking Ward identity*.

If β^λ commutes with α_t, then, by the invariance of the ground state under α_t, one has

$$\lim_{n\to\infty} < [Q_n(t), F] >= \lim_{n\to\infty} < [\alpha_t(Q_n), F] >=$$

$$\lim_{n\to\infty} < [Q_n, \alpha_{-t}(F)] >= -id < \beta^\lambda(\alpha_{-t}(F)) > /d\lambda|_{\lambda=0} = -id < \beta^\lambda(F) > /d\lambda|_{\lambda=0}$$

$$= \lim_{n\to\infty} < [Q_n, F] > .$$

It is worthwhile to stress that such a time independence of the Ward identity holds also in the more general case in which the symmetry does not commute with the Hamiltonian, $\lim_{n\to\infty}[Q_n(t), H] = A \neq 0$, but $< [A, F] >= 0$ (in analogy with the so-called anomaly occurring in the infinite-dimensional case). This is, e.g., the case in which the Hamiltonian is invariant up to a time derivative which commutes with F (see the example of the Bloch electron discussed below).

Theorem 4.3 *Let β^λ, $\lambda \in \mathbf{R}$, be a one-parameter group of automorphisms of the algebra $\mathcal{A}$, α_t the one-parameter group of time translations and π the representation defined by a ground state ω_0. If for some $F \in \pi(\mathcal{A})$,*

$$< \delta F > \equiv d < \beta^\lambda(F) > /d\lambda|_{\lambda=0} \neq 0,$$

$$< \delta F >= i \lim_{n\to\infty} < [Q_n, F] >= i \lim_{n\to\infty} < [Q_n(t), F] >, \tag{2.4.7}$$

for a suitable sequence of $Q_n = Q_n^*$, $Q_n(t) \equiv \alpha_t(Q_n)$, the limit being understood in the sense of convergence of tempered distributions in the variable t, then there is no energy gap above the ground state. Actually, there is a state (Goldstone-like state) orthogonal to the ground state, with the ground state energy.

Proof It is enough to consider the case in which $F = F^*$, since if $F = F_1 + iF_2$, $F_i = F_i^*$, $i = 1, 2$, by linearity the symmetry breaking condition must hold for at least one F_i. Since the representation is defined by a ground state, $\alpha(t)$ is implemented by a one-parameter group of unitary operators $U(t)$, $t \in \mathbf{R}$, with the generator normalized so that the ground state has zero energy. Without loss of generality one can assume that $< Q_n >= 0$, since Eq. (2.4.7) holds also for $\tilde{Q}_n \equiv Q_n - < Q_n >$. Then, one has

$$2\,\mathrm{Im} \lim_{n\to\infty} < Q_n U(-t) F >= i \lim_{n\to\infty} < [\,Q_n,\, F\,] > \neq 0. \tag{2.4.8}$$

The distributional convergence of $J_n(t) \equiv 2\,\mathrm{Im} < Q_n U(-t) F >$ and Eq. (2.4.7) imply the following distributional convergence of the Fourier transforms $\tilde{J}_n(\omega)$

$$\lim_{n\to\infty} \tilde{J}_n(\omega) =< \delta F > \ \delta(\omega).$$

Then, by using the spectral representation $U(t) = \int e^{-i\omega t}\, dE(\omega)$, one concludes that the energy spectral measure contains a $\delta(\omega)$.
The ground state ω_0 cannot be responsible for such a point spectrum, since its contribution as intermediate state in the right hand side of Eq. (2.4.7) vanishes as a consequence of $< Q_n >= 0$; hence there is a zero energy eigenvector orthogonal to the ground state vector.

Remarks A few remarks may be useful.

The statement that the infinitesimal variations under the symmetry transformations is given by a limit of commutators with charges Q_n does not require that, in the given ground state representation, β^λ is implemented by a weakly continuous group of unitary operators.

In the infinite-dimensional cases of quantum field theory and of many body theory, the generation of the symmetry is through the commutator of local charges, typically the integrals of the charge density $j_0(\mathbf{x}, t)$ of a conserved current $j_\mu(\mathbf{x}, t)$ $(\partial^\mu j_\mu = 0)$:

$$\delta F = i \lim_{R\to\infty} [\,Q_R,\, F\,], \quad Q_R = \int_{|\mathbf{x}|\leq R} dx\, j_0(\mathbf{x}, t)$$

and $< \delta F > \neq 0$ implies that the commutator $[\,Q_R,\, F\,]$ does not converge to the commutator of a charge Q; therefore Q_R is not weakly convergent to a well defined global charge Q. Hence, the generation of the symmetry can only be expected to occur as a limit of commutators of (not weakly converging) local charges.

It is worthwhile to stress that the non-invariance of the ground state expectation of a field F does not guarantee that one can write a corresponding Ward identity, a crucial ingredient for the Goldstone theorem.

The interplay between gauge invariance and the breaking of a continuous symmetry provides a mechanism for evading the conclusions of the Goldstone theorem, i.e. for allowing an energy gap in the presence of symmetry breaking.

In fact, let us consider the case in which

i) the continuous symmetry β^λ is defined by a (one-parameter) subgroup of the Heisenberg group G_H, which does not commute with the gauge transformations,

ii) in the ground state irreducible representation π of the observable algebra $\mathcal{A}$ there is a gauge invariant operator $F \in \pi(\mathcal{A})$, which yields a non-symmetric order parameter $< \delta F > \neq 0$, and
iii) the Hamiltonian is invariant under β^λ up to a time derivative which commutes with F,

then the conclusions of the Goldstone theorem do not apply by the following mechanism.

In the irreducible regular representation π_r of the field algebra $\mathcal{F}_W$, β^λ is implemented by a (weakly continuous) group of unitary operators $T(\lambda)$, all the matrix elements are invariant, but there is no gauge invariant (proper) vector state invariant under time translations. The symmetry gets broken by the direct integral decomposition of $\mathcal{H}_{\pi_r}$ over the spectrum of $\mathcal{Z}$, but one cannot write a symmetry breaking Ward identity for the expectation on the gauge invariant ground state.

On the other side, in the representation of $\mathcal{F}_W$ defined by a gauge invariant ground state ω_θ, the one-parameter group $T(\lambda)$ is not regularly represented. Therefore its generator cannot be defined as an operator in $\mathcal{H}_{\omega_0}$ and $\omega_\theta(\delta F) \neq 0$ cannot be written in terms of a limit of commutators of charges. In conclusion, the symmetry breaking Ward identity cannot be written in terms of expectations on θ states.

Such a mechanism is active in several quantum mechanics gauge models discussed in Chap. 3, as well as in the interesting case of chiral symmetry breaking in QCD, as discussed in Chap. 4.

4.5 Bloch Electron as a Gauge Model

The field algebra $\mathcal{F}_W$ is generated by the Weyl operators $U(\alpha)$, $V(\beta)$, $\alpha, \beta \in \mathbf{R}$ (we keep considering the one-dimensional case).

The periodic structure of the system leads to consider as observable C^*-algebra $\mathcal{A}$ the sub-algebra generated by $V(\beta)$ and by the periodic functions of the position $U(2\pi n/a)$, $n \in \mathbf{Z}$. The center $\mathcal{Z}$ of $\mathcal{A}$ is generated by the translations $V(a)$ and the irreducible representations of $\mathcal{A}$ are defined by the subspaces $\mathcal{H}_\theta$ (θ *sectors*).

The operators $U(\alpha/a), \alpha \neq 2\pi n$ intertwine between the inequivalent representations π_θ and $\pi_{\theta+\alpha}$ and the corresponding one-parameter group is non-regularly represented in the representation of $\mathcal{F}_W$ defined by the gauge invariant ground state Ψ_0.

The Bloch model has been discussed in order to clarify structures and mechanisms argued to characterize Quantum Chromodynamics (QCD).[12] For the analogies and correspondences we remark that the lattice translations $V(na)$ play the role of the large gauge transformations T_n and the θ sectors $\mathcal{H}_\theta$ correspond to the

[12] R. Jackiw, Topological Investigations of Quantized Gauge Theories, in S.B. Treiman, R. Jackiw, B. Zumino and E. Witten, *Current Algebra and Anomalies*, World Scientific 1985, p. 211–359, Sect. 3.5.

representations defined by the θ vacua, (here all the states $\Psi_\theta = U(\theta)\Psi_0$ have higher energy than Ψ_0). The transformations on $\mathcal{F}_W$ defined by the one-parameter group $\tilde{U}(\alpha) \equiv U(\alpha/a)$, $\alpha \in \mathbf{R}$: $\gamma_\alpha(F) \equiv \tilde{U}(\alpha)F\tilde{U}(\alpha)^{-1}$, $\forall F \in \mathcal{F}_W$, correspond to the chiral transformations.

The transformations γ_α are implemented by unitary operators in the space $\mathcal{H}$ carrying an irreducible representation of the (gauge dependent) field algebra $\mathcal{F}_W$; therefore they define Wigner symmetries there, but they do not leave the θ sectors invariant and therefore they are not implemented by unitary operators there. The corresponding symmetry is spontaneously broken in each θ sector.

An explicit symmetry breaking order parameter is provided by p or by $V(na)$, since $\gamma^\alpha(p) = p - \alpha$ and $(\Psi_\theta, \delta p\, \Psi_\theta) = \alpha \neq 0$. The Hamiltonian is invariant up to a time derivative:

$$\gamma^\alpha(H) = H - \alpha p/m.$$

The equations expected to hold in QCD are rigorously reproduced here

$$\tilde{U}(\alpha)\Psi_\theta = \Psi_{\theta+\alpha}, \quad V(na)\tilde{U}(\alpha)V(na)^{-1} = e^{i\alpha n}\,\tilde{U}(\alpha).$$

In the QCD context, the last equation is usually written in terms of the chiral charge Q^5, which is assumed to generate the chiral transformations,

$$T_n Q^5 T_n^{-1} = Q^5 + n,$$

however, it should be stressed that the generator of the "chiral" transformations does not exist, not only in the θ sectors, but not even in the large Hilbert space $\mathcal{H}$, because $\tilde{U}(\alpha)$ is non-regularly represented.

Thus, one cannot write a symmetry breaking Ward identity for the expectations on θ states. The overlooking of this subtle point is at the basis of problems and paradoxes affecting the use of Ward identities in the temporal gauge of QCD.

The Bloch model clearly displays the fact that the crucial ingredient for the breaking of chiral symmetry with energy gap (the so-called $U(1)$ problem) is the existence of a non-trivial center in the algebra of observables and its pointwise instability under chiral transformations.[13]

[13]For the realization and relevance of this structure see F. Strocchi, *Selected Topics on the General Properties of Quantum Field Theory*, World Scientific 1994, Sect. 7.4 and refs. therein; G. Morchio and F. Strocchi, J. Phys. A: Math. Theor. **40**, 3173 (2007); Ann. Phys. **324**, 2236 (2009).

5 Quantum Hall Electron: Zak States

The quantum mechanical behavior of an electron in a periodic lattice in the presence of a constant magnetic field is particularly interesting also in connection with the quantum Hall effect.

a) **Bloch electron in a constant magnetic field**

The Hamiltonian has the following form

$$H = \frac{\Pi^2}{2M} + W(\mathbf{x}), \quad \Pi_i \equiv p_i - \frac{e}{c} A_i, \quad i = 1, 2, 3, \tag{2.5.1}$$

where M denotes the electron mass and $W(\mathbf{x})$ is a bounded measurable periodic potential reflecting the lattice periodic structure in the xy plane.

We adopt the symmetric gauge, so that the electromagnetic potential A_i is given by $A_i = -\frac{1}{2}\varepsilon_{ijk} H_j x_k$, we take the magnetic field $\mathbf{H}$ in the z-direction and consider the motion in the xy-plane. For simplicity, we shall use units such that $\hbar = 1 = M = e|\mathbf{H}|/c$, so that the cyclotron frequency $\omega_c = e|\mathbf{H}|/Mc$ and the magnetic length $l = (\hbar c/e|\mathbf{H}|)^{1/2}$ are both equal to one. Then, one has

$$\Pi_x = p_x - y/2, \quad \Pi_y = p_y + x/2, \quad [\,\Pi_y,\, \Pi_x\,] = i. \tag{2.5.2}$$

$\boldsymbol{\Pi}$ has the meaning of the (gauge invariant) velocity.

The lattice translations on the xy-plane are described by the operators

$$T(\mathbf{a}_j) \equiv e^{i\boldsymbol{\Pi}_c\cdot\mathbf{a}_j}, \quad j = 1, 2, \quad \boldsymbol{\Pi}_c \equiv \mathbf{p} + e\mathbf{A}/c, \tag{2.5.3}$$

$$[\,\Pi_{cx},\, \Pi_{cy}\,] = i, \quad [\,\boldsymbol{\Pi}_c,\, \boldsymbol{\Pi}\,] = 0, \tag{2.5.4}$$

where the vectors $\mathbf{a}_j$ are the lattice basis. The operators $T(\mathbf{a}_j)$, also called *magnetic translations*, commute with the Hamiltonian and satisfy the following commutation relation

$$T(\mathbf{a}_1)\, T(\mathbf{a}_2) = T(\mathbf{a}_2)\, T(\mathbf{a}_1)\, e^{i(a_{1x} a_{2y} - a_{1y} a_{2x})}. \tag{2.5.5}$$

Thus, they commute if the lattice cell satisfies the “rationality condition”[14]

$$a_{1x}\, a_{2y} - a_{1y}\, a_{2x} = 2\pi k, \quad k \in \mathbf{Z}. \tag{2.5.6}$$

In the following for simplicity we shall consider a square lattice with unit lattice spacing.

[14]For a detailed excellent discussion of the magnetic translation group see E. Brown, Aspects of Group Theory in Electron Dynamics, in *Solid State Physics*, F. Seitz et al. eds., Academic Press 1968, pp. 313–408.

The abelian group generated by the $T_j \equiv T(\alpha_j)$, with the condition of Eq. (2.5.6), will be denoted by $\mathcal{G}$. It plays the same role of the group of lattice translations $V(na)$ of the Bloch electron without magnetic field (see Sect. 4.5). It may therefore be given the meaning of an abelian *gauge group*.

In view of the above symmetry properties, it is convenient to describe the systems in terms of the two pairs of canonical (independent) variables

$$q \equiv \Pi_y, \quad p \equiv \Pi_x; \qquad Q \equiv \Pi_{cx}, \quad P \equiv \Pi_{cy}. \tag{2.5.7}$$

The corresponding Heisenberg group G_H is generated by the exponentials

$$u(\alpha) \equiv e^{i\alpha q}, \quad v(\beta) \equiv e^{i\beta p}; \qquad U(\gamma) \equiv e^{i\gamma Q}, \quad V(\delta) \equiv e^{i\delta P}.$$

We denote by $\mathcal{F}_W$ the corresponding *field C^*-algebra*. The elements $u(\alpha)$, $v(\beta)$, $U(n)$, $V(2\pi m)$ generate the *gauge invariant subgroup G_{obs}*, which can be interpreted as the observable subgroup. The corresponding C^*-algebra is denoted by $\mathcal{A}$, with the meaning of the *C^*-algebra of observables*. $\mathcal{A}$ has a non-trivial *center $\mathcal{Z}$* generated by the elements of the gauge group $\mathcal{G}$, which play the role of the large gauge transformations of QCD.

We start by considering the case of $W = 0$. In terms of the above canonical variables one has

$$H_0 = \tfrac{1}{2}(p^2 + q^2) = H_{osc} + \tfrac{1}{2}L_z, \tag{2.5.8}$$

$$H_{osc} \equiv \tfrac{1}{2}\,[p_x^2 + p_y^2 + \tfrac{1}{4}(x^2 + y^2)\,] = \tfrac{1}{4}\,(p^2 + q^2 + P^2 + Q^2), \tag{2.5.9}$$

$$L_z \equiv x p_y - y p_x = \tfrac{1}{2}(p^2 + q^2) - \tfrac{1}{2}(P^2 + Q^2). \tag{2.5.10}$$

L_z is conserved, but it does not commute with the gauge group $\mathcal{G}$ generated by the large gauge transformations $T_1 \equiv e^{i\sqrt{2\pi}Q}$, $T_2 \equiv e^{i\sqrt{2\pi}P}$.

The spectrum of H_0 is the familiar quantum oscillator spectrum, each level being now infinitely degenerate. For a very large magnetic field one may restrict the attention to the first Landau level (LL) corresponding to the lowest energy states of H. For the description of the degeneracy of the first LL one has many options.

One possibility, used for the discussion of Quantum Hall Effect, is to describe such a degeneracy in terms of eigenstates of L_z or of H_{osc}. With such a choice, a state (of the first LL) with maximum xy localization is

$$\Psi_{0\,0} \equiv (2\pi)^{-1/2} e^{-(x^2+y^2)/4}, \quad L_z\Psi_{0\,0} = 0$$

and it is also the ground state of the harmonic oscillator Hamiltonian $H_{osc}(Q, P) \equiv \tfrac{1}{2}(P^2 + Q^2)$.

A complete set of states for the first LL level if obtained by acting on $\Psi_{0\,0}$ by the magnetic translations

$$\Psi_{m\,n}(x, y) \equiv T(\mathbf{a}_1)^m\, T(\mathbf{a}_2)^n\, \Psi_{0\,0}(x, y).$$

Since, apart from (xy dependent phases) the magnetic translations act as lattice translations on the wave functions, the Ψ_{mn} defined above are peaked at the lattice points.

In this way, one gets a regular representation of the Heisenberg group G_H which, however, does not contain gauge invariant states, i.e. states invariant under the gauge group $\mathcal{G}$.

By an argument similar to that repeatedly used before (see e.g. Proposition 2.1) a *gauge invariant state* ω defines a non-regular representation of the Heisenberg group G_H (or of the exponential field algebra $\mathcal{F}_W$). Its representative cyclic vector Ψ_ω is an eigenstate of T_j

$$T_j\,\Psi_\omega = e^{i\theta_j}\,\Psi_\omega, \quad \theta_j \in [0, 2\pi), \quad j = 1, 2. \tag{2.5.11}$$

Such states are the so-called *Zak states*[15] $\omega_{\theta_1\,\theta_2}$

$$\begin{aligned} \omega_{\theta_1\,\theta_2}(U(\gamma)\,V(\delta)) &= e^{in\theta_1}\,e^{im\theta_2}, \quad \text{if} \;\; (\gamma, \delta) = \sqrt{2\pi}(n, m) \\ &= 0, \qquad\qquad\quad \text{if} \;\; (\gamma, \delta) \notin \sqrt{2\pi}(\mathbf{Z}, \mathbf{Z}). \end{aligned} \tag{2.5.12}$$

Clearly, the introduction of the periodic potential does not change such conclusions and actually strengthens the interpretation of the lattice translations as the generators of a gauge group, with a picture which is very close to the case of the Bloch electron without magnetic field. The first LL is not stable under the application of $u(\alpha), v(\beta)$ and of the potential W.

As in case of zero magnetic field, the GNS representation space $\mathcal{K}$ defined by a gauge invariant state has an orthogonal decomposition over the spectrum of the generators of the gauge group

$$\mathcal{K} = \sum_{\theta_1,\,\theta_2} \oplus\, \mathcal{H}_{\theta_1,\,\theta_2}, \tag{2.5.13}$$

each $\mathcal{H}_{\theta_1,\,\theta_2}$ being the carrier of an irreducible representation of the gauge invariant algebra of observables $\mathcal{A}$.

The operators $U(\gamma)$, $V(\delta)$ intertwine between the θ sectors

$$U(\gamma)\,V(\delta)\,\mathcal{H}_{\theta_1,\theta_2} = \mathcal{H}_{\theta_1-\tilde{\delta},\,\theta_2+\tilde{\gamma}}, \qquad (\tilde{\gamma},\,\tilde{\delta}) \equiv \sqrt{2\pi}(\gamma, \delta).$$

They play the same role of the $U(\alpha)$ discussed in Sect. 4.5 and as such are the analogs of the chiral transformations in QCD; they commute with the Hamiltonian if the potential W (which plays the role of the fermion mass term in QCD) vanishes.

[15] J. Zak, Phys. Rev. **168**, 686 (1968).

They define Wigner symmetries in $\mathcal{K}$ which are broken in each irreducible representation $\mathcal{H}_{\theta_1,\theta_2}$ of the observable algebra. Such a breaking does not require the existence of Goldstone-like states by the same mechanism discussed in Sects. 4.3, 4.5.

The potential W is a periodic function of $x = q - P$ and $y = Q - p$ and since it commutes with T_j it must be a function of them[16] and therefore in each sector $\mathcal{H}_{\theta_1,\theta_2}$ it reduces to a periodic function of $q - \theta_2$, $Q - \theta_1$.

As in the case of vanishing magnetic field, in each θ sector W is infinitesimally smaller than $H_0 \equiv \frac{1}{2}(p^2 + q^2)$ in the sense of Kato; therefore, in each sector, since the spectrum of H_0 is discrete, so is the spectrum of $H = H_0 + W$, and, by a generalized Perron-Frobenius theorem as in the case of zero magnetic field (Sect. 3, Proposition 3.1), the ground state is unique. This implies that the spectrum of H is discrete in $\mathcal{K}$ and there is a (possibly degenerate) ground state in $\mathcal{K}$.

Particular instructive is the simple case of a periodic potential described by

$$W = \lambda \cos(\sqrt{2\pi}\, x)\, \cos(\sqrt{2\pi}\, y) = \lambda \cos(\sqrt{2\pi}(q - P)) \cos(\sqrt{2\pi}(Q - p)).$$

In the sector $\mathcal{H}_{\theta_1,\theta_2}$, W reduces to $\lambda \cos(\sqrt{2\pi} q - \theta_2) + \cos(\sqrt{2\pi} p - \theta_1)$ and to first order in λ the inf of the spectrum of H in $\mathcal{H}_{\theta_1,\theta_2}$ is

$$E_0(\theta_1, \theta_2) = \tfrac{1}{2} + \lambda e^{-\pi}\, (\cos(\theta_1) + \cos(\theta_2)). \tag{2.5.14}$$

Thus, for negative λ the minimum in $\mathcal{K}$ is obtained for $\theta_i = 0$, and for positive λ for $\theta_i = \pi$, $i = 1, 2$.

In conclusion one has

Proposition 5.1 *A gauge invariant state ω, i.e. a state invariant under magnetic translations, T_1, T_2, defines an irreducible representation $(\mathcal{H}_{\theta_1,\theta_2}, \pi_{\theta_1,\theta_2})$ of the observable algebra $\mathcal{A}$, labeled by the eigenvalues θ_1, θ_2 of the magnetic translations and non-regular representation $(\mathcal{K}, \pi)$ of the Heisenberg group G_H (and of the field algebra $\mathcal{F}_W$).*

The (non-separable) Hilbert space $\mathcal{K}$ has an orthogonal decomposition over the spectrum of the magnetic translations, Eq. (2.5.13).

The Hamiltonian H, Eq. (2.5.1), has a discrete spectrum in $\mathcal{K}$ and, at least for small periodic potential, a unique ground state belonging to the $\theta_1 = \theta_2 = 0$ sector.

It it worthwhile to remark that in the representation defined by a gauge invariant state the one-parameter groups of unitary operators $U(\gamma)$, $V(\delta)$ are not regularly represented, so that Q, P cannot be defined as operators in $\mathcal{K}$, only their exponentials exists in $\mathcal{K}$.[17]

[16] J. Zak, Phys. Rev. **168**, 686 (1968).

[17] With such a proviso, some of the paradoxes raised in the literature (see, e.g., R. Ferrari, Int. Jour. Mod. Phys. **12**, 1105 (1998)) disappear. In particular, the derivatives with respect to the angles θ_1, θ_2 correspond to the momenta canonically conjugated to Q, P respectively, and therefore cannot be defined in $\mathcal{K}$.

In our opinion, as in the case of zero magnetic field (Sect. 3), the description in terms of the representation given by the Hilbert space $\mathcal{K}$ clarifies the meaning and the role of the boundary conditions used in the literature for the wave function restricted to the primitive cell,[18] as a substitute of Eq. (2.5.11); such boundary conditions are unstable under the action of the unitary operators $U(\gamma)$, $V(\delta)$, which, instead, are well defined in $\mathcal{K}$. Thus, the description in terms of the states of $\mathcal{K}$ already takes the infinite volume limit into account.

b) **Quantum Hall electron**

In the Quantum Hall Effect (QHE) each electron lives in a periodic lattice under the influence of both a constant strong magnetic field, say in the z direction, and a constant electric field $\mathbf{E}$ in the xy plane.

It is convenient to choose the symmetric temporal gauge for the vector potential

$$A_i = -\tfrac{1}{2}\varepsilon_{ijk} H_j x_k - eE_i,$$

so that the motion of an electron in the xy plane is described by the following time dependent Hamiltonian

$$H(t) = \tfrac{1}{2}\tilde{\mathbf{\Pi}}^2 + W(\mathbf{x}), \qquad \tilde{\Pi}_i \equiv p_i - eA_i/c = \Pi_i - eE_i. \tag{2.5.15}$$

By means of a Galilei transformation[19] one can shift the dependence on the electric field from the "kinetic term" to the periodic potential, obtaining the following new Hamiltonian (by introducing the dual $\mathbf{E}^*$ of $\mathbf{E}$, $E_1^* \equiv -E_2$, $E_2^* \equiv E_1$)

$$H'(t) = \tfrac{1}{2}\Pi^2 + W(\mathbf{x} - e(\mathbf{E} + \mathbf{E}^* t)). \tag{2.5.16}$$

The only difference with respect to the case of zero electric field is that the periodic potential has become time dependent.

Thus, most of the previous analysis applies. In particular, the algebraic structure of the Heisenberg group G_H, of the exponential field algebra $\mathcal{F}_W$, of the gauge group $\mathcal{G}$ and of the gauge invariant or observable algebra $\mathcal{A}$ remain the same.

Gauge invariant states are analogously defined and the representations defined by them have the same properties as in Proposition 5.1.

[18] See, e.g. J. Zak, Phys. Rev. **168**, 686 (1968); E. Brown, Aspects of Group Theory in Electron Dynamics, in *Solid State Physics*, F. Seitz et al. eds., Academic Press 1968, pp. 313–408; F.D.M. Haldane, Phys. Rev. Lett. **55**, 2095 (1985).

[19] J. Belissard, Quantum systems periodically perturbed in time, in *Fundamental aspects of quantum theory*, V. Gorini and A. Frigerio eds., Plenum Press 1986, p. 163–171, and references therein; R. Ferrari, Int. Jour. Mod. Phys. **12**, 1105 (1998). For a comprehensive updated discussion of the quantum Hall effect and of the physical principles underlying it see: S. Bieri and J.M. Fröhlich, Comptes Rendus Physique, **12**, 332–346 (2011).

Chapter 3
Quantum Mechanical Gauge Models

1 Quantum Particle on a Circle

The canonical variables which describe a quantum particle on a circle (of radius $R = 1$) are the angle $\varphi \in [0, 2\pi)$ and its conjugate momentum p. They are not genuine Heisenberg variables, since one cannot write the corresponding canonical commutation relations in Heisenberg form. In fact, $[\,\varphi,\, p\,] = i$, would imply $[\,\varphi,\, e^{i\beta p}\,] = i\beta e^{i\beta p}$, (the existence of $e^{i\beta p}$, $\forall \beta \in \mathbf{R}$ is given by the self-adjointness of p), which yields $||\varphi|| \geq \beta/2, \forall \beta$, contrary to the constraint $\varphi \in [0,\, 2\pi\,]$. One is then led to consider a Weyl quantization based on the C^*-subalgebra $\mathcal{A}$ of the standard Weyl (field) algebra $\mathcal{F}_W$, generated by $U(n) = e^{in\varphi}$, $n \in \mathbf{Z}$, and $V(\beta) = e^{i\beta p}$, $\beta \in \mathbf{R}$.

The canonical commutation relations read

$$U(n)\, V(\beta) = e^{-in\beta}\, V(\beta)\, U(n). \tag{3.1.1}$$

Actually, $\mathcal{A}$ may be characterized as the subalgebra of $\mathcal{F}_W$ invariant under the translations γ^n of $2\pi n$, $n \in \mathbf{Z}$, which, therefore, get the meaning of *gauge transformations*.

The structure is similar to that of the Bloch model and fits into the general discussion of Heisenberg group G_H, observable subgroup G_{obs} and gauge group $\mathcal{G}$, with corresponding C^*-algebras $\mathcal{F}_W$, $\mathcal{A}$ and a non-trivial center $\mathcal{Z}$ of $\mathcal{A}$ (see Chap. 2, Sect. 4.1). A representation π of $\mathcal{A}$ is regular if $\pi(V(\beta))$ is a weakly continuous group of unitary operators.

In each irreducible representation of $\mathcal{A}$, the element $V(2\pi) \in \mathcal{Z}$ is a multiple of the identity, say $e^{i\theta}$, $\theta \in [0,\, 2\pi)$.

Theorem 1.1 *For any given θ, all the irreducible regular representations π_θ of $\mathcal{A}$ with $\pi_\theta(e^{i2\pi p}) = e^{i\theta}$ are unitary equivalent.*

Proof We start by proving the Theorem for $\theta = 0$. Any corresponding representation π_0 satisfies

F. Strocchi, *Gauge Invariance and Weyl-polymer Quantization*,
Lecture Notes in Physics 904, DOI 10.1007/978-3-319-17695-6_3

$$\pi_0(W(0,\beta)) \equiv \pi_0(V(\beta)) = \pi_0(V(\beta + 2\pi)).$$

Then, the operator

$$P \equiv (2\pi)^{-1} \int_0^{2\pi} d\beta \, \pi_0(V(\beta)) = P^*,$$

is well defined, satisfies

$$\pi_0(V(\beta))\, P = P$$

as a consequence of the periodicity of the operator valued function $\pi_0(V(\beta))$ and cannot vanish because it would imply the vanishing of

$$\pi_0(U(-n))PU(n)) = (2\pi)^{-1} \int_0^{2\pi} d\beta \, e^{in\beta} \pi_0(V(\beta)), \ \forall n \in \mathbf{Z},$$

i.e. the vanishing of the periodic unitary operator-valued function $\pi_0(V(\beta))$. Furthermore, one has

$$P\pi_0(U(n)V(\beta))P = (2\pi)^{-2} \int_0^{2\pi} d\delta \int_0^{2\pi} d\gamma \, e^{in\delta} \pi_0(U(n)V(\delta + \beta + \gamma)) =$$

$$(2\pi)^{-1} \int_0^{2\pi} d\delta e^{in\delta} \pi_0(U(n))\, P = \delta_{n,0} P. \tag{3.1.2}$$

Thus, P is a projection and the representation space $\mathcal{H}_{\pi_0}$ contains a (cyclic) vector Ψ_0 with the property

$$(\Psi_0, \, \pi_0(U(n)\, V(\beta))\, \Psi_0) = \delta_{n,0}. \tag{3.1.3}$$

By the GNS theorem (see Chap. 1, Sect. 3) all such representations are unitarily equivalent.

For $\theta \neq 0$, we note that the automorphisms ρ_θ defined by

$$\rho_\theta(U(n)) = U(n), \quad \rho_\theta(V(\beta)) = e^{i\tilde{\theta}\beta}\, V(\beta), \quad \tilde{\theta} \equiv \theta/2\pi,$$

intertwines between π_θ and π_0: $\pi_\theta(A) = \pi_0(\rho_\theta(A))$. Then, given two representations $\pi_\theta^{(1)}$, $\pi_\theta^{(2)}$, their unitary equivalence follows from that of the corresponding $\pi_0^{(i)}$:

$$\pi_\theta^{(1)}(A) = \pi_0^{(1)}(\rho_\theta(A)) = U\pi_0^{(2)}(\rho_\theta(A))U^{-1} = U\pi_\theta^{(2)}(A)U^{-1}.$$

The representations π_θ are the GNS representations defined by the states ω_θ on $\mathcal{A}$ defined by

$$\omega_\theta(U(n)V(\beta)) = \delta_{n,0}\, e^{i\beta\tilde{\theta}}. \tag{3.1.4}$$

The field algebra $\mathcal{F}_W$ can be obtained as the enlargement of the observable algebra $\mathcal{A}$, such that the automorphisms ρ_θ become inner, i.e. are described by (unitary) elements $U(\tilde{\theta})$, $\tilde{\theta} \in [0, 1)$, of the enlarged algebra. In fact, $U(\alpha) \equiv U(\tilde{\theta} + n) = U(\tilde{\theta})\, U(n)$, $\alpha \in \mathbf{R}$ and $V(\beta)$ generate the Heisenberg group, and the corresponding C^*-algebra $\mathcal{F}_W$.

Proposition 1.2 *In the representation π_θ of $\mathcal{A}$, the Hamiltonian is well defined as a strong limit of elements of $\mathcal{A}$ and given by*

$$H_\theta = p_\theta^2/2m,$$

where $p_\theta \equiv \pi_\theta(p)$. The spectrum of H_θ is discrete:

$$E_n^\theta = E_0(n + \tilde{\theta})^2, \quad E_0 \equiv (2m)^{-1}, \quad n \in \mathbf{Z}. \tag{3.1.5}$$

*Each θ sector contains a lowest energy state, called **θ vacuum**, and the ground state belongs to the $\theta = 0$ sector.*

Each θ vacuum defines a non-regular representation of the field algebra $\mathcal{F}_W$ and for $\theta \in [0, \pi)$ it is given by

$$\omega_\theta(U(\alpha)\, V(\beta)) = 0, \quad \textit{if} \quad \alpha \neq 0, \quad \omega_\theta(V(\beta)) = e^{i\beta\tilde{\theta}}. \tag{3.1.6}$$

Proof We start with the simpler case $\theta = 0$. Then, Eq. (3.1.3) implies $(\pi_0(V(\gamma)\, U(n))\, \Psi_0,\, [\pi_0(V(\beta) - \mathbf{1}]\, \Psi_0) = 0$, and since the linear span of the vectors $\pi_0(V(\gamma)\, U(n))\, \Psi_0$, $\gamma \in \mathbf{R}$, $n \in \mathbf{Z}$, is dense,

$$\pi_0(V(\beta)\, \Psi_0 = \Psi_0, \quad \pi_0(p)\, \Psi_0 = 0, \quad H_0\, \Psi_0 = 0.$$

Furthermore, Eq. (3.1.1) gives

$$\pi_0(p)\, \pi(V(\gamma)\, U(n))\, \Psi_0 = -n\, \pi(V(\gamma)\, U(n))\, \Psi_0,$$

so that the spectrum of H is given by Eq. (3.1.5), with $\tilde{\theta} = 0$.

In the representation π_θ, the representative of the state ω_θ defined by Eq. (3.1.4), satisfies $\pi(V(\beta))\Psi_\theta = e^{i\beta\tilde{\theta}}\, \Psi_\theta$ and therefore, for its extension to $\mathcal{F}_W$, one has, $\forall \gamma \in \mathbf{R}$,

$$\omega_\theta(U(\alpha)\, V(\beta)) = \omega_\theta(V(\gamma)^* U(\alpha) V(\beta) V(\gamma)) = e^{-i\gamma\alpha}\, \omega_\theta(U(\alpha)\, V(\beta))$$

and Eq. (3.1.6) follows (for $\theta \in [0, \pi)$ the ω_θ are the lowest energy states).

By Eq. (3.1.3) and the automorphism ρ_θ, the representation π_θ of $\mathcal{A}$ is the same as that of the gauge invariant algebra of the Bloch model (Sect. 4.5) with $a = 2\pi$. The elements of $\mathcal{H}_\theta$ can be represented by quasi periodic wave functions as in

Eq. (2.3.6):

$$\psi_\theta(\varphi) = e^{i\tilde{\theta}\varphi} \sum_{n \in \mathbf{Z}} a_n e^{in\varphi}, \quad \{a_n\} \in l^2,$$

and the generator p_θ of $\pi_\theta(V(\beta))$ is the differential operator $-id/d\varphi$ on quasi periodic functions; the spectrum is well known and given by Eq. (3.1.5).

The strict correspondence with the Bloch model allows to draw similar analogies with the structures argued to characterize the vacuum structure of quantum chromodynamics (QCD). The operators $T^n = V(2\pi n)$ play the role of the *large gauge transformations*, the automorphisms ρ_θ play the role of the *chiral transformations* and the θ vacua structure accounts for the *chiral symmetry breaking with energy gap* (absence of Goldstone states), by the same mechanism discussed in Chap. 2, Sect. 4.4, i.e. the impossibility of writing the symmetry breaking Ward identities in terms of expectations on θ vacua.

2 Jackiw Model of Gauss Law Constraint

A characteristic feature of gauge theories is that one of the equations is the so-called *Gauss law constraint*, e.g. in electrodynamics one has

$$G \equiv \operatorname{div} \mathbf{E} - \rho = 0, \quad \dot{G} = 0.$$

The time independent operator G is the so-called *Gauss law operator*.

The formulation of gauge theories, in particular the canonical formulation, is conveniently done in terms of a gauge dependent field algebra $\mathcal{F}_W$ and its observable subalgebra $\mathcal{A}$ is characterized by its pointwise invariance under gauge transformations, which are generated by the Gauss (law) operator G.

In abelian gauge theories, like electrodynamics, the gauge transformations are related to an abelian Lie group, typically $U(1)$, and one has only one Gauss law operator, which therefore belongs to the observable algebra. Thus, one has the same structure discussed before, namely a field algebra $\mathcal{F}_W$, an observable subalgebra $\mathcal{A}$ and a non-trivial center $\mathcal{Z}$ of $\mathcal{A}$. Hence, the canonical quantization of such a structure meets the same general problems discussed in the previous Chapter. In particular the *Gauss law constraint is incompatible with canonical commutation relations in Heisenberg form*. This has been the origin of several discussion and proposal in the literature for overcoming this apparent difficulty at the basis of the quantization of gauge theories.

In his authoritative and excellent lectures on topological structures in gauge theories[1] Jackiw proposed quantum mechanical models for illustrating the role and the implementation of the "Gauss law" constraint, implicitly suggesting the same strategy advocated for the two-body problem he had discussed before.

As for the two-body problem, a mathematically more acceptable treatment of such models is obtained by using non-regular representations rather than non-normalizable (cyclic) vectors. This shall be argued below, by showing that Jackiw models display the same general features discussed in Chap. 2, and therefore require to adopt a non-regular Weyl quantization, the standard Heisenberg quantization being mathematically precluded.

We discuss the following Jackiw model which mimics an interaction of "two particles" with an electromagnetic potential A in $0 + 1$ space-time dimensions and in order to strengthen this interpretation we add the kinetic term for A. Thus, the model is defined by the following Lagrangian:

$$L = \tfrac{1}{2}(\dot{q}_1 + eA)^2 + \tfrac{1}{2}(\dot{q}_2 + eA)^2 - V(q_1 - q_2) + \tfrac{1}{2}\dot{A}^2. \tag{3.2.1}$$

The canonical variables are the two particle positions q_1, q_2, the corresponding conjugate momenta $p_i = \dot{q}_i + eA$, $i = 1, 2$, the "electromagnetic potential" A and its conjugate momentum $\pi = \dot{q} \equiv E$, playing the role of the "electric field". The Hamiltonian and the corresponding Hamilton equations are

$$H = \tfrac{1}{2}(p_1^2 + p_2^2) - e(p_1 + p_2)A + \tfrac{1}{2}E^2 + V(q_1 - q_2),$$
$$\dot{q}_i = p_i - eA, \quad \dot{p}_1 = -\partial V/\partial q_1 = -\dot{p}_2, \quad \dot{q} = E, \quad \dot{E} = e(p_1 + p_2).$$

The Lagrangian is invariant under the following *gauge transformations* $\gamma^{(a+bt)}$, $a, b, t \in \mathbf{R}$: $\gamma^{(a+bt)}(q_i) = q_i + a + b\,t$, $\gamma^{(a+bt)}(A) = A - b/e$. They imply the invariance of the conjugate variables p_i, E as well as of the equations of motion; the Hamiltonian is invariant up to the time derivative Eb/e.

The *field algebra* $\mathcal{F}_W$ is generated by the Weyl operators $U(\alpha), V(\beta), u(\alpha), v(\beta), U_A(\alpha), V_A(\beta), \alpha, \beta, \in \mathbf{R}$, corresponding to the exponentials of $q \equiv (q_1 - q_2)$, $p \equiv \frac{1}{2}(p_1 - p_2)$, $Q \equiv \frac{1}{2}(q_1 + q_2)$, $P \equiv p_1 + p_2, A$ and E, respectively. The *observable subalgebra* $\mathcal{A}$ is generated by (the exponentials of) q, p, P, E.

The gauge transformations γ^a and $\gamma^{b\,t}$ are, respectively, generated by P and by the time independent Gauss operator

$$G(t) \equiv E(t)/e - t(p_1 + p_2) = E(0)/e, \quad dG/dt = 0, \tag{3.2.2}$$

(by the explicit appearance of the time variable $G(t)$ does not transform covariantly under time translations). Clearly, the Gauss law constraint on the states in the form

[1] R. Jackiw, Topological investigations of quantized gauge theories, in S.B. Treiman, R. Jackiw, B. Zumino and E. Witten, *Current Algebra and Anomalies*, World Scientific 1985, pp. 211–359.

$G\Psi = 0$, is incompatible with the canonical commutation relations in Heisenberg form and therefore one must use Weyl quantization.

Proposition 2.1 *Any state ω on the canonical field algebra $\mathcal{F}_W$ satisfying the Gauss law constraint $\omega(FG) = 0$, $\forall F \in \mathcal{F}_W$, equivalently invariant under the gauge transformations generated by the Gauss operator G, defines a non-regular representation of $\mathcal{F}_W$:*

$$\omega(u(\alpha)\, v(\beta) U_A(\delta)\, V_A(\gamma)) = 0, \quad \textit{if} \quad \alpha \neq 0, \quad \textit{or if} \quad \delta \neq 0. \tag{3.2.3}$$

The one-parameter groups $U(\alpha)$, $V(\beta)$, $v(\beta)$, $V_A(\gamma)$, generated by q, p, P, E, are regularly represented (so that the expectations of the gauge invariant variables are well defined), but $u(\alpha)$ and $U_A(\alpha)$ are not, so that the corresponding gauge dependent generators Q and A cannot be defined as operators in the GNS representation defined by ω.

Proof Let $F(\alpha, \beta, \delta, \gamma)$ denote the operator in the expectation (3.2.3). For a gauge invariant state ω, the corresponding (GNS) representative vector Ψ_ω must be an eigenstate of G, then by using the canonical commutation relations in Weyl form, one has $(T(b) \equiv e^{i\,b\,G})$, $\forall b, t, \in \mathbf{R}$,

$$\omega(F(\alpha, \beta, \delta, \gamma)) = \omega(T(b)^* F(\alpha, \beta, \delta, \gamma) T(b)) = e^{-ib(\alpha t+\delta)}\omega(F(\alpha, \beta, \delta, \gamma))$$

and Eq. (3.2.3) follows.

The Hamiltonian can also be written in the following form

$$H = \tfrac{1}{2}(p^2 + P^2) + V(q) - ePA + \tfrac{1}{2}E^2 \equiv H(q,p) + H(P,A,E).$$

The time evolution α_t of the observable variables q, p is the standard time evolution of a "particle" in a potential V and for the gauge invariant variables P, E one has $\alpha_t(P) = P$, $\alpha_t(E) = E(0) + e\,P\,t$. Thus, as expected and needed, α_t maps the observable algebra into itself.

For a gauge invariant state, the invariance under time translations requires

$$e^2\, t^2\, ||P\, \Psi_\omega||^2 + 2\, e\, t\, (\Psi_\omega, E(0)\, P\, \Psi_\omega) = 0,$$

i.e. $P\,\Psi_\omega = 0$. Then, if $\pi_\omega(\mathcal{A})$ is irreducible, since the algebra generated by P, E is abelian (and commutes with q, p), one has $\pi_\omega(P) = 0$, $\pi_\omega(E) = e\pi_\omega(G) = \lambda \mathbf{1}$. In such a representation the generator of time translations reduces to $H(q, p)$.

In any irreducible representation of the gauge invariant algebra the automorphisms defined by $u(\alpha)$, $U_A(\alpha)$ are broken by the same mechanism discussed in Chap. 2, Sect. 4.4.

3 Christ-Lee Model

The problems of canonical quantization arising from the invariance of the Lagrangian under time dependent gauge transformations have been discussed by Christ and Lee[2] in the analysis of the relation between the different gauges. In order to clarify the nature of such problems they proposed a simple quantum mechanical gauge model exhibiting the basic interplay between gauge invariance and canonical quantization. The model has also been discussed by Jackiw in his lectures quoted above.

In order to strengthen its relation with gauge theories we add a kinetic term for the "electromagnetic potential". Then, the model is defined by the following Lagrangian ($q^2 \equiv q_1^2 + q_2^2$)

$$L = \tfrac{1}{2}(\dot{q}_1^2 + \dot{q}_2^2) - e(q_1\dot{q}_2 - q_2\dot{q}_1)A + \tfrac{1}{2}e^2A^2q^2 + \tfrac{1}{2}\dot{A}^2 - V(q). \qquad (3.3.1)$$

By introducing the complex variables $\varphi \equiv (q_1 + i\,q_2)/\sqrt{2}$, the Lagrangian takes the following form

$$L = (D_t\varphi)^* D_t\varphi + \tfrac{1}{2}\dot{A}^2 - V(\varphi^*\varphi), \quad D_t\varphi \equiv (\partial_t - i\,eA)\varphi,$$

which somewhat mimics the abelian Higgs-Kibble model in 0 +1 space-time dimensions (with the term $\dot{E}$ corresponding to the gauge fixing term $\frac{1}{2}\partial_\mu A^\mu$).

The Lagrangian is invariant under the following time dependent *gauge transformations* $\gamma^{(a+b\,t)}$,

$$\varphi \to e^{ie\,(a+b\,t)}\varphi, \quad A \to A + b, \quad a, b, t \in \mathbf{R}, . \qquad (3.3.2)$$

As a function of the canonical variables $q_i, p_1 = \dot{q}_1 + eAq_2$, $p_2 = \dot{q}_2 - eAq_1$, A, $E = \dot{A}$, the Hamiltonian reads

$$H = \tfrac{1}{2}(p_1^2 + p_2^2) + eA(q_1p_2 - q_2p_1) + \tfrac{1}{2}E^2 + V(q).$$

The operators

$$\rho \equiv 2q = \varphi^*\,\varphi, \quad j \equiv (q_1p_2 - q_2p_1) = i[(D_t\varphi)^*\,\varphi - \varphi^*\,D_t\varphi]$$

play, respectively, the role of a charge density and of a current density (with Hamiltonian coupling $e\,A\,j$). It is easy to see that

$$\frac{dj}{dt} = i[H, j] = 0, \quad \frac{dE}{dt} = -ej.$$

[2]N. Christ and T.D. Lee, Phys. Rev. **D 22**, 939 (1980); T.D. Lee, *Particle Physics and Introduction to Field Theory*, Harwood Academic 1990, Chap. 18, Sect. 1A.

The gauge transformations γ^a and γ^{bt} are generated, respectively, by ej and by the time independent Gauss operator

$$G(t) = E(t) + e j t = E(0).$$

The *field algebra* $\mathcal{F}_W$ is generated by the exponentials of the canonical variables and a candidate for the *observable algebra* $\mathcal{A}$ is the algebra generated by (the exponentials of) the gauge invariant operators $p_1^2 + p_2^2$, E, ρ, j, $D_t\varphi^* D_t\varphi$. Thus, as it happens in gauge quantum field theories, the observable algebra $\mathcal{A}$ is not a subalgebra of the canonical (exponential) field algebra $\mathcal{F}_W$ and therefore it is convenient to introduce a *larger field algebra*, still denoted by $\mathcal{F}_W$, generated by the exponentials of the canonical variables and by the exponentials of a set of their gauge invariant polynomials, briefly called *observable exponentials*.

A representation π of $\mathcal{F}_W$ is *regular* if all the one-parameter groups defined by such exponentials are weakly continuous, otherwise it is said to be non-regular. A representation of $\mathcal{F}_W$ is *physical* if the one-parameter groups defined by the observable exponentials are weakly continuous, so that the corresponding observable generators are well defined.

As it is typical in gauge field theories,[3] the canonical Hamiltonian is gauge invariant up to a time derivative, $\gamma^{at}(H) = H - a\dot{E}/e$, and in fact the equations of motion are gauge invariant.

The Gauss law constraint on the states, $G\Psi = 0$, is incompatible with canonical commutations relations in Heisenberg form and, as before, one must use Weyl quantization.[4]

Proposition 3.1 *A gauge invariant state ω on $\mathcal{F}_W$ defines a non-regular representation of $\mathcal{F}_W$, with the following properties:*

i) the one-parameter group generated by the exponential of the "electromagnetic potential" A is not weakly continuous and therefore A cannot be a well defined operator in $\mathcal{H}_\omega$

ii) if ω is invariant under time translations and provides a physical representation of $\mathcal{F}_W$, then φ, φ^* cannot be applied to Ψ_ω, and therefore ω cannot provide a representation of the polynomial algebra generated by such fields.

Proof The first statement follows from the gauge invariance of ω and the canonical commutation relations (in Weyl form) for the exponentials of A and G as in the proof

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[3]See, e.g., the canonical Hamiltonian for a complex scalar field interacting with the electromagnetic field; G. Wentzel, *Quantum Theory of Fields*, Dover 2004, p. 68.

[4]In the original form of the Christ-Lee model, without the term $\frac{1}{2}\dot{A}^2$, the Lagrangian reduces to $L = \frac{1}{2}(\dot{r}^2 + r^2(\dot{\theta} - \xi)^2) - V(r)$, where ξ has the meaning of a gauge parameter. The role of the Gauss law constraint is taken by the canonical momentum $p_\theta \equiv q^2(\dot{\theta} - \xi)$, conjugated to θ, and gauge invariance would require $p_\theta = 0$, incompatibly with the canonical commutation relations in Heisenberg form. The strategy advocated above of adopting Weyl quantization provides a mathematically acceptable treatment also in this case.

of Eq. (3.2.3) of Proposition 2.1. For the proof of the second statement we consider the scalar product

$$W(t_2 - t_1) \equiv (\varphi(0)\,\Psi_\omega,\, e^{iH\,(t_2-t_1)}\,\varphi(0)\,\Psi_\omega) = (\varphi_{t_1}\,\Psi_\omega,\, \varphi_{t_2}\,\Psi_\omega)$$

and use the gauge invariance of ω, i.e.

$$e^{ib\,G(t)}\,\Psi_\omega = e^{i\lambda}\Psi_\omega, \quad \lambda \in \mathbf{R}.$$

Then, by using the transformation properties of $\varphi(t)$ under the time dependent gauge transformations, Eq. (3.3.2), we get

$$W(t_2 - t_1) = e^{ib\,(t_2-t_1)}\,W(t_2 - t_1), \quad \forall b \in \mathbf{R}.$$

This implies $W(\tau) = 0$, if $\tau \neq 0$; on the other hand, $W(0)$ cannot vanish, because otherwise φ, φ^* and the corresponding polynomial algebra would annihilate Ψ_ω and one would not get an acceptable representation of the gauge invariant algebra. Then, one has $W(\tau) = 0$, for $\tau \neq 0$, $W(0) \neq 0$, which imply that the unitary group of time translations would not be weakly continuous, i.e. the Hamiltonian could not be defined on $\mathcal{H}_\omega$. Hence, the existence of the Hamiltonian requires that φ and φ^* cannot be applied to Ψ_ω and ω cannot provide a representation of the polynomial algebra generated by φ, φ^*.

4 A QM Model of QCD Structures and of Josephson Effect

We discuss a quantum mechanical gauge model exhibiting an interesting interplay between gauge invariance and symmetry breaking, namely the mechanism by which a symmetry generated by a gauge dependent operator gets broken in the irreducible representations of the observable algebra compatibly with an energy gap.

The model is described by the canonical pairs φ, p, with φ an angle, and A, E

$$[\varphi, p] = i, \quad [A, E] = i, \tag{3.4.1}$$

with the following Hamiltonian

$$H = \tfrac{1}{2}(p - eA)^2 + \tfrac{1}{2}E^2 - M\cos(\varphi - \theta_M). \tag{3.4.2}$$

The model can be considered as describing a quantum particle on a circle interacting with an "electromagnetic potential" A, $E = \dot{A}$ playing the role of the "electric field". The model also coincides with the model which describes the Josephson effect taking into account inductance terms, φ being in this case the Josephson phase, A the charge carried by the "normal" current (identified by $\dot{A} = E$).[5]

The model can also be interpreted as the bosonized version of the *massive* Schwinger model in $0 + 1$ dimensions, with $M \cos(\varphi - \theta_M)$ and θ_M playing the role of the fermion mass term and the mass angle, respectively. From this point of view the model provides a non-perturbative realization and control of mechanisms which have been argued to play a crucial role in Quantum Chromodynamics (QCD).[6] We briefly mention that the so called $U(1)$ and strong *CP* problem of QCD arise from the fact that the QCD Lagrangian is invariant under an axial $U(1)$ symmetry which is not realized in nature (absence of parity doublets) and for the spontaneous breaking of which, without Goldstone bosons, one cannot invoke the Higgs mechanism.

The standard explanation is that the gauge invariant axial current associated to the axial symmetry has an anomaly, but the non-trivial contribution of the anomaly to the commutator with the order parameter has only been argued on the basis of semiclassical approximations. On the other hand, as clarified by Bardeen on the basis of perturbative renormalization (in local gauges),[7] the time independent axial $U(1)$ transformations define a time independent symmetry of the field algebra and of its observable subalgebra and they are generated by the conserved gauge dependent axial current. Thus, the Ward identities related to axial symmetry should be written for such a current, and the absence of the corresponding Goldstone bosons requires an explanation.

A very deep insight on the vacuum structure of QCD, with consequences on the $U(1)$ problem, has been obtained by functional integral methods and semiclassical approximations[8]: the topological structure of the smooth configurations of gauge fields gives rise to the so called large gauge transformations, the spectrum of which label the gauge invariant ground states or θ vacua. Since the set of such smooth configurations are expected to have zero functional measure, a mathematical control of this structure is at issue.

The above problems become even more relevant in the case of massive fermions ($M \neq 0$), since the invariance of QCD under *CP* requires the alignment of the vacuum angle θ and the mass angle θ_M, and the question arises of obtaining it without fine tuning in a way stable under radiative corrections.

One of the interests of the model is that all the above problems are reproduced and their solution is under mathematical control, shedding light on the expected mechanisms of QCD.

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[5] S.M. Apenko, Phys. Lett. A ***142***, 277 (1989).

[6] J. Löffelholz, G. Morchio and F. Strocchi, Ann. Phys. **250**, 367 (1996).

[7] W.A. Bardeen, Nucl. Phys. **75**, 246 (1974).

[8] G. t' Hooft, Phys. Rev. Lett. **37**, 8 (1976); R. Jackiw and C. Rebbi, Phys. Rev. Lett. **37**, 172 (1976); C.G. Callan, R.F. Dashen and D. Gross, Phys. Rev. Lett. **B 63**, 34 (1976). For an excellent review see the quoted lectures by R. Jackiw in *Current Algebra and Anomalies*, 1985, pp. 211–359. For a mathematical control see Chapter 4, Sect. 4.

i) *The observable algebra. Large gauge transformations*
The meaning of φ as an angle requires to introduce only the periodic functions of φ, as for the particle on a circle; therefore, also for a better control of the mathematical problem, we consider the C^*-algebra $\mathcal{F}_W$ generated by $e^{i\varphi}$, $e^{i\beta p}$, $e^{i\gamma A}$, $e^{i\delta E}$, $\beta, \gamma, \delta \in \mathbf{R}$, hereafter also called *field algebra*.

The *gauge transformations*

$$\varphi \to \varphi, \;\; p \to p + \lambda, \;\; A \to A + \lambda/e, \;\; E \to E,$$

leave the canonical structure and the Hamiltonian invariant, but irreducible representations π of $\mathcal{F}_W$ are obtained by fixing the gauge, namely by fixing the value of the non-trivial center $\tilde{\mathcal{Z}}$ of $\mathcal{F}_W$, generated by $e^{i2\pi p}$, namely $\pi(e^{i2\pi p}) = e^{i2\pi\theta_F}\mathbf{1}$, $\theta_F \in [0,\,1)$. In the following, for simplicity, we shall take $\theta_F = 0$. The following group of gauge transformations (hereafter called *large gauge transformations*)

$$\varphi \to \varphi, \;\; p \to p + n, \;\; A \to A + n/e, \;\; E \to E, \;\; n \in \mathbf{N}, \tag{3.4.3}$$

survive the gauge fixing and are implemented by the unitary operators

$$T^n \equiv e^{in(\varphi - E/e)}, \qquad [H, T] = 0. \tag{3.4.4}$$

The exponential field algebra $\mathcal{F}_W$ contains a gauge invariant subalgebra $\mathcal{A}$, with the physical interpretation of *observable algebra*, generated by the gauge invariant variables $e^{i\varphi}$, $P \equiv p - eA$, E. $\mathcal{A}$ has a non-trivial center $\mathcal{Z}$, generated by the periodic variable $q \equiv \varphi - E/e$, equivalently by T; hence, the irreducible representations π_θ of $\mathcal{A}$ are labeled by the points $e^{i\theta}$, $\theta \in [0,\,2\pi)$, of the spectrum of T; the corresponding Hilbert spaces $\mathcal{H}_\theta$ are called θ *sectors*.

The Hamiltonian takes a simple form in terms of the new canonical variables $Q \equiv E/e$, P, e^{iq}, and p:

$$H = \tfrac{1}{2}(P^2 + m^2 Q^2) - M\cos(Q + q - \theta_M), \qquad m^2 \equiv e^2. \tag{3.4.5}$$

The quantum problem is to study the irreducible representations of $\mathcal{A}$ and the corresponding spectrum of H there, i.e. with $q = \theta$ in Eq. (3.4.5). It is convenient to discuss separately the massless and the massive case.

A. **M = 0** ("massless fermion")

ii) *n-vacua, θ-vacua and symmetry breaking*

In the discussion of the non-perturbative structures of QCD, the construction of θ vacua is done in terms of the so called *n-states* Ψ_n, which correspond to classical field configurations of zero energy; however, their existence and status beyond the semiclassical approximation is not completely under mathematical control and the model provides a clear picture of the mechanism argued for QCD.

The lesson that the model teaches us is that there are two possible quantum descriptions. One choice corresponds to the regular irreducible representation of the field algebra $\mathcal{F}_W$: the Hilbert space is $\mathcal{H}_r = L^2([0, 2\pi) \times \mathbf{R}, d\varphi\, dA)$, with states described by wave functions $\psi(\varphi, A)$, on which the fields φ, A act as multiplication operators, and $p,\ E$ as derivatives $-i\partial/\partial\varphi$ (on periodic functions of φ), and $-i\partial/\partial A$, respectively.

The spectrum of the large gauge transformation $T = e^{iq}$ is purely continuous in $\mathcal{H}_r$ and therefore gauge invariant states (i.e. eigenstates of T) are not proper vectors of $\mathcal{H}_r$. The irreducible representations $(\pi_\theta,\ \mathcal{H}_\theta)$ of the gauge invariant (observable) algebra can be obtained by decomposing $\mathcal{H}_r$ according to the spectrum of T

$$\mathcal{H}_r = \int d\mu(\theta)\, \mathcal{H}_\theta, \quad \theta \in [0,\ 2\pi), \tag{3.4.6}$$

where $\mathcal{H}_\theta$ are improper subspaces of $\mathcal{H}_r$.

This is the representation typically obtained by the perturbative expansion or by the functional integral with functional measure $\mathcal{D}\varphi\, \mathcal{D}A$. It is also the representation which allows to work with the (gauge dependent) fields φ, A and not only with their exponentials (see below).

Another important feature of such a representation is its symmetry properties. In the case $M = 0$, the Hamiltonian is invariant under the one-parameter group β^μ, $\mu \in \mathbf{R}$, of transformations

$$\beta^\mu(\varphi) = \varphi + \mu \mod 2\pi, \tag{3.4.7}$$

all the other variables remaining unchanged, which will be called *chiral transformations*. The generator of such transformations is p, which satisfy a conservation law $dp/dt = 0$, but it is not gauge invariant.

On the other hand the gauge invariant operator $P = p - eA$ generates the chiral transformations on the fields φ, A, at equal times, but not at any time, since

$$dP/dt = -eE \tag{3.4.8}$$

i.e. P is affected by an *anomaly*.

In the regular representation space $\mathcal{H}_r$ the Hamiltonian has the pure point spectrum of the harmonic oscillator $E_k = m(k + \frac{1}{2})$, $k \in \mathbf{N}$, with infinite multiplicities, labeled by $n \in \mathbf{Z}$. Putting $H_{ren} \equiv H - m/2$ one has

$$H_{ren}\Psi_{k,n} = mk\Psi_{k,n}\,, \quad e^{ip}\Psi_{k,n} = e^{in}\Psi_{k,n}\,, \quad e^{iq}\,\Psi_{k,n} = \Psi_{k,n+1}. \tag{3.4.9}$$

Thus, the ground states $\Psi_{0,n}$, also called *n-vacua, are invariant under chiral transformations* and one has $< \delta F >_{0,n} \equiv (\Psi_{0,n}, \delta F\, \Psi_{0,n}) = 0$, $\forall F \in \mathcal{F}_W$, since

$$< e^{i\varphi} >_{0,n} = < e^{iq} >_{0,n} < e^{iQ} >_{0,n} = e^{-1/4e} \ < e^{iq} >_{0,n} = 0,$$

having used that the ground state of $P^2 + Q^2$ gives the expectation $< e^{i(\gamma Q + \delta P)} > = e^{-(\gamma^2+\delta^2)/4}$ and the last of Eqs. (3.4.9). This indicates that the n-vacua correlation functions of the fields calculated either by the functional integral with the field functional measure or by the perturbative expansion are chirally symmetric.

However, each n vacuum $\Psi_{0,n}$ defines a reducible representation of the observable subalgebra with an integral decomposition over the continuous spectrum of T, Eq. (3.4.6).

Each θ sector has a lowest energy state Ψ^0_θ, called θ-*vacuum*, which by definition is an eigenstate of T, $T\Psi^0_\theta = e^{i\theta}\,\Psi^0_\theta$. Thus, an alternative to the regular (or Heisenberg) quantization is to use from the start a gauge invariant state, typically a θ vacuum ω_θ. As we already learned from the quantum gauge models discussed before, the representation of the exponential field algebra $\mathcal{F}_W$ defined by a θ vacuum is non-regular, namely the gauge dependent fields cannot be defined in the corresponding representation space $\mathcal{K}$. As repeatedly stressed before, this is the mathematically consistent way of solving the conflict between gauge invariance and canonical quantization.

A characteristic feature of the state space $\mathcal{K}$ is that the spectrum of T is a purely point spectrum, so that eigenstates of T are proper vectors of $\mathcal{K}$ and in the decomposition

$$K = \sum_{\theta\in[0,2\pi)} \oplus\,\mathcal{H}_\theta \tag{3.4.10}$$

$\mathcal{H}_\theta$ appear as proper subspaces. In $\mathcal{K}$ the standard formal equations get a rigorous status:

$$\Psi^0_\theta = \sum_n e^{in\theta}T^n\Psi_{0,0}, \quad e^{i\mu p}\Psi^0_\theta = \Psi^0_{\theta+\mu}, \quad (\Psi^0_\theta, \Psi^0_{\theta'})_{\mathcal{K}} = \delta_{\theta,\theta'}. \tag{3.4.11}$$

Such equations display the mechanism of chiral symmetry breaking in the θ sectors by the instability of the gauge invariant θ vacua.

The operator $e^{i\mu p}$, which implements the time independent chiral transformations, is a well define operator with continuous spectrum in the regular representation space $\mathcal{H}_r$ of the exponential field algebra and it is weakly continuous in μ. It has a pure point spectrum in $\mathcal{K}$, but it does not define a one-parameter weakly continuous group there, since

$$(\Psi^0_\theta,\, e^{i\mu p}\Psi^0_\theta) = \delta_{\theta,\,\theta+\mu} = \delta_{\mu,0}.$$

Therefore, the chiral symmetry breaking in each θ sector does not imply the existence of a Goldstone state, because the analog of Eq. (2.4.7) cannot be written, since the would be generator at all times p becomes a non-regular variable and does not define an operator in $\mathcal{H}_\theta$. As a matter of fact no Ward identity expressing the breaking of the chiral symmetry can be written in $\mathcal{H}_\theta$; a Ward identity can be written in $\mathcal{H}_r$, where however the chiral symmetry is unbroken.

In each θ sector, the compatibility of chiral symmetry breaking and an energy gap $m > 0$, in the subspace $e^{i\varphi}\,\Psi^0_\theta$ can be easily checked by computing ($< >$ denotes the expectation on Ψ^0_θ)

$$< [\,P(t),\, e^{i\varphi}\,] >_\theta = e^{i\theta} < [\,P(t),\, e^{iQ}\,] >_\theta = e^{i\theta} \cos(mt) < e^{iQ} >_\theta = \\ = e^{i\theta}\, \cos(mt) e^{-1/4e}. \tag{3.4.12}$$

It is worthwhile to note the essential singularity at $e = 0$, i.e. a severe lack of analyticity.

Once it has been realized that the symmetry breaking Ward identity cannot be written in the θ sector, and therefore one cannot get information on the energy spectrum in this way, the following simple computation suggests a possible general strategy for getting an estimate about the energy gap, in more general cases. Such information can be obtained in terms of energy sum rules, by computing the moments of the energy spectral measure in the symmetry breaking channel, here $Q\Psi_\theta$. The calculation involves only equal time commutators:

$$< QH^{n+1}Q >_\theta = i^{n+1}\,(d^n/dt^n) 2 < PU(t)Q >_\theta \,|_{t=0} = \\ = i^{n+1}\,(d^n/dt^n) < [\,P(t),\, Q(0)\,] >_\theta \,|_{t=0}.$$

In this simple model the computation of such energy moments yields

$$\int (\omega^2 - m^2)\, \omega < Q\Psi^0_\theta, dE(\omega)\, Q\Psi^0_\theta >= 0,$$

i.e. the existence of a $\delta(\omega - m)$ in the energy spectral measure.

The overall picture emerging from the $M = 0$ case substantially agrees with the conventional wisdom on QCD structures in the case of massless fermions. In this case the lesson of the model is a mathematical control of delicate points and a clarification of the mechanisms argued to hold in QCD.

A relevant message from the model is that, despite the anomaly of P (corresponding to the gauge invariant axial current in QCD), chiral symmetry is a well defined *time independent* symmetry of the field algebra and of the observable subalgebra.[9] Its spontaneous breaking in each θ sector is due to the fact that the center of the observable algebra is not pointwise invariant.[10]

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[9]For different statements in the literature, see e.g. S. Coleman, *Aspects of Symmetry*,, Cambridge Univ. Press 1985, Chap. 7.

[10]Such a mechanism of spontaneous symmetry breaking is compatible with an energy gap; relevant examples are the breaking of Galilei symmetry in Coulomb systems, accompanied by the plasmon energy gap, the breaking of the $U(1)$ symmetry in the BCS model of superconductivity. For a general discussion of the mechanism see: G. Morchio and F. Strocchi, Removal of the infrared cutoff, seizing of the vacuum and symmetry breaking in many body and in gauge theories, invited talk at the *IX Int. Conf. on Mathematical Physics*, Swansea 1988, B. Simon et al eds., Adam Hilger Publ. 1989, p. 490; F. Strocchi, Long range dynamics and spontaneous symmetry breaking in many body systems, lectures at the Workshop on *Fractals, Quasicrystals, Knots and Algebraic Quantum mechanics*, A. Amman et al. eds., Kluwer 1988.

Such a breaking is not accompanied by a Goldstone state, because first there is no such a state in the large Hilbert space in which the fields are regularly represented, since the symmetry is unbroken there. On the other hand, no genuine symmetry breaking Ward identity can be written in each θ sector, because the time independent generator p cannot be defined there. Even if $i[P(t), e^{i\varphi}] \neq \delta(e^{i\varphi})$, one can nevertheless consider the expectation $< [P(t), e^{i\varphi}] >_\theta$ and show its time dependence, as we did in Eq. (3.4.12), but such an expectation does not give a genuine symmetry breaking Ward identity, since it is not directly related to the spontaneous breaking of chiral symmetry.

With the insight so gained, we are in a better position for posing the much more interesting and debated questions of the $U(1)$ and strong CP problems in the realistic case of massive fermions.

B. $\mathbf{M} \neq \mathbf{0}$ ("massive fermion")

The general question is what happens of the infinite ground state degeneracy when the "fermion mass term" is switched on and in particular what happens of the construction of the θ vacua in terms of the n-vacua.

A strictly related issue is the strong CP problem: the Hamiltonian H, Eq. (3.4.2), is invariant under the following *CP transformation*

$$\varphi \to -\varphi + 2\,\theta_M, \quad A \to -A, \tag{3.4.13}$$

crucially dependent on the mass angle θ_M. As displayed by Eq. (3.4.5), in a θ sector one has $q = \theta$ and the corresponding Hamiltonian is no longer invariant, except in the special case $\theta = \theta_M$. This equality requires a so called fine tuning, and in QCD it has been argued that its validity at the tree level will not be stable under radiative corrections. Thus, the experimental evidence that the strong interactions, supposed to be described by QCD, are invariant under the CP symmetry defined by the fermion mass term, requires an explanation.

iii) *Hamiltonian spectrum and ground state*

In the regular representation of the fields the Hamiltonian (3.4.2) has a purely continuous spectrum, so that there is no proper ground state in $\mathcal{H}_r$ and the θ vacua cannot be constructed as in Eq. (3.4.11).

Theorem 4.1 [11] *For any fixed $\theta_F \in [0, 2\pi)$, let π_0 denote the GNS representation of $\mathcal{F}_W$ defined by the following gauge invariant state*

$$\omega_\theta(e^{inq}\, e^{i\beta p}\, e^{i\gamma Q + \delta P}) = e^{i\theta}\, e^{-(\gamma^2 + e^2\delta^2)/4e}, \quad \textit{if} \;\; \beta/2\pi \in \mathbf{Z},$$

$$= 0, \quad \textit{otherwise}. \tag{3.4.14}$$

[11] J. Löffelholz, G. Morchio and F. Strocchi, Ann. Phys. **250**, 367 (1996).

Then

i) *the Hamiltonian H is well defined and gauge invariant states exists in the corresponding Hilbert space* $\mathcal{K}$*; moreover, for* $M \neq 0$ *this is the only irreducible representation of* $\mathcal{F}_W$ *in which H has a ground state,*
ii) *the spectrum of* $T = e^{iq}$ *is discrete in* $\mathcal{K}$*, which may therefore be decomposed as in Eq.* (3.4.10)*; in each proper subspace* $\mathcal{H}_\theta$ *the Hamiltonian has a pure point spectrum, with no degeneracy and therefore a unique ground state (*θ *vacuum),*
iii) *in* $\mathcal{K}$ *there is a lowest energy state corresponding to* $\theta = \theta_M$ *and therefore CP invariant.*

Proof The requirement that the Hamiltonian be well defined implies that so is also $P^2 + e^2Q^2$, since the mass term is a bounded perturbation. This condition selects the Fock representation for the Weyl algebra $\mathcal{A}(Q, P)$ generated by the canonical variables P, Q.
Since T commutes with H, one can reduce the spectrum of H with respect to the spectrum of T, i.e. discuss the spectrum $E(\theta)$ of H as a function of θ. Now, one can prove (see below) that the function $\mathrm{Inf}\, E(\theta)$ has only one minimum, corresponding to $\theta = \theta_M$. Therefore, the existence of a ground state requires that in the decomposition of the representation space over the spectrum of T, of the form of Eq. (3.4.6), the Hilbert space $\mathcal{H}_{\theta=\theta_M}$ should be a proper subspace, i.e. $e^{i\theta_M}$ should be an eigenvalue of T.
Since e^{ip} acts transitively on the spectrum of q, the irreducibility of the representation implies that the entire spectrum of q is a pure point spectrum, with no multiplicity and $\mathcal{K}$ can be decomposed as in Eq. (3.4.10). Any vector of the proper subspace $\mathcal{H}_\theta$ is an eigenvector of q and since π_0 is a Fock representation of $\mathcal{A}(Q, P)$, one can find a vector $\psi^0_\theta \in \mathcal{H}_\theta$, which gives the expectation (3.4.14) for $n = 0 = \beta$. Finally, since $q e^{i\beta p} \psi^0_\theta = (\theta - \beta)\, \psi^0_\theta$, for $\beta/2\pi \notin \mathbf{Z}$, $e^{i\beta p}\psi^0_\theta$ is orthogonal to ψ^0_θ and Eq. (3.4.14) follows.
The uniqueness of the lowest energy state in each $\mathcal{H}_\theta$ follows from the strict positivity of the kernel of $e^{-H_\theta \tau}$ in the variable Q and by a generalized Perron-Frobenius argument.[12] The corresponding energy eigenvalue E^0_θ can be computed, for fixed $e > 0$ and M small, to first order in M:

$$E^0(\theta) = \tfrac{1}{2} e - M e^{-1/4e} \cos(\theta - \theta_M). \qquad (3.4.15)$$

The absolute minimum is attained for $\theta = \theta_M$. It is worthwhile to note the essential singularity at $e = 0$, signaling a non-perturbative effect.

Thus, the model suggests a possible "dynamical solution" of the strong *CP* problem, the absolute ground state being $\Psi^0_{\theta=\theta_M}$. It is worthwhile to remark, that an approach to the massive case which first chooses a θ sector and then adds the

[12] J. Glimm and A. Jaffe, *Quantum Physics. A Functional Integral Point of View*, Springer 1987, Sect. 3.3; for a simple account F. Strocchi, *An Introduction to the Mathematical Structure of Quantum Mechanics*, World Scientific 2005, Sect. 6.4.

fermion mass perturbation would yield a *CP* non symmetric θ vacuum, if $\theta \neq \theta_M$. In the analogy with the Bloch electron, this would correspond to choose a band higher than the ground state band. Furthermore, the energy spectrum has an essential singularity for vanishing gauge coupling constant.

Since no (local) observable can make transition between different θ sectors, it is of some interest to investigate a mechanism for reaching the *CP* symmetric vacuum if one starts from a "metastable" *CP* violating θ vacuum.

A possible selection of the absolute ground state emerges in a careful discussion of the thermodynamical limit performed on the functional integral.[13] In particular the limit $M \to 0$ and the thermodynamical limit do not commute. The alignment of θ to θ_M in the thermodynamical limit occurs by the same mechanism by which in the Heisenberg-Weiss model the mean field spin gets aligned to an external magnetic field.[14]

[13] J. Löffelholz, G. Morchio and F. Strocchi, Ann. Phys. **250**, 367 (1996).

[14] G. Morchio and F. Strocchi, Boundary Terms, Long Range Effects, and Chiral Symmetry Breaking, Lectures at the Schladming School 1990, *Fields and Particles*, H. Mitter and W. Schweigen eds., Springer 1990, p. 171.

Chapter 4
Non-regular Representations in Quantum Field Theory

1 Quantum Field Algebras and Quantizations

The traditional (historically the first) approach to field quantization[1] mimics very closely the standard quantum mechanical case, by realizing a field as a mechanical system of infinite degrees of freedom and by adopting the canonical formalism and quantization.

For example, for a real scalar field $\varphi(\mathbf{x}, t)$, by choosing a complete set $\{f_n(\mathbf{x})\}$, $n \in \mathbf{N}$, of real orthonormal smooth functions, a real scalar field is completely identified by a denumerable set of Lagrangian variables

$$q_n(t) \equiv \varphi(f_n, t) \equiv \int d^3x f_n(\mathbf{x})\varphi(\mathbf{x}, t), \quad \dot{q}_n(t) \equiv \dot{\varphi}(f_n, t), \tag{4.1.1}$$

since

$$\varphi(\mathbf{x}, t) = \sum_n q_n(t) f_n(\mathbf{x}), \quad \dot{\varphi}(\mathbf{x}, t) = \sum_n \dot{q}_n(t) f_n(\mathbf{x}).$$

A canonical formalism may be introduced by defining the canonical momenta $p_i(t)$ by

$$p_n(t) = \partial L/\partial \dot{q}_n(t) = \int d^3x\, \pi(\mathbf{x}, t) f_n(\mathbf{x}), \quad \pi(\mathbf{x}, t) \equiv \partial \mathcal{L}/\partial \dot{\varphi}(\mathbf{x}, t),$$

[1]See G. Wentzel, *Quantum Theory of Fields*, Dover 1949; S. Schweber, *An Introduction to Relativistic Quantum Field Theory*, Harper and Row 1961, Part Two.

F. Strocchi, *Gauge Invariance and Weyl-polymer Quantization*,
Lecture Notes in Physics 904, DOI 10.1007/978-3-319-17695-6_4

where

$$L \equiv \int d^3x \, \mathcal{L}(\varphi(\mathbf{x}, t), \partial_i \varphi(\mathbf{x}, t), \dot{\varphi}(\mathbf{x}, t))$$

is the Lagrangian function written as an integral over the Lagrangian density $\mathcal{L}$. Then the Hamiltonian is defined by

$$H = \sum_n p_n(t) \, \dot{q}_n(t) - L = \int d^3x \, (\partial \mathcal{L}(x)/\partial \dot{\varphi}(x) \;\; \dot{\varphi}(x) - \mathcal{L}(x)).$$

The canonical (classical) Poisson brackets read

$$\{q_i(t), \, p_j(t)\} = \delta_{ij}, \tag{4.1.2}$$

all other Poisson brackets vanishing. They are equivalent to the following Poisson brackets for the fields

$$\{\varphi(\mathbf{x}, t), \, \pi(\mathbf{y}, t)\} = \delta^3(\mathbf{x} - \mathbf{y}), \tag{4.1.3}$$

all other Poisson brackets vanishing.

Canonical quantization is obtained by Dirac prescription of replacing the Poisson brackets by $-i$ times commutators. Thus, the Heisenberg Lie algebra and the Heisenberg group become infinitely dimensional and the corresponding Weyl algebra infinitely generated.

It should be mentioned that canonical quantization may be in conflict with a non-trivial interaction. In fact, it requires that the fields allow sharp time restrictions, at least as (tempered) distributions in the space variables and perturbative and non-perturbative results indicate that generically this is not the case in 3+1 space-time dimensions, except in the free case.[2]

This conflict arises whenever the so-called wave function renormalization constant Z^{-1} is divergent as a consequence of short distance or ultraviolet (UV) singularities. In fact, in this case canonical commutation relations may be required to hold for the ill defined unrenormalized fields, whereas the canonical commutation relations of the renormalized fields are proportional Z^{-1}.[3] Thus, in order to safely use canonical quantization one must introduce an UV cutoff.

[2]For (irreducible) fermion fields, R.T. Powers (Comm. Math. Phys. **4**, 145 (1967)) has shown that canonical anticommutation relation are compatible only with a free theory in $s + 1$ dimensions with $s > 1$, under very general conditions. No interaction theorems for the bosonic case have been proved by K. Baumann (Jour. Math. Phys. **28**, 697 (1987); Lecture at the Schladming School 1987, in *Recent Developments in Mathematical Physics*, H. Mitter and L. Pittner eds., Springer 1987).

[3]A.S. Wightman, Phys. Rev. **101**, 105 (1956); S. Schweber, *An Introduction to Relativistic Quantum Field Theory*, Harper and Row 1961, Sect. 7b; for a brief discussion see e.g. F. Strocchi, *An introduction to non-perturbative foundations of quantum field theory*, Oxford Univ. Press 2013.

For these reasons, following Wightman,[4] a quantum field theory (QFT) is defined in terms of a **field *-algebra** $\mathcal{F}$, briefly called **field algebra**, generated by the polynomials of the smeared fields

$$\varphi(f) = \int dx\, \varphi(x) f(x), \quad f \in \mathcal{S}(\mathbf{R}^d),$$

d denoting the space-time dimension and $\mathcal{S}(\mathbf{R}^d)$ the C^∞ functions of fast decrease. The field algebra is called *local* if it satisfies microcausality or locality; e.g., for a hermitian scalar field $\varphi(x)$, this means

$$[\,\varphi(x),\, \varphi(y)\,] = 0, \quad \text{if} \quad (x-y)^2 < 0. \tag{4.1.4}$$

It will be understood that $\mathcal{F}$ will also include the sharp time restrictions of the fields, whenever they exist as distributions in the space variables. The space-time translations are assumed to define automorphisms α_a, $a = (t, \mathbf{a}) \in \mathbf{R}^d$, of $\mathcal{F}$.

In the following, $\mathcal{F}_W$ denotes the **exponential field algebra** formally generated by the exponentials of the (smeared) fields $\varphi(f)$; $\mathcal{A}$ and $\mathcal{A}_W$ will denote the **observable (field) subalgebras** of $\mathcal{F}$ and of $\mathcal{F}_W$, respectively. It should be mentioned that the definition of the exponential of a smeared field requires its self-adjointness, a property which is not obvious in QFT, in contrast with the quantum mechanical case; for this purpose general conditions have been discussed in constructive QFT, which guarantee the exponentiability of fields.[5]

Definition 1.1 *A representation of a field algebra $\mathcal{F}$ is a homomorphism π into a *-algebra of operators acting on a common dense domain of a Hilbert space $\mathcal{H}$.*

By an easy adaptation of the GNS theorem, a positive linear functional ω on $\mathcal{F}$ defines a representation of $\mathcal{F}$ in a Hilbert space $\mathcal{H}$, with a cyclic vector Ψ_ω such that

$$\omega(F) = (\Psi_\omega,\, \pi_\omega(F)\, \Psi_\omega), \quad \forall F \in \mathcal{F}.$$

Such a result in QFT is called the Wightman reconstruction theorem (see Streater and Wightman book).

Definition 1.2 *A quantum field theory is a representation of a field algebra $\mathcal{F}$ defined by a positive linear functional W on $\mathcal{F}$, invariant under space-time translations (d the spacetime dimensions)*

$$W(\alpha_a(F)) = W(F), \quad \forall a \in \mathbf{R}^d, \;\; \forall F \in \mathcal{F},$$

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[4] R.F. Streater and A.S. Wightman, *P C T, Spin and Statistics and All That*, Benjamin 1980; R. Jost, *The General Theory of Quantized Fields*, Am. Math. Soc. 1965.

[5] J. Fröhlich, Comm. Math. Phys.**54**, 135 (1977).

and satisfying the positive energy spectral condition

$$\int dt\, e^{-itE}\, W(F\alpha_t(G)) = 0, \quad \textit{if} \quad E < 0, \tag{4.1.5}$$

where α_t denotes the time translations.

A relativistic quantum field theory is defined by a local field algebra $\mathcal{F}$, transforming covariantly under the Poincaré transformations $\alpha_{a,\Lambda}$, and by a state invariant under the Poincaré group.

It should be remarked that the field algebra $\mathcal{F}$ involved for a formulation of a quantum field theory does not need to consist of observable fields only. Therefore, the locality of $\mathcal{F}$ and its covariance under the Lorentz transformations are not a must, for a relativistic quantum mechanical interpretation of the theory; clearly, all what is needed for an acceptable relativistic interpretation is that locality and covariance are fulfilled by the observable subalgebra $\mathcal{A}$. Such a structure characterizes gauge quantum field theories and closely resembles the quantum mechanical structures discussed in Chap. 2, Sect. 4.

Since $\mathcal{F}$ is not a C^*-algebra, just as it happens for the polynomial algebra generated by the Heisenberg canonical variables, a positive linear functional W provides a (natural) generalization of the concept of state, and, with a slight abuse of language, shall be called a *vacuum state* on $\mathcal{F}$, if it is invariant under space-time translations and satisfies the positive energy spectral condition.

Definition 1.3 *A vacuum state ω defines a* **regular** *representation π of the exponential field algebra $\mathcal{F}_W$ if ω has a well defined (positive) extension to the field algebra $\mathcal{F}$; otherwise, the representation defined by it is said to be a* **non-regular** *representation.*

Under general technical conditions the correlation function of a (smeared) field $\varphi(f)$ can be obtained from a representation π of $\mathcal{F}_W$ if the one-parameter group defined by the exponentials of $\varphi(f)$ is weakly continuous. A representation of $\mathcal{F}_W$ is *physical* if the representation of the observable subalgebra $\mathcal{A}_W$ is regular, in order to allow for the existence of the correlation functions of observable fields, as required by a reasonable physical interpretation. However, non-regular representations of the non-observable (exponential) field algebras are allowed, as in the quantum mechanical examples discussed before.

As discussed in Chap. 2, Sect. 4, a general mechanism for the need of non-regular representations of the field algebra $\mathcal{F}_W$ is when i) the observable algebra $\mathcal{A}$ is a proper subalgebra of the field algebra characterized by its pointwise invariance under a group $\mathcal{G}$ of automorphisms of $\mathcal{F}_W$ defined by elements of $\mathcal{F}_W$, and ii) one considers representations defined by gauge invariant vacuum states.

As we shall discuss below, the characteristic property of *gauge quantum field theories* is the Gauss law[6] and the Gauss law operator generates gauge transformations. Hence, the validity of the Gauss law constraint, as an operator equation satisfied by the physical states, requires a non-regular representation of the field algebra $\mathcal{F}$, unless positivity of the vacuum functional is abandoned.

An explicit analysis shall be presented in Sect. 3 below for the QED case and in Sect. 4 for the QCD case. As we shall see, the use of non-regular representations will not only provide a mathematically acceptable formulation, but it will also prove useful for the discussion and solution of structural problems.

2 Massless Scalar Field in 1+1 Dimensions

The quantum theory of a relativistic massless hermitian scalar field in two space-time dimensions has attracted much attention in the last years, not only as a gauge model, but also because it is one of the building blocks of very instructive two-dimensional models, like the Schwinger model, which mimics very closely some basic structure expected to characterize QCD, like quark confinement and chiral symmetry breaking.[7]

The massless scalar field is also at the basis of the construction of two-dimensional conformal models, which have received much attention in connection with important developments of QFT[8] as well as for string theory.[9]

At the classical level, the relativistic massless real scalar field $\varphi(x, t)$ in 1+1 dimensions satisfies the free wave equation and the classical canonical Poisson brackets

$$\Box\varphi = 0, \qquad \{\varphi(x,t),\, \partial_0\varphi(y,t)\,\} = \delta(x-y), \tag{4.2.1}$$

all the other Poisson brackets vanishing.

The quantization of such a system meets non-trivial problems. If one tries to adopt canonical quantization, the so obtained field algebra does not admit a positive linear functional ω_0 invariant under the Poincaré group $\mathcal{P}$ (space-time translations

[6]For a focusing of this property see e.g. F. Strocchi and A.S. Wightman, Jour. Math. Phys. **15**, 2198 (1974); F. Strocchi, Gauss law in local quantum field theory, in *Field Theory, Quantization and Statistical Physics* D. Reidel 1981, p. 227–236; F. Strocchi, *Elements of Quantum Mechanics of Infinite Systems*, World Scientific 1985, Part C, Chap. II; F. Strocchi, *An Introduction to Non-perturbative Foundations of Quantum Field Theory*, Oxford Univ. Press 2013.

[7]For an analysis of the Schwinger model which takes into account the delicate issue of the quantization of the massless scalar field in two-dimensions, see G. Morchio, D. Pierotti and F. Strocchi, Ann. Phys. **188**, 317 (1988); F. Strocchi, *Selected Topics on the General Properties of Quantum Field Theory*, World Scientific 1993, Chapter VII, Section 7.4.

[8]See e.g. P. Furlan, G.M. Sotkov and I.T. Todorov, two-dimensional Quantum Field Theory, Riv. Nuovo Cim. **12**, 1 (1989).

[9]B.F. Hatfield, *Quantum Field Theory of Point Particles and Strings*, Addison-Wesley 1992.

and Lorentz transformations). Actually, quite generally, the two point function of φ defined by the expectation of a Poincaré invariant vacuum ω_0 violates positivity.[10]

Such a result has led to the statement that there is no quantum field theory for the massless scalar field. Actually, as we shall see, the result means that there is no regular representation of the exponential field algebra $\mathcal{F}_W$ invariant under $\mathcal{P}$, but a Poincaré invariant non-regular representation of $\mathcal{F}_W$ exists.

For this purpose, we first note that, since the field is a free field, its sharp time restrictions exist and $\mathcal{F}_W$ may be assumed to contain the **Weyl field algebra** generated by the exponentials $U(h_1)$, $V(h_2)$ of $\varphi(h_1, t)$, $\pi(h_2, t) \equiv \dot{\varphi}(h_2, t)$, $h_i \in \mathcal{S}(\mathbf{R})$, respectively, satisfying the Weyl commutation relations

$$U(h_1)\, V(h_2) = V(h_2)\, U(h_1)\, e^{i(h_1, h_2)}, \tag{4.2.2}$$

where $(h_1,\, h_2)$ denotes the $L^2(\mathbf{R}, dx)$ scalar product.

A (unique) Poincaré invariant vacuum state Ω exists for the subalgebra $\mathcal{A}_W$ generated by the exponentials of $\varphi(f)$, with $\tilde{f}(0) = 0$, briefly $f \in \mathcal{S}_0(\mathbf{R}^2)$, defined by the two point function

$$\Omega(\partial_\mu \varphi(x)\, \partial_\nu \varphi(y)) = \partial_\mu \partial_\nu (4\pi)^{-1} \log[-(x-y)^2 + i\varepsilon(x_0 - y_0)\,], \tag{4.2.3}$$

(we recall that the elements of $\mathcal{S}_0(\mathbf{R})$ may be written as the derivatives of elements of $\mathcal{S}(\mathbf{R})$). Such a state also satisfies the relativistic spectral condition, i.e. one has $\forall A,\, B \in \mathcal{A}_W$

$$\int da\, e^{-ipa}\, \Omega(A\, U(a)\, B) = 0, \quad \text{if} \quad p \notin \overline{V}_+ \equiv \{p; p^2 \geq 0,\ p_0 \geq 0\,\}. \tag{4.2.4}$$

Thus, $\mathcal{A}_W$ qualifies as the **observable subalgebra of** $\mathcal{F}_W$.

$\mathcal{A}_W$ can be characterized as the subalgebra of $\mathcal{F}_W$ which is pointwise invariant under the *gauge transformations*

$$\beta^\lambda : \varphi \to \varphi + \lambda, \qquad \lambda \in \mathbf{R}.$$

Thus, the model shares the general structure discussed in Chap. 2, Sect. 4.

Proposition 2.1 *Let Ω be a positive (pure) vacuum state on $\mathcal{A}_W$, then*

i) Ω does not have a positive extension to the field algebra $\mathcal{F}$, generated by $\varphi(g)$, $g \in \mathcal{S}(\mathbf{R}^2)$, invariant under space-time translations and satisfying the positive energy spectral condition,

[10] A.S. Wightman, Introduction to some aspects of the relativistic dynamics of quantized fields, in *High Energy Electromagnetic Interactions and Field Theory*, Cargèse lectures in Theoretical Physics (1964), M. Lévy ed., Gordon and Breach 1967, esp. Sect. I.4; for a general analysis of the quantization in terms of a non-positive vacuum state, see G. Morchio, D. Pierotti and F. Strocchi, Jour. Math. Phys. **31**, 147 (1990).

ii) Ω *has a unique positive extension* ω *to* $\mathcal{F}_W$, *invariant under gauge transformations and therefore given by*

$$\omega(U(h)\,V(g)) = \Omega(U(h)\,V(g)), \quad \textit{if} \quad \tilde{h}(0) = 0,$$
$$= 0, \quad \textit{otherwise}. \tag{4.2.5}$$

The representation π *of* $\mathcal{F}_W$ *defined by the gauge invariant vacuum state* ω *has the property that its representation space* $\mathcal{K}$ *is non-separable and has the following orthogonal decomposition into disjoint irreducible representations of* $\mathcal{A}_W$

$$\mathcal{K} = \sum_{\alpha \in \mathbf{R}} \oplus \mathcal{H}_\alpha, \qquad \mathcal{H}_\alpha = \overline{\pi(\mathcal{A}_W)\,U(h)\,\Psi_0}, \quad \tilde{h}(0) = \alpha, \tag{4.2.6}$$

where Ψ_0 *is the representative vector of* ω.

Proof The free wave equation in two space-time dimensions implies that the Fourier transform of the two point function $\omega(\varphi(x)\,\varphi(y))$ defined by a translationally invariant state ω is of the form

$$A(p_+)\,\delta(p_-) + B(p_-)\,\delta(p_+), \quad p_\pm \equiv p_0 \pm p_1, \quad A, B \in \mathcal{S}'(\mathbf{R}).$$

The energy spectral condition requires that A and B be of the form $A(p_+) = a(p_+)\theta(p_+)$, $B(p_-) = b(p_-)\theta(p_-)$, and the condition of providing an extension of the unique two point function (4.2.3) implies that the distributions $a(p_+)$, $b(p_-)$ satisfy

$$p_+\,a(p_+)\theta(p_+) = 1, \quad p_-\,b(p_-)\,\theta(p_-) = 1$$

There are no positive distributions or measures which satisfy such equations except $\delta(p_\pm)$, which lead to a trivial two point function, (incompatibly with the non-triviality of Ω on $\mathcal{A}_W$). Thus, Ω does not have a positive extension to $\mathcal{F}$.

A positive extension to $\mathcal{F}_W$ is provided by the gauge invariant state defined by Eq. (4.2.5). It is easy to see that gauge invariance uniquely selects such an extension. In fact, a gauge invariant extension ω of Ω to $\mathcal{F}_W$ satisfies

$$\omega(U(h)V(g)) = \omega(\beta^\lambda(U(h)\,V(g))) = e^{i\,\lambda \tilde{h}(0)}\,\omega(U(h)V(g))),$$

i.e. Eq. (4.2.5) holds, since for $\tilde{h}(0) = 0$, $U(h)\,V(g) \in \mathcal{A}_W$ and ω coincides with Ω there. One can actually show that the state ω defined by Eq. (4.2.5) is the only positive extension of Ω.[11]

[11] See F. Acerbi, G. Morchio and F. Strocchi, Jour. Math. Phys. **34**, 899 (1993); F. Acerbi, Master thesis, SISSA Trieste, 1990.

The fields $U(h)$ have the interpretation of *charged fields* with charge $\alpha \equiv \int dx\, h(x) = \tilde{h}(0)$:

$$\beta^{\lambda}(U(h)) = e^{i\alpha\lambda}\, U(h).$$

The gauge invariance of ω implies that the gauge group is unbroken in $\mathcal{K}$ and the decomposition of Eq. (4.2.6) follows.

In conclusion, the model reproduces most of the structural features discussed in Chap. 2, for a general quantum mechanical model described by a canonical algebra. Thus, one has the analog of the non-regular representations of the Heisenberg group, which has now become infinite dimensional.

An important structure associated to the quantization of the massless scalar field is obtained by considering the extension $\mathcal{A}_F$ of the observable algebra $\mathcal{A}_W$ generated by the (unitary) operators

$$U(h), \;\; V(g), \;\; h \in \mathcal{S}(\mathbf{R}), \quad g \in \partial^{-1}\mathcal{S}(\mathbf{R}) \equiv \{g \in C^{\infty}(\mathbf{R}), \;\; \partial g \in \mathcal{S}(\mathbf{R})\}.$$

The algebra $\mathcal{A}_F$ contains anticommuting fields, which can be interpreted as *fermionic fields*, and its construction in terms of a scalar (bosonic) field is called *fermion bosonization*. The (positive) extension of Ω to $\mathcal{A}_F$ defines a non-regular representation with an unbroken $U(1)$ group generated by the fermionic charges.[12]

It may be worthwhile to mention that the non-regular representation of $\mathcal{A}_F$ provides a simple mechanism for the proof of confinement in the bosonized Schwinger model, since in this case the representation of the time translations is not weakly continuous; this implies that in the charged sectors one cannot have a finite energy and therefore the corresponding states are not physically realizable.[13]

3 Temporal Gauge in QED

For the analysis of non-perturbative aspects of QED, typically by the functional integral approach, and in particular for the discussion of the existence of a symmetry breaking order parameter in the Higgs phenomenon, the temporal gauge has been widely used, but the textbook presentations of such a gauge have somewhat neglected its peculiar mathematical features and in particular the mathematical consistency of the proposed realizations.[14]

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[12]For an analysis of this structure, see F. Acerbi, G. Morchio and F. Strocchi, Lett. Math. Phys. **26**, 13 (1992).

[13]F. Acerbi, G. Morchio and F. Strocchi, Lett. Math. Phys. **27**, 1 (1993).

[14]See e.g. the comprehensive book by A. Bassetto, G. Nardelli and R. Soldati, *Yang-Mills Theories in Algebraic Non-covariant Gauges*, World Scientific 1991.

The *temporal gauge* is defined by the gauge condition $A_0 = 0$ (both in the free as well as in the interacting case), without requiring the transversality of A_i. Thus, in the free case it is similar to the Coulomb gauge, but longitudinal photons are allowed. Manifest covariance is obviously lost, but, in contrast with the Coulomb gauge, locality holds in the free case and has been argued to persist in the interacting case.

The variation with respect to A_i of the gauge invariant Lagrangian

$$\mathcal{L} = -\tfrac{1}{4} F_{\mu\nu}\, F^{\mu\nu} + \mathcal{L}_{matter},$$

where $\mathcal{L}_{matter}$ is the gauge invariant Lagrangian for the matter fields (for simplicity taken to be Dirac fermions $\psi, \bar{\psi}$), typically obtained from the free Lagrangian by the minimal coupling prescription, yields the following equations of motion

$$\partial_0^2 A_i - \Delta A_i + \partial_i \mathrm{div} A = j_i, \tag{4.3.1}$$

where j_μ is the conserved gauge invariant electromagnetic current constructed in terms of the charged fields.

The canonical commutation relations for the vector potential

$$[\,A_i(\mathbf{x}, 0),\; \partial_0 A_j(\mathbf{y}, 0)\,] = i\,\delta_{ij}\delta(\mathbf{x} - \mathbf{y}), \quad [A_i(\mathbf{x}, 0),\, A_j(\mathbf{y}, 0)] = 0, \tag{4.3.2}$$

are clearly incompatible with the Gauss law, which in the free case reads $\mathrm{div} E = 0$, $E_i = \partial_0 A_i$ and such an incompatibility persists in the interacting case.

In fact, the conflict between quantization and the validity of the Gauss law $\mathrm{div} E(x) - j_0(x) = 0$ can be argued quite generally, independently of the existence of equal time restrictions of the fields and of a full quantum canonical structure. Eqs. (4.3.1) and the current continuity equation imply that $G(\mathbf{x}, x_0) \equiv \mathrm{div} E(x) - j_0(x)$ is time independent, so that

$$G(f, x_0) \equiv G(f, h), \quad f \in \mathcal{S}(\mathbf{R}^3),\; h \in \mathcal{S}(\mathbf{R}), \quad \int dt\, h(t) = 1,$$

is a well (densely) defined operator and its equal time commutators with the fields are well defined operator valued distributions, fixed by the condition, which can be taken as part of the definition of the temporal gauge, that G generates the time independent gauge transformations. Thus, quite generally, one has

$$[\,A_i(\mathbf{x}, t), G(\mathbf{y}, t)\,] = -i\,\partial_i \delta(\mathbf{x} - \mathbf{y}) \tag{4.3.3}$$

and, by the time independence of $G(f, t)$

$$[\,A_i(g, h),\; G(f, t)\,] = i(\partial_i g, f)\,\tilde{h}(0),$$

where $(\cdot,\, \cdot)$ denotes the $L^2(\mathbf{R}^3, dx)$ scalar product.

By taking the vacuum expectation of the above equation, one gets a conflict with the validity of the Gauss law on the vacuum:

$$G(f)\Psi_0 = 0.$$

There are only two (mathematically consistent) alternative solutions[15] of such a conflict between Eq. (4.3.3) and the Gauss law, with peculiar (widely unnoticed) mathematical properties.

One alternative, characterized by the requirement of existence of the two point function (or of the propagator) of the vector potential A_i, requires to abandon positivity and to weaken the Gauss law. In the second alternative, the vector potential cannot be defined as a field operator, only its exponentials being definable.[16]

For the second (positive) alternative, it is convenient to introduce the following field algebras (for simplicity, we consider the case in which the charged fields are the Dirac fermion fields ψ, $\overline{\psi}$):

i) *the field algebra* $\mathcal{F}$, generated by the smeared fields $A(f) \equiv A_i(f_i)$, and $\psi(g), \overline{\psi}(g), f_i,\ g \in \mathcal{S}(\mathbf{R}^4)$;
ii) *the observable field subalgebra* $\mathcal{A}$ characterized by its pointwise invariance under time independent gauge transformations:

$$[\,G(f),\, A\,] = 0, \quad \forall A \in \mathcal{A};$$

iii) *the exponential field algebra* $\mathcal{F}_W$ generated by $\exp[iA(f)]$, and $\psi(g)$, $\overline{\psi}(g)$, $f, g \in \mathcal{S}(\mathbf{R}^4)$;
iv) *the exponential longitudinal algebra* $\mathcal{F}_L$ generated by $\exp[iG(f)]$, $\exp[iA(\partial h)]$, $A(\partial h) \equiv A_i(\partial_i h)$, $h \in \mathcal{S}(\mathbf{R}^4)$.

The inevitable violation of positivity for the regular quantizations and the general properties of the positive quantizations are stated by the following

Proposition 3.1 *Let* Ω *be a Poincaré invariant positive vacuum state on the observable subalgebra* $\mathcal{A}$*, then one has*

i) any extension of Ω *to the field algebra* $\mathcal{F}$ *violates positivity*
ii) all the positive extensions ω *to the exponential field algebra* $\mathcal{F}_W$ *are invariant under time independent gauge transformations, in particular, if* Ψ_ω *denotes the GNS cyclic vector which represents the state* ω*, one has*

$$G(f)\Psi_\omega = 0. \tag{4.3.4}$$

[15] J. Löffelholz, G. Morchio and F. Strocchi, Jour. Math. Phys. **44**, 5095 (2003), hereafter referred to as LMS.

[16] The non-regularity of the positive quantizations of the temporal gauge in the free case has been discussed by D. Buchholz and K. Fredenhagen, Comm. Math. Phys. **84**, 1 (1982); H. Grundling and C.A. Hurst, Lett. Math. Phys. **15**, 205 (1988); F. Acerbi, G. Morchio and F. Strocchi, J. Math. Phys. **34**, 899 (1993). The unique selection of the non-regular representation by the condition of positivity of the energy spectrum has been proved in LMS.

iii) *any positive extension ω of Ω to the exponential longitudinal algebra $\mathcal{F}_L$ is **non-regular**, since it must satisfy*

$$\omega(e^{i\alpha A(\partial h)}) = 0, \quad \textit{if } \alpha \neq 0; \\ = 1, \quad \textit{if } \alpha = 0 \tag{4.3.5}$$

iv) *in the GNS representation defined by the a positive extension ω to $\mathcal{F}_W$, the unitary operators $U(\mathbf{a})$ which implement the space translations are not strongly continuous, and therefore their generator, the momentum, cannot be defined in the whole representation space $\mathcal{H}_\omega$.*

Proof

i) Since $G(f)$ commutes with any element of $\mathcal{A}$, by Theorem 4.5 of Streater and Wightman book (referred to in footnote 4), if Ψ_Ω denotes the cyclic vector which represents the state Ω on $\mathcal{A}$,

$$G(f)\Psi_\Omega = c(f)\Psi_\Omega, \quad c(f) \in \mathbf{C},$$

and by the Lorentz invariance of Ω one has $c(f) = 0$. Hence, by Schwarz' inequality, a positive extension ω to $\mathcal{F}$ should satisfy

$$\omega(G(f)F) = 0, \quad \forall F \in \mathcal{F}, \tag{4.3.6}$$

which is incompatible with the ω-expectation of Eq. (4.3.3). Hence, there is no positive extension to $\mathcal{F}$.

ii) Any positive extension to $\mathcal{F}_W$ satisfies Eq. (4.3.6), with F replaced by any $B \in \mathcal{F}_W$, so that Eq. (4.3.4) follows by the cyclicity of Ψ_ω.

iii) Putting $V(f) \equiv \exp iG(f)$, one has

$$\omega(e^{i\alpha A(\partial h)}) = \omega(V(-f)\, e^{i\alpha A(\partial h)}\, V(f)) = e^{i\alpha \int d^4x h \Delta f}\, \omega(e^{i\alpha A(\partial h)})$$

and Eq. (4.3.5) follows.

iv) Putting $h_{\mathbf{a}}(x) \equiv h(x + \mathbf{a})$ one has

$$\omega(e^{iA(\partial h)}U(-\mathbf{a})e^{-iA(\partial h)}) = \omega(e^{iA(\partial h - \partial h_{\mathbf{a}})}) = 0, \quad \text{if} \quad \mathbf{a} \neq 0; \\ = 1, \quad \text{if} \quad \mathbf{a} = 0.$$

The invoked escape of a non-normalizable vacuum vector is mathematically unacceptable because, if the vacuum is not normalizable, the divergent expectations are not confined to those of the gauge dependent fields, but affect also the observable fields.

As far as we know, the only mathematically acceptable solution is to keep the normalizability of the vacuum and give up the well definiteness of the gauge dependent fields $A(f)$, only their exponentials having well defined vacuum correlation functions. This strategy has a well founded mathematical status and it is under complete control for the quantization of canonical systems as discussed before.

In the free field case the positive non-regular representation of the exponential field algebra $\mathcal{F}_W$ is uniquely determined solely by the condition of positivity of the energy spectrum.[17]

4 Temporal Gauge in QCD: Chiral Symmetry Breaking

For the discussion of the non-perturbative aspects of QCD, in particular the θ vacuum structure, its relation with the topology of the gauge field configurations and its role in chiral symmetry breaking, the temporal gauge has proved to be particularly convenient,[18] because the residual gauge group $\mathcal{G}$ consists of the time independent gauge transformations and its non-trivial topology emerges clearly. However, as noted before for the abelian (QED) case, the conflict between the Gauss law constraint and canonical quantization raises problems of mathematical consistency, which also have a non-trivial impact on the derivation of the general structures leading to significant physical implications.

4.1 Gauss Law and Gauge Transformations

For simplicity, we start by considering the case with only vector fields (no fermion or scalar field being present). Then, at the classical level the QCD Lagrangian density reduces to the Yang-Mills form

$$\mathcal{L} = -\tfrac{1}{4} \sum_a F^a_{\mu\nu} F^{\mu\nu\, a} = \tfrac{1}{2} \sum_a (\mathbf{E}_a^2 - \mathbf{B}_a^2), \tag{4.4.1}$$

where in the temporal gauge, defined by $A^0_a = 0$,

$$\mathbf{E}_a = -\dot{\mathbf{A}}_a, \qquad \mathbf{B}_a = \nabla_a \times \mathbf{A}_a - \tfrac{1}{2} g f_{abc}\, \mathbf{A}_b \times \mathbf{A}_c, \tag{4.4.2}$$

(a is a color index and f_{abc} are the structure constants of the Lie algebra of the color gauge group). The corresponding equations of motion, obtained by variations with

[17] LMS, Proposition 3.3.

[18] See the excellent lectures by R. Jackiw, Topological investigations of quantized gauge theories, in S.B. Treiman, R. Jackiw, B. Zumino and E. Witten, *Current Algebra and Anomalies*, World Scientific 1985, pp. 211–359. This section relies on G. Morchio and F. Strocchi, Ann. Phys. **324**, 2236 (2009), hereafter referred to as Ref. MS; for an expository account, see F. Strocchi, *An introduction to non-perturbative foundations of quantum field theory*, Oxford Univ. Press 2013.

respect to $\mathbf{A}_a$,

$$\partial_t \mathbf{E}_a = \nabla \times \mathbf{B}_a + g f_{abc} \mathbf{A}_b \times \mathbf{B}_c \equiv (\mathbf{D} \times \mathbf{B})_a, \tag{4.4.3}$$

imply

$$\partial_t G_a = 0, \quad G_a \equiv \nabla \cdot \mathbf{E}_a + g f_{abc} \mathbf{A}_b \cdot \mathbf{E}_c \equiv (\mathbf{D} \cdot \mathbf{E})_a. \tag{4.4.4}$$

The operators G_a are called the *Gauss law operators*.

In the standard quantum version of the temporal gauge it is assumed that the fields $\mathbf{A}_a$ and their powers can be defined and the quantization is given by the canonical commutation relations. In particular one has the following commutation relations

$$-i\,[\,\mathbf{D} \cdot \mathbf{E}_a(\mathbf{x}, t),\ \mathbf{A}_b(\mathbf{y}, t)\,] = \delta_{ab}\, \nabla \delta(\mathbf{x} - \mathbf{y}) + g f_{abc} \mathbf{A}_c(\mathbf{x}, t)\, \delta(\mathbf{x} - \mathbf{y}). \tag{4.4.5}$$
$$-i\,[\,\mathbf{D} \cdot \mathbf{E}_a(\mathbf{x}, t),\ \mathbf{E}_b(\mathbf{y}, t)\,] = g f_{abc}\, \mathbf{E}_c(\mathbf{x}, t)\, \delta(\mathbf{x} - \mathbf{y}).$$

They state that the Gauss operators generate the infinitesimal time independent gauge transformations, δ^Λ, with the c-number gauge function $\Lambda^a(\mathbf{x}) \in \mathcal{S}(\mathbf{R}^3)$

$$\delta^\Lambda \mathbf{A}_b(x) = \delta_{a\,b}\, \nabla \Lambda^a(\mathbf{x}) + g f_{abc} \mathbf{A}^b(x)\, \Lambda^c(\mathbf{x})). \tag{4.4.6}$$

Since the variables A^0_a are missing in the Lagrangian, one cannot exploit the stationarity of the action with respect to them and therefore one does not get the Gauss law $G_a = 0$. Actually, the Gauss law is incompatible with Eq. (4.4.5) and therefore with canonical quantization and more crucially with the Gauss operator being the generator of the time independent gauge transformations, Eq. (4.4.6).

A proposed solution of this conflict, widely adopted in the literature and in textbook discussions of the temporal gauge, is to require the Gauss law constraint as an operator equation on the (subspace of) physical states and, in particular, on the vacuum state. However, such a solution is not free of paradoxes and mathematical inconsistency. In fact, the vacuum expectation of Eq. (4.4.5) gives zero on the left hand side and non-zero on the right hand side.

It has been proposed[19] to cope with this paradox by admitting that the vacuum vector is not normalizable. In our opinion, such a solution is not acceptable, because it does not yield a representation of a field algebra containing both gauge dependent and gauge independent fields, since the non-normalizability of the vacuum vector would give divergent expectations also of the observable algebra (which in particular contains the identity).

A mathematically acceptable solution for the Gauss law constraint is to adopt a Weyl quantization and admit non-regular representations.

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[19] R. Jackiw, loc cit.; A. Bassetto, G. Nardelli and R. Soldati, *Yang-Mills Theories in Algebraic Non-covariant Gauges*, World Scientific 1991.

To this purpose, we consider the *local field algebra* $\mathcal{F}$ generated by the polynomials of the smeared fields

$$A^i_a(f) = \int d^4x A^i_a(x) f(x), \quad f \in \mathcal{S}(\mathbf{R}^4),$$

transforming covariantly under the space time translations α_y, $y \in \mathbf{R}^4$, i.e. $\alpha_y(A^i_a(f)) = A^i_a(f_y)$, $f_y(x) = f(x-y)$. We shall assume that by a suitable point splitting procedure one can define as local space-time covariant fields belonging to $\mathcal{F}$ the powers of $\mathbf{A}_a(x)$ and its derivatives, like e.g. the Gauss operator $G_a(g)$, the "magnetic" field $B^i_a(g)$, $g \in \mathcal{S}(\mathbf{R}^4)$, etc.

For a simpler bookkeeping of the indices, it is convenient to introduce the following notations:

i) T^a denote the hermitian representation matrices of the Lie algebra of the gauge group, normalized so that $\mathrm{Tr}(T^a T^b) = \delta_{ab}$;

ii) the fields $A^i(x) = \sum_a A^i_a(x) T^a$ are Lie algebra valued distributions on Lie algebra valued test functions $f^i(x) = \sum_a f^i_a(x)\, T^a, f^i_a \in \mathcal{S}(\mathbf{R}^4)$, (the sum over repeated indices is understood) and

$$\mathbf{A}(\mathbf{f}) \equiv \int d^4x \mathrm{Tr}\,[\,\mathbf{A}(x)\,\mathbf{f}(x)\,] = \int d^4x \sum_{i,a} A^i_a(x) f^i_a(x);$$

iii) $\mathcal{G}$ denotes the **group of time independent gauge transformations** $\alpha_{\mathcal{U}}$ labeled by *space localized gauge functions* $\mathcal{U}(\mathbf{x})$ which are gauge group valued C^∞ functions differing from the identity only on a (space) compact set $K_{\mathcal{U}}$; the space-time support of $\mathcal{U} - \mathbf{1}$ is given by the cylinder $C_{\mathcal{U}} \equiv K_{\mathcal{U}} \times \mathbf{R}$ and

$$\alpha_{\mathcal{U}}(\mathbf{A}(\mathbf{f})) = \mathbf{A}(\mathcal{U}\mathbf{f}\mathcal{U}^{-1}) + \mathcal{U}\boldsymbol{\partial}\mathcal{U}^{-1}(\mathbf{f}), \tag{4.4.7}$$

$$\mathcal{U}\boldsymbol{\partial}\mathcal{U}^{-1}(\mathbf{f}) \equiv \int d^4x\ \mathrm{Tr}\,[\sum_i \mathcal{U}(\mathbf{x})\partial^i\mathcal{U}^{-1}(\mathbf{x}) f^i(x)\,] \equiv D_{\mathcal{U}}(\mathbf{f})$$

iv) $\mathcal{U}^\lambda$, $\lambda \in \mathbf{R}$ denote the gauge functions corresponding to one-parameter subgroups of $\mathcal{G}$; they are of the form $\mathcal{U}^\lambda(\mathbf{x}) = e^{i\lambda\, g(\mathbf{x})}$, with $g(\mathbf{x}) = \sum_a g_a(\mathbf{x})\, T^a$ a Lie algebra valued function, infinitely differentiable and of compact support ($g_a \in \mathcal{D}(\mathbf{R}^3)$). All gauge transformations of compact support in a neighborhood of the identity, in the C^∞ topology, are of this form and generate the **Gauss subgroup** $\mathcal{G}_0$.

For the Weyl quantization one has to consider the **exponential field algebra** $\mathcal{F}_W$ generated by the unitary operators $W(\mathbf{f})$, with $f^i(x) = \sum_a f^i_a(x)\, T^a, f^i_a \in \mathcal{S}(\mathbf{R}^4)$, formally the exponentials $e^{i\mathbf{A}(\mathbf{f})}$, and by the unitary operators $V(\mathcal{U}^\lambda)$, which represent $\mathcal{G}_0$, briefly called *Gauss implementers*, formally the exponentials of the Gauss operators,

$$V(\mathcal{U}^\lambda) = e^{i\lambda\, G(g)}, \quad G(g) = \sum_a G_a(g_a), \quad g_a \in \mathcal{D}(\mathbf{R}^3). \tag{4.4.8}$$

The operators $V(\mathcal{U}^\lambda)$ transform covariantly under space translations. As it is standard we assume that the dynamics α_t may be obtained as a limit of localized dynamics α_t^R defined by local gauge invariant Hamiltonians H_R (e.g. by Hamiltonian densities integrated over spheres of radius R), i.e. $\alpha_t(F) = \lim_{R\to\infty} \alpha_t^R(F)$.

Then, the gauge invariance of the local Hamiltonians implies the time independence of the $V(\mathcal{U}^\lambda)$

$$(d/dt)\alpha_t(V(\mathcal{U}^\lambda)) = i \lim_{R\to\infty} [\, H_R,\, V(\mathcal{U}^\lambda)\,] = 0.$$

A representation of $\mathcal{F}_W$ also defines a representation of $\mathcal{F}$, and in this case it is called *regular*, if (the representatives of) the field exponentials $W(\lambda\mathbf{f})$, $\lambda \in \mathbf{R}$, define weakly continuous one-parameter groups.

A *vacuum state* is invariant under spacetime translations and therefore in the corresponding representation the space-time translations α_a, $a \in \mathbf{R}^4$, are implemented by unitary operators $U(a)$.

A state ω on the exponential field algebra $\mathcal{F}_W$, in particular a vacuum state, satisfies the **Gauss law** in exponential form if $\omega(V(\mathcal{U}^\lambda)) = 1$, equivalently if its representative vector Ψ_ω, in the (GNS) Hilbert space $\mathcal{H}_\omega$ defined by the expectations $\omega(\mathcal{F}_W)$, satisfies

$$V(\mathcal{U}^\lambda)\, \Psi_\omega = \Psi_\omega, \quad \forall\, \mathcal{U}^\lambda.$$

Briefly, a vector state $\Psi \in \mathcal{H}_\omega$ satisfying $V(\mathcal{U}^\lambda)\Psi = \Psi$, $\forall\, \mathcal{U}^\lambda$, is said to be **Gauss invariant** and $\mathcal{H}' \subset \mathcal{H}_\omega$ denotes the subspace of Gauss invariant vectors. An operator in $\mathcal{H}_\omega$ is *Gauss invariant* if it commutes with all the $V(\mathcal{U}_\lambda)$.

Proposition 4.1 *A vacuum state ω on the exponential field algebra $\mathcal{F}_W$, satisfying the Gauss law, defines a non-regular representation of $\mathcal{F}_W$, since :*

$$\omega(W(\mathbf{f})) = 0, \qquad \textit{if} \quad \mathbf{f}(x) \neq 0. \tag{4.4.9}$$

The fields $\mathbf{A}$*, formally the generators of the* $W(\mathbf{f})$*, cannot be defined in the GNS Hilbert space defined by the vacuum expectations and in particular the two point function of the gauge potential does not exist, only (the vacuum expectations of) the exponential functions (and of course the gauge invariant functions) of* $\mathbf{A}$ *can be defined.*

In the free case, the exponential field algebra becomes a Weyl field algebra and Eqs. (4.4.9) uniquely determine its representation as a **non-regular Weyl quantization**.

Proof For each $\mathbf{f}$ there is a one-parameter subgroup $\mathcal{U}^\lambda$ such that $\mathcal{U}^\lambda\, \mathbf{f}\, \mathcal{U}^{-\lambda} = \mathbf{f}$, and $\exp iD_{\mathcal{U}^\lambda}(\mathbf{f}) = \exp(i\mathcal{U}^\lambda \boldsymbol{\partial}\, \mathcal{U}^{-\lambda}(\mathbf{f})) \neq 1$; therefore, by the Gauss invariance of ω, one has

$$\omega(W(\mathbf{f}) = \omega(V(\mathcal{U}^\lambda)\, W(\mathbf{f})\, V(\mathcal{U}^\lambda)^*) = e^{iD_{\mathcal{U}^\lambda}(\mathbf{f})}\, \omega(W(\mathbf{f})),$$

and Eqs. (4.4.9) follow.

Clearly, the one-parameter groups defined by $W(\mathbf{f})$ cannot be weakly continuous and therefore the corresponding generators, i.e. the fields $A^i_a(f)$ do not exist as operators in the GNS Hilbert space $\mathcal{H}_\omega$ defined by the expectations of $\mathcal{F}_W$ on ω. In particular the "gluon" propagator does not exists. The free case can be worked out as for the abelian case.[20]

The one-parameter groups $V(U_\lambda)$ are not assumed to be weakly continuous in λ; actually, continuity cannot hold if the global gauge group is simple and has rank at least two (as in the case of color $SU(3)$) (see Ref. MS).

4.2 Large Gauge Transformations

By definition, Eq. (4.4.7), the gauge functions considered are localized and, therefore, they extend to the one-point compactification $\dot{\mathbf{R}}^3$ of $\mathbf{R}^3$, which is isomorphic to the three-sphere S^3:

$$\mathcal{U}(\mathbf{x}) : \dot{\mathbf{R}}^3 \sim S^3 \to \mathcal{G}.$$

Such maps fall into disjoint homotopy classes labeled by "winding" numbers n

$$n(\mathcal{U}) = (24\pi^2)^{-1} \int d^3x\, \varepsilon^{ijk}\, \mathrm{Tr}\,(\mathcal{U}_i(\mathbf{x})\, \mathcal{U}_j(\mathbf{x})\, \mathcal{U}_k(\mathbf{x})) \equiv \int d^3x\, n_{\mathcal{U}}(\mathbf{x}), \qquad (4.4.10)$$

where $\mathcal{U}_i(\mathbf{x}) \equiv \mathcal{U}(\mathbf{x})^{-1}\, \partial_i\, \mathcal{U}(\mathbf{x})$.

The gauge transformations with $n \neq 0$ are called *large gauge transformations*. Those with zero winding number are called *small*; since they are contractible to the identity, they are products of $\mathcal{U}(\mathbf{x})$ which are close to the identity (in the C^∞ topology) and therefore are expressible as products of $\mathcal{U}^\lambda$. Clearly, all the gauge transformations of this form have zero winding number.[21]

Thank to the space localization of the gauge functions the Gauss invariance of the vacuum implies its invariance under the whole group $\mathcal{G}$ of (time independent)

[20] J. Löffelholz, G. Morchio and F. Strocchi, Jour. Math. Phys. **44**, 5095 (2003).

[21] For the geometry of the gauge transformations see S. Coleman, *Aspects of Symmetry*, Cambridge Univ. Press 1985, Chapter 7, Sect. 3; R. Jackiw, loc. cit.; T. Frankel, *The Geometry of Physics. An Introduction*, 2nd ed. Cambridge Univ. Press 2004.

gauge transformations, (see Ref. MS). Then, the $\alpha_{\mathcal{U}}$ are implementable by (time independent) unitary operators $V(\mathcal{U})$.

The physically crucial question is the implications of the non-trivial topology of the large gauge transformations on the classification of the physical states. Actually, the non-triviality of the large gauge transformations on the physical space turns out to be a rather subtle question, not fully appreciated in the standard treatment, and, as we shall see below, the presence of fermions plays a crucial role.

An important ingredient is the exploitation of the properties of the so-called *topological current*, formally defined by

$$C^{\mu}(x) = -(16\pi^2)^{-1}\varepsilon^{\mu\nu\rho\sigma}\mathrm{Tr}(F_{\nu\rho}(x)\,A_{\sigma}(x) - \tfrac{2}{3}A_{\nu}(x)\,A_{\rho}(x)\,A_{\sigma}(x)). \qquad (4.4.11)$$

$$\partial_{\mu}C^{\mu}(x) = -(16\pi^2)^{-1}\,\mathrm{Tr}\,{}^{*}F_{\mu\nu}(x)\,F_{\mu\nu}(x) \equiv \mathcal{P},$$

where $A_{\mu} = (0, A_i)$, ${}^{*}F_{\mu\nu} \equiv \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}\,F^{\rho\sigma}$. In the mathematical literature, for classical fields, $\mathcal{P}$ is called the "Pontryagin density" and C_{μ} the "Chern-Simons secondary characteristic class".

At the classical level, one can prove the following transformation law of $C_0(x)$ under gauge transformations $\alpha_{\mathcal{U}}$, defined by Eq. (4.4.7)

$$\alpha_{\mathcal{U}}(C^0(x)) = C^0(x) - (8\pi^2)^{-1}\partial_i[\varepsilon^{ijk}\mathrm{Tr}(\partial_j\mathcal{U}(\mathbf{x})\,\mathcal{U}(\mathbf{x})^{-1}A_k(x))] + n_{\mathcal{U}}(\mathbf{x}). \qquad (4.4.12)$$

Therefore, at the classical level the space integral of $C_0(\mathbf{x}, x_0)$ is invariant under small transformations, but it gets shifted by n under gauge transformations with winding number n.

In the quantum case, one meets non-trivial mathematical problems. First of all, the formal expression in the right hand side of Eq. (4.4.11) requires a point splitting regularization. It is very reasonable to assume that this can be done by keeping the transformation properties of the formal expression under large gauge transformations, Eq. (4.4.12).

The next problem is the space integral of $C_0(x)$. Even for conserved currents, the space integral of the charge density is known to diverge and suitable regularizations are needed, including a time smearing. Under some general conditions one may obtain the convergence of a suitably regularized integral of the charge density of a conserved current, in matrix elements on states with some localization[22]; but in the general case of non-conserved currents the problem is seriously open.

For these reasons, one is led to consider a suitably regularized space integral of $C_0(x)$

$$C^0(f_R\alpha_R) \equiv \int d^4x f_R(\mathbf{x})\,\alpha_R(x_0)\,C^0(x), \qquad (4.4.13)$$

[22] B. Schroer and P. Stichel, Comm. Math. Phys. **3**, 258 (1966); H.D. Maison, Nuovo Cim. **11A**, 389 (1972); M. Requardt, Comm. Math. Phys. **50**, 259 (1976); G. Morchio and F. Strocchi, Jour. Math. Phys. **44**, 5569 (2003).

where $f_R(\mathbf{x}) = f(|\mathbf{x}|/R), f(x) = 1$, for $|x| \leq 1, = 0$, for $|x| \geq 1 + \varepsilon$, $\alpha_R(x_0) = \alpha(x_0/R)/R$, $\int dx_0\, \alpha(x_0) = \tilde{\alpha}(0) = 1$.

Actually, an even more serious problem arises in the quantum case, as a consequence of the gauge dependence of $C_\mu(x)$, i.e. only its exponentials exist. To this purpose, we consider the unitary operators $V^C(f_R\alpha_R)$, formally the exponentials $\exp i\, C^0(f_R\alpha_R)$, with properties inferred from those of such exponentials. In particular, for any f_R with $f_R = 1$ on the space support $K_\mathcal{U}$ of $\mathcal{U}$, so that $f_R\partial_j\mathcal{U} = \partial_j\mathcal{U}$, $\partial_i f_R\, \partial_j\mathcal{U} = 0$, from the gauge transformations of $C^0(x)$, Eq. (4.4.12), one has that

$$\alpha_\mathcal{U}(V^C(f_R\alpha_R)) = e^{in_\mathcal{U}}\, V^C(f_R\alpha_R). \tag{4.4.14}$$

Proposition 4.2 *The operators $V^C(\lambda f_R\alpha_R)$, $\lambda \in \mathbf{R}$, are not weakly continuous in λ and therefore the field $C_0(f_R\alpha_R)$ cannot be defined. Furthermore, for all Gauss invariant vectors Ψ, Φ, one has*

$$(\Psi,\ V^C(f_R\alpha_R)\, \Phi) = 0. \tag{4.4.15}$$

Proof In fact, if $C_0(f), f \in \mathcal{D}(\mathbf{R}^4)$, exists, by using the Gauss gauge invariance of the vacuum state ω, the vanishing of $\omega(A_k)$ by rotational invariance and Eq. (4.4.12), one has

$$\omega(C_0(f)) = \omega(V(\mathcal{U}^\lambda)C_0(f)V(\mathcal{U}^\lambda)^{-1}) = \omega(C_0(f)) + \int d^3x f(\mathbf{x}, t) n_{\mathcal{U}^\lambda}(\mathbf{x}).$$

Since for any f there is at least one $\mathcal{U}^\lambda(\mathbf{x})$ such that the integral on the right hand side does not vanish, one gets a contradiction. Thus, only the exponential of $C_0(f)$ can be defined.

Moreover, given f_R one can find a small gauge transformation $\mathcal{U}(\mathbf{x})$, with

$$\mathcal{U}(\mathbf{x}) = \mathcal{U}_1(\mathbf{x})\,\mathcal{U}_2(\mathbf{x}), \quad n_{\mathcal{U}_1} + n_{\mathcal{U}_2} = 0, \quad n_{\mathcal{U}_1} \neq 0,$$
$$f_R\,\mathcal{U}_2 = \mathbf{1}, \quad f_R\,\mathcal{U}_1 = \mathcal{U}_1.$$

Then, $\partial_i f_R \mathcal{U}_1 = 0$, $\partial_i f_R\, \partial_j\mathcal{U} = 0$ and the second term on the right hand side of Eq. (4.4.12) vanishes; furthermore $\int d^3x\, n_\mathcal{U}(\mathbf{x}) f_R(\mathbf{x}) = n_{\mathcal{U}_1}$. Hence, one has

$$(\Psi,\ V^C(f_R\alpha_R)\, \Phi) = (\Psi,\ V(\mathcal{U})\, V^C(f_R\alpha_R)\, V(\mathcal{U})^{-1}\, \Phi) =$$
$$= e^{in_{\mathcal{U}_1}}\, (\Psi,\ V^C(f_R\alpha_R)\, \Phi),$$

and therefore $(\Psi,\ V^C(f_R\alpha_R)\, \Phi) = 0$.

Thus, if only vector fields are present, the topological current and its non invariance under large gauge transformations cannot give rise to any relevant structure on the physical states (as claimed in the literature).

4.3 Fermions and Chiral Symmetry Breaking

The situation changes substantially in the presence of massless fermions, since the role of the topological current is taken by a *conserved* current; hence, there is a symmetry associated to it and the crucial point is its relations with the large gauge transformations and with their implementers.

In this case, the Lagrangian of Eq. (4.4.1) gets modified by the addition of the (gauge invariant) fermion Lagrangian and the Gauss operators become

$$G_a = (\mathbf{D} \cdot \mathbf{E})_a - j_a^0, \quad j_\mu^a = ig\bar{\psi}\gamma_\mu t^a \psi.$$

The time independent gauge transformations of the fermion fields in the fundamental representation of the gauge group are

$$\alpha_{\mathcal{U}}(\psi(x)) = \mathcal{U}(\mathbf{x})\psi(x). \tag{4.4.16}$$

At the classical level, the Lagrangian is invariant under the one-parameter group of *chiral transformations* β^λ, $\lambda \in \mathbf{R}$,

$$\beta^\lambda(\psi) = e^{\lambda\gamma_5}\,\psi, \quad \beta^\lambda\,(\bar{\psi}) = \bar{\psi}\; e^{\lambda\gamma_5}, \quad \gamma_5^* = -\gamma_5, \quad \beta^\lambda(\mathbf{A}) = \mathbf{A}. \tag{4.4.17}$$

Correspondingly, there is a conserved current $j_\mu^5 = ig\bar{\psi}\gamma^5\gamma_\mu\psi$, the gauge invariant fermion axial current.

In the quantum case, a gauge invariant point splitting regularization is needed for the definition of j_μ^5 and this inevitably leads to an *anomaly*,[23]

$$\partial^\mu j_\mu^5 = -2\partial^\mu C_\mu = -2\mathcal{P}.$$

The conserved axial current is now the gauge dependent current

$$J_\mu^5(x) = j_\mu^5(x) + 2C_\mu,$$

its conservation being equivalent to the anomaly equation for j_μ^5. For the discussion of the Weyl quantization, we take as **local exponential field algebra** $\mathcal{F}_W$ the algebra generated by the operators $W(\mathbf{f})$, by the local implementers of the gauge transformations $V(\mathcal{U})$, by the gauge invariant bilinear functions of the fermion fields and by the unitary operators $V^5(f)$, formally given by $\exp i J_0^5(f)$.

[23]For a general review of this phenomenon, see R. Jackiw, Field Theoretical Investigations in Current Algebra, in S.B. Treiman, R. Jackiw, B. Zumino and E. Witten, *Current Algebra and Anomalies*, World Scientific 1985, pp. 81–210.

As shown by Bardeen[24] on the basis of perturbative renormalization in local gauges, the chiral transformations are generated by the conserved gauge dependent current J^5_μ (not by j^5_μ); the continuity equation of J^5_μ plays a crucial role in Bardeen analysis.

In fact, since J^5_μ is conserved, most of the standard wisdom is available[25]; in particular, the commutation relations of $V^5_R(\lambda)$ with the local fields are governed by the canonical (anti)commutation relations and the extension to unequal times is provided by the current conservation. Then, one has that

$$\lim_{R\to\infty} V^5_R(\lambda)\, F\, V^5_R(-\lambda) = \beta^\lambda(F), \qquad \forall F \in \mathcal{F}_W. \tag{4.4.18}$$

It is important to stress that the limit $R \to \infty$ exists thanks to locality, the limit being actually reached for finite values of R; furthermore the limit preserves locality and gauge invariance, i.e. it defines an automorphism of the local exponential algebra $\mathcal{F}_W$.

Thus, contrary to statements appeared in the literature, the presence of the chiral anomaly does not prevent the chiral symmetry from being a well defined time independent automorphism of the observable algebra, which is identified as the gauge invariant subalgebra of the field algebra $\mathcal{F}_W$.

Furthermore, the chiral symmetry β^λ is locally generated by the unitary operators $V^5_R(\lambda) \equiv V^5(\lambda f_R \alpha_R)$, the formal exponentials of $J^5_0(f_R\alpha_R)$, which act as local implementers of β^λ on the local field algebra $\mathcal{F}_W$.

In conclusion, the above argument shows that: i) *the chiral symmetry defines an automorphism of the observable algebra* ii) and it is *locally generated* by (gauge dependent) unitary operators.

The loss of chiral symmetry is therefore a genuine phenomenon of spontaneous symmetry breaking and the confrontation with the Goldstone theorem becomes a crucial issue, the so-called $U(1)$ problem. The gauge dependence of the local implementers does not allow to *a priori* dismiss the problem, since also non abelian symmetries are generated by gauge dependent currents and the evasion of the Goldstone theorem is not trivially solved by this property.

Actually, the crucial point is the interplay between large gauge transformations and chiral symmetry. To this purpose we that the gauge transformations of $\mathcal{F}_W$ are given by Eqs. (4.4.7), (4.4.15). Therefore, since j^5_0 is gauge invariant, Eq. (4.4.14) implies that for R large enough, so that $f_R(\mathbf{x}) = 1$ on the localization region of $\mathcal{U}_n(\mathbf{x})$,

$$\alpha_{\mathcal{U}_n}(V^5_R(\lambda)) = e^{i\lambda\, 2n}\, V^5_R(\lambda). \tag{4.4.19}$$

It is worthwhile to stress that Eq. (4.4.18) is merely a consequence of the localizability of the large gauge transformations and that $V^5_R(\lambda)$ has the same transformation

[24] W.A. Bardeen, Nucl. Phys. **B 75**, 246 (1974).

[25] G. Morchio and F. Strocchi, Jour. Phys. A: Math. Theor. **40**, 3173 (2007).

properties under them as the formal exponential $\exp[i\lambda\, J_0^5(f_R\alpha_R)]$. It codifies the crucial consequence of the axial anomaly, namely that the implementers $V(\mathcal{U}_n)$ of the large gauge transformations have a non-trivial relation with the local implementers of the chiral transformations.

The absence of parity doublets requires that the chiral symmetry be broken in QCD and the $U(1)$ problem amounts to explaining the absence of the corresponding Goldstone massless bosons. Now, as discussed above, one of the basic assumptions of the Goldstone theorem, namely the existence of a one-parameter group of automorphisms of the algebra of observables, which commute with space and time translations is satisfied.

The second crucial property, needed the proof of the theorem,[26] is the link between the vacuum expectation of the infinitesimal variation of the symmetry breaking order parameter and the local commutator of a conserved current

i.e.

$$< \delta A > \equiv \frac{d}{d\lambda} < \beta^\lambda(A) >_{\lambda=0} = i \lim_{R\to\infty} < [\, J_0^5(f_R\alpha_R),\, A\,] > . \qquad (4.4.20)$$

Since the chiral automorphism β^λ is C^∞ in the group parameter λ, its generator is well defined, but the problem is its relation with the formal generator of the one-parameter group of unitary operators $V_R^5(\lambda)$, which implement the symmetry on the field algebra $\mathcal{F}_W$.

As a matter of fact, even if β^λ can be described by the action of the local operators $V_R^5(\lambda)$ on $\mathcal{F}_W$, Eq. (4.4.17), the non-regularity of the one-parameter unitary group $V_R^5(\lambda)$, prevents the existence of the corresponding generator $J_0^5(f_R\alpha_R)$, so that one cannot write the symmetry breaking Ward identities and obtain the Goldstone energy-momentum spectrum of $< J_0^5(x)\, A >$.

Proposition 4.3 *If ω is a Gauss invariant vacuum state and A is a gauge invariant order parameter for chiral symmetry breaking, i.e.*

$$\omega(\beta^\lambda(A)) \neq \omega(A) \neq 0,$$

(the standard candidate for A being $\bar\psi\psi$), then the vacuum expectations $\omega(J_0^5(f_R\alpha_R)\, A)$ cannot be defined and Eq. (4.4.19) does not hold.

Proof In fact, by exploiting the local properties of the gauge transformations $\mathcal{U}_n$ and its space translated one $\mathcal{U}_n^a$ by $\mathbf{a}$, one has that for any local operator F, for $|\mathbf{a}|$ large enough, $\alpha_{\mathcal{U}_n}(F) = \alpha_{\mathcal{U}_0}\alpha_{\mathcal{U}_n^a}(F) = \alpha_{\mathcal{U}_0}(F)$, with α_{U_0} the Gauss transformation labeled by $\mathcal{U}_n\, (\mathcal{U}_n^a)^{-1}$. Then, the Gauss invariance of ω implies its gauge invariance. Hence, if the expectations $\omega(J_0^5(f_R\alpha_R)\, A)$ are defined, by the invariance of ω under gauge transformations and by Eq. (4.4.18), for R sufficiently large, one would get a

[26]For a discussion of the role of this property, see e.g. F. Strocchi, *Symmetry Breaking*, 2nd ed. Springer 2008, esp. Part II, Chapter 15.

contradiction

$$\omega(J_0^5(f_R\alpha_R)A) = \omega(\alpha_{\mathcal{U}_n}(J_0^5(f_R\alpha_R)A)) = \omega(J_0^5(f_R\alpha_R)A) + 2n\,\omega(A).$$

The impossibility of writing expectations involving J^5_μ on a gauge invariant vacuum state, solves the the $U(1)$ problem and the problems raised by R.J. Crewther in his analysis of chiral Ward identities.[27]

By exploiting the non-regular Weyl quantization one can prove that the non-trivial topology of $\mathcal{G}$ gives rise to the θ vacuum structure and forces the chiral symmetry breaking in each θ sector.[28]

It is worthwhile to remark that, for the evasion of the Goldstone theorem discussed above, the occurrence of the so-called chiral anomaly (which is present also in the abelian case) is not enough; the crucial ingredient is Eq. (4.4.18), which directly implies the non-regularity of the unitary operators $V^5_R(\lambda)$ and the non-existence of the local charges $J^5_0(f_R\alpha_R)$ generating the chiral symmetry, in expectations on a gauge invariant vacuum state.

If $V(\mathcal{U}_n)$ denote local implementers of the large gauge transformations, Eq. (4.4.17) gives a relation of exactly the same form of the Weyl relation arising in the description of a quantum particle on a circle (see Chap. 3, Sect. 1), with $V(\mathcal{U}_n)$ playing the role of $e^{i\,2\pi n p_\phi}$ and $V^5_R(\lambda)$ playing the role of formal exponential $e^{i\lambda\phi}$. The emerging picture is also the same as in the quantum mechanical model of QCD structures (see Chap. 3, Sect. 4). It is worthwhile to note that the Schwinger model in the temporal gauge exactly reproduces the general results discussed above [28].

5 Abelian Chern-Simons Theory

Another interesting example of the need of Weyl quantization is provided by the abelian Chern-Simons free field theory[29] defined by the following Lagrangian

$$\mathcal{L} = -(1/4)F^{\mu\nu}F_{\mu\nu} + (\mu/4)\,\varepsilon^{\mu\nu\sigma}F_{\mu\nu\sigma}, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \tag{4.5.1}$$

with A_μ a field in 2+1 spacetime dimensions. The equations of motion

$$\partial_\mu F^{\mu\nu} + \tfrac{1}{2}\mu\varepsilon^{\nu\sigma\rho}F_{\sigma\rho} = 0, \tag{4.5.2}$$

[27] R.J. Crewther, Chiral Properties on Quantum Chromodynamics, in *Field Theoretical Methods in Particle Physics*, W. Rühl ed., Reidel 1980, pp. 529–590.

[28] See G. Morchio and F. Strocchi, Ann. Phys. **324**, 2236 (2009); F. Strocchi, *An introduction to non-perturbative foundations of quantum field theory*, Oxford Univ. Press 2013.

[29] F. Nill, A constructive approach to abelian Chern-Simons theory, in *Theory of Elementary Particles*, G. Weigt ed., Inst. Hochenenergiephys. 1991, p. 78; Int. Jour. Mod. Phys. **6**, 2159 (1992).

imply $(\Box + \mu^2)F_{\mu\nu} = 0$. They are invariant under the gauge transformations $A_\mu \to A_\mu + \partial_\mu \Lambda$, whereas the Lagrangian changes by a total derivative. It is convenient to discuss the quantization in the temporal gauge, defined by $A_0 = 0$.[30] The *Gauss operator*

$$G = \partial_i \Pi^i + \tfrac{1}{2}\, \mu \varepsilon_{ij} \partial_j A^j, \quad \Pi_i = F_{0\,i} + \tfrac{1}{2} \mu \varepsilon_{ij} A^j \tag{4.5.3}$$

is independent of time. As a consequence of the canonical (equal time) commutation relations

$$[A_i(\mathbf{x}),\ \Pi_j(\mathbf{y})] = i\delta_{ij}\delta(\mathbf{x} - \mathbf{y}), \tag{4.5.4}$$

all other equal time commutators vanishing, the Gauss operator generates the *time independent gauge transformations*.

As for the cases discussed in Sects. 3, 4, the Gauss law must be imposed as a condition of the physical states. We consider the local (exponential) field algebra $\mathcal{F}_W$ generated by the Weyl operators $W(\mathbf{f}), f^i(x) \in \mathcal{S}(\mathbf{R}^3)$, formally the exponentials of $\mathbf{A}(\mathbf{f}) = A_i(f^i)$, and by the unitary operators $V(\mathcal{U}^\lambda)$, $\mathcal{U}^\lambda = e^{i\lambda G(g)}$, $g \in \mathcal{D}(\mathbf{R}^2)$, formally the exponentials of the Gauss operator, representing the group $\mathcal{G}$ of time independent gauge transformations

$$V(\mathcal{U}^\lambda)\, W(\mathbf{f}) V(\mathcal{U}^\lambda)^{-1} = e^{i \int d^2x f_i \partial_i g}\, W(\mathbf{f}). \tag{4.5.5}$$

The algebraic structure of $\mathcal{F}_W$ is defined by the canonical (equal time) commutation relations and by the time evolution generated by the Hamiltonian

$$H = \tfrac{1}{2} \int d^2x [\mathbf{E}^2 + B^2], \quad E_i = F_{0\,i}, \ \ B = \varepsilon^{ij} \partial_i A_j. \tag{4.5.6}$$

A Gauss invariant vacuum state ω on $\mathcal{F}_W$ satisfies the Gauss law $\omega(V(\mathcal{U}^\lambda)) = 1$ and the physical vector states Ψ in the corresponding (GNS) Hilbert space $\mathcal{H}_\omega$ are characterized by the **Gauss law**

$$V(\mathcal{U}^\lambda)\Psi = \Psi. \tag{4.5.7}$$

Similarly to the previous cases, we have

Proposition 5.1 *A vacuum state ω on $\mathcal{F}_W$, satisfying the Gauss law defines a non-regular representation of $\mathcal{F}_W$, since*

$$\omega(W(\mathbf{f})) = 0, \quad \textit{if} \ \ \partial_i f^i \neq 0. \tag{4.5.8}$$

[30] S. Deser, R. Jackiv and S. Templeton, Ann. Phys. **140**, 372 (1982).

Thus, the fields $\mathbf{A}(\mathbf{f})$*, cannot be defined in the Hilbert space* $\mathcal{H}_\omega$*, if* $\partial_i f^i \neq 0$ *(***non-regular Weyl quantization***).*

Proof As before,

$$\omega(W(\mathbf{f})) = \omega(V(\mathcal{U}^\lambda)\, W(\mathbf{f})\, V(\mathcal{U}^{-\lambda})) = e^{i\int d^2x f_i\, \partial_i g}\, \omega(W(\mathbf{f}))$$

and Eq. (4.5.8) follows.

The non-regularity of the representation defined by a vacuum state satisfying the (suitably modified) Gauss law may be similarly proved in the presence of (interacting) fermions, with interaction term

$$\mathcal{L}_{int} = A_\mu j^\mu, \quad j_\mu = e\bar{\psi}\gamma_\mu\, \psi.$$

Clearly, the same argument of non-regular representation applies to the pure non-abelian Chern-Simons theory in 2+1 dimension defined in the temporal gauge by the Lagrangian $\mathcal{L} = \frac{1}{2}\mu\, \varepsilon^{ij}\dot{A}_i^a\, A_j^a$.[31]

[31] G.V.Dunne, R. Jackiw and C.A. Trugenberger, Ann. Phys. **194**, 197 (1989).

Chapter 5
Diffeomorphism Invariance and Weyl Polymer Quantization

1 Diffeomorphism Invariance and Weyl Polymer Quantization

Especially in connection with Quantum Gravity, great attention has been paid to the problem of combining diffeomorphism invariance and quantization.

According to Wigner theorem on quantum symmetries, diffeomorphism symmetry, or equivalently diffeomorphism covariance, of a quantum theory requires the implementation of the elements of the diffeomorphism group by unitary operators in the Hilbert space of the (quantum) states.

Now, the diffeomorphism group of a manifold is infinite dimensional and the representation of infinite dimensional groups is not yet under mathematical control; this is one of the reasons of the relevance and difficulty of the problem.

For the infinite dimensional local gauge group the difficulty of quantization has been overcome through the choice of the gauge fixing, which breaks the local gauge symmetry and reduce it to the identity on the physical states characterized by the fulfillment of a subsidiary condition; a crucial point is the independence of the expectation values of the observables from the choice of the gauge fixing.

For the case of diffeomorphism symmetry such a route is made difficult by the lack of a full control on the representations of the diffeomorphism group. To this purpose it has been suggested, in particular in connection with Loop Quantum Gravity (LQG) and String Theory (ST), to use a *non-regular Weyl* (also called *polymer*) *quantization*[1] and to guarantee the unitary implementation of the elements

[1]For the vast literature on this subject we refer to the Bibliography of Publications related to Classical Self-dual variables and Loop Quantum Gravity, see: A. Corichi and A. Hauser, arXiv:gr-qc/0509039v2. For very recent reviews of LQG, see H. Sahlmann, Loop quantum gravity-a short review, arXiv:1001.4188v1 [gr-qc]; S. Mercuri, Introduction to Loop Quantum Gravity, arXiv:1001.1330v1 [gr-qc].

F. Strocchi, *Gauge Invariance and Weyl-polymer Quantization*,
Lecture Notes in Physics 904, DOI 10.1007/978-3-319-17695-6_5

of the diffeomorphism group by the stronger condition that the Hilbert space of the (quantum) states is the GNS representation space defined by a diffeomorphism invariant (vacuum) state.

Such a proposal has raised both interest and discussions and, in our opinion, it may be useful to focus the problem starting with the question of diffeomorphism invariance and quantization in general. As one should expect on the basis of quantum gauge models, the diffeomorphism invariance of the ground state leads to a non-regular Weyl quantization. To get some insight, in the following we shall start by considering the case of diffeomorphism covariance in quantum mechanics.

2 Quantum Mechanics on a Manifold and Diffeomorphism Invariance

The problem of the formulation of quantum mechanics on a manifold $\mathcal{M}$ as been largely discussed in the literature under suitable restrictions or choices and recently a general approach which implements diffeomorphism covariance has been proposed.[2] We shall briefly review it, also because it shows that a diffeomorphism covariant quantization may be obtained in a rather simple and natural way, once the question of the independent degrees of freedom is properly settled. In fact, by exploiting the relations induced on the Lie algebra of vector fields of the manifold by multiplications by C^∞ functions, locally the number of independent vector fields reduces to the dimension of the manifold.

i) *Algebraic description of a manifold and its diffeomorphisms*

The geometry of $\mathcal{M}$ is completely described by the algebra of the C^∞ real functions on $\mathcal{M}$ and by the group *Diff* $(\mathcal{M})$ of diffeomorphisms of $\mathcal{M}$. At the local level the relevant group is the subgroup Diff $(\mathcal{M})$ of the connected component of the identity of *Diff*$(\mathcal{M})$, generated by the one-parameter groups $g_{\lambda v}$, $\lambda \in \mathbf{R}$, $v \in \text{Vect}(\mathcal{M}) \equiv$ the Lie algebra of C^∞ vector fields of compact support.

Thus, as a basic geometrical structure one has

i) the **algebra** $C^\infty(\mathcal{M})$ generated by the real C^∞ functions of compact support and by the constant functions on $\mathcal{M}$ and

ii) the space Vect$(\mathcal{M})$ of C^∞ real **vector fields of compact support** in $\mathcal{M}$. Vect$(\mathcal{M})$) is a Lie algebra of derivations $v \in \text{Vect}(\mathcal{M})$) on $C^\infty(\mathcal{M})$, $v : C^\infty(\mathcal{M}) \ni f \to v(f)$, with Lie product $\{v, w\}$ defined by

$$\{v, w\}(f) = v(w(f)) - w(v(f)). \tag{5.2.1}$$

[2] G. Morchio and F. Strocchi, Lett. Math. Phys. **82**, 219 (2007).

The action of Vect($\mathcal{M}$) on $C^\infty(\mathcal{M})$ can also be interpreted as an extension of the Lie product from Vect($\mathcal{M}$) to $C^\infty(\mathcal{M}) + \text{Vect}(\mathcal{M}) \equiv \mathcal{L}(\mathcal{M})$. Putting

$$\{ v, f \} \equiv v(f), \quad \{f, g\} \equiv 0, \tag{5.2.2}$$

$\mathcal{L}(\mathcal{M})$ gets the structure of a Lie algebra.

In order to discuss the representations of such an algebraic structure it is convenient to introduce the analogs of the Weyl operators, namely the operators $W(f), f \in C^\infty(\mathcal{M})$, formally the exponentials e^{if}, and the operators $V(v)$, $v \in \text{Vect}(\mathcal{M})$, formally the exponentials e^{iv}. The operator $V(\lambda v)$, $\lambda \in \mathbf{R}$, can also be labeled by the corresponding element of the one-parameter group $g(\lambda v)$ generated by v; hence, with a slight abuse of notations we shall denote $V(\lambda v)$ also by $V(g(\lambda v))$. In general, by $V(g)$ we shall denote the operator labeled by the generic group element $g \in \text{Diff}(\mathcal{M})$.

We introduce the following notations

$$f_{g(v)}(x) \equiv f(g(v)^{-1}x) \equiv (\alpha_v(f))(x),$$
$$\alpha_v(w) \equiv g(v)(w) = \text{the adjoint action of Diff}(\mathcal{M}) \text{ on Vect}(\mathcal{M}).$$

Then, the analogs of the Weyl relations read

$$W(f)\, W(g) = W(f+g), \quad V(v)\, W(f)\, V(v)^{-1} = W(f_{g(v)}), \tag{5.2.3}$$

$$V(v)\, V(w)\, V(v)^{-1} = V(g(v)w), \;\; V(g(v))\, V((g(w)) = V(g(v)g(w)). \tag{5.2.4}$$

The reality of $C^\infty(\mathcal{M})$ and of Vect($\mathcal{M}$) induce the following involution on the above defined Weyl operators

$$W(f)^* = W(-f), \quad V(v)^* = V(-v) \tag{5.2.5}$$

and the *polynomial algebra $\mathcal{A}$ generated by the Weyl operators* becomes a *-algebra.

By construction, Eqs. (5.2.3–5.2.4), Diff($\mathcal{M}$) defines a group of *-automorphisms α_h, $h \in \text{Diff}(\mathcal{M})$ of $\mathcal{A}$

$$\alpha_h(W(f)) = W(f_h), \quad \alpha_h(V(v)) = V(hv).$$

Since Vect($\mathcal{M}$) is generated by an infinite number of linearly independent vector fields, Diff($\mathcal{M}$) is an infinite dimensional group and the algebra $\mathcal{A}$ is infinite-dimensional. Thus, it is not easy to analyze the positive functionals on it and its representations are not under complete mathematical control.

ii) *Quantum mechanics with unitary implementation of* Diff($\mathcal{M}$)

To overcome this difficulty, it has been suggested, also on the basis of physical considerations,[3] to take into account that different vector fields may be functionally dependent through multiplication by elements of $C^\infty(\mathcal{M})$. Indeed, as a module over $C^\infty(\mathcal{M})$, Vect($\mathcal{M}$) is locally generated by n vector fields, with n the dimension of $\mathcal{M}$.

We recall that an algebra $\mathcal{A}$ is a *module* over a commutative ring C if there is a product $C \circ \mathcal{A} \subset \mathcal{A}$, which is distributive in both factors and associative in the first, i.e. $\forall f, g \in C,\ a, b \in \mathcal{A}$,

$$f \circ (a + b) = f \circ a + f \circ b, \quad (f + g) \circ a = f \circ a + g \circ a,$$
$$(fg) \circ a = f \circ (g \circ a). \tag{5.2.6}$$

If $\mathcal{L}$ is a (real) Lie algebra, with Lie product denoted by $\{\cdot, \cdot\}$, which acts as derivations on a (real) commutative algebra $\mathcal{L}_0$, namely $\forall v \in \mathcal{L}$, $\mathcal{L}_0 \ni f \to v(f)$, and $\mathcal{L}$ is a module over $\mathcal{L}_0$ with product satisfying

$$\{f \circ v,\, g\} = f \circ \{v,\, g\}, \quad f \circ (g \circ v) = (fg) \circ v,$$
$$\{v, f \circ w\} = v(f) \circ w + f \circ \{v,\, w\}, \tag{5.2.7}$$

then $\circ$ is a *Lie-Rinehart (LR) product* and the pair $(\mathcal{L}_0, \mathcal{L})$ is (called) a **Lie-Rinehart (LR) algebra**. A LR algebra is said to have an identity if $\mathcal{L}_0$ has an identity $\mathbf{1}$, satisfying $\mathbf{1} \circ v = v,\ \forall v \in \mathcal{L}$.

A module structure of Vect($\mathcal{M}$) on $C^\infty(\mathcal{M})$ is realized by the product $f \circ v = w \in$ Vect($\mathcal{M}$), where w is the vector field with components $(f \partial/\partial x_i)$, if $v_i = \partial/\partial x_i$. It is easy to check that the so-defined product satisfies Eqs. (5.2.6), (5.2.7)

Thus, the pair $(C^\infty(\mathcal{M}), \mathbf{Vect}\,\mathcal{M})$ **is a LR algebra with identity 1** given by the function $\mathbf{1}(x) = 1$; the crucial consequence is that, *as a module over* $C^\infty(\mathcal{M})$, *Vect* $(\mathcal{M})$ *is locally generated by n vector fields*, with n the dimension of $\mathcal{M}$, so that the infinite dimensional Diff($\mathcal{M}$) and its Lie algebra may be described by a finite number of generators. This means that, for any region $\mathcal{O}$ diffeomorphic to a disc, one can find n vector fields p_i, supp $p_i \supset \mathcal{O}$, so that for any v, with supp $v \subset \mathcal{O}$, there exist n functions $f_i \in C^\infty(\mathcal{M})$ such that $v = \sum_i f_i \circ p_i$. This means that any vector v with supp $v \subset \mathcal{O}$, is functionally dependent on the n vector fields p_i, through the LR product.

Since a quantum particle on a manifold has a finite number of degrees of freedom, the reduction of the algebra $\mathcal{A}$ to finite dimensions is necessary for its quantum mechanical use and the natural way is to take into account the Lie-Rinehart structure of $(C^\infty(\mathcal{M}), \mathrm{Vect}\,\mathcal{M})$.

[3] G. Morchio and F. Strocchi, Lett. Math. Phys. **82**, 219 (2007).

Definition 2.1 *A* **quantum particle regular representation** π *of the* $*-$*algebra* $\mathcal{A}$*, generated by the Weyl operators* $W(f)$*,* $V(v)$*, Eqs. (5.2.3–5.2.5), is a* $*-$*homomorphism of* $\mathcal{A}$ *into the* $*-$*algebra of bounded operators in a Hilbert space* $\mathcal{H}_\pi$ *with the following properties:*

*i) (***regular representation***)* $\pi(W(\lambda f))$*,* $\pi(V(\lambda v))$*,* $\lambda \in \mathbf{R}$*, define weakly continuous one-parameter unitary groups and the algebra generated by the* $W(f)$ *is not trivially represented; the corresponding generators* $\pi(f) \equiv T_f$*,* $\pi(v) \equiv T_v$ *exist on a common dense domain D invariant under* $\pi(\mathcal{A})$*,*

*ii) (***Lie-Rinehart structure***) the generators represent the Lie-Rinehart algebra* $(C^\infty(\mathcal{M}), Vect\,\mathcal{M})$ *on D, i.e. the Lie algebra relations of the vector fields and of their action over* $C^\infty(\mathcal{M})$*, with the Lie-Rinehart product represented by*

$$\pi(f \circ v) = \tfrac{1}{2}[\pi(f)\pi(v) + \pi(v)\pi(f)]. \tag{5.2.8}$$

Clearly, a Hilbert space representation of $\mathcal{A}$ provides a quantization in which the diffeomorphism group is implemented by unitary operators, i.e. one has a *diffeomorphism covariant quantum system.*

Quantum particle regular representations of $\mathcal{A}$ have been shown to exists, to be locally unitarily equivalent to the Schrödinger representation in $L^2(\mathcal{M}, d\mu)$, where $d\mu$ is the Lebesgue measure, $\pi(W(f))$ act as multiplication operators and $\pi(V(v))$ act in the following way, $\forall \psi \in L^2(\mathcal{M}, d\mu)$,

$$\pi(V(v))\psi(x) = \psi(g(v)^{-1}(x))\, J(g(v), x),$$
$$J(g(v), x) \equiv [d\mu(g(v)^{-1}(x))/d\mu(x)]^{1/2}. \tag{5.2.9}$$

Globally, such representations are in one-to-one correspondence to the unitary irreducible representations of the first homotopy group of $\mathcal{M}$.[4]

iii) *Diffeomorphism invariant states*

The algebra $\mathcal{A}$ is generated by the monomials $W(f)\,V(v)$, $f \in C^\infty(\mathcal{M})$, $v \in$ Vect$(\mathcal{M})$, since, as a consequence of the generalized Weyl relations, Eqs. (5.2.3–5.2.4), products of $W(f), V(v)$ reduce to monomials of this form. Hence, a linear functional ω on $\mathcal{A}$ is completely determined by the expectations $\omega(W(f)\,V(v))$.

A *state* ω on $\mathcal{A}$ is *diffeomorphism invariant* if

$$\omega(\alpha_v(A)) = \omega(A), \quad \forall v \in \text{Vect}(\mathcal{M}),\ \forall A \in \mathcal{A}. \tag{5.2.10}$$

By a standard argument, in the GNS representation defined by a diffeomorphism invariant state ω, Diff$(\mathcal{M})$ is unitarily implemented by the operators $U(\lambda v)$

[4]G. Morchio and F. Strocchi, Lett. Math. Phys. **82**, 219 (2007).

defined by

$$U(\lambda v)A\Psi_\omega = \alpha_{\lambda v}(A)\,\Psi_\omega, \quad \forall A \in \mathcal{A}, \quad U(\lambda v)\,\Psi_\omega = \Psi_\omega, \tag{5.2.11}$$

with Ψ_ω the representative vector of ω.

Then, by Eqs. (5.2.3–5.2.4) the operators $U(\lambda v)\,V(\lambda v)^{-1}$ commute with $\mathcal{A}$ and therefore, if the representation is irreducible, they are multiple of the identity; since Diff($\mathcal{M}$) has no non-trivial one-dimensional representation, without loss of generality, one may put $V(\lambda v)\Psi_\omega = \Psi_\omega$, which implies $\forall f \in C^\infty(\mathcal{M})$, $v, w \in$ Vect($\mathcal{M}$)

$$\omega(W(f)V(v)) = \omega(W(f)) = \omega(W(g(w)(f)). \tag{5.2.12}$$

Clearly, the requirement that a representation be defined by a diffeomorphism invariant state is a much stronger requirement than the realization of diffeomorphism symmetry.

More generally, it is worthwhile to stress the crucial difference, not sufficiently emphasized in the literature, between the implementation of the diffeomorphism group by unitary operators, the existence of a diffeomorphism invariant vector state, the coincidence of the unitary implementers with the Weyl operators $V(\lambda v)$ and the weak continuity of the latter.

Proposition 2.2 *The GNS representation defined by a diffeomorphism invariant state cannot be a regular quantum particle representation.*

Proof In fact, the regularity of the representation implies the existence of the generators and diffeomorphism invariance of the state ω gives

$$\omega([v,\, W(f)]) = 0, \quad \forall f \in C^\infty(\mathcal{M})\,, \quad \forall v \in \text{Vect}(\mathcal{M}).$$

This is incompatible with local equivalence to the Schrödinger representation, Eq. (5.2.9), and therefore with regularity.

The next non-trivial question is the existence and the explicit realization of diffeomorphism invariant states. For this purpose, we start by considering diffeomorphism invariant (hermitian) functionals, characterized by Eq. (5.2.10); then one has to find the conditions which guarantee the property of positivity.

Proposition 2.3 *A diffeomorphism invariant hermitian functional ω on $\mathcal{A}$ is positive iff $|\omega(W(f))| \leq 1$, $\forall f \in C^\infty(\mathcal{M})$.*

Proof In fact, putting $\mathcal{W}(f, v) \equiv W(f)\,V(v)$, one has

$$\omega((a\mathcal{W}(f, v) + b\mathcal{W}(g, w))^*\,(a\mathcal{W}(f, v) + b\mathcal{W}(g, w)) =$$
$$= |a|^2 + |b|^2 + 2\text{Re}\,(a^* b\,\omega(W(g - f))$$

and positivity follows.

On the other hand, if ω is positive, in the corresponding GNS representation π_ω, $\pi_\omega(W(f))$ is a unitary operator and therefore $|\omega(W(f)| = |(\Psi_\omega, W(f)\, \Psi_\omega)| \leq 1$.

For simplicity, we shall discuss the following two extreme cases

1) $|\omega(W(f))| = 1, \forall f \in C^\infty(\mathcal{M})$
2) $\omega(W(f) = 0$, if f is not a constant function, i.e. if f is not a multiple of the identity, $f \neq \text{const}\, \mathbf{1}$.

Proposition 2.4 *The GNS representation space defined by a diffeomorphism invariant state ω with*

$$|\omega(W(f))| = 1, \quad \forall f \in C^\infty(\mathcal{M}),$$

is one-dimensional.

Proof In fact, $|\omega(W(f))| = 1$ implies $W(f)\Psi_\omega = e^{\phi(f)}\, \Psi_\omega$ and since Ψ_ω is invariant under the $V(v)$, $\phi(f)$ must yield a representation of Diff($\mathcal{M}$). Since the only one-dimensional representation is the trivial one, $\phi(f)$ must be a constant independent of f and without loss of generality one can take $W(f)\, \Psi_\omega = \Psi_\omega$. Hence, the generating monomials $\mathcal{W}(f, v) \equiv W(f)\, V(v)$ leave Ψ_ω invariant and by the cyclicity of Ψ_ω the Hilbert space $\mathcal{H}_\omega$ is one-dimensional.

Proposition 2.5 *The GNS representation π defined by a diffeomorphism invariant state ω with*

$$\omega(W(f)) = 0, \quad \text{if } f \neq \text{const}\, \mathbf{1}$$

has the following properties

a) Ψ_ω is a cyclic vector for the algebra $\mathcal{A}_W$ generated by the Weyl operators $W(f)$ and any two vectors $W(f)\Psi_\omega$, $W(g)\Psi_\omega$ are orthogonal iff $f - g \neq$ const **1**,
b) even if Ψ_ω is diffeomorphism invariant, the one-parameter groups $\pi(V(g(\lambda v)))$ are not weakly continuous and therefore the corresponding generators do not exist as operators in $\mathcal{H}_\omega$

Proof a) follows immediately from $\omega(W(f)) = 0$, if $f \neq \text{const}\, \mathbf{1}$. Moreover, one has

$$(\pi(W(f))\Psi_\omega, \pi(V(g(\lambda v)))\, \pi(\mathcal{W}(f))\, \Psi_\omega) = \omega(W(f_{g(\lambda v)} - f)).$$

and the right hand side is equal to 1 if $\lambda = 0$ and vanishes otherwise. Hence, weak continuity fails.

In connection with Loop Quantum Gravity (LQG) and String Theory (ST), it has been suggested to use diffeomorphism invariant states with very strong invariance properties, the so-called *polymer states*.[5] Their analog for the algebra $\mathcal{A}$ is the state

$$\omega(W(f)V(v)) = 0, \quad \text{if} f \neq \text{const}\,\mathbf{1}, \quad \text{or} \quad v \neq 0.$$

The invariance under diffeomorphisms is obvious; the representation is highly reducible and corresponds to a thermal state in the limit of infinite temperature.[6] Both unitary groups $\pi(W(\lambda f))$, $\pi(V\lambda v))$ are not weakly continuous and the corresponding generators do not exist.

An instructive example of diffeomorphism invariant state may be worked out in the case of $\mathcal{M} = S^1$, where the topology plays a non-trivial role.

[5]Such a kind of states were proposed by A. Ashtekar, S. Fairhurst and J.L. Willis, Class. Quantum Grav. **20**, 1031 (2003); T. Thiemann, Class. Quantum Grav. **23** 1923 (2006).

[6]F. Acerbi, G. Morchio and F. Strocchi, Jour. Math. Phys. **34**, 899 (1993).

Chapter 6
* A Generalization of the Stone-von Neumann Theorem

1 Zak Algebra

The relevance of non-regular representations of the Heisenberg group (or of the Weyl C^*-algebra $\mathcal{A}_W$) raises the question of a possible classification of them, which generalizes Stone-von Neumann (SvN) theorem. For this purpose, a possible strategy is to consider a maximal abelian $\mathcal{A}$ subalgebra of $\mathcal{A}_W$, identify its Gelfand spectrum $\Sigma(\mathcal{A})$ and classify the realizations of such an abelian algebra in terms of multiplication operators on $L^2(\Sigma(\mathcal{A}), d\mu)$, with $d\mu$ a (Borel) measure on $\Sigma(\mathcal{A})$.

In the Schrödinger choice, such an abelian algebra is the C^*-algebra $\mathcal{A}_q$ generated by the Weyl operators $U(\alpha)$, $\alpha \in \mathbf{R}^d$, (d the space dimension), formally corresponding to $e^{i\alpha q}$, (equivalently one may consider the algebra $\mathcal{A}_p$ generated by the $V(\beta)$, $\beta \in \mathbf{R}^d$ formally given by $e^{i\beta p}$). $\mathcal{A}_q$ is isomorphic to the algebra of almost periodic functions $f(x)$, $x \in \mathbf{R}^d$; by Gelfand theorem it is also isomorphic to the algebra of continuous functions on the compact space $\Sigma(\mathcal{A}_q)$, whose description is however not simple.[1]

For these reasons, it is convenient to consider the maximal abelian algebras $\mathcal{A}_Z(\lambda)$ generated by the pair of elements $U_i(-2\pi/\lambda)$, $V_i(\lambda)$, $i = 1, \dots d$, for a fixed positive λ (formally corresponding to $\exp -i\,(2\pi/\lambda)q_i$, $\exp i\,\lambda p_i$,, respectively).

Up to isomorphisms we can choose $\lambda = 1$; we shall call such an algebra the **Zak algebra** and denote it by $\mathcal{A}_Z$.

For simplicity, in the following we shall consider the one-dimensional case; then $\mathcal{A}_Z$ is generated by only two elements, with spectra given by the unit circle, so that

[1] J. Löffelholz, G. Morchio and F. Strocchi, Lett. Math. Phys. **35**, 251 (1995).

F. Strocchi, *Gauge Invariance and Weyl-polymer Quantization*,
Lecture Notes in Physics 904, DOI 10.1007/978-3-319-17695-6_6

its Gelfand spectrum is very simple

Proposition 1.1 *The Gelfand spectrum of $\mathcal{A}_Z$ is homeomorphic to the two dimensional torus*

$$\mathbf{T}^2 = \{(\lambda_1, \lambda_2) \in \mathbf{C}^2; |\lambda_1| = |\lambda_2| = 1\}. \tag{6.1.1}$$

In the following we shall identify $\mathbf{T}^2$ with $\mathbf{R}/\mathbf{Z} \times \mathbf{R}/(2\pi\mathbf{Z}) = [0,\ 1) \times [0,\ 2\pi)$ and a point $(\alpha, \beta) \in \mathbf{T}^2$ with the multiplicative linear functional

$$m_{\alpha\,\beta}(U(-2\pi)\,V(1)) = e^{-i2\pi\alpha}\,e^{i\beta}.$$

By the Gelfand-Naimark characterization of abelian C^*-algebras, $\mathcal{A}_Z$ is isomorphic to the algebra of continuous functions on $\mathbf{T}^2$, the isomorphisms being given by the Gelfand transform, $A \to \hat{A}$.

Proposition 1.2 *The elements $T(a, b) \equiv U(-a)\,V(b)\,e^{-iab/2}$, a, $b \in \mathbf{R}$ define automorphisms of $\mathcal{A}_Z$: $\tau_{ab}(A) = T(a, b)^* A\, T(a, b)$, $A \in \mathcal{A}_Z$, which correspond to translations of the Gelfand transforms.*

The Schrödinger representation π_S is isometrically isomorphic to the following representation π_Z in terms of L^2 functions on (the Gelfand spectrum of $\mathcal{A}_Z$) $\mathbf{T}^2$: $\forall \phi \in L^2(\mathbf{T}^2, d\alpha\, d\beta)$,

$$(\pi_Z(W(a, 0))\phi)(\alpha, \beta) = e^{i[\alpha+a]\beta}\,\phi((\alpha + a) \bmod 1, \beta), \tag{6.1.2}$$

$$(\pi_Z(W(0, b))\phi)(\alpha, \beta) = e^{-ib\alpha}\,\phi(\alpha, (\beta + b) \bmod 2\pi). \tag{6.1.3}$$

The isometry $\mathcal{U} : L^2(\mathbf{R}) \ni \psi \leftrightarrow \phi \in L^2(\mathbf{T}^2, d\alpha\, d\beta)$ is given by

$$(\mathcal{U}\psi)(\alpha, \beta) = \sum_{n\in\mathbf{Z}} \psi(n + \alpha)e^{-in\beta}, \tag{6.1.4}$$

$$(\mathcal{U}^{-1}\phi)(x) = (2\pi)^{-1} \int_0^{2\pi} d\beta\, \phi(x \bmod 1, \beta)\, e^{i[x]\beta}, \tag{6.1.5}$$

where $[x]$ denotes the integer part of x. In the following, we shall analyze the representations of $\mathcal{A}_W$ in terms of representations of $\mathcal{A}_Z$.

2 A Generalization of Stone-von Neumann Theorem

For a generalization of SvN theorem which yields a classification of non regular representations of the Weyl algebra (equivalently of non-regular unitary representation of the Heisenberg group), one has to find a weaker counterpart of irreducibility and regularity.

As we shall see, a useful generalization is provided by the class of representations which are spectrally multiplicity free (a generalization of irreducibility) and satisfy a notion of strongly measurability, which applies also to non-separable spaces, replaces regularity and it is satisfied by all non-regular representations of physical interest discussed in the previous Chapters.

A representation of the Weyl algebra satisfying such conditions will be shown to be unitarily equivalent to the representation given by Eqs. (6.1.2), (6.1.3) on functions ϕ on $\mathbf{T}^2$ which are square integrable with respect to a (positive) translationally invariant measure μ, which reduces to the Lebesgue measure on $\mathbf{T}^2$ iff the representation is regular, and otherwise is a general (positive) translationaly invariant Borel measure on $\mathbf{T}^2$.

To this purpose, we start by recalling that a *Borel set* in a topological space is any set which can be obtained by taking countable unions, countable intersections and complements of open sets. The family of Borel sets form a *Borel σ-algebra*. A Borel measure μ is a measure defined on the Borel σ algebra and it is regular if for any Borel set S it satisfies: $\mu(S) = \sup_C\{\mu(C); C \subset S, C$ compact and Borel$\} = \inf_O\{O \supset S,\ O$ open$\}$. The Baire σ-algebra on a compact topological space is the minimal σ-algebra of sets needed for the measurability of the continuous functions; a Baire measure is a measure defined on the Baire σ-algebra.

Given a representation π of a C^*-algebra $\mathcal{A}$ in a Hilbert space $\mathcal{H}$, the Baire *-algebra is the smallest C^*-algebra of operators in $\mathcal{H}$ which contains $\pi(\mathcal{A})$ and the limit of each weakly convergent monotone sequence.

As in the standard case, the strategy is to classify representations in which the maximal abelian subalgebra $\mathcal{A}_Z$ is represented by multiplicative operators on $L^2(\mathbf{T}^2, d\mu)$, with μ a Borel measure of $\mathbf{T}^2$, so that and element A of $\mathcal{A}_Z$ is represented by the multiplication operator given by the continuous function defined by its Gelfand transform $\hat{A}$. The advantage of the Zak algebra is that the space on which such continuous functions are defined is the two-dimensional torus.

Thus, as a first step, one has to find a condition on the representation which guarantees that its representation space is isomorphic to a space $L^2(\mathbf{T}^2, d\mu)$, with μ a Borel measure.

Definition 2.1 *A representation π of a (unital) abelian C^*-algebra $\mathcal{A}$ is* **spectrally multiplicity free** *if there exists a positive measure μ on the Baire sets of (the Gelfand spectrum) $\hat{\mathcal{A}} \equiv \Sigma(\mathcal{A})$ and an isometric isomorphism U of the representation space $\mathcal{H}_\pi$ onto $L^2(\hat{\mathcal{A}}, d\mu)$, such that $U\pi(A)U^{-1}$ is the multiplication operator by the Gelfand transform $\hat{A}$.*

A representation π of a C^*-algebra $\mathcal{A}$ in $\mathcal{H}_\pi$ is *non-degenerate* iff $\mathcal{H}_0 \equiv \{x \in \mathcal{H}_\pi,\ \pi(A)x = 0,\ \forall A \in \mathcal{A}\} = \{0\}$.

Then, the property of being spectrally multiplicity free is in some way the generalized counterpart of irreducibility and non-degeneracy for regular representations.

Proposition 2.2 *A regular representation π of the Weyl algebra $\mathcal{A}_W$ is non-degenerate and irreducible if and only if it is spectrally multiplicity free as a representation of $\mathcal{A}_Z$.*

Proof By an explicit control, one can see that the Schrödinger representation, and therefore all the regular (non-degenerate) irreducible representations of $\mathcal{A}_W$, are spectrally multiplicity free for $\mathcal{A}_Z$. For the converse, spectral multiplicity free implies non-degeneracy and by the Stone-von Neumann theorem the representation is unitarily equivalent to a direct sum of Schrödinger representations. Multiplicity free, implied by spectral multiplicity free, requires that such a sum contains only one term.[2]

Furthermore, one has[3]

Theorem 2.3 *Let π be a representation of a (unital) abelian C^*-algebra $\mathcal{A}$, then the following statements are equivalent*

i) the representation is spectrally multiplicity free
*ii) for each vector $x \in \mathcal{H}_\pi$ the projection on the cyclic subspace $\mathcal{H}_x = \pi(\mathcal{A})x$ belongs to the Baire *-algebra generated by $\pi(\mathcal{A})$ (i.e. the smallest Baire *-algebra in $\mathcal{H}_\pi$ containing $\pi(\mathcal{A})$)*
iii) for every $x \in \mathcal{H}_\pi$, there exists a Baire subset S_x of $\hat{\mathcal{A}}$, such that $\mu_x(\hat{\mathcal{A}}/S_x) = 0$, and $\mu_y(S_x) = 0$, $\forall y \in \mathcal{H}_x^\perp$, where μ_x denotes the positive Baire measure on the Gelfand spectrum $\hat{A}$ defined by x via the Riesz-Markov representation theorem (called the spectral measure of x).

The next problem is to find a generalized substitute of regularity. For this purpose, we recall that, for representations in a separable Hilbert space, strong measurability is equivalent to strong continuity and therefore to regularity. Now, there is a notion of strong measurability in non-separable Hilbert spaces, which provides the needed generalization of regularity.[4]

Definition 2.4 *Let $(X, \mathcal{M}, \mu)$ be a positive σ-finite measure space and $\mathcal{H}$ a Hilbert space.*

A function $F : X \to \mathcal{H}$ is said to be countably-valued if it assumes at most a countable set of values in $\mathcal{H}$, each value being taken on a measurable set. A function $F : X \to \mathcal{H}$ is called **measurable with respect to** *μ if there exists a sequence of countably-valued functions converging μ-almost everywhere to it.*

An operator valued function $F : X \ni a \to F(a) \in \mathcal{B}(\mathcal{H})$ is **strongly measurable w.r.t.** *μ if, $\forall x \in \mathcal{H}$, the vector-valued function $F(a)x$ is measurable with respect to μ.*

Theorem 2.5 *Let $(X, \mathcal{M}, \mu)$ be a positive σ-finite measure space and $\mathcal{H}$ a Hilbert space.*

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[2]The detailed argument is given in S. Cavallaro, Ph. D Thesis, academic year 1996/97, ISAS, Trieste, Chapter V, Sect. 3.

[3]S. Cavallaro, Algebras and Representation Theory, **3**, 175 (2000).

[4]E. Hille and R.S. Phillips, *Functional Analysis and Semigroups*, Amer. Math. Soc. Colloq. Pub. Vol. 31, New York 1948, Theors. 3.5.3, 3.5.5.

An operator valued function $F : X \to \mathcal{B}(\mathcal{H})$ is strongly measurable w.r.t. to μ iff

i) it is weakly measurable
ii) $F(a)x$, $a \in X$, is μ-almost separably-valued for every $x \in \mathcal{H}$ (i.e. there is a μ-null measurable subset $N \subset X$ such that $\{F(a)x; a \in X/N\}$ is separable.

The following Theorem provides a generalization of Stone-von Neumann theorem by giving a characterization of all representations of the Weyl algebra which are spectrally multiplicity free on $\mathcal{A}_Z$ and satisfy a condition of strong measurability

Theorem 2.6 [5] *Let π be a representation of the Weyl algebra $\mathcal{A}_W$ with the following properties*

1) π is spectrally multiplicity free as a representation of the Zak subalgebra $\mathcal{A}_Z$
2) the operator-valued function $\mathbf{T}^2 \to \pi(W(a,b))$ is strongly measurable w.r.t. every positive spectral measure μ_y, $y \in \mathcal{H}_\pi$.

Then, π is an irreducible representation of $\mathcal{A}_W$ and there exists a positive translationally invariant measure μ on the Borel σ-algebra of $\mathbf{T}^2$, such that π is unitarily equivalent to the following representation on $L^2(\mathbf{T}^2, d\mu)$, (see Eqs. (6.1.2)*,* (6.1.3)*),*

$$(\pi(W(a,0))\phi)(\alpha,\beta) = e^{i[\alpha+a]\beta}\,\phi((\alpha+a)\,mod\,1,\beta), \tag{6.2.1}$$

$$(\pi(W(0,b))\phi)(\alpha,\beta) = e^{-ib\alpha}\,\phi(\alpha,(\beta+b)\,mod\,2\pi), \tag{6.2.2}$$

where $[\alpha + a]$ denotes the integer part of $\alpha + a$.

Moreover, the measure μ can be written as the sum of a family of finite positive Borel measures concentrated on disjoint sets.

A comment may be helpful concerning the strong measurability condition. In general, for each $y \in \mathcal{H}_\pi$ the set $\{\pi(W(a,b))\,y; (a,b) \in \mathbf{T}^2\}$ is in general non-separable, but by Theorem 2.6, strong measurability with respect to every positive spectral measure μ_x, implies that for every μ_x there exists a Borel set N of $\mathbf{T}^2$ such that $\mu_x(N) = 0$ and $\{\pi(W(a,b))\,y; (a,b) \in \mathbf{T}^2/N\}$ is separable. This allows to make "local" use of standard results which hold only in separable spaces (and for σ-finite measures), even if $\mathcal{H}_\pi$ is non-separable.

It is worthwhile to remark that all the non-regular representations, arising in the discussion of the physical problems discussed in the previous Chapters, are covered by such a classification theorem.

The Schrödinger representation is given by the Lebesgue measure on $\mathbf{T}^2$, with representation space $L^2(\mathbf{T}^2, (2\pi)^{-1}d\alpha,\, d\beta)$, see Eqs. (6.1.2)–(6.1.5).

The representations appearing in the examples of Chaps. 2, Sect. 2 (translationally invariant state), Sect. 3 (Bloch electron), and Chap. 3, Sect. 2 (Jackiw model),

[5] S. Cavallaro, G. Morchio and F. Strocchi, Lett, Math. Phys. **47**, 307 (1999). At the moment, we cannot offer a simpler version of the rather technical proof presented there.

Sect. 3 (Christ-Lee model), Sect. 4 (QM model of QCD structures) correspond to the measure $\mu = \sum_{\theta \in [0,2\pi)} d\alpha_\theta$, where $d\alpha_\theta$ denotes for each $\theta \in [0, 2\pi)$ the one-dimensional Lebesgue measure concentrated on the segment $\{(\alpha, \theta); \alpha \in [0, 1)\} \subseteq \{\mathbf{T}^2\}$.

The representation defined by the Zak states in the discussion of the quantum Hall electron (Chap. 2, Sect. 5), corresponds to the counting measure on $\mathbf{T}^2$, with representation space $l^2(\mathbf{T}^2)$.

Since, as pointed out by G. Mackey,[6] the Stone-von Neumann theorem can be generalized in the direction of representations of locally compact groups, the above classification of non-regular representations may provide an extension of the standard strategy for the classification of the representations of such groups.

[6] G. Mackey, Duke Math. J. **16**, 313 (1949).

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