Wolfgang Nolting

Theoretical Physics 2

Analytical Mechanics

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General Preface

The seven volumes of the series *Basic Course: Theoretical Physics* are thought to be textbook material for the study of university-level physics. They are aimed to impart, in a compact form, the most important skills of theoretical physics which can be used as basis for handling more sophisticated topics and problems in the advanced study of physics as well as in the subsequent physics research. The conceptual design of the presentation is organized in such a way that

Classical Mechanics (volume 1)
Analytical Mechanics (volume 2)
Electrodynamics (volume 3)
Special Theory of Relativity (volume 4)
Thermodynamics (volume 5)

are considered as the theory part of an *integrated course* of experimental and theoretical physics as is being offered at many universities starting from the first semester. Therefore, the presentation is consciously chosen to be very elaborate and self-contained, sometimes surely at the cost of certain elegance, so that the course is suitable even for self-study, at first without any need of secondary literature. At any stage, no material is used which has not been dealt with earlier in the text. This holds in particular for the mathematical tools, which have been comprehensively developed starting from the school level, of course more or less in the form of recipes, such that right from the beginning of the study, one can solve problems in theoretical physics. The mathematical insertions are always then plugged in when they become indispensable to proceed further in the programme of theoretical physics. It goes without saying that in such a context, not all the mathematical statements can be proved and derived with absolute rigour. Instead, sometimes a reference must be made to an appropriate course in mathematics or to an advanced textbook in mathematics. Nevertheless, I have tried for a reasonably balanced representation so that the mathematical tools are not only applicable but also appear at least 'plausible'.

The mathematical interludes are of course necessary only in the first volumes of this series, which incorporate more or less the material of a bachelor programme.

In the second part of the series which comprises the modern aspects of theoretical physics,

Quantum Mechanics: Basics (volume 6)
Quantum Mechanics: Methods and Applications (volume 7)
Statistical Physics (volume 8)
Many-Body Theory (volume 9),

mathematical insertions are no longer necessary. This is partly because, by the time one comes to this stage, the obligatory mathematics courses one has to take in order to study physics would have provided the required tools. The fact that training in theory has already started in the first semester itself permits inclusion of parts of quantum mechanics and statistical physics in the bachelor programme itself. It is clear that the content of the last three volumes cannot be part of an *integrated course* but rather the subject matter of pure theory lectures. This holds in particular for *Many-Body Theory* which is offered, sometimes under different names as, e.g., *advanced quantum mechanics*, in the eighth or so semester of study. In this part, new methods and concepts beyond basic studies are introduced and discussed which are developed in particular for correlated many particle systems which in the meantime have become indispensable for a student pursuing master's or a higher degree and for being able to read current research literature.

In all the volumes of the series *Basic Course: Theoretical Physics*, numerous exercises are included to deepen the understanding and to help correctly apply the abstractly acquired knowledge. It is obligatory for a student to attempt on his own to adapt and apply the abstract concepts of theoretical physics to solve realistic problems. Detailed solutions to the exercises are given at the end of each volume. The idea is to help a student to overcome any difficulty at a particular step of the solution or to check one's own effort. Importantly these solutions should not seduce the student to follow the *easy way out* as a substitute for his own effort. At the end of each bigger chapter, I have added self-examination questions which shall serve as a self-test and may be useful while preparing for examinations.

I should not forget to thank all the people who have contributed one way or an other to the success of the book series. The single volumes arose mainly from lectures which I gave at the universities of Muenster, Wuerzburg, Osnabrueck, and Berlin in Germany, Valladolid in Spain, and Warangal in India. The interest and constructive criticism of the students provided me the decisive motivation for preparing the rather extensive manuscripts. After the publication of the German version, I received a lot of suggestions from numerous colleagues for improvement, and this helped to further develop and enhance the concept and the performance of the series. In particular, I appreciate very much the support by Prof. Dr. A. Ramakanth, a long-standing scientific partner and friend, who helped me in many respects, e.g. what concerns the checking of the translation of the German text into the present English version.

Special thanks are due to the Springer company, in particular to Dr. Th. Schneider and his team. I remember many useful motivations and stimulations. I have the feeling that my books are well taken care of.

Berlin, Germany Wolfgang Nolting
May 2015

Preface to Volume 2

The concern of classical mechanics consists in the setting up and solving of equations of motion for

mass points, system of mass points, rigid bodies

on the basis of as few as possible

axioms and principles.

The latter are mathematically not strictly provable but represent merely up to now self-consistent facts of everyday experience. One might of course ask why one even today still deals with classical mechanics although this discipline may have a direct relationship to current research only in very rare cases. On the other hand, classical mechanics represents the indispensable basis for the modern trends of theoretical physics, which means they cannot be put across without a deep understanding of classical mechanics. Furthermore, as a side effect, mechanics permits in connection with relatively familiar problems a certain *habituation* to mathematical algorithms. So we have exercised intensively in the first volume of this *Basic Course: Theoretical Physics* in connection with *Newton's Mechanics* the input of vector algebra.

Why, however, are we dealing in this second volume once more with classical mechanics? The *analytical mechanics* of the underlying second volume treats the formulations according to *Lagrange, Hamilton*, and *Hamilton-Jacobi*, which, strictly speaking, do not present *any new physics* compared to the *Newtonian version* being, however, methodically much more elegant and, what is more, revealing a more direct reference to advanced courses in theoretical physics such as the *quantum mechanics*.

The main goal of this volume 2 corresponds exactly to that of the total *Ground Course: Theoretical Physics*. It is thought to be an accompanying textbook material for the study of university-level physics. It is aimed to impart, in a compact form, the most important skills of theoretical physics which can be used as basis for handling more sophisticated topics and problems in the advanced study of physics as well as in the subsequent physics research. It is presented in such a way that it enables self-study without the need for a demanding and laborious reference to secondary literature. For the understanding of this volume, familiarity with the material presented in volume 1 is the only precondition. Mathematical interludes are always then presented in a compact and functional form and practiced when it appears indispensable for further development of the theory. For the whole text, it holds that I had to focus on the essentials, presenting them in a detailed and elaborate form, sometimes consciously sacrificing certain elegance. It goes without saying that after the basic course, secondary literature is needed to deepen the understanding of physics and mathematics.

This volume on *classical mechanics* arose from relevant lectures I gave at the German Universities in Münster and Berlin. The animating interest of the students in my lecture notes has induced me to prepare the text with special care. This volume as well as the subsequent volumes is thought to be a textbook material for the study of basic physics, primarily intended for the students rather than for the teachers.

I am thankful to the Springer company, especially to Dr. Th. Schneider, for accepting and supporting the concept of my proposal. The collaboration was always delightful and very professional. A decisive contribution to the book was provided by Prof. Dr. A. Ramakanth from the Kakatiya University of Warangal (India). Many thanks for it!

Berlin, Germany Wolfgang Nolting
October 2015

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Chapter 1
Lagrange Mechanics

1.1 Constraints, Generalized Coordinates

The Newtonian mechanics, which was the subject matter of the considerations in the first volume of the series **Basic Course: Theoretical Physics**, deals with systems of particles (*mass points*), where each particle follows an equation of motion of the form:

$$m_i \ddot{\mathbf{r}}_i = \mathbf{F}_i^{(\mathrm{ex})} + \sum_{j \neq i} \mathbf{F}_{ij} \tag{1.1}$$

$\mathbf{F}_i^{(\mathrm{ex})}$ is the *external* force acting on particle i, $\mathbf{F}_{ij}$ the (*internal*) force executed by particle j on particle i. In the case of N particles we get from (1.1) a coupled system of $3N$ differential equations of second order the solution of which requires the knowledge of a sufficiently large number of initial conditions. Typical physical systems of our environment are, however, very often not typical particle systems. Let us consider as an example the model of a piston machine (Fig. 1.1). The machine itself consists of almost infinitely many particles. The state of the machine is, however, in general already reasonably characterized by a specification of the angle φ. Forces and tensions, for instance within the piston rod, are normally not of interest. They cause certain *geometric constraints* between the particles. Because of these the particle movements of a macroscopic system are as a rule not completely free. It is said that they are restricted by certain

forces of constraint

To take them in detail into consideration by the internal forces $\mathbf{F}_{ij}$ in (1.1) practically always means a hopeless endeavor.

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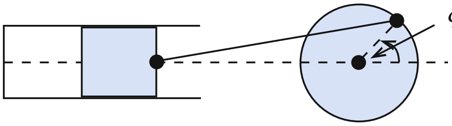


Fig. 1.1 Model of a piston machine

We introduce two for the following very important terms:

Definition 1.1.1

1. **'Constraints'** are conditions which limit the free motion of the particles of a physical system (*geometric bounds*).
2. **'Forces of constraint'** are forces which cause the constraints impeding the free particle movement (tracking force, thread tensions, ...).

In the description of a mechanical system there arise two profound problems:

(a) Forces of constraint are in general unknown. Only their impact is known. The system (1.1) of coupled equations of motions is therefore hardly ever possible to formulate, let alone to solve. We thus try to restate the mechanics in such a way that the forces of constraint are not included anymore. Exactly this idea leads to the Lagrange-version of Classical Mechanics!

(b) The particle coordinates

$$\mathbf{r}_i = (x_i, y_i, z_i) \ , \qquad i = 1, 2, \ldots, N$$

are, because of the forces of constraint, not independent of each other. We therefore intend to replace them later by linearly independent generalized coordinates. As a consequence these generalized coordinates will be in general rather unimaginative, on the other hand, however, mathematically simpler to handle.

It is immediately clear that the constraints play an important role in the concrete solution of a mechanical problem. A classification of mechanical systems with respect to nature and type of their constraints thus surely appears reasonable.

1.1.1 Holonomic Constraints

By these one understands connections between particle coordinates and possibly even the time of the following form:

$$f_\nu (\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N, t) = 0 \ , \qquad \nu = 1, 2, \ldots, p \ . \tag{1.2}$$

(1) Holonomic-Scleronomic Constraints

These are holonomic constraints which do **not** depend explicitly on time, i.e. conditions of the form (1.2) for which additionally holds:

$$\frac{\partial f_\nu}{\partial t} = 0 , \qquad \nu = 1, \ldots, p \tag{1.3}$$

Examples

(1) Dumbbell

The constraint concerns the constant distance between the two masses m_1 and m_2 (Fig. 1.2):

$$(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2 = l^2 . \tag{1.4}$$

(2) Rigid body

A rigid body is characterized by constant inter-particle distances ((4.1), Vol. 1). That corresponds to the constraints:

$$(\mathbf{r}_i - \mathbf{r}_j)^2 - c_{ij}^2 = 0 , \qquad i, j = 1, 2, \ldots, N , \quad c_{ij} = \text{const} . \tag{1.5}$$

(3) Particle on the surface of a sphere

The mass m is bound to the surface of a sphere by the constraint (Fig. 1.3):

$$x^2 + y^2 + z^2 - R^2 = 0 \tag{1.6}$$

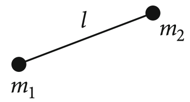


Fig. 1.2 Schematic representation of a dumbbell consisting of two masses m_1 and m_2 which are kept at a constant distance by a massless rod

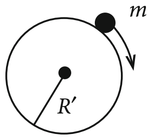


Fig. 1.3 Particle of mass m on the surface of a sphere

(2) Holonomic-Rheonomic Constraints

These are holonomic constraints with an explicit time-dependence:

$$\frac{\partial f_\nu}{\partial t} \neq 0 \,. \tag{1.7}$$

We want to illustrate this term by some examples:

Examples

(1) Particle in an elevator
The particle can freely move only within the xy plane while for the z coordinate the constraint

$$z(t) = v_0(t - t_0) + z_0 \,, \tag{1.8}$$

holds, because the elevator shifts upwards with constant velocity v_0 (Fig. 1.4).

(2) Mass on an inclined plane with variable slope
The time variation of the inclination of the plane (Fig. 1.5) causes a holonomic-rheonomic constraint:

$$\frac{z}{x} - \tan\varphi(t) = 0 \,. \tag{1.9}$$

Holonomic constraints do reduce the number of degrees of freedom. An N particle system without constraints has $3N$ degrees of freedom, but in the presence of p holonomic constraints the number of degrees of freedom is only

$$S = 3N - p \,. \tag{1.10}$$

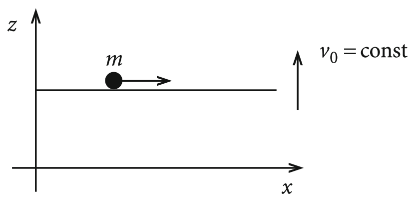


Fig. 1.4 Particle of mass m on a plane that moves with velocity v_0 in z direction

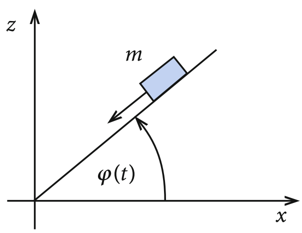


Fig. 1.5 Mass m on an inclined plane whose angle of slope changes with time

A possible numerical procedure can be to eliminate p of the $3N$ Cartesian coordinates by exploiting the constraints (1.2) and to integrate for the rest Newton's equations of motion. However, it is more elegant and more efficient to introduce

'generalized coordinates' $q_1, q_2, \ldots, q_S$**,**

which have to fulfill two conditions:

1. The current configuration of the physical system is **uniquely** fixed by $q_1, \ldots, q_S$. In particular, the transformation formulas
$$\mathbf{r}_i = \mathbf{r}_i\,(q_1, \ldots, q_S, t) \ , \qquad i = 1, 2, \ldots, N \ , \tag{1.11}$$
must implicitly include the constraints.
2. The q_j are independent of each other, i.e. there does **not** exist a relation of the type $F(q_1, \ldots, q_S, t) = 0$.

The concept of the generalized coordinates will play an important role in the following. We therefore add to the above definition some additional remarks:

(a) By the

configuration space

one understands the S-dimensional space which is spanned by the generalized coordinates $q_1, \ldots, q_S$. Each point of the configuration space (*configuration vector*)
$$\mathbf{q} = (q_1, q_2, \ldots, q_S) \tag{1.12}$$
corresponds to a possible state of the system.

(b) One denotes
$$\dot{q}_1, \dot{q}_2, \ldots, \dot{q}_S \quad \text{\textit{'generalized velocities'}}\ .$$

(c) With known initial conditions
$$\mathbf{q}_0 = \mathbf{q}(t_0) \equiv (q_1(t_0), \ldots, q_S(t_0)) \ ,$$
$$\dot{\mathbf{q}}_0 = \dot{\mathbf{q}}(t_0) \equiv (\dot{q}_1(t_0), \ldots, \dot{q}_S(t_0))$$
the state of the system in the configuration space is determinable by equations of motion which are still to be derived.

(d) The choice of the quantities $q_1, \ldots, q_S$ is not unique, only their number S is fixed. One chooses the coordinates according to expediency, which in most cases is clearly predetermined by the physical problem under question.
(e) The quantities q_j are arbitrary. They are not necessarily quantities with the dimension *'length'*. They characterize **'in their entirety'** the system and do no longer describe unconditionally single particles. As a disadvantage it may be considered that then the problem becomes a bit less illustrative.

Examples

(1) Particle on the surface of a sphere

There is one holonomic-scleronomic constraint:

$$x^2 + y^2 + z^2 - R^2 = 0 \; .$$

That means for the number of degrees of freedom:

$$S = 3 - 1 = 2 \; .$$

As generalized coordinates two angles would be appropriate (Fig. 1.6):

$$q_1 = \vartheta \; ; \quad q_2 = \varphi \; .$$

The transformation formulas

$$x = R \sin q_1 \, \cos q_2 \; ,$$
$$y = R \sin q_1 \, \sin q_2 \; ,$$
$$z = R \cos q_1$$

include implicity the constraint. $q_1, \; q_2$ uniquely codify the *state* of the system.

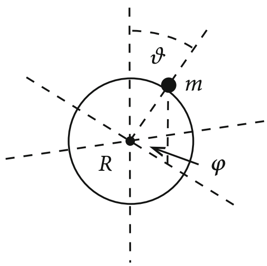


Fig. 1.6 Generalized coordinates for a particle of mass m bound to the surface of a sphere

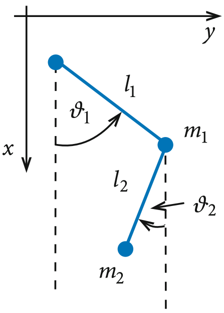


Fig. 1.7 Generalized coordinates for the planar double pendulum

(2) Planar double pendulum

There are altogether four holonomic-scleronomic constraints (Fig. 1.7):

$$z_1 = z_2 = \text{const} ,$$
$$x_1^2 + y_1^2 - l_1^2 = 0 ,$$
$$(x_2 - x_1)^2 + (y_2 - y_1)^2 - l_2^2 = 0 .$$

Hence the number of degrees of freedom amounts to:

$$S = 6 - 4 = 2 .$$

'Convenient' generalized coordinates are obviously in this case:

$$q_1 = \vartheta_1 ; \quad q_2 = \vartheta_2 .$$

The transformation formulas

$$x_1 = l_1 \cos q_1 ; \quad y_1 = l_1 \sin q_1 ; \quad z_1 = 0 ,$$
$$x_2 = l_1 \cos q_1 + l_2 \cos q_2 ; \quad y_2 = l_1 \sin q_1 - l_2 \sin q_2 ; \quad z_2 = 0$$

include again implicitly the constraints.

(3) Particle in the central field

In this case there are no constraints. Nevertheless, the introduction of generalized coordinates can be expedient:

$$S = 3 - 0 = 3 .$$

'Convenient' generalized coordinates are now:

$$q_1 = r ; \quad q_2 = \vartheta ; \quad q_3 = \varphi .$$

The transformation formulas ((1.389), Vol. 1)

$$x = q_1 \sin q_2 \, \cos q_3 \; ,$$
$$y = q_1 \sin q_2 \, \sin q_3 \; ,$$
$$z = q_1 \cos q_2$$

are already known to us from many applications (see Vol. 1). They illustrate that the use of generalized coordinates can be reasonable also in systems **without** constraints, namely when because of certain symmetries the integration of the equations of motion is simplified by a *point transformation* onto curvilinear coordinates.

1.1.2 Non-holonomic Constraints

Therewith one understands connections between the particle coordinates which can **not** be represented as in (1.2) so that they cannot be used to eliminate dispensable coordinates. For systems with non-holonomic constraints there does not exist a general numerical procedure. Special methods will be discussed at a later stage.

(1) Constraints as Inequalities

If the constraints are on hand only as inequalities then using them it is obviously impossible to reduce the number of variables.

Examples

(1) Pearls of an abacus (counting frame)
The pearls (mass points) perform one-dimensional movements only between two fixed limits. The constraints are then partly holonomic,

$$z_i = \text{const} \; ; \quad y_i = \text{const} \; , \qquad i = 1, 2, \ldots, N \; ,$$

but partly also non-holonomic:

$$a \le x_i \le b \; , \qquad i = 1, 2, \ldots, N \; .$$

(2) Particle on a sphere in the earth's gravitational field
The constraint

$$\left(x^2 + y^2 + z^2\right) - R^2 \ge 0$$

confines the free motion of the mass m, but cannot be used to eliminate superfluous coordinates (Fig. 1.8).

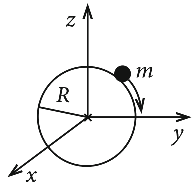


Fig. 1.8 Particle of mass m on the surface of a sphere in the earth's gravitational field as an example for non-holonomic constraints

(2) Constraints in Differential, but Not Integrable Form

These are special constraints which contain the particle velocities. They have the general form

$$\sum_{m=1}^{3N} f_{im}\, dx_m + f_{it}\, dt = 0 \,, \qquad i = 1, \ldots, p \,, \tag{1.13}$$

where the left-hand side can **not** be integrated. It does not represent a total differential. That means that there does not exist a function F_i with

$$f_{im} = \frac{\partial F_i}{\partial x_m} \quad \forall m \,; \quad \frac{\partial F_i}{\partial t} = f_{it} \,.$$

If such a function F_i existed, then it would follow from (1.13)

$$F_i\,(x_i, \ldots, x_{3N}, t) = \text{const}$$

and the corresponding constraint would thus be holonomic.

Example **'Rolling' wheel on a rough undersurface**
The movement of the wheel disc (Radius R) happens so that the disc plane always stands vertically. The movement is completely described by

1. the momentary support point (x, y),
2. the angles φ, ϑ.

Hence the problem is solved if these quantities are known as functions of time (Fig. 1.9).

The constraint *'rolling'* concerns the direction and the magnitude of the velocity of the support point:

$$\begin{aligned} \text{magnitude:} \quad & |\mathbf{v}| = R\dot{\varphi} \,, \\ \text{direction:} \quad & \mathbf{v} \ \text{perpendicular to the wheel axis} \,, \\ & \dot{x} = v_x = v \cos\vartheta \,, \\ & \dot{y} = v_y = v \sin\vartheta \,. \end{aligned}$$

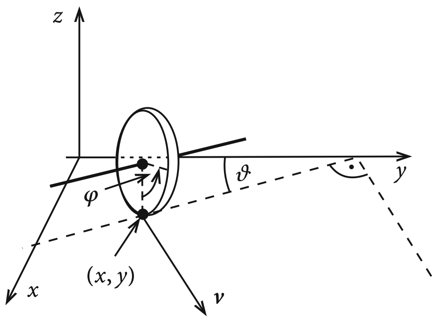


Fig. 1.9 Coordinates for the description of a rolling wheel on a rough undersurface

The combination of the constraints yields

$$\dot{x} - R\dot{\varphi}\,\cos\vartheta = 0\,;\quad \dot{y} - R\dot{\varphi}\,\sin\vartheta = 0$$

or

$$dx - R\cos\vartheta\, d\varphi = 0\,;\quad dy - R\sin\vartheta\, d\varphi = 0\,. \tag{1.14}$$

These conditions are not integrable since the knowledge of the full time-dependence of $\vartheta = \vartheta(t)$ would be necessary which, however, is available not before the full solution of the problem. Hence the constraint *'rolling'* does not lead to a reduction of the number of coordinates. In a certain sense it delimitates *microscopically* the degrees of freedom of the wheel, while *macroscopically* the number remains unchanged. Empirically we know that the wheel can reach every point of the plane by proper transposition manoeuvres.

1.2 The d'Alembert's Principle

1.2.1 Lagrange Equations

According to the considerations of the last section the most urgent objective must be to eliminate the in general not explicitly known constraint forces out of the equations of motion. Exactly that is the new aspect of the *Lagrange mechanics* compared to the Newtonian version. We start with the introduction of another important concept:

Definition 1.2.1 'virtual displacement' $\delta \mathbf{r}_i$

This is the arbitrary (virtual), infinitesimal coordinate change which has to be compatible with the constraints and is instantaneously executed. The latter means:

$$\delta t = 0 \,. \tag{1.15}$$

The quantities $\delta \mathbf{r}_i$ are not necessarily related to the real course of motion. They are therefore to be distinguished from the real displacements $d\mathbf{r}_i$ in the time interval dt, in which the forces and the constraint forces can change:

$$\delta \longleftrightarrow \textit{virtual} \,; \quad d \longleftrightarrow \textit{real} \,.$$

Mathematically we treat the symbol δ like the normal differential d. We elucidate the matter by a simple example (Fig. 1.10):

Example: Particle in an Elevator

The constraint (holonomic-rheonomic) has already been given in (1.8). A suitable generalized coordinate is $q = x$. But then it holds because of $\delta t = 0$:

$$\begin{array}{ll} \text{real displacement:} & d\mathbf{r} = (dx,\, dz) = (dq, v_0\, dt) \,, \\ \text{virtual displacement} & \delta\mathbf{r} = (\delta x, \delta z) = (\delta q, 0) \,. \end{array}$$

Definition 1.2.2 Virtual work

$$\delta W_i = -\mathbf{F}_i \cdot \delta \mathbf{r}_i \,. \tag{1.16}$$

$\mathbf{F}_i$ is the force acting on particle i:

$$\mathbf{F}_i = \mathbf{K}_i + \mathbf{Z}_i = m_i \ddot{\mathbf{r}}_i \,. \tag{1.17}$$

$\mathbf{K}_i$ is the *driving force* acting on the mass point which is somewhat limited in its mobility because of certain constraints. $\mathbf{Z}_i$ is the constraint force.

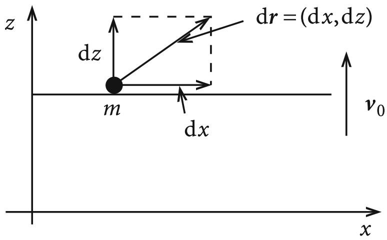


Fig. 1.10 To the distinction between real and virtual displacements by the example of a particle on a plane which moves upwardly with constant velocity v_0

Obviously it holds:

$$\sum_i (\mathbf{K}_i - m_i \ddot{\mathbf{r}}_i) \cdot \delta \mathbf{r}_i + \sum_i \mathbf{Z}_i \cdot \delta \mathbf{r}_i = 0 \, . \tag{1.18}$$

The fundamental
Principle of virtual work

$$\sum_i \mathbf{Z}_i \cdot \delta \mathbf{r}_i = 0 \tag{1.19}$$

will not be mathematically derived being, however, considered as unambiguously *empirically proven*. It expresses the fact that for each *thought movement*, which is compatible with the constraints, the constraint forces do not execute any work. One should notice that in (1.19) only the sum, not necessarily each summand, has to be zero.

Examples

(1) Particle on a 'smooth' curve (Fig. 1.11)
'Smooth' means that there does not exist any component of the constraint force **Z** along the path line. Without any concrete knowledge about **Z** we conclude therewith that **Z** must be perpendicular to the path line and thus also to the virtual displacement $\delta\mathbf{r}$:

$$\mathbf{Z} \cdot \delta \mathbf{r} = 0 \, .$$

(2) Dumbbell (Fig. 1.12)
It holds:

$$\mathbf{Z}_1 = -\mathbf{Z}_2 \, .$$

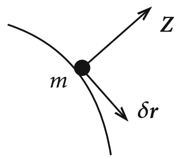


Fig. 1.11 Constraint force for a particle on a smooth curve

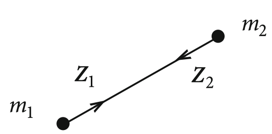


Fig. 1.12 Constraint forces for a dumbbell consisting of two masses m_1 and m_2

The virtual displacements of the two masses can be written as a common translation $\delta\mathbf{s}$ plus an additional rotation $\delta\mathbf{x}_R$ of mass m_2 around the already shifted mass m_1:

$$\delta\mathbf{r}_1 = \delta\mathbf{s}\,; \quad \delta\mathbf{r}_2 = \delta\mathbf{s} + \delta\mathbf{x}_R\,.$$

Inserted into (1.19) it results,

$$\delta W = -\mathbf{Z}_1 \cdot \delta\mathbf{r}_1 - \mathbf{Z}_2 \cdot \delta\mathbf{r}_2 = -(\mathbf{Z}_1 + \mathbf{Z}_2) \cdot \delta\mathbf{s} - \mathbf{Z}_2 \cdot \delta\mathbf{x}_R = 0\,,$$

since $\delta\mathbf{x}_R$ is perpendicular to $\mathbf{Z}_2$ and the sum $(\mathbf{Z}_1 + \mathbf{Z}_2)$ vanishes. We recognize with this example, which can directly be generalized to the rigid body, that only the sum of the contributions in (1.19) must be zero, not necessarily each single summand.

(3) Atwood's free-fall machine (Fig. 1.13)

For the *thread tensions* $\mathbf{Z}_1, \mathbf{Z}_2$ we will find (see (1.49)):

$$\mathbf{Z}_1 = \mathbf{Z}_2$$

That means for the virtual work δW:

$$\begin{aligned}\delta W &= -\mathbf{Z}_1 \cdot \delta\mathbf{x}_1 - \mathbf{Z}_2 \cdot \delta\mathbf{x}_2 \\ &= Z_1(\delta x_1 + \delta x_2) \\ &= Z_1\,\delta\underbrace{(x_1 + x_2)}_{\text{const}} = 0\,.\end{aligned}$$

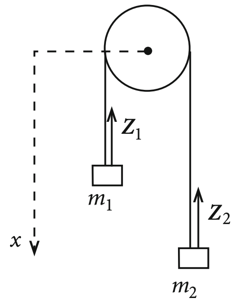


Fig. 1.13 Forces of constraint appearing in Atwood's free-fall machine

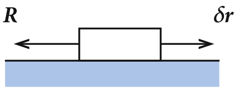


Fig. 1.14 Demonstration of the virtual work of the friction force R

(4) Frictional forces (Fig. 1.14)

Friction forces are **not** considered as constraint forces because they violate the *principle of virtual work*:

$$\delta W = -\mathbf{R} \cdot \delta \mathbf{r} = R\,\delta r \neq 0\,.$$

Therefore, the friction forces will demand special attention in the following.

The principle of virtual work (1.19) can be reformulated by use of (1.18) and is then denoted as

d'Alembert's principle

$$\sum_{i=1}^{N} (\mathbf{K}_i - m_i \ddot{\mathbf{r}}_i) \cdot \delta \mathbf{r}_i = 0\,. \tag{1.20}$$

Hence, the virtual work of the *'lost forces'* is zero. So a first provisional goal is reached. The constraint forces do no longer appear. Indeed, simple mechanical problems can already be solved with (1.20). However, it still remains a disadvantage: The virtual displacements $\delta \mathbf{r}_i$ are because of the constraints not independent of each other. That is why Eq. (1.20) is not yet suitable to derive expedient equations of motion using it. Therefore we try to transform the quantities $\delta \mathbf{r}_i$ into generalized coordinates. From

$$\mathbf{r}_i = \mathbf{r}_i\,(q_1, q_2, \ldots, q_S, t)\ , \qquad i = 1, 2, \ldots, N \tag{1.21}$$

we have:

$$\dot{\mathbf{r}}_i = \sum_{j=1}^{S} \frac{\partial \mathbf{r}_i}{\partial q_j} \dot{q}_j + \frac{\partial \mathbf{r}_i}{\partial t} = \dot{\mathbf{r}}_i\,(q_1, \ldots, q_S, \dot{q}_1, \ldots, \dot{q}_S, t)\ . \tag{1.22}$$

This means in particular:

$$\frac{\partial \dot{\mathbf{r}}_i}{\partial \dot{q}_j} = \frac{\partial \mathbf{r}_i}{\partial q_j}\,. \tag{1.23}$$

For the virtual displacements equation (1.22) reads because of $\delta t = 0$:

$$\delta \mathbf{r}_i = \sum_{j=1}^{S} \frac{\partial \mathbf{r}_i}{\partial q_j} \delta q_j\,. \tag{1.24}$$

That yields for the first summand in (1.20):

$$-\,\delta W_K = \sum_i \mathbf{K}_i \cdot \delta \mathbf{r}_i = \sum_{i=1}^{N} \sum_{j=1}^{S} \mathbf{K}_i \frac{\partial \mathbf{r}_i}{\partial q_j} \delta q_j \equiv \sum_{j=1}^{S} Q_j \, \delta q_j \; . \tag{1.25}$$

We have introduced a further *'generalized quantity'* by the definition:

Definition 1.2.3 'Generalized force components'

$$Q_j = \sum_{i=1}^{N} \mathbf{K}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} \; . \tag{1.26}$$

Since the terms q_j are not necessarily *'lengths'*, the quantities Q_j also do not necessarily have the dimension of a *'force'*. However, it is always true that

$$\left[Q_j \, q_j\right] = energy \; .$$

The **conservative systems** represent an important special case, since they possess a potential ((2.233), Vol. 1),

$$V = V\left(\mathbf{r}_1, \ldots, \mathbf{r}_N\right) \; , \tag{1.27}$$

which in particular does not depend on velocities $\dot{\mathbf{r}}_i$ and is closely related to the forces:

$$\mathbf{K}_i = -\nabla_i V \tag{1.28}$$

In such a case it holds for the generalized force components:

$$Q_j = \sum_{i=1}^{N} (-\nabla_i V) \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} = -\frac{\partial V}{\partial q_j} \; , \qquad j = 1, 2, \ldots, S \; . \tag{1.29}$$

We now analyze the second summand in (1.20). Thereby we use:

$$\begin{aligned} \frac{d}{dt} \frac{\partial \mathbf{r}_i}{\partial q_j} &= \sum_{l=1}^{S} \frac{\partial^2 \mathbf{r}_i}{\partial q_l \, \partial q_j} \dot{q}_l + \frac{\partial^2 \mathbf{r}_i}{\partial t \, \partial q_j} \\ &= \frac{\partial}{\partial q_j} \left\{ \sum_{l=1}^{S} \frac{\partial \mathbf{r}_i}{\partial q_l} \dot{q}_l + \frac{\partial \mathbf{r}_i}{\partial t} \right\} = \frac{\partial \dot{\mathbf{r}}_i}{\partial q_j} \; . \end{aligned} \tag{1.30}$$

It is assumed here that the transformation formulas (1.21) have continuous partial derivatives at least up to second order ((1.257), Vol. 1):

$$\begin{aligned}
\sum_i m_i \ddot{\mathbf{r}}_i \cdot \delta\mathbf{r}_i &= \sum_{i=1}^{N}\sum_{j=1}^{S} m_i \ddot{\mathbf{r}}_i \frac{\partial \mathbf{r}_i}{\partial q_j}\delta q_j \\
&= \sum_{i=1}^{N}\sum_{j=1}^{S} m_i \left\{ \frac{d}{dt}\left(\dot{\mathbf{r}}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j}\right) - \dot{\mathbf{r}}_i \frac{d}{dt}\frac{\partial \mathbf{r}_i}{\partial q_j} \right\} \delta q_j \\
&\overset{(1.23,\,1.30)}{=} \sum_{i=1}^{N}\sum_{j=1}^{S} m_i \left\{ \frac{d}{dt}\left(\dot{\mathbf{r}}_i \cdot \frac{\partial \dot{\mathbf{r}}_i}{\partial \dot{q}_j}\right) - \dot{\mathbf{r}}_i \cdot \frac{\partial \dot{\mathbf{r}}_i}{\partial q_j} \right\} \delta q_j \\
&= \sum_{i=1}^{N}\sum_{j=1}^{S} m_i \left\{ \frac{d}{dt}\left[\frac{\partial}{\partial \dot{q}_j}\left(\frac{1}{2}\dot{\mathbf{r}}_i^2\right)\right] - \frac{\partial}{\partial q_j}\left(\frac{1}{2}\dot{\mathbf{r}}_i^2\right) \right\} \delta q_j \\
&= \sum_{j=1}^{S} \left\{ \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} \right\} \delta q_j \; . \qquad (1.31)
\end{aligned}$$

$T = 1/2 \sum_i m_i \dot{\mathbf{r}}_i^2$ is the kinetic energy of the particle system. We insert (1.31) and (1.25) into (1.20) and then get the

d'Alembert's principle

$$\sum_{j=1}^{S} \left\{ \left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} \right] - Q_j \right\} \delta q_j = 0 \; . \qquad (1.32)$$

In this form the principle still holds very generally. The following specializations are important:

(1) Holonomic Constraints

In this case the coordinates q_j are independent of each other, the quantities δq_j are accordingly freely selectable. We could, e.g., put to zero all δq_j except for one. That has the consequence that in (1.32) not only the sum but even each summand vanishes:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = Q_j \; , \qquad j = 1, 2, \ldots, S \; . \qquad (1.33)$$

(2) Conservative System

In this case (1.29) is valid. Moreover, V does not depend on the generalized velocities $\dot{q}_j$ so that we can write instead of (1.32):

$$\sum_{j=1}^{S}\left[\frac{d}{dt}\frac{\partial}{\partial \dot{q}_j}(T-V)-\frac{\partial}{\partial q_j}(T-V)\right]\delta q_j = 0\ .$$

With the **definition** of the fundamental
Lagrangian function

$$L\left(q_1,\ldots,q_S,\,\dot{q}_1,\ldots,\dot{q}_S,\,t\right) = T\left(q_1,\ldots,q_S,\,\dot{q}_1,\ldots,\dot{q}_S,\,t\right) - V\left(q_1,\ldots,q_S,t\right)\ , \tag{1.34}$$

which is very important for the further considerations, it follows then:

$$\sum_{j=1}^{S}\left[\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j}-\frac{\partial L}{\partial q_j}\right]\delta q_j = 0\ . \tag{1.35}$$

(3) Conservative System with Holonomic Constraints

This is the case which will be discussed most frequently in the following:
Lagrange equations of motion (of second kind)

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j}-\frac{\partial L}{\partial q_j} = 0\ , \qquad j = 1,2,\ldots,S\ . \tag{1.36}$$

The dominant quantities of the Newtonian mechanics are *momentum* and *force* and these are vectors. On the other hand, *energy* and *work* play the corresponding role in the Lagrangian version of mechanics, and these are scalars. That may be considered as a certain advantage. The Lagrange equation (1.36) replace Newton's equations of motion (1.1). They are differential equations of second order, for the complete solution of which

$2S$ **initial conditions**

must be given. The constraint forces are eliminated, they do no longer appear in the equations of motion.

We investigate the Lagrange equations for arbitrary, general coordinates. With (1.22) the kinetic energy can be written as

$$T = \frac{1}{2}\sum_{i=1}^{N} m_i\,\dot{\mathbf{r}}_i^2 = \frac{1}{2}\sum_{j,l=1}^{S}\mu_{jl}\,\dot{q}_j\,\dot{q}_l + \sum_{j=1}^{S}\alpha_j\,\dot{q}_j + \alpha\ , \tag{1.37}$$

with the following abbreviations:

$$\alpha = \frac{1}{2}\sum_{i=1}^{N} m_i \left(\frac{\partial \mathbf{r}_i}{\partial t}\right)^2 , \qquad (1.38)$$

$$\alpha_j = \sum_{i=1}^{N} m_i \left(\frac{\partial \mathbf{r}_i}{\partial t}\right) \cdot \left(\frac{\partial \mathbf{r}_i}{\partial q_j}\right) , \qquad (1.39)$$

$$\mu_{jl} = \sum_{i=1}^{N} m_i \left(\frac{\partial \mathbf{r}_i}{\partial q_j}\right) \cdot \left(\frac{\partial \mathbf{r}_i}{\partial q_l}\right) \quad : \quad \textit{'generalized masses'}. \qquad (1.40)$$

The Lagrangian function therefore has the following general structure:

$$L = T - V = L_2 + L_1 + L_0 , \qquad (1.41)$$

$$L_2 = \frac{1}{2}\sum_{j,l=1}^{S} \mu_{jl}\, \dot{q}_j\, \dot{q}_l , \qquad (1.42)$$

$$L_1 = \sum_{j=1}^{S} \alpha_j\, \dot{q}_j , \qquad (1.43)$$

$$L_0 = \alpha - V(q_1, \dots, q_S, t) \ . \qquad (1.44)$$

The quantities L_n are *homogeneous functions* of the generalized velocities of order $n = 2, 1, 0$. Homogeneous functions are generally defined as follows:

Definition 1.2.4 $f(x_1, \dots, x_m)$ is homogeneous of order n if it holds:

$$f(ax_1, \dots, ax_m) = a^n f(x_1, \dots, x_m) \qquad \forall\ a \in \mathbb{R} \ . \qquad (1.45)$$

At an earlier stage we stated that the choice of the generalized coordinates is more or less arbitrary, only their total number S is fixed. We now demonstrate that

Lagrange equations are forminvariant
under (differentiable) point transformations

$$(q_1, \dots, q_S) \ \longleftrightarrow \ (\bar{q}_1, \dots, \bar{q}_S)$$

We start with the transformation formulas

$$\left.\begin{aligned} \bar{q}_j &= \bar{q}_j(q_1, \dots, q_S, t) \\ q_l &= q_l(\bar{q}_1, \dots, \bar{q}_S, t) \end{aligned}\right\} \qquad j, l = 1, \dots, S \ .$$

Under the presumption

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = 0 \qquad \text{for } j = 1, 2, \ldots, S$$

it follows for

$$\widetilde{L}\left(\bar{\mathbf{q}}, \dot{\bar{\mathbf{q}}}, t\right) = L\left(\mathbf{q}\left(\bar{\mathbf{q}}, t\right), \dot{\mathbf{q}}\left(\bar{\mathbf{q}}, \dot{\bar{\mathbf{q}}}, t\right), t\right)$$

the assertion:

$$\frac{d}{dt}\frac{\partial \widetilde{L}}{\partial \dot{\bar{q}}_l} - \frac{\partial \widetilde{L}}{\partial \bar{q}_l} = 0 \, , \qquad l = 1, 2, \ldots, S \, . \tag{1.46}$$

Proof

$$\dot{q}_j = \sum_l \frac{\partial q_j}{\partial \bar{q}_l} \dot{\bar{q}}_l + \frac{\partial q_j}{\partial t} \implies \frac{\partial \dot{q}_j}{\partial \dot{\bar{q}}_l} = \frac{\partial q_j}{\partial \bar{q}_l} \, ,$$

$$\frac{\partial \widetilde{L}}{\partial \bar{q}_l} = \sum_j \left(\frac{\partial L}{\partial q_j} \frac{\partial q_j}{\partial \bar{q}_l} + \frac{\partial L}{\partial \dot{q}_j} \frac{\partial \dot{q}_j}{\partial \bar{q}_l} \right) \, ,$$

$$\frac{\partial \widetilde{L}}{\partial \dot{\bar{q}}_l} = \sum_j \frac{\partial L}{\partial \dot{q}_j} \frac{\partial \dot{q}_j}{\partial \dot{\bar{q}}_l} = \sum_j \frac{\partial L}{\partial \dot{q}_j} \frac{\partial q_j}{\partial \bar{q}_l}$$

$$\implies \frac{d}{dt}\frac{\partial \widetilde{L}}{\partial \dot{\bar{q}}_l} = \sum_j \left\{ \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} \right) \frac{\partial q_j}{\partial \bar{q}_l} + \frac{\partial L}{\partial \dot{q}_j} \left(\frac{d}{dt}\frac{\partial q_j}{\partial \bar{q}_l} \right) \right\}$$

$$= \sum_j \left\{ \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} \right) \frac{\partial q_j}{\partial \bar{q}_l} + \frac{\partial L}{\partial \dot{q}_j} \frac{\partial \dot{q}_j}{\partial \bar{q}_l} \right\}$$

$$\implies \frac{d}{dt}\frac{\partial \widetilde{L}}{\partial \dot{\bar{q}}_l} - \frac{\partial \widetilde{L}}{\partial \bar{q}_l} = \sum_j \left\{ \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} \right\} \frac{\partial q_j}{\partial \bar{q}_l} = 0 \, .$$

For the term *'form invariance'* it is not really decisive that $\widetilde{L}$ arises from L simply by inserting the transformation formulas. It is only important that there does exist at all for $L(\mathbf{q}, \dot{\mathbf{q}}, t)$ a unique $\widetilde{L}(\bar{\mathbf{q}}, \dot{\bar{\mathbf{q}}}, t)$ so that the Langrange equations are formally identical in both systems of coordinates.

1.2.2 Simple Applications

In this section we want to demonstrate and practice extensively the algorithm which is usually applied for the solution of mechanical problems by exploiting the Lagrange equations. Throughout the following considerations we will presume

holonomic constraints, conservative forces

The solution method then consists of six sub-steps:

1. Formulate the constraints.
2. Choose proper generalized coordinates **q**.
3. Find the transformation formulas.
4. Write down the Lagrangian function $L = T - V = L(\mathbf{q}, \dot{\mathbf{q}}, t)$.
5. Derive and solve the Lagrange equation (1.36).
6. Back transformation to the original, *'illustrative'* coordinates.

We want to exercise this procedure with some typical examples.

(1) Atwood's Free-Fall Machine

It is about a conservative system with holonomic-scleronomic constraints (Fig. 1.15):

$$x_1 + x_2 = l - \pi R = \text{const} ,$$
$$y_1 = z_1 = z_2 = 0 , \quad y_2 = 2R .$$

There thus remains

$$S = 6 - 5 = 1$$

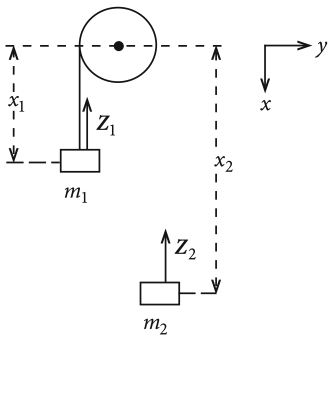


Fig. 1.15 Atwood's free-fall machine

degree of freedom. As a suitable generalized coordinate we may choose:

$$q = x_1 \quad (\Longrightarrow \; x_2 = l - \pi R - q) \; .$$

Therewith the transformation formulas are already known.

With the kinetic energy

$$T = \frac{1}{2}\left(m_1\dot{x}_1^2 + m_2\dot{x}_2^2\right) = \frac{1}{2}\left(m_1 + m_2\right)\dot{q}^2$$

and the potential energy

$$V = -m_1 g\, x_1 - m_2 g\, x_2 = -m_1 g\, q - m_2 g(l - \pi R - q)$$

we have the Lagrangian function

$$L = \frac{1}{2}\left(m_1 + m_2\right)\dot{q}^2 + \left(m_1 - m_2\right) g\, q + m_2 g(l - \pi R) \; . \tag{1.47}$$

By differentiating the Lagrangian

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = \left(m_1 + m_2\right)\ddot{q} \; ; \qquad \frac{\partial L}{\partial q} = \left(m_1 - m_2\right) g$$

one finds with (1.36) the following simple equation of motion:

$$\ddot{q} = \frac{m_1 - m_2}{m_1 + m_2}\, g \; . \tag{1.48}$$

That is just the *'delayed'* free fall. With the presetting of two initial conditions equation (1.48) can easily be integrated. Therewith the problem is solved.

We now have even the possibility by comparison with Newton's equations of motion

$$m_1\ddot{x}_1 = m_1 g + Z_1 \; ; \qquad m_2\ddot{x}_2 = m_2 g + Z_2$$

to determine explicitly the constraint forces (*thread tensions*). Because of

$$\ddot{x}_1 = -\ddot{x}_2 = \ddot{q}$$

it holds:

$$m_1\ddot{x}_1 - m_2\ddot{x}_2 = \left(m_1 - m_2\right) g + \left(Z_1 - Z_2\right) = \left(m_1 + m_2\right)\ddot{q} = \left(m_1 - m_2\right) g \; .$$

This means:

$$Z_1 = Z_2 = Z \ .$$

Therewith it follows further:

$$m_1\ddot{x}_1 + m_2\ddot{x}_2 = (m_1 + m_2)\, g + 2Z = (m_1 - m_2)\, \ddot{q} \ .$$

The thread tension Z hence reads:

$$Z = -2g\, \frac{m_1\, m_2}{m_1 + m_2} \ . \tag{1.49}$$

(2) Gliding Bead on a Uniformly Rotating Rod

The conservative system possesses two holonomic constraints; one of them is scleronomic, the other rheonomic:

$$z = 0 \ ,$$
$$y = x \ \tan \omega t \ .$$

As generalized coordinate the distance between bead and the center of rotation suggests itself (Fig. 1.16):

$$q = r \ .$$

With the transformation formulas

$$x = q \ \cos \omega t \ ; \quad y = q \ \sin \omega t \ ; \quad z = 0$$

one finds the kinetic energy

$$T = \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2\right) = \frac{m}{2}\left(\dot{q}^2 + q^2\omega^2\right) \ ,$$

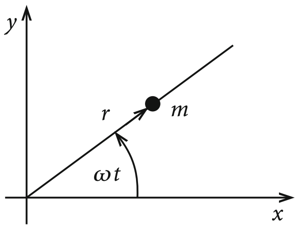


Fig. 1.16 Gliding bead on a rod that rotates with constant angular velocity ω

which because of $V \equiv 0$ is identical to the Lagrangian:

$$L = T - V = \frac{m}{2}\left(\dot{q}^2 + q^2\omega^2\right) = L_2 + L_0 \ . \tag{1.50}$$

The function L_1 does not appear in spite of rheonomic constraints. However, that is purely accidental. Normally the function L_1 (1.43) shows up explicitly in such a case. On the other hand, the function L_0 is here indeed a consequence of the rheonomic constraint.

The equation of motion

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = m\,\ddot{q} = \frac{\partial L}{\partial q} = m\,q\,\omega^2$$

leads to:

$$\ddot{q} = \omega^2 q \ .$$

The general solution reads:

$$q(t) = A\,e^{\omega t} + B\,e^{-\omega t} \ .$$

With the initial conditions

$$q(t=0) = r_0 > 0 \ ; \quad \dot{q}(t=0) = 0$$

one gets $A = B = r_0/2$ and therewith

$$q(t) = \frac{1}{2} r_0 \left(e^{\omega t} + e^{-\omega t}\right) \ .$$

The bead thus moves outwards with growing acceleration for $t \to \infty$. Thereby the energy of the bead steadily increases since the constraint force carries out work. At first glance that appears to be a contradiction to the principle of virtual work (1.19). However, that is not the case! The real displacement of the mass m in the time interval dt is not identical to the virtual displacement $\delta \mathbf{r}$ since the latter is done at fixed time. Thus the work really executed by the constraint force

$$dW_Z = \mathbf{Z} \cdot d\mathbf{r} \neq 0$$

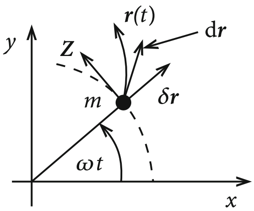


Fig. 1.17 Demonstration of the difference between real and virtual work using the example of the gliding pearl on a rotating rod

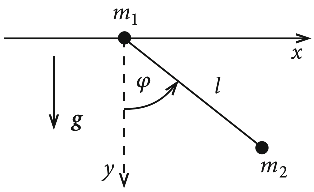


Fig. 1.18 In the earth's gravitational field oscillating dumbbell where one of its masses m_1 can move frictionlessly in x direction

is to distinguish from the virtual work (Fig. 1.17)

$$\delta W_Z = \mathbf{Z} \cdot \delta \mathbf{r} = 0 \,, \quad \text{since } \mathbf{Z} \perp \delta \mathbf{r} \,,$$

(3) Oscillating Dumbbell

The mass m_1 of a dumbbell of length l can move frictionlessly along a horizontal straight line (Fig. 1.18). We ask ourselves which curves will be described by the masses m_1 and m_2 under the influence of the gravitational force.

There are on hand four holonomic-scleronomic constraints:

$$z_1 = z_2 = 0 \,; \quad y_1 = 0 \,; \quad (x_1 - x_2)^2 + y_2^2 - l^2 = 0 \,.$$

Thus there are left

$$S = 6 - 4 = 2$$

degrees of freedom. Convenient generalized coordinates are then most probably:

$$q_1 = x_1 \,; \quad q_2 = \varphi$$

That yields as transformation formulas:

$$\begin{aligned} x_1 &= q_1 \,; & y_1 &= z_1 = 0 \,, \\ x_2 &= q_1 + l \sin q_2 \,; & y_2 &= l \cos q_2 \,; \; z_2 = 0 \,. \end{aligned}$$

Therewith we calculate the kinetic energy:

$$T = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\left(\dot{x}_2^2 + \dot{y}_2^2\right)$$
$$= \frac{1}{2}(m_1 + m_2)\,\dot{q}_1^2 + \frac{1}{2}m_2\left(l^2\dot{q}_2^2 + 2l\,\dot{q}_1\dot{q}_2\,\cos q_2\right) .$$

For the potential energy we find:

$$V_1 \equiv 0 ; \quad V_2 = -m_2\,g\,l\,\cos\varphi ; \quad V = -m_2\,g\,l\,\cos q_2 .$$

This leads to the following Lagrangian:

$$L = \frac{1}{2}(m_1 + m_2)\,\dot{q}_1^2 + \frac{1}{2}m_2\left(l^2\,\dot{q}_2^2 + 2\,l\,\dot{q}_1\,\dot{q}_2\,\cos q_2\right) + m_2\,g\,l\,\cos q_2 . \tag{1.51}$$

Before we continue to consider the concrete procedure of solution we want to introduce two terms which are eminently important for following discussions.

Definition 1.2.5 ***Generalized momentum***

$$p_i = \frac{\partial L}{\partial \dot{q}_i} . \tag{1.52}$$

Definition 1.2.6 ***Cyclic coordinate***

$$q_j \textit{ cyclic} \iff \frac{\partial L}{\partial q_j} = 0 \iff p_j = \frac{\partial L}{\partial \dot{q}_j} = \text{const} . \tag{1.53}$$

Each cyclic coordinate automatically leads to a conservation law. For this reason one should always choose the generalized coordinates such that a maximal number of them are already cyclic.

In our example here q_1 is cyclic. That means:

$$p_1 = \frac{\partial L}{\partial \dot{q}_1} = (m_1 + m_2)\,\dot{q}_1 + m_2\,l\,\dot{q}_2\,\cos q_2 = \text{const} .$$

We solve this equation for $\dot{q}_1$,

$$\dot{q}_1 = c - \frac{m_2\,l}{m_1 + m_2}\,\dot{q}_2\,\cos q_2 ,$$

and integrate:

$$q_1(t) = c\,t - \frac{m_2\,l}{m_1 + m_2}\,\sin q_2(t) + a .$$

We need four initial conditions:

$$q_1(t=0) = 0\ ; \quad q_2(t=0) = 0\ ;$$
$$\dot{q}_1(t=0) = -\frac{m_2}{m_1+m_2}\, l\,\omega_0\ ; \quad \dot{q}_2(t=0) = \omega_0\ . \tag{1.54}$$

A first consequence herefrom is:

$$a = 0\ , \quad c = 0\ .$$

Therewith we have the interim solution:

$$q_1(t) = -\frac{m_2}{m_1+m_2}\, l\, \sin q_2(t)\ .$$

For the motion of mass m_1 it therefore holds:

$$x_1(t) = -\frac{m_2}{m_1+m_2}\, l\, \sin\varphi(t)\ ; \quad y_1(t) = z_1(t) = 0\ . \tag{1.55}$$

With the transformation formulas it follows for mass m_2:

$$x_2(t) = \frac{m_1}{m_1+m_2}\, l\, \sin\varphi(t)\ ; \quad y_2(t) = l\,\cos\varphi(t)\ ; \quad z_2(t) = 0\ . \tag{1.56}$$

In a combined form that can be written as the midpoint equation of an ellipse:

$$\frac{x_2^2}{\left(\frac{m_1\, l}{m_1+m_2}\right)^2} + \frac{y_2^2}{l^2} = 1\ . \tag{1.57}$$

The mass m_2 is thus running through a part of an ellipse with the horizontal semiaxis $m_1\, l/(m_1+m_2)$ and the vertical semiaxis l. In the limit $m_1 \to \infty$ that reduces to the simple mathematical pendulum (Sect. 2.3.4, Vol. 1).

With (1.55) and (1.56) the problem is not yet completely solved since $\varphi(t)$ is still unknown. However, we still have at our disposal a further Lagrange equation:

$$\frac{\partial L}{\partial \dot{q}_2} = m_2\left(l^2\,\dot{q}_2 + l\,\dot{q}_1\,\cos q_2\right),$$
$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_2} = m_2\left(l^2\,\ddot{q}_2 + l\,\ddot{q}_1\,\cos q_2 - l\,\dot{q}_1\,\dot{q}_2\,\sin q_2\right),$$
$$\frac{\partial L}{\partial q_2} = m_2\left(-l\,\dot{q}_1\,\dot{q}_2\,\sin q_2 - g\,l\,\sin q_2\right).$$

Insertion into (1.36) yields the following equation of motion:

$$l^2\,\ddot{q}_2 + l\,\ddot{q}_1\,\cos q_2 + g\,l\,\sin q_2 = 0\,. \tag{1.58}$$

For 'small' values of $q_2 = \varphi$ we can assume

$$\cos q_2 \approx 1\,; \quad \sin q_2 \approx q_2$$

whereby (1.58) is simplified to

$$l\,\ddot{q}_2 + \ddot{q}_1 + g\,q_2 \approx 0$$

From (1.55) we read out:

$$q_1 \approx -\frac{m_2}{m_1 + m_2}\,l\,q_2 \implies \ddot{q}_1 \approx -\frac{m_2\,l}{m_1 + m_2}\,\ddot{q}_2\,.$$

This yields for q_2 the following equation of motion:

$$\ddot{q}_2 + \frac{g}{l}\,\frac{m_1 + m_2}{m_1}\,q_2 \approx 0\,.$$

It appears recommendable as solution to propose:

$$q_2 = A\,\cos\omega t + B\,\sin\omega t\,.$$

The chosen initial conditions (1.54) require $A = 0$ and $B\omega = \omega_0$. Therewith it finally follows:

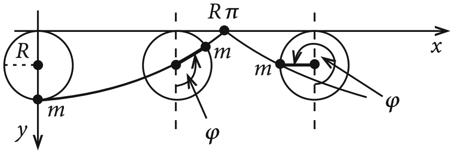
$$\varphi(t) = \frac{\omega_0}{\omega}\,\sin\omega t\,; \quad \omega = \sqrt{\frac{g}{l}\,\frac{m_1 + m_2}{m_1}}\,. \tag{1.59}$$

(4) Cycloidal Pendulum

A particle of mass m moves in the earth's gravitational field on a *cycloidal curve*. The latter is realized by the rolling of a wheel (radius R) on a flat plane without slipping. It has the following parameter representation (Fig. 1.19):

$$\begin{aligned} x &= R\,\varphi + R\,\sin\varphi = R(\varphi + \sin\varphi)\,, \\ y &= 2\,R - R(1 - \cos\varphi) = R(1 + \cos\varphi)\,. \end{aligned} \tag{1.60}$$

Fig. 1.19 Realization of a cycloidal curve by rolling (without slipping) a wheel on a plane



The first term for x is just the unrolling condition, the second is due to the rotation of the wheel. We can solve the y equation for φ and insert the result into the x equation. Equation (1.60) therewith brings about **one** constraint. A further one is $z \equiv 0$. So there is left for the mass point m only $S = 3 - 2 = 1$ degree of freedom. A recommendable generalized coordinate q is surely the angle φ. With

$$\dot{x} = R\,\dot{q}(1 + \cos q)\;; \quad \dot{y} = -R\,\dot{q}\,\sin q$$

we now calculate the kinetic energy:

$$T = \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2\right) = m\,R^2\dot{q}^2(1 + \cos q)\;.$$

For the potential energy we have:

$$V = -m\,g\,y = -m\,g\,R(1 + \cos q)\;.$$

That leads to the Lagrangian function:

$$L = T - V = m\,R\,(1 + \cos q)\,\left(R\,\dot{q}^2 + g\right)\;. \tag{1.61}$$

From L we get by differentiation:

$$\frac{\partial L}{\partial \dot{q}} = 2\,m\,R^2(1 + \cos q)\dot{q}\;,$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = 2\,m\,R^2\left[\ddot{q}(1 + \cos q) - \dot{q}^2\sin q\right]\;,$$

$$\frac{\partial L}{\partial q} = -m\,R\,\sin q\,\left(R\,\dot{q}^2 + g\right)\;.$$

With $(1 + \cos q) = 2\,\cos^2(q/2)$ and $\sin q = 2\,\sin(q/2)\,\cos(q/2)$ we find the equation of motion which is to solve:

$$2\,\ddot{q}\,\cos\frac{q}{2} - \dot{q}^2\,\sin\frac{q}{2} + \frac{g}{R}\,\sin\frac{q}{2} = 0\;.$$

That can be reformulated as:

$$\frac{d^2}{dt^2}\sin\frac{q}{2}+\frac{g}{4R}\sin\frac{q}{2}=0\,. \tag{1.62}$$

The cycloidal pendulum thus obeys an oscillation equation for $\sin(q/2)=\sin(\varphi/2)$ with the frequency

$$\omega=\frac{1}{2}\sqrt{\frac{g}{R}}\,. \tag{1.63}$$

The general solution reads:

$$\varphi(t)=2\,\arcsin\left[A\,e^{i\omega t}+B\,e^{-i\omega t}\right]\,, \tag{1.64}$$

where A, B are to be fixed by initial conditions.

In the case of the simple pendulum (Sect. 2.3.4, Vol. 1) the oscillation frequency does depend on the amplitude of the oscillation. The usual assumption $\sin\varphi\approx\varphi$ which leads to the oscillation equation is clearly allowed only for small amplitudes and is wrong for large amplitudes. In the present case we have found a geometric motion of the mass point for which the oscillation period is strictly independent of the oscillation amplitude.

(5) *N*-Particle System Without Constraints

We have to expect that in this special case the Lagrange equations are identical to Newton's equations of motion. Because of the absence of constraints there are $S=3N$ degrees of freedom, and as generalized coordinates one could think for instance of the Cartesian ones. From the Lagrangian

$$L=T-V=\sum_{i=1}^{N}\frac{m_i}{2}\left(\dot{x}_i^2+\dot{y}_i^2+\dot{z}_i^2\right)-V\left(x_1,\ldots,z_N,t\right) \tag{1.65}$$

it follows:

$$\frac{d}{dt}\frac{\partial L}{\partial\dot{x}_i}=m_i\,\ddot{x}_i\,;\qquad\frac{\partial L}{\partial x_i}=-\frac{\partial V}{\partial x_i}=F_{x_i}\,.$$

The Lagrange equations of motion,

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i} = \frac{\partial L}{\partial x_i} \iff m_i \ddot{x}_i = F_{x_i} \;,$$

indeed lead directly to Newton's equations of motion. Adding the form invariance in connection with point transformations proved in (1.46) the Lagrange-Newton equivalence does hold also for arbitrary curvilinear coordinates.

(6) Kepler Problem

We consider the motion of a particle of mass m in the central field (see Example 3 in Sect. 1.1.1) with the potential energy ((2.259) in Vol. 1):

$$V(x, y, z) = \frac{-\alpha}{\sqrt{x^2 + y^2 + z^2}} \quad (\text{z. B. } \alpha = \gamma\, m M) \;.$$

Cartesian coordinates lead to rather complicated equations of motion. In connection with Example 3 in Sect. 1.1.1 we have already recognized that spherical coordinates would be a much more convenient starting point. By using these coordinates the Lagrangian reads:

$$L\big(r, \vartheta, \varphi, \dot{r}, \dot{\vartheta}, \dot{\varphi}\big) = \frac{m}{2}\big(\dot{r}^2 + r^2\dot{\vartheta}^2 + r^2 \sin^2\vartheta\, \dot{\varphi}^2\big) + \frac{\alpha}{r} \;. \tag{1.66}$$

Simply the immediate observation that the coordinate φ is cyclic results in some important physical consequences:

$$p_\varphi = \frac{\partial L}{\partial \dot{\varphi}} = m\, r^2 \sin^2\vartheta\, \dot{\varphi} = L_z = \text{const} \;. \tag{1.67}$$

The z component of the angular momentum $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ is a constant of motion. Since the z direction is by no means special we have to conclude that even the full angular momentum must be constant:

$$\mathbf{L} = m\, \mathbf{r} \times \dot{\mathbf{r}} = \mathbf{const} \;. \tag{1.68}$$

(One has to distinguish the vector $\mathbf{L}$ (angular momentum) from the scalar L (Lagrangian function)). Without any restriction of generality we can position the z axis of our system of coordinates parallel to the angular momentum $\mathbf{L}$ so that automatically must be $L_x = L_y \equiv 0$. The orbital plane spanned by $[\mathbf{r} \times \dot{\mathbf{r}}]$ is then the xy plane. That brings about $\vartheta \equiv \pi/2$ and therewith $\dot{\vartheta} \equiv 0$ which simplifies the Lagrangian

$$L = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\varphi}^2\right) + \frac{\alpha}{r} \tag{1.69}$$

Then one has to only discuss the equation of motion

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = m\,\ddot{r} \stackrel{!}{=} \frac{\partial L}{\partial r} = m\,r\,\dot{\varphi}^2 - \frac{\alpha}{r^2} \tag{1.70}$$

1.2.3 Generalized Potentials

The simple examples of the last section presume the validity of the Lagrange equations in the form (1.36). They concern conservative systems with holonomic constraints. For non-conservative systems, but with holonomic constraints, instead of that, the starting point must be (1.33):

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} = Q_j \ , \qquad j = 1, 2, \ldots, S \ .$$

However, we come to formally unchanged Lagrange equations for the so-called
Generalized potentials

$$U = U\left(q_1, \ldots, q_S, \dot{q}_1, \ldots, \dot{q}_S, t\right) \ ,$$

if the generalized forces Q_j are derivable from U by:

$$Q_j = \frac{d}{dt}\frac{\partial U}{\partial \dot{q}_j} - \frac{\partial U}{\partial q_j} \ , \qquad j = 1, 2, \ldots, S \ . \tag{1.71}$$

The first term on the right-hand side is new compared to the case of a conservative system. For the
Generalized Lagrangian function

$$L = T - U \tag{1.72}$$

the equations of motion are obviously valid, because of (1.71), in the formally unchanged version (1.36). On the other hand, the requirement (1.71) appears to be very special. However, there does exist a very important application example:

Charged particle in an electromagnetic field

In Vol. 3 we will learn that a particle with charge $\bar{q}$ which moves with the velocity $\mathbf{v}$ in an electromagnetic field (electrical field $\mathbf{E}$, magnetic induction $\mathbf{B}$) experiences the so-called *'Lorentz force'*

$$\mathbf{F} = \bar{q}\ \left[\mathbf{E} + (\mathbf{v} \times \mathbf{B})\right] \tag{1.73}$$

This force is **not** conservative. It possesses, though, a generalized potential U in the sense of (1.71). To show this we first express **F** by the *electromagnetic potentials*,

$$\varphi(\mathbf{r}, t) \;:\; \text{scalar potential} \; ; \quad \mathbf{A}(\mathbf{r}, t) \;:\; \text{vector potential} \; ,$$

They are chosen in such a way that in the *Maxwell equations*, which in electrodynamics take over the same fundamental role as Newton's axioms in mechanics,

$$\nabla \times \mathbf{E} + \frac{\partial}{\partial t}\mathbf{B} = 0 \; ; \qquad \nabla \cdot \mathbf{B} = 0 \; ; \tag{1.74}$$

$$\nabla \times \mathbf{H} - \frac{\partial}{\partial t}\mathbf{D} = \mathbf{j} \; ; \qquad \nabla \cdot \mathbf{D} = \rho \; , \tag{1.75}$$

the two *homogeneous equations* (1.74) are automatically fulfilled:

$$\mathbf{B} = \nabla \times \mathbf{A} \; ; \quad \mathbf{E} = -\nabla\varphi - \frac{\partial}{\partial t}\mathbf{A} \; . \tag{1.76}$$

In the *inhomogeneous equations* (1.75), which we do not need in the following, **H** denotes the magnetic field, **D** the dielectric displacement, **j** the current density, and ρ the charge density. Further details will be discussed in Vol. 3.

With (1.76) the Lorentz force reads

$$\mathbf{F} = \bar{q}\left[-\nabla\varphi - \frac{\partial}{\partial t}\mathbf{A} + (\mathbf{v} \times (\nabla \times \mathbf{A}))\right] \; . \tag{1.77}$$

Corresponding to this force we try to find a generalized potential

$$U = U\left(x, y, z, \dot{x}, \dot{y}, \dot{z}, t\right)$$

taking as generalized coordinates just the Cartesian coordinates of the charged particle. Therewith the generalized force components become identical to $F_{x,y,z}$:

$$\begin{aligned}
(\mathbf{v} \times (\nabla \times \mathbf{A}))_x &= \dot{y}(\nabla \times \mathbf{A})_z - \dot{z}(\nabla \times \mathbf{A})_y \\
&= \dot{y}\left(\frac{\partial}{\partial x}A_y - \frac{\partial}{\partial y}A_x\right) - \dot{z}\left(\frac{\partial}{\partial z}A_x - \frac{\partial}{\partial x}A_z\right) \\
&= \dot{y}\,\frac{\partial}{\partial x}A_y + \dot{z}\frac{\partial}{\partial x}A_z + \dot{x}\,\frac{\partial}{\partial x}A_x \\
&\quad -\dot{x}\,\frac{\partial}{\partial x}A_x - \dot{y}\,\frac{\partial}{\partial y}A_x - \dot{z}\,\frac{\partial}{\partial z}A_x \\
&= \frac{\partial}{\partial x}(\mathbf{v} \cdot \mathbf{A}) - \left(\frac{d}{dt}A_x - \frac{\partial}{\partial t}A_x\right) \; .
\end{aligned}$$

Therewith the x component of the Lorentz force reads:

$$F_x = \bar{q}\left[-\frac{\partial\varphi}{\partial x} - \frac{d}{dt}A_x + \frac{\partial}{\partial x}(\mathbf{v}\cdot\mathbf{A})\right] .$$

We still use

$$\frac{d}{dt}A_x = \frac{d}{dt}\left[\frac{\partial}{\partial\dot{x}}(\mathbf{A}\cdot\mathbf{v})\right] ; \quad \frac{d}{dt}\frac{\partial}{\partial\dot{x}}\varphi = 0$$

finding therewith:

$$F_x = \bar{q}\left[-\frac{\partial}{\partial x}(\varphi - \mathbf{v}\cdot\mathbf{A}) + \frac{d}{dt}\frac{\partial}{\partial\dot{x}}(\varphi - \mathbf{v}\cdot\mathbf{A})\right] .$$

We define the
generalized potential of the Lorentz force

$$U = \bar{q}(\varphi - \mathbf{v}\cdot\mathbf{A}) , \tag{1.78}$$

which fulfills for the x component the requested relation (1.71):

$$F_x = \frac{d}{dt}\frac{\partial U}{\partial\dot{x}} - \frac{\partial U}{\partial x} = Q_x .$$

The same can be shown analogously for the other two components $F_y,\ F_z$. Therewith we have found as an important result the
Lagrangian function of a particle with mass m and charge $\bar{q}$ in the electromagnetic field:

$$L(\mathbf{r},\dot{\mathbf{r}},t) = \frac{m}{2}\dot{\mathbf{r}}^2 + \bar{q}(\dot{\mathbf{r}}\cdot\mathbf{A}) - \bar{q}\varphi . \tag{1.79}$$

Although we have chosen as generalized coordinates the Cartesian spatial coordinates the generalized momenta $\mathbf{p}$ are not identical to the mechanical momenta $m\,\mathbf{v}$. According to (1.52) it holds instead:

$$p_x = \frac{\partial L}{\partial\dot{x}} = m\dot{x} + \bar{q}A_x ; \quad p_y = \frac{\partial L}{\partial\dot{y}} = m\dot{y} + \bar{q}A_y ; \quad p_z = \frac{\partial L}{\partial\dot{z}} = m\dot{z} + \bar{q}A_z . \tag{1.80}$$

The real experimentally measured quantities are the electromagnetic fields $\mathbf{E}$ and $\mathbf{B}$. The potentials $\varphi,\ \mathbf{A}$ are, however, only auxiliary quantities. **Gauge transformations** of the form

$$\mathbf{A} \longrightarrow \mathbf{A} + \nabla\chi ; \quad \varphi \longrightarrow \varphi - \frac{\partial}{\partial t}\chi , \tag{1.81}$$

are therefore allowed, where χ may be an arbitrary scalar function, since thereby according to (1.76) the fields **E** and **B** do not change by the transformation. However, as the Lagrangian (1.79) directly depends on the potentials φ, **A** this function can **not** be gauge-invariant. In contrast, however, the Lagrange-equation of motion

$$m\ddot{\mathbf{r}} = \bar{q}\,[\mathbf{E} + (\dot{\mathbf{r}} \times \mathbf{B})] \tag{1.82}$$

is gauge-invariant because here only the fields **E** and **B** and not the potentials appear. The Lagrangian itself changes according to

$$L \longrightarrow L + \bar{q}\left(\dot{\mathbf{r}} \cdot \nabla\chi + \frac{\partial}{\partial t}\,\chi\right) = L + \bar{q}\,\frac{d}{dt}\chi(\mathbf{r}, t)\ . \tag{1.83}$$

Now one can show very generally that for a
mechanical gauge transformation

$$L \longrightarrow L + L_0\ ; \quad L_0(\mathbf{q},\ \dot{\mathbf{q}},\ t) = \frac{d}{dt}f(\mathbf{q}, t) \tag{1.84}$$

the equations of motion do not change if f is an almost arbitrary but sufficiently often differentiable function exclusively depending on **q** and t. That is because:

$$\begin{aligned}
\frac{\partial L_0}{\partial q_j} &= \frac{\partial}{\partial q_j}\frac{df}{dt} = \frac{\partial^2 f}{\partial q_j\,\partial t} + \sum_l \frac{\partial^2 f}{\partial q_j\,\partial q_l}\,\dot{q}_l\ ,\\
\frac{d}{dt}\frac{\partial L_0}{\partial \dot{q}_j} &= \frac{d}{dt}\frac{\partial}{\partial \dot{q}_j}\frac{df}{dt} = \frac{d}{dt}\left[\frac{\partial}{\partial \dot{q}_j}\left(\frac{\partial f}{\partial t} + \sum_l \frac{\partial f}{\partial q_l}\,\dot{q}_l\right)\right]\\
&= \frac{d}{dt}\frac{\partial f}{\partial q_j} = \frac{\partial^2 f}{\partial t\,\partial q_j} + \sum_l \frac{\partial^2 f}{\partial q_l\,\partial q_j}\,\dot{q}_l\ .
\end{aligned}$$

Using this it follows with

$$\frac{d}{dt}\frac{\partial L_0}{\partial \dot{q}_j} - \frac{\partial L_0}{\partial q_j} = 0 \qquad \forall\, j \tag{1.85}$$

just the assertion. Gauging the Lagrangian according to (1.84) leaves the equation of motion and therewith the path line $\mathbf{q}(t)$ in the configuration space invariant. Note that only $\mathbf{q}(t)$ is empirically observable. Therefore the electromagnetic gauge transformation (1.81) turns out to be irrelevant in this sense.

1.2.4 Friction

Frictional forces cannot be derived as in (1.71) from any generalized potential U. Thus we have to incorporate them in a special manner into the equations of motion. They cannot be considered as constraint forces in the literal sense. They do not fulfill the d'Alembert's principle.

According to (1.33) it holds in case of holonomic constraints:

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} = Q_j \equiv \sum_{i=1}^{N} \mathbf{K}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} = Q_j^{(V)} + Q_j^{(R)} . \tag{1.86}$$

Thereby, the part $Q_j^{(V)}$ is derivable from a potential $\left(\mathbf{K}_i^{(V)} \equiv -\nabla_i V\right)$, while $Q_j^{(R)}$ provides the influence of the friction force.

The Lagrangian

$$L = T - V \qquad \left(V \text{ from } Q_j^{(V)}\right)$$

then obeys the equations of motion:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = Q_j^{(R)} , \qquad j = 1, 2, \ldots, S . \tag{1.87}$$

The following expression represents a suitable, rather general phenomenological ansatz for the friction forces (see (2.59), Vol. 1):

$$Q_j^{(R)} = -\sum_{l=1}^{S} \beta_{jl}\, \dot{q}_l \qquad \left(\beta_{jl} = \beta_{lj}\right) \tag{1.88}$$

Forces of this kind are described by
Rayleigh's dissipation function

$$D = \frac{1}{2} \sum_{l,m=1}^{S} \beta_{lm} \dot{q}_l \dot{q}_m \tag{1.89}$$

Therewith we get **'modified'** Lagrange equations of the form:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} + \frac{\partial D}{\partial \dot{q}_j} = 0 , \qquad j = 1, 2, \ldots, S . \tag{1.90}$$

For a detailed formulation of the equations of motion two scalar functions L and D must be known.

We still want to amplify the physical meaning of the dissipation function. In systems with friction the sum of kinetic and potential energy is no longer a conserved quantity since the system has to do work against the friction:

$$dW^{(R)} = -\sum_j Q_j^{(R)}\, dq_j = \sum_{j,l} \beta_{jl}\, \dot{q}_l\, dq_j \,.$$

Thus:

$$\frac{dW^{(R)}}{dt} = 2D \quad (\textbf{energy dissipation}) \,. \tag{1.91}$$

The energy dissipation corresponds to the temporal change of the total energy $(T + V)$:

$$\frac{d}{dt}(T+V) = \sum_{j=1}^{S} \left(\frac{\partial T}{\partial q_j}\dot{q}_j + \frac{\partial T}{\partial \dot{q}_j}\ddot{q}_j \right) + \frac{dV}{dt} \,,$$

$$\sum_{j=1}^{S} \frac{\partial T}{\partial \dot{q}_j}\ddot{q}_j = \frac{d}{dt}\left(\sum_{j=1}^{S} \frac{\partial T}{\partial \dot{q}_j}\dot{q}_j \right) - \sum_{j=1}^{S} \dot{q}_j \frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j} \,.$$

We presume scleronomic constraints. The kinetic energy T is then according to (1.37) a homogeneous function of the generalized velocities of second order. Furthermore, except for the friction terms, the system shall be conservative:

$$\sum_{j=1}^{S} \frac{\partial T}{\partial \dot{q}_j}\ddot{q}_j = \frac{d}{dt}(2T) - \sum_{j=1}^{S} \dot{q}_j \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} \,.$$

From this it follows with (1.90):

$$\begin{aligned}
\frac{d}{dt}(T+V) &= \sum_{j=1}^{S} \frac{\partial T}{\partial q_j}\dot{q}_j + \frac{d}{dt}(2T) + \frac{dV}{dt} - \sum_{j=1}^{S} \dot{q}_j \left(\frac{\partial L}{\partial q_j} - \frac{\partial D}{\partial \dot{q}_j} \right) \\
&= \sum_{j=1}^{S} \frac{\partial V}{\partial q_j}\dot{q}_j + \frac{d}{dt}(2T + V) + 2D \\
&= \frac{d}{dt}(2T + 2V) + 2D \,.
\end{aligned}$$

That means:

$$\frac{d}{dt}(T+V) = -2D \,. \tag{1.92}$$

Example A particle of mass m may fall vertically under the influence of gravity where friction forces occur according to a dissipation function (*Stokes' friction*, (2.59), Vol. 1):

$$D = \frac{1}{2}\alpha\, v^2$$

With $v = -\dot{z}$ (one-dimensional motion!) we get the Lagrangian:

$$L = T - V = \frac{m}{2}\dot{z}^2 - m\, g\, z\ .$$

After (1.90) we have to solve the following modified Lagrange equation:

$$m\ddot{z} + m\, g + \alpha\, \dot{z} = 0\ .$$

Rewriting

$$\frac{d}{dt}v = g - \frac{\alpha}{m}v \implies dt = \frac{dv}{g - \frac{\alpha}{m}v}\ .$$

this can easily be integrated:

$$t - t_0 = -\frac{m}{\alpha}\ln\frac{\alpha\, v - m\, g}{\alpha\, v_0 - m\, g}\ .$$

We choose as initial conditions

$$t_0 = 0\ ; \quad v_0 = 0$$

ending up with the familiar result ((2.119), Vol. 1):

$$v = \frac{m\, g}{\alpha}\left[1 - \exp\left(-\frac{\alpha}{m}t\right)\right]\ .$$

Because of the friction the velocity v remains finite even for $t \to \infty$!

1.2.5 Non-holonomic Systems

Holonomic constraints can be written as in (1.2). By introduction of $S = 3N - p$ (p = number of the holonomic constraints) generalized coordinates $q_1, \ldots, q_S$, which are independent of each other and do uniquely fix the configuration of the system, it is pointed out, inter alia, that the holonomic constraints reduce the number of degrees of freedom from $3N$ to $S = 3N - p$.

In the case of non-holonomic constraints such a reduction is no longer possible. One cannot find a set of **independent** generalized coordinates so that their number is identical to the number of degrees of freedom. In particular the Lagrange equations are no longer applicable in the form (1.36). After the considerations in Sect. 1.1 non-holonomic constraints may be given as inequalities or as differential non-integrable relations. For the latter case there does exist an algorithm, namely the

method of the Lagrange multipliers

which we now want to introduce. For this purpose we consider a system that is subject to $\bar{p}$ constraints. Among these $p \leq \bar{p}$ shall be given in the following non-holonomic form:

$$\sum_{m=1}^{3N} f_{im}(x_1, \dots, x_{3N}, t)\, dx_m + f_{it}\,(x_1, \dots, x_{3N}, t)\, dt = 0\,, \quad i = 1, \dots, p\,. \tag{1.93}$$

Let us develop the *'recipe of solution'* step-by-step:

1. In general the system will possess both holonomic as well as non-holonomic constraints. The holonomic ones we use for a partial reduction of the number of coordinates $3N$ to

   $$j = 3N - (\bar{p} - p)\;.$$

   Hence we express the particle positions $\mathbf{r}_i$ by j *generalized* coordinates $q_1, \dots, q_j$:

   $$\mathbf{r}_i = \mathbf{r}_i\left(q_1, \dots, q_j, t\right)\;. \tag{1.94}$$

   It is clear that the coordinates q_j cannot all be independent of each other.
2. The conditions (1.93) will be adapted to the $q_1, \dots, q_j$:

   $$\sum_{m=1}^{j} a_{im}\, dq_m + b_{it}\, dt = 0\,, \qquad i = 1, \dots, p\,. \tag{1.95}$$

3. The constraints are then formulated for virtual displacements ($\delta t = 0$):

   $$\sum_{m=1}^{j} a_{im}\, \delta q_m = 0\,, \qquad i = 1, \dots, p\,. \tag{1.96}$$

4. We now introduce so-called *'Lagrange multipliers'* λ_i which shall be independent of $\mathbf{q}$ but possibly may depend on t. In trivial consequence of (1.96) it is:

   $$\sum_{i=1}^{p} \lambda_i \sum_{m=1}^{j} a_{im}\, \delta q_m = 0\,. \tag{1.97}$$

5. The system shall be conservative so that a Lagrangian is definable for which the equations of motion are of the type (1.35):

$$\sum_{m=1}^{j}\left(\frac{\partial L}{\partial q_m}-\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_m}\right)\delta q_m = 0\,. \tag{1.98}$$

These equations we now combine with (1.97):

$$\sum_{m=1}^{j}\left\{\frac{\partial L}{\partial q_m}-\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_m}+\sum_{i=1}^{p}\lambda_i\,a_{im}\right\}\,\delta q_m = 0\,. \tag{1.99}$$

The δq_m are not independent of each other, i.e. we can not conclude that already each summand is equal to zero.
6. Because of the constraints only $j-p=3N-\bar{p}$ coordinates are actually freely selectable. We specify:

$$\begin{aligned} q_m: &\quad m=1,\ldots,j-p: &&\text{independent}\,,\\ q_m: &\quad m=j-p+1,\ldots,j: &&\text{dependent}\,. \end{aligned}$$

The p Lagrange multipliers λ_i are still undetermined. We choose them so that it holds:

$$\frac{\partial L}{\partial q_m}-\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_m}+\sum_{i=1}^{p}\lambda_i\,a_{im}\stackrel{!}{=}0\,,\qquad m=j-p+1,\ldots,j\,. \tag{1.100}$$

These are p equations for p unknown λ_i which are now fixed by (1.100). Insertion into (1.99) then leads to

$$\sum_{m=1}^{j-p}\left\{\frac{\partial L}{\partial q_m}-\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_m}+\sum_{i=1}^{p}\lambda_i\,a_{im}\right\}\,\delta q_m = 0\,.$$

These quantities q_m, however, are now independent of each other so that each summand separately must already be zero:

$$\frac{\partial L}{\partial q_m}-\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_m}+\sum_{i=1}^{p}\lambda_i\,a_{im}=0\,,\qquad m=1,2,\ldots,j-p\,. \tag{1.101}$$

7. Eventually we combine the equations (1.101) and (1.100) getting therewith the important

   Lagrange equations of motion of the first kind

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_m} - \frac{\partial L}{\partial q_m} = \sum_{i=1}^{p} \lambda_i \, a_{im} \,, \qquad m = 1, \ldots, j \,. \tag{1.102}$$

   These are j equations for a total of $j + p$ unknowns, namely j coordinates q_m and p multipliers λ_i. The missing conditional equations are the p constraints (1.95):

$$\sum_{m=1}^{j} a_{im} \, \dot{q}_m + b_{it} = 0 \,, \qquad i = 1, \ldots, p \,. \tag{1.103}$$

   These constraints can not directly be integrated but possibly in conjunction with the above equations of motion. That we will demonstrate later by examples. Using this procedure we get more information than originally intended, namely besides the q_m additionally we get the λ_i.

What is the physical meaning of the λ_i? If one compares (1.102) with (1.33) then it becomes clear that

$$\overline{Q}_m = \sum_{i=1}^{p} \lambda_i \, a_{im} \tag{1.104}$$

can be interpreted as a component of a *generalized constraint force* which realizes the non-holonomic constraints. Hence we can write (1.97) also as follows:

$$\sum_{m=1}^{j} \overline{Q}_m \, \delta q_m = 0 \,. \tag{1.105}$$

In a certain sense that can be seen as d'Alembert's principle for the generalized constraint force.

One can apply the method of Lagrange multiplier of course also to systems with solely holonomic constraints. To show this we rewrite the p holonomic constraints (out of altogether $\bar{p} \geq p$),

$$f_i(\mathbf{r}_1, \ldots, \mathbf{r}_N, t) = 0 \,, \qquad i = 1, \ldots, p \,,$$

as functions of the generalized coordinates $q_1, \ldots, q_j$:

$$\bar{f}_i\left(q_1, \ldots, q_j, t\right) = 0 \,, \qquad i = 1, \ldots, p \,.$$

Therewith we also have the relation

$$d\bar{f}_i = \sum_{m=1}^{j} \frac{\partial \bar{f}_i}{\partial q_m} dq_m + \frac{\partial \bar{f}_i}{\partial t} dt = 0 \, ,$$

which is formally identical to (1.95) and (1.103), respectively, with

$$a_{im} = \frac{\partial \bar{f}_i}{\partial q_m} \quad \text{and} \quad b_{it} = \frac{\partial \bar{f}_i}{\partial t}$$

Therewith (1.102) and (1.103) are to be solved. This method provides now additional information about constraint forces at the same time, however, it is also more complicated since instead of $j - p$ now $j + p$ equations must be handled. The examples in Sect. 1.2.6 will help to become familiar with the abstract formalism presented here.

But before that we want to present an alternate and somewhat more direct method for treating the above special case. In the method described so far holonomic constraints at hand are **not** used only for reducing the number of variables but to get additionally access to the constraint forces which are otherwise pretty difficult to approach .

Let us consider an N particle system with p holonomic constraints:

$$f_\nu(\mathbf{r}_1, \ldots, \mathbf{r}_N, t) = 0 \; ; \quad \nu = 1, \ldots, p \, . \tag{1.106}$$

Since none of the constraints shall be used to eliminate certain variables it stands to reason to choose as *'generalized'* coordinates just the components (Cartesian, cylindrical, spherical,...) of the particle positions:

$$\mathbf{r}_i = (x_{i1}, x_{i2}, x_{i3}) \tag{1.107}$$

The constraints (1.106) react on **real** displacements as follows:

$$df_\nu = \sum_{i=1}^{N} \sum_{j=1}^{3} \frac{\partial f_\nu}{\partial x_{ij}} dx_{ij} + \frac{\partial f_\nu}{\partial t} dt = \sum_{i=1}^{N} \nabla_i f_\nu \cdot d\mathbf{r}_i + \frac{\partial f_\nu}{\partial t} dt = 0$$

$$\curvearrowright \quad \sum_{i=1}^{N} \dot{\mathbf{r}}_i \cdot \nabla_i f_\nu = -\frac{\partial f_\nu}{\partial t} \, . \tag{1.108}$$

On the other hand, it holds for virtual displacements because of $\delta t = 0$:

$$\sum_{i=1}^{N} \nabla_i f_\nu \cdot \delta \mathbf{r}_i = 0 \, . \tag{1.109}$$

The principle of virtual work (1.19) leads with (1.18) to the equations of motion (1.20) *'to which we couple'* the p side conditions (1.109) by use of p Lagrange multipliers λ_ν:

$$\sum_{i=1}^{N}\left(\mathbf{K}_i - m_i\ddot{\mathbf{r}}_i + \sum_{\nu=1}^{p}\lambda_\nu\nabla_i f_\nu\right)\cdot\delta\mathbf{r}_i = 0\,. \tag{1.110}$$

$\mathbf{K}_i$ is the *driving force* on particle i. By the third summand we simply added zero to the equation of motion (1.20). Because of the constraints not all of the $\delta\mathbf{r}_i$ are independent of each other. With the same justification as that for (1.100) we consider $3N-p$ of the variations δx_{ij} as independent. For the remaining p variations we choose the p multipliers λ_ν in such a way that each of the corresponding brackets in (1.110) already vanishes. Then we are left with the $3N-p$ independent (!) variations so that we can conclude eventually:

$$m_i\ddot{\mathbf{r}}_i = \mathbf{K}_i + \sum_{\nu=1}^{p}\lambda_\nu\nabla_i f_\nu\,. \tag{1.111}$$

That can be interpreted as *'Newton's analogue'* to the Lagrange equations of motion of the first kind (1.102). In this case, too, there are $3N$ equations for $(3N + p)$ variables $(x_{11},\dots,x_{N3};\ \lambda_1,\dots,\lambda_p)$. The missing conditional equations we take from the p constraints (1.106).

The comparison with (1.17) now permits an explicit specification of the constraint force $\mathbf{Z}_i$ that acts on the i-th particle in the real three-dimensional space:

$$\mathbf{Z}_i = \sum_{\nu=1}^{p}\lambda_\nu\nabla_i f_\nu\,. \tag{1.112}$$

The possibility of such a direct determination of the usually not so descriptive constraint forces can be considered as a weighty advantage of the procedure presented here.

With the aid of the method of Lagrange multiplier the influence of the constraint forces on the energy law of the particle system can be immediately understood. Let us assume that the driving force can be derived from a potential V:

$$\mathbf{K}_i = -\nabla_i V\,.$$

Therewith for the temporal change of the kinetic energy T we have:

$$\frac{d}{dt}T = \frac{d}{dt}\frac{1}{2}\sum_{i=1}^{N} m_i\dot{\mathbf{r}}_i^2 = \sum_{i=1}^{N}\dot{\mathbf{r}}_i \cdot (m_i\ddot{\mathbf{r}}_i)$$
$$= \sum_{i=1}^{N}\dot{\mathbf{r}}_i \cdot (\mathbf{K}_i + \mathbf{Z}_i) = -\frac{\mathrm{d}}{\mathrm{d}t}V + \sum_{i=1}^{N}\dot{\mathbf{r}}_i \cdot \mathbf{Z}_i$$
$$\curvearrowright \quad \frac{d}{dt}(T+V) = \sum_{i=1}^{N}\dot{\mathbf{r}}_i \cdot \mathbf{Z}_i \overset{(1.112)}{=} \sum_{i=1}^{N}\sum_{\nu=1}^{p}\lambda_\nu \dot{\mathbf{r}}_i \cdot \nabla_i f_\nu \ .$$

If we still exploit (1.108) then we come to:

$$\frac{d}{dt}(T+V) = -\sum_{\nu=1}^{p}\lambda_\nu \frac{\partial f_\nu}{\partial t} \ . \tag{1.113}$$

In case of exclusively scleronomic constraints f_ν the energy conservation law therefore holds. On the other hand, in case of rheonomic constraints the constraint forces in general carry out work on the system. Energy conservation is then no longer valid (see Example 2 in Sect. 1.2.2).

1.2.6 Applications of the Method of Lagrange Multipliers

We discuss three simple physical problems.

(1) Atwood's Free-Fall Machine

As an in principle holonomic system we discussed the fall machine already as application Example (1) in Sect. 1.2.2. Here it serves as illustration of the method of Lagrange multipliers (Fig. 1.20).

There are five constraints

$$y_1 = z_1 = z_2 = 0 \ , \quad y_2 = 2R$$
$$x_1 + x_2 - l + \pi R = 0$$

from which we take only the first four for a reduction of the number of coordinates:

$$j = 6 - 4 = 2 \ .$$

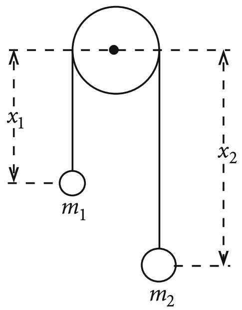


Fig. 1.20 Atwood's free-fall machine

As *generalized* coordinates we choose:

$$q_1 = x_1 \; ; \quad q_2 = x_2 \; .$$

The remaining constraint then reads:

$$f(q_1, q_2, t) = q_1 + q_2 - l + \pi R = 0 \qquad (p = 1)$$
$$\Longrightarrow \; df = dq_1 + dq_2 = 0 \; .$$

The comparison with (1.95) yields

$$a_{11} = a_{12} = 1 \; .$$

Because of $p = 1$ only one Lagrange multiplier λ is necessary:

$$\overline{Q}_1 = \overline{Q}_2 = \lambda \qquad \textbf{thread tension} \; .$$

With the Lagrangian

$$L = \frac{1}{2}\left(m_1 \dot{q}_1^2 + m_2 \dot{q}_2^2\right) + g\,(m_1 q_1 + m_2 q_2)$$

we get according to (1.102) the equations of motion:

$$m_1 \ddot{q}_1 - m_1 g = \lambda \; ; \quad m_2 \ddot{q}_2 - m_2 g = \lambda \; .$$

Furthermore, the constraint contributes corresponding to (1.103)

$$\dot{q}_1 + \dot{q}_2 = 0 \; .$$

These are now three equations to be solved instead of one previously as the free-fall machine was treated as a pure holonomic system. In return, however, we now get additional information about the constraint force. As solution of the above system of equations we are able to confirm the previous results (1.48) and (1.49):

$$\ddot{q}_1 = -\ddot{q}_2 = \frac{m_1 - m_2}{m_1 + m_2}\, g\,; \quad \lambda = -2g\,\frac{m_1\, m_2}{m_1 + m_2}\,.$$

(2) Rolling Barrel on an Inclined Plane

The *'barrel'* is a hollow cylinder of mass M whose moment of inertia J has been calculated in Sect. 4.3 of Vol. 1:

$$J = \int \rho(\mathbf{r}) r^2\, d^3r = M\, R^2 \tag{1.114}$$

$\rho(\mathbf{r})$ is the mass density of the hollow cylinder. As a repeating exercise, the reader should verify the expression (1.114). It is here again about a holonomic problem. We consider as *generalized* coordinates (see Fig. 1.21)

$$q_1 = x\,; \quad q_2 = \vartheta \quad (j = 2)$$

with the *rolling off condition*

$$R\, d\vartheta = dx$$

as constraint. This is of course integrable and therewith holonomic. But intentionally, that shall not be exploited here. From

$$R\, dq_2 - dq_1 = 0 \quad (p = 1)$$

it follows:

$$a_{11} = -1\,; \quad a_{12} = R\,.$$

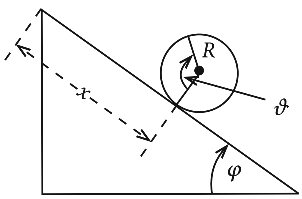


Fig. 1.21 Rolling hollow cylinder on an inclined plane

The rolling barrel possesses the Lagrangian:

$$L = \frac{M}{2}\,\dot{q}_1^2 + \frac{1}{2}\,J\,\dot{q}_2^2 - M\,g\,(l - q_1)\,\sin\varphi\ .$$

Because of $p = 1$ we need **one** Lagrange multiplier λ. According to (1.102) we then have:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_1} - \frac{\partial L}{\partial q_1} = M\,\ddot{q}_1 - M\,g\,\sin\varphi = \lambda\,a_{11} = -\lambda\ ,$$
$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_2} - \frac{\partial L}{\partial q_2} = J\,\ddot{q}_2 = \lambda\,a_{12} = R\,\lambda\ .$$

The coordinate q_2 seems to be cyclic. But this fact does not lead to a conservation law since q_1 and q_2 are not independent of each other. The constraint provides, according to (1.103), still a third conditional equation:

$$-\dot{q}_1 + R\,\dot{q}_2 = 0\ .$$

One easily finds as preliminary solution:

$$\ddot{q}_1 = \ddot{x} = \frac{1}{2}\,g\,\sin\varphi\ ,$$
$$\ddot{q}_2 = \ddot{\vartheta} = \frac{1}{2R}\,g\,\sin\varphi\ ,$$
$$\lambda = \frac{M}{2}\,g\,\sin\varphi\ .$$

The linear acceleration of the rolling cylinder thus is only half as large as that of a body which glides frictionless on the inclined plane (cf. (4.36), Vol. 1). For the generalized constraint forces we find

$$\overline{Q}_1 = \lambda\,a_{11} = -\frac{M}{2}\,g\,\sin\varphi\ ;\quad \overline{Q}_2 = \lambda\,a_{12} = \frac{1}{2}\,M\,g\,R\,\sin\varphi\ .$$

$\overline{Q}_1$ can be identified with the x component of the constraint force that results from the *'roughness'* of the undersurface causing the *'rolling'* of the barrel. It diminishes the actually acting gravitational force from $Mg\sin\varphi$ to $(1/2)Mg\sin\varphi$. $\overline{Q}_2$ corresponds to the torque on the cylinder also imposed by the *'roughness'* of the undersurface.

(3) Rolling of a Wheel on a Rough Undersurface

We discussed this system already in Sect. 1.1 as an application example for non-holonomic constraints. Let us adopt the notation of example (B,2) in Sect. 1.1 and choose as *'generalized'* coordinates:

$$q_1 = x \; ; \quad q_2 = y \; ; \quad q_3 = \varphi \; ; \quad q_4 = \vartheta \; .$$

The constraint *'rolling'* is given according to (1.14) by

$$\dot{x} - R\, \cos\vartheta\, \dot{\varphi} = 0 \; ; \quad \dot{y} - R\, \sin\vartheta\, \dot{\varphi} = 0$$

This means after (1.95) ($p = 2$):

$$\begin{aligned} a_{11} &= 1 \; ; & a_{12} &= 0 \; ; & a_{13} &= -R\, \cos\vartheta \; ; & a_{14} &= 0 \; ; \\ a_{21} &= 0 \; ; & a_{22} &= 1 \; ; & a_{23} &= -R\, \sin\vartheta \; ; & a_{24} &= 0 \; . \end{aligned}$$

We need two Lagrange multipliers λ_1 and λ_2. Following (1.104), the generalized constraint forces then read:

$$\overline{Q}_1 = \lambda_1 \; ; \quad \overline{Q}_2 = \lambda_2 \; ; \quad \overline{Q}_3 = -R\, \cos\vartheta\, \lambda_1 - R\, \sin\vartheta\, \lambda_2 \; ; \quad \overline{Q}_4 = 0 \; .$$

The wheel disc shall move in a force-free space, therefore possesses only kinetic energy:

$$L = T = \frac{M}{2}\left(\dot{x}^2 + \dot{y}^2\right) + \frac{1}{2} J_1\, \dot{\varphi}^2 + \frac{1}{2} J_2\, \dot{\vartheta}^2 \; .$$

J_1 is the moment of inertia with respect to the wheel axis and J_2 that with respect to the axis which goes through the disc center and the support point. As an exercise, verify that $J_1 = (1/2)MR^2$, $J_2 = (1/4)M(R^2 + (1/3)d^2)$ where d is the disc thickness. The Lagrange equation (1.102) now appear as follows:

$$M\, \ddot{x} = \lambda_1 \; ; \quad M\, \ddot{y} = \lambda_2 \; ; \quad J_1\, \ddot{\varphi} = -R\, \lambda_1\, \cos\vartheta - R\, \lambda_2\, \sin\vartheta \; ; \quad J_2\, \ddot{\vartheta} = 0 \; .$$

Together with the above constraints there are now six equations for six unknowns. In a first step it follows from $\ddot{\vartheta} = 0$ with $\vartheta(t = 0) = 0$:

$$\vartheta = \omega t \quad (\omega = \text{const}) \; .$$

We differentiate the constraints once more with respect to the time:

$$\begin{aligned} \ddot{x} &= -R\, \omega \dot{\varphi}\, \sin\omega t + R\, \ddot{\varphi}\, \cos\omega t \; , \\ \ddot{y} &= R\, \omega\, \dot{\varphi}\, \cos\omega t + R\, \ddot{\varphi}\, \sin\omega t \; . \end{aligned}$$

Therewith also the multipliers λ_1 and λ_2 are fixed:

$$\lambda_1 = -M R \omega \sin \omega t \, \dot{\varphi} + M R \cos \omega t \, \ddot{\varphi} \; ,$$
$$\lambda_2 = M R \omega \cos \omega t \, \dot{\varphi} + M R \sin \omega t \, \ddot{\varphi} \; .$$

The last, up to now not yet exploited Lagrange equation yields then after insertion of λ_1 and λ_2:

$$\begin{aligned} J_1 \, \ddot{\varphi} &= M R^2 \omega \sin \omega t \cos \omega t \, \dot{\varphi} - M R^2 \cos^2 \omega t \, \ddot{\varphi} \\ &\quad - M R^2 \omega \cos \omega t \sin \omega t \, \dot{\varphi} - M R^2 \sin^2 \omega t \, \ddot{\varphi} \\ &= -M R^2 \, \ddot{\varphi} \; . \end{aligned}$$

This equation, however, can have only the solution

$$\ddot{\varphi} \equiv 0 \iff \dot{\varphi} = \dot{\varphi}_0 = \text{const}$$

It remains to integrate:

$$\ddot{x} = -R \, \omega \, \dot{\varphi}_0 \sin \omega t \; ; \quad \ddot{y} = R \, \omega \, \dot{\varphi}_0 \cos \omega t \; ,$$

That is easily possible with the given initial conditions for $x(t)$ and $y(t)$. The constants ω and $\dot{\varphi}_0$ follow from the initial conditions for $\vartheta(t)$ and $\varphi(t)$. Even the constraint forces are now completely determined:

$$\overline{Q}_1 = -M R \, \omega \, \dot{\varphi}_0 \sin \omega t \; ; \quad \overline{Q}_2 = M R \, \omega \, \dot{\varphi}_0 \cos \omega t \; ; \quad \overline{Q}_3 = \overline{Q}_4 = 0 \; . \tag{1.115}$$

They take care for the disc to roll vertically on the xy plane. If the wheel moves solely straight ahead, ω is zero so that all the constraint forces disappear.

1.2.7 Exercises

Exercise 1.2.1 Discuss the motion of a bead that glides frictionlessly on a uniformly rotating wire. r is its distance from the center of rotation. Given are the initial conditions

$$r(t = 0) = r_0 \; ; \quad \dot{r}(t = 0) = -r_0 \omega$$

ω: constant angular velocity of the wire.

Exercise 1.2.2 We consider once more, as in Exercise 1.2.1 or in the application Example (2) of Sect. 1.2.2, a bead of mass m which is gliding frictionlessly on a wire that rotates with constant angular velocity ω (Fig. 1.17). For the interpretation of the results we made in Sect. 1.2.2 the plausible assumption that the acting constraint force $\mathbf{Z}(t)$ is oriented always perpendicular to the rotating wire.

1. Confirm this *plausible* assumption by an explicit derivation of $\mathbf{Z}(t)$. Use the same initial conditions as in Sect. 1.2.2.

$$r(t=0) = r_0 \; ; \quad \dot{r}(t=0) = 0 \; .$$

2. Discuss the energy law:

$$\frac{d}{dt}(T+V) = ?$$

Exercise 1.2.3 Consider again a bead of mass m frictionlessly gliding on a wire which rotates with constant angular velocity ω. Different from Exercise 1.2.1 the bead shall now additionally move in the earth's gravitational field (Fig. 1.22).

1. Which constraint forces are present?
2. Formulate the Lagrangian function for the bead!
3. Determine the Lagrange equation of motion and find its general solution!
4. Use the initial conditions

$$r(t=0) = r_0 \; ; \quad \dot{r}(t=0) = 0 \; .$$

   How large must ω be at the least to force the bead to move outwards for $t \to \infty$?
5. How would we have to treat the problem in Newton's mechanics?

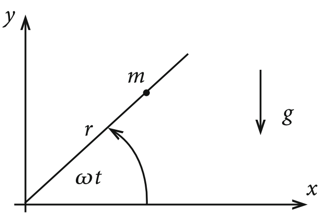


Fig. 1.22 Bead on a rotating wire in the earth's gravitational field

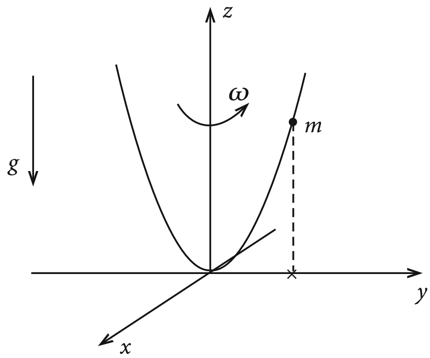


Fig. 1.23 Bead on a rotating parabolic wire in the earth's gravitational field

Exercise 1.2.4 A parabolically curved wire rotates with constant angular velocity ω around the z axis. On this rotating wire a bead of mass m moves frictionlessly in the earth's gravitational field ($\mathbf{g} = -g\mathbf{e}_z$). If the wire is just within the yz plane (Fig. 1.23) then it holds for the position of the mass

$$z = \alpha y^2 \quad (\alpha > 0) \ .$$

1. Find the constraints! How many degrees of freedom are left?
2. Use cylindrical coordinates (ρ, φ, z) to represent the Lagrangian!
3. Calculate for the special case

   $$\omega = \sqrt{2\alpha g}$$

   the Lagrange equations of the second kind and show that

   $$\left(1 + 4\alpha^2\rho^2\right) \dot{\rho}^2$$

   is an integral of motion (conserved quantity)!

Exercise 1.2.5 Let the position of a particle be described by cylindrical coordinates (ρ, φ, z). The potential energy of the particle is given as

$$V(\rho) = V_0 \ln \frac{\rho}{\rho_0} \ , \qquad V_0 = \text{const} \ , \ \ \rho_0 = \text{const} \ .$$

1. Write down the Lagrangian!
2. Formulate the Lagrange equations of motion!
3. Find and interpret at least two conservation laws!

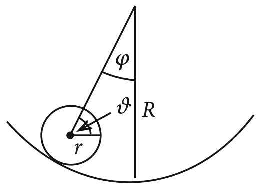


Fig. 1.24 Cylinder rolling on the inner surface of the side wall of another ('larger') cylinder

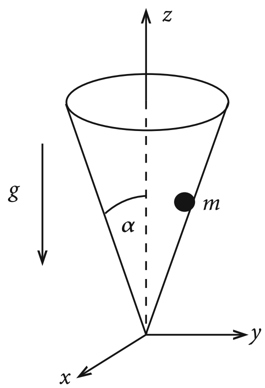


Fig. 1.25 Point mass m on the inner surface of a circular cone in the earth's gravitational field

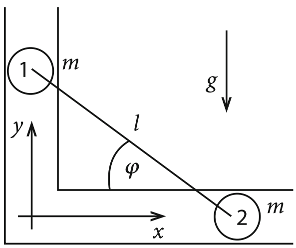


Fig. 1.26 Two connected spheres gliding under the influence of the gravitational force in different wells

Exercise 1.2.6 On the inner surface of a cylinder side wall (radius R) rolls another cylinder (radius r, mass density $\rho = \text{const}$) (Fig. 1.24).

1. What is the Lagrangian of the system?
2. Formulate the Lagrange equations of motion!
3. Integrate the equation of motion for small *'deflections'* φ.

Exercise 1.2.7 A mass point m rolls frictionlessly on the inner surface of a circular cone (cone angle α) in the gravitational field of the earth (Fig. 1.25).

1. Formulate the constraints and choose appropriate generalized coordinates.
2. Seek the Lagrangian and write down the equations of motion of the second kind!
3. Which coordinate is cyclic? Specify the corresponding conservation law!

Exercise 1.2.8 Two spheres of equal masses m are connected with each other by a (mass-less) rod of length l. As indicated in Fig. 1.26 the spheres move frictionlessly in two given wells under the influence of the gravitational force.

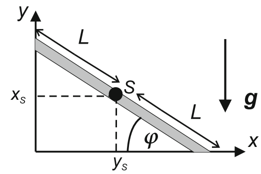


Fig. 1.27 A rod slipping on a wall in the earth's gravitational field

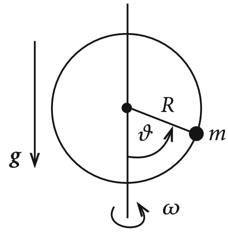


Fig. 1.28 Bead on a rotating wire-ring under the influence of the gravitational force

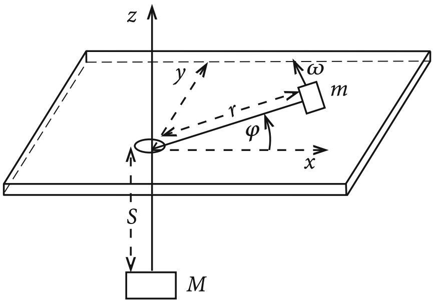


Fig. 1.29 On a table frictionlessly rotating mass m being connected by a thread to another mass M which experiences the gravitational force

1. Introduce suitable generalized coordinates and find the Lagrangian!
2. Solve the Lagrange equations of motion!

Exercise 1.2.9 A rod of the length $2L$ with circular cross-section πR^2 is slipping down a wall (y axis) because of the gravitational force (Fig. 1.27). The rod possesses a homogeneous mass distribution (mass M, homogeneous density ρ_0). φ is the time-dependent inclination angle with respect to the ground (x axis). Discuss the time-dependence of φ!

Exercise 1.2.10 A bead of mass m is gliding frictionlessly on a wire-hoop with the radius R. The hoop rotates with constant angular velocity ω around its diameter in the gravitational field $\mathbf{g}$ (Fig. 1.28).

1. Formulate and classify the constraints!
2. Find the Lagrange equation of motion!
3. Integrate the equation of motion for $\vartheta \ll 1$!

Exercise 1.2.11 A mass m rotates frictionlessly on a tabletop. Via a thread of the length l ($l = r + s$) it is connected through a hole in the table with another mass M (Fig. 1.29). How does M move under the influence of the gravitational force?

1. Formulate and classify the constraints!
2. Find the Lagrangian and its equations of motion!
3. Under what conditions does the mass M slip upwards or downwards?
4. Discuss the special case $\omega = 0$.

Exercise 1.2.12 Consider a planar thread pendulum with the thread-length l in the homogeneous gravitational field (Fig. 1.30). Only small deflections of the pendulum are to be discussed.

1. Find the Lagrangian and the equation of motion! Choose the initial conditions such that at time $t = 0$ the pendulum swings through its equilibrium position. How big is the frequency ω_0 of the oscillation?
2. Calculate the thread tension!

Exercise 1.2.13 A particle with the mass m oscillates in the xy plane on a mass-less thread of length l in the earth's gravitational field (Fig. 1.31).

1. Formulate the constraints!
2. Do **not** use all the constraints for eliminating variables, but choose φ **and** r as generalized coordinates. Calculate the Lagrangian

$$L = L(r, \varphi, \dot{r}, \dot{\varphi})$$

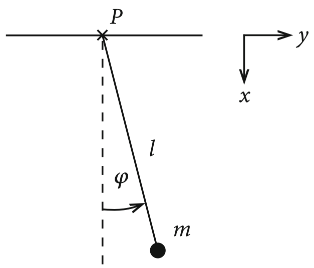


Fig. 1.30 Thread pendulum of length l in the earth's gravitational field

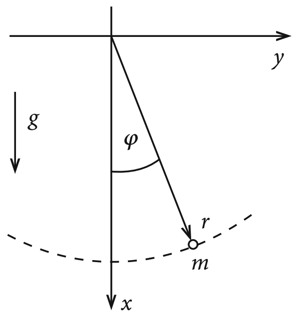


Fig. 1.31 Oscillating mass in the earth's gravitational field

3. Introduce a proper Lagrange multiplier and derive for φ and r the Lagrange equations of the **first** kind. Use the equation of motion for r for the determination of the *'thread tension'* Q_r.
4. Solve the equation of motion for φ with the initial conditions

$$\varphi(0) = 0 \qquad\qquad \dot{\varphi}(0) = \sqrt{\frac{g}{l}}\varphi_0 \quad (\varphi_0 \ll 1)$$

under the presumption of only small pendulum deflections ($\varphi \ll 1$)!

Exercise 1.2.14 A block of mass M is gliding frictionlessly under the influence of the gravitational force on an inclined plane with the inclination angle α. A mass m is attached to the center of gravity of the block by a mass-less thread of length l (Fig. 1.32).

1. What is the Lagrangian $L(\varphi, s, \dot{\varphi}, \dot{s})$?
2. Show that a solution $\varphi(t) = \varphi_0 = \text{const}$ does exist!
3. Find a closed differential equation for φ. Solve this equation for $M \gg m$ and for small angle deflections ($\varphi \approx \alpha$)!

Exercise 1.2.15 Two masses m_1 and m_2 are moving frictionlessly on a wedge under the influence of the gravitational force (Fig. 1.33). They are connected with each other by a massless thread of length $l = l_1 + l_2$.

1. Formulate the constraints! Of which type are these? How many degrees of freedom S does the system have?
2. Choose suitable generalized coordinates and write down the transformation formulas!
3. Find the Lagrangian!

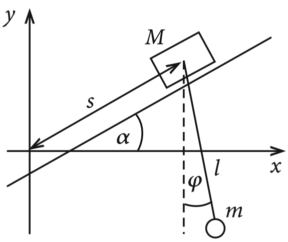


Fig. 1.32 Thread pendulum coupled to a mass on an inclined plane

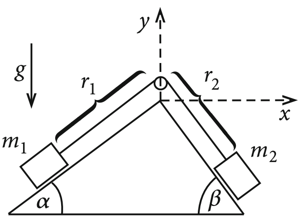


Fig. 1.33 Two masses connected with each other by a thread on a wedge in the earth's gravitational field

4. Write down the Lagrange equations of motion and solve them! Determine $r_1(t)$ with the initial conditions:

$$r_1(t=0) = r_0 \; ; \qquad \dot{r}_1(t=0) = 0 \; .$$

   Find the equilibrium conditions!
5. Do **not** use the (holonomic) constraint *'constant thread length'* to eliminate variables. Apply instead a Lagrange multiplier λ to find the *'thread tension'*! How large is this in equilibrium?

Exercise 1.2.16 Two homogeneous cylinders with masses M_1, M_2 and radiuses R_1, R_2 are wrapped by a thread and therewith connected with each other. The first cylinder is firmly horizontally pivoted but can be rotated frictionlessly. The second cylinder drops down in x direction due to the earth's gravitational field while on both the cylinders the thread is unwinding (Fig. 1.34).

1. Use the angular-momentum law to find the equation of motion and in particular to determine the thread tensions $\mathbf{F}_1$ and $\mathbf{F}_2$! (This problem has already been treated as Exercise 4.5.4 in Vol. 1)
2. Formulate the Lagrangian! To this use φ_1 and φ_2 (see Fig. 1.34) as generalized coordinates.
3. Determine $x_2(t)$ with the initial conditions:

$$x_2(0) = 0 \; ; \; \dot{x}_2(0) = 0 \; .$$

4. Verify the result for the thread tension from part 1.!

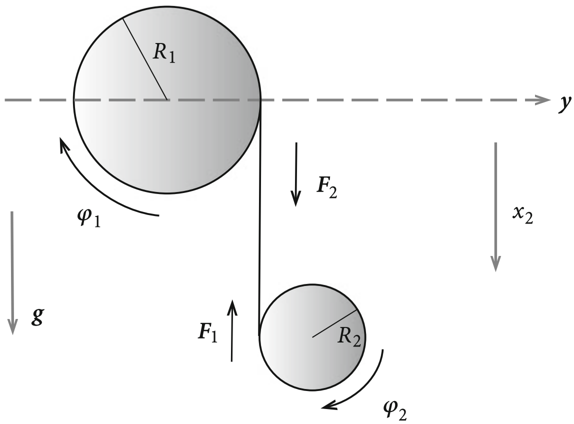


Fig. 1.34 Two rotatable cylinders coupled by a thread

Exercise 1.2.17 A homogeneous hollow cylinder (mass M, radius R) is pivoted around a horizontal axis through the center of mass P in the gravitational field $\mathbf{g} = -g\mathbf{e}_z$. Within this hollow cylinder a homogeneous solid cylinder (mass m, radius r) can roll without any gliding. The two cylinder axes are parallel (Fig. 1.35).

Figure 1.35 is to be read as follows: O and P are points fixed in space while A, B, C, and S are body fixed, i.e. connected to the rolling cylinder. Thus in equilibrium C coincides with O, B with O, and S is on the line $\overline{OP}$. ψ describes the deflection of the hollow cylinder from the equilibrium position. χ is a measure of the displacement of the solid cylinder from its equilibrium position, while φ gives the angular position of the center of gravity of the solid cylinder.

1. List the constraints and fix generalized coordinates!
2. Determine the Lagrangian!
3. Find the equations of motion!
4. Calculate the eigen-frequencies in case of small deflections!

Exercise 1.2.18 Two point masses $m_1 = m_2 = m$ are connected by a mass-less rod of length l to build a dumbbell (Fig. 1.36). They are moving in the xy plane being thereby subject to a friction force which is proportional to the velocities of the masses. ($\mathbf{F}_i = -\alpha\,\dot{\mathbf{r}}_i, i = 1, 2$)

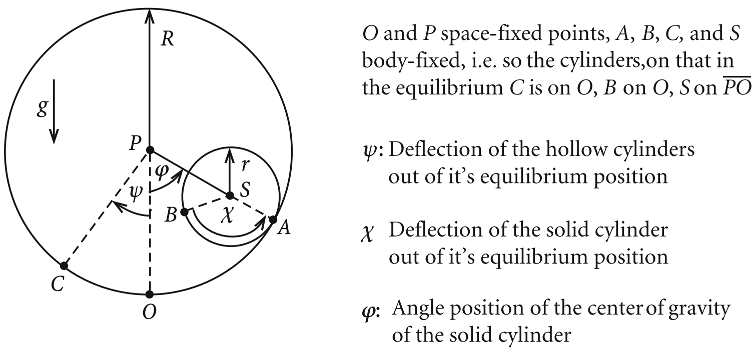


Fig. 1.35 A solid cylinder that rolls on the inner surface of a hollow cylinder driven by the earth's gravitational field

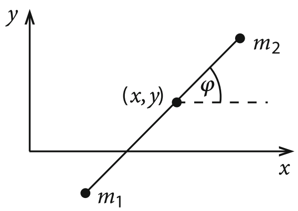


Fig. 1.36 Motion of a dumbbell under the influence of a friction force

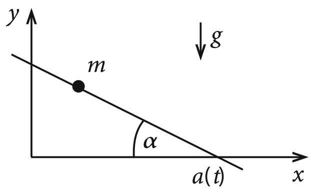


Fig. 1.37 Mass point on an inclined plane in the earth's gravitational field

1. Formulate the constraints and choose suitable generalized coordinates!
2. Friction forces are not conservative. Derive the corresponding generalized forces Q_j!
3. Which equations of motion are to be solved?
4. Solve these equations by use of the initial conditions:

$$\text{center of gravity:} \qquad x(0) = y(0) = 0\,; \quad \dot{x}(0) = v_x\,; \quad \dot{y}(0) = v_y$$
$$\text{angle:} \qquad \varphi(0) = 0\,; \quad \dot{\varphi}(0) = \omega$$

Exercise 1.2.19 A mass point slips without friction down an inclined plane experiencing the earth's gravitational field (Fig. 1.37). The plane moves with constant inclination α in x direction where the intersection point with the x axis exhibits the time-dependence

$$a(t) = \frac{1}{2}ct^2 \quad (c > 0)$$

1. Write down the constraints and the Lagrangian L. Thereby, do **not** exploit the constraint for the motion of the inclined plane for a reduction of the number of coordinates.
2. Determine the Lagrange equations of the first kind and use these to fix the generalized constraint forces Q_x, Q_y.
3. Solve the equations of motion with the initial conditions

$$\dot{x}(0) = 0\,; \quad x(0) = x_0\,.$$

Exercise 1.2.20 A mass point m is on a spherical surface of radius R and experiences the earth's homogeneous gravitational field.

1. Specify the constraints and formulate the Lagrangian!
2. Set up the Lagrange equation of the second kind and find an integral of motion!
3. What is the height z_0 at which the mass point hops from the sphere if it was initially in an unstable equilibrium state at the highest point of the sphere and then gets an infinitesimal initial velocity?

Exercise 1.2.21 A particle of mass m moves in a plane under the influence of a force that acts in the direction to a force center. For the magnitude F of the non-conservative force it holds if r is the distance from the center of force:

$$F = \frac{\alpha}{r^2}\left(1 - \frac{\dot{r}^2 - 2r\ddot{r}}{c^2}\right) \qquad (\alpha\text{: constant of appropriate dimension})\ .$$

Determine the generalized potential

$$U = U(r, \dot{r})$$

and therewith the Lagrangian for planar motion!

Exercise 1.2.22 A homogeneous circular disc (radius R, mass M) with a point-shaped mass fixed on its edge,

$$m = \frac{1}{2}M$$

moves without gliding frictionlessly on a horizontal straight line under the influence of the gravitational force (Fig. 1.38).

1. Calculate the coordinates x_M, y_M of the center of the disc as functions of the roll angle φ. Fix them so that $\varphi = 0$ for $x_M = 0$.
2. Calculate as functions of φ the coordinates x_m, y_m of the mass point as well as the coordinates x_S, y_S of the common center of gravity of the disc and the mass point! Of which type are the trajectories?
3. Calculate the kinetic energy $T(\varphi, \dot{\varphi})$ and the potential energy $V(\varphi)$ of the total system!
4. Find the Lagrangian $L(\varphi, \dot{\varphi})$ and the corresponding equation of motion for φ. What is the value of the frequency for small oscillations around the equilibrium position $\varphi = 0$?
5. Determine the constraint force $\mathbf{Z}(\varphi, \dot{\varphi}, \ddot{\varphi})$ that is exerted by the horizontal straight line onto the disc!

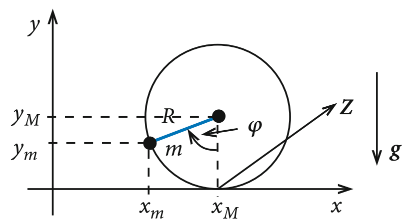


Fig. 1.38 Homogeneous circular disc with a mass point m on its edge

6. For sufficiently high initial velocity $v = \dot{x}_M$, referred to the support point at $\varphi = 0$, the disc is able to take off from the horizontal straight line because of the additional mass m. How large must v be in order to guarantee the *'take off'* at $\varphi = 2\pi/3$?
7. Finally demonstrate the equivalence of Newton and Lagrange mechanics! Generally we describe the movement of a solid body by α) the translation of the center of gravity and β) the rotation around the center of gravity. Derive with the constraint force $\mathbf{Z}(\varphi, \dot{\varphi}, \ddot{\varphi})$ from part 5. the equation of motion for β). It should be identical to that of part 4).

Exercise 1.2.23 A mass point P is kept by a thread on a circular path with the initial radius R_0 (no gravitational field!). The thread is then shortened, e.g. by putting the thread through a pipe constructed just in the midpoint of the circle and vertical to the plane of the circle and then pulling on the thread (Fig. 1.39). At first the thread shortening takes place so slowly that the respective radial kinetic energy can be neglected.

1. Determine an integral of motion!
2. Which work W is executed on the system when the path radius changes from R_0 to $R < R_0$?
3. Now let the thread be shortened with finite velocity,

$$\dot{r}(t) = -b\,t\ , \qquad (b > 0)\ ,$$

   starting with the thread length R_0 at $t = 0$. Is the integral of motion from 1. still valid?
4. How does the constraint force $\mathbf{Z}$, which produces the constraint $\dot{r}(t) = -b\,t$, look like?
5. How large is now the work to be done by the system in order to shorten the thread length from R_0 to $R < R_0$?

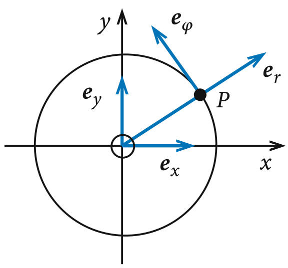


Fig. 1.39 Mass point on a circular path

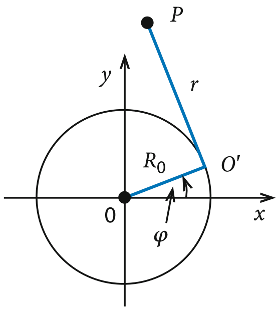


Fig. 1.40 Mass point P on a thread which is attached to a cylinder jacket

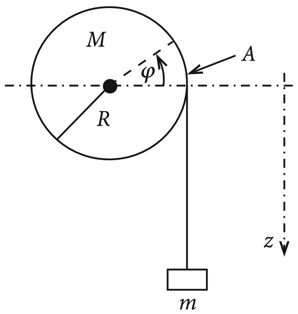


Fig. 1.41 Mass m hanging on a thread attached to a roller which is rotatable around a horizontal axis

Exercise 1.2.24 A thread of total length l is fixed on a cylinder which has a radius R_0 and stands vertically to the drawing plane in Fig. 1.40. A revolution of the mass point P around the firm cylinder means that the thread is wrapped up and the *'free'* thread length $r = \overline{PO'}$ is correspondingly shortened.

1. Determine an integral of motion and compare it with part 1. of Exercise 1.2.23.
2. Derive the equation of motion for the angle φ and solve it with the initial conditions

$$\varphi(t = 0) = 0 \; ; \qquad l\,\dot{\varphi}(t = 0) = v_0$$

   ($\varphi = 0$ means the completely unwounded thread). After which time is the thread completely wrapped up?
3. Demonstrate that the generalized momentum p_φ belonging to φ is just the magnitude of the angular momentum of the mass point with respect to $\mathcal{O}$.

Exercise 1.2.25 On a cylindrical roller (radius R, mass M), rotatable around a horizontal axis, a thread of length $l \gg R$ is wrapped up. One end of the thread is fixed on the roller while a mass m is hanging on the other free end (Fig. 1.41). The mass density of the roller increases linearly with the radius, starting with zero at the axis. The coordinate z of the mass m starts at the roller axis and is counted downwards (Fig. 1.41).

1. Find the equation of motion of the system for $0 \leq z \leq l$ and integrate it with the initial condition that the mass m is released at time $t = 0$ at the height of the roller axis.
2. What can be said about the course of motion in the region $l \leq z \leq R + l$ if $M \gg m$ is assumed? How does the motion continue after the minimum point is reached?
3. How strong is the thread tension in the regions $0 \leq z \leq l$ and $l \leq z \leq R + l$, respectively?

Exercise 1.2.26 Consider the planar motion of a dumbbell in the gravitational field defined by the potential energy

$$V = -\gamma \frac{m}{r} \qquad (\gamma > 0)$$

which belongs to a point mass m at the distance r from the field center P. The dumbbell consists of two mass points of equal mass m which are connected by a mass-less rod of length $2a$ (Fig. 1.42).

1. Introduce besides r two suitable angles as generalized coordinates, set up the Lagrangian and derive therewith the equations of motion of the dumbbell.
2. Find the conservation law for the total angular momentum of the dumbbell. Define in a proper way an orbital angular momentum and an intrinsic (eigen) angular momentum.
3. Expand the Lagrange equations in ascending powers of (a/r) up to the order $(a/r)^2$. Show that for $(a/r) \ll 1$ the orbital motion can approximately be decoupled from the intrinsic self-rotation!
4. Investigate the two special motions for which the center of gravity S moves uniformly on a circle with radius R around P while the dumbbell-rod, in the first case, points steadily to the center P and, in the second case, lies always tangential to this circle. Show that these two cases are possible special solutions of the Lagrange equations. How large are thereby the angular velocities ω_1 and ω_2, respectively, of the motions of S? (Accuracy up to $(a/r)^2$ is sufficient!) Is the fact that ω_1 and ω_2 are found to be different a contradiction to the general rule that the center of mass of a system moves as if the total mass were concentrated in the center and all external forces were acting on it?

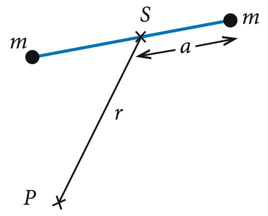


Fig. 1.42 Dumbbell consisting of two equal masses with constant distance $2a$ in the earth's gravitational field

Exercise 1.2.27 Which of the two Lagrangians

$$L_1 = \frac{m}{2}\dot{\mathbf{r}}^2 + q\mathbf{E}\cdot\mathbf{r}$$
$$L_2 = \frac{m}{2}\dot{\mathbf{r}}^2 - q\mathbf{E}\cdot\dot{\mathbf{r}}t$$

describes a charged particle (mass m, charge q) in the constant homogeneous electric field $\mathbf{E}$?

1.3 The Hamilton Principle

In this section we will become familiar with a new fundamental principle of Classical Mechanics which turns out to be at least equivalent to the up to now discussed principles (Newton, d'Alembert). The physical laws, rules and theorems of Classical Mechanics can be derived from two different types of **variational principles**. By the

(1) differential principle (d'Alembert)

one compares a present state of the system with small (virtual) deviations from this state. As result one gets fundamental equations of motion. In contrast, an

(2) integral principle (Hamilton)

concerns a finite (!) path element between fixed times t_1 and t_2 which is related to small (virtual) displacements of the total path from the real course of the system. Here, too, the principle leads to equations of motion.

1.3.1 Formulation of the Principle

For a better understanding of the integral principle let us recall once more two previous definitions. We understand by the

configuration space

the S-dimensional space the axes of which are given by the generalized coordinates $q_1, \ldots, q_S$. Each point of the configuration space specifies a possible state of the **total system**. There need not necessarily exist a compelling link between the configuration space and the three-dimensional physical space. The curve in the configuration space which the system runs through in the course of time is called

configuration path: $\mathbf{q}(t) = (q_1(t), \ldots, q_S(t))$.

On this path the system moves as a whole. Therefore, it may be that the configuration path does not have the slightest similarity to the actual particle paths.

We restrict the following considerations at first to

holonomic, conservative systems.

Generalizations are discussed later in the text. If one inserts into the Lagrangian the configuration path $\mathbf{q}(t)$ and its time-derivative $\dot{\mathbf{q}}(t)$ then L becomes a pure time-function:

$$L\left(\mathbf{q}(t),\ \dot{\mathbf{q}}(t),\ t\right) \equiv \widetilde{L}(t)\ . \tag{1.116}$$

We define:

$$S\left\{\mathbf{q}(t)\right\} = \int\limits_{t_1}^{t_2} \widetilde{L}(t)\, dt\ . \tag{1.117}$$

S has the dimension of *'action'(=energy* $\times$ *time)* and is dependent on the times t_1, t_2 as well as on the path $\mathbf{q}(t)$. For fixed t_1, t_2 to each path $\mathbf{q}(t)$ a pure **number** $S\{\mathbf{q}(t)\}$ is ascribed. This is called a **'functional'**. To each point of the system's path a manifold of virtual displacements $\delta\mathbf{q}$ exists which along the path form something like a continuum. One can now consider virtual displacements to be composed in such a way that they, on their own part, represent a continuously differentiable **'variational orbit'**. Such a composition of virtual displacements may be done in totally different manners so that a full manifold of variational orbits can be found.

Definition 1.3.1

$$M \equiv \left\{\mathbf{q}(t) : \mathbf{q}(t_1) = \mathbf{q}_{\mathrm{a}}\ ; \quad \mathbf{q}(t_2) = \mathbf{q}_{\mathrm{e}}\right\} \tag{1.118}$$

is the ensemble of configuration paths (**'competitive set'**) with the following properties:

1. Equal endpoints of time t_1, t_2, i.e. equal *'pass-through times'* for the system.
2. Each path arises by virtual displacements from the real one, being therefore compatible with the constraints.
3. The virtual displacements of the endpoints $\mathbf{q}_{\mathrm{a}}$, $\mathbf{q}_{\mathrm{e}}$ are for all paths equal to zero:

$$\delta\mathbf{q}_{\mathrm{a}} = \delta\mathbf{q}_{\mathrm{e}} = 0\ . \tag{1.119}$$

Figure 1.43 shows a one-dimensional illustration of the competitive set M. The solid line represents the *'real'* path.

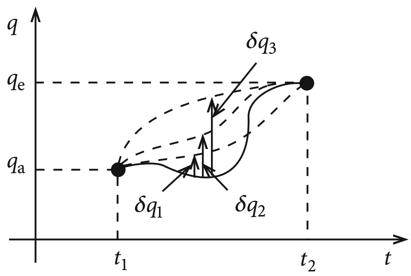


Fig. 1.43 One-dimensional illustration of the competitive set of configuration paths introduced in Definition 1.3.1

We define as

$$\textbf{action functional:} \qquad S\{\mathbf{q}(t)\} = \int_{t_1}^{t_2} L\left(\mathbf{q}(t),\ \dot{\mathbf{q}}(t), t\right)\, dt\,, \tag{1.120}$$

which helps us to formulate the Hamilton principle.

Hamilton's principle

(A) The motion of the system takes place such that $S\{\mathbf{q}(t)\}$ becomes **extremal (stationary)** among the competitive set M, defined in (1.118), just for the real path.

(B) The motion of the system takes place such that the variation of S on M vanishes for the real path $\mathbf{q}(t)$:

$$\delta S = \delta \int_{t_1}^{t_2} L\left(\mathbf{q}(t),\ \dot{\mathbf{q}}(t), t\right)\, dt \overset{!}{=} 0\,. \tag{1.121}$$

(A) and (B) are of course equivalent statements. How to perform explicitly the variation in (1.121) that we will learn in the next section. The result will be again the Lagrange equations of motion in the form (1.36). The Hamilton principle, however, shows up some remarkable advantages:

1. It is a very *'elegant'* modeling which in a nutshell contains the whole Classical Mechanics of conservative, holonomic systems.
2. The principle is not only applicable to typical mechanical systems being actually a superordinate principle.
3. It is independent of the system of coordinates by which L is expressed.

We show next the

equivalence to d'Alembert's principle

The latter we have formulated in (1.20):

$$\sum_{i=1}^{N} (m_i \ddot{\mathbf{r}}_i - \mathbf{K}_i) \cdot \delta \mathbf{r}_i = 0 \ . \tag{1.122}$$

The virtual displacements $\delta \mathbf{r}_i$ are differentiable functions of time:

$$\ddot{\mathbf{r}}_i \cdot \delta \mathbf{r}_i = \frac{d}{dt} (\dot{\mathbf{r}}_i \cdot \delta \mathbf{r}_i) - \dot{\mathbf{r}}_i \cdot \delta \dot{\mathbf{r}}_i = \frac{d}{dt} (\dot{\mathbf{r}}_i \cdot \delta \mathbf{r}_i) - \frac{1}{2} \delta \left(\dot{\mathbf{r}}_i^2 \right) \ .$$

We remember that with respect to *'calculatory terms'* we can work with the symbol $'\delta'$ exactly as we do it with the total differential $'d'$. We now integrate (1.122) between two fixed times t_1 and t_2:

$$\int\limits_{t_1}^{t_2} \left(\sum_{i=1}^{N} (m_i \ddot{\mathbf{r}}_i - \mathbf{K}_i) \cdot \delta \mathbf{r}_i \right) dt$$

$$= \int\limits_{t_1}^{t_2} \left(\sum_{i=1}^{N} \left[\frac{d}{dt} (m_i \, \dot{\mathbf{r}}_i \cdot \delta \mathbf{r}_i) - \frac{m_i}{2} \delta \left(\dot{\mathbf{r}}_i^2 \right) - \mathbf{K}_i \cdot \delta \mathbf{r}_i \right] \right) dt = 0 \ .$$

The first summand can directly be evaluated:

$$\int\limits_{t_1}^{t_2} \sum_{i=1}^{N} \frac{d}{dt} (m_i \, \dot{\mathbf{r}}_i \cdot \delta \mathbf{r}_i) \ dt = \sum_{i=1}^{N} m_i \, \dot{\mathbf{r}}_i \cdot \delta \mathbf{r}_i \Bigg|_{t_1}^{t_2} = 0 \ .$$

This expression vanishes since only such paths are allowed for the variation which coincide at the endpoints with the real path:

$$\delta \mathbf{r}_i \big|_{t=t_1,\,t_2} = \sum_{j=1}^{S} \frac{\partial \mathbf{r}_i}{\partial q_j} \delta q_j \Bigg|_{t=t_1,\,t_2} = 0 \ . \tag{1.123}$$

Therefore what is left from the d'Alembert's principle is:

$$\int\limits_{t_1}^{t_2} \sum_{i=1}^{N} \left[\delta \left(\frac{m_i}{2} \dot{\mathbf{r}}_i^2 \right) + \mathbf{K}_i \cdot \delta \mathbf{r}_i \right] dt = 0 \ . \tag{1.124}$$

By the use of the transformation formulas

$$\mathbf{r}_i = \mathbf{r}_i\,(q_1, \ldots, q_S, t) \ , \qquad i = 1, 2, \ldots, N$$

we can rewrite this expression using generalized coordinates. According to (1.26) and (1.29) it holds for a conservative system:

$$\sum_{i=1}^{N} \mathbf{K}_i \cdot \delta\mathbf{r}_i = \sum_{j=1}^{S} Q_j\,\delta q_j = -\sum_{j=1}^{S} \frac{\partial V}{\partial q_j}\delta q_j = -\delta V \ .$$

Therewith (1.124) reads:

$$\int_{t_1}^{t_2} \delta(T - V)\,dt = \delta \int_{t_1}^{t_2} (T - V)\,dt = \delta \int_{t_1}^{t_2} L\,dt = 0 \ . \tag{1.125}$$

For the last two relations we have exploited that for virtual displacements the times are not varied at all ($\delta t = 0$) so that we could draw, for instance, the variation δ to the front of the integral.

Equation (1.125) is the **Hamilton principle**. For all processes going on in nature the time integral of the Lagrangian will attain an extreme value in comparison to all virtual neighboring paths which connect the same time points t_1 and t_2 and the same end-configurations q_a, q_e.

The Hamilton principle can be transferred into a system of differential equations by applying methods from the calculus of variations. We therefore will deal in the next section, as an insertion, a little bit with the calculus of variations.

1.3.2 Elements of the Calculus of Variations

How can we exploit in practice the Hamilton's principle, i.e. how can we infer the *'stationary'* path from the action functional $S\{\mathbf{q}(t)\}$? The task to find the curve for which a definite line integral becomes extremal represents a typical

variational problem

At first we will outline the main features for a **one-dimensional problem** (Fig. 1.44).

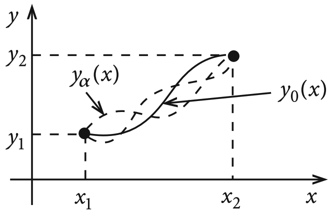


Fig. 1.44 One-dimensional illustration of the ensemble of paths permitted for the variational problem

We define as

'competing ensemble M'

$M \equiv \{y(x)\,;$ at least twofold differentiable with $y\,(x_1) = y_1$ and $y(x_2) = y_2\}$,
and on this the functional:

$$J\,\{y(x)\} = \int\limits_{x_1}^{x_2} f(x, y, y')\,dx = \int\limits_{x_1}^{x_2} \tilde{f}(x)\,dx\ , \qquad (1.126)$$

where $y' = dy/dx$, while $f(u, v, w)$ represents a differentiable function with continuous partial derivatives.

The problem is to find out for which $y(x)$ the functional $J\{y(x)\}$ becomes extremal, i.e. *'stationary'*. This particular problem reminds us of an elementary extreme value problem and is indeed treated correspondingly. We characterize and distinguish the curves $y(x)$ under consideration from M by an *ensemble parameter* α, which may be chosen in such a way that

$$y_{\alpha=0}(x) = y_0(x)$$

is the extremal path we want to find. This succeeds, e.g., by the following *parameter representation*:

$$y_\alpha(x) = y_0(x) + \gamma_\alpha(x)\ . \qquad (1.127)$$

$\gamma_\alpha(x)$ is thereby an *'almost arbitrary'* function which shall be sufficiently often differentiable and has to fulfill:

$$\begin{aligned} \gamma_\alpha\,(x_1) = \gamma_\alpha\,(x_2) &\equiv 0 \qquad \forall\,\alpha\ , \\ \gamma_{\alpha=0}(x) &\equiv 0 \qquad \forall\,x \end{aligned} \qquad (1.128)$$

A possible, very simple choice for $\gamma_\alpha(x)$ could be, for instance:

$$\gamma_\alpha(x) = \alpha\,\eta(x) \quad \text{with} \quad \eta\,(x_1) = \eta\,(x_2) = 0\ .$$

$\gamma_\alpha(x)$ and therewith also $y_\alpha(x)$ are for a given x quite normal functions of α which can be expanded in a Taylor series:

$$\gamma_\alpha(x) = \alpha \left(\frac{\partial \gamma_\alpha(x)}{\partial \alpha} \right)_{\alpha=0} + \frac{\alpha^2}{2} \left(\frac{\partial^2 \gamma_\alpha(x)}{\partial \alpha^2} \right)_{\alpha=0} + \cdots ,$$
$$y_\alpha(x) = y_0(x) + \alpha \left(\frac{\partial \gamma_\alpha(x)}{\partial \alpha} \right)_{\alpha=0} + \cdots .$$

We denote as

variation of the path $y_\alpha(x)$

the *displacement* δy of the path that appears as a consequence of a parameter change from $\alpha = 0$ to $d\alpha$:

$$\delta y = y_{d\alpha}(x) - y_0(x) = d\alpha \left(\frac{\partial \gamma_\alpha(x)}{\partial \alpha} \right)_{\alpha=0} . \tag{1.129}$$

This displacement is done for a fixed x, therefore reminds of a virtual displacement which is performed at a constant time.

Fully analogously we define the

variation of the functional $J\{y(x)\}$

$$\delta J = J\{y_{d\alpha}(x)\} - J\{y_0(x)\} = \left(\frac{dJ(\alpha)}{d\alpha} \right)_{\alpha=0} d\alpha$$
$$= \int_{x_1}^{x_2} dx \left(f\left(x, y_{d\alpha}, y'_{d\alpha}\right) - f\left(x, y_0, y'_0\right)\right) . \tag{1.130}$$

If one manages to determine a $y_0(x)$ such that $J(\alpha)$ becomes extremal at $\alpha = 0$ for **all (!)** $\gamma_\alpha(x)$ then $y_0(x)$ will obviously be the required *stationary* path. Thus the **extreme value condition** reads:

Choose $y_0(x)$ so that it holds

$$\left(\frac{dJ(\alpha)}{d\alpha} \right)_{\alpha=0} = 0 \quad \text{for } \textbf{arbitrary (!)} \ \gamma_\alpha(x)$$

That means according to (1.130):

$$\text{'stationary' path} \iff \delta J \stackrel{!}{=} 0 . \tag{1.131}$$

We now can further evaluate this prescription:

$$\frac{d}{d\alpha}J(\alpha) = \int_{x_1}^{x_2} dx \left(\frac{\partial f}{\partial y} \frac{\partial y}{\partial \alpha} + \frac{\partial f}{\partial y'} \frac{\partial y'}{\partial \alpha} \right) .$$

The endpoints x_1, x_2 as well as the variable x itself are uninfluenced by the variation. Therefore, the α-differentiations can be drawn into the integrand:

$$\int_{x_1}^{x_2} dx \frac{\partial f}{\partial y'} \frac{\partial y'}{\partial \alpha} = \int_{x_1}^{x_2} dx \frac{\partial f}{\partial y'} \frac{d}{dx} \left(\frac{\partial y}{\partial \alpha} \right)$$

$$= \frac{\partial f}{\partial y'} \frac{\partial y}{\partial \alpha} \bigg|_{x_1}^{x_2} - \int_{x_1}^{x_2} dx \left(\frac{d}{dx} \frac{\partial f}{\partial y'} \right) \frac{\partial y}{\partial \alpha} .$$

The first summand disappears because of (1.128). It remains:

$$\frac{d}{d\alpha}J(\alpha) = \int_{x_1}^{x_2} dx \left(\frac{\partial f}{\partial y} - \frac{d}{dx} \frac{\partial f}{\partial y'} \right) \frac{\partial y}{\partial \alpha} ,$$

That means with (1.129) and (1.130):

$$\delta J = \int_{x_1}^{x_2} dx \left(\frac{\partial f}{\partial y} - \frac{d}{dx} \frac{\partial f}{\partial y'} \right) \delta y .$$

Except for being zero at the integration limits the variation δy is arbitrary. Thus the requirement (1.131) is satisfiable only if

Euler's equation:
$$\frac{\partial f}{\partial y} - \frac{d}{dx} \frac{\partial f}{\partial y'} = 0 \tag{1.132}$$

is fulfilled! Let us add some remarks:

1. The requirement $\delta J = 0$ is realizable by minima, maxima or inflection points. The decision what is really on hand is given by the second variation $\delta^2 J$. However, that is rather uninteresting in our context here, because Hamilton's principle requires only $\delta S = 0$. S is thereby mostly minimal, in some cases, however, also maximal.

2. Euler's equation is a differential equation of second order which written in detail reads:

$$\frac{\partial f}{\partial y} - \frac{\partial^2 f}{\partial x\, \partial y'} - \frac{\partial^2 f}{\partial y\, \partial y'}\, y' - \frac{\partial^2 f}{\partial y'^2}\, y'' = 0 \tag{1.133}$$

$y(x)$ thus must be at least two times differentiable.

3. In this context one could ask of course whether it were not possible that even a function $y(x)$ which is only one time differentiable could make the functional $J(x)$ extremal. This is not true. The proof, however, turns out to be a rather involved problem of the functional analysis and therefore will not be outlined here.

Let us exercise the formalism with three typical **application examples**:

(1) Shortest Connection Between Two Points in the Plane

For the element of the arc length in the xy plane it holds:

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + y'^2}\, dx\,.$$

The full path length is then given by:

$$J = \int_1^2 ds = \int_{x_1}^{x_2} \sqrt{1 + y'^2}\, dx\,. \tag{1.134}$$

We seek the shortest connection and therewith the minimum of J for which as necessary condition $\delta J = 0$ must be fulfilled (Fig. 1.45). That corresponds to the above treated situation so that the Euler's equation (1.132) can be applied. It must be formulated for

$$f(x, y, y') = \sqrt{1 + y'^2}\,.$$

Because of

$$\frac{\partial f}{\partial y} \equiv 0\,; \quad \frac{\partial f}{\partial y'} = \frac{y'}{\sqrt{1 + y'^2}}$$

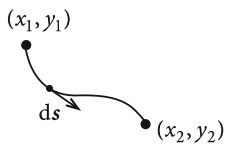


Fig. 1.45 To the calculation of the shortest connection between two points in the plane by use of the variational technique

it is to require:

$$\frac{d}{dx}\frac{y'}{\sqrt{1+y'^2}} = 0 \iff \frac{y'}{\sqrt{1+y'^2}} = \text{const}$$

This means $y' = a = \text{const}$. Hence, no surprise, the shortest connection is a straight line:

$$y(x) = a\,x + b\,. \tag{1.135}$$

The constants a, b are fixed by the prescription that $y(x)$ should run through the points (x_1, y_1), (x_2, y_2).

(2) Minimum Area of Rotation

We ask ourselves how the connecting line between the points (x_1, y_1) and (x_2, y_2) must be shaped to make the lateral surface of the hollow body, which is generated by a rotation of the connecting line around the y axis (Fig. 1.46), minimal. The stripe area of width ds, indicated in Fig. 1.46, amounts to

$$2\pi x\,ds = 2\pi x\sqrt{1+y'^2}\,dx\,.$$

That yields as total area

$$J = 2\pi\int_{x_1}^{x_2} x\sqrt{1+y'^2}\,dx\,. \tag{1.136}$$

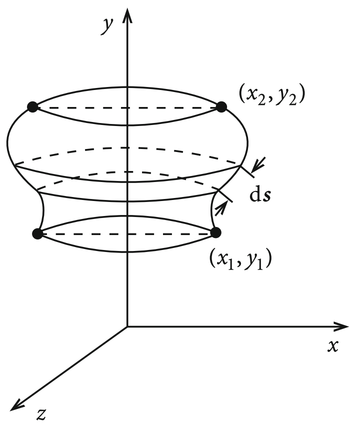


Fig. 1.46 To the calculation of the connecting line between two points of a plane where this line shall lead to a minimal lateral surface of the hollow body that arises by a rotation of the connecting line around the y axis

We require $\delta J = 0$ so that the function

$$f\left(x, y, y'\right) = x\sqrt{1 + y'^2}$$

has to fulfill the Euler's equation (1.132). Because of

$$\frac{\partial f}{\partial y} \equiv 0\,; \quad \frac{\partial f}{\partial y'} = \frac{x y'}{\sqrt{1 + y'^2}}$$

that means:

$$\frac{x y'}{\sqrt{1 + y'^2}} = a = \text{const} \iff y' = \frac{a}{\sqrt{x^2 - a^2}}\,.$$

In case of a minimal rotation area it thus holds:

$$y(x) = a \,\text{arccosh}\left(\frac{x}{a}\right) + b \iff x = a \cosh\left(\frac{y - b}{a}\right)\,. \tag{1.137}$$

The constants a and b are uniquely fixed by the endpoints of the connecting line.

(3) Brachistochrone Problem

What is the way $y(x)$ on which a frictionlessly gliding mass point m under the influence of the gravitational force needs the shortest time to get from $(x_1, 0)$ to (x_2, y_2) (Fig. 1.47)? Let the initial velocity be zero:

$$J = \int_{t_1}^{t_2} dt = \int_1^2 \frac{ds}{v} \stackrel{!}{=} \text{minimum} \iff \delta J \stackrel{!}{=} 0\,.$$

The velocity v we take from the energy law:

$$\frac{m}{2}\, v^2 - m\, g\, y = \text{const} = \frac{m}{2}\, v_1^2 - m\, g\, y_1 = 0\,.$$

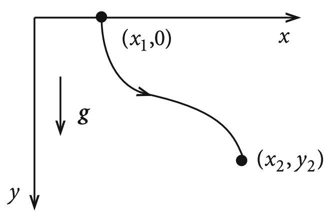


Fig. 1.47 Geometry for the brachistochrone problem

Thus it is:

$$v = \sqrt{2\,g\,y}\,.$$

With $ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + y'^2}\,dx$ one has to calculate:

$$\delta \int \sqrt{\frac{1 + y'^2}{y}}\,dx \overset{!}{=} 0\,. \tag{1.138}$$

Hence the function

$$f\left(x, y, y'\right) = \sqrt{\frac{1 + y'^2}{y}}$$

must fulfill the Euler's equation (1.132):

$$\begin{aligned}
\frac{\partial f}{\partial y'} &= \frac{y'}{\sqrt{y\,(1 + y'^2)}} = \hat{f}\left(y, y'\right)\,,\\
\frac{d}{dx}\frac{\partial f}{\partial y'} &= \frac{\partial \hat{f}}{\partial x} + \frac{\partial \hat{f}}{\partial y}\,y' + \frac{\partial \hat{f}}{\partial y'}\,y''\\
&= -\frac{y'^2}{2y^{3/2}\sqrt{1 + y'^2}} + \frac{y''}{\sqrt{y\,(1 + y'^2)}} - \frac{y'^2 y''}{(1 + y'^2)^{3/2}\,\sqrt{y}}\,,\\
\frac{\partial f}{\partial y} &= -\frac{\sqrt{1 + y'^2}}{2y^{3/2}}\,.
\end{aligned}$$

Insertion into the Euler's equation leads to:

$$\left(1 + y'^2\right) = -2y\,y'' + y'^2 + \frac{2y\,y'^2 y''}{1 + y'^2}\,.$$

That is equivalent to:

$$1 + y'^2 + 2y\,y'' = 0 \iff \frac{d}{dx}y\left(1 + y'^2\right) = 0\,.$$

It follows with a constant a which is to be determined later:

$$y'^2 = \frac{a - y}{y}\,; \quad dx = \sqrt{\frac{y}{a - y}}\,dy\,.$$

We substitute

$$y = a\,\sin^2\varphi \implies dy = 2a\,\sin\varphi\,\cos\varphi\,d\varphi$$

and integrate therewith the above equation:

$$\begin{aligned} x - x_1 &= \int\limits_0^y d\bar{y}\sqrt{\frac{\bar{y}}{a-\bar{y}}} = 2a\int\limits_0^\varphi d\overline{\varphi}\,\sin\overline{\varphi}\,\cos\overline{\varphi}\frac{\sin\overline{\varphi}}{\cos\overline{\varphi}} \\ &= 2a\,\frac{1}{2}(\varphi - \sin\varphi\,\cos\varphi)\ . \end{aligned}$$

So we have found:

$$\begin{aligned} x &= a\left(\varphi - \frac{1}{2}\,\sin 2\varphi\right) + x_1\ , \\ y &= a\,\sin^2\varphi = \frac{a}{2}\,(1 - \cos 2\varphi)\ . \end{aligned}$$

We still replace

$$R = \frac{a}{2}\ ; \quad x_1 = R\,\pi\ ; \quad \psi = 2\varphi + \pi\ . \tag{1.139}$$

Therewith we arrive at:

$$x = R\,(\psi + \sin\psi)\ ; \quad y = R\,(1 + \cos\psi)\ . \tag{1.140}$$

The comparison with (1.60) shows that the curve we sought for represents a cycloid with a peak at the initial point $(x_1, 0)$.

1.3.3 Lagrange Equations

We have first to generalize the variational calculus to more than one variable. From the requirement

$$\delta J = \delta\int\limits_{x_1}^{x_2} dx f\left(x,\, y_1(x), \ldots, y_S(x),\, y_1'(x), \ldots, y_S'(x)\right) \stackrel{!}{=} 0 \tag{1.141}$$

the extremal (stationary) path $\mathbf{y}(x) = (y_1(x), \ldots, y_S(x))$ is to be derived. For each single component we define a **'competing ensemble'** M_i by:

$$M_i = \{y_i(x)\,;\ \text{at least two times differentiable with } y_i(x_1) = y_{i1} \text{ and } y_i(x_2) = y_{i2}\}\,.$$

We use here again a *parameter representation* for the component function $y_i(x)$:

$$y_{i\alpha}(x) = y_{i0}(x) + \gamma_{i\alpha}(x)\,, \qquad i = 1, 2, \ldots, S\,. \tag{1.142}$$

$y_{i0}(x)$ are here the solutions of the extreme-value problem and $\gamma_{i\alpha}(x)$ *'almost arbitrary'*, but sufficiently often differentiable functions with

$$\begin{aligned} \gamma_{i\alpha}(x_1) = \gamma_{i\alpha}(x_2) &= 0 \qquad \forall\, \alpha, i\,, \\ \gamma_{i\alpha=0}(x) &= 0 \qquad \forall\, x, i\,. \end{aligned} \tag{1.143}$$

The variations δy_i of the path components,

$$\delta y_i = \left(\frac{\partial y_{i\alpha}}{\partial \alpha}\right)_{x,\,\alpha=0} d\alpha\,, \tag{1.144}$$

and the variation δJ of the functional,

$$\delta J = \left(\frac{dJ(\alpha)}{d\alpha}\right)_{\alpha=0} d\alpha = \int_{x_1}^{x_2} dx \sum_{i=1}^{S} \left(\frac{\partial f}{\partial y_i}\frac{\partial y_i}{\partial \alpha} + \frac{\partial f}{\partial y_i'}\frac{\partial y_i'}{\partial \alpha}\right)_{\alpha=0} d\alpha\,, \tag{1.145}$$

are defined analogously to the special cases $(S = 1)$ (1.129) and (1.130), respectively. A partial integration of the second term in (1.145) yields:

$$\int_{x_1}^{x_2} dx\, \frac{\partial f}{\partial y_i'}\frac{\partial y_i'}{\partial \alpha} = \int_{x_1}^{x_2} dx\, \frac{\partial f}{\partial y_i'}\frac{d}{dx}\frac{\partial y_i}{\partial \alpha} = \frac{\partial f}{\partial y_i'}\frac{\partial y_i}{\partial \alpha}\bigg|_{x_1}^{x_2} - \int_{x_1}^{x_2} dx \left(\frac{d}{dx}\frac{\partial f}{\partial y_i'}\right)\frac{\partial y_i}{\partial \alpha}\,.$$

The first summand disappears because of (1.143) so that it remains in (1.145):

$$\delta J = \int_{x_1}^{x_2} dx \sum_{i=1}^{S} \left(\frac{\partial f}{\partial y_i} - \frac{d}{dx}\frac{\partial f}{\partial y_i'}\right)\delta y_i \overset{!}{=} 0\,. \tag{1.146}$$

According to the assumption the δy_i are freely selectable except for being zero at the integration limits. That means that Eq. (1.141) is exactly then fulfilled when the following relations hold:

Euler-Lagrange differential equations

$$\frac{d}{dx}\frac{\partial f}{\partial y_i'} - \frac{\partial f}{\partial y_i} = 0\,, \qquad i = 1, 2, \ldots, S \tag{1.147}$$

We now come back to our original task, namely the evaluation of Hamilton's principle (1.121). For this purpose we substitute in (1.147):

$$x \Longrightarrow t\,; \quad y_i \Longrightarrow q_i\,; \quad y_i' \Longrightarrow \dot{q}_i\,; \quad f(x, \mathbf{y}, \mathbf{y}') \Longrightarrow L\,(t, \mathbf{q}, \dot{\mathbf{q}})$$

getting therewith immediately from the Hamilton principle the

Lagrange equations of motion of the second kind

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0\,, \qquad i = 1, 2, \ldots, S\,. \tag{1.148}$$

Let us remember once more the preconditions which were necessary for the above derivation of these equations. They hold for **conservative systems**, because otherwise the Lagrangian $L = T - V$ is not definable. Furthermore, their derivation presumed **holonomic constraints** so that the δq_i are independent of each other. Under these preconditions d'Alembert's and Hamilton's principle are equivalent as we have shown.

We now want to relax these preconditions a bit. What follows from the Hamilton principle in case of

conservative systems with non-holonomic constraints in differential form

$$\sum_{m=1}^{j} a_{im}\dot{q}_m + b_{it} = 0\,, \qquad i = 1, \ldots, p \tag{1.149}$$

The Lagrangian $L = T - V$ in such a case is still definable but the conclusion from (1.146) to (1.147) is no longer allowed because of the non-holonomic constraints. The Hamilton principle (1.121) leads at first only to (1.146):

$$\int_{t_1}^{t_2} dt \sum_{m=1}^{j} \left(\frac{\partial L}{\partial q_m} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_m} \right) \delta q_m = 0 \tag{1.150}$$

We rewrite the constraints (1.149) for virtual displacements ($\delta t = 0$) as we did in (1.96),

$$\sum_{m=1}^{j} a_{im}\delta q_m = 0 \ , \qquad i = 1, 2, \ldots, p \ ,$$

coupling them via Lagrange multipliers λ_i,

$$\int_{t_1}^{t_2} dt \left(\sum_{i=1}^{p} \lambda_i \sum_{m=1}^{j} a_{im}\delta q_m \right) = 0 \ ,$$

to Eq. (1.150):

$$\int_{t_1}^{t_2} dt \sum_{m=1}^{j} \left(\frac{\partial L}{\partial q_m} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_m} + \sum_{i=1}^{p} \lambda_i a_{im} \right) \delta q_m = 0 \ . \tag{1.151}$$

Using exactly the same considerations as the ones following (1.99) we can choose the multipliers λ_i in such a way that already each summand in (1.151) is zero. Because of the constraints (1.149) only $j - p$ coordinates are freely selectable. Therefore we set:

$$\begin{aligned} q_m &: \quad m = 1, \ldots, j-p \qquad &&\text{independent} \ , \\ q_m &: \quad m = j-p+1, \ldots, j \qquad &&\text{dependent} \ . \end{aligned}$$

The p multipliers λ_i are then chosen so that the bracket in the sum in (1.151) becomes identical to zero for each of the *'dependent'* q_m. That means then all in all:

Lagrange equations of motion of the first kind

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_m} - \frac{\partial L}{\partial q_m} = \sum_{i=1}^{p} \lambda_i a_{im} \tag{1.152}$$

Together with (1.149) there are now $(j + p)$ equations for the determination of j coordinates q_m and p multipliers λ_i. We conclude that for conservative systems even with non-holonomic constraints d'Alembert's and Hamilton's principle prove to be equivalent.

1.3.4 Extension of the Hamilton Principle

We want to further relax the hitherto existing precondition

conservative systems with holonomic constraints

and modify the Hamilton principle so that it becomes applicable also for

non-conservative systems

That means, we now allow that the *driving* forces $\mathbf{K}_i$ are **not** derivable from a scalar potential. The suitably extended principle should of course be formulated so that it reduces to (1.121) for the special case of conservative systems. For this purpose we define a modified
action functional

$$\widetilde{S}\{\mathbf{q}(t)\} = \int_{t_1}^{t_2} (T - W)\, dt \,, \tag{1.153}$$

$$W = -\sum_{i=1}^{N} \mathbf{K}_i \cdot \mathbf{r}_i \,. \tag{1.154}$$

The *'extended'* Hamilton's principle tells us that the *'real'* path can be derived by the requirement that

$$\delta \widetilde{S} \stackrel{!}{=} 0 \tag{1.155}$$

holds on the *'competing ensemble'*

$$M = \{\mathbf{q}(t) \;:\quad \mathbf{q}\,(t_1) = \mathbf{q}_\mathrm{a}\,, \quad \mathbf{q}\,(t_2) = \mathbf{q}_\mathrm{e}\} \tag{1.156}$$

The set M of the paths which are admitted to the variational procedure is defined exactly as in (1.118). Since the time is not co-varied we can also write instead of (1.155):

$$\int_{t_1}^{t_2} \delta(T - W)\, dt \stackrel{!}{=} 0 \tag{1.157}$$

The extended Hamilton principle thus indicates that the variation of the time integral over the sum of kinetic energy and virtual work, to be done due to the path variation, must be zero. As in (1.26) we introduce generalized force components Q_j:

$$Q_j = \sum_{i=1}^{N} \mathbf{K}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} \,. \tag{1.158}$$

Because of

$$\mathbf{r}_i = \mathbf{r}_i(\mathbf{q}, t) \implies d\mathbf{r}_i = \sum_{j=1}^{S} \frac{\partial \mathbf{r}_i}{\partial q_j} dq_j + \frac{\partial \mathbf{r}_i}{\partial t} dt$$

$$\implies \delta\mathbf{r}_i = \sum_{j=1}^{S} \frac{\partial \mathbf{r}_i}{\partial q_j} \delta q_j \qquad (\delta t = 0)$$

the virtual work reads:

$$-\delta W = \sum_{i=1}^{N} \mathbf{K}_i \cdot \delta\mathbf{r}_i = \sum_{i=1}^{N}\sum_{j=1}^{S} \mathbf{K}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} \delta q_j = \sum_{j=1}^{S} Q_j \delta q_j \ . \tag{1.159}$$

As contribution of the kinetic energy T we find:

$$\int_{t_1}^{t_2} \delta T \, dt = \int_{t_1}^{t_2} \sum_{j=1}^{S} \left(\frac{\partial T}{\partial q_j} \delta q_j + \frac{\partial T}{\partial \dot{q}_j} \delta \dot{q}_j \right) dt \ ,$$

$$\int_{t_1}^{t_2} \frac{\partial T}{\partial \dot{q}_j} \delta \dot{q}_j \, dt = \int_{t_1}^{t_2} \frac{\partial T}{\partial \dot{q}_j} \left(\frac{d}{dt} \delta q_j \right) dt = \frac{\partial T}{\partial \dot{q}_j} \underbrace{\delta q_j \Big|_{t_1}^{t_2}}_{=0} - \int_{t_1}^{t_2} \left(\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_j} \right) \delta q_j \, dt \ .$$

That means:

$$\int_{t_1}^{t_2} \delta T \, dt = \int_{t_1}^{t_2} \sum_{j=1}^{S} \left(\frac{\partial T}{\partial q_j} - \frac{d}{dt} \frac{\partial T}{\partial \dot{q}_j} \right) \delta q_j \, dt \ . \tag{1.160}$$

This we insert together with (1.159) into (1.157):

$$\sum_{j=1}^{S} \int_{t_1}^{t_2} \left(\frac{\partial T}{\partial q_j} - \frac{d}{dt} \frac{\partial T}{\partial \dot{q}_j} + Q_j \right) \delta q_j \, dt = 0 \ .$$

Because of the holonomic constraints the δq_j are independent of each other. Hence it follows with

$$\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} = Q_j \ , \qquad j = 1, 2, \ldots, S \tag{1.161}$$

exactly the same result as in (1.33) which we found on the basis of d'Alembert's principle.

Finally we still investigate the special case of a conservative system:

$$\mathbf{K}_i = -\nabla_i V \implies Q_j = -\sum_{i=1}^{N} \nabla_i V \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} = -\frac{\partial V}{\partial q_j} \,.$$

For the virtual work δW it therewith follows:

$$\delta W = -\sum_{j=1}^{S} Q_j \,\delta q_j = \sum_{j=1}^{S} \frac{\partial V}{\partial q_j} \,\delta q_j = \delta V \,.$$

The postulation (1.157) then reads:

$$\delta \widetilde{S} = \int_{t_1}^{t_2} \delta(T - V)\, dt = \int_{t_1}^{t_2} \delta L \, dt = \delta S \stackrel{!}{=} 0 \,.$$

We see that the *extended* Hamilton principle (1.155) is for conservative systems identical to the original principle (1.121).

Therewith we have shown that all statements of the d'Alembert's principle follow also in identical manner from the Hamilton's principle. The two principles are obviously completely equivalent.

1.3.5 Exercises

Exercise 1.3.1 Determine by use of the variational calculus the shortest link between a given point A of the xy plane and a straight line g parallel to the y axis which does not run through A (Fig. 1.48).

1. Show that the shortest distance between A and a fixed point B of the straight line g its just the line segment $\overline{AB}$.
2. Investigate then **all** line segments from A to any points on g.

Exercise 1.3.2 Determine the shortest connection between two points P_1 and P_2 on a cylinder barrel!

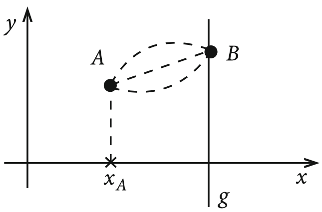


Fig. 1.48 Arrangement for the determination of the shortest distance between a point and a straight line within the *xy plane*

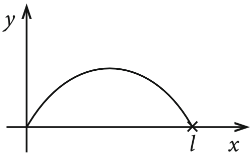


Fig. 1.49 Deflection of an oscillating string

Exercise 1.3.3 It is sought the displacement $y(x,t)$ of an oscillating string with a mass distribution $m(x)$ $\left(= \frac{dm}{dx}(x)\right)$ (Fig. 1.49).

1. What is the kinetic energy T?
2. Find an expression for the potential energy V if this is proportional to the elongation of the string during the oscillation.
3. Derive for small displacements of the string by use of Hamilton's principle a differential equation for $y(x,t)$!

Exercise 1.3.4 A particle of mass m moves in the earth's gravitational field ($\mathbf{g} = -g\mathbf{e}_z$). It thereby performs a one-dimensional motion $z = z(t)$. Calculate the action functional

$$S = \int_{t_1}^{t_2} L(z, \dot{z})\, dt$$

for the path

$$z(t) = -\frac{1}{2}gt^2 + f(t) \; .$$

Thereby $f(t)$ may be an in principle arbitrary but continuously differentiable function with $f(t_1) = f(t_2) = 0$. Show that S becomes minimal for $f(t) \equiv 0$!

Exercise 1.3.5 Try to find the function $y(x)$ for which the functional

$$J\{y(x)\} = \int_{x_1}^{x_2} f(x, y, y')dx$$

will be extremal. Show that for the case that f does not explicitly depend on x, ($f = f(y, y')$), the solution complies with

$$f - y'\frac{\partial f}{\partial y'} = \text{const} \; .$$

Exercise 1.3.6 A rope of length l lies in the xy plane being fixed at $P_1 = (-d, 0)$ and $P_2 = (d, 0)$. For which position of the rope will the area F between rope and x axis become maximal?

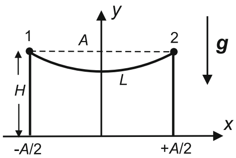


Fig. 1.50 High-voltage cable between two posts of height H with a distance A in the earth's gravitational field

Hint: Couple the constraint with a Lagrange multiplier λ to the variational task, i.e., investigate

$$\delta(F - \lambda l) \stackrel{!}{=} 0 \, .$$

The result of Exercise 1.3.5 might be helpful!

Exercise 1.3.7 A high-voltage cable is hanging between two posts of height H and with a distance A (Fig. 1.50). It possesses a constant mass density

$$\frac{dm}{ds} = \alpha = \text{const} \qquad (ds\text{: line element of the cable})$$

Because of the gravitational force ($\propto \mathbf{g}$) the cable tends to sag. If the cable length L were just equal to the pole distance A then there would act on the two poles strong side tensions which would make the system unstable towards other strains as, e.g., external weather conditions. Thus one has to choose from the beginning $L > A$.

1. Which kind of curve $y(x)$ will the cable take for given $L > A$ if one assumes that it corresponds to the minimum of the potential energy?
2. How to find the optimal cable length?

1.4 Conservation Laws

During the motion of a mechanical system the $2S$ quantities q_j, $\dot{q}_j$ ($j = 1, 2, \ldots, S$) in general change in the course of time. However, occasionally one finds certain functions F_r of the q_j, $\dot{q}_j$, which remain constant during the motion being fixed exclusively by the initial conditions of the system. Among these functions F_r there are some whose constancies are connected to basic properties of time and space (homogeneity, isotropy). One calls

$$\mathbf{F_r}: \quad \textbf{integrals (constants) of motion (conserved quantities)}\, , \qquad r = 1, 2, \ldots\, ,$$

if they are functions of the q_j, $\dot{q}_j$, but **not** of $\ddot{q}_j$, having a constant value c_r for the full path of the system:

$$F_r = F_r\,(q_1, \ldots, q_S, \dot{q}_1, \ldots, \dot{q}_S, t) = c_r \, , \qquad r = 1, 2, \ldots \, . \tag{1.162}$$

A system with S degrees of freedom is described by S differential equations of second order the solution of which requires the knowledge of $2S$ initial conditions. In case that $2S$ integrals of motion are known then the problem would already be solved:

$$q_j = q_j\,(c_1, c_2, \ldots, c_{2S}, t) \ , \qquad j = 1, 2, \ldots, S \ .$$

Normally of course not all the $2S$ constants c_r will be on hand. However, already the knowledge of some of these c_r can help us to learn much about the physical properties of the system and can considerably the integration of the equations of motion. Thus it is recommendable to detect before the explicit evaluation of a physical problem as many integrals of motion as possible.

Certain integrals of motion follow immediately from the **cyclic coordinates** introduced in (1.53). The **generalized momenta** p_j attributed to the cyclic coordinates q_j are the first integrals of motion. One should therefore choose the generalized coordinates always such that as many q_j as possible are cyclic. We illustrate that by an example:

two-body problem

For a two-body interaction which depends only on the particle distance

$$V\,(\mathbf{r}_1, \mathbf{r}_2) = V\,(|\mathbf{r}_1 - \mathbf{r}_2|)$$

the splitting into a relative and a center-of-gravity motion (Fig. 1.51) appears advisable (see Sect. 3.2, Vol. 1):

total mass: $M = m_1 + m_2$,

reduced mass: $\mu = \frac{m_1\, m_2}{n_1 + m_2}$,

center of gravity: $\mathbf{R} = \frac{1}{M}\,(m_1\mathbf{r}_1 + m_2\mathbf{r}_2) \equiv (X, Y, Z)$,

relative coordinate: $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2 = r(\sin\vartheta\ \cos\varphi,\ \sin\vartheta\ \sin\varphi,\ \cos\vartheta)$.

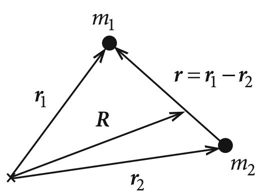


Fig. 1.51 Center-of-gravity and relative coordinates for the two-body problem

The relative motion takes place as if the reduced mass μ moves in the central field $V(\mathbf{r}) = V(r)$ (see Sect. 3.2.1, Vol. 1). With the generalized coordinates

$$q_1 = X\ ,\quad q_2 = Y\ ,\quad q_3 = Z\ ,\quad q_4 = r\ ,\quad q_5 = \vartheta\ ,\quad q_6 = \varphi \tag{1.163}$$

the Lagrangian thus reads:

$$L = \frac{M}{2}\left(\dot{q}_1^2 + \dot{q}_2^2 + \dot{q}_3^2\right) + \frac{\mu}{2}\left(\dot{q}_4^2 + q_4^2\,\dot{q}_5^2 + q_4^2\,\sin^2 q_5\,\dot{q}_6^2\right) - V\left(q_4\right)\ . \tag{1.164}$$

One recognizes immediately that

$$q_1,\ q_2,\ q_3,\ q_6$$

are cyclic coordinates leading directly to four integrals of motion. The first three,

$$\begin{aligned} p_1 &= \frac{\partial L}{\partial \dot{q}_1} = M\,\dot{q}_1 = M\,\dot{X} = \text{const}\ ,\\ p_2 &= \frac{\partial L}{\partial \dot{q}_2} = M\,\dot{q}_2 = M\,\dot{Y} = \text{const}\ ,\\ p_3 &= \frac{\partial L}{\partial \dot{q}_3} = M\,\dot{q}_3 = M\,\dot{Z} = \text{const}\ , \end{aligned}$$

result combined in the center-of-mass theorem for closed systems ((3.48), Vol. 1):

$$\mathbf{P} = M\,\dot{\mathbf{R}} = \mathbf{const}\ . \tag{1.165}$$

The fourth integral of motion:

$$p_6 = \frac{\partial L}{\partial \dot{q}_6} = \mu\,q_4^2\sin^2 q_5\,\dot{q}_6 = \mu r^2\,\sin^2\vartheta\,\dot{\varphi} = L_r^{(z)} = \text{const}$$

concerns the z-component of the relative angular momentum. Since no space direction is specified, we can even conclude:

$$\mathbf{L}_r = \mathbf{const}\ . \tag{1.166}$$

Had we formulated the problem by Cartesian coordinates,

$$L = \frac{m_1}{2}\left(\dot{x}_1^2 + \dot{y}_1^2 + \dot{z}_1^2\right) + \frac{m_2}{2}\left(\dot{x}_2^2 + \dot{y}_2^2 + \dot{z}_2^2\right) - V\left[(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2\right] ,$$

none of the coordinates would have been cyclic although, of course, nothing of the system had changed. The conservation laws (1.165) and (1.166) of course would still be valid, but to recognize that would have turned out to be very much more complicated.

In the framework of Newton's version of Classical Mechanics (see Vol. 1) we got to know a series of physically fundamental conservation laws (for energy, for linear momentum, for angular momentum, etc.). These we find of course also in the Lagrangian formulation. They have then, however, sometimes a somewhat different shape and can lead to new aspects with their interpretations. In the subsequent sections we will be able to interpret them as direct consequences of basic symmetries of the physical system (**Noether's theorems**). Thereby we will assume, without mentioning it always explicitly,

holonomic, conservative systems.

1.4.1 Homogeneity of Time

We call a system *'temporally homogeneous'* if its properties prove to be invariant under time translations. The results of measurements performed under exactly the same boundary conditions are independent of the point in time of the measurement. The ensemble of all possible paths which start at a certain given time is independent of the choice of this initial time but only dependent on the initial configuration $\mathbf{q}_a$. If $\mathbf{q}(t)$ is the configuration path which the system is passing through between the times t_a and t_e with the initial- and end-configurations

$$\mathbf{q}(t_a) = \mathbf{q}_a \qquad \text{and} \qquad \mathbf{q}(t_e) = \mathbf{q}_e ,$$

then the *'temporally shifted'* configuration path between $t_a + \Delta t$ and $t_e + \Delta t$ will go through, provided the temporal homogeneity is given, exactly the same points of the configuration space if only the initial- and end-configurations are the same:

$$\mathbf{q}(t_a + \Delta t) = \mathbf{q}_a ; \qquad \mathbf{q}(t_e + \Delta t) = \mathbf{q}_e .$$

This means, however, that the Lagrangian L of the system by which we calculate its path can not explicitly depend on the time:

$$\textit{homogeneity of time} \iff \frac{\partial L}{\partial t} = 0 . \tag{1.167}$$

That we now want to analyze a bit more precisely. At first it follows for the total time differential

$$\frac{d}{dt}L = \sum_{j=1}^{S}\left(\frac{\partial L}{\partial q_j}\dot{q}_j + \frac{\partial L}{\partial \dot{q}_j}\ddot{q}_j\right) = \sum_{j=1}^{S}\left[\left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j}\right)\dot{q}_j + \frac{\partial L}{\partial \dot{q}_j}\ddot{q}_j\right]$$
$$= \frac{d}{dt}\sum_{j=1}^{S}\frac{\partial L}{\partial \dot{q}_j}\dot{q}_j \; ,$$

where we have exploited in the second step the Lagrange equations of motion (1.36):

$$\frac{d}{dt}\left(L - \sum_{j=1}^{S}\frac{\partial L}{\partial \dot{q}_j}\dot{q}_j\right) = 0 \; . \tag{1.168}$$

According to (1.52) $\partial L/\partial \dot{q}_j$ is just the generalized momentum p_j. We define already at this stage the so-called

Hamiltonian function

$$H = \sum_{j=1}^{S} p_j \dot{q}_j - L \; , \tag{1.169}$$

which we will deal with in a greater detail in the next section. It obviously represents according to (1.168), in case of temporal homogeneity of the system, an integral of motion:

$$\textbf{homogeneity of time} \iff \frac{\partial L}{\partial t} = 0 \; ,$$

$$\textbf{`system motion such that'} \quad H = \text{const} \; . \tag{1.170}$$

How can we interpret this conservation law? If we presume scleronomic constraints, or more precisely, transformation formulas $\mathbf{r}_i(\mathbf{q}, t)$ of the particle coordinates which do **not** explicitly depend on time,

$$\frac{\partial \mathbf{r}_i}{\partial t} \equiv 0 \; , \qquad i = 1, 2, \ldots, N \; ,$$

then the kinetic energy T according to (1.37) and (1.39) is a homogeneous function of second order of the generalized velocities $\dot{q}_j$, i.e.

$$T\left(a\dot{q}_1, \ldots, a\dot{q}_S\right) \equiv a^2 T\left(\dot{q}_1, \ldots, \dot{q}_S\right) \; .$$

That means for arbitrary real a:

$$\frac{\partial T}{\partial a} = \sum_{j=1}^{S} \frac{\partial T}{\partial \left(a\dot{q}_j\right)} \dot{q}_j = 2a\,T$$

or especially for $a = 1$:

$$\sum_{j=1}^{S} \frac{\partial T}{\partial \dot{q}_j} \dot{q}_j = 2T \; . \tag{1.171}$$

Since the considered system is assumed to be also conservative it holds additionally

$$\frac{\partial V}{\partial \dot{q}_j} = 0 \; , \qquad j = 1, \ldots, S \; . \tag{1.172}$$

It follows therewith

$$2T = \sum_j \frac{\partial T}{\partial \dot{q}_j} \dot{q}_j = \sum_j \frac{\partial L}{\partial \dot{q}_j} \dot{q}_j = \sum_j p_j \dot{q}_j \; .$$

In this case it thus holds for the Hamiltonian function:

$$H = T + V = E \qquad \Longleftrightarrow \qquad \textbf{total energy} \; .$$

Equation (1.170) then states that the energy conservation law for holonomic-scleronomic conservative systems is, in the last analysis, a consequence of the homogeneity of time.

Why was it necessary here to presume *scleronomic constraints*? Let us remember the characteristic difference between Newton's and Lagrange's version of mechanics. In Newton's mechanics all forces appear in the equations of motion, the constraint forces included, while in Lagrange's mechanics the constraint forces are eliminated. According to d'Alembert's principle constraint forces do not do work for virtual displacements. Virtual displacements differ from real ones by the additional requirement $\delta t = 0$. For scleronomic constraints therefore holds *virtual* = *real*, but not for rheonomic constraints. In the latter case constraint forces can *indeed* execute work which, however, do not appear in H since constraint forces are eliminated in the Lagrange formalism. The conservation law holds then only in the form of (1.170) H = const, but H can not be interpreted as total energy.

We illustrate the issue for the Example (2) in Sect. 1.2.2, the **gliding bead on a uniformly rotating rod** (Fig. 1.52). Besides the holonomic-scleronomic constraint

$$z = 0$$

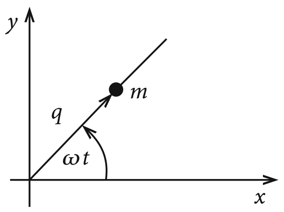


Fig. 1.52 Gliding bead on a rotating rod

there is also a holonomic-rheonomic condition:

$$y = x \tan \omega t \ .$$

Nevertheless, the Lagrangian (1.50)

$$L = T = \frac{m}{2} \left(\dot{q}^2 + q^2 \omega^2 \right)$$

is not explicitly time-dependent so that we have:

$$\frac{\partial L}{\partial t} = 0$$

That gives the conservation law:

$$H = p \, \dot{q} - L = \text{const} \ .$$

On the other hand, one finds:

$$H = \frac{\partial L}{\partial \dot{q}} \dot{q} - L = m \, \dot{q}^2 - \frac{1}{2} m \left(\dot{q}^2 + q^2 \omega^2 \right) = \frac{1}{2} m \left(\dot{q}^2 - q^2 \omega^2 \right)$$
$$\neq T = T + V = E \ .$$

Hence the above conservation law is **not** identical to the energy law!

1.4.2 Homogeneity of Space

A system is called *'spatially homogeneous'* if its properties are independent of its position, i.e. if a rigid shift of the system as a whole does not change the results of measurement. That, for instance, is the case when the considered system is subject exclusively to (internal) forces which depend only on the interparticle-**distances**.

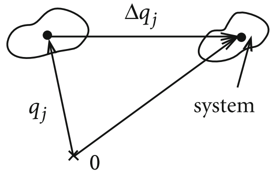


Fig. 1.53 Illustration of the cyclic coordinate which corresponds to the homogeneity of space

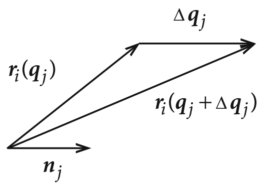


Fig. 1.54 Change of particle coordinates in consequence of a shift of the total system by $\Delta \mathbf{q}_j$ in the direction $\mathbf{n}_j$

The generalized coordinate q_j may be chosen in such a way that Δq_j means a translation of the total system (Fig. 1.53). That can be realized, e.g., by the Cartesian components of the center of mass. Then it follows as sufficient condition for spatial homogeneity:

$$\frac{\partial L}{\partial q_j} = 0 \,. \tag{1.173}$$

q_j is therefore cyclic leading to the conservation law:

$$p_j = \frac{\partial L}{\partial \dot{q}_j} = \text{const} \,. \tag{1.174}$$

But what is the physical meaning of p_j? Since the system is thought to be conservative it holds:

$$\frac{\partial V}{\partial \dot{q}_j} = 0$$

and therewith also:

$$p_j = \frac{\partial L}{\partial \dot{q}_j} = \frac{\partial T}{\partial \dot{q}_j} = \sum_{i=1}^{N} m_i \, \dot{\mathbf{r}}_i \, \frac{\partial \dot{\mathbf{r}}_i}{\partial \dot{q}_j} = \sum_{i=1}^{N} m_i \, \dot{\mathbf{r}}_i \, \frac{\partial \mathbf{r}_i}{\partial q_j} \,. \tag{1.175}$$

In the last step we have used (1.23).

$\mathbf{n}_j$ shall be the unit vector in the direction of the translation (Fig. 1.54). All particle coordinates are changed by the same constant vector:

$$\Delta \mathbf{q}_j = \Delta q_j \mathbf{n}_j \,.$$

It follows:

$$\frac{\partial \mathbf{r}_i}{\partial q_j} = \lim_{\Delta q_j \to 0} \frac{\mathbf{r}_i\left(q_j + \Delta q_j\right) - \mathbf{r}_i\left(q_j\right)}{\Delta q_j} = \lim_{\Delta q_j \to 0} \frac{\Delta q_j\, \mathbf{n}_j}{\Delta q_j} = \mathbf{n}_j \; . \tag{1.176}$$

Hence p_j is the component of the total momentum belonging to q_j in the direction of the translation $\mathbf{n}_j$:

$$p_j = \mathbf{n}_j \sum_{i=1}^{N} m_i\, \dot{\mathbf{r}}_i = \mathbf{n}_j \cdot \mathbf{P} \; . \tag{1.177}$$

Since $\mathbf{n}_j$ can be chosen arbitrarily it holds the following conservation law:
homogeneity of space $\Longleftrightarrow$ momentum conservation law

$$\mathbf{P} = \sum_{i=1}^{N} m_i\, \dot{\mathbf{r}}_i = \mathbf{const} \; . \tag{1.178}$$

Let us add a short discussion:

1. The generalized force component Q_j is related to the coordinate q_j by:

$$Q_j = \sum_{i=1}^{N} \mathbf{F}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} = \mathbf{n}_j \cdot \sum_{i=1}^{N} \mathbf{F}_i = \mathbf{n}_j \cdot \mathbf{F} \; . \tag{1.179}$$

Because of *'action* $=$ *reaction'* the *internal* forces (particle interactions) cancel each other so that $\mathbf{F}$ represents the total *external* force. In a conservative system it holds (1.29):

$$Q_j = -\frac{\partial V}{\partial q_j} \; .$$

Furthermore, (1.176) yields:

$$\frac{\partial \dot{\mathbf{r}}_i}{\partial q_j} = \frac{d}{dt}\frac{\partial \mathbf{r}_i}{\partial q_j} = \frac{d}{dt}\mathbf{n}_j = 0 \; .$$

This means:

$$\frac{\partial T}{\partial q_j} = 0$$

and therewith:

$$Q_j = \frac{\partial L}{\partial q_j} \; .$$

Because of (1.173) we have then:

$$Q_j = \mathbf{n}_j \cdot \mathbf{F} = \dot{p}_j = 0 \; . \tag{1.180}$$

This relation is fulfilled if

$$\mathbf{F} \equiv 0 \qquad \text{or} \qquad \mathbf{F} \perp \mathbf{n}_j$$

2. For external fields, which are finite but with given symmetries, the coordinate q_j can be cyclic for translations in certain space directions, namely for those for which $\mathbf{n}_j$ is orthogonal to $\mathbf{F}$ (see (1.172)). We have found therewith an important relationship:

momentum conservation in symmetry directions

Examples

(1) Field of an infinite homogeneous plane

Each point of the plane is the source of a spherically symmetric field so that after superposition of all contributions only a resultant z component remains finite. The force on particle i, executed by all points of the infinite (xy) plane, has therefore only a non-zero z component. That holds then of course also for the total force:

$$\mathbf{F} = \sum_{i=1}^{N} \mathbf{F}_i \equiv (0, 0, F) \; . \tag{1.181}$$

For $\mathbf{n}_j = \mathbf{e}_x, \mathbf{e}_y$ (1.180) is obviously fulfilled. That yields the integrals of motion:

$$P_x = \text{const} \; ; \qquad P_y = \text{const} \; . \tag{1.182}$$

(2) Field of an infinite homogeneous circular cylinder

The rotational symmetry around the cylinder axis suggests the use of cylindrical coordinates (see Sect. 1.7.3, Vol. 1):

$$\rho, \varphi, z \; : \qquad x = \rho \cos\varphi \; ; \quad y = \rho \sin\varphi \; ; \quad z = z \; ,$$

$$\begin{aligned} \mathbf{e}_\rho &= (\cos\varphi, \sin\varphi, 0) \; , \\ \mathbf{e}_\varphi &= (-\sin\varphi, \cos\varphi, 0) \; , \\ \mathbf{e}_z &= (0, 0, 1) \; . \end{aligned}$$

Since the circular cylinder shall be infinitely long and homogeneous the field will be independent of φ and z:

$$\mathbf{F}_i = F_i \, \mathbf{e}_{\rho_i} \implies \mathbf{F} = \sum_i \mathbf{F}_i = \left(F_x, F_y, 0\right) \ . \tag{1.183}$$

This means according to (1.177):

$$P_z = \text{const} \ . \tag{1.184}$$

1.4.3 Isotropy of Space

One calls a system *'spatially isotropic'* if the system properties do not change on arbitrary rotations. We now choose the generalized coordinate q_j such that Δq_j represents a rotation of the system by an angle $\Delta\varphi$ around the axial direction $\mathbf{n}_j$ (see Fig. 1.55):

$$|\Delta \mathbf{r}_i| = \Delta q_j \, r_i \, \sin \vartheta_i \ .$$

$\Delta \mathbf{r}_i$ is orthogonal to $\mathbf{r}_i$ and to $\mathbf{n}_j$. It therefore holds:

$$\Delta \mathbf{r}_i = \Delta q_j \, \mathbf{n}_j \times \mathbf{r}_i \ . \tag{1.185}$$

It follows as sufficient condition for spatial isotropy:

$$\frac{\partial L}{\partial q_j} = 0 \ . \tag{1.186}$$

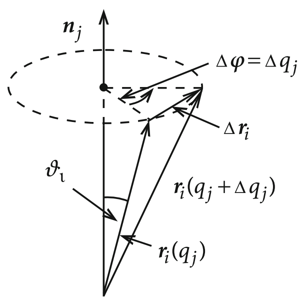


Fig. 1.55 Illustration of the cyclic coordinate due to the isotropy of space

The so defined coordinate q_j is therefore cyclic and leads to the conservation law:

$$p_j = \frac{\partial L}{\partial \dot{q}_j} = \text{const} \; . \tag{1.187}$$

What is the meaning of p_j? Since the system is again thought to be conservative equation (1.175) is valid. With

$$\frac{\partial \mathbf{r}_i}{\partial q_j} = \lim_{\Delta q_j \to 0} \frac{\Delta \mathbf{r}_i}{\Delta q_j} = \mathbf{n}_j \times \mathbf{r}_i \tag{1.188}$$

it follows:

$$p_j = \sum_{i=1}^{N} m_i \, \dot{\mathbf{r}}_i \cdot \left(\mathbf{n}_j \times \mathbf{r}_i\right) = \mathbf{n}_j \cdot \sum_{i=1}^{N} \left(\mathbf{r}_i \times m_i \, \dot{\mathbf{r}}_i\right) \; .$$

Hence p_j is the component of the total angular momentum $\mathbf{L}$ in n_j direction:

$$p_j = \mathbf{n}_j \cdot \sum_{i=1}^{N} \mathbf{L}_i = \mathbf{n}_j \cdot \mathbf{L} \; . \tag{1.189}$$

Since the axial direction $\mathbf{n}_j$ can be chosen arbitrarily we come to the conclusion:
isotropy of space $\Longleftrightarrow$ angular momentum conservation law

$$\mathbf{L} = \sum_{i=1}^{N} m_i \, \mathbf{r}_i \times \dot{\mathbf{r}}_i = \mathbf{const} \; . \tag{1.190}$$

We want to further comment briefly on this result:

1. The coordinate q_j is related to the force component Q_j for which one finds with (1.188):

$$Q_j = \sum_i \mathbf{F}_i \cdot \left(\mathbf{n}_j \times \mathbf{r}_i\right) = \mathbf{n}_j \cdot \sum_i \left(\mathbf{r}_i \times \mathbf{F}_i\right) = \mathbf{n}_j \cdot \sum_i \mathbf{M}_i = \mathbf{n}_j \cdot \mathbf{M} \tag{1.191}$$

   Thus it is the component of the total torque in rotational direction $\mathbf{n}_j$.
   Because of

$$\frac{\partial T}{\partial q_j} = \sum_i m_i \, \dot{\mathbf{r}}_i \cdot \frac{\partial \dot{\mathbf{r}}_i}{\partial q_j} = \sum_i m_i \, \dot{\mathbf{r}}_i \left(\frac{d}{dt} \frac{\partial \mathbf{r}_i}{\partial q_j}\right) = \sum_i m_i \, \dot{\mathbf{r}}_i \cdot \left(\mathbf{n}_j \times \dot{\mathbf{r}}_i\right) = 0$$

it follows with (1.186):

$$Q_j = -\frac{\partial V}{\partial q_j} = \frac{\partial L}{\partial q_j} = 0 \ . \tag{1.192}$$

We see that according to (1.191) and (1.192) spatial isotropy is equivalent to the disappearance of the total moment of rotation (torque) $\mathbf{M}$ acting on the system.

2. In case of incomplete spatial isotropy equation (1.192) can nevertheless be fulfilled, namely when the external fields exhibit symmetries so that $\mathbf{M}$ is orthogonal to certain directions in space $\mathbf{n}_j$. Again we illustrate this fact by some examples:

(a) Field of an Infinite Homogeneous Plane

As explained before (see (1.181)) the force on particle i reads:

$$\mathbf{F}_i \equiv (0, 0, F_i) \ .$$

That leads to

$$\mathbf{M}_i = \mathbf{r}_i \times \mathbf{F}_i \perp \mathbf{e}_z$$

yielding eventually the conservation law:

$$L_z = \text{const} \ . \tag{1.193}$$

(b) Field of an Infinite Homogeneous Circular Cylinder

As in (1.183) we use cylindrical coordinates for the representation of the force $\mathbf{F}_i$ which acts on particle i:

$$\mathbf{r}_i = (\rho_i \cos\varphi_i,\ \rho_i \sin\varphi_i,\ z_i) = \rho_i \mathbf{e}_{\rho_i} + z_i \mathbf{e}_z \ , \tag{1.194}$$

$$\mathbf{F}_i = F_{i\rho} \mathbf{e}_{\rho_i} = F_{i\rho} (\cos\varphi_i,\ \sin\varphi_i,\ 0) \ . \tag{1.195}$$

Though the torque $\mathbf{M}$ is not at all zero,

$$\mathbf{M} = \sum_i (\mathbf{r}_i \times \mathbf{F}_i) = \sum_i z_i F_{i\rho} (-\sin\varphi_i,\ \cos\varphi_i,\ 0) \ ,$$

its z component vanishes:

$$\mathbf{e}_z \cdot \mathbf{M} = 0 \ . \tag{1.196}$$

That leads to the conservation law:

$$L_z = \text{const} \, . \tag{1.197}$$

(c) Field of a Homogeneous Circular Ring (Annulus)

We choose the axis of the ring as z axis. Then the field must be rotationally symmetric to the z axis so that again cylindrical coordinates are recommendable. The force $\mathbf{F}_i$ acting on particle i cannot have a finite φ component:

$$\mathbf{F}_i = F_{i\rho}\mathbf{e}_\rho + F_{iz}\mathbf{e}_z = \left(F_{i\rho}\,\cos\varphi_i,\, F_{i\rho}\,\sin\varphi_i,\, F_{iz}\right) \, . \tag{1.198}$$

It holds with (1.194):

$$\begin{aligned}\mathbf{r}_i \times \mathbf{F}_i &= \left(\rho_i\mathbf{e}_{\rho_i} + z_i\mathbf{e}_z\right) \times \left(F_{i\rho}\mathbf{e}_{\rho_i} + F_{iz}\mathbf{e}_z\right)\\ &= \left(-\rho_i F_{iz} + z_i F_{i\rho}\right)\, \mathbf{e}_{\varphi_i}\end{aligned}$$

That means for the torque:

$$\mathbf{M} = \sum_i \left(\mathbf{r}_i \times \mathbf{F}_i\right) \equiv \left(M_x, M_y, 0\right) \, , \tag{1.199}$$

so that also in this case (1.196) and (1.197) are valid.

1.4.4 Exercises

Exercise 1.4.1 Consider a conservative system with holonomic constraints. Furthermore let there exist a one-to-one mapping of coordinates:

$$\begin{aligned}\mathbf{q} &\longrightarrow \mathbf{q}' = \mathbf{q}'(\mathbf{q}, t, \alpha)\\ \mathbf{q}' &\longrightarrow \mathbf{q} = \mathbf{q}(\mathbf{q}', t, \alpha) \, .\end{aligned}$$

Thereby α may be a continuously adjustable parameter. The transformation formulas are continuously differentiable with respect to this parameter. For $\alpha = 0$ we have the identity transformation $\mathbf{q}'(\mathbf{q}, t, \alpha = 0) = \mathbf{q}$. Insertion of the transformation formulas into the Lagrangian yields:

$$L\left(\mathbf{q}, \dot{\mathbf{q}}, t\right) = L\left(\mathbf{q}(\mathbf{q}', t, \alpha), \dot{\mathbf{q}}(\mathbf{q}', \dot{\mathbf{q}}', t, \alpha), t\right) \equiv L'\left(\mathbf{q}', \dot{\mathbf{q}}', t, \alpha\right) \, .$$

We now require that the transformation is just such that the Lagrangian remains invariant, i.e.

$$L'\left(\mathbf{q}', \dot{\mathbf{q}}', t, \alpha\right) = L\left(\mathbf{q}', \dot{\mathbf{q}}', t\right) \qquad \forall\, \alpha\ .$$

Show then that

$$I\left(\mathbf{q}, \dot{\mathbf{q}}, t\right) = \sum_{j=1}^{S} \frac{\partial L}{\partial \dot{q}_j} \left.\frac{\partial q_j\left(\mathbf{q}', t, \alpha\right)}{\partial \alpha}\right|_{\alpha=0}$$

represents an integral of motion (Noether's theorem)!

Exercise 1.4.2 A particle of mass m is described by the Lagrangian:

$$L\left(\mathbf{q}, \dot{\mathbf{q}}\right) = \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - V\left(x^2 + y^2, z\right)$$

Show that L remains invariant on a rotation around the z axis (rotation angle α). Find then with the result of Exercise 1.4.1 an integral of motion!

Exercise 1.4.3 As in Exercise 1.4.1 we consider a conservative system with holonomic constraints. There exists again a one-to-one transformation of coordinates with a continuous parameter α:

$$\begin{aligned} \mathbf{q} &\longrightarrow \quad \mathbf{q}' = \mathbf{q}'(\mathbf{q}, t, \alpha) \\ \mathbf{q}' &\longrightarrow \quad \mathbf{q} = \mathbf{q}(\mathbf{q}', t, \alpha) \end{aligned}$$

The transformation formulas are continuously differentiable with respect to α. For $\alpha = 0$ it is the identity transformation. Insertion of the transformation formulas into the Lagrangian yields:

$$L\left(\mathbf{q}, \dot{\mathbf{q}}, t\right) = L\left(\mathbf{q}\left(\mathbf{q}', t, \alpha\right), \dot{\mathbf{q}}\left(\mathbf{q}', \dot{\mathbf{q}}', t, \alpha\right), t\right) \equiv L'\left(\mathbf{q}', \dot{\mathbf{q}}', t, \alpha\right)\ .$$

1. Now let the transformation change the Lagrangian in the following manner:

$$L'\left(\mathbf{q}', \dot{\mathbf{q}}', t, \alpha\right) = L\left(\mathbf{q}', \dot{\mathbf{q}}', t\right) + \frac{d}{dt} f\left(\mathbf{q}', t, \alpha\right)\ .$$

Thereby $f(\mathbf{q}', t, \alpha)$ can be an arbitrary, but sufficiently often differentiable function in all variables (*mechanical gauge transformation* (1.84)). Show then that

$$\widehat{I}(\mathbf{q}, \dot{\mathbf{q}}, t) = \sum_{j=1}^{S} \frac{\partial L}{\partial \dot{q}_j} \left.\frac{\partial q_j(\mathbf{q}', t, \alpha)}{\partial \alpha}\right|_{\alpha=0} - \left.\frac{\partial}{\partial \alpha} f(\mathbf{q}', t, \alpha)\right|_{\alpha=0}$$

represents an integral of motion!

2. Consider as an application example the free fall of the mass m in the homogeneous earth's gravitational field: $L(x, \dot{x}) = \frac{m}{2}\dot{x}^2 - mgx$. Show that the Galilei transformation

$$x \longrightarrow x' = x + \alpha t$$

fulfills the preconditions of part 1. and find the corresponding conserved quantity!

1.5 Self-examination Questions

To Section 1.1

1. What does one understand by constraints, what are constraint forces?
2. Which difficulties arise by the existence of constraints when treating a mechanical problem?
3. What are holonomic, holonomic-scleronomic, holonomic-rheonomic, non-holonomic constraints?
4. Which conditions are to be fulfilled by generalized coordinates?
5. How is the configuration space defined?

To Section 1.2

1. What is understood by a virtual displacement and by virtual work?
2. Formulate the principle of virtual work!
3. Why are friction forces not counted as constraint forces?
4. What are generalized force components?
5. What does the d'Alembert's principle state?
6. Under which conditions do the Lagrange equations of the second kind follow from the d'Alembert's principle?
7. How do the Lagrange equations behave under point transformations?
8. How is a generalized momentum defined?
9. What is a cyclic coordinate?
10. How does the parameter representation of the cycloid look like?
11. Of which type are the equations of motion which follow from the d'Alembert's principle for non-conservative systems with holonomic constraints?

12. Which conditions must be fulfilled by *'generalized potentials'*? Can they depend even on generalized velocities?
13. Which Lagrangian is found for the charged particle in an electromagnetic field? What can be said about its generalized momentum?
14. How does the Lagrangian of a charged particle behave when being subject to a gauge transformation $\varphi \to \varphi - (\partial/\partial t)\chi$; $\mathbf{A} \to \mathbf{A} + \nabla\chi$? What does thereby happen to the equations of motion?
15. What do we understand by a mechanical gauge transformation?
16. How are systems to be described which are influenced by friction forces?
17. Which physical meaning is ascribed to the dissipation function?
18. Explain the method of the Lagrange multipliers!
19. What are Lagrange equations of motion of the first kind?
20. Which physical meaning can be ascribed to the Lagrange multipliers?

To Section 1.3

1. Comment on the difference between differential and integral principles!
2. What do we understand by a configuration path?
3. Formulate Hamilton's principle! What kind of paths are admitted to the variational process?
4. What is an action functional?
5. Explain the term *variation of the functional* $J\{y(x)\}$.
6. Write down Euler's equations and outline their derivation!
7. What is the brachistochrone problem?
8. How does one derive from the Hamilton's principle the Lagrange equations of motion of the first kind for conservative systems with non-holonomic constraints?
9. How does the Hamilton's principle read for non-conservative systems? Which action functional is then to be varied?
10. Of which kind are the equations of motion which follow from the *'extended'* Hamilton principle?

To Section 1.4

1. What is an *integral of motion*?
2. Why is it convenient in the Lagrange formulation of a problem in physics to choose the generalized coordinates such that as many of them as possible are cyclic?
3. When is a system called *temporally homogeneous*? What does it then hold for the Lagrangian?
4. How is the Hamilton function defined?
5. Which conservation law follows from the temporal homogeneity of a physical system?
6. Under which conditions is the Hamilton function identical to the total energy of the system?
7. When can a system be termed *spatially homogeneous*? What does it then hold for the Lagrangian?

8. Which conservation law is a consequence of the homogeneity of space?
9. Which connection does exist between momentum conservation and symmetry directions?
10. How does *spatial isotropy* manifest itself in the Lagrangian of a physical system?
11. Which conservation law is due to spatial isotropy? What must hold for the total torque?
12. Which symmetry condition must be addressed to the force acting on particle i in order to guarantee that the x component of the angular momentum is an integral of motion?

Chapter 2
Hamilton Mechanics

This chapter goes in for a

further development of the theory of Classical Mechanics

The main goal is thereby not so much the evolution and presentation of new auxiliary calculation tools. Furthermore, we will see that the Hamiltonian version of Classical Mechanics does *not provide any new physics*. Its range of validity and application corresponds namely rather exactly to that of the Lagrangian version. What it is about is rather to gain a deeper insight into the formal mathematical structure of the physical theory, and to investigate all thinkable reformulations of the basic principles. Aside from that we have to bear in mind that Classical Mechanics as any other physical theories possesses only a restricted range of validity which is not 'a priori' clear, however, the representation will turn out to be especially convenient for subsequent generalizations. Concept formations and mathematical correlations of the Hamiltonian formalism will prove to be helpful for a connection to the principles of the superordinate Quantum Mechanics. In the last analysis, that is the decisive motivation to deal with the Hamiltonian version of Classical Mechanics.

As a certain *'review of situation'*, let us contrast the various concepts which we discussed so far. The *Newtonian Mechanics* represents a very general concept. All types of forces are admitted and involved. The solutions of the equations of motion manifest themselves very descriptively as *particle trajectories*. The Newtonian Mechanics is, however, valid only in inertial systems. In non-inertial systems suitable *pseudo forces* must be introduced. The rather *'cumbersome'* constraint forces are to be considered explicitly in the equations of motion. Furthermore, the Newton equations turn out not to be form-invariant with respect to coordinate transformations.

The *Lagrangian Mechanics*, on the other hand, is valid in all systems of coordinates. Its special advantage lies in the fact that the *'cumbersome'* constraint forces are eliminated. The Lagrange equations of motion turn out to be form-invariant with respect to point transformations. They are derived from basic principles, the

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differential principle of d'Alembert or the integral principle of Hamilton, which replace Newton's axioms. In holonomic and conservative systems they are about S differential equations of second order for S generalized coordinates $q_1, \ldots, q_S$, for the solution of which $2S$ initial conditions must be given. Since the generalized coordinates can be arbitrary physical quantities, i.e. not necessarily with the dimension *'length'*, the solutions of the equations of motion are correspondingly less descriptive than those of the Newton equations. Only after a back transformation to particle coordinates $\mathbf{r}_1, \ldots, \mathbf{r}_N$ one gets the classical *particle trajectories*. That may be seen as a certain disadvantage of the Lagrangian formalism, as well as the fact that there does not exist a unique prescription for the treatment of all thinkable types of constraints.

The *Hamiltonian Mechanics*, which we are now going to discuss, shall construct a bridge between the classical theories and the non-classical ones (Quantum Mechanics, Statistical Mechanics). The most important result will be the finding that Classical Mechanics and Quantum Mechanics can be regarded as different realizations of one and the same superordinate abstract mathematical structure. When going over from the Lagrangian to the Hamiltonian formalism the generalized velocities are replaced by the generalized momenta:

$$(\mathbf{q}, \dot{\mathbf{q}}, t) \;\Rightarrow\; (\mathbf{q}, \mathbf{p}, t) \;.$$

$\mathbf{q}$ and $\mathbf{p}$ are considered as variables which are *independent* of each other. A consequence of these transformations will be a set of $2S$ differential equations of **first** order for S generalized coordinates $q_1, \ldots, q_S$ and S generalized momenta $p_1, \ldots, p_S$. The number of initial conditions necessary for a full solution thus remains unchanged to be $2S$. As method for the change of coordinates the so-called *Legendre transformation* is chosen. Its technique will be introduced in the next section.

2.1 Legendre Transformation

We discuss as mathematical interlude a procedure for the transformation of the variables which is important for theoretical physics:

Let a function $f = f(x)$ be given with the differential

$$df = \frac{df}{dx} dx = u\, dx \;.$$

and we look for a function $g = g(u)$ for which it holds:

$$\frac{dg}{du} = \pm x$$

This is found very easily as follows:

$$\begin{aligned} df &= u\,dx = d(ux) - x\,du \\ \Longrightarrow\ d(f-ux) &= -x\,du \ \Longrightarrow\ \frac{d}{du}(f-ux) = -x\,. \end{aligned}$$

One therefore defines:
Legendre transform of $f(x)$:

$$g(u) = f(x) - ux = f(x) - x\frac{df}{dx}\,. \tag{2.1}$$

Why is the transformation of the variables not simply carried out *'by insertion'*? The following example makes clear that then the transformation would not be a one-to-one mapping. The transformation

$$\frac{df}{dx} = u(x) \ \Longrightarrow\ x = x(u) \ \Longrightarrow\ \tilde{f}(u) = f\,(x(u))$$

would for instance mean that the two functions

$$f(x) = \alpha\,x^2 \qquad \text{and} \qquad \bar{f}(x) = \alpha(x+c)^2$$

both have the same transform $\tilde{f}(u)$:

$$\left.\begin{array}{l} u = \dfrac{df}{dx} = 2\,\alpha\,x \\[2ex] \bar{u} = \dfrac{d\bar{f}}{dx} = 2\,\alpha(x+c) \end{array}\right\} \Longrightarrow \left.\begin{array}{l} x = \dfrac{u}{2\alpha} \\[2ex] x = \dfrac{\bar{u}}{2\alpha} - c \end{array}\right\} \Longrightarrow \begin{array}{l} \tilde{f}(u) = \dfrac{u^2}{4\alpha} \\[2ex] \tilde{f}(\bar{u}) = \dfrac{\bar{u}^2}{4\alpha} \end{array}\,.$$

Hence the back-transformation cannot be unique. In contrast, a Legendre transformation is unique as can be read off from the following pattern:

$$\begin{array}{ccccc} & f(x) & = & g(u)-u\dfrac{dg}{du} & \\ & \downarrow & & \uparrow & \nwarrow \\ x=x(u) \longleftarrow & u=\dfrac{df}{dx} & & -x=\dfrac{dg}{du} & \longrightarrow u=u(x) \\ \searrow & \downarrow & & \uparrow & \\ & f(x)-x\dfrac{df}{dx} & = & g(u) & \end{array} \tag{2.2}$$

Obviously this pattern is applicable only if additionally

$$\frac{d^2 f}{dx^2} \neq 0 \tag{2.3}$$

holds because otherwise u cannot really be a variable. From $\left(d^2 f\right)/\left(dx^2\right) = 0$ it would namely follow $(df)/(dx) = u = \text{const.}$ In the above pattern (2.2) no point is special. The back-transformation is therefore unique.

Let us extend the theory to functions of two variables. It is now given the function

$$f = f(x, y) \implies df = u(x, y)\, dx + v(x, y)\, dy \;,$$

where it holds:

$$u(x, y) = \left(\frac{\partial f}{\partial x}\right)_y \;, \quad v(x, y) = \left(\frac{\partial f}{\partial y}\right)_x \;. \tag{2.4}$$

We want to find another function

$$g = g(x, v) \implies dg = u\, dx - y\, dv$$

with

$$u\,(x, y(x, v)) = \left(\frac{\partial g}{\partial x}\right)_v \;, \quad y(x, v) = -\left(\frac{\partial g}{\partial v}\right)_x \;. \tag{2.5}$$

One denotes x as the *passive* and y as the *active* variable. The function $g(x, v)$ we are looking for is found as follows:

$$\begin{aligned} df &= u\, dx + v\, dy = u\, dx + d(vy) - y\, dv \\ &\implies d(f - vy) = u\, dx - y\, dv \\ &\implies \left(\frac{\partial(f - vy)}{\partial x}\right)_v = u \;, \quad \left(\frac{\partial(f - vy)}{\partial v}\right)_x = -y \;. \end{aligned}$$

One now defines:

Legendre transform of $f(x,y)$ with respect to y :

$$g(x, v) = f(x, y) - vy = f(x, y) - y\left(\frac{\partial f}{\partial y}\right)_x \tag{2.6}$$

The transformation pattern (2.2) is only slightly to be changed:

$$\begin{array}{ccccc}
 & f(x,y) & = & g(x,v) - v\left(\dfrac{\partial g}{\partial v}\right)_x & \\
 & \downarrow & & \uparrow & \nwarrow \\
y = y(x,v) \longleftarrow & v = \left(\dfrac{\partial f}{\partial y}\right)_x & & -y = \left(\dfrac{\partial g}{\partial v}\right)_x & \longrightarrow v = v(x,y) \\
\searrow & \downarrow & & \uparrow & \\
 & f(x,y) - y\left(\dfrac{\partial f}{\partial y}\right)_x & = & g(x,v) &
\end{array} \tag{2.7}$$

The generalization of the algorithm to more than two variables is obvious.

2.1.1 Exercises

Exercise 2.1.1 Determine the Legendre transform

1. $g(u)$ of the function $f(x) = \alpha\, x^2$,
2. $g(x, v)$ of the function $f(x, y) = \alpha\, x^2 y^3$.

Exercise 2.1.2 Determine the Legendre transform

1. $g(u)$ of the function $f(x) = \alpha(x + \beta)^2 \quad (\alpha, \beta :$ constants)
2. $g(x, v)$ of the function

$$f(x, y) = \alpha x^3 y^5 \ .$$

For checking purposes perform the back-transformation!

Exercise 2.1.3 A frequent application of the Legendre transformation is found in thermodynamics (see Vol. 4 of this basic course), e.g. in connection with calculation and disposition of the *'thermodynamic potentials'*. These are energy quantities which as functions of their so-called *'natural variables'* exhibit some useful and special properties. The *internal energy* of a gas U, e.g., possesses as natural variables the entropy S and the volume V. A change of the internal energy is calculated according to

$$dU = TdS - pdV \ .$$

p is the pressure and T the temperature of the gas. Since S and V are not always optimal variables for experimental targets, alternative potentials are brought into play:

1. free energy: $F = F(T, V)$
2. enthalpy: $H = H(S, p)$
3. free enthalpy: $G = G(T, p)$

These potentials differ from each other and from U by proper Legendre transformations. Find out the connections of the potentials F, H, G with U and determine the partial derivatives with respect to their natural variables!

2.2 Canonical Equations of Motion

2.2.1 Hamilton Function

We transform the Lagrangian,

$$L = L\,(q_1, \ldots, q_S,\, \dot{q}_1, \ldots, \dot{q}_S, t) \ ,$$

considering the generalized velocities $\dot{q}_1, \ldots, \dot{q}_S$ as active variables and replace them by the generalized momenta

$$p_i = \frac{\partial L}{\partial \dot{q}_i} \ , \qquad i = 1, \ldots, S$$

The Legendre-transform we got to know already in (1.169), except for the sign, as **Hamilton function**

$$H\,(q_1, \ldots, q_S,\, p_1, \ldots, p_S,\, t) = \sum_{i=1}^{S} p_i \dot{q}_i - L\,(q_1, \ldots, q_S,\, \dot{q}_1, \ldots, \dot{q}_S,\, t) \ . \tag{2.8}$$

We have seen in Sect. 1.4.1 that there exists a close relationship between this function and the energy of the system. But before we come back to this point let us first derive the equations of motion which are connected to the Hamilton function H. For this purpose we build the total differential:

$$\begin{aligned} dH &= \sum_{i=1}^{S} (dp_i\, \dot{q}_i + p_i\, d\dot{q}_i) - \sum_{i=1}^{S} \left(\frac{\partial L}{\partial q_i}\, dq_i + \frac{\partial L}{\partial \dot{q}_i}\, d\dot{q}_i \right) - \frac{\partial L}{\partial t}\, dt \\ &= \sum_{i=1}^{S} \left(\dot{q}_i\, dp_i - \frac{\partial L}{\partial q_i}\, dq_i \right) - \frac{\partial L}{\partial t}\, dt \ . \end{aligned}$$

We still exploit the Lagrange equations:

$$dH = \sum_{i=1}^{S} (\dot{q}_i \, dp_i - \dot{p}_i \, dq_i) - \frac{\partial L}{\partial t} \, dt \; . \tag{2.9}$$

On the other hand it of course also holds:

$$dH = \sum_{i=1}^{S} \left(\frac{\partial H}{\partial p_i} \, dp_i + \frac{\partial H}{\partial q_i} \, dq_i \right) + \frac{\partial H}{\partial t} \, dt \; . \tag{2.10}$$

Since q_i, p_i, t are independent coordinates the direct comparison of (2.9) and (2.10) yields:

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \; , \qquad i = 1, \ldots, S \; , \tag{2.11}$$

$$\dot{p}_i = -\frac{\partial H}{\partial q_i} \; , \qquad i = 1, \ldots, S \; , \tag{2.12}$$

$$-\frac{\partial L}{\partial t} = \frac{\partial H}{\partial t} \; . \tag{2.13}$$

We have therewith found

Hamilton's equations of motion

which are also called

canonical equations (of motion)

These are $2S$ equations of motion, of first order in the time, which supersede the S Lagrange equations which are of second order. One should keep in mind the remarkable symmetry of the equations with respect to the q_i and the p_i. They describe the motion of the system in the $2S$-dimensional

phase space

whose axes are defined by the generalized coordinates q_i and the generalized momenta p_i.

We should concern ourselves a little bit more with the physical meaning of the Hamilton function. Thereto we remember the general structure (1.41) of the Lagrangian L:

$$L = T - V = L_2 + L_1 + L_0 \; .$$

The L_i are here homogeneous functions of the generalized velocities $\dot{q}_j$ of order i (1.45). That means (see (1.171)):

$$\sum_{j=1}^{S} \frac{\partial L}{\partial \dot{q}_j} \dot{q}_j = 2L_2 + L_1 \; . \tag{2.14}$$

It follows then with (2.8) for the Hamilton function:

$$H = L_2 - L_0 \; . \tag{2.15}$$

Hence it does not contain the term L_1. In case of **scleronomic constraints** (more exactly, for $\partial \mathbf{r}_i / \partial t \equiv 0$) one finds according to (1.38) and (1.39) $\alpha = \alpha_j = 0$. That leads to:

$$L_0 = -V \; , \quad L_1 = 0 \; , \quad L_2 = T \; . \tag{2.16}$$

H is then identical to the total energy:

$$H = T + V = E \; . \tag{2.17}$$

Because of the missing term L_1 this is no longer true in case of **rheonomic constraints**, which give rise to $\partial \mathbf{r}_i / \partial t \neq 0$.
For the total time differential of H we get:

$$\frac{dH}{dt} = \sum_{j=1}^{S} \left\{ \frac{\partial H}{\partial q_j} \dot{q}_j + \frac{\partial H}{\partial p_j} \dot{p}_j \right\} + \frac{\partial H}{\partial t} = \sum_{j=1}^{S} \left\{ \frac{\partial H}{\partial q_j} \frac{\partial H}{\partial p_j} - \frac{\partial H}{\partial p_j} \frac{\partial H}{\partial q_j} \right\} + \frac{\partial H}{\partial t} \; .$$

Total and partial derivatives of H with respect to the time are obviously the same:

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t} \; . \tag{2.18}$$

H is thus an *integral of motion* if there is no explicit time-dependence:

$$H = \text{const} \iff \frac{\partial H}{\partial t} = 0 \; . \tag{2.19}$$

According to (2.17) this is the energy conservation law if there are no rheonomic constraints. If, however, they do exist then we have $L_1 \neq 0$ with the consequence that H is not the total energy.

The Hamiltonian formalism becomes especially expedient when cyclic coordinates are present. We remember:

$$q_j \quad \text{cyclic} \iff \frac{\partial L}{\partial q_j} = 0 \iff p_j = \text{const} = c_j \ . \tag{2.20}$$

But this also brings about

$$\dot{p}_j = 0 = -\frac{\partial H}{\partial q_j} \ , \tag{2.21}$$

so that a cyclic coordinate q_j does not appear in H, either. The corresponding momentum $p_j = c_j$ is not an actual variable being fixed instead by initial conditions. H just contains only $(2S-2)$ variables, the number of degrees of freedom has practically dropped from S to $(S-1)$:

$$H = H\left(q_1, \ldots, q_{j-1},\, q_{j+1}, \ldots, q_S,\, p_1, \ldots, p_{j-1}, p_{j+1}, \ldots, p_S,\, t|c_j\right) \ . \tag{2.22}$$

In contrast, the Lagrangian L still contains **all** $\dot{q}_j$, the number of the degrees of freedom remains unchanged:

$$L = L\left(q_1, \ldots, q_{j-1},\, q_{j+1}, \ldots, q_S,\, \dot{q}_1, \ldots, \dot{q}_S,\, t\right) \ . \tag{2.23}$$

As regards the computational aspect one can say that the Hamiltonian formalism offers a real advantage compared to the Lagrangian formalism, strictly speaking, only if cyclic coordinates are present. The so-called

Routh-formalism

thus takes in a certain sense an intermediate position between Lagrangian and Hamiltonian formalism because it performs the Legendre transformation $\{\mathbf{q}, \dot{\mathbf{q}}, t\} \rightarrow \{\mathbf{q}, \mathbf{p}, t\}$ only for cyclic coordinates since only then an advantage is in evidence. Let

$$q_1, q_2, \ldots, q_n \qquad \text{be cyclic coordinates} \ ,$$

then $\dot{q}_1, \ldots, \dot{q}_n$ are the active and $q_1, \ldots, q_S, \dot{q}_{n+1}, \ldots, \dot{q}_S, t$ the passive transformation variables. That leads to the
Routh function

$$\begin{aligned} &R\left(q_1, \ldots, q_S,\, p_1, \ldots, p_n,\, \dot{q}_{n+1}, \ldots, \dot{q}_S,\, t\right) \\ &= \sum_{i=1}^{n} \left(\frac{\partial L}{\partial \dot{q}_i}\right) \dot{q}_i - L = \sum_{i=1}^{n} p_i \dot{q}_i - L = H - \sum_{i=n+1}^{S} p_i \dot{q}_i \ . \end{aligned} \tag{2.24}$$

For $n = S$ of course $R = H$ and for $n = 0$ $R = -L$. We determine the equations of motion again via the total differential, now of the Routh function:

$$dR = \sum_{i=1}^{S} \left(\frac{\partial R}{\partial q_i}\right) dq_i + \sum_{i=1}^{n} \left(\frac{\partial R}{\partial p_i}\right) dp_i + \sum_{i=n+1}^{S} \left(\frac{\partial R}{\partial \dot{q}_i}\right) d\dot{q}_i + \left(\frac{\partial R}{\partial t}\right) dt$$

$$= \sum_{i=1}^{n} (p_i\, d\dot{q}_i + \dot{q}_i\, dp_i) - \sum_{i=1}^{S} \left(\frac{\partial L}{\partial q_i}\right) dq_i - \sum_{i=1}^{S} \left(\frac{\partial L}{\partial \dot{q}_i}\right) d\dot{q}_i - \left(\frac{\partial L}{\partial t}\right) dt$$

$$= \sum_{i=1}^{n} \dot{q}_i\, dp_i - \sum_{i=1}^{S} \left(\frac{\partial L}{\partial q_i}\right) dq_i - \sum_{i=n+1}^{S} \left(\frac{\partial L}{\partial \dot{q}_i}\right) d\dot{q}_i - \left(\frac{\partial L}{\partial t}\right) dt\ .$$

By equating coefficients we get:

$$\frac{\partial R}{\partial p_i} = \dot{q}_i\ , \qquad i = 1, \ldots, n\ , \tag{2.25}$$

$$\frac{\partial R}{\partial q_i} = -\frac{\partial L}{\partial q_i} = -\dot{p}_i\ , \qquad i = 1, \ldots, n\ , \tag{2.26}$$

$$\frac{\partial R}{\partial t} = -\frac{\partial L}{\partial t}\ . \tag{2.27}$$

That corresponds for the cyclic coordinates to Hamilton's equations of motion.

$$\frac{\partial R}{\partial q_i} = -\frac{\partial L}{\partial q_i} = -\dot{p}_i\ , \qquad i = n+1, \ldots, S\ , \tag{2.28}$$

$$\frac{\partial R}{\partial \dot{q}_i} = -\frac{\partial L}{\partial \dot{q}_i} = -p_i\ , \qquad i = n+1, \ldots, S\ . \tag{2.29}$$

The last two equations can be combined to

$$\frac{d}{dt}\frac{\partial R}{\partial \dot{q}_i} - \frac{\partial R}{\partial q_i} = 0\ , \qquad i = n+1, \ldots, S \tag{2.30}$$

Thus the non-cyclic coordinates obey Lagrange's equations of motion.

Since it holds $\partial L/\partial q_i = 0$ for $i = 1, \ldots, n$ it is also true that

$$\frac{\partial R}{\partial q_i} = -\dot{p}_i = 0 \iff p_i = \text{const}_i = c_i\ . \tag{2.31}$$

Thus cyclic coordinates do not appear neither in L or H nor in R. The corresponding momenta occur only as parameters fixed by initial conditions:

$$R = R\left(q_{n+1}, \ldots, q_S,\ \dot{q}_{n+1}, \ldots, \dot{q}_S,\ t | c_1, \ldots, c_n\right)\ . \tag{2.32}$$

The Routh formalism does not bring about any decisive computational advantage compared with the Hamiltonian version. It therefore did not become generally accepted. In the framework of our theory course here we will not further go into it.

2.2.2 *Simple Examples*

The theory of the last section for the solution of problems in mechanics in the framework of the Hamiltonian formalism can be summarized by the following scheme:

1. Selection of proper generalized coordinates:

$$\mathbf{q} \equiv (q_1, q_2, \dots, q_S) \ .$$

2. Preparation of the transformation formulas:

$$\begin{aligned} \mathbf{r}_i &= \mathbf{r}_i(q_1, \dots, q_S, t) \ , \\ \dot{\mathbf{r}}_i &= \dot{\mathbf{r}}_i(\mathbf{q}, \dot{\mathbf{q}}, t) \ . \end{aligned} \qquad i = 1, 2, \dots, N \ .$$

3. Formulation of kinetic and potential energy as functions of the particle coordinates, then insertion of 2.:

$$L(\mathbf{q}, \dot{\mathbf{q}}, t) = T(\mathbf{q}, \dot{\mathbf{q}}, t) - V(\mathbf{q}, t) \qquad \text{(conservative system)} \ .$$

4. Derivation of the generalized momenta:

$$p_j = \frac{\partial L}{\partial \dot{q}_j} \implies p_j = p_j(\mathbf{q}, \dot{\mathbf{q}}, t) \ , \qquad j = 1, 2, \dots, S \ .$$

5. Solving for $\dot{q}_j$:

$$\dot{q}_j = \dot{q}_j(\mathbf{q}, \mathbf{p}, t) \ , \qquad j = 1, 2, \dots, S \ .$$

6. Lagrangian:

$$L(\mathbf{q}, \dot{\mathbf{q}}(\mathbf{q}, \mathbf{p}, t), t) = \tilde{L}(\mathbf{q}, \mathbf{p}, t) \ .$$

7. Legendre transformation:

$$H(\mathbf{q}, \mathbf{p}, t) = \sum_{j=1}^{S} p_j \, \dot{q}_j(\mathbf{q}, \mathbf{p}, t) - \tilde{L}(\mathbf{q}, \mathbf{p}, t) \ .$$

8. Formulation and integration of the canonical equations.

For practicing the procedure let us derive according to this scheme the Hamilton function and its equations of motion for some rather simple examples.

(1) Pendulum Oscillation

The mass point m is subject to the constraints (Fig. 2.1)

$$z = \text{const} = 0 \,,$$
$$x^2 + y^2 = l^2 = \text{const} \,,$$

therewith having exactly one degree of freedom ($S = 1$). With the generalized coordinate

$$q = \varphi$$

the transformation formulas come out as follows:

$$x = l \sin q \,; \quad y = l \cos q \,,$$
$$\dot{x} = l \dot{q} \cos q \,, \quad \dot{y} = -l \dot{q} \sin q \,.$$

Kinetic and potential energy then read:

$$T = \frac{1}{2} m \left(\dot{x}^2 + \dot{y}^2\right) = \frac{1}{2} m \, l^2 \dot{q}^2 \,,$$
$$V = -m \, g \, y = -m \, g \, l \cos q$$
$$\Longrightarrow \; L = T - V = \frac{1}{2} m \, l^2 \dot{q}^2 + m \, g \, l \cos q \,.$$

From that we derive the generalized momentum p:

$$p = \frac{\partial L}{\partial \dot{q}} = m \, l^2 \, \dot{q} \Longrightarrow \dot{q} = \frac{p}{m \, l^2} \,.$$

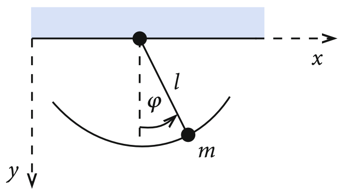

Fig. 2.1 Pendulum oscillation of the mass m as one-dimensional problem of motion

This we insert into $L(\mathbf{q}, \dot{\mathbf{q}})$,

$$\tilde{L}(\mathbf{q}, \mathbf{p}) = \frac{p^2}{2m\,l^2} + m\,g\,l\,\cos q\,,$$

and perform therewith the Legendre transformation:

$$H = p\,\dot{q} - L = \frac{p^2}{m\,l^2} - \tilde{L}(\mathbf{q}, \mathbf{p})$$

$$\Longrightarrow\ H = \frac{p^2}{2m\,l^2} - m\,g\,l\,\cos q\,. \tag{2.33}$$

Hamilton's equations of motion

$$\dot{q} = \frac{\partial H}{\partial p} = \frac{p}{m\,l^2} \ \Longrightarrow\ \dot{p} = m\,l^2\ddot{q}\,,$$

$$\dot{p} = -\frac{\partial H}{\partial q} = -m\,g\,l\,\sin q$$

yield, when combined, the well-known *oscillation equation*:

$$\ddot{q} + \frac{g}{l}\sin q = 0\,. \tag{2.34}$$

(2) Harmonic Oscillator

We think of a mass m on a spring with the spring constant k which obeys Hooke's law

$$F = -k\,x$$

where x represents the displacement of the mass from its rest position (Fig. 2.2). The constraints

$$y = z \equiv 0$$

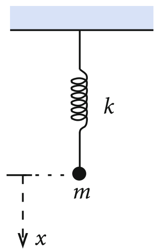

Fig. 2.2 Spring within Hooke's law as a realization of the harmonic oscillator

take care for a one-dimensional motion of the mass m. With the generalized coordinate

$$q = x$$

we find immediately:

$$T = \frac{1}{2} m \dot{q}^2 , \quad V = \frac{1}{2} k q^2 , \quad L = \frac{1}{2} m \dot{q}^2 - \frac{1}{2} k q^2 .$$

In the last equation we replace $\dot{q}$ by the generalized momentum

$$p = \frac{\partial L}{\partial \dot{q}} = m \dot{q} .$$

With

$$\tilde{L}(q, p) = \frac{p^2}{2m} - \frac{1}{2} k q^2$$

we get the Hamilton function $H = p \dot{q} - \tilde{L}$ of the harmonic oscillator:

$$H = \frac{p^2}{2m} + \frac{1}{2} m \omega_0^2 q^2 , \qquad \omega_0^2 = \frac{k}{m} . \tag{2.35}$$

It is the case of a conservative system with scleronomic constraints. Because of

$$\frac{\partial H}{\partial t} = 0 \iff H = E = \text{const}$$

H is identical to the total energy E. Reformulating Eq. (2.35) still a bit,

$$\frac{p^2}{2mE} + \frac{q^2}{\frac{2E}{m \omega_0^2}} = 1 , \tag{2.36}$$

then yields the midpoint equation of an ellipse. The path of the system in the (q, p)-phase space is thus an ellipse with the semiaxes

$$a = \sqrt{2mE} \qquad \text{and} \qquad b = \sqrt{\frac{2E}{m \omega_0^2}} .$$

The canonical equations

$$\dot{p} = -\frac{\partial H}{\partial q} = -m\,\omega_0^2 q \ ,$$

$$\dot{q} = \frac{\partial H}{\partial p} = \frac{p}{m} \implies \dot{p} = m\,\ddot{q}$$

lead directly to the oscillation equation:

$$\ddot{q} + \omega_0^2 q = 0 \ . \tag{2.37}$$

(3) Charged Particle in the Electromagnetic Field

We have already investigated in Sect. 1.2.3 the motion of a particle with the mass m and the charge $\bar{q}$ in the electromagnetic field. The particle is subject to the non-conservative *Lorentz force*

$$\mathbf{F} = \bar{q}(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \ ,$$

where $\mathbf{v}$ is its velocity. We had derived in (1.78) the generalized potential of the Lorentz force

$$U = \bar{q}(\varphi - \mathbf{v} \cdot \mathbf{A})$$

for which we have, if the Cartesian coordinates are chosen as generalized coordinates:

$$Q_j = \mathbf{F} \cdot \frac{\partial \mathbf{r}}{\partial q_j} = F_j = \frac{d}{dt}\frac{\partial U}{\partial \dot{q}_j} - \frac{\partial U}{\partial q_j}$$

For the Lagrangian we found in (1.79):

$$L = \frac{1}{2} m\,\dot{\mathbf{r}}^2 + \bar{q}(\dot{\mathbf{r}} \cdot \mathbf{A}) - \bar{q}\,\varphi = T - U \ .$$

As generalized momentum, which differs from the mechanical momentum, we get:

$$\mathbf{p} = m\,\dot{\mathbf{r}} + \bar{q}\,\mathbf{A}(\mathbf{r}, t) \ . \tag{2.38}$$

That leads via

$$H = \mathbf{p} \cdot \dot{\mathbf{r}} - L = m\,\dot{\mathbf{r}}^2 + \bar{q}\,\mathbf{A} \cdot \dot{\mathbf{r}} - \frac{1}{2} m\,\dot{\mathbf{r}}^2 - \bar{q}\,(\dot{\mathbf{r}} \cdot \mathbf{A}) + \bar{q}\,\varphi$$

to the Hamilton function

$$H = \frac{1}{2\,m}\left(\mathbf{p} - \bar{q}\,\mathbf{A}(\mathbf{r},t)\right)^2 + \bar{q}\,\varphi(\mathbf{r},t)\;, \tag{2.39}$$

which turns out to be identical to the total energy, which is not at all always a matter of course for generalized potentials. With the expression (2.39) we will extensively deal in Quantum Mechanics (Vol. 5), and there as Hamilton **operator**.

(4) Particle Without Constraint

Even if the particle does not experience any constraint the application of special curvilinear coordinates can be advised by the symmetry of the problem, e.g. in order to let as many coordinates as possible to be cyclic. We therefore want to write down now for a conservative system the Hamilton function by use of three of the most common systems of coordinates.

(a) Cartesian coordinates (x,y,z)

Since no constraint is present it holds of course $H = T + V$ and $L = T - V$, respectively:

$$H = \frac{1}{2\,m}\left(p_x^2 + p_y^2 + p_z^2\right) + V(x,y,z)\;. \tag{2.40}$$

The generalized momenta are in this case identical to the mechanical ones:

$$p_x = \frac{\partial L}{\partial \dot{x}} = m\,\dot{x}\;;\quad p_y = \frac{\partial L}{\partial \dot{y}} = m\,\dot{y}\;,\quad p_z = \frac{\partial L}{\partial \dot{z}} = m\,\dot{z}\;. \tag{2.41}$$

(b) Cylindrical coordinates (ρ,φ,z)

According to ((1.381), Vol. 1) the transformation formulas read:

$$x = \rho\,\cos\varphi\;;\quad y = \rho\,\sin\varphi\;;\quad z = z\;.$$

Herefrom one gets the velocities:

$$\dot{x} = \dot{\rho}\,\cos\varphi - \rho\,\dot{\varphi}\,\sin\varphi\;;\quad \dot{y} = \dot{\rho}\,\sin\varphi + \rho\,\dot{\varphi}\,\cos\varphi\;;\quad \dot{z} = \dot{z}\;.$$

Kinetic and potential energy,

$$T = \frac{1}{2}\, m \left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) = \frac{1}{2}\, m \,\left(\dot{\rho}^2 + \rho^2 \dot{\varphi}^2 + \dot{z}^2\right) \,,$$
$$V = V(\rho, \varphi, z) \,,$$

lead by the Lagrangian $L = T - V$ to the generalized momenta:

$$p_\rho = \frac{\partial L}{\partial \dot{\rho}} = m\,\dot{\rho}\,; \quad p_\varphi = \frac{\partial L}{\partial \dot{\varphi}} = m\,\rho^2\,\dot{\varphi}\,; \quad p_z = \frac{\partial L}{\partial \dot{z}} = m\,\dot{z}\,. \tag{2.42}$$

With

$$H = p_\rho\,\dot{\rho} + p_\varphi\,\dot{\varphi} + p_z\,\dot{z} - L$$

it follows for the Hamilton function:

$$H = \frac{1}{2\,m}\left(p_\rho^2 + \frac{p_\varphi^2}{\rho^2} + p_z^2\right) + V(\rho, \varphi, z)\,. \tag{2.43}$$

(c) Spherical coordinates (r,ϑ,φ)

According to ((1.389), Vol. 1) the transformation formulas are given by

$$x = r\,\sin\vartheta\,\cos\varphi\,; \quad y = r\,\sin\vartheta\,\sin\varphi\,; \quad z = r\,\cos\vartheta\,.$$

Therewith one easily calculates:

$$T = \frac{1}{2}\, m \left(\dot{r}^2 + r^2\dot{\vartheta}^2 + r^2\sin^2\vartheta\,\dot{\varphi}^2\right)\,; \quad V = V(r, \vartheta, \varphi)\,.$$

This yields via $L = T - V$ the generalized momenta:

$$p_r = \frac{\partial L}{\partial \dot{r}} = m\,\dot{r}\,; \quad p_\vartheta = \frac{\partial L}{\partial \dot{\vartheta}} = m\,r^2\dot{\vartheta}\,; \quad p_\varphi = \frac{\partial L}{\partial \dot{\varphi}} = m\,r^2\sin^2\vartheta\,\,\dot{\varphi}\,. \tag{2.44}$$

Eventually, the Hamilton function reads:

$$H = \frac{1}{2\,m}\left(p_r^2 + \frac{p_\vartheta^2}{r^2} + \frac{p_\varphi^2}{r^2\,\sin^2\vartheta}\right) + V(r, \vartheta, \varphi)\,. \tag{2.45}$$

2.2.3 Exercises

Exercise 2.2.1 Given the Hamilton function $H = H(\mathbf{q}, \mathbf{p}, t)$ of a mechanical system and its equations of motion (2.11), (2.12) and (2.13). The Lagrangian is the negative Legendre-transform of the Hamilton function:

$$L(\mathbf{q}, \dot{\mathbf{q}}, t) = \sum_{j=1}^{S} p_j \frac{\partial H}{\partial p_j} - H$$

Use Hamilton's equations of motion in order to derive with this relation the Lagrange equations of motion of the second kind!

Exercise 2.2.2 Determine the Routh function and its equations of motion for the two-body problem already treated in Sect. 1.4 (masses m_1, m_2 with distance-dependent pair interaction in the otherwise force-free space).

Exercise 2.2.3 A particle of mass m performs a two-dimensional motion in the xy-plane under the influence of the force

$$\mathbf{F}(\mathbf{r}) = \mathbf{F}(x, y) = -\left(\alpha + \frac{\beta}{r}\right)\mathbf{r} \qquad \alpha, \beta\text{: positive constants .}$$

Choose plane polar coordinates (ρ, φ) as generalized coordinates!

1. Write down the kinetic and the potential energy in plane polar coordinates!
2. Calculate the generalized momenta p_ρ and p_φ!
3. Formulate the Hamilton function! Find and interpret two integrals of motion (conservation laws)!

Exercise 2.2.4 The potential energy of a particle of mass m is given in cylindrical coordinates (ρ, φ, z):

$$V(\rho) = V_0 \ln \frac{\rho}{\rho_0}\ ; \qquad V_0 = \text{const}\ , \quad \rho_0 = \text{const}\ .$$

1. What is the Hamilton function?
2. Derive Hamilton's equations of motion!
3. Find three conservation laws!

Exercise 2.2.5 A box is gliding without friction along the x axis with constant velocity $\mathbf{v}_0$. On the bottom of the box there oscillates, also in x direction and frictionlessly, a mass m being fixed by a spring (spring constant: k) at the back-wall of the box (Fig. 2.3).

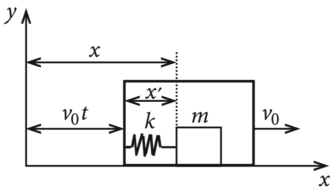

Fig. 2.3 Oscillating mass in a with constant velocity frictionlessly gliding box

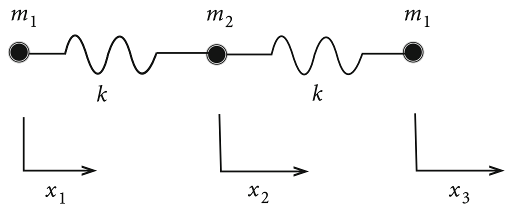

Fig. 2.4 Oscillations of a three-atom molecule

1. Find the Hamilton function in the rest system of coordinates Σ. Is H a conserved quantity? Is H identical to the total energy E? Derive Hamilton's equations of motion!
2. Investigate the same problem in the co-moving system of coordinates Σ'!

Exercise 2.2.6 Consider a three-atom molecule. It exhibits one-dimensional oscillations with equal *spring constants* k. x_1, x_2, x_3 are the displacements out of the rest positions. The two outer atoms have the same mass m_1 (Fig. 2.4). Write down the Hamilton function, find the equations of motion and solve them!

Exercise 2.2.7 A mass point moves in a cylindrically symmetric potential $V(\rho)$. Determine the Hamilton function and the canonical equations with respect to a system of coordinates that rotates with constant angular velocity ω around the symmetry axis, and that

1. in Cartesian coordinates,
2. in cylindrical coordinates.

Exercise 2.2.8 A particle of mass m moves within a plane under the influence of a non-conservative force which acts in the direction towards the center of force:

$$\mathbf{F}(\mathbf{r}) = F(r, \dot{r}, d\dot{r})\,\mathbf{e}_r \; ; \quad F(r, \dot{r}, d\dot{r}) = \frac{\alpha}{r}\left(1 - \frac{\dot{r}^2 - 2rd\dot{r}}{c^2}\right)$$

r is the distance to the force center; α and c are constants of proper dimension. Determine the Hamilton function of the particle! Note that the Lagrangian of the particle has already been calculated as Exercise 1.2.21!

2.3 Action Principles

We got to know in Sect. 1.3.3 the integral principle of Hamilton from which we were able to derive the basic Lagrange equations. It is typical for integral principles to compare **finite** path elements, which the system traverses in a **finite** time span, with their related *thought* (*'virtual'*) neighboring path elements. According to the type of this relationship one distinguishes different integral principles. The most important ones of them will be discussed and contrasted with each other in this section.

2.3.1 Modified Hamilton's Principle

A weighty advantage, among others, of Hamilton's principle considered in Sect. 1.3 lies in the fact that it is applicable also to systems which are not of typical mechanical nature. We now want to reformulate it in such a way that the equivalence to the canonical equations becomes evident. For this purpose we briefly recall the essential elements of this principle. It states that the system movement always takes place such that the **action functional**

$$S\{\mathbf{q}(t)\} = \int\limits_{t_1}^{t_2} L\left(\mathbf{q}(t),\, \dot{\mathbf{q}}(t),\, t\right)\, dt \tag{2.46}$$

becomes extremal on the set M of the admitted configuration paths $\mathbf{q}(t)$ (Fig. 2.5),

$$M \equiv \{\mathbf{q}(t) :\ \mathbf{q}\,(t_1) = \mathbf{q}_\mathrm{a},\ \mathbf{q}\,(t_2) = \mathbf{q}_\mathrm{e}\}\ , \tag{2.47}$$

for the actual path we are interested in:

$$(\delta S)_\mathrm{M} \overset{!}{=} 0\ . \tag{2.48}$$

Of decisive importance for the evaluation of the principle is the **variational condition**: The variation of the action functional S is done by a variation of the

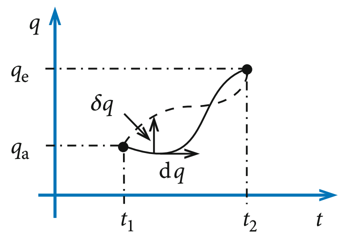

Fig. 2.5 One-dimensional illustration of the ensemble of configuration paths admitted to the variational problem of Hamilton's principle

path-piece between fixed end-configurations $\mathbf{q}_{\mathrm{a}} = \mathbf{q}(t_1)$ and $\mathbf{q}_{\mathrm{e}} = \mathbf{q}(t_2)$. The individual points of the different paths arise out of each other by virtual displacements $\delta\mathbf{q}$ which are done always at fixed time ($\delta t = 0$) and therefore need not necessarily agree with the actual displacements $d\mathbf{q}$. The evaluation of the Hamilton principle is performed by use of a *parameter representation* of the competing paths:

$$q_{j\alpha}(t) = q_j(t) + \gamma_{j\alpha}(t) \, , \qquad j = 1, 2, \ldots, S \, . \tag{2.49}$$

$q_j(t)$ is the actual path and $\gamma_{j\alpha}(t)$ a sufficiently often differentiable function with

$$\gamma_{j\alpha}(t_1) = \gamma_{j\alpha}(t_2) = 0 \quad \forall\alpha \, , \tag{2.50}$$

$$\gamma_{j\alpha=0}(t) \equiv 0 \, . \tag{2.51}$$

Therewith we must then calculate:

$$\delta S = S\{\mathbf{q}_{d\alpha}(t)\} - S\{\mathbf{q}_0(t)\} = \left(\frac{dS(\alpha)}{d\alpha}\right)_{\alpha=0} d\alpha \, , \tag{2.52}$$

$$\delta\mathbf{q} = \left(\frac{\partial\mathbf{q}_\alpha}{\partial\alpha}\right)_{\alpha=0} d\alpha \, . \tag{2.53}$$

Hence the δ-variation is representable by normal differentiating:

$$\delta \iff d\alpha\frac{\partial}{\partial\alpha} \, . \tag{2.54}$$

In this manner we have derived the Lagrange equations of motion from Hamilton's principle.

We now formally replace in the action functional S the Lagrangian by the Hamilton function using the expression (2.8):

Modified Hamilton's principle

$$\delta S = \delta\int_{t_1}^{t_2} dt\left(\sum_{j=1}^{S} p_j\,\dot{q}_j - H(\mathbf{p}, \mathbf{q}, t)\right) \stackrel{!}{=} 0 \, . \tag{2.55}$$

The new feature is that the momenta p_j, besides the q_j, are independent variables on an equal footing. The variation of the path has therefore to be done in the

phase space

which is spanned by the q_j and the p_j:

$$S = S\{\mathbf{q}(t),\ \mathbf{p}(t)\}\ . \tag{2.56}$$

As far as the coordinates q_j are concerned the same conditions hold as in the previous version (2.47). In analogy to (2.49) we now introduce also for the momenta a *parameter representation*:

$$p_{j\alpha}(t) = p_j(t) + \hat{\gamma}_{j\alpha}(t)\ , \qquad j = 1, 2, \ldots, S\ . \tag{2.57}$$

The projections of the *'admitted'* paths in the phase space on the $(\mathbf{q}, t)$-plane must all coincide for t_1 and t_2. On the other hand, it need **not** necessarily be $\hat{\gamma}_{j\alpha}(t_1) = \hat{\gamma}_{j\alpha}(t_2) = 0$, but only

$$\hat{\gamma}_{j\alpha=0}(t) \equiv 0 \tag{2.58}$$

is to be required (Fig. 2.6).

According to (2.54) and (2.55) we have to evaluate now:

$$\delta S = d\alpha \left\{ \frac{\partial}{\partial \alpha} \int\limits_{t_1}^{t_2} dt \left(\sum_{j=1}^{S} p_{j\alpha}\, \dot{q}_{j\alpha} - H\left(\mathbf{p}_\alpha, \mathbf{q}_\alpha, t\right) \right) \right\}_{\alpha=0} \overset{!}{=} 0\ . \tag{2.59}$$

The times are uninfluenced by the variation so that the differentiation with respect to α can be drawn into the integrand:

$$0 = \delta S = d\alpha \int\limits_{t_1}^{t_2} dt \sum_{j=1}^{S} \left(\frac{\partial p_{j\alpha}}{\partial \alpha} \dot{q}_{j\alpha} + p_{j\alpha} \frac{\partial \dot{q}_{j\alpha}}{\partial \alpha} - \frac{\partial H}{\partial q_{j\alpha}} \frac{\partial q_{j\alpha}}{\partial \alpha} - \frac{\partial H}{\partial p_{j\alpha}} \frac{\partial p_{j\alpha}}{\partial \alpha} \right)_{\alpha=0}\ . \tag{2.60}$$

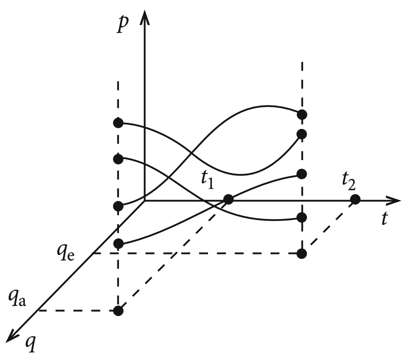

Fig. 2.6 One-dimensional illustration of the configuration paths which are admitted to the variation process in the modified Hamilton principle

We exploit

$$\frac{\partial \dot{q}_{j\alpha}}{\partial \alpha} = \frac{d}{dt}\frac{\partial q_{j\alpha}}{\partial \alpha}$$

and perform an integration by parts:

$$d\alpha \left\{ \int_{t_1}^{t_2} dt\, p_{j\alpha} \frac{\partial \dot{q}_{j\alpha}}{\partial \alpha} \right\}_{\alpha=0} = d\alpha \left\{ p_{j\alpha} \frac{\partial q_{j\alpha}}{\partial \alpha} \right\}_{\alpha=0} \Bigg|_{t_1}^{t_2} - d\alpha \left\{ \int_{t_1}^{t_2} dt\, \dot{p}_{j\alpha} \frac{\partial q_{j\alpha}}{\partial \alpha} \right\}_{\alpha=0} .$$

Since the virtual displacements δq_j are assumed to be zero at the endpoints the first term vanishes. With (2.53) and the analogous expression for the momenta

$$\delta p_j = \left(\frac{\partial p_{j\alpha}}{\partial \alpha} \right)_{\alpha=0} d\alpha \tag{2.61}$$

it then follows from (2.59):

$$0 \stackrel{!}{=} \delta S = \int_{t_1}^{t_2} dt \sum_{j=1}^{S} \left[\delta p_j \left(\dot{q}_j - \frac{\partial H}{\partial p_j} \right) - \delta q_j \left(\dot{p}_j + \frac{\partial H}{\partial q_j} \right) \right] .$$

δq_j, δp_j are freely selectable. Therefore we can read off from this expression Hamilton's equations of motion (2.11) and (2.12):

$$\dot{q}_j = \frac{\partial H}{\partial p_j} \; ; \quad \dot{p}_j = -\frac{\partial H}{\partial q_j} \, , \qquad j = 1, 2, \ldots, S . \tag{2.62}$$

2.3.2 Principle of Least Action

A further principle dates back to Maupertuis (1747) which has the same explanatory power as Hamilton's principle. We will formulate it and prove its equivalence to the principle of Hamilton. Let us define:

Definition 2.3.1 'action'

$$A = \int_{t_1}^{t_2} \sum_{j=1}^{S} p_j \, \dot{q}_j \, dt . \tag{2.63}$$

A has the dimension *'energy · time'*. We express the *'principle of least action'* as

Theorem 2.3.1 *For conservative systems with*

$$H = T + V = E = \text{const} \tag{2.64}$$

it holds:

$$\Delta A = \Delta \int_{t_1}^{t_2} dt \sum_{j=1}^{S} p_j \dot{q}_j = 0 \tag{2.65}$$

for the path in phase space actually chosen by the system.

In order to correctly understand the theorem the new path variation Δ must be defined very carefully. In Hamilton's principle (1.121) and (2.55), respectively, the paths which are admitted to the

δ-variation

arise from each other by the virtual displacements δq, which are performed at constant time. All paths assume for t_1, t_2 the same end-configurations q_a, q_e. Common characteristic of all paths is thus the same pass-through time. For the

Δ-variation ,

too, the end-configurations shall be fixed:

$$\Delta \mathbf{q}_a = \Delta \mathbf{q}_e = 0 \,. \tag{2.66}$$

The common characteristic of all paths admitted to this variation is now an identical Hamilton function:

$$\Delta H = 0 \iff \Delta T = -\Delta V \,. \tag{2.67}$$

The *pass-through times* for the various trajectories, however, must not necessarily be the same (Fig. 2.7).

It is quite possible that certain paths are admitted to both variation procedures (δ, Δ), where, however, the system moves along the paths with different velocities, on the one hand, in order to realize a given pass-through time (δ), and on the other hand, in order to ensure $H = \text{const}$ (Δ).

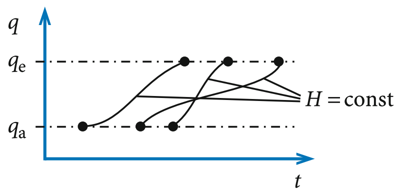

Fig. 2.7 One-dimensional illustration of the paths admitted to the variation in the principle of least action

Since for the Δ-variation the pass-through times need no longer be the same, the time, too, has to be varied now. We use again a *'parameter representation'* for the paths which are admitted to the variation:

$$\begin{aligned}\mathbf{q}_\alpha\left(t_\alpha\right) &: t_{1\alpha} \leq t_\alpha \leq t_{2\alpha}\ ,\\ \mathbf{q}(t) &: \textit{actual}\ \text{path}\ .\end{aligned} \tag{2.68}$$

The paths fulfill the boundary conditions:

$$\begin{aligned}\mathbf{q}_\alpha\left(t_{1\alpha}\right) &= \mathbf{q}\left(t_1\right) = \mathbf{q}_\mathrm{a} \quad \forall\alpha,\\ \mathbf{q}_\alpha\left(t_{2\alpha}\right) &= \mathbf{q}\left(t_2\right) = \mathbf{q}_\mathrm{e} \quad \forall\alpha\ .\end{aligned} \tag{2.69}$$

With the parameter representation the path variations can be written down explicitly:

$$\delta\text{-procedure}\ :\ \delta q = d\alpha\left(\frac{\partial q_\alpha}{\partial\alpha}\right)_{\alpha=0}\ , \tag{2.70}$$

$$\Delta\text{-procedure}\ :\ \Delta q = d\alpha\left(\frac{dq_\alpha}{d\alpha}\right)_{\alpha=0} = d\alpha\left(\frac{\partial q_\alpha}{\partial\alpha} + \dot{q}_\alpha\frac{dt_\alpha}{d\alpha}\right)_{\alpha=0}\ . \tag{2.71}$$

This can be combined as follows:

$$\Delta q = \delta q + \dot{q}\,\Delta t \qquad \text{with} \qquad \Delta t = d\alpha\left.\frac{dt_\alpha}{d\alpha}\right|_{\alpha=0}\ . \tag{2.72}$$

We have often exploited previously that the δ-variation and the time differentiations can be interchanged:

$$\delta\,\frac{d}{dt} \equiv \frac{d}{dt}\,\delta\ . \tag{2.73}$$

That was allowed because the time was not co-varied. But this does no longer apply to the Δ-variation. In general we have to accept:

$$\Delta\,\frac{d}{dt} \neq \frac{d}{dt}\,\Delta \tag{2.74}$$

That has to be taken carefully into consideration. Apart from that the symbol Δ is treated as quite a normal differential:

$$\begin{aligned} f = f(\mathbf{q},t) \implies \Delta f &= \sum_{j=1}^{S}\frac{\partial f}{\partial q_j}\,\Delta q_j + \frac{\partial f}{\partial t}\,\Delta t\\ &= \sum_{j=1}^{S}\frac{\partial f}{\partial q_j}\,\delta q_j + \left(\sum_{j=1}^{S}\frac{\partial f}{\partial q_j}\,\dot{q}_j + \frac{\partial f}{\partial t}\right)\Delta t\ .\end{aligned}$$

From this one reads off:

$$\Delta f = \delta f + \dot{f}\,\Delta t\ . \tag{2.75}$$

With these preparations we are now able to prove the principle of least action (2.65) if we presume that Hamilton's principle is known and valid. At first it holds:

$$A = \int_{t_1}^{t_2} dt \sum_{j=1}^{S} p_j\,\dot{q}_j = \int_{t_1}^{t_2} (L + H)\,dt = \int_{t_1}^{t_2} L\,dt + H\,(t_2 - t_1)\ . \tag{2.76}$$

One has to bear in mind that for different paths the end-times t_1 and t_2 are also different. We now prove that A becomes extremal on the actual path. For the line of argument the actual path is thereby the path for which Hamilton's principle is fulfilled:

$$\Delta A = \Delta \int_{t_1}^{t_2} L\,dt + H\ (\Delta t_2 - \Delta t_1)\ . \tag{2.77}$$

In the first term Δ can not be taken simply into the integral because t_1, t_2 have to be co-varied. On the other hand, H is the same for all the paths of the competitive set. We put:

$$\int_{t_1}^{t_2} L\,dt = I\,(\mathbf{q}, t_2) - I\,(\mathbf{q}, t_1)\ .$$

For a preset path I is a pure time function. With (2.75) now follows:

$$\begin{aligned} \Delta \int_{t_1}^{t_2} L\,dt &= \Delta I\,(\mathbf{q}, t_2) - \Delta I\,(\mathbf{q}, t_1) \\ &= \delta I\,(\mathbf{q}, t_2) - \delta I\,(\mathbf{q}, t_1) + \dot{I}\,(\mathbf{q}, t_2)\ \Delta t_2 - \dot{I}\,(\mathbf{q}, t_1)\ \Delta t_1 \\ &= \delta \int_{t_1}^{t_2} L\,dt + [L(t)\Delta t]_{t_1}^{t_2}\ . \end{aligned} \tag{2.78}$$

The first term is not at all zero as one could perhaps argue misleadingly from Hamilton's principle. The latter requires for the end-points $\delta \mathbf{q}_{\mathrm{a,\,e}} = 0$, while here $\Delta \mathbf{q}_{\mathrm{a,\,e}} = 0$ holds. It is in fact:

$$\begin{aligned}
\delta \int_{t_1}^{t_2} L\,dt &= \int_{t_1}^{t_2} \delta L\,dt = \int_{t_1}^{t_2} \sum_{j=1}^{S} \left(\frac{\partial L}{\partial q_j} \delta q_j + \frac{\partial L}{\partial \dot{q}_j} \delta \dot{q}_j \right) dt \\
&= \sum_{j=1}^{S} \int_{t_1}^{t_2} \left[\left(\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_j} \right) \delta q_j + \frac{\partial L}{\partial \dot{q}_j} \frac{d}{dt} \delta q_j \right] dt \\
&= \sum_{j=1}^{S} \int_{t_1}^{t_2} \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_j} \delta q_j \right) dt = \sum_{j=1}^{S} \frac{\partial L}{\partial \dot{q}_j} \delta q_j \Bigg|_{t_1}^{t_2} \\
&= \sum_{j=1}^{S} \left(\frac{\partial L}{\partial \dot{q}_j} \Delta q_j - \frac{\partial L}{\partial \dot{q}_j} \dot{q}_j \, \Delta t \right) \Bigg|_{t_1}^{t_2} .
\end{aligned}$$

Hence it follows with $\Delta q_j \big|_{t_1}^{t_2} = 0$:

$$\delta \int_{t_1}^{t_2} L\,dt = -\sum_{j=1}^{S} \frac{\partial L}{\partial \dot{q}_j} \dot{q}_j \, \Delta t \Bigg|_{t_1}^{t_2} .$$

This we insert into (2.78):

$$\Delta \int_{t_1}^{t_2} L\,dt = \left(L - \sum_{j=1}^{S} \frac{\partial L}{\partial \dot{q}_j} \dot{q}_j \right) \Delta t \Bigg|_{t_1}^{t_2} .$$

With (2.77) it eventually results in:

$$\Delta A = \left(L - \sum_{j=1}^{S} p_j \, \dot{q}_j + H \right) \Delta t \Bigg|_{t_1}^{t_2} . \tag{2.79}$$

If we still insert the definition (2.8) of the Hamilton function H then the assertion $\Delta A = 0$ is proven. Under the presumption, that the Lagrange equations of motion are valid, which we have exploited in the above chain of proof, the principle of least action (2.65) fixes the actual path of the system. Therefore it possesses the same informative power as the Hamilton principle!

2.3.3 Fermat's Principle

We want to apply the just discussed principle of least action further to a special case, namely to the

$$\textit{force-free}\text{ motion} \iff V = \text{const} \; ,$$

Since already $H = T + V = \text{const}$ was presumed it now even holds:

$$\sum_{j=1}^{S} p_j \, \dot{q}_j = H + L = 2\,T = \text{const} \; . \tag{2.80}$$

On all the admitted paths the kinetic energy is therefore a conserved quantity. The principle (2.65) then simplifies to the statement:

$$\Delta \int_{t_1}^{t_2} dt = \Delta\,(t_2 - t_1) \overset{!}{=} 0 \; . \tag{2.81}$$

In case of a force-free movement the system always seeks the path for which the pass-through time becomes extremal (minimal). That is the

principle of least time

first formulated by Fermat that is known in geometrical optics as *Fermat's principle*. It says there that the light ray moves between two points of space such that the pass-through time becomes minimal. This principle can successfully be applied, for instance, to light refraction leading to the law of reflexion.

If we specialize furtheron to a

force-free mass point

then we have because of $T = \text{const}$ even $v = \text{const}$ and (2.81) reads:

$$\Delta \int_{t_1}^{t_2} dt = \Delta \int_{t_1}^{t_2} v dt = \Delta \int_{1}^{2} ds \overset{!}{=} 0 \; . \tag{2.82}$$

This is the

principle of the shortest path

It determines the force-free movement of a mass point on a curved plane along a so-called *geodesic line*. In general one understands by this the shortest line connecting two points on a given plane.

2.3.4 Jacobi's Principle

Sometimes it appears reasonable to eliminate the time completely out of the principle of least action so that the variation does not refer to anything else except the spatial character of the path of the system. According to (2.65) first we have:

$$\Delta \int_{t_1}^{t_2} dt \sum_{j=1}^{S} p_j \dot{q}_j = \Delta \int_{t_1}^{t_2} 2\, T\, dt \stackrel{!}{=} 0\,. \tag{2.83}$$

For an N particle system the kinetic energy reads:

$$T = \frac{1}{2} \sum_{i=1}^{N} m_i \left(\frac{d\mathbf{r}_i}{dt}\right)^2 \implies dt = \frac{1}{\sqrt{2T}} \sqrt{\sum_i m_i\, (d\mathbf{r}_i)^2}\,.$$

With $T = E - V$ it follows then from (2.83):

$$\Delta \int_{1}^{2} \sqrt{2(E-V)} \sqrt{\sum_i m_i\, (d\mathbf{r}_i)^2} \stackrel{!}{=} 0\,. \tag{2.84}$$

In this version the variation affects indeed only the spatial course of the path; pass-through times do no longer play any role. Δ-variation and δ-variation are then identical.

Let us still look for a somewhat more general representation. Because of $H = E =$ const, which in particular means scleronomic constraints, it holds according to (1.38) to (1.42) for the kinetic energy T:

$$T = \frac{1}{2} \sum_{j,l} \mu_{jl}\, \dot{q}_j\, \dot{q}_l\,. \tag{2.85}$$

μ_{jl} are the *generalized masses* (1.40). We define:

$$(d\rho)^2 = \sum_{j,l} \mu_{jl}\, dq_j\, dq_l\,. \tag{2.86}$$

$d\rho$ is the most general form of the line element in the S-dimensional configuration space the axes of which are given by the generalized coordinates $q_1, \ldots, q_S$. In this sense the μ_{jl} are the elements of the so-called

metric tensor

By this the differential geometry understands the transformation matrix between the square $(d\rho)^2$ of the line element in the S-dimensional space and the infinitesimal changes of the coordinates. We illustrate this by well-known examples of the three-dimensional space of our experience:

$$(d\rho)^2 = (d\mathbf{r})^2 \implies \mu_{jl} = \frac{\partial \mathbf{r}}{\partial q_j} \cdot \frac{\partial \mathbf{r}}{\partial q_l} \ . \tag{2.87}$$

(1) Cartesian:

$$q_1 = x\ ; \quad q_2 = y\ ; \quad q_3 = z\ ; \quad \mathbf{r} = (x, y, z)$$
$$\implies \mu_{jl} = \delta_{jl}\ . \tag{2.88}$$

(2) Cylindrical:

$$\begin{aligned} & q_1 = \rho\ ; \quad q_2 = \varphi\ ; \quad q_3 = z\ ; \quad \mathbf{r} \equiv (\rho\,\cos\varphi,\ \rho\,\sin\varphi,\ z) \\ \implies & \frac{\partial \mathbf{r}}{\partial \rho} = (\cos\varphi,\ \sin\varphi,\ 0)\ , \\ & \frac{\partial \mathbf{r}}{\partial \varphi} = \rho\,(-\sin\varphi,\ \cos\varphi,\ 0)\ , \\ & \frac{\partial \mathbf{r}}{\partial z} = (0, 0, 1)\ . \end{aligned}$$

The off-diagonal elements of the metric tensor obviously disappear. The system of coordinates is *curvilinear-orthogonal*:

$$\mu_{\rho\rho} = 1\ ; \quad \mu_{\varphi\varphi} = \rho^2\ ; \quad \mu_{zz} = 1\ . \tag{2.89}$$

That means:

$$(d\mathbf{r})^2 = (d\rho)^2 + \rho^2 (d\varphi)^2 + (dz)^2\ . \tag{2.90}$$

(3) Spherical:

$$\begin{aligned} & q_1 = r\ ; \quad q_2 = \vartheta\ ; \quad q_3 = \varphi\ , \\ & \mathbf{r} \equiv r(\sin\vartheta\ \cos\varphi,\ \sin\vartheta\ \sin\varphi,\ \cos\vartheta) \\ \implies & \frac{\partial \mathbf{r}}{\partial r} = (\sin\vartheta\ \cos\varphi,\ \sin\vartheta\ \sin\varphi,\ \cos\vartheta)\ , \\ & \frac{\partial \mathbf{r}}{\partial \vartheta} = r(\cos\vartheta\ \cos\varphi,\ \cos\vartheta\ \sin\varphi,\ -\sin\vartheta)\ , \\ & \frac{\partial \mathbf{r}}{\partial \varphi} = r(-\sin\vartheta\ \sin\varphi,\ \sin\vartheta\ \cos\varphi,\ 0)\ . \end{aligned}$$

The spherical coordinates, too, represent a *curvilinear-orthogonal* system. The off-diagonal elements of the metric tensor are thus equal to zero:

$$\mu_{rr} = 1 \; ; \quad \mu_{\vartheta\vartheta} = r^2 \; ; \quad \mu_{\varphi\varphi} = r^2 \sin^2 \vartheta \; . \tag{2.91}$$

The square of the line element therefore reads:

$$(d\mathbf{r})^2 = (dr)^2 + r^2 (d\vartheta)^2 + r^2 \sin^2 \vartheta \, (d\varphi)^2 \; . \tag{2.92}$$

The metric of the configuration space is normally non-Cartesian but curvilinear with in general non-zero off-diagonal elements. According to (2.85) and (2.86) it holds:

$$T = \frac{1}{2} \frac{(d\rho)^2}{(dt)^2} \iff dt = \frac{d\rho}{\sqrt{2T}} \; . \tag{2.93}$$

Therewith (2.83) becomes
Jacobi's principle

$$\Delta \int\limits_1^2 \sqrt{E - V(\mathbf{q})} \, d\rho \stackrel{!}{=} 0 \; . \tag{2.94}$$

For the special case of the *force-free* motion one finds:

$$\Delta \int\limits_1^2 d\rho \stackrel{!}{=} 0 \; . \tag{2.95}$$

The system seeks the shortest configuration path, i.e. it moves along a *geodesic line* in the configuration space. That need not necessarily mean *straight-lined* in this abstract space.

2.3.4.1 Application Examples

(1) Path of a force-free particle in the three-dimensional space of experience

Since time does not appear in the Jacobi's principle the Δ- and δ-variational procedures are identical:

$$\Delta \int\limits_1^2 d\rho = \delta \int\limits_1^2 d\rho \stackrel{!}{=} 0 \; . \tag{2.96}$$

We have therefore to calculate:

$$\delta \int_1^2 \sqrt{m(dx^2 + dy^2 + dz^2)} \overset{!}{=} 0 \,.$$

This is equivalent to

$$\delta \int_{x_1}^{x_2} \sqrt{1 + y'^2 + z'^2}\, dx \overset{!}{=} 0 \,.$$

We perform the variation by use of the Euler-Lagrange differential equation (1.147):

$$f\left(x, y, z, y', z'\right) \equiv \sqrt{1 + y'^2 + z'^2}$$

$$\Longrightarrow \quad \frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} \overset{!}{=} 0 = -\frac{d}{dx}\frac{y'}{\sqrt{1 + y'^2 + z'^2}} \,,$$

$$\frac{\partial f}{\partial z} - \frac{d}{dx}\frac{\partial f}{\partial z'} \overset{!}{=} 0 = -\frac{d}{dx}\frac{z'}{\sqrt{1 + y'^2 + z'^2}} \,.$$

From this one reads off:

$$y'^2 = c_1\left(1 + z'^2\right) \;; \quad z'^2 = c_2\left(1 + y'^2\right)$$

$$\Longrightarrow \quad y'^2 = \text{const}_1 \;; \quad z'^2 = \text{const}_2 \,.$$

The particle path is therefore, not surprisingly, a straight line:

$$\begin{aligned} y(x) &= c\,x + \bar{c}\,, \\ z(x) &= d\,x + \bar{d}\,. \end{aligned} \qquad \left(c, d, \bar{c}, \bar{d} = \text{const}\right) \tag{2.97}$$

(2) Electron-optical law of refraction

The x-axis may be the place of a potential jump from $V_1 = \text{const}$ to $V_2 = \text{const}$. In both half planes the electron performs a force-free motion, which according to Example (1) must be straight-lined (Fig. 2.8). We ask ourselves how should C and x

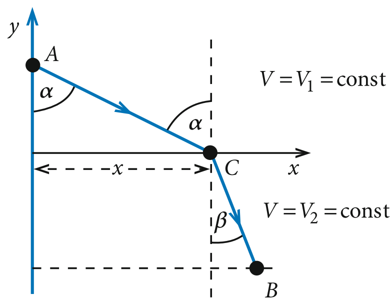

Fig. 2.8 Force-free motion of an electron in two regions which border on each other having constant but different potentials (electron-optical law of refraction)

be chosen in order to guarantee that the electron arrives at B when starting at A? That we clarify with (2.94):

$$\begin{aligned}
\Delta \int\limits_A^B \sqrt{2mT}\,\sqrt{dx^2+dy^2} &= \Delta \int\limits_A^C \sqrt{2\,m\,(E-V_1)}\;ds + \Delta \int\limits_C^B \sqrt{2\,m\,(E-V_2)}\;ds \\
&= \sqrt{2\,m\,(E-V_1)}\,\Delta\left(\sqrt{x^2+y_A^2}\right) \\
&\quad + \sqrt{2\,m\,(E-V_2)}\;\Delta\left(\sqrt{(x_B-x)^2+y_B^2}\right) \\
&= \sqrt{2\,m\,(E-V_1)}\left(\frac{d}{dx}\sqrt{x^2+y_A^2}\right)\Delta x \\
&\quad + \sqrt{2\,m\,(E-V_2)}\left(\frac{d}{dx}\sqrt{(x_B-x)^2+y_B^2}\right)\Delta x \\
&\overset{!}{=} 0\,.
\end{aligned}$$

Because of $\Delta x \neq 0$ it follows:

$$\begin{aligned}
0 &= \sqrt{E-V_1}\,\frac{x}{\sqrt{x^2+y_A^2}} - \sqrt{E-V_2}\,\frac{x_B-x}{\sqrt{(x_B-x)^2+y_B^2}} \\
&= \sqrt{E-V_1}\,\sin\alpha - \sqrt{E-V_2}\,\sin\beta\,.
\end{aligned}$$

Therewith we find eventually:

$$\frac{\sin\alpha}{\sin\beta} = \sqrt{\frac{E-V_2}{E-V_1}} = \sqrt{\frac{T_2}{T_1}} = \frac{v_2}{v_1}\,. \tag{2.98}$$

2.4 Poisson Brackets

2.4.1 Representation Spaces

We want to discuss in this section some abstract terms which will turn out to be useful for our further considerations. Some of them we have already repeatedly used. We start with a classification of the representation spaces.

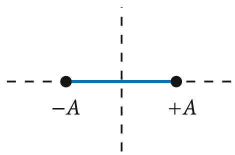

Fig. 2.9 Path of the linear harmonic oscillator in the configuration space

(1) Configuration Space

This representation space is already known to us. It has the

$$\text{dimension: } S$$

and as

$$\text{axes:} \quad \mathbf{q} = (q_1, q_2, \ldots, q_S) \ .$$

Example: Linear Harmonic Oscillator
(see Example 2 in Sect. 2.2.2)

The configuration space is here the x-axis. The *configuration path* is built by all x for which $|x| \leq A$.

By specification of the configuration path the mechanical problem is not yet solved because it remains unclear where the system finds itself at a certain point of time (Fig. 2.9).

(2) Event Space

$$\text{dimension:} \qquad S + 1 \ ,$$

$$\text{axes:} \qquad \mathbf{q} = (q_1, q_2, \ldots, q_S) \quad \text{and} \quad t \ .$$

The *path in the event space* $(\mathbf{q}, t)$ is definitely calculable if $2S$ initial conditions are given. These can be the configurations at two different times, $(\mathbf{q}(t_1),\ \mathbf{q}(t_2))$, or S generalized coordinates and the corresponding S generalized velocities at a certain point of time t_0, $(\mathbf{q}(t_0),\ \dot{\mathbf{q}}(t_0))$:

Lagrange formalism $\Longleftrightarrow$ event space.

Example: Linear Harmonic Oscillator

Because of $S = 1$ two initial conditions are necessary in order to fix uniquely the path in the event space (Fig. 2.10).

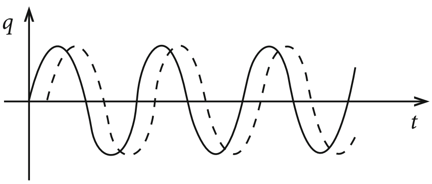

Fig. 2.10 Path of the linear harmonic oscillator in the event space

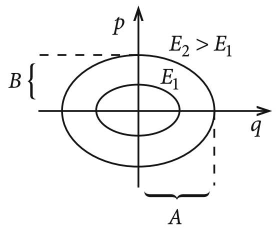

Fig. 2.11 Path of the linear harmonic oscillator in the phase space for two different energies

(3) Phase Space

$$\text{dimension}: \quad 2S\,,$$

$$\text{axes}: \quad \mathbf{q} = (q_1, q_2, \ldots, q_S)\;; \quad \mathbf{p} = (p_1, p_2, \ldots, p_S)\;.$$

Since coordinates q_j and momenta p_j are to be seen as equitable variables one occasionally merges them to a single *phase* and to a *phase vector*, respectively:

$$\boldsymbol{\imath} = (\pi_1, \pi_2, \ldots, \pi_{2S}) \equiv (q_1, \ldots, q_S, p_1, \ldots, p_S)\;. \tag{2.99}$$

As *phase curve* or *phase trajectory* one denotes the set of all phases $\boldsymbol{\imath}$ which the physical system can assume in the course of time.

Example: Linear Harmonic Oscillator

According to (2.36) the phase curves are now ellipses (Fig. 2.11)

$$\frac{p^2}{2\,m\,E} + \frac{q^2}{\frac{2\,E}{m\,\omega_0^2}} = 1$$

with energy-dependent semiaxes:

$$A = \sqrt{\frac{2\,E}{m\,\omega_0^2}}\;; \quad B = \sqrt{2\,m\,E}\;.$$

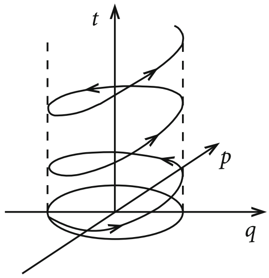

Fig. 2.12 Trajectory of the linear harmonic oscillator in the state space

(4) State Space

$$\text{dimension}: \qquad 2S+1 ,$$

$$\text{axes}: \qquad \mathbf{q}=(q_1,\ldots,q_S) \ ; \quad \mathbf{p}=(p_1,\ldots,p_S) \quad \text{and} \quad t \ .$$

This is the most general representation space (*phase space plus time axis*). All the other spaces are special cases, i.e. projections of the state space onto certain planes or axes.

Example: Linear Harmonic Oscillator

The path $\boldsymbol{\pi}(t)$ is now a helical line (Fig. 2.12) which by a preset

$$\textbf{initial phase:} \qquad \boldsymbol{\pi}_0=\left(q_1^{(0)},\ldots,p_S^{(0)}\right)=\boldsymbol{\pi}(t_0)$$

is uniquely fixed for all times.

Since the phase trajectory $\boldsymbol{\pi}(t)$ is found by solving Hamilton's equations of motion, i.e. from a set of differential equations of **first** order, it is sufficient to know the phase point of the mechanical system at a single point of time in order to fix the phase $\boldsymbol{\pi}(t)$ for all times:

Hamilton formalism $\Longleftrightarrow$ state space.

With the introduction of the state space we encounter a term that is of great importance for almost all branches of physics:

Definition 2.4.1 'state'

Minimal but complete set of determinants (parameters) which is sufficient to describe all properties of the system

This is a very abstract definition which for each physical theory must be substantiated and interpreted since for different disciplines the actually interesting properties may be different.

Which *'minimal information'* determines the mechanical properties of a mass point? Statements about position, velocity, momentum, angular momentum, energy etc. would be interesting. But it is obviously not necessary to measure all of them at the same time. Position and momentum suffice to fix simultaneously the other quantities. On the other hand, **both** quantities must really be measured, one alone is not enough:

$$\begin{array}{l}\textit{each mechanical property}\\ \textit{of the mass point}\end{array} \iff f(\mathbf{r},\mathbf{p}) \ .$$

In the same manner the mechanical properties of a general N-particle system are established by generalized coordinates and generalized momenta:

$$\begin{array}{l}\textit{each mechanical property}\\ \textit{of a physical system}\end{array} \iff f(\mathbf{q},\mathbf{p}) = f(\boldsymbol{\pi}) \ .$$

This means:

$$\begin{array}{l}\textit{state } \psi \textit{ of a}\\ \textit{mechanical system}\end{array} \iff \begin{array}{l}\textit{point } \boldsymbol{\pi} \textit{ in the}\\ \textit{state space .}\end{array}$$

According to our definition of the term *'state'* even its time evolution must uniquely be determined already by the fixation of a minimal set of determinants at any given point of time t_0 $(\psi_0 = \psi(t_0))$:

$$\psi(t) = \psi\,(t;\psi_0) \ . \tag{2.100}$$

From a mathematical point of view $\psi(t)$ must therefore arise from a differential equation of first order with respect to the time:

$$\dot{\psi}(t) = \tilde{f}\,(\psi(t)) \ . \tag{2.101}$$

That means for mechanics:

$$\dot{\boldsymbol{\pi}}(t) = \tilde{f}\,(\boldsymbol{\pi}(t)) \ . \tag{2.102}$$

Hamilton's equations of motion are indeed of this kind. On the other hand, it is then also clear that the configuration $\mathbf{q}(t)$ itself can not be a *state* yet because its time evolution obeys differential equations of second order with respect to the time (Lagrange's equations of motion).

2.4.2 Fundamental Poisson Brackets

We now want to introduce the concept of the Poisson brackets. This concept allows for an especially concise formulation of the classical equations of motion and the conservation laws. That will be demonstrated in the following.

After the pre-considerations of the last section we know that any arbitrary mechanical observable can be represented as a phase function:

$$f(\boldsymbol{\pi}, t) = f(\mathbf{q}, \mathbf{p}, t)$$

Let us investigate its equation of motion:

$$\begin{aligned} \frac{df}{dt} &= \sum_{j=1}^{S}\left(\frac{\partial f}{\partial q_j}\dot{q}_j + \frac{\partial f}{\partial p_j}\dot{p}_j\right) + \frac{\partial f}{\partial t} \\ &= \sum_{j=1}^{S}\left(\frac{\partial f}{\partial q_j}\frac{\partial H}{\partial p_j} - \frac{\partial f}{\partial p_j}\frac{\partial H}{\partial q_j}\right) + \frac{\partial f}{\partial t}\,. \end{aligned} \tag{2.103}$$

Definition 2.4.2 $f = f(\mathbf{q}, \mathbf{p}, t)$, $g = g(\mathbf{q}, \mathbf{p}, t)$: scalar functions of the vector pairs $\mathbf{q} = (q_1, \ldots, q_S)$, $\mathbf{p} = (p_1, \ldots, p_S)$.

$$\{f, g\}_{\mathbf{q},\mathbf{p}} \equiv \sum_{j=1}^{S}\left(\frac{\partial f}{\partial q_j}\frac{\partial g}{\partial p_j} - \frac{\partial f}{\partial p_j}\frac{\partial g}{\partial q_j}\right)\,. \tag{2.104}$$

'Poisson bracket' of f with g.

By the subscripts at the bracket-symbol on the left-hand side it is referred to the variables with respect to which the differentiations have to be performed. However, later we will recognize that this is in principle unnecessary. As an important result it will turn out that the Poisson bracket is independent of the choice of canonical variables by which the differentiations are explicitly done.

The equation of motion (2.103) now reads:

$$\frac{df}{dt} = \{f, H\}_{\mathbf{q},\mathbf{p}} + \frac{\partial f}{\partial t}\,. \tag{2.105}$$

At this stage it is about only an abbreviating notation. This result will become important only after we show that the Poisson bracket is independent of the $(\mathbf{q}, \mathbf{p})$-choice.

For (2.104) and (2.105) one easily realizes the following special cases:

$$\dot{q}_j = \left\{q_j, H\right\}_{\mathbf{q},\mathbf{p}}\,, \tag{2.106}$$

$$\dot{p}_j = \left\{p_j, H\right\}_{\mathbf{q},\mathbf{p}}\,. \tag{2.107}$$

The following three relations are denoted as
fundamental Poisson brackets

$$\{q_i, q_j\}_{\mathbf{q},\mathbf{p}} = 0 , \tag{2.108}$$

$$\{p_i, p_j\}_{\mathbf{q},\mathbf{p}} = 0 , \tag{2.109}$$

$$\{q_i, p_j\}_{\mathbf{q},\mathbf{p}} = \delta_{ij} . \tag{2.110}$$

We justify only (2.110). For that purpose we insert $f = q_i$ and $g = p_j$ into the definition (2.104):

$$\begin{aligned}
\{q_i, p_j\}_{\mathbf{q},\mathbf{p}} &= \sum_{k=1}^{S} \left(\frac{\partial q_i}{\partial q_k} \frac{\partial p_j}{\partial p_k} - \frac{\partial q_i}{\partial p_k} \frac{\partial p_j}{\partial q_k} \right) \\
&= \sum_{k=1}^{S} \left(\delta_{ik}\, \delta_{jk} - 0 \right) = \delta_{ij} \qquad \text{q.e.d.}
\end{aligned}$$

In the next step we now show that the fundamental brackets are independent of the special choice of the canonical variables.

Theorem 2.4.1 $(\mathbf{q}, \mathbf{p})$ *and* $(\mathbf{Q}, \mathbf{P})$ *may be two sets of canonical variables for both of which Hamilton's equations of motion are valid with respect to:*

$$H(\mathbf{q}, \mathbf{p}) = \widetilde{H}(\mathbf{Q}, \mathbf{P}) .$$

Thereby $\widetilde{H}(\mathbf{Q}, \mathbf{P})$ *shall result from* $H(\mathbf{q}, \mathbf{p})$ *simply by insertion of* $\mathbf{q} = \mathbf{q}(\mathbf{Q}, \mathbf{P})$ *and* $\mathbf{p} = \mathbf{p}(\mathbf{Q}, \mathbf{P})$. *Then it holds:*

$$\{Q_i, Q_j\}_{\mathbf{q},\mathbf{p}} = 0 ; \quad \{P_i, P_j\}_{\mathbf{q},\mathbf{p}} = 0 , \tag{2.111}$$

$$\{Q_i, P_j\}_{\mathbf{q},\mathbf{p}} = \delta_{ij} . \tag{2.112}$$

Proof

$$\begin{aligned}
\dot{Q}_i &= \frac{d}{dt} Q_i(\mathbf{q}, \mathbf{p}) = \sum_{k=1}^{S} \left(\frac{\partial Q_i}{\partial q_k} \dot{q}_k + \frac{\partial Q_i}{\partial p_k} \dot{p}_k \right) = \sum_{k=1}^{S} \left(\frac{\partial Q_i}{\partial q_k} \frac{\partial H}{\partial p_k} - \frac{\partial Q_i}{\partial p_k} \frac{\partial H}{\partial q_k} \right) \\
&= \sum_{k,l} \left[\frac{\partial Q_i}{\partial q_k} \left(\frac{\partial \widetilde{H}}{\partial Q_l} \frac{\partial Q_l}{\partial p_k} + \frac{\partial \widetilde{H}}{\partial P_l} \frac{\partial P_l}{\partial p_k} \right) - \frac{\partial Q_i}{\partial p_k} \left(\frac{\partial \widetilde{H}}{\partial Q_l} \frac{\partial Q_l}{\partial q_k} + \frac{\partial \widetilde{H}}{\partial P_l} \frac{\partial P_l}{\partial q_k} \right) \right] \\
&= \sum_{k,l} \left[\frac{\partial \widetilde{H}}{\partial Q_l} \left(\frac{\partial Q_i}{\partial q_k} \frac{\partial Q_l}{\partial p_k} - \frac{\partial Q_i}{\partial p_k} \frac{\partial Q_l}{\partial q_k} \right) + \frac{\partial \tilde{H}}{\partial P_l} \left(\frac{\partial Q_i}{\partial q_k} \frac{\partial P_l}{\partial p_k} - \frac{\partial Q_i}{\partial p_k} \frac{\partial P_l}{\partial q_k} \right) \right] \\
&= \sum_{l} \left(-\dot{P}_l \{Q_i, Q_l\}_{\mathbf{q},\mathbf{p}} + \dot{Q}_l \{Q_i, P_l\}_{\mathbf{q},\mathbf{p}} \right) .
\end{aligned}$$

The comparison yields:

$$\{Q_i,\, Q_l\}_{\mathbf{q},\,\mathbf{p}} = 0\;; \qquad \{Q_i,\, P_l\}_{\mathbf{q},\,\mathbf{p}} = \delta_{il}\;.$$

By $\dot{P}_i$ one finds analogously the third bracket.

Theorem 2.4.2 *The value of a Poisson bracket is independent of the set of canonical coordinates with respect to which the differentiation processes are performed*

Proof Let F and G be arbitrary phase functions and $(\mathbf{q}, \mathbf{p})$, $(\mathbf{Q}, \mathbf{P})$ two sets of canonical variables for which one has:

$$\mathbf{q} = \mathbf{q}(\mathbf{Q},\, \mathbf{P})\;; \qquad \mathbf{p} = \mathbf{p}(\mathbf{Q},\, \mathbf{P})\;,$$
$$\mathbf{Q} = \mathbf{Q}(\mathbf{q},\, \mathbf{p})\;; \qquad \mathbf{P} = \mathbf{P}(\mathbf{q},\, \mathbf{p})$$

Therewith we calculate:

$$\begin{aligned}
\{F, G\}_{\mathbf{q},\,\mathbf{p}} &= \sum_{j=1}^{S}\left(\frac{\partial F}{\partial q_j}\frac{\partial G}{\partial p_j} - \frac{\partial F}{\partial p_j}\frac{\partial G}{\partial q_j}\right)\\
&= \sum_{j,l}\left[\frac{\partial F}{\partial q_j}\left(\frac{\partial G}{\partial Q_l}\frac{\partial Q_l}{\partial p_j} + \frac{\partial G}{\partial P_l}\frac{\partial P_l}{\partial p_j}\right) - \frac{\partial F}{\partial p_j}\left(\frac{\partial G}{\partial Q_l}\frac{\partial Q_l}{\partial q_j} + \frac{\partial G}{\partial P_l}\frac{\partial P_l}{\partial q_j}\right)\right]\\
&= \sum_{l}\left(\frac{\partial G}{\partial Q_l}\{F,\, Q_l\}_{\mathbf{q},\,\mathbf{p}} + \frac{\partial G}{\partial P_l}\{F,\, P_l\}_{\mathbf{q},\,\mathbf{p}}\right)\;.
\end{aligned}$$

From this expression we can read off two useful intermediate results. If we put especially $F = Q_k$ and exploit (2.111) as well as (2.112) then it follows:

$$\{G,\, Q_k\}_{\mathbf{q},\,\mathbf{p}} = -\frac{\partial G}{\partial P_k}\;. \tag{2.113}$$

Choosing, on the other hand, $F = P_k$, leads to:

$$\{G,\, P_k\}_{\mathbf{q},\,\mathbf{p}} = \frac{\partial G}{\partial Q_k}\;. \tag{2.114}$$

These two intermediate results are used in the above expression:

$$\{F,\, G\}_{\mathbf{q},\,\mathbf{p}} = \sum_{l}\left(\frac{\partial G}{\partial Q_l}\left(-\frac{\partial F}{\partial P_l}\right) + \frac{\partial G}{\partial P_l}\frac{\partial F}{\partial Q_l}\right) = \{F,\, G\}_{\mathbf{Q},\,\mathbf{P}}\;.$$

That was to be proven. Hence from now on we can skip the indexes at the bracket symbol. The basis can be built by **arbitrary** sets of canonical variables.

2.4.3 *Formal Properties*

Up to now the introduction of the Poisson bracket implied solely a simplification of the scientific notation and does not bring us a step further in the solution of a practical problem. Important are, however, some algebraic properties which allow for a generalization going distinctly beyond the framework of classical mechanics. First we list these properties and bring the proof, where it is not obvious, afterwards:
antisymmetry

$$\{f,\, g\} = -\{g, f\}\ ; \qquad \{f, f\} = 0 \qquad \forall f\ . \tag{2.115}$$

linearity

$$\{c_1 f_1 + c_2 f_2,\, g\} = c_1\{f_1,\, g\} + c_2\{f_2,\, g\}\ , \qquad c_1, c_2\ \ \text{: constants}\ . \tag{2.116}$$

zero (null) element

$$\{c,\, g\} = 0 \qquad \forall g = g(\mathbf{q}, \mathbf{p})\ , \qquad c\ \ \text{: constant}\ . \tag{2.117}$$

product rule

$$\{f,\, g\, h\} = g\,\{f,\, h\} + \{f,\, g\}\, h\ . \tag{2.118}$$

Jacobi identity

$$\{f,\, \{g,\, h\}\} + \{g,\, \{h, f\}\} + \{h,\, \{f,\, g\}\} = 0\ . \tag{2.119}$$

Equations (2.115)–(2.117) follow directly from the Definition (2.104) of the bracket. The same holds for (2.118), too, when the product rule for differentiations is applied. The proof of Eq. (2.119) follows, even though a bit lengthy, simply by insertion or more elegantly as follows:

At first we express the Poisson bracket by a differential operator,

$$\{g, h\} = D_g\, h\ ,$$

where

$$D_g = \sum_{j=1}^{S} \left(\frac{\partial g}{\partial q_j} \frac{\partial}{\partial p_j} - \frac{\partial g}{\partial p_j} \frac{\partial}{\partial q_j} \right) \equiv \sum_{i=1}^{2S} \alpha_i(g) \frac{\partial}{\partial \pi_i}\ .$$

Therewith we can write:

$$
\begin{aligned}
&\{f,\{g,h\}\} + \{g,\{h,f\}\} \\
&= \{f,\{g,h\}\} - \{g,\{f,h\}\} = D_f(D_g\,h) - D_g(D_f\,h) \\
&= \sum_{i,j}\left[\beta_i(f)\frac{\partial}{\partial\pi_i}\left(\alpha_j(g)\frac{\partial h}{\partial\pi_j}\right) - \alpha_j(g)\frac{\partial}{\partial\pi_j}\left(\beta_i(f)\frac{\partial h}{\partial\pi_i}\right)\right] \\
&= \sum_j\left\{\sum_i\left[\left(\beta_i(f)\frac{\partial}{\partial\pi_i}\alpha_j(g)\right) - \left(\alpha_i(g)\frac{\partial}{\partial\pi_i}\beta_j(f)\right)\right]\right\}\frac{\partial h}{\partial\pi_j}\,.
\end{aligned}
$$

The expression in the square bracket depends on f and g being, however, independent of h. We can therefore write,

$$
\{f,\{g,h\}\} + \{g,\{h,f\}\} = \sum_{j=1}^{S}\left(A_j\frac{\partial h}{\partial q_j} + B_j\frac{\partial h}{\partial p_j}\right)\,,
$$

where A_j, B_j are independent of h. Hence we can determine these terms by special choices of h:

$h = q_i$:

$$
A_i = \{f,\{g,q_i\}\} + \{g,\{q_i,f\}\} = -\left\{f,\frac{\partial g}{\partial p_i}\right\} + \left\{g,\frac{\partial f}{\partial p_i}\right\} = -\frac{\partial}{\partial p_i}\{f,g\}\,.
$$

Here we have used (2.113). In the next step we apply (2.114):

$h = p_i$:

$$
B_i = \{f,\{g,p_i\}\} + \{g,\{p_i,f\}\} = \left\{f,\frac{\partial g}{\partial q_i}\right\} - \left\{g,\frac{\partial f}{\partial q_i}\right\} = \frac{\partial}{\partial q_i}\{f,g\}\,.
$$

We insert these results for A_j and B_j into the above equation:

$$
\begin{aligned}
\{f,\{g,h\}\} + \{g,\{h,f\}\} &= \sum_{j=1}^{S}\left(-\frac{\partial}{\partial p_j}\{f,g\}\frac{\partial h}{\partial q_j} + \frac{\partial}{\partial q_j}\{f,g\}\frac{\partial h}{\partial p_j}\right) \\
&= \{\{f,g\},\,h\}\,.
\end{aligned}
$$

That proves the assertion!

2.4.4 Integrals of Motion

According to (2.105) the temporal change of a state variable is essentially given by the Poisson bracket of this quantity with the Hamilton function H. That stresses once more the importance of H. The Hamilton function determines the time evolution of mechanical observables.

Let us consider a physical quantity

$$F = F(\mathbf{q}, \mathbf{p}, t)$$

which has for all times the same value:

$$\frac{dF}{dt} = 0 \iff F : \textit{integral of motion} \; . \tag{2.120}$$

According to (2.105) this is exactly then fulfilled if it holds

$$\{H, F\} \stackrel{!}{=} \frac{\partial F}{\partial t} \tag{2.121}$$

We see that the constant of motion itself can absolutely depend explicitly on time. If this is not the case then the Poisson bracket of H with F vanishes. Therewith we have found a rather *concise* criterion for the decision whether or not an integral of motion is on hand. Compare this with the original definition (1.162) for the motion in the event space.

For the Hamilton function H particularly one finds:

$$\frac{dH}{dt} = \{H, H\} + \frac{\partial H}{\partial t} = \frac{\partial H}{\partial t} \; . \tag{2.122}$$

That agrees with the previous result (2.18). If H does not depend explicitly on time then it is an integral of motion, which, as we know, in case of scleronomic constraints is identical to the energy conservation law.

Poisson's theorem *The Poisson bracket of two integrals of motion is itself again an integral of motion.*

Proof Let f, g be integrals of motion. That means because of (2.121):

$$\{H, f\} = \frac{\partial f}{\partial t} \; ; \quad \{H, g\} = \frac{\partial g}{\partial t} \; .$$

We exploit the Jacobi identity (2.119):

$$\begin{aligned} 0 &= \{f, \{g, H\}\} + \{g, \{H, f\}\} + \{H, \{f, g\}\} \\ &= -\left\{f, \frac{\partial g}{\partial t}\right\} + \left\{g, \frac{\partial f}{\partial t}\right\} + \{H, \{f, g\}\} \; . \end{aligned}$$

This reads with the result from Exercise 2.4.4, part (1):

$$\{H, \{f, g\}\} = \frac{\partial}{\partial t}\{f, g\} \; ,$$

With (2.121) the assertion is verified. $\{f, g\}$ is again an integral of motion.

Sometimes it is possible to construct by application of Poisson's theorem a whole sequence of integrals of motion. That would mean of course an important step towards the full solution of the mechanical problem. However, occasionally the Poisson bracket of two integrals of motion leads only to a trivial constant or simply to a function of the initial integrals of motion. That does of course not represent a new integral of motion.

2.4.5 Relationship to Quantum Mechanics

Let us forget for the moment the actual definition of the classical Poisson bracket and relate the

abstract bracket $\{\ldots,\ldots\}$

with the properties (2.115) to (2.119) to a

system of axioms of an abstract mathematical structure

A possible **concrete** realization would then be the classical Poisson bracket (2.104). But there are also further thinkable realizations. One important possibility concerns

linear operators $\widehat{A},\ \widehat{B},\ \widehat{C},\ldots$**,**
represented by square matrices.

One defines for these operators the so-called
commutator

$$\left[\widehat{A},\ \widehat{B}\right]_{-} \equiv \widehat{A}\widehat{B}-\widehat{B}\widehat{A}\ . \tag{2.123}$$

Since the order of operators is not arbitrary the commutator is normally unequal zero and is itself again an operator. If one understands by *'constant'* $\widehat{A}$ a multiple of the unit matrix and strictly respects in (2.118) the order of the operators then the commutator fulfills the axioms (2.115) to (2.119). The replacement of the abstract bracket by the commutator (2.123) decisively rules the so-called

quantum mechanics

In this sense classical mechanics and quantum mechanics are due to the same superordinate abstract mathematical structure. They are *'merely'* different realizations of the *abstract bracket*. The realization of *quantum mechanics* can be substantiated by the following

correspondence principle :

1. measurable physical quantity A (*observable*) $\Longleftrightarrow$ Hermitean linear operator $\widehat{A}$, represented by a square matrix in a special vector space (*Hilbert space*).
2. measured values $\Longleftrightarrow$ eigenvalues or expectation values of these operators.
3. $\{\ldots,\ldots\} \;\Longleftrightarrow\; \frac{1}{i\hbar}\left[\widehat{A},\ \widehat{B}\right]_{-}$,
   where $\hbar=\frac{h}{2\pi}$ and $h = 6{,}626\cdot 10^{-34}\,\mathrm{J\,s}$: Planck's constant.

4. fundamental brackets:

$$\left[\hat{q}_i, \hat{p}_j\right]_- = i\hbar\,\delta_{ij}\;, \tag{2.124}$$

$$\left[\hat{q}_i, \hat{q}_j\right]_- = \left[\hat{p}_i, \hat{p}_j\right]_- = 0\;. \tag{2.125}$$

5. Hamilton function $H(\mathbf{q}, \mathbf{p}, t) \iff$ Hamilton operator $\widehat{H}(\hat{\mathbf{q}}, \hat{\mathbf{p}}, t)$.
6. equation of motion:

$$\frac{d}{dt}\widehat{A} = \frac{1}{i\hbar}\left[\widehat{A}, \widehat{H}\right]_- + \frac{\partial}{\partial t}\widehat{A}. \tag{2.126}$$

Let us finally demonstrate by a simple example how physical problems can be solved by use of the *abstract bracket* without referring to a special representation of the bracket.

We seek the equation of motion of the *harmonic oscillator*, which according to (2.35) is defined by

$$H = \frac{p^2}{2m} + \frac{1}{2}m\,\omega_0^2 q^2\;.$$

Because of $\partial H/\partial t = 0$ it holds at first:

$$\begin{aligned}\dot{p} = \{p, H\} &= \frac{1}{2m}\left\{p, p^2\right\} + \frac{1}{2}m\,\omega_0^2\left\{p, q^2\right\}\\ &= \frac{1}{2m}\left(p\,\{p, p\} + \{p, p\}\,p\right) + \frac{1}{2}m\,\omega_0^2\left(q\{p, q\} + \{p, q\}\,q\right)\\ &= -m\,\omega_0^2 q\;.\end{aligned}$$

Analogously one finds

$$\dot{q} = \{q, H\} = \frac{p}{m}\;.$$

These are, however, just the Hamilton's equations of motion,

$$\dot{p} = -\frac{\partial H}{\partial q}\;; \qquad \dot{q} = \frac{\partial H}{\partial p}\;,$$

being derived without the need at any point of the special definition of the abstract bracket as a classical Poisson bracket. Consequentially

$$\dot{\hat{p}} = -m\,\omega_0^2\hat{q}\;, \tag{2.127}$$

$$\dot{\hat{q}} = \frac{1}{m}\hat{p}\;. \tag{2.128}$$

must be valid in quantum mechanics, too, as the equations of motion of the harmonic oscillator if one interprets $\hat{q}$, $\hat{p}$ as operators according to the prescriptions of quantum mechanics.

2.4.6 Exercises

Exercise 2.4.1

1. Determine the Poisson brackets which are built by the Cartesian components of the linear momentum $\mathbf{p}$ and the angular momentum $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ of a mass point.
2. Calculate the Poisson brackets which consist of the components of $\mathbf{L}$.

Exercise 2.4.2 Given is the angular momentum $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ of a mass point m.

1. Let it be:

$$\mathbf{L}^2 = L_x^2 + L_y^2 + L_z^2 \ .$$

   Calculate:

$$\left\{\mathbf{L}^2, L_{x,y,z}\right\} \ .$$

2. Demonstrate that if two components of $\mathbf{L}$ are integrals of motion then the same holds also for the third component. Thereby we assume that L_x, L_y, L_z are not explicitly time-dependent!

Exercise 2.4.3 A particle of mass m moves in a central field.

1. What is the Hamilton function? Which generalized coordinates are convenient?
2. Demonstrate by use of the Poisson bracket that the z-component L_z of the angular momentum is an integral of motion!

Exercise 2.4.4 Show that for the functions

$$f = f(\mathbf{q}, \mathbf{p}, t) \ ; \quad g = g(\mathbf{q}, \mathbf{p}, t) \ ; \quad h = h(\mathbf{q}, \mathbf{p}, t)$$

the following relations are valid:

$$1) \quad \frac{\partial}{\partial t}\{f, g\} = \left\{\frac{\partial f}{\partial t}, g\right\} + \left\{f, \frac{\partial g}{\partial t}\right\} \ ,$$

$$2) \quad \frac{d}{dt}\{f, g\} = \left\{\frac{df}{dt}, g\right\} + \left\{f, \frac{dg}{dt}\right\} \ ,$$

$$3) \quad \{f, g \cdot h\} = g\{f, h\} + \{f, g\}h \ .$$

Exercise 2.4.5

1. Give the expression for the Poisson bracket of the angular momentum with an arbitrary vector $\mathbf{A}$ that depends on $\mathbf{r}$ and $\mathbf{p}$.
2. Calculate therewith in particular $\{L_i, x_j\}$,
3. $\{L_i, p_j\}$,
4. $\{L_i, L_j\}$ and
5. $\{\mathbf{A}^2, L_j\}$.

Exercise 2.4.6 Two particles of masses m_1 and m_2 move in an arbitrary force field without any constraints. They have the angular momenta $\mathbf{L}_1$ and $\mathbf{L}_2$, respectively.

1. Why must the Poisson brackets between these two observables fulfill:

$$\{\mathbf{L}_1, \mathbf{L}_2\} = 0 \; .$$

2. Verify the following relation:

$$\{\mathbf{L}_1, \mathbf{L}_1 \cdot \mathbf{L}_2\} = -\left(\mathbf{L}_1 \times \mathbf{L}_2\right) \; .$$

3. Prove with part 2.:

$$\{\mathbf{L}_1, (\mathbf{L}_1 \cdot \mathbf{L}_2)^n\} = -n\,(\mathbf{L}_1 \cdot \mathbf{L}_2)^{n-1}\;(\mathbf{L}_1 \times \mathbf{L}_2) \qquad\qquad n = 1, 2, 3 \ldots$$

Exercise 2.4.7

1. Let the mechanical observable $f(\mathbf{q}, \mathbf{p}, t)$ as well as the Hamilton function H be integrals of motion. Show that $\partial f / \partial t$, too, is an integral of motion.
2. Consider the linear force-free motion of a particle of mass m. Show that H is an integral of motion and verify for the observable

$$f(q, p, t) = q - \frac{pt}{m}$$

   the statement of part 1. That means, demonstrate that f as well as $\partial f / \partial t$ are integrals of motion.

Exercise 2.4.8 Check whether for the linear harmonic oscillator the mechanical observable

$$f(q, p, t) = p \sin \omega t - m\omega q \cos \omega t$$

is an integral of motion.

Validate the result by a direct calculation of df/dt!

Exercise 2.4.9 Let

$$A = A\left(\mathbf{q}(t), \mathbf{p}(t)\right)$$

be a not explicitly time-dependent phase-space function:

$$\frac{\partial A}{\partial t} = 0 \; .$$

The Hamilton function H of the system shall be not explicitly time-dependent, either:

$$\frac{\partial H}{\partial t} = 0 \; .$$

Express the time-dependence of A by H and $A(0) = A(\mathbf{q}(0), \mathbf{p}(0))$!

2.5 Canonical Transformations

2.5.1 Motivation

Classical Mechanics appears in four equivalent formulations:

1. Newton (Vol. 1),
2. Lagrange (Sect. 1),
3. Hamilton (Sect. 2),
4. Hamilton-Jacobi (Sect. 3).

The transition from the Lagrange to the Hamilton formalism was mathematically carried out by use of a Legendre transformation. In the next section the Hamilton-Jacobi theory will be based on the Hamilton mechanics, discussed in this section with the aid of a so-called *'canonical transformation'*. For this purpose, some preliminary considerations are certainly advisable.

We have shown previously that in the Lagrange formalism the choice of the generalized coordinates $q_1, \ldots, q_S$ is in principle arbitrary, only the total number S is fixed. That is because the Lagrange equations

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = 0 \; , \qquad j = 1, 2, \ldots, S \; ,$$

are in the configuration space *forminvariant with respect to point transformations*. That we proved in Sect. 1.2.1. For the transformation

$$(q_1, \ldots, q_S) \iff (\bar{q}_1, \ldots, \bar{q}_S)$$

with

$$\bar{q}_j = \bar{q}_j(\mathbf{q}, t) \; ; \quad q_j = q_j\left(\bar{\mathbf{q}}, t\right) \; , \qquad j = 1, 2, \ldots, S$$

formally unchanged Lagrange equations arise,

$$\frac{d}{dt}\frac{\partial \overline{L}}{\partial \dot{\bar{q}}_j} - \frac{\partial \overline{L}}{\partial \bar{q}_j} = 0\,, \qquad j = 1, 2, \ldots, S\,,$$

where the *new* Lagrangian $\overline{L}$ emerges from the *old* one simply by insertion of the transformation formulas:

$$\overline{L} = L\left(\mathbf{q}\left(\bar{\mathbf{q}}, t\right), \dot{\mathbf{q}}\left(\bar{\mathbf{q}}, \dot{\bar{\mathbf{q}}}, t\right)\; t\right) = \overline{L}\left(\bar{\mathbf{q}}, \dot{\bar{\mathbf{q}}}, t\right)\;.$$

In addition, the Lagrange equations are also invariant with respect to so-called *mechanical gauge transformations* (1.84):

$$L \;\Rightarrow\; L + L_0\,; \qquad L_0 = \frac{d}{dt} f(\mathbf{q}, t)\;.$$

Thereby f is allowed to be an almost arbitrary function of $\mathbf{q}$ and t. The actual reason for these invariances stems from the action functional $S\{\mathbf{q}(t)\}$ (1.120), which becomes always extremal for the same path in M (1.118) independently of the special choice of coordinates. On the other hand, the Lagrange equations of motion result from the requirement $\delta S = 0$.

With the *modified Hamilton's principle* (2.48) we got to know such a formulation from which Hamilton's equations of motion are derivable if one only treats the coordinates $\mathbf{q}$ and the momenta $\mathbf{p}$ as autonomous variables which are to be varied independently of each other. Consequentially the canonical equations, too, are forminvariant with respect to point transformations if one properly co-varies the momenta according to their definition

$$p_j = \frac{\partial L}{\partial \dot{q}_j}$$

Also the *mechanical gauge transformation* (1.84) can be shown to provide not only an equivalent Lagrangian but also an equivalent Hamilton function. Thereby '*equivalent*' is to be understood in such a manner that the canonical equations, which determine the dynamics of the system, remain forminvariant under this gauge transformation as the Lagrange equations do. That one can see as follows: Because of $\overline{q}_j = q_j$ for all j and

$$\begin{aligned}\bar{p}_j = \frac{\partial \overline{L}}{\partial \dot{q}_j} &= \frac{\partial L}{\partial \dot{q}_j} + \frac{\partial}{\partial \dot{q}_j}\frac{d}{dt} f(\mathbf{q}, t) \\ &= \frac{\partial L}{\partial \dot{q}_j} + \frac{\partial}{\partial \dot{q}_j}\left(\frac{\partial}{\partial t} f(\mathbf{q}, t) + \sum_{l=1}^{S} \frac{\partial f}{\partial q_l}\dot{q}_l\right) = \frac{\partial L}{\partial \dot{q}_j} + \frac{\partial f}{\partial q_j}\end{aligned}$$

the mechanical gauge transformation leads to the following new variables:

$$\bar{q}_j = q_j\,; \qquad \bar{p}_j = p_j + \frac{\partial f}{\partial q_j}\,. \tag{2.129}$$

Therewith we construct the new Hamilton function:

$$\begin{aligned}\overline{H} &= \sum_j \bar{p}_j\,\dot{\bar{q}}_j - \overline{L} = \sum_j \left(p_j + \frac{\partial f}{\partial q_j}\right)\dot{q}_j - L - \frac{d}{dt}f \\ &= H + \sum_j \frac{\partial f}{\partial q_j}\dot{q}_j - \sum_l \frac{\partial f}{\partial q_l}\dot{q}_l - \frac{\partial f}{\partial t}\,.\end{aligned}$$

With the so transformed Hamilton function,

$$\overline{H} = H\left(\mathbf{q}, \mathbf{p}\left(\bar{\mathbf{p}}, \mathbf{q}, t\right), t\right) - \frac{\partial f(\mathbf{q}, t)}{\partial t}\,, \tag{2.130}$$

we check the canonical equations:

$$\begin{aligned}\frac{\partial \overline{H}}{\partial \bar{q}_j} = \frac{\partial \overline{H}}{\partial q_j} &= \frac{\partial H}{\partial q_j} + \sum_l \frac{\partial H}{\partial p_l}\frac{\partial p_l}{\partial q_j} - \frac{\partial^2 f}{\partial q_j \partial t} \\ &= -\dot{p}_j - \sum_l \dot{q}_l \frac{\partial^2 f}{\partial q_j \partial q_l} - \frac{\partial^2 f}{\partial q_j \partial t} \\ &= -\dot{p}_j - \frac{d}{dt}\frac{\partial}{\partial q_j} f(\mathbf{q}, t)\,.\end{aligned}$$

Eventually it remains with (2.129):

$$\frac{\partial \overline{H}}{\partial \bar{q}_j} = -\dot{\bar{p}}_j\,. \tag{2.131}$$

Analogously one finds:

$$\begin{aligned}\frac{\partial \overline{H}}{\partial \bar{p}_j} &= \sum_{l=1}^{S} \frac{\partial H}{\partial p_l}\frac{\partial p_l}{\partial \bar{p}_j} = \sum_{l=1}^{S} \frac{\partial H}{\partial p_l}\delta_{jl} = \dot{q}_j\,, \\ \frac{\partial \overline{H}}{\partial \bar{p}_j} &= \dot{\bar{q}}_j\,.\end{aligned} \tag{2.132}$$

Equations (2.131) and (2.132) show the form-invariance of the canonical equations. The above derivation contains an important detail, namely, we could show that besides the set of variables

$$q_j, p_j\,, \qquad j = 1, 2, \ldots, S$$

also

$$q_j, p_j + \frac{\partial}{\partial q_j} f(\mathbf{q}, t) \;, \qquad j = 1, 2, \ldots, S$$

with arbitrary $f(\mathbf{q}, t)$ represent a *'canonically conjugate'* pair of variables. The presetting of $\mathbf{q}$ thus does not at all uniquely fix the corresponding canonically conjugate momenta.

This is typical for Hamilton's formulation of Classical Mechanics for which the momenta p_j are on an equal footing with the coordinates q_j. The set of *allowed* transformations for which the fundamental equations of motion remain forminvariant is therefore in the Hamilton mechanics substantially larger than in the version of Lagrange. That is an advantage of the Hamilton formalism which we will investigate and exploit in the following in more detail.

By a
'phase transformation'

$$\bar{q}_j = \bar{q}_j(\mathbf{q}, \mathbf{p}, t) \;; \qquad \bar{p}_j = \bar{p}_j(\mathbf{q}, \mathbf{p}, t) \;, \qquad j = 1, 2, \ldots, S \tag{2.133}$$

one understands a point transformation in the phase space. While **all** point transformations in the configuration space result in an equivalent Lagrangian, not all phase transformations let the Hamilton's equations of motion to be forminvariant. On the other hand, only those transformations of the Hamilton mechanics are interesting which do not change the form of the equations of motion. They are denoted as *canonical transformations*.

Definition 2.5.1 The phase transformation

$$(\mathbf{q}, \mathbf{p}) \longrightarrow (\bar{\mathbf{q}}, \bar{\mathbf{p}})$$

is called **canonical** if a function

$$\overline{H} = \overline{H}\left(\bar{\mathbf{q}}, \bar{\mathbf{p}}, t\right) \tag{2.134}$$

does exist for which we have:

$$\dot{\bar{q}}_j = \frac{\partial \overline{H}}{\partial \bar{p}_j} \;; \quad \dot{\bar{p}}_j = -\frac{\partial \overline{H}}{\partial \bar{q}_j} \;, \qquad j = 1, 2, \ldots, S \tag{2.135}$$

In this context how $\overline{H}$ arises from H is in principle dispensable. For the proof of the form-invariance of the Lagrange equations $\overline{L}$ was found from L simply by insertion of the transformation formulas. If that is the case also for the Hamilton function $\overline{H}$, i.e.

$$\overline{H} = H\left(\mathbf{q}\left(\bar{\mathbf{q}}, \bar{\mathbf{p}}, t\right), \; \mathbf{p}\left(\bar{\mathbf{q}}, \bar{\mathbf{p}}, t\right), t\right) \;, \tag{2.136}$$

then one calls the transformation *canonical in the narrower sense*.

Before we try to work out practical criteria for canonical transformations two special examples will give us an idea what canonical transformations are able to accomplish.

(1) Interchange of Coordinates and Momenta

The phase transformation

$$\bar{q}_j = \bar{q}_j(\mathbf{q}, \mathbf{p}, t) = -p_j \ , \tag{2.137}$$

$$\bar{p}_j = \bar{p}_j(\mathbf{q}, \mathbf{p}, t) = q_j \tag{2.138}$$

is *canonical in the narrower sense* since with

$$H = H(\mathbf{q}, \mathbf{p}, t) \ ,$$
$$\overline{H} = \overline{H}\left(\bar{\mathbf{q}}, \bar{\mathbf{p}}, t\right) = H\left(\bar{\mathbf{p}}, -\bar{\mathbf{q}}, t\right) \tag{2.139}$$

we find:

$$\frac{\partial \overline{H}}{\partial \bar{p}_j} = \frac{\partial H\left(\bar{\mathbf{p}}, -\bar{\mathbf{q}}, t\right)}{\partial \bar{p}_j} = \frac{\partial H(\mathbf{q}, \mathbf{p}, t)}{\partial q_j} = -\dot{p}_j = \dot{\bar{q}}_j \ ,$$

$$\frac{\partial \overline{H}}{\partial \bar{q}_j} = \frac{\partial H\left(\bar{\mathbf{p}}, -\bar{\mathbf{q}}, t\right)}{\partial \bar{q}_j} = -\frac{\partial H(\mathbf{q}, \mathbf{p}, t)}{\partial p_j} = -\dot{q}_j = -\dot{\bar{p}}_j \ .$$

The canonical equations thus remain forminvariant after the transformation (2.137) and (2.138). This phase transformation interchanges *positions* and *momenta* therewith impressively indicating that the conceptual assignment $\mathbf{q} \iff$ *position* and $\mathbf{p} \iff$ *momentum* becomes rather worthless within the framework of Hamilton's mechanics. One should consider $\mathbf{q}$ and $\mathbf{p}$ as abstract, completely equitable independent variables.

(2) Cyclic Coordinates

Already several times we realized that the *'right'* choice of the generalized coordinates q_j can be decisively important for the practical handling of a mechanical problem. If we succeeded to make the choice such that

all q_j are cyclic

then the solution of the problem would be trivial if in addition one can assume

$$\frac{\partial H}{\partial t} = 0 \qquad (H\text{: constant of motion}) \ .$$

'All q_j cyclic' means:

$$\frac{\partial H}{\partial q_j} = 0 \qquad \forall j \iff H = H(\mathbf{p}) \; . \tag{2.140}$$

It holds then:

$$\dot{p}_j = 0 \qquad \forall j \iff p_j = \text{const} = c_j \; . \tag{2.141}$$

It follows from the other canonical equation:

$$\dot{q}_j = \frac{\partial H}{\partial p_j} = \dot{q}_j\,(p_1, \ldots, p_S) = \dot{q}_j\,(c_1, \ldots, c_S) \; .$$

But that means

$$\dot{q}_j = \text{const} = \alpha_j \qquad \forall j \; , \tag{2.142}$$

what can easily be integrated:

$$q_j = \alpha_j\, t + d_j \; , \qquad j = 1, 2, \ldots, S \; . \tag{2.143}$$

The α_j are known because of (2.142) and the c_j, d_j by initial conditions. With (2.141) and (2.143) the problem is therefore elementarily solved.

The decisive question is of course whether the above assumption *'all q_j cyclic'* can indeed be realized. That is found to be in principle possible and is worked out by the Hamilton-Jacobi theory (Chap. 3) to a mighty method of solution. It is to be expected, however, that the physically *plausible*, obvious coordinates do not fulfill this condition. At first they have to be canonically transformed in a proper manner. The careful investigation of canonical transformations is therefore sufficiently motivated.

2.5.2 The Generating Function

Starting point for the following considerations is the *modified Hamilton's principle* (2.55). This states that the motion of the system takes place in such a way that the action functional

$$S\{\mathbf{q}(t), \mathbf{p}(t)\} = \int_{t_1}^{t_2} dt \left(\sum_{j=1}^{S} p_j \dot{q}_j - H(\mathbf{p}, \mathbf{q}, t) \right) \tag{2.144}$$

becomes extremal for the actual path on the competing set M of admitted phase paths:

$$M = \{(\mathbf{q}(t), \mathbf{p}(t)) : \mathbf{q}(t_1) = \mathbf{q}_\mathrm{a}, \mathbf{q}(t_2) = \mathbf{q}_\mathrm{e}; \; \mathbf{p}(t_1), \mathbf{p}(t_2) \text{ arbitrary}\} \tag{2.145}$$

What is now to be taken into consideration with respect to this principle when we perform a phase transformation

$$(\mathbf{q}, \mathbf{p}) \longrightarrow (\bar{\mathbf{q}}, \bar{\mathbf{p}}) \ ?$$

1. The boundary conditions will possibly change! After the transformation the paths which belong to M do not necessarily have the same initial and end-configurations since

$$\bar{\mathbf{q}}\,(t_1) = \bar{\mathbf{q}}\,(\mathbf{q}_\mathrm{a},\ \mathbf{p}(t_1),\ t_1)\ , \tag{2.146}$$

$$\bar{\mathbf{q}}\,(t_2) = \bar{\mathbf{q}}\,(\mathbf{q}_\mathrm{e},\ \mathbf{p}(t_2),\ t_2) \tag{2.147}$$

can depend on $\mathbf{p}(t_1)$ and $\mathbf{p}(t_2)$, respectively, and therefore might be different for different paths.
2. If, additionally, the transformation shall be canonical then there must exist for the new variables, too, a modified Hamilton's principle:

$$\delta \int_{t_1}^{t_2} dt \left(\sum_{j=1}^{S} \bar{p}_j \dot{\bar{q}}_j - \overline{H}\,(\bar{\mathbf{q}}, \bar{\mathbf{p}}, t) \right) \stackrel{!}{=} 0\ . \tag{2.148}$$

Thereby the variation possibly concerns paths others than those of the original competing set (2.145), namely such paths which have in common the fixed initial and end-configurations $\bar{\mathbf{q}}_\mathrm{a}$ and $\bar{\mathbf{q}}_\mathrm{e}$.

Hereto we prove the following assertion:

Theorem 2.5.1 *The phase transformation* $(\mathbf{q}, \mathbf{p}) \longrightarrow (\bar{\mathbf{q}}, \bar{\mathbf{p}})$ *is* ***canonical****, if*

$$\sum_{j=1}^{S} p_j \dot{q}_j - H = \sum_{j=1}^{S} \bar{p}_j \dot{\bar{q}}_j - \overline{H} + \frac{dF_1}{dt}\ . \tag{2.149}$$

Thereby

$$F_1 = F_1\,(\mathbf{q}, \bar{\mathbf{q}}, t) \tag{2.150}$$

is an arbitrary but sufficiently often differentiable function of the 'old' *and the* 'new' *coordinates.*

Proof We show at first that F_1 completely determines the transformation and also $\overline{H}$ so that the denomination

$$F_1 = F_1\,(\mathbf{q}, \bar{\mathbf{q}}, t) \iff \textbf{generating function of the transformation}$$

appears to be justifiable. We start with:

$$dF_1 = \sum_{j=1}^{S} \left(\frac{\partial F_1}{\partial q_j}\, dq_j + \frac{\partial F_1}{\partial \bar{q}_j}\, d\bar{q}_j \right) + \frac{\partial F_1}{\partial t}\, dt \; .$$

For comparison we rewrite (2.149):

$$dF_1 = \sum_{j=1}^{S} \left(p_j\, dq_j - \bar{p}_j\, d\bar{q}_j \right) + \left(\overline{H} - H \right)\, dt \; .$$

With respect to F_1 the variables $\mathbf{q}$, $\bar{\mathbf{q}}$ and t are to be taken as independent; thus it follows by equating coefficients:

$$p_j = \frac{\partial F_1}{\partial q_j} \; ; \quad \bar{p}_j = -\frac{\partial F_1}{\partial \bar{q}_j} \; ; \quad \overline{H} = H + \frac{\partial F_1}{\partial t} \; . \tag{2.151}$$

Thereby the transformation is already completely determined. If $\mathbf{q}$, $\mathbf{p}$ and F_1 are preset then one solves

$$p_j = \frac{\partial F_1}{\partial q_j} = p_j\,(\mathbf{q}, \bar{\mathbf{q}}, t)$$

with respect to $\bar{q}$ and gets therewith the first half of the transformation equations:

$$\bar{q}_j = \bar{q}_j(\mathbf{q}, \mathbf{p}, t) \; .$$

That we insert into

$$\bar{p}_j = -\frac{\partial F_1}{\partial \bar{q}_j} = \bar{p}_j(\mathbf{q}, \bar{\mathbf{q}}, t)$$

and obtain:

$$\bar{p}_j = \bar{p}_j(\mathbf{q}, \mathbf{p}, t) \; .$$

This consideration presumes as usual that the function $F_1(\mathbf{q}, \bar{\mathbf{q}}, t)$ fulfills all necessary requirements concerning the differentiability and invertibility.

The new Hamilton function, too, is completely fixed by $F_1(\mathbf{q}, \bar{\mathbf{q}}, t)$:

$$\overline{H}\,(\bar{\mathbf{q}}, \bar{\mathbf{p}}, t) = H\,(\mathbf{q}\,(\bar{\mathbf{q}}, \bar{\mathbf{p}}, t)\,,\; \mathbf{p}\,(\bar{\mathbf{q}}, \bar{\mathbf{p}}, t)\,, t) + \frac{\partial}{\partial t} F_1\,(\mathbf{q}\,(\bar{\mathbf{q}}, \bar{\mathbf{p}}, t)\,, \bar{\mathbf{q}}, t) \; . \tag{2.152}$$

We now show as second step that the phase transformation *generated* by $F_1(\mathbf{q}, \bar{\mathbf{q}}, t)$ is indeed canonical. For this purpose we consider the action functional:

$$S = \int_{t_1}^{t_2} dt \left(\sum_{j=1}^{S} p_j \dot{q}_j - H(\mathbf{q}, \mathbf{p}, t) \right) = \int_{t_1}^{t_2} dt \left(\sum_{j=1}^{S} \bar{p}_j \dot{\bar{q}}_j - \overline{H}\left(\bar{\mathbf{q}}, \bar{\mathbf{p}}, t\right) + \frac{dF_1}{dt} \right)$$

$$= \int_{t_1}^{t_2} dt \left(\sum_{j=1}^{S} \bar{p}_j \dot{\bar{q}}_j - \overline{H}\left(\bar{\mathbf{q}}, \bar{\mathbf{p}}, t\right) \right) + F_1\left(\mathbf{q}_\mathrm{e}, \bar{\mathbf{q}}(t_2), t_2\right) - F_1\left(\mathbf{q}_\mathrm{a}, \bar{\mathbf{q}}\left(t_1\right), t_1\right) \ .$$

S must now be varied with respect to $\bar{\mathbf{q}}$ and $\bar{\mathbf{p}}$ instead of $\mathbf{q}$ and $\mathbf{p}$, where one has to take into consideration what is said under the above point 1.:

$$0 \stackrel{!}{=} \delta S = \delta \left\{ F_1\left(\mathbf{q}_\mathrm{e}, \bar{\mathbf{q}}(t_2), t_2\right) - F_1\left(\mathbf{q}_\mathrm{a}, \bar{\mathbf{q}}(t_1), t_1\right) \right\}$$
$$+ \int_{t_1}^{t_2} dt \left[\sum_{j=1}^{S} \left(\delta \bar{p}_j \dot{\bar{q}}_j + \bar{p}_j\, \delta \dot{\bar{q}}_j - \frac{\partial \overline{H}}{\partial \bar{q}_j}\, \delta \bar{q}_j - \frac{\partial \overline{H}}{\partial \bar{p}_j}\, \delta \bar{p}_j \right) \right] \ .$$

Since $\mathbf{q}_\mathrm{a}$, $\mathbf{q}_\mathrm{e}$ are uninfluenced by the variation it holds:

$$\delta \left\{ F_1\left(\mathbf{q}_\mathrm{e}, \bar{\mathbf{q}}(t_2), t_2\right) - F_1\left(\mathbf{q}_\mathrm{a}, \bar{\mathbf{q}}(t_1), t_1\right) \right\} = \sum_{j=1}^{S} \frac{\partial F_1}{\partial \bar{q}_j}\, \delta \bar{q}_j \Bigg|_{t_1}^{t_2} \ .$$

If we now perform an integration by parts,

$$\int_{t_1}^{t_2} dt\, \bar{p}_j\, \delta \dot{\bar{q}}_j = \bar{p}_j\, \delta \bar{q}_j \Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} dt\, \dot{\bar{p}}_j\, \delta \bar{q}_j \ ,$$

then we are left with:

$$0 \stackrel{!}{=} \delta S = \sum_{j=1}^{S} \left(\bar{p}_j + \frac{\partial F_1}{\partial \bar{q}_j} \right) \delta \bar{q}_j \Bigg|_{t_1}^{t_2} + \int_{t_1}^{t_2} dt \sum_{j=1}^{S} \left[\delta \bar{p}_j \left(\dot{\bar{q}}_j - \frac{\partial \overline{H}}{\partial \bar{p}_j} \right) - \delta \bar{q}_j \left(\dot{\bar{p}}_j + \frac{\partial \overline{H}}{\partial \bar{q}_j} \right) \right] \ . \tag{2.153}$$

Following our foregoing considerations we can **not** conclude for the first summand that $\delta \bar{q}_j$ vanishes at t_1 and t_2, respectively. Because of (2.151), however, the bracket is already equal to zero. Using the independency of the new variables $\bar{q}_j, \bar{p}_j$, (2.153) eventually leads to Hamilton's equations of motion:

$$\dot{\bar{q}}_j = \frac{\partial \overline{H}}{\partial \bar{p}_j} \ ; \qquad \dot{\bar{p}}_j = -\frac{\partial \overline{H}}{\partial \bar{q}_j} \ . \tag{2.154}$$

The transformation, *generated* by $F_1(\mathbf{q}, \bar{\mathbf{q}}, t)$, is thus indeed canonical.

2.5.3 Equivalent Forms of the Generating Function

The $(\mathbf{q}, \bar{\mathbf{q}})$-dependence of the generating function F_1 is in principle by no means selfevident. Applying Legendre transformations one can find three further types of generating functions:

$$F_2 = F_2(\mathbf{q}, \bar{\mathbf{p}}, t) \ , \tag{2.155}$$

$$F_3 = F_3(\mathbf{p}, \bar{\mathbf{q}}, t) \ , \tag{2.156}$$

$$F_4 = F_4(\mathbf{p}, \bar{\mathbf{p}}, t) \ . \tag{2.157}$$

The *generating functions* always combine a *new* and an *old* coordinate. The ongoing statement of the problem decides which form is most convenient. For all the three functions there exists a theorem as that for F_1 in (2.149) and (2.150) which we have proved in the last section. Let us inspect this point in the following in a little bit more detail.

$$\boxed{F_2 = F_2(\mathbf{q}, \bar{\mathbf{p}}, t)}$$

F_2 one obtains from F_1 by a Legendre transformation with respect to $\bar{\mathbf{q}}$:

$$F_2(\mathbf{q}, \bar{\mathbf{p}}, t) = F_1(\mathbf{q}, \bar{\mathbf{q}}, t) - \sum_{j=1}^{S} \frac{\partial F_1}{\partial \bar{q}_j} \bar{q}_j = F_1(\mathbf{q}, \bar{\mathbf{q}}, t) + \sum_{j=1}^{S} \bar{p}_j \bar{q}_j \ . \tag{2.158}$$

From the relation, already used to find (2.151),

$$dF_1 = \sum_{j=1}^{S} \left(p_j\, dq_j - \bar{p}_j\, d\bar{q}_j\right) + \left(\overline{H} - H\right)\, dt \tag{2.159}$$

it follows for F_2:

$$dF_2 = dF_1 + \sum_{j=1}^{S} \left(\bar{p}_j\, d\bar{q}_j + \bar{q}_j\, d\bar{p}_j\right) = \sum_{j=1}^{S} \left(p_j\, dq_j + \bar{q}_j\, d\bar{p}_j\right) + \left(\overline{H} - H\right)\, dt \ . \tag{2.160}$$

That means:

$$p_j = \frac{\partial F_2}{\partial q_j} \ ; \quad \bar{q}_j = \frac{\partial F_2}{\partial \bar{p}_j} \ ; \quad \overline{H} = H + \frac{\partial F_2}{\partial t} \ . \tag{2.161}$$

By inverting and resolving one shows, as explicitly demonstrated in the last section for F_1, that from (2.161) the transformation formulas $(\mathbf{q}, \mathbf{p}) \longrightarrow (\bar{\mathbf{q}}, \bar{\mathbf{p}})$ are uniquely derivable.

To demonstrate that F_2, too, provides a canonical transformation we first have to rearrange the expression (2.149) according to (2.160):

$$\begin{aligned}\sum_{j=1}^{S} p_j \dot{q}_j - H &= \sum_{j=1}^{S} \bar{p}_j \dot{\bar{q}}_j - \overline{H} + \frac{dF_1}{dt} \\ &= \sum_{j=1}^{S} \bar{p}_j \dot{\bar{q}}_j - \overline{H} + \frac{dF_2}{dt} - \sum_{j=1}^{S} \left(\bar{p}_j \dot{\bar{q}}_j + \bar{q}_j \dot{\bar{p}}_j\right) \ .\end{aligned}$$

By putting instead of (2.149) now the relation

$$\sum_{j=1}^{S} p_j \dot{q}_j - H = -\sum_{j=1}^{S} \bar{q}_j \dot{\bar{p}}_j - \overline{H} + \frac{dF_2}{dt} \tag{2.162}$$

into the modified Hamilton's principle and performing the variation with respect to $\bar{\mathbf{q}}$ and $\bar{\mathbf{p}}$ one finds out that F_2, too, generates a canonical phase transformation.

$$\boxed{F_3 = F_3\left(\mathbf{p}, \bar{\mathbf{q}}, t\right)}$$

F_3 one gets from F_1 by a Legendre transformation with respect to $\mathbf{q}$:

$$F_3\left(\mathbf{p}, \bar{\mathbf{q}}, t\right) = F_1\left(\mathbf{q}, \bar{\mathbf{q}}, t\right) - \sum_{j=1}^{S} \frac{\partial F_1}{\partial q_j} q_j = F_1\left(\mathbf{q}, \bar{\mathbf{q}}, t\right) - \sum_{j=1}^{S} p_j q_j \ . \tag{2.163}$$

We build again the total differential:

$$dF_3 = dF_1 - \sum_{j=1}^{S} \left(dp_j\, q_j + p_j\, dq_j\right) \ .$$

Inserting (2.159) for dF_1 yields:

$$dF_3 = -\sum_{j=1}^{S} \left(q_j\, dp_j + \bar{p}_j\, d\bar{q}_j\right) + \left(\overline{H} - H\right)\, dt \ . \tag{2.164}$$

From this one reads off:

$$q_j = -\frac{\partial F_3}{\partial p_j} \ ; \quad \bar{p}_j = -\frac{\partial F_3}{\partial \bar{q}_j} \ ; \quad \overline{H} = H + \frac{\partial F_3}{\partial t} \ . \tag{2.165}$$

When we invert these expressions and solve them for $\bar{\mathbf{q}}$, $\bar{\mathbf{p}}$ then we obtain the explicit transformation formulas mediated by F_3.

We rearrange (2.149) with (2.163):

$$\sum_{j=1}^{S} p_j\,\dot{q}_j - H = \sum_{j=1}^{S} \bar{p}_j\,\dot{\bar{q}}_j - \overline{H} + \frac{dF_1}{dt}$$
$$= \sum_{j=1}^{S} \bar{p}_j\,\dot{\bar{q}}_j - \overline{H} + \frac{dF_3}{dt} + \sum_{j=1}^{S}\left(\dot{p}_j\,q_j + p_j\dot{q}_j\right) \ .$$

If one uses now

$$\sum_{j=1}^{S} p_j\,\dot{q}_j - H = \sum_{j=1}^{S}\left(\bar{p}_j\,\dot{\bar{q}}_j + \dot{p}_j\,q_j + p_j\,\dot{q}_j\right) - \overline{H} + \frac{dF_3}{dt} \tag{2.166}$$

instead of (2.149) in the modified Hamilton's principle and performs the variation of the action functional with respect to $\bar{\mathbf{q}}$, $\bar{\mathbf{p}}$ then one gets again Hamilton's equations of motion in the form of (2.154).

$$\boxed{F_4 = F_4\left(\mathbf{p}, \bar{\mathbf{p}}, t\right)}$$

F_4 follows from F_1 by a double Legendre transformation with respect to the two variables $\mathbf{q}$ and $\bar{\mathbf{q}}$:

$$F_4\left(\mathbf{p}, \bar{\mathbf{p}}, t\right) = F_1\left(\mathbf{q}, \bar{\mathbf{q}}, t\right) - \sum_{j=1}^{S}\left(\frac{\partial F_1}{\partial q_j} q_j + \frac{\partial F_1}{\partial \bar{q}_j}\,\bar{q}_j\right)$$
$$= F_1\left(\mathbf{q}, \bar{\mathbf{q}}, t\right) + \sum_{j=1}^{S}\left(\bar{p}_j\,\bar{q}_j - p_j\,q_j\right) \ . \tag{2.167}$$

From the total differential

$$dF_4 = dF_1 + \sum_j\left(d\bar{p}_j\,\bar{q}_j + \bar{p}_j\,d\bar{q}_j - dp_j\,q_j - p_j\,dq_j\right)$$
$$= \sum_{j=1}^{S}\left(p_j\,dq_j - \bar{p}_j\,d\bar{q}_j\right) + \left(\overline{H} - H\right)\,dt \sum_j\left(d\bar{p}_j\,\bar{q}_j + \bar{p}_j\,d\bar{q}_j - dp_j\,q_j - p_j\,dq_j\right)$$
$$= \sum_{j=1}^{S}\left(\bar{q}_j\,d\bar{p}_j - q_j\,dp_j\right) + \left(\overline{H} - H\right)\,dt \tag{2.168}$$

we can again read off the partial derivatives:

$$\bar{q}_j = \frac{\partial F_4}{\partial \bar{p}_j}\ ; \quad q_j = -\frac{\partial F_4}{\partial p_j}\ ; \quad \overline{H} = H + \frac{\partial F_4}{\partial t}\ . \tag{2.169}$$

The transformation formulas, mediated by F_4, follow in this case by inverting and resolving with respect to $\bar{\mathbf{q}}$, $\bar{\mathbf{p}}$. For proving the canonicity of the phase transformation we now use

$$\sum_j p_j\,\dot{q}_j - H = \sum_j \bar{p}_j\,\dot{\bar{q}}_j - \overline{H} + \frac{dF_1}{dt}$$
$$= \sum_j \bar{p}_j\,\dot{\bar{q}}_j - \overline{H} + \frac{dF_4}{dt} - \sum_j \left(\dot{\bar{p}}_j\,\bar{q}_j + \bar{p}_j\,\dot{\bar{q}}_j - \dot{p}_j\,q_j - p_j\,\dot{q}_j\right)$$

and therewith

$$\sum_{j=1}^{S} p_j\dot{q}_j - H = \sum_{j=1}^{S}\left(\dot{p}_j q_j + p_j\dot{q}_j - \dot{\bar{p}}_j\bar{q}_j\right) - \overline{H} + \frac{dF_4}{dt} \tag{2.170}$$

for the modified Hamilton's principle. The resulting expression is now to be varied with respect to $\bar{\mathbf{q}}$, $\bar{\mathbf{p}}$ in order to verify therewith the Hamilton's equations of motion. (This is explicitly performed as Exercise 2.5.1!)

For a better overview we gather the derived transformation formulas once more:

(1) $\boxed{F_1(\mathbf{q}, \bar{\mathbf{q}}, t)}$

$$p_j = \frac{\partial F_1}{\partial q_j}\ ;\ \bar{p}_j = -\frac{\partial F_1}{\partial \bar{q}_j}$$

(2) $\boxed{F_2(\mathbf{q}, \bar{\mathbf{p}}, t)}$

$$p_j = \frac{\partial F_2}{\partial q_j}\ ;\ \bar{q}_j = \frac{\partial F_2}{\partial \bar{p}_j}$$

(3) $\boxed{F_3(\mathbf{p}, \bar{\mathbf{q}}, t)}$

$$q_j = -\frac{\partial F_3}{\partial p_j}\ ;\ \bar{p}_j = -\frac{\partial F_3}{\partial \bar{q}_j}$$

(4) $\boxed{F_4(\mathbf{p}, \bar{\mathbf{p}}, t)}$

$$q_j = -\frac{\partial F_4}{\partial p_j}\ ;\ \bar{q}_j = \frac{\partial F_4}{\partial \bar{p}_j}$$

The time-dependence is in all the four cases the same:

$$\overline{H} = H + \frac{\partial F_i}{\partial t}\ , \qquad i = 1, 2, 3, 4\ . \tag{2.171}$$

2.5.4 Examples of Canonical Transformations

We want to discuss some characteristic applications of the up to now still rather abstract formalism.

(1) Interchange of Momenta and Coordinates

We choose

$$F_1\,(\mathbf{q},\bar{\mathbf{q}},t) = -\sum_{j=1}^{S} q_j\,\bar{q}_j \tag{2.172}$$

and have then produced with

$$p_j = \frac{\partial F_1}{\partial q_j} = -\bar{q}_j\ ; \qquad \bar{p}_j = -\frac{\partial F_1}{\partial \bar{q}_j} = q_j \tag{2.173}$$

an interchange of *momenta* and *coordinates (sites)*:

$$(\mathbf{q},\mathbf{p}) \xrightarrow{F_1} (\bar{\mathbf{p}},-\bar{\mathbf{q}})\ . \tag{2.174}$$

This transformation we got to know already as preparative example with (2.137) and (2.138). Obviously the same effect can be achieved by use of

$$F_4\,(\mathbf{p},\bar{\mathbf{p}},t) = -\sum_{j=1}^{S} p_j \bar{p}_j\ . \tag{2.175}$$

(2) Identity Transformation

We choose

$$F_2\,(\mathbf{q},\bar{\mathbf{p}},t) = \sum_{j=1}^{S} q_j \bar{p}_j \tag{2.176}$$

finding then with (2.161):

$$p_j = \frac{\partial F_2}{\partial q_j} = \bar{p}_j\ ; \quad \bar{q}_j = \frac{\partial F_2}{\partial \bar{p}_j} = q_j\ . \tag{2.177}$$

Thus it is obviously the identity transformation which we could have generated also by

$$F_3\,(\mathbf{p}, \bar{\mathbf{q}}, t) = -\sum_{j=1}^{S} p_j \bar{q}_j \,. \tag{2.178}$$

(3) Point Transformation

If we choose

$$F_2\,(\mathbf{q}, \bar{\mathbf{p}}, t) = \sum_{j=1}^{S} f_j(\mathbf{q}, t)\bar{p}_j \,, \tag{2.179}$$

it follows:

$$\bar{q}_j = \frac{\partial F_2}{\partial \bar{p}_j} = f_j(\mathbf{q}, t) \,, \tag{2.180}$$

which corresponds to a *point transformation* in the configuration space and we have already argued in Sect. 2.5.1 that it is canonical.

As canonically conjugate variables the momenta are of course also affected by the point transformation:

$$p_j = \frac{\partial F_2}{\partial q_j} = \sum_{l=1}^{S} \frac{\partial f_l}{\partial q_j} \bar{p}_l \,. \tag{2.181}$$

These relations are to be solved for the $\bar{p}_l$!

(4) Harmonic Oscillator

We want to demonstrate with this example that a properly chosen canonical transformation can indeed greatly simplify the integration of the equations of motion, sometimes even make them redundant.

According to (2.35) the Hamilton function of the harmonic oscillator reads:

$$H = \frac{p^2}{2m} + \frac{1}{2} m\,\omega_0^2 q^2 \,.$$

We choose the following *generating function*:

$$F_1\,(q, \bar{q}) = \frac{1}{2} m\,\omega_0 q^2\, \cot \bar{q} \,. \tag{2.182}$$

This function we use in (2.151) to get:

$$p = \frac{\partial F_1}{\partial q} = m\,\omega_0 q\,\cot\bar{q}\;, \tag{2.183}$$

$$\bar{p} = -\frac{\partial F_1}{\partial \bar{q}} = \frac{1}{2} m\,\omega_0 q^2 \frac{1}{\sin^2\bar{q}}\;. \tag{2.184}$$

The actual transformation formulas we find by resolving with respect to q and p:

$$q = \sqrt{\frac{2\bar{p}}{m\,\omega_0}}\;\sin\bar{q}\;, \tag{2.185}$$

$$p = \sqrt{2\bar{p}\,m\,\omega_0}\;\cos\bar{q}\;. \tag{2.186}$$

Because of $\partial F_1/\partial t = 0$ we find for the *new* Hamilton function:

$$\begin{aligned}\overline{H}\left(\bar{q},\bar{p}\right) &= H\left(q\left(\bar{q},\bar{p}\right),\,p\left(\bar{q},\bar{p}\right)\right)\\ &= \frac{1}{2m}2\bar{p}\,m\,\omega_0\,\cos^2\bar{q} + \frac{1}{2}m\,\omega_0^2\frac{2\bar{p}}{m\,\omega_0}\,\sin^2\bar{q}\;.\end{aligned}$$

Therewith it takes an especially simple form:

$$\overline{H}\left(\bar{q},\bar{p}\right) = \omega_0\bar{p}\;. \tag{2.187}$$

The coordinate $\bar{q}$ is now cyclic. That means:

$$\bar{p}(t) = \bar{p}_0 = \text{const}\;. \tag{2.188}$$

One has in addition:

$$\begin{aligned}\dot{\bar{q}} &= \frac{\partial\overline{H}}{\partial\bar{p}} = \omega_0\;,\\ \bar{q}(t) &= \omega_0 t + \bar{q}_0\;.\end{aligned} \tag{2.189}$$

The solution is complete if we further insert (2.188) and (2.189) into the transformation formulas (2.185) and (2.186):

$$q(t) = \sqrt{\frac{2\bar{p}_0}{m\,\omega_0}}\;\sin\left(\omega_0 t + \bar{q}_0\right)\;, \tag{2.190}$$

$$p(t) = \sqrt{2\bar{p}_0\,m\,\omega_0}\;\cos\left(\omega_0 t + \bar{q}_0\right)\;. \tag{2.191}$$

That is the already known solution of the harmonic oscillator. $\bar{q}_0$ and $\bar{p}_0$ are fixed by initial conditions. This example illustrates that a physical problem can be decisively simplified by a suitably chosen canonical transformation if this, for instance, makes all coordinates to be cyclic. The new momenta then are all integrals of motion. The far from being a trivial problem, however, consists of course in finding the right generating function (2.182). This, by the way, is just the statement of the central problem of the Hamilton-Jacobi theory which we are going to work out in the next chapter.

(5) Mechanical Gauge Transformation

For this transformation we have already shown in Sect. 2.5.1 that it is canonical. It leads with (2.129) to the following transformation formulas:

$$\bar{q}_j = q_j ; \quad \bar{p}_j = p_j + \frac{\partial f}{\partial q_j} ; \quad \overline{H} = H - \frac{\partial f}{\partial t} . \tag{2.192}$$

Thereby

$$f = f(\mathbf{q}, t)$$

is an arbitrary function of the coordinates and the time. The transformation (2.192) corresponds to the already several times discussed re-calibration of the Lagrangian,

$$L \longrightarrow \overline{L} = L + \frac{df}{dt} ,$$

which lets the Lagrange's equations of motion invariant. It is generated by:

$$F_2\,(\mathbf{q}, \bar{\mathbf{p}}) = \sum_{j=1}^{S} q_j \bar{p}_j - f(\mathbf{q}, t) . \tag{2.193}$$

since it follows with (2.161):

$$\bar{q}_j = \frac{\partial F_2}{\partial \bar{p}_j} = q_j ; \quad p_j = \frac{\partial F_2}{\partial q_j} = \bar{p}_j - \frac{\partial f}{\partial q_j} ,$$

$$\overline{H} = H + \frac{\partial F_2}{\partial t} = H - \frac{\partial f}{\partial t} .$$

This is exactly related to (2.192).

2.5.5 Criteria for Canonicity

How can we recognize whether or not a given phase transformation

$$\bar{q}_j = \bar{q}_j(\mathbf{q}, \mathbf{p}, t)\ ; \quad \bar{p}_j = \bar{p}_j(\mathbf{q}, \mathbf{p}, t)\ , \qquad j = 1, 2, \ldots, S \tag{2.194}$$

is canonical if the corresponding generating function is not explicitly known? We discuss two different procedures.

(1) We solve (2.194) for $\mathbf{p}$ and $\bar{\mathbf{p}}$:

$$p_j = p_j\left(\mathbf{q}, \bar{\mathbf{q}}, t\right)\ ; \quad \bar{p}_j = \bar{p}_j\left(\mathbf{q}, \bar{\mathbf{q}}, t\right)\ . \tag{2.195}$$

In the case that the transformation is canonical there must exist a generating function $F_1(\mathbf{q}, \bar{\mathbf{q}}, t)$ with

$$p_j = \frac{\partial F_1}{\partial q_j}; \quad \bar{p}_j = -\frac{\partial F_1}{\partial \bar{q}_j}, \qquad j = 1, 2, \ldots, S\ .$$

But that also means:

$$\frac{\partial p_j}{\partial \bar{q}_m} = \frac{\partial^2 F_1}{\partial \bar{q}_m \partial q_j} = \frac{\partial^2 F_1}{\partial q_j \partial \bar{q}_m} = -\frac{\partial \bar{p}_m}{\partial q_j}\ .$$

We thus investigate whether

$$\left(\frac{\partial p_j}{\partial \bar{q}_m}\right)_{\substack{q,t \\ \bar{q}_{l,l\neq m}}} \overset{!}{=} -\left(\frac{\partial \bar{p}_m}{\partial q_j}\right)_{\substack{q,t \\ \bar{q}_{l,l\neq j}}} \tag{2.196}$$

is valid for all pairs of indexes (j, m). Analogously thereto it must of course also hold:

$$\left(\frac{\partial p_j}{\partial q_m}\right)_{\substack{\bar{q},t \\ q_{l,l\neq m}}} \overset{!}{=} \left(\frac{\partial p_m}{\partial q_j}\right)_{\substack{\bar{q},t \\ q_{l,l\neq j}}}\ , \tag{2.197}$$

$$\left(\frac{\partial \bar{p}_j}{\partial \bar{q}_m}\right)_{\substack{q,t \\ \bar{q}_{l,l\neq m}}} \overset{!}{=} \left(\frac{\partial \bar{p}_m}{\partial \bar{q}_j}\right)_{\substack{q,t \\ \bar{q}_{l,l\neq j}}}\ . \tag{2.198}$$

It stands immediately to reason that in spite of the simple concept the practical handling of these formulas will be rather cumbersome. The procedure to be discussed next will turn out to be essentially more convenient.

The solution of the transformation formulas (2.194) for $\mathbf{p}$ and $\bar{\mathbf{p}}$ as in (2.195) is of course not at all compulsory. Important for the solution is only that it is done for an *old* in combination with a *new* coordinate. Further possible combinations are therefore:

$$q_j = q_j(\mathbf{p}, \bar{\mathbf{p}}, t) \;\; ; \quad \bar{q}_j = \bar{q}_j(\mathbf{p}, \bar{\mathbf{p}}, t) \iff F_4(\mathbf{p}, \bar{\mathbf{p}}, t) \tag{2.199}$$

$$q_j = q_j(\mathbf{p}, \bar{\mathbf{q}}, t) \;\; ; \quad \bar{p}_j = \bar{p}_j(\mathbf{p}, \bar{\mathbf{q}}, t) \iff F_3(\mathbf{p}, \bar{\mathbf{q}}, t) \;, \tag{2.200}$$

$$p_j = p_j(\mathbf{q}, \bar{\mathbf{p}}, t) \;\; ; \quad \bar{q}_j = \bar{q}_j(\mathbf{q}, \bar{\mathbf{p}}, t) \iff F_2(\mathbf{q}, \bar{\mathbf{p}}, t) \;. \tag{2.201}$$

(2) We introduce the second method for examining the canonicity of a phase transformation in the form of a theorem:

Theorem 2.5.2 *The phase transformation (2.194) is canonical if and only if the fundamental Poisson brackets in the* new *variables,*

$$\left\{\bar{q}_i, \bar{p}_j\right\} = \delta_{ij} \;, \tag{2.202}$$

$$\left\{\bar{q}_i, \bar{q}_j\right\} = \left\{\bar{p}_i, \bar{p}_j\right\} = 0 \;, \tag{2.203}$$

are fulfilled.

Proof We present the proof for **not** explicitly time-dependent phase transformations,

$$\frac{\partial F}{\partial t} = 0 \iff \overline{H}(\bar{\mathbf{q}}, \bar{\mathbf{p}}, t) = H\left(\mathbf{q}(\bar{\mathbf{q}}, \bar{\mathbf{p}}) \,,\, \mathbf{p}(\bar{\mathbf{q}}, \bar{\mathbf{p}}) \,,\, t\right) \;,$$

which we therefore investigate for *canonicity in the narrower sense*. After the theorem proven in Sect. 2.4.2 the Poisson bracket is independent of the set of canonical coordinates which are chosen as basis. Let us take here the *old* variables $\mathbf{q}$ and $\mathbf{p}$. According to the general equation of motion (2.105) it holds at first:

$$\dot{\bar{q}}_j = \left\{\bar{q}_j, H\right\}_{\mathbf{q},\mathbf{p}} = \sum_{l=1}^{S}\left(\frac{\partial \bar{q}_j}{\partial q_l}\frac{\partial H}{\partial p_l} - \frac{\partial \bar{q}_j}{\partial p_l}\frac{\partial H}{\partial q_l}\right) \;,$$

$$\dot{\bar{p}}_j = \left\{\bar{p}_j, H\right\}_{\mathbf{q},\mathbf{p}} = \sum_{l=1}^{S}\left(\frac{\partial \bar{p}_j}{\partial q_l}\frac{\partial H}{\partial p_l} - \frac{\partial \bar{p}_j}{\partial p_l}\frac{\partial H}{\partial q_l}\right) \;.$$

The partial derivatives of the Hamilton function H can be written as follows:

$$\frac{\partial H}{\partial p_l} = \sum_{k=1}^{S}\left(\frac{\partial \overline{H}}{\partial \bar{q}_k}\frac{\partial \bar{q}_k}{\partial p_l} + \frac{\partial \overline{H}}{\partial \bar{p}_k}\frac{\partial \bar{p}_k}{\partial p_l}\right) \;,$$

$$\frac{\partial H}{\partial q_l} = \sum_{k=1}^{S}\left(\frac{\partial \overline{H}}{\partial \bar{q}_k}\frac{\partial \bar{q}_k}{\partial q_l} + \frac{\partial \overline{H}}{\partial \bar{p}_k}\frac{\partial \bar{p}_k}{\partial q_l}\right) \;.$$

That we insert into the above expression for $\dot{\bar{q}}_j$:

$$\dot{\bar{q}}_j = \sum_{l,k}\left[\frac{\partial \bar{q}_j}{\partial q_l}\left(\frac{\partial \overline{H}}{\partial \bar{q}_k}\frac{\partial \bar{q}_k}{\partial p_l}+\frac{\partial \overline{H}}{\partial \bar{p}_k}\frac{\partial \bar{p}_k}{\partial p_l}\right)-\frac{\partial \bar{q}_j}{\partial p_l}\left(\frac{\partial \overline{H}}{\partial \bar{q}_k}\frac{\partial \bar{q}_k}{\partial q_l}+\frac{\partial \overline{H}}{\partial \bar{p}_k}\frac{\partial \bar{p}_k}{\partial q_l}\right)\right].$$

This can be summarized as follows:

$$\dot{\bar{q}}_j = \sum_k\left[\frac{\partial \overline{H}}{\partial \bar{q}_k}\left\{\bar{q}_j,\bar{q}_k\right\}_{\mathbf{q},\mathbf{p}}+\frac{\partial \overline{H}}{\partial \bar{p}_k}\left\{\bar{q}_j,\bar{p}_k\right\}_{\mathbf{q},\mathbf{p}}\right]. \tag{2.204}$$

In the same manner one finds:

$$\dot{\bar{p}}_j = \sum_k\left[-\frac{\partial \overline{H}}{\partial \bar{q}_k}\left\{\bar{q}_k,\bar{p}_j\right\}_{\mathbf{q},\mathbf{p}}+\frac{\partial \overline{H}}{\partial \bar{p}_k}\left\{\bar{p}_j,\bar{p}_k\right\}_{\mathbf{q},\mathbf{p}}\right]. \tag{2.205}$$

Hamilton's equations of motion,

$$\dot{\bar{q}}_j = \frac{\partial \overline{H}}{\partial \bar{p}_j}\ ; \quad \dot{\bar{p}}_j = -\frac{\partial \overline{H}}{\partial \bar{q}_j}\ , \tag{2.206}$$

are thus valid if and only if (2.202) and (2.203) are fulfilled. Exactly this was to be proven.

Theorem 2.5.2 turns out to be a rather handy criterion for the canonicity of a given phase transformation.

2.5.6 Exercises

Exercise 2.5.1 Let $(\mathbf{q},\mathbf{p}) \longrightarrow (\bar{\mathbf{q}},\bar{\mathbf{p}})$ be a phase transformation for which

$$\sum_{j=1}^{S} p_j\dot{q}_j - H = \sum_{j=1}^{S}\left(\dot{p}_j q_j + p_j\dot{q}_j - \dot{\bar{p}}_j\bar{q}_j\right) - \overline{H} + \frac{dF_4}{dt}$$

is valid where $F_4 = F_4(\mathbf{p},\bar{\mathbf{p}},t)$ is an arbitrary function of the *old* and the *new* momenta. Show that

1. $\overline{H}$ and the phase transformation

$$\bar{q}_j = \bar{q}_j(\mathbf{q},\mathbf{p},t)\ ; \quad \bar{p}_j = \bar{p}_j(\mathbf{q},\mathbf{p},t)$$

   are completely determined by the *generating function* F_4.
2. The transformation mediated by F_4 is canonical.

Exercise 2.5.2 Is it possible that two components of the angular momentum, e.g. L_x, L_y, appear simultaneously as canonical momenta? Explain

Exercise 2.5.3 Investigate whether the following transformation is canonical:

$$\bar{q} = \ln\left(\frac{\sin p}{q}\right) \; ; \quad \bar{p} = q \cot p \; .$$

Exercise 2.5.4 q, p are canonically conjugate variables. By the transformation

$$\bar{q} = \ln\left(1 + \sqrt{q}\cos p\right) \; ,$$
$$\bar{p} = 2\left(1 + \sqrt{q}\cos p\right)\sqrt{q}\sin p$$

new coordinates $\bar{q}, \bar{p}$ are defined.

1. Show that the transformation is canonical.
2. Show that the transformation is generated by

$$F_3\left(p, \bar{q}, t\right) = -\left(\mathrm{e}^{\bar{q}} - 1\right)^2 \tan p$$

Exercise 2.5.5

1. Let p and q be canonically conjugate variables. Is the following transformation canonical?

$$\hat{q} = \hat{q}\left(p, q\right) = \arcsin \frac{q}{\sqrt{q^2 + \dfrac{p^2}{\alpha^2}}}$$

$$\hat{p} = \hat{p}\left(p, q\right) = \frac{1}{2}\left(\alpha q^2 + \frac{p^2}{\alpha}\right)$$

   α is a real constant.
2. Which transformation is provided by the following generating function?

$$F_1\left(q, \hat{q}\right) = \frac{1}{2}\alpha q^2 \cot \hat{q}$$

Exercise 2.5.6 A mechanical system with the Hamilton function

$$H = \frac{1}{2m}p^2 q^4 + \frac{k}{2q^2}$$

is given as well as the *generating function* of a canonical transformation:

$$F_1\left(q, \bar{q}\right) = -\sqrt{m\,k}\;\frac{\bar{q}}{q} \; .$$

1. What are the transformation formulas

$$p = p\,(\bar{q}, \bar{p}) \;\; ; \quad q = q\,(\bar{q}, \bar{p}) \; ?$$

2. What is the new Hamilton function

$$\overline{H} = \overline{H}\,(\bar{q}, \bar{p}) \; ?$$

3. Find the solution of the problem for the variables $\bar{q}, \bar{p}$!

Exercise 2.5.7 A system is described by the Hamilton function

$$H\,(q,p) = \frac{3}{2}\beta q p \qquad (\beta \in \mathbb{R})$$

where q and p are conjugate variables.

1. Which phase transformation $(q,p) \;\to\; (\hat{q}, \hat{p})$ is provided by the generating function

$$F_2\,(q, \hat{p}) = \alpha q^2 \hat{p}^3 \qquad (\alpha \in \mathbb{R}) \; ?$$

2. Show that the transformation is indeed canonical.
3. Write down

$$\hat{H}\,(\hat{q}, \hat{p})$$

4. What are the equations of motion for the *'new'* variables $\hat{q}$ and $\hat{p}$?

Exercise 2.5.8 For which values α and β is the phase transformation

$$\bar{q} = q^{\alpha} \; \cos\,(\beta\, p) \;\; ; \quad \bar{p} = q^{\alpha} \; \sin\,(\beta\, p)$$

canonical?

Exercise 2.5.9 For a one-dimensional system a transformation of variables is performed,

$$(q,p) \;\longrightarrow\; (\bar{q}, \bar{p}) \; ,$$

where it is found:

$$\bar{q} = q^k p^l \; ; \quad \bar{p} = q^m p^n \; ; \qquad k, l, m, n \in \mathbb{R} \; .$$

1. How are k, l, m, n to be chosen in order to make the transformation canonical?
2. Which (canonical) transformation comes out in particular for $m = 0$?

3. Determine the generating functions of the canonical transformations from 1. if these are of the type

$$F_1 = F_1(q, \bar{q}) \; !$$

4. How would a generating function of the type

$$F_2 = F_2(q, \bar{p})$$

look like?

Exercise 2.5.10 An electron (mass m, charge $-e$) moves in a homogeneous magnetic field

$$\mathbf{B} = (0, 0, B) = \operatorname{rot} \mathbf{A} \; .$$

For the vector potential $\mathbf{A}$ the Coulomb-gauge shall be valid:

$$\operatorname{div} \mathbf{A} = 0 \; .$$

1. Show that

$$\mathbf{A}(\mathbf{r}) = \frac{1}{2} B(-y, x, 0)$$

is a thinkable representation of the multiple-valued vector potential.
2. Take

$$q_1 = x \; , \quad q_2 = y \; , \quad q_3 = z \; .$$

and verify the following form of the Hamilton function

$$H = \frac{p_3^2}{2m} + H_0$$
$$H_0 = \frac{1}{2m} \left(p_1 - \frac{1}{2} m \omega_c q_2 \right)^2 + \frac{1}{2m} \left(p_2 + \frac{1}{2} m \omega_c q_1 \right)^2$$

with

$$\omega_c = \frac{eB}{m} \; .$$

3. Consider from now on exclusively $H_0 (= H(p_3 \equiv 0))$. Let a phase transformation

$$(\mathbf{q}, \mathbf{p}) \longrightarrow (\hat{\mathbf{q}}, \hat{\mathbf{p}})$$

be due to the generating function

$$F_1(\mathbf{q}, \hat{\mathbf{q}}) = m\omega_c \left(q_1\hat{q}_1 + q_2\hat{q}_2 - \hat{q}_1\hat{q}_2 - \frac{1}{2}q_1q_2 \right)$$

Calculate the transformation formulas

$$\mathbf{q} = \mathbf{q}(\hat{\mathbf{q}}, \hat{\mathbf{p}}) \ , \quad \mathbf{p} = \mathbf{p}(\hat{\mathbf{q}}, \hat{\mathbf{p}}) \ ,$$
$$\hat{\mathbf{q}} = \hat{\mathbf{q}}(\mathbf{q}, \mathbf{p}) \ , \quad \hat{\mathbf{p}} = \hat{\mathbf{p}}(\mathbf{q}, \mathbf{p}) \ .$$

4. How does the transformed Hamilton function

$$\hat{H} = \hat{H}(\hat{\mathbf{q}}, \hat{\mathbf{p}})$$

look like? What kind of problem of motion remains to be solved?

5. Try to trace back the transformation formulas from part 3. to a generating function of type

$$F_2 = F_2(\mathbf{q}, \hat{\mathbf{p}}) \ .$$

2.6 Self-examination Questions

To Section 2.1

1. What is the objective of the Hamilton mechanics?
2. Contrast the advantages and disadvantages of Newton's formulation of classical mechanics with those of the Lagrange version!
3. Which transformation of variables does produce the transition from Lagrange to Hamilton mechanics?
4. How is the Legendre-transform of the function $f(x, y)$ with respect to the variable y defined?

To Section 2.2

1. Which are the active and which are the passive variables of the transformation from the Lagrangian to the Hamilton function?
2. Formulate Hamilton's equations of motion!
3. Under which conditions is H identical to the total energy of the system?
4. Demonstrate that total and partial time derivative of H are equal.
5. Which advantage is brought about in the Hamilton-formalism by cyclic coordinates?
6. What is the idea of the Routh-formalism? How is the Routh function related to the Hamilton function?
7. How does one find the Hamilton function of a physical system?

8. What is the Hamilton function of the harmonic oscillator?
9. Which Hamilton function does describe the motion of a particle of mass m and charge $\bar{q}$ in an electromagnetic field?
10. Formulate the Hamilton function in cylindrical and in spherical coordinates for a particle of mass m that is subject to a conservative force but not to constraints.

To Section 2.3

1. List and comment the most important integral principles of classical mechanics!
2. What do we understand by the *modified Hamilton's principle*?
3. Formulate precisely the variational prescription for the *modified Hamilton's principle*!
4. Which are the characteristic differences between the *original* and the *modified Hamilton's principle*?
5. How is the *action A* defined?
6. What does the *principle of least action* say?
7. By what do the variational prescriptions for the Hamilton's principle and the principle of least action differ from each other?
8. Which special case is treated by *Fermat's principle*?
9. What do we understand by the *principle of least time* and what by the *principle of the shortest way*?
10. How does the Jacobi's principle differ from the *principle of least action*?
11. What do we understand by the *metric tensor*?
12. Formulate the Jacobi's principle for the force-free movement!

To Section 2.4

1. Is a mechanical problem solved with the specification of the configuration path? Give reasons for your answer!
2. What do we understand by the *event space*?
3. In which representation spaces do, respectively, Lagrange and Hamilton formalism work?
4. How does the *path in the event space* of the linear harmonic oscillator look like, how is its phase trajectory?
5. Define the state space!
6. What does the term *state* ψ mean?
7. Which minimum information must be given in order to fix all mechanical properties of a general N-particle system?
8. Why must the time evolution of a state ψ necessarily follow from a differential equation of first order?
9. Why is the configuration $\mathbf{q}(t)$ of a mechanical system not yet a *state*?
10. What is the definition of the Poisson bracket?
11. Formulate the equation of motion of an arbitrary phase function $f(\mathbf{q}, \mathbf{p}, t)$ by the use of the Poisson bracket!
12. In which manner does the Poisson bracket depend on the choice of the canonical variables $(\mathbf{q}, \mathbf{p})$?
13. What are the *fundamental Poisson brackets*?

14. List some formal properties of the Poisson bracket!
15. What is the Jacobi identity?
16. How can we use the Poisson bracket in order to check whether $F(\mathbf{q}, \mathbf{p}, t)$ represents an integral of motion?
17. What does *Poisson's theorem* state?
18. Give reasons why classical mechanics and quantum mechanics can be understood as different realizations of the same superordinate abstract mathematical structure!

To Section 2.5

1. What is the actual reason for the invariance of the Lagrange's equations of motion with respect to point transformations in the configuration space and with respect to mechanical gauge transformations?
2. How does the Hamilton function change when a mechanical gauge transformation is performed? What does thereby happen to the canonical equations?
3. Are by the generalized coordinates $q_1, \ldots, q_s$ the corresponding generalized momenta p_j uniquely fixed?
4. What is a phase transformation?
5. What is the meaning of a canonical transformation? When is it called *canonical in the narrower sense*?
6. Name, in connection with the modified Hamilton's principle, a first criterion for a phase transformation $(\mathbf{q}, \mathbf{p}) \longrightarrow (\bar{\mathbf{q}}, \bar{\mathbf{p}})$ to be canonical!
7. What do we understand by the *generating function* of a canonical transformation?
8. Which types of generating functions for canonical transformations do you know? What is their common feature?
9. Give at least two generating functions for a phase transformation that interchanges momenta and positions!
10. Which *generating function* does provide the identity transformation?
11. How does the *generating function* for a point transformation in the configuration space look like? What happens thereby to the canonical momenta?
12. Find at least two criteria for the canonicity of a phase transformation!

Chapter 3
Hamilton-Jacobi Theory

The considerations of the last section of the last chapter concerning canonical transformations reveal such a manifold of transformation possibilities that it should in fact be possible to construct by it efficient

general methods for the solution of mechanical problems

We therefore investigate now which way a Hamilton function H has to be transformed in order to make the solution of the physical problem as simple as possible, may be even trivial. The following methods, for instance, may offer themselves:

1. One chooses the transformation in such a way that in the new variables $\bar{\mathbf{q}}$, $\bar{\mathbf{p}}$ the transformed Hamilton function $\overline{H}$ constitutes a known, already solved problem (e.g. harmonic oscillator, see Exercise 2.5.6!).
2. One chooses the transformation in such a way that all new variables $\bar{q}_j$ are cyclic. In Sect. 2.5.1 we had already shown that then the integration of the equations of motion becomes trivial, at least if we can additionally assume:

$$\frac{\partial H}{\partial t} = 0$$

What then simply remains is:

$$\bar{p}_j = \alpha_j = \text{const} , \qquad j = 1, \ldots, S ,$$

$$\overline{H} = \overline{H}(\alpha) ; \qquad \omega_j = \frac{\partial \overline{H}}{\partial \alpha_j} ,$$

$$\bar{q}_j = \omega_j \, t + \beta_j , \qquad j = 1, 2, \ldots, S .$$

The $2S$ constants α_j, β_j are eventually fixed by initial conditions.

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3. One chooses the transformation in such a way that

$$\bar{q}_j = \beta_j = \text{const} \; ; \quad \bar{p}_j = \alpha_j = \text{const} \; , \quad j = 1, 2, \ldots, S \; .$$

The solution is then *'simply'* found by inversion of the transformation,

$$\mathbf{q} = \mathbf{q}(\boldsymbol{\beta}, \boldsymbol{\alpha}, t) \; ; \quad \mathbf{p} = \mathbf{p}(\boldsymbol{\beta}, \boldsymbol{\alpha}, t) \; ,$$

where the β_j, α_j are again fixed by initial conditions.

The problem is, however, in finding the canonical transformations which fit 1., 2., 3..

3.1 Hamilton-Jacobi Equation

The procedure 1. appears of course rather special, realizable surely only for a few special cases. The procedures 2. and 3. are more general, where 3. has the advantage over 2. to be applicable also for systems with an explicit time-dependence of the Hamilton function. We will therefore concentrate ourselves here on method 3..

So we seek a canonical transformation after which the *new* variables $\bar{\mathbf{q}}$ and $\bar{\mathbf{p}}$ are constant in time. That surely holds if the transformation achieves for the *new* Hamilton function

$$\overline{H} = H + \frac{\partial F}{\partial t} \equiv 0 \; . \tag{3.1}$$

That would namely trivially mean:

$$\dot{\bar{q}}_j = \frac{\partial \overline{H}}{\partial \bar{p}_j} = 0 \implies \bar{q}_j = \beta_j = \text{const} \; , \qquad j = 1, 2, \ldots, S \; ,$$

$$\dot{\bar{p}}_j = -\frac{\partial \overline{H}}{\partial \bar{q}_j} = 0 \implies \bar{p}_j = \alpha_j = \text{const} \; , \qquad j = 1, 2, \ldots, S$$

It is advisable, but not at all necessary, to choose the generating function F as of the type F_2,

$$F_2 = F_2 \left(\mathbf{q}, \bar{\mathbf{p}}, t \right) \; ,$$

Then it holds according to (2.161):

$$p_j = \frac{\partial F_2}{\partial q_j} \; ; \quad \bar{q}_j = \frac{\partial F_2}{\partial \bar{p}_j} \; .$$

If we insert this into (3.1) it results in the

Hamilton-Jacobi differential equation

$$H\left(q_1, \dots, q_S, \frac{\partial F_2}{\partial q_1}, \dots, \frac{\partial F_2}{\partial q_S}, t\right) + \frac{\partial F_2}{\partial t} = 0 \,. \tag{3.2}$$

From this equation the generating function F_2 must be determined. We want to discuss this equation a bit in order to be able to develop a practicable method of solution.

1. The solution is called, from reasons which later become clear,

   Hamilton's action function $F_2 = S$.

2. The Hamilton-Jacobi differential equation (HJD) represents a

   non-linear partial differential equation of first order for F_2 in $(S+1)$ variables $q_1, \dots, q_S, t$

   being therewith in general mathematically not so easy to deal with. It is *non-linear* since H depends quadratically on the momenta and therewith on $\partial F_2 / \partial q_j$. There appear only partial derivatives of first order with respect to q_j and t.
3. The HJD contains $(S+1)$ different derivatives of the required function F_2. After the integration one therefore finds $(S+1)$ constants of integration. But since F_2 appears in the HJD only in the form $\partial F_2/\partial q_j$ or $\partial F_2/\partial t$ $F_2 + C$ the solution F_2 is unique only up to an additive constant. One of the integration constants is therefore trivially additive:

   $$\textbf{Solution:} \qquad F_2\left(q_1, \dots, q_S, t \,|\, \alpha_1, \alpha_2, \dots, \alpha_S\right) + \alpha_{S+1} \,. \tag{3.3}$$

   α_{S+1} is unimportant since in the transformation formulas (2.161) only the derivatives of F_2 play a role. One calls (3.3) a **complete solution** of the HJD.
4. The HJD determines only the $\mathbf{q}$ and the t dependences of the solution $F_2 = F_2(\mathbf{q}, \bar{\mathbf{p}}, t)$ without making any statement about the momenta $\bar{p}_j$. However, we intend to have the $\bar{p}_j = $ const having therewith the freedom to identify the integration constants with the *new* momenta:

   $$\bar{p}_j = \alpha_j \,, \qquad j = 1, 2, \dots, S \,. \tag{3.4}$$

   Following these considerations we now construct a **method of solution**:

   (a) Formulate $H = H(\mathbf{q}, \mathbf{p}, t)$, insert $p_j = (\partial F_2/\partial q_j)$, and establish the HJD.
   (b) Solve the HJD for F_2,

   $$F_2 = S\left(q_1, \dots, q_S, t \,|\, \alpha_1, \dots, \alpha_S\right) \,, \tag{3.5}$$

   and identify the integration constants with the *new* momenta:

   $$\bar{p}_j = \alpha_j \,, \qquad j = 1, \dots, S \,. \tag{3.6}$$

(c) Take

$$\bar{q}_j = \frac{\partial S\,(\mathbf{q},t\,|\,\boldsymbol{\alpha})}{\partial \alpha_j} = \bar{q}_j\,(\mathbf{q},t\,|\,\boldsymbol{\alpha}) = \beta_j\;, \qquad j=1,\ldots,S\,. \tag{3.7}$$

These are S equations which are to be solved for the coordinates $q_1,\ldots,q_S$:

$$q_j = q_j\,(t\,|\,\beta_1,\ldots,\beta_S,\,\alpha_1,\ldots,\alpha_S) = q_j\,(t\,|\,\boldsymbol{\beta},\boldsymbol{\alpha})\;\;, \qquad j=1,\ldots,S\,. \tag{3.8}$$

(d) Calculate the momenta from

$$p_j = \frac{\partial S(\mathbf{q},t\,|\,\boldsymbol{\alpha})}{\partial q_j} = p_j(\mathbf{q},t\,|\,\boldsymbol{\alpha})\;, \qquad j=1,\ldots,S \tag{3.9}$$

and insert therein the coordinates from (3.8):

$$p_j = p_j\,(t\,|\,\beta_1,\ldots,\beta_S,\,\alpha_1,\ldots,\alpha_S) = p_j(t\,|\,\boldsymbol{\beta},\boldsymbol{\alpha})\;, \quad j=1,\ldots,S\,. \tag{3.10}$$

(e) The initial conditions

$$q_j^{(0)} = q_j\,(t=t_0)\;\;; \quad p_j^{(0)} = p_j\,(t=t_0)\;\;, \qquad j=1,\ldots,S$$

yield with (3.9):

$$\boldsymbol{\alpha} = \boldsymbol{\alpha}\left(t_0;\mathbf{p}^{(0)},\,\mathbf{q}^{(0)}\right)\;. \tag{3.11}$$

With (3.8) $\boldsymbol{\beta}$, too, is then determined:

$$\boldsymbol{\beta} = \boldsymbol{\beta}\left(t_0;\mathbf{p}^{(0)},\,\mathbf{q}^{(0)}\right)\;. \tag{3.12}$$

(f) The so derived $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are inserted into (3.8) and (3.10) so that the mechanical problem is solved.

Finally we want to discuss the

physical meaning of the HJD solution.

Until now $F_2 = S(\mathbf{q},\bar{\mathbf{p}},t)$ is merely the generating function of a special canonical transformation which takes care for $\overline{H} \equiv 0$ and consequently for the desired transition

$$\boldsymbol{\pi} \equiv (\mathbf{q},\mathbf{p}) \;\stackrel{S}{\longrightarrow}\; \overline{\boldsymbol{\pi}} \equiv (\boldsymbol{\beta},\boldsymbol{\alpha}) = \text{const}\,. \tag{3.13}$$

The total temporal derivative of S makes the physical meaning clearer:

$$\frac{dF_2}{dt} = \sum_{j=1}^{S} \left(\frac{\partial F_2}{\partial q_j} \dot{q}_j + \frac{\partial F_2}{\partial \bar{p}_j} \dot{\bar{p}}_j \right) + \frac{\partial F_2}{\partial t} .$$

For $F_2 = S$ it particularly holds:

$$\frac{\partial F_2}{\partial q_j} = p_j ; \quad \dot{\bar{p}}_j \equiv 0 ; \quad \frac{\partial F_2}{\partial t} = \overline{H} - H = -H .$$

Hence it remains:

$$\frac{dS}{dt} = \sum_{j=1}^{S} p_j \dot{q}_j - H = L . \tag{3.14}$$

S is therefore just the *action functional* that we got to know in connection with the Hamilton's principle,

$$S = \int L \, dt + \text{const} , \tag{3.15}$$

and that for a system which at the time $t = t_0$ fulfills the initial conditions $\mathbf{q} = \mathbf{q}^{(0)}$, $\mathbf{p} = \mathbf{p}^{(0)}$. Equation (3.15) serves here of course only as the physical interpretation of the HJD solution. It can not at all be used for the determination of S. For such a case $\mathbf{q}(t)$ and $\dot{\mathbf{q}}(t)$ have to be known for the actual motion of the system in order to be inserted into L. But then the problem would already have been completely solved.

It is interesting to remember that previously we could derive the Lagrange- and Hamilton-equations of motion from Hamilton's principle by the use of the **definite** action integral

$$\int_{t_1}^{t_2} L \, dt .$$

These equations, together with an initial phase $\boldsymbol{\pi}^{(0)}$, fix the total phase trajectory $\boldsymbol{\pi}(t)$:

$$\boldsymbol{\pi}^{(0)} \equiv \left(\mathbf{q}^{(0)}, \mathbf{p}^{(0)}\right) \xrightarrow{\int_{t_1}^{t_2} L\, dt} \boldsymbol{\pi}(t) \equiv (\mathbf{q}(t), \mathbf{p}(t)) , \tag{3.16}$$

The solution of the HJD, in contrast, is the **indefinite** action integral (3.15) which now can be interpreted practically as the generating function of the transformation reverse to (3.16):

$$\boldsymbol{\pi}(t) \equiv (\mathbf{q}(t),\ \mathbf{p}(t)) \stackrel{\int L\,dt + \text{const}}{\longrightarrow} \overline{\boldsymbol{\pi}} \equiv (\boldsymbol{\beta}, \boldsymbol{\alpha})\ . \tag{3.17}$$

3.2 The Method of Solution

By the simple example of the linear harmonic oscillator we want to illustrate the Hamilton-Jacobi procedure which was developed in the last section. To make it as clear as possible we strictly proceed according to the formal scheme presented there.
To (a):
The Hamilton function of the harmonic oscillator reads (2.35):

$$H = \frac{p^2}{2m} + \frac{1}{2}\, m\, \omega_0^2 q^2\ . \tag{3.18}$$

We seek a canonical transformation which makes $\overline{H} = 0$. Let the corresponding generating function be of the form

$$F_2 = F_2\,(q, \bar{p}, t) = S\,(q, \bar{p}, t) \tag{3.19}$$

with

$$p = \frac{\partial S}{\partial q}\ . \tag{3.20}$$

According to (3.2) one then finds the Hamilton-Jacobi differential equation:

$$\frac{1}{2m}\left(\frac{\partial S}{\partial q}\right)^2 + \frac{1}{2} m\, \omega_0^2 q^2 + \frac{\partial S}{\partial t} = 0\ . \tag{3.21}$$

To (b):
We choose the following solution approach:

$$S\,(q, \bar{p}, t) = W\,(q\,|\,\bar{p}) + V\,(t\,|\,\bar{p})\ \ . \tag{3.22}$$

Insertion into the HJD yields:

$$\frac{1}{2m}\left(\frac{\partial W}{\partial q}\right)^2 + \frac{1}{2} m\, \omega_0^2 q^2 = -\frac{\partial V}{\partial t}\ .$$

The solution approach (*'ansatz'*) (3.22), which one calls a **'separation approach (ansatz)'**, allows to split the HJD into a part that depends only on q (left side) and another one that depends only on t (right side). Therefore each side of the equation must necessarily be constant separately equal to the same. The original partial differential equation decomposes therewith into two ordinary differential equations:

$$\frac{1}{2m}\left(\frac{dW}{dq}\right)^2 + \frac{1}{2}m\,\omega_0^2 q^2 = \alpha \ , \tag{3.23}$$

$$\frac{dV}{dt} = -\alpha \ . \tag{3.24}$$

Equation (3.24) yields immediately

$$V(t) = -\alpha\, t + V_0 \ , \tag{3.25}$$

where the additive constant V_0 is unimportant. From (3.23) we get:

$$\left(\frac{dW}{dq}\right)^2 = m^2\omega_0^2\left(\frac{2\alpha}{m\,\omega_0^2} - q^2\right) \ . \tag{3.26}$$

The generating function we are looking for is therewith:

$$S(q,\alpha,t) = m\,\omega_0 \int dq\,\sqrt{\frac{2\alpha}{m\,\omega_0^2} - q^2} - \alpha\, t \ . \tag{3.27}$$

The indefinite integral delivers the unessential constant $C = \alpha_{S+1}$. It is a standard-integral so that the integration can be carried out without any difficulty:

$$S(q,\alpha,t) = m\,\omega_0\left[\frac{1}{2}q\,\sqrt{\frac{2\alpha}{m\,\omega_0^2} - q^2} + \frac{\alpha}{m\,\omega_0^2}\ \arcsin\left(q\sqrt{\frac{m\,\omega_0^2}{2|\alpha|}}\right)\right] - \alpha\, t + C \tag{3.28}$$

At this stage, however, the integration is in principle not necessary because later we will be interested only in the partial derivatives of S.

We identify the constant α with the *new* momentum:

$$\bar{p} = \alpha \ . \tag{3.29}$$

To (c):
We take:

$$\bar{q} = \frac{\partial S}{\partial \alpha} \stackrel{!}{=} \text{const} = \beta \ . \tag{3.30}$$

This means according to (3.27):

$$\beta = \frac{1}{\omega_0} \int dq \left\{ \frac{2\alpha}{m\,\omega_0^2} - q^2 \right\}^{-1/2} - t\,.$$

This is again a standard-integral:

$$\beta + t = \frac{1}{\omega_0}\, \arcsin\left(q\,\omega_0 \sqrt{\frac{m}{2\alpha}} \right)\,. \tag{3.31}$$

The solution for q reads:

$$q = \frac{1}{\omega_0} \sqrt{\frac{2\alpha}{m}}\, \sin\left(\omega_0(t+\beta)\right) = q(t\,|\,\beta,\alpha)\,. \tag{3.32}$$

The *new* coordinate $\bar{q} = \beta$ obviously has the dimension *'time'*.
To (d):
We now use (3.9) and (3.26),

$$p = \frac{\partial S}{\partial q} = \frac{dW}{dq} = m\,\omega_0 \sqrt{\frac{2\alpha}{m\,\omega_0^2} - q^2}\,, \tag{3.33}$$

and insert (3.32):

$$p = \sqrt{2\,\alpha\,m}\, \cos\left(\omega_0(t+\beta)\right) = p(t\,|\,\beta,\alpha)\,. \tag{3.34}$$

To (e):
We want to be precise and therefore choose the following concrete initial conditions:

$$t = t_0 = 0\ :\quad p^{(0)} = 0\,;\quad q^{(0)} = q_0 \neq 0\,. \tag{3.35}$$

Therewith we can fix by (3.33) the constant α:

$$\alpha = \frac{1}{2}\, m\,\omega_0^2 q_0^2\,. \tag{3.36}$$

Since the system possesses at the inversion point ($p^{(0)} = 0$), point of maximum amplitude q_0, only potential energy we can conclude

$$\alpha = E = \text{total energy}\,.$$

We now insert (3.35) and (3.36) into (3.31):

$$\beta = \frac{1}{\omega_0}\, \arcsin(1) = \frac{\pi}{2\,\omega_0}\,. \tag{3.37}$$

The action function S thus generates a canonical transformation that leads to a generalized momentum $\bar{p} = \alpha$ which is identical to the total energy E and to a generalized coordinate $\bar{q} = \beta$ which represents a (constant) time. **Energy and time are obviously canonically conjugate variables!**

To (f):

Eventually we get the complete solution by inserting α and β into (3.32) and (3.34), respectively:

$$q(t) = \sqrt{\frac{2E}{m\,\omega_0^2}}\cos\omega_0 t\,; \quad p(t) = -\sqrt{2Em}\,\sin\omega_0 t\,. \tag{3.38}$$

That is the well-known result for the harmonic oscillator!

Let us add to our discussion still two further considerations:

1. The solution of the HJD is a generating function of the type $F_2(q, \bar{p}, t)$. Let us demonstrate by use of the above results the construction of another type of generating function, for instance $F_1 = F_1(q, \bar{q}, t)$. At first we have with (3.29) and (3.32):

$$\bar{p} = \alpha = \frac{1}{2}\,m\,\omega_0^2 q^2\,\sin^{-2}\left(\omega_0\left(t + \bar{q}\right)\right) \stackrel{!}{=} -\frac{\partial F_1}{\partial \bar{q}}\,. \tag{3.39}$$

   This we insert into (3.34):

$$p = m\,\omega_0\,q\,\cot\left(\omega_0\left(t + \bar{q}\right)\right) \stackrel{!}{=} \frac{\partial F_1}{\partial q}\,. \tag{3.40}$$

   A first integration of (3.40) yields:

$$F_1\left(q, \bar{q}, t\right) = \frac{1}{2}\,m\,\omega_0\,q^2\,\cot\left(\omega_0\left(t + \bar{q}\right)\right) + f_1\left(\bar{q}, t\right)\,. \tag{3.41}$$

   We differentiate this expression partially with respect to $\bar{q}$ and compare the result with (3.39). Then it follows necessarily:

$$\frac{\partial f_1}{\partial \bar{q}} = 0\,.$$

   Furthermore, F_1 has to fulfill the HJD (3.2):

$$-\frac{\partial F_1}{\partial t} = \frac{1}{2m}\left(\frac{\partial F_1}{\partial q}\right)^2 + \frac{1}{2}\,m\,\omega_0^2 q^2\,.$$

This means with (3.40) and (3.41),

$$-\frac{\partial f_1}{\partial t} + \frac{\frac{1}{2}\, m\, \omega_0^2 q^2}{\sin^2\left(\omega_0\,(t+\bar{q})\right)} = \frac{1}{2}\, m\, \omega_0^2 q^2 \left(\cot^2\left(\omega_0\,(t+\bar{q})\right) + 1\right) ,$$

which can be satisfied again only by

$$\frac{\partial f_1}{\partial t} = 0$$

Except for an inessential additive constant we are therefore left with the generating function:

$$F_1\,(q, \bar{q}, t) = \frac{1}{2}\, m\, \omega_0 q^2\, \cot\left(\omega_0\,(t+\bar{q})\right) . \tag{3.42}$$

Except for the time dependence we got to know this expression already in Eq. (2.182) as the generating function of a canonical transformation for the harmonic oscillator.

2. We had realized in (3.15) that the solution of the HJD is identical to the indefinite action integral. Let us check this statement for the just discussed harmonic oscillator. With (3.33) in (3.27) it holds at first:

$$S(q, \alpha, t) = \int dq\, p - \alpha t .$$

We insert (3.38):

$$S(q, \alpha, t) = 2E \int dt\, \sin^2 \omega_0 t - E\, t , \qquad (\alpha = E) . \tag{3.43}$$

On the other hand, the oscillator-trajectory (3.38) yields the Lagrangian:

$$L = T - V = \frac{p^2}{2m} - \frac{1}{2}\, m\, \omega_0^2 q^2 = E\left(\sin^2 \omega_0 t - \cos^2 \omega_0 t\right) = 2E\, \sin^2 \omega_0 t - E .$$

Therewith it follows from (3.43) the expected result:

$$S = \int L\, dt + C .$$

3.3 Hamilton's Characteristic Function

The integration of the Hamilton-Jacobi differential equation of the harmonic oscillator in the last section became possible first of all by the use of a so-called *separation approach (ansatz)* (3.22) which separates the q- and t-dependences additively from each other. Such a separation is then always reasonable when the *old* Hamilton function does not explicitly contain time:

$$\frac{\partial H}{\partial t} = 0 \iff H \; : \textbf{integral of motion} \; .$$

Then the HJD (3.2) reads:

$$H\left(\mathbf{q}, \frac{\partial S}{\partial q_1}, \ldots, \frac{\partial S}{\partial q_S}\right) + \frac{\partial S}{\partial t} = 0 \; . \tag{3.44}$$

The total time-dependence is now due to the second summand so that the ansatz

$$S(\mathbf{q}, \bar{\mathbf{p}}, t) = W(\mathbf{q} \,|\, \bar{\mathbf{p}}) - E\,t \tag{3.45}$$

seems to be obvious by which the time-dependence is completely eliminated from the HJD (3.44):

$$H\left(\mathbf{q}, \frac{\partial W}{\partial q_1}, \ldots, \frac{\partial W}{\partial q_S}\right) = E \; . \tag{3.46}$$

In *normal* cases, i.e. for scleronomic constraints, the constant E is just the total energy of the system. The function $W(\mathbf{q} \,|\, \bar{\mathbf{p}})$ is called

Hamilton's characteristic function

E is of course via W dependent on the *new* momenta $\bar{p}_j = \alpha_j$:

$$E = E(\alpha_1, \ldots, \alpha_S) \; . \tag{3.47}$$

The canonical transformation generated by the function S from (3.45) is then given by

$$\bar{q}_j = \frac{\partial W}{\partial \alpha_j} - \frac{\partial E}{\partial \alpha_j} t \; ; \quad p_j = \frac{\partial W}{\partial q_j} \tag{3.48}$$

But one can consider $W(\mathbf{q} \,|\, \bar{\mathbf{p}})$ also on its own as the generating function of a canonical transformation (*in the narrower sense*), i.e. no longer only as part of $S(\mathbf{q}, \bar{\mathbf{p}}, t)$. W is of the type F_2 generating therewith the transformation

$$p_j = \frac{\partial W}{\partial q_j} \; ; \quad \bar{q}_j = \frac{\partial W}{\partial \bar{p}_j} \; ; \quad \overline{H} = H \; . \tag{3.49}$$

Here we presume

$$\frac{\partial H}{\partial t} = 0 \iff H = E = \text{const} \tag{3.50}$$

We demand from the generating function W that it leads to

$$\text{all } \bar{q}_j \text{ cyclic} \iff \text{all } \bar{p}_j = \alpha_j = \text{const} \tag{3.51}$$

That corresponds to the solution procedure 2. presented at the beginning of this chapter. From (3.50) then simply by insertion:

$$H\left(q_1, \ldots, q_S, \frac{\partial W}{\partial q_1}, \ldots, \frac{\partial W}{\partial q_S}\right) = E\left(\alpha_1, \ldots, \alpha_S\right) \ , \tag{3.52}$$

i.e. in spite of the now somewhat different objective (3.51), the same differential equation as in (3.46) follows.

Since by construction

$$\overline{H} = H = E\left(\bar{\mathbf{p}}\right) = \overline{H}\left(\bar{\mathbf{p}}\right) \tag{3.53}$$

the canonical equations of motion can be trivially integrated:

$$\dot{\bar{q}}_j = \frac{\partial \overline{H}}{\partial \bar{p}_j} = \frac{\partial E}{\partial \alpha_j} = \omega_j \ , \tag{3.54}$$

$$\bar{q}_j(t) = \omega_j t + \beta_j = \frac{\partial W}{\partial \bar{p}_j} \ . \tag{3.55}$$

For a clarification we want to sketch here in detail, as in the last section, the respective **method of solution**:

(a) We set up the HJD in the form (3.52)!
(b) We look for the complete solution for W with parameters $\alpha_1, \ldots, \alpha_S$:

$$W = W\left(q_1, \ldots, q_S, \alpha_1, \ldots, \alpha_S\right) \ . \tag{3.56}$$

(c) We identify:

$$\bar{p}_j = \alpha_j \ , \qquad j = 1, 2, \ldots, S \ . \tag{3.57}$$

(d) We resolve the HJD (3.52) for

$$p_j = \frac{\partial W}{\partial q_j} = p_j\left(\mathbf{q}, \alpha_1, \ldots, \alpha_S\right) \tag{3.58}$$

or differentiate the solution W accordingly.

(e) We set

$$E = E(\boldsymbol{\alpha}) \tag{3.59}$$

and calculate:

$$\omega_j = \frac{\partial E}{\partial \alpha_j}\,, \qquad j = 1, 2, \ldots, S\,. \tag{3.60}$$

Equation (3.59) is arranged according to aspects of pure convenience. We clarify this by two plausible examples:

1. With the ansatz

$$E(\boldsymbol{\alpha}) = \sum_{j=1}^{S} \frac{\alpha_j^2}{2m} \tag{3.61}$$

the respective transformation leads because of

$$\overline{H} = H = \sum_{j=1}^{S} \frac{\bar{p}_j^2}{2m} \tag{3.62}$$

to the Hamilton function $\overline{H}$ of a system of free mass points. The interaction present in H is thus transformed away and the solutions of the problem then according to (3.55) have the known form of those for the force-free movement of mass points:

$$\bar{q}_j(t) = \frac{\alpha_j}{m}\,t + \beta_j\,. \tag{3.63}$$

By the way, this corresponds to the procedure 1. as indicated at the beginning of this chapter.

2. One might also opt to take simply

$$E\,(\alpha_1, \ldots, \alpha_S) = \alpha_1\,. \tag{3.64}$$

One identifies then the *new* momentum $\bar{p}_1$ with α_1 and the other $S-1$ momenta $\bar{p}_j$ with the $S-1$ essential integration constants of the complete solution W of the HJD (3.52). Then we have

$$\omega_j = \delta_{j1} \tag{3.65}$$

and the new coordinates are:

$$\bar{q}_1 = t + \beta_1\,; \quad \bar{q}_j = \beta_j\,, \qquad j = 2, \ldots, S\,. \tag{3.66}$$

(f) We solve

$$\bar{q}_j = \omega_j(\boldsymbol{\alpha})\, t + \beta_j = \frac{\partial W}{\partial \alpha_j}(\mathbf{q}, \boldsymbol{\alpha}) \tag{3.67}$$

for

$$q_j = q_j(t, \boldsymbol{\alpha}, \boldsymbol{\beta}) \tag{3.68}$$

and insert the solution then into (3.58):

$$p_j = p_j(t, \boldsymbol{\alpha}, \boldsymbol{\beta}) \; . \tag{3.69}$$

(g) With the initial conditions

$$q_j^{(0)} = q_j(t = t_0) \; ; \quad p_j^{(0)} = p_j(t = t_0)$$

it follows from (3.58):

$$\boldsymbol{\alpha} = \boldsymbol{\alpha}\left(\mathbf{p}^{(0)}, \mathbf{q}^{(0)}\right) \; . \tag{3.70}$$

With (3.68) and (3.69) one further finds:

$$\boldsymbol{\beta} = \boldsymbol{\beta}\left(\mathbf{p}^{(0)}, \mathbf{q}^{(0)}\right) \; . \tag{3.71}$$

By insertion of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ into (3.68) and (3.69) the problem is completely solved.

Let us reflect at the end a bit more on the physical meaning of Hamilton's characteristic function W. We had seen in (3.15) that the solution of the *full* HJD (3.2) is identical to the indefinite action integral $\int L\, dt$. We can ascribe to W, too, a similar interpretation.

$$W = W(\mathbf{q}, \bar{\mathbf{p}}) \implies \frac{dW}{dt} = \sum_{j=1}^{S} \left(\frac{\partial W}{\partial q_j}\, \dot{q}_j + \frac{\partial W}{\partial \bar{p}_j}\, \dot{\bar{p}}_j \right) = \sum_{j=1}^{S} p_j \dot{q}_j \; . \tag{3.72}$$

Thus W corresponds to the *action* A that was used in (2.65) to formulate the *principle of least action*:

$$W = \int \sum_{j=1}^{S} p_j \dot{q}_j \, dt = \int \sum_{j=1}^{S} p_j \, dq_j \; . \tag{3.73}$$

A is the definite and W the indefinite integral.

3.4 Separation of the Variables

We have to ask ourselves whether the Hamilton-Jacobi procedure in the form up to now presented is helpful at all. In the end one replaces $2S$ ordinary (Hamilton's) differential equations by a partial differential equation. The latter is in general very much more difficult to solve. The method represents therefore a mighty auxiliary means, second to none of the other procedures, only if the HJD can be *separated*. What this means we will make plain to us in this section.

We presume that H is not explicitly time-dependent and therefore represents an integral of motion. The canonical transformation is carried out by the characteristic function $W(\mathbf{q}, \bar{\mathbf{p}})$ studied in the last section. We can then apply the Hamilton-Jacobi differential equation in the form of (3.52):

$$H\left(q_1, \ldots, q_S, \frac{\partial W}{\partial q_1}, \ldots, \frac{\partial W}{\partial q_S}\right) = E\,. \tag{3.74}$$

Let us assume that q_1 and $\partial W/\partial q_1$ appear in H only as

$$f\left(q_1, \frac{\partial W}{\partial q_1}\right)\,,$$

an expression that does not contain any other q_j, $\partial W/\partial q_j$, $j > 1$ so that (3.74) can be cast into the form:

$$H\left(q_2, \ldots, q_S, \frac{\partial W}{\partial q_2}, \ldots, \frac{\partial W}{\partial q_S}, f\left(q_1, \frac{\partial W}{\partial q_1}\right)\right) = E\,. \tag{3.75}$$

Then the following ansatz suggests itself:

$$W(\mathbf{q}, \bar{\mathbf{p}}) = \overline{W}\left(q_2, \ldots, q_S, \bar{\mathbf{p}}\right) + W_1\left(q_1, \bar{\mathbf{p}}\right)\,. \tag{3.76}$$

Insertion into (3.75) yields:

$$H\left(q_2, \ldots, q_S, \frac{\partial \overline{W}}{\partial q_2}, \ldots, \frac{\partial \overline{W}}{\partial q_S}, f\left(q_1, \frac{\partial W_1}{\partial q_1}\right)\right) = E\,. \tag{3.77}$$

Let us assume that we had already found the solution for W. Then (3.77) must become an identity after the insertion of (3.76), i.e. must be fulfilled for arbitrary q_1. A change of the coordinate q_1 should not become noticeable with respect to H. Since

q_1, however, appears only in f, f itself must be constant:

$$f\left(q_1, \frac{dW_1}{dq_1}\right) = C_1 \ , \tag{3.78}$$

$$H\left(q_2, \dots, q_S, \frac{\partial \overline{W}}{\partial q_2}, \dots, \frac{\partial \overline{W}}{\partial q_S}; C_1\right) = E \ . \tag{3.79}$$

Since the new momenta $\bar{p}_j$ are by construction all constant, W_1 is only dependent on q_1. We can therefore replace in (3.78) the partial by the corresponding total derivative. What did we achieve with (3.78) and (3.79)? Equation (3.78) is an **ordinary** differential equation for W_1 while (3.79) is still a partial differential equation but with a smaller by one number of independent variables.

In certain cases all the coordinates can successively be separated in this manner and in the generalization of (3.76) the full solution of the HJD may be approached as follows:

$$W = \sum_{j=1}^{S} W_j\left(q_j;\ \alpha_1, \dots, \alpha_S\right) \ . \tag{3.80}$$

Therewith the HJD is then decomposed into S **ordinary** differential equations of the form

$$H_j\left(q_j, \frac{dW_j}{dq_j}, \alpha_1, \dots, \alpha_S\right) = \alpha_j \tag{3.81}$$

One says in such a case that the HJD is **separable** in the coordinates $\{q_j\}$. Each equation in (3.81) contains only one coordinate q_j and the corresponding derivative of W_j with respect to q_j, and should therefore be normally simply solvable for dW_j/dq_j. The subsequent integration should be possible. Whether or not a separation of the form (3.80) is indeed possible depends very strongly on the choice of the generalized coordinates q_j, though, and of course also, on the type of the Hamilton function. If one succeeds, for instance, to find the coordinates for which the various terms of the Hamilton function can additively be grouped as follows,

$$H(\mathbf{q}, \mathbf{p}) = \sum_{j=1}^{S} H_j(q_j, p_j) \ ,$$

then the ansatz (3.80) apparently leads directly to (3.81).

For the *special case* where only one coordinate is non-cyclic a separation is always possible:

$$\left.\begin{array}{l} q_1 \ \text{ non-cyclic} \\ q_{j,j>1} \ \text{ cyclic} \end{array}\right\} \quad \Longrightarrow \ p_j = \frac{\partial W}{\partial q_j} = \alpha_j = \text{const} \ , \quad j > 1 \ . \tag{3.82}$$

Which ansatz in such a case is now advisable? By construction W is a transformation on exclusively cyclic coordinates. $q_2, \ldots, q_S$ are, however, already cyclic. For these W should therefore be the **identity transformation** (2.176):

$$F_2\left(\mathbf{q}, \bar{\mathbf{p}}\right) = \sum_{j=2}^{S} q_j \bar{p}_j \; . \tag{3.83}$$

With $\bar{p}_j = \alpha_j$ the following ansatz for W then suggests itself:

$$W = W_1(q_1) + \sum_{j=2}^{S} \alpha_j q_j \; . \tag{3.84}$$

The HJD (3.74) therewith becomes an ordinary differential equation of first order for W_1:

$$H\left(q_1, \frac{dW_1}{dq_1}, \alpha_2, \ldots, \alpha_S\right) = E \; . \tag{3.85}$$

Equation (3.84) can naturally be generalized insofar as that such an ansatz is not only applied for the case where all the coordinates q_j except one are cyclic but, very generally, **each** cyclic coordinate q_i is separable by an ansatz of the form

$$W = \overline{W}\left(q_{j, j \neq i}, \bar{\mathbf{p}}\right) + \alpha_i q_i \; . \tag{3.86}$$

For non-cyclic coordinates there does not exist a general procedure for separation. Nevertheless, the Hamilton-Jacobi method may be considered as the most successful means for finding general solutions of equations of motion. That will finally be demonstrated by two examples:

(1) Planar motion of a particle in a central field

'Central field' means $V(\mathbf{r}) = V(r)$. As generalized coordinates the spherical coordinates offer themselves, where, in addition, the *planar motion* provides $\vartheta =$ const. Thus there are left

$$q_1 = r \; ; \quad q_2 = \varphi \; . \tag{3.87}$$

So the Hamilton function (2.45) reads:

$$H = \frac{1}{2m}\left(p_r^2 + \frac{p_\varphi^2}{r^2}\right) + V(r) \; . \tag{3.88}$$

φ is obviously cyclic and therewith

$$p_\varphi = \alpha_\varphi = \text{const} \qquad (\textit{orbital angular momentum}) \; . \tag{3.89}$$

According to (3.86) it recommends itself as ansatz for the characteristic function W:

$$W = W_1(r) + \alpha_\varphi \, \varphi \; . \tag{3.90}$$

Since for this example $\partial H/\partial t = 0$ and furthermore the constraint (*motion in a plane*) is scleronomic means the HJD to be solved reads:

$$\frac{1}{2m}\left\{\left(\frac{\partial W}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial W}{\partial \varphi}\right)^2\right\} + V(r) = \frac{1}{2m}\left\{\left(\frac{dW_1}{dr}\right)^2 + \frac{\alpha_\varphi^2}{r^2}\right\} + V(r) = E \; . \tag{3.91}$$

From that it follows immediately:

$$\frac{dW_1}{dr} = \sqrt{2m\,(E - V(r)) - \frac{\alpha_\varphi^2}{r^2}} \; . \tag{3.92}$$

The characteristic function W is then:

$$W = \int dr \sqrt{2m\,(E - V(r)) - \frac{\alpha_\varphi^2}{r^2}} + \alpha_\varphi \, \varphi \; . \tag{3.93}$$

Thereby, in the first summand, we have of course an *indefinite* integral.

As in (3.65) we now choose

$$E = \alpha_1 \iff \omega_j = \delta_{j1} \; , \tag{3.94}$$

getting therewith from the equations of transformation (3.66) and (3.67):

$$t + \beta_1 = \bar{q}_1 = \frac{\partial W}{\partial \alpha_1} = \frac{\partial W}{\partial E} = \int dr \frac{m}{\sqrt{2m\,(E - V(r)) - \frac{\alpha_\varphi^2}{r^2}}} \; . \tag{3.95}$$

The reversal then yields $r = r(t; \boldsymbol{\alpha}, \boldsymbol{\beta})$.

$$\beta_2 = \bar{q}_2 = \frac{\partial W}{\partial \alpha_2} = \frac{\partial W}{\partial \alpha_\varphi} = -\int dr \frac{\frac{\alpha_\varphi}{r^2}}{\sqrt{2m\,(E - V(r)) - \frac{\alpha_\varphi^2}{r^2}}} + \varphi \; . \tag{3.96}$$

We further take

$$\beta_2 = \varphi_0 \; , \quad x = \frac{1}{r} \; , \quad \alpha_\varphi = L$$

and have then found with

$$\varphi = \varphi_0 - \int \frac{dx}{\sqrt{\frac{2m}{L^2}(E - V(1/x)) - x^2}} \tag{3.97}$$

after inversion the known orbital equation $r = r(\varphi)$ of the central-force problem. L is identical to the orbital angular momentum. The results (3.95) and (3.97) needed very much more computational effort in the Newton's mechanics ((2.256), (2.257), Vol. 1). Initial conditions fix β_1, φ_0, E, L.

(2) Particle in the gravitational field

The Hamilton function $H = T + V = E$ is known:

$$H = \frac{1}{2m}\left(p_x^2 + p_y^2 + p_z^2\right) + m\,g\,z\,. \tag{3.98}$$

x and y are cyclic and therewith the corresponding momenta constant:

$$p_x = \alpha_x = \text{const}\,; \quad p_y = \alpha_y = \text{const}\,. \tag{3.99}$$

A suitable approach for the characteristic function W is then:

$$W = W_1(z) + \alpha_x\,x + \alpha_y\,y\,. \tag{3.100}$$

Therewith the HJD reads:

$$\frac{1}{2m}\left\{\left(\frac{dW_1}{dz}\right)^2 + \alpha_x^2 + \alpha_y^2\right\} + m\,g\,z = E\,.$$

It follows then immediately:

$$\begin{aligned} W_1(z) &= \int dz\,\sqrt{2m(E - m\,g\,z) - \alpha_x^2 - \alpha_y^2} \\ &= -\frac{1}{3m^2 g}\left\{2m(E - m\,g\,z) - \alpha_x^2 - \alpha_y^2\right\}^{3/2} + C\,. \end{aligned}$$

For the characteristic function we thus have:

$$W = -\frac{1}{3m^2 g}\left\{2m(E - m\,g\,z) - \alpha_x^2 - \alpha_y^2\right\}^{3/2} + \alpha_x\,x + \alpha_y\,y\,. \tag{3.101}$$

We take again $E = \alpha_1$ getting then according to (3.66):

$$\begin{aligned}
\bar{q}_1 &= t + \beta_1 = \frac{\partial W}{\partial E} = -\frac{1}{m\,g}\left\{2m(E - m\,g\,z) - \alpha_x^2 - \alpha_y^2\right\}^{1/2} \;, \\
\bar{q}_2 &= \beta_2 = \frac{\partial W}{\partial \alpha_x} = x + \frac{\alpha_x}{m^2 g}\left\{2m(E - m\,g\,z) - \alpha_x^2 - \alpha_y^2\right\}^{1/2} \;, \\
\bar{q}_3 &= \beta_3 = \frac{\partial W}{\partial \alpha_y} = y + \frac{\alpha_y}{m^2 g}\left\{2m(E - m\,g\,z) - \alpha_x^2 - \alpha_y^2\right\}^{1/2} .
\end{aligned}$$

From the first line we derive:

$$z(t) = -\frac{1}{2}\,g\,\left(t + \beta_1\right)^2 + \frac{2mE - \left(\alpha_x^2 + \alpha_y^2\right)}{2m^2 g} \;. \tag{3.102}$$

Inserting the first line into the two other lines yields furtheron:

$$x(t) = \beta_2 + \frac{\alpha_x}{m}\left(t + \beta_1\right) \;, \tag{3.103}$$

$$y(t) = \beta_3 + \frac{\alpha_y}{m}\left(t + \beta_1\right) \;. \tag{3.104}$$

The rest is settled by initial conditions. We choose for $t = 0$:

$$\begin{aligned}
&x(0) = y(0) = z(0) = 0 \;; \\
&p_x(0) = p_0 \;; \quad p_y(0) = p_z(0) = 0 \;.
\end{aligned} \tag{3.105}$$

With these conditions we deduce:

$$\begin{aligned}
p_x &= \frac{\partial W}{\partial x} = \alpha_x = \text{const} = p_0 \;, \\
p_y &= \frac{\partial W}{\partial y} = \alpha_y = \text{const} = 0 \;, \\
p_z &= \frac{\partial W}{\partial z} = \left\{2m(E - m\,g\,z) - \alpha_x^2 - \alpha_y^2\right\}^{1/2} \\
&= \left\{2m(E - m\,g\,z) - p_0^2\right\}^{1/2} \;, \\
p_z(0) = 0 &\quad \Longrightarrow \quad E = \frac{1}{2m} p_0^2 \;.
\end{aligned}$$

With Eqs. (3.102) up to (3.105) we still have:

$$\beta_1 = \beta_2 = \beta_3 = 0 \;.$$

This yields the well-known solution:

$$z(t) = -\frac{1}{2}\, g\, t^2 \; ; \quad x(t) = \frac{p_0}{m}\, t \; ; \quad y(t) \equiv 0 \; . \tag{3.106}$$

3.5 The Action and Angle Variable

3.5.1 Periodic Systems

We now discuss an important modification of the Hamilton-Jacobi method that is especially designed for

periodic systems

for which we are often more interested in the characteristic frequencies of the movement than, for instance, in the actual shape of the trajectory.

What do we understand by **'periodic'**?

In case of only one degree of freedom ($S = 1$) the answer is immediately clear. After a certain time τ, the *'period'*, the system again reaches its initial state. The phase space is the two-dimensional (p, q)-plane. In this context one distinguishes two types of periodicities:

(1) Libration

The phase trajectory is a closed curve:

$$\begin{aligned} q(t+\tau) &= q(t) \; , \\ p(t+\tau) &= p(t) \; . \end{aligned} \tag{3.107}$$

q and p are periodic with the same frequency. That is typical for oscillating systems such as pendulum, spring etc., which move between two states of vanishing kinetic energy (Figs. 3.1 and 3.2).

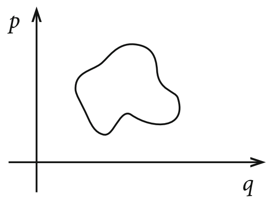


Fig. 3.1 Simple example of a libration

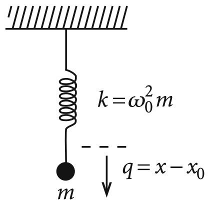


Fig. 3.2 The linear harmonic oscillator as an example of a periodic system (libration)

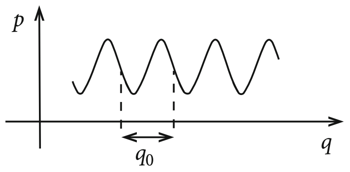


Fig. 3.3 Simple example of a rotation

Example: 'Linear Harmonic Oscillator'

The phase trajectory is an ellipse as we have discussed as example to (2.99):

$$1 = \frac{p^2}{2mE} + \frac{q^2}{\frac{2E}{m\,\omega_0^2}} \;, \qquad H = E \;.$$

(2) Rotation

p is periodic in this case, too,

$$p(t+\tau) = p(t) \;, \tag{3.108}$$

but q is not. In fact, the coordinate changes within the period τ by a constant value q_0:

$$q(t+\tau) = q(t) + q_0 \;. \tag{3.109}$$

The phase trajectory in this case is open where, however, p is a periodic function of q (Fig. 3.3).

Example: 'Axial Rotation of a Rigid Body'

$$q = \varphi \;; \quad q_0 = 2\pi \;.$$

Sometimes both types of periodic movement can be observed for one and the same system, as for instance for the **pendulum** (Fig. 3.4). We have derived the Hamilton

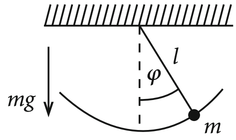


Fig. 3.4 The pendulum as an example of a periodic system with both libration and rotation

function of the pendulum as Eq. (2.33) in Sect. 2.2.2:

$$H = \frac{p_\varphi^2}{2ml^2} - m\,g\,l\,\cos\varphi = E\ .$$

As generalized momentum we found:

$$p_\varphi = m\,l^2\,\dot{\varphi} = \pm\sqrt{2m\,l^2(E + m\,g\,l\,\cos\varphi)}\ .$$

p_φ is the angular momentum of the pendulum and therefore a real quantity. The radicand must therefore be positive:

$$\cos\varphi \geq -\frac{E}{m\,g\,l}\ .$$

(a) $E > m\,g\,l$: All angles φ are possible. The pendulum *overturns*. It is about a **rotation**.
(b) $-m\,g\,l < E < m\,g\,l$: Only a limited region of the angle $[-\varphi_0, \varphi_0]$ with $\cos\varphi_0 = -(E/m\,g\,l)$ is allowed. In this case it obviously is about a **libration**.

For systems with $S > 1$ degrees of freedom the motion is called **periodic** if the projection of the phase trajectory onto each (q_j, p_j)-plane is periodic in the above sense. Thereby it is not required that all (q_j, p_j) sets are periodic with the same frequency so that the path line in the $2S$-dimensional phase space is then not necessarily a simple periodic curve. If the quotients of the two frequencies one each of the projected trajectories are not rational numbers then even an open trajectory arises. In such a case one speaks of a **conditional periodic motion**.

For systems for which the Hamilton-Jacobi differential equation can be completely separated, so that (3.80) as well as (3.81) hold, the periodicity can easily be checked.

$$W = \sum_{j=1}^{S} W_j\left(q_j; \boldsymbol{\alpha}\right)\ ,$$

$$p_j = \frac{\partial W}{\partial q_j} = \frac{dW_j}{dq_j} = p_j\left(q_j; \boldsymbol{\alpha}\right)\ . \tag{3.110}$$

In such a case the projected trajectories are independent of each other. If $p_j(q_j)$ represents for each $j = 1, \ldots, S$ a closed curve or a periodic function as given in (3.108) and (3.109) then the movement of the system as a whole is periodic.

3.5.2 The Action and Angle Variable

The considerations of this section are concerned exclusively with periodic systems. Let us first summarize the essentials of the Hamilton-Jacobi method:

We look for a canonical transformation

$$(\mathbf{q}, \mathbf{p}) \longrightarrow (\bar{\mathbf{q}}, \bar{\mathbf{p}})$$

so that we get:

$$\bar{p}_j = \text{const} \quad \forall j ,$$

$$\bar{q}_j = \begin{cases} \text{const} \;\; \forall j \Longleftrightarrow S(\mathbf{q}, \bar{\mathbf{p}}, t) \;\; , \\ \text{cyclic} \; \forall j \Longleftrightarrow W(\mathbf{q}, \bar{\mathbf{p}}) \;\; . \end{cases}$$

The generating functions S and W are here the solutions of the HJD with the constants of integration $\alpha_1, \ldots, \alpha_S$, which are identified with the *new* momenta:

$$\bar{p}_j = \alpha_j \qquad \forall j .$$

One could have just as well equated *arbitrary functions* of the α_j with the $\bar{p}_j$. The

'action variables' J_j

are rather special functions of the α_j:

$$J_j = \oint p_j \, dq_j \; , \qquad j = 1, 2, \ldots, S \; . \tag{3.111}$$

The integration is extended over a full period of the libration and rotation, as the case may be (Fig. 3.5).

We presume as (3.110) a separable system and can then write for (3.111):

$$J_j = \oint \frac{dW_j\left(q_j; \boldsymbol{\alpha}\right)}{dq_j} dq_j = J_j(\boldsymbol{\alpha}) \; . \tag{3.112}$$

J_j thus represents the **increase** of the generating function W, experienced by W per q_j-circle. In (3.112) q_j is nothing but a variable of integration so that the action variable J_j, too, does indeed depend only on the constants $\alpha_1, \alpha_2, \ldots, \alpha_S$ being therefore appropriate to be chosen as *new* momenta $\bar{p}_j$. Since the pairs of variables

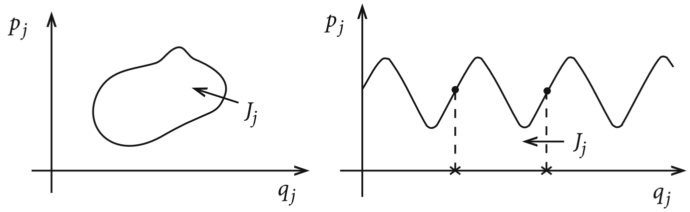


Fig. 3.5 To the interpretation of the action variable of a libration (*left*) and a rotation (*right*)

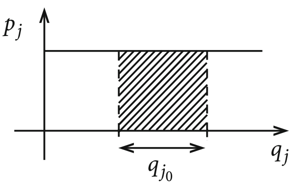


Fig. 3.6 Action variable for the special case of a cyclic coordinate

(q_j, p_j) are independent of each other, the same holds of course also for the J_j. The reversal of (3.112) yields:

$$\alpha_j = \alpha_j (J_1, \ldots, J_S) \ , \qquad j = 1, 2, \ldots, S \ . \tag{3.113}$$

Therewith, the Hamilton's characteristic function W becomes dependent on the $J_1, \ldots, J_S$:

$$W = W (q_1, \ldots, q_S; J_1, \ldots, J_S) \ . \tag{3.114}$$

Because of (3.64)

$$H = \overline{H} = \alpha_1(\mathbf{J}) \tag{3.115}$$

also the *new* Hamilton function is then a function exclusively of the J_j:

$$\overline{H} = \overline{H} (J_1, \ldots, J_S) \ . \tag{3.116}$$

A special case is given by

$$q_j \quad \text{cyclic} \quad \Longleftrightarrow \quad p_j = \text{const}$$

in which case the phase trajectory runs parallel to the q_j axis (Fig. 3.6). To this limiting case of a periodic motion a period q_{j0} can be ascribed at random. Since for rotations q_j represents mostly an angle, one agrees upon $q_{j0} = 2\pi$. This means for the related action variable:

$$J_j = 2\pi\, p_j \;, \quad \text{if} \quad q_j \quad \text{cyclic} \;. \tag{3.117}$$

We now come to the

'angle variables' ω_j

which can be introduced as the conjugate variables to the J_j:

$$\bar{p}_j = J_j \iff \bar{q}_j = \omega_j \;, \qquad j = 1, 2, \ldots, S \;.$$

By construction (see (3.116)) all $\bar{q}_j$ are cyclic. The ω_j can be derived from W:

$$\omega_j = \frac{\partial W}{\partial J_j} \;, \qquad j = 1, 2, \ldots, S \;. \tag{3.118}$$

With Hamilton's equation of motion for $\dot{\bar{q}}_j$ it follows:

$$\dot{\omega}_j = \frac{\partial}{\partial J_j} \overline{H}(\mathbf{J}) = \nu_j \,(J_1, \ldots, J_S) = \text{const} \;. \tag{3.119}$$

The integration is then trivial:

$$\omega_j = \nu_j\, t + \beta_j \;, \qquad j = 1, 2, \ldots, S \;. \tag{3.120}$$

This corresponds to the procedure presented in Sect. 3.3. But the special advantage of the method does not consist in that. This becomes clear when one inspects the physical meaning of the action and angle variables. For this purpose let us calculate the change of ω_i owing to the change of the coordinate q_j over a full cycle:

$$\begin{aligned} \Delta_j \omega_i &= \oint_j d\omega_i = \oint \frac{\partial \omega_i}{\partial q_j}\, dq_j = \oint \frac{\partial^2 W}{\partial q_j \partial J_i}\, dq_j \\ &= \frac{\partial}{\partial J_i} \oint \frac{\partial W}{\partial q_j} dq_j = \frac{\partial}{\partial J_i} J_j \;. \end{aligned}$$

Thus ω_i undergoes a change only for $q_j = q_i$, and then just by the value 1:

$$\Delta_j \omega_i = \delta_{ij} \;. \tag{3.121}$$

This means with (3.120), if τ_i is the period of q_i:

$$\Delta_i \omega_i = \nu_i\, \tau_i = 1 \;. \tag{3.122}$$

We therefore have:

$$\nu_i = \frac{1}{\tau_i} \quad : \quad \textit{frequency of the periodic motion belonging to } q_i \; .$$

Herein we find the actual importance of the method of action and angle variables because it allows for a determination of the frequencies of periodic motions without being obliged to work out the full solution for the motion of the system. One can directly calculate the frequencies ν without a back transformation to the actual coordinates. Our standard example the

linear harmonic oscillator

will again serve to illustrate the method.

The phase trajectory is an ellipse, the system is thus periodic. By

$$\overline{H} = H = \frac{1}{2m} p^2 + \frac{1}{2} m\,\omega_0^2 q^2 = \alpha_1$$

we have:

$$p = \pm m\,\omega_0 \sqrt{\frac{2\alpha_1}{m\,\omega_0^2} - q^2} = \frac{dW}{dq} \; .$$

The zeros of the radicand define the reversal points:

$$q_\pm = \pm \sqrt{\frac{2\alpha_1}{m\,\omega_0^2}} \; .$$

Because of $\dot{q} = \frac{\partial H}{\partial p}$ it is $p > 0$ on the way $q_- \longrightarrow q_+$ but $p < 0$ on the way back $q_+ \longrightarrow q_-$. We can calculate therewith the action variable:

$$J = \oint p\,dq = 2 \int\limits_{q_-}^{q_+} p\,dq = 2m\,\omega_0 \int\limits_{q_-}^{q_+} \sqrt{\frac{2\alpha_1}{m\,\omega_0^2} - q^2}\; dq$$

$$= 2m\,\omega_0 \left[\frac{1}{2} q \sqrt{\frac{2\alpha_1}{m\,\omega_0^2} - q^2} + \frac{\alpha_1}{m\,\omega_0^2} \arcsin \frac{q}{\sqrt{\frac{2\alpha_1}{m\,\omega_0^2}}} \right] \Bigg|_{q_-}^{q_+} = \frac{2\pi}{\omega_0} \alpha_1 \; .$$

Hence the *new* Hamilton function takes the simple form:

$$\overline{H} = \alpha_1 = \frac{\omega_0}{2\pi} J \; . \tag{3.123}$$

For the frequency ν of the periodic motion, then follows the expected result:

$$\nu = \frac{\partial \overline{H}}{\partial J} = \frac{1}{2\pi}\,\omega_0\,. \tag{3.124}$$

3.5.3 The Kepler Problem

The just discussed example of the harmonic oscillator served only as an illustration. The full usefulness of the method manifests itself more noticeably in connection with the rather sophisticated problems of the planetary and atomic mechanics.

The Kepler problem is defined by the potential

$$V(\mathbf{r}) = -\frac{k}{r} \quad (k > 0) \tag{3.125}$$

Concrete realizations are for instance:

$$\begin{aligned} k &= \gamma\, m M \iff \text{gravitation } ((2.261), \text{Vol. 1})\,, \\ k &= \frac{q_1\, q_2}{4\pi\,\varepsilon_0} \iff \text{Coulomb} \quad ((2.258), \text{Vol. 1}) \end{aligned} \tag{3.126}$$

Because of (3.125) spherical coordinates are appropriate. The Hamilton function then reads according to (2.45):

$$H = \frac{1}{2m}\left(p_r^2 + \frac{1}{r^2}\,p_\vartheta^2 + \frac{1}{r^2 \sin^2\vartheta}\,p_\varphi^2\right) - \frac{k}{r}\,. \tag{3.127}$$

For the generalized momenta we found already in (2.44):

$$p_r = m\,\dot{r}\,, \tag{3.128}$$

$$p_\vartheta = m\,r^2 \dot{\vartheta}\,, \tag{3.129}$$

$$p_\varphi = m\,r^2 \sin^2\vartheta\,\dot{\varphi} = L_z = \text{const}\,. \tag{3.130}$$

φ is cyclic. The z-component of the angular momentum $p_\varphi = L_z$ is therefore a constant of motion. We have as HJD:

$$\frac{1}{2m}\left[\left(\frac{\partial W}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial W}{\partial \vartheta}\right)^2 + \frac{1}{r^2\sin^2\vartheta}\left(\frac{\partial W}{\partial \varphi}\right)^2\right] - \frac{k}{r} = \alpha_1 = E\,. \tag{3.131}$$

The problem is separable:

$$W = W_r(r) + W_\vartheta(\vartheta) + W_\varphi(\varphi)\,. \tag{3.132}$$

Since φ is cyclic we choose for W_φ the identity transformation:

$$W_\varphi = \alpha_\varphi \, \varphi \; , \tag{3.133}$$

$$\alpha_\varphi = p_\varphi = L_z = \text{const} \; . \tag{3.134}$$

We properly rearrange the HJD (3.131):

$$\frac{1}{2m} r^2 \left(\frac{dW_r}{dr}\right)^2 - k\,r - E\,r^2 = -\frac{1}{2m}\left[\left(\frac{dW_\vartheta}{d\vartheta}\right)^2 + \frac{\alpha_\varphi^2}{\sin^2\vartheta}\right] .$$

The left-hand side does depend only on r, the right-hand side only on ϑ. Consequently, each side itself must already be equal to the same constant:

$$\left(\frac{dW_\vartheta}{d\vartheta}\right)^2 + \frac{\alpha_\varphi^2}{\sin^2\vartheta} = \alpha_\vartheta^2 = \text{const} \tag{3.135}$$

$$\left(\frac{dW_r}{dr}\right)^2 + \frac{\alpha_\vartheta^2}{r^2} = 2m\left(E + \frac{k}{r}\right) \tag{3.136}$$

α_1, α_ϑ, α_φ are the three required constants of integration. One easily sees that α_ϑ^2 is just the square of the total angular momentum:

$$L_x = y\,p_z - z\,p_y = -m\,r^2\left(\sin\varphi\,\dot{\vartheta} + \sin\vartheta\,\cos\vartheta\,\cos\varphi\,\dot{\varphi}\right) ,$$

$$L_y = z\,p_x - x\,p_z = m\,r^2\left(\cos\varphi\,\dot{\vartheta} - \sin\vartheta\,\cos\vartheta\,\sin\varphi\,\dot{\varphi}\right) ,$$

$$L_z = x\,p_y - y\,p_x = m\,r^2\sin^2\vartheta\,\dot{\varphi} \; .$$

Therewith we have:

$$|\mathbf{L}|^2 = L_x^2 + L_y^2 + L_z^2 = m^2 r^4\left(\dot{\vartheta}^2 + \sin^2\vartheta\,\dot{\varphi}^2\right) = p_\vartheta^2 + \frac{p_\varphi^2}{\sin^2\vartheta} \; . \tag{3.137}$$

The comparison with (3.135) shows:

$$\alpha_\vartheta^2 = |\mathbf{L}|^2 \; . \tag{3.138}$$

α_1 (3.131), α_φ (3.134) and α_ϑ (3.138) are thus constants of integration with fundamental physical meanings.

Let us now concentrate on the calculation of the action variables:

$$J_\varphi = \oint p_\varphi \, d\varphi = \oint \frac{dW_\varphi}{d\varphi} \, d\varphi = \alpha_\varphi \oint d\varphi \, , \tag{3.139}$$

$$J_\vartheta = \oint p_\vartheta \, d\vartheta = \oint \frac{dW_\vartheta}{d\vartheta} \, d\vartheta = \oint \sqrt{\alpha_\vartheta^2 - \frac{\alpha_\varphi^2}{\sin^2 \vartheta}} \, d\vartheta \, , \tag{3.140}$$

$$J_r = \oint p_r \, dr = \oint \frac{dW_r}{dr} \, dr = \oint \sqrt{2m\left(E + \frac{k}{r}\right) - \frac{\alpha_\vartheta^2}{r^2}} \, dr \, . \tag{3.141}$$

We want to evaluate these expressions step by step and very detailedly. J_φ turns out to be simple:

$$J_\varphi = 2\pi \, \alpha_\varphi \, . \tag{3.142}$$

When calculating J_ϑ it is to be borne in mind that p_ϑ as a *'physical'* momentum must be real.

$$p_\vartheta = \alpha_\varphi \sqrt{a^2 - \frac{1}{\sin^2 \vartheta}} \, ,$$

$$a^2 = \frac{\alpha_\vartheta^2}{\alpha_\varphi^2} \geq 1 \, .$$

Therefore there are reversal points (Fig. 3.7) with

$$\sin \vartheta_{1,2} = |a|^{-1} \leq 1 \, .$$

In (3.140) we have therefore to take for a certain path the positive root and for the back path the negative one:

$$\begin{aligned} \vartheta_1 \longrightarrow \vartheta_2 &: p_\vartheta = m\, r^2 \dot{\vartheta} > 0 \; : +\sqrt{a^2 - \tfrac{1}{\sin^2 \vartheta}} \, , \\ \vartheta_2 \longrightarrow \vartheta_1 &: p_\vartheta < 0 \; : \qquad\qquad\;\; -\sqrt{a^2 - \tfrac{1}{\sin^2 \vartheta}} \, . \end{aligned}$$

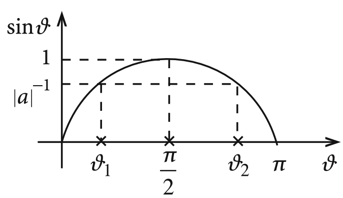


Fig. 3.7 Specification of the limits of integration for the determination of the action variable J_ϑ of the Kepler problem

It remains then to be calculated:

$$J_\vartheta = 2\alpha_\varphi \int_{\vartheta_1}^{\vartheta_2} + \sqrt{a^2 - \frac{1}{\sin^2\vartheta}}\, d\vartheta = 2i\,\alpha_\varphi \int_{\vartheta_1}^{\vartheta_2} \frac{\Delta\, d\vartheta}{\sin\vartheta} ,$$

$$\Delta = \sqrt{1 - a^2 \sin^2\vartheta} .$$

In a good table of integrals we find:

$$\int_{\vartheta_1}^{\vartheta_2} \frac{\Delta}{\sin\vartheta} d\vartheta = \left[-\frac{1}{2}\ln(\Delta + \cos\vartheta) + \frac{1}{2}\ln(\Delta - \cos\vartheta) + a\ln(a\cos\vartheta + \Delta) \right] \Bigg|_{\vartheta_1}^{\vartheta_2} .$$

One verifies by differentiation that the right-hand side is indeed the antiderivative of the integrand $\Delta / \sin\vartheta$. With

$$\vartheta_2 = \pi - \vartheta_1 \; ; \quad \cos\vartheta_1 = -\cos\vartheta_2 > 0 \; ; \quad \Delta(\vartheta_1) = \Delta(\vartheta_2) = 0 ,$$

and

$$\ln(\cos\vartheta_2) = \ln(-\cos\vartheta_1) = \ln(\cos\vartheta_1) \pm i\pi$$

it further follows:

$$\int_{\vartheta_1}^{\vartheta_2} \frac{\Delta}{\sin\vartheta} d\vartheta = \pm i\pi(a - 1)$$

In any case J_ϑ must be positive, therefore the lower sign is valid here:

$$J_\vartheta = 2\pi \left(\alpha_\vartheta - \alpha_\varphi \right) . \tag{3.143}$$

Notice that the angle-dependent parts J_φ, J_ϑ are still completely independent of the type of the central field. The actual form (3.125) does not yet enter anywhere our calculation. That happens only for the still to be determined J_r-integral.

However, before doing that let us derive the result (3.143) once more in another, maybe somewhat *more elegant* manner. We exploit from the beginning that the motion occurs in a fixed orbital plane since L_z = const where the z direction is not at all fixed by any means. Therefore we have even to conclude that $\mathbf{L} = \mathbf{const}$. But then we can calculate the increment dW of the generating function in two different sets of coordinates.

1. Spherical coordinates (r, ϑ, φ):

$$p_\varphi = L_z = \alpha_\varphi = \text{const} .$$

2. Planar polar coordinates of the orbital plane $(\rho, \overline{\varphi})$:

$$p_{\overline{\varphi}} = \overline{L}_z = |\mathbf{L}| = \alpha_\vartheta = \text{const} .$$

In the last step we still used (3.138):

$$\frac{dW}{dt} = \sum_j \left(\frac{\partial W}{\partial q_j} \dot{q}_j + \frac{\partial W}{\partial \bar{p}_j} \underbrace{\dot{\bar{p}}_j}_{=0} \right) = \sum_j p_j \dot{q} .$$

By this relation we can write for dW:

$$dW = p_r\, dr + p_\varphi\, d_\varphi + p_\vartheta\, d\vartheta = p_\rho\, d\rho + p_{\overline{\varphi}}\, d\overline{\varphi} .$$

Since the radial parts are of course the same in both systems of coordinates we further have:

$$p_\vartheta\, d\vartheta = p_{\overline{\varphi}}\, d\overline{\varphi} - p_\varphi\, d\varphi = \alpha_\vartheta\, d\overline{\varphi} - \alpha_\varphi\, d\varphi .$$

Therewith the action variable J_ϑ is found as in (3.143):

$$J_\vartheta = \oint p_\vartheta\, d\vartheta = \alpha_\vartheta \oint d\overline{\varphi} - \alpha_\varphi \oint d\varphi = 2\pi(\alpha_\vartheta - \alpha_\varphi) .$$

Finally we are left with the task of calculating the J_r integral. For this we have according to (3.141)–(3.143):

$$\begin{aligned} J_r &= \oint \sqrt{2m\left(E + \frac{k}{r}\right) - \frac{\left(J_\varphi + J_\vartheta\right)^2}{4\pi^2 r^2}}\, dr \\ &= \oint \sqrt{2m\left(E - V_{\text{eff}}(r)\right)}\, dr , \end{aligned} \tag{3.144}$$

$$V_{\text{eff}}(r) = -\frac{k}{r} + \frac{\left(J_\varphi + J_\vartheta\right)^2}{8\pi^2\, m\, r^2} . \tag{3.145}$$

For bound states, which we want to presume here (periodic motion!), it must hold:

$$E < 0$$

The reversal points $r_{1,2}$ are found as zeros of the radicand in (3.144):

$$0 < r_1 \leq r \leq r_2 < \infty .$$

Again we have to take into consideration that

$$p_r = m\dot{r} \begin{cases} > 0 & \text{for} \quad r_1 \longrightarrow r_2 \ , \\ < 0 & \text{for} \quad r_2 \longrightarrow r_1 \ . \end{cases} \tag{3.146}$$

Therefore one has to choose in (3.144) on one way the positive and on the way back the negative root. The direct integration turns out to be rather cumbersome. It is therefore recommendable to perform a so-called **'complex integration'**. This technique will be presented in detail in Sect. 4.4 of Vol. 3. Unfortunately and inconsistent with the general idea of this basic course, we have here to anticipate a bit. The reader who is not yet familiar with the *complex integration* should skip the following discussion until Eq. (3.150).

We choose a path of integration in the complex r plane as indicated in Fig. 3.8. Since the values of the function are positive *on the way there* ($r_1 \to r_2$) and negative *on the way back* ($r_2 \to r_1$) the integration contains both branches of the two-valued square root function. In the region **on the left-hand side** of the directed integration path (Fig. 3.8, lower part), however, the function is unique. Only at the cutting line between the two branching points $r_1,\ r_2$ the function values have a discontinuity. If we contract the integration path onto the $\overline{r_1\, r_2}$ line we get by use of Cauchy's residue theorem ((4.386), Vol. 3):

$$J_r = 2\pi\, i \cdot \textit{(residues of the poles to the left of the directed integration path)}$$

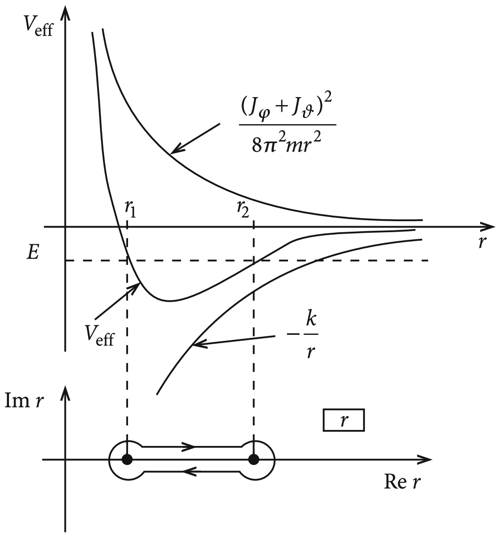


Fig. 3.8 Illustration of the path of integration for the calculation of the action variable J_r of the Kepler problem

The integrand in (3.144) has poles in the interesting region at $r = 0$ and $r = \infty$:

$$J_r = 2\pi\, i\, \left(a_{-1}(0) + a_{-1}(\infty)\right) \tag{3.147}$$

(a_{-1} : symbol for the residue). Let us first inspect the pole at $r = 0$. There the integrand behaves like $\left[(1+x)^{-1/2} = 1 - (1/2)x + 0(x^2)\right]$:

$$\frac{1}{r}\sqrt{-\alpha_\vartheta^2}\,\sqrt{1 - \frac{2m}{\alpha_\vartheta^2}(E\,r^2 + k\,r)} = \frac{i\,\alpha_\vartheta}{r}\left[1 - \frac{m}{\alpha_\vartheta^2}(E\,r^2 + k\,r) + \mathcal{O}(r^2)\right] .$$

The residue thus is:

$$a_{-1}(0) = i\,\alpha_\vartheta \;. \tag{3.148}$$

In order to discuss the point $r = \infty$ we transform:

$$r = \frac{1}{u} \implies dr = -\frac{1}{u^2}\,du \;.$$

$r \to \infty$ thus means $u \to 0$. The integrand in (3.144) now reads:

$$-\frac{1}{u^2}\sqrt{2mE}\,\sqrt{1 + \frac{k}{E}u - \frac{\alpha_\vartheta^2 u^2}{2mE}} = -\frac{1}{u^2}\sqrt{2mE}\left(1 + \frac{k}{2E}u - \frac{\alpha_\vartheta^2 u^2}{4mE} + \mathcal{O}(u^2)\right) .$$

The residue is the coefficient of the $(1/u)$ term:

$$a_{-1}(\infty) = -\sqrt{2m\,E}\,\frac{1}{2}\,\frac{k}{E} = -\frac{i}{2}\,k\sqrt{\frac{2m}{-E}} \;. \tag{3.149}$$

We insert (3.148) and (3.149) into (3.147) getting therewith:

$$J_r = -2\pi\,\alpha_\vartheta + \pi\,k\sqrt{\frac{2m}{-E}} \;. \tag{3.150}$$

Because $\alpha_\vartheta = (1/2\pi)(J_\vartheta + J_\varphi)$ we can also write:

$$J_r = -(J_\vartheta + J_\varphi) + \pi\,k\sqrt{\frac{2m}{-E}} \;. \tag{3.151}$$

Since $\overline{H} = E = \alpha_1$ we can now express the *new* Hamilton function by the action variables:

$$\overline{H}\left(J_r, J_\vartheta, J_\varphi\right) = -\frac{2\pi^2 m\,k^2}{\left(J_r + J_\vartheta + J_\varphi\right)^2} \;. \tag{3.152}$$

The three frequencies of the periodic motion,

$$\nu_j = \frac{\partial \overline{H}}{\partial J_j} , \qquad j = r,\, \vartheta,\, \varphi ,$$

are obviously all the same:

$$\nu = \frac{4\pi^2 m\, k^2}{\left(J_r + J_\vartheta + J_\varphi\right)^3} . \tag{3.153}$$

One says that the motion is **completely degenerate** and **simple-periodic**. For a potential of the form (3.125) the trajectory is closed for negative total energy E. After a period the angles ϑ, φ and the radius r retain again their original initial values. Notice that the degeneracy with respect to the angles ϑ and φ is already a characteristic of **all** central fields. This one realizes when inspecting (3.144) where E is connected with J_φ, J_ϑ in the form $(J_\varphi + J_\vartheta)$ without that we had to specify $V(\mathbf{r}) = V(r)$.

We are still able to derive with (3.151) and (3.153) an interesting secondary result:

$$\left(J_r + J_\vartheta + J_\varphi\right)^3 = \pi^3 k^3 \frac{(2m)^{3/2}}{(-E)^{3/2}} ,$$

$$\tau = \frac{1}{\nu} = \pi\, k \sqrt{\frac{m}{-2E^3}} . \tag{3.154}$$

The period of the motion is therefore related to the semi-major axis ((2.270), Vol. 1),

$$a = -\frac{k}{2E} ,$$

in the following manner:

$$\tau^2 \sim a^3 . \tag{3.155}$$

This relationship we got to know as the **Kepler's third law** ((2.278), Vol. 1).

3.5.4 Degeneracy

In Sect. 3.5.1 we have referred to the movement of a system in the $2S$-dimensional phase space as **periodic** if the projection of the trajectory onto each of the S (q_j, p_j)-planes is periodic in terms of libration and rotation, respectively, where the frequencies

$$\nu_j = \frac{1}{\tau_j} , \qquad j = 1, 2, \ldots, S \tag{3.156}$$

can certainly be distinct. The phase trajectory is called **simple-periodic** in the case when after a sufficiently long time the phase comes back to its initial value. This requires, however, that the frequencies ν_j are rational multiples of each other. If not then the phase trajectory is **conditional-periodic**.

If the frequencies ν_j are indeed rational multiples of each other,

$$p_1\nu_1 = p_2\nu_2 \; ; \quad p_2\nu_2 = p_3\nu_3 \; ; \quad \ldots \; ; \quad p_{S-1}\nu_{S-1} = p_S\nu_S \; ; \quad p_i \in \mathbb{N} \; ,$$

then there obviously exist $(S-1)$ independent relations of the type:

$$\sum_{j=1}^{S} n_j^{(l)} \nu_j = 0 \; , \qquad l = 1, 2, \ldots, S-1 \; , \quad n_j^{(l)} \in \mathbb{Z} \; . \tag{3.157}$$

One speaks of an **'m-fold degenerate'** system if there are $m \le (S-1)$ relations of this kind. The motion is **'completely degenerate'** if $m = S-1$. That is, for instance, the case if, as for the Kepler problem, all ν_j are equal. A *simple-periodic*, i.e. closed phase trajectory is therefore always completely degenerate.

In the case of an m-fold degeneracy, the m degeneracy conditions can be used in order to describe the periodic motion by $S-m$ instead of S frequencies. That can be done as follows: One performs a canonical transformation

$$(\boldsymbol{\omega}, \boldsymbol{J}) \longrightarrow (\overline{\boldsymbol{\omega}}, \overline{\boldsymbol{J}}) \tag{3.158}$$

with the generating function

$$F_2\left(\boldsymbol{\omega}, \overline{\boldsymbol{J}}\right) = \sum_{l=1}^{m} \sum_{j=1}^{S} \overline{J}_l \, n_j^{(l)} \, \omega_j + \sum_{l=m+1}^{S} \overline{J}_l \, \omega_l \; . \tag{3.159}$$

The second summand corresponds to the identity transformation (2.176):

$$\overline{\omega}_l = \frac{\partial F_2}{\partial \overline{J}_l} = \begin{cases} \sum_{j=1}^{S} n_j^{(l)} \, \omega_j & \text{for } l = 1, \ldots, m \; , \\ \omega_l & \text{for } l = m+1, \ldots, S \; . \end{cases} \tag{3.160}$$

For the *new* frequencies we then have:

$$\overline{\nu}_l = \dot{\overline{\omega}}_l = \begin{cases} \sum_{j=1}^{S} n_j^{(l)} \, \nu_j = 0 & \text{for } l = 1, \ldots, m \; , \\ \nu_l & \text{for } l = m+1, \ldots, S \; . \end{cases} \tag{3.161}$$

After the transformation there are left only $S-m$ independent, non-zero, distinct frequencies. In the first line in (3.161) we find just the m degeneracy conditions (3.157). Since, on the other hand, it must generally hold

$$\overline{\nu}_j = \frac{\partial \overline{H}}{\partial \overline{J}_j} \; ,$$

the Hamilton function can always be written such that it depends only on $S-m$ action variables:

$$\overline{H} = \overline{H}\left(\overline{J}_{m+1}, \ldots, \overline{J}_S\right) \; . \tag{3.162}$$

For the *Kepler motion* $S = 3$, and there are on hand two degeneracy conditions:

$$\nu_\varphi - \nu_\vartheta = 0 \; ; \quad \nu_\vartheta - \nu_r = 0 \; . \tag{3.163}$$

That leads according to (3.159) to the following generating function:

$$F_2 = \left(\omega_\varphi - \omega_\vartheta\right)\overline{J}_1 + \left(\omega_\vartheta - \omega_r\right)\overline{J}_2 + \omega_r \overline{J}_3 \; . \tag{3.164}$$

After (3.160) the next step yields:

$$\overline{\omega}_1 = \omega_\varphi - \omega_\vartheta \; ; \quad \overline{\omega}_2 = \omega_\vartheta - \omega_r \; ; \quad \overline{\omega}_3 = \omega_r \; . \tag{3.165}$$

Because of (3.163) that means for the frequencies:

$$\overline{\nu}_1 = \overline{\nu}_2 = 0 \; ; \quad \overline{\nu}_3 = \nu_r \; . \tag{3.166}$$

The generating function F_2 fixes also the new action variables, because

$$J_j = \frac{\partial F_2}{\partial \omega_j}$$

leads with (3.164) to:

$$J_\varphi = \overline{J}_1 \; ; \quad J_\vartheta = -\overline{J}_1 + \overline{J}_2 \; ; \quad J_r = -\overline{J}_2 + \overline{J}_3 \; .$$

Resolved for $\overline{J}_j$ it is:

$$\overline{J}_1 = J_\varphi \; ; \quad \overline{J}_2 = J_\varphi + J_\vartheta \; ; \quad \overline{J}_3 = J_\varphi + J_\vartheta + J_r \; . \tag{3.167}$$

The transformed Hamilton function $\overline{H}$ eventually depends only on one action variable (3.152):

$$\overline{H} = -\frac{2\pi^2 m\, k^2}{\overline{J}_3^2} = \overline{H}\left(\overline{J}_3\right) \;, \tag{3.168}$$

$$\nu = \frac{\partial \overline{H}}{\partial \overline{J}_3} = \frac{4\pi^2 m\, k^2}{\overline{J}_3^3} \;. \tag{3.169}$$

3.5.5 Bohr-Sommerfeld Atom Theory

The perhaps most spectacular success of the method of action and angle variables represents Bohr's atom theory whose quantum hypothesis can be formulated most simply via action variables.

Definition 3.5.1 J is an 'eigen-action variable' if the corresponding frequency is non-zero and non-degenerate

In the example of the last section $\overline{J}_3$ is such an eigen-action variable. In classical mechanics there is no restriction on the range of values for J. Experimental observations in the framework of atomic physics, however, require the setting up of the classically not provable

quantum hypothesis

If J is an eigen-action variable then the actual motion of the system is permitted only on those orbits for which holds:

$$J = n\, h \;, \tag{3.170}$$

$n \in \mathbb{N}$, $h = 6{,}626176 \cdot 10^{-34}\, J\, s$ *(Planck's quantum of action).*

Let us consider as an example the

hydrogen atom $\Longleftrightarrow$ Kepler problem with $k = \dfrac{e^2}{4\pi\, \varepsilon_0}$.

According to (3.168) the energy of the shell electron amounts to:

$$E = -\frac{2\pi^2 m\, e^4}{(4\pi\, \varepsilon_0)^2\, \overline{J}_3^2} \;. \tag{3.171}$$

It is **quantized** because $\overline{J}_3$ is an eigen-action variable.

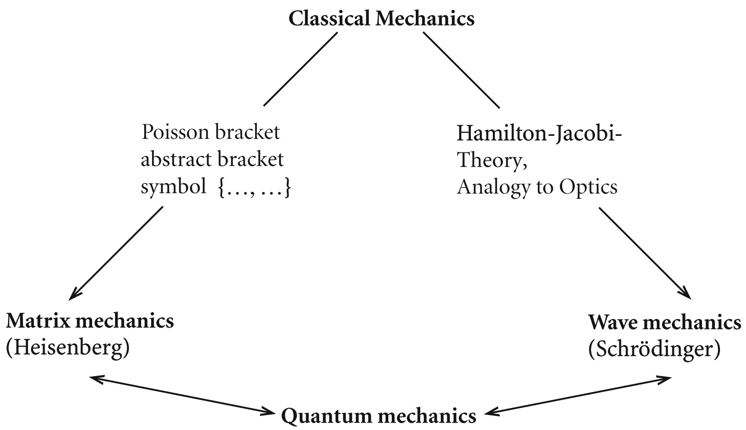
$$E_n = -\frac{E_\mathrm{R}}{n^2}\,, \qquad n = 1, 2, \ldots\,, \tag{3.172}$$

$$E_\mathrm{R} = \frac{2\pi^2 m\, e^4}{(4\pi\,\varepsilon_0)^2\; h^2} = 13{,}61\,\mathrm{eV}\,, \quad \textbf{Rydberg energy}\,. \tag{3.173}$$

n is the so-called **principal quantum number**. Equation (3.172) corresponds exactly to the correct quantum mechanical result.

3.6 The Transition to Wave (Quantum) Mechanics

The application of classical mechanics to atomic problems has led to spectacular successes by the Bohr-Sommerfeld atom theory, it however leaves behind also serious discrepancies between theory and experiment. In particular it is based on hypotheses which appear *rather arbitrary*. What we need is something like a generalization of the macroscopically correct classical mechanics in order to be able to describe also microscopic (atomic) systems. This problem was already touched upon briefly in Sect. 2.4.5 where we inferred from the classical Poisson bracket that there should exist a super-ordinate mathematical structure which permits besides classical mechanics further realizations such as for instance, the quantum mechanics in the form of the so-called **matrix mechanics (Heisenberg)**. We will now exploit an analogy reflection on optics in order to interpret classical mechanics as a limiting case of quantum mechanics in the form of the so-called **wave mechanics (Schrödinger)**:



3.6.1 The Wave Equation of Classical Mechanics

The following considerations are valid for systems with

$$H = T + V = E = \text{const} \,, \tag{3.174}$$

i.e., the Hamilton function is not explicitly time-dependent, and there are no rheonomic constraints. According to (3.45) we can then separate the time-dependence of the **action function**:

$$S\,(\mathbf{q}, \bar{\mathbf{p}}, t) = W\,(\mathbf{q}, \bar{\mathbf{p}}) - E\,t \,. \tag{3.175}$$

Let us remember: S is a generating function of the type F_2, which takes care for $\overline{H} = 0$ and therewith leads to $\bar{\mathbf{p}} = \mathbf{const}$, $\bar{\mathbf{q}} = \mathbf{const}$. The characteristic function $W(\mathbf{q}, \bar{\mathbf{p}})$ is time-independent, and because of $\bar{\mathbf{p}} = \boldsymbol{\alpha} = \mathbf{const}$ we can conclude:

$$\mathbf{W = const} \iff \textbf{fixed plane in the configuration space}\,.$$

The planes $S = \text{const}$, on the other hand, are moving in the configuration space, shifting themselves in the course of time t over the fixed W-planes (Fig. 3.9). They build within the configuration space propagating wave fronts of the so-called

waves of action

We ask ourselves:

1. What is the velocity of the propagating S-planes?
2. What is the physical meaning of the motion of the waves of action?

For simplicity we assume that the considered system consists of one single particle,

$$\mathbf{q} = (x, y, z) \,, \tag{3.176}$$

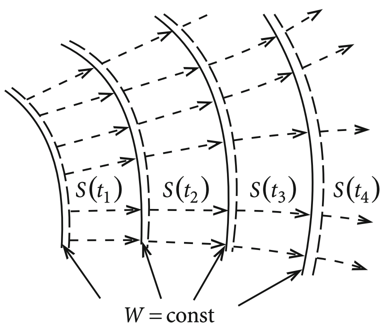


Fig. 3.9 Action-wave fronts in the configuration space

so that the configuration space coincides with the three-dimensional visual space. The

wave velocity u

is the propagation-velocity of a certain point of the front of the wave of action. Since the area of constant S will change its shape as a function of time, the wave velocity will in general not be the same for all points of the wave front. Let us consider two neighboring points in the configuration space and event space, respectively:

$$\begin{aligned} A &= (x, y, z) && \text{at time } t \,, \\ B &= (x + dx,\ y + dy,\ z + dz) && \text{at time } t + dt \,. \end{aligned}$$

From A to B the action function changes by dS:

$$\begin{aligned} dS &= \frac{\partial S}{\partial t}\, dt + \frac{\partial S}{\partial x}\, dx + \frac{\partial S}{\partial y}\, dy + \frac{\partial S}{\partial z}\, dz \\ &= -E\, dt + \nabla W \cdot d\mathbf{r} \,. \end{aligned} \tag{3.177}$$

How fast have we to move from A to B so that the observed action S **does not** change, i.e. in order to co-move with the wave of action? From the requirement

$$dS \overset{!}{=} 0 = -E\, dt + (\nabla W \cdot \mathbf{u})\, dt$$

we have with (3.177):

$$\nabla W \cdot \mathbf{u} = E \,. \tag{3.178}$$

$\mathbf{u}$ is oriented perpendicularly to the wave front. ∇W stands perpendicularly on the area $W =$ const being therewith parallel or antiparallel to $\mathbf{u}$:

$$|\mathbf{u}| = \frac{|E|}{|\nabla W|} \,. \tag{3.179}$$

W is a generating function of the type F_2. According to the general transformation formulas (2.161) it therefore holds in the present case for the momentum of the particle $\mathbf{p}$:

$$\mathbf{p} = \nabla W \,. \tag{3.180}$$

The particle momentum and therewith the total trajectory of the particle are also taking course perpendicularly to the wave front $S =$ const and $W =$ const, respectively. **The velocity of the wave of action and the velocity of the particle**

are thus (anti-)parallel! For the magnitudes we have

$$u = \frac{|E|}{|\nabla W|} = \frac{|E|}{p} = \frac{|E|}{m\,v}$$

and therewith:

$$u\,v = \frac{|E|}{m} = \text{const}\,. \tag{3.181}$$

Particle and action-wave velocity are therefore oriented (anti-)parallelly where their magnitudes are inversely proportional to each other.
Limiting cases:

$$E = T \implies u = \frac{v}{2}\,, \tag{3.182}$$

$$E = V \implies u = \infty,\ \text{da}\ v = 0\,. \tag{3.183}$$

We come to a first conclusion: There exist obviously two types of motion which turn out to be completely equivalent for the description of the system:

(1) **actual particle motion,**
(2) **wave of action.**

Here we have a first hint of the

particle-wave dualism

which will become fundamentally important for quantum mechanics. In order to further deepen this aspect we reformulate the familiar Hamilton-Jacobi differential equation for the particle motion into a wave equation for the wave of action:

$$u = \frac{|E|}{p} = \frac{|E|}{\sqrt{2m\,T}} = \frac{|E|}{\sqrt{2m(E-V)}}\,. \tag{3.184}$$

The HJD reads

$$\frac{1}{2m}\left[\left(\frac{\partial W}{\partial x}\right)^2 + \left(\frac{\partial W}{\partial y}\right)^2 + \left(\frac{\partial W}{\partial z}\right)^2\right] + V = E \tag{3.185}$$

or in short:

$$|\nabla W|^2 = 2m(E-V)\,. \tag{3.186}$$

Comparison with (3.184) yields:

$$|\nabla W|^2 = \frac{E^2}{u^2} \, . \tag{3.187}$$

This is of course in accordance with (3.179). Waves of action and particle movement both are therefore solutions of the HJD. Because of

$$\nabla W = \nabla S \quad \text{and} \quad -E = \frac{\partial S}{\partial t}$$

we obtain from (3.187) the
wave equation of classical mechanics

$$(\nabla S)^2 = \frac{1}{u^2}\left(\frac{\partial S}{\partial t}\right)^2 \, . \tag{3.188}$$

What did we achieve? The wave equation (3.188) is surely a very adequate formulation of classical mechanics, at least for the description of atomic systems. **However, the wave equation is not in all respects exact!**

We therefore seek a new theory which has a broader region of validity than the classical mechanics and which contains classical mechanics as a certain limiting case. Such a theory of course **can not be derived** from our hitherto existing knowledge about classical mechanics.

We are in fact forced to construct an as plausible as possible approach whose justification must be taken in the final analysis by a comparison of its results with experimental data.

Thereby the above formulation of classical mechanics in the form of a wave equation will substantially help us. Namely, a rather analogous problem has been managed in optics.
Idea:

Is the classical mechanics in the framework of the to be found super-ordinate theory possibly something like the geometrical optics in relation to the general theory of light waves?

For many optical problems one does not need to apply the full electromagnetic theory of light waves. Sometimes the auxiliary concept of

‘light rays’ $\cong$ **paths of ‘light particles’**

is sufficient in order to reach reasonable results with quasi-geometrical considerations. However, to this approach limits are set, namely, for instance, when diffraction phenomena become relevant. Then light has to be considered as wave motion for which planes of constant phase are propagating with the velocity $\mathbf{u}$ through the space. We will investigate in the following the indicated analogy in some more detail.

3.6.2 Insertion About Light Waves

Today one knows that 'light' is an electromagnetic process which is described by the

scalar wave equation of optics

$$\nabla^2\varphi - \frac{n^2}{c^2}\frac{\partial^2\varphi}{\partial t^2} = 0 \qquad (3.189)$$

Here:

φ :	scalar electromagnetic potential ,
$c = 3 \cdot 10^{10}\,\mathrm{cm\,s^{-1}}$:	speed of light in vacuum ,
n :	index of refraction, generally $n = n(\mathbf{r})$,
$u = c/n$:	speed of light in matter .

We are looking for simple solutions of the wave equation. For this purpose first we assume

$$n = \text{const}$$

Then the following ansatz (*plane wave*)

$$\varphi = \varphi_0 \mathrm{e}^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)} \qquad (3.190)$$

obviously solves the wave equation provided:

$$k = \omega\frac{n}{c} = \frac{2\pi\,\nu}{u} = \frac{2\pi}{\lambda} \qquad (3.191)$$

Thereby we have exploited

$$\omega = 2\pi\,\nu\ ; \quad u = \nu\,\lambda \qquad (3.192)$$

The direction of $\mathbf{k}$ (*wave vector*) may define the z-axis. Let $\mathbf{k}_0$ be the wave vector in vacuum ($n = 1$):

$$\mathbf{k} = n\,\mathbf{k}_0\ ; \quad \omega = c\,k_0\ . \qquad (3.193)$$

Therewith we can write the solution (3.190) also as follows:

$$\varphi = \varphi_0\,\mathrm{e}^{i\,k_0(nz - ct)}\ . \qquad (3.194)$$

In the next step we now assume

$$n = n(\mathbf{r}) \neq \text{const} .$$

The space-dependence of the index of refraction gives rise to perturbations (*diffractions*) of the light wave; the plane wave (3.194) is no longer solution of (3.189). Strictly speaking, even the wave equation has no longer the form of (3.189). In order that this form of the wave equation remains valid to a good approximation, though, and the solution φ at least still possesses the structure (3.194), it is required that:

$$|\nabla n(\mathbf{r})| \ll \frac{n(\mathbf{r})}{\lambda}$$

This characterizes the **limiting case of geometrical optics** (Sect. 4.3.16, Vol. 3):

$n(\mathbf{r})$ is only weakly space-dependent so that $n \approx$ const may be assumed over regions of the extension λ

But then φ also should still have approximately a form as in (3.194). One therefore tries the following **ansatz**:

$$\varphi = \exp\left(A(\mathbf{r}) + i\,k_0\,(L(\mathbf{r}) - c\,t)\right) \ . \tag{3.195}$$

The first term fixes the amplitude to be of course constant for $n =$ const. One calls

$L(\mathbf{r})$: **'light (optical) path', 'eikonal'**

where $L(\mathbf{r}) = nz$ if $n =$ const. We now insert the ansatz (3.195) into the wave equation (3.189):

$$\begin{aligned}
\nabla\varphi &= \varphi\left[\nabla\left(A(\mathbf{r}) + i\,k_0 L(\mathbf{r})\right)\right] \ , \\
\nabla^2\varphi &= \varphi\left[\left(\nabla\left(A(\mathbf{r}) + i\,k_0 L(\mathbf{r})\right)\right)^2 + \nabla^2\left(A(\mathbf{r}) + i\,k_0 L(\mathbf{r})\right)\right] \\
&= \varphi\Big[\left(\nabla A(\mathbf{r})\right)^2 - k_0^2\left(\nabla L(\mathbf{r})\right)^2 + 2i\,k_0\left(\nabla A(\mathbf{r})\right)\cdot\left(\nabla L(\mathbf{r})\right) \\
&\quad + \nabla^2 A(\mathbf{r}) + i\,k_0 \nabla^2 L(\mathbf{r})\Big] \ .
\end{aligned}$$

The wave equation (3.189) hence yields:

$$\begin{aligned}
0 = {}& i\,k_0\left[\nabla^2 L(\mathbf{r}) + 2\left(\nabla A(\mathbf{r})\right)\cdot\left(\nabla L(\mathbf{r})\right)\right] \\
&+ \left[\nabla^2 A(\mathbf{r}) + \left(\nabla A(\mathbf{r})\right)^2 - k_0^2\left(\nabla L(\mathbf{r})\right)^2 + n^2 k_0^2\right] \ .
\end{aligned}$$

Real and imaginary parts of this equation must already separately vanish:

$$\nabla^2 L(\mathbf{r}) + 2\,(\nabla A(\mathbf{r})) \cdot (\nabla L(\mathbf{r})) = 0\,, \tag{3.196}$$

$$\nabla^2 A(\mathbf{r}) + (\nabla A(\mathbf{r}))^2 + k_0^2 \left(n^2 - (\nabla L(\mathbf{r}))^2\right) = 0\,. \tag{3.197}$$

Up to now everything is still exact. The presumptions of the

geometrical optics

can now be formulated as follows:

$$A(\mathbf{r}) \;:\; \text{weakly } \mathbf{r}\text{-dependent}\,,$$
$$\lambda_0 \;\ll\; \text{changes in the medium}\,.$$

λ_0 is the wave length of the light in vacuum. Because of $k_0^2 = 4\pi^2/\lambda_0^2$ the last term in (3.197) dominates. That yields to a good approximation the so-called
eikonal equation of geometrical optics

$$(\nabla L(\mathbf{r}))^2 = n^2 = \frac{c^2}{u^2}\,. \tag{3.198}$$

The solutions define planes of constant phases ($L =$ const), i.e. wave fronts. The ray trajectories are running perpendicularly to these wave fronts.

The eikonal equation (3.198) is formally identical to the wave equation (3.188) of classical mechanics. Between classical mechanics and geometrical optics an analogy exists insofar as the classical mechanics comes to the same statements about the action function S and W, respectively, as the geometrical optics about the eikonal L.

3.6.3 The Ansatz of Wave Mechanics

The considerations of the last section suggest the following attempt for a generalization of classical mechanics:

classical mechanics $\Longleftrightarrow$ geometrical-optical limiting case of a wave mechanics

We extend the hitherto theory in the sense that we now interpret the

particle motion as wave motion.

We can draw the final justification of this idea of course, if at all, only from a later comparison between theory and experiment. We use, at first only tentatively, the following assignments:

$$(\nabla W)^2 = \frac{E^2}{u^2} \iff (\nabla L)^2 = n^2 \tag{3.199}$$

$$W \iff L \tag{3.200}$$

$$\frac{|E|}{u} = \sqrt{2m(E-V)} \iff n = \frac{c}{u} \, . \tag{3.201}$$

That does not at all mean that the single terms were exactly equal. They only correspond to each other. For instance, they might be proportional to each other.

If the particle can really be interpreted as a wave then it should also be possible to ascribe to it a wave length λ and a frequency ν. After (3.200) W is analogous to L. But then

$$S = W - E\,t$$

should correspond in (3.195) to the total phase:

$$k_0(L - c\,t)$$

That means $E \sim c\,k_0$ and therewith

$$E \sim \nu \, .$$

The proportionality factor must have the dimension of an *'action'*:

$$E = h\,\nu \, . \tag{3.202}$$

This is the **energy spectrum of the particle wave**. Furtheron it holds:

$$u = \lambda\nu \implies \lambda = \frac{u}{\nu} = \frac{E/p}{E/h} \, .$$

The **wave length of the particle** can therefore be fixed as

$$\lambda = \frac{h}{p} \tag{3.203}$$

The experiment impressively confirms these relations provided

h**: Planck's quantum of action** (3.170).

We see that energy and momentum of the particle define frequency and wave length of the **particle wave**. Just in this sense the particle motion can be interpreted as wave motion.

We now want to upgrade classical mechanics to a super-ordinate wave mechanics in a similar manner as the geometrical optics has been supplemented to the wave optics. The wave optics is reached by exactly solving the wave equation (3.189). With

$$\varphi \approx \mathrm{e}^{-i\omega t} \tag{3.204}$$

we get a **time-independent wave equation**

$$\nabla^2\varphi + \frac{\omega^2}{u^2}\varphi = \nabla^2\varphi + \frac{4\pi^2}{\lambda^2}\varphi = 0 \ . \tag{3.205}$$

φ describes in a certain sense the *state* of the light wave. Analogously the **state** of the particle may be described by the

wave function $\psi = \psi(\mathbf{r}, t)$

where a more deepened physical interpretation may be left over to the field of *quantum mechanics*. It follows with

$$\frac{4\pi^2}{\lambda^2} = \frac{4\pi^2}{h^2}p^2 = \frac{1}{\hbar^2}2m(E-V) \ ; \quad \hbar = \frac{h}{2\pi}$$

by analogy from the wave equation (3.205)

$$\Delta\psi + \frac{2m}{\hbar^2}(E-V)\psi = 0 \ . \tag{3.206}$$

This is the famous

time-independent Schrödinger equation

which fundamentally dominates the whole of quantum mechanics.

We still multiply, eventually, Eq. (3.206) by $\hbar^2/2m$:

$$\left(-\frac{\hbar^2}{2m}\Delta + V(\mathbf{r})\right)\psi(\mathbf{r}, t) = E\,\psi(\mathbf{r}, t) \ . \tag{3.207}$$

This is an **eigen-value equation** (see (4.64), Vol. 1) of the so-called

Hamilton operator

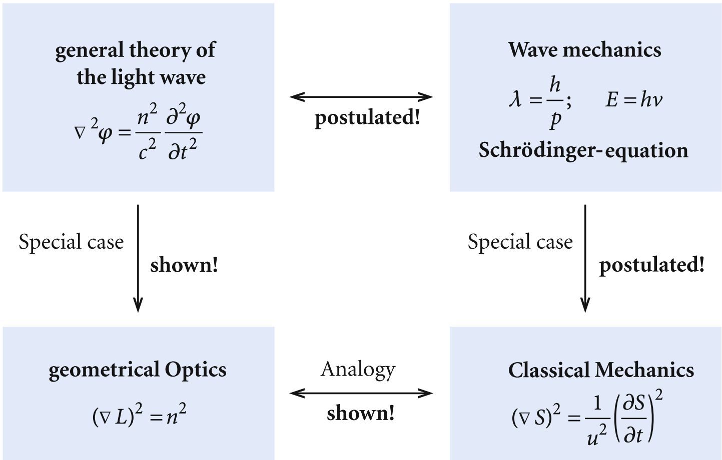
$$\overline{H} = -\frac{\hbar^2}{2m}\Delta + V(\mathbf{r}) \ , \tag{3.208}$$

$$\overline{H}\,\psi(\mathbf{r}, t) = E\,\psi(\mathbf{r}, t) \ . \tag{3.209}$$

This operator arises from the classical Hamilton function $H(q, p)$ by replacing the dynamical variables q, p by corresponding operators. Obviously it holds the following assignment (*'position representation'*):

$$\hat{\mathbf{q}} \Longrightarrow \mathbf{r}\,; \quad \hat{\mathbf{p}} \Longrightarrow \frac{\hbar}{i}\nabla\,. \tag{3.210}$$

We close this section with a schematic summary of our conclusions:



3.7 Exercises

Exercise 3.7.1 Formulate the Hamilton-Jacobi differential equation for a force-free particle and solve it for the characteristic function W.

Exercise 3.7.2 Find the Hamilton-Jacobi differential equation for the one-dimensional movement of a particle of mass m in the potential

$$V(x) = -b\,x$$

and solve the problem with the initial conditions

$$x(t=0) = x_0\,; \quad \dot{x}(t=0) = v_0\,.$$

Exercise 3.7.3 A particle of mass m performs a one-dimensional movement in the potential:

$$V(q) = c\,\mathrm{e}^{\gamma q} \qquad\qquad c, \gamma \in \mathbb{R}$$

Calculate $q(t)$ and $p(t)$ by use of the Hamilton-Jacobi method. Apply as *characteristic* function $W(q, \hat{p})$ a generating function of type $F_2(q, \hat{p})$.

Exercise 3.7.4 A particle of mass m executes a two-dimensional movement in the xy-plane in the potential

$$V(x,y) = c(x-y) \quad (c = \text{const})$$

Solve the Hamilton-Jacobi differential equation for the characteristic function $W(x, y, \hat{p}_x, \hat{p}_y)$. Use the initial conditions:

$$x(t=0) = y(t=0) = 0$$
$$\dot{x}(t=0) = v_{0x} > 0$$
$$\dot{y}(t=0) = 0 \, .$$

Exercise 3.7.5 Write down the Hamilton-Jacobi differential equation for the two-dimensional harmonic oscillator in Cartesian coordinates. Find $x(t)$ and $y(t)$!

Exercise 3.7.6 Given the linear harmonic oscillator:

$$H = \frac{p^2}{2m} + \frac{1}{2} m \omega_0^2 q^2$$

Perform in the framework of the Hamilton-Jacobi theory a canonical transformation in such a manner that the *'new'* coordinate $\hat{q}$ and the *'new'* momentum $\hat{p}$ are constants of motion:

$$(q,p) \xrightarrow{S} (\alpha, \beta) \, .$$

The generating function S may be of the type $F_3 = F_3(p, \hat{q}, t)$. Calculate $q(t)$ and $p(t)$ with the initial conditions:

$$t = 0 \ : \ p(0) = p_0 = 0 \, , \quad q(0) = q_0 > 0 \, .$$

How is the generating function S related to the action functional of Hamilton's principle?

Exercise 3.7.7 Apply the method of the action and angle variables to determine the frequencies of a three-dimensional harmonic oscillator with pairwise different force constants.

Exercise 3.7.8 Consider the three-dimensional harmonic oscillator of the last exercise, but now for the case that all the force constants are equal. Transform the result of the last exercise to eigen-action variables!

3.8 Self-examination Questions

To Section 3.1

1. Which canonical transformations do you know that makes the integration of the Hamilton equations of motion quasi-trivial?
2. How does the Hamilton-Jacobi differential equation read? Sketch briefly its motivation and its derivation!
3. Which type of differential equation does the HJD represent? Which function shall be determined with the HJD?
4. How is the solution of the HJD classified? Give reasons for this classification!
5. Outline the procedure of solution by which problems of classical mechanics can be treated via the HJD!

To Section 3.2

1. What is the Hamilton-Jacobi differential equation of the linear harmonic oscillator?
2. What is to be understood by separation ansatz (approach)? Find such an ansatz for the HJD of the harmonic oscillator!
3. Of which type must the *generating function* be which fulfills the HJD?

To Section 3.3

1. When is a separation ansatz for the HJD-solution, which separates **q**- and t-dependencies, reasonable?
2. How is Hamilton's characteristic function defined?
3. One can consider Hamilton's characteristic function as a generating function of a canonical transformation. Of which type is this transformation? What shall be provided by it?
4. Under which conditions does the transformation, caused by Hamilton's characteristic function, lead to the Hamilton function of a system of free mass points?
5. Describe the physical meaning of Hamilton's characteristic function!

To Section 3.4

1. When is the Hamilton-Jacobi method useful at all?
2. Under which conditions does a separation ansatz for the solution of the HJD appear reasonable?
3. When do we denote the HJD as *separable* in the coordinates q_j?
4. Which form does the Hamilton's characteristic function have in the case of cyclic coordinates?

To Section 3.5

1. What is to be understood by a libration, what by a rotation?
2. When is the movement of the pendulum a libration and when is it a rotation?
3. When is a multi-dimensional movement ($S > 1$) periodic? When is it called *conditional-periodic*?

4. How can one check easily the periodicity in case of completely separable systems?
5. How are the action and angle variables defined?
6. Outline the advantages of the action and angle variables! For which systems is the method applicable?
7. How does the angle variable ω_i change when the coordinate q_j runs through its full period?
8. How does the Hamilton function of the linear harmonic oscillator look like after transformation on action variables?
9. How does one calculate the frequency ν_i of the periodic q_i-movement?
10. Find the HJD for the Kepler problem!
11. Show that the Kepler problem is completely separable by the use of spherical coordinates!
12. Do the action variables J_φ, J_ϑ of the Kepler problem depend in any way on the type of the central field?
13. How does the transformed Hamilton function $\overline{H}$ of the Kepler problem depend on the action variables J_r, J_ϑ, J_φ?
14. When is the Kepler movement called *completely degenerate*?
15. When is a motion called *simple-periodic* and when *conditional-periodic*?
16. What does an *'m-fold degenerate system'* mean?
17. When is a *simple-periodic* phase trajectory *completely degenerate*?
18. How many independent frequencies are necessary to describe an m-fold degenerate, S-dimensional trajectory?
19. What are the degeneracy conditions for the Kepler movement?
20. What is an eigen-action variable?
21. Formulate the quantum hypothesis!
22. Apply the quantum hypothesis to the motion of the electron in the hydrogen atom!

To Section 3.6

1. Explain the term *action wave*!
2. What does one understand by the velocity $\mathbf{u}$ of the action wave?
3. Which direction does $\mathbf{u}$ have?
4. For a system that consists only of one single particle, what can you say about the direction and magnitude of the particle velocity and the action wave velocity? What does concretely hold when the total energy consists only of kinetic energy?
5. Interpret the term *particle-wave dualism* in classical mechanics!
6. What do we consider as the wave equation of classical mechanics?
7. How does the scalar wave equation of the optics look like? What is its solution in the case of a constant index of refraction?
8. Define the term *'eikonal'*!
9. List the assumptions which guarantee the validity of *geometrical optics*!
10. What is the *'eikonal equation'* of geometrical optics?

11. Discuss the analogy between the eikonal equation of geometrical optics and the scalar wave equation of classical mechanics!
12. How can we ascribe frequency and wavelength to a *'mechanical'* particle?
13. Which analogy-consideration can introduce the wave function of a particle?
14. What is the Schrödinger equation of a particle?
15. Which relation does exist between Hamilton function (classical mechanics) and Hamilton operator (quantum mechanics)?

Appendix A
Solutions of the Exercises

Section 1.2.7

Solution 1.2.1 We follow Example (2) in Sect. 1.2.2. There we had derived as general solution:

$$r(t) = A\,\mathrm{e}^{\omega t} + B\,\mathrm{e}^{-\omega t}$$

The initial conditions supply for A and B the equations of determination:

$$r_0 = A + B\,; \quad -r_0\omega = (A - B)\omega\,.$$

This leads to $A = 0$ and $B = r_0$. The solution thus reads:

$$r(t) = r_0\,\mathrm{e}^{-\omega t}\,.$$

In this special case the bead is moving with decreasing velocity towards the center of rotation, finally to come to rest there.

Solution 1.2.2

1. We calculate the constraint force, conveniently according to (1.112). The application of cylindrical coordinates r, φ, z appears reasonable. Transformation formulas ((1.381), Vol. 1):

   $$x = r\cos\varphi\,; \quad y = r\sin\varphi\,; \quad z = z\,.$$

   Constraints:

   $$f_1(\mathbf{r}, t) = z = 0$$
   $$f_2(\mathbf{r}, t) = \varphi - \omega t = 0\,.$$

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Gradient expressed by cylindric coordinates ((1.388), Vol. 1):

$$\nabla \equiv \left(\frac{\partial}{\partial r}, \frac{1}{r}\frac{\partial}{\partial \varphi}, \frac{\partial}{\partial z}\right) = \mathbf{e}_r\frac{\partial}{\partial r} + \mathbf{e}_\varphi\frac{1}{r}\frac{\partial}{\partial \varphi} + \mathbf{e}_z\frac{\partial}{\partial z} \, .$$

That means:

$$\nabla f_1 = (0, 0, 1) = \mathbf{e}_z \, ; \quad \nabla f_2 = \left(0, \frac{1}{r}, 0\right) = \frac{1}{r}\mathbf{e}_\varphi \, .$$

Constraint force according to (1.112):

$$\mathbf{Z} = \sum_{\nu=1}^{2} \lambda_\nu \nabla f_\nu = \lambda_1 \mathbf{e}_z + \lambda_2 \frac{1}{r}\mathbf{e}_\varphi \, .$$

λ_1 and λ_2 are at first undetermined Lagrange multipliers. Since in this example no *driving* force is present ($\mathbf{K} \equiv 0$) we have according to (1.111) by use of cylindric coordinates ((2.19), Vol. 1):

$$m\ddot{\mathbf{r}} = m\left(\ddot{r} - r\dot{\varphi}^2\right)\mathbf{e}_r + m\left(r\ddot{\varphi} + 2\dot{r}\dot{\varphi}\right)\mathbf{e}_\varphi + m\ddot{z}\mathbf{e}_z \overset{!}{=} \mathbf{Z} \, .$$

The constraints lead to:

$$\ddot{z} = 0 \, ; \quad \dot{\varphi} = \omega \, ; \quad \ddot{\varphi} = 0 \, .$$

It remains to solve (component-by-component):

$$\begin{aligned} m\left(\ddot{r} - r\omega^2\right) &= 0 \\ 2m\dot{r}\omega &= \lambda_2 \frac{1}{r} \\ 0 &= \lambda_1 \, . \end{aligned}$$

Therewith the Lagrange multipliers are fixed:

$$\lambda_1 = 0 \, ; \quad \lambda_2 = 2mr\dot{r}\omega \, .$$

The constraint force

$$\mathbf{Z} = 2m\dot{r}\omega \, \mathbf{e}_\varphi$$

therefore indeed fulfills the previous (Sect. 1.2.2) *plausible* assumption: $\mathbf{Z} \perp \mathbf{e}_r$. We now have the equation of motion (as previously),

$$\ddot{r} = r\,\omega^2 \, ,$$

with the general solution:

$$r(t) = A\,\mathrm{e}^{\omega t} + B\,\mathrm{e}^{-\omega t}\ .$$

The given initial conditions lead to $A = B = r_0/2$ and therewith:

$$r(t) = \frac{r_0}{2}\left(\mathrm{e}^{\omega t} + \mathrm{e}^{-\omega t}\right) = r_0 \cosh(\omega t) \curvearrowright \dot{r}(t) = r_0\,\omega \sinh(\omega t)\ .$$

Thereby we have also determined the explicit time-dependence of the constraint force:

$$\mathbf{Z}(t) = 2mr_0\,\omega^2 \sinh(\omega t)\,\mathbf{e}_\varphi\ .$$

2. With (1.113) we calculate:

$$\begin{aligned}\frac{d}{dt}(T+V) = -\lambda_1\frac{\partial f_1}{\partial t} - \lambda_2\frac{\partial f_2}{\partial t} &= +2mr\dot{r}\omega^2 \\ &= 2mr_0^2\omega^3 \cosh(\omega t)\sinh(\omega t)\ .\end{aligned}$$

Solution 1.2.3

1. Constraints:

$$\begin{aligned} z &= 0 \qquad \text{(scleronomic)} \\ y - x\tan\omega t &= 0 \qquad \text{(rheonomic)}\ . \end{aligned}$$

   Both constraints are holonomic and therewith the number of degrees of freedom $S = 3 - 2 = 1$. A proper generalized coordinate is of course the distance $q = r$ of the bead from the center of rotation.
2. With the transformation formulas

$$x = q\cos\omega t\ ; \quad y = q\sin\omega t$$

   we have:

$$\begin{aligned}\dot{x} &= \dot{q}\cos\omega t - q\omega\sin\omega t \\ \dot{y} &= \dot{q}\sin\omega t + q\omega\cos\omega t\end{aligned}$$

   Kinetic energy:

$$T = \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2\right) = \frac{m}{2}\left(\dot{q}^2 + q^2\omega^2\right) = T(q,\dot{q})$$

Potential energy:

$$V = m\,g\,y = mgq\sin\omega t = V(q,t)$$

Lagrangian:

$$L = T - V = \frac{m}{2}\left(\dot{q}^2 + q^2\omega^2\right) - mgq\sin\omega t$$

3. Equation of motion:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = m\ddot{q}\ ; \qquad \frac{\partial L}{\partial q} = mq\omega^2 - mg\sin\omega t$$

to solve:

$$\ddot{q} - q\omega^2 + g\sin\omega t = 0$$

We start with the respective homogeneous equation:

$$\ddot{q} - q\omega^2 = 0$$

General solution

$$q_0(t) = \alpha \mathrm{e}^{\omega t} + \beta \mathrm{e}^{-\omega t}\ .$$

Ansatz for a special solution of the inhomogeneous equation:

$$q_s(t) = \gamma \sin\omega t \qquad (\omega \neq 0)\ .$$

Insertion into the equation of motion:

$$-\gamma\omega^2 \sin\omega t - \omega^2\gamma \sin\omega t + g\sin\omega t = 0$$

$$-2\gamma\omega^2 + g = 0 \ \curvearrowright\ \gamma = \frac{g}{2\omega^2}\ .$$

Therewith we have the general solution of the inhomogeneous differential equation:

$$q(t) = \alpha \mathrm{e}^{\omega t} + \beta \mathrm{e}^{-\omega t} + \frac{g}{2\omega^2}\sin\omega t\ .$$

4. We use the given initial conditions:

$$q(t=0) = r_0 \ \curvearrowright\ r_0 = \alpha + \beta$$

$$\dot{q}(t=0) = 0 \ \curvearrowright\ \omega(\alpha - \beta) + \frac{g}{2\omega} = 0$$

$$\curvearrowright \; 2\alpha = r_0 - \frac{g}{2\omega^2}$$

$$\curvearrowright \; \alpha = \frac{r_0}{2} - \frac{g}{4\omega^2} \; ; \quad \beta = \frac{r_0}{2} + \frac{g}{4\omega^2}$$

That yields the complete solution:

$$q(t) = \frac{r_0}{2}\left(\mathrm{e}^{\omega t} + \mathrm{e}^{-\omega t}\right) - \frac{g}{4\omega^2}\left(\mathrm{e}^{\omega t} - \mathrm{e}^{-\omega t}\right) + \frac{g}{2\omega^2}\sin\omega t \; .$$

We calculate the first derivative wit respect to the time

$$\dot{q}(t) = \frac{r_0\omega}{2}\left(\mathrm{e}^{\omega t} - \mathrm{e}^{-\omega t}\right) - \frac{g}{4\omega}\left(\mathrm{e}^{\omega t} + \mathrm{e}^{-\omega t}\right) + \frac{g}{2\omega}\cos\omega t \; .$$

For large times it holds:

$$\dot{q}(t) \overset{t\to\infty}{\longrightarrow} \left(\frac{1}{2}r_0\omega - \frac{g}{4\omega}\right)\mathrm{e}^{\omega t} \; .$$

The bead moves outwards if $\dot{q}(t \to \infty) > 0$, i.e.

$$\omega^2 > \frac{g}{2r_0} \; .$$

5. The constraint is in principle rather difficult to be explicitly found. However, in the co-rotating coordinate system it is about an effectively one-dimensional problem:

$$m\ddot{r} = -F_g + F_z \; .$$

F_g is the component of the gravitational force acting into the direction of the wire:

$$F_g = mg\sin\omega t \; .$$

F_z is the centrifugal force for which we generally have ((2.79), Vol. 1):

$$\mathbf{F}_z = -m\left(\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})\right) = -m\left(\boldsymbol{\omega}(\boldsymbol{\omega}\cdot\mathbf{r}) - \mathbf{r}\omega^2\right) = m\mathbf{r}\omega^2 \; .$$

The last step is correct since $\boldsymbol{\omega}$ and $\mathbf{r}$ are orthogonal to each other. If we insert $F_z = m\omega^2 r$ into the above equation of motion we find after eliminating the mass m:

$$\ddot{r} - \omega^2 r + g\sin\omega t = 0 \; .$$

That is identical to the Lagrange equation from part 3.

Solution 1.2.4

1. Constraints:

$$\varphi - \omega t = 0\,; \quad z - \alpha\rho^2 = 0 \;\curvearrowright\; S = 3 - 2 = 1 \quad \text{degrees of freedom}\,.$$

2. Transformation formulas:

$$\begin{aligned} x &= \rho\,\cos\varphi \;\curvearrowright\; \dot{x} = \dot{\rho}\cos\varphi - \rho\dot{\varphi}\sin\varphi \\ y &= \rho\,\sin\varphi \;\curvearrowright\; \dot{y} = \dot{\rho}\sin\varphi + \rho\dot{\varphi}\cos\varphi \\ z &= \alpha\rho^2 \;\curvearrowright\; \dot{z} = 2\alpha\rho\dot{\rho}\,. \end{aligned}$$

Kinetic and potential energy:

$$\begin{aligned} T &= \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) \\ &= \frac{1}{2}m\left(\dot{\rho}^2\left(\cos^2\varphi + \sin^2\varphi\right) + \rho^2\dot{\varphi}^2\left(\cos^2\varphi + \sin^2\varphi\right) + 4\alpha^2\rho^2\dot{\rho}^2\right) \\ &= \frac{1}{2}m\left(1 + 4\alpha^2\rho^2\right)\dot{\rho}^2 + \frac{1}{2}m\rho^2\omega^2 \\ V &= mgz = mg\alpha\rho^2\,. \end{aligned}$$

Lagrangian:

$$L = T - V = \frac{1}{2}m\left(1 + 4\alpha^2\rho^2\right)\dot{\rho}^2 + \frac{1}{2}m\left(\omega^2 - 2\alpha g\right)\rho^2\,.$$

3. Special case:

$$\omega = \sqrt{2\alpha g}\,.$$

The Lagrangian simplifies to:

$$L = \frac{1}{2}m\left(1 + 4\alpha^2\rho^2\right)\dot{\rho}^2\,.$$

Therewith we have:

$$\begin{aligned} \frac{\partial L}{\partial\dot{\rho}} &= m\left(1 + 4\alpha^2\rho^2\right)\dot{\rho} \\ \frac{d}{dt}\frac{\partial L}{\partial\dot{\rho}} &= m\left(\left(1 + 4\alpha^2\rho^2\right)\ddot{\rho} + 8\alpha^2\rho\dot{\rho}^2\right) \\ \frac{\partial L}{\partial\rho} &= 4m\alpha^2\rho\dot{\rho}^2\,. \end{aligned}$$

Equation of motion

$$\left(1 + 4\alpha^2\rho^2\right)\ddot{\rho} + 4\alpha^2\rho\dot{\rho}^2 = 0\ .$$

Multiplication by $\dot{\rho}$ yields:

$$\left(1 + 4\alpha^2\rho^2\right)\dot{\rho}\ddot{\rho} + 4\alpha^2\rho\dot{\rho}^3 = \frac{d}{dt}\left(1 + 4\alpha^2\rho^2\right)\dot{\rho}^2 \stackrel{!}{=} 0\ .$$

That means that $(1 + 4\alpha^2\rho^2)\dot{\rho}^2$ represents an integral of motion.

Solution 1.2.5

1.

$$\begin{aligned} &x = \rho\,\cos\varphi\ ; \quad y = \rho\,\sin\varphi\ ; \quad z = z\ , \\ &\dot{x} = \dot{\rho}\,\cos\varphi - \rho\,\dot{\varphi}\,\sin\varphi\ ; \quad \dot{y} = \dot{\rho}\,\sin\varphi + \rho\,\dot{\varphi}\,\cos\varphi \\ \Longrightarrow\ &\dot{x}^2 + \dot{y}^2 = \dot{\rho}^2 + \rho^2\dot{\varphi}^2\ . \end{aligned}$$

Lagrangian:

$$L = T - V = \frac{m}{2}\left(\dot{\rho}^2 + \rho^2\dot{\varphi}^2 + \dot{z}^2\right) - V_0\,\ln\frac{\rho}{\rho_0}\ .$$

2.

$$\begin{aligned} \frac{d}{dt}\frac{\partial L}{\partial\dot{\rho}} - \frac{\partial L}{\partial\rho} &= 0 = m\,\ddot{\rho} - m\,\rho\,\dot{\varphi}^2 + \frac{V_0}{\rho}\ , \\ \frac{d}{dt}\frac{\partial L}{\partial\dot{\varphi}} - \frac{\partial L}{\partial\varphi} &= 0 = m\,\rho^2\ddot{\varphi} + 2m\,\rho\,\dot{\rho}\,\dot{\varphi}\ , \\ \frac{d}{dt}\frac{\partial L}{\partial\dot{z}} - \frac{\partial L}{\partial z} &= 0 = m\,\ddot{z}\ . \end{aligned}$$

3. φ and z are cyclic $\Longrightarrow$

$$\begin{aligned} p_\varphi = \frac{\partial L}{\partial\dot{\varphi}} &= m\,\rho^2\dot{\varphi} = \text{const}: \quad z\text{-component of the angular momentum}\ , \\ p_z = \frac{\partial L}{\partial\dot{z}} &= m\,\dot{z} = \text{const}: \quad z\text{-component of the (linear) momentum}\ . \end{aligned}$$

Solution 1.2.6

1. The kinetic energy

$$T = T_{\text{trans}} + T_{\text{rot}}\ ,$$

is composed of a translational part (m: mass of the cylinder)

$$T_{\text{trans}} = \frac{1}{2}m(R-r)^2\dot{\varphi}^2 \ ,$$

and a rotational part

$$T_{\text{rot}} = \frac{1}{2}J\left(\dot{\varphi} + \dot{\vartheta}\right)^2 \ ,$$

where

$$J = \frac{1}{2}m\,r^2$$

is the moment of inertia of the cylinder. The rolling off condition reads:

$$\begin{aligned} R\,d\varphi &= \ r\,d\vartheta \\ \Longrightarrow \ \dot{\vartheta} &= \ -\frac{R}{r}\dot{\varphi} \ . \end{aligned}$$

The potential energy is:

$$V = m\,g(R-r)\,(1-\cos\varphi) \ .$$

That yields the Lagrangian:

$$\begin{aligned} L &= T - V \\ &= \frac{1}{2}m(R-r)^2\dot{\varphi}^2 + \frac{1}{4}mr^2\left(\dot{\varphi} + \dot{\vartheta}\right)^2 - m\,g(R-r)(1-\cos\varphi) \\ &= \frac{1}{2}m\left((R-r)^2 + \frac{1}{2}r^2\left(1-\frac{R}{r}\right)^2\right)\dot{\varphi}^2 - m\,g(R-r)(1-\cos\varphi) \\ &= \frac{3}{4}m(R-r)^2\dot{\varphi}^2 + m\,g(R-r)\cos\varphi - m\,g(R-r) \ . \end{aligned}$$

2. With

$$\frac{\partial L}{\partial \varphi} = -m\,g(R-r)\sin\varphi$$

and

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\varphi}} = \frac{3}{2}m(R-r)^2\ddot{\varphi}$$

we have the equation of motion for φ:

$$\ddot{\varphi} = -\frac{2}{3}\frac{g}{R-r}\sin\varphi \ .$$

3. For small deflections $\varphi \ll 1$ one can approach:

$$\sin\varphi \approx \varphi$$
$$\Longrightarrow \quad \ddot{\varphi} = -\frac{2}{3}\frac{g}{R-r}\varphi \ .$$

With

$$\omega = \sqrt{\frac{2}{3}\frac{g}{R-r}}$$

the general solution is then

$$\varphi(t) = a\,\cos\omega t + b\,\sin\omega t \ ,$$

where a and b are fixed by initial conditions.

Solution 1.2.7

1. Cylindrical coordinates (r, φ, z) appear to be obviously convenient.
Constraint:

$$\tan\alpha = \frac{r}{z}$$
$$\Longleftrightarrow z = r\cot\alpha$$

Degrees of freedom:

$$S = 3 - 1 = 2$$

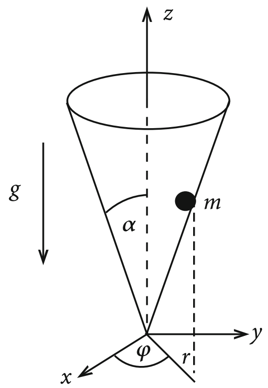
Generalized coordinates:

$$q_1 = r \ ; \quad q_2 = \varphi$$

Transformation formulas (Fig. A.1):

$$x = r\cos\varphi$$
$$y = r\sin\varphi$$
$$z = r\cot\alpha$$

Fig. A.1



2. Lagrangian:

$$\begin{aligned}\dot{x} &= \dot{r}\cos\varphi - r\dot{\varphi}\sin\varphi \\ \dot{y} &= \dot{r}\sin\varphi + r\dot{\varphi}\cos\varphi \\ \dot{z} &= \dot{r}\cot\alpha\end{aligned}$$

$$\begin{aligned}\Longrightarrow \quad \dot{x}^2 + \dot{y}^2 + \dot{z}^2 &= \dot{r}^2\cos^2\varphi + r^2\dot{\varphi}^2\sin^2\varphi - 2\dot{r}r\dot{\varphi}\cos\varphi\sin\varphi + \dot{r}^2\sin^2\varphi \\ &\quad + r^2\dot{\varphi}^2\cos^2\varphi + 2\dot{r}r\dot{\varphi}\sin\varphi\cos\varphi + \dot{r}^2\cot^2\alpha \\ \Longrightarrow \quad T &= \frac{m}{2}\left\{\left(1+\cot^2\alpha\right)\dot{r}^2 + r^2\dot{\varphi}^2\right\} \\ V &= mgz = mgr\cot\alpha \\ \Longrightarrow \quad L = T - V &= \frac{m}{2}\left[\left(1+\cot^2\alpha\right)\dot{r}^2 + r^2\dot{\varphi}^2\right] - mgr\cot\alpha\end{aligned}$$

Equations of motion:

$$\begin{aligned}\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} &= m\left(1+\cot^2\alpha\right)\ddot{r} \\ \frac{\partial L}{\partial r} &= m\left(r\dot{\varphi}^2 - g\cot\alpha\right) \\ \frac{d}{dt}\frac{\partial L}{\partial \dot{\varphi}} &= m\left(r^2\ddot{\varphi} + 2r\dot{r}\dot{\varphi}\right)\end{aligned}$$

$$\frac{\partial L}{\partial \varphi} = 0$$

$$\Longrightarrow \quad \begin{array}{c} \left(1+\cot^2\alpha\right)\ddot{r} - r\dot{\varphi}^2 + g\cot\alpha = 0 \\ r\ddot{\varphi} + 2\dot{r}\dot{\varphi} = 0 \end{array} \quad (r \neq 0)$$

3. φ is cyclic

$$\Longrightarrow \quad p_\varphi = \frac{\partial L}{\partial \dot{\varphi}} = mr^2\dot{\varphi} = \text{const}$$

Angular-momentum conservation!

Solution 1.2.8

1. The constraints are:

$$z_1 = z_2 = 0$$
$$x_1 = y_2 = 0$$
$$x_2^2 + y_1^2 = l^2$$

So the system possesses one degree of freedom. With φ as generalized coordinate one finds the transformation formulas:

$$x_2 = l\cos\varphi$$
$$y_1 = l\sin\varphi$$

Kinetic energy:

$$T = \frac{m}{2}\dot{x}_2^2 + \frac{m}{2}\dot{y}_1^2 = \frac{m}{2}l^2\dot{\varphi}^2\left(\sin^2\varphi + \cos^2\varphi\right)$$
$$= \frac{1}{2}ml^2\dot{\varphi}^2$$

Potential energy:

$$V = mgy_1 + 0 = mgl\sin\varphi$$

The Lagrangian is then:

$$L = T - V = \frac{1}{2}ml^2\dot{\varphi}^2 - mgl\sin\varphi$$

2. The Lagrange equation of motion for φ reads:

$$0 = \frac{d}{dt}\frac{\partial L}{\partial \dot{\varphi}} - \frac{\partial L}{\partial \varphi} = ml^2\ddot{\varphi} + mgl\cos\varphi$$

$$\Longrightarrow \quad 0 = \ddot{\varphi} + \frac{g}{l}\cos\varphi$$

Multiplication by $\dot{\varphi}$ and integration:

$$\ddot{\varphi}\dot{\varphi} + \frac{g}{l}\cos\varphi\dot{\varphi} = 0$$

$$\Longrightarrow \quad \frac{1}{2}\dot{\varphi}^2 + \frac{g}{l}\sin\varphi = c = \text{const}$$

$$\Longrightarrow \quad \dot{\varphi} = \sqrt{2\left(c - \frac{g}{l}\sin\varphi\right)}$$

This can be further integrated after separation of variables:

$$dt = \frac{d\varphi}{\sqrt{2\left(c - \frac{g}{l}\sin\varphi\right)}}$$

$$\Longrightarrow \quad t - t_0 = \int\limits_{\varphi_0}^{\varphi} \frac{d\varphi'}{\sqrt{2\left(c - \frac{g}{l}\sin\varphi'\right)}}$$

The constants of integration $\varphi_0 = \varphi(t_0)$ and c follow by exploiting the initial conditions.
For *'small angles'* the solution simplifies to:

$$t - t_0 = -\frac{l}{g}\left(\sqrt{2\left(c - \frac{g}{l}\varphi\right)} - \sqrt{2\left(c - \frac{g}{l}\varphi_0\right)}\right) .$$

Solution 1.2.9 The motion can be decomposed into a translation of the center of gravity S and a rotation of the rod around its center of gravity within the xy-plane. For the latter we need the moment of inertia of the rod (see (4.11), Vol. 1):

$$J = \int d^3r\rho(\mathbf{r})(\mathbf{n} \times \mathbf{r})^2 .$$

$\mathbf{n}$ is the unit vector in the direction of the rotation axis. For the calculation of J we use cylindrical coordinates $\bar{\rho}, \bar{\varphi}, z$. The z-axis may be identical to the axis of the rod. Then the rotation axis will be the x-axis (or equivalently the y-axis):

$$(\mathbf{n} \times \mathbf{r}) = (1,0,0) \times (\bar{\rho}\cos\bar{\varphi}, \bar{\rho}\sin\bar{\varphi}, z) = (0, -z, \bar{\rho}\sin\bar{\varphi}) .$$

It remains to calculate:

$$J = \rho_0 \int_0^R \bar{\rho} d\bar{\rho} \int_0^{2\pi} d\bar{\varphi} \int_{-L}^{+L} dz \left(z^2 + \bar{\rho}^2 \sin^2 \bar{\varphi}\right)$$

$$\curvearrowright \; J = \frac{1}{3} ML^2 \left(1 + \frac{3}{4}\left(\frac{R}{L}\right)^2\right) \stackrel{R \ll L}{\longrightarrow} \frac{1}{3} ML^2 \; .$$

Kinetic energy:
Translation:

$$T_S = \frac{1}{2} M \left(\dot{x}_S^2 + \dot{y}_S^2\right) = \frac{1}{2} M \left((-L\dot{\varphi}\sin\varphi)^2 + (L\dot{\varphi}\cos\varphi)^2\right) = \frac{1}{2} ML^2 \, \dot{\varphi}^2 \; .$$

Rotation:

$$T_R = \frac{1}{2} J \dot{\varphi}^2 = \frac{1}{6} ML^2 \left(1 + \frac{3}{4}\left(\frac{R}{L}\right)^2\right) \dot{\varphi}^2 \stackrel{R \ll L}{\longrightarrow} \frac{1}{6} ML^2 \, \dot{\varphi}^2 \; .$$

Potential energy:

$$V = Mgy = MgL \sin\varphi \; .$$

Lagrangian:

$$\hat{L} = T - V = \frac{2}{3} ML^2 \, \dot{\varphi}^2 - MgL \sin\varphi \; .$$

Lagrange equation of motion of the second kind:

$$\frac{d}{dt} \frac{\partial \hat{L}}{\partial \dot{\varphi}} = \frac{4}{3} ML^2 \, \ddot{\varphi} \; ; \quad \frac{\partial \hat{L}}{\partial \varphi} = -MgL \cos\varphi$$

$$\curvearrowright \; \ddot{\varphi} + \frac{3}{4} \frac{g}{L} \cos\varphi = 0 \; .$$

With the substitution $\varphi \to \varphi - \pi/2$ we get a differential equation of the oscillation type as for the mathematical pendulum (2.124, Vol.1). However, it does not come to a real oscillation since the rod *'is falling'* only from $\varphi = \pi/2$ to $\varphi = 0$ in order to hit at $\varphi = 0$ the bottom of the earth. The solution of the equation of motion turns out to be an elliptical integral.

Solution 1.2.10

1. Constraints:

$$\begin{aligned} &x^2 + y^2 + z^2 - R^2 = 0 : \text{holonomic-scleronomic} \, , \\ &\frac{y}{x} - \tan\omega t = 0 : \qquad \text{holonomic-rheonomic} \, . \end{aligned}$$

2. $q = \vartheta$

$$\begin{aligned} x &= R \sin\vartheta \ \cos\omega t \, , \\ y &= R \sin\vartheta \ \sin\omega t \, , \\ z &= R \cos\vartheta \ . \end{aligned}$$

$$T = \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) = \frac{m}{2}\left(R^2 \sin^2\vartheta \ \omega^2 + R^2\dot{\vartheta}^2\right) \ .$$

The first summand results from the rotation of the ring, the second from the motion on the ring.

$$V = m\,g\,R(1 - \cos\vartheta) \ .$$

Lagrangian:

$$L = \frac{m}{2}R^2\left(\omega^2 \sin^2\vartheta + \dot{\vartheta}^2\right) - m\,g\,R(1 - \cos\vartheta) \ .$$

Equation of motion:

$$\begin{aligned} \frac{d}{dt}\frac{\partial L}{\partial\dot{\vartheta}} &= m\,R^2\ddot{\vartheta} \ , \\ \frac{\partial L}{\partial\vartheta} &= m\,R^2\omega^2 \sin\vartheta \ \cos\vartheta - m\,g\,R \ \sin\vartheta \end{aligned}$$

$$\Longrightarrow \quad \ddot{\vartheta} + \left(\frac{g}{R} - \omega^2 \cos\vartheta\right)\sin\vartheta \ = 0 \ .$$

3. $\vartheta \ll 1 : \ \cos\vartheta \approx 1, \ \sin\vartheta \approx \vartheta.$
   Therewith the equation of motion simplifies to

$$\begin{aligned} \ddot{\vartheta} + \overline{\omega}^2\vartheta &= 0 \ , \\ \overline{\omega}^2 &= \frac{g}{R} - \omega^2 \end{aligned}$$

   with the general solution:

$$\vartheta(t) = A \ \cos\overline{\omega}t + B \ \sin\overline{\omega}t \ .$$

Solution 1.2.11

1. There are four holonomic-scleronomic constraints:

$$\begin{aligned} l &= r + S\,, \qquad \textit{(length of the thread)}\,,\\ z(m) &= 0\,,\\ x(M) &= 0\,,\\ y(M) &= 0\,. \end{aligned}$$

2. Because of the four constraints there remain $6-4=2$ degrees of freedom. Thus we need two generalized coordinates

$$q_1 = \varphi\,; \quad q_2 = S\,.$$

From Fig. 1.29 we read off the transformation formulas:

$$\begin{aligned} x(m) &= r\,\cos\varphi = (l-S)\cos\varphi\,,\\ y(m) &= r\,\sin\varphi = (l-S)\sin\varphi\,,\\ z(M) &= -S\\ \Longrightarrow\quad \dot{x}(m) &= -\dot{S}\,\cos\varphi - (l-S)\dot{\varphi}\,\sin\varphi\,,\\ \dot{y}(m) &= -\dot{S}\,\sin\varphi + (l-S)\dot{\varphi}\,\cos\varphi\,,\\ \dot{z}(M) &= -\dot{S}\,. \end{aligned}$$

Kinetic energy:

$$T = \frac{1}{2}m\left(\dot{x}^2(m) + \dot{y}^2(m)\right) + \frac{1}{2}M\,\dot{z}^2(M) = \frac{1}{2}(m+M)\dot{S}^2 + \frac{1}{2}m(l-S)^2\dot{\varphi}^2\,.$$

Potential energy:

$$V = M\,g\,z(M) = -M\,g\,S\,.$$

Lagrangian:

$$L = T - V = \frac{1}{2}(m+M)\dot{S}^2 + \frac{1}{2}m(l-S)^2\dot{\varphi}^2 + M\,g\,S\,.$$

We realize that the coordinate φ is cyclic. That means:

$$\frac{\partial L}{\partial\dot{\varphi}} = m(l-S)^2\dot{\varphi} = \text{const} = J\,\dot{\varphi} = L_0\,.$$

This is the angular-momentum conservation law. The quantities

$$J = J(t) \qquad \text{and} \qquad \dot{\varphi} = \dot{\varphi}(t)$$

change in course of time but the product remains constant.
For the second coordinate $q_2 = S$ we have the Lagrange equation of motion:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{S}} = (m+M)\ddot{S} \,,$$

$$\frac{\partial L}{\partial S} = -m(l-S)\dot{\varphi}^2 + M\,g = -\frac{L_0^2}{m(l-S)^3} + M\,g$$

$$\Longrightarrow \qquad (m+M)\ddot{S} + \frac{L_0^2}{m(l-S)^3} - M\,g = 0 \,.$$

We multiply this equation by $\dot{S}$ and integrate:

$$\frac{1}{2}(m+M)\dot{S}^2 + \frac{L_0^2}{2m(l-S)^2} - M\,g\,S = \text{const} \,.$$

But that is the energy conservation law:

$$T + V = E = \text{const} \,.$$

3. Equilibrium means:

$$\ddot{S} = 0 \,.$$

But then it must also be valid:

$$\frac{L_0^2}{m(l-S)^3} = M\,g \implies S = S_0 = \text{const} \,,$$

$$\omega_0 = \dot{\varphi}_0 = \frac{L_0}{m(l-S_0)^2} = \sqrt{\frac{M\,g}{m(l-S_0)}} \,.$$

We read off from the equation of motion:

$$\dot{\varphi} > \omega_0 \iff \ddot{S} < 0 \iff \ddot{z}(M) > 0 : \quad M \ \text{slips upwards !}$$

$$\dot{\varphi} < \omega_0 \iff \ddot{S} > 0 \iff \ddot{z}(M) < 0 : \quad M \ \text{slips downwards !}$$

4. For the special case $\omega = \dot{\varphi} = 0$ the equation of motion yields:

$$\ddot{S} = \frac{M}{m+M} g \; .$$

   This is just the delayed free-fall of the mass M.

Solution 1.2.12

1. Lagrangian:

$$L = T - V = \frac{1}{2} m\, l^2 \dot{\varphi}^2 - m\, g (1 - \cos\varphi) l \; ,$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\varphi}} = m\, l^2 \ddot{\varphi} \; ; \quad \frac{\partial L}{\partial \varphi} = -m\, g\, l\, \sin\varphi \underbrace{\approx}_{\text{small deflections}} -m\, g\, l\, \varphi$$

   $\Longrightarrow$ oscillation equation:

$$\ddot{\varphi} + \frac{g}{l}\varphi = 0 \; .$$

   General solution:

$$\varphi(t) = A\, \cos\omega_0 t + B\, \sin\omega_0 t \; ; \quad \omega_0 = \sqrt{\frac{g}{l}} \; .$$

   Special boundary condition $\varphi(0) = 0$:

$$\varphi(t) = \varphi_0 \sin\omega_0 t \; .$$

2. The thread tension is the constraint force which guarantees the constant length of the thread.

$$m\, \ddot{\mathbf{r}}(t) \quad : \quad \text{force which acts on the mass } m.$$

$$\mathbf{r}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = l \begin{pmatrix} \cos\varphi(t) \\ \sin\varphi(t) \end{pmatrix}$$

$$\Longrightarrow \quad \dot{\mathbf{r}}(t) = l\, \dot{\varphi}(t) \begin{pmatrix} -\sin\varphi(t) \\ \cos\varphi(t) \end{pmatrix}$$

$$\Longrightarrow \quad m\ddot{\mathbf{r}}(t) = m\, l\, \ddot{\varphi}(t) \begin{pmatrix} -\sin\varphi(t) \\ \cos\varphi(t) \end{pmatrix} + m\, l\, \dot{\varphi}^2(t) \begin{pmatrix} -\cos\varphi(t) \\ -\sin\varphi(t) \end{pmatrix}$$

$$= m\, g\, \mathbf{e}_x - Z\, \mathbf{e}_r \; .$$

From that we determine the thread tension:

$$
\begin{aligned}
\mathbf{Z} &= Z\,\mathbf{e}_r\ , \\
Z &= m\,g(\mathbf{e}_x\cdot\mathbf{e}_r) - m\,\ddot{\mathbf{r}}(t)\cdot\mathbf{e}_r\ , \\
\mathbf{e}_x &= \begin{pmatrix}1\\0\end{pmatrix}\ ;\quad \mathbf{e}_r = \begin{pmatrix}\cos\varphi(t)\\ \sin\varphi(t)\end{pmatrix} \implies \mathbf{e}_x\cdot\mathbf{e}_r = \cos\varphi(t) \\
\implies Z &= m\,g\,\cos\varphi(t) + m\,l\,\dot\varphi^2(t)\ .
\end{aligned}
$$

Small pendulum displacements: $\cos\varphi(t)\approx 1-\frac{1}{2}\varphi^2(t)$:

$$
\begin{aligned}
\implies Z &= m\,g\left(1-\frac{1}{2}\varphi_0^2\sin^2\omega_0 t\right) + m\,l\,\omega_0^2\varphi_0^2\cos^2\omega_0 t \\
&= m\,g\left(1-\frac{1}{2}\varphi_0^2+\frac{1}{2}\varphi_0^2\cos^2\omega_0 t+\varphi_0^2\cos^2\omega_0 t\right) \\
\implies Z &= m\,g\left(1-\frac{1}{2}\varphi_0^2+\frac{3}{2}\varphi_0^2\cos^2\omega_0 t\right)\ .
\end{aligned}
$$

Solution 1.2.13

1. Constraints:

$$
\begin{aligned}
z &= 0 \\
x^2+y^2+z^2-l^2 &= 0
\end{aligned}
$$

$\implies\quad s = 3-2 = 1$ degree of freedom

2. Transformation formulas:

$$
\begin{aligned}
x &= r\cos\varphi & y &= r\sin\varphi \\
\implies \dot x &= \dot r\cos\varphi - \dot\varphi r\sin\varphi & \dot y &= \dot r\sin\varphi + \dot\varphi r\cos\varphi
\end{aligned}
$$

Kinetic energy:

$$
T = \frac{m}{2}\left(\dot x^2+\dot y^2\right) = \frac{m}{2}\left(\dot r^2 + r^2\dot\varphi^2\right)
$$

Potential energy:

$$
V = -mgx = -mgr\cos\varphi
$$

Lagrangian:

$$
L\left(r,\varphi,\dot r,\dot\varphi\right) = T - V = \frac{m}{2}\left(\dot r^2 + r^2\dot\varphi^2\right) + mgr\cos\varphi
$$

3. Given:

$$q_1 := \varphi\ , \quad q_2 := r\ .$$

Generally it holds for m variables and p holonomic constraints:

$$\sum_{j=1}^{m} a_{ij} dq_j + b_{it} dt = 0 \qquad i = 1, l \ldots, p$$

here: $m = 2$ variables ($q_1 = \varphi$, $q_2 = r$) and $p = 1$ constraints:

$$\begin{aligned} & q_2 = r = l = \text{const} \\ \Longrightarrow \quad & dq_2 = 0 \\ \Longrightarrow \quad & a_{11} = 0\ , \quad a_{12} = 1\ , \quad b_{1t} = 0 \end{aligned}$$

Generally we have for the constraint forces:

$$Q_j = \sum_{i=1}^{p} \lambda_i a_{ij}$$

where the λ_i are Lagrange multipliers. Here:

$$Q_1 = Q_\varphi = 0\ , \quad Q_2 = Q_r = \lambda$$

Therewith we get the equations of motion:

$$\begin{aligned} 0 = Q_\varphi &= \frac{d}{dt}\frac{\partial L}{\partial \dot\varphi} - \frac{\partial L}{\partial \varphi} \\ &= \frac{d}{dt}\left(mr^2\dot\varphi\right) + mgr\sin\varphi \\ &= mr^2\ddot\varphi + 2mr\dot r\dot\varphi + mgr\sin\varphi \\ \lambda = Q_r &= \frac{d}{dt}\frac{\partial L}{\partial \dot r} - \frac{\partial L}{\partial r} \\ &= \frac{d}{dt}(m\dot r) - mr\dot\varphi^2 - mg\cos\varphi \\ &= m\ddot r - mr\dot\varphi^2 - mg\cos\varphi \\ \Longrightarrow \quad 0 &= \ddot\varphi + \frac{2\dot r}{r}\dot\varphi + \frac{g}{r}\sin\varphi \\ \frac{Q_r}{m} &= \ddot r - r\dot\varphi^2 - g\cos\varphi \end{aligned}$$

With $r = l$ and $\ddot{r} = 0$ it follows for the thread tension Q_r from the equation of motion for r:

$$Q_r = -m\left(l\dot{\varphi}^2 + g\cos\varphi\right)$$

(see solution to Exercise 1.2.12, part (1))

4. With $\dot{r} = 0$, the approach of small angles

$$\varphi \ll 1 \quad \Longrightarrow \quad \sin\varphi \approx \varphi$$

and the definition

$$\omega_0 := \sqrt{\frac{g}{l}}$$

we find the equation of motion:

$$\ddot{\varphi} + \omega_0^2\varphi = 0$$

The general solution is:

$$\varphi(t) = \alpha\sin\omega_0 t + \beta\cos\omega_0 t$$

With the initial conditions

$$\begin{aligned} \varphi(0) &= 0 && \Longrightarrow \beta = 0 \\ \dot{\varphi}(0) &= \alpha\omega_0 = \sqrt{g/l}\,\varphi_0 && \Longrightarrow \alpha = \varphi_0 \end{aligned}$$

we have the solution:

$$\varphi(t) = \varphi_0 \sin\omega_0 t$$

Solution 1.2.14

1. By use of the Cartesian coordinates of the block of mass M

$$X = s\cos\alpha\ , \quad Y = s\sin\alpha$$

and the coordinates of the mass m

$$x = s\cos\alpha + l\sin\varphi\ , \quad y = s\sin\alpha - l\cos\varphi$$

the kinetic energy can be expressed by s and φ:

$$
\begin{aligned}
T &= \frac{M}{2}\left(\dot{X}^2 + \dot{Y}^2\right) + \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2\right) \\
&= \frac{M}{2}\left(\dot{s}^2\cos^2\alpha + \dot{s}^2\sin^2\alpha\right) + \frac{m}{2}\left(\dot{s}^2\cos^2\alpha + 2l\dot{s}\dot{\varphi}\cos\alpha\cos\varphi + l^2\dot{\varphi}^2\cos^2\varphi\right. \\
&\qquad \left. + \dot{s}^2\sin^2\alpha + 2l\dot{s}\dot{\varphi}\sin\alpha\sin\varphi + l^2\dot{\varphi}^2\sin^2\varphi\right)
\end{aligned}
$$

With the addition theorem ((1.61), Vol. 1)

$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

it simplifies to:

$$T = \frac{1}{2}(M+m)\dot{s}^2 + \frac{m}{2}l^2\dot{\varphi}^2 + ml\dot{s}\dot{\varphi}\cos(\alpha - \varphi)\ .$$

With the potential energy

$$V = Mgs\sin\alpha + mg\left(s\sin\alpha - l\cos\varphi\right)\ .$$

we then get for the Lagrangian:

$$
\begin{aligned}
L &= T - V \\
&= \frac{1}{2}(M+m)\dot{s}^2 + \frac{1}{2}ml^2\dot{\varphi}^2 + ml\dot{s}\dot{\varphi}\cos(\alpha-\varphi) - (M+m)gs\sin\alpha + mgl\cos\varphi
\end{aligned}
$$

2. Lagrange equation of motion for s:

$$
\begin{aligned}
0 &= (M+m)\ddot{s} + ml\left(\ddot{\varphi}\cos(\alpha-\varphi) + \dot{\varphi}^2\sin(\alpha-\varphi)\right) + (M+m)g\sin\alpha \\
\Longrightarrow \quad \ddot{s} &= -g\sin\alpha - \frac{ml}{M+m}\left(\ddot{\varphi}\cos(\alpha-\varphi) + \dot{\varphi}^2\sin(\alpha-\varphi)\right)
\end{aligned}
$$

Lagrange equation of motion for φ:

$$
\begin{aligned}
0 &= ml^2\ddot{\varphi} + ml\dot{s}\dot{\varphi}\sin(\alpha-\varphi) \\
&\qquad + ml\ddot{s}\cos(\alpha-\varphi) - ml\dot{s}\dot{\varphi}\sin(\alpha-\varphi) + mgl\sin\varphi \\
\Longrightarrow \quad \ddot{\varphi} &= -\frac{g}{l}\sin\varphi - \frac{\ddot{s}}{l}\cos(\alpha-\varphi)
\end{aligned}
$$

Special solution:

$$\varphi = \varphi_0 = \text{const}$$
$$\Longrightarrow \quad \ddot{s} = -g\sin\alpha = -g\frac{\sin\varphi_0}{\cos(\alpha-\varphi_0)}$$
$$\Longrightarrow \quad s(t) = s_0 + v_0 t - \frac{g}{2}t^2\sin\alpha$$
$$\varphi(t) = \alpha$$

(s_0 and v_0 by initial conditions)

3. Insertion of the differential equation for $\ddot{s}$ into the differential equation for $\ddot{\varphi}$ yields:

$$\begin{aligned}\ddot{\varphi} &= \frac{g}{l}\left(\sin\alpha\cos(\alpha-\varphi) - \sin\varphi\right)\\ &\quad + \frac{m}{M+m}\big(\ddot{\varphi}\cos(\alpha-\varphi) + \dot{\varphi}^2\sin(\alpha-\varphi)\big)\cos(\alpha-\varphi)\\ &= \frac{g}{l}\cos\alpha\sin(\alpha-\varphi)\\ &\quad + \frac{m}{M+m}\big(\ddot{\varphi}\cos^2(\alpha-\varphi) + \dot{\varphi}^2\sin(\alpha-\varphi)\cos(\alpha-\varphi)\big)\end{aligned}$$

Thereby we exploited the addition theorem ((1.60), Vol. 1):

$$\sin(\alpha-\varphi) = \sin\alpha\cos\varphi - \cos\alpha\sin\varphi$$

For $M \gg m$ it can approximately be assumed:

$$\frac{m}{M+m} \approx 0$$
$$\Longrightarrow \quad \ddot{\varphi} \approx -\frac{g}{l}\cos\alpha\sin(\varphi-\alpha)$$

With the abbreviation

$$\omega = \sqrt{\frac{g}{l}\cos\alpha}$$

and the approximation for small angle deflections ($\varphi \approx \alpha$)

$$\sin(\varphi-\alpha) \approx \varphi - \alpha$$

it comes out as oscillation equation:

$$\ddot{\varphi} = -\omega^2(\varphi - \alpha)$$
$$\Longrightarrow \quad \varphi(t) = \alpha + \hat{\varphi}\sin(\omega t + delta)$$

($\hat{\varphi}$ and δ from initial conditions)

Solution 1.2.15

1. There are five holonomic-scleronomic constraints:
   1. $z_1 = 0$
   2. $z_2 = 0$
   3. $-y_1 \,/\, -x_1 = \tan\alpha$
   4. $-y_2 \,/\, x_2 = \tan\beta$
   5. $r_1 + r_2 = l$

   The system therewith possesses $s = 2 \cdot 3 - 5 = 1$ degrees of freedom.
2. $s = 1 \Longrightarrow$ one generalized coordinate, e.g.:

$$q = r_1$$

   Transformation formulas:

$$\begin{aligned} x_1 &= -q\cos\alpha \\ y_1 &= -q\sin\alpha \\ z_1 &= 0 \\ x_2 &= (l-q)\cos\beta \\ y_2 &= -(l-q)\sin\beta \\ z_2 &= 0 \end{aligned}$$

3. Kinetic energy:

$$\begin{aligned} T &= \frac{m_1}{2}\left(\dot{x}_1^2 + \dot{y}_1^2\right) + \frac{m_2}{2}\left(\dot{x}_2^2 + \dot{y}_2^2\right) \\ &= \frac{m_1}{2}\dot{q}^2\left(\cos^2\alpha + \sin^2\alpha\right) + \frac{m_2}{2}\dot{q}^2\left(\cos^2\beta + \sin^2\beta\right) \\ &= \frac{1}{2}(m_1 + m_2)\,\dot{q}^2 \end{aligned}$$

   Potential energy:

$$\begin{aligned} V &= m_1 g y_1 + m_2 g y_2 \\ &= -m_1 g q\sin\alpha - m_2 g(l-q)\sin\beta \end{aligned}$$

$\Longrightarrow$ Lagrangian:

$$\begin{aligned} L &= T - V \\ &= \frac{1}{2}(m_1 + m_2)\dot{q}^2 + m_1 g q \sin\alpha + m_2 g(l - q)\sin\beta \end{aligned}$$

4. Lagrange equation of motion:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = (m_1 + m_2)\ddot{q} \overset{!}{=} \frac{\partial L}{\partial q} = (m_1 \sin\alpha - m_2 \sin\beta)g$$

$$\Longrightarrow \quad \ddot{q} = \frac{m_1 \sin\alpha - m_2 \sin\beta}{m_1 + m_2} g \qquad \text{'delayed' free fall}$$

Integration of the equation of motion and application of the initial conditions:

$$q(t) = r_1(t) = \frac{1}{2}\frac{m_1 \sin\alpha - m_2 \sin\beta}{m_1 + m_2} g t^2 + r_0$$

'System in its equilibrium' means:

$$\begin{aligned} q(t) &= \text{const} \\ \Longrightarrow \quad 0 &= m_1 \sin\alpha - m_2 \sin\beta \\ \Longrightarrow \quad \frac{m_1}{m_2} &= \frac{\sin\beta}{\sin\alpha} \end{aligned}$$

5. Now the 5. constraint will **not** be used to eliminate a variable. Therefore two generalized coordinates are necessary:

$$q_1 = r_1 \; ; \quad q_2 = r_2$$

Because of $r_1 + r_2 = l = \text{const}$ it follows:

$$\begin{aligned} dq_1 + dq_2 &= 0 \\ \Longrightarrow \quad a_{11} = a_{12} &= 1 \end{aligned}$$

$\Longrightarrow$ generalized constraint forces:

$$Q_1 = Q_2 = \lambda$$

Lagrangian:

$$L = \frac{1}{2}\left(m_1 \dot{q}_1^2 + m_2 \dot{q}_2^2\right) + m_1 g q_1 \sin\alpha + m_2 q_2 \sin\beta$$

$\Longrightarrow$ equation of motion:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = Q_i = \lambda \qquad i = 1, 2$$

$$\Longrightarrow \quad m_1 \ddot{q}_1 - m_1 g \sin\alpha = \lambda$$

$$m_2 \ddot{q}_2 - m_2 g \sin\beta = \lambda$$

From the 5. constraint it follows:

$$\dot{q}_1 + \dot{q}_2 = 0 \quad \Longrightarrow \quad \ddot{q}_1 = -\ddot{q}_2$$

$$\Longrightarrow \quad \ddot{q}_1 - g \sin\alpha = \frac{\lambda}{m_1}$$

$$-\ddot{q}_1 - g \sin\beta = \frac{\lambda}{m_2}$$

$$\Longrightarrow \quad -g\,(\sin\alpha + \sin\beta) = \lambda \left(\frac{1}{m_1} + \frac{1}{m_2}\right)$$

Therewith we have the constraint force *'thread tension'*:

$$Q = \lambda = -g \frac{m_1 m_2}{m_1 + m_2} (\sin\alpha + \sin\beta)$$

In the equilibrium it holds (see the preceding part of this exercise):

$$\frac{m_1}{m_2} = \frac{\sin\beta}{\sin\alpha}$$

$$\Longrightarrow \quad \sin\alpha + \sin\beta = \left(1 + \frac{m_1}{m_2}\right) \sin\alpha$$

$$= \frac{m_1 + m_2}{m_2} \sin\alpha$$

Finally we have the thread tension in the equilibrium:

$$Q_0 = -m_1 g \sin\alpha = -m_2 g \sin\beta$$

Solution 1.2.16

1. Starting point may be two body-fixed systems of Cartesian coordinates with parallel axes, as indicated in Fig. 1.34. The origins of coordinates are in the middle of the respective cylinder axis. Rotation axes:

$$\mathbf{n}_1 = \mathbf{n}_2 = -\mathbf{e}_z \, .$$

$\mathbf{r}_1$, $\mathbf{r}_2$ may be the points of support of the thread, i.e. the contact points of the thread tension:

$$\mathbf{r}_1 = (0, R_1, z_1) \;; \quad \mathbf{r}_2 = (0, -R_2, z_2) \;.$$

Thread tensions:

$$\mathbf{F}_1 = (F, 0, 0) = -\mathbf{F}_2 \;.$$

Torque moments:

$$\begin{aligned}
\mathbf{M}_{\text{ex}}^{(1)} &= (0, R_1, z_1) \times (F, 0, 0) = (0, z_1 F, -R_1 F) \\
\mathbf{M}_{\text{ex}}^{(2)} &= (0, -R_2, z_2) \times (-F, 0, 0) = (0, -z_2 F, -R_2 F) \;.
\end{aligned}$$

Paraxial components:

$$\mathbf{M}_{\text{ex}}^{(1)} \cdot \mathbf{n}_1 = R_1 F \;; \quad \mathbf{M}_{\text{ex}}^{(2)} \cdot \mathbf{n}_2 = R_2 F \;.$$

Angular momentum law ((4.17), Vol. 1):

$$J_1 \ddot{\varphi}_1 = R_1 F \;; \quad J_2 \ddot{\varphi}_2 = R_2 F \;.$$

Momenta of inertia of the cylinders with homogeneous mass density according to ((4.13), Vol. 1):

$$J_1 = \frac{1}{2} M_1 R_1^2 \;; \quad J_2 = \frac{1}{2} M_2 R_2^2 \,.$$

Rolling off condition:

$$\begin{aligned}
& x_2 = \text{const} + R_1 \varphi_1 + R_2 \varphi_2 \\
\curvearrowright \; & \ddot{x}_2 = R_1 \ddot{\varphi}_1 + R_2 \ddot{\varphi}_2 \,.
\end{aligned}$$

Translation of cylinder 2 according to the center of mass theorem:

$$M_2 \ddot{x}_2 = M_2 g - F \;.$$

Therewith it follows by insertion:

$$\begin{aligned}
& M_2 R_1 \ddot{\varphi}_1 + M_2 R_2 \ddot{\varphi}_2 = M_2 g - F \\
\curvearrowright \; & M_2 R_1 \frac{R_1 F}{J_1} + M_2 R_2 \frac{R_2 F}{J_2} = M_2 g - F
\end{aligned}$$

$$\curvearrowright \; F\left(1+\frac{R_1^2M_2}{J_1}+\frac{R_2^2M_2}{J_2}\right)=M_2g$$
$$\curvearrowright \; F\left(1+2\frac{M_2}{M_1}+2\right)=M_2g\ .$$

So it holds for the thread tension:

$$F=\frac{M_1M_2}{3M_1+2M_2}\,g\ .$$

2. Generalized coordinates: φ_1, φ_2
Constraint: Winding up the thread:

$$\begin{aligned} x_2 &= \text{const} + R_1\varphi_1 + R_2\varphi_2 \\ \curvearrowright \; \dot{x}_2 &= R_1\dot{\varphi}_1 + R_2\dot{\varphi}_2 \\ \ddot{x}_2 &= R_1\ddot{\varphi}_1 + R_2\ddot{\varphi}_2\ . \end{aligned}$$

Kinetic and potential energy:

$$\begin{aligned} T &= \tfrac{1}{2}J_1\dot{\varphi}_1^2 + \tfrac{1}{2}J_2\dot{\varphi}_2^2 + \tfrac{1}{2}M_2\dot{x}_2^2 \\ V &= -M_2g(x_2-\text{const})\ . \end{aligned}$$

Lagrangian:

$$L=\frac{1}{2}J_1\dot{\varphi}_1^2+\frac{1}{2}J_2\dot{\varphi}_2^2+\frac{1}{2}M_2\left(R_1\dot{\varphi}_1+R_2\dot{\varphi}_2\right)^2+M_2g\left(R_1\varphi_1+R_2\varphi_2\right)\ .$$

3. Equations of motion:

$$\frac{d}{dt}\frac{\partial L}{\partial\dot{\varphi}_i}-\frac{\partial L}{\partial\varphi_i}=0 \qquad i=1,2\ .$$

We calculate the $i=1$-equation:

$$\begin{aligned} \frac{\partial L}{\partial\dot{\varphi}_1} &= J_1\dot{\varphi}_1 + M_2R_1\left(R_1\dot{\varphi}_1+R_2\dot{\varphi}_2\right) \\ \curvearrowright \; \frac{d}{dt}\frac{\partial L}{\partial\dot{\varphi}_1} &= J_1\ddot{\varphi}_1 + M_2R_1\left(R_1\ddot{\varphi}_1+R_2\ddot{\varphi}_2\right) \\ &= \left(\frac{1}{2}M_1+M_2\right)R_1^2\ddot{\varphi}_1 + M_2R_1R_2\ddot{\varphi}_2 \\ \frac{\partial L}{\partial\varphi_1} &= M_2gR_1\ . \end{aligned}$$

That leads to the first equation of motion:

$$\left(\frac{1}{2}M_1 + M_2\right) R_1\ddot{\varphi}_1 + M_2R_2\ddot{\varphi}_2 = M_2g \; . \tag{A.1}$$

We calculate the $i = 2$-equation:

$$\begin{aligned}
\frac{\partial L}{\partial \dot{\varphi}_2} &= J_2\dot{\varphi}_2 + M_2R_2\left(R_1\dot{\varphi}_1 + R_2\dot{\varphi}_2\right) \\
\curvearrowright \quad \frac{d}{dt}\frac{\partial L}{\partial \dot{\varphi}_2} &= J_2\ddot{\varphi}_2 + M_2R_2\left(R_1\ddot{\varphi}_1 + R_2\ddot{\varphi}_2\right) \\
&= \frac{3}{2}M_2R_2^2\ddot{\varphi}_2 + M_2R_2R_1\ddot{\varphi}_1 \\
\frac{\partial L}{\partial \varphi_2} &= M_2gR_2 \; .
\end{aligned}$$

That yields the second equation of motion:

$$\frac{3}{2}R_2\ddot{\varphi}_2 + R_1\ddot{\varphi}_1 = g \; . \tag{A.2}$$

We insert $R_2^2\ddot{\varphi}_2$ from (A.2) into (A.1):

$$\begin{aligned}
\left(\frac{1}{2}M_1 + M_2\right) R_1\ddot{\varphi}_1 + \frac{2}{3}M_2g - \frac{2}{3}M_2R_1\ddot{\varphi}_1 &= M_2g \\
\curvearrowright \quad \left(\frac{1}{2}M_1 + \frac{1}{3}M_2\right) R_1\ddot{\varphi}_1 &= \frac{1}{3}M_2g \; .
\end{aligned}$$

This is the equation of motion for φ_1:

$$R_1\ddot{\varphi}_1 = \frac{2M_2}{3M_1 + 2M_2}\,g \; . \tag{A.3}$$

That for φ_2 follows immediately:

$$\begin{aligned}
R_2\ddot{\varphi}_2 &= \frac{2}{3}g - \frac{2}{3}\frac{2M_2}{3M_1 + 2M_2}\,g = \frac{2}{3}g\,\frac{3M_1 + 2M_2 - 2M_2}{3M_1 + 2M_2} \\
\curvearrowright \quad R_2\ddot{\varphi}_2 &= \frac{2M_1}{3M_1 + 2M_2}\,g \; .
\end{aligned}$$

Trivially integrable with given initial conditions:

$$\ddot{x}_2 = R_1\ddot{\varphi}_1 + R_2\ddot{\varphi}_2 = 2g\,\frac{M_1 + M_2}{3M_1 + 2M_2} .$$

That corresponds to the *'delayed'* free fall. With the given initial conditions one easily finds:

$$x_2(t) = \frac{M_1 + M_2}{3M_1 + 2M_2}\,gt^2 .$$

4. Newton's mechanics delivers according to part 1. for the thread tension:

$$M_2\ddot{x}_2 = M_2 g - F \;\curvearrowright\; F = M_2 g\left(1 - \frac{2M_1 + 2M_2}{3M_1 + 2M_2}\right) .$$

That is the *'old'* result of the Newton's mechanics from part 1.

$$F = \frac{M_1 M_2}{3M_1 + 2M_1}\,g .$$

Solution 1.2.17

1. The constraint is the rolling off of the solid cylinder within the hollow cylinder, i.e.:

$$\begin{aligned} &\textit{arc of circle} \quad \widehat{\mathrm{AB}} = \textit{arc of circle} \quad \widehat{\mathrm{AC}} \\ \Longleftrightarrow \quad &R(\psi + \varphi) = r\chi \end{aligned}$$

Proper generalized coordinates are:

$$q_1 := \varphi \quad q_2 := \psi$$

2. The hollow cylinder swings with fixed axis around its center of gravity. Its potential energy is therefore constant ($= 0$). The potential energy of the solid cylinder is:

$$V_S = -mg(R - r)\cos\varphi$$

The kinetic energy of the cylinders is composed by the translation of the center of gravity S of the solid cylinder

$$T_t = \frac{1}{2}m(R - r)^2\dot{\varphi}^2$$

and the rotations of both cylinders:

$$T_r = \frac{1}{2} J_H \dot{\psi}^2 + \frac{1}{2} J_V \left(\dot{\chi} - \dot{\varphi} \right)^2 \; .$$

Here are

$$J_H = MR^2 \qquad \text{and} \qquad J_S = \frac{1}{2} mr^2$$

the moment of inertia of the hollow cylinder and the solid cylinder, respectively (check it!). With the rolling off condition

$$\chi = \frac{R}{r} (\psi + \varphi)$$

we find:

$$\begin{aligned}
T_r &= \frac{1}{2} MR^2 \dot{\psi}^2 + \frac{1}{4} mr^2 \left(\frac{R}{r} \left(\dot{\psi} + \dot{\varphi} \right) - \dot{\varphi} \right)^2 \\
&= \frac{1}{2} MR^2 \dot{\psi}^2 + \frac{1}{4} mr^2 \left(\frac{R^2}{r^2} \left(\dot{\psi}^2 + 2 \dot{\psi} \dot{\varphi} + \dot{\varphi}^2 \right) - 2 \frac{R}{r} \left(\dot{\psi} \dot{\varphi} + \dot{\varphi}^2 \right) + \dot{\varphi}^2 \right) \\
&= \frac{1}{2} R^2 \left(M + \frac{1}{2} m \right) \dot{\psi}^2 + \frac{1}{2} mR(R-r) \dot{\varphi} \dot{\psi} + \frac{1}{4} m(R-r)^2 \dot{\varphi}^2
\end{aligned}$$

That yields the Lagrangian:

$$\begin{aligned}
L &= T - V = T_r + T_t - V_V \\
&= \frac{1}{2} R^2 \left(M + \frac{1}{2} m \right) \dot{\psi}^2 + \frac{1}{2} mR(R-r) \dot{\varphi} \dot{\psi} + \frac{3}{4} m(R-r)^2 \dot{\varphi}^2 + mg(R-r) \cos \varphi
\end{aligned}$$

3. ψ is cyclic:

$$\begin{aligned}
& \frac{\partial L}{\partial \psi} = 0 \\
\Longrightarrow \quad & p_\psi = \frac{\partial L}{\partial \dot{\psi}} = R^2 \left(M + \frac{1}{2} m \right) \dot{\psi} + \frac{1}{2} mR(R-r) \dot{\varphi} = \text{const} \\
\Longrightarrow \quad & 0 = R^2 \left(M + \frac{1}{2} m \right) \ddot{\psi} + \frac{1}{2} mR(R-r) \ddot{\varphi} \\
\Longrightarrow \quad & \ddot{\psi} = - \frac{m(R-r)}{2R \left(M + \frac{1}{2} m \right)} \ddot{\varphi}
\end{aligned}$$

With

$$\frac{\partial L}{\partial \varphi} = -mg(R-r)\sin\varphi$$
$$\frac{\partial L}{\partial \dot{\varphi}} = \frac{3}{2}m(R-r)^2\dot{\varphi} + \frac{1}{2}mR(R-r)\dot{\psi}$$
$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\varphi}} = \frac{3}{2}m(R-r)^2\ddot{\varphi} + \frac{1}{2}mR(R-r)\ddot{\psi}$$

we have:

$$0 = \frac{3}{2}m(R-r)^2\ddot{\varphi} + \frac{1}{2}mR(R-r)\ddot{\psi} + mg(R-r)\sin\varphi$$
$$\Longrightarrow \quad 0 = \frac{3}{2}(R-r)\ddot{\varphi} + \frac{1}{2}R\ddot{\psi} + g\sin\varphi$$

After replacing $\ddot{\psi}$ it follows:

$$0 = \frac{1}{2}(R-r)\left[3 - \frac{mR}{2R\left(M + \frac{1}{2}m\right)}\right]\ddot{\varphi} + g\sin\varphi$$
$$= \frac{1}{2}(R-r)\frac{3M+m}{M+\frac{1}{2}m}\ddot{\varphi} + g\sin\varphi$$

That yields the equation of motion for φ:

$$0 = \ddot{\varphi} + \frac{2M+m}{3M+m}\frac{g}{R-r}\sin\varphi$$

4. The usual approximation $\sin\varphi \approx \varphi$ for small deflections leads to the oscillation equation

$$\ddot{\varphi} + \omega_\varphi^2\varphi = 0$$

with the eigen-frequency

$$\omega_\varphi = \sqrt{\frac{2M+m}{3M+m}\frac{g}{R-r}}$$

and known solution $\varphi(t)$. Furthermore, $\psi(t)$ is found from

$$\ddot{\psi} = \frac{m(R-r)}{2R\left(M+\frac{1}{2}m\right)} \frac{2M+m}{3M+m} \frac{g}{R-r}\varphi(t)$$
$$= \frac{mg}{R(3M+m)}\varphi(t)$$

by integration with given initial conditions.

Solution 1.2.18

1. Constraints:

$$z_1 = z_2 = 0$$
$$(x_1-x_2)^2 + (y_1-y_2)^2 = l^2$$
$$\Longrightarrow \quad S = 6-3 = 3 \text{ degrees of freedom}$$

generalized coordinates:
x, y: coordinates of the center of gravity; φ: angle of the dumbbell relative to the x-axis
Transformation formulas:

$$x_1 = x - \frac{l}{2}\cos\varphi\,; \quad y_1 = y - \frac{l}{2}\sin\varphi$$
$$x_2 = x + \frac{l}{2}\cos\varphi\,; \quad y_2 = y + \frac{l}{2}\sin\varphi$$

2. Friction forces:

$$F_{x_1} = -\alpha\dot{x}_1 = -\alpha\left(\dot{x} + \frac{l}{2}\dot{\varphi}\sin\varphi\right)$$
$$F_{y_1} = -\alpha\dot{y}_1 = -\alpha\left(\dot{y} - \frac{l}{2}\dot{\varphi}\cos\varphi\right)$$
$$F_{x_2} = -\alpha\dot{x}_2 = -\alpha\left(\dot{x} - \frac{l}{2}\dot{\varphi}\sin\varphi\right)$$
$$F_{y_2} = -\alpha\dot{y}_2 = -\alpha\left(\dot{y} + \frac{l}{2}\dot{\varphi}\cos\varphi\right)$$

Generalized forces:

$$Q_j = F_{x_1}\frac{\partial x_1}{\partial q_j} + F_{x_2}\frac{\partial x_2}{\partial q_j} + F_{y_1}\frac{\partial y_1}{\partial q_j} + F_{y_2}\frac{\partial y_2}{\partial q_j}$$

$$\frac{\partial x_1}{\partial x} = 1\,; \quad \frac{\partial x_1}{\partial y} = 0\,; \quad \frac{\partial x_1}{\partial \varphi} = \frac{l}{2}\sin\varphi$$
$$\frac{\partial x_2}{\partial x} = 1\,; \quad \frac{\partial x_2}{\partial y} = 0\,; \quad \frac{\partial x_2}{\partial \varphi} = -\frac{l}{2}\sin\varphi$$
$$\frac{\partial y_1}{\partial x} = 0\,; \quad \frac{\partial y_1}{\partial y} = 1\,; \quad \frac{\partial y_1}{\partial \varphi} = -\frac{l}{2}\cos\varphi$$
$$\frac{\partial y_2}{\partial x} = 0\,; \quad \frac{\partial y_2}{\partial y} = 1\,; \quad \frac{\partial y_2}{\partial \varphi} = \frac{l}{2}\cos\varphi$$

$$\begin{aligned}
Q_x &= F_{x_1} + F_{x_2} = -2\alpha\dot{x} \\
Q_y &= F_{y_1} + F_{y_2} = -2\alpha\dot{y} \\
Q_\varphi &= \frac{l}{2}\left(F_{x_1}\sin\varphi - F_{x_2}\sin\varphi - F_{y_1}\cos\varphi + F_{y_2}\cos\varphi\right) \\
&= \frac{l}{2}\sin\varphi\,\underbrace{(F_{x_1} - F_{x_2})}_{-\alpha l\dot{\varphi}\sin\varphi} - \frac{l}{2}\cos\varphi\,\underbrace{\left(F_{y_1} - F_{y_2}\right)}_{+\alpha l\dot{\varphi}\cos\varphi} \\
&= -\frac{1}{2}\alpha l^2\dot{\varphi}
\end{aligned}$$

3. Equations of motion:
   holonomic constraints; non-conservative forces (1.33):

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} = Q_j \qquad j = x, y, \varphi$$

$$\dot{x}_1 = \dot{x} + \frac{l}{2}\dot{\varphi}\sin\varphi\,; \quad \dot{y}_1 = \dot{y} - \frac{l}{2}\dot{\varphi}\cos\varphi$$
$$\dot{x}_2 = \dot{x} - \frac{l}{2}\dot{\varphi}\sin\varphi\,; \quad \dot{y}_2 = \dot{y} + \frac{l}{2}\dot{\varphi}\cos\varphi$$

$$\begin{aligned}
\Longrightarrow \quad T &= \frac{1}{2}m\left(\dot{x}_1^2 + \dot{x}_2^2 + \dot{y}_1^2 + \dot{y}_2^2\right) \\
&= m\left(\dot{x}^2 + \dot{y}^2\right) + m\frac{l^2}{4}\dot{\varphi}^2
\end{aligned}$$

$$\begin{aligned}
\Longrightarrow \quad m\ddot{x} &= -\alpha\dot{x} \\
m\ddot{y} &= -\alpha\dot{y} \\
m\ddot{\varphi} &= -\alpha\dot{\varphi}
\end{aligned}$$

4. General solutions:

$$q_i(t) = \alpha_i + \beta_i \exp\left(-\frac{\alpha t}{m}\right)$$

α_i, β_i from initial conditions:

$$\begin{aligned} & x(0) = y(0) = 0\,; \quad \varphi(0) = 0 \\ \Longrightarrow \quad & \alpha_i = -\beta_i \\ \Longrightarrow \quad & q_i(t) = \alpha_i \left(1 - \exp\left(-\frac{\alpha t}{m}\right)\right) \end{aligned}$$

$$\dot{q}_i(0) = \frac{\alpha_i \alpha}{m}$$

$$\begin{aligned} \Longrightarrow \quad \alpha_x &= v_x \frac{m}{\alpha} \\ \alpha_y &= v_y \frac{m}{\alpha} \\ \alpha_\varphi &= \omega\, \frac{m}{\alpha} \end{aligned}$$

Solution 1.2.19

1. Constraints:

$$\begin{aligned} z &= \text{const} = 0 \\ y &= (a(t) - x) \tan\alpha \end{aligned}$$

The constraint for y shall **not** be used for a reduction of the number of coordinates:

$$q_1 = x\,; \quad q_2 = y$$

Lagrangian:

$$L = T - V = \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2\right) - mgy$$

2. Constraint in differential form:

$$0 = (\dot{a}(t)dt - dx)\sin\alpha - dy\cos\alpha$$

With the notation after (1.95):

$$a_{11} = -\sin\alpha \; ; \quad a_{12} = -\cos\alpha$$

generalized constraint forces:

$$Q_x = -\lambda \sin\alpha \; ; \quad Q_y = -\lambda \cos\alpha$$

λ: Lagrange multiplier
Lagrange equation of the first kind:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\alpha}} - \frac{\partial L}{\partial \alpha} = Q_\alpha \; ; \quad \alpha = x, y$$
$$m\ddot{x} = -\lambda \sin\alpha$$
$$m\ddot{y} + mg = -\lambda \cos\alpha$$

An additional equation of determination is found from the constraint 1. by twofold differentiation with respect to time:

$$\ddot{y} = (c - \ddot{x}) \tan\alpha$$

Insert $\ddot{x}$, $\ddot{y}$:

$$-g - \frac{\lambda}{m} \quad \cos\alpha = \left(c + \frac{\lambda}{m}\sin\alpha\right)\tan\alpha$$
$$-g\cos\alpha \quad -\frac{\lambda}{m}\left(\cos^2\alpha + \sin^2\alpha\right) = c\sin\alpha$$
$$\Longrightarrow \quad \lambda = -mg\cos\alpha - cm\sin\alpha$$
$$Q_x = m\sin\alpha\,(g\cos\alpha + c\sin\alpha)$$
$$Q_y = m\cos\alpha\,(g\cos\alpha + c\sin\alpha)$$

3. Equation of motion for $x(t)$:

$$m\ddot{x} = Q_x$$
$$\Longrightarrow \quad \ddot{x} = \sin\alpha\,(g\cos\alpha + c\cos\alpha) = \text{const}$$

with initial conditions:

$$x(t) = \frac{1}{2}\sin\alpha\,(g\cos\alpha + c\sin\alpha)\,t^2 + x_0$$

$y(t)$ by integration of the equation of motion or directly from the constraint:

$$y(t) = \left(\frac{1}{2}ct^2 - x(t)\right)\tan\alpha$$

Solution 1.2.20

1. As long as the mass point is on the spherical surface the constraint reads:

$$R \geq z \geq z_0 \ : \quad x^2 + y^2 + z^2 - R^2 = 0 \ .$$

Below the *'jump height'* z_0, however, the constraint is:

$$z_0 \geq z \ : \quad x^2 + y^2 + z^2 - R^2 > 0 \ .$$

We see that in the general case the constraints are not suitable to reduce the number of variables.
With spherical coordinates (r, φ, ϑ) as generalized coordinates and the transformation formulas

$$x = r\sin\vartheta\cos\varphi$$
$$y = r\sin\vartheta\sin\varphi$$
$$z = r\cos\vartheta$$

the kinetic energy is

$$T = \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right)$$
$$= \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\vartheta}^2 + r^2\sin^2\vartheta\,\dot{\varphi}^2\right)$$

and the potential energy:

$$V = mgz = mgr\cos\vartheta \ .$$

The general form of the Lagrangian is therewith:

$$L = T - V = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\vartheta}^2 + r^2\sin^2\vartheta\,\dot{\varphi}^2\right) - mgr\cos\vartheta \ .$$

2. On the spherical surface, i.e. in the region $R \geq z \geq z_0$ the constraints are holonomic:

$$r = R = \text{const}$$

It remains:

$$L = L(\vartheta, \varphi) = \frac{m}{2}\left(R^2\dot{\vartheta}^2 + R^2\sin^2\vartheta\dot{\varphi}^2\right) - mgR\cos\vartheta \ .$$

The equations of motion for ϑ and φ then read:

$$\frac{d}{dt}\frac{\partial L}{\partial\dot{\vartheta}} = mR^2\ddot{\vartheta} = \frac{\partial L}{\partial\vartheta} = mR^2\sin\vartheta\cos\vartheta\dot{\varphi}^2 + mgR\sin\vartheta$$
$$\frac{d}{dt}\frac{\partial L}{\partial\dot{\varphi}} = \frac{d}{dt}\left(mR^2\sin^2\vartheta\dot{\varphi}\right) = \frac{\partial L}{\partial\varphi} = 0 \ .$$

The coordinate φ is cyclic, therefore:

$$p_\varphi = \frac{\partial L}{\partial\dot{\varphi}} = mR^2\sin^2\vartheta\dot{\varphi} = \text{const} \ .$$

The z-component of the angular momentum related to the origin is an integral of motion since the gravitational force acts in z-direction so that no torque appears in z-direction.

3. Constraint for $R \geq z \geq z_0$:

$$r - R = 0$$
$$\Longrightarrow \quad dr = 0 \ , \quad \dot{r} = 0 \ .$$

Lagrange equation of the first kind for r with the constraint force Q_r:

$$\frac{d}{dt}\frac{\partial L}{\partial\dot{r}} - \frac{\partial L}{\partial r} = Q_r$$
$$m\ddot{r} - mr\left(\dot{\vartheta}^2 + \sin^2\vartheta\dot{\varphi}^2\right) + mg\cos\vartheta = Q_r \ .$$

With

$$r = R = \text{const} \ , \quad \dot{r} = 0$$

and the velocity

$$v^2 = R^2\left(\dot{\vartheta}^2 + \sin^2\vartheta\dot{\varphi}^2\right)$$

we have:

$$Q_r = mg\cos\vartheta - \frac{mv^2}{R}$$

The *'hopping point'* $z_0 = R\cos\vartheta_0$ is characterized by:

$$Q_r(z_0) = 0$$
$$\Longrightarrow \quad \frac{mgz_0}{R} = \frac{mv_0^2}{R}$$
$$\Longrightarrow \quad z_0 = \frac{v_0^2}{g}$$

where v_0 can be derived from the energy conservation law:

$$\frac{m}{2}v_0^2 + mgz_0 = mgR$$
$$\Longrightarrow \quad v_0^2 = 2g\,(R - z_0)\ .$$

Then we have the equation of determination for z_0:

$$z_0 = 2\,(R - z_0)$$
$$\Longrightarrow \quad z_0 = \frac{2}{3}R\ .$$

Accordingly the mass point moves in the free fall with the initial conditions:

$$z = z_0 = \frac{2}{3}R$$
$$v = v_0 = \sqrt{\frac{2}{3}gR}\ .$$

Solution 1.2.21 **F** has only a radial component:

$$F_r = \frac{\alpha}{r^2}\left(1 - \frac{\dot{r}^2 - 2r\ddot{r}}{c^2}\right)$$

We take

$$U(r,\dot{r}) = \frac{\alpha}{r}\left(1 + \frac{\dot{r}^2}{c^2}\right)$$

and verify by insertion that:

$$Q_r = F_r = \frac{d}{dt}\frac{\partial U}{\partial \dot{r}} - \frac{\partial U}{\partial r}$$

$$\frac{\partial}{\partial r}\left[\frac{1}{r}\left(1 + \frac{\dot{r}^2}{c^2}\right)\right] = -\frac{1}{r^2}\left(1 + \frac{\dot{r}^2}{c^2}\right)\ ,$$

$$\frac{\partial}{\partial\dot{r}}\left[\frac{1}{r}\left(1+\frac{\dot{r}^2}{c^2}\right)\right] = 2\frac{\dot{r}}{r\,c^2}\ ,$$

$$\frac{d}{dt}\frac{\partial}{\partial\dot{r}}\left[\frac{1}{r}\left(1+\frac{\dot{r}^2}{c^2}\right)\right] = \frac{2}{r^2c^2}(r\ddot{r}-\dot{r}^2)\ ,$$

$$\left(\frac{d}{dt}\frac{\partial}{\partial\dot{r}}-\frac{\partial}{\partial r}\right)\left[\frac{\alpha}{r}\left(1+\frac{\dot{r}^2}{c^2}\right)\right] = \frac{\alpha}{r^2}\left(\frac{2r\ddot{r}}{c^2}-\frac{2\dot{r}^2}{c^2}+1+\frac{\dot{r}^2}{c^2}\right)$$

$$= \frac{\alpha}{r^2}\left(1-\frac{1}{c^2}(\dot{r}^2-2r\ddot{r})\right) = F_r\ .$$

Thus the above $U(r,\dot{r})$ is indeed the generalized potential of the force $\mathbf{F}$. Since the motion takes place in the plane,

$$x = r\cos\varphi\ ;\quad y = r\sin\varphi\ ,$$

it holds for the kinetic energy:

$$T = \frac{1}{2}m(\dot{x}^2+\dot{y}^2) = \frac{1}{2}m(\dot{r}^2+r^2\dot{\varphi}^2)\ .$$

The Lagrangian therefore reads:

$$L = \frac{1}{2}m(\dot{r}^2+r^2\dot{\varphi}^2)-\frac{\alpha}{r}\left(1+\frac{\dot{r}^2}{c^2}\right)\ .$$

Let us look for an alternative solution which appears a bit more systematical:

For the generalized potential $U(r,\dot{r})$ it must be required:

$$F = Q_r = \frac{d}{dt}\frac{\partial U}{\partial\dot{r}}-\frac{\partial U}{\partial r}$$

The solution can only be guessed. Because of the second derivative in F the following ansatz appears plausible:

$$U(r,\dot{r}) = \alpha f(r)+\alpha g(r)\dot{r}^2$$

Insertion of F and this ansatz for U into the above equation and then arranging according to the time derivatives of r:

$$F = \frac{d}{dt}\frac{\partial U}{\partial\dot{r}}-\frac{\partial U}{\partial r}$$

$$\Longrightarrow\quad \frac{1}{r^2}\left(1-\frac{\dot{r}^2-2r\ddot{r}}{c^2}\right) = \frac{d}{dt}(2g\dot{r})-\left(f'+g'\dot{r}^2\right)$$

$$= \left(2g\ddot{r} + 2g'\dot{r}^2\right) - \left(f' + g'\dot{r}^2\right)$$

$$\frac{2}{c^2 r}\ddot{r} - \frac{1}{c^2 r^2}\dot{r}^2 + \frac{1}{r^2} = 2g\ddot{r} + g'\dot{r}^2 - f' \; .$$

Comparison of the coefficients of $\ddot{r}$ yields:

$$g(r) = \frac{1}{c^2 r} \; . \quad \Longrightarrow \quad g'(r) = -\frac{1}{c^2 r^2} \; ,$$

being consistent with the coefficients of $\dot{r}^2$. By comparing the remaining terms we have:

$$f'(r) = -\frac{1}{r^2} \quad \Longrightarrow \quad f(r) = \frac{1}{r} + \text{const} \; .$$

The constant of integration is arbitrarily choosable being set to zero for simplicity. Then we have the generalized potential:

$$U(r, \dot{r}) = \frac{\alpha}{r} + \frac{\alpha}{c^2 r}\dot{r}^2 = \frac{\alpha}{r}\left(1 + \frac{\dot{r}^2}{c^2}\right) \; .$$

Solution 1.2.22

1. $x_M = R\,\varphi$ (rolling condition!) ; $\quad y_M = R$.
2. Mass point:

$$x_m = x_M - R\,\sin\varphi = R(\varphi - \sin\varphi) \; ,$$

$$y_m = y_M - R\,\cos\varphi = R(1 - \cos\varphi) \; .$$

This is the *ordinary* cycloid (see Example 4 in Sect. 1.2.2).
Center of gravity:

$$\mathbf{R}_S = \frac{M\,\mathbf{r}_M + \frac{1}{2}M\,\mathbf{r}_m}{M + \frac{1}{2}M} = \frac{2}{3}\mathbf{r}_M + \frac{1}{3}\mathbf{r}_m$$

$$\Longrightarrow \; x_S = x_M - \frac{1}{3}R\,\sin\varphi = R\left(\varphi - \frac{1}{3}\sin\varphi\right) \; ,$$

$$y_S = y_M - \frac{1}{3}R\,\cos\varphi = R\left(1 - \frac{1}{3}\cos\varphi\right) \; .$$

This is the so-called *'shortened'* cycloid.

3. T_m: kinetic energy of the mass point

$$\begin{aligned} T_m &= \frac{1}{2} m \left(\dot{x}_m^2 + \dot{y}_m^2 \right) \ , \\ \dot{x}_m &= R\,\dot{\varphi}(1 - \cos\varphi)\ ; \quad \dot{y}_m = R\dot{\varphi}\ \sin\varphi \\ \Longrightarrow \quad T_m &= m\,R^2 \dot{\varphi}^2 (1 - \cos\varphi) \end{aligned}$$

T_M: kinetic energy of the disc being composed by a rotational and a translational part:

$$\begin{aligned} T_M &= T_M^{\mathrm{rot}} + T_M^{\mathrm{tr}} \ , \\ T_M^{\mathrm{tr}} &= \frac{1}{2} M \left(\dot{x}_M^2 + \dot{y}_M^2 \right) = \frac{1}{2} M\,R^2 \dot{\varphi}^2 \ . \end{aligned}$$

For the rotational part we need the moment of inertia J of the disc with respect to an axis through the center of the disc ($D =$ thickness of the disc):

$$\begin{aligned} J &= \int r^2\,dm = \rho_0 \int r^2\,d^3r = \frac{M}{\pi\,R^2\,D} \iiint dz\,r^3\,dr\,d\varphi \\ &= \frac{M}{\pi\,R^2 D} D\,2\pi \int_0^R r^3 dr = \frac{1}{2} M\,R^2 \ . \end{aligned}$$

Therewith we have:

$$\begin{aligned} T_M^{\mathrm{rot}} &= \frac{1}{2} J\,\dot{\varphi}^2 = \frac{1}{4} M\,R^2 \dot{\varphi}^2 \\ \Longrightarrow \quad T_M &= \frac{3}{4} M\,R^2\,\dot{\varphi}^2 \ . \end{aligned}$$

The total kinetic energy is then:

$$T(\varphi, \dot{\varphi}) = \frac{1}{2} M\,R^2\,\dot{\varphi}^2 \left[\frac{3}{2} + (1 - \cos\varphi) \right] \ .$$

The potential energy V can also be divided into contributions of the mass point and the disc:

$$\begin{aligned} V(\varphi) &= V_m + V_M = m\,g\,y_m + C_m + V_M \\ &= -\frac{1}{2} M\,g\,R\ \cos\varphi + \frac{1}{2} M\,g\,R + C_m + V_M \ . \end{aligned}$$

The contribution of the disc is constant. The choice of the origin is free. Then we can choose the constant C_m of course so that

$$V(\varphi) = -\frac{1}{2} M\, g\, R\, \cos\varphi$$

4.

$$L = T(\varphi, \dot{\varphi}) - V(\varphi) = \frac{1}{2} M \left[R^2 \dot{\varphi}^2 \left(\frac{5}{2} - \cos\varphi \right) + g\, R\, \cos\varphi \right] .$$

Equation of motion:

$$\frac{\partial L}{\partial \dot{\varphi}} = M\, R^2\, \dot{\varphi} \left(\frac{5}{2} - \cos\varphi \right) ,$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\varphi}} = M\, R^2 \ddot{\varphi} \left(\frac{5}{2} - \cos\varphi \right) + M\, R^2 \dot{\varphi}^2 \sin\varphi ,$$

$$\frac{\partial L}{\partial \varphi} = \frac{1}{2} M \left[R^2 \dot{\varphi}^2 \sin\varphi - g\, R\, \sin\varphi \right]$$

$$\Longrightarrow \qquad \ddot{\varphi}(5 - 2\, \cos\varphi) + \left(\dot{\varphi}^2 + \frac{g}{R} \right) \sin\varphi = 0 .$$

Simplification for small oscillations:

$$\varphi \ll 1 \; : \quad \cos\varphi \approx 1 , \quad \sin\varphi \approx \varphi , \quad \dot{\varphi}^2 \approx 0$$

$$\Longleftrightarrow \; \ddot{\varphi} + \frac{g}{3R} \varphi \approx 0 \; \Longrightarrow \; \omega^2 \approx \frac{g}{3R} .$$

5. The motion of the total mass concentrated at the center of gravity

$$M_{\text{tot}} = M + m = \frac{3}{2} M$$

is caused by the total force:

$$\mathbf{F} = \mathbf{Z} - \frac{3}{2} M\, g\, \mathbf{e}_y$$

Newton's equations of motion therefore read:

$$\frac{3}{2} M\, (\ddot{x}_S, \ddot{y}_S) = \left(Z_x, Z_y - \frac{3}{2} M\, g \right) .$$

x_S, y_S we have already calculated in part 2.

$$\dot{x}_S = R\,\dot{\varphi}\left(1 - \frac{1}{3}\cos\varphi\right)\,, \quad \dot{y}_S = \frac{1}{3}R\,\dot{\varphi}\,\sin\varphi$$

$$\Longrightarrow\; \ddot{x}_S = \frac{1}{3}R\left(\ddot{\varphi}(3-\cos\varphi) + \dot{\varphi}^2\sin\varphi\right)\,,$$

$$\ddot{y}_S = \frac{1}{3}R\,\left(\ddot{\varphi}\,\sin\varphi + \dot{\varphi}^2\cos\varphi\right)\,.$$

The constraint force has therefore the components:

$$Z_x = \frac{1}{2}M\,R\left(\ddot{\varphi}(3-\cos\varphi) + \dot{\varphi}^2\sin\varphi\right)\,,$$

$$Z_y = \frac{1}{2}M\,R\left(\ddot{\varphi}\,\sin\varphi + \dot{\varphi}^2\cos\varphi + \frac{3g}{R}\right)\,.$$

6. Condition for the *'taking off'*: $Z_y \stackrel{!}{=} 0$
Because of $\partial L/\partial t = 0$ and scleronomic constraints the energy conservation law is valid:

$$E = T + V = \frac{1}{2}M\left[R^2\dot{\varphi}^2\left(\frac{5}{2} - \cos\varphi\right) - g\,R\,\cos\varphi\right] = \text{const}\,.$$

We express E by the initial velocity v:

$$E = \frac{1}{2}M\left[v^2\left(\frac{5}{2} - 1\right) - g\,R\right]\,, \quad v = R\,\dot{\varphi}|_{\varphi=0} = \dot{x}_M(\varphi = 0)\,.$$

Thus we have for arbitrary φ:

$$\frac{3}{2}v^2 - g\,R = R^2\dot{\varphi}^2\left(\frac{5}{2} - \cos\varphi\right) - g\,R\,\cos\varphi\,.$$

We determine v from the condition $Z_y = 0$ at $\varphi = 2\pi/3$. So we need according to part 5. $\dot{\varphi}$, $\ddot{\varphi}$ at $\varphi = 2\pi/3$:

$$\varphi = \frac{2\pi}{3} \;\Longrightarrow\; \sin\varphi = \frac{1}{2}\sqrt{3}\,; \quad \cos\varphi = -\frac{1}{2}\,,$$

$$\dot{\varphi}^2\left(\varphi = \frac{2\pi}{3}\right) = \frac{1}{2}\left(\frac{v^2}{R^2} - \frac{g}{R}\right)\,.$$

According to part 4. it is:

$$6\ddot{\varphi}\left(\varphi = \frac{2\pi}{3}\right) + \frac{1}{2}\sqrt{3}\left(\frac{1}{2}\frac{v^2}{R^2} + \frac{1}{2}\frac{g}{R}\right) = 0$$

$$\Longrightarrow \ddot{\varphi}\left(\varphi = \frac{2\pi}{3}\right) = -\frac{\sqrt{3}}{24}\left(\frac{v^2}{R^2} + \frac{g}{R}\right) .$$

Equation of determination for v:

$$\begin{aligned}
0 &\overset{!}{=} Z_y\left(\varphi = \frac{2\pi}{3}\right) \\
&= \frac{1}{2}MR\left(\frac{1}{2}\sqrt{3}\ddot{\varphi}\left(\varphi = \frac{2\pi}{3}\right) - \frac{1}{2}\dot{\varphi}^2\left(\varphi = \frac{2\pi}{3}\right) + \frac{3g}{R}\right) \\
\Longleftrightarrow \quad 0 &= -\frac{1}{16}\left(\frac{v^2}{R^2} + \frac{g}{R}\right) - \frac{1}{4}\left(\frac{v^2}{R^2} - \frac{g}{R}\right) + \frac{3g}{R} \\
\Longrightarrow \quad v^2 &= \frac{51}{5}gR .
\end{aligned}$$

7. Moment of inertia:

$$J_S = J_{mS} + J_{MS} ,$$

J_S: Moment of inertia of the total system with respect to the center of gravity S,
J_{mS}: Contribution of the additional mass,
J_{MS}: Contribution of the disc.

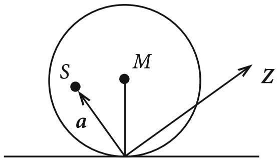
After Steiner's theorem (Sect. 4.2.4, Vol. 1) it holds:

$$J_{MS} = J + M\left[(x_M - x_S)^2 + (y_M - y_S)^2\right] .$$

J is the moment of inertia of the disc related to an axis through the center of the disc as calculated in 3.

$$\begin{aligned}
J_{MS} &= \frac{1}{2}MR^2 + M\left(\frac{1}{9}R^2\sin^2\varphi + \frac{1}{9}R^2\cos^2\varphi\right) = \frac{11}{18}MR^2 , \\
J_{mS} &= \frac{1}{2}M\left[(x_m - x_S)^2 + (y_m - y_S)^2\right] = \\
&= \frac{1}{2}M\left(\frac{4}{9}R^2\sin^2\varphi + \frac{4}{9}R^2\cos^2\varphi\right) = \frac{4}{18}MR^2 \\
\Longrightarrow \quad J_S &= \frac{5}{6}MR^2 .
\end{aligned}$$

Fig. A.2



The constraint force $\mathbf{Z}$ acts on the point of support. It gives rise to a torque around S leading therewith to a rotation of the disc (Fig. A.2):

$$\mathbf{M} = \mathbf{a} \times \mathbf{Z} = \left(a_x Z_y - a_y Z_x\right) \mathbf{e}_z \; .$$

Since the rotational movement is exclusively caused by the constraint force $\mathbf{Z}$ the equation of motion reads:

$$J_S \ddot{\varphi} = a_x Z_y - a_y Z_x \; .$$

For the vector $\mathbf{a}$ it holds:

$$\mathbf{a} = \left(-\left(x_M - x_S\right), \, y_S \, , \; 0\right) = \left(-\frac{1}{3} R \, \sin\varphi \, , \; R\left(1 - \frac{1}{3}\cos\varphi\right), \, 0\right)$$

$$\Longrightarrow \quad J_S \ddot{\varphi} = \left(-\frac{1}{3} R \, \sin\varphi\right) \frac{1}{2} M R \left(\ddot{\varphi} \, \sin\varphi + \dot{\varphi}^2 \cos\varphi + \frac{3g}{R}\right)$$
$$-R\left(1 - \frac{1}{3}\cos\varphi\right) \frac{1}{2} M R \left(\ddot{\varphi}(3 - \cos\varphi) + \dot{\varphi}^2 \sin\varphi\right)$$

$$\Longrightarrow \quad 5\ddot{\varphi} = -\ddot{\varphi} \, \sin^2\varphi - \dot{\varphi}^2 \sin\varphi \, \cos\varphi - \frac{3g}{R} \sin\varphi$$
$$-\ddot{\varphi}\left(9 - 6\, \cos\varphi + \cos^2\varphi\right) - \dot{\varphi}^2 (3 \, \sin\varphi - \cos\varphi \, \sin\varphi)$$

$$\Longrightarrow \quad \ddot{\varphi}(15 - 6 \, \cos\varphi) + 3\, \dot{\varphi}^2 \sin\varphi + \frac{3g}{R} \sin\varphi = 0 \; ,$$

$$\Longrightarrow \quad \ddot{\varphi}(5 - 2 \, \cos\varphi) + \left(\dot{\varphi}^2 + \frac{g}{R}\right) \sin\varphi = 0 \; .$$

This agrees with the equation of motion in part 4.!

Solution 1.2.23

1. Lagrangian:

$$L = T - V = T = \frac{1}{2} m \left(\dot{r}^2 + r^2 \dot{\varphi}^2\right) \; .$$

The coordinate φ is cyclic:

$$p_\varphi = \frac{\partial L}{\partial \dot{\varphi}} = m\,r^2\dot{\varphi} = L_z = \text{const} .$$

The angular momentum is an integral of motion.

2. Because of the disregard of the kinetic energy in radial direction it is $\dot{r}^2 \approx 0$:

$$T = \frac{1}{2} m\,r^2\dot{\varphi}^2 = \frac{L_z^2}{2m\,r^2} .$$

The work W carried out corresponds to the change of the kinetic energy (energy theorem!):

$$W = T(r = R) - T(r = R_0) = \frac{L_z^2}{2m}\left(\frac{1}{R^2} - \frac{1}{R_0^2}\right) .$$

3. Yes! The Lagrangian is the same as that in part 1., φ is still cyclic.
4. From $\dot{r}(t) = -b\,t$ we have the constraint:

$$r(t) = -\frac{1}{2} b\,t^2 + R_0 \qquad \text{(holonomic-rheonomic)} .$$

The constraint force $\mathbf{Z}$ which causes this time-dependence is the only acting force. Therefore it holds:

$$m\,\ddot{\mathbf{r}} = \mathbf{Z} .$$

In planar polar coordinates (see (2.13), Vol. 1) it is:

$$\ddot{\mathbf{r}} = \left(\ddot{r} - r\,\dot{\varphi}^2\right)\,\mathbf{e}_r + \left(r\,\ddot{\varphi} + 2\dot{r}\,\dot{\varphi}\right)\,\mathbf{e}_\varphi .$$

The conservation of angular momentum leads to:

$$\begin{aligned}
\left(r\,\ddot{\varphi} + 2\dot{r}\,\dot{\varphi}\right) &= \frac{1}{r}\frac{d}{dt} r^2\dot{\varphi} = \frac{1}{r\,m}\frac{d}{dt} L_z = 0 \\
\Longrightarrow\ \mathbf{Z} &= m\left(\ddot{r} - r\,\dot{\varphi}^2\right)\,\mathbf{e}_r = -m\left(b + r\,\dot{\varphi}^2\right)\,\mathbf{e}_r \\
&= -\left(m\,b + \frac{L_z^2}{m\,r^3(t)}\right)\,\mathbf{e}_r .
\end{aligned}$$

5.

$$T = \frac{1}{2} m\left(\dot{r}^2 + r^2\dot{\varphi}^2\right) = \frac{1}{2} m\,b^2 t^2 + \frac{L_z^2}{2m\,r^2} = -m\,b\,\left(r - R_0\right) + \frac{L_z^2}{2m\,r^2}$$

$$\Longrightarrow\ W = -m\,b\,(R - R_0) + \frac{L_z^2}{2m}\left(\frac{1}{R^2} - \frac{1}{R_0^2}\right) .$$

Solution 1.2.24

1. Constraint:

$$r = l - R_0\varphi \qquad \text{(holonomic-scleronomic)}\,.$$

Position vector of the mass point:

$$\mathbf{r}(P) = \mathbf{R}_0 + \bar{\mathbf{r}}\,,$$

where $\mathbf{R}_0 = R_0(\cos\varphi,\ \sin\varphi)$ and $\bar{\mathbf{r}} = r\,\mathbf{e}_\varphi = r(-\sin\varphi,\ \cos\varphi)$.

$$\begin{aligned}\Longrightarrow\ \mathbf{r}(P) &= (R_0\cos\varphi - (l - R_0\varphi)\sin\varphi, R_0\sin\varphi + (l - R_0\varphi)\cos\varphi)\ ,\\ \dot{\mathbf{r}}(P) &= (-R_0\dot{\varphi}\ \sin\varphi + R_0\dot{\varphi}\ \sin\varphi - (l - R_0\varphi)\,\dot{\varphi}\ \cos\varphi,\\ &\qquad R_0\dot{\varphi}\ \cos\varphi - R_0\dot{\varphi}\ \cos\varphi - (l - R_0\varphi)\,\dot{\varphi}\ \sin\varphi)\\ &= -\,(l - R_0\varphi)\,\dot{\varphi}\,\mathbf{e}_r\ .\end{aligned}$$

Lagrangian:

$$L = T - V = T = \frac{1}{2}m\dot{\mathbf{r}}^2(P) = \frac{1}{2}m\,(l - R_0\varphi)^2\,\dot{\varphi}^2\ .$$

The coordinate φ is **not** cyclic, different from the preceding exercise. The angular momentum L_z is therefore **not** a conserved quantity. However, because of the holonomic-scleronomic constraint and because of $\partial L/\partial t = 0$ energy conservation holds:

$$E = \text{const} = T\ .$$

2. The energy conservation law saves already one integration:

$$\begin{aligned}\dot{\varphi} &= \frac{\sqrt{\frac{2E}{m}}}{l - R_0\varphi}\ ; \quad t = 0\ :\quad v_0 = \frac{l\sqrt{\frac{2E}{m}}}{l} = \sqrt{\frac{2E}{m}}\\ \Longrightarrow\ \dot{\varphi} &= \frac{v_0}{l - R_0\varphi}\ .\end{aligned}$$

This can be rewritten:

$$v_0 = l\,\dot{\varphi} - R_0\varphi\,\dot{\varphi}\ \Longrightarrow\ v_0 t = l\,\varphi - \frac{1}{2}R_0\varphi^2 + C\ .$$

It follows immediately from the initial conditions that $C = 0$. We solve for φ:

$$\varphi = \frac{l}{R_0} \pm \sqrt{\frac{l^2}{R_0^2} - \frac{2}{R_0} v_0 t} \; .$$

Because of $\varphi(0) = 0$ only the minus sign can be valid:

$$\varphi(t) = \frac{l}{R_0} \left(1 - \sqrt{1 - \frac{2R_0}{l^2} v_0 t} \right) \; .$$

After the time t_0 the thread is fully wound up, i.e.:

$$R_0 \varphi \left(t = t_0 \right) = l \; .$$

That means:

$$t_0 = \frac{1}{2} \frac{l^2}{R_0 v_0} \; .$$

3.

$$p_\varphi = \frac{\partial L}{\partial \dot{\varphi}} = m \left(l - R_0 \varphi \right)^2 \dot{\varphi} \; .$$

Angular momentum with respect to $\mathcal{O}$:

$$\begin{aligned} \mathbf{L} &= m\mathbf{r}(P) \times \dot{\mathbf{r}}(P) \\ &= m\big(R_0 \mathbf{e}_r + r\mathbf{e}_\varphi\big) \times \big(-(l - R_0\varphi)\dot{\varphi} \big)\mathbf{e}_r \\ &= -mr(l - R_0\varphi)\dot{\varphi} \left(\mathbf{e}_\varphi \times \mathbf{e}_r \right) \\ &= m \left(l - R_0 \varphi \right)^2 \dot{\varphi} \mathbf{e}_z \end{aligned}$$

Solution 1.2.25

1. We firstly calculate the moment of inertia of the roller. For that we use cylindrical coordinates r, φ, $\bar{z}$. The $\bar{z}$-direction may coincide with the axis of the cylinder. For the mass density it is assumed:

$$\rho(r, \varphi, \bar{z}) = \alpha \, r \; .$$

What is α? We express α by the mass M:

$$M = \int\limits_{\text{roller}} d^3 r \, \rho(\mathbf{r}) = 2\pi \, h \, \alpha \int\limits_0^R r^2 dr = 2\pi \, h \, \alpha \frac{1}{3} R^3$$

$$\Longrightarrow \quad \alpha = \frac{3M}{2\pi \, h \, R^3} \; .$$

Moment of inertia with respect to the $\bar{z}$-axis:

$$J = \int\limits_{\text{roller}} r^2\, dm = \int\limits_{\text{roller}} r^2 \rho(\mathbf{r})\, d^3r = 2\pi\, h\, \alpha \int\limits_0^R r^4\, dr = \frac{3M}{R^3}\, \frac{1}{5} R^5 = \frac{3}{5} M\, R^2 \;.$$

In the region $0 \le z \le l$ the mass m performs a one-dimensional motion, i.e. without any side-deviation:

$$\textit{generalized coordinate:} \quad z \;,$$
$$\textit{constraint:} \quad z = R\,\varphi \;.$$

Kinetic energy:

$$T = \frac{1}{2} J\, \dot{\varphi}^2 + \frac{1}{2} m\, \dot{z}^2 = \frac{1}{2}\left(\frac{3}{5}M + m\right)\dot{z}^2 \;.$$

Potential energy:

$$V = m\, g(l + R - z) \quad \text{(minimum when the thread is fully wound up)} \;.$$

Lagrangian:

$$L\,(z, \dot{z}) = \frac{1}{2}\left(\frac{3}{5}M + m\right)\dot{z}^2 - m\, g(l + R - z) \;.$$

Equation of motion:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{z}} = \left(\frac{3}{5}M + m\right)\ddot{z} \overset{!}{=} \frac{\partial L}{\partial z} = m\, g$$
$$\Longrightarrow \; \ddot{z} = \frac{m}{m + \frac{3}{5}M}\, g \;.$$

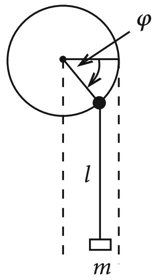
The mass m performs a uniformly accelerated motion (*delayed free-fall!*).
With the initial conditions

$$z(t = 0) = 0 \;; \quad \dot{z}(t = 0) = 0$$

we find:

$$z(t) = \frac{1}{2}\frac{m}{m + \frac{3}{5}M}\, g\, t^2 \;.$$

Fig. A.3



2. For $z > l$ the side-movement comes additionally into play. From Fig. A.3 we take the position vector $\mathbf{r}_m$ of the mass m:

$$\mathbf{r}_m = (R\cos\varphi,\ l + R\sin\varphi)$$
$$\Longrightarrow\ \dot{\mathbf{r}}_m = R\dot{\varphi}(-\sin\varphi,\ \cos\varphi)\ .$$

$R\dot{\varphi}$ is of course no longer equal to $\dot{z}$!
Kinetic energy:

$$T = \frac{1}{2}mR^2\dot{\varphi}^2 + \frac{1}{2}J\dot{\varphi}^2 = \frac{1}{2}\left(m + \frac{3}{5}M\right)R^2\dot{\varphi}^2\ .$$

Potential energy:

$$V = m\,g\,R(1 - \sin\varphi)\ .$$

Lagrangian:

$$L(\varphi, \dot{\varphi}) = \frac{1}{2}\left(m + \frac{3}{5}M\right)R^2\dot{\varphi}^2 - m\,g\,R(1 - \sin\varphi)\ .$$

Equation of motion:

$$\frac{d}{dt}\frac{\partial L}{\partial\dot{\varphi}} - \frac{\partial L}{\partial\varphi} = \left(m + \frac{3}{5}M\right)R^2\ddot{\varphi} - m\,g\,R\cos\varphi \stackrel{!}{=} 0$$
$$\Longrightarrow\ \ddot{\varphi} = \frac{1}{R}\frac{m}{m + \frac{3}{5}M}g\cos\varphi\ .$$

One may compare this result with that from part 1. From $z = l$ ($\varphi = 0$) to $z = l + R$ ($\varphi = \pi/2$) $\ddot{\varphi}$ decreases monotonically to zero. If then in addition $M \gg m$ may be assumed then $\ddot{\varphi} \approx 0$. That means:

$$\dot{\varphi} \approx \dot{\varphi}_l = \text{const} \qquad (\dot{\varphi}_l \text{ known from part 1. !})$$

It follows:

$$z = l + R\,\sin\varphi \approx l + R\,\sin\left[\dot{\varphi}_l\,(t - t_l)\right]\ ,$$
$$\dot{z} = R\,\dot{\varphi}\,\cos\varphi \approx R\,\dot{\varphi}_l\cos\left[\dot{\varphi}_l\,(t - t_l)\right]\ ,$$
$$\ddot{z} = R\,\ddot{\varphi}\,\cos\varphi - R\,\dot{\varphi}^2\sin\varphi \approx -R\,\dot{\varphi}_l^2\sin\left[\dot{\varphi}_l\,(t - t_l)\right]\ .$$

t_l is the time after which the thread is wound up to its full length. It can be determined with the result from part 1.:

$$l = \frac{1}{2}\frac{m}{m + \frac{3}{5}M}g\,t_l^2 \implies t_l = \sqrt{\frac{2l\left(m + \frac{3}{5}M\right)}{m\,g}}\ .$$

In the region $l \le z \le l + R$ it is $0 \le \varphi \le \pi/2$ and therewith $\ddot{z} < 0$. Obviously a deceleration takes place.
We still have to discuss the side-movement:

$$x = R\,\cos\varphi \approx R\,\cos\left[\dot{\varphi}_l\,(t - t_l)\right]\ ,$$
$$\dot{x} = -R\,\dot{\varphi}\,\sin\varphi \approx -R\,\dot{\varphi}_l\sin\left[\varphi_l\,(t - t_l)\right]\ ,$$
$$\ddot{x} = -R\,\ddot{\varphi}\,\sin\varphi - R\,\dot{\varphi}^2\cos\varphi \approx -R\,\dot{\varphi}_l^2\cos\left[\varphi_l\,(t - t_l)\right]\ .$$

3.

$$m\ddot{z} = m\,g - Z \implies Z = m\,\left(g - \ddot{z}\right)\ .$$

$0 \le z \le l$:

$$Z = m\,g\left(1 - \frac{m}{m + \frac{3}{5}M}\right) = m\frac{3M}{3M + 5m}\,g = \text{const} \approx m\,g\ .$$

$l \le z \le l + R$:

$$\ddot{z} \approx -R\,\dot{\varphi}_l^2\sin\varphi\ .$$

According to part 1. we have:

$$\dot{\varphi}_l = \frac{1}{R}\dot{z}\,(t = t_l) = \frac{1}{R}\frac{m}{m + \frac{3}{5}M}\,g\sqrt{\frac{2l\left(m + \frac{3}{5}M\right)}{m\,g}}$$

$$\implies R\,\dot{\varphi}_l^2 = \frac{1}{R}\frac{m}{m + \frac{3}{5}M}\,g\,2l$$

$$\implies \ddot{z} \approx -\frac{2}{R}g\,l\,\frac{5m}{3M}\sin\varphi$$

$$\implies Z \approx m\,g\left(1 + \frac{10l\,m}{3M\,R}\sin\varphi\right) .$$

Solution 1.2.26

1. Constraints:

$$z_1 = z_2 = 0 \qquad \text{(planar motion)} ,$$

$$(x_2 - x_1)^2 + (y_2 - y_1)^2 = (2a)^2 \qquad \text{(constant distance)} .$$

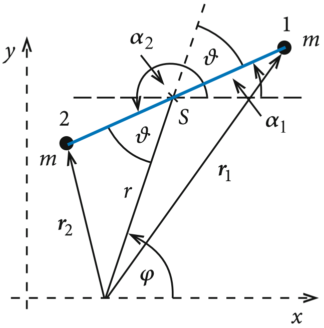
p=3: number of constraints $\Longrightarrow$ number of degrees of freedom:

$$S = 3N - p = 6 - 3 = 3 .$$

Thus we need three generalized coordinates (Fig. A.4):

$$q_1 = r\,; \quad q_2 = \varphi\,; \quad q_3 = \vartheta .$$

Fig. A.4



Kinetic energy:

$$T = T_S + T_E \ ,$$

T_S: motion of the center of gravity, *'orbital motion'*; T_E: self-rotation around S.

$$\text{center of gravity:} \qquad \mathbf{R} = \frac{1}{M}\sum_{i=1}^{2} m_i \mathbf{r}_i = \frac{1}{2}(\mathbf{r}_1 + \mathbf{r}_2) \ ,$$
$$\text{total mass:} \qquad M = m_1 + m_2 = 2m \ ,$$

$$T_S = \frac{1}{2} M \dot{\mathbf{R}}^2 = m\left(\dot{R}_x^2 + \dot{R}_y^2\right) \ ,$$
$$R_x = r\cos\varphi \implies \dot{R}_x = \dot{r}\cos\varphi - r\dot{\varphi}\sin\varphi \ ,$$
$$R_y = r\sin\varphi \implies \dot{R}_y = \dot{r}\sin\varphi + r\dot{\varphi}\cos\varphi$$
$$\implies T_S = m\left(\dot{r}^2 + r^2\dot{\varphi}^2\right) \ .$$

Self-rotation:

$$T_E = \frac{1}{2} m_1 a^2 \dot{\alpha}_1^2 + \frac{1}{2} m_2 a^2 \dot{\alpha}_2^2 \ ;$$
$$\alpha_1 = \varphi - \vartheta \ ; \quad \alpha_2 = \pi + \alpha_1 = \pi + \varphi - \vartheta$$
$$\implies T_E = m a^2 \left(\dot{\varphi} - \dot{\vartheta}\right)^2 \ .$$

Potential energy:

$$V = -m\gamma\left(\frac{1}{r_1} + \frac{1}{r_2}\right) \ ,$$
$$r_2 = \sqrt{r^2 + a^2 - 2ra\cos(\pi - \vartheta)} = \sqrt{r^2 + a^2 + 2ra\cos\vartheta} \ ,$$
$$r_1 = \sqrt{r^2 + a^2 - 2ra\cos\vartheta} \ .$$

Lagrangian:

$$L = T_S + T_E - V = m\left(\dot{r}^2 + r^2\dot{\varphi}^2\right) + m a^2\left(\dot{\varphi} - \dot{\vartheta}\right)^2$$
$$+ m\gamma\left[\left(r^2 + a^2 + 2ra\cos\vartheta\right)^{-1/2} + \left(r^2 + a^2 - 2ra\cos\vartheta\right)^{-1/2}\right] \ .$$

Equations of motion:
$q_1 = r$:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = 2m\ddot{r}\,,$$

$$\frac{\partial L}{\partial r} = 2m\,r\,\dot{\varphi}^2 - m\,\gamma\left[\frac{r + a\,\cos\vartheta}{(r^2 + a^2 + 2r\,a\,\cos\vartheta)^{3/2}} + \frac{r - a\,\cos\vartheta}{(r^2 + a^2 - 2r\,a\,\cos\vartheta)^{3/2}}\right]$$

$$\Longrightarrow\; \ddot{r} - r\dot{\varphi}^2 = -\frac{1}{2}\gamma\left[\frac{r + a\,\cos\vartheta}{(r^2 + a^2 + 2r\,a\,\cos\vartheta)^{3/2}} + \frac{r - a\,\cos\vartheta}{(r^2 + a^2 - 2r\,a\,\cos\vartheta)^{3/2}}\right].$$

$q_2 = \varphi$: φ is cyclic!

$$p_\varphi = \frac{\partial L}{\partial \dot{\varphi}} = 2m\,r^2\dot{\varphi} + 2m\,a^2\left(\dot{\varphi} - \dot{\vartheta}\right) = \text{const}\,.$$

$q_3 = \vartheta$:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\vartheta}} = -2m\,a^2\left(\ddot{\varphi} - \ddot{\vartheta}\right)\,,$$

$$\frac{\partial L}{\partial \vartheta} = m\,\gamma\,r\,a\left[\frac{\sin\vartheta}{\left(r^2 + a^2 + 2r\,a\,\cos\vartheta\right)^{3/2}} - \frac{\sin\vartheta}{\left(r^2 + a^2 - 2r\,a\,\cos\vartheta\right)^{3/2}}\right]$$

$$\Longrightarrow\; \ddot{\varphi} = \ddot{\vartheta} - \frac{\gamma\,r}{2a}\sin\vartheta\left[\frac{1}{\left(r^2 + a^2 + 2r\,a\,\cos\vartheta\right)^{3/2}} - \frac{1}{\left(r^2 + a^2 - 2r\,a\,\cos\vartheta\right)^{3/2}}\right].$$

2. *Orbital angular momentum* $\cong$ angular momentum of the center of gravity:

$$\mathbf{L}_O = \mathbf{R}\times\mathbf{P} = \begin{pmatrix} r\,\cos\varphi \\ r\,\sin\varphi \end{pmatrix}\times M\begin{pmatrix} \dot{r}\,\cos\varphi - r\,\dot{\varphi}\,\sin\varphi \\ \dot{r}\,\sin\varphi + r\,\dot{\varphi}\,\cos\varphi \end{pmatrix}$$
$$= \left(2m\,r^2\dot{\varphi}\right)\,\mathbf{e}_z\,.$$

Intrinsic (eigen) angular momentum $\cong$ angular momentum with respect to S:

$$\mathbf{L}_I = \sum_{i=1}^{2} m_i a^2\begin{pmatrix} \cos\alpha_i \\ \sin\alpha_i \end{pmatrix}\times\dot{\alpha}_i\begin{pmatrix} -\sin\alpha_i \\ \cos\alpha_i \end{pmatrix}$$
$$= \sum_{i=1}^{2} m_i a^2\dot{\alpha}_i\left(\cos^2\alpha_i + \sin^2\alpha_i\right)\,\mathbf{e}_z = 2m\,a^2\left(\dot{\varphi} - \dot{\vartheta}\right)\,\mathbf{e}_z\,.$$

Total angular momentum of the dumbbell:

$$\mathbf{L} = \mathbf{L}_O + \mathbf{L}_I = 2m\left[r^2\dot{\varphi} + a^2\left(\dot{\varphi} - \dot{\vartheta}\right)\right]\mathbf{e}_z = p_\varphi \mathbf{e}_z$$

$$\Longrightarrow \mathbf{L} = \mathbf{const}\,, \quad \text{since } \varphi \text{ is cyclic}\,.$$

3.

$$(1+x)^{-3/2} = 1 - \frac{3}{2}x + \frac{15}{8}x^2 - \frac{35}{16}x^3 + \dots$$

$$\Longrightarrow \frac{1}{r_{1,2}^3} = \frac{1}{r^3}\left[1 + \left(\frac{a^2}{r^2} \pm 2\frac{a}{r}\cos\vartheta\right)\right]^{-3/2}$$

$$= \frac{1}{r^3}\left[1 - \frac{3}{2}\left(\frac{a^2}{r^2} \pm 2\frac{a}{r}\cos\vartheta\right)\right.$$

$$+\frac{15}{8}\left(\frac{a^4}{r^4} + 4\frac{a^2}{r^2}\cos^2\vartheta \pm 4\frac{a^3}{r^3}\cos\vartheta\right)$$

$$\left.-\frac{35}{16}\left(\frac{a^6}{r^6} \pm 6\frac{a^5}{r^5}\cos\vartheta + 12\frac{a^4}{r^4}\cos^2\vartheta \pm 8\frac{a^3}{r^3}\cos^3\vartheta\right) + \dots\right]$$

$$\approx \frac{1}{r^3}\left[1 \mp 3\frac{a}{r}\cos\vartheta + \frac{3}{2}\frac{a^2}{r^2}(5\cos^2\vartheta - 1)\right.$$

$$\left.\pm \frac{5}{2}\frac{a^3}{r^3}\cos\vartheta\left(3 - 7\cos^2\vartheta\right)\right]$$

$$\Longrightarrow \frac{r + a\cos\vartheta}{r_1^3} + \frac{r - a\cos\vartheta}{r_2^3}$$

$$\approx \frac{1}{r^3}\left[2r + 3\frac{a^2}{r}\left(5\cos^2\vartheta - 1\right)\right.$$

$$\left.-6\,\frac{a^2}{r}\cos^2\vartheta + 5\frac{a^4}{r^3}\cos^2\vartheta\left(3 - 7\cos^2\vartheta\right)\right]$$

$$= \frac{1}{r^2}\left[2 + 3\frac{a^2}{r^2}\left(3\cos^2\vartheta - 1\right) + 5\frac{a^4}{r^4}\cos^2\vartheta\left(3 - 7\cos^2\vartheta\right)\right].$$

$$\Longrightarrow \frac{1}{r_1^3} - \frac{1}{r_2^3} = \frac{1}{r^3}\left(-\frac{6a}{r}\cos\vartheta + 5\frac{a^3}{r^3}\cos\vartheta(3 - 7\cos^2\vartheta) + \dots\right)$$

$$= -\frac{\cos\vartheta}{r^3}\frac{a}{r}\left(6 - 5\frac{a^2}{r^2}(3 - 7\cos^2\vartheta) + \dots\right)$$

Therewith we rewrite the equations of motion from part 1.:
$q_1 = r$:

$$\ddot{r} - r\dot{\varphi}^2 = -\frac{\gamma}{r^2}\left[1 + \frac{3}{2}\left(\frac{a^2}{r^2}\right)\left(3\cos^2\vartheta - 1\right) + \dots\right] .$$

$q_2 = \varphi$ (unchanged):

$$\frac{d}{dt}\left[r^2\dot{\varphi} + a^2(\dot{\varphi} - \dot{\vartheta})\right] = 0 .$$

$q_3 = \vartheta$:

$$\ddot{\varphi} = \ddot{\vartheta} + \frac{3}{2}\frac{\gamma}{r^3}\sin 2\vartheta - \frac{5}{4}\frac{\gamma}{r^3}\left(\frac{a}{r}\right)^2 \sin 2\vartheta(3 - 7\cos^2\vartheta) + \dots$$

We have used $\sin 2\vartheta = 2\sin\vartheta\cos\vartheta$.
For $a/r \longrightarrow 0$ these equations simplify further:

$$\ddot{r} - r\dot{\varphi}^2 + \frac{\gamma}{r^2} \approx 0 ,$$

$$\frac{d}{dt}\left(r^2\dot{\varphi}\right) \approx 0 ,$$

$$\ddot{\varphi} - \ddot{\vartheta} - \frac{3}{2}\frac{\gamma}{r^3}\sin 2\vartheta \approx 0 .$$

The first two equations do **not** contain any ϑ-contributions. The orbital motion $r = r(\varphi)$ is therefore decoupled from the intrinsic motion which is labeled by ϑ.

4. **Case 1:**
   The dumbbell-rod may be directed always onto P (Fig. A.5)

$$\Longrightarrow \quad \vartheta = 0 = \text{const} \quad \Longrightarrow \quad \dot{\vartheta} = 0 .$$

Uniform circular movement:

$$r = R = \text{const} ; \quad \dot{\varphi} = \omega_1 = \text{const} .$$

Fig. A.5

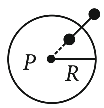
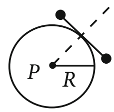


Fig. A.6



Lagrange equations:

$$\begin{aligned} q_1 = r &: \; -R\,\omega_1^2 = -\frac{\gamma}{R^2}\left[1 + 3\frac{a^2}{R^2} + \dots\right] , \\ q_2 = \varphi &: \; \frac{d}{dt}\left[R^2\omega_1 + a^2\omega_1\right] = 0 , \\ q_3 = \vartheta &: 0 = 0 . \end{aligned}$$

The last two equations are trivially fulfilled, while the first yields:

$$\omega_1^2 = \frac{\gamma}{R^3}\left[1 + 3\left(\frac{a}{R}\right)^2\right] .$$

Case 2:
The dumbbell-rod is oriented always tangentially on the circle (Fig. A.6):

$$\vartheta = \frac{\pi}{2} = \text{const} \implies \dot{\vartheta} = 0 .$$

Uniform circular motion:

$$r = R = \text{const} ; \quad \dot{\varphi} = \omega_2 = \text{const} .$$

Lagrange equations:

$$\begin{aligned} q_1 = r &: \; -R\,\omega_2^2 = -\frac{\gamma}{R^2}\left[1 - \frac{3}{2}\frac{a^2}{R^2} + \dots\right] , \\ q_2 = \varphi &: \; \frac{d}{dt}\left[R^2\omega_2 + a^2\omega_2\right] = 0 , \\ q_3 = \vartheta &: 0 = 0 . \end{aligned}$$

The last two equations are again trivially fulfilled, while the first now yields:

$$\omega_2^2 = \frac{\gamma}{R^3}\left(1 - \frac{3}{2}\frac{a^2}{R^2}\right) .$$

The cited theorem holds of course also for the dumbbell motion. However, because of the inhomogeneity of the gravitational field the total force is different for the two above discussed special cases!

Solution 1.2.27 Equations of motion for L_1:

$$\frac{d}{dt}\frac{\partial L_1}{\partial \dot{x}_i} - \frac{\partial L_1}{\partial x_i} = 0 \curvearrowright m\ddot{x}_i = qE_i \qquad (i = 1, 2, 3) .$$

Equations of motion for L_2:

$$\frac{d}{dt}\frac{\partial L_2}{\partial \dot{x}_i} - \frac{\partial L_2}{\partial x_i} = 0 \curvearrowright m\ddot{x}_i = qE_i \qquad (i = 1, 2, 3) .$$

Both Lagrangians lead to the same equations of motion being therefore equivalent. Both describe the motion of a charged particle in a constant homogeneous electrical field $\mathbf{E}$. This result becomes understandable if one realizes that

$$\frac{d}{dt}(q\mathbf{E}\cdot\mathbf{r}t) = q\mathbf{E}\cdot\dot{\mathbf{r}}t + q\mathbf{E}\cdot\mathbf{r} .$$

Thus both functions are related to each other by a mechanical gauge transformation (1.84):

$$L_1 = L_2 + \frac{d}{dt}(q\mathbf{E}\cdot\mathbf{r}t) \equiv L_2 + \frac{d}{dt}f(\mathbf{r}, t) .$$

That explains the equivalence!

Section 1.3.5

Solution 1.3.1 With the notation from Sect. 1.3.2 we have:

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + y'^2}dx .$$

Thus the functional

$$J = \int_A^B ds = \int_{x_A}^{x_B} \sqrt{1 + y'^2}dx .$$

is to be varied. We take

$$f(x, y, y') = f(y') = \sqrt{1 + y'^2}$$
$$\Longrightarrow \quad \frac{\partial f}{\partial y} = 0 ; \quad \frac{\partial f}{\partial y'} = \frac{y'}{\sqrt{1 + y'^2}} .$$

For the variation δJ we had derived subsequent to Eq. (1.123):

$$\delta J = \frac{\partial f}{\partial y'}\delta y\Big|_A^B + \int_A^B \left(\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'}\right)\delta y\, dx \,.$$

That means here:

$$\delta J = \frac{y'}{\sqrt{1+y'^2}}\delta y\Big|_A^B - \int_A^B \left(\frac{d}{dx}\frac{y'}{\sqrt{1+y'^2}}\right)\delta y\, dx \,.$$

1. A and B are at first fixed for **all** curves of the competitive set. It holds therefore:

   $$\delta y(A) = \delta y(B) = 0 \,.$$

   The first summand in the above expression for δJ thus vanishes. The requirement $\delta J = 0$ leads for otherwise arbitrary δy to

   $$\frac{d}{dx}\frac{y'}{\sqrt{1+y'^2}} = 0 \iff \frac{y'}{\sqrt{1+y'^2}} = \text{const} \iff y' = m = \text{const} \,.$$

   The shortest connection between A and B is therefore the line $\overline{AB}$ (see Example 1 in Sect. 1.3.2).
2. The competitive set now consists of all **lines** from A to **arbitrary** points B on the straight line g, which must be oriented parallelly to the y-axis in order that for **all** lines x_A and x_B are fixed. For each curve which is admitted to the variation one therefore has $y' = \text{const}$ so that now the second summand in the above δJ-expression vanishes. The first summand, however, is unequal zero since now only A is fixed:

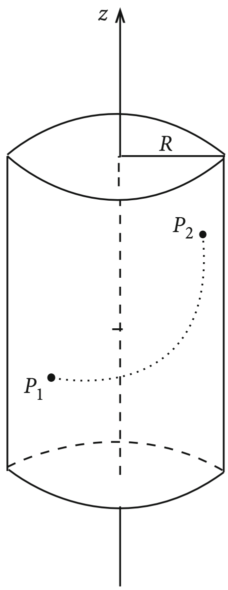
   $$\delta y(A) = 0 \,; \quad \delta y(B) \neq 0 \,.$$

   This means:

   $$0 \stackrel{!}{=} \delta J = \frac{y'(B)}{\sqrt{1+y'^2(B)}}\delta y(B)$$
   $$\implies y'(B) = 0 \,.$$

   The stationary path has then a zero-slope. It is just the straight line which starts at A and ends perpendicularly on g.

Fig. A.7



Solution 1.3.2 Vector line element in cylindrical coordinates:

$$d\mathbf{r} = d\rho\,\mathbf{e}_\rho + \rho\,d\varphi\,\mathbf{e}_\varphi + dz\,\mathbf{e}_z\ .$$

In our case it is $d\rho = 0$ since $\rho \equiv R = \text{const}$ (Fig. A.7). Arc length:

$$ds = \sqrt{d\mathbf{r}\cdot d\mathbf{r}} = \sqrt{R^2 d\varphi^2 + dz^2} = \sqrt{R^2 + z'^2}\,d\varphi\ ; \quad z = z(\varphi)\ .$$

The length of the connection line can be calculated by:

$$S = \int_1^2 ds = \int_{\varphi_1}^{\varphi_2} \sqrt{R^2 + z'^2} d\varphi \ \curvearrowright\ f\left(\varphi, z, z'\right) = \sqrt{R^2 + z'^2}\ .$$

The requirement that S is minimal leads to Euler's equation:

$$\delta S \overset{!}{=} 0 \iff \frac{\partial f}{\partial z} - \frac{d}{d\varphi}\frac{\partial f}{\partial z'} \overset{!}{=} 0\ .$$

Because of

$$\frac{\partial f}{\partial z} = 0$$

we get

$$\frac{\partial f}{\partial z'} = \text{const} = c = \frac{z'}{\sqrt{R^2 + z'^2}}$$

and therewith

$$z' = \frac{cR}{1 - c^2} = d = \text{const} \ .$$

The shortest connection is therefore a helical line:

$$z(\varphi) = d\,\varphi + \hat{d} \ .$$

Solution 1.3.3

1. By *mass distribution* it is understood *mass per length*:

$$m(x) = \frac{dm}{dx} \ .$$

For the kinetic energy T we then have

$$T = \frac{1}{2} \int_0^l m(x)\, \dot{y}^2 dx$$

with

$$\dot{y} = \frac{\partial y}{\partial t} = \dot{y}(x,t) \ .$$

2. We try:

$$V = \alpha \left(\int_0^l ds - l \right) \ ; \quad ds = \sqrt{dx^2 + dy^2} \ .$$

It follows with $y' = dy/dx$:

$$V = \alpha \left(\int_0^l \sqrt{1 + y'^2} dx - l \right) \ .$$

3. 'Small deflections' means also *small* y',

$$\sqrt{1+y'^2} \approx 1 + \frac{1}{2}y'^2$$

$$\Longrightarrow \quad V \approx \frac{\alpha}{2}\int\limits_0^l y'^2 dx\ .$$

Action functional:

$$S = \int\limits_{t_1}^{t_2} L\,dt = \frac{1}{2}\int\limits_{t_1}^{t_2}\left[\int\limits_0^l \left(m(x)\dot{y}^2 - \alpha\, y'^2\right) dx\right] dt\ .$$

The competitive set consists of curves whose deflections vanish at the points $x = 0$ and $x = l$ (constraints!) and which are fixedly preset at the times t_1 and t_2 (Hamilton's principle!).

$$0 \overset{!}{=} \delta S = \int\limits_{t_1}^{t_2}\int\limits_0^l \left(m(x)\dot{y}\,\delta\dot{y} - \alpha y'\,\delta y'\right)\,dx\,dt$$

$$= \int\limits_0^l m(x)\ \left[\dot{y}\,\delta y\right]\Big|_{t_1}^{t_2} dx - \alpha\int\limits_{t_1}^{t_2}\left[y'\,\delta y\right]\Big|_0^l\ dt - \int\limits_{t_1}^{t_2}\int\limits_0^l \left(m(x)\ddot{y} - \alpha y''\right)\delta y\,dx\,dt\ .$$

Since δy vanishes at the limits it remains:

$$0 = -\int\limits_{t_1}^{t_2}\int\limits_0^l \left(m(x)\ddot{y} - \alpha y''\right)\delta y\,dx\,dt\ .$$

Apart from that δy is freely choosable so that it must hold already

$$m(x)\frac{\partial^2 y}{\partial t^2} = \alpha\frac{\partial^2 y}{\partial x^2}$$

This is the required differential equation. For the special case of a *homogeneous* mass distribution $m(x) = m/l$ we get the *simple* oscillation equation.

Solution 1.3.4 Lagrangian:

$$L = T - V = \frac{m}{2}\dot{z}^2 - mgz$$

It holds for the given trajectory:

$$\dot{z}(t) = -gt + \dot{f}$$

Action functional:

$$S = \int_{t_1}^{t_2} dt \left(\frac{m}{2} \left(-gt + \dot{f}\right)^2 - mg \left(-\frac{1}{2} g t^2 + f \right) \right)$$

$$= \int_{t_1}^{t_2} dt \left(\frac{m}{2} g^2 t^2 - mgt\dot{f} + \frac{m}{2} \dot{f}^2 + \frac{1}{2} m g^2 t^2 - mgf \right)$$

$$= mg^2 \int_{t_1}^{t_2} dt\, t^2 + \frac{m}{2} \int_{t_1}^{t_2} dt \dot{f}(t)^2 - mg \int_{t_1}^{t_2} dt \left(t\dot{f} + f \right)$$

Integration by parts:

$$\int_{t_1}^{t_2} dt\, t\dot{f} = \underbrace{tf\Big|_{t_1}^{t_2}}_{=0\text{, because of } f(t_1) = f(t_2) = 0} - \int_{t_1}^{t_2} dt f \, .$$

We are left with:

$$S = mg^2 \cdot \frac{1}{3} \left(t_2^3 - t_1^3 \right) + \frac{m}{2} \int_{t_1}^{t_2} dt \dot{f}(t)^2 \, .$$

The first summand is independent of $f(t)$. The second is minimal at

$$\dot{f}(t) = 0 \quad \Longrightarrow \quad f(t) = \text{const} \, .$$

Because of $f(t_1) = f(t_2) = 0$ it must therefore be:

$$f(t) \equiv 0$$

Solution 1.3.5 Take:

$$g(y, y') \equiv f - y' \frac{\partial f}{\partial y'} \; ; \quad f = f(y, y')$$

Therewith:

$$\frac{dg}{dx} = \underbrace{\frac{\partial}{\partial x} f}_{=0} - \underbrace{\frac{\partial}{\partial x} \left(y' \frac{\partial f}{\partial y'} \right)}_{=0} + \frac{\partial f}{\partial y} y' - y' \frac{\partial^2 f}{\partial y \partial y'} y' + \frac{\partial f}{\partial y'} y'' - \frac{\partial f}{\partial y'} y'' - y' \frac{\partial^2 f}{\partial y'^2} y''$$

$$= y' \left(\frac{\partial f}{\partial y} - \frac{\partial^2 f}{\partial y \partial y'} y' - \frac{\partial^2 f}{\partial y'^2} y'' \right)$$

$$= y' \left(\frac{\partial f}{\partial y} - \frac{d}{dx} \frac{\partial f}{\partial y'} \right)$$
$$= 0$$

According to Euler's equation (1.124) the bracket is zero.
So we have:

$$g(y, y') = f - y' \frac{\partial f}{\partial y'} = \text{const} \,.$$

Solution 1.3.6

$$y(x) \ : \quad \text{position of the rope in the } xy\text{-plane}$$

$$F = \int_{-d}^{+d} dx\, y(x) \ : \quad \text{area between rope and } x\text{-axis}$$

$$l = \int_1^2 dS = \int_1^2 \sqrt{dx^2 + dy^2} = \int_1^2 dx \sqrt{1 + y'^2} \ : \quad \text{length of the rope !}$$

Variational task:

$$\delta\,(F - \lambda l) = \int_{-d}^{+d} dx \underbrace{\left(y(x) - \lambda \sqrt{1 + y'^2}\right)}_{f = f(y, y')}$$

Preconditions of Exercise 1.3.5 are fulfilled:

$$f - y' \frac{\partial f}{\partial y'} = \text{const}$$

$$\Longrightarrow \quad y - \lambda \sqrt{1 + y'^2} + \lambda \frac{y'^2}{\sqrt{1 + y'^2}} \stackrel{!}{=} a = \text{const}$$

$$\Longleftrightarrow \quad y - \frac{\lambda}{\sqrt{1 + y'^2}} = a$$

Solving for y'^2:

$$(y - a)^2 = \frac{\lambda^2}{1 + y'^2} \quad \Longrightarrow \quad 1 + y'^2 = \frac{\lambda^2}{(y - a)^2}$$

$$\Longrightarrow \quad y'^2 = \frac{\lambda^2}{(y - a)^2} - 1 = \frac{\lambda^2 - (y - a)^2}{(y - a)^2}$$

$$\Longrightarrow \quad \frac{dy}{dx} = \frac{\sqrt{\lambda^2 - (y-a)^2}}{y-a}$$

$$\Longrightarrow \quad dx = \frac{y-a}{\sqrt{\lambda^2 - (y-a)^2}} dy = \frac{d}{dy}\left(-\sqrt{\lambda^2 - (y-a)^2}\right) dy$$

$$\Longrightarrow \quad x - b = -\sqrt{\lambda^2 - (y-a)^2} \; ; \quad b = \text{const} \, .$$

It follows:

$$(x-b)^2 + (y-a)^2 = \lambda^2 \quad \Longrightarrow \quad \textbf{circle} \text{ with radius } \lambda \text{ and its center at} (b, a)$$

a, b and λ from boundary points and constraint.

$$P_1 : \quad (-d-b)^2 + (-a)^2 = \lambda^2$$
$$P_2 : \quad (d-b)^2 + (-a)^2 = \lambda^2$$

Subtraction:

$$(d+b)^2 - (d-b)^2 = 0 \quad \Longleftrightarrow \quad 4db = 0 \quad \Longrightarrow \quad \underline{b=0}$$

$$\Longrightarrow \quad a = \sqrt{\lambda^2 - d^2}$$

Length of the rope:

$$l = \int_{-d}^{+d} dx \sqrt{1 + y'^2}$$

$$\text{see above} \qquad y - a = \sqrt{\lambda^2 - x^2} \quad \Longrightarrow \quad y' = \frac{-x}{\sqrt{\lambda^2 - x^2}}$$

$$\Longrightarrow \quad 1 + y'^2 = 1 + \frac{x^2}{\lambda^2 - x^2} = \frac{\lambda^2}{\lambda^2 - x^2}$$

$$\Longrightarrow \quad \sqrt{1 + y'^2} = \frac{\lambda}{\sqrt{\lambda^2 - x^2}}$$

Therewith it holds:

$$l = \int_{-d}^{+d} dx \frac{\lambda}{\sqrt{\lambda^2 - x^2}} = \lambda \int_{-d}^{+d} dx \frac{d}{dx} \arcsin \frac{x}{\lambda} = 2\lambda \arcsin \frac{d}{\lambda}$$

$\Longrightarrow \quad \lambda$ determined by l and $d \Longrightarrow a$ is found!

Solution 1.3.7

1. Potential energy of the cable in the earth's gravitational field:

$$V = \int_1^2 dmgy = \alpha g \int_1^2 dsy = \alpha g \int_{-(A/2)}^{+(A/2)} dx\sqrt{dx^2 + y'^2}y \,.$$

Each type of curve which is admitted to the variation has its endpoints at 1 and 2 and possesses, to begin with, a firmly given length L:

$$L = \int_1^2 ds = \int_{-(A/2)}^{+(A/2)} dx\sqrt{1 + y'^2} \,. \tag{A.4}$$

This constraint is included in the variation as in Exercise 1.3.6 by a Lagrange multiplier λ:

$$\delta J \overset{!}{=} 0 \,; \quad J = \int_{-(A/2)}^{+(A/2)} dx(\alpha g y - \lambda)\sqrt{1 + y'^2} \,.$$

The functional to be used in Euler's equation,

$$f(x, y, y') = (\alpha g y - \lambda)\sqrt{1 + y'^2} \equiv f(y, y')$$

is not explicitly x-dependent. So we have according to Exercise 1.3.5:

$$f - y'\frac{\partial f}{\partial y'} = c = \text{const} \,.$$

That means:

$$(\alpha g y - \lambda)\sqrt{1 + y'^2} - (\alpha g y - \lambda)\frac{y'^2}{\sqrt{1 + y'^2}} = c$$

$$\Leftrightarrow \; c\sqrt{1 + y'^2} = \alpha g y - \lambda$$

$$\Leftrightarrow \; y'^2 = \frac{(\alpha g\, y - \lambda)^2}{c^2} - 1$$

$$\Leftrightarrow \; dx = \frac{dy}{\sqrt{\frac{(\alpha g y - \lambda)^2}{c^2} - 1}}$$

$$\Leftrightarrow \; x = \int \frac{dy}{\sqrt{\frac{(\alpha g y-\lambda)^2}{c^2} - 1}}$$

$$\Leftrightarrow \; x = \frac{c}{\alpha g} \int \frac{dz}{\sqrt{z^2 - 1}} \qquad \left(z = \frac{\alpha g\, y - \lambda}{c} \right) .$$

Thus we have found:

$$x(y) = \frac{c}{\alpha g} \operatorname{arc\,cosh} \left(\frac{\alpha g y - \lambda}{c} \right) + d \,.$$

The reversal yields the required curve form:

$$y(x) = \frac{c}{\alpha g} \cosh \left(\frac{\alpha g}{c} (x - d) \right) + \frac{\lambda}{\alpha g} \,.$$

This equation must be fulfilled by the suspension points $(-A/2, H)$ and $(+A/2, H)$. Together with the constraint (A.4) this leads to three equations of determination for the three still unknown quantities c, d and λ. Therewith the energetically most convenient shape of curve of the cable for a given L is fixed.

2. With the solution $y(x)$ from part 1. for a given L we calculate the potential energy V of the cable. From the minimum of V as a function of L we then get the optimal length of the cable!

Section 1.4.4

Solution 1.4.1

$$L' = L' \left(\mathbf{q}', \dot{\mathbf{q}}', t, \alpha \right) = L \left(\mathbf{q} \left(\mathbf{q}', t, \alpha \right), \dot{\mathbf{q}} \left(\mathbf{q}', \dot{\mathbf{q}}', t, \alpha \right), t \right) \,.$$

With this expression we calculate:

$$\frac{\partial L'}{\partial \alpha} = \sum_{j=1}^{S} \left(\frac{\partial L}{\partial q_j} \cdot \frac{\partial q_j}{\partial \alpha} + \frac{\partial L}{\partial \dot{q}_j} \cdot \frac{\partial \dot{q}_j}{\partial \alpha} \right)$$

$$= \sum_{j=1}^{S} \left(\left(\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_j} \right) \frac{\partial q_j}{\partial \alpha} + \frac{\partial L}{\partial \dot{q}_j} \frac{d}{dt} \frac{\partial q_j}{\partial \alpha} \right)$$

$$= \frac{d}{dt} \left(\sum_j \frac{\partial L}{\partial \dot{q}_j} \cdot \frac{\partial q_j}{\partial \alpha} \right) .$$

In the second step we have exploited the Lagrange equations of motion and the continuous differentiability of the q_j. This expression is valid for arbitrary α, i.e. also for $\alpha = 0$:

$$\left.\frac{\partial L'}{\partial \alpha}\right|_{\alpha=0} = \frac{d}{dt}\left(\sum_j \frac{\partial L}{\partial \dot{q}_j} \cdot \frac{\partial q_j}{\partial \alpha}\right)_{\alpha=0} .$$

According to the presumption, however, the Lagrangian shall be invariant with respect to the transformation of coordinates. Therefore L' can not explicitly depend on α:

$$\left.\frac{\partial L'}{\partial \alpha}\right|_{\alpha=0} = 0 .$$

This yields immediately the Noether's theorem:

$$I(\mathbf{q}, \dot{\mathbf{q}}, t) = \sum_{j=1}^{S} \frac{\partial L}{\partial \dot{q}_j} \left.\frac{\partial q_j(\mathbf{q}', t, \alpha)}{\partial \alpha}\right|_{\alpha=0} = \text{const} .$$

According to that, each transformation that lets L to be invariant leads to a conserved quantity.

Solution 1.4.2 Rotation around the z-axis ((1.320), Vol. 1):

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} x'\cos\alpha + y'\sin\alpha \\ -x'\sin\alpha + y'\cos\alpha \\ z' \end{pmatrix} .$$

$\alpha = 0$ means the identical mapping. Further on it follows:

$$\dot{x} = \dot{x}'\cos\alpha + \dot{y}'\sin\alpha$$
$$\dot{y} = -\dot{x}'\sin\alpha + \dot{y}'\cos\alpha$$
$$\dot{z} = \dot{z}' .$$

So we have:

$$\dot{x}^2 + \dot{y}^2 + \dot{z}^2 = \dot{x}'^2 + \dot{y}'^2 + \dot{z}'^2 .$$

Analogously one finds:

$$x^2 + y^2 = x'^2 + y'^2$$
$$z = z' .$$

The Lagrangian is thus invariant with respect to the here performed transformation of coordinates:

$$L(\mathbf{q}, \dot{\mathbf{q}}) = L\left(\mathbf{q}', \dot{\mathbf{q}}'\right) \ .$$

Therewith the preconditions for the Noether's theorem from Exercise 1.4.1 are fulfilled.
For the calculation of the integral of motion we need from Exercise 1.4.1:

$$\begin{aligned}
\frac{\partial x\left(\mathbf{q}', \alpha\right)}{\partial \alpha} &= -x' \sin\alpha + y' \cos\alpha = y \\
\frac{\partial y\left(\mathbf{q}', \alpha\right)}{\partial \alpha} &= -x' \cos\alpha - y' \sin\alpha = -x \\
\frac{\partial z}{\partial \alpha} &= 0 \ .
\end{aligned}$$

Hence we find the following conserved quantity,

$$I = \frac{\partial L}{\partial \dot{x}} \cdot y + \frac{\partial L}{\partial \dot{y}} \cdot (-x) = m\dot{x}y - m\dot{y}x = p_x \cdot y - p_y \cdot x = L_z = \text{const} \, .$$

which turns out to be the z-component of the angular momentum!

Solution 1.4.3

1. In this case, too, it holds of course as in Exercise 1.4.1:

$$\frac{\partial L'}{\partial \alpha} = \frac{d}{dt}\left(\sum_j \frac{\partial L}{\partial \dot{q}_j} \cdot \frac{\partial q_j}{\partial \alpha}\right) \ .$$

However, it is now:

$$\frac{\partial L'}{\partial \alpha} = \frac{\partial}{\partial \alpha}\frac{d}{dt} f\left(\mathbf{q}', t, \alpha\right) = \frac{d}{dt}\frac{\partial}{\partial \alpha} f\left(\mathbf{q}', t, \alpha\right) \ .$$

The integral of motion therefore reads:

$$\widehat{I}(\mathbf{q}, \dot{\mathbf{q}}, t) = \sum_{j=1}^{S} \frac{\partial L}{\partial \dot{q}_j} \left.\frac{\partial q_j\left(\mathbf{q}', t, \alpha\right)}{\partial \alpha}\right|_{\alpha=0} - \left.\frac{\partial}{\partial \alpha} f\left(\mathbf{q}', t, \alpha\right)\right|_{\alpha=0} = \text{const} \ .$$

2.

$$L(x, \dot{x}) = \frac{m}{2}\dot{x}^2 - mgx \ .$$

The Galilean transformation

$$x \longrightarrow x' = x + \alpha t$$

fulfills the preconditions which we had to require. With $\dot{x} = \dot{x}' - \alpha$ it follows for the *'new'* Lagrangian:

$$L'\left(x', \dot{x}', t, \alpha\right) = \frac{m}{2}\left(\dot{x}' - \alpha\right)^2 - mg\left(x' - \alpha t\right) = L\left(x', \dot{x}'\right) + \frac{d}{dt} f\left(x', t, \alpha\right) \;.$$

Thereby we have defined:

$$\begin{aligned} \frac{d}{dt} f\left(x', t, \alpha\right) &= -\alpha m\dot{x}' + \frac{m}{2}\alpha^2 + mg\alpha t \\ \curvearrowright\; f\left(x', t, \alpha\right) &= -\alpha m x' + \frac{m}{2}\alpha^2 t + \frac{1}{2} mg\alpha t^2 \\ \curvearrowright\; \left.\frac{\partial f}{\partial \alpha}\right|_{\alpha=0} &= -mx + \frac{1}{2} mgt^2 \,. \end{aligned}$$

With

$$\left.\frac{\partial L}{\partial \dot{x}} \cdot \frac{\partial x}{\partial \alpha}\right|_{\alpha=0} = -m\dot{x}t$$

and part 1. it follows:

$$\widehat{I}\,(x, \dot{x}, t) = -m\dot{x}t + mx - \frac{1}{2} mgt^2 = m\left(x - \dot{x}t - \frac{1}{2} gt^2\right) \;.$$

That is the well-known result for the free fall. With

$$\begin{aligned} x(t) &= x_0 + v_0 t - \frac{1}{2} gt^2 \\ \dot{x}(t) &= v_0 - gt \end{aligned}$$

thus

$$\widehat{I} = m x_0$$

is surely a conserved quantity, even though a trivial one.

Section 2.1.1

Solution 2.1.1

1.

$$f(x) = \alpha\, x^2 \implies u = \frac{df}{dx} = 2\alpha\, x \implies x = \frac{u}{2\alpha}$$
$$\implies f(x) - x\frac{df}{dx} = -\alpha\, x^2$$
$$\implies g(u) = -\frac{u^2}{4\alpha}\ .$$

2.

$$f(x,y) = \alpha\, x^2 y^3 \implies v = \left(\frac{\partial f}{\partial y}\right)_x = 3\alpha\, x^2 y^2$$
$$\implies y^2 = \frac{v}{3\alpha x^2}$$
$$\implies f(x,y) - y\left(\frac{\partial f}{\partial y}\right)_x = -2\alpha\, x^2 y^3 = -2\alpha\, x^2 \frac{v^{3/2}}{(3\alpha\, x^2)^{3/2}}$$
$$\implies g(x,v) = -\frac{2}{3}\frac{v^{3/2}}{(3\alpha\, x^2)^{1/2}}\ .$$

Solution 2.1.2

1.

$$f(x) = \alpha\,(x+\beta)^2$$
$$\implies \quad u = \frac{df}{dx} = 2\alpha\,(x+\beta) \quad \implies \quad x = \frac{u}{2\alpha} - \beta$$
$$\implies \quad f(x) - x\frac{df}{dx} = \alpha\left(\frac{u}{2\alpha}\right)^2 - u\left(\frac{u}{2\alpha} - \beta\right)$$
$$= \beta u - \frac{u^2}{4\alpha}$$
$$= g(u)$$

Back-transformation:

$$g(u) = \beta u - \frac{u^2}{4\alpha}$$
$$\implies \quad -x = \frac{dg}{du} = \beta - \frac{u}{2\alpha} \quad \implies \quad u = (\beta + x)\,2\alpha$$

$$\Longrightarrow \quad g(u) - u\frac{dg}{du} = \beta u - \frac{u^2}{4\alpha} - \beta u + \frac{u^2}{2\alpha}$$
$$= \frac{u^2}{4\alpha} = \alpha\,(\beta + x)^2$$
$$= f(x)$$

2.

$$f(x,y) = \alpha x^3 y^5$$
$$\Longrightarrow \quad v = \left(\frac{\partial f}{\partial y}\right)_x = 5\alpha x^3 y^4 \quad \Longrightarrow \quad y^4 = \frac{v}{5\alpha x^3}$$
$$\Longrightarrow \quad f(x,y) - y\left(\frac{\partial f}{\partial y}\right)_x = \alpha x^3 y^5 - 5\alpha x^3 y^5$$
$$= -4\alpha x^3 y^5$$
$$\Longrightarrow \quad g(x,v) = -4\alpha x^3 \frac{v^{5/4}}{(5\alpha x^3)^{5/4}}$$

Back-transformation:

$$y = -\left(\frac{\partial g}{\partial v}\right)_x = +5\alpha x^3 \frac{v^{1/4}}{(5\alpha x^3)^{5/4}}$$
$$\Longrightarrow \quad v^{5/4} = y^5\left(5\alpha x^3\right)^{5/4}$$

$$\Longrightarrow \quad g(x,v) - v\left(\frac{\partial g}{\partial v}\right)_x = -4\alpha x^3 \frac{v^{5/4}}{(5\alpha x^3)^{5/4}} + 5\alpha x^3 \frac{v^{5/4}}{(5\alpha x^3)^{5/4}}$$
$$= \alpha x^3 \frac{v^{5/4}}{(5\alpha x^3)^{5/4}}$$
$$= \alpha x^3 y^5 = f(x,y)$$

Solution 2.1.3

1. Legendre transformation with S as active variable:

$$F(T,V) = U - S\left(\frac{\partial U}{\partial S}\right)_V = U - TS\,.$$

By the differential one recognizes that F is a function of T and V:

$$dF = dU - SdT - TdS = -SdT - pdV \; .$$

Partial derivatives:

$$\left(\frac{\partial F}{\partial T}\right)_V = -S \; ; \quad \left(\frac{\partial F}{\partial V}\right)_T = -p \; .$$

2. Legendre transformation with V as active variable:

$$H(S,p) = U - V\left(\frac{\partial U}{\partial V}\right)_S = U + pV \; .$$

By the differential one recognizes that H is a function of S and p:

$$dH = dU + pdV + Vdp = TdS + Vdp \; .$$

Partial derivatives:

$$\left(\frac{\partial H}{\partial S}\right)_p = T \; ; \quad \left(\frac{\partial H}{\partial p}\right)_S = V \; .$$

3. Legendre transformation with S and V as active variables:

$$G(T,p) = U - S\left(\frac{\partial U}{\partial S}\right)_V - V\left(\frac{\partial U}{\partial V}\right)_S = U - TS + pV \; .$$

By the differential one recognizes that G is a function of T and p:

$$dG = dU - SdT - TdS + pdV + Vdp = -SdT + Vdp \; .$$

Partial derivatives:

$$\left(\frac{\partial G}{\partial T}\right)_p = -S \; ; \quad \left(\frac{\partial G}{\partial p}\right)_T = V \; .$$

Section 2.2.3

Solution 2.2.1 Given

$$H = H(\mathbf{q}, \mathbf{p}, t) \; ; \quad \dot{q}_j = \frac{\partial H}{\partial p_j} \; ; \quad \dot{p}_j = -\frac{\partial H}{\partial q_j} \; .$$

For the differential dL of the Lagrangian,

$$L(\mathbf{q}, \dot{\mathbf{q}}, t) = \sum_{j=1}^{S} p_j \frac{\partial H}{\partial p_j} - H = \sum_{j=1}^{S} p_j \dot{q}_j - H \quad ,$$

one finds with the Hamilton's equations of motion:

$$\begin{aligned} dL &= \sum_{j=1}^{S} \left(p_j d\dot{q}_j + dp_j \dot{q}_j - \frac{\partial H}{\partial q_j} dq_j - \frac{\partial H}{\partial p_j} dp_j \right) - \frac{\partial H}{\partial t} dt \\ &= \sum_{j=1}^{S} \left(p_j d\dot{q}_j + \dot{p}_j dq_j \right) - \frac{\partial H}{\partial t} dt \ . \end{aligned}$$

For the total differential dL it must of course also be valid:

$$dL = \sum_{j=1}^{S} \left(\frac{\partial L}{\partial q_j} dq_j + \frac{\partial L}{\partial \dot{q}_j} d\dot{q}_j \right) + \frac{\partial L}{\partial t} dt \ .$$

The comparison yields:

$$p_j = \frac{\partial L}{\partial \dot{q}_j} \ ; \quad \dot{p}_j = \frac{\partial L}{\partial q_j} \ ; \quad \frac{\partial L}{\partial t} = -\frac{\partial H}{\partial t} \ .$$

From the first two equations we get:

$$\dot{p}_j = \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_j} = \frac{\partial L}{\partial q_j} \ , \quad j = 1, l \ldots, S \ .$$

That was to be shown.

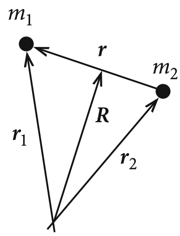
Solution 2.2.2

total mass:	$M = m_1 + m_2$,
reduced mass:	$\mu = (m_1 \, m_2) \, / M$,
relative coordinate:	$\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$,
center of mass:	$\mathbf{R} = \frac{1}{M} \, (m_1 \mathbf{r}_1 + m_2 \mathbf{r}_2) = (X, Y, Z)$,
generalized coordinates:	$X, Y, Z, r, \vartheta, \varphi$.

Lagrangian according to (1.164) (Fig. A.8):

$$L = \frac{1}{2} M (\dot{X}^2 + \dot{Y}^2 + \dot{Z}^2) + \frac{1}{2} \mu (\dot{r}^2 + r^2 \dot{\vartheta}^2 + r^2 \sin^2 \vartheta \, \dot{\varphi}^2) - V(r) \ .$$

Fig. A.8



X, Y, Z, φ are cyclic. Therefore:

$$P_x = M\dot{X} = \text{const} = C_x\ ,$$
$$P_y = M\dot{Y} = \text{const} = C_y\ ,$$
$$P_z = M\dot{Z} = \text{const} = C_z\ ,$$
$$P_\varphi = \mu\, r^2 \sin^2\vartheta\ \dot{\varphi} = \text{const} = C_\varphi\ .$$

Legendre transformation with respect to $\dot{X}, \dot{Y}, \dot{Z}, \dot{\varphi}$:

$$R\ \left(X, Y, Z, r, \vartheta, \varphi, \dot{r}, \dot{\vartheta}, P_x, P_y, P_z, p_\varphi\right)$$
$$= \frac{1}{2M}\left(C_x^2 + C_y^2 + C_z^2\right) + \frac{C_\varphi^2}{2\mu\, r^2 \sin^2\vartheta} - \frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\vartheta}^2\right) + V(r)$$
$$= R\left(r, \vartheta, \dot{r}, \dot{\vartheta}\ \middle|\ C_x, C_y, C_z, C_\varphi\right)\ .$$

Equations of motion:
r, ϑ non-cyclic:

$$\frac{d}{dt}\frac{\partial R}{\partial \dot{q}_j} = \frac{\partial R}{\partial q_j}$$

$$q_j = r\ :\ -\mu\,\ddot{r} = -\frac{C_\varphi^2}{\mu\, r^3 \sin^2\vartheta} - \mu\, r\,\dot{\vartheta}^2 + \frac{\partial V}{\partial r}\ ,$$
$$q_j = \vartheta\ :\ -\mu\, r^2\ddot{\vartheta} = -\frac{C_\varphi^2\,\cos\vartheta}{\mu\, r^2 \sin^3\vartheta}$$

X, Y, Z, φ cyclic:

$$\dot{X} = \frac{\partial R}{\partial P_x} = \frac{\partial R}{\partial C_x} = \frac{C_x}{M}\ ,$$

$$\dot{Y} = \frac{\partial R}{\partial P_y} = \frac{\partial R}{\partial C_y} = \frac{C_y}{M} \; ,$$

$$\dot{Z} = \frac{\partial R}{\partial P_z} = \frac{\partial R}{\partial C_z} = \frac{C_z}{M} \; ,$$

$$\dot{\varphi} = \frac{\partial R}{\partial p_\varphi} = \frac{\partial R}{\partial C_\varphi} = \frac{C_\varphi}{\mu\, r^2 \sin^2 \vartheta} \; ,$$

$$\dot{P}_x = -\frac{\partial R}{\partial X} = 0 \; ; \quad \dot{P}_y = -\frac{\partial R}{\partial Y} = 0 \; ; \quad \dot{P}_z = -\frac{\partial R}{\partial Z} = 0 \; ,$$

$$\dot{p}_\varphi = -\frac{\partial R}{\partial \varphi} = 0 \; .$$

Solution 2.2.3

1.

$$\mathbf{r} = (x, y) = \rho(\cos\varphi, \sin\varphi) = \rho\mathbf{e}_\rho \;\curvearrowright\; r = |\mathbf{r}| = \rho$$

$$\dot{\mathbf{r}} = \dot{\rho}(\cos\varphi, \sin\varphi) + \rho\dot{\varphi}(-\sin\varphi, \cos\varphi) = \dot{\rho}\mathbf{e}_\rho + \rho\dot{\varphi}\,\mathbf{e}_\varphi \; .$$

Kinetic energy:

$$T = \frac{m}{2}\,\dot{\mathbf{r}}^2 = \frac{m}{2}\left(\dot{\rho}\mathbf{e}_\rho + \rho\dot{\varphi}\mathbf{e}_\varphi\right)^2 = \frac{m}{2}\left(\dot{\rho}^2 + \rho^2\dot{\varphi}^2\right) \; .$$

Potential energy:

$$\mathbf{F}(\mathbf{r}) = -\left(\alpha + \frac{\beta}{r}\right)\mathbf{r} = -(\alpha\rho + \beta)\mathbf{e}_\rho$$

$$\stackrel{!}{=} -\nabla V \stackrel{((1.388),\,\text{Vol.1})}{=} -\left(\mathbf{e}_\rho \frac{\partial V}{\partial \rho} + \mathbf{e}_\varphi \frac{1}{\rho}\frac{\partial V}{\partial \varphi}\right)$$

$$\curvearrowright\; \frac{\partial V}{\partial \rho} = \alpha\rho + \beta \; ; \quad \frac{\partial V}{\partial \varphi} \equiv 0$$

$$\curvearrowright\; V(\rho) = \frac{1}{2}\alpha\rho^2 + \beta\rho + \gamma \; .$$

The constant γ is unimportant.

2. Lagrangian:

$$L = T - V = \frac{m}{2}\left(\dot{\rho}^2 + \rho^2\dot{\varphi}^2\right) - \frac{1}{2}\alpha\rho^2 - \beta\rho$$

Generalized momenta:

$$p_\rho = \frac{\partial L}{\partial \dot{\rho}} = m\dot{\rho}\ ; \quad p_\varphi = \frac{\partial L}{\partial \dot{\varphi}} = m\rho^2\dot{\varphi}\ .$$

3.

$$\dot{\rho} = \frac{p_\rho}{m}\ ; \quad \dot{\varphi} = \frac{p_\varphi}{m\rho^2}$$

$$\curvearrowright\ L^\star(\rho, \varphi, p_\rho, p_\varphi) = \frac{1}{2m}\left(p_\rho^2 + \frac{p_\varphi^2}{\rho^2}\right) - \frac{1}{2}\alpha\rho^2 - \beta\rho\ .$$

Hamilton function:

$$H = p_\rho\dot{\rho} + p_\varphi\dot{\varphi} - L^\star = \frac{1}{2m}\left(p_\rho^2 + \frac{p_\varphi^2}{\rho^2}\right) + \frac{1}{2}\alpha\rho^2 + \beta\rho\ .$$

Integrals of motion:

- $\frac{\partial H}{\partial t} = 0$ and scleronomic constraints (planar motion) $\curvearrowright$

$$H = T + V = E = \text{const} \qquad \textit{energy conservation}\ .$$

- $\frac{\partial H}{\partial \varphi} = 0 \curvearrowright \varphi$: cyclic $\curvearrowright$

$$p_\varphi = m\rho^2\dot{\varphi} = \text{const} \qquad \textit{angular-momentum conservation}\ .$$

Solution 2.2.4

1. In Exercise 1.2.5 we have calculated the Lagrangian:

$$L = \frac{1}{2}m\left(\dot{\rho}^2 + \rho^2\dot{\varphi}^2 + \dot{z}^2\right) - V_0 \ln\frac{\rho}{\rho_0}\ .$$

The generalized momenta then read:

$$p_\rho = \frac{\partial L}{\partial \dot{\rho}} = m\,\dot{\rho}\ ; \quad p_\varphi = \frac{\partial L}{\partial \dot{\varphi}} = m\,\rho^2\dot{\varphi}\ ; \quad p_z = \frac{\partial L}{\partial \dot{z}} = m\,\dot{z}$$

$$\Longrightarrow\ H = p_\rho\dot{\rho} + p_\varphi\dot{\varphi} + p_z\dot{z} - L = \frac{1}{2}m(\dot{\rho}^2 + \rho^2\dot{\varphi}^2 + \dot{z}^2) + V_0 \ln\frac{\rho}{\rho_0}\ .$$

This yields the Hamilton function:

$$H = \frac{1}{2m}\left(p_\rho^2 + \frac{p_\varphi^2}{\rho^2} + p_z^2\right) + V_0 \ln\frac{\rho}{\rho_0}\ .$$

2. Hamilton's equations of motion:

$$\dot{p}_\rho = -\frac{\partial H}{\partial \rho} = \frac{p_\varphi^2}{m\,\rho^3} - \frac{V_0}{\rho}\ ,$$

$$\dot{p}_\varphi = -\frac{\partial H}{\partial \varphi} = 0\ ; \quad \dot{p}_z = -\frac{\partial H}{\partial z} = 0\ ,$$

$$\dot{\rho} = \frac{\partial H}{\partial p_\rho} = \frac{p_\rho}{m}\ ; \quad \dot{\varphi} = \frac{\partial H}{\partial p_\varphi} = \frac{p_\varphi}{m\,\rho^2}\ ; \quad \dot{z} = \frac{\partial H}{\partial p_z} = \frac{p_z}{m}\ ,$$

$$\frac{\partial H}{\partial t} = 0\ .$$

3. Conservation laws:
   φ, z are cyclic. Consequently:

$$p_\varphi = m\,\rho^2\dot{\varphi} = \text{const} : \textit{angular-momentum conservation law}\ ,$$
$$p_z = m\,\dot{z} = \text{const} : \quad \textit{principle of conservation of linear momentum}\ .$$

$\frac{\partial H}{\partial t} = 0$ and $\frac{\partial}{\partial t}\mathbf{r}(\mathbf{q}, t) = 0$. From that it follows:

$$H = E = \text{const} : \qquad \textit{energy conservation law}\ .$$

Solution 2.2.5

1. Σ: rest system of coordinates
   constraints:
   one-dimensional motion $\Longrightarrow \quad z = y = 0 \quad \Longrightarrow \quad q = x$
   spring *'relaxed'* for $x' = d$
   kinetic energy:

$$T = \frac{m}{2}\dot{x}^2$$

   potential energy:

$$V = \frac{k}{2}\left(x' - d\right)^2 = \frac{k}{2}\left(x - v_0 t - d\right)^2$$

Lagrangian:

$$L = T - V = \frac{m}{2}\dot{x}^2 - \frac{k}{2}(x - v_0 t - d)^2$$

generalized momentum:

$$p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x} \quad \Longrightarrow \quad \dot{x} = \frac{p_x}{m}$$

$$\Longrightarrow \quad L^*(x, p_x, t) = \frac{p_x^2}{2m} - \frac{k}{2}(x - v_0 t - d)^2$$

$\Longrightarrow$ Hamilton function:

$$H = p_x\dot{x} - L = \frac{p_x^2}{m} - L^*$$

$$= \frac{p_x^2}{2m} + \frac{k}{2}(x - v_0 t - d)^2$$

obviously:

$$H = T + V = E$$

but:

$$\frac{\partial L}{\partial t} \neq 0\,; \quad \frac{\partial H}{\partial t} \neq 0$$

$\Longrightarrow$ H is **not** a conserved quantity
Equations of motion:

$$\dot{x} = \frac{\partial H}{\partial p_x} = \frac{p_x}{m} \quad \Longrightarrow \quad \dot{p}_x = m\ddot{x}$$

$$\dot{p}_x = -\frac{\partial H}{\partial x} = -k(x - v_0 t - d)$$

$$\Longrightarrow \quad m\ddot{x} + kx = k(v_0 t + d)$$

2. Σ': co-moving system of coordinates

$$x' = x - v_0 t \quad \Longrightarrow \quad \dot{x}' = \dot{x} - v_0$$

$$\Longrightarrow \quad \overline{L}\left(x', \dot{x}'\right) = \frac{m}{2}\left(\dot{x}' + v_0\right)^2 - \frac{k}{2}\left(x' - d\right)^2$$

$$\Longrightarrow \quad p_x' = \frac{\partial \overline{L}}{\partial \dot{x}'} = m\left(\dot{x}' + v_0\right)$$

$$\Longrightarrow \quad \overline{L}^*\left(x', p_x'\right) = \frac{p_x'^2}{2m} - \frac{k}{2}\left(x' - d\right)^2$$

$$\Longrightarrow \quad \overline{H} = p_x'\dot{x}' - \overline{L}^*\left(x', p_x'\right)$$

$$= p_x'\left(\frac{p_x'}{m} - v_0\right) - \frac{p_x'^2}{2m} + \frac{k}{2}\left(x' - d\right)^2$$

$$\Longrightarrow \quad \overline{H} = \frac{p_x'^2}{2m} - p_x' v_0 + \frac{k}{2}\left(x' - d\right)^2$$

Reverse situation compared to that from part 1.:

$$\overline{H} \neq E = \frac{p_x'^2}{2m} + \frac{k}{2}\left(x' - d\right)^2$$

$\overline{H}$ is **not** the total energy, **but**:

$$\frac{\partial \overline{H}}{\partial t} = 0 \quad \Longrightarrow \quad \overline{H}\ : \qquad \text{integral of motion}$$

Equations of motion:

$$\dot{x}' = \frac{\partial \overline{H}}{\partial p_x'} = \frac{p_x'}{m} - v_0$$

$$\Longrightarrow \quad p_x' = m\left(\dot{x}' + v_0\right) \quad \Longrightarrow \quad \dot{p}_x' = m\ddot{x}'$$

$$\dot{p}_x' = -\frac{\partial \overline{H}}{\partial x'} = -k\left(x' - d\right)$$

Combination:

$$m\ddot{x}' + kx' = kd$$

That is the undamped harmonic oscillator with a time-independent external force!

Solution 2.2.6 The system is conservative and subject to holonomic-scleronomic constraints. Thus the Hamilton function is identical to the total energy and an integral of motion:

$$H = T + V = E = \text{const}\,.$$

All forces are conservative:

$$\mathbf{F}_1 = +k\left(x_2 - x_1\right)\,\mathbf{e}_x = -\nabla_1 V$$

$$\mathbf{F}_2 = +k\,(x_3 - x_2)\ \mathbf{e}_x - k\,(x_2 - x_1)\ \mathbf{e}_x = -\nabla_2 V$$
$$\mathbf{F}_3 = -k\,(x_3 - x_2)\ \mathbf{e}_x = -\nabla_3 V\ .$$

This leads to the potential energy:

$$V = \frac{1}{2}k\,\left((x_1 - x_2)^2 + (x_3 - x_2)^2\right)\ .$$

Kinetic energy:

$$T = \frac{1}{2}m_1\,\left(\dot{x}_1^2 + \dot{x}_3^2\right) + \frac{1}{2}m_2\,\dot{x}_2^2\,.$$

Lagrangian:

$$L = T - V = \frac{1}{2}m_1\,\left(\dot{x}_1^2 + \dot{x}_3^2\right) + \frac{1}{2}m_2\,\dot{x}_2^2 - \frac{1}{2}k\,\left((x_1 - x_2)^2 + (x_3 - x_2)^2\right)\ .$$

Generalized momenta:

$$p_1 = \frac{\partial L}{\partial \dot{x}_1} = m_1\,\dot{x}_1 \ \curvearrowright\ \dot{x}_1 = \frac{p_1}{m_1}$$
$$p_2 = \frac{\partial L}{\partial \dot{x}_2} = m_2\,\dot{x}_2 \ \curvearrowright\ \dot{x}_2 = \frac{p_2}{m_2}$$
$$p_3 = \frac{\partial L}{\partial \dot{x}_3} = m_1\,\dot{x}_3 \ \curvearrowright\ \dot{x}_3 = \frac{p_3}{m_1}\ .$$

Inserting the momenta into the Lagrangian:

$$L^*\,(x_1, x_2, x_3, p_1, p_2, p_3) = \frac{1}{2m_1}\left(p_1^2 + p_3^2\right) + \frac{1}{2m_2}p_2^2 - \frac{1}{2}k\,\left((x_1 - x_2)^2 + (x_3 - x_2)^2\right)\ .$$

Hamilton function:

$$H = \sum_{j=1}^{3} p_j \dot{x}_j - L^*\,(x_1, x_2, x_3, p_1, p_2, p_3)$$
$$= \frac{1}{2m_1}\left(p_1^2 + p_3^2\right) + \frac{1}{2m_2}p_2^2 + \frac{1}{2}k\,\left((x_1 - x_2)^2 + (x_3 - x_2)^2\right)\ .$$

Equations of motion:

•

$$\dot{x}_1 = \frac{\partial H}{\partial p_1} = \frac{p_1}{m_1}$$

$$\dot{p}_1 = -\frac{\partial H}{\partial x_1} = -k\,(x_1 - x_2)$$
$$\curvearrowright\ \ddot{x}_1 = \frac{1}{m_1}\dot{p}_1 = -\frac{k}{m_1}\,(x_1 - x_2)$$

$\Longrightarrow$

$$\ddot{x}_1 + \frac{k}{m_1}\,(x_1 - x_2) = 0\,. \tag{A.5}$$

•

$$\dot{x}_2 = \frac{\partial H}{\partial p_2} = \frac{p_2}{m_2}$$
$$\dot{p}_2 = -\frac{\partial H}{\partial x_2} = k\,(x_1 - x_2) + k\,(x_3 - x_2)$$
$$\curvearrowright\ \ddot{x}_2 = \frac{1}{m_2}\dot{p}_2 = \frac{k}{m_2}\,(x_1 - 2x_2 + x_3)$$

$\Longrightarrow$

$$\ddot{x}_2 + \frac{k}{m_2}\,(-x_1 + 2x_2 - x_3) = 0\,. \tag{A.6}$$

•

$$\dot{x}_3 = \frac{\partial H}{\partial p_3} = \frac{p_3}{m_1}$$
$$\dot{p}_3 = -\frac{\partial H}{\partial x_3} = -k\,(x_3 - x_2)$$
$$\curvearrowright\ \ddot{x}_3 = \frac{1}{m_1}\dot{p}_3 = -\frac{k}{m_1}\,(x_3 - x_2)$$

$\Longrightarrow$

$$\ddot{x}_3 + \frac{k}{m_1}\,(x_3 - x_2) = 0\,. \tag{A.7}$$

Solution approach:

$$x_i(t) = \alpha_i\,\mathrm{e}^{i\omega t} \qquad i = 1, 2, 3\,.$$

This we insert into (A.5)–(A.7) getting therewith a homogeneous system of equations for the coefficients α_i:

$$-\alpha_1\omega^2 + \frac{k}{m_1}(\alpha_1 - \alpha_2) = 0$$

$$-\alpha_2\omega^2 + \frac{k}{m_2}(2\alpha_2 - \alpha_1 - \alpha_3) = 0$$

$$-\alpha_3\omega^2 + \frac{k}{m_1}(\alpha_3 - \alpha_2) = 0\,.$$

A non-trivial solution requires a vanishing secular determinant:

$$\begin{vmatrix} -\omega^2 + \frac{k}{m_1} & -\frac{k}{m_1} & 0 \\ -\frac{k}{m_2} & -\omega^2 + 2\frac{k}{m_2} & -\frac{k}{m_2} \\ 0 & -\frac{k}{m_1} & -\omega^2 + \frac{k}{m_1} \end{vmatrix} \stackrel{!}{=} 0\,.$$

That is equivalent to

$$\left(-\omega^2 + \frac{k}{m_1}\right)^2 \left(-\omega^2 + 2\frac{k}{m_2}\right) - \frac{2k^2}{m_1 m_2}\left(-\omega^2 + \frac{k}{m_1}\right) \stackrel{!}{=} 0\,.$$

One of the possible solutions (*eigen-frequencies*) can be directly read off:

$$\omega_1 = \sqrt{\frac{k}{m_1}}\,. \tag{A.8}$$

The other solutions follow from

$$\left(-\omega^2 + \frac{k}{m_1}\right)\left(-\omega^2 + 2\frac{k}{m_2}\right) - \frac{2k^2}{m_1 m_2} \stackrel{!}{=} 0\,,$$

or

$$\omega^4 - \omega^2\left(\frac{k}{m_1} + 2\frac{k}{m_2}\right) \stackrel{!}{=} 0\,.$$

The other *eigen-frequencies* are then:

$$\omega_2 = 0 \tag{A.9}$$

$$\omega_3 = \sqrt{k\left(\frac{1}{m_1} + \frac{2}{m_2}\right)}\,. \tag{A.10}$$

Note that the negative roots in (A.8) and (A.10) are mathematically possible solutions but physically not allowed because the frequencies cannot be negative. In order to find the amplitudes α_i we insert the solutions (A.8)–(A.10) into the homogeneous system of equations.

- $\omega = \omega_1$
  One finds immediately:
  $$\alpha_2 = 0\,; \quad \alpha_1 = -\alpha_3\,.$$
  The middle atom is at rest while the two outer atoms oscillate in opposing phases with equal amplitudes.
- $\omega = \omega_2$
  Again the solution is simple:
  $$\alpha_1 = \alpha_2 = \alpha_3\,.$$
  This corresponds to a simple (in phase) translation of the three atoms without any relative motion.
- $\omega = \omega_3$
  It holds:
  $$-\alpha_1 k\left(\frac{1}{m_1}+\frac{2}{m_2}\right)+\frac{k}{m_1}(\alpha_1-\alpha_2)=0 \curvearrowright \alpha_1 = -\frac{1}{2}\frac{m_2}{m_1}\alpha_2\,.$$
  The second equation of the homogeneous system of equations reads for $\omega = \omega_3$:
  $$-\alpha_2 k\left(\frac{1}{m_1}+\frac{2}{m_2}\right)+\frac{k}{m_2}(2\alpha_2-\alpha_1-\alpha_3)=0 \curvearrowright \alpha_1 = \alpha_3\,.$$
  The two outer atoms oscillate in phase with equal amplitudes, while the middle atom oscillates in opposing phase to the two outer atoms, and with modified amplitude.

The general oscillation is then a superposition of the three fundamental oscillations discussed here.

Solution 2.2.7

Σ': inertial system; axes: x', y', z'
Σ: accelerated (rotating) non-inertial system; axes: x, y, z
rotation around the $z = z'$-axis with $\boldsymbol{\omega} = \omega\mathbf{e_z}$

Equation of motion ((2.77), Vol. 1):

$$m\ddot{\mathbf{r}} = \mathbf{F} - \underbrace{m(\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}))}_{\text{centrifugal force}} - \underbrace{2m(\boldsymbol{\omega}\times\dot{\mathbf{r}})}_{\text{Coriolis force}}$$

in addition:

$$\mathbf{F} = -\nabla V$$

Does there exist a potential V_{tot} from which even the pseudo forces can be derived? If yes, how does it look like?

(a) Centrifugal force:

$$\begin{aligned}
\mathbf{F}^{(cf)} &= -m(\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})) = -m\left\{\boldsymbol{\omega}(\boldsymbol{\omega} \cdot \mathbf{r}) - \mathbf{r}\omega^2\right\} \\
&= -m\left\{\omega^2 z\mathbf{e_z} - \omega^2\mathbf{r}\right\} \\
&= m\omega^2\left(x\mathbf{e_x} + y\mathbf{e_y}\right) \\
\Longrightarrow \quad V^{(cf)} &= -\frac{m\omega^2}{2}\left(x^2 + y^2\right)
\end{aligned}$$

Verification:

$$\mathbf{F}^{(cf)} = -\left(\mathbf{e_x}\frac{\partial}{\partial x} + \mathbf{e_y}\frac{\partial}{\partial y}\right) V^{(cf)} = m\omega^2\left(x\mathbf{e_x} + y\mathbf{e_y}\right)$$

(b) Coriolis force

$$\mathbf{F}^{(co)} = -2m\left(\boldsymbol{\omega} \times \dot{\mathbf{r}}\right) = 2m\left(\dot{\mathbf{r}} \times \boldsymbol{\omega}\right)$$

compare with the Lorentz force:

$$\begin{aligned}
2m &\longleftrightarrow \hat{q} \quad \text{charge} \\
\boldsymbol{\omega} &\longleftrightarrow \mathbf{B} = \operatorname{rot}\mathbf{A}
\end{aligned}$$

$\Longrightarrow$ generalized potential according to (1.78):

$$V^{(co)} = -2m\dot{\mathbf{r}} \cdot \mathbf{A}$$

with:

$$\operatorname{rot}\mathbf{A} = \boldsymbol{\omega} \quad \Longrightarrow \quad \mathbf{A} = \frac{\omega}{2}(-y, x, 0) = \frac{1}{2}(\boldsymbol{\omega} \times \mathbf{r})$$

$$\begin{aligned}
\Longrightarrow \quad V^{(co)}(\mathbf{r}, \dot{\mathbf{r}}, t) &= -m\left(\dot{\mathbf{r}} \cdot (\boldsymbol{\omega} \times \mathbf{r})\right) \\
&= -m\omega\left(-\dot{x}y + \dot{y}x\right) \\
&= -m\omega\left(x\dot{y} - y\dot{x}\right)
\end{aligned}$$

$\Longrightarrow$ generalized Lagrangian:

$$L = \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - V(\rho) - V^{(cf)} - V^{(co)}$$
$$= T - V\left(\sqrt{x^2+y^2}\right) + \frac{m\omega^2}{2}\left(x^2+y^2\right) + m\omega\left(x\dot{y} - y\dot{x}\right)$$

1. Cartesian coordinates:
   generalized momenta:

$$p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x} - m\omega y \quad \Longrightarrow \quad \dot{x} = \frac{p_x}{m} + \omega y$$
$$p_y = \frac{\partial L}{\partial \dot{y}} = m\dot{y} + m\omega x \quad \Longrightarrow \quad \dot{y} = \frac{p_y}{m} - \omega x$$
$$p_z = \frac{\partial L}{\partial \dot{z}} = m\dot{z} \quad \Longrightarrow \quad \dot{z} = \frac{p_z}{m}$$

$$L^* = \frac{m}{2}\left(\frac{p_x}{m} + \omega y\right)^2 + \frac{m}{2}\left(\frac{p_y}{m} - \omega x\right)^2 + \frac{p_z^2}{2m} + \frac{m\omega^2}{2}\left(x^2+y^2\right)$$
$$+ m\omega\left(\frac{p_y x}{m} - \omega x^2 - \frac{p_x y}{m} - \omega y^2\right) - V\left(\sqrt{x^2+y^2}\right)$$
$$= \frac{1}{2m}\left(p_x^2 + p_y^2 + p_z^2\right) + \left(p_x\omega y - p_y\omega x\right) + \frac{m}{2}\omega^2\left(x^2+y^2\right)$$
$$+ \frac{m}{2}\omega^2\left(x^2+y^2\right) - m\omega^2\left(x^2+y^2\right) + \omega\left(p_y x - p_x y\right) - V\left(\sqrt{x^2+y^2}\right)$$
$$L^* = \frac{1}{2m}\left(p_x^2 + p_y^2 + p_z^2\right) - V\left(\sqrt{x^2+y^2}\right)$$

With

$$\dot{x}p_x + \dot{y}p_y + \dot{z}p_z = \frac{1}{m}\left(p_x^2 + p_y^2 + p_z^2\right) + \omega y p_x - \omega x p_y$$

it follows then for the Hamilton function:

$$H = \frac{1}{2m}\left(p_x^2 + p_y^2 + p_z^2\right) - \omega\left(x p_y - y p_x\right) + V\left(\sqrt{x^2+y^2}\right)$$

Equations of motion:

$$\dot{x} = \frac{\partial H}{\partial p_x} = \frac{p_x}{m} + \omega y\,; \quad \dot{p}_x = -\frac{\partial H}{\partial x} = \omega p_y - \frac{\partial V}{\partial x}$$
$$\dot{y} = \frac{\partial H}{\partial p_y} = \frac{p_y}{m} - \omega x\,; \quad \dot{p}_y = -\frac{\partial H}{\partial y} = -\omega p_x - \frac{\partial V}{\partial y}$$
$$\dot{z} = \frac{\partial H}{\partial p_z} = \frac{p_z}{m}\,; \quad \dot{p}_z = -\frac{\partial H}{\partial z} = 0$$

z is cyclic, p_z therefore a conserved quantity!

2. Cylindrical coordinates:

$$x = \rho\cos\varphi\,; \quad y = \rho\sin\varphi\,; \quad z = z$$
$$\Longrightarrow \quad \dot{x} = \dot{\rho}\cos\varphi - \rho\dot{\varphi}\sin\varphi\,; \quad \dot{y} = \dot{\rho}\sin\varphi + \rho\dot{\varphi}\cos\varphi$$

$$\begin{aligned}\Longrightarrow \quad x\dot{y} - y\dot{x} &= \rho\dot{\rho}\cos\varphi\sin\varphi + \rho^2\dot{\varphi}\cos^2\varphi \\ &\quad -\rho\dot{\rho}\cos\varphi\sin\varphi + \rho^2\dot{\varphi}\sin^2\varphi \\ &= \rho^2\dot{\varphi}\end{aligned}$$

$\Longrightarrow$ Lagrangian:

$$L = \frac{m}{2}\left(\dot{\rho}^2 + \rho^2\dot{\varphi}^2 + \dot{z}^2\right) - V(\rho) + \frac{1}{2}m\omega^2\rho^2 + m\omega\rho^2\dot{\varphi}$$

generalized momenta:

$$p_\rho = \frac{\partial L}{\partial\dot{\rho}} = m\dot{\rho} \quad \Longrightarrow \quad \dot{\rho} = \frac{p_\rho}{m}$$
$$p_\varphi = \frac{\partial L}{\partial\dot{\varphi}} = m\rho^2\dot{\varphi} + m\omega\rho^2 \quad \Longrightarrow \quad \dot{\varphi} = \frac{p_\varphi}{m\rho^2} - \omega$$
$$p_z = \frac{\partial L}{\partial\dot{z}} = m\dot{z} \quad \Longrightarrow \quad \dot{z} = \frac{p_z}{m}$$

therewith:

$$\begin{aligned}L^* = \frac{m}{2}&\left(\frac{p_\rho^2}{m^2} + \rho^2\left(\frac{p_\varphi}{m\rho^2} - \omega\right)^2 + \frac{p_z^2}{m^2}\right) \\ &-V(\rho) + \frac{1}{2}m\omega^2\rho^2 + m\omega\rho^2\left(\frac{p_\varphi}{m\rho^2} - \omega\right)\end{aligned}$$

$$= \frac{1}{2m}\left(p_\rho^2 + \frac{p_\varphi^2}{\rho^2} + p_z^2\right) - \rho^2 \frac{p_\varphi \omega}{\rho^2}$$
$$+\frac{m}{2}\rho^2\omega^2 + \frac{1}{2}m\omega^2\rho^2 + \omega p_\varphi - m\omega^2\rho^2 - V(\rho)$$
$$= \frac{1}{2m}\left(p_\rho^2 + \frac{p_\varphi^2}{\rho^2} + p_z^2\right) - V(\rho)$$

in addition:

$$p_\rho \dot{\rho} + p_\varphi \dot{\phi} + p_z \dot{z} = \frac{1}{m}\left(p_\rho^2 + \frac{p_\varphi^2}{\rho^2} + p_z^2\right) - p_\varphi \omega$$

$\Longrightarrow$

$$H = \frac{1}{2m}\left(p_\rho^2 + \frac{p_\varphi^2}{\rho^2} + p_z^2\right) - \omega p_\varphi + V(\rho)$$

Equations of motion:

$$\dot{\rho} = \frac{\partial H}{\partial p_\rho} = \frac{p_\rho}{m}\ ; \quad \dot{p}_\rho = -\frac{\partial H}{\partial \rho} = \frac{p_\varphi^2}{m\rho^3} - \frac{\partial V}{\partial \rho}$$
$$\dot{\varphi} = \frac{\partial H}{\partial p_\varphi} = \frac{p_\varphi}{m\rho^2} - \omega\ ; \quad \dot{p}_\varphi = -\frac{\partial H}{\partial \varphi} = 0$$
$$\dot{z} = \frac{\partial H}{\partial p_z} = \frac{p_z}{m}\ ; \quad \dot{p}_z = -\frac{\partial H}{\partial z} = 0$$

$\Longrightarrow$ z and φ are cyclic
$\Longrightarrow$ 2 conservation laws:

$$p_\varphi = m\rho^2\dot{\varphi} + m\omega\rho^2 = \text{const}\ ; \quad p_z = m\dot{z} = \text{const}\ .$$

Solution 2.2.8 The motion is restricted to a plane, e.g. the xy-plane, without further constraints. As generalized coordinates plane polar coordinates r, φ are therefore recommendable:

$$x = r\cos\varphi\ ; \quad y = r\sin\varphi\ .$$

The generalized potential U of the non-conservative force $\mathbf{F}$ has been determined in Exercise 1.2.21 having led there to the Lagrangian:

$$L = T - U = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\varphi}^2\right) - \frac{\alpha}{r}\left(1 + \frac{\dot{r}^2}{c^2}\right)\ .$$

If one sums up the velocity-dependent terms to a *'generalized'* kinetic energy $T^\star$ then it can also be written:

$$L = T^\star(r, \varphi, \dot{r}, \dot{\varphi}) - V(r)$$
$$T^\star(r, \varphi, \dot{r}, \dot{\varphi}) = \frac{1}{2}\left(m - \frac{2\alpha}{rc^2}\right)\dot{r}^2 + \frac{1}{2}mr^2\dot{\varphi}^2$$
$$V(r) = \frac{\alpha}{r}\ .$$

Generalized momenta:

$$p_r = \frac{\partial L}{\partial \dot{r}} = \frac{\partial T^\star}{\partial \dot{r}} = \left(m - \frac{2\alpha}{rc^2}\right)\dot{r} \curvearrowright \dot{r} = \frac{p_r}{m - \frac{2\alpha}{rc^2}}$$
$$p_\varphi = \frac{\partial L}{\partial \dot{\varphi}} = \frac{\partial T^\star}{\partial \dot{\varphi}} = mr^2\dot{\varphi} \curvearrowright \dot{\varphi} = \frac{p_\varphi}{mr^2}$$
$$p_r\,\dot{r} = \frac{p_r^2}{m - \frac{2\alpha}{rc^2}}\ ; \quad p_\varphi\dot{\varphi} = \frac{p_\varphi^2}{mr^2}\ .$$

Lagrangian as function of the coordinates and the momenta:

$$L^\star(r, \varphi, p_r, p_\varphi) = \frac{1}{2}\left(m - \frac{2\alpha}{rc^2}\right)\left(\frac{p_r}{m - \frac{2\alpha}{rc^2}}\right)^2 + \frac{1}{2}mr^2\left(\frac{p_\varphi}{mr^2}\right)^2 - \frac{\alpha}{r}$$
$$= \frac{p_r^2}{2\left(m - \frac{2\alpha}{rc^2}\right)} + \frac{p_\varphi^2}{2mr^2} - \frac{\alpha}{r}$$

Hamilton function:

$$H(r, \varphi, p_r, p_\varphi) = p_r\dot{r} + p_\varphi\dot{\varphi} - L^\star(r, \varphi, p_r, p_\varphi) = T^\star(r, \varphi, p_r, p_\varphi) + V(r)$$
$$= \frac{p_r^2}{2\left(m - \frac{2\alpha}{rc^2}\right)} + \frac{p_\varphi^2}{2mr^2} + \frac{\alpha}{r}\ .$$

Section 2.4.6

Solution 2.4.1

1. For arbitrary phase functions $f(\mathbf{q}, \mathbf{p}, t)$ we have according to (2.114):

$$\{f, p_j\} = \frac{\partial f}{\partial q_j}\ .$$

That we use here:

$$\{L_x, p_x\} = \frac{\partial}{\partial x}\left(y p_z - z p_y\right) = 0\ ,$$
$$\left\{L_x, p_y\right\} = \frac{\partial}{\partial y}\left(y p_z - z p_y\right) = p_z\ ,$$
$$\{L_x, p_z\} = \frac{\partial}{\partial z}\left(y p_z - z p_y\right) = -p_y\ .$$

Analogously one finds the other brackets:

$$\left\{L_i, p_j\right\} = \varepsilon_{ijl} p_l\ ,$$

where $(i, j, l) = (x, y, z)$ and cyclic, ε_{ijl} : fully antisymmetric unit tensor of third rank ((1.192),Vol. 1).

2.

$$\begin{aligned}
\{L_x, L_x\} &= \left\{L_y, L_y\right\} = \{L_z, L_z\} = 0\ ,\\
\left\{L_x, L_y\right\} &= \left\{y p_z - z p_y,\ z p_x - x p_z\right\}\\
&= \{y p_z,\ z p_x\} - \underbrace{\left\{z p_y,\ z p_x\right\}}_{=0} - \underbrace{\left\{y p_z,\ x p_z\right\}}_{=0} + \left\{z p_y,\ x p_z\right\}\\
&= y\,\{p_z,\ z\}\,p_x + x\,\{z,\ p_z\}\,p_y = -y p_x + x p_y\\
&= L_z\ .
\end{aligned}$$

The other brackets are calculated completely analogously:

$$\left\{L_i, L_j\right\} = \varepsilon_{ijl}\, L_l\ ,$$

where $(i, j, l) = (x, y, z)$ and cyclic.

Solution 2.4.2

1. Use Exercise 2.4.1, part 2.:

$$\begin{aligned}
\left\{\mathbf{L}^2, L_x\right\} &= \left\{L_x^2 + L_y^2 + L_z^2, L_x\right\} = \left\{L_y^2 + L_z^2, L_x\right\}\\
&= L_y\{L_y, L_x\} + \{L_y, L_x\}L_y + L_z\{L_z, L_x\} + \{L_z, L_x\}L_z\\
&= -L_y L_z - L_z L_y + L_z L_y + L_y L_z\\
&= 0
\end{aligned}$$

analogously:

$$\left\{\mathbf{L}^2, L_y\right\} = \left\{\mathbf{L}^2, L_z\right\} = 0$$

2. The statement follows directly from Poisson's theorem. As an example the case may be investigated that L_x and L_y are integrals of motion. Because of

$$\frac{\partial L_x}{\partial t} = \frac{\partial L_y}{\partial t} = 0$$

L_x and L_y are indeed integrals of motion if it holds:

$$\{H, L_x\} = \{H, L_y\} = 0$$

Jacobi identity:

$$\begin{aligned} 0 &= \{L_x, \{L_y, H\}\} + \{H, \{L_x, L_y\}\} + \{L_y, \{H, L_x\}\} \\ &= 0 + \{H, \{L_x, L_y\}\} + 0 \\ &= \{H, L_z\} \end{aligned}$$

With

$$\frac{\partial L_z}{\partial t} = 0$$

we have found that L_z, too, is an integral of motion.

Solution 2.4.3

1. Particle without constraint in the central field:

$$V(\mathbf{r}) = V(r)\,.$$

Spherical coordinates are obviously convenient. The Hamilton function was already derived in 2.45:

$$H = \frac{1}{2m}\left(p_r^2 + \frac{p_\vartheta^2}{r^2} + \frac{p_\varphi^2}{r^2 \sin^2\vartheta}\right) + V(r)\,.$$

For the momenta thereby Eq. (2.44) holds:

$$\begin{aligned} p_r &= \frac{\partial L}{\partial \dot{r}} = m\dot{r} \\ p_\vartheta &= \frac{\partial L}{\partial \dot{\vartheta}} = mr^2\dot{\vartheta} \\ p_\varphi &= \frac{\partial L}{\partial \dot{\varphi}} = mr^2 \sin^2\vartheta\,\dot{\varphi}\,. \end{aligned}$$

Of course one recognizes already here that φ is cyclic, the corresponding momentum is therefore an integral of motion.

2. Because of

$$\begin{aligned}
L_z &= xp_y - yp_x = m(x\dot{y} - y\dot{x}) \\
&= m\Big(r\sin\vartheta\cos\varphi\left(\dot{r}\sin\vartheta\sin\varphi + r\dot{\vartheta}\cos\vartheta\sin\varphi + r\sin\vartheta\dot{\varphi}\cos\varphi\right) \\
&\qquad - r\sin\vartheta\sin\varphi\left(\dot{r}\sin\vartheta\cos\varphi + r\dot{\vartheta}\cos\vartheta\cos\varphi - r\sin\vartheta\dot{\varphi}\sin\varphi\right)\Big) \\
&= m\left(r^2\sin^2\vartheta\dot{\varphi}\cos^2\varphi + r^2\sin^2\vartheta\dot{\varphi}\sin^2\varphi\right) \\
&= mr^2\sin^2\vartheta\,\dot{\varphi}
\end{aligned}$$

we have:

$$p_\varphi = L_z \ .$$

Poisson bracket:

$$\{H, L_z\} = \{H, p_\varphi\} = \frac{\partial H}{\partial\varphi} = 0$$

L_z is not explicitly time-dependent: $\partial L_z/\partial t = 0$. According to (2.121) L_z is therewith an integral of motion!

Solution 2.4.4

1.

$$\begin{aligned}
\frac{\partial}{\partial t}\{f, g\} &= \frac{\partial}{\partial t}\sum_{j=1}^{S}\left(\frac{\partial f}{\partial q_j}\frac{\partial g}{\partial p_j} - \frac{\partial f}{\partial p_j}\frac{\partial g}{\partial q_j}\right) \\
&= \sum_{j=1}^{S}\left(\frac{\partial^2 f}{\partial t\,\partial q_j}\frac{\partial g}{\partial p_j} + \frac{\partial f}{\partial q_j}\frac{\partial^2 g}{\partial t\,\partial p_j} - \frac{\partial^2 f}{\partial t\,\partial p_j}\frac{\partial g}{\partial q_j} - \frac{\partial f}{\partial p_j}\frac{\partial^2 g}{\partial t\,\partial q_j}\right) \\
&= \sum_{j=1}^{S}\left[\left(\frac{\partial}{\partial q_j}\frac{\partial f}{\partial t}\right)\frac{\partial g}{\partial p_j} - \left(\frac{\partial}{\partial p_j}\frac{\partial f}{\partial t}\right)\frac{\partial g}{\partial q_j} + \frac{\partial f}{\partial q_j}\left(\frac{\partial}{\partial p_j}\frac{\partial g}{\partial t}\right)\right. \\
&\qquad \left. - \frac{\partial f}{\partial p_j}\left(\frac{\partial}{\partial q_j}\frac{\partial g}{\partial t}\right)\right] \\
&= \left\{\frac{\partial f}{\partial t}, g\right\} + \left\{f, \frac{\partial g}{\partial t}\right\} \ .
\end{aligned}$$

2. Equation of motion:

$$\begin{aligned}
\frac{d}{dt}\{f,g\} &= \{\{f,g\},H\} + \frac{\partial}{\partial t}\{f,g\} \\
&= -\{\{g,H,\},f\} - \{\{H,f\},g\} + \left\{\frac{\partial f}{\partial t},g\right\} + \left\{f,\frac{\partial g}{\partial t}\right\} \quad \text{(Jacobi identity)} \\
&= \left\{f,\{g,H\}+\frac{\partial g}{\partial t}\right\} + \left\{\{f,H\}+\frac{\partial f}{\partial t},g\right\} \\
&= \left\{f,\frac{dg}{dt}\right\} + \left\{\frac{df}{dt},g\right\} \ .
\end{aligned}$$

3.

$$\begin{aligned}
\{f,g\,h\} &= \sum_{j=1}^{S}\left(\frac{\partial f}{\partial q_j}\frac{\partial}{\partial p_j}(g\,h) - \frac{\partial f}{\partial p_j}\frac{\partial}{\partial q_j}(g\,h)\right) \\
&= \sum_{j=1}^{S}\left(h\,\frac{\partial f}{\partial q_j}\frac{\partial g}{\partial p_j} + g\,\frac{\partial f}{\partial q_j}\frac{\partial h}{\partial p_j} - g\,\frac{\partial f}{\partial p_j}\frac{\partial h}{\partial q_j} - h\,\frac{\partial f}{\partial p_j}\frac{\partial g}{\partial q_j}\right) \\
&= h\sum_{j=1}^{S}\left(\frac{\partial f}{\partial q_j}\frac{\partial g}{\partial p_j} - \frac{\partial f}{\partial p_j}\frac{\partial g}{\partial q_j}\right) + g\sum_{j=1}^{S}\left(\frac{\partial f}{\partial q_j}\frac{\partial h}{\partial p_j} - \frac{\partial f}{\partial p_j}\frac{\partial h}{\partial q_j}\right) \\
&= h\,\{f,g\} + g\,\{f,h\} \ .
\end{aligned}$$

Solution 2.4.5

1. For an arbitrary phase function $f(\mathbf{q},\mathbf{p},t)$ Eqs. (2.113) and (2.114) are valid:

$$\begin{aligned}
\{f,p_j\} &= \frac{\partial f}{\partial q_j} \\
\{f,q_j\} &= -\frac{\partial f}{\partial p_j}
\end{aligned}$$

With $\mathbf{A} = \mathbf{A}(\mathbf{r},\mathbf{p})$ and $\mathbf{L} = \mathbf{r}\times\mathbf{p}$ we then have:

$$\begin{aligned}
L_i &= \sum_{jk}\varepsilon_{ijk}x_jp_k \\
\{L_i,A_m\} &= \sum_{jk}\varepsilon_{ijk}\{x_jp_k,A_m\}
\end{aligned}$$

$$= \sum_{jk} \varepsilon_{ijk} \left(x_j \{p_k, A_m\} + \{x_j, A_m\} p_k \right)$$

$$= \sum_{jk} \varepsilon_{ijk} \left(\frac{\partial A_m}{\partial p_j} p_k - x_j \frac{\partial A_m}{\partial x_k} \right)$$

2. In particular for x_m we have:

$$\{L_i, x_m\} = \sum_{jk} \varepsilon_{ijk} \left(\frac{\partial x_m}{\partial p_j} p_k - x_j \frac{\partial x_m}{\partial x_k} \right) = \sum_{jk} \varepsilon_{ijk} \left(-x_j \delta_{mk} \right)$$

$$= \sum_{j} \varepsilon_{imj}\, x_j$$

3. Analogously:

$$\{L_i, p_m\} = \sum_{jk} \varepsilon_{ijk} \delta_{mj} p_k$$

$$= \sum_{k} \varepsilon_{imk}\, p_k$$

4. For the components of the angular momentum we have with the formula from part 1.:

$$\{L_i, L_j\} = \sum_{kl} \varepsilon_{ikl} \left(\frac{\partial L_j}{\partial p_k} p_l - x_k \frac{\partial L_j}{\partial x_l} \right)$$

Insertion of

$$L_j = \sum_{mn} \varepsilon_{jmn}\, x_m\, p_n$$

$$\frac{\partial L_j}{\partial p_k} = \sum_{mn} \varepsilon_{jmn}\, x_m \delta_{nk} = \sum_{m} \varepsilon_{jmk}\, x_m = -\sum_{m} \varepsilon_{jkm}\, x_m$$

$$\frac{\partial L_j}{\partial x_l} = \sum_{mn} \varepsilon_{jmn} \delta_{ml}\, p_n = \sum_{m} \varepsilon_{jlm}\, p_m$$

yields:

$$\{L_i, L_j\} = -\sum_{klm} \varepsilon_{ikl} \left(\varepsilon_{jkm} x_m p_l + \varepsilon_{jlm} x_k p_m \right)$$

With

$$\sum_k \varepsilon_{ikl}\varepsilon_{jkm} = \delta_{ij}\delta_{lm} - \delta_{im}\delta_{lj}$$

$$\sum_l \varepsilon_{ikl}\varepsilon_{jlm} = -\delta_{ij}\delta_{km} + \delta_{im}\delta_{kj}$$

we have:

$$\begin{aligned}\{L_i, L_j\} &= -\delta_{ij}\sum_l x_l p_l + x_i p_j + \delta_{ij}\sum_k x_k p_k - x_j p_i \\ &= x_i p_j - x_j p_i \\ &= \sum_k \varepsilon_{ijk} L_k\end{aligned}$$

5. With

$$\mathbf{A}^2 = \sum_m A_m^2$$

it is:

$$\{L_i, \mathbf{A}^2\} = 2\sum_{jkm} \varepsilon_{ijk} A_m \left(\frac{\partial A_m}{\partial p_j} p_k - x_j \frac{\partial A_m}{\partial x_k}\right)$$

Solution 2.4.6

1. The two-particle system possesses six degrees of freedom and is therefore described by six Cartesian coordinates $(x_{\alpha 1}, x_{\alpha 2}, x_{\alpha 3}; \alpha = 1, 2)$ and the corresponding six momenta $(p_{\alpha 1}, p_{\alpha 2}, p_{\alpha 3}; \alpha = 1, 2)$. The angular momenta are:

$$\mathbf{L}_\alpha = \mathbf{L}_\alpha\left(\mathbf{x}_\alpha, \mathbf{p}_\alpha\right) \qquad \alpha = 1, 2\ .$$

Then the Poisson bracket reads:

$$\{\mathbf{L}_1, \mathbf{L}_2\} = \sum_{\alpha=1}^{2}\sum_{j=1}^{3}\left(\frac{\partial \mathbf{L}_1\left(\mathbf{x}_1, \mathbf{p}_1\right)}{\partial x_{\alpha j}}\frac{\partial \mathbf{L}_2\left(\mathbf{x}_2, \mathbf{p}_2\right)}{\partial p_{\alpha j}} - \frac{\partial \mathbf{L}_1\left(\mathbf{x}_1, \mathbf{p}_1\right)}{\partial p_{\alpha j}}\frac{\partial \mathbf{L}_2\left(\mathbf{x}_2, \mathbf{p}_2\right)}{\partial x_{\alpha j}}\right)\ .$$

For each $\alpha = 1, 2$ the two products within the bracket contain one factor which is zero. For $\alpha = 1$ in both products the second factor, for $\alpha = 2$ the first factor is zero. That proves the assertion!

2. We use the result of Exercise 2.4.1 and that from part 1.:

$$\begin{aligned}\left\{L_{11}, \mathbf{L}_1 \cdot \mathbf{L}_2\right\} &= \sum_j \left\{L_{11}, L_{1j} L_{2j}\right\} = \sum_j \left\{L_{11}, L_{1j}\right\} L_{2j} \\ &= \sum_{jk} \varepsilon_{1jk} L_{1k} L_{2j} = \sum_{jk} \varepsilon_{1jk} L_{2j} L_{1k} \\ &= (\mathbf{L}_2 \times \mathbf{L}_1)_1 = -(\mathbf{L}_1 \times \mathbf{L}_2)_1 \ .\end{aligned}$$

That is also valid for the two other components so that the assertion is proven:

$$\left\{\mathbf{L}_1, \mathbf{L}_1 \cdot \mathbf{L}_2\right\} = -\left(\mathbf{L}_1 \times \mathbf{L}_2\right) \ .$$

3. Proof by complete induction with the result from part 2. as induction base. The assertion may be true for $n = k$. We draw the conclusion for $k + 1$:

$$\begin{aligned}\left\{\mathbf{L}_1, (\mathbf{L}_1 \cdot \mathbf{L}_2)^{k+1}\right\} &= \left\{\mathbf{L}_1, (\mathbf{L}_1 \cdot \mathbf{L}_2)^k\right\} (\mathbf{L}_1 \cdot \mathbf{L}_2) \\ &\quad + (\mathbf{L}_1 \cdot \mathbf{L}_2)^k \left\{\mathbf{L}_1, (\mathbf{L}_1 \cdot \mathbf{L}_2)\right\} \\ &= -k\,(\mathbf{L}_1 \cdot \mathbf{L}_2)^{k-1} (\mathbf{L}_1 \times \mathbf{L}_2)(\mathbf{L}_1 \cdot \mathbf{L}_2) \\ &\quad - (\mathbf{L}_1 \cdot \mathbf{L}_2)^k (\mathbf{L}_1 \times \mathbf{L}_2) \\ &= -(k+1)\,(\mathbf{L}_1 \cdot \mathbf{L}_2)^k (\mathbf{L}_1 \times \mathbf{L}_2) \ .\end{aligned}$$

That proves the assertion!

Solution 2.4.7

1. Equation of motion for the observable f:

$$\frac{df}{dt} = \{f, H\} + \frac{\partial f}{\partial t}$$

With

$$f \text{ integral of motion} \quad \Longleftrightarrow \quad \{f, H\} = -\frac{\partial f}{\partial t}$$

and

$$H \text{ integral of motion} \quad \Longleftrightarrow \quad \{H, H\} = -\frac{\partial H}{\partial t} \quad \Longleftrightarrow \quad \frac{\partial H}{\partial t} = 0$$

we have then:

$$\left\{\frac{\partial f}{\partial t}, H\right\} = \frac{\partial}{\partial t}\{f, H\} - \left\{f, \frac{\partial H}{\partial t}\right\}$$
$$= -\frac{\partial}{\partial t}\left(\frac{\partial f}{\partial t}\right)$$
$$\Longrightarrow \quad \frac{\partial f}{\partial t} \qquad \text{integral of motion}$$

2.

$$H = \frac{p^2}{2m} \quad \Longrightarrow \quad \frac{\partial H}{\partial t} = 0 \quad \Longrightarrow \quad H \qquad \text{integral of motion}$$

$$\{f, H\} = \left\{q - \frac{pt}{m}, \frac{p^2}{2m}\right\} = \left\{q, \frac{p^2}{2m}\right\} - \left\{\frac{pt}{m}, \frac{p^2}{2m}\right\}$$
$$= \frac{1}{2m}\left\{q, p^2\right\} - \frac{t}{2m^2}\left\{p, p^2\right\}$$
$$= \frac{1}{2m}\Big(p\underbrace{\{q, p\}}_{=1} + \underbrace{\{q, p\}}_{=1}p\Big) - \frac{t}{2m^2}\Big(p\underbrace{\{p, p\}}_{=0} + \underbrace{\{p, p\}}_{=0}p\Big)$$
$$= \frac{p}{m} = -\frac{\partial f}{\partial t}$$

Although explicitly time-dependent f turns out to be an integral of motion. Thus $\partial f/\partial t$, too, should be an integral of motion:

$$\left\{\frac{\partial f}{\partial t}, H\right\} = \left\{-\frac{p}{m}, \frac{p^2}{2m}\right\} = -\frac{1}{2m^2}\underbrace{\left\{p, p^2\right\}}_{=0} = 0 = -\frac{\partial^2 f}{\partial t^2}$$

Therewith, $\partial f/\partial t$ is an integral of motion.

Solution 2.4.8 Hamilton function:

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 q^2 \,.$$

We exploit the linearity of the Poisson bracket in order to calculate:

$$\{H, f\} = \left\{\frac{p^2}{2m} + \frac{1}{2}m\omega^2 q^2, p\sin\omega t - m\omega q\cos\omega t\right\}$$
$$= \left\{\frac{p^2}{2m}, p\sin\omega t\right\} - \left\{\frac{p^2}{2m}, m\omega q\cos\omega t\right\} + \left\{\frac{1}{2}m\omega^2 q^2, p\sin\omega t\right\}$$

$$
\begin{aligned}
&\quad -\left\{\frac{1}{2}m\omega^2 q^2, m\omega q\cos\omega t\right\}\\
&= \frac{1}{2m}\sin\omega t\left\{p^2,p\right\} - \frac{1}{2}\omega\cos\omega t\left\{p^2,q\right\} + \frac{1}{2}m\omega^2\sin\omega t\left\{q^2,p\right\}\\
&\quad -\frac{1}{2}m^2\omega^3\cos\omega t\left\{q^2,q\right\}\ .
\end{aligned}
$$

Fundamental brackets (2.108) to (2.110) and product rule (2.118):

$$
\begin{aligned}
\{p^2,p\} &= p\,\underbrace{\{p,p\}}_{0} + \underbrace{\{p,p\}}_{0}\,p = 0\\
\{p^2,q\} &= p\,\underbrace{\{p,q\}}_{-1} + \underbrace{\{p,q\}}_{-1}\,p = -2p\\
\{q^2,p\} &= q\,\underbrace{\{q,p\}}_{+1} + \underbrace{\{q,p\}}_{+1}\,q = 2q\\
\{q^2,q\} &= q\,\underbrace{\{q,q\}}_{0} + \underbrace{\{q,q\}}_{0}\,q = 0\ .
\end{aligned}
$$

It remains:

$$
\{H,f\} = \omega\cos\omega t\cdot p + m\omega^2\sin\omega t\cdot q
$$

On the other hand it is:

$$
\frac{\partial f}{\partial t} = p\omega\cos\omega t + m\omega^2 q\sin\omega t \ \Rightarrow\ \{H,f\} = \frac{\partial f}{\partial t}\ \Rightarrow\ \frac{df}{dt} = 0
$$

$f(q,p,t)$ is thus an *integral of motion*!

We confirm this statement by a direct calculation of the total time derivative of f, where we exploit at appropriate places Hamilton's equations of motion:

$$
\begin{aligned}
\frac{df}{dt} &= \dot{p}\sin\omega t + p\omega\cos\omega t - m\omega\dot{q}\cos\omega t + m\omega^2 q\sin\omega t\\
&= \left(-\frac{\partial H}{\partial q}\right)\sin\omega t + p\omega\cos\omega t - m\omega\left(\frac{\partial H}{\partial p}\right)\cos\omega t + m\omega^2 q\sin\omega t\\
&= -m\omega^2 q\sin\omega t + p\omega\cos\omega t - p\omega\cos\omega t + m\omega^2 q\sin\omega t\\
&= 0
\end{aligned}
$$

Solution 2.4.9 Taylor expansion:

$$
A(t) = A(0) + \frac{1}{1!}\dot{A}(0)\,t + \frac{1}{2!}\ddot{A}(0)\,t^2 + l\ldots
$$

Because of

$$\frac{\partial A}{\partial t} = 0$$

it holds:

$$\dot{A}(0) = \{A(0), H\} \ .$$

Because of

$$\frac{\partial H}{\partial t} = 0$$

it is in addition

$$\frac{\partial}{\partial t}\{A(0), H\} = 0 \ .$$

That means

$$\ddot{A}(0) = \{\{A(0), H\}, H\} \ .$$

This procedure can be continued leading eventually to:

$$A(t) = A(0) + \frac{1}{1!}\{A(0), H\}\ t + \frac{1}{2!}\{\{A(0), H\}, H\}\ t^2 + \ldots$$

Section 2.5.6

Solution 2.5.1

1.

$$dF_4 = \left(\overline{H} - H\right) dt + \sum_{j=1}^{S}\left(p_j dq_j - dp_j q_j - p_j dq_j + d\bar{p}_j \bar{q}_j\right) \ .$$

One reads off:

$$\frac{\partial F_4}{\partial t} = \overline{H} - H \ ; \quad \frac{\partial F_4}{\partial p_j} = -q_j \ ; \quad \frac{\partial F_4}{\partial \bar{p}_j} = \bar{q}_j \ .$$

and solves

$$q_j = -\frac{\partial F_4}{\partial p_j} = q_j\left(\mathbf{p}, \bar{\mathbf{p}}, t\right)$$

for $\bar{p}_j$ getting therewith the first part of the transformation:

$$\bar{p}_j = \bar{p}_j(\mathbf{q}, \mathbf{p}, t) \ .$$

Into the second relation

$$\bar{q}_j = \frac{\partial F_4}{\partial \bar{p}_j} = \bar{q}_j\,(\mathbf{p}, \bar{\mathbf{p}}, t)$$

we insert the just found $\bar{\mathbf{p}}$:

$$\bar{q}_j = \bar{q}_j(\mathbf{q}, \mathbf{p}, t) \ .$$

For the *new* Hamilton function we find:

$$\overline{H}\,(\bar{\mathbf{q}}, \bar{\mathbf{p}}, t) = H\,(\mathbf{q}(\bar{\mathbf{q}}, \bar{\mathbf{p}}, t), \mathbf{p}(\bar{\mathbf{q}}, \bar{\mathbf{p}}, t), t) + \frac{\partial}{\partial t} F_4\,(\mathbf{p}\,(\bar{\mathbf{q}}, \bar{\mathbf{p}}, t)\,, \bar{\mathbf{p}}, t) \ .$$

2. Modified Hamilton's principle:

$$\begin{aligned} 0 \overset{!}{=} \delta S &= \delta \int_{t_1}^{t_2} dt \left(\sum_j p_j \dot{q}_j - H \right) \\ &= \delta \int_{t_1}^{t_2} dt \left[\sum_j \left(\dot{p}_j q_j + p_j \dot{q}_j - \dot{\bar{p}}_j \bar{q}_j \right) - \overline{H} \right] \\ &\quad + \delta \left\{ F_4\,(\bar{\mathbf{p}}\,(t_2)\,, \mathbf{p}\,(t_2)\,, t_2) - F_4\,(\bar{\mathbf{p}}\,(t_1)\,, \mathbf{p}\,(t_1)\,, t_1) \right\} \ , \end{aligned}$$

Note that $\bar{\mathbf{p}}(t_{1,2})$ and $\mathbf{p}(t_{1,2})$ are **not** fixed. In fact it holds:

$$\delta \left\{ F_4\,(\bar{\mathbf{p}}\,(t_2)\,, \mathbf{p}\,(t_2)\,, t_2) - F_4\,(\bar{\mathbf{p}}\,(t_1)\,, \mathbf{p}\,(t_1)\,, t_1) \right\} = \sum_{j=1}^{S} \left(\frac{\partial F_4}{\partial p_j} \delta p_j + \frac{\partial F_4}{\partial \bar{p}_j} \delta \bar{p}_j \right) \Bigg|_{t_1}^{t_2} \ .$$

It remains therewith:

$$\begin{aligned} 0 \overset{!}{=} & \sum_{j=1}^{S} \left(\frac{\partial F_4}{\partial p_j} \delta p_j + \frac{\partial F_4}{\partial \bar{p}_j} \delta \bar{p}_j \right) \Bigg|_{t_1}^{t_2} \\ & + \int_{t_1}^{t_2} dt \sum_{j=1}^{S} \left(\delta \dot{p}_j q_j + \dot{p}_j \delta q_j + \delta p_j \dot{q}_j + p_j \delta \dot{q}_j \ - \ \delta \dot{\bar{p}}_j \bar{q}_j - \dot{\bar{p}}_j \delta \bar{q}_j - \frac{\partial \overline{H}}{\partial \bar{q}_j} \delta \bar{q}_j - \frac{\partial \overline{H}}{\partial \bar{p}_j} \delta \bar{p}_j \right) \ . \end{aligned}$$

We perform some integrations by parts:

$$\int_{t_1}^{t_2} dt\, q_j \delta \dot{p}_j = q_j \delta p_j \Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} dt\, \dot{q}_j \delta p_j \; ,$$

$$\int_{t_1}^{t_2} dt\, p_j \delta \dot{q}_j = \underbrace{p_j \delta q_j}_{=0} \Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} dt\, \dot{p}_j \delta q_j \; ,$$

$$\int_{t_1}^{t_2} dt\, \bar{q}_j \delta \dot{\bar{p}}_j = \bar{q}_j \delta \bar{p}_j \Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} dt\, \dot{\bar{q}}_j \delta \bar{p}_j \; .$$

That leads to:

$$0 \stackrel{!}{=} \sum_{j=1}^{S} \left[\underbrace{\left(\frac{\partial F_4}{\partial p_j} + q_j \right)}_{=0} \delta p_j \Big|_{t_1}^{t_2} + \underbrace{\left(\frac{\partial F_4}{\partial \bar{p}_j} - \bar{q}_j \right)}_{=0} \delta \bar{p}_j \Big|_{t_1}^{t_2} \right]$$

$$+ \int_{t_1}^{t_2} dt \sum_{j=1}^{S} \Bigg[\left(-\dot{q}_j + \dot{q}_j \right) \delta p_j + \left(\dot{p}_j - \dot{p}_j \right) \delta q_j$$

$$+ \left(\dot{\bar{q}}_j - \frac{\partial \overline{H}}{\partial \bar{p}_j} \right) \delta \bar{p}_j - \left(\dot{\bar{p}}_j + \frac{\partial \overline{H}}{\partial \bar{q}_j} \right) \delta \bar{q}_j \Bigg] \; .$$

Since $\delta \bar{p}_j$, $\delta \bar{q}_j$ are independent quantities it follows eventually:

$$\dot{\bar{q}}_j = \frac{\partial \overline{H}}{\partial \bar{p}_j} \; ; \quad \dot{\bar{p}}_j = -\frac{\partial \overline{H}}{\partial \bar{q}_j} \; .$$

Solution 2.5.2 According to (2.203) it should then hold

$$\left\{ L_i, L_j \right\} = 0 \; .$$

But in Exercise 2.4.1, part (2) we have shown:

$$\left\{ L_i, L_j \right\} = \varepsilon_{ijl} L_l \; .$$

That means in particular:

$$\left\{ L_x, L_y \right\} = L_z \; .$$

Therefore, L_x and L_y cannot simultaneously occur as canonical momenta.

Solution 2.5.3 Trivially we have $\{\bar{q},\bar{q}\} = \{\bar{p},\bar{p}\} = 0$. Thus we have still to prove

$$\{\bar{q},\bar{p}\}_{q,p} = 1 \ .$$

$$\frac{\partial \bar{q}}{\partial q} = \frac{q}{\sin p}\left(-\frac{\sin p}{q^2}\right) = -\frac{1}{q} \ ,$$

$$\frac{\partial \bar{q}}{\partial p} = \frac{q}{\sin p}\left(\frac{\cos p}{q}\right) = \cot p \ ,$$

$$\frac{\partial \bar{p}}{\partial q} = \cot p \ ,$$

$$\frac{\partial \bar{p}}{\partial p} = -\frac{q}{\sin^2 p} \ .$$

It follows therewith:

$$\{\bar{q},\bar{p}\} = \frac{\partial \bar{q}}{\partial q}\frac{\partial \bar{p}}{\partial p} - \frac{\partial \bar{q}}{\partial p}\frac{\partial \bar{p}}{\partial q} = \frac{1}{\sin^2 p} - \cot^2 p = \frac{1-\cos^2 p}{\sin^2 p} = 1 \ .$$

Solution 2.5.4

1. We show

$$\{\bar{q},\bar{p}\} = 1 \ .$$

For that we need:

$$\begin{aligned}
\frac{\partial \bar{q}}{\partial q}\frac{\partial \bar{p}}{\partial p} &= \frac{\frac{1}{2}q^{-1/2}\cos p}{1+q^{1/2}\cos p}\left[2\left(1+q^{1/2}\cos p\right)q^{1/2}\cos p - 2q^{1/2}\sin p \ q^{1/2}\sin p\right] \\
&= \frac{\frac{1}{2}q^{-1/2}\cos p}{1+q^{1/2}\cos p}\left[2q^{1/2}\cos p + 2q(\cos^2 p - \sin^2 p)\right] \\
&= \cos^2 p - \sin^2 p \frac{q^{1/2}\cos p}{1+q^{1/2}\cos p} \ , \\
\frac{\partial \bar{q}}{\partial p}\frac{\partial \bar{p}}{\partial q} &= \frac{-q^{1/2}\sin p}{1+q^{1/2}\cos p}\left[\left(1+q^{1/2}\cos p\right)q^{-1/2}\sin p + q^{-1/2}\cos p \, q^{1/2}\sin p\right] \\
&= -\sin^2 p - \sin^2 p \ \frac{q^{1/2}\cos p}{1+q^{1/2}\cos p} \ .
\end{aligned}$$

Therefrom we get:

$$\frac{\partial \bar{q}}{\partial q}\frac{\partial \bar{p}}{\partial p} - \frac{\partial \bar{q}}{\partial p}\frac{\partial \bar{p}}{\partial q} = \cos^2 p + \sin^2 p = 1 \ .$$

Hence the transformation is canonical!

2. If $F_3(p, \bar{q})$ is the *generating function* then it must be

$$q = -\frac{\partial F_3}{\partial p}\ ; \quad \bar{p} = -\frac{\partial F_3}{\partial \bar{q}}\ .$$

Let us check:

$$\frac{\partial F_3}{\partial p} = -\left(\mathrm{e}^{\bar{q}} - 1\right)^2 \frac{1}{\cos^2 p} \stackrel{!}{=} -q$$

$$\Longleftrightarrow \ \mathrm{e}^{\bar{q}} = 1 + q^{1/2}\cos p \ \Longleftrightarrow \ \bar{q} = \ln\left(1 + q^{1/2}\cos p\right) \quad \text{q. e. d.}$$

$$\frac{\partial F_3}{\partial \bar{q}} = -2\left(\mathrm{e}^{\bar{q}} - 1\right)\mathrm{e}^{\bar{q}}\tan p = -2\left(1 + q^{1/2}\cos p - 1\right)\left(1 + q^{1/2}\cos p\right)\tan p$$

$$= -2q^{1/2}\sin p\left(1 + q^{1/2}\cos p\right) \stackrel{!}{=} -\bar{p}$$

$$\Longleftrightarrow \ \bar{p} = 2q^{1/2}\sin p\left(1 + q^{1/2}\cos p\right) \quad \text{q. e. d.}$$

Solution 2.5.5

1. We check the fundamental Poisson brackets. The brackets

$$\{\hat{q}, \hat{q}\}_{q,p} = \{\hat{p}, \hat{p}\}_{q,p} = 0$$

are trivial. With

$$\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1 - x^2}} \qquad \text{for} \qquad |x| < 1$$

it follows:

$$\frac{\partial \hat{q}}{\partial q} = \frac{1}{\sqrt{1 - \dfrac{q^2}{q^2 + \dfrac{p^2}{\alpha^2}}}}\left(\frac{1}{\sqrt{q^2 + \dfrac{p^2}{\alpha^2}}} - \frac{\dfrac{1}{2}q \cdot 2q}{\left(q^2 + \dfrac{p^2}{\alpha^2}\right)^{3/2}}\right)$$

$$= \sqrt{\frac{\alpha^2}{p^2}}\left(1 - \frac{q^2}{q^2 + \dfrac{p^2}{\alpha^2}}\right) = \sqrt{\frac{\alpha^2}{p^2}}\frac{\dfrac{p^2}{\alpha^2}}{q^2 + \dfrac{p^2}{\alpha^2}}$$

$$= \frac{\sqrt{\frac{p^2}{\alpha^2}}}{q^2 + \frac{p^2}{\alpha^2}}$$

$$\frac{\partial \hat{q}}{\partial p} = \frac{1}{\sqrt{1 - \frac{q^2}{q^2 + \frac{p^2}{\alpha^2}}}} \frac{-\frac{1}{2} q \frac{2p}{\alpha^2}}{\left(q^2 + \frac{p^2}{\alpha^2}\right)^{3/2}}$$

$$= -\sqrt{\frac{\alpha^2}{p^2}} \frac{qp}{\alpha^2 q^2 + p^2}$$

$$\frac{\partial \hat{p}}{\partial q} = \alpha q$$

$$\frac{\partial \hat{p}}{\partial p} = \frac{p}{\alpha} \; .$$

Therewith we have for the Poisson bracket:

$$\{\hat{q}, \hat{p}\}_{q,p} = \frac{\partial \hat{q}}{\partial q} \frac{\partial \hat{p}}{\partial p} - \frac{\partial \hat{q}}{\partial p} \frac{\partial \hat{p}}{\partial q} = \frac{\frac{p^2}{\alpha^2}}{q^2 + \frac{p^2}{\alpha^2}} + \frac{\alpha}{p} \frac{pq}{\alpha^2 q^2 + p^2} \alpha q$$

$$= \frac{1}{q^2 + \frac{p^2}{\alpha^2}} \left(\frac{p^2}{\alpha^2} + q^2\right) = 1 \; .$$

2. It is:

$$p = \frac{\partial F}{\partial q} = \alpha q \cot \hat{q}$$

$$\hat{p} = -\frac{\partial F}{\partial \hat{q}} = \frac{1}{2} \alpha q^2 \frac{1}{\sin^2 \hat{q}} \; .$$

It follows therefrom:

$$\hat{p} = \frac{1}{2} \alpha q^2 \frac{1}{\sin^2 \hat{q}} = \frac{1}{2} \alpha q^2 \left(1 + \cot^2 \hat{q}\right)$$

$$= \frac{1}{2} \alpha q^2 \left(1 + \frac{p^2}{\alpha^2 q^2}\right)$$

or rather:

$$\sin^2\hat{q} = \frac{1}{2}\alpha q^2 \frac{1}{\hat{p}} = \frac{\alpha q^2}{\alpha q^2 + \dfrac{p^2}{\alpha}} = \frac{q^2}{q^2 + \dfrac{p^2}{\alpha^2}}$$

$$\Longrightarrow \quad \hat{q} = \hat{q}(p,q) = \arcsin\frac{q}{\sqrt{q^2 + \dfrac{p^2}{\alpha^2}}} \, .$$

The transformation is exactly the same as that from part 1.

Solution 2.5.6

1.

$$p = \frac{\partial F_1}{\partial q} = \sqrt{m\,k}\frac{\bar{q}}{q^2} \, ,$$

$$\bar{p} = -\frac{\partial F_1}{\partial \bar{q}} = \frac{\sqrt{m\,k}}{q} \quad \Longrightarrow \quad q = \frac{\sqrt{m\,k}}{\bar{p}} \, ,$$

$$p = \sqrt{m\,k}\;\bar{q}\,\frac{\bar{p}^2}{m\,k} = \frac{1}{\sqrt{m\,k}}\bar{q}\,\bar{p}^2 \, .$$

2. Because of $\partial F_1/\partial t = 0$ we have:

$$\overline{H}\,(\bar{q},\bar{p}) = H\,(q\,(\bar{q},\bar{p})\,, p\,(\bar{q},\bar{p})) = \frac{1}{2m}\,\frac{1}{m\,k}\bar{q}^2\bar{p}^4\,\frac{(m\,k)^2}{\bar{p}^4} + \frac{1}{2}k\frac{\bar{p}^2}{m\,k}$$

$$\Longrightarrow \; \overline{H}\,(\bar{q},\bar{p}) = \frac{\bar{p}^2}{2m} + \frac{1}{2}m\,\omega^2\bar{q}^2 \, , \qquad \omega^2 = \frac{k}{m} \, .$$

3. $\overline{H}$ is according to part 2. the Hamilton function of the harmonic oscillator. The solution is thus known.

Solution 2.5.7

1. Transformation formulas:

$$p = \frac{\partial F_2}{\partial q} = 2\alpha q \hat{p}^3$$

$$\hat{q} = \frac{\partial F_2}{\partial \hat{p}} = 3\alpha q^2 \hat{p}^2$$

$$\Longrightarrow \quad \hat{p} = \left(\frac{p}{2\alpha q}\right)^{1/3}$$

$$\hat{q} = 3\left(\frac{1}{4}\alpha q^4 p^2\right)^{1/3} \, .$$

2. The fundamental Poisson brackets are to be checked. The brackets

$$\{\hat{q},\hat{q}\}_{q,p} = 0\ , \quad \{\hat{p},\hat{p}\}_{q,p} = 0$$

are trivially fulfilled. It remains to show:

$$\{\hat{q},\hat{p}\}_{q,p} = 1\ .$$

With

$$\begin{aligned}\frac{\partial\hat{q}}{\partial q} &= 4\left(\frac{1}{4}\alpha p^2\right)^{1/3} q^{1/3} & \frac{\partial\hat{q}}{\partial p} &= 2\left(\frac{1}{4}\alpha q^4\right)^{1/3} p^{-1/3}\\ \frac{\partial\hat{p}}{\partial q} &= -\frac{1}{3}\left(\frac{p}{2\alpha}\right)^{1/3} q^{-4/3} & \frac{\partial\hat{p}}{\partial p} &= \frac{1}{3}\left(\frac{1}{2\alpha q}\right)^{1/3} p^{-2/3}\end{aligned}$$

it follows

$$\begin{aligned}\{\hat{q},\hat{p}\}_{q,p} &= \frac{\partial\hat{q}}{\partial q}\frac{\partial\hat{p}}{\partial p} - \frac{\partial\hat{q}}{\partial p}\frac{\partial\hat{p}}{\partial q} = 4\left(\frac{\alpha p^2}{4}\right)^{1/3} q^{1/3}\frac{1}{3}\left(\frac{1}{2\alpha q}\right)^{1/3} p^{-2/3}\\ &+2\left(\frac{\alpha q^4}{4}\right)^{1/3} p^{-1/3}\frac{1}{3}\left(\frac{p}{2\alpha}\right)^{1/3} q^{-4/3} = \frac{2}{3} + \frac{1}{3} = 1\end{aligned}$$

and therewith the canonicity of the transformation.

3. The new Hamilton function results from *the old one* by:

$$\widehat{H}(\hat{q},\hat{p}) = H\left(q(\hat{q},\hat{p}),p(\hat{q},\hat{p})\right) + \frac{\partial F_2\left(q(\hat{q},\hat{p}),\hat{p},t\right)}{\partial t}\ .$$

With

$$\frac{\partial F_2}{\partial t} = 0$$

and

$$\hat{q}\hat{p} = 3\left(\frac{1}{4}\alpha q^4 p^2\right)^{1/3}\left(\frac{p}{2\alpha q}\right)^{1/3} = \frac{3}{2}qp$$

it follows:

$$\widehat{H}(\hat{q},\hat{p}) = \frac{3}{2}\beta q(\hat{q},\hat{p})\,p(\hat{q},\hat{p}) = \beta\hat{q}\hat{p}\ .$$

4. Therewith the new equations of motion read:

$$\dot{\hat{q}} = \frac{\partial\hat{H}}{\partial\hat{p}} = \beta\hat{q} \qquad \dot{\hat{p}} = -\frac{\partial\hat{H}}{\partial\hat{q}} = -\beta\hat{p}\ .$$

Solution 2.5.8

$$\frac{\partial \bar{q}}{\partial q}\frac{\partial \bar{p}}{\partial p} = \alpha q^{\alpha-1}\cos(\beta p)\,\beta q^{\alpha}\cos(\beta p) = \alpha\beta q^{2\alpha-1}\cos^2(\beta p)\ ,$$
$$\frac{\partial \bar{q}}{\partial p}\frac{\partial \bar{p}}{\partial q} = -\beta q^{\alpha}\sin(\beta p)\,\alpha q^{\alpha-1}\sin(\beta p) = -\alpha\beta q^{2\alpha-1}\sin^2(\beta p)\ ,$$

It follows:

$$\{\bar{q},\bar{p}\} = \alpha\beta q^{2\alpha-1} \overset{!}{=} 1\ .$$

It is about a canonical transformation only for $\alpha = 1/2$ and $\beta = 2$.

Solution 2.5.9

1. Since the fundamental Poisson brackets $\{\bar{q},\bar{q}\} = 0$ and $\{\bar{p},\bar{p}\} = 0$ are trivially fulfilled it remains to investigate:

$$\begin{aligned}\{\bar{q},\bar{p}\} &= \left\{q^k p^l, q^m p^n\right\} = q^k\left\{p^l, q^m\right\}p^n + q^m\left\{q^k, p^n\right\}p^l \\ &= q^k\left(0 - \frac{\partial p^l}{\partial p}\frac{\partial q^m}{\partial q}\right)p^n + q^m\left(\frac{\partial q^k}{\partial q}\frac{\partial p^n}{\partial p} + 0\right)p^l \\ &= q^k\left(-lmp^{l-1}q^{m-1}\right)p^n + q^m\left(knq^{k-1}p^{n-1}\right)p^l \\ &= (kn - lm)q^{k+m-1}p^{l+n-1} \\ &\overset{!}{=} 1\ .\end{aligned}$$

   Thus we have to require:

$$k + m = 1\ ;\quad l + n = 1\ ;\quad kn - lm = 1\ .$$

   Hence for any arbitrary m the transformation is canonical if it holds:

$$k = 1 - m\ ;\quad l = -m\ ;\quad n = 1 + m$$
$$\bar{q} = q^{1-m}p^{-m}\ ;\quad \bar{p} = q^m p^{1+m}\ .$$

2. For $m = 0$ it is $\bar{q} = q$ and $\bar{p} = p$. It is then the identity transformation!
3. For the generating function $F_1 = F_1(q,\bar{q})$ it must be valid according to (2.151):

$$p = p(q,\bar{q}) = \frac{\partial F_1}{\partial q}\ ;\quad \bar{p} = \bar{p}(q,\bar{q}) = -\frac{\partial F_1}{\partial \bar{q}}\ .$$

From part 1. one takes:

$$p(q,\bar{q}) = \bar{q}^{-\frac{1}{m}} q^{\frac{1-m}{m}} = \frac{1}{q}\left(\frac{q}{\bar{q}}\right)^{\frac{1}{m}} ; \quad \bar{p}(q,\bar{q}) = q^m \left(\frac{q}{\bar{q}}\right)^{\frac{1+m}{m}} \frac{1}{q^{1+m}} = \frac{1}{\bar{q}}\left(\frac{q}{\bar{q}}\right)^{\frac{1}{m}}$$

That leads to:

$$F_1(q,\bar{q}) = m\left(\frac{q}{\bar{q}}\right)^{\frac{1}{m}} + f(\bar{q}) \curvearrowright \frac{\partial F_1}{\partial \bar{q}} = -\frac{1}{\bar{q}}\left(\frac{q}{\bar{q}}\right)^{\frac{1}{m}} + \frac{df}{d\bar{q}} .$$

The comparison to $\bar{p}$ shows that f can be only an unimportant constant. So the required generating function is:

$$F_1(q,\bar{q}) = m\left(\frac{q}{\bar{q}}\right)^{\frac{1}{m}} .$$

4. Similar considerations as in part 3. are now performed for a generating function of the type $F_2(q,\bar{p})$ with (2.161):

$$p = p(q,\bar{p}) = \frac{\partial F_2}{\partial q} ; \qquad \bar{q} = \bar{q}(q,\bar{p}) = \frac{\partial F_2}{\partial \bar{p}} .$$

It follows with part 1.:

$$p(q,\bar{p}) = \bar{p}^{\frac{1}{1+m}} q^{-\frac{m}{1+m}} ; \quad \bar{q}(q,\bar{p}) = q^{1-m}\bar{p}^{-\frac{m}{1+m}} q^{\frac{m^2}{1+m}} = q^{\frac{1}{1+m}}\bar{p}^{-\frac{m}{1+m}} .$$

From the first equation we get:

$$F_2(q,\bar{p}) = (1+m)\,\bar{p}^{\frac{1}{1+m}} q^{\frac{1}{1+m}} + g(\bar{p}) \curvearrowright \frac{\partial F_2}{\partial \bar{p}} = q^{\frac{1}{1+m}}\bar{p}^{-\frac{m}{1+m}} + \frac{dg}{d\bar{p}}$$

The comparison to the second equation reveals that $g(\bar{p})$, too, can be only an unimportant constant. Thus it remains:

$$F_2(q,\bar{p}) = (1+m)\ (\bar{p}\,q)^{\frac{1}{1+m}}$$

Solution 2.5.10

1. With

$$\operatorname{div}\mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} = 0$$

A is subject to the Coulomb-gauge. Furthermore **A** yields the correct magnetic induction:

$$\operatorname{rot}\mathbf{A} = \frac{1}{2}B \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & x & 0 \end{vmatrix} = \frac{1}{2}B\left(\mathbf{e}_z + \mathbf{e}_z\right) = (0,0,B)\ .$$

2. Generally the Lagrangian for a particle with the charge $\hat{q}$ in the electromagnetic field with the electrical potential $\varphi(\mathbf{q})$ and the vector potential $\mathbf{A}(\mathbf{q})$ reads according to (1.78):

$$L\left(\mathbf{q},\dot{\mathbf{q}}\right) = \tfrac{m}{2}\dot{\mathbf{q}}^2 + \hat{q}\left(\dot{\mathbf{q}}\cdot\mathbf{A}(\mathbf{q}) - \varphi(\mathbf{q})\right)\ .$$

Therefore, the generalized momenta

$$p_j = \frac{\partial L}{\partial \dot{q}_j} = m\dot{q}_j + \hat{q}A_j(\mathbf{q})$$

are different from the components of the mechanical momentum:

$$\mathbf{p}_{\text{mech}} = m\dot{\mathbf{q}} = \mathbf{p} - \hat{q}\mathbf{A}(\mathbf{q})\ .$$

With

$$\dot{\mathbf{q}} = \frac{\mathbf{p}_{\text{mech}}}{m} = \frac{1}{m}\left(\mathbf{p} - \hat{q}\mathbf{A}(\mathbf{q})\right)$$

one gets as Hamilton function in the general case:

$$\begin{aligned} H\left(\mathbf{p},\mathbf{q}\right) &= \mathbf{p}\,\dot{\mathbf{q}}(\mathbf{p},\mathbf{q}) - L\left(\mathbf{q},\dot{\mathbf{q}}(\mathbf{p},\mathbf{q})\right) \\ &= \mathbf{p}\,\dot{\mathbf{q}} - \frac{m}{2}\dot{\mathbf{q}}^2 - \hat{q}\mathbf{A}\cdot\dot{\mathbf{q}} + \hat{q}\varphi \\ &= \frac{1}{m}\mathbf{p}\left(\mathbf{p} - \hat{q}\mathbf{A}\right) - \frac{1}{2m}\left(\mathbf{p} - \hat{q}\mathbf{A}\right)^2 - \frac{1}{m}\hat{q}\mathbf{A}\cdot\left(\mathbf{p} - \hat{q}\mathbf{A}\right) + \hat{q}\varphi \\ &= \frac{1}{2m}\left(\mathbf{p} - \hat{q}\mathbf{A}(\mathbf{q})\right)^2 + \hat{q}\varphi(\mathbf{q})\ . \end{aligned}$$

Here it holds in particular:

$$\begin{aligned} \hat{q} &= -e \\ \varphi(\mathbf{q}) &\equiv 0 \\ \mathbf{A}(\mathbf{q}) &= (-q_2, q_1, 0) \end{aligned}$$

Hamilton function:

$$\begin{aligned} H(\mathbf{q},\mathbf{p}) &= \frac{1}{2m}\left(\mathbf{p}+e\,\mathbf{A}\right)^2 \\ &= \frac{1}{2m}\left[\left(p_1-\frac{1}{2}eBq_2\right)^2+\left(p_2+\frac{1}{2}eBq_1\right)^2+p_3^2\right] \\ &= \frac{p_3^2}{2m}+\frac{1}{2m}\left[\left(p_1-\frac{1}{2}m\omega_c q_2\right)^2+\left(p_2+\frac{1}{2}m\omega_c q_1\right)^2\right] \\ &= \frac{p_3^2}{2m}+H_0\ . \end{aligned}$$

3. In general it holds:

$$p_j=\frac{\partial F_1}{\partial q_j} \qquad \text{and} \qquad \hat{p}_j=-\frac{\partial F_1}{\partial \hat{q}_j}\ .$$

Here we have:

$$\begin{aligned} p_1 &= \frac{\partial F_1}{\partial q_1}=m\omega_c\left(\hat{q}_1-\frac{1}{2}q_2\right) \\ p_2 &= \frac{\partial F_1}{\partial q_2}=m\omega_c\left(\hat{q}_2-\frac{1}{2}q_1\right) \\ \hat{p}_1 &= -\frac{\partial F_1}{\partial \hat{q}_1}=-m\omega_c\left(q_1-\hat{q}_2\right) \\ \hat{p}_2 &= -\frac{\partial F_1}{\partial \hat{q}_2}=-m\omega_c\left(q_2-\hat{q}_1\right)\ . \end{aligned}$$

That leads to the first set of transformation formulas:

$$\begin{aligned} \hat{q}_1(\mathbf{q},\mathbf{p}) &= \frac{1}{m\omega_c}p_1+\frac{1}{2}q_2 \\ \hat{q}_2(\mathbf{q},\mathbf{p}) &= \frac{1}{m\omega_c}p_2+\frac{1}{2}q_1 \\ \hat{p}_1(\mathbf{q},\mathbf{p}) &= p_2-\frac{1}{2}m\omega_c q_1 \\ \hat{p}_2(\mathbf{q},\mathbf{p}) &= p_1-\frac{1}{2}m\omega_c q_2\ . \end{aligned}$$

The reversal yields:

$$\begin{aligned} q_1(\hat{\mathbf{q}},\hat{\mathbf{p}}) &= \hat{q}_2 - \frac{1}{m\omega_c}\hat{p}_1 \\ q_2(\hat{\mathbf{q}},\hat{\mathbf{p}}) &= \hat{q}_1 - \frac{1}{m\omega_c}\hat{p}_2 \\ p_1(\hat{\mathbf{q}},\hat{\mathbf{p}}) &= \frac{1}{2}m\omega_c\hat{q}_1 + \frac{1}{2}\hat{p}_2 \\ p_2(\hat{\mathbf{q}},\hat{\mathbf{p}}) &= \frac{1}{2}m\omega_c\hat{q}_2 + \frac{1}{2}\hat{p}_1 \ . \end{aligned}$$

4. From the above transformation formulas it results:

$$p_1 - \frac{1}{2}m\omega_c q_2 = \hat{p}_2$$

and

$$p_2 + \frac{1}{2}m\omega_c q_1 = m\omega_c\hat{q}_2 \ .$$

Therewith we get the transformed Hamilton function

$$\begin{aligned} \hat{H}_0(\hat{\mathbf{p}},\hat{\mathbf{q}}) &= H_0\left(\mathbf{q}(\hat{\mathbf{p}},\hat{\mathbf{q}}), \mathbf{p}(\hat{\mathbf{p}},\hat{\mathbf{q}})\right) \\ &= \frac{1}{2m}\left(\hat{p}_2^2 + m^2\omega_c^2\hat{q}_2^2\right) \\ &= \frac{\hat{p}_2^2}{2m} + \frac{1}{2}m\omega_c^2\hat{q}_2^2 \end{aligned}$$

which turns out to be formally identical to that of the harmonic oscillator whose equations of motion are known to us:

$$\hat{q}_2(t) \ , \quad \hat{p}_2(t) \ .$$

Furthermore, $\hat{q}_1$ and $\hat{p}_1$ are cyclic and therewith both constants of motion:

$$\begin{aligned} \dot{\hat{q}}_1 &= \frac{\partial H_0}{\partial \hat{p}_1} = 0 \quad \Longrightarrow \quad \hat{q}_1 = \text{const} \\ \dot{\hat{p}}_1 &= -\frac{\partial H_0}{\partial \hat{q}_1} = 0 \quad \Longrightarrow \quad \hat{p}_1 = \text{const} \ . \end{aligned}$$

By means of the transformation formulas from part 3. one then finds the equations of motion for the *old* variables with the corresponding initial conditions.

5. We have:

$$p_j = \frac{\partial F_2(\mathbf{q}, \hat{\mathbf{p}})}{\partial q_j} \qquad \hat{q}_j = \frac{\partial F_2(\mathbf{q}, \hat{\mathbf{p}})}{\partial \hat{p}_j} \ .$$

By means of the transformation formulas from part 3. the p_j and $\hat{q}_j$ can be expressed as functions of the q_j and $\hat{p}_j$:

$$\begin{aligned}
p_1 &= \frac{1}{2} m\omega_c q_2 + \hat{p}_2 \\
p_2 &= \frac{1}{2} m\omega_c q_1 + \hat{p}_1 \\
\hat{q}_1 &= \frac{1}{m\omega_c} \hat{p}_2 + q_2 \\
\hat{q}_2 &= \frac{1}{m\omega_c} \hat{p}_1 + q_1 \ .
\end{aligned}$$

By integration and differentiation one obtains:

$$\begin{aligned}
& p_1 = \frac{\partial F_2}{\partial q_1} = \frac{1}{2} m\omega_c q_2 + \hat{p}_2 \\
\Longrightarrow \quad & F_2 = \frac{1}{2} m\omega_c q_2 q_1 + \hat{p}_2 q_1 + f(q_2, \hat{p}_1, \hat{p}_2) \\
& p_2 = \frac{\partial F_2}{\partial q_2} = \frac{1}{2} m\omega_c q_1 + \frac{\partial f}{\partial q_2} = \frac{1}{2} m\omega_c q_1 + \hat{p}_1 \\
\Longrightarrow \quad & F_2 = \frac{1}{2} m\omega_c q_2 q_1 + \hat{p}_2 q_1 + \hat{p}_1 q_2 + g(\hat{p}_1, \hat{p}_2) \\
& \hat{q}_1 = \frac{\partial F_2}{\partial \hat{p}_1} = q_2 + \frac{\partial g}{\partial \hat{p}_1} = \frac{1}{m\omega_c} \hat{p}_2 + q_2 \\
\Longrightarrow \quad & F_2 = \frac{1}{2} m\omega_c q_2 q_1 + \hat{p}_2 q_1 + \hat{p}_1 q_2 + \frac{1}{m\omega_c} \hat{p}_1 \hat{p}_2 + h(\hat{p}_2) \\
& \hat{q}_2 = \frac{\partial F_2}{\partial \hat{p}_2} = q_1 + \frac{1}{m\omega_c} \hat{p}_1 + \frac{dh}{d\hat{p}_2} = \frac{1}{m\omega_c} \hat{p}_1 + q_1 \\
\Longrightarrow \quad & F_2 = \frac{1}{2} m\omega_c q_2 q_1 + \hat{p}_2 q_1 + \hat{p}_1 q_2 + \frac{1}{m\omega_c} \hat{p}_1 \hat{p}_2 + \text{const} \ .
\end{aligned}$$

Therewith F_2 is determined except for an arbitrary constant:

$$F_2 = \frac{1}{2} m\omega_c q_2 q_1 + \hat{p}_2 q_1 + \hat{p}_1 q_2 + \frac{1}{m\omega_c} \hat{p}_1 \hat{p}_2 \ .$$

Section 3.7

Solution 3.7.1

$$H = \frac{1}{2m}\left(p_x^2 + p_y^2 + p_z^2\right) \implies \frac{\partial H}{\partial t} = 0\,; \qquad H = E\,.$$

Therewith the HJD reads:

$$\frac{1}{2m}\left[\left(\frac{\partial W}{\partial x}\right)^2 + \left(\frac{\partial W}{\partial y}\right)^2 + \left(\frac{\partial W}{\partial z}\right)^2\right] = E\,.$$

Since x, y, z are cyclic the HJD is trivially separable:

$$W = \alpha_x\, x + \alpha_y\, y + \alpha_z\, z\,; \qquad (\boldsymbol{\alpha} = \mathbf{p} = \bar{\mathbf{p}})\ .$$

W is thus just the identity transformation.

Solution 3.7.2

$$H = \frac{p^2}{2m} - bx \implies \frac{\partial H}{\partial t} = 0\,; \qquad H = E\,.$$

Therewith it follows the HJD:

$$\frac{1}{2m}\left(\frac{\partial W}{\partial x}\right)^2 - bx = E \implies \frac{dW}{dx} = \pm\sqrt{2m(E + bx)}\,.$$

Except for the trivial additive constant we then have:

$$W(x) = \pm\frac{1}{3mb}\,[2m(E + bx)]^{3/2}\ .$$

We take $E = \alpha$ and obtain then from (3.67):

$$t + \beta = \frac{\partial W}{\partial \alpha} = \pm\frac{1}{b}\,[2m(\alpha + bx)]^{1/2}$$

$$\implies x(t) = \frac{b}{2m}(t + \beta)^2 - \frac{\alpha}{b}\,.$$

With the initial conditions the solution is:

$$x(t) = \frac{b}{2m}\left(t + \frac{m\,v_0}{b}\right)^2 - \frac{1}{2}\,\frac{m}{b}v_0^2 + x_0\,.$$

Solution 3.7.3 Hamilton function:

$$H = \frac{p^2}{2m} + c\,\mathrm{e}^{\gamma q}\,.$$

Because of

$$\frac{\partial H}{\partial t} = 0$$

we get

$$H = E = \text{const}\,.$$

The transformation

$$(q,p) \implies (\hat{q},\hat{p}) \qquad H \implies \widehat{H}$$

may be so that $\hat{q}$ is cyclic. Generating function:

$$W = F_2(q,\hat{p}) = W(q,\hat{p})\;.$$

Because of

$$\frac{\partial W}{\partial t} = 0 \qquad \text{follows} \qquad \widehat{H} = H = \widehat{H}(\hat{p})\;.$$

Since $\hat{q}$ is cyclic the new momentum is $\hat{p} = \alpha = \text{const}$.
Hamilton-Jacobi differential equation:

$$\frac{1}{2m}\left(\frac{\partial W}{\partial q}\right)^2 + c\,\mathrm{e}^{\gamma q} = E = E(\alpha)\;.$$

Solving for W leads to:

$$W(q,\hat{p}) = \sqrt{2m}\int dq\sqrt{E - c\,\mathrm{e}^{\gamma q}}\;.$$

Choose

$$E = E(\alpha) = \alpha \;\curvearrowright\; \frac{\partial E}{\partial \alpha} = 1\;.$$

That yields for the *new* coordinate the trivial result:

$$\hat{q}(t) = t + \beta \qquad \beta = \text{const}\;.$$

Otherwise it must also be valid:

$$\hat{q} = \frac{\partial W}{\partial \alpha} = \frac{1}{2}\sqrt{2m}\int dq\,\frac{1}{\sqrt{\alpha - c\,\mathrm{e}^{\gamma q}}}\ .$$

Substitution:

$$x = \sqrt{c}\,\mathrm{e}^{\frac{1}{2}\gamma q} \curvearrowright \frac{dx}{dq} = \frac{1}{2}\gamma\,x \curvearrowright dq = \frac{2}{\gamma}\frac{dx}{x}\ .$$

Therewith we calculate:

$$\begin{aligned}
\hat{q} &= \frac{\sqrt{2m}}{\gamma}\int \frac{1}{\sqrt{\alpha - x^2}}\,\frac{1}{x}\,dx \\
&= \frac{\sqrt{2m}}{\gamma}\frac{1}{\sqrt{\alpha}}\ \ln\left(\frac{x}{\sqrt{\alpha}+\sqrt{\alpha - x^2}}\right) \\
&= -\frac{\sqrt{2m}}{\gamma}\frac{1}{\sqrt{\alpha}}\ \ln\left(\sqrt{\frac{\alpha}{x^2}} + \sqrt{\frac{\alpha}{x^2}-1}\right) \\
&= -\frac{\sqrt{2m}}{\gamma}\frac{1}{\sqrt{\alpha}}\,\mathrm{arccosh}\left(\frac{\sqrt{\alpha}}{x}\right)\ .
\end{aligned}$$

That can be solved for x:

$$\begin{aligned}
\frac{\sqrt{\alpha}}{x} &= \cosh\left(-\gamma\sqrt{\frac{\alpha}{2m}}\,(t+\beta)\right) = \sqrt{\frac{\alpha}{c}}\,\mathrm{e}^{-\frac{1}{2}\gamma q} \\
\curvearrowright\ \mathrm{e}^{\frac{1}{2}\gamma q} &= \sqrt{\frac{\alpha}{c}}\,\frac{1}{\cosh\left(\gamma\sqrt{\frac{\alpha}{2m}}\,(t+\beta)\right)}
\end{aligned}$$

($\cosh x = \cosh(-x)$). The generalized coordinate is therewith already determined:

$$q(t) = \frac{2}{\gamma}\ \ln\left\{\sqrt{\frac{\alpha}{c}}\,\frac{1}{\cosh\left(\gamma\sqrt{\frac{\alpha}{2m}}\,(t+\beta)\right)}\right\}\ .$$

The generalized momentum is derivable from (see above):

$$\begin{aligned}
p^2 = 2m\,(\alpha - c\,\mathrm{e}^{\gamma q}) &= 2m\alpha - 2mc\frac{\alpha}{c}\,\frac{1}{\cosh^2\left(\gamma\sqrt{\frac{\alpha}{2m}}\,(t+\beta)\right)} \\
&= 2m\alpha\left(1 - \frac{1}{\cosh^2\left(\gamma\sqrt{\frac{\alpha}{2m}}\,(t+\beta)\right)}\right) \\
&= 2m\alpha\tanh^2\left(\gamma\sqrt{\frac{\alpha}{2m}}\,(t+\beta)\right)\ .
\end{aligned}$$

Therewith we have the result:

$$p(t) = \sqrt{2m\alpha}\tanh\left(\gamma\sqrt{\frac{\alpha}{2m}}\,(t+\beta)\right) .$$

The solution is now complete. α and β are due to initial conditions.

Solution 3.7.4 Hamilton function:

$$H = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + c(x-y) \;, \qquad \frac{\partial H}{\partial t} = 0 \;.$$

The generating function $W(x, y, \hat{p}_x, \hat{p}_y)$ for the transformation

$$\left(x, y, p_x, p_y\right) \xrightarrow{W} \left(\hat{x}, \hat{y}, \hat{p}_x, \hat{p}_y\right)$$

shall be of such a kind that the new coordinates are all cyclic. The generating function W is of the type F_2:

$$p_x = \frac{\partial W}{\partial x} \;, \quad p_y = \frac{\partial W}{\partial y} \;, \quad \hat{x} = \frac{\partial W}{\partial \hat{p}_x} \;, \quad \hat{y} = \frac{\partial W}{\partial \hat{p}_y} \;.$$

The HJD is then:

$$\frac{1}{2m}\left(\left(\frac{\partial W}{\partial x}\right)^2 + \left(\frac{\partial W}{\partial y}\right)^2\right) + c(x-y) = E \;.$$

For the solution a separation approach (ansatz) appears recommendable:

$$W(x, y, \hat{p}_x, \hat{p}_y) = W_x(x, \hat{p}_x, \hat{p}_y) + W_y(y, \hat{p}_x, \hat{p}_y) \;.$$

Then the HJD reads:

$$\underbrace{\frac{1}{2m}\left(\frac{dW_x}{dx}\right)^2 + cx}_{\text{only } x\text{-dependent}} = \underbrace{E - \frac{1}{2m}\left(\frac{dW_y}{dy}\right)^2 + cy}_{\text{only } y\text{-dependent}}$$

$$\Longrightarrow \quad \frac{1}{2m}\left(\frac{dW_x}{dx}\right)^2 + cx = \alpha_1$$

$$E - \frac{1}{2m}\left(\frac{dW_y}{dy}\right)^2 + cy = \alpha_1$$

$$\Longrightarrow \quad \frac{dW_x}{dx} = \pm\sqrt{2m\,(\alpha_1 - cx)}$$

$$\frac{dW_y}{dy} = \pm\sqrt{2m\,(E - \alpha_1 + cy)}$$

$$\Longrightarrow \quad W_x = \mp\frac{1}{3mc}\,(2m(\alpha_1 - cx))^{3/2}$$

$$W_y = \pm\frac{1}{3mc}\,(2m(E - \alpha_1 + cy))^{3/2} \ .$$

So the total characteristic function is:

$$W = \mp\frac{1}{3mc}(2m)^{3/2}\left\{(\alpha_1 - cx)^{3/2} - (E - \alpha_1 + cy)^{3/2}\right\} \ .$$

We identify the new momenta with the constants:

$$\hat{p}_j = \alpha_j = \text{const}$$

where α_2 still remains undetermined. Therewith we have:

$$p_x = \frac{\partial W}{\partial x} = \pm\sqrt{2m(\alpha_1 - cx)}$$

$$p_y = \frac{\partial W}{\partial y} = \mp\sqrt{2m(E - \alpha_1 + cy)} \ .$$

Choose, for convenience:

$$E = E(\alpha_1, \alpha_2) = \alpha_2 \ .$$

Then we have:

$$\dot{\hat{x}} = \frac{\partial \hat{H}}{\partial \alpha_1} = \frac{\partial E}{\partial \alpha_1} = 0 \quad \Longrightarrow \quad \hat{x} = \beta_1$$

$$\dot{\hat{y}} = \frac{\partial \hat{H}}{\partial \alpha_2} = \frac{\partial E}{\partial \alpha_2} = 1 \quad \Longrightarrow \quad \hat{y} = t + \beta_2 \ .$$

Solving of

$$\beta_1 = \frac{\partial W}{\partial \alpha_1} = \mp\frac{1}{c}\left(\sqrt{2m(\alpha_1 - cx)} + \sqrt{2m(\alpha_2 - \alpha_1 + cy)}\right)$$

$$t + \beta_2 = \frac{\partial W}{\partial \alpha_2} = \pm\frac{1}{c}\sqrt{2m(\alpha_2 - \alpha_1 + cy)}$$

leads to:

$$
\begin{aligned}
y(t) &= \frac{c}{2m}(t+\beta_2)^2 - \frac{\alpha_2 - \alpha_1}{c} \\
x(t) &= -\frac{c}{2m}(t+\beta_1+\beta_2)^2 + \frac{\alpha_1}{c} \\
p_x &= \pm\sqrt{c^2(t+\beta_1+\beta_2)^2} = \pm c(t+\beta_1+\beta_2) \\
p_y &= \mp\sqrt{c^2(t+\beta_2)^2} = \mp c(t+\beta_2)\ .
\end{aligned}
$$

Equations of motion:

$$
\begin{aligned}
\dot{x} &= \frac{\partial H}{\partial p_x} = \frac{p_x}{m} \\
\dot{p}_x &= -\frac{\partial H}{\partial x} = -c \quad \Longrightarrow \quad p_x(t) = -c(t+\beta_1+\beta_2) \\
\dot{y} &= \frac{\partial H}{\partial p_y} = \frac{p_y}{m} \\
\dot{p}_y &= -\frac{\partial H}{\partial y} = c \quad \Longrightarrow \quad p_y(t) = c(t+\beta_2)\ .
\end{aligned}
$$

The initial conditions yield:

$$
\begin{aligned}
p_y(0) = 0 \quad &\Longrightarrow \quad \beta_2 = 0 \\
p_x(0) = mv_{0x} \quad &\Longrightarrow \quad \beta_1 = -\frac{mv_{0x}}{c} \\
y(0) = 0 \quad &\Longrightarrow \quad \alpha_1 = \alpha_2 \\
x(0) = 0 \quad &\Longrightarrow \quad \alpha_1 = \frac{c^2}{2m}\left(-\frac{mv_{0x}}{c}\right)^2 = \frac{1}{2}mv_{0x}^2 = E\ .
\end{aligned}
$$

So we have found the solution:

$$
\begin{aligned}
x(t) &= -\frac{c}{2m}\left(t - \frac{mv_{0x}}{c}\right)^2 + \frac{m}{2c}v_{0x}^2 \\
y(t) &= \frac{c}{2m}t^2\ .
\end{aligned}
$$

Solution 3.7.5

$$
H = \frac{1}{2m}\left(p_x^2 + p_y^2\right) + \frac{1}{2}m\left(\omega_x^2 x^2 + \omega_y^2 y^2\right) \Longrightarrow \frac{\partial H}{\partial t} = 0\ ; \qquad H = E\ .
$$

The HJD reads:

$$\frac{1}{2m}\left[\left(\frac{\partial W}{\partial x}\right)^2 + \left(\frac{\partial W}{\partial y}\right)^2\right] + \frac{1}{2}m\left(\omega_x^2\, x^2 + \omega_y^2\, y^2\right) = E\ .$$

Separation approach (ansatz):

$$W = W(x, y;\, \boldsymbol{\alpha}) = W_x(x;\, \boldsymbol{\alpha}) + W_y(y;\, \boldsymbol{\alpha})\ .$$

This is inserted into the HJD:

$$\frac{1}{2m}\left(\frac{dW_x}{dx}\right)^2 + \frac{1}{2}m\omega_x^2\, x^2 = E - \frac{1}{2m}\left(\frac{dW_y}{dy}\right)^2 - \frac{1}{2}m\omega_y^2\, y^2\ .$$

Both sides separately must already be constant. We take $E = \alpha_1$:

$$\frac{1}{2m}\left(\frac{dW_x}{dx}\right)^2 + \frac{1}{2}m\omega_x^2\, x^2 = \alpha_2 = \text{const}$$

$$\frac{1}{2m}\left(\frac{dW_y}{dy}\right)^2 + \frac{1}{2}m\omega_y^2\, y^2 = \alpha_1 - \alpha_2 = \text{const}$$

$$\Longrightarrow \quad \frac{dW_x}{dx} = m\omega_x\sqrt{\frac{2\alpha_2}{m\omega_x^2} - x^2}\ ,$$

$$\frac{dW_y}{dy} = m\omega_y\sqrt{\frac{2(\alpha_1 - \alpha_2)}{m\omega_y^2} - y^2}\ .$$

For the characteristic function we obtain eventually:

$$\begin{aligned} W(x, y, \boldsymbol{\alpha}) &= \int \frac{dW_x}{dx}dx + \int \frac{dW_y}{dy}dy \\ &= m\omega_x\left[\frac{x}{2}\sqrt{\frac{2\alpha_2}{m\omega_x^2} - x^2} + \frac{\alpha_2}{m\omega_x^2}\arcsin\left(x\sqrt{\frac{m\omega_x^2}{2\alpha_2}}\right)\right] \\ &\quad + m\omega_y\left[\frac{y}{2}\sqrt{\frac{2(\alpha_1 - \alpha_2)}{m\omega_y^2} - y^2} + \frac{\alpha_1 - \alpha_2}{m\omega_y^2}\arcsin\left(y\sqrt{\frac{m\omega_y^2}{2(\alpha_1 - \alpha_2)}}\right)\right]\ . \end{aligned}$$

It holds furtheron:

$$\beta_1 + t = \frac{\partial W}{\partial \alpha_1} = \frac{\partial}{\partial \alpha_1} \int \frac{dW_y}{dy} dy$$

$$= \frac{1}{\omega_y} \int dy \left[\frac{2(\alpha_1 - \alpha_2)}{m\omega_y^2} - y^2 \right]^{-1/2} = \frac{1}{\omega_y} \arcsin \left[y \sqrt{\frac{m\omega_y^2}{2(\alpha_1 - \alpha_2)}} \right]$$

$$\Longrightarrow \; y(t) = \sqrt{\frac{2(\alpha_1 - \alpha_2)}{m\omega_y^2}} \sin \left[\omega_y(\beta_1 + t) \right] ,$$

$$\beta_2 = \frac{\partial W}{\partial \alpha_2} = \frac{\partial}{\partial \alpha_2} \int \frac{dW_x}{dx} dx + \frac{\partial}{\partial \alpha_2} \int \frac{dW_y}{dy} dy$$

$$= \frac{1}{\omega_x} \int dx \left(\frac{2\alpha_2}{m\omega_x^2} - x^2 \right)^{-1/2} - \frac{1}{\omega_y} \int dy \left[\frac{2(\alpha_1 - \alpha_2)}{m\omega_y^2} - y^2 \right]^{-1/2}$$

$$= \frac{1}{\omega_x} \arcsin \left(x \sqrt{\frac{m\omega_x^2}{2\alpha_2}} \right) - \beta_1 - t$$

$$\Longrightarrow \; x(t) = \sqrt{\frac{2\alpha_2}{m\omega_x^2}} \sin \left[\omega_x(\beta_1 + \beta_2 + t) \right] .$$

$\beta_1, \beta_2, \alpha_1, \alpha_2$ are fixed by initial conditions!

Solution 3.7.6 With the generating function $F_3 = F_3(p, \hat{q}, t)$,

$$q = -\frac{\partial F_3}{\partial p} , \qquad \hat{p} = -\frac{\partial F_3}{\partial \hat{q}}$$

the transformation shall be done so that the *'new'* coordinate and the *'new'* momentum are constant:

$$(q, p) \xrightarrow{F_3} (\hat{q} = \alpha = \text{const}, \hat{p} = \beta = \text{const}) .$$

It succeeds by use of the generating function $S(p, \hat{q}, t) = F_3(p, \hat{q}, t)$ by which $\hat{H} \equiv 0$:

$$0 \stackrel{!}{=} \widehat{H}(\hat{q}, \hat{p}, t) = H(p, q, t) + \frac{\partial S}{\partial t} = H\left(p, -\frac{\partial S}{\partial p}, t\right) + \frac{\partial S}{\partial t} \qquad \text{(HJD)} .$$

This differential equation for S reads explicitly:

$$\frac{1}{2m}p^2 + \frac{1}{2}m\omega_0^2\left(\frac{\partial S}{\partial p}\right)^2 + \frac{\partial S}{\partial t} = 0\ .$$

For the solution a separation ansatz is chosen:

$$S(p,\hat{q},t) = W(p,\hat{q}) + V(t,\hat{q})\ .$$

The HJD does not say anything about the $\hat{q}$-dependence of S. However, it must come out $\hat{q} = \beta =$ const which, for instance, can be achieved by equating it to one of the integration constants. The HJD is now:

$$\frac{1}{2m}p^2 + \frac{1}{2}m\omega_0^2\left(\frac{dW}{dp}\right)^2 = -\frac{dV}{dt}$$

The left-hand side depends only on p, the right-hand side only on t. Each side separately must already be constant. We therefore take except for an unimportant additive constant:

$$\beta = -\frac{dV}{dt} \quad \Longrightarrow \quad V(t) = -\beta t$$

Hence it follows for the left-hand side of the W-equation:

$$\left(\frac{dW}{dp}\right)^2 = \frac{2}{m\omega_0^2}\left(\beta - \frac{p^2}{2m}\right) = \frac{1}{m^2\omega_0^2}\left(2m\beta - p^2\right)\ .$$

That leads to the following solution of the HJD:

$$S(\beta,p,t) = \pm\frac{1}{m\omega_0}\int dp\sqrt{2m\beta - p^2} - \beta t\ .$$

Now we take:

$$\alpha = \hat{p} = -\frac{\partial S}{\partial \beta} = t \mp \frac{1}{m\omega_0}\int dp\frac{m}{\sqrt{2m\beta - p^2}} = t \mp \frac{1}{\omega_0}\arcsin\frac{p}{\sqrt{2m\beta}}$$

Therewith the *'old'* momentum is:

$$p = \pm\sqrt{2m\beta}\,\sin\left(\omega_0(t-\alpha)\right)$$

For the *'old'* coordinate we get:

$$q = -\frac{\partial S}{\partial p} = -\frac{\partial W}{\partial p} = \mp \frac{1}{m\omega_0}\sqrt{2m\beta - p^2} =$$
$$= \mp \frac{\sqrt{2m\beta}}{m\omega_0}\cos\left(\omega_0(t-\alpha)\right) \ .$$

With the initial conditions

$$p_0 = p(t=0) = 0 \quad \Longrightarrow \quad \alpha = 0$$
$$q_0 = q(t=0) > 0$$

it follows then:

$$q(t) = \sqrt{\frac{2\beta}{m\omega_0^2}}\cos\left(\omega_0 t\right)$$
$$p(t) = -\sqrt{2m\beta}\sin\left(\omega_0 t\right) \ .$$

In the preceding equations always the lower sign is valid because of $q(t=0) > 0$. With

$$q_0 = \sqrt{\frac{2\beta}{m\omega_0^2}} \quad \Longrightarrow \quad \beta = \frac{1}{2}m\omega_0^2 q_0^2 = E$$

it follows after insertion of q_0:

$$q(t) = q_0 \cos\omega_0 t$$
$$p(t) = -m\omega_0 q_0 \sin\omega_0 t \ .$$

Let us conclude with some remarks on the physical meaning:

$$S(\beta, p, t) = -\frac{\sqrt{2m\beta}}{m\omega_0}\int \sqrt{1-\sin^2\omega_0 t}\, dp - \beta t \ .$$

With

$$dp = -m\omega_0^2\, q_0 \cos\left(\omega_0 t\right) dt$$

it follows then:

$$S(\beta, p, t) = \frac{\sqrt{2m\beta}}{m\omega_0} m\omega_0^2 \sqrt{\frac{2\beta}{m\omega_0^2}} \int \cos^2 \omega_0 t \, dt - \beta t$$
$$= 2\beta \int \cos^2(\omega_0 t) dt - \beta t \, .$$

On the other hand it is

$$L = T - V = \frac{1}{2m} m^2 \omega_0^2 q_0^2 \sin^2 \omega_0 t - \frac{1}{2} m\omega_0^2 q_0^2 \cos^2 \omega_0 t$$
$$= \beta \left(\sin^2 \omega_0 t - \cos^2 \omega_0 t \right)$$
$$= -2\beta \cos^2(\omega_0 t) + \beta$$

So the generating function is just the negative indefinite action functional:

$$S(\beta, p, t) = -\int L dt$$

Solution 3.7.7 From

$$\overline{H} = H = \frac{1}{2m} \left(p_x^2 + p_y^2 + p_z^2 \right) + \frac{1}{2} m \left(\omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2 \right) = \alpha_1$$

it follows by rearranging:

$$\frac{1}{2m} \left(p_x^2 + p_y^2 \right) + \frac{1}{2} m \left(\omega_x^2 x^2 + \omega_y^2 y^2 \right) = \alpha_1 - \frac{1}{2m} p_z^2 - \frac{1}{2} m\omega_z^2 z^2 \, .$$

Separation approach:

$$W = W_x(x, \boldsymbol{\alpha}) + W_y(y, \boldsymbol{\alpha}) + W_z(z, \boldsymbol{\alpha})$$
$$\Longrightarrow \; p_x = \frac{dW_x}{dx} \; ; \quad p_y = \frac{dW_y}{dy} \; ; \quad p_z = \frac{dW_z}{dz} \, .$$

Insertion into the above equation means that the right-hand side depends only on z, while the left-hand side is only a function of x and y. Therefore it must hold:

$$\alpha_1 - \frac{1}{2m} \left(\frac{dW_z}{dz} \right)^2 - \frac{1}{2} m\omega_z^2 z^2 = \text{const} = \alpha_z$$
$$\Longrightarrow \; p_z = \frac{dW_z}{dz} = m\omega_z \sqrt{\frac{2(\alpha_1 - \alpha_z)}{m\omega_z^2} - z^2} \, .$$

Reversal points:

$$z_{\pm} = \pm\sqrt{\frac{2(\alpha_1 - \alpha_z)}{m\omega_z^2}} \; .$$

$$\begin{aligned}
J_z &= \oint p_z dz = 2m\omega_z \int\limits_{z_-}^{z_+} \sqrt{\frac{2(\alpha_1 - \alpha_z)}{m\omega_z^2} - z^2}\; dz \\
&= 2m\omega_z \left[\frac{1}{2} z \sqrt{\frac{2(\alpha_1 - \alpha_z)}{m\omega_z^2} - z^2} + \frac{\alpha_1 - \alpha_z}{m\omega_z^2} \arcsin \frac{z}{\sqrt{\frac{2(\alpha_1 - \alpha_z)}{m\omega_z^2}}} \right] \Bigg|_{z_-}^{z_+} \\
&= 2m\omega_z \frac{\alpha_1 - \alpha_z}{m\omega_z^2} \pi \\
&\Longrightarrow \; J_z = \frac{2\pi}{\omega_z} (\alpha_1 - \alpha_z) \; .
\end{aligned}$$

Furtheron it holds:

$$\frac{1}{2m} \left(\frac{dW_x}{dx} \right)^2 + \frac{1}{2} m\omega_x^2 x^2 = \alpha_z - \frac{1}{2m} \left(\frac{dW_y}{dy} \right)^2 - \frac{1}{2} m\omega_y^2 y^2 \stackrel{!}{=} \alpha_x$$

$$\Longrightarrow \; p_x = \frac{dW_x}{dx} = m\omega_x \sqrt{\frac{2\alpha_x}{m\omega_x^2} - x^2} \; .$$

Reversal points:

$$x_{\pm} = \pm\sqrt{\frac{2\alpha_x}{m\omega_x^2}} \; .$$

This means:

$$J_x = 2m\omega_x \int\limits_{x_-}^{x_+} \sqrt{\frac{2\alpha_x}{m\omega_x^2} - x^2}\, dx \; .$$

The same calculation as the above one yields:

$$J_x = \frac{2\pi}{\omega_x} \alpha_x \; .$$

Eventually we are still left with:

$$\frac{1}{2m}\left(\frac{dW_y}{dy}\right)^2 + \frac{1}{2}m\omega_y^2 = \alpha_z - \alpha_x \ .$$

The same considerations as those above lead now to:

$$J_y = \frac{2\pi}{\omega_y}\left(\alpha_z - \alpha_x\right) \ .$$

Finally it follows:

$$\overline{H} = \alpha_1 = \frac{\omega_z}{2\pi}J_z + \alpha_z = \frac{\omega_z}{2\pi}J_z + \frac{\omega_y}{2\pi}J_y + \alpha_x$$
$$\Longrightarrow \overline{H}(\mathbf{J}) = \frac{1}{2\pi}\left(\omega_x J_x + \omega_y J_y + \omega_z J_z\right) \ .$$

Frequencies:

$$\nu_\alpha = \frac{\partial \overline{H}}{\partial J_\alpha} = \frac{1}{2\pi}\omega_\alpha \ ; \quad \alpha = x, y, z \ .$$

Solution 3.7.8 Degeneracy conditions:

$$\nu_x - \nu_y = 0 \ ; \quad \nu_y - \nu_z = 0 \ .$$

This yields according to (3.159) the generating function:

$$F_2 = \left(\omega_x - \omega_y\right)\overline{J}_1 + \left(\omega_y - \omega_z\right)\overline{J}_2 + \omega_z\overline{J}_3$$
$$\Longrightarrow \bar{\omega}_1 = \frac{\partial F_2}{\partial \overline{J}_1} = \omega_x - \omega_y \ ; \quad \bar{\omega}_2 = \frac{\partial F_2}{\partial \overline{J}_2} = \omega_y - \omega_z \ ; \quad \bar{\omega}_3 = \frac{\partial F_2}{\partial \overline{J}_3} = \omega_z \ .$$

This means:

$$\bar{\nu}_1 = \bar{\nu}_2 = 0 \ ; \qquad \bar{\nu}_3 = \nu_z \ .$$

From F_2 it follows also:

$$J_x = \frac{\partial F_2}{\partial \omega_x} = \overline{J}_1 \ ; \quad J_y = \frac{\partial F_2}{\partial \omega_y} = -\overline{J}_1 + \overline{J}_2 \ ; \quad J_z = \frac{\partial F_2}{\partial \omega_z} = -\overline{J}_2 + \overline{J}_3$$
$$\Longrightarrow J_x + J_y + J_z = \overline{J}_3 = \overline{J} \ .$$

This means:

$$\overline{H} = \frac{\omega}{2\pi}\overline{J} \ .$$

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