Springer Proceedings in Physics 208

Piero Nicolini · Matthias Kaminski
Jonas Mureika · Marcus Bleicher *Editors*

2nd Karl Schwarzschild Meeting on Gravitational Physics

Springer

Springer Proceedings in Physics

Volume 208

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ISSN 0930-8989 ISSN 1867-4941 (electronic)
Springer Proceedings in Physics
ISBN 978-3-319-94255-1 ISBN 978-3-319-94256-8 (eBook)
https://doi.org/10.1007/978-3-319-94256-8

Library of Congress Control Number: 2018951701

This Springer imprint is published by the registered company Springer Nature Switzerland AG
The registered company address is: Gewerbestrasse 11, 6330 Cham, Switzerland

We dedicate this meeting to the memory of Jakob Bekenstein, a true pioneer in fundamental physics and a highly respected colleague, who passed away on August 16, 2015. In addition to this incarnation of the meeting, Dr. Bekenstein also served on the advisory board for the 2013 KSM and was an enthusiastic supporter of our mission, particularly our focus on the next generation of gravitational physicists. His contributions to theoretical physics are groundbreaking and will undoubtedly mark his legacy. Although he is gone, he will never be forgotten.

Frankfurt, Germany
July 2015

Piero Nicolini
Jonas Mureika
Matthias Kaminski
Marcus Bleicher

Preface

The 2015 Karl Schwarzschild Meeting on Gravitational Physics (KSM2015), held at the Frankfurt Institute for Advanced Studies, was a top international event involving the world's leading scientific researchers in the field of black hole physics, general relativity, information theory, and related topics. In 2013, the inaugural event named after Schwarzschild was a very successful meeting, considered by many participants as a benchmark for the international community active in the physics of black holes and their ramifications. This meet had two important goals: We aimed to consolidate the scientific collaborations that emerged after the KSM2013 and also create new knowledge in terms of exchange of ideas, publications, and research projects.

This year marked a milestone in gravitational physics: the 100th anniversary of Schwarzschild's derivation of the famous static, spherically symmetric black hole solution, which was published in February 1916 on *Sitzungsberichte der Königlich-Preussischen Akademie der Wissenschaften*. Black holes have since become a central theme in several areas of physics and over the last decade have gained a solid reception in the field of observational astronomy due to the improved technology of modern radio telescopes. They have also emerged as a fundamental feature of attempts to address the hierarchy problem through the introduction of extra spatial dimensions. In such frameworks, it has become commonplace in high energy physics to suppose that microscopic black holes could be produced in current and future accelerator experiments. This explains why astronomers and theoretical particle physicists have found in black holes a common discussion topic supported by international research networks. More importantly, black holes constitute one of the primary testbeds of the ultimate theory of nature: quantum gravity. This is accompanied by the key role that black holes are playing in the full understanding of fundamental interactions.

The KSM2015 saw the participation of 86 people from 22 countries. This included 65 junior and senior scientists and 21 students. A highlight of the meeting was the Karl Schwarzschild Memorial Lecture given by the 1999 Physics Nobel Laureate Gerard 't Hooft. In addition to this keynote address, 't Hooft also moderated the lively panel discussion, a format that has become a staple of the KSM. In this context, this time, the lively debate between Steve Giddings and Carlo Rovelli was an illuminating presentation of the future prospects of two competing theories—loop quantum gravity and string theory—which was aptly nicknamed “The Duel”.

Once again, the Karl Schwarzschild Prize competition for best student talk and best junior talk resulted in a wonderful display of talent from the rising generation of gravitational physicists. The winner of the best student talk was Antonia Frassino (Goethe University/FIAS) for her presentation “Lovelock Black Hole Thermodynamics”. Honorable mentions in this category were awarded to Andrea Giugno (Bologna), Eugene Kur (UC Berkeley), and Supakchai Ponglertsakul (University of Sheffield). The Schwarzschild Prize for best junior scientist talk was awarded to Francesca Vidotto (Radboud University Nijmegen) for her presentation “Quantum gravity phenomenology with primordial black holes”. Honorable mentions included Mirah Gary (TU Wien), Tigran Kalaydzhyan (Stony Brook University), and Christi Stoica (Horia Hulubei National Institute, Bucharest).

With its focus on top-quality keynote speakers, small participant numbers, and plenary sessions for juniors and students, the Karl Schwarzschild Meeting is not a venue for mere exchange of information. It is the place where new ideas are developed through complementary knowledge and encouraged interactions of the participants, and the future of gravitational physics is born. This volume represents the culmination of our efforts to synthesize these activities. We look forward to carrying on this new tradition for years to come.

Frankfurt, Germany

Piero Nicolini
Jonas Mureika
Matthias Kaminski
Marcus Bleicher

The participants and organizers of the Karl Schwarzschild Meeting 2015 in the FIAS Lecture Hall


Keynote Speakers:
Bernard Carr (QMU London)
Georgi Dvali (LMU/MPI Munich/NYU)
George Ellis (Cape Town)
Valeria Ferrari (Sapienza Rome)
Steve Giddings (UC Santa Barbara)
Daniel Grumiller (TU Vienna)
Achim Kempf (Waterloo/PI/IQC)
Claus Lämmerzahl (ZARM/Bremen)
Robert Mann (Waterloo/PI/IQC)
Frans Pretorius (Princeton)
Luciano Rezzolla* (Frankfurt)
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KSM 2015

Acknowledgements

The organizers of the 2015 Karl Schwarzschild Meeting on Gravitational Physics are grateful for the generosity of the Frankfurt Institute for Advanced Studies for providing the spectacular venue at which the meeting took place. We thank the Deutsche Forschungsgemeinschaft (DFG), the Helmholtz International Center for FAIR, and the Freunde und Förderer der Goethe-Universität Frankfurt for their financial support, as well as the Physikalischer Verein Frankfurt for hosting a visit at the Taunus Observatorium on the Kleiner Feldberg. A special thanks goes to Springer for providing the prizes for Best Student and Best Junior Scientist presentations, the basis for an integral part of our philosophy.

For their help and support of our efforts, we extend a special acknowledgment to our International Advisory Board: Jacob Bekenstein (Hebrew University of Jerusalem), Juan Maldacena (IAS, Princeton), Donald Marolf (University of California, Santa Barbara), Martin Reuter (Johannes Gutenberg University of Mainz), Carlo Rovelli (CPT/Aix-Marseille University), Dam T. Son (University of Chicago).

Finally, we are indebted to all the participants who helped to make the KSM a success, and hopefully the first of many such meetings to come.

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Chapter 1
Singularities, Horizons, Firewalls, and Local Conformal Symmetry

G. 't Hooft

Abstract The Einstein–Hilbert theory of gravity can be rephrased by focusing on local conformal symmetry as an exact, but spontaneously broken symmetry of nature. The conformal component of the metric field is then treated as a dilaton field with only renormalizable interactions. This imposes constraints on the theory, which can also be viewed as demanding regularity of the action as the dilaton field variable tends to 0. In other words, we have constraints on the small distance behaviour. It is not known whether theories can be constructed that obey these constraints; if so, all interaction parameters that are normally freely adjustable, would become computable.

1.1 Introduction

The modern representation of Karl Schwarzschild's spherically symmetric solution of Einstein's equations reads[1]

$$\mathrm{d}s^2 = -\left(1 - \tfrac{2M}{r}\right)\mathrm{d}t^2 + \frac{1}{1-2M/r}\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2). \qquad (1.1)$$

[1]In Schwarzschild's original work [1], the coordinate r in (1.1) was called R, while he chose an other radial coordinate r such that the point $R = 2M$ corresponds to $r = 0$, since it seemed to be obvious to expect a singular mass distribution at the origin of the coordinate frame. Today, we know that this was unnecessary, for two reasons: first, one is free to choose the most convenient coordinate system anyway, and secondly, the surface $r = 2M$ does not represent a physical singularity at all, but just a coordinate singularity, much like the north pole of the Earth. It is the black hole horizon.

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P. Nicolini et al. (eds.), *2nd Karl Schwarzschild Meeting on Gravitational Physics*, Springer Proceedings in Physics 208,
https://doi.org/10.1007/978-3-319-94256-8_1

As we now know very well, matter can enter the black hole through the horizon, defined by the surface $r = 2M$, while in the standard, unquantised theory, nothing can emerge out of it. The horizon is a one way door.[2] In the coordinates of (1.1), the point $r = 0$ is a real physical singularity.

Even though the horizon appears to be a regular region of space-time, we do have a problem with it. According to Hawking's well-known result [2], it is due to vacuum fluctuations that a distant observer will observe particles leaving the black hole: Hawking radiation. These particles appear to have a thermal spectrum, independent of the black hole formation process.

Hawking's original conclusion was that this result must imply that a black hole as a physical object violates the laws of quantum mechanics: even if it originates from matter in a single quantum state, it ends up in a thermal, that is, a quantum mechanically mixed state. How could it be that a derivation that uses quantum mechanics can yield a result violating the laws of this theory? Hawking particles are now understood to be formed at the horizon, not, as was originally thought, somewhere near the $r = 0$ singularity in its past.

According to the present author's understanding of quantum mechanics [3], however, all states in which the Hawking particles fluctuate differently, are different ontological states of the system, and they should be treated as different quantum states as well. Thus, the vacuum state, which is a single quantum state, emerges at the horizon as a collection (superposition) of infinitely many ontological states, and it should be treated as such. One can then understand how particles entering a black hole, can affect these ontological states in spite of the fact that their probabilistic distribution remains unaltered. This effect can actually be calculated [4].

Note added: This paper is the written version of the talk presented by the author at the 2nd Schwarzschild Meeting on Gravitational Physics, on 23 July 2015. Since then, the author's views and insights in the matter of quantized black holes evolved significantly, see Ref. [5].

1.2 Local Conformal Symmetry

It is to be noted that the main features that went into our description of the back reaction to Hawking radiation only requires knowledge of light-like geodesics. These depend on all components of the metric tensor $g_{\mu\nu}(x)$, except for one overall factor. This is because the equation for light-like geodesics,

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = 0, \tag{1.2}$$

[2]On some web pages, these facts are still being disputed, which we can only attribute to ignorance. Schwarzschild, who wrote his paper in less than two months after Einstein's discovery, could be excused for not immediately realising the rather subtle features of black hole horizons, which required several years to be cleared up, but today's experts cannot afford to make such mistakes.

is unaltered by the substitution

$$g_{\mu\nu}(x) \to \Omega^2(x)\, g_{\mu\nu}(x) \tag{1.3}$$

(take into consideration that this equation alone, without higher derivatives, determines the shapes of all light cones). Theories invariant under (1.3) are said to be locally conformally invariant. By adding a dilaton field, as will be explained shortly, even the Einstein–Hilbert action can be made invariant under (1.3), because the entire metric tensor, including its common factor, consists of dynamical variables. $\sqrt{-g}$ is not invariant, but covariant. This implies that flat space time, that is, the vacuum state, breaks the symmetry. Thus we say that conformal symmetry is not explicitly, but spontaneously broken in Enstein–Hilbert gravity, just as local $SU(2) \times U(1)$ gauge symmetry is spontaneously broken by the BEH mechanism.

We write the standard Lagrangian for gravity interacting with matter as

$$\mathscr{L} = \mathscr{L}^{\rm EM} + \mathscr{L}^{\rm matter}; \qquad \mathscr{L}^{\rm EM} = \tfrac{1}{16\pi G}\sqrt{-g}\,(R - 2\Lambda), \tag{1.4}$$

$$\mathscr{L}^{\rm matter} = \mathscr{L}^{YM}(A) + \mathscr{L}^{\rm bos}(A,\, \phi,\, g_{\mu\nu}) + \mathscr{L}^{\rm ferm}(A,\, \psi,\, \phi,\, g_{\mu\nu}), \tag{1.5}$$

where $\phi(x)$ represents the scalar matter fields, and $\psi(x)$ the fermionic ones. A stands for $A_\mu(x)$, the Yang–Mills fields in the matter Lagrangian. Now define

$$g_{\mu\nu} = \omega^2(\mathbf{x}, t)\, \hat{g}_{\mu\nu}\,; \qquad \mathscr{L} = \mathscr{L}(\omega,\, \hat{g}_{\mu\nu},\, A_\mu, \psi, \phi). \tag{1.6}$$

This contains the 'trivial' conformal symmetry

$$\hat{g}_{\mu\nu} \to \Omega^2(\mathbf{x}, t)\hat{g}_{\mu\nu}, \quad \omega \to \Omega^{-1}\omega, \quad A_\mu \to A_\mu, \\ \phi \to \Omega^{-1}\phi, \quad \psi \to \Omega^{-3/2}\psi. \tag{1.7}$$

We shall refer to the field $\omega(\mathbf{x}, t)$ as the *dilaton field*.

Working out the Einstein–Hilbert Lagrangian and the matter Lagrangian a bit more explicitly gives

$$\mathscr{L}^{\rm EM} = \sqrt{-\hat{g}}\left(\tfrac{1}{16\pi G}(\omega^2\, \hat{R} + 6\hat{g}^{\mu\nu}\partial_\mu\omega\partial_\nu\omega) - \tfrac{\Lambda}{8\pi G}\,\omega^4\right)\,; \tag{1.8}$$

$$\mathscr{L}^{\rm matter} = -\tfrac{1}{4}F_{\mu\nu}F_{\mu\nu} + \\ \sqrt{\hat{g}}\left(-\frac{1}{2}\hat{g}^{\mu\nu}D_\mu\phi D_\nu\phi - \frac{1}{2}m^2\omega^2\phi^2 - \tfrac{1}{12}\hat{R}\phi^2 - \tfrac{\lambda}{8}\phi^4\right) + \mathscr{L}^{\rm ferm}. \tag{1.9}$$

Here, we included the $\hat{R}\,\phi^2$ term for restoring conformal invariance of $\mathscr{L}^{\rm matter}$, where $\hat{R}$ is the scalar curvature associated to $\hat{g}_{\mu\nu}$.

Now, several remarks are of order:

- surprisingly perhaps, the Einstein–Hilbert action appears to be an entirely renormalizable Lagrangian for the dilaton field $\omega(x)$.

- With the cosmological term acting as a quartic coupling term, the matter Lagrangian for the scalar field $\phi(x)$ has the same form as $\mathscr{L}^{\mathrm{EM}}$, apart from a factor $-4\pi G/3$.
- This factor can easily be taken care of by rescaling the ω field, but its sign is curious. Since it so happens that the Standard Model neither contains explicit mass terms for the fermions, nor cubic couplings among the saclar fields,[3] we can include a factor i in the redefinition of ω without any obvious violation of unitarity.
- Due to the necessary rescaling of ω, all physical constants (including mass terms and the cosmological term) eventually emerge as dimensionless combinations of Newton's constant G and the Standard Model parameters.

Nevertheless, the theory is non renormalizable. This is because a kinetic term for the $\hat{g}_{\mu\nu}$ field is missing. Normally, theories cease to be renormalizable if a kinetic term is missing. The theory would be ill-defined altogether, but by inspecting the way one would normally handle the Einstein equations in perturbation expansions, one finds the following formal prescription for solving the classical equations:

Find the total energy-momentum-stress tensor $T^{\mathrm{tot}}_{\mu\nu}$ for the matter fields, including the ω field. Note that the original, ω-independent Einstein–Hilbert action disappeared, so that Einstein's equation is to be replaced by one where Newton's constant is infinite. Therefore, the equations are:

$$T^{\mathrm{tot}}_{\mu\nu} \;=\; T^{\mathrm{matter}}_{\mu\nu} - T_{\mu\nu}(\omega) \;=\; 0 \;=\; T^{\mathrm{matter}}_{\mu\nu} - \tfrac{1}{8\pi G}\, G_{\mu\nu}. \tag{1.10}$$

We kept the minus sign in the contribution of the ω field; it disappears when the factor i mentioned above is employed. We recognise Einstein's original equation of course; however, in the conformally symmetric notation, we should say that the condition that the total energy-momentum-stress tensor vanishes is a constraint. It has exactly the right dimension to enforce the equations for the $\hat{g}_{\mu\nu}$ field (the conformal stress-energy momentum tensor is traceless).

If our aim were to restore renormalizability, all we had to do now would be to collect all divergent diagrams and determine their general form. This should provide us with terms to be added to the original bare Lagrangian of the theory, as is usually done. In this case, we find that all divergent expressions unaccounted for, contain external lines for $\hat{g}_{\mu\nu}$ and factors k^4 because they are quartically divergent. Since all calculations should be performed while respecting local conformal invariance, and all diagrams are polynomials in the fields, one expects that the only terms that should be added in the Lagrangian are locally conformally invariant expressions with four derivatives in the metric fields $\hat{g}_{\mu\nu}$. There exists only one such term that is invariant under local conformal transformations, the Weyl action:

$$\mathscr{L}^{\mathrm{kin}} = -\tfrac{\lambda^W}{2}\, C_{\mu\nu\alpha\beta}\, C_{\mu\nu\alpha\beta} \quad\to\quad -\tfrac{\lambda^W}{4}\,(\partial^2 \hat{g}^{\mathrm{transverse}}_{\mu\nu})^2. \tag{1.11}$$

With this term added, the theory indeed becomes renormalizable, as is well-known, but there appear to be two complications: first, the Weyl term would generate prop-

[3] Such terms would come with a factor $i\omega$, and hence appear to violate unitarity.

agators for $\hat{g}_{\mu\nu}$ that are quartic in the momenta. This is not in accordance with standard prescriptions in renormalization theory. Propagators ought to be quadratic in the momenta, in a carefully prescribed way, in order to comply with unitarity, causality, and positivity of the energy. Does this mean that our theory is not unitary, or is its energy not bounded from below? Note that the energy momentum tensor is required to obey (1.10), so that the total energy vanishes strictly, but that was before we added the Weyl action. What is the unitarity/energy condition in the case of conformal invariance?

Secondly, there is an other mystery. When the required renormalization counter term is computed without keeping track of conformal symmetry, one finds [6] that it does not take the form (1.11), since also $\sqrt{-g}\, R^2$ terms appear. Now, if we do use the conformal notation, this would generate $\partial_\mu\omega/\omega$ and $\partial_\mu\omega^2/\omega^2$ terms, which of course cannot come from symmetric diagrams. This anomaly is the well-known conformal anomaly. It actually ruines the renormalizability of gravity with Weyl term added [6]. It is caused by the fact that the system cannot be regularised with scale-invariant regulators (The author thanks M. Duff for a discussion on this point).

The same remains true for the theory when the terms (1.8) and (1.9) are added to the Lagrangian. Together, these anomalies generate the renormalization group β coefficients. In the usual theories, this is not considered to be a flaw of the theory but just an interesting feature. Here, however, we are dealing with a *local* gauge symmetry; the anomalies are a fatal flaw of the principle of *local* conformal invariance that is required for renormalizability. In our present case, therefore, *all conformal anomalies must be demanded to cancel out*. This means that all renormalization group β coefficients must be demanded to vanish. Consequently, all coupling parameters must be adjusted such that they are at a zero of their β functions. This generates at least as many constraints as there are coupling parameters of the theory. We are lead to an exciting speculation: In gravity theories with conformal invariance, *all* physical constants, including the masses and even the cosmological constant, are constrained to values that in principle must be computable.

In short, we propose that conformal symmetry is not just an accident that vaguely applies to some branches of physics, but that it may play a very important role as an absolutely exact transformation rule. It will then be an essential instrument that might lead us towards calculating parameters that otherwise would have been freely adjustable, and a crucial ingredient of the description of black holes, as we will see.

1.3 Black Holes

In a nut shell, the *black hole information problem* is the question how information concerning the state of matter entering the hole, can be seen to be present in the particles coming out, as was mentioned in the Introduction. One way of phrasing this difficulty is the question how to avoid that the information entering the hole disappears into the central singularity, see Fig. (1.1a). A related difficulty arises if one considers the entanglement of particles entering the hole and others that emerge.

The contradictions appear to be strong enough to make some researchers [7] believe that a *firewall* should emerge at the horizon, prohibiting particles to enter (dotted lines in Fig. 1.1a and 1.1b).

Conformal symmetry will be of help here: the central singularity disappears,[4] and the horizon will become "fuzzy", see Fig. (1.1b) and (1.1c). Consider the mass M of the black hole. An observer A falling in passes the horizon, experiencing the metric $g_{\mu\nu}$ associated to the mass M. An outside observer B, however, may observe the Hawking radiation that causes the mass to shrink. While the ingoing observer still hovers over the horizon, seeing a fixed mass M, the outside observer sees the mass shrink to zero. Who is right?

The answer may be *Black hole complementarity* [8]: both observers are right, but they should use the metric $\hat{g}_{\mu\nu}$, and its conformal factor depends on who is looking. Both observers may describe their metric as

$$\mathrm{d}\hat{s}^2 = M^2(\tilde{t})\left(-\mathrm{d}t^2(1-\tfrac{2}{r}) + \frac{\mathrm{d}r^2}{1-2/r} + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2)\right). \quad (1.12)$$

Here, $M(\tilde{t})$ may depend on the retarded time $\tilde{t}$, and be different for the different observers. For the observer A entering the hole, $M(\tilde{t}) = M$ is constant, but for the outside observer B, it goes to zero. It may also depend on the advanced time. Since the black hole has a finite life time, there is, strictly speaking, no horizon.[5]

With 'black hole complementarity', the distant observer B sees matter going in and matter going out, The observer A, going in, sees the locally clear horizon, while she cannot detect the Hawking particles. Observer B sees that the mass M vanishes during the final explosion. For this observer, the horizon produces matter, as if the imploding matter contained some sort of dynamite, causing an explosion at exactly the right moment. This observer sees an almost singular concentration of Ricci scalar curvature[6] at the horizon, see Fig. 1.2. The curvature is strong where the future event horizon meets the past horizon. One could call this a 'firewall', but the firewall is invisible for the ingoing observer A. The observers use different ways to fix the 'conformal gauge'. Hence, they also have different perceptions of the energy-momentum tensor of the matter present. They do both agree what the vacuum expectation value of the ω field should be, but they do not agree about what the vacuum state is. Note that, this disagreement about the vacuum state has always been a standard concept in the derivation of Hawking radiation [2, 4].

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[4]This happens as follows: near the singularity, we can stretch the coordinates so much, that the curvature-squared will no longer be singular, but instead, the singularity moves to the infinite future. In a sense, that is where it belongs anyway. A good exercise is to multiply the metric with an overall factor such as $1/r^4$, to see how this makes the singularity move towards the infinite future, where space-time becomes locally flat.

[5]But keep in mind that, for most of all practical purposes, there still is a horizon, as its 'fuzziness' is almost imperceptible. Only for black holes close to the Planck size, this fussiness becomes important.

[6]Note that we now use the metric $\hat{g}_{\mu\nu}$ throughout; the dilaton ω is just an 'ordinary' renormalizable field, that happens to hover around its vacuum value.

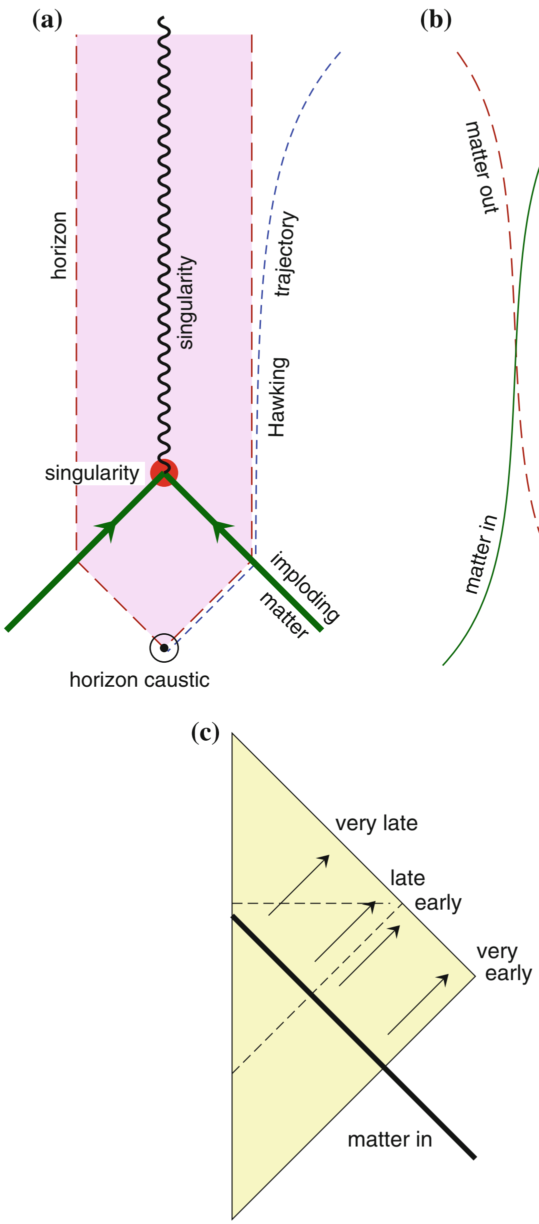

Fig. 1.1 **a** Standard view of space-time of a black hole formed by imploding matter; **b** Conformal distortion of a, such that time reversal symmetry is restored. **c** Penrose diagram of symmetric black hole showing entangled states of Hawking radiation

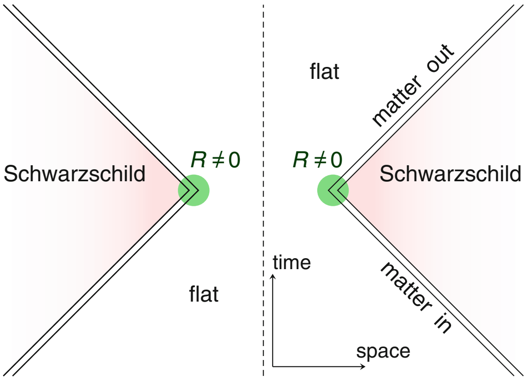

Fig. 1.2 The black hole metric as seen by a distant observer B, see text

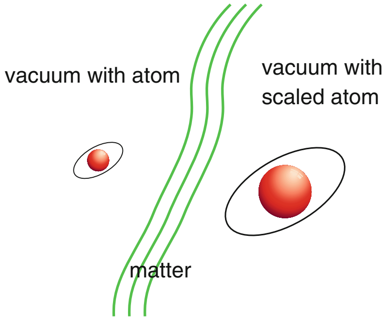

Fig. 1.3 Observer B experiences a firewall, formed by Hawking particles, while A sees no such thing. Therefore, they disagree about sizes of things inside the black hole, see text

The outside observer B sees Hawking particles emerge from a highly concentrated 'curtain' along the past event horizon. If he computes the corresponding metric, he will disagree with observer A, by finding an extra conformal factor. The Hawking particles cause a sharp jump in the gradients of this conformal factor. Consequently, observers A and B disagree about this conformal factor when discussing the interior region of the black hole, see Fig. 1.3.

The comparison between our spontaneously broken local conformal symmetry and the Brout-Englert-Higgs mechanism gives striking similarities. For one, this mechanism improves considerably the convergence features of the theory in the far ultra-violet. If the Weyl term (1.11) would really be allowed then we would indeed have a renormalizable theory of gravity, as is well known. The other similarity concerns the topologically non trivial soliton solutions: before invoking the BEH mechanism, Maxwell's theory cannot allow for a singularity-free description of magnetic monopoles, while in some versions of the BEH models one can have regular monopoles; similarly, with local conformal symmetry, black holes can be made singularity free.

Our demand that all conformal anomalies cancel is a severe constraint on the theory, but this may actually be a welcome feature; it may imply that constants that are not normally computable may now be found to obey (interesting) equations.

A difficulty here is that all couplings may emerge as being large, in which case we cannot perform perturbative calculations. By carefully choosing the algebra, depending on a large integer N, one sometimes can search for solutions with couplings proportional to $1/N$ or $1/N^2$, allowing us to do $1/N$ expansions. A very preliminary search was only partially successful, as it did allow for $1/N$ expansions, but it did not lead to physically interesting solutions.

1.4 Features and Limitations

We do observe that spontaneously broken local conformal symmetry holds the promise that the unknown parameters of the matter Lagrangian, today described by the Standard Model, are not freely adjustable but can be computed. The numbers, however, will depend on the algebra of the matter theory. For sure, the algebra of the Standard Model will need considerable extensions in order for it to be applicable all the way to the Planck scale.

The most fundamental obstacle was found to be the *hierarchy problem*: ratios of physical constants in the real world contain very large or small numbers such as 10^{-122} for the cosmological constant, whereas in principle our models should turn up numbers of order 1 in Planck units. Our universe owes its complexity to the existence of exotic large numbers. One might bring this forward as an objection to our theory but it has to be remembered that the hierarchy problem is the source of headaches for many other theories as well. Barring the "anthropic principle", no theory is known that can account for the complexity of our universe.

An other mystery is the apparent lack of unitarity. This feature was further investigated. If one chooses the coefficient λ^W in the Weyl term (1.11) large, the theory allows for a perturbative analysis. If we add this extra term to the original Einstein Hilbert action (in the old, non conformal notation), one finds that λ^W has the dimensionality of an inverse mass squared,

$$\lambda^W \equiv 1/M^2, \tag{1.13}$$

where M locates poles in the complex momentum plane, and it is small compared to the Planck mass (in a renormalizable theory, a Planck mass large compared to the mass scale M of the theory indicates small gravitational couplings, hence the usefulness of perturbative expansions).

We then look at plane waves of the theory, finding that the wave equations indeed contain new poles. Identifying the quantum numbers of these poles, we found:

- one massless pole, describing the familiar graviton. This was to be expected because the far infrared region should not be affected by the Weyl term, as it contains extra derivatives. The graviton has spin 2, but, being massless, has only two physical helicities, as in the usual theory;

- poles of the form $1/(k^2 + M^2 - i\varepsilon)$. They all turned out to be at the same mass value M. One pole has helicities ± 2, one has helicities ± 1 and one pole has helicity 0. We recognise this as the five helicities of a single, massive spin 2 particle. The problem with these poles is that they all have the wrong overall sign in the propagator. The fact that this wrong sign is inevitable can easily be seen from the far ultraviolet limit. There, only the Weyl term contributes to the graviton propagator. It was inevitable that we have there:

$$1/(k^2 + \lambda^W k^4) = 1/k^2 - 1/(k^2 + M^2). \tag{1.14}$$

- The scalar pole that might be generated by the conformal factor, as usual in perturbative gravity, is a ghost, so that it can be ignored, it is not a physical particle, while the others seem to be real.

Thus we have a single, negative metric, spin 2 companion of the graviton, with 5 possible helicity states. We propose the name "gravitello" for that, the mysterious companion of the graviton. Having such a particle seems to be inevitable.

We do not know how to accommodate for it in a unitary theory, but one could consider the following notions.

Our fields have oscillation modes with opposite signs. The energies can be written as

$$H = |\mathbf{k}|(p_1^2 + x_1^2) - \sqrt{\mathbf{k}^2 + m^2}\,(p_2^2 + x_2^2), \tag{1.15}$$

where $\mathbf{k}$ is the spacelike momentum of the wave, while x_i are the fields and p_i are the canonical momenta of these fields, at wave number $\mathbf{k}$. They obey the usual commutation rules

$$[x_i,\ x_j] = [p_i,\ p_j] = 0, \qquad [x_i,\ p_j] = i\,\delta_{ij}. \tag{1.16}$$

We can write (1.15) as

$$H = A\,a^\dagger a - B\,b^\dagger b + C, \tag{1.17}$$

where a and $a^\dagger$ are the annihilation and creation operators of the graviton, and $b,\ b^\dagger$ those of the gravitello, both having momentum $\mathbf{k}$.

One approach is the following. We could impose a lower bound to the energies of the modes with the wrong sign, by putting a limit on the occupation numbers of the $b,\ b^\dagger$ operators. This requires the interchange $b \leftrightarrow b^\dagger$, which is possible if we can rearrange and renormalise the quantum states reached by these operators. Effectively, this switches the sign of the commutator of b and $b^\dagger$. The associated replacement in the operators x_2 and p_2 implies that they commute as purely imaginary fields would:

$$x_2 \to i x_2, \qquad p \to i p, \tag{1.18}$$

that is, the field of the gravitello should be chosen to be purely imaginary. This is the same operation as was required in the quantisation of ordinary gravitation; there also, the overall conformal factor, which would contribute with the wrong sign to the Einstein Hilbert action, must be replaced by a purely imaginary Lagrange multiplier field.

The procedure described here is related to the author's proposals for the interpretation of quantum mechanics [3]. All harmonic oscillators that we encounter in the physical world, should be associated with processes in an ontological underlying world that are *periodic* in time. When harmonic oscillators interact, they cease to be exactly periodic, and this means that, also in the ontological underlying world, the associated processes are no longer exactly periodic.

To identify the quantum states of the oscillator with the classical states of the ontological world [3], we have to discretise them. This is achieved in 'cogwheel models', which are periodic but have only a finite number of states. The annihilation and creation operators a and $a^\dagger$ are then replaced by the operators L_- and L_+ in a large ℓ representation of the SU(2) rotation algebra of angular momenta. These decrease or increase the quantum number $m = L_3$, which has both a lower and an uper bound: $|m| \le \ell$.

Regarding the gravitello, we should hasten to add that this approach has not yet been elaborated in a satisfactory way, since the gravitello would couple with an imaginary coupling constant to gravitational sources, and we have not succeeded in showing how this can be squared with unitarity.

1.5 Conclusion

Local conformal symmetry shines a new light on problems where gravity couples to matter, both in the domain of the Standard Model and in our understanding of black holes. It is always our intention to make the smallest possible modifications in our models of physics, because pure Einstein–Hilbert gravity, coupling just with the Standard Model particles, appears to agree with observations extremely well, and lessons learned from past experiences suggest that one should not abandon known facts in the natural world too easily.

It so happens that exact local conformal invariance does play a role already in Einstein–Hilbert gravity itself, by isolating the dilaton component of the metric, so the only 'new' thing we add to this is the demand that this symmetry should be exact, rather than having it as an accidental, approximate feature.

It may be noted that our approach is related to the theory of 'asymptotic safety' [9]. In this theory, the UV limit of gravity theory tends to a fixed point. At this fixed point, such models should also enjoy scale invariance, *ergo* local conformal invariance. There, however, one is confronted with strong interactions (since the fixed point values of the coupling parameters are not close to zero). In our models, we offer perturbative accessability by adding explicit interactions (the Weyl term). This could be an advantage, but it would also introduce wrong metric states that we have to handle, somehow. In fact, whether asymptotically safe models contain states with the wrong sigm of the metric and/or the energy, is not known.

Our procedure appears to suggest that all freely adjustable parameters, being both the masses and the coupling parameters of the Standard Model, including eventually the cosmological constant, must be computable. Our point is that, with the dilaton field ω added, the matter component of our particle system should be completely conformally invariant, so that the physical parameters, all starting out as being dimensionless, must be exactly at the fixed point, that is, the point where all β coefficients vanish.

The inclusion of the Weyl interaction, which on the one hand seems to be inevitable, does generate severe problems of negative metric states, or equivalently, negative energy states. This may simply mean that we have not yet fully understood what local conformal symmetry really is [10].

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Chapter 2
Panel Discussion, "The Duel": The Good, the Bad, and the Ugly of Gravity and Information

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Two hundred thousand dollars is a lot of money. We're gonna have to earn it.
— Blondie, "The Good, the Bad and the Ugly", 1966

Abstract Various contenders for a complete theory of quantum gravity are at odds with each other. This is in particular seen in the ways they relate to information and black holes, and how to effectively treat quantization of the background spacetime. Modern perspectives on black hole evaporation suggest that quantum gravity effects in the near-horizon region can perturb the local geometry. The approaches differ,

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P. Nicolini et al. (eds.), *2nd Karl Schwarzschild Meeting on Gravitational Physics*, Springer Proceedings in Physics 208,
https://doi.org/10.1007/978-3-319-94256-8_2

however, in the time scale on which one can expect these effects to become important. This panel session presents three points of view on these problems, and considers the ultimate prospect of observational tests in the near future.

2.1 Introduction

In 1975 Hawking obtained a ground breaking result about the fundamental nature of black holes [1] that highlighted three crucial characteristics. Firstly, quantum mechanics allows for particle emission from black holes. Secondly, the spectrum of such an emission is thermal[1] in the sense of black body radiation. Lastly, the temperature of the emitted particles is proportional to the black hole's surface gravity. Although relations between horizon area, surface gravity, and black hole mass resembling the laws of thermodynamics were already known at that time [2], the idea of black hole thermodynamics was only taken seriously after Hawking's derivation of "black hole evaporation". The thermal nature of black holes has since stimulated an immense number of investigations, and more importantly, intersected several research fields, such as particle physics, cosmology, statistical physics, and information theory. Since for (asymptotically flat Schwarzschild) black holes[2] the temperatures increase as their masses decrease, soon after Hawking's discovery, it became clear that a complete description of the evaporation process would ultimately require a consistent quantum theory of gravity. This is necessary as the semiclassical formulation of the emission process breaks down during the final stages of the evaporation as characterized by Planckian values of the temperature and spacetime curvature.

[1]The spectrum at the event horizon is thermal in an extremely accurate approximation according to Hawking's argument. Small corrections come from the decreasing black hole mass due to evaporation, as well as finite size and shape effects during the emission. An asymptotic observer measures deviations from a thermal spectrum induced by the curved geometry outside the horizon, i.e. greybody factors.

[2]This is the system which we consider in this paper if not stated otherwise.

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More than 40 years after Hawking's discovery the situation remains unclear and a variety of issues unresolved. A quantum theory of gravity is not only expected to provide an ultraviolet completion of general relativity, but also to describe consistently the microscopic degrees of freedom at the basis of the statistical interpretation of black hole thermodynamic variables [3–7]. For instance, both the heat capacity and the entropy of black holes exhibit anomalous behavior. Black holes have negative heat capacity throughout the entire evaporation process, while their entropy is proportional to the area of the event horizon rather than the interior volume, as is the case in classical thermodynamic systems. As acceptance of the area-entropy law as a realization of the holographic principle [8–11] is still debated between communities, also black hole thermodynamics still posits important issues related to our understanding of fundamental physics. At the classical level, the formation of an event horizon as a result of a gravitational collapse implies information loss of the star's initial microstates. Quantum mechanical effects worsen the situation. Thermal radiation is a mixed quantum mechanical state, and therefore black hole evaporation challenges one of the basic principles of quantum mechanics, i.e., the impossibility for pure states to evolve into such mixed states.

Several formulations have been proposed in order to address issues raised by black hole evaporation, which we term the Good, the Bad, and the Ugly:

- **The Good**: There exist some model-independent characteristics, or at least an agreement on how the semiclassical description of black hole thermodynamics has to be improved. For instance, the nature of the black hole entropy is often interpreted in terms of entanglement entropy [4, 12, 13]. But ultimately some modifications are expected in the vicinity of the horizon. It is expected that the usual notions of locality and causality will be violated when both gravitational and quantum mechanical effects are simultaneously taken into account. These violations might allow information to leak out of the horizon (see e.g. [14–20]). Traditionally, the above issues were approached by maintaining the dogmas of gravity at the expense of local quantum field theory principles. However, it is also possible to postulate the breakdown of gravity while maintaining quantum field theory. In this way, the possibility of a black hole firewall [21] might be considered.
- **The Bad**: To date, we do not have any experimental/observational data in order to discriminate among the plethora of possible scenarios, even though there exists a new generation of facilities, e.g. the Event Horizon Telescope [22], Advanced LIGO, [23, 24] as well as future runs at the Large Hadron Collider (LHC, scheduled until 2035) [25], that have the potential to disclose crucial clues.
- **The Ugly**: We need to dig deep into a complicated mess of mathematical formalism to understand the relationship between gravity and information and to extrapolate reliable phenomenological scenarios.

In the following pages, we expand on these topics to show the ways in which they conflict, but also the ways in which they complement one another.

2.2 The Loop Quantum Gravity Perspective

Contribution by C. Rovelli

The convergence of ideas we have witnessed in this conference is surprising, and is good news. I was surprised how Steve and I, coming from different theoretical paths, have come to similar conclusions.

The first convergence point is the growing conviction that quantum gravitational phenomena can violate the basic assumptions of standard Local Quantum Field Theory (LQFT). They can violate the causality dictated by the background geometry. On the other hand, I understand that Gerard disagrees with this, and does not expect quantum gravity to violate the LQFT basics.

There is a difference, however, between Steve's and my views on the violation of LQFT causality. For Steve, this is a shocking new phenomenon that points to some mysterious new physics and demands some deep revision of current physics and some courageous new speculation. For me — in fact for a large community of people that have been working on quantum gravity for decades — this is the natural consequence to be expected when general relativity and quantum theory are taken together.

Let me explain: we have learned from general relativity that spacetime geometry — therefore the causal structure — are determined by the gravitational field. The gravitational field is a quantum field, therefore the geometry it determines is not going to be sharp: it undergoes quantum fluctuations, can be in superpositions, and entangled. Therefore the causal structure of a single fixed background geometry can be violated.

To have such violations, one must exit the regime of validity of perturbation theory, because perturbation theory can be formulated over a background configuration respecting the causality of the background. But of course there are phenomena in quantum mechanics that escape perturbation theory: quantum tunnelling is a prominent example. Therefore tunnelling phenomena can easily violate the causality of conventional LQFT, because LQFT does not allow a violation of the causality determined by the background geometry.

What is the time scale for these violations? There is a simple dimensional argument that gives an indication. To have quantum gravitational phenomena, you need something to get to Planckian scale, which is to say to get to order 1 in Planck units. One well known possibility is to have a very high curvature $R \sim 1$: this happens near the region where classical General Relativity (GR) predicts a singularity, indicating that classical GR fails there. But there is another possibility: small quantum corrections can pile up and give radical departures from classicality over a long time T. Therefore in a region of low curvature we might already see quantum phenomena after a time T, as soon as $RT \sim 1$. Around a mass m the curvature goes like $R \sim m/r^3$, where r is the radius, and in the region outside the horizon of a black hole $r \sim m$. This gives

$$T \sim m^2 ,$$

as a possible time to see quantum gravitational effects. For a macroscopic hole, this is a huge time: it is the Hubble time for a millimeter-size black hole. But it is still enormously smaller than the immense Hawking evaporation time, which is $T \sim m^3$, meaning that it takes 10^{35} Hubble times to evaporate the same millimetre size black hole. This indicates that for a black hole there can be *other* quantum phenomena than the Hawking radiation, taking a shorter time than the Hawking time, which are not seen by LQFT.

There is another way of seeing that quantum gravity allows for surprising violations of causality. Suppose for a moment that causality could be violated at the distance of a single Planck length. For instance, suppose information could be fast transmitted one Planck length away in a spacelike direction. Take an arbitrary point P in the classical metric of a collapsed star, sufficiently after the collapse. It is a fact that the distance between P and the singularity is smaller than a Planck length. This is completely counter-intuitive for a Newtonian intuition, but it is a simple consequence of special relativity: a light ray emitted from a point just before P can reach the singularity, therefore there are points of the singularity at an arbitrarily small spacial distance from P. If quantum effects can spread information a Planck distance away, they can move information from the singularity to anywhere in space. No wild speculation about new physics is required for this: general relativity and quantum theory suffice. But we must go beyond LQFT.

A simple possibility of a quantum gravitational phenomenon that can happen as soon as classical causality does not constrain quantum gravity phenomena, is the following: a macroscopic black hole can "explode", namely tunnell-out to a white-hole, while still macroscopic, without having to wait for the end of Hawking's evaporation [26–52]. This is akin to standard nuclear decay, which is a prototypical tunnelling phenomenon. Remarkably, it has been shown in [53] that this is possible *without* violating the classical equations of motion outside a compact spacetime region. A black hole can thus quantum gravitationally tunnel into a white hole and explode. This is a standard quantum tunneling phenomenon, and therefore there is no plausible reason for it not to happen. The relevant physical question is how long it takes. If it takes too long, it is not of astrophysical relevance (but it *still* shows that LQFT is going to be violated). If, on the other hand, the dimensional estimate above ($T \sim m^2$) is correct, then this phenomenon can have astrophysical relevance because millimeter size primordial black holes could be exploding today, leading to observable cosmic rays. A millimeter size black hole has the mass of a planet. In the sudden explosion triggered by quantum gravitational tunneling the huge corresponding energy is projected out. It has been conjectured that some Fast Radio Bursts and high energy gamma rays could have this origin [54, 55].

This is the second remarkable point of convergence between Steve and I: we are now talking of potentially observable quantum gravity. We even have calculations trying to compute the distance from the horizon where violations of the classical could be expected [56, 57]. For a field long in search of observations [58, 59], this is again good news. Maybe black holes could 'reveal their inner secrets' [60] after all, thanks to quantum theory.

Of course in order to describe this kind of phenomenon we need a formulation of quantum general relativity which is not in the form of a LQFT over a fixed geometry. It is in this sense that we need something radical to understand quantum black holes. That is, not in the sense that we need mysterious new physics; but in the sense that we have to accept the idea that combining general relativity and quantum theory requires us to abandon the framework of LQFT.

Loop quantum gravity [61–65] does provide a formulation of quantum gravity which is background free, and it does indicate that non-perturbative phenomena violating the causality of the background geometry are possible. Explicit calculations are in course to use LQG to compute the lifetime of a black hole under this kind of decay [66]. In the bounce region, the 'architecture' [67] of the quantum geometry is fully non-classical.

I close with a physical picture of the causality violation. The simplest way to interpret black hole entropy is in terms of quantum field entanglement across the horizon. The traditional difficulty of this interpretation is the "species problem", namely the naive expectation that the amount of entanglement entropy should depend on the number of existing fields and not have the universal character of the Bekenstein-Hawking entropy. This difficulty has been brilliantly solved by Bianchi, in [68] by showing that $\mathrm{d}S = \mathrm{d}A/4$ is independent from the ultraviolet cut-off and from the number of species; it is an infrared phenomenon that follows simply from the Einstein equations and standard QFT.[3] Now, the interpretation of black hole entropy as entanglement entropy may seem to support the solidity of a physical picture where the background geometry can be considered fixed. But this would be wrong: among the entangled fields is the gravitational field itself, which means that the geometry on the horizon fluctuates, which means that the causal structure on the horizon fluctuates. This is a physical process allowing information to escape, of course. Preliminary calculations in Loop Quantum Gravity [69] show that the amplitudes for black hole explosion come precisely from interference between different eigenstates of the horizon geometry.

Overall, I feel that we are making excellent progress in understanding quantum gravity and quantum black holes. What restrains understanding is excessive trust in the validity of LQFT for describing quantum gravitational physics. This is still widely diffused among theoretical physicists.

[3] As a side remark: the famous $1/4$ factor of the Bekenstein-Hawking entropy $S = A/4$ is confusing: if instead of $G_{\text{Newton}} = 1$ we use units where the proper coupling constant of GR is taken to be unit, namely $8\pi G_{\text{Newton}} = 1$, then the coefficient of the Bekenstein-Hawking entropy looks far more conventional: $S = 2\pi A$.

2.3 Black Holes and Correspondence as Guides to New Principles

Contribution by S. Giddings

In my talk I outlined arguments for important quantum modifications to a description of black holes that has historically been based on perturbative local quantum field theory (LQFT) on a semiclassical background, which for example might be closely approximated by Schwarzschild’s original solution. Here, I’ll review some of these arguments and respond to some of the comments made by others in our discussion, and also comment further on the prospects for observation of or observational constraints on black hole quantum structure. This will summarize more in-depth discussion from a series of papers I’ve written; references will be given to guide the reader to more detailed arguments.

2.3.1 Quantum Modifications to Standard Locality

There has been a growing sense in the theoretical community that the only way we can consistently describe black hole evolution is to accept that there are significant modifications to the treatment based on perturbative quantization of matter and metric fluctuations on a background geometry, which can be semiclassically corrected to account for Hawking flux. In particular, a central issue appears to be the role of locality in quantum gravity; it is the usual locality of LQFT that forbids transfer of information (signaling) from the interior of a black hole to its exterior, but transfer of information from black hole “internal” states to exterior is apparently exactly what is ultimately needed for a unitary description of evolution. An important point is that while everyone expects a breakdown of locality at very short (e.g. Planckian) scales, we have now realized that what is apparently needed is a modification of the LQFT description of locality at *long* distance scales – comparable to the horizon radius scale of even very large black holes. It is hard to see how short-distance modifications to locality yield this result.

Of course the question of locality in quantum gravity is a tricky one. Indeed, in gravity there is an obstacle to formulating local gauge (diffeomorphism)-invariant observables; such local observables are used in non-gravitational LQFT to sharply characterize locality (see, e.g., [70]). As has been made even more precise recently [71–73], the “gravitational dressing” required to satisfy the gravitational constraint equations, or their quantum version, produces an obstacle to this LQFT locality.

There has been a lot of discussion over time whether effects related to this observation actually save us in the black hole context. For example, the proposal that black holes carry quantum gravitational hair [18] appears closely related, as do Gerard’s suggestions in the discussion that gravitational backreaction communicates the needed quantum information from ingoing particles to outgoing Hawking particles [19]. However, so far it has been very hard for many of us to see that such effects can be strong enough to restore unitarity to black hole evaporation; certain toy models for black hole evaporation, e.g. [74], seem to reinforce the case that gravitational dressing or backreaction is not enough.[4] I would make the same comment in response to Carlo's suggestion that fluctuations in the metric and causal structure are sufficient – certainly there are perturbative studies of such fluctuations, and while there have been arguments that these can become important [76] at long times, I don't know of clear arguments that their proper treatment can achieve the needed unitarization of Hawking's original story by sufficiently delocalizing information or effectively transferring information from black hole states to the black hole exterior.

So in short, given the profound conflict we have encountered between the principles of relativity, of quantum mechanics, and of locality in describing black holes, and the ensuing "unitarity crisis," I am proposing that we need to consider fundamentally new quantum effects that do not respect the locality principle as formulated in LQFT on a semiclassical geometry, and that do not arise from a naïve quantization of general relativity. Such effects seem necessary, in order to save unitarity and quantum mechanics in the black hole context. I don't see an easier way out of our quandary, and this explains the origin of my talk title, "Beyond Schwarzschild."

In fact, here we encounter the "ugly" of the story: we don't presently seem to have a set of foundational principles to describe quantum gravity. Of course many string theorists believe that AdS/CFT or related dualities provide such fundamental formulation, but many puzzles about how this could work remain, and skepticism [77] has grown in the community. My own point of view is that we need to think more generally, but that the "good" includes the statement that at least quantum mechanics, in a suitably general formulation (see, e.g., [78]), should be an essential element of the foundation of the theory. Another part of the "good" is the notion of correspondence for quantum gravity [73, 79, 80] – whatever the more fundamental formulation is, it should match on to LQFT on semiclassical spacetime in the context of weak gravitational fields. This should be an important guide helping us to infer the necessary additional mathematical structure that is needed beyond basic quantum postulates such as the existence of a Hilbert space. Quantum mechanics and correspondence together appear to be very powerful constraints.

In the absence of the complete fundamental structure, but assuming that it fits within the framework of quantum mechanics, I have taken the pragmatic viewpoint that we need to model and parameterize the relevant dynamics; doing so is in turn hoped to furnish important clues about this fundamental structure. Specifically, since the perturbative gravitational dressing appears to be a "weak" effect, I'll begin with the assumption that at the least we have a localization of information [81] into quantum subsystems such as "black hole," "black hole atmosphere," and "asymptotic spacetime;" far from a black hole we expect this localization of information to be described, to a very good approximation, just like in LQFT. Then, we can investigate

[4]Note for example that there is a cancellation between the interaction of infalling matter with outgoing Hawking particles and with Hawking "partners" behind the horizon [75], which appears to eliminate large effects like those proposed by Gerard.

the kinds of interactions between these subsystems that are needed to provide a unitary description of evolution.[5]

In such a subsystem picture, the problem is that the Hawking process builds up entanglement between the black hole and the exterior, as is easily seen in a description where entangled Hawking particles and their partners are produced, with the latter residing inside the black hole. It is this growth of entanglement that ultimately contradicts unitarity, once a black hole evaporates completely; to avoid this, the entanglement with the black hole states must be transferred out [17, 82, 83] as the black hole decays. Here is where the novel effects, which do not respect locality of LQFT, are needed.

I will also make what I consider to be the reasonable assumption that we should look for the "minimal," or most conservative, departures from the usual story of LQFT. So, while transfer of information from the black hole states to the exterior is needed, we might expect this information transfers to the states in the immediate vicinity of the black hole, and not, for example, to states at a million times the Schwarzschild radius R_S. Likewise, while we could consider drastic departures from the semiclassical spacetime geometry near the horizon, I will explore the assumption that we seek physics that produces minimal such departure, so, for example, an observer could still sail into a large enough black hole without being harmed at the point where he or she would expect to be crossing the horizon.

Here, by the way, I have evolved in my thinking. When I first proposed [15] that transfer of information that is nonlocal with respect to the semiclassical geometry is the way out of this crisis, this was part of a "violent" picture in which the information escapes through formation of a new kind of object – a massive remnant, which provides a sharp interface to the exterior vacuum spacetime. Of course, the most unorthodox part of this was the nonlocality of the transfer of information from inside the horizon. – I think many of my colleagues thought I was crazy to propose this, and I wondered if I indeed was. But, I couldn't see an easier way. Since then, it has been gratifying to see a number of other authors have ultimately come around to a similar viewpoint. In particular, it is nice to see the amount of agreement with Carlo; as I show in Fig. 2.1, if one redraws his picture it clearly looks like such a massive remnant scenario. Of course, in saying this, I am assuming that we should declare victory over the crisis once we see how the massive remnant forms; the puzzling attachment of the upper half of the picture, where one transitions to something more like a white hole, is less clearly necessary. Likewise, proposals such as gravastars [43] and fuzzballs [84] (at least in certain versions), and firewalls, appear to invoke the same crazy proposal – information must transfer in a way that appears spacelike when described with respect to the semiclassical geometry, in order to form a new quantum "object" with the information at or outside the would-be horizon.

But, perhaps due to advancing age, I've become more conservative – now I'm suggesting that there could be a less-violent, but still "nonlocal" (with respect to the semiclassical geometry) transfer of information, that can for example save the

[5]Note that if nonlocality from gravitational dressing indeed is found to play a central role, its effect might also possibly be parameterizable in such a fashion.

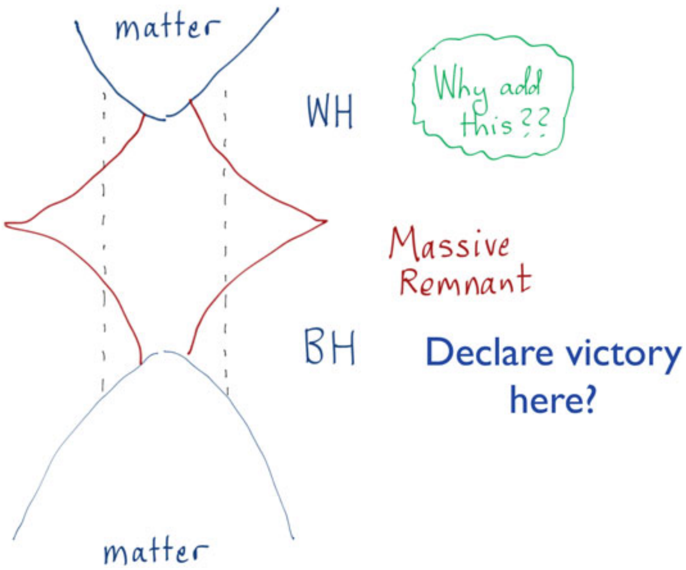

Fig. 2.1 A spacetime diagram corresponding to the scenario Carlo described in his talk. The Planckian region crosses the black hole horizon in a spacelike, "nonlocal," fashion, like in the massive remnant scenario of [15]. Once this is allowed, in principle we can describe information transfer from inside the black hole to outside. Carlo also symmetrically adds a white hole piece to his spacetime

infalling observer from an untimely death. Whatever the correct picture is, and given our uncertainty about the more fundamental dynamics, we should try to parameterize and constrain the relevant physics. One relevant parameter is the time scale on which LQFT and semiclassical spacetime break down and new dynamics manifests itself; we know that this must happen by a time $\sim R_S^3$, and should not happen before a time $\sim R_S \log R_S$. The longer time scale is that on which the black hole appreciably shrinks, and by which information must start to emerge for a unitary description; the latter time scale is inferred from gedanken experiments where an outside observer collects information and then enters the black hole to try to compare with information inside the black hole. We can also parameterize the range over which the dynamics extends, and the "hardness" (e.g. characteristic momentum scales) of the new dynamics. All of these characteristics are important to describe what happens to an infalling observer, and also to describe possible observational signatures of such quantum modifications to black hole structure.

I'll close this section with a final note about departures from the standard locality of LQFT. As we know, in LQFT locality and causality are intimately linked; if an observer can send a signal to a spacelike separated event, we can perform a Lorentz transformation and find that signal propagates back in time. And, by combining two such signals, the observer can send a signal into their own backward lightcone, clearly violating causality – and causing paradoxes. Indeed, in his comments at the meeting, Gerard (see Sect. 2.4) indicated there is reluctance in the community to give up locality and causality, possibly having this argument in mind.

The interesting thing is that if there are departures from standard locality that only really become relevant in the black hole context, these don’t necessarily lead to anything that an observer would describe as acausality [85]. Specifically, information can fall into a black hole, and then be transmitted out at a later time, without any observer seeing a violation of causality. The loophole in the preceding argument arises because the background black hole spacetime specifies a definite frame of reference, so the argument about constructing a signal propagating into your past light cone can’t be made. Moreover, with such a delay, with respect to the black hole’s frame, one can satisfy an alternate macroscopic test of causality, which requires that in scattering outgoing signals should not precede the corresponding ingoing ones. Thus, despite such nonlocality, it is still perfectly possible for the S-matrix to have the expected time delays [80, 86].

2.3.2 Violence Is Not the Answer?

Despite the fact that this session has the underlying theme of a “duel”, I’m going to seek a nonviolent alternative – for the physics! Specifically, I propose to investigate interactions that represent a “minimal” departure from the usual LQFT description of the immediate surroundings of a black hole. This suggests that we work within an effective field theory framework, but consider adding new interaction terms that couple the black hole quantum state to the quantum fields in the immediate vicinity of the black hole [87, 88].

In this spirit of minimality, I’ll assume that these new interactions only extend over a range of size $\sim R_S$ outside the black hole horizon. An important question is which LQFT modes they couple to. For example, one could model a firewall by introducing interactions transferring information [89] to very short wavelength (as seen by the infalling observer) modes that are very near the horizon. A more benign scenario is, apparently, to transfer information to modes whose wavelength grows with the size R_S of the black hole; a simplest example is to simply take this wavelength to be $\sim R_S$.

A priori, these interactions could transfer information (and correspondingly energy) to any of the quantum fields in nature [87, 88]; for example just to fermions or just to gauge bosons. However, generic such interactions would spoil a particularly beautiful part of our current account of black holes: black holes would no longer obey the laws of black hole thermodynamics. It may be that there is no absolute necessity that they must respect these laws [4], with the entropy formula given by Bekenstein and Hawking, but it seems desirable to preserve, at least approximately, such a nice story.

One reason generic interactions spoil black hole thermodynamics is because they would provide channels for information, and thus energy, to escape a black hole that are not universal; then, a black hole could not be brought into equilibrium with a thermal bath at the Hawking temperature – detailed balance would fail. An obvious way to achieve such universality is if the couplings to the black hole internal states are through the stress tensor [90]. Such couplings have another feature – they make a story of information flow from a black hole more robust to "mining" scenarios, where one introduces a cosmic string, or other mining apparatus, to increase the evaporation rate [91–94]. Such interactions through the stress tensor would also universally couple to the mining apparatus. One might also be concerned that these stress-tensor couplings would violate the Bekenstein-Hawking formula because they increase the energy flux from a black hole. But, recall that emission from a black body is given by the Stefan-Boltzmann law, and in particular is proportional to the area of the emitting surface; in the black hole context, the new interactions may effectively be increasing that area [95].

By following such a "conservative" path, we have greatly limited our alternatives. The new interactions universally couple to soft modes (wavelength $\sim R_S$) in the immediate vicinity of the black hole, and the number of such modes that escape the black hole is limited. And, these interactions have a job to do: to unitarize black hole decay, they must transfer of order one qubit of information per time R_S, so that the entanglement entropy of the black hole with its surroundings decreases to zero by the time it has disappeared. This constrains the strength of the new couplings; the simplest way to achieve such information transfer is if they provide an order one correction to the Hawking radiation.

Thus, a rather simple and "conservative" set of assumptions has led us to an interesting conclusion. First, note that black hole state-dependent couplings to the stress tensor can be though of as a black hole state-dependent modification of the metric in the vicinity of the black hole. Then, if such a coupling is of order unity, that corresponds to a metric deviation that is of order unity. So, while the fluctuations in the "effective metric" can be soft (long wavelength), they are strong. One might be initially concerned that such fluctuations would greatly alter the experience of the infalling observer. However, with the softness scale set by R_S, typical curvatures measured by an infalling observer are only of size $R \sim 1/R_S^2$ – the same size such an observer would see in Schwarzschild/Kerr. So, and this addresses another concern expressed by Gerard in Sect. 2.4, the infalling observer doesn't need to see a drastic near-horizon departure from the experience expected for infall into a classical black hole.

Before turning to the question of observation, it is worth reviewing the assumptions we have made, to emphasize their simplicity and paucity. We need to reconcile black hole disintegration with quantum mechanics. Assuming information can be localized to begin with, this implies that information must transfer from black hole states to outgoing radiation, and at a certain rate, of rough size one qubit per time R_S. This can be accomplished with new quantum interactions. To avoid a violent departure from the semiclassical picture, these interactions should couple to external modes with large wavelengths; the relevant scale could be R_S^p for some $p > 0$, but a simplest assumption is $p = 1$. We also assume that such new interactions don't extend beyond the immediate vicinity of the black hole, that extends to a distance $\sim R_S$ from the horizon. Next, if we want to preserve black hole thermodynamics and in particular (at least approximately) the Bekenstein-Hawking formula for the entropy, and address mining, these interactions should couple universally to all modes, through the stress tensor. Thus, they behave like black hole state-dependent metric fluctuations. Finally, the size of these perturbations is determined by the statement that they need to provide an $\mathscr{O}(1)$ alteration to the Hawking radiation, imprinting information at the necessary rate.

2.3.3 Observation via EHT

While it is clearly important to further sharpen the preceding arguments, the prospect that order one departures from the Schwarzschild (or Kerr) metric in the immediate vicinity of the horizon are present raises the extremely interesting question of direct observation.

In short, there has been a growing realization that quantum modifications to Schwarzschild/Kerr are needed on scales of order the horizon size (not just near the singularity); we have now entered the era where observations can be made of the near-horizon geometry, and so we should seek to observe or constrain such quantum structure.

In particular, as we heard at the meeting [96], the Event Horizon Telescope (EHT) is now probing the structure of Sgr A* on the event horizon scale. What might we look for, in a story where there is new quantum structure? A preliminary discussion of this was given in [97]: near-horizon perturbations in the effective metric seen by matter would lead to deviations from the geodesics predicted for the Schwarzschild/Kerr solutions. This, in turn, would alter the observed images, and specifically could alter the shape of the black hole shadow and photon ring that is seen just outside it. A careful treatment of the possible effects on EHT images involves modeling the accretion flow – which emits the light that EHT observes – and numerical ray tracing to see the effects of the perturbations on the collection of light trajectories. Such a treatment is in progress [98]. But, as I pointed out in the discussion, generic expectations from such perturbations are a smaller, fuzzier shadow, and distortion of the photon ring. (Since this meeting and the original version of this writeup, this analysis has appeared [99].)

2.3.4 Post-LIGO Update

Observation of gravitational radiation from inspiraling black holes also provides a possible probe of black hole structure, as was pointed out in [97] and preliminarily explored in [56]. In short, if black holes have quantum structure not described by Schwarzschild/Kerr, then this is expected to alter their dynamics, particularly in the plunge/merger phase. No large deviations were seen in the first LIGO detection [23], which already indicates constraints on such scenarios. Providing sharp constraints requires modeling the effects on the gravity wave signal due to quantum black hole structure, which in particular probably requires treatment via numerical relativity, given the strong nonlinearities. Improvement of LIGO sensitivity to possible deviations combined with further theoretical work thus offers the exciting possibility of learning more about such effects. One might expect that ultimately the photon observations of EHT can more cleanly resolve deviations from general relativity, but LIGO may also ultimately offer the advantage of statistics of multiple events, so that remains to be seen.[6]

2.4 A Resolution to the Duel: Finding Common Ground in Quantum Gravity Research

Contribution by G. 't Hooft

The central question we face is how black holes can be properly incorporated in a grand scheme of quantum gravity.

It is of tantamount importance to hold on as much as possible to the principles and symmetries that we are dealing with in today's theories. Starting from what we know, black holes will then display quantum features that seem to be physically quite acceptable: they radiate particles of all sorts, with, on the average, thermal spectra. These properties appear to obey the usual conditions of causality and locality. An observer will not see any violations of these principles. Yet, when we try to find good descriptions of these phenomena, it seems some of us are inclined to throw locality and causality overboard, replace pure states by mixed states, and so on – sometimes a bit too easily. A classical observer should still think all (s)he sees agrees with the classical theories.

On the other hand, could it be that a more 'primitive' theory does not have general invariance – or even quantum mechanics? In this respect it would be interesting to cite an observation by a computer scientist: as far as we know, all classical solutions of Einstein's equations have the property that signals are *slowed down*, in comparison with any featureless vacuum solution. The computer scientist commented that this would be what one should expect if nature would be modelled as a computer: near a heavy body, this computer must process more information inside a smaller volume, so this goes slower.

In such a world view [19, 100–105, 127], what we call quantum mechanics today, may eventually become more like what thermodynamics is now: a description of some deterministic system in a statistical language. Note: thermodynamics can still be applied to single atoms, and similarly, quantum mechanics will still be true for small-mass black holes, but it could be that a deterministic theory will explain where these laws come from. That theory may turn out to be as deterministic as one's laptop.

Ideally, what should come out of an advanced theory is that, in the very end, black holes act just as heavy radioactive nuclei, or objects like a bucket of water: they

[6]For a different point of view on this subject, see the footnote at the end of Sect. 2.4.

absorb and emit particles, and, in practice, one can’t distinguish whether they are in a pure quantum state or a mixed state. We should be able to describe them accurately. This means that, just as a bucket of water, one should be able to describe them as pure quantum states. Twenty years ago, this view, coming from particle physicists, was not very popular among the traditional practitioners of general relativity.

There will be entangled components in these states. In particle physics, the lowest energy states are very well distinguishable. In the early days, the 1960s, complete lists of all particles and their properties were published yearly: the so-called Rosenfeld tables. Nowadays, such tables consist of thousands of pages. In principle, we should be able to repeat this for all black hole states, after which we should be able to decipher Nature’s book keeping system.

Of particular interest would be black holes whose mass is 10 times to 1000 times the Planck mass, in the transition region between pure quantum objects and classical objects Note that, even if the size of such black holes is much smaller than the size scales in the Standard Model, they may still be in the classical regime. This is because, as soon as velocities approach that of light, the momentum p of such a black hole can easily become much more than the Planck momentum. Therefore, the product of the uncertainties Δp and Δx will be much greater than $\hbar$. Therefore, black holes considerably heavier and bigger than the Planckian dimensions, may in all respects be handled as classical objects. For them, Einstein’s equations make perfectly sense. The most challenging problem of black holes in physics is the regime where black holes, and the particles they interact with, all reside in, or near, the Planckian domain. Classical black holes obey a “no-hair” theorem, but the quantum black hole has hair. One can see this if there is a scalar field present. At the horizon, such a field does not change under a Lorentz transformation, so its values on the horizon are conserved in time for the outside observer.

It is not true that a typical black hole horizon can only be defined if one waits for very long time periods. As soon as a time of the order of $m \log m$ has gone by, in Planck units, one sees the typical horizon features such as Hawking radiation. That is a robust property of a horizon, and this amount of time is short. In all black hole theories I am considering, time scales are as short as $m \log m$. We then have the internal degrees of freedom of the black hole, and those of particles surrounding it.

What one finds, in general, is that the gravitational back reaction of particles can in principle impart their information onto the out-going Hawking particles. The starting point is the Aichelburg–Sexl solution of the gravitational field of a fast moving particle [106, 107]. It is a common misconception that this effect will be small and unimportant. To the contrary, it diverges exponentially with time delay, such that, at time scales of the order of, and beyond, $m \log m$ this effect becomes dominant, and can completely explain how information re-emerges from a black hole. The price one pays is, that particles at the Planck scale cannot have the quantum numbers we are used to in the Standard Model: *only* the geometric properties of particles should suffice to describe the information they carry. This is because they return information by the way they interact with gravity. This is indeed as it is in string theories. Globally conserved quantum numbers such as baryon number will be untenable, since we know that a black hole can be formed from much more baryons than whatever can be returned when they decay.

As a note added in proof: in a recent investigation [20, 108], a clearer picture is drawn of how all information can reappear, which is by a topological twist in space-time, a twist that cannot be observed directly by outside observers. It is a necessary result of the fact that the Penrose diagram of a stationary black hole (a hole that is much older than $m \log m$ in Planck units) has *two* asymptotic regions; one must have a good understanding of what they both mean. As soon as we accept the notion that particles in the Planck regime only have geometrical properties, the method I have been advocating for years, does away with the firewalls. This is because it gives a unitary evolution law just as the Schroedinger equation does, and the solutions of this law usually consist of entangled states. The gravitational drag effect, the central engine in this domain, forces us to rephrase these states as soon as the time delays surpass $m \log m$, which is where the older, more primitive arguments would give us firewalls. The firewalls are transformed away now, as I've tried to explain in my most recent papers.

Yet other problems remain. The total set of quantum states available for a black hole is discrete, all of space and time here seems to be discrete, and therefore we will have to give up some of our sacred notions that only make sense in a continuum, such as strict locality. Even the notion of probability will have to be reconsidered. The problem is not how to imagine crazy scenarios, the problem is how to arrive at the correct scenario by making only small steps, without having to make outrageous assumptions. There is no lack of information: we have special and general relativity, and the entire Standard Model of the elementary particles with some 25 parameters, coupling strengths that we do not know yet how to derive, but which can be measured accurately.

The Standard Model now, is the prototype of a successful development in science. It is based on Fock space approaches. In black hole physics, as in gravity in general, Fock space will not be good enough, so that I think gravity, and black holes, will put new constraints on the Standard Model. A theory better than Fock space may help us understand, and calculate, those 25+ parameters.

This information is notoriously difficult to implement, however. All today's theories are based on *real numbers*: positions, momenta, energies, as well as constants of nature. But every single real number carries an infinity of information, and that's probably more than can be accommodated for inside volumes as small as the Planck volume. How do we make a theory without real numbers? Even the set of *integers* might be too large. Should we exclusively employ bits and bytes?

Note, that eventually, when working out any theory, real numbers such as π and *e will* show up soon enough, but only when you integrate things over larger volumes and distance scales.

How exactly to incorporate any most useful version of locality will be a difficult problem. In particular, locality in the quantum mechanical sense is an issue to which special meetings are devoted.[7]

2.5 Summary and Observational Prospects

When brought to a final showdown, the Good, the Bad, and the Ugly of gravity and information do leave room for non-violent resolution and redemption. We witness a congregation of representatives from different prominent approaches, converging in similar ideas. Most notably, theoretical problems with local quantum field theory (LQFT), as well as those concerning current concepts of locality in general, are acknowledged across various communities. When venturing into the uncharted territory in the Wild West of our current knowledge, revisions of these concepts are considered foundational to the construction of quantum gravity. Further consideration is required to demonstrate which technique will lead to a resolution of the current stand-off: On one side, loop quantum gravity claims that the standard causal structure of spacetime fluctuates, leading to a necessary revision of how causality is defined. This is based on the idea that, according to general relativity (GR), the geometry of spacetime is determined by the gravitational field, which is a quantum field, undergoing quantum fluctuations and non-perturbative quantum effects. On the other side, a convincing argument has not been given that *short distance* fluctuations in the causal structure leads to the needed *long distance* modifications to a picture based on semiclassical spacetime. So, the need for unitary evolution suggests that new non-local physical effects are needed. This physics may be of a "violent" nature, such as with massive remnants or firewalls, or of a "non-violent" nature as with soft quantum structure of black holes. Both sides seem to agree that significant modifications to mere perturbative quantization of matter and metric fluctuations on a background geometry are necessary. There is hope that these modifications are non-violent and could be added systematically as corrections to known physics. Effective field theory descriptions are proposed as a systematic and also the most conservative approach in order to capture the leading quantum gravity corrections to our classical gravity understanding. One obvious example of such an effective description may be to take into account the backreaction of the Hawking radiation and its interaction with the infalling matter [19, 109, 110]. Potentially, we could also keep locality, causality, and the concept of everything being a pure state (even black holes from their formation until their evaporation) intact. This may require a re-interpretation of quantum mechanics (and LQFT) as an effective description, similar to statisti-

[7]Regarding observational prospects and in direct response to Steve (see Sect. 2.3.4), the by far most likely scenario is that quantum effects will leave no trace in the behavior of kilometer-sized black holes, since we expect, like everywhere else in quantum mechanics, that all phenomena where the length scale, the time scale and the mass scale are way beyond the quantum regime, will be described by classical laws. In this case, these will be Einstein's equations, so that no deviations from the standard GR results will be seen to occur.

cal mechanics. In this analogy, black holes are like buckets of water: we could try to follow all its constituents through the causal local evolution, but describing the system in the analog of a statistical ensemble would be much more appropriate. In such a description, globally conserved quantities (baryon number) seem untenable and unnecessary, because only the geometric properties (mass, energy-momentum tensor, topological information) of particles should matter to gravity, dictating the symmetries for a putative effective field theory description of quantum black holes and quantum gravity. Taking an entirely different route, AdS/CFT may provide a fundamental formulation for quantum gravity, mapping a strongly curved regime of a gravitational theory to a weakly coupled field theory that can be described perturbatively.

And so, we reach the end of our story detailing the Good, the Bad, and the Ugly of gravitation and information with good news. Currently, we are standing at the precipice of the strong gravity regime. Over the next decade we will open the window and gaze in. As of February 2016, the LIGO Scientific Collaboration has confirmed the observation of gravitational waves due to merging black hole binaries, giving a first real test of general relativity in this limit [23]. In the not too distant future, the Event Horizon Telescope promises to provide the first image of a black hole's horizon at an incredible resolution. On the theorists' wish list there are also observations of high curvature effects, long-time pile up of perturbative corrections, or obviously direct observation of deviations of the near-horizon geometry from Schwarzschild or Kerr geometry. Also of interest is the possibility that the Event Horizon Telescope (EHT) may discover altered (non-GR) photon rings, and distorted shadows compared to GR. LIGO may observe altered dynamics during the plunge or merger phase, or maybe a quasinormal mode ringdown that deviates from the one predicted by GR.

In parallel to this, there is still a great expectation for the results the LHC will provide in future runs. In the presence of extra dimensions [111–115] gravitational collapse of colliding particles to form a microscopic (evaporating) black hole is possible at LHC working energies [116–119]. Other particle physics testbeds include the formation of microscopic black holes in ultra high energy cosmic ray showers that might be observed at the Pierre Auger Observatory [120] or in AMANDA/IceCube, ANTARES neutrino telescopes [121, 122]. Interestingly non-classical effects might drastically change the production rates [123] and the signature in detectors [124–126]. The above experiments aim to the observation of the evaporation end-point of a microscopic black hole, a fact that could disclose crucial insights about how information is ultimately exchanged and preserved beyond what is known on the ground of standard semiclassical arguments.

The gravitational physics community looks forward to these, and future discoveries, with tempered anticipation, as LIGO, EHT and the LHC are staking new claims, eager to uncover observational gold mines.

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Part I
Black Holes in Classical General Relativity, Numerical Relativity, Astrophysics, Cosmology and Alternative Theories of Gravity

Chapter 3
A Menagerie of Hairy Black Holes

E. Winstanley

Abstract According to the no-hair conjecture, equilibrium black holes are simple objects, completely determined by global charges which can be measured at infinity. This is the case in Einstein-Maxwell theory due to beautiful uniqueness theorems. However, the no-hair conjecture is not true in general, and there is now a plethora of matter models possessing hairy black hole solutions. In this note we focus on one such matter model: Einstein-Yang-Mills (EYM) theory, and restrict our attention to four-dimensional, static, non-rotating black holes for simplicity. We outline some of the menagerie of EYM solutions in both asymptotically flat and asymptotically anti-de Sitter space. We attempt to make sense of this black hole zoo in terms of Bizon's modified no-hair conjecture.

3.1 The "No-Hair" Conjecture

Static, spherically symmetric, asymptotically flat, four-dimensional black hole solutions of the Einstein equations in vacuum or coupled to an electromagnetic field are very simple (see, for example, [1] for a review). The metric must be a member of the Reissner-Nordström family, determined by just two parameters. These parameters correspond to the mass and charge of the black hole, which are global conserved quantities, measurable (at least in principle) far from the black hole. A natural question is whether this simplicity remains when some of the assumptions leading to the electrovac uniqueness theorems are relaxed. We phrase this question as the following conjecture, known as the "no-hair conjecture" [2]:

> A static, spherically symmetric, four-dimensional black hole is uniquely determined by global charges.

In this note we explore this conjecture when the matter content of the theory is no longer simply an electromagnetic field. We consider Einstein-Yang-Mills (EYM)

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P. Nicolini et al. (eds.), *2nd Karl Schwarzschild Meeting on Gravitational Physics*, Springer Proceedings in Physics 208,
https://doi.org/10.1007/978-3-319-94256-8_3

theory, which has been extensively studied over the past twenty-five years. This theory is sufficiently complicated to have a rich space of black hole solutions, yet simple enough that it is possible to analytically prove at least some results concerning these black holes.

3.2 $\mathfrak{su}(N)$ Einstein-Yang-Mills Theory

We consider the following action for four-dimensional Einstein gravity, with a cosmological constant Λ, coupled to an $\mathfrak{su}(N)$ nonabelian gauge field:

$$S = \frac{1}{2}\int \mathrm{d}^4x \sqrt{-g}\left[R - 2\Lambda - \mathrm{Tr}\, F_{\alpha\beta}F^{\alpha\beta}\right], \tag{3.1}$$

where R is the Ricci scalar, $F_{\alpha\beta}$ is the Yang-Mills (YM) gauge field strength and we have set the gauge coupling equal to unity. Varying the action (3.1) gives the field equations

$$\begin{aligned} R_{\alpha\beta} - \frac{1}{2}Rg_{\alpha\beta} + \Lambda g_{\alpha\beta} &= T_{\alpha\beta}, \\ D_\alpha F^\alpha{}_\beta = \nabla_\alpha F^\alpha{}_\beta + \left[A_\alpha, F^\alpha{}_\beta\right] &= 0, \end{aligned} \tag{3.2}$$

where A_α is the YM gauge field potential and the stress-energy tensor of the YM field is

$$T_{\alpha\beta} = \mathrm{Tr}\, F_{\alpha\lambda}F^\lambda{}_\beta - \frac{1}{4}g_{\alpha\beta}\mathrm{Tr}\, F_{\lambda\sigma}F^{\lambda\sigma}. \tag{3.3}$$

We consider static, spherically symmetric, black holes with line element

$$\mathrm{d}s^2 = -\nu(r)S(r)^2\,\mathrm{d}t^2 + [\nu(r)]^{-1}\,\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right), \tag{3.4}$$

where the metric functions $\nu(r)$ and $S(r)$ depend on the radial co-ordinate r only and $\nu(r)$ has the following form, in terms of an alternative metric function $m(r)$,

$$\nu(r) = 1 - \frac{2m(r)}{r} - \frac{\Lambda r^2}{3}. \tag{3.5}$$

With a suitable choice of gauge, an appropriate static, spherically symmetric ansatz for the $\mathfrak{su}(N)$ YM gauge potential is [3]

$$A_\alpha\,\mathrm{d}x^\alpha = \mathscr{A}\,\mathrm{d}t + \frac{1}{2}\left(C - C^H\right)\mathrm{d}\theta - \frac{i}{2}\left[\left(C + C^H\right)\sin\theta + D\cos\theta\right]\mathrm{d}\phi, \tag{3.6}$$

where $\mathscr{A}$, C and D are $N \times N$ matrices. The matrix $\mathscr{A}$ depends on $N-1$ electric gauge field functions $h_j(r)$; the matrix C depends on $N-1$ magnetic gauge field functions $\omega_j(r)$ and the matrix D is constant.

There is now an extensive literature on the EYM system and this short note cannot do justice to all aspects, nor make reference to all relevant articles. Instead we focus on work of the author and collaborators and a few selected other papers. We refer the reader to the reviews [4, 5] for wider coverage of the field and more complete bibliographies.

Let us for the moment restrict attention to purely magnetic configurations for which all electric gauge field functions $h_j(r)$ vanish identically. We will return to solutions with nontrivial $h_j(r)$ in Sect. 3.3.2. The first EYM black holes to be found were asymptotically flat, with vanishing cosmological constant $\Lambda = 0$ and gauge group $\mathfrak{su}(2)$, and are known as "coloured black holes" [6]. With this gauge group, the purely magnetic YM field is described by a single function $\omega_1(r)$, which has at least one zero. The requirement that the space-time is asymptotically flat constrains $\omega_1(r)$ to tend to ± 1 as $r \to \infty$. As a result, the "coloured" black holes have no global magnetic charge (see Sect. 3.3.1). They are therefore indistinguishable at infinity from a Schwarzschild black hole, although the metric exterior to the event horizon is not the same. Thus the "coloured" black holes are counter-examples to the "no-hair" conjecture as stated above. However, there is a very general result that all purely magnetic, spherically symmetric, asymptotically flat, $\mathfrak{su}(N)$ EYM black holes are unstable [7]. Physically, it is natural to focus on stable equilibrium configurations, so we consider the following modification of the "no-hair" conjecture [8]:

> For a fixed matter model, a *stable* static, spherically symmetric, four-dimensional black hole is uniquely determined by global charges.

The "coloured" black holes do not contradict this conjecture due to their instability.

If we include a positive cosmological constant $\Lambda > 0$ in the action (3.1), then "cosmic coloured black holes" exist [9] when the gauge group is $\mathfrak{su}(2)$. Like their asymptotically flat counterparts, these too are unstable, and so the modified "no-hair" conjecture holds, at least for the EYM model with $\Lambda \geq 0$.

3.3 Asymptotically adS $\mathfrak{su}(N)$ EYM Black Holes

In this section we consider whether the modified "no-hair" conjecture also holds for EYM black holes when the cosmological constant Λ is negative, and the space-time is asymptotically anti-de Sitter (adS).

3.3.1 Purely Magnetic Black Holes

Static, spherically symmetric, asymptotically adS black hole solutions of $\mathfrak{su}(2)$ EYM with a purely magnetic gauge field have been found numerically [10–12]. In addition, a very rich phase space of asymptotically adS black hole solutions has been found when the larger $\mathfrak{su}(N)$ gauge group is considered [13, 14].

These asymptotically adS solutions differ significantly from those in asymptotically flat space. Notably, there exist black hole solutions for which all the magnetic gauge field functions $\omega_j(r)$ have no zeros, provided $|\Lambda|$ is sufficiently large [15], which have no counterpart in asymptotically flat space. These nodeless solutions are of particular interest because it can be proven that at least some of them are linearly stable under spherically symmetric perturbations [16]. When the gauge group is $\mathfrak{su}(N)$, the gauge field is described by $N-1$ independent functions $\omega_j(r)$, corresponding to $N-1$ matter degrees of freedom. Since there are stable solutions for any N, there is therefore no limit on the amount of stable gauge field "hair" with which a black hole in adS can be dressed.

The question is then whether these stable EYM black holes satisfy the modified "no-hair" conjecture, in other words, are stable, asymptotically adS, $\mathfrak{su}(N)$ EYM black holes uniquely determined by global charges? To answer this question, we first define magnetic YM charges as follows [17]

$$Q(X) = \frac{1}{4\pi}\mathscr{K}\left(X, \int_{S_\infty} F\right), \tag{3.7}$$

where F is the YM field strength, S_∞ the two-sphere at space-like infinity, X is an element of the Cartan subalgebra of the YM Lie algebra, and $\mathscr{K}$ is the Lie algebra Killing form. Since $\mathfrak{su}(N)$ has rank $N-1$, the definition (3.7) gives $N-1$ independent magnetic charges Q_j. The charges Q_j depend on the values of the magnetic gauge field functions $\omega_j(r)$ as $r \to \infty$. For example, for $\mathfrak{su}(2)$, the single charge Q_1 is given by

$$Q_1 = 1 - \omega_1^2(\infty), \tag{3.8}$$

and for $\mathfrak{su}(3)$, the two charges are

$$Q_1 = 1 - \omega_1^2(\infty) + \frac{1}{2}\omega_2^2(\infty), \qquad Q_2 = \sqrt{3}\left[1 - \frac{1}{2}\omega_2^2(\infty)\right]. \tag{3.9}$$

The asymptotically flat "coloured" black holes in $\mathfrak{su}(2)$ EYM must have $\omega_1 \to \pm 1$ as $r \to \infty$ in order that the space-time is asymptotically Minkowskian, leading to vanishing magnetic charge. However, in asymptotically adS space-time, the boundary conditions as $r \to \infty$ imply that each magnetic gauge field function $\omega_j(r)$ must tend towards a constant, but do not constrain the values of these constants. In general, asymptotically adS EYM black holes have nonzero magnetic charges Q_j. In [18], we presented numerical evidence and an analytic argument that at least a subset of the $\mathfrak{su}(N)$ EYM black hole solutions which are linearly stable are uniquely characterized by the cosmological constant Λ, black hole mass M (which is the finite limit as $r \to \infty$ of the function $m(r)$ in the metric (3.5)) and the set of $N-1$ global nonabelian magnetic charges Q_j.

Therefore stable black holes in $\mathfrak{su}(N)$ EYM in adS, while possessing potentially unlimited amounts of stable gauge field hair, satisfy the modified "no-hair" conjecture as they are uniquely determined by global charges.

3.3.2 Dyonic Black Holes

So far we have considered only purely magnetic gauge field configurations. For $\mathfrak{su}(2)$ EYM in asymptotically flat space-time, nontrivial black holes must have a purely magnetic gauge field [19, 20]; the only black hole solution having a nontrivial electric gauge field component being the embedded abelian Reissner-Nordström solution. This is no longer the case when the space-time is asymptotically adS.

Dyonic (that is, having nontrivial electric and magnetic gauge field components) black hole solutions of $\mathfrak{su}(2)$ EYM in adS were found numerically soon after the corresponding purely magnetic black holes [10, 11]. These black holes have a single electric gauge field function $h_1(r)$ and a single magnetic gauge field function $\omega_1(r)$. The electric gauge field function $h_1(r)$ is always nodeless, and there exist solutions for which the magnetic gauge field function $\omega_1(r)$ also has no zeros [10, 11, 21]. As in the purely magnetic case, at least a subset of these nodeless solutions are stable under linear, spherically symmetric perturbations when $|\Lambda|$ is sufficiently large [22].

Enlarging the gauge group to $\mathfrak{su}(N)$, a rich phase space of dyonic black hole solutions is found [23]. As with the $\mathfrak{su}(2)$ solutions, the electric gauge field functions $h_j(r)$ always have no zeros, and, for sufficiently large $|\Lambda|$, there are solutions for which the magnetic gauge field functions $\omega_j(r)$ are all nodeless [24]. The stability of dyonic black holes with the larger gauge group remains an open question, but one might conjecture the existence of stable dyonic black holes for sufficiently large $|\Lambda|$. The question of whether these dyonic black holes are uniquely characterized by global charges also remains uninvestigated at the time of writing.

3.4 Topological Black Holes

In four-dimensional adS, black hole event horizons do not necessarily have spherical topology, which is the only possibility in asymptotically flat space-time. We now consider static $\mathfrak{su}(N)$ EYM black holes in adS having event horizons with nonspherical topology. In this case the metric takes the form

$$\mathrm{d}s^2 = -\nu(r)S(r)^2\,\mathrm{d}t^2 + [\nu(r)]^{-1}\,\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + f_k^2(\theta)\,\mathrm{d}\phi^2\right), \tag{3.10}$$

and the metric function $\nu(r)$ is modified to be

$$\nu(r) = k - \frac{2m(r)}{r} - \frac{\Lambda r^2}{3}. \tag{3.11}$$

In (3.10), the form of the function $f_k(\theta)$ depends on the constant k as follows:

$$f_k(\theta) = \begin{cases} \sin\theta, & k = 1, \\ \theta, & k = 0, \\ \sinh\theta, & k = -1, \end{cases} \tag{3.12}$$

where $k = 1$ denotes spherical event horizon topology; $k = 0$ for planar event horizon topology, and for $k = -1$ the event horizon is a surface of constant negative curvature. For topological black holes with $k \neq 1$ the gauge potential ansatz (3.6) is generalized to [25, 26]

$$A_\alpha \, \mathrm{d}x^\alpha = \mathscr{A} \, \mathrm{d}t + \frac{1}{2}\left(C - C^H\right) \mathrm{d}\theta - \frac{i}{2}\left[\left(C + C^H\right) f_k(\theta) + D\frac{\mathrm{d}f_k(\theta)}{\mathrm{d}\theta}\right] \mathrm{d}\phi. \tag{3.13}$$

Purely magnetic topological black holes with gauge group $\mathfrak{su}(2)$ were found in [26]. All the solutions are such that the single magnetic gauge field function $\omega_1(r)$ has no zeros if $k \neq 1$. This is in contrast to the situation when $k = 1$ and the black hole is spherically symmetric, when, as described in Sect. 3.3.1, there exist solutions for which $\omega_1(r)$ is nodeless, but there are also black holes for which $\omega_1(r)$ has zeros.

Enlarging the gauge group to $\mathfrak{su}(N)$, it is no longer the case that all the magnetic gauge field functions $\omega_j(r)$ are nodeless for purely magnetic configurations [27], although it can be shown for any N that there are purely magnetic black holes for which all the $\omega_j(r)$ have no zeros [25], if $|\Lambda|$ is sufficiently large.

Dyonic topological black holes also exist. Those with planar event horizons ($k = 0$) have attracted great attention in the recent literature as models of p-wave holographic superconductors (see [28] for a review and references). Planar black holes with $\mathfrak{su}(2)$ gauge group have been found numerically [29], as have their counterparts with the larger $\mathfrak{su}(N)$ gauge group [30]. For both $k = 0$ and $k = -1$, there exist topological dyonic black hole solutions for which the magnetic gauge field functions $\omega_j(r)$ have no zeros, for any value of N and $|\Lambda|$ sufficiently large [24].

The stability of topological EYM black holes has been investigated thus far only in the purely magnetic case. As might be anticipated from the discussion in Sect. 3.3.1, there exist nodeless purely magnetic topological black holes in $\mathfrak{su}(N)$ EYM in adS are which stable under linear perturbations [26, 31]. Whether or not it is possible to uniquely characterize these stable topological black holes by global charges at infinity has yet to be investigated.

3.5 Understanding the EYM adS Black Hole Menagerie

In this note we have briefly reviewed some aspects of the veritable zoo of hairy black hole solutions of $\mathfrak{su}(N)$ EYM in adS, restricting our attention to four-dimensional, static, spherically symmetric and topological black holes. We have considered solutions with a purely magnetic gauge field, and also dyonic black holes whose gauge field has nontrivial electric and magnetic components. In the literature, the existence of nontrivial black hole solutions has been proven for all N, with the rich solution space explored numerically for smaller values of N. Given the abundance of solutions, we have explored whether these black holes satisfy the modified "no-hair" conjecture, namely whether stable black holes in this model are uniquely determined by global charges.

Table 3.1 Summary of the $\mathfrak{su}(N)$ EYM adS black hole menagerie

	Existence of stable solutions?	Characterization by global charges?
Spherically symmetric, Purely magnetic	Yes [16]	Yes [18]
Spherically symmetric, Dyonic	Yes[a] [22]	?
Topological, Purely magnetic	Yes [31]	?
Topological, Dyonic	?	?

[a]Results only for $\mathfrak{su}(2)$

In Table 3.1, we have listed the different types of solutions considered in this note, and summarized what is known about their stability and characterization by global charges. A question mark ? means that this aspect has yet to be investigated in the literature. Most is known about spherically symmetric, purely magnetic black holes, for which there is analytic and numerical evidence that at least a subset of stable hairy black holes are characterized by global charges, for any N and $|\Lambda|$ sufficiently large [18]. Recently the existence of stable topological black holes with purely magnetic $\mathfrak{su}(N)$ gauge field has been proven [31], but it is not known whether these can be characterized by global charges. For dyonic black holes with nontrivial electric and magnetic gauge field components, rather less is known, with the existence of stable spherically symmetric dyonic black holes with $\mathfrak{su}(2)$ gauge group only recently proven [22]. Characterization by global charges in the dyonic case remains an open question.

To conclude, stable black hole solutions of $\mathfrak{su}(N)$ EYM theory in adS can be arbitrarily complicated, in the sense that they are dressed with gauge field hair with unbounded numbers of degrees of freedom. However, work to date indicates that despite their complexity, these black holes can be uniquely characterized by global charges defined at infinity. Hence the modified “no-hair” conjecture [8] seems to be valid for black holes in $\mathfrak{su}(N)$ EYM in adS.

Acknowledgements Many thanks to my collaborators, joint work with whom is the subject of this note: Erik Baxter, Marc Helbling, Brien Nolan and Ben Shepherd. This work is supported by the Lancaster-Manchester-Sheffield Consortium for Fundamental Physics under STFC grant ST/L000520/1.

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Chapter 4
Black Holes Sourced by a Massless Scalar

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Abstract We construct asymptotically flat black hole solutions of Einstein-scalar gravity sourced by a nontrivial scalar field with $1/r$ asymptotic behaviour. Near the singularity the black hole behaves as the Janis-Newmann-Winicour-Wyman solution. The hairy black hole solutions allow for consistent thermodynamical description. At large mass they have the same thermodynamical behaviour of the Schwarzschild black hole, whereas for small masses they differ substantially from the latter.

4.1 Introduction and Motivations

Static, spherically symmetric solutions of Einstein gravity sourced by scalar fields have played an important role for the development of black hole physics. The simplest solution of this kind, describing a asymptotically flat (AF) spherically symmetric solution with no horizon, sourced by a scalar with vanishing potential are known since a long time [1, 2]. They are called the Janis-Newmann-Winicour or Wyman (JNWW) solutions. Initially, the search for AF black holes (BHs) with scalar hair was motivated by the issue of the uniqueness of the Schwarzschild solution and related "old" no-hair theorems [3, 4], which forbid the existence of BHs if the scalar potential V is convex or semipositive definite.

In the early nineties it was discovered that low-energy string models may allow for black hole solutions with scalar hair [5–8]. But, in this case non-minimal couplings between the scalar field and the electromagnetic field.

In recent times, the quest for hairy black hole and black brane solutions has been motivated by the application of the AdS/CFT correspondence to condensed matter

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P. Nicolini et al. (eds.), *2nd Karl Schwarzschild Meeting on Gravitational Physics*, Springer Proceedings in Physics 208,
https://doi.org/10.1007/978-3-319-94256-8_4

systems [9–17]. In holographic applications the scalar field has a nice interpretation as an order parameter triggering symmetry breaking/phase transitions in the dual field theory.

Several numerical and analytical, black hole and black brane with AdS asymptotics solutions with scalar hair have been found in this context [9–13, 16, 18, 19].

Shifting from AF to anti de Sitter (AdS) black holes allows to circumvent standard no-hair theorems because in AdS the scalar field may have tachyonic excitations without destabilizing the vacuum [20]. This led to the formulation of "new" no-hair theorems [21]. The violation of the positivity energy theorem (PET) [22]. being identified as a necessary condition for the existence of BH with scalar hair.

In this note, which is based on [23], we will show as the expertise achieved in the holographic context can be successfully used to find AF BH solutions with scalar hair. Extension to asymptotically flat BF is an important issue because we know that scalar fields play a crucial role in gravitational and particle physics. Experimental discovery of the Higgs particle at LHC has confirmed that there is a fundamental scalar particle [24]. Observation of the Plank 2013–2015 satellite gives striking confirmation of cosmological inflation driven by scalar field coupled to gravity [25]. Moreover, scalar field give a way to describe dark energy.

The main result presented here is that the solution generating techniques developed in the holographic context in [16] can be also successfully used to construct AF BH solutions sourced by a scalar behaving at $r = \infty$ as an harmonic function, $\phi = 1/r$.

The structure of the paper is as follows. In Sect. 4.2 we present the review the solution-generating technique of [16]. In Sect. 4.3 we rederive the JNWW solutions and discuss their main features. The boundary conditions on the scalar field and the corresponding asymptotic behavior for $V(\phi)$ are discussed in Sect. 4.4. In Sect. 4.5 we present our hairy BH solutions. The thermodynamical behaviour of our solutions is discussed in Sect. 4.6. Finally, in Sect. 4.7 we present our conclusions.

4.2 The Solution-Generating Technique

We consider Einstein gravity in four spacetime dimensions minimally coupled to a scalar ϕ ($\mathscr{R}$ is the scalar curvature),

$$A = \int \mathrm{d}^4x\sqrt{-g}\left(\mathscr{R} - 2(\partial\phi)^2 - V(\phi)\right) \tag{4.1}$$

and static, spherically symmetric solutions of the field equations,

$$\mathrm{d}s^2 = -U(r)\mathrm{d}t^2 + U^{-1}(r)\mathrm{d}r^2 + R^2(r)\mathrm{d}\Omega^2, \tag{4.2}$$

where $\mathrm{d}\Omega^2$ is the metric element of the two-sphere S^2.

Finding exact solutions of the field equations stemming from the action (4.1) is a very difficult task even for simple forms of the potential V. To solve the fields equation (FE) we use the solution generating technique developed in [16] to find asymptotically AdS solutions once the scalar field profile $\phi = \phi(r)$ is given. Using the variables introduced in [16]

$$R = e^{\int Y}, \quad u = UR^2, \tag{4.3}$$

the field equations take the simple form.

$$Y' + Y^2 = -(\phi')^2, \tag{4.4}$$
$$(u\phi')' = \frac{1}{4}\frac{\partial V}{\partial \phi} e^{2\int Y}, \tag{4.5}$$
$$u'' - 4(uY)' = -2, \tag{4.6}$$
$$u'' = 2 - 2Ve^{2\int Y}. \tag{4.7}$$

Equations (4.4) for Y (Riccati equation) and (4.6) for u are universal, they do not depend on the potential. One starts from a given scalar field profile $\phi(r)$ and solves the Riccati equation for Y. Once Y is known can easily integrate the linear equation for u, (4.6) to obtain

$$u = R^4 \left[-\int \mathrm{d}r \left(\frac{2r + C_1}{R^4} \right) + C_2 \right], \tag{4.8}$$

where $C_{1,2}$ are integration constants.

The last step is to determinate the potential using (4.7)

$$V = \frac{1}{R^2}\left(1 - \frac{u''}{2}\right). \tag{4.9}$$

This is a very efficient solving method, very useful in the holographic context, allowing to find exact solutions of Einstein-scalar gravity in which the potential is not an input but an output of the theory.

4.3 The JNWW Solutions

The parametrization (4.3), allows a simple (re)derivation of solutions for $V = 0$ (the JNWW solutions). Equation (4.7) gives u as a quadratic function of r, (4.5) and (4.6) give $\phi(r)$ and $R(r)$, whereas the Riccati equation simply constrains the parameters,

$$U = \left(1 - \frac{r_0}{r}\right)^{2w-1}, \; R^2 = r^2\left(1 - \frac{r_0}{r}\right)^{2(1-w)}, \; \phi = -\gamma \ln\left(1 - \frac{r_0}{r}\right) + \phi_0, \; w - w^2 = \gamma^2. \tag{4.10}$$

According to old no-hair theorems, for $0 < w < 1$ the solution is not a BH (V=0) but interpolates between Minkowski space at $r = \infty$ and a naked singularity at $r = r_0$ (or $r = 0$). Nevertheless the solution is of interest for several reasons. The BH mass is $M = 8\pi(2w - 1)r_0$. We can have a solution with zero or positive mass even in the presence of a naked singularity. In particular for $w = 1/2$ we have $M = 0$, a degeneracy of the Minkowski vacuum. The JNWW appears as the zero charge limit of charged dilatonic black holes. Near to the singularity the solution has a scaling behavior typical of hyperscaling violation [17].

4.4 Asymptotic Behavior of the Scalar Field and of the Potential

We are looking for AF BH solutions sourced by scalar field, which decays as $1/r$. We also assume that the Minkowki vacuum is at $\phi = 0$ and that it is an extremum of the potential with zero mass: $V(0) = V'(0) = V''(0) = 0$. These conditions imply that near $\phi = 0$ the potential behaves as $V(\phi) = \mu\phi^n$ with $n \geq 3$. The corresponding asymptotic behavior for the scalar is determined by using the boundary conditions at $r = \infty$ $u = r^2$, $R = r$ in the FE. For $n = 5$ we get $\phi = \frac{\beta}{r} + \mathcal{O}(1/r^2)$. Hence, an harmonic decay of the scalar field requires a quintic behavior for the potential V.

4.5 Black Hole Solutions

Let us now use the solution-generating method of Sect. 4.2. We need an ansatz for the scalar. We use the JNWW scalar profile (also previously used to in the literature to derive AdS BHs): $\phi = -\gamma \ln(1 - r_0/r)$. The Riccati equation gives the form of the metric function R: $R^2 = r^2 (1 - r_0/r)^{2(1-w)}$, $w - w^2 = \gamma^2$, $1/2 \leq w < 1$. We get three different class of solutions ($X = 1 - r_0/r$),

$$U(r) = X^{2w-1}\left[1 - \Lambda\left(r^2 + (4w-3)rr_0 + (2w-1)(4w-3)r_0^2\right)\right] + \frac{\Lambda r^2}{X^{2(w-1)}} \tag{4.11}$$

$$U(r) = \frac{r^2}{r_0^2} X\left[\left(1 + r_0^2\Lambda\right)X - 2r_0^2\Lambda \ln X + \left(1 - r_0^2\Lambda\right)X^{-1} - 2\right], \tag{4.12}$$

$$U(r) = \frac{r^2}{r_0^2} X^{1/2}\left[\left(1 + \frac{r_0^2\Lambda}{2}\right)X^2 - 2\left(1 + r_0^2\Lambda\right)X + r_0^2\Lambda \ln X + 1 + \frac{3r_0^2\Lambda}{2}\right] \tag{4.13}$$

respectively for $1/2 < w < 1$, ($w \neq 3/4$), $w = 1/2$ and $w = 3/4$. The corresponding potentials are given by,

$$V(\phi) = 4\Lambda \left[w(4w-1)\sinh\frac{(2w-2)\phi}{\gamma} + 8\gamma^2 \sinh\frac{(2w-1)\phi}{\gamma} + \right.$$

$$\left. +(1-w)(3-4w)\ \sinh\frac{2w\phi}{\gamma} \right], \tag{4.14}$$

$$V(\phi) = 4\Lambda\left[3\sinh 2\phi - 2\phi\left(\cosh 2\phi + 2\right)\right], \tag{4.15}$$

$$V(\phi) = \Lambda\left(8\sqrt{3}\phi\cosh\frac{2\phi}{\sqrt{3}} - 9\sinh\frac{2\phi}{\sqrt{3}} - \sinh 2\sqrt{3}\phi\right) \tag{4.16}$$

The previous solutions describe a one parameter family of AF black holes sourced by a scalar field behaving asymptotically as $1/r$ and with a curvature singularity at $r = r_0$ (or $r = 0$) and a regular event horizon at $r = r_h$. The scalar charge is not independent. Near to the singularity the solution have the same scaling behavior of the JNWW solutions. As expected near $\phi = 0$ the potential has always a quintic behavior. The existence of these BH solution represent a way to circumvent old and new no-hair theorems. In fact the potential V is not semipositive definite, it has an inflection point at $\phi = 0$ and is unlimited from below. The ADM mass is not semipositive definite (the PET is violated).

4.6 Black Hole Thermodynamics

Scalar charge σ is not independent from the mass but determined by the BH mass M, implying the absence of an associate thermodynamical potential. The First principle has therefore the form $dM = TdS$, where the temperature T and the entropy S are given by the usual forms $T = \frac{U'}{4\pi}\Big|_{r=r_h}$, $S = 16\pi^2 R^2|_{r=r_h}$. For $w = 1/2$ we have

$$T(\omega) = \frac{\sqrt{\Lambda}}{4\pi\sqrt{l}}\left[2\left(1-\frac{2}{\omega}\right)\ln(1-\omega) - 4\right], \quad S(\omega) = \frac{16\pi^2}{\Lambda l}\left(\frac{1}{\omega^2} - \frac{1}{\omega}\right) \tag{4.17}$$

where l is a function of ω defined implicitly by $2(1-\omega)\ln(1-\omega) - \omega^2(1+l) + 2\omega = 0$. We have an extremal low-mass state with non vanishing mass M_min, zero entropy and infinite temperature. In the large mass (small temperature) limit we get the Schwarzschild behavior for the thermodynamical potentials: $M = 2/T,\ S = 1/T^2,\ F = M - TS = 1/T$. For $w = 3/4$ we have for T and S a different behaviour (see [23]). Both the low and large mass regimes have the Schwarzschild behavior. The extremal state has $M = S = 0$ and $T = \infty$. The thermodynamical behaviour of the solutions with $1/2 < w < 3/4$ and $3/4 < w < 1$ are similar respectively to the cases $w = 1/2$ and $w = 3/4$.

4.7 Concluding Remarks

AF BH solution sourced by a scalar field with $1/r$ fall-off do exist but require a potential unlimited from below. Because $\phi = 0$ is an inflection point for V, the $\phi = 0$ Schwarzschild black hole is unstable. For $3/4 \leq w < 1$ BH thermodynamics is similar to Schwarzschild. For $1/2 \leq w < 3/4$ the low-mass regime drastically different. Near to the $\phi = 0$ Minkowski vacuum V has a quintic behaviour. The corresponding Field theory is not renormalizable. It cannot be fundamental. However it could represent an effective description arising from renormalization group flow.

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Chapter 5
Rotating Black Hole Solutions in $f(R)$-Gravity

M. De Laurentis and R. Farinelli

Abstract We present a strategy to get axially symmetric solutions in $f(R)$ gravity by starting from spherically symmetric space-times. To do so, we assume the validity of a complex coordinate transformation, which acts on the spherically symmetric metric and permits one to infer the corresponding $f(R)$ modification. The consequences of this recipe are here described, giving particular emphasis to define a class of compatible axially symmetric solutions, which fairly well describes the motion in cylindrical geometries in the field of $f(R)$, in two different classes of coordinates. We demonstrate that our approach is general and may be applied for several cases of interest. We also show that our treatment is compatible with the standard approach of general relativity, evaluating the motion of a freely falling particle in the context of our metric.

5.1 Introduction

Alternative theories of gravity pose the problem to recover or extend the well-established results of General Relativity (GR) as the initial value problem, the stability of solutions and, in particular, the issue of finding out new solutions [1]. As it is well known, beside cosmological solutions, spherically and axially symmetric solutions play a fundamental role in several astrophysical problems ranging from black holes to active galactic nuclei. Alternative gravities, to be consistent with results of GR,

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P. Nicolini et al. (eds.), *2nd Karl Schwarzschild Meeting on Gravitational Physics*, Springer Proceedings in Physics 208,
https://doi.org/10.1007/978-3-319-94256-8_5

should comprise solutions like Schwarzschild and Kerr ones but present, in general, new solutions that could be physically interesting. Due to this reason, methods to find out exact and approximate solutions are particularly relevant in order to check if observations can be framed in Extended Theories of Gravity [2].

Recently, the interest in spherically and axially symmetric solutions of $f(R)$-gravity is growing up [3–6].

In this paper, we want to seek for a general method to find out axially symmetric solutions by performing a complex coordinate transformation. Newman and Janis showed that it is possible to obtain an axially symmetric solution (like the Kerr metric) by making an elementary complex transformation on the Schwarzschild solution [7]. This same method has been used to obtain a new stationary and axially symmetric solution known as the Kerr-Newman metric [8]. The Kerr-Newman space-time is associated to the exterior geometry of a rotating massive and charged black-hole. For a review on the Newman-Janis method to obtain both the Kerr and Kerr-Newman metrics see [9].

By means of very elegant mathematical arguments, Schiffer et al. [10] have given a rigorous proof to show how the Kerr metric can be derived starting from a complex transformation on the Schwarzschild solution. We will not go into the details of this demonstration, but point out that the proof relies on two main assumptions. The first is that the metric belongs to the same algebraic class of the Kerr-Newman solution, namely the Kerr-Schild class [11]. The second assumption is that the metric corresponds to an empty solution of the Einstein field equations. Gürses and Gürsey, in 1975 [12], showed that if a metric can be written in the Kerr-Schild form, then a complex transformation "is allowed in General Relativity." In this paper, we will show that such a transformation can be extended to $f(R)$-gravity.

The paper is structured as follows. In Sect. 5.2, we describe the method and we highlight its fundamental properties. To do so, we consider the general treatment and we specialize it to the case of pure spherically symmetric solutions. We therefore obtain the corresponding modifications to the standard Kerr metric in the context of $f(R)$ gravity and we describe some dynamical properties of this solution, by means of circular orbits in the framework of the Hamiltonian formalism. We therefore demonstrate that our strategy is general and may be extended to the case of fourth order gravities without stability problems. In Sect. 5.3, we summarize our results and we propose possible perspectives of our method.

5.2 From Spherical Symmetry to Axially Symmetric Solutions in $f(R)$ Gravity

In the framework of $f(R)$ gravity, the action takes the simple form

$$S = \int d^4x\sqrt{-g}\left[f(R) + \mathscr{X}\mathscr{L}_m\right].$$

By varying it, in terms of the metric $g_{\mu\nu}$, one argues the corresponding field equations:

$$f'(R)R_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu} - f'(R)_{;\mu\nu} + g_{\mu\nu}\Box f'(R) = \mathscr{X} T_{\mu\nu}\,,$$
$$3\Box f'(R) + f'(R)R - 2f(R) = \mathscr{X} T\,, \tag{5.1}$$

where $T_{\mu\nu}$ represents the standard energy-momentum tensor for dust-like matter, which can be expressed in the form: $T_{\mu\nu} = \frac{-2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathscr{L}_m)}{\delta g^{\mu\nu}}$. The constant $\mathscr{X}$ contains the gravitational constant G, since $\mathscr{X} = \frac{8\pi G}{c^4}$, while g is the metric determinant.

Our formalism involves the use of spherically symmetric space-time as starting point. In fact, we set up our treatment by assuming the most general spherically symmetric space-time below:

$$\mathrm{d}s^2 = g_{tt}(t,r)\mathrm{d}t^2 - g_{rr}(t,r)\mathrm{d}r^2 - r^2\mathrm{d}\Omega\,, \tag{5.2}$$

in which $\mathrm{d}\Omega$ represents the solid angle. The basic demands consists in employing on it a transformation that maps (5.2), providing that the off-diagonal terms vanish. Hence, the spherically symmetric space-time may be obtained by assuming that (5.2) satisfies particular cosmic symmetries. Here, we consider the Noether symmetries and so, after several calculations, we can write down the simplest spherically symmetric space-time as:

$$\mathrm{d}s^2 = (\alpha + \beta r)\mathrm{d}t^2 - \frac{1}{2}\frac{\beta r}{\alpha + \beta r}\mathrm{d}r^2 - r^2\mathrm{d}\Omega\,, \tag{5.3}$$

where we assumed α as a combination of auxiliary constants, e.g. Σ_0 and k and $\beta = k_1$ [4].

Here, we demonstrate how it is possible to get an axially symmetric solution adopting the Newman-Janis procedure, extending their treatment in the context of $f(R)$ gravities and going beyond the standard usage of using the Newman-Janis procedure in general relativity only. To this end, as we already stressed before, we employ the existence of Noether symmetries which make the $f(R)$ model consistent with the corresponding field equations. For our purposes, let us recast the spherically symmetric metric as $\mathrm{d}s^2 = e^{2\phi(r)}\mathrm{d}t^2 - e^{2\lambda(r)}\mathrm{d}r^2 - r^2\mathrm{d}\Omega$, with $g_{tt}(t,r) = e^{2\phi(r)}$ and $g_{rr}(t,r) = e^{2\lambda(r)}$. Hereafter, our convention is to refer to time-like components as tt or 00, whereas space-like as rr or ii, with i running from $i = 0$–3.

Considering the suitable Eddington–Finkelstein coordinates, i.e. (u, r, θ, ϕ), which represent a viable choice for our coordinate representation, after simple algebra, we definitively get $\mathrm{d}s^2 = e^{2\phi(r)}\mathrm{d}u^2 \pm 2e^{\lambda(r)+\phi(r)}\mathrm{d}u\mathrm{d}r - r^2\mathrm{d}\Omega$. Thus, the matrix associated to the metric is rewritable in terms of a null tetrad as:

$$g^{\mu\nu} = l^\mu n^\nu + l^\nu n^\mu - m^\mu\bar{m}^\nu - m^\nu\bar{m}^\mu\,, \tag{5.4}$$

where l^μ, n^μ, m^μ and $\bar{m}^\mu$ should satisfy

$$l_\mu l^\mu = m_\mu m^\mu = n_\mu n^\mu = 0\,, \tag{5.5}$$

$$l_\mu n^\mu = -m_\mu \bar{m}^\mu = 1\,, \tag{5.6}$$

$$l_\mu m^\mu = n_\mu m^\mu = 0\,, \tag{5.7}$$

where we assumed the bars as indication of the complex conjugation.

In our case, a generic space-time event becomes

$$x^\mu \to \tilde{x}^\mu = x^\mu + i y^\mu(x^\sigma)\,, \tag{5.8}$$

in which we notice that $y^\mu(x^\sigma)$ are functions of the real coordinates x^σ. Analogously, the null tetrad vectors $Z_a^\mu = (l^\mu, n^\mu, m^\mu, \bar{m}^\mu)$, with $a = 1, 2, 3, 4$, should satisfy

$$Z_a^\mu \to \tilde{Z}_a^\mu(\tilde{x}^\sigma, \bar{\tilde{x}}^\sigma) = Z_a^\rho \frac{\partial \tilde{x}^\mu}{\partial x^\rho}\,. \tag{5.9}$$

All this procedure provides a net effect which consists in generating a new metric. The component of such a space-time are real and depend upon complex variables. We have:

$$g_{\mu\nu} \to \tilde{g}_{\mu\nu} \,:\, \tilde{\mathbf{x}} \times \tilde{\mathbf{x}} \mapsto \mathbb{R}\,, \tag{5.10}$$

where we consider:

$$\tilde{Z}_a^\mu(\tilde{x}^\sigma, \bar{\tilde{x}}^\sigma)|_{\mathbf{x}=\tilde{\mathbf{x}}} = Z_a^\mu(x^\sigma)\,. \tag{5.11}$$

From the transformed null tetrad vectors, a new metric is therefore obtained. So, assuming the covariant form, we can list the corresponding metric components as:

$$\begin{aligned}
g_{00} &= e^{2\phi(\tilde{r},\theta)}\,,\\
g_{01} &= e^{\lambda(\tilde{r},\theta)+\phi(\tilde{r},\theta)}\,,\\
g_{03} &= a e^{\phi(\tilde{r},\theta)}[e^{\lambda(\tilde{r},\theta)} - e^{\phi(\tilde{r},\theta)}]\sin^2\theta\,,\\
g_{13} &= -a e^{\phi(\tilde{r},\theta)+\lambda(\tilde{r},\theta)}\sin^2\theta\,,\\
g_{22} &= -\Sigma^2\,,\\
g_{33} &= -[\Sigma^2 + a^2\sin^2\theta e^{\phi(\tilde{r},\theta)}(2e^{\lambda(\tilde{r},\theta)} - e^{\phi(\tilde{r},\theta)})]\sin^2\theta.
\end{aligned}$$

where we assumed that all the other components, i.e. the components that we did not report above, are zero.

This procedure is circumscribed to the use of the particular choice of coordinates. However, one can also perform the Newman-Janis algorithm on any static spherically symmetric solutions, by means of the more practically Boyer-Lindquist coordinates. So, evaluating the same steps performed above and the analogous strategy to get the tetrad null vectors in the case of axially symmetric space-time, we simply obtain:

$$
\begin{aligned}
\mathrm{d}s^2 = &\frac{r(\alpha+\beta r)+a^2\beta\cos^2\theta}{\Sigma}\mathrm{d}t^2 + 2\frac{a(-2\alpha r - 2\beta\Sigma^2+\sqrt{2\beta}\Sigma^{3/2})\sin^2\theta}{2\Sigma}\mathrm{d}t\mathrm{d}\phi + \\
&-\frac{\beta\Sigma^2}{2\alpha r+\beta(a^2+r^2+\Sigma^2)}\mathrm{d}r^2 \\
&-\Sigma^2\mathrm{d}\theta^2 - \left[\Sigma^2 - \frac{a^2(\alpha r+\beta\Sigma^2-\sqrt{2\beta}\Sigma^{3/2})\sin^2\theta}{\Sigma}\right]\sin^2\theta\mathrm{d}\phi^2
\end{aligned}
$$

As in standard general relativity, our treatment should be compatible with the motion of a freely falling particle. Hence, we can treat a physical example which accounts for a freely falling particle moving in our so-obtained metric. To do so, we make extensive use of the Hamiltonian formalism, which has the advantage not to show any sign ambiguity which may come from turning points in the orbits [13]. The reduced Hamiltonian, linearly reported in terms of momenta, is:

$$
H == \left[\frac{p_i g^{0i}}{g^{00}} + \left[\left(\frac{p_i g^{0i}}{g^{00}}\right)^2 - \frac{m^2+p_i p_j g^{ij}}{g^{00}}\right]^{1/2}\right], \tag{5.12}
$$

providing $H = -p_0$ and even satisfying the following motion equations:

$$
\frac{\mathrm{d}x^i}{\mathrm{d}t} = \frac{\partial H}{\partial p_i}, \quad \frac{\mathrm{d}p_i}{\mathrm{d}t} = -\frac{\partial H}{\partial x^i}, \tag{5.13}
$$

which permit to numerically obtain the requested orbits. In particular, in the equatorial plane, which corresponds to the case $\theta = \frac{\pi}{2}$, $\dot{\theta} = 0$, we conventionally employ $\alpha = 10$ and $\beta = 5$, without losing generality and we consider the dependence on ϕ and on the conjugate momentum p_ϕ, which represents an integral of motion. As a consequence, we find out that the coupled equations for $\{r, \theta, \phi, p_r, p_\theta\}$ may be numerically integrated, giving compatible trajectories with respect to the ones inferred from the standard Kerr space-time. To better clarify this statement, we explicitly report below the geodesic equations:

$$
\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda} = \frac{\partial\mathscr{H}}{\partial p_\mu} = g^{\mu\nu}p_\nu = p^\mu, \tag{5.14}
$$

$$
\frac{\mathrm{d}p_\mu}{\mathrm{d}\lambda} = -\frac{\partial\mathscr{H}}{\partial x\mu} = -\frac{1}{2}\frac{\partial g^{\alpha\beta}}{\partial x^\mu}p_\alpha p_\beta = g^{\gamma\beta}\Gamma^\alpha_{\mu\gamma}p_\alpha p_\beta, \tag{5.15}
$$

In Fig. 5.1, the relative trajectories are sketched.

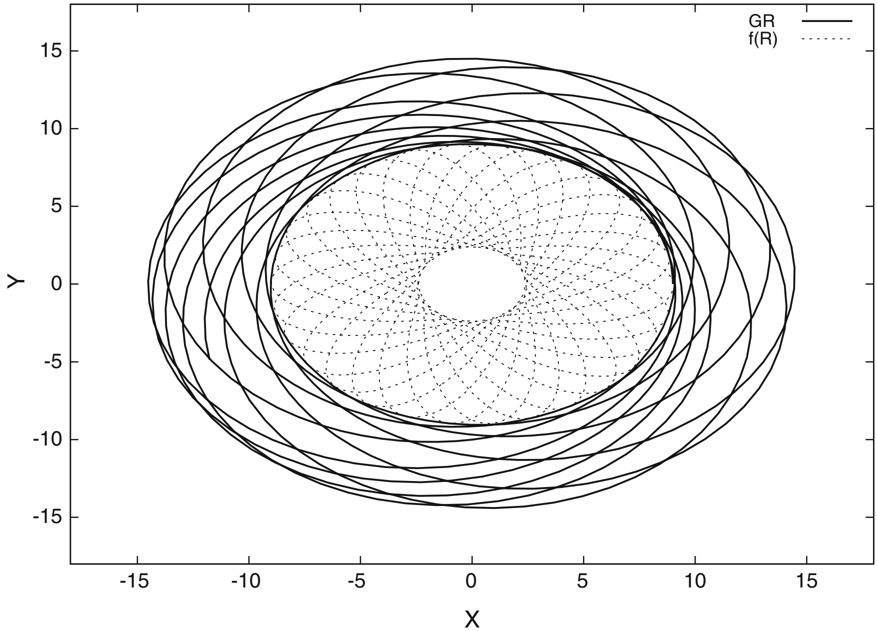

Fig. 5.1 Example of massive particle equatorial trajectories in a Kerr and axially-simmetric $f(R)$ metric, obtained from the solution of the Hamilton-Jacboy equations (5.13). In both cases the BH spin is $a = 0.5$, and for $f(R)$ we employed $\alpha = 10$ and $\beta = 5$ for representative purposes. The test mass at the beginning has a pure tangential velocity component $\mathrm{d}\phi/\mathrm{d}t = 0.03$ and is placed at $9R_g$

5.3 Final Outlooks and Perspectives

In this paper, we considered the framework of $f(R)$ gravity to describe a technique able to get axially symmetric solutions from spherical ones. This treatment has been extensively described by Newman-Janis in a precise algorithm, which takes into account complex transformations. In particular, assuming a spherically symmetric expression for the space-time, we demonstrated that it is possible to extend the complex transformations in the context of $f(R)$ gravity. To do so, we evaluated the null tetrad associated to this method in two different classes of coordinates and we found out the corresponding axially symmetric metrics. In order to understand if the thus obtained space-time works well in the field of particle motion, we considered a freely falling particle and we showed that its motion is perfectly compatible with the expected standard Kerr metric, which corresponds to the simplest axially symmetric solution in general relativity. Further investigations will be carried forward in order to describe different symmetries by means of the Newman-Janis strategy. In particular, measuring possible corrections due to $f(R)$ around compact objects, e.g. evaluating possible discrepancies from the standard cases of accretion disks, one would constrain the $f(R)$ functions at astrophysical regimes. This would open new challenges for the problem of $f(R)$ reconstructions.

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Chapter 6
Symplectic Structure of Extremal Black Holes

K. Hajian and A. Seraj

Abstract We review the construction of phase space for the near horizon extremal geometries (NHEG) as solutions to Einstein gravity. We study the symplectic symmetries of this phase space and compute their corresponding conserved charges. We show that the symmetry algebra is an interesting generalization of Virasoro algebra. The analysis is based on covariant phase space method.

6.1 Introduction

Black holes (BHs) are solutions to theories of gravity, specified by having an event horizon in their geometry. They also usually have a singularity behind the horizon. Although these solutions were known from the early stages of development of general relativity by Karl Schwarzschild, their thermodynamic behaviors were unraveled in early 70s by seminal works of Bekenstein and Hawking [1, 2] in which entropy and temperature were associated to BHs. In Einstein-Hilbert theory of gravity, BH entropy is related to the area of the horizon $S = \frac{A}{4G}$, while Hawking temperature can be read from the surface gravity of the black hole κ, through the relation $T_{\text{H}} = \frac{\kappa}{2\pi}$. Also for stationary BHs in d-dimensions, with a number of commuting and compact $U(1)$ axial isometries, labelled by index i, one can associate conserved angular momenta J_i as well as mass M (due to time translation symmetry). Thermodynamic conjugates to the angular momenta are angular velocities of the horizon, denoted by Ω^i_{H} (index H for *Horizon*). In addition, dynamics of BHs satisfy laws which are analogous to usual laws of thermodynamic [3]. Specifically, the first law of BH thermodynamics is $\delta M = T_{\text{H}}\delta S + \Omega^i_{\text{H}}\delta J_i$ [3, 4]. During the past four decades, an active line of research aims at describing microstates underlying these thermodynamic behaviors. The present work would also be in the same line.

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P. Nicolini et al. (eds.), *2nd Karl Schwarzschild Meeting on Gravitational Physics*, Springer Proceedings in Physics 208,
https://doi.org/10.1007/978-3-319-94256-8_6

Motivated by the microstate counting for usual thermodynamic systems (e.g. an ideal gas) which is based on their phase spaces, here we try to build the classical symplectic structure for the set of *extremal* (vanishing temperature) BHs. Interested reader can refer to the original papers [5, 6] for detailed analysis, or to [7] as a pedagogical text. In this analysis, we keep the spacetime dimension to be arbitrary d and the theory to be given by the Einstein-Hilbert Lagrangian $\mathscr{L} = \frac{1}{16\pi G}R$, where R is the Ricci scalar. In order to make the analysis simpler, we will concentrate on extremal BHs with $d-3$ number of commuting $U(1)$ axial isometries, denoted by $U(1)^{d-3}$.

Significantly, thermodynamic properties of the BHs are encoded in their near horizon region. The temperature and other chemical potentials, in addition to BHs conserved charges can be read directly from that region. Interestingly, Iyer and Wald have shown that BH entropy is the conserved charge associated to the Killing vector of the horizon, which is calculated on the horizon [4, 8]. Hence, one expects to find the microstates of black holes by focusing on their near horizon region. Therefore, we study the phase space of near horizon geometries of extremal black holes (NHEG). These solutions share some interesting features:

- Taking the near horizon limit of an extremal BH as a solution to a given theory leads to a near horizon extremal geometry (NHEG) which is a solution to the same theory [9] (because they are found by a limiting process instead of approximation process [7]).
- Stationarity of BH is enhanced to $SL(2,\mathbb{R})$ in NHEG, therefore the symmetries of NHEG with the above mentioned properties is $SL(2,\mathbb{R})\times U(1)^{d-3}$.
- NHEGs are uniquely identified by $d-3$ number of angular momenta J_i [10].
- Under appropriate isometry and boundary conditions, perturbations on NHEGs are restricted (*upto infinitesimal diffeomophisms*) to parametric variations, i.e. infinitesimal variations of the solution identified by J_i to an adjacent solution identified by $J_i+\delta J_i$ [11].

The metric of the considered NHEGs can be written in a suitable coordinate system as [12, 13]:

$$\mathrm{d}s^2 = \Gamma(\theta)\Big[-r^2\mathrm{d}t^2+\frac{\mathrm{d}r^2}{r^2}+\mathrm{d}\theta^2+\gamma_{ij}(\theta)(\mathrm{d}\varphi^i+k^i r\,\mathrm{d}t)(\mathrm{d}\varphi^j+k^j r\,\mathrm{d}t)\Big]. \tag{6.1}$$

$\Gamma(\theta)$ and $\gamma_{ij}(\theta)$ might be determined by imposing the equation of motion over the above ansatz, or by taking the near horizon limit of a given BH. In the coordinate which the metric is represented, the Killing vectors of $SL(2,\mathbb{R})\times U(1)^{d-3}$ isometry are explicitly as

$$\xi_-=\partial_t\,,\qquad \xi_0=t\partial_t-r\partial_r,\qquad \xi_+=\frac{1}{2}\Big(t^2+\frac{1}{r^2}\Big)\partial_t-tr\partial_r-\frac{k^i}{r}\partial_{\varphi^i}\,,\qquad \mathrm{m}_i=\partial_{\varphi^i}\,.$$

Their commutation relation is

$$[\xi_0, \xi_-] = -\xi_-, \qquad [\xi_0, \xi_+] = \xi_+, \qquad [\xi_-, \xi_+] = \xi_0\,, \qquad [\xi_a, m_i] = 0, \tag{6.2}$$

in which $a \in \{-, 0, +\}$ and $i \in \{1, \ldots, d-3\}$.

A significant property of NHEG geometry is that any surface of constant (t, r) is the bifurcation point of a Killing horizon [5, 6], which we denote by $\mathscr{H}$. Explicitly, the $d-2$ surface $\mathscr{H}$ determined by $t = t_{\mathscr{H}}, r = r_{\mathscr{H}}$, is the intersection of the following $d-1$-dimensional null hypersurfaces

$$\mathscr{N}_{\mathscr{H}+} :\ t + \frac{1}{r} = t_{\mathscr{H}} + \frac{1}{r_{\mathscr{H}}}\,, \qquad \mathscr{N}_{\mathscr{H}-} :\ t - \frac{1}{r} = t_{\mathscr{H}} - \frac{1}{r_{\mathscr{H}}}\,. \tag{6.3}$$

The magical Killing vector generating the above null two hypersurfaces is

$$\zeta_{\mathscr{H}} = n^a_{\mathscr{H}}\, \xi_a - k^i m_i\,, \tag{6.4}$$

in which

$$n^-_{\mathscr{H}} = -\frac{t^2_{\mathscr{H}} r^2_{\mathscr{H}} - 1}{2r_{\mathscr{H}}}\,, \qquad n^0_{\mathscr{H}} = t_{\mathscr{H}} r_{\mathscr{H}}\,, \qquad n^+_{\mathscr{H}} = -r_{\mathscr{H}}\,. \tag{6.5}$$

Therefore $\mathscr{N} = \{\mathscr{N}_{\mathscr{H}+} \cup \mathscr{N}_{\mathscr{H}-}\}$ is a Killing horizon and their intersection $\mathscr{H}$ is the bifurcation surface. It is shown [14] that entropy of the NHEG (which is equal to the entropy of original BH) is conserved charge associated to the $\zeta_{\mathscr{H}}$, calculated on $\mathscr{H}$.

6.2 A Review on Covariant Phase Space Method

By definition, a phase space is a manifold consisting of a set of allowed configurations of a system equipped with a symplectic form Ω_{ab} (i.e a nondegenerate closed 2 form). For a given gauge theory like Einstein gravity, and a given collection of geometries viewed as an abstract manifold, there is a well established method for defining the symplectic structure over this manifold, and thereby obtain a phase space [15]. This is known as covariant phase space method, since the construction of phase space does not involve with breaking of covariance of the theory (unlike what happens in ADM construction). Here we give the general method for construction of the symplectic structure. In next section, we specify exactly what is the set of geometries relevant for the construction of NHEG phase space (Fig. 6.1).

In covariant phase space method, the manifold $\mathscr{M}$ is built up of a set of metric configurations $g_{\mu\nu}(x^\alpha)$. Therefore vectors tangent to the phase space are indeed perturbations of the metric. For Einstein gravity, the symplectic 2-form acting on two vectors $\delta_1 g, \delta_2 g$ is given by

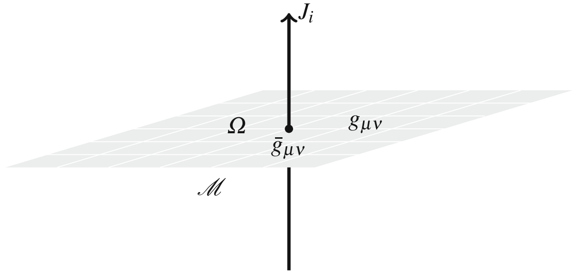

Fig. 6.1 A schematic of NHEG phase space, in terms of angular momenta $\mathbf{J}$. The manifold $\mathscr{M}$ is comprised of some metric configurations $g_{\mu\nu}(x^\alpha)$. The point $\bar{g}_{\mu\nu}$ is the known NHEG solution. Symplectic 2-form Ω, is the Lee-Wald form, upto $\mathbf{Y}$ ambiguities

$$\Omega(\delta_1 g, \delta_2 g, g) = \int_\Sigma \boldsymbol{\omega}(\delta_1 g, \delta_2 g, g) \tag{6.6}$$

where the symplectic current $\boldsymbol{\omega}$ is

$$\boldsymbol{\omega}(\delta_1 g, \delta_2 g, g) \equiv \delta_1 \boldsymbol{\Theta}(\delta_2 g, g) - \delta_2 \boldsymbol{\Theta}(\delta_1 g, g)\,. \tag{6.7}$$

The integration surface Σ in (6.6) is a $d-1$-dim hypersurface. The $d-1$-form $\boldsymbol{\Theta}$ is defined through the variation of the Lagrangian (as a top form) after using the equations of motion, i.e $\delta\mathbf{L} \approx \mathrm{d}\boldsymbol{\Theta}$ (In this paper $\approx$ means on shell equality). For the Einstein gravity [8]

$$\boldsymbol{\Theta}(\delta g_{\mu\nu}, g_{\mu\nu}) = \frac{\sqrt{-g}}{(d-1)!}\ \varepsilon_{\mu\mu_1\cdots\mu_{d-1}}\ \frac{1}{16\pi G}(\nabla_\alpha h^{\mu\alpha} - \nabla^\mu h)\ \mathrm{d}x^{\mu_1}\wedge\cdots\wedge \mathrm{d}x^{\mu_{d-1}} \tag{6.8}$$

where $h^{\mu\nu} \equiv g^{\mu\sigma} g^{\nu\tau}\delta g_{\sigma\tau}$ and $h \equiv h^\alpha_{\ \alpha}$ and $\varepsilon_{\mu_1\cdots\mu_d}$ is the Levi-Civita symbol. If the variation δg satisfies the linearized equation of motion, then it can be shown that the symplectic current is closed on-shell, i.e. $\mathrm{d}\boldsymbol{\omega}(\delta_1 g, \delta_2 g, g) \approx 0$.

Now we turn to the definition of symmetries of the covariant phase space and their corresponding conserved charges. An infinitesimal symmetry of a phase space, is an infinitesimal coordinate transformation $x \to x - \xi$ such that any metric configuration in the phase space is sent to another configuration in the phase space. In other words, although the configurations are transformed under the symmetry action, but the whole phase space is closed under the symmetry action. Now the corresponding conserved charge H_ξ which is the generator of the symmetry transformation is defined through the contraction of $\mathscr{L}_\xi g$ with the symplectic form

$$\delta H_\xi \equiv \Omega(\delta g, \delta_\xi g, g) = \int_\Sigma \delta\boldsymbol{\Theta}(\mathscr{L}_\xi g, g) - \mathscr{L}_\xi \boldsymbol{\Theta}(\delta g, g). \tag{6.9}$$

It can be shown (e.g. see Appendix C.2 in [7]) that the integrand is on-shell an exact form $\mathrm{d}\boldsymbol{k}_\xi(\delta g, g)$. Therefore the charges can alternatively be defined (using Stoke's theorem) through the integration of $\boldsymbol{k}_\xi$ over $\partial\Sigma$ which is a codimension 2 closed surface. The latter is even more fundamental for geometries with more than one boundaries. In Einstein gravity $\boldsymbol{k}_\xi$ is [16]

$$\boldsymbol{k}_\xi(\delta g_{\mu\nu}, g_{\mu\nu}) = \frac{\sqrt{-g}}{(d-2)!\,2!}\ \varepsilon_{\mu\nu\mu_1\cdots\mu_{d-2}}\, k_\xi^{\mu\nu}\ \mathrm{d}x^{\mu_1}\wedge\cdots\wedge \mathrm{d}x^{\mu_{d-2}} \tag{6.10}$$

where

$$k_\xi^{\mu\nu} = \frac{1}{16\pi G}\left(\left[\xi^\nu\nabla^\mu h - \xi^\nu\nabla_\sigma h^{\mu\sigma} + \xi_\sigma\nabla^\nu h^{\mu\sigma} + \frac{1}{2}h\nabla^\nu\xi^\mu - h^{\rho\nu}\nabla_\rho\xi^\mu\right] - [\mu\leftrightarrow\nu]\right). \tag{6.11}$$

6.3 The NHEG Phase Space

Now in order to construct the NHEG phase space, we need to specify the set of relevant geometrie. The rough idea is that phase space configurations can serve as the microstates of extremal black hole. According to the uniqueness of dynamical perturbations, the set of relevant geometries are obtained by coordinate transformations of the background NHEG geometry. These transformations are infinitesimally obtained by a vector field χ through $x \to x - \chi$. We refer the interested reader to the original papers for the arguments for determination of χ, and state the result here. The vector field χ is given by

$$\chi[\varepsilon(\boldsymbol{\varphi})] = -\mathbf{k}\cdot\boldsymbol{\partial}_\varphi\varepsilon\left(\frac{1}{r}\partial_t + r\partial_r\right) + \varepsilon\mathbf{k}\cdot\boldsymbol{\partial}_\varphi\,, \tag{6.12}$$

where ε can be any periodic smooth function of the coordinates φ^i. Hence, χ generates the infinitesimal perturbations tangent to the phase space around the background by $\delta g[\varepsilon(\boldsymbol{\varphi})] = \mathcal{L}_\chi\bar{g}$. Exponentiation of this infinitesimal transformation produces the finite coordinate transformations which transfer $\bar{g}_{\mu\nu}$ to arbitrary configurations $g_{\mu\nu}$ of the phase space $\mathscr{M}$. The finite coordinate transformation is

$$\bar{t} = t - \frac{1}{r}(e^\Psi - 1), \qquad \bar{r} = re^{-\Psi}, \qquad \bar{\theta} = \theta\,, \qquad \bar{\varphi}^i = \varphi^i + k^iF\,. \tag{6.13}$$

We call $F(\boldsymbol{\varphi})$ the *wiggle function* which is periodic in all its arguments and Ψ is given by $e^\Psi = 1 + \mathbf{k}\cdot\boldsymbol{\partial}_\varphi F$. Therefore, corresponding to any function F, a configuration over $\mathscr{M}$ with the following metric is identified

$$\mathrm{d}s^2 = \Gamma(\theta)\left[-(\boldsymbol{\sigma} - \mathrm{d}\Psi)^2 + \left(\frac{\mathrm{d}r}{r} - \mathrm{d}\Psi\right)^2 + \mathrm{d}\theta^2 + \gamma_{ij}(\mathrm{d}\tilde{\varphi}^i + k^i\boldsymbol{\sigma})(\mathrm{d}\tilde{\varphi}^j + k^j\boldsymbol{\sigma})\right], \tag{6.14}$$

in which $\boldsymbol{\sigma} = e^{-\Psi}r\,\mathrm{d}(t+\frac{1}{r}) + \frac{\mathrm{d}r}{r}$ and $\tilde{\varphi}^i = \varphi^i + k^i(F-\Psi)$.

By construction, the infinitesimal transformations generated by χ are symmetries of the NHEG phase space. However, χ has also another important significance, i.e that χ is the *symplectic symmety* of the NHEG phase space.[1] The notion of symplectic symmetry is defined as

Definition 6.1 The vector field χ is the generator of a *symplectic symmetry generators* iff [5]

1. $\boldsymbol{\omega}(\delta g, \delta_\chi g, g) \approx 0 \qquad \forall g \in \mathscr{M}$ and $\delta g \in T\mathscr{M}$,
2. δH_χ be integrable, and H_χ be finite over the $\mathscr{M}$,

Thanks to the properties of diffeomorphisms, any point of the phase space has complete $SL(2,\mathbb{R}) \times U(1)^{d-3}$ isometry. It can be checked that any configuration has the same angular momenta $\mathbf{J}$ and entropy S as the NHEG metric $\bar{g}_{\mu\nu}$ background.

The symplectic symmetries form a closed algebra. To see this we expand χ in its Fourier modes

$$\chi_{\mathbf{n}} = -e^{-i(\mathbf{n}\cdot\boldsymbol{\varphi})}\left(i(\mathbf{n}\cdot\mathbf{k})\left(\frac{1}{r}\partial_t + r\partial_r\right) + \mathbf{k}\cdot\boldsymbol{\partial}_{\varphi}\right). \tag{6.15}$$

Then, the commutator of these vectors is

$$[\chi_{\mathbf{n}}, \chi_{\mathbf{m}}] = i\,\mathbf{k}\cdot(\mathbf{n}-\mathbf{m})\,\chi_{\mathbf{n}+\mathbf{m}} \tag{6.16}$$

which is a nice generalization of Witt algebra. It can be shown that the corresponding Hamiltonian generators are [5, 6]

$$H_{\mathbf{n}} = \oint_{\mathscr{H}} \boldsymbol{\varepsilon}_{\mathscr{H}}\; T[\Psi]e^{-i\mathbf{n}\cdot\boldsymbol{\varphi}}, \tag{6.17}$$

where

$$T[\Psi] = \frac{1}{16\pi G}\Big((\Psi')^2 - 2\Psi'' + 2e^{2\Psi}\Big) \tag{6.18}$$

and primes are directional derivatives along the vector $\mathbf{k}$. The function $T[\Psi]$ transforms under infinitesimal phase space transformations as

$$\delta_\varepsilon T = \varepsilon T' + 2\varepsilon' T - \frac{1}{8\pi G}\varepsilon'''. \tag{6.19}$$

Therefore the function $T[\Psi]$ resembles a Liouville type stress tensor.

The Poisson bracket of conserved charges have the same commutation relations as (6.16) up to a central extension. Significantly, the central extension turns out to be the entropy of the NHEG. Explicitly [5, 6]

$$\{H_{\mathbf{m}}, H_{\mathbf{n}}\} = -i\mathbf{k}\cdot(\mathbf{m}-\mathbf{n})H_{\mathbf{m}+\mathbf{n}} - i(\mathbf{k}\cdot\mathbf{m})^3\frac{S}{2\pi}\,\delta_{\mathbf{m}+\mathbf{n},0}. \tag{6.20}$$

[1]The definition of a consistent symplectic form on NHEG phase space, however involves a suitable fixing of ambiguities in the symplectic current. For details, see [5, 6].

Using the Dirac quantization rules $\{\ \ \} \to \frac{1}{i}[\ \ ]$ and $H_{\mathbf{n}} \to L_{\mathbf{n}}$, the symmetry algebra promotes to an operator algebra, the *NHEG algebra* $\widehat{\mathscr{V}_{\mathbf{k},S}}$

$$[L_{\mathbf{m}}, L_{\mathbf{n}}] = \mathbf{k}\cdot(\mathbf{m}-\mathbf{n})L_{\mathbf{m}+\mathbf{n}} + \frac{S}{2\pi}(\mathbf{k}\cdot\mathbf{m})^3\delta_{\mathbf{m}+\mathbf{n},0}\,. \tag{6.21}$$

The J_i and H_{ξ_a} commute with $L_{\mathbf{n}}$, and are therefore central elements of the NHEG algebra $\widehat{\mathscr{V}_{\mathbf{k},S}}$. Also by Definition 6.1, they are symplectic symmetry generators. Hence, the *full symplectic symmetry of the phase space* is

$$\text{NHEG Symplectic Symmetry Algebra} = \widehat{\mathscr{V}_{\mathbf{k},S}} \oplus \ \mathrm{sl}(2,\mathbb{R}) \underbrace{\oplus\ \mathrm{u}(1)}_{(d-3\text{ times})}\,. \tag{6.22}$$

We stress again that all geometries in the phase space have vanishing $SL(2,\mathbb{R})$ charges, and $U(1)$ charges equal to J_i.

Yet many different mathematical and physical aspects of the NHEG phase space and its algebra are yet to be understood and analyzed.

Acknowledgements The authors would like to thank M.M. Sheikh Jabbari and Geoffrey Compère for the collaboration. They also acknowledge financial support of Allameh Tabatabai grant of Bonyad Melli Nokhbegan of Iran.

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Chapter 7
Einstein-Charged Scalar Field Theory: Black Hole Solutions and Their Stability

S. Ponglertsakul, S. Dolan and E. Winstanley

Abstract A complex scalar field on a charged black hole in a cavity is known to experience a superradiant instability. We investigate possible final states of this instability. We find hairy black hole solutions of a fully coupled system of Einstein gravity and a charged scalar field. The black holes are surrounded by a reflecting mirror. We also investigate the stability of these black holes.

7.1 Introduction

In black hole physics, there is a mechanism where rotational (electromagnetic) energy can be extracted from a rotating (charged) black hole. This is called superradiant scattering. More specifically, the amplitude of a scalar field around a black hole will be amplified if the frequency σ of the field satisfies [1] $\sigma < m\Omega_{\rm H} + q\Phi_{\rm H}$, where $m, q, \Omega_{\rm H}$ and $\Phi_{\rm H}$ are the azimuthal quantum number, scalar field charge, angular velocity and electric potential at the outer horizon respectively. By setting $\Omega_{\rm H} = 0$, the charged version of the superradiant condition is obtained.

One can create an instability of the spacetime background via a superradiant scattering process, if there is some mechanism to confine the bosonic field within the vicinity of the black hole. Then wave modes will be repeatedly scattered off the black hole and their amplitude will be intensified. The back-reaction of the field modes on the background will eventually become significant. In the charged case, the trapping mechanism can be induced by either (i) a reflecting mirror [2] or (ii) anti-de Sitter

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P. Nicolini et al. (eds.), *2nd Karl Schwarzschild Meeting on Gravitational Physics*, Springer Proceedings in Physics 208,
https://doi.org/10.1007/978-3-319-94256-8_7

boundary condition [4]. By having either of these with a charged black hole, the superradiant instability can be triggered.

An interesting question that one might ask is, what is the end-point of this charged-scalar superradiant instability? To tackle this problem, a fully non-linear analysis is required. Hence in this talk, we investigate possible end-points of the superradiant instability for charged black holes with a reflecting mirror. More specifically, a coupled system of gravity and a massless complex scalar field with a mirror-like boundary condition is studied. Numerical black hole solutions with a non-trivial scalar field are obtained. By considering linearised perturbations of these black holes, a numerical analysis of the black hole's stability is undertaken. Here we present a selection of plots to illustrate our numerical results. More details of this work can be found in [3].

7.2 Linear Perturbations in Electrovacuum

In this section, a massless complex scalar field ϕ on the Reissner–Nordström (RN) background in a cavity is considered. In the test-field limit, the dynamics of a scalar field on RN spacetime is described by the Klein–Gordon (KG) equation. By substituting the ansatz $\phi \sim e^{-i\sigma t} R(r)$, where σ and $R(r)$ are respectively the frequency and the radial part of the scalar field, into the KG equation, we obtain a second order differential equation in terms of the radial part. To solve this equation, we apply the following boundary conditions: (i) an ingoing wave near the horizon $r \to r_{\rm h}$ and (ii) at the mirror the scalar field vanishes, so $R(r_{\rm m}) = 0$, where $r_{\rm m}$ is the location of the mirror. Then a numerical technique called the shooting method is implemented. We scan for corresponding frequencies σ such that the perturbations satisfy the boundary conditions.

With the black hole mass fixed to be $M = 1$, the example plot below (Fig. 7.1) illustrates the frequency as a function of the location of mirror. It is clear from Fig. 7.1a, b that this system experiences a superradiant instability as there are regions where $\mathrm{Im}(\sigma) > 0$, indicating an unstable mode. One can learn the following from these plots: when the location of mirror $r_{\rm m}$ is small (close to black hole), the field mode decays exponentially in time; instability occurs when $r_{\rm m}$ reaches a certain value.

We find that a massless charged scalar field on the RN background in a cavity experiences a superradiant instability. These results are in agreement with previous work done by Herdeiro et al. [2], where they studied a massive complex scalar field on a charged black hole with a mirror. To fully understand what could happen at the end-point of this instability, a non-linear system of gravity and a charged scalar field must be investigated. In the next section, a fully coupled Einstein-charged scalar system will be considered.

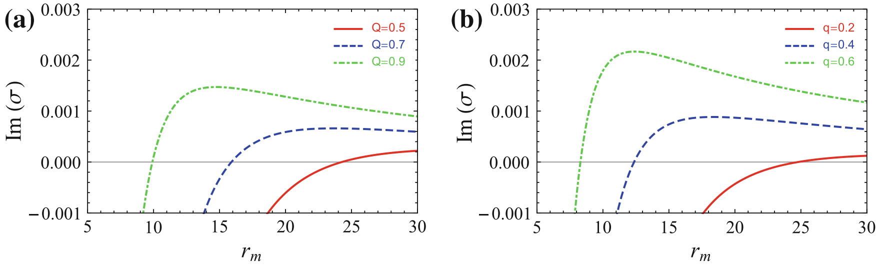

Fig. 7.1 The imaginary part of σ is plotted as a function of the location of the mirror r_m, **a** for fixed scalar charge $q = 0.5$ and different values of the black hole charge Q, **b** for fixed $Q = 0.9$ and different values of q. Taken from [3]

7.3 Static Black Holes

The self-gravitating system of a charged scalar field is described by the following action

$$S = \int \mathrm{d}^4x\sqrt{-g}\left[\frac{R}{16\pi G} - \frac{1}{4}F_{ab}F^{ab} - \frac{1}{2}g^{ab}D^*_{(a}\phi^* D_{b)}\phi\right]. \tag{7.1}$$

In (7.1) the Faraday tensor is defined by $F_{ab} = \nabla_a A_b - \nabla_b A_a$, where A_a is the electromagnetic potential, $D_a = \nabla_a - iqA_a$, q is the scalar field charge and $X_{(ab)} = \frac{1}{2}(X_{ab} + X_{ba})$. Varying (7.1), we obtain three equations of motion

$$G_{ab} = 8\pi G\left(T^F_{ab} + T^\phi_{ab}\right), \tag{7.2}$$

$$\nabla_a F^{ab} = \frac{iq}{2}\left(\phi^* D^b\phi - \phi(D^b\phi)^*\right), \tag{7.3}$$

$$D_a D^a\phi = 0. \tag{7.4}$$

We consider a static spherically symmetry black hole spacetime with line element

$$\mathrm{d}s^2 = -f(r)h(r)\mathrm{d}t^2 + f(r)^{-1}\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2\right). \tag{7.5}$$

In addition, the electromagnetic vector potential is $A_a \equiv [A_0(r), 0, 0, 0]$ and the scalar field depends on r only $\phi = \phi(r)$. By inserting these ansatzes into (7.2)–(7.4), we obtain three coupled ordinary differential equations. By imposing appropriate boundary conditions at the event horizon and at the mirror, these coupled equations can be solved numerically. We also require that the scalar field must vanish at the location of the mirror.

To obtain static solutions, three parameters must be specified: ϕ_h, the value of the scalar field on the horizon; $E_\mathrm{h} \equiv A_0'(r_\mathrm{h})$, the electric field on the horizon; and q. In Fig. 7.2, we show some example solutions where the black hole radius is fixed at $r_h = 1$. The nontrivial structure of the scalar field can be seen. Because these

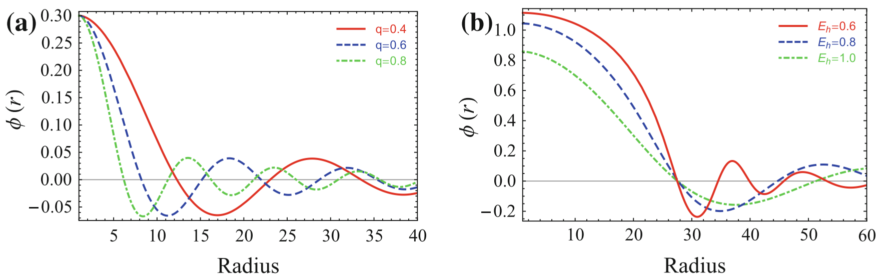

Fig. 7.2 The scalar field $\phi(r)$ is plotted as a function of radius **a** for fixed $\phi_{\rm h} = 0.3$, $E_{\rm h} = 0.6$ and different values of q, **b** for fixed $q = 0.1$ and different values of $E_{\rm h}$. Taken from [3]

scalar fields oscillate around zero therefore, one can put the mirror at any zero of ϕ, however, in this work, we only consider the case where the mirror is located at the first zero, since these solutions are expected to be stable.

By varying the three parameters $\phi_{\rm h}$, $E_{\rm h}$ and q, we obtain different hairy black hole solutions. In Fig. 7.2a, three distinct black hole solutions with three different scalar charges are displayed. Note that these solutions possess three different mirror radii. However, it is possible that different static solutions can share the same mirror location as illustrated in Fig. 7.2b.

7.4 Stability of the Hairy Black Holes

In the previous section, we have shown that Einstein-charged scalar field theory in a cavity allows the existence of hairy black holes. Our next important question is, are these solutions stable or unstable? If they are shown to be stable, they could represent a possible end-point of the superradiant instability for a massless charged scalar perturbation on the RN background with a mirror.

We consider linear spherically symmetric perturbations, where the four field variables (f, h, A_0, ϕ) are rewritten as follows, $f = \bar{f}(r) + \delta f(t, r)$ and similarly for the other three quantities. In this notation, $\bar{f}$ is the equilibrium quantity and δf is the perturbed part. By linearising the field equations, we arrive at three coupled perturbation equations in terms of δA_0 and the real and imaginary parts of $\delta\phi$. These perturbation equations are very complicated and lengthy, full details of these equations can be found in [3]. The three perturbation equations consist of two dynamical equations describing the real and imaginary parts of the scalar field respectively, the other one is a constraint equation. By substituting the ansatz $\delta\phi \sim e^{-i\sigma t}\tilde{\phi}(r)$ (and similarly for the other perturbations) into the equations, we can integrate the perturbation equations numerically. The perturbation modes are required to satisfy the boundary conditions that $\tilde{\phi}(r)$ and other perturbation modes have an ingoing wave-like condition near the horizon, and at the mirror the scalar field perturbations must vanish ($\tilde{\phi}(r_{\rm m}) = 0$).

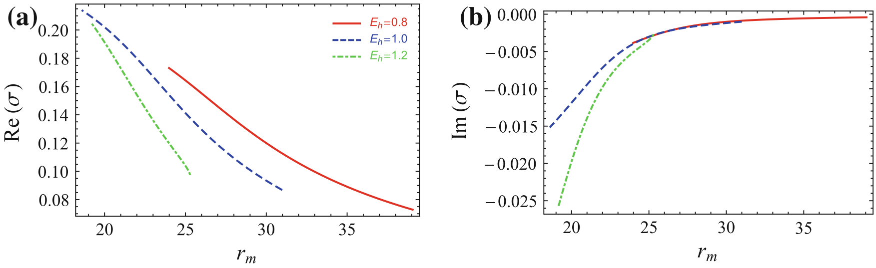

Fig. 7.3 The real **a** and imaginary parts **b** of the perturbation frequency σ are plotted against the location of the mirror r_{m} with the scalar charge fixed to be $q = 0.2$, ϕ_{h} varying from 0.1 to 1.3 and different values of E_{h}. We find that $\mathrm{Im}(\sigma) < 0$. Taken from [3]

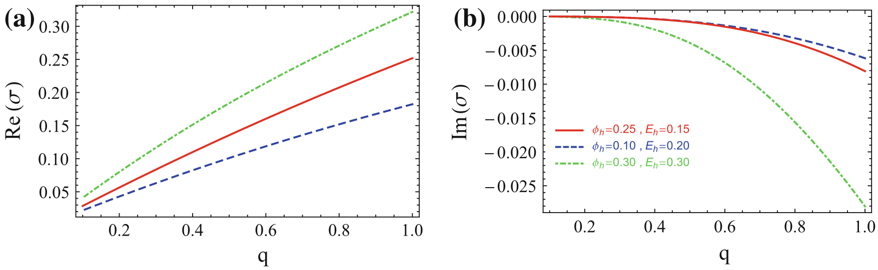

Fig. 7.4 The real **a** and imaginary parts **b** of the perturbation frequency σ are plotted against the scalar charge q with a selection of various values of ϕ_{h} and E_{h}. Taken from [3]

The numerical scheme is as follows. Firstly, static background parameters are specified, q, ϕ_{h} and E_{h}, then the equilibrium field equations are integrated. From the solution, we locate the first zero of the equilibrium scalar field, setting this to be the location of the mirror r_{m}. Then the coupled perturbation equations are solved by scanning for frequencies σ such that the perturbations satisfy the required boundary conditions.

In Fig. 7.3, the real and imaginary parts of σ are plotted as functions of the mirror radius r_{m} for scalar field charge $q = 0.2$. In each plot, the parameter describing static solution varies between $\phi_{\mathrm{h}} = 0.1 - 1.3$. Thus each point in this plot refers to the perturbation frequency for one distinct hairy black hole. With the mirror at the first zero of the static scalar field ϕ, we find one value of σ for which the perturbations satisfy the boundary conditions. Figure 7.3b shows that the perturbation modes decay exponentially in time since we find that $\mathrm{Im}(\sigma) < 0$. In addition, the real and imaginary parts of the perturbation frequency σ are plotted as functions of the scalar charge q are displayed in Fig. 7.4. Here in this example, for each curve, a selection of values of static black hole parameters ϕ_{h} and E_{h} are fixed. Figure 7.4b illustrates that the perturbation modes are exponentially decaying in time. We refer the reader to [3] where we find $\mathrm{Im}(\sigma) < 0$ for all black hole solutions investigated.

7.5 Summary

We have studied a coupled system involving Einstein gravity and a complex charged scalar field in the presence of a mirror-like boundary condition. Numerical hairy black hole solutions were obtained by the shooting method. By putting a mirror at the first node of the equilibrium scalar field, we showed these black hole solutions are stable under spherically symmetric linear perturbations. Therefore, we conclude that these black holes could represent an end-point of the superradiant instability of Reissner–Nordstörm black holes to charged scalar field perturbations.

Acknowledgements The work of SRD and EW is supported by the Lancaster-Manchester-Sheffield Consortium for Fundamental Physics under STFC grant ST/L000520/1. The work of SRD is also supported by EPSRC grant EP/M025802/1. The work of SP is supported by the 90th Anniversary of Chulalongkorn University Fund (Ratchadaphiseksomphot Endowment Fund).

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Chapter 8
The Good Properties of Schwarzschild's Singularity

O. C. Stoica

Abstract The most notable problems of General Relativity (GR), such as the occurrence of singularities and the information paradox, were initially found on the background provided by Schwarzschild's solution. The reason is that this solution has singularities, widely regarded as a big problem of GR. While the event horizon singularity can be removed by moving to non-singular coordinates, not the same is true about the $r = 0$ singularity. However, I will present coordinates which make the metric finite and analytic at the singularity $r = 0$. The metric becomes degenerate at $r = 0$, so the singularity still exists, but it is of a type that can be described geometrically by referring to finite quantities only. Also, the topology of the causal structure is shown to remain intact, and the solution is globally hyperbolic. This suggests a possible solution to the black hole information paradox, in the framework of GR. As a side effect, the Schwarzschild singularity belongs to a class of singularities accompanied by dimensional reduction effects, which are hoped to cure the infinities in perturbative Quantum Gravity.

8.1 Extending the Schwarzschild Solution Beyond the Singularity

As it is well known, the Schwarzschild solution of Einstein's equation is

$$\mathrm{d}s^2 = -\left(1 - \frac{2m}{r}\right)\mathrm{d}t^2 + \left(1 - \frac{2m}{r}\right)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\sigma^2, \tag{8.1}$$

where

$$\mathrm{d}\sigma^2 = \mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2 \tag{8.2}$$

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P. Nicolini et al. (eds.), *2nd Karl Schwarzschild Meeting on Gravitational Physics*, Springer Proceedings in Physics 208,
https://doi.org/10.1007/978-3-319-94256-8_8

is the metric of the unit sphere S^2, m the mass of the body, and the units are chosen such that $c = 1$ and $G = 1$ (see for example [1], p. 149). It represents a spherically symmetric static and electrically neutral black hole. The metric (8.1) has singularities at $r = 0$ and $r = 2m$, which puzzled Schwarzschild, who decided to replace the coordinate r with $R = r - 2m$, so that the only singularity is in the new origin $R = 0$.

However, there are other coordinates for the Schwarzschild black hole, which remove the singularity $r = 2m$, for example the Eddington–Finkelstein coordinates [2, 3]. This shows that the event horizon singularity is due to the coordinates, which themselves are singular.

Unfortunately, changing the coordinates cannot be used to remove the singularity $r = 0$, as can be seen from the fact that the Kretschmann scalar $R_{abcd}R^{abcd}$ is infinite at $r = 0$ in any coordinates.

Fortunately, coordinate transformations can remove "half" of the singularity, so that the metric g_{ab} is made finite and extends analytically beyond the singularity [4].

Theorem 8.1 *The Schwarzschild metric can be extended analytically at* $r = 0$.

Proof To see this, let us apply the coordinate transformation

$$\begin{cases} r = \tau^2 \\ t = \xi\tau^T \end{cases} \tag{8.3}$$

where $T \geq 2$ is an integer. Then, the components of the Jacobian matrix are

$$\frac{\partial r}{\partial \tau} = 2\tau, \ \frac{\partial r}{\partial \xi} = 0, \ \frac{\partial t}{\partial \tau} = T\xi\tau^{T-1}, \ \frac{\partial t}{\partial \xi} = \tau^T. \tag{8.4}$$

In the new coordinates, the components of the metric metric become

$$g_{\tau\tau} = -\frac{4\tau^4}{2m - \tau^2} + T^2\xi^2(2m - \tau^2)\tau^{2T-4} \tag{8.5}$$

$$g_{\tau\xi} = T\xi(2m - \tau^2)\tau^{2T-3} \tag{8.6}$$

$$g_{\xi\xi} = (2m - \tau^2)\tau^{2T-2} \tag{8.7}$$

and its determinant

$$\det g = -4\tau^{2T+2}. \tag{8.8}$$

Then, the four-metric becomes

$$\mathrm{d}s^2 = -\frac{4\tau^4}{2m - \tau^2}\mathrm{d}\tau^2 + (2m - \tau^2)\tau^{2T-4}(T\xi\mathrm{d}\tau + \tau\mathrm{d}\xi)^2 + \tau^4\mathrm{d}\sigma^2, \tag{8.9}$$

which remains finite and is in fact analytic at $r = 0$. □

Given that T can be any integer $T \geq 2$, we have obtained an infinite number of solutions. However, a unique solution among them has special properties, as I will explain in the following.

8.2 The Most Regular Extension

From geometric point of view, the problem with singular metrics is the following. In semi-Riemannian geometry (where the metric is regular), one can define in a natural way a unique connection which preserves the metric and is torsionless. Then, we can define covariant derivatives for tensor fields, which enable us to write field equations. Also, the curvature tensor, needed for the Einstein equation, can be defined and is unique. But if the metric becomes singular, there is no way to define a covariant derivative and curvature tensor by usual means. The reason is that both the metric tensor g_{ab} and its reciprocal g^{ab} are used in the construction of these objects. When g_{ab} has infinite components–as it happens in the Schwarzschild metric (8.1), the connection and curvature can no longer be defined. Even if all of the components of g_{ab} are finite, but it is degenerate (its determinant vanishes), g^{ab} is not defined or is singular, and one cannot define the connection and curvature. If the metric is degenerate with constant signature, Kupeli showed one can define a sort of connection and curvature, but his construction is not invariant and not unique, relying on choosing at each point a subspace of the tangent space complementary to the isotropic subspace [5, 6]. But in [7, 8] it was shown that we can do this in an invariant way, and it also works for a large class of metrics with variable signature (which are the ones needed in GR). For this kind of metrics (named *semi-regular* in [7]) the covariant derivatives can be defined for a large class of differential forms. Also, a differential operator which plays the same role as the covariant derivative can be defined for vector fields. It turns out that for semi-regular metrics we can also define the Riemann curvature tensor R_{abcd} (although $R^a{}_{bcd}$ usually is still singular). Moreover, the Einstein equation can be cast in a form which has the same content outside the singularities, but also works at semi-regular singularities [7, 9].

If the spacetime events where the metric is regular form a dense subset of the spacetime, the metric is semi-regular if the contractions $g^{st}\Gamma_{abs}\Gamma_{cdt}$ are smooth [7], where $\Gamma_{abc} = \frac{1}{2}(\partial_a g_{bc} + \partial_b g_{ca} - \partial_c g_{ab})$ are Christoffel's symbols of the first kind. The reciprocal metric g^{st} becomes infinite at the singularity, but $g^{st}\Gamma_{abs}\Gamma_{cdt}$ remains smooth. For a general and invariant definition of semi-regular metrics see [7].

Among the solutions (8.9), there is only one with semi-regular metric [4].

Theorem 8.2 *The solution (8.9) can be extended analytically so that the singularity at $r = 0$ is semi-regular, if and only if $T = 4$.*

Proof In the coordinate system (8.3), Christoffel's symbols of the first kind Γ_{abc} are also smooth. Since at $r = 0$ the determinant of the metric vanishes, g^{st} is singular. But we can find T so that $\Gamma_{abs}\Gamma_{cdt}$ compensates this singularity and the contractions $g^{st}\Gamma_{abs}\Gamma_{cdt}$ are smooth.

The reciprocal metric has the components

$$g^{\tau\tau} = -\frac{1}{4}(2m - \tau^2)\tau^{-4} \tag{8.10}$$

$$g^{\tau\xi} = \frac{1}{4}T\xi(2m - \tau^2)\tau^{-5} \tag{8.11}$$

$$g^{\xi\xi} = \frac{\tau^{-2T+2}}{2m - \tau^2} - \frac{1}{4}T^2\xi^2(2m - \tau^2)\tau^{-6} \tag{8.12}$$

The partial derivatives of the coefficients of the metric are

$$\partial_\tau g_{\tau\tau} = 8\frac{\tau^5 - 4m\tau^3}{(2m - \tau^2)^2} + 2T^2(2T - 4)m\xi^2\tau^{2T-5} - T^2(2T - 2)\xi^2\tau^{2T-3}, \tag{8.13}$$

$$\partial_\tau g_{\tau\xi} = 2T(2T - 3)m\xi\tau^{2T-4} - T(2T - 1)\xi\tau^{2T-2}, \tag{8.14}$$

$$\partial_\tau g_{\xi\xi} = 2m(2T - 2)\tau^{2T-3} - 2T\tau^{2T-1}, \tag{8.15}$$

$$\partial_\xi g_{\tau\tau} = 2T^2\xi(2m - \tau^2)\tau^{2T-4}, \tag{8.16}$$

$$\partial_\xi g_{\tau\xi} = T(2m - \tau^2)\tau^{2T-3}, \tag{8.17}$$

$$\partial_\xi g_{\xi\xi} = 0. \tag{8.18}$$

From (8.13)–(8.18) we find that the least power of τ in the partial derivatives of the metric is $\min(3, 2T - 5)$. From the (8.10)–(8.12), the least power of τ in the reciprocal metric is $\min(-6, -2T + 2)$. Hence, the least power of τ in $g^{st}\Gamma_{abs}\Gamma_{cdt}$ is non-negative only if

$$-1 - 2T + 3\min(3, 2T - 5) \geq 0. \tag{8.19}$$

Therefore $g^{st}\Gamma_{abs}\Gamma_{cdt}$ are smooth only for $T = 4$, and the metric in two dimensions (τ, ξ) is semi-regular. The metric in all four dimensions is the warped product between the two-dimensional space (τ, ξ) and the sphere S^2, with warping function τ^2, which according to [10], is semi-regular. □

The geodesics of the extended solution are given by

$$\frac{\mathrm{d}\xi}{\mathrm{d}\tau} = -\frac{4\xi}{\tau} \pm \frac{2}{(2m - \tau^2)\tau}, \tag{8.20}$$

which become tangent to the hypersurface $\tau = 0$. The causal structure is represented in Fig. 8.1. We can see that, although the lightcones at events from the singularity are flattened, they have the same topology as those outside the singularity [11].

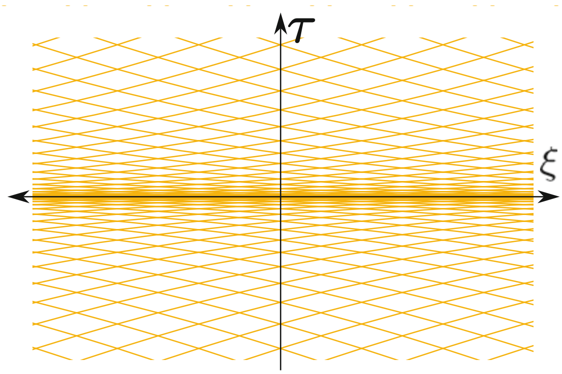

Fig. 8.1 The causal structure of the extended Schwarzschild solution. The lightcones at the singularity are flattened, but they have the same topology as the lightcones outside the singularity

8.3 Globally Hyperbolic Spacetimes with Black Holes

The analytic extension of the Schwarzschild metric from (8.9) is symmetric at the time reversal $\tau \mapsto -\tau$, which means that it extends beyond the singularity as a white hole. If we modify the Schwarzschild solution to describe a black hole which forms by gravitational collapse, for example as in the *Oppenheimer–Snyder model* [12], then the solution extends beyond the singularity as an evaporating black hole (Fig. 8.2b). It is interesting that, while one would normally expect that spacetime ends at the singularity (Fig. 8.2a), solution (8.9) does not behave like this, and is compatible with globally hyperbolic spacetimes like that in Fig. 8.2b [13].

In order for information to be preserved, this is not enough. The field equations normally involve covariant derivatives, which are not defined in general when the metric is degenerate. But the solution (8.9) with $T = 4$ allows us to define covariant derivatives and even to rewrite the Einstein equation without infinities [7, 9]. What about other fields? In [14] it is shown how we can write the Maxwell and Yang–Mills equations when the metric is semi-regular. There are still some open problems related to this, in particular how to formulate the Dirac equation for semi-regular metrics.

8.4 Implications of Singularities to Quantum Gravity

The dimension of Newton's constant is $2 - D = -2$ in mass units, where D is the dimension of spacetime. This makes Quantum Gravity (QG) perturbatively non-renormalizable even without matter, at two loops [15, 16], by requiring an infinite number of higher derivative counterterms, with their coupling constants. Various approaches to make QG perturbatively renormalizable indicate that in the UV limit a *dimensional reduction* to two dimensions takes place (for a review, see [17]), either as a consequence of other hypotheses, or by being directly postulated to obtain the desired result.

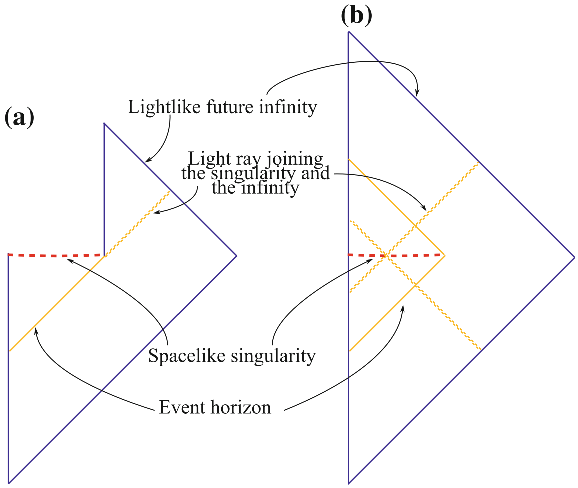

Fig. 8.2 **a** Standard evaporating black hole, whose singularity destroys the information. **b** Evaporating black hole extended through the singularity preserves information

At a semi-regular singularity, the metric becomes degenerate, behaving like a lower-dimension metric. Moreover, the Weyl curvature tensor also becomes lower-dimensional, and for this reason it vanishes [18]. Such effects happen at the singularities of the Schwarzschild, but also of the charged and rotating black holes [8]. In particular they accompany pointlike particles. Some of the dimensional reduction effects postulated in several approaches to QG follow naturally at singularities [19]. This suggests that when we use perturbative methods, by taking into consideration the corrections introduced by the singularities, the desired dimensional reduction occurs naturally.

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Part II
Black Holes in Quantum Gravity and String Theory

Chapter 9
Quantum Black Holes as the Link Between Microphysics and Macrophysics

B. J. Carr

Abstract There appears to be a duality between elementary particles, which span the mass range below the Planck scale, and black holes, which span the mass range range above it. In particular, the Black Hole Uncertainty Principle correspondence posits a smooth transition between the Compton and Schwarzschild scales as a function of mass. This suggests that all black holes are in some sense quantum, that elementary particles can be interpreted as sub-Planckian black holes, and that there is a subtle connection between quantum and classical physics.

9.1 Classical Versus Quantum Black Holes

At the previous Karl Schwarzschild meeting, I spoke about some quantum aspects of primordial black holes [1] and what I term the Black Hole Uncertainty Principle correspondence [2]. My contribution this year will involve an amalgamation of these two ideas and is therefore a natural follow-up. It will also allow me to discuss some recent work with two of the organisers of this meeting.

Black holes could exist over a wide range of mass scales. Those larger than several solar masses would form at the endpoint of evolution of ordinary stars and there should be billions of these even in the disc of our own galaxy. "Intermediate Mass Black Holes" (IMBHs) would derive from stars bigger than 100 $M_{\odot}$, which are radiation-dominated and collapse due to an instability during oxygen-burning, and the first primordial stars may have been in this range. "Supermassive Black Holes" (SMBHs), with masses from 10^6 $M_{\odot}$ to 10^{10} $M_{\odot}$, are thought to reside in galactic nuclei, with our own galaxy harbouring one of 4×10^6 $M_{\odot}$ and quasars being powered by ones of around 10^8 $M_{\odot}$. All these black holes might be described as "macroscopic" since they are larger than a kilometre in radius.

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P. Nicolini et al. (eds.), *2nd Karl Schwarzschild Meeting on Gravitational Physics*, Springer Proceedings in Physics 208,
https://doi.org/10.1007/978-3-319-94256-8_9

Black holes smaller than a solar mass could have formed in the early universe, the density being $\rho \sim 1/(Gt^2)$ at a time t after the Big Bang. Since a region of mass M requires a density $\rho \sim c^6/(G^3M^2)$ to form an event horizon, such "Primordial Black Holes" (PBHs) would initially have of order the horizon mass, $M_H \sim c^3t/G$, so those forming at the Planck time ($t_P \sim 10^{-43}$ s) would have the Planck mass ($M_P \sim 10^{-5}$ g), while those forming at $t \sim 1$s would have a mass of $10^5 M_\odot$. Therefore PBHs could span an enormous mass range. Those initially lighter than $M_* \sim 10^{15}$ g would be smaller than a proton and have evaporated by now due to Hawking radiation, the temperature and evaporation time of a black hole of mass M being $T \sim 10^{12}(M/10^{15}\text{g})^{-1}$ K and $\tau \sim 10^{10}(M/10^{15}\text{g})^3$y, respectively [3]. I will classify black holes smaller than M_* as "quantum", although I will argue later that *all* black holes are in a sense quantum. Those smaller than a lunar mass, 10^{24} g, will be classified as "microscopic", since their size is less than a micron. Coincidentally, this is also the mass above which T falls below the CMB temperature.

A theory of quantum gravity would be required to understand the evaporation process as the black hole mass falls to M_P and this might even allow stable Planck-mass relics. The existence of extra spatial dimensions, beyond the three macroscopic ones, may also come into play. These dimensions are usually assumed to be compactified on the Planck length ($R_P \sim 10^{-33}$ cm) but they can be much larger than this in some models. This would imply that gravity grows more strongly at short distances than implied by the inverse-square law [4], leading to the possibility of TeV quantum gravity and black hole production at accelerators. Such holes are not themselves primordial but this would have crucial implications for PBH formation.

The wide range of masses of black holes and their crucial role in linking macrophysics and microphysics is summarized in Fig. 9.1. This shows the Cosmic Uroborus (the snake eating its own tail), with the various scales of structure in the universe indicated along the side. It can be regarded as a sort of "clock" in which the scale changes by a factor of 10 for each minute – from the Planck scale at the top left to the scale of the observable universe at the top right. The head meets the tail at the Big Bang because at the horizon distance one is peering back to an epoch when the universe was very small, so the very large meets the very small there. The various types of black holes discussed above are indicated on the outside of the Urobrous. They are labelled by their mass, this being proportional to their size if there are three spatial dimensions. On the right are the well established astrophysical black holes. On the left – and possibly extending somewhat to the right – are the more speculative PBHs. The vertical line between the bottom of the Uroborus (planetary mass black holes) and the top (Planck mass black holes and extra dimensions) provides a convenient division between the microphysical and macrophysical domains.

Although the length-scale λ decreases as one approaches the top of the Uroborus from the left, the mass of the associated particle $m \sim \hbar/(\lambda c)$ increases. So Fig. 9.1 can also be used to represent elementary particles. On the inside of the Uroborus are indicated the positions of the Higgs boson (250 GeV) and proton (1 GeV) on the left, the dark energy mass-scale (10^{-4} eV) at the bottom, and the (possible) mass of the graviton (10^{-32} eV) at the top. Note that the inner scale also gives the temperature of a black hole with mass indicated by the outer scale.

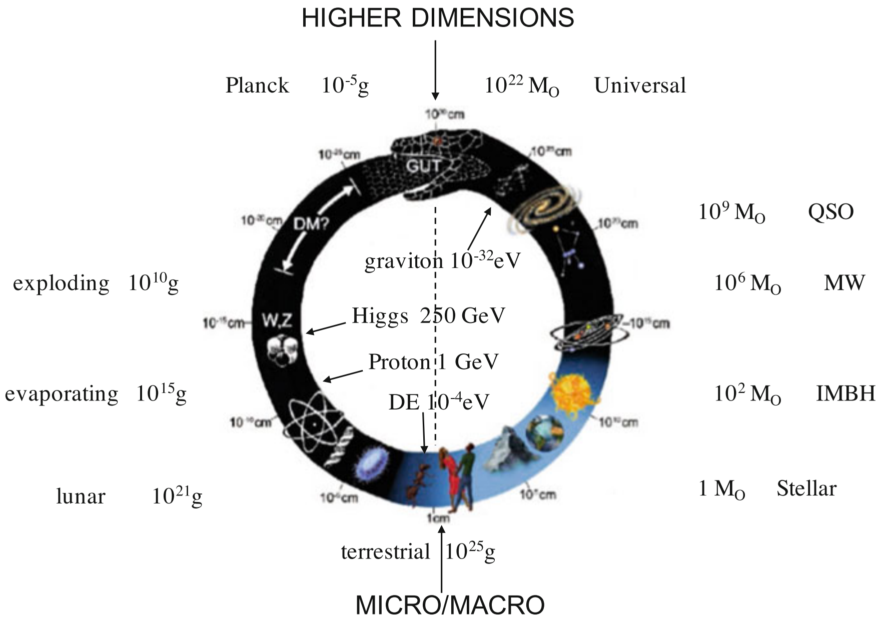

Fig. 9.1 The Cosmic Uroboros is used to indicate that mass the various types of black holes and elementary particles, the division between the micro and macro domains being indicated by the vertical line. QSO stands for "Quasi-Stellar Object", MW for "Milky Way", IMBH for "Intermediate Mass Black Hole", LHC for "Large Hadron Collider", and "DE" for "Dark Energy"

9.2 The Black Hole Uncertainty Principle Correspondence

A key feature of the microscopic domain is the (reduced) Compton wavelength for a particle of rest mass M, which is $R_C = \hbar/(Mc)$. In the (M, R) diagram of Fig. 9.2, the region corresponding to $R < R_C$ might be regarded as the "quantum domain", in the sense that the classical description breaks down there. A key feature of the macroscopic domain is the Schwarzschild radius for a body of mass M, $R_S = 2GM/c^2$, which corresponds to the size of the event horizon. The region $R < R_S$ might be regarded as the "relativistic domain", in the sense that there is no stable classical configuration in this part of Fig. 9.2.

The Compton and Schwarzschild lines intersect at around the Planck scales, $R_P = \sqrt{\hbar G/c^3} \sim 10^{-33}$ cm, $M_P = \sqrt{\hbar c/G} \sim 10^{-5}$g, and divide the (M, R) diagram in Fig. 9.2 into three regimes, which we label quantum, relativistic and classical. There are several other interesting lines in the figure. The vertical line $M = M_P$ marks the division between elementary particles ($M < M_P$) and black holes ($M > M_P$), since the size of a black hole is usually required to be larger than the Compton wavelength associated with its mass. The horizontal line $R = R_P$ is significant because quantum fluctuations in the metric should become important below this [5]. Quantum gravity effects should also be important whenever the density exceeds the Planck value,

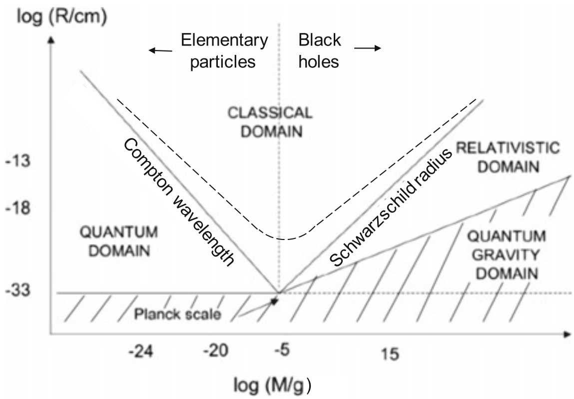

Fig. 9.2 The division of the (M, R) diagram into the classical, quantum, relativistic and quantum gravity domains. The boundaries are specified by the Planck density, the Compton wavelength and the Schwarzschild radius

$\rho_P = c^5/(G^2\hbar) \sim 10^{94}\,\mathrm{g\,cm^{-3}}$, corresponding to the sorts of curvature singularities associated with the big bang or the centres of black holes [6]. This implies $R < R_P(M/M_P)^{1/3}$, which is well above the $R = R_P$ line in Fig. 9.2 for $M \gg M_P$, so one might regard the shaded region as specifying the 'quantum gravity' domain. This point has recently been invoked to support the notion of Planck stars [7] and could have important implications for the detection of evaporating black holes [8]. Note that the Compton and Schwarzschild lines transform into one another under the T-duality transformation $M \rightarrow M_P^2/M$. This interchanges sub-Planckian and super-Planckian mass scales and corresponds to a reflection in the line $M = M_P$ in Fig. 9.2. T-dualities arise naturally in string theory and are known to map momentum-carrying string states to winding states and vice-versa [9].

Although the Compton and Schwarzschild boundaries correspond to straight lines in the logarithmic plot of Fig. 9.2, this form presumably breaks down near the Planck point due to quantum gravity effects. One might envisage two possibilities: either there is a smooth minimum, as indicated by the broken line in Fig. 9.2, so that the Compton and Schwarzschild lines in some sense merge, or there is some form of phase transition or critical point at the Planck scale, so that the separation between particles and black holes is maintained. Which alternative applies has important implications for the relationship between elementary particles and black holes [10]. This may also relate to the issue of T-duality since this purports to play some role in linking point particles and black holes. Such a link is also suggested by Fig. 9.1.

One way of smoothing the transition between the Compton and Schwarzschild lines is to invoke some connection between the Uncertainty Principle on microscopic scales and black holes on macroscopic scales. This is termed the Black Hole Uncertainty Principle (BHUP) correspondence [1] and also the Compton-Schwarzschild correspondence when discussing an interpretation in terms of extended de Broglie relations [11]. It is manifested in a unified expression for the Compton wavelength and Schwarzschild radius. The simplest expression of this kind would be

$$R_{CS} = \frac{\beta\hbar}{Mc} + \frac{2GM}{c^2}\,, \tag{9.1}$$

where β is the (somewhat arbitrary) constant appearing in the Compton wavelength expression. In the sub-Planckian regime, this can be written as

$$R'_C = \frac{\beta\hbar}{Mc}\left[1 + \frac{2}{\beta}\left(\frac{M}{M_P}\right)^2\right] \quad (M \ll M_P)\,, \tag{9.2}$$

with the second term corresponding to a small correction of the kind invoked by the Generalised Uncertainty Principle [12]. In the super-Planckian regime, it becomes

$$R'_S = \frac{2GM}{c^2}\left[1 + \frac{\beta}{2}\left(\frac{M_P}{M}\right)^2\right] \quad (M \gg M_P)\,. \tag{9.3}$$

This is termed the Generalised Event Horizon [1], with the second term corresponding to a small correction to the usual Schwarzschild expression. More generally, the BHUP correspondence might allow any unified expression $R'_C(M) \equiv R'_S(M)$ which has the asymptotic behaviour $\beta\hbar/(Mc)$ for $M \ll M_P$ and $2GM/c^2$ for $M \gg M_P$. One could envisage many such expressions but we are particularly interested in those which – like (9.1) – exhibit T-duality.

At the last meeting, I discussed some of the consequences of the BHUP correspondence, with particular emphasis on the implied black hole temperature, the link with Loop Quantum Gravity [6] and the effect of extra dimensions [13, 14]. The implication is that in some sense elementary particles are sub-Planckian black holes. Next I discuss some developments arising out of recent work with my collaborators.

9.3 Carr–Mureika–Nicolini work

The results of [10] are now summarised. In the standard picture, the mass in the Schwarzschild solution is obtained by matching the metric coefficients with the Newtonian potential and this gives the Komar integral

$$M \equiv \frac{1}{4\pi G}\int_{\partial\Sigma} \mathrm{d}^2x\sqrt{\gamma^{(2)}}\, n_\mu \sigma_\nu \nabla^\mu K^\nu\,, \tag{9.4}$$

where K^ν is a timelike vector, Σ is a spacelike surface with unit normal n^μ, and $\partial\Sigma$ is the boundary of Σ (typically a 2-sphere at spatial infinity) with metric $\gamma^{(2)ij}$ and outward normal σ^μ. For $M \gg M_\mathrm{P}$, quantum effects are negligible and one finds the usual Schwarzschild mass. For $M < M_\mathrm{P}$, however, the expression can simultaneously refer to a particle and a black hole. One usually considers the particle case and writes (9.4) as

$$M \equiv \int_\Sigma \mathrm{d}^3x\sqrt{\gamma}\, n_\mu K_\nu T^{\mu\nu} \approx -4\pi\int_0^{R_\mathrm{C}} \mathrm{d}r\, r^2 T_0{}^0\,, \tag{9.5}$$

where γ is the determinant of the spatially induced metric γ^{ij}, $T^{\mu\nu}$ is the stress-energy tensor and $T_0{}^0$ accounts for the particle distribution on a scale of order R_C. This corresponds to the mass appearing in the expression for the Compton wavelength. When the black hole reaches the final stages of evaporation, the major contribution to integral (9.4) becomes

$$M = -4\pi \int_0^{R_P} \mathrm{d}r\, r^2 T_0{}^0 , \tag{9.6}$$

where $T_0{}^0$ accounts for an unspecified quantum-mechanical distribution of matter and energy. Integral (9.6) is then unknown and might lead to a completely different definition of the Komar energy.

Inspired by the dual role of M in the BHUP correspondence we explore a variant of the last scenario, based on the existence of sub-Planckian black holes, *i.e.* quantum mechanical objects that are simultaneously black holes and elementary particles. In this context, we suggest that the Arnowitt–Deser–Misner (ADM) mass, which coincides with the Komar mass in the stationary case, should be

$$M_{\mathrm{ADM}} = M \left(1 + \frac{\beta}{2} \frac{M_P^2}{M^2}\right) , \tag{9.7}$$

which is equivalent to (9.3). We thus posit a quantum-corrected Schwarzschild metric, like the usual one but with M replaced by M_{ADM}. We note a possible connection with the energy-dependent metric proposed in the framework of "gravity's rainbow" [15]. It may also relate to the distinction between the bare and renormalized mass in QFT in the presence of stochastic metric fluctuations [16].

The horizon size for the modified metric is given by

$$R'_S = \frac{2M_{\mathrm{ADM}}}{M_P^2} \approx \begin{cases} 2M/M_P^2 & (M \gg M_P) \\ (2+\beta)/M_P & (M \approx M_P) \\ \beta/M & (M \ll M_P) , \end{cases} \tag{9.8}$$

where we use units with $\hbar = c = 1$. The first expression is the standard Schwarzschild radius. The intermediate expression gives a minimum of order R_P, so the Planck scale is never actually reached for $\beta > 0$ and the singularity remains inaccessible. The last expression resembles the Compton wavelength. If the temperature is determined by the black hole's surface gravity [3], one has

$$T = \frac{M_P^2}{8\pi M_{\mathrm{ADM}}} \approx \begin{cases} M_P^2/(8\pi M) & (M \gg M_P) \\ M_P/(8\pi(1+\beta/2)) & (M \approx M_P) \\ M/(4\pi\beta) & (M \ll M_P) . \end{cases} \tag{9.9}$$

This temperature is plotted in Fig. 9.3. The large M limit is the usual Hawking temperature with a small correction. However, as the black hole evaporates, the temperature reaches a maximum at around T_P and then decreases to zero as $M \to 0$.

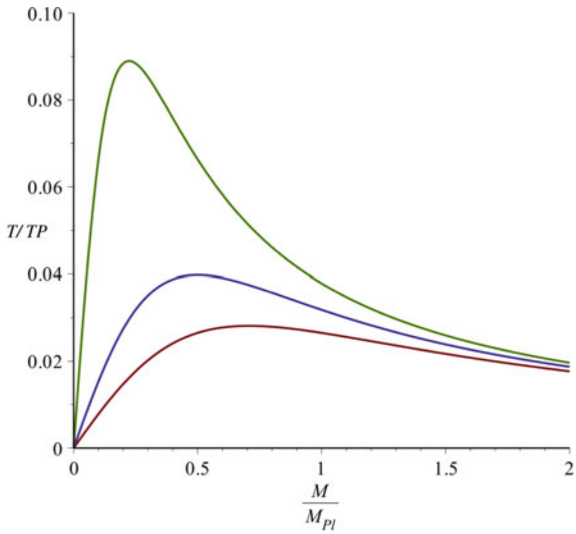

Fig. 9.3 Hawking temperature (9.9) implied by the surface gravity argument as a function of M/M_{P} for $\beta = 1$ (bottom), $\beta = 0.5$ (middle) and $\beta = 0.1$ (top). As M decreases, T reaches a maximum below T_P and then falls to zero

A possible explanation for the $M \ll M_{Pl}$ behaviour is that a decaying black hole makes a temporary transition to a (1+1)-D dilaton black hole when approaching the Planck scale, since this naturally encodes a $1/M$ term in its gravitational radius. For according to 't Hooft [17], gravity might experience a (1+1)-D phase at the Planck scale due to spontaneous dimensional reduction, such a conjecture being further supported by studies of the fractal properties of a quantum spacetime at the Planck scale. At this point the Komar mass can be defined as for dilaton black holes by [18]

$$M \sim \int \mathrm{d}x \sqrt{g^{(1)}}\, n_i^{(2)} T_0{}^i \,, \tag{9.10}$$

where $g^{(1)}$ is the determinant of the spatial section of g^{ij}, the effective 2D quantum spacetime metric, and ${}^{(2)}T_0{}^i$ is the dimensionally reduced energy-momentum tensor.

The black hole luminosity in this model is $L = \gamma^{-1} M_{\text{ADM}}^{-2}$ where $\gamma \sim t_{\text{P}}/M_{\text{P}}^3$. Although the black hole loses mass on a timescale $\tau \sim M/L \sim \gamma M^3 (1 + \beta M_{\text{P}}^2/2M^2)^2$, it never evaporates entirely because the mass loss rate decreases when M falls below M_P. There are two values of M for which τ is comparable to the age of the Universe ($t_0 \sim 10^{17}$ s). One is super-Planckian, $M_* \sim (t_0/\gamma)^{1/3} \sim (t_0/t_{\text{P}})^{1/3} M_{\text{P}} \sim 10^{15}$g, this being the standard expression for the mass of a PBH evaporating at the present epoch, and the other is sub-Planckian, $M_{**} \sim \beta^2 (t_{\text{P}}/t_o) M_{\text{P}} \sim 10^{-65}$ g. The usual Hawking lifetime ($\tau \propto M^3$) gives the time for this mass to decrease to M_{P}, after which it quickly falls to the value M_{**}. Although this mass-scale is very tiny, it arises naturally in some estimates for the photon or graviton mass [19]. Note that the PBH mass cannot actually reach M_{**} at the present epoch because the black hole temperature is less than the CMB temperature below $M_{\text{CMB}} \sim 10^{-36}$ g, leading to *effectively* stable relics of this mass which might provide the dark matter.

To summarise the advantages of our proposal: it encodes the BHUP duality in the expression for the mass; it smooths the $M(R)$ curve, so that there is no critical point; it cures the thermodynamic instability of evaporating black holes; it exhibits dimensional reduction in the sub-Planckian regime; and it gives a consistent theory of gravity in different spacetime dimensions without needing two regimes governed by different theories (GR and QM). Indeed, in some sense, the BHUP correspondence implies that all black holes are quantum and that the Uncertainty Principle has a gravitational explanation.

9.4 Lake–Carr Work

Canonical (non-gravitational) quantum mechanics is based on the concept of wave-particle duality, encapsulated in the de Broglie relations $E = \hbar\omega$ and $p = \hbar k$. When combined with the energy-momentum relation for a non-relativistic point particle, these lead to the dispersion relation $\omega = (\hbar/2m)k^2$. However, these relations break down near the Planck scale, since they correspond to wavelengths $\lambda \ll R_P$ or periods $t \ll t_P$. Reference [11] therefore proposes modified forms for the de Broglie relations which may be applied even for $E \gg M_P c^2$, with the additional terms being interpreted as representing the self-gravitation of the wave packet.

The simplest such relations are $E = \hbar\Omega$ and $p = \hbar\kappa$ with

$$\Omega = \begin{cases} \omega_P^2 \left(\omega + \omega_P^2/\omega\right)^{-1} & (m < M_P) \\ \beta\left(\omega + \omega_P^2/\omega\right) & (m > M_P) \end{cases}, \quad \kappa = \begin{cases} k_P^2 \left(k + k_P^2/k\right)^{-1} & (m < M_P) \\ \beta\left(k + k_P^2/k\right) & (m > M_P). \end{cases} \tag{9.11}$$

Continuity of E, p, $\mathrm{d}E/\mathrm{d}\omega$ and $\mathrm{d}p/\mathrm{d}k$ at $\omega = \omega_P$ and $k = k_P$ is ensured by setting $\beta = 1/4$. The relation $\Omega = (\hbar/2m)\kappa^2$ then leads to new dispersion relations, quadratic in ω, which can be solved for both $E \ll M_p c^2$ and $E \gg M_p c^2$. The two solution branches, $\omega_\pm(k, m)$, are shown as functions of k for three values of m in Fig. 9.4a. The solutions are dual under the transformation $m \to M_P'^2/m$ where $M_P' \equiv (\pi/2)M_P$. Canonical non-relativistic quantum mechanics is recovered in the bottom left region, where $\omega_- \approx (\hbar/2m)k^2$. The branches meet at $\omega_\pm(k_P) = \omega_P$ for the critical case $m = M_P'$ but there is a gap in the allowed values of k for $m \neq M_P'$. The limiting values for a given mass, $k_\pm(m)$, are shown in Fig. 9.4b and these also exhibit duality. These values correspond to the Schwarzschild formula for $E \gg M_P c^2$ and the Compton formula for $E \ll M_P c^2$. So this is another way of interpreting the BHUP correspondence.

In our second paper [20] we discuss the preservation of T-duality in higher dimensions. In three spatial dimensions, the Compton wavelength and Schwarzschild radius are dual under the transformation $M \to M_P^2/M$. In the presence of n extra dimensions, compactified on some scale R_E, it is usually assumed that $R_S \propto M^{1/(1+n)}$ [21] and $R_C \propto M^{-1}$ (as in three domensions) for $R < R_E$, which breaks the duality. This situation is illustrated in Fig. 9.5a and gives the standard scenario in which the effec-

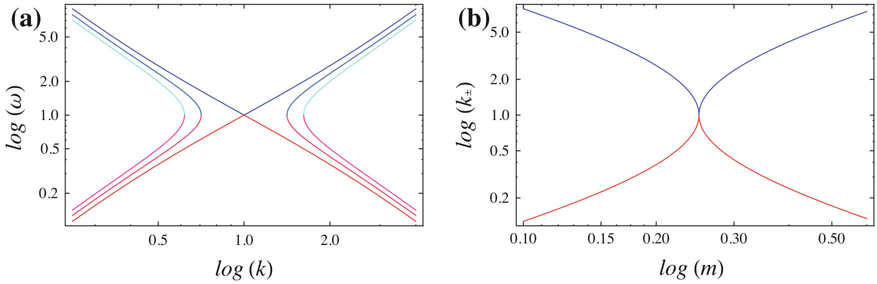

Fig. 9.4 Illustrating how **a** the dispersion relations $\omega_{\pm}(m, k)$ for three values of m and **b** the limiting wavenumbers $k_{\pm}(m)$ are changed in the proposed model

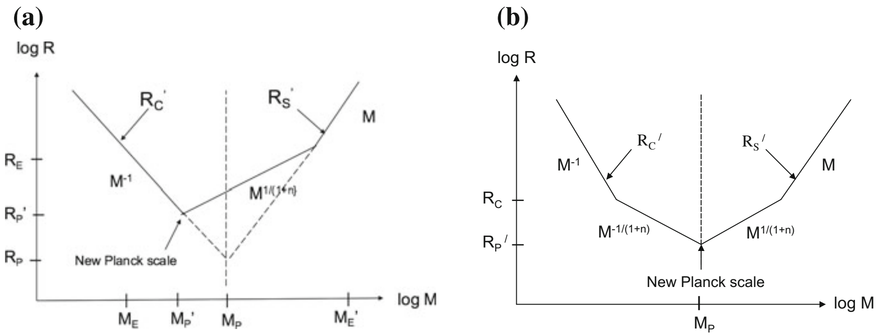

Fig. 9.5 Showing change in Planck scales for large extra dimensions if **a** only the Schwarzschild radius is modified and **b** the Compotn wavelength is also modified, preserving T-duality

tive Planck length is increased and the Planck mass reduced, allowing the possibility of black hole production at the LHC.

Currently there is no evidence for such production. However, the effective Compton wavelength depends on the form of the $(3+n)$-dimensional wavefunction. If this is spherically symmetric, then one indeed has $R_C \propto M^{-1}$. But if the wave function is pancaked in the extra dimensions and maximally asymmetric, then $R_C \propto M^{-1/(1+n)}$, so that the duality between R_C and R_S is preserved. This situation is illustrated in Fig. 9.5b, which shows that the effective Planck length is reduced even more but the Planck mass is unchanged. So TeV quantum gravity is precluded in this case and black holes *cannot* be generated in collider experiments. Nevertheless, the extra dimensions could still have consequences for the detectability of black hole evaporations and the enhancement of pair-production at accelerators on scales below R_E.

Acknowledgements I thank my collaborators in the work reported here: Matthew Lake, Jonas Mureika and Piero Nicolini.

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Chapter 10
Free Energy of Topologically Massive Gravity and Flat Space Holography

D. Grumiller and W. Merbis

Abstract We calculate the free energy from the on-shell action for topologically massive gravity with negative and vanishing cosmological constant, thereby providing a first principles derivation of the free energy of Bañados–Teitelboim–Zanelli black holes and flat space cosmologies. We summarize related recent checks of flat space holography.

10.1 Introduction

The Schwarzschild solution was found a few weeks after Einstein's theory of general relativity was finished and has engendered a century worth of interesting research results. One indirect outcome of Schwarzschild's remarkable discovery is black hole holography [1], which is at the core of numerous current research avenues, not just in classical and quantum gravity, but even in neighboring fields such as quantum field theory or condensed matter physics (specifically at strong coupling).

The anti-de Sitter/conformal field theory (AdS/CFT) correspondence [2] provides a concrete realization of holography. A specific set of applications and checks is the determination of correlation functions on the gravity side [3]. The 0-point function, or on-shell action, gives the free energy and should therefore capture all features of the free energy of the dual CFT [4]. The 1-point functions, or vacuum expectation values, allow to determine conserved charges like mass or angular momentum [5–7]. The 2- and 3-point functions are highly constrained by symmetries and allow basic checks of the correspondence, while the higher n-point functions provide further applications and consistency checks, see [8] and references therein.

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P. Nicolini et al. (eds.), *2nd Karl Schwarzschild Meeting on Gravitational Physics*, Springer Proceedings in Physics 208,
https://doi.org/10.1007/978-3-319-94256-8_10

If holography is a true aspect of Nature it must also work beyond AdS/CFT, particularly in flat space. After some early progress on extracting features of the S-matrix as a limit from AdS/CFT correlators [9–11] it took a while to come up with the first precise proposal for a holographic correspondence between a specific quantum theory of gravity in flat space, dubbed flat space chiral gravity [12], and a specific quantum field theory. This proposal is based on a scaling limit of topologically massive gravity (TMG) [13], which is a three-dimensional gravity theory that consists of the Einstein–Hilbert action and a gravitational Chern–Simons term. In the limit of interest only the gravitational Chern–Simons term remains and the ensuing theory, known as conformal Chern–Simons gravity (CSG), has interesting holographic properties [12, 14, 15].

One particular check of the flat space chiral gravity proposal and more generally of flat space holography is the microscopic derivation of the entropy [16] of flat space cosmology solutions using properties of the dual Galilean CFT [17]. For flat space Einstein gravity the microscopic result matches the expected Bekenstein–Hawking law, which can be derived from first principles using the Euclidean path integral formulation [18] to determine the free energy and extract from it the entropy using standard thermodynamical relations.

Naively applying the same methods to TMG, including its limiting case CSG, appears to fail. Indeed, inserting for instance the BTZ line-element into the bulk-plus-boundary action constructed in [19] and used in [14] yields the correct 1-, 2- and 3-point functions, but not the correct 0-point function or free energy. One can avoid this issue by directly calculating the entropy [20], e.g. using Solodukhin's method of conical deficits [21] or Tachikawa's generalization of the Wald entropy [22]. If one then postulates the first law and integrates it (essentially Legendre transforming entropy) one can extract free energy. However, this indirect derivation of free energy is not completely satisfactory since it uses the first law as an input rather than providing it as a result.

It is of interest to directly calculate free energy for TMG from first principles, since this provides a check of the validity of the first law. In this work we achieve this goal for TMG and its CSG limit, both in AdS and in flat space.

This proceedings contribution is organized as follows. In Sect. 10.2 we display the bulk action in a Chern–Simons like formulation and clarify our notation. In Sect. 10.3 we review the Euclidean BTZ solution. In Sect. 10.4 we determine the BTZ free energy from the on-shell action. In Sect. 10.5 we apply our results to flat space cosmology solutions. In Sect. 10.6 we comment on correlation functions in flat space holography. In Sect. 10.7 we conclude with mentioning a further possible check of the flat space chiral gravity proposal, namely holographic entanglement entropy.

10.2 Action and Notation

It turns out to be useful to employ the Chern–Simons-like formulation of TMG, whose bulk action is given by (see [23] and references therein)

$$I_{\text{TMG}} = \frac{1}{4\pi G} \int \text{tr}\Big[-\sigma e \wedge R + \tfrac{\Lambda_0}{3} e \wedge e \wedge e + f \wedge T - \tfrac{1}{2\mu}\omega \wedge \big(\mathrm{d}\omega + \tfrac{2}{3}\omega \wedge \omega\big)\Big]. \tag{10.1}$$

Here e is the dreibein, ω the (dualized) spin-connection and f is an auxiliary $so(2, 1)$ valued one-form field. R denotes the Riemann curvature two-form and T is the torsion two-form. In addition, $\sigma = \pm 1$ is a sign parameter and the cosmological parameter Λ_0 is related to the cosmological constant as $\Lambda_0 = \sigma\Lambda$. In this work we assume that the cosmological constant is either negative (AdS) or vanishes (flat space). The quantity G is Newton's constant.

10.3 Bañados–Teitelboim–Zanelli Solution

The Euclidean BTZ black hole [24] has the topology of a solid torus, which we coordinatize by a radial coordinate $\rho \in [0, \infty)$, a contractible cycle coordinate $t \sim t + 1$ and a non-contractible cycle coordinate $\phi \sim \phi + 2\pi$. Since we work with fixed coordinate ranges the chemical potentials, essentially temperature and angular velocity, appear explicitly in the solutions.

The BTZ solution (in the basis of $sl(2, \mathbb{R})$ generators L_+, L_0, L_-) can be parametrized by the dreibein $e = e_\rho \mathrm{d}\rho + e_\phi \mathrm{d}\phi + e_t \mathrm{d}t$ as

$$e_\rho = \ell L_0 \qquad e_\phi = \frac{\ell}{2}\left(e^\rho(L_+ - L_-) - \frac{8\pi G}{\ell} e^{-\rho}(\mathscr{L}^+ L_- - \mathscr{L}^- L_+)\right) \tag{10.2}$$

$$e_t = \frac{\ell}{2}\left(e^\rho(\mu^+ L_+ + \mu^- L_-) - \frac{8\pi G}{\ell} e^{-\rho}(\mu^+ \mathscr{L}^+ L_- + \mu^- \mathscr{L}^- L_+)\right). \tag{10.3}$$

Regularity of the solution requires a relation between charges $\mathscr{L}^\pm$ and chemical potentials $\mu^\pm$

$$\mathscr{L}^\pm = \frac{\pi\ell}{8G(\mu^\pm)^2} \tag{10.4}$$

where $\mu^\pm$ are related to the inverse temperature β and the angular velocity Ω as

$$\beta = \frac{\ell}{2}(\mu^+ + \mu^-) \qquad \beta\Omega = -\frac{1}{2}(\mu^+ - \mu^-). \tag{10.5}$$

In addition, the charges $\mathscr{L}^{\pm}$ are related to the mass M and angular momentum J as $\mathscr{L}^{\pm} = \frac{1}{4\pi}(M\ell \mp J)$ and to the loci of the BTZ Killing horizons $r_{\pm}$ as

$$r_{\pm} = \sqrt{8\pi G\ell\mathscr{L}^{-}} \pm \sqrt{8\pi G\ell\mathscr{L}^{+}}\,. \tag{10.6}$$

The solution for ω follows from the constraint of vanishing torsion, which is one of the TMG equations of motion (EOM).

$$\omega_{\rho} = 0 \qquad \omega_{\phi} = \frac{1}{2}\left(e^{\rho}(L_{+} + L_{-}) - \frac{8\pi G}{\ell}e^{-\rho}(\mathscr{L}^{+}L_{-} + \mathscr{L}^{-}L_{+})\right) \tag{10.7}$$

$$\omega_{t} = \frac{1}{2}\left(e^{\rho}(\mu^{+}L_{+} - \mu^{-}L_{-}) - \frac{8\pi G}{\ell}e^{-\rho}(\mu^{+}\mathscr{L}^{+}L_{-} - \mu^{-}\mathscr{L}^{-}L_{+})\right) \tag{10.8}$$

The auxiliary one-form f also follows from the TMG EOM and is simply related to the dreibein e by

$$f = -\frac{1}{2\ell^{2}\mu}\,e\,. \tag{10.9}$$

10.4 On-Shell Action

Following Bañados and Mendez [25] we compute the on-shell action using so-called ‘angular quantization’. The essence is to slice the solid torus into constant ϕ slices. Then on the constant ϕ slice we can use (any) regular coordinate system, while close to the boundary we use the ρ, t, ϕ Schwarzschild-like coordinates. The full action $\Gamma_{\text{TMG}} = I_{\text{TMG}} + B$ consists of the bulk action (10.1) and a boundary term B. In the angular decomposition for one-forms, $a = a_{\alpha}\mathrm{d}x^{\alpha} + a_{\phi}\mathrm{d}\phi$ where $a = e, \omega, f$ and x^{α} are the coordinates on the disks of constant ϕ, the full action is given by

$$\begin{aligned}\Gamma_{\text{TMG}} = {}& \frac{1}{4\pi G}\int \mathrm{d}\phi\mathrm{d}^{2}x\,\varepsilon^{\alpha\beta}\mathrm{tr}\Big[\sum_{a} a_{\phi}(\text{EOM})_{\alpha\beta} - \sigma e_{\beta}\partial_{\phi}\omega_{\alpha} + f_{\beta}\partial_{\phi}e_{\alpha} - \frac{1}{2\mu}\omega_{\beta}\partial_{\phi}\omega_{\alpha}\Big] \\ & + \frac{1}{4\pi G}\int_{\rho\to\infty}\mathrm{d}t\mathrm{d}\phi\,\mathrm{tr}\left[-\sigma e_{t}\omega_{\phi} + f_{t}e_{\phi} - \frac{1}{2\mu}\omega_{t}\omega_{\phi}\right] + B\,.\end{aligned} \tag{10.10}$$

The boundary term B that gives a well-defined variational principle is one-half [18, 26, 27] the Gibbons–Hawking–York boundary term.

$$B = \frac{\sigma}{8\pi G}\int_{\rho\to\infty}\mathrm{d}t\mathrm{d}\phi\,\mathrm{tr}(\omega_{\phi}e_{t} - \omega_{t}e_{\phi}) \tag{10.11}$$

The bulk part of the action (10.10) vanishes on-shell for spherically symmetric (ϕ-independent) fields and we are left with the boundary terms in the second line of (10.10). Due to the compensating contribution from the boundary term B the result is finite on the solutions given in the last section and it equals to βF_{BTZ}, where F_{BTZ} is the BTZ free energy. Expressing everything in terms of temperature $T = \beta^{-1}$ and angular velocity Ω through (10.5) gives the free energy of the BTZ black hole in TMG.

$$F_{\text{BTZ}} = \frac{1}{\beta}(I_{\text{TMG}}^{\text{EOM}} + B) = -\frac{\pi^2\ell^2T^2}{2G(1-\Omega^2\ell^2)}\left(\sigma + \mu^{-1}\Omega\right) \tag{10.12}$$

This result agrees with the free energy obtained by Legendre transforming entropy[1] [20–22]

$$S_{\text{BTZ}} = -\frac{\partial F_{\text{BTZ}}}{\partial T}\Big|_{\Omega} = \sigma\,\frac{2\pi r_+}{4G} + \frac{2\pi r_-}{4G\mu\ell} \tag{10.13}$$

using the first law. We have thus succeeded in a first principles derivation of the BTZ free energy for TMG.

Our derivation of the BTZ free energy (10.12) from evaluating on-shell the full action (10.10) readily generalizes to other cases. In the next section we focus particularly on asymptotically flat space, $\Lambda = 0$, by considering the free energy of flat space cosmology solutions.

10.5 Flat-Space Solutions

Consider locally flat line-elements with constant chemical potentials μ_M, μ_L [28].

$$\begin{aligned} \mathrm{d}s^2 &= \left(r^2\mu_L^2 + \mathscr{M}(1+\mu_M)^2 + 2\mathscr{N}(1+\mu_M)\mu_L\right)\mathrm{d}u^2 \\ &\quad + \left(r^2\mu_L + \mathscr{N}(1+\mu_M)\right)2\mathrm{d}u\mathrm{d}\phi - (1+\mu_M)2\mathrm{d}r\mathrm{d}u + r^2\mathrm{d}\phi^2 \end{aligned} \tag{10.14}$$

Analogously to the BTZ case discussed in the previous two sections, regularity of the flat space cosmology solutions (10.14) relates the charges $\mathscr{M}$, $\mathscr{N}$ to the chemical potentials μ_M, μ_L and to the temperature T and angular velocity Ω. We find

$$\mathscr{M} = \frac{4\pi^2}{\mu_L^2} \qquad \mathscr{N} = -\mathscr{M}\frac{1+\mu_M}{\mu_L} \qquad T = \frac{1}{2\pi}\frac{\mathscr{M}^{3/2}}{|\mathscr{N}|} \qquad \Omega = \frac{\mathscr{M}}{\mathscr{N}}\,. \tag{10.15}$$

The easiest way to compute the free energy of these solutions in TMG is to write the dreibein which squares to (10.14) in the $sl(2,\mathbb{R})$ basis of the previous section and compute the corresponding spin-connection and auxiliary field from the TMG EOM. Then we can plug these solutions into the angularly decomposed action (10.10) and eventually find the free energy.

[1] We stress that for finite μ entropy (10.13) does not obey the Bekenstein–Hawking area law. Nevertheless, it is compatible with the Cardy formula in the presence of a gravitational anomaly, i.e., the left- and right-moving central charges are not equal, $c - \bar{c} = 3/(\mu G)$ [20].

Such a dreibein can be written as $e = e_r \mathrm{d}r + e_\phi \mathrm{d}\phi + e_u \mathrm{d}u$ with

$$e_r = \frac{1}{2} L_- \qquad e_\phi = -\frac{\mathscr{N}}{2} L_- + r L_0 \tag{10.16}$$

$$e_u = (1+\mu_M) L_+ - \Big(\frac{1}{4}\mathscr{M}(1+\mu_M) + \frac{1}{2}\mathscr{N}\mu_L\Big) L_- + r\mu_L L_0\,. \tag{10.17}$$

By solving the TMG EOM with this dreibein one finds that $f = 0$ and

$$\omega_r = 0 \qquad \omega_\phi = L_+ - \frac{\mathscr{M}}{4} L_- \qquad \omega_u = \mu_L\Big(L_+ - \frac{\mathscr{M}}{4} L_-\Big)\,. \tag{10.18}$$

After plugging this into the full action (10.10) and using (10.15) to write everything in terms of temperature and angular velocity we obtain the free energy.

$$F_{\text{FSC}} = \frac{1}{\beta}(I^{\text{EOM}}_{\text{TMG}} + B) = -\frac{\pi^2 T^2}{2G\Omega^2}\left(\sigma + \frac{\Omega}{\mu}\right) \tag{10.19}$$

This result agrees with the one derived in [29] by Legendre transforming the entropy of TMG with asymptotically flat boundary conditions. In particular, for flat space chiral gravity we obtain the entropy

$$S = -\frac{\partial F_{\text{FSC}}(\sigma = 0)}{\partial T}\Big|_\Omega = \frac{\pi^2 T}{G\mu\Omega} = 2\pi\sqrt{\frac{ch}{6}} \tag{10.20}$$

with $k = 1/(8G\mu) > 0$, $c = 24k$ and $h = k\mathscr{M}$ [12]. The last equality in (10.20) shows consistency with the chiral Cardy formula, as observed first in [29], compatible with the flat space chiral gravity conjecture [12].

10.6 Correlation Functions

Let us now move on from 0- to 1-point functions. They provide the first entries in the flat space holographic dictionary by identifying sources and vacuum expectation values as non-normalizable and normalizable solutions of the linearized EOM on the gravity side, respectively. The second order formulation [18, 30, 31] reproduces the canonical results for the conserved charges in flat space Einstein gravity [32]. It is slightly easier to obtain these results in the first order formulation [33]. It would be interesting to generalize them to TMG in order to provide another check of the flat space chiral gravity conjecture. This would require either an extension from our calculations in the Chern–Simons formulation to Chern–Simons-like theories such as TMG or an application of the Horne–Witten formulation of CSG [34].

If the 1-point functions match, as we expect them to do, one can actually go much further and check n-point correlation functions of the flat space holographic stress tensor in flat space chiral gravity. The procedure would follow the steps of the recent derivation of all n-point correlation functions in flat space Einstein gravity [33], which we now summarize briefly:

- Instead of directly calculating the n-th variation of the full action and inserting non-normalizable solutions to the linearized EOM we calculate the 1-point function on an arbitrary background, deformed by a chemical potential. On the field theory side this corresponds to a deformation of the original action Γ_0 to a deformed action Γ_μ with

$$\Gamma_\mu = \Gamma_0 - \int \mathrm{d}^2 z\, \mu(z, \bar{z}) \mathscr{O}(z, \bar{z}) \tag{10.21}$$

  where the chemical potential for the operator $\mathscr{O}$ is localized at $n-1$ points, $\mu = \sum_{i=2}^{n} \varepsilon_i \delta(z - z_i, \bar{z} - \bar{z}_i)$; the coefficients ε_i are a convenient book keeping device and $z, \bar{z}$ are some coordinates used in the 2-dimensional field theory.
- The 1-point function on the deformed background then yields the n-point function for the original background, e.g. for $n = 2$ we get

$$\langle \mathscr{O}(z_1, \bar{z}_1) \rangle_\mu = \langle \mathscr{O}(z_1, \bar{z}_1) \rangle_0 + \varepsilon_2 \langle \mathscr{O}(z_1, \bar{z}_1) \mathscr{O}(z_2, \bar{z}_2) \rangle_0 + \cdots \tag{10.22}$$

  The term to linear order in ε_2 yields the 2-point correlator, both on gravity and field theory sides. The same procedure works for arbitrary n-point correlators.
- In order to show the equivalence of all correlations functions on gravity and field theory sides one can use complete induction by proving recursion relations between n- and $(n-1)$-point correlation functions, analogous to the BPZ-recursion relations for the stress tensor in a CFT [35]. For flat space Einstein gravity and Galilean CFTs these recursion relations were established recently [33], thus showing the equivalence of all flat space holographic stress tensor correlation functions with corresponding Galilean CFT correlation functions.

This procedure provides a fairly non-trivial check of flat space holography in three dimensions. It would be great to generalize it to flat space chiral gravity.

10.7 Flat Space Holographic Entanglement Entropy

As concluding part of this proceedings contribution we focus on a further check of flat space holography. One particularly interesting part of the AdS/CFT developments was the insight by Ryu and Takayanagi a decade ago that entanglement entropy can be calculated by elementary methods on the gravity side, through minimizing the area of certain hypersurfaces, depending on the entangling region for which entanglement entropy is calculated [36]. With methods similar to the ones used in the CFT derivation [37–39] one can also derive entanglement entropy for 2-dimensional Galilean CFTs [40] and, following the holographic computation of entanglement entropy in the presence of a gravitational Chern–Simons term [41], we expect that this result should match with the flat space chiral gravity prediction

$$S_{\rm EE} = \frac{c}{6} \ln \frac{L}{a} \tag{10.23}$$

where $c = 24k$, L is the length of the entangling region and a is an ultraviolet cutoff. Also this prediction of flat space chiral gravity was confirmed recently [42].

Acknowledgements We thank Arjun Bagchi for collaboration on n-point correlation functions in flat space holography. DG additionally thanks Arjun Bagchi, Stephane Detournay, Max Riegler, Jan Rosseel and Joan Simon for a wonderful long-term collaboration on numerous aspects of flat space holography.

DG was supported by projects of the Austrian Science Fund (FWF) Y 435-N16, I 952-N16 and I 1030-N27, and by the program Science without Borders, project CNPq-401180/2014-0. WM was supported by the FWF project P 27182-N27.

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Chapter 11
Super-Entropic Black Holes

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Abstract Super-entropic black holes form a new class of exact solutions to the Einstein–Maxwell–AdS equations. They are obtained from taking a new ultraspinning limit to the class of Kerr-AdS metrics, and have a number of intriguing properties. Their event horizons are not compact but have finite area, they have no closed timelike curves, and their entropy is larger than expected from general considerations of their thermodynamic volume. Here a brief overview of the construction and basic physical properties of these metrics is presented.

11.1 Introduction

One of the early fundamental results in the study of black holes concerns the topology of their horizons. Four-dimensional asymptotically flat stationary black holes necessarily have horizons of topology S^2 provided the dominant energy condition is satisfied [1]. A preponderance of black hole solutions with differing horizon topologies can be obtained by relaxing the assumptions underlying this theorem. For example, if the black hole is asymptotically anti de Sitter then the asymptotic topology in the domain of outer communication can be something other than S^2 and a whole class of black holes with a variety of horizon topologies can result [2–7] .

A new class of black holes has recently entered the scene. Known as super-entropic black holes [8], they were first constructed in [9] and further elucidated in both $\mathcal{N} = 2$ gauged supergravity coupled to vector multiplets [10] and in Einstein–Maxwell-Λ theory [8]. Their event horizons are not compact yet have finite area (and therefore finite entropy). Topologically, the event horizon is a sphere with two punctures, and this class of solutions can be obtained from a certain limit of the Kerr-solution [11].

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P. Nicolini et al. (eds.), *2nd Karl Schwarzschild Meeting on Gravitational Physics*, Springer Proceedings in Physics 208,
https://doi.org/10.1007/978-3-319-94256-8_11

Apart from the fact that this demonstrates an even richer landscape for event horizon topologies than was previously known, these black holes have another interesting feature from the perspective of extended phase space thermodynamics [12, 13]: they provided the first counterexample [8] to the conjectured '*Reverse Isoperimetric Inequality*' [14], which asserted that for a black hole of given thermodynamic volume the entropy will be maximal for the (charged) Schwarzschild AdS black hole. In this sense they exceed their expected maximal entropy and so have been called '*super-entropic*' [8].

11.2 Construction

The basic procedure for constructing super-entropic black holes is to take a rotating asymptotically AdS black hole and set its rotation parameter a equal to its AdS length l. The basic steps are as follows.

1. Eliminate any possible terms that would result in divergences in the $a \to l$ limit of a given rotating AdS black hole.
2. Recast the metric in a coordinate system that allows rescaling of the azimuthal coordinate in this limit.
3. Take the $a \to l$ limit; this effectively 'boosts' the asymptotic rotation to the speed of light.
4. Compactify the corresponding azimuthal direction. This qualitatively changes the structure of the spacetime since it is no longer possible to return to a frame that does not rotate at infinity.

The resulting structure is a black hole that has a non-compact horizon whose topology is that of a sphere with two punctures.

The simplest example is obtained from the Kerr–Newman-AdS black hole in four dimensions [15], whose metric and electromagnetic gauge potential are

$$
\begin{aligned}
\mathrm{d}s^2 &= -\frac{\Delta_a}{\Sigma_a}\left[\mathrm{d}t - \frac{a\sin^2\theta}{\Xi}\mathrm{d}\phi\right]^2 + \frac{\Sigma_a}{\Delta_a}\mathrm{d}r^2 + \frac{\Sigma_a}{S}\mathrm{d}\theta^2 + \frac{S\sin^2\theta}{\Sigma_a}\left[a\mathrm{d}t - \frac{r^2+a^2}{\Xi}\mathrm{d}\phi\right]^2 \\
\mathscr{A} &= -\frac{qr}{\Sigma_a}\left(\mathrm{d}t - \frac{a\sin^2\theta}{\Xi}\mathrm{d}\phi\right) \qquad (11.1)
\end{aligned}
$$

in standard Boyer–Lindquist form, where

$$
\begin{aligned}
\Sigma_a &= r^2 + a^2\cos^2\theta\,, \quad \Xi = 1 - \frac{a^2}{l^2}\,, \quad S = 1 - \frac{a^2}{l^2}\cos^2\theta\,, \\
\Delta_a &= (r^2+a^2)\Big(1 + \frac{r^2}{l^2}\Big) - 2mr + q^2\,, \qquad (11.2)
\end{aligned}
$$

with the horizon r_h defined by $\Delta_a(r_h) = 0$. This coordinate system rotates at infinity with an angular velocity $\Omega_\infty = -a/l^2$.

To carry out step 1, define $\psi = \phi/\Xi$. Rewriting the metric using this new coordinate (step 2), a conical singularity is avoided by identifying ψ with period $2\pi/\Xi$, and the $a \to l$ limit (step 3) yields

$$\mathrm{d}s^2 = -\frac{\Delta}{\Sigma}\left[\mathrm{d}t - l\sin^2\theta\mathrm{d}\psi\right]^2 + \frac{\Sigma}{\Delta}\mathrm{d}r^2 + \frac{\Sigma}{\sin^2\theta}\mathrm{d}\theta^2 + \frac{\sin^4\theta}{\Sigma}\left[l\mathrm{d}t - (r^2 + l^2)\mathrm{d}\psi\right]^2$$
$$\mathscr{A} = -\frac{qr}{\Sigma}\left(\mathrm{d}t - l\sin^2\theta\mathrm{d}\psi\right) \tag{11.3}$$

where

$$\Sigma = r^2 + l^2\cos^2\theta\,, \quad \Delta = \left(l + \frac{r^2}{l}\right)^2 - 2mr + q^2 \tag{11.4}$$

and ψ is now a noncompact azimuthal coordinate; by requiring that $\psi \sim \psi + \mu$ it is compactified (step 4) and the procedure is complete. The metric (11.3) is a four-dimensional charged super-entropic black hole.

11.3 Basic Properties

The black hole described by the metric and gauge potential (11.3) has a number of interesting properties.

Minimal mass Examining the roots of Δ in (11.4) it is easy to see that

$$m \geq m_0 \equiv 2r_0\left(\frac{r_0^2}{l^2} + 1\right), \qquad r_0^2 \equiv \frac{l^2}{3}\left[-1 + \left(4 + \frac{3q^2}{l^2}\right)^{\frac{1}{2}}\right] \tag{11.5}$$

and so there is a minimum value $m > m_0$ of the mass required for horizons to exist. When $m = m_0$ the two roots of Δ coincide and the black hole is extremal. For $m < m_0$ there is a naked singularity.

No Closed Timelike Curves It is straightforward to see that

$$g_{\psi\psi} = \frac{l^4\sin^4\theta}{l^2\cos^2\theta + r^2}\left(2mr - q^2\right) > 0 \tag{11.6}$$

provided $m > m_0$ and $r_+ > r_0$. The spacetime (11.3) is free of closed timelike curves.

Non-compact Horizon There is an infinite amount of proper distance from any point to the $\theta = 0, \pi$ axis, and so this axis is removed from the spacetime. The topology of the horizon is that of a sphere with two punctures (i.e. a cylinder). Setting $\kappa = l(1 - \cos\theta)$ in (11.3) yields

$$\mathrm{d}s^2 = (r^2 + l^2)\left[\frac{\mathrm{d}\kappa^2}{4\kappa^2} + \frac{4(2mr - q^2)}{(r^2 + l^2)^2}\kappa^2 \mathrm{d}\psi^2\right], \tag{11.7}$$

for small κ. The metric (11.7) is a metric of constant negative curvature on a quotient of the hyperbolic space $\mathbb{H}^2$. Any fixed (r, t) sections have the same topology, approaching a Lobachevsky space near the axis.

Ergosphere The region for which the Killing vector ∂_t is no longer timelike is

$$\Delta - l^2 \sin^4\theta < 0 \tag{11.8}$$

and forms the ergosphere of the black hole. The outer boundary of this region is where $\Delta = l^2 \sin^4\theta$.

Horizon Shape The geometry of the horizon can be visualized by embedding it in Euclidean 3-space [9]

$$\mathrm{d}s_3^2 = \mathrm{d}z^2 + rmdR^2 + R^2 d\phi^2 = g_{\psi\psi}(r = r_+)\mathrm{d}\psi^2 + g_{\theta\theta}(r = r_+)\mathrm{d}\theta^2\Big|_{r=r_+} \tag{11.9}$$

where the latter equality holds provided

$$R^2(\theta) = \left(\frac{\mu}{2\pi}\right) g_{\psi\psi}(r = r_+) \qquad \left(\frac{\mathrm{d}z(\theta)}{\mathrm{d}\theta}\right)^2 = g_{\theta\theta}(r = r_+) - \left(\frac{\mathrm{d}R(\theta)}{\mathrm{d}\theta}\right)^2 \tag{11.10}$$

These equations can be integrated numerically for various values of r_+, l and q; a typical result is depicted in Fig. 11.1. Note that the fact that $z(\theta)$ extends to $\pm\infty$ at the poles does not imply that the horizon extends to spatial infinity; rather the horizon should be understood as a Lobachevsky space of spherical shape, with any point on the sphere an infinite proper distance from either pole.

Asymptotic Structure Taking $r \to \infty$ and rescaling the metric by the conformal factor l^2/r^2 yields

$$\mathrm{d}s^2_{\mathrm{bdry}} = -\mathrm{d}t^2 - 2l\sin^2\theta \mathrm{d}t\mathrm{d}\psi + \frac{l^2}{\sin^2\theta}\mathrm{d}\theta^2 \tag{11.11}$$

which is the metric of the conformal boundary of (11.3). The ψ coordinate becomes null here; essentially the spacetime is rotating at the speed of light at infinity.

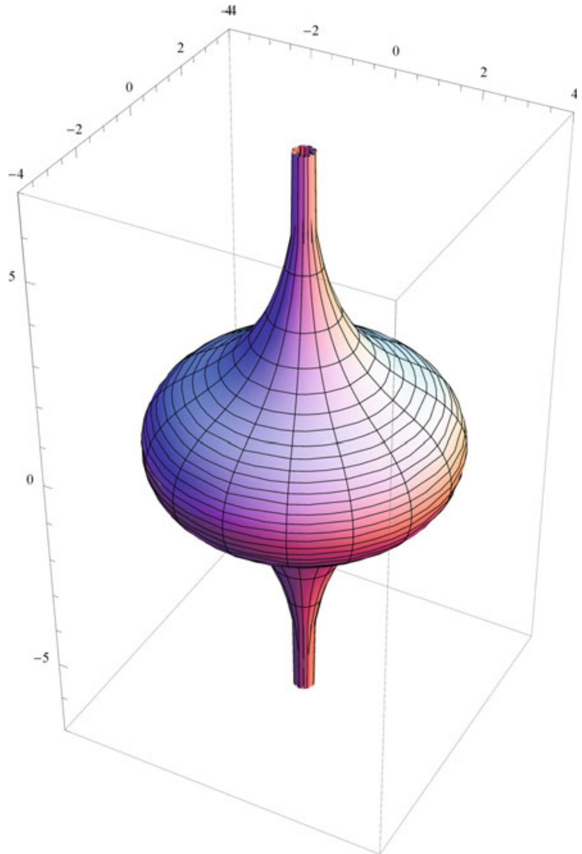

Fig. 11.1 Horizon embedding. The horizon geometry of a 4d super-entropic black hole embedded in $\mathbb{E}^3$ for the following choice of parameters: $q = 0, l = 1$, $r_+ = \sqrt{10}$ and $\mu = 2\pi$

11.4 Thermodynamics

In the framework of extended phase space thermodynamics, a subject that has come to be known as *Black Hole Chemistry* [13]. The thermodynamic quantities associated with the metric (11.3) are [8, 10]

$$M = \frac{\mu m}{2\pi} \quad J = Ml \quad \Omega = \frac{l}{r_+^2 + l^2}, \quad A = 2\mu(l^2 + r_+^2)$$
$$S = \frac{A}{4} \quad T = \frac{1}{4\pi r_+}\left(3\frac{r_+^2}{l^2} - 1 - \frac{q^2}{l^2 + r_+^2}\right)$$
$$Q = \frac{\mu q}{2\pi} \quad \Phi = \frac{q r_+}{r_+^2 + l^2} \quad V = \frac{r_+ A}{3} \quad P = -\frac{\Lambda}{8\pi} = \frac{3}{8\pi l^2} \tag{11.12}$$

where M is the black hole enthalpy, J its angular momentum, Ω its horizon angular velocity, T its temperature, S its entropy, Q its charge, Φ its conjugate gauge potential, V its thermodynamic volume, and P its conjugate pressure. Note that the (negative) cosmological constant is identified with pressure – the basic postulate of black hole chemistry [12]. Both Ω and Φ are measured with respect to infinity. The thermodynamic quantities in (11.12) cannot be obtained by taking the $a \to l$ limit of their Kerr–Newman-AdS counterparts due to the singular nature of the ultraspinning limit.

It is straightforward to show that the first law of black hole thermodynamics

$$\mathrm{d}M = T\mathrm{d}S + V\mathrm{d}P + \Omega\mathrm{d}J + \Phi\mathrm{d}Q + K\mathrm{d}\mu. \tag{11.13}$$

is satisfied for the quantities in (11.12) where

$$K = \frac{(l^2 - r_+^2)\left[(r_+^2 + l^2)^2 + q^2 l^2\right]}{8\pi l^2 r_+ (r_+^2 + l^2)} \tag{11.14}$$

is the thermodynamic conjugate to μ, which can be interpreted as a chemical potential analogous to that done in asymptotically Schrödinger geometries [16]. The conserved charge M is no longer interpreted the energy (or mass) of the black hole but rather as its chemical enthalpy [17].

Shortly after the introduction of the quantities P and V into blackhole thermodynamics the d-dimensional quantity

$$\mathscr{R} \equiv \left(\frac{(d-1)V}{\omega_{d-2}}\right)^{\frac{1}{d-1}} \left(\frac{\omega_{d-2}}{A}\right)^{\frac{1}{d-2}} \tag{11.15}$$

was noted to satisfy $\mathscr{R} \geq 1$ for all black holes known at that time [14]. The quantity A is the horizon area and ω_d is the unit area of the space orthogonal to constant (t, r) surfaces. The relation $\mathscr{R} \geq 1$ was conjectured to hold for all black holes [14] and is referred to as the *reverse isoperimetric inequality*: in physical terms it is the statement that for a black hole of a given thermodynamic volume, the entropy will be maximal for the (charged) Schwarzschild-AdS black hole.

Due to the compactification of ψ we have from (11.3) $\omega_2 = \mu\pi^{\frac{1}{2}}/\Gamma\left(\frac{3}{2}\right) = 2\mu$, and so (11.15) becomes

$$\mathscr{R} = \left(\frac{r_+ A}{2\mu}\right)^{1/3} \left(\frac{2\mu}{A}\right)^{1/2} = \left(\frac{r_+^2}{r_+^2 + l^2}\right)^{1/6} < 1 \tag{11.16}$$

in violation of the conjecture. For a given thermodynamic volume the entropy of the black hole (11.3) exceeds that of the Schwarzschild-AdS black hole; this is why they are called ‘super-entropic’ [8].

11.5 Multispinning Super-Entropic Black Holes

In dimensions larger than four more possibilities emerge for constructing super-entropic black holes. It is straightforward to construct higher-dimensional versions of the metric (11.3) from the singly-spinning d-dimensional version of (11.1). However it is also possible to have black holes with rotations about more than one axis, and these lead to interesting new black hole spacetimes [11].

An interesting case with 2 rotations is the $d = 5$ solution of minimal gauged supergravity [18], whose metric and gauge potential A are

$$\mathrm{d}s^2 = \mathrm{d}\gamma^2 - \frac{2q\nu\omega}{\Sigma} + \frac{f\omega^2}{\Sigma^2} + \frac{\Sigma \mathrm{d}r^2}{\Delta} + \frac{\Sigma \mathrm{d}\theta^2}{S} \qquad A = \frac{\sqrt{3}q\omega}{\Sigma} \tag{11.17}$$

where

$$\begin{aligned} \mathrm{d}\gamma^2 &= -\frac{S\rho^2 \mathrm{d}t^2}{\Xi_a \Xi_b l^2} + \frac{r^2+a^2}{\Xi_a}\sin^2\theta d\phi^2 + \frac{r^2+b^2}{\Xi_b}\cos^2\theta \mathrm{d}\psi^2 \\ \nu = b\sin^2\theta d\phi + a\cos^2\theta \mathrm{d}\psi \qquad \omega &= \frac{S\mathrm{d}t}{\Xi_a \Xi_b} - a\sin^2\theta\frac{\mathrm{d}\phi}{\Xi_a} - b\cos^2\theta\frac{\mathrm{d}\psi}{\Xi_b} \end{aligned} \tag{11.18}$$

and

$$\begin{aligned} S &= \Xi_a \cos^2\theta + \Xi_b \sin^2\theta \qquad f = 2m\Sigma - q^2 + \frac{2abq}{l^2}\Sigma \\ \Delta &= \frac{(r^2+a^2)(r^2+b^2)\rho^2/l^2 + q^2 + 2abq}{r^2} - 2m \qquad \rho^2 = r^2 + l^2 \\ \Sigma &= r^2 + a^2\cos^2\theta + b^2\sin^2\theta \qquad \Xi_a = 1 - \frac{a^2}{l^2} \quad \Xi_b = 1 - \frac{b^2}{l^2} \end{aligned} \tag{11.19}$$

with a and b the two rotation parameters and q proportional to the black hole charge.

To obtain a super-entropic black hole, we can write

$$\phi = \phi_R + \frac{a}{l^2}t \quad \psi = \psi_R + \frac{b}{l^2}t \tag{11.20}$$

and then set $\varphi = \phi_R/\Xi_a$. Taking the limit $a \to l$ yields the doubly-spinning charged super-entropic black hole metric and gauge potential

$$\mathrm{d}s^2 = \mathrm{d}\gamma_s^2 - \frac{2q\nu_s\omega_s}{\Sigma} + \frac{f\omega_s^2}{\Sigma^2} + \frac{\Sigma \mathrm{d}r^2}{\Delta} + \frac{\Sigma \mathrm{d}\theta^2}{\Xi_b \sin^2\theta} \qquad \mathscr{A} = \frac{\sqrt{3}q\omega_s}{\Sigma} \tag{11.21}$$

where

$$\Delta = \frac{\rho^4(r^2+b^2)/l^2+q^2+2lbq}{r^2} - 2m \quad \omega_s = \mathrm{d}t - l\sin^2\theta\mathrm{d}\varphi - \frac{b\cos^2\theta\mathrm{d}\psi_R}{\Xi_b}$$

$$\nu_s = \frac{b}{l}\mathrm{d}t + l\cos^2\theta\mathrm{d}\psi_R \quad \Sigma = r^2 + l^2\cos^2\theta + b^2\sin^2\theta \quad f = 2m\Sigma - q^2 + \frac{2bq}{l}\Sigma$$

$$\begin{aligned} \mathrm{d}\gamma_s^2 &= -\frac{\sin^2\theta}{l^2}\Big[(\rho^2+l^2)\mathrm{d}t^2 - 2l\rho^2\mathrm{d}td\varphi\Big] \\ &\quad + \frac{\cos^2\theta}{\Xi_b}\Big[(r^2+b^2)d\psi_R^2 + \frac{2b}{l^2}(r^2+b^2)\mathrm{d}td\psi_R - \frac{\mathrm{d}t^2}{l^2}\Big(\rho^2-(r^2+b^2)\frac{b^2}{l^2}\Big)\Big] \end{aligned} \tag{11.22}$$

and the coordinate φ is identified with period μ: $\varphi \sim \varphi + \mu$. The metric (11.22) satisfies the Einstein–Maxwell-AdS equations and horizons exist provided $\Delta'(r_+) > 0$.

Many other multiply-rotating cases can be constructed [11]. However once the super-entropic limit in one azimuthal direction is taken it is no longer possible to perform an additional super-entropic limit. This is easily seen in the metric (11.22): the $1/\Xi_b$ factor will diverge as $b \to l$ and this divergence cannot be absorbed into a new azimuthal coordinate. It is also not possible to set several rotation parameters equal and then perform simultaneously the super-entropic limit in all such directions. However it is possible to combine the super-entropic limit in one direction with the hyperboloid membrane limit in another direction [11].

11.6 Summary

Super-entropic black holes result from taking a new ultraspinning limit from the class of Kerr-AdS metrics. Topologically, the event horizons are doubly-punctured spheres (i.e. cylinders), and so can be considered as the AdS generalization of asymptotically flat black cylinders [19, 20], despite the lack of a flat-space limit. They form the first known examples of black holes that violate the conjectured Reverse Isoperimetric Inequality [14], suggesting that this conjecture may apply only to black holes with compact horizons under (perhaps) other restrictions. While recent work has suggested a connection with an upper bound of butterfly velocities for AdS black holes [21], the proof of this (restricted) conjecture remains an interesting open problem.

Acknowledgements This work was supported in part by the Natural Sciences and Engineering Research Council of Canada. I am grateful to Robie Hennigar, David Kubiznak, and Nathan Muskoe, who collaborated with me in this work.

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Chapter 12
Aspects of Quantum Chaos Inside Black Holes

A. Addazi

12.1 Introduction and Conclusions

Theoretical physicists are all agreed that Semiclassical Black holes are paradoxical objects (as nicely reconfirmed by several discussions during the Karl Schwarzschild meeting 2015). However, a clear strategy in order to solve this problem is still unknown.

In this paper, we would like to suggest that infalling information could be chaotized inside a black hole. Our claim is related to a different picture about quantum black holes' nature: we retained unmotivated to think seriously about a quantum black hole as a conformal Penrose's diagram, i.e as a smoothed semiclassical geometry with a singularity in its center (eventually cutoff at the planck scale). In particular, one could expect that, in a "window" of length scales among the Schwarzschild radius and the Planck scale, there is a non-topologically trivial region of space-time rather than a smoothed one. A realistic black hole could be a superposition of different horizonless solutions, perhaps associated GR gravitational instantons or "exotic" gravitational instantons.[1] In this picture, a black hole' horizon is an approximated Chauchy null-like surface (for energy scales closed to an inverse Schwarzschild radius). However, for lenght scales L in the range $l_{Pl} << L << R$, geometrical deviations and asperities with respect to semiclassical smoothed geometries are reasonable expected. In this regime, gravitational interactions among horizonless geometries can be neglected as well as microscopical exchanges of matter and gauge fields among their surfaces. In

[1] In string theory, the class of instantons is much larger than in field theories. Applications of a particular class of these solutions in particle physics were recently studied in [1–10].

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P. Nicolini et al. (eds.), *2nd Karl Schwarzschild Meeting on Gravitational Physics*, Springer Proceedings in Physics 208,
https://doi.org/10.1007/978-3-319-94256-8_12

this sense, a black hole cannot be described by a single Penrose's diagram at all the lenght scales. In particular, in the "middle region" a black hole would be described by a superposition of a large number of Penrose's diagrams.

Such a black hole can be rigorously defined in an euclidean path integral formulation. It emits a thermal radiation like a semiclassical BH, with small corrections on Bekenstein–Hawking entropy [1].

At this point, a further question is the following: what happen to infalling informations in such a "scale variant" system? Let us consider the usual thought experiment of a infalling radiation in a quantum pure state, with a very small initial frequency $\omega \simeq R^{-1}$. Such a radiation will start to probe a smoothed semiclassical geometry of a black hole, near the horizon. However, radiation will be inevitably blueshifted inside the gravitational potential of a black hole, i.e its De Broglie wave length starts to be smaller than the Schwarzschild's radius. So that, infalling radiation will start to probe the middle region before than the full quantum quantum regime. In the middle region, radiation is scattered back and forth among asperities that usually are not present at all in semiclassical BH solutions. As a consequence, radiation will be chaotically diffracted inside this system. At that middle scales, a black hole is a sort of *space-temporal chaotic Sinai billiard* rather than a smoothed manifold. Usually, in simpler classical chaotic billiards than our one, chaotic zones of unstable orbits trapped forever in the system are formed. Simple examples of such a trapped paths: (i) an orbit trapped in back and forth scatterings among the asperity A and the asperity B (AB segments); (ii) one trapped among A, B, C asperities (triangular orbits); and so on. Considering quantum fields rather than classical trajectories, one has also to consider quantum transitions induced by inelastic scatterings on gravitational backgrounds $< g, \ldots, g >$ (thought as a vacuum expectation value of gravitons). $\phi + < g, \ldots, g > \rightarrow X + < g, \ldots, g >$ where ϕ is a generic gauge/matter field, and X is a collection of N fields. For example a process like a photon-background scattering

$$\gamma + < g, \ldots, g > \rightarrow q\bar{q} + < g, \ldots, g > \rightarrow \text{hadronization} + < g, \ldots, g >$$

will lead to a complicated hadronic cascade of entangled fields. As a consequence, such a system is even more chaotic than classical one. So that, a part of the initial infalling information is effectively fractioned in a "forever" (black hole lifetime or so) trapped part and another one, so that

$$|\mathrm{IN}\rangle = a|\mathrm{OUT}\rangle + b|\mathrm{TRAPPED}\rangle$$

where a, b parametrize our ignorance about the space-time billiard, $|\mathrm{OUT}\rangle$ is emitted as Bekestein–Hawking radiation. As a consequence, the in-going information is a linear combination of outgoing informations and trapped informations during $0 << t << t_{Evaporation}$. In this picture, information paradoxes are understood as an *apparent* losing of unitarity. In fact, $|\mathrm{IN}\rangle \rightarrow |\mathrm{OUT}\rangle$ is not allowed by quantum mechanics: $|\mathrm{IN}\rangle$ is a pure state, while $|\mathrm{OUT}\rangle$ is a mixed one. However, also $|\mathrm{TRAPPED}\rangle$ is a mixed state, and a linear combination of two mixed states can be a pure one. In this approach, a $|\mathrm{IN}\rangle \to |\mathrm{OUT}\rangle$ transition can be effectively described in a density matrix approach, with an effective non-unitary evolution. However, unitarity is not lost at fundamental level because of the real transition $|\mathrm{IN}\rangle \to a|\mathrm{OUT}\rangle + b|\mathrm{TRAPPED}\rangle$ is not contradicting unitarity. Let us consider, for example, a (famous) Bekenstein–Hawking particle-antiparticle pair created nearby the black hole horizon. As usual, one of the two is captured inside the black hole space-like interior, while the second one can tunnel outside the horizon. As well known, the two particles are entangled, and this will lead to the undesired fire-wall paradox. However, in a frizzy black hole, the infalling particle will start to be blueshifted so that it will start to scatter back and forth inside the system, giving rise to an exponentially growing cascade of N particles continuing to scatter and to scatter in the billiard. The process will be even more chaotic in a realistic case in which a large number of infalling partners from a large number of Bekenstein–Hawking pairs have to be considered. As a consequence, P outgoing pairs will be entangled with a total number $N >> P$ of particles inside the system. This practically disentangles the P outgoing pairs from the original ones, as a quantum decoherence effect induced by the non-trivial space-time topology. In other words, the space-time topology is collapsing the entangled wave function as a quantum decoherence phenomena, as well as two entangled pairs are disentangled by a an experimental apparatus. The entanglement entropy is linearly growing with the number of back and forth scatterings n of a particle, because of the density matrix of the internal black states are exponentially growing with n:

$$S_{\mathrm{interior}} = -Tr\rho_{\mathrm{interior}}\log S_{\mathrm{interior}} \sim n$$

so that is growing with time. On the other hand, for P Bekenstein–Hawking particles $S_{\mathrm{int}} \sim n \log P$. Our model predicts $S_{B.H.} \sim P$ from entanglement entropy definition.

However, if a frizzy black hole emits a Bekenstein–Hawking radiation with small deviations from thermality, it cannot have an infinite life-time. On the other hand, the non-trivial topological space-time configuration of a frizzy black hole is sourced by the black hole mass. The final configuration after the complete black hole evaporation is a Minkowski space-time with a dilute residual radiation. As a consequence, a space-time phase transition from the "frizzy" topology to the Minkowski space-time is expected at the Page time or so. As a consequence, chaotic saddles of trapped information will be emitted in the environment as a *final information burst*. For this motivation, the S-matrix describing BH evolution from the initial collapse/formation to its complete evaporation is unitary:

$$\langle \mathrm{COLLAPSE|S|EVAPORATION}\rangle$$

$$= \langle \mathrm{TOTAL\ INFALLING}|\ \mathrm{S}|\ (\mathrm{a}|\mathrm{TRAPPED}\rangle + \mathrm{b}|\mathrm{OUT}\rangle)$$

The trapped probability density $\rho(T)$ is approximately described by

$$\frac{d\rho(T)}{dT} \sim -\frac{1}{T^2}e^{-\Gamma T}$$

In fact, $\rho(T)$ is dependent by the number of asperities N_s as $\rho \sim N_s e^{-\Gamma T}$, where Γ is proportional to *effective average deepness* of asperities (trapping ρ). But the number of asperities is depending by the Black hole mass. In turn, the black hole mass decreases with the temperature as $dM/dT = -1/8\pi T^2$.

To conclude, chaotic aspects of quantum black holes could be relevantly connected to the information paradoxes. In particular, a semiclassical black hole could be reinterpreted as a superposition of horizonless geometries, chaotizing infalling informations. Such an approach could have surprising connections with recent results in contest of AdS/CFT correspondence [11–13].

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Chapter 13
Black Hole Entropy in the Presence of Chern–Simons Term and Holography

T. Azeyanagi

Abstract We propose a manifestly covariant formulation of the differential Noether charge for higher-dimensional Chern–Simons terms. With our differential Noether charge, we provide a covariant proof of the black hole entropy formula for gravitational and mixed $U(1)$-gravitational Chern–Simons terms. By evaluating the charge on the rotating charged AdS black hole background constructed by the fluid/gravity derivative expansion, we show that the Chern–Simons contribution to black hole entropy agrees with the anomaly-induced entropy current in the dual conformal field theory at the leading order of the derivative expansion.

13.1 Chern–Simons Term and Anomaly Polynomial

The setup we are interested in here is $(2n+1)$-dimensional Einstein–Maxwell–Chern–Simons theory with a negative cosmological constant. We consider gravitational and/or mixed $U(1)$-gravitational Chern–Simons terms which, as will be seen later, contribute nontrivially to black hole entropy. For example, the five-dimensional mixed $U(1)$-gravitational Chern–Simons term and seven-dimensional (double trace) gravitational Chern–Simons term are given respectively by

$$A \wedge \mathrm{tr}\,(R \wedge R)\ , \qquad \mathrm{tr}\left(\Gamma \wedge R - \frac{1}{3}\Gamma \wedge \Gamma \wedge \Gamma\right) \wedge \mathrm{tr}\,(R \wedge R)\ . \qquad (13.1)$$

Here $A = A_a \mathrm{d}x^a$ is the $U(1)$ gauge potential one-form, $\Gamma^a{}_b = \Gamma^a{}_{bc}\mathrm{d}x^c$ is the Christoffel connection one-form and $R^a{}_b = (1/2)R^a{}_{bcd}\mathrm{d}x^c \wedge \mathrm{d}x^d$ is the Riemann curvature two-form (A_a: $U(1)$ gauge potential, $\Gamma^a{}_{bc}$: Christoffel connection, $R^a{}_{bcd}$: Riemann curvature). In general, the Chern–Simons term I_{CS} depends on covariant quantities, F and $R^a{}_b$, as well as non-covariant quantities, A and $\Gamma^a{}_b$ ($F = \mathrm{d}A$:

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P. Nicolini et al. (eds.), *2nd Karl Schwarzschild Meeting on Gravitational Physics*, Springer Proceedings in Physics 208,
https://doi.org/10.1007/978-3-319-94256-8_13

field strength two-form for A). In addition to this, the Chern–Simons term has the following three important properties:

1. *Non-Covariance of Chern–Simons Term*
   Under gauge transformation/diffeomorphism labeled by $\chi = \{\Lambda, \xi\}$, the Chern–Simons term transforms covariantly up to a total derivative term:
   $$\delta_\chi I_{CS} = \mathcal{L}_\xi I_{CS} + \mathrm{d}(\ldots)\,, \tag{13.2}$$
   where $\mathcal{L}_\xi$ is the Lie derivative generated by ξ.
2. *Anomaly Polynomial*
   For each Chern–Simons term, one can define a formal $(2n+2)$-form called the anomaly polynomial. The anomaly polynomial is defined by taking the derivative of I_{CS}: $P_{\text{anom}} = \mathrm{d}I_{CS}$. The anomaly polynomial depends only on covariant quantities, F and $R^a{}_b$, and thus is covariant. For example, the anomaly polynomials corresponding to the Chern–Simons terms in (13.1) are
   $$F \wedge \mathrm{tr}(R \wedge R)\,, \qquad \mathrm{tr}(R \wedge R) \wedge \mathrm{tr}(R \wedge R)\,. \tag{13.3}$$
3. *Covariance of Equation of Motion*
   Although the Chern–Simons term is not covariant under gauge transformation/diffeomorphism, Chern–Simons contribution to the equation of motion is written in terms of (a covariant derivative of) a derivative of the anomaly polynomial. Therefore, the equation of motion is covariant even in the presence of the Chern–Simons term in the Lagrangian.

13.2 Noether Charge Formalism for Chern–Simons Term

In general, a black hole entropy formula depends on gravitational theory one is considering. Entropy of a black hole in Einstein gravity is given by the well-known Bekenstein-Hawking formula. For a general gravitational theory with a covariant Lagrangian L_{cov}, the Wald formalism provides a covariant way to derive the corresponding black hole entropy formula [1–4]. In this formalism, variation of black hole entropy is given as a differential Noether charge evaluated at the bifurcation horizon of the black hole. More precisely, for a stationary rotating black hole with the Killing vector $\xi = (\partial_t + \Omega_H \partial_\phi)/T_H$ (Here ∂_t and ∂_ϕ respectively are the Killing vectors corresponding to time translation and a rotation, Ω_H is the associated angular velocity at the bifurcation horizon and T_H is the Hawking temperature), variation of black hole entropy is given by

$$\delta S_{\text{Wald}} = \int_{Bif} \delta Q_\chi\,. \tag{13.4}$$

Here the differential Noether charge δQ_χ associated with χ is defined by

$$\mathrm{d}\delta Q_\chi = -\delta\delta_\chi \Omega_{PS} - i_\xi \delta E - \delta(\star N_\chi)\,. \tag{13.5}$$

($\delta\delta\Omega_{PS}$: pre-symplectic current, δE: equation of motion, N_χ: Noether current associated with χ, i_ξ: interior product with ξ, $\star$: Hodge dual. For more details of the definitions, please refer to [5], for example.) From (13.4), the celebrated Wald formula is obtained:

$$S_{\text{Wald}} = 2\pi \int_{\text{Bif}} \sqrt{h} \frac{\delta L_{\text{cov}}}{\delta R_{abcd}} \varepsilon_{ab}\varepsilon_{cd}\,, \tag{13.6}$$

where $\sqrt{h}$ and ε_{ab} respectively are the area element and binormal at the bifurcation horizon. We note that the right hand side of equation (13.6) evaluated at a general horizon slice also gives the correct black hole entropy [4].

The original Wald formalism is not directly applicable to the Chern–Simons term because of its non-covariant transformation property under gauge transformation/diffeomorphism. In [6], this point is taken into account and a generalization of the Wald formalism to the Chern–Simons term is proposed. By using this formalism, the black hole entropy formula for the Chern–Simons term is derived (See also [7, 8]):

$$S_{CS} = (4\pi) \int_{\text{Bif}} \sum_{k=1}^{\infty} (2k)\Gamma_N \wedge R_N^{2k-2} \wedge \frac{\partial P_{\text{anom}}}{\partial \text{tr}(R^{2k})}\,. \tag{13.7}$$

Here $\Gamma_N = (-1/2)\varepsilon^a{}_b\Gamma^b{}_a$ is the normal bundle connection one-form at the bifurcation horizon and $R_N = d\Gamma_N$ is the curvature two-form associated with it. However, [9] pointed out recently that, for gravitational and mixed gravitational-$U(1)$ Chern–Simons terms in more than three-dimensions, this black hole entropy formula can be obtained from the generalization of the Wald formalism in [6] only when gauge and coordinates are chosen appropriately. This implies that Tachikawa's formalism seems to break covariance somewhere.

In [5], we pointed out that the formalism breaks covariance at the level of the pre-symplectic current. This non-covariance is then inherited to the differential Noether charge, ending up with the subtle result pointed out in [9]. To overcome this subtlety and to realize a manifestly covariant formulation of the differential Noether charge for the Chern–Simons term, we provided a covariant way to construct a pre-symplectic current for the Chern–Simons term. First of all, let us recall the defining equation of the pre-symplectic current $\delta\delta\Omega_{PS}$ based on equation of motion δE:

$$\mathrm{d}(\delta_1\delta_2\Omega_{PS}) = \delta_1\delta_2 E - \delta_2\delta_1 E\,. \tag{13.8}$$

We note that, by definition, the pre-symplectic current has an ambiguity to add total derivative terms. One of the key points of the Wald formalism is that, by starting with the Lagrangian description and variation of the Lagrangian $\delta L_{\text{cov}} = \delta E + \mathrm{d}\delta\Theta$, this defining equation can be rewritten as $\mathrm{d}(\delta_1\delta_2\Omega_{PS}) = -\mathrm{d}(\delta_1\delta_2\Theta - \delta_2\delta_1\Theta)$. Then the total derivative ambiguity of the pre-symplectic current can be fixed as $\Omega_{PS}^{\text{Wald}} = -\delta_1\delta_2\Theta + \delta_2\delta_1\Theta$. This is the definition of the pre-symplectic current for the Wald formalism. The same definition of the pre-symplectic current is used in [6]. For the Chern–Simons terms, however, the pre-symplectic current defined in this way breaks covariance. Essentially, this breakdown originates in the non-covariance of the boundary term $\delta\Theta$ for the Chern–Simons term [5].

For the Chern–Simons term, in fact, we can construct a covariant pre-symplectic current in a much easier way. As mentioned above, the equation of motion for the Chern–Simons term is written in terms of (a covariant derivative of) a derivative of the anomaly polynomial. Thus, by starting from the equation of motion and the defining equation (13.8), one can directly carry out integration by part to rewrite the right hand side of equation (13.8) into a total derivative form. It is straightforward to keep track of covariance in the intermediate steps because of covariance of the equation of motion. We can then construct a manifestly covariant pre-symplectic current. Once this covariant pre-symplectic current is constructed, we can follow the same step as [6] to obtain a manifestly covariant differential Noether charge. By evaluating this manifestly covariant differential Noether charge for the Killing vector $\xi = (\partial_t + \Omega_H \partial_\phi)/T_H$ at the bifurcation horizon, the entropy formula (13.7) is proved covariantly [5].

13.3 Application: Replacement Rule from Fluid/Gravity Correspondence

One of the motivations to investigate higher-dimensional Chern–Simons terms is that these terms are needed to setup AdS/CFT duality for even-dimensional CFTs with some class of quantum anomalies. More precisely, to induce gravitational/mixed anomaly on the CFT side, the gravitational/mixed Chern–Simons term needs to be added on the dual gravity side. With deep connections to AdS/CFT, even-dimensional CFT at finite temperature with these anomalies has been investigated intensively in the hydrodynamic limit. As a consequence of these continuous anomalies in the underlying microscopic theory, hydrodynamic currents and stress tensor acquire extra terms called anomaly-induced transports [10–12]. Recently, it is proved by using an argument based on Euclidean thermal partition function in the hydrodynamic limit that the leading order anomaly-induced transport coefficients are completely determined from the corresponding anomaly polynomial through what-is-called the replacement rule [13–18]. In particular, the anomaly-induced entropy current is given by

$$s^\mu_{\text{anom}} = -\frac{\partial \mathscr{F}}{\partial T} V^\mu + \cdots , \quad \text{with} \quad \mathscr{F} = P_{\text{anom}}[F \to \mu,\ \mathrm{tr}(R^{2k}) = 2(2\pi T)^{2k}] , \tag{13.9}$$

where $V^{\mu} = \varepsilon^{\mu\alpha_1\alpha_2\ldots\alpha_{2n-1}} u_{\alpha_1} \omega_{\alpha_2\alpha_3} \omega_{\alpha_4\alpha_5} \ldots \omega_{\alpha_{2n-2}\alpha_{2n-1}}$ (u_α: fluid velocity, $\omega_{\alpha\beta}$: vorticity).

Now a natural question is whether we can reproduce the replacement rule for the anomaly-induced transport systematically from the dual gravity side. (Please refer to [19–23] for some analysis of anomaly-induced transports in the 4d CFT with mixed anomaly and its 5d gravity dual.) In [24], we started with Einstein-Maxwell-Chern–Simons theory with a negative cosmological constant as the simplest setup and carried out systematic analysis of anomaly-induced entropy current in even-dimensional CFTs from the dual gravity side. For Einstein-Maxwell theory with a negative cosmological constant, rotating charged AdS black hole can be constructed in the fluid/gravity derivative expansion and back reaction from the Chern–Simons terms is also taken into account [25]. By evaluating the differential Noether charge $\delta Q_\chi = \delta Q_\chi^{EM} + \delta Q_\chi^{CS}$ for the full theory at the horizon $r = r_H$ of this background, we have shown that the replacement rule for the anomaly-induced entropy current can be reproduced from the dual gravity side for any gravitational/mixed Chern–Simons term in any odd-dimensions [24]:

$$\int_{r=r_H} \delta Q_\chi|_{V^\mu-\text{linear}} = \int_{r=r_H} \delta Q_\chi^{EM}|_{V^\mu-\text{linear}} + \int_{r=r_H} \delta Q_\chi^{CS}|_{V^\mu-\text{linear}}\,, \tag{13.10}$$

$$\text{with} \qquad \int_H \delta Q_\chi^{EM}|_{V^\mu-\text{linear}} = 0\,, \quad \int_{r=r_H} \delta Q_\chi^{CS}|_{V^\mu-\text{linear}} = -\delta \int \frac{\partial \mathscr{F}}{\partial T} V_\mu\,.$$

Our result has a significant meaning for black hole microstate counting. Traditionally, black hole microstate counting based on AdS/CFT relies heavily on the Cardy(-like) formula in 2d CFT. One therefore needs to stick to some setups to which AdS_3/CFT_2 (or something similar) is applicable. Our result shows that, even for non-extremal black holes approaching asymptotically to higher-dimensional AdS, so far as we restrict ourselves to the Chern–Simons contribution, black hole entropy can be reproduced from the dual CFT side through the replacement rule. This is a new direction for black hole microstate counting business. We expect that our result is meaningful for the investigation of Cardy-like formulas in higher-dimensional CFTs and their application to black hole microstate counting.

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Chapter 14
A Quantum Cosmic Conjecture

R. Casadio and O. Micu

Abstract For a quantum mechanically Gaussian shaped, electrically charged, massive particle, we compute the Horizon Wave-function(s) in order to study (a) the existence of the inner Cauchy horizon of the corresponding Reissner–Nordström space-time when the charge-to-mass ratio $0 < \alpha < 1$ and (b) the survival of a naked singularity when the charge-to-mass ratio $\alpha > 1$. Our results suggest that any semi-classical instability one expects near the inner horizon may not occur in quantum black holes, with a mass around the Planck scale, and that no states with charge-to-mass ratio greater than a critical value (of the order of $\sqrt{2}$) should exist.

14.1 Introduction

There is a general consensus that black holes might play the same role in quantum gravity as the hydrogen atom does in the quantum theory of ordinary matter. In fact, a black hole realises the strongest non-perturbative effect that gravity can have, namely a total causal confinement. In order to investigate this feature in a quantum context, the Horizon Wave Function (HWF) formalism was proposed and developed in [1, 2], which is built on the quantised Einstein equation relating the size of the gravitational radius to the (quantum) state of matter.

The construction of the HWF starts from the spectral decomposition of the quantum mechanical state for a spherically symmetric matter source which is localised in space and static in time. By expressing the energy in terms of the gravitational

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P. Nicolini et al. (eds.), *2nd Karl Schwarzschild Meeting on Gravitational Physics*, Springer Proceedings in Physics 208,
https://doi.org/10.1007/978-3-319-94256-8_14

(Schwarzschild) radius, the spectral decomposition then directly yields the (unnormalised) HWF. The normalised HWF supplies the probability for an observer to detect a gravitational radius of a certain size (areal radius) around the source in the quantum state that was used in the first place. The gravitational radius can then be interpreted as a horizon if the probability of finding the particle inside of it is reasonably high. According to this quantum picture, the horizon appears necessarily a fuzzy location in space, precisely for the same reason the position of the particle that sources the geometry is intrinsically uncertain. This formalism has been applied to several case studies [3, 4], yielding sensible results in agreement with (semi)classical expectations, and there is therefore hope that it will help our understanding of the quantum nature of black holes.

In this talk, we will in particular summarise the results obtained for charged sources in [5, 6].

14.2 Electrically Charged Spherical Source

We start from the Reissner–Nordström metric,

$$\mathrm{d}s^2 = -f\,\mathrm{d}t^2 + f^{-1}\,\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right) , \tag{14.1}$$

with $f = 1 - \frac{2\,\ell_{\mathrm{p}}\,M}{m_{\mathrm{p}}\,r} + \frac{Q^2}{r^2}$, where M and Q respectively represent the ADM mass and charge of the source. It is convenient to introduce the specific charge $\alpha = |Q|\,m_{\mathrm{p}}/\ell_{\mathrm{p}}\,M$. The case $\alpha = 0$ then reduces to the neutral Schwarzschild metric with one horizon of radius $R_{\mathrm{H}} = 2\,\ell_{\mathrm{p}}\,M/m_{\mathrm{p}}$. For $0 < \alpha \leq 1$, the space-time contains two horizons, namely

$$R_{\pm} = \ell_{\mathrm{p}}\,\frac{M}{m_{\mathrm{p}}}\left(1 \pm \sqrt{1-\alpha^2}\right) , \tag{14.2}$$

which overlap for the extremal case $\alpha = 1$. Finally, for $\alpha > 1$ no horizon exists and the central singularity is therefore “naked”, or accessible to outer observers.

We next consider a spherically symmetric Gaussian source

$$\psi_{\mathrm{S}}(r) = \frac{e^{-\frac{r^2}{2\,\ell^2}}}{\ell^{3/2}\,\pi^{3/4}} , \tag{14.3}$$

whose width ℓ is assumed to be the minimum compatible with the Heisenberg uncertainty principle,

$$\ell = \lambda_m \simeq \ell_{\mathrm{p}}\,\frac{m_{\mathrm{p}}}{m} , \tag{14.4}$$

where λ_m is the Compton length of the particle of rest mass m [2]. The spectral decomposition of (14.3) is easily obtained from assuming the relativistic mass-shell relation in flat space, $M^2 = p^2 + m^2$, and by going to momentum space,

$$\psi_{\mathrm{S}}(p) = \frac{e^{-\frac{p^2}{2\Delta^2}}}{\Delta^{3/2}\,\pi^{3/4}} \ , \tag{14.5}$$

where $p^2 = \mathbf{p}\cdot\mathbf{p}$ is the square modulus of the spatial momentum, and the width $\Delta = m_{\mathrm{p}}\,\ell_{\mathrm{p}}/\ell \simeq m$.

14.2.1 Inner Horizon and Mass Inflation

For $0 < \alpha < 1$, one can write a HWF for each of the two horizons (14.2), namely [5]

$$\psi_{\mathrm{H}}(R_\pm) = \mathscr{N}_\pm\,\Theta\,(R_\pm - R_{\mathrm{min}\pm})\exp\left\{-\frac{m_{\mathrm{p}}^2\,R_\pm^2}{2\,\Delta^2\,\ell_{\mathrm{p}}^2\,(1\pm\sqrt{1-\alpha^2})^2}\right\} \ , \tag{14.6}$$

where the step function accounts for the minimum energy $M = m$, corresponding to $R_{\mathrm{min}\pm} = \ell_{\mathrm{p}}\,\frac{m}{m_{\mathrm{p}}}\left(1\pm\sqrt{1-\alpha^2}\right)$, and the normalisations $\mathscr{N}_\pm$ are fixed by using the Schrödinger scalar product.

The probability density that the particle lies inside its own gravitational radius of size $r = R_\pm$ can now be calculated starting from the wave-functions (14.6) as

$$\mathscr{P}_{<\pm}(r < R_\pm) = P_{\mathrm{S}}(r < R_\pm)\,\mathscr{P}_{\mathrm{H}}(R_\pm) \ , \tag{14.7}$$

where $P_{\mathrm{S}}(r < R_\pm) = 4\,\pi\,\int_0^{R_\pm}|\psi_{\mathrm{S}}(r)|^2\,r^2\,\mathrm{d}r$ is the probability that the particle is inside the sphere $r = R_\pm$, and $\mathscr{P}_{\mathrm{H}}(R_\pm) = 4\,\pi\,R_\pm^2\,|\psi_{\mathrm{H}}(R_\pm)|^2$ is the probability density that the sphere $r = R_\pm$ is the gravitational radius. Finally, one can integrate (14.7) over all possible values of R_+ to find the probability that the particle is a BH, namely

$$P_{\mathrm{BH}+} = \int_{R_{\mathrm{min}+}}^{\infty}\mathscr{P}_{<+}(r < R_+)\,\mathrm{d}R_+ \ . \tag{14.8}$$

The analogous quantity for R_-,

$$P_{\mathrm{BH}-} = \int_{R_{\mathrm{min}-}}^{\infty}\mathscr{P}_{<-}(r < R_-)\,\mathrm{d}R_- \ , \tag{14.9}$$

will instead be viewed as the probability that the particle lies further inside its inner horizon. It is obvious that $P_{\mathrm{BH}-} < P_{\mathrm{BH}+}$, and that only when $P_{\mathrm{BH}-} \simeq 1$ we can say that both horizons are physically realised at $\langle\,\hat{R}_-\,\rangle$ and $\langle\,\hat{R}_+\,\rangle$.

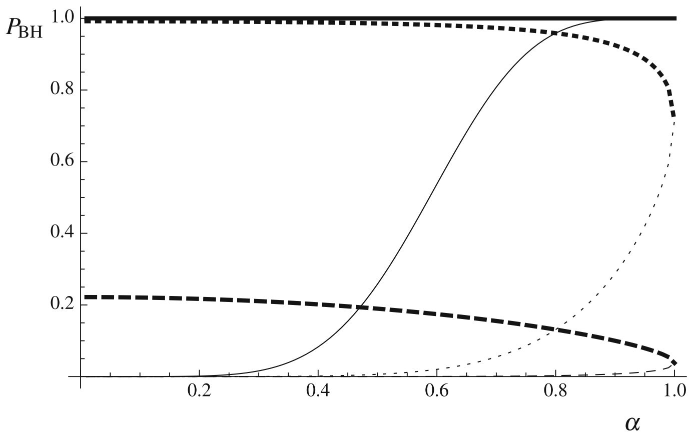

Fig. 14.1 Probabilities $P_{\rm BH+}$ (thick lines) and $P_{\rm BH-}$ (thin lines) as functions of α for $m = 2\,m_{\rm p}$ (continuous line), $m = m_{\rm p}$ (dotted line) and $m = 0.5\,m_{\rm p}$ (dashed line)

The plot in Fig. 14.1 shows the probabilities $P_{\rm BH\pm}$ as functions of α for values of the particle mass above, equal to and below the Planck mass. One can notice that $P_{\rm BH+}$ stays very close to one for masses larger than the Planck scale, whereas, for $m < m_{\rm p}$, it clearly decreases as the specific charge increases to one. For instance, if $m = 0.5\,m_{\rm p}$ ($\ell = 2\,\ell_{\rm p}$), $P_{\rm BH+} \simeq 0.2$ for a sizable range of α, and it only decreases below 0.1 when $\alpha \to 1$ and the source is nearly maximally charged. The situation is however different for the inner horizon. The probability $P_{\rm BH-}$ starts out negligible for small values of the charge-to-mass ratio and increases with α – the larger m, the smaller the value of α for which the probability becomes significant. There is a considerable range of α for which the probability for the inner horizon to exist is approximately zero, while $P_{\rm BH+} \simeq 1$ and the object is a black hole. Figure 14.2 shows the probabilities $P_{\rm BH\pm}$ as functions of the mass m for $\alpha = 0.3$, 0.8 and 1. It is clear that for smaller values of α, the probability $P_{\rm BH+}$ starts to increase from zero to one at smaller values of m, but the opposite occurs for $P_{\rm BH-}$. For $\alpha = 0.3$, it is only around a particle mass $m \simeq 6\,m_{\rm p}$ that $P_{\rm BH-} \simeq 1$, while $P_{\rm BH+} \simeq 1$ already around $m_{\rm p}$. This means that for $m_{\rm p} \leq m \leq 6\,m_{\rm p}$, the probability $P_{\rm BH+} \simeq 1$ while $P_{\rm BH-} \ll 1$. This mass range broadens up even more for smaller values of α, but decreases to zero in the maximally charged limit $\alpha = 1$. Our main finding is therefore that there exists a considerable parameter space for m (around the Planck scale) and $\alpha < 1$ in which

$$P_{\rm BH+} \simeq 1 \quad \text{and} \quad P_{\rm BH-} \simeq 0\,. \tag{14.10}$$

In this range the particle is (most likely) a black hole, but the inner horizon at $r = \langle\,\hat{R}_-\,\rangle$ is suppressed by quantum mechanical fluctuations. This conclusion is important in light of the instability usually referred to as the “mass inflation” [7].

Let us conclude this part by deriving a generalised uncertainty relation analogous to the neutral case [1]. We first note that the expectation value

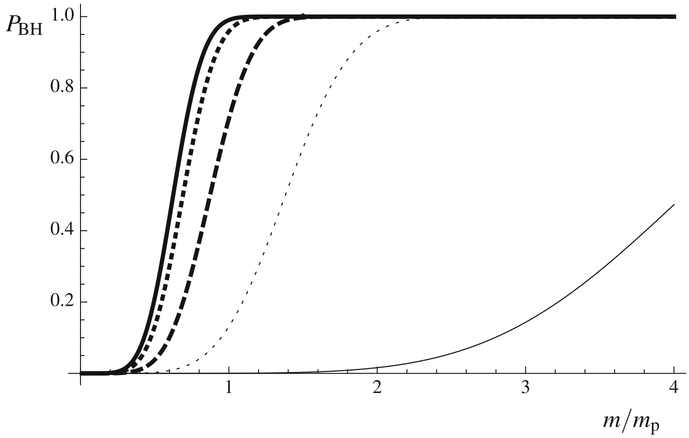

Fig. 14.2 Probabilities $P_{\mathrm{BH}+}$ (thick lines) and $P_{\mathrm{BH}-}$ (thin lines) as functions of the mass for $\alpha = 0.3$ (continuous line), $\alpha = 0.8$ (dotted line) and $\alpha = 1$ (dashed line)

$$\langle\, \hat{R}_+ \,\rangle = 4\,\pi \int_{R_{\mathrm{min}+}}^{\infty} |\psi_{\mathrm{H}}(R_+)|^2\, R_+^3\, \mathrm{d}R_+ = R_+(\bar{M})\ , \tag{14.11}$$

reproduces exactly the classical expression of R_+ in (14.2) for $\ell = \lambda_m \sim m^{-1}$ and $\bar{M} = 4\,m/[2 + e\,\sqrt{\pi}\,\mathrm{erfc}(1)] \simeq 1.45\,m$ (in agreement with the wave-function ψ_{S} containing energy contributions from momenta $p > 0$). From $\langle\, \hat{R}_+^2 \,\rangle \simeq R_+^2(\bar{M})$ one can then calculate the uncertainty

$$\Delta R_+ = \sqrt{\langle\, \hat{R}_+^2 \,\rangle - \langle\, \hat{R}_+ \,\rangle^2} \simeq R_+ \sim m\ , \tag{14.12}$$

which, like in the neutral Schwarzschild case, grows linearly with the mass m of the source.[1] If we now combine the horizon uncertainty (14.12) with the usual Heisenberg uncertainty in the radial size of the source, $\Delta r^2 \simeq \ell^2$, we finally obtain a total uncertainty

$$\Delta r = \sqrt{\langle\, \Delta r^2 \,\rangle} + \gamma\,\sqrt{\langle\, \Delta R_+^2 \,\rangle} \simeq \ell_{\mathrm{p}}\,\frac{m_{\mathrm{p}}}{\Delta p} + \gamma\,\ell_{\mathrm{p}}\,\frac{\Delta p}{m_{\mathrm{p}}}\ , \tag{14.13}$$

where γ is a coefficient of order one. We can therefore conclude that the outer horizon behaves qualitatively like the neutral Schwarzschild radius.

[1] Such objects would remain quantum mechanical even in astrophysical regimes, where we expect the horizon has a sharp location. This result therefore supports alternative models of black holes as extended quantum objects, like the ones in [4, 8].

14.2.2 Naked Singularity and Cosmic Censorship

We next move on to overcharged sources, represented by $\alpha > 1$ [6]. The Cosmic Censorship Conjecture [9] was formulated in order to forbid the existence of such naked singularities in the classical theory of gravity, so it is interesting to investigate whether quantum physics leads to any predictions therein. Our guiding principle will be to assume that the quantum states for $\alpha > 1$ can be obtained by extending continuously the HWF obtained for $\alpha < 1$. Of course, this is by no means a compelling choice, but we hope that it leads to consistent predictions for charges not too much larger than the classical limiting value of $\alpha = 1$.

The classical expressions of $R_\pm$ are complex for $\alpha > 1$, hence we lift only the real parts of $R_\pm$ to quantum observables. The modulus squared of the two HWFs (14.6), for $R_\pm > R_{\min\pm}$, then merge into

$$|\psi_{\rm H}(R)|^2 = \mathcal{N}^2 \exp\left\{-\frac{2-\alpha^2}{\alpha^4}\,\frac{m_{\rm p}^2 R^2}{\Delta^2 \ell_{\rm p}^2}\right\} , \tag{14.14}$$

where R has replaced $R_\pm$. This HWF is still normalizable if R is real and

$$1 < \alpha^2 < 2 . \tag{14.15}$$

We deduce that no normalisable quantum state with $\alpha^2 > 2$ is allowed. We must also consider what happens to the Heaviside function in (14.6) in the superextremal regime. First we note that the real parts of the minimum values of R_+ and R_- are again the same for $\alpha > 1$, and our continuity principle requires

$$R \geq R_{\rm min} = {\rm Re}\left[\ell_{\rm p}\,\frac{m}{m_{\rm p}}\left(1 \pm \sqrt{1-\alpha^2}\right)\right] = \ell_{\rm p}\,\frac{m}{m_{\rm p}} . \tag{14.16}$$

The expectation value for $\hat{R}$ then matches exactly the corresponding expressions for $\alpha < 1$ [5],

$$\lim_{\alpha \searrow 1} \langle\, \hat{R}\,\rangle = \frac{4\,\ell_{\rm p}^2/\ell}{2 + e\sqrt{\pi}\,{\rm erfc}(1)} = \lim_{\alpha \nearrow 1} \langle\, \hat{R}_\pm\,\rangle , \tag{14.17}$$

like the uncertainty $\Delta R(\ell, \alpha > 1)$ matches the uncertainties $\Delta R_\pm(\ell, \alpha < 1)$ at $\alpha = 1$. We remark that, for $\alpha = 1$, the width $\ell > \langle\, \hat{R}\,\rangle$ for $m < \sqrt{2 + e\sqrt{\pi}\,{\rm erfc}(1)}\, m_{\rm p}/2 \simeq 0.8\, m_{\rm p}$, and quantum fluctuations of the source will dominate for masses $m \ll m_{\rm p}$ (like in the neutral case [1, 2]). It is important to further note that the ratio

$$\frac{\langle\, \hat{R}\,\rangle}{\ell} \simeq \frac{2^{5/4}\,\ell_{\rm p}^2}{\sqrt{\pi\,\left(\sqrt{2}-\alpha\right)}} \tag{14.18}$$

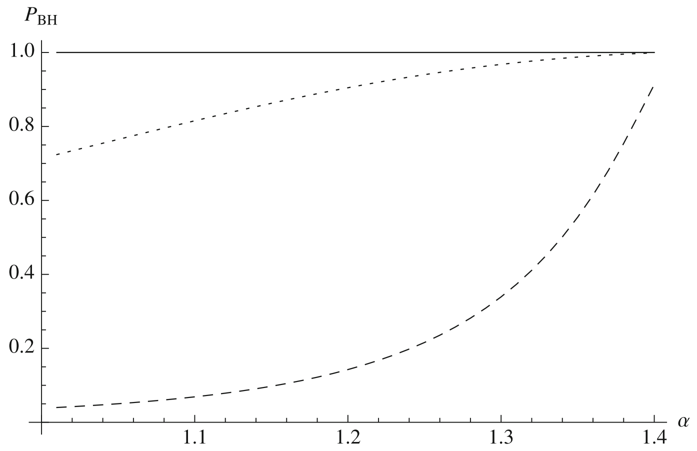

Fig. 14.3 P_{BH} as a function of α for $m = 2\,m_p$ (solid line), $m = m_p$ (dotted line) and $m = 0.5\,m_p$ (dashed line). Cases with $m \gg m_p$ are not plotted since they behave the same as $m = 2\,m_p$, i.e. an object with $1 < \alpha^2 < 2$ must be a BH

blows up for all values of the mass $m \sim 1/\ell$ in the limit $\alpha^2 \to 2$, and so does its uncertainty, since $\Delta R \simeq \sqrt{3\pi/8 - 1}\,\langle\,\hat{R}\,\rangle \simeq 0.4\,\langle\,\hat{R}\,\rangle$.

Using (14.8), one can calculate the probability P_{BH} that the particle is a black hole for α in the range (14.15). Figure 14.3 shows that, for a mass above the Planck scale, $P_{BH} \simeq 1$ throughout the entire range of α (extending the similar result for $\alpha < 1$). Moreover, even for m significantly less than m_p, P_{BH} approaches one in the limit $\alpha^2 \to 2$. We recall that $P_{BH} \ll 1$ for small m is related to $\ell \gg \langle\,\hat{R}\,\rangle$, and quantum fluctuations in the source's size dominate. On the other end, since both $\langle\,\hat{R}\,\rangle$ and ΔR blow up at $\alpha^2 = 2$, the superextremal configurations with a significant probability of being black holes display strong quantum fluctuations in the horizon's size.

14.3 Conclusions and Outlook

We extended the HWF formalism from neutral to electrically charged sources and considered separately the analogues of the classical Reissner–Nordström space-times with two horizons or a naked singularity. In the former case, with $0 < \alpha \leq 1$, we have shown that quantum fluctuations can cover the inner horizon, thus helping to avoid the instability known as mass inflation, at least for black holes not much heavier than the Planck scale. In the latter, we have found that quantum black holes extend into the range of classical naked singularities $\alpha > 1$, but a quantum obstruction occurs at $\alpha^2 = 2$.

Future developments involve extending the HWF to spinning sources and black hole formation by colliding particles with a non-vanishing impact parameter.

Acknowledgements We would like to thank D. Stojkovic for his contribution to the work reported here. R.C. was supported in part by the INFN grant FLAG. O.M. was supported in part by research grant UEFISCDI project PN-II-RU-TE-2011-3-0184.

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Chapter 15
Phase Transitions of Regular Schwarzschild-Anti-deSitter Black Holes

A. M. Frassino

Abstract We study a solution of the Einstein's equations generated by a self-gravitating, anisotropic, static, non-singular matter fluid. The resulting Schwarzschild-like solution is regular and accounts for smearing effects of noncommutative fluctuations of the geometry. We call this solution regular Schwarzschild spacetime. In the presence of an Anti-deSitter cosmological term, the regularized metric offers an extension of the Hawking–Page transition into a van der Waals-like phase diagram. Specifically, the regular Schwarzschild-Anti-deSitter geometry undergoes a first order small/large black hole transition similar to the liquid/gas transition of a real fluid. In the present analysis, we have considered the cosmological constant as a dynamical quantity, and its variation is included in the first law of black hole thermodynamics.

15.1 Regular Schwarzschild-Anti-deSitter Spacetime

The regular Schwarzschild anti-deSitter (AdS) metric is a static, spherically symmetric solution of the Einstein's equations with negative cosmological constant $\Lambda = -3/b^2$ and a Gaussian matter source [1–4]. To obtain this metric we replace the vacuum with a Gaussian distribution having variance equivalent to the parameter $\sqrt{\theta}$

$$\rho(r) \equiv \frac{M}{(4\pi\theta)^{3/2}} e^{-r^2/4\theta} . \tag{15.1}$$

This type of matter distribution emulates non-commutativity of space-time through the parameter θ that corresponds to the area of the elementary quantum cell, account-

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P. Nicolini et al. (eds.), *2nd Karl Schwarzschild Meeting on Gravitational Physics*, Springer Proceedings in Physics 208,
https://doi.org/10.1007/978-3-319-94256-8_15

ing for a natural ultraviolet spacetime cut-off (see [2] and references therein). The resulting energy momentum tensor describes an anisotropic fluid, whose components, fixed by $\nabla_\mu T^{\mu\nu} = 0$ and the condition $g_{00} = -g_{rr}^{-1}$, read

$$T_0^0 = T_r^r = -\rho(r) \qquad T_\phi^\phi = T_\theta^\theta = -\rho(r) - \frac{r}{2}\frac{\partial\rho(r)}{\partial r} \,. \tag{15.2}$$

The spherically symmetric solution of the Einstein's equations with this energy momentum tensor and the cosmological constant Λ is given by the line element

$$ds^2 = -V(r)\,dt^2 + \frac{dr^2}{V(r)} + r^2 d\Omega^2 \tag{15.3}$$

where $d\Omega^2 = d\vartheta^2 + \sin^2\vartheta d\varphi^2$ and

$$V(r) = 1 + \frac{r^2}{b^2} - \frac{\omega M}{r}\gamma\left(\frac{3}{2}, \frac{r^2}{4\theta}\right) . \tag{15.4}$$

Here $\omega = 2G_N/\Gamma(3/2)$, G_N is the four-dimensional Newton's constant and b is the curvature radius of the AdS space. The function $\gamma\left(\frac{3}{2}, \frac{r^2}{4\theta}\right)$ is the incomplete gamma function defined as $\gamma(n, x) \equiv \int_0^x dt\, t^{n-1}e^{-t}$. The line element (15.3) has an event horizon at $r = r_+$, where r_+ is solution of the horizon equation $V(r) = 0$. The event horizon radius coincides with the Schwarzschild radius in the limit $\sqrt{\theta}/r_+ \to 0$. The metric (15.3) admits an inner horizon $r_- < r_+$, that coalesces with r_+ in the extremal black hole configuration at $r_0 = r_+ = r_-$. Such a degenerate horizon occurs even without charge or angular momentum.

15.2 Thermodynamics and Equation of State

The temperature associated to the event horizon r_+ can be computed through the formula $T = \frac{1}{4\pi} V'(r)\big|_{r=r_+}$ and reads

$$T = \frac{1}{4\pi r_+}\left\{1 + \frac{r_+^2}{b^2}\left(3 - r_+\frac{\gamma'(r_+)}{\gamma(r_+)}\right) - r_+\frac{\gamma'(r_+)}{\gamma(r_+)}\right\} , \tag{15.5}$$

where $\gamma(r_+) \equiv \gamma\left(\frac{3}{2}, \frac{r_+^2}{4\theta}\right)$, $\gamma'(r_+) = \frac{r_+^2}{4\theta^{3/2}}e^{-r_+^2/4\theta}$ is its derivative with respect to r_+. In contrast to the standard Schwarzschild-anti-deSitter case, extremal solution exists with vanishing Hawking temperature (15.5).

Recently, the idea of including the variation of the cosmological constant in the first law of black hole thermodynamics has been considered [5–7] with interesting consequences: If the cosmological constant Λ behaves like a pressure, we have that

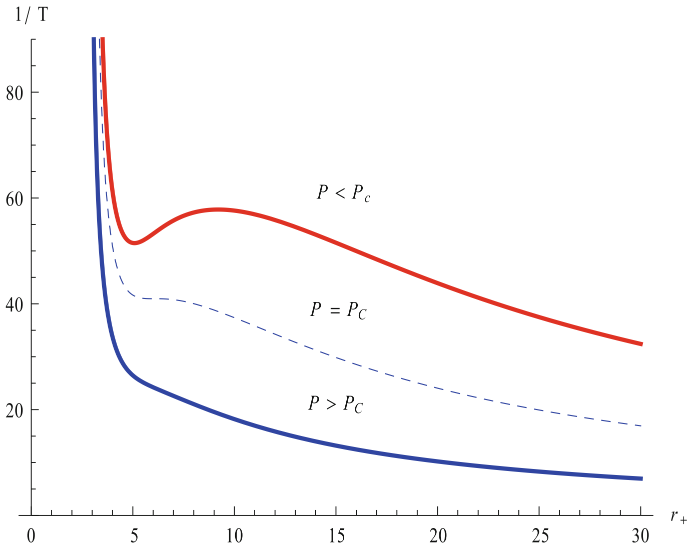

Fig. 15.1 The inverse temperature as function of r_+ (with $\theta = 1$). When $P < P_c$, there are three branches. The middle branch is unstable, while the branch with the smaller radii and the one with bigger radii are stable. This graph reproduces the pressure-volume diagram of the van der Waals theory, provided one identifies the black hole thermodynamic variables $\beta \equiv 1/T$, r_+ and P respectively with pressure pressure, volume and temperature of the van der Waals gas

for negative cosmological constant the pressure turns to be positive [6, 8], i.e.,

$$\frac{1}{b^2} = -\frac{\Lambda}{3} \equiv \frac{8\pi P}{3} , \tag{15.6}$$

giving rise to several effects (see for example [9–11]). In such a case the equation of state $P(V, T)$ for the regular AdS black hole becomes

$$P = \frac{3\gamma(r_+)}{(3\gamma(r_+) - r_+\gamma'(r_+))} \left\{ \frac{T}{2r_+} - \frac{1}{8\pi r_+^2} + \frac{\gamma'(r_+)}{8\pi r_+ \gamma(r_+)} \right\} . \tag{15.7}$$

Here T is the Hawking temperature of the black hole, i.e. (15.5). Using the equation of state (15.7) it is possible to plot the isotherm functions in a $P - V$ diagram for a regular black hole that resembles the van der Waals pressure-volume diagram (Fig. 15.1).

15.2.1 Gibbs Free Energy

In order to complete the analogy between the regular black hole and a van der Waals gas, we proceed by calculating the Gibbs free energy [6, 7]. This can be done by calculating the action of the Euclidean metric (see for example [13]). Such action provides the Gibbs free energy via $G = I/\beta$ where β is the period of the imaginary time $\beta \equiv 1/T$. Then, the Gibbs free energy can be expressed as a function of pressure and temperature. The Hawking–Page transition [14] for the standard Schwarzschild-AdS black hole is first order phase transition between a large black hole phase and the purely thermal AdS spacetime. Such a transition takes place when the Gibbs energy changes its sign from positive to negative. In the regular black hole case and considering the cosmological constant as a pressure we find

$$G = \frac{r_+}{12\,G_N}\left[3 - 8P\pi r_+^2 + \frac{r_+\left(3 + 8P\pi r_+^2\right)\gamma'\left(r_+\right)}{\gamma\left(r_+\right)}\right] \tag{15.8}$$

and the Gibbs free energy (15.8) exhibits a characteristic swallowtail behavior (see Fig. 15.2). This usually corresponds to a small black hole/large black hole first-order phase transition [7, 15]. By performing the classical limit for $r \gg \theta$ we get the usual result for a classical uncharged Schwarzschild-AdS black hole that is $G\left(T, P\right) = \left(1/4G_N\right)\left(r_+ - \frac{8\pi}{3} P\, r_+^3\right)$ [7]. Remarkably, in the regular Schwarzschild-AdS black hole case, as in the Reissner–Nordström-AdS (RN-AdS) black hole spacetime, there is a phase transition that occurs at positive Gibbs energy. This fact is visible from the presence of the swallowtail in Fig. 15.2. To investigate this aspect, we need to study the sign of the heat capacity. As underlined in [6], the specific heat related to the black hole is calculated at constant pressure

$$C_p = \left(\frac{\partial H}{\partial T}\right)_P = \left(\frac{\partial H}{\partial r_+}\right)_P \left(\frac{\partial r_+}{\partial T}\right)_P , \tag{15.9}$$

where the enthalpy H is identified with the black hole mass M [6]. Now one can study the phase transitions from the change of the sign of the specific heat: the stability requires that the specific heat at fixed pressure is $C_p \geq 0$ and the specific heat at fixed volume is $C_v \geq 0$. In the case under investigation C_v is always equal to zero because the entropy is only volume dependent. This means that the heat capacity C_v does not diverge at the critical point and its critical exponent is $\alpha = 0$. By studying the sign of the function C_p, we can see that for $P > P_c$ the quantity C_p is always positive, and the black hole is stable. In the limit $P \to P_c$ there is a critical value for r_+ for which C_p diverges. For $P < P_c$ there are two discontinuities of the specific heat and the situation is the same as in the Reissner–Nordström-AdS black holes [12]. Thus, in the regular Schwarzschild-AdS case for $P < P_c$ it seems that a different phase transition is allowed because the heat capacity changes again from positive values to negative values. For large r_+ we have the Hawking–Page behavior in which the branch with negative specific heat has lower mass and thus falls in an unstable phase, while the

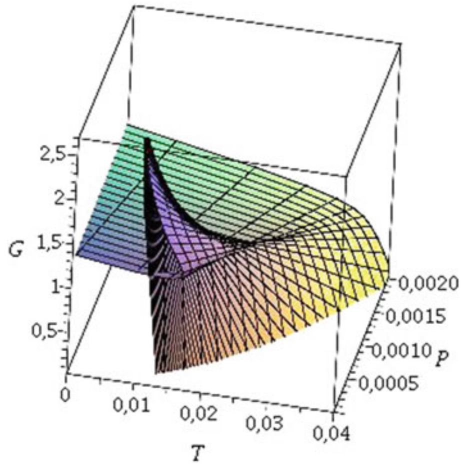

Fig. 15.2 Gibbs free energy as function of the black hole pressure and temperature. The Gibbs free energy G changes its sign at a specific T and P (intersection of the function with the $T - P$-plane). As in the van der Waals case, the phases are controlled by the universal 'cusp', typical of the theory of discontinuous transitions [15]. The Gibbs free energy shows the "swallowtail" shape, a region where $G(T, P)$ is a multivalued function. This region ends in a point (T_c, P_c). In the region with $P < P_c$ and $T < T_c$ we can see a transition between small black hole/large black hole. Note that r_+ is a function of temperature and pressure via the equation of state (15.7). For large value of P (or T) there is only one branch allowed

branch with larger mass is locally stable and corresponds to a positive specific heat. Thus, the resulting phase diagram presents a critical point at a critical cosmological constant value in Plank units and a smooth crossover thereafter.

15.2.2 *Critical Exponent*

We already determined $\alpha = 0$ in the previous section. Now, by defining the variable $t \equiv (T - T_\mathrm{c}) / T_\mathrm{c}$, we can compute the critical exponent of C_p by evaluating the ratio $\ln \left(C_p(t)\right) / \ln(t)$ in the limit $t \to 0$. We find that the limit exists and the critical exponent is $\gamma = 1$. This result implies that the heat capacity diverges near the critical point like $C_p \propto |t|^{-1}$. Then using the scaling relations

$$\alpha + 2\beta + \gamma = 2 \tag{15.10}$$

$$\alpha + \beta(1 + \delta) = 2 \tag{15.11}$$

is possible to calculate the other two exponents, i.e., δ that determines the behaviour of the isothermal compressibility of a VdW system and β that describes the behaviour of the difference between of the volume of the gas phase and the liquid phase. For the regular black hole, the scaling relations give $\delta = 3$ and $\beta = 1/2$, result that coincides with the case of charged black holes [7]. These critical exponents are consistent with the Ising mean field values $(\alpha, \beta, \gamma, \delta) = (0, 1/2, 1, 3)$ allowing for an efficient mean field theory description. Since it is believed that the determination of critical exponents define universality classes, i.e., they do not depend on the details of the physical system (except the number of dimensions), we can say that the phase transitions in the regular Schwarzschild-AdS black holes and the RN-AdS black holes in four-dimensional spacetime have the same nature.

15.3 Final Remarks

After almost hundred years since the Karl Schwarzschild's exact solution of Einstein's equation, black hole physics is nowadays at the forefront of current research in several branches of theoretical physics. Specific interest has been developed in the thermodynamics of charged black holes in asymptotically AdS spacetime, largely because they admit a gauge duality description via a dual thermal field theory [13]. In recent studies, it has been shown that charged Reissner-Nordström AdS black holes exhibit critical behavior similar to a van der Waals liquid–gas phase transition [7]. This analogy becomes "complete" if the cosmological constant Λ is considered as a dynamical quantity and its variation is included in the first law of black hole thermodynamics [7, 9]. This extended phase space shows new insights with respect to the conventional phase space of a four-dimensional black hole in AdS background consisting only of two variables: entropy and temperature. In this work, the cosmological constant has been considered as a thermodynamical pressure and its conjugate quantity as a thermodynamical volume. The black hole equation of state (15.7) obtained by considering the regular Schwarzschild-AdS solution shows an analogy with the van der Waals liquid-gas system where the parameter θ plays an analog role of the charge. Note that a detailed description of the not-extended phase space has been presented in [3].

Acknowledgements This work has been supported by the Helmholtz Research School for Quark Matter Studies (H-QM). The author is grateful to P. Nicolini and D. Kubiznak for having carefully read the draft and provided valuable comments.

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Chapter 16
Generalized Uncertainty Principle and Extra Dimensions

S. Köppel, M. Knipfer, M. Isi, J. Mureika and P. Nicolini

Abstract The generalized uncertainty principle (GUP) is a modification of standard quantum mechanics due to Planck scale effects. The GUP has recently been used to improve the short distance behaviour of classical black hole spacetimes by invoking nonlocal modifications of the gravity action. We present the problem of extending such a GUP scenario to higher dimensional spacetimes and we critically review the existing literature on the topic.

16.1 Generalized Uncertainty Principle and Black Holes

Gravitation plays no conventional role in quantum mechanical systems. Atoms are dominated by electromagnetic forces, with nuclear forces becoming relevant at the smaller sub-atomic scales. It is, however, interesting to ask how quantum mechanics deviates from its standard formulation if the above systems were subject to gravitational interactions. A full answer to this question would require a quantum theory of gravity, whose formulation is probably one of the biggest problems in fundamental physics. There is nevertheless a "side effect" of quantum gravity that one

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P. Nicolini et al. (eds.), *2nd Karl Schwarzschild Meeting on Gravitational Physics*, Springer Proceedings in Physics 208,
https://doi.org/10.1007/978-3-319-94256-8_16

can estimate in a semiclassical way. For instance, one can consider a non-vanishing Newtonian gravitational interaction between the photon and the electron in Heisenberg's microscope *Gedankenexperiment* [1]. As a result one finds a modification of standard commutation relations [2–6]

$$[x^i, p_j] = i\,\hbar\,\delta^i_{\ j}\,(1 + f(\mathbf{p}^2)), \tag{16.1}$$

where the function f is customarily assumed as $f(\mathbf{p}^2) \simeq \beta\mathbf{p}^2 + \ldots$ to first order. Interestingly, the parameter β turns out to be a natural ultraviolet cutoff, since the corresponding uncertainty relations prevent better spatial resolution than $\sqrt{\beta}$,

$$\Delta x \Delta p \geq \frac{\hbar}{2}(1 + \beta(\Delta p)^2). \tag{16.2}$$

The above relation is the Generalized Uncertainty Principle (GUP), represented in Fig. 16.1.

The GUP has been studied in a variety of physical systems (for reviews see [7–9]), and applied most notably to black holes and their evaporation [5, 10–12]. By assuming the emitted particles have momenta uncertainty proportional to the black hole temperature, $\Delta p \sim T$, and position uncertainty proportional to the black hole size, $\Delta x \sim GM$, one ends up with a non-divergent increase of the black hole temperature and vanishing heat capacity at the Planck scale. Such a scenario for the final stage of the evaporation would suggest the formation of a black hole remnant – a Planckian size, neutral object that might be considered as a dark matter candidate [11, 13]. Unfortunately this particular GUP temperature profile cannot be associated to a surface gravity of any known black hole metric. In addition such remnants would turn to be hot, since their temperature is of the order of the Planck temperature $T_{\mathrm{P}} \sim 10^{32}$ K.

Against this background, it has recently been noted that GUP effects can be implemented at the level of the spacetime metric by a non-local deformation of the gravitational action [14]. This approach allows for calculating corrections to black hole thermodynamics by a genuine modification of the surface gravity. In the case of a spherically symmetric, static, and neutral black hole, one can solve the non-local equations and obtain the metric

$$\mathrm{d}s^2 = -\left(1 - \frac{2G\mathscr{M}(r)}{r}\right)\mathrm{d}t^2 + \left(1 - \frac{2G\mathscr{M}(r)}{r}\right)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2. \tag{16.3}$$

Here $\mathscr{M}(r)$ takes into account the spread of matter that is no longer concentrated in a point, as conventionally occurs in the Schwarzschild case [15–19]. For a specific profile of the nonlocal action, the mass distribution $\mathscr{M}(r)$ reproduces the ultraviolet smearing predicted by the GUP. The final result then reads

$$\mathscr{M}(r) = M\,\gamma(2,\ r/\sqrt{\beta}) = M\,\left(1 - e^{-r/\sqrt{\beta}} - (r/\sqrt{\beta})e^{-r/\sqrt{\beta}}\right), \tag{16.4}$$

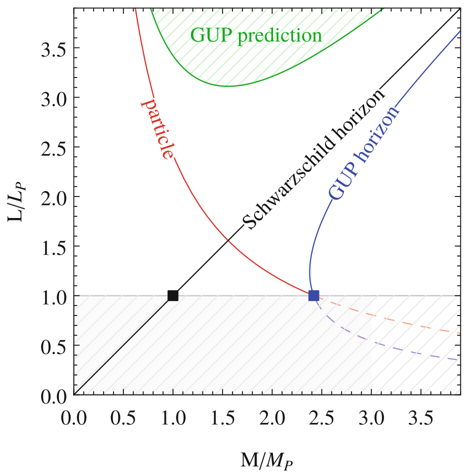

Fig. 16.1 Length versus mass in the Planck region: The black line shows the Schwarzschild horizon length scale $L \sim M$, the red line the Compton wavelength $L \sim 1/M$ in Planck units. The GUP green curve interpolates the two curves and predicts a minimal length but does not resolve the phase ambiguity (black spot). The GUP-inspired black holes [14] modify the intersection of the Compton/horizon curves due the introduction of a cold remnant (blue spot). In the resulting diagram, the two phases (particles and black holes) are unambiguously separated

with M the ADM mass and $\gamma(s,\ x)$ the lower incomplete gamma function. A compelling feature of the GUP is that it implies a tail of the mass distribution trespassing the event horizon, in close analogy with the leakage of quantum mechanical effects that has been recently invoked to overcome the black hole information paradox [20]. The above metric features a two-horizon structure and an extremal configuration. The latter is a zero temperature state that nicely fits into the cold dark matter paradigm. Interestingly, the presence of the remnant makes the metric consistent with the expected self-complete character of gravity [21–32]. Rather than a complete evaporation, the black hole asymptotically approaches the remnant configuration, preventing the exposure of length scales smaller of $\sqrt{\beta}$ (see Fig. 16.1).

16.2 Extra Dimensions and the Heisenberg Microscope

Terascale quantum gravity is a formulation proposed to address the weak hierarchy problem of the Standard Model by assuming the existence of additional spatial dimensions [33–44]. According to the Arkani–Hamed/Dimopoulos/Dvali (ADD) model, the spacetime is endowed with $N - 3$ extra dimensions which are compactified at a length scale $R \sim 1$ mm or smaller. Gravity is the only fundamental interaction able to "see" the additional dimensions, which would become relevant only for high energy events (i.e., at the TeV or higher). If this is the case, GUP effects should be expected to set in at these energy scales rather than at the usual Planck scale. It is therefore natural to consider the higher dimensional extension of the non-local action proposed in [14] and derive the related black hole solution. To reach this goal, however, one has to face a potential ambiguity that we illustrate below.

According to the Kempf–Mangano–Mann (KMM) model [6], the Hilbert space representation of the identity reads

$$1 = \int \frac{\mathrm{d}^N p}{1+\beta \mathbf{p}^2} \mid p\rangle\langle p \mid, \tag{16.5}$$

with $\mathbf{p}$ an N-dimensional spatial vector. This can be interpreted as saying that, while momentum operators preserve their standard character, position operators do not have physical eigenstates, as one expects in the presence of a minimal length $\sqrt{\beta}$. Accordingly the integration measure in momentum space is squeezed in the ultraviolet regime as follows

$$\mathrm{d}\mathscr{V}_p \equiv \frac{\mathrm{d}^N p}{1+\beta \mathbf{p}^2} \underset{\beta \mathbf{p}^2 \gg 1}{\approx} \mathbf{p}^{N-3}\, \mathrm{d}p. \tag{16.6}$$

We note the GUP correction becomes less and less important with increasing N.

The above profile of GUP corrections can be used to improve the higher dimensional Newtonian potential. This can offer a first taste of the repercussions of GUP, even before extending the action proposed in [14] to the higher dimensional case. To reach this goal one has to consider the exchange of virtual massless scalars between two static bodies at distance $r = |\mathbf{x}|$. The static gravitational potential is the Fourier transform of the massless scalar propagator,

$$\Phi(r) = -\frac{G_{(N)} M}{(2\pi)^N} \int \mathrm{d}\mathscr{V}_p \;\; D(p)\, |_{p_0=0} \;\; \exp(i\mathbf{p}\cdot\mathbf{x}), \tag{16.7}$$

where the integration measure has been deformed as in (16.6). The net result reads

$$\frac{\Phi(r)}{G_{(N)}M} = \underbrace{\pi^{1-\frac{N}{2}}\Gamma\left(\frac{N}{2}-1\right)\left(\frac{1}{r}\right)^{N-2}}_{\text{classical potential}} - \underbrace{\frac{2}{(2\pi\, r\,\sqrt{\beta})^{\frac{N}{2}-1}}\, K_{\frac{N-2}{2}}\left(\frac{r}{\sqrt{\beta}}\right)}_{\text{GUP corrections}} \tag{16.8}$$

for $r < R$. The short distance behaviour of the above potential is regular at the origin only for $N = 3$, i.e., $\Phi = 1/\sqrt{\beta}$ as $r \to 0$. For $N > 3$ GUP corrections are suppressed and cannot improve the potential.

There are, however, other proposals. As stated in the introduction, the GUP arises from the inclusion of gravitational effects in quantum mechanics. This is the case of Heisenberg's microscope. Additional dimensions should not disrupt the reasoning that leads to the GUP [1, 28]. As for $N = 3$ we identify two terms for the spatial uncertainty $\Delta x \sim \Delta x_{\mathrm{C}} + \Delta x_{\mathrm{g}}$, one coming from the Compton wavelength of a particle $\Delta x_C \sim \lambda \sim 1/\Delta p < R$ and one due to the gravitational potential in the compact higher dimensional space. Specifically, the photon not only illuminates the electron but also exerts a gravitational force on it. The resulting acceleration causes the electron to be displaced by

$$\Delta x_g \sim G_{(N)} \frac{M_{\text{eff}}}{r^{N-1}} \left(\frac{r^2}{c^2}\right) \sim G_{(N)} \frac{\Delta p}{r^{N-3}} \rightsquigarrow \Delta x_g^{N-2} \sim L_{(N)}^{N-1} \Delta p, \tag{16.9}$$

where $M_{\text{eff}} = h/(\lambda c)$ is the photon effective mass, $G_{(N)}$ is the higher dimensional Newtonian constant, and $L_{(N)}$ is the new fundamental length scale that replaces the Planck length. In the above derivation we assumed the interaction distance $r \sim \Delta x_g < R$. The above Gedankenexperiment motivates a *modified* higher dimensional GUP

$$\Delta x \Delta p \geq \frac{\hbar}{2} \left(1 + \left(\sqrt{\beta}\, \Delta p\right)^{\frac{N-1}{N-2}}\right), \tag{16.10}$$

where we assumed $\sqrt{\beta} \sim L_{(N)}$. Equation (16.10) is consistent with what proposed in [28, 45–49] and cleanly reproduces the higher dimensional Schwarzschild radii for energies above the terascale. On the other hand, such a proposal fails to improve the Newtonian potential and predicts GUP corrections even milder than those of the KMM model for $N > 3$.

Alternatively, one can revise the basic reasoning behind Heisenberg's microscope in higher dimensional space. Following Fig. 16.1, one can approach the quantum gravity scale from the left. In such a sub-Planckian regime, the gravitational corrections are still sub-leading, i.e., $\Delta x_g < \Delta x_C$. Accordingly, one can assume that at the leading order, the typical interaction distance is controlled by the Compton wavelength $r \sim \Delta x_C \sim \lambda < R$. As a result, one finds a spatial uncertainty

$$\Delta x_g \sim G_{(N)} \frac{M_{\text{eff}}}{r^{N-1}} \left(\frac{r^2}{c^2}\right) \sim G_{(N)} \frac{\Delta p}{r^{N-3}} \sim L_{(N)}^{N-1} \Delta p^{N-2} \tag{16.11}$$

that is consistent with that proposed in [50–52]. Interestingly enough, the above relation can also improve the asymptotic behaviour of the momentum integration (16.6), as follows

$$\mathrm{d}\mathscr{V}_p \equiv \frac{\mathrm{d}^N p}{1 + (\beta \mathbf{p}^2)^{\frac{N-1}{2}}} \underset{\beta \mathbf{p}^2 \gg 1}{\approx} \mathrm{d}p. \tag{16.12}$$

The repercussions of the GUP are no longer dependent on N, and the suppression of higher momenta is consistent with the original derivation for $N = 3$. The above integration measure (16.12) relaxes the condition of reproducing the Schwarzschild radius in the trans-Planckian regime, even if the condition remains valid for distances $r > R$. Such a deviation of the curve from the Schwarzschild radius for $r < R$ is fully legitimate, since the quadratic correction in (16.6) is known to be the lowest energy correction to standard quantum mechanics and makes sense only for the four dimensional spacetime at scales $r > R$.

It is natural to expect that at scales $r < R$ where gravity becomes so strong as to probe additional dimensions, the GUP effects also become stronger as in (16.10). In other words, the picture based on matching a length scale dictated by general relativity [28, 45–49] loses its meaning at an energy regime characterized by string/p-brane effects [53], noncommutative geometry [54, 55] or a variety of non-classical effects [56]. Such a vision is also consistent with a recent proposal aiming to interpret black holes in terms of a pre-geometric, purely quantum mechanical formulation [19, 57, 58].

16.3 Conclusions

In this paper, we have reviewed the basic properties of the GUP in three or more dimensions. We have presented a black hole metric derived by a nonlocal action able to reproduce GUP effects [14]. Such a metric overcomes the usual limitations one encounters when considering GUP effects in Hawking radiation [11]. Specifically, the new metric allows for the presence of cold remnants and the derivation of the black hole temperature in terms of surface gravity. In the second part of the paper, we provided an analysis of the GUP in higher dimensional spacetime. We showed there is a potential ambiguity in the deformation of the measure in momentum space. We highlighted that current proposals are unable to reproduce a consistent cutoff to improve the bad short distance behaviour of gravity in higher dimensional spacetimes [28, 45–49]. As a possible resolution to such a issue, we revised the reasoning of Heisenberg's microscope in higher dimensional spacetimes. We proposed an improved version of the GUP valid at length scales below the extra-dimensional compactification radius. Our findings are consistent with previous approaches in the literature [50–52]. We plan to study the repercussions of such a proposal on black hole physics in a future investigation.

Acknowledgements This work has partially been supported by the project "Evaporation of the microscopic black holes" of the German Research Foundation (DFG) under the grant NI 1282/2-2.

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Chapter 17
Perihelion Precession and Generalized Uncertainty Principle

F. Scardigli and R. Casadio

Abstract We compute the corrections to the Schwarzschild metric necessary to reproduce the Hawking temperature derived from a Generalized Uncertainty Principle (GUP), so that the GUP deformation parameter is directly linked to the deformation of the metric. Using this modified Schwarzschild metric, we compute corrections to the standard General Relativistic predictions for the perihelion precession for planets in the solar system. This analysis allows us to set bounds for the GUP deformation parameter from well-known astronomical measurements.

17.1 Introduction

Research on generalizations of the uncertainty principle of quantum mechanics has nowadays a long history [1]. One of the main lines of investigation focuses on understanding how the Heisenberg Uncertainty Principle (HUP) should be modified once gravity is taken into account. Given the pivotal rôle played by gravitation in these arguments, it is not surprising that the most relevant modifications to the HUP have been proposed in string theory, loop quantum gravity, deformed special relativity, and studies of black hole physics [2–7], just to mention some of the most notable frameworks.

Studies that aim at putting bounds on the dimensionless deforming parameter of the GUP, heretofore denoted by β, date back at least to Brau [8], and can be roughly divided into three different categories (actually, only two, as we will see). In the first group one finds papers such as those of Brau [8], Vagenas [9], Nozari [10], which

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P. Nicolini et al. (eds.), *2nd Karl Schwarzschild Meeting on Gravitational Physics*, Springer Proceedings in Physics 208,
https://doi.org/10.1007/978-3-319-94256-8_17

use a specific (in general, non linear) representation of the operators in the deformed fundamental commutator[1] $\left[\hat{X}, \hat{P}\right] = i\,\hbar(1 + \beta\,\hat{P}^2/m_\mathrm{p}^2)$ in order to compute corrections to quantum mechanical predictions, such as energy shifts in the spectrum of the hydrogen atom, or to the Lamb shift, the Landau levels, Scanning Tunneling Microscope, charmonium levels, etc. The bounds so obtained on β are quite stringent, but the drawback of this approach is a potentially strong dependence of the expected shifts on the specific (non linear) representation chosen for the operators $\hat{X}$ and $\hat{P}$ in the fundamental commutator.

In the second group, we can find the works of, e.g., Chang [11], Nozari and Pedram [12], where a deformation of classical Newtonian mechanics is introduced by modifying the standard Poisson brackets in a way that resembles the quantum commutator, $\left[\hat{x}, \hat{p}\right] = i\,\hbar\left(1 + \beta_0\,\hat{p}^2\right) \quad \Rightarrow \quad \{X, P\} = \left(1 + \beta_0\,P^2\right)$, where $\beta_0 = \beta/m_\mathrm{p}^2$. In particular, Chang in [11] computes the precession of the perihelion of Mercury directly from this GUP-deformed Newtonian mechanics, and interprets it as an extra contribution to the well known precession of 43"/century due to General Relativity (GR). He then compares this global result with the observational data, and the very accurate agreement between the GR prediction and observations leaves Chang not much room for possible extra contributions to the precession. In fact, he obtains the tremendously small bound $\beta \lesssim 10^{-66}$. A problem with this approach is that a GUP-deformed Newtonian mechanics is simply superposed linearly to the usual GR theory. One may argue that a modification of GR at order β should likewise be considered, but this is however omitted in [11]. In other words, it is not clear why the two structures, GR and GUP-modified Newtonian mechanics, should coexist independently, and why the two different precession errors add into a final single precession angle. Most important, as a matter of fact, in the limit $\beta \to 0$, [11] recovers *only* the Newtonian mechanics but not GR, and GR corrections must be added as an extra structure. Clearly, the physical relevance of this approach and the bound that follows for β, remain therefore questionable.

Finally, a third group of works on the evaluation of β contains, for example, papers by Ghosh [13] and Pramanik [14]. They use a covariant formalism, first defined in Minkowski space, with the metric $\eta_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$, which can be easily generalized to curved space-times via the standard procedure $\eta_{\mu\nu} \to g_{\mu\nu}$. These papers should however be considered as belonging to the second group. In fact, a closer look reveals that they also start from a deformation of classical Poisson brackets, although posited in covariant form. From the deformed covariant Poisson brackets, they obtain interesting consequences, like a β-deformed geodesic equation, which leads to a violation of the Equivalence Principle. They do not deform the field equations or the metric. In [15], however, we show that this violation of the Equivalence Principle is completely due to the postulate of deformed Poisson brackets, and has nothing to do with the covariant formalism, or with a deformation of the GR field equations or solutions, or of the geodesic equation. Nonetheless, the

[1] We shall work with $c = k_B = 1$, but explicitly show the Newton constant G_N and Planck constant $\hbar$. We also recall that the Planck length is defined as $\ell_\mathrm{p}^2 = G_\mathrm{N}\,\hbar/c^3$, the Planck energy as $\mathcal{E}_\mathrm{p}\,\ell_\mathrm{p} = \hbar\,c/2$, and the Planck mass as $m_\mathrm{p} = \mathcal{E}_\mathrm{p}/c^2$, so that $G_\mathrm{N} = \ell_\mathrm{p}/2\,m_\mathrm{p}$ and $\hbar = 2\,\ell_\mathrm{p}\,m_\mathrm{p}$.

Ghosh–Pramanik formalism remains covariant when $\beta \to 0$ and reproduces standard GR results in the limit $\beta \to 0$ (this differs, in general, from the results obtained by papers in the second group).

The novelties of our approach, when compared with the previous ones, are many and various. The main point is to start directly from a quantum mechanical effect, the Hawking evaporation, for which the GUP is necessarily relevant, rather than postulating specific representations of canonical operators or modifications of the classical equations of motion. We connect the deformation of the Schwarzschild metric directly to the uncertainty relation, without relying on a specific representation of commutators. We leave the Poisson brackets and classical Newtonian mechanics untouched, and recover GR, and standard quantum mechanics, in the limit $\beta \to 0$. In particular, we preserve the Equivalence Principle, and the equation of motion of a test particle is still given by the standard geodesic equation. In the present work, this is obtained by deforming a specific solution of the standard GR field equations, namely the Schwarzschild metric.

17.2 Deforming the Schwarzschild Metric

In this section, we start from a known way of deriving the Hawking temperature directly from the metric of a black hole, and then show how the GUP modifies the Hawking temperature. These two steps will pave the road to a deformation of the Schwarzschild metric, constructed so as to reproduce the GUP-modified Hawking temperature. We consider here a space-time with a metric that locally has the form

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^{\mu}\mathrm{d}x^{\nu} = F(r)\,\mathrm{d}t^2 - F(r)^{-1}\,\mathrm{d}r^2 - r^2\,\mathrm{d}\Omega^2, \tag{17.1}$$

where $\mathrm{d}\Omega^2 = \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2$. The horizons (if any), are located at the positive zeros of the function $F(r)$ (see, for example, [16]).

We loosely follow a standard derivation, as for example that in [17]. Suppose $r = r_{\mathrm{H}}$ is an horizon, so that $F(r_{\mathrm{H}}) = 0$, and consider $r \geq r_{\mathrm{H}}$. Then, a quantized scalar field outside the horizon lives in a heat bath with temperature

$$T = \hbar\,\frac{F'(r_{\mathrm{H}})}{4\pi}\,. \tag{17.2}$$

Therefore the temperature of the black hole horizon as seen by a distant observer is in general given by formula (17.2). In particular, for a Schwarzschild black hole the function $F(r)$ is given by $(1 - 2G_{\mathrm{N}}M/r)$, the horizon is at $r_H = 2\,G_{\mathrm{N}}\,M$, and we get $T_{\mathrm{H}} = \hbar/(8\pi\,G_{\mathrm{N}}\,M)$, which is the well-known Hawking temperature.

We now give here a derivation of the mass-temperature relation starting directly from the uncertainty relations. The most common form of deformation of the Heisenberg uncertainty relation (and the form of GUP that we are going to study in this paper) is without doubt the following

$$\Delta x \, \Delta p \geq \frac{\hbar}{2} \left(1 \, + \, \beta \, \frac{4 \, \ell_{\mathrm{p}}^2}{\hbar^2} \, \Delta p^{\,2} \right) = \frac{\hbar}{2} \left[1 + \beta \left(\frac{\Delta p}{m_{\mathrm{p}}} \right)^2 \right] . \tag{17.3}$$

The dimensionless parameter β is usually assumed to be of order one, in the most common quantum gravity formulations. Following the arguments of [18–23], we promptly arrive to translate relation (17.3) into a mass-temperature relation for a Schwarzschild black hole

$$M = \frac{\hbar}{8\pi \, G_{\mathrm{N}} \, T} + \beta \, \frac{T}{2\pi} \, . \tag{17.4}$$

To zero order in β, we recover the usual Hawking formula. Let us note that in this work we assume that the correction induced by the GUP has a thermal character, and therefore it can be cast in the form of a shift of the Hawking temperature. Of course, there are also different approaches (see e.g. [24]), where the corrections do not respect the exact thermality of the spectrum, and thus need not be reducible to a simple shift of the temperature.

We can legitimately wonder what kind of (deformed) metric would predict a Hawking temperature like the one inferred from the GUP relation (17.4), for a given β. Since we are interested only in small corrections to the Hawking formula, we can consider a deformation of the Schwarzschild metric of the kind

$$F(r) = 1 - \frac{2 \, G_{\mathrm{N}} \, M}{r} + \varepsilon \, \frac{G_{\mathrm{N}}^2 \, M^2}{r^2} \, , \tag{17.5}$$

and we shall look for the lowest order correction in ε. We see that (17.5) is actually the simplest mathematical form, if one supposes that the metric can be expanded in powers of $1/r$. This is nothing else than the well known Eddington–Robertson expansion of a spherically symmetric metric. Note however that, since $R_{\mathrm{H}}/r \sim 10^{-5}$ on the surface of the Sun, the term proportional to ε can still be considered small even if ε is relatively large. The temperature predicted by this deformed Schwarzschild metric is

$$T(\varepsilon) = \hbar \, \frac{F'(r_{\mathrm{H}})}{4\pi} = \frac{\hbar}{2\pi \, G_{\mathrm{N}} \, M} \, \frac{\sqrt{1-\varepsilon}}{\left(1 + \sqrt{1-\varepsilon}\right)^2} \, , \tag{17.6}$$

which must coincides with the temperature $T(\beta)$ predicted by (17.4), for any given β. This yields a relation between β and ε,

$$\beta(\varepsilon) = -\pi^2 \, \frac{G_{\mathrm{N}} \, M^2}{\hbar} \, \frac{\varepsilon^2}{1-\varepsilon} \, . \tag{17.7}$$

For $|\varepsilon| \ll 1$, to the lowest order in ε, we thus get $\beta = -\pi^2 M^2 \varepsilon^2 / (4 \, m_{\mathrm{p}}^2)$ where we notice that both β and ε are dimensionless. It is now of great interest to observe that (17.7) forces us to admit that $\beta < 0$, since $\varepsilon \leq 1$. Although quite unexpected, this might be a suggestion of fundamental importance. It seems that a metric is able to reproduce the GUP-deformed Hawking temperature *only if* the deforming parameter β is *negative*. We already encountered a situation like this when we studied the uncertainty relation formulated on a crystal lattice [25]. This could be a further hint that the physical space-time has actually a lattice or granular structure at the level of the Planck scale.

17.3 Perihelion Precession by Deformed Schwarzschild Metric

Having established a connection between the GUP parameter β and the deformation ε of the Schwarzschild metric, we are now in a position to compute the physical (possible observable) consequences of such a deformed metric. Here, we consider a particle bound in a orbit around a massive body, typically a planet around the Sun. Again, we roughly follow the treatment of [26]. The relevant geometrical parameters for an elliptic orbit in a polar coordinates system, with the radial coordinate r which at aphelia and perihelia takes, respectively, the maximum value r_+ and minimum value r_-, are the eccentricity e, the semi-major axis a, and the *semilatus rectum* L. These geometrical parameters are related by $r_\pm = (1 \pm e)\, a$, $L = (1 - e^2)\, a$, $\frac{2}{L} = \frac{1}{r_+} + \frac{1}{r_-}$. The angle swept out by the position vector when it increases from r_- to r is then given by the integral

$$\phi(r) - \phi(r_-) = \int_{r_-}^{r} \left[\frac{r_-^2 \left(\frac{1}{F(r)} - \frac{1}{F(r_-)} \right) - r_+^2 \left(\frac{1}{F(r)} - \frac{1}{F(r_+)} \right)}{r_-^2 r_+^2 \left(\frac{1}{F(r_+)} - \frac{1}{F(r_-)} \right)} - \frac{1}{r^2} \right]^{-1/2} \frac{\mathrm{d}r}{r^2 \sqrt{F(r)}} \,. \tag{17.8}$$

The total change in ϕ at every lap is just twice the change as r increases from r_- to r_+. This would equal 2π if the orbit were a closed ellipse, so the total orbital precession in each revolution is given by $\Delta\phi = 2\,|\phi(r_+) - \phi(r_-)| - 2\pi$. We expand the integrand before integrating, and the small parameter is given by R_{H}/r_-, or better R_{H}/L. Finally the total precession after a single lap, to first order in R_{H}/L, is given by

$$\Delta\phi \simeq \frac{6\pi\, G_{\mathrm{N}}\, M}{L} \left(1 - \frac{\varepsilon}{6} \right) , \tag{17.9}$$

which, of course, reproduces the usual GR prediction in the limit $\varepsilon \to 0$. This relation should now be compared with known observational data.

The perihelion precession for Mercury is by far the best known and measured GR precession in the Solar system. Referring to [27] for the latest most accurate and comprehensive data, we can report the relation

$$\langle\dot{\omega}\rangle = \frac{6\pi\, G_{\mathrm{N}}\, M}{L} \left[\frac{1}{3}(2 + 2\gamma - \bar{\beta}) + 3 \cdot 10^{-4}\, \frac{J_2}{10^{-7}} \right], \tag{17.10}$$

where $\langle\dot{\omega}\rangle$ is the measured perihelion shift, J_2 a dimensionless measure of the quadrupole moment of the Sun, and γ and $\bar{\beta}$ are the usual Eddington–Robertson expansion parameters. The latest data from helioseismology give $J_2 = (2.2 \pm 0.1) \cdot 10^{-7}$. The measured perihelion shift of Mercury is known to about 0.1% from radar observations of Mercury between 1966 and 1990 [28]. The solar oblateness effect due to the quadrupole moment is then smaller than the observational error, so it can be neglected. Substituting standard orbital elements and physical constants for Mercury and the Sun, we obtain

$$\langle\dot{\omega}\rangle = \left(1 + \frac{2\gamma - \bar{\beta} - 1}{3}\right) 42.98\text{''/century} \,, \tag{17.11}$$

where we can place a bound of $|2\,\gamma - \bar{\beta} - 1| \lesssim 3 \cdot 10^{-3}$. Comparing with $\Delta\phi$ from (17.9), we get $|\varepsilon| \lesssim 6 \cdot 10^{-3}$ which, replaced in (17.7), yields the lower bound

$$|\beta| = \frac{M^2}{4\,m_{\mathrm{p}}^2}\,\frac{\pi^2\,\varepsilon^2}{1-\varepsilon} \lesssim 3 \cdot 10^{72} \,. \tag{17.12}$$

We can also consider the most recent data from the Messenger spacecraft [29], which orbited Mercury in 2011–2013, and improved very much the knowledge of its orbit. Then we can push this bound even lower, to $|2\gamma - \bar{\beta} - 1| \lesssim 7.8 \cdot 10^{-5}$, although the knowledge of J_2 would have to improve simultaneously. If just the error in $|2\gamma - \bar{\beta} - 1|$ were taken into account, this would imply $|\varepsilon| = 2\,\left|2\gamma - \bar{\beta} - 1\right| \lesssim 1.56 \cdot 10^{-4}$ and therefore

$$|\beta| \lesssim 2 \cdot 10^{69} \,. \tag{17.13}$$

But of course this limit should not be considered completely reliable in this contest, since the less accurate bound on J_2 cannot be brutally neglected, at least in principle. Once again the perihelion shift appears to be one of the most precise tests of GR, a true GR effect not present at all in Newtonian gravity (as it is well known).

17.4 Conclusions

We have shown that a suitable deformation of the Schwarz-schild metric can reproduce the Hawking temperature for a black hole, when this is computed from a Generalized Uncertainty Principle. We have found in this way an analytic relation between the deformation parameter of the metric ε and the usual GUP deformation parameter β. In particular, when $\beta \to 0$, we correctly recover GR, and standard quantum mechanics. Neither the geodesic equation, nor the equivalence principle are violated, for any value of β or ε. Well-known astronomical measurements, in the Solar system as well as in binary pulsar systems, allowed us to put constraints on the parameter β. This direction seems to point towards promising research: at present we just deformed the Schwarzschild solution, but a future possibility is to deform the full field equations of GR, in order to get, among other things, a more stringent bound on the GUP parameter β. We would like to conclude by emphasizing once again that, although in the existing literature one can find bounds on β much tighter than those obtained in this paper, they seem to depend, at least partially, either on a specific (non linear) representation of the deformed commutator, or on the hypothesis of a deformation of Poisson brackets, which implies a violation of the equivalence principle. The line of reasoning presented in this paper avoids these possible difficulties.

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Chapter 18
Quantum-Gravity Phenomenology with Primordial Black Holes

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Abstract Quantum gravity may allow black holes to tunnel into white holes. If so, the lifetime of a black hole could be shorter than the one given by Hawking evaporation, solving the information paradox. More interestingly, this could open to a new window for quantum-gravity phenomenology, in connection with the existence of primordial black holes (PBH). We discuss in particular the power of the associated explosion and the possibility to observe an astrophysical signal in the radio and in the gamma wavelengths.

18.1 A New Theoretical Framework for Quantum Black Holes

The idea that black holes may explode dates back to Hawking's original paper [1]. But Hawking evaporation may not be the primary cause of for black holes to explode. In fact, since then various mechanisms have been proposed that disrupts the horizon so that matter can be released, possibly in an explosive event [2, 3]. The framework is generic and relies on the possibility that quantum gravity effects would forbid

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P. Nicolini et al. (eds.), *2nd Karl Schwarzschild Meeting on Gravitational Physics*, Springer Proceedings in Physics 208,
https://doi.org/10.1007/978-3-319-94256-8_18

curvature singularity to develop. The works on resolution of cosmological singularity in the context of Loop Quantum Gravity [4, 5] have motivated a recent model for regular black holes [6]. The model imports the main ideas of loop cosmology, in particular that quantum effects can be described at an effective level as a repulsive force. The threshold of the quantum gravitational regime is governed by the energy density rather than by a length, implying that the minimal size that a collapsing object can reach is typically many order of magnitude greater than the Planck length [6].

These quantum gravity effects are expected to dominate over Hawking radiation, which can be disregarded in a first order approximation. In such an approximation the equation of General Relativity are invariant under time reversal. Therefore the black-hole evolution is then described by gluing together a collapsing and an expanding solution of the Einstein equations via a quantum region, where those equations are not satisfied as quantum effect modifies the classical geometry. The process of passing trough a classically forbidden region can be thought as a tunneling process. In other words, quantum gravity may allow black holes to "decay" in a white holes [7–9].

18.2 How Long Does a Black Hole Live?

For an observer comoving with the collapse, the process is very short: it is just the time light takes to travel in a distance equal to the black holes size. For a solar mass black holes, this is of the order of the milliseconds. On the other hand, for an observer sitting out of the black hole, the process appears redshifted: this redshift, that in the classical theory is infinite, is finite here.[1] The value of such a redshift is governed by quantum gravity effects, and can be given in terms of a probability distribution rather than as an exact value. The phenomenological properties of this process depends on this time, the black holes lifetime, that can be expressed as a function of the black-hole total mass.

The lifetime τ of the black hole can be constrained by the following heuristic arguments. On the one hand, the "firewall" argument [12] provides a time upper bound. This can be see as a no-go theorem involving the following hypothesis: the unitarity of the quantum evolution, the equivalence principle at the base of general relativity and the validity of quantum field theory on a (fixed) curved background. At the Page time (that can be roughly identified with the time after which the mass of the black hole has half evaporated, and is therefore of the order $\sim m^3$ in natural units) the three hypothesis cannot hold together: a signal that the approximation of a fixed background should be abandoned for a fully dynamical theory of the quantum gravitational field. Therefore quantum gravity should manifest, in the form of the decay of the black hole into a white hole, no later than a time $\tau_{\max} \sim m^3$.

[1] The relation between the time inside the horizon and the time outside is coded in the metric. There exist a one-parameter family of metrics modeling the black-to-white process. The extreme case for which the time inside is equal to the time inside has been studied in [10, 11].

On the other hand, quantum gravity effects require a minimal time to manifest. In particular, we want the time to be long enough for quantum effect to manifest outside the horizon in order to modify its classical behaviour. The fact that quantum gravity effects can manifest outside of the horizon may sound surprising: we usually identify the quantum gravity regime with a regime of Planckian curvature ($R \sim \ell_{\text{Planck}}^{-2}$) but at the horizon the curvature may be small. Consider instead the combination of the curvature and a time: $R\tau \sim \ell_{\text{Planck}}^{-1}$. If now we substitute the value of the curvature near the horizon $R = (2m)^{-2}$ we find that the hole lifetime τ must be longer or of the order of $\tau_{\text{min}} \sim m^2$.

Notice that to determine the black hole lifetime it is required a fully non-perturbative quantum gravity computation. Preliminary results have being obtained in the context of the covariant formalism of Loop Quantum Gravity (spinfoam) [13] and they seems to indicate that the probability for the black hole decay should be peaked on its shortest permitted value, i.e. $\tau = m^2$, as expected for other known decaying phenomena.

For the following analysis of the phenomenology associated to black-to-white hole decays, we have considered the full window of possible lifetimes $m^2 \lesssim \tau \lesssim m^3$. We parametrize this interval by introducing a parameter k such that $\tau = 4kM^2$.

18.3 Primordial Black Holes and Their Signature

The framework of the black-to-white transition is expected to apply to any kind of black hole, irrespectively of the mass or the way it formed. As the lifetime depends on the mass, stellar black hole will explode in the future. To observe an explosion today, we need black holes that are sufficiently small and sufficiently old. PBH satisfy these conditions. The formations of black holes in the early universe can be achieved by a variety of models (see [14] for a review). Here we consider in particular the case of overdense regions collapsing at the time of reheating, but our picture will not qualitatively change for other models of formation, as for instance during the contracting phase in the pre-big-bang epoch [15], as the long cosmological times dominate over short differences in the exact time of formation.

But if PBH explode, how can they be distinguished from all the other astrophysical sources? Remarkably, their signal carries a characteristic signature. The wavelength of the signal depends on the mass of the exploding black hole.[2] Smaller PBH should have exploded earlier: smaller black holes produce a signal of shorter wavelength, but this get redshifted as we can observe them as an earlier (and therefore distant) explosion. It is possible to compute how the wavelength scales with the distance (in terms of the redshift). Standard astrophysical objects scale with a simple linear law. Instead, we find a peculiar flat curve [16] where the shorter wavelengths get

[2]Notice that this differs from the case of black holes exploding via Hawking evaporation, as in that case they would all explodes when they reach the Planck size irrespectively of their initial mass.

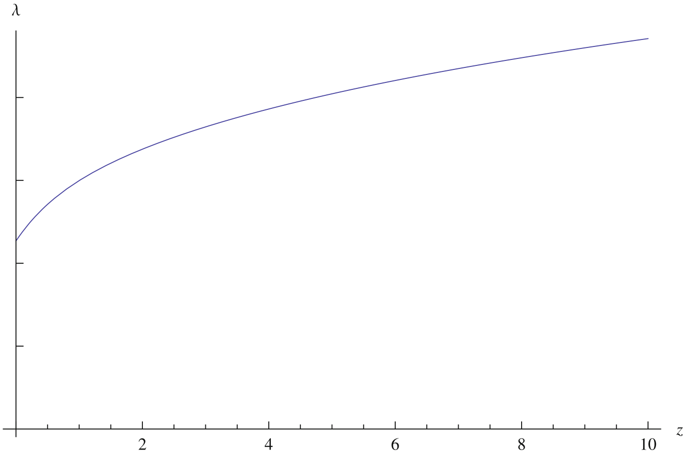

Fig. 18.1 The expected wavelength (unspecified units) of the signal from black hole explosions as a function of the redshift z. The curve flattens at large distance: the shorter wavelength from smaller black holes exploding earlier get compensated by the redshift

compensated by the amount of redshirt (Fig. 18.1). Ideally, we would like to detect very energetic burst for which the distance of the source can be known, in order to fit this curve.

18.3.1 Description of the Expected Signals

The model of black-to-white tunneling provides a concrete mechanism for the explosion, but lacks of any detail of the precise astrophysical process. Heuristic arguments lead us to consider two possible signal channels, that may concur together to the total emission. Both depend on the initial mass of the black hole, but for different reasons.

The first one, that we call the *high energy* channel, is given by matter (mainly photons) that is re-emitted at the same temperature it had at the time it collapsed forming the black hole, as its co-moving bouncing time is very short. In the simplest model, PBH form at different sizes corresponding to the Hubble horizon as the universe expands and cools. This happens typically at the reheating, therefore at a temperature of the order of TeV. This suggests a high energy component of the signal in the order of TeV. This is a very interesting observational window, as TeV astronomy is expected to develop in the forthcoming year [17]. On the other hand, our observational horizon is limited: cosmic rays at such an high energy interact with the cosmic background radiation. Therefore the high energy channel would be observable only for events happening in our galaxy or nearby.

The second one, denoted as *low energy* channel, assumes that the signal will carry a mode that corresponds to the size (i.e. the Schwarzschild radius, that is just

$R = 2\text{m}$) of the exploding object. Knowing the PBH lifetime τ, we can estimate the mass of black holes exploding today. For $\tau \approx m^3$, the emitted signal would be in the GeV, but a detailed analysis [19] of such a signal have shown that MeV photons will have higher density and are more likely to be detected. Transient signals in this range are Short Gamma Ray Bursts [20], whose origin is still unclear. As their energy is so high, the dispersion due to the cosmic background limits our observational horizon . For $\tau \approx m^2$, the estimated signal is expected in the millimeter range of the radio spectrum. Interesting, very energetic transients (Fast Radio Bursts [21]) have recently being discovered in the radio frequency; these may candidate as detection of PBH explosions. Given the approximations taken in the present model, the energy of the Fast Radio Bursts are intriguingly close to those predicted by this model. The radio window allows for observational depth, giving virtually access to events in the entire observable universe. Large antenna available on earth would detect even faint signals in the longer radio wavelength. On the other hand, the (sub)millimetrer wavelengths are shielded by the atmosphere: for them we relay on space telescope, whose detection technology is not sensitive to transients. A more promising strategy seems to be to study the integrated emission, i.e. the relic radiation from all the detectable past explosions.

18.3.2 Diffuse Emission

A detailed study of the diffuse emission from PBH has been carried in [22] considering the full range of possible PBH lifetime, for both the proposed channels of emission. The resulting spectrum carries a distortion from the expected black-body spectrum due to the characteristic redshift-wavelength relation of Fig. 18.1. We report here (Fig. 18.2) a sample of our results. Units in the ordinate axis are not better specified as the normalization of the spectrum depends on the amount of PBH contributing to dark matter.

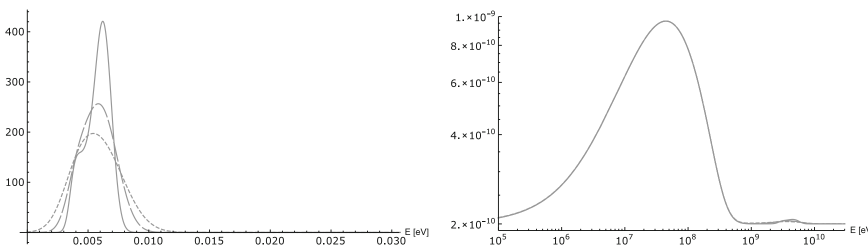

Fig. 18.2 The diffuse emission of the low energy channel plotted for the minimal k, i.e. for the shortest lifetime (left), and for the maximal k, i.e. for the longest lifetime (right). The plots have being obtained using the PYTHIA code [18], that gives the particle production for a process at a given initial energy

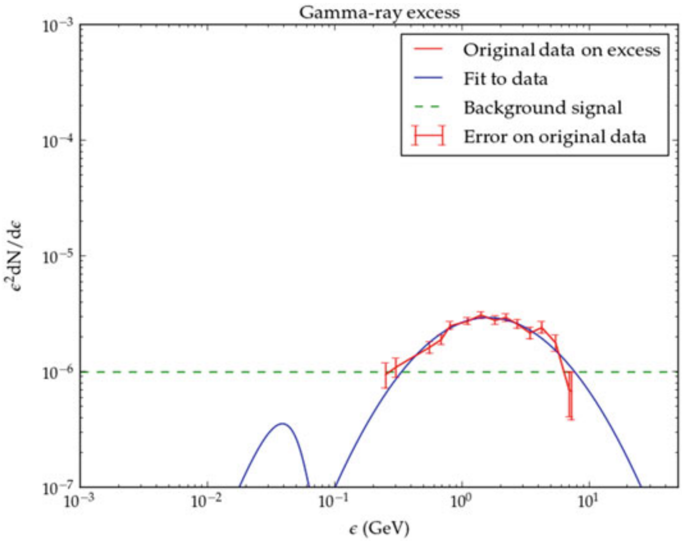

Fig. 18.3 Spectral energy density ($\varepsilon^2 \mathrm{d}N/\mathrm{d}\varepsilon$) fitting the Fermi-LAT data by exploding PBH. The agreement with data is good, with a χ^2 per degree of freedom of 1.05. The bump on the left is given by the secondary gamma-rays, whose spectral energy density is much lower than the one of primary photons and remains below the background (dash line)

The study of the diffuse emission provides a promising tool to constrain the model and may lead to some unexpected surprises. One may ask whether the measured value of the background radiation can be explained by considering a contribution from exploding PBH. We have tested this possibility for the excess of gamma rays coming from the galactic center measured by the Large Area Telescope (LAT) installed in the *Fermi* satellite [23]. We found [24] a remarkable good fit for the case of PBH with a lifetime close to the maximal allowed by the model (Fig. 18.3).

The phenomenology of PBH exploding via a quantum gravity black-to-white transition has just started to be explored. It presents peculiar new features that have consequences for PBH dark matter models. They differs from the ones given by explosions via Hawking evaporations and they provide a new windows for quantum gravity phenomenology.

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Part III
Other Topics in Contemporary Gravitation

Chapter 19
Self Sustained Traversable Phantom Wormholes and Gravity's Rainbow

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Abstract A Self Sustained Traversable Wormhole is a wormhole which is powered by their own gravitational quantum fluctuations. We consider the effects of introducing Gravity's Rainbow on such a configuration to determine the form of the shape function. This last one can be obtained by imposing the equation of state $p_r = \omega\rho$. We investigate the size of the wormhole as a function of the parameter ω in the phantom region. We show that a wormhole which is traversable in principle, but not in practice, can be produced.

19.1 Introduction

In these last years, modifications of gravity at Planckian or Transplanckian energy captured the attention of the scientific community of theoretical physics. The purpose of such a modification is to include quantum gravitational effects in the description of physical phenomena keeping under control the usual Ultraviolet (UV) divergences. This could avoid the application of a standard regularization/renormalization scheme. To this aim, Noncommutative geometry, Gravity's Rainbow and Generalized Uncertainty Principle (GUP) represent some possibilities to cure the divergences that appear in general relativity [1, 2]. In particular, in Gravity's Rainbow, two arbitrary functions $g_1(E/E_P)$ and $g_2(E/E_P)$ having the following properties

$$\lim_{E/E_P \to 0} g_1(E/E_P) = 1 \quad \text{and} \quad \lim_{E/E_P \to 0} g_2(E/E_P) = 1 \tag{19.1}$$

are introduced. $g_1(E/E_P)$ and $g_2(E/E_P)$ appear into the solutions of the modified Einstein's Field Equations [3]

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P. Nicolini et al. (eds.), *2nd Karl Schwarzschild Meeting on Gravitational Physics*, Springer Proceedings in Physics 208,
https://doi.org/10.1007/978-3-319-94256-8_19

$$G_{\mu\nu}\left(E/E_P\right) = 8\pi G\left(E/E_P\right) T_{\mu\nu}\left(E/E_P\right) + g_{\mu\nu}\Lambda\left(E/E_P\right), \tag{19.2}$$

where $G\left(E/E_P\right)$ is an energy dependent Newton's constant, defined so that $G\left(0\right)$ is the low-energy Newton's constant and $\Lambda\left(E/E_P\right)$ is an energy dependent cosmological constant. E is usually interpreted as the energy of a particle deforming the spacetime geometry. However this deformation begins at the Planck scale where we expect that even spacetime begins to show quantum fluctuations realizing a Zero Point Energy (ZPE). When matter fields are absent, we can invoke only one particle compatible with the deformation of the Einstein's gravity: the graviton. The ZPE calculation is strictly connected with the Casimir effect. One of the features of the Casimir effect is represented by its negative energy density, which allows it to candidate as a source for a traversable wormhole. A wormhole is often termed Einstein-Rosen bridge because a "*bridge*" connecting two "sheets" was the result obtained by A. Einstein and N. Rosen in attempting to build a geometrical model of a physical elementary "particle" that was everywhere finite and singularity free [4]. It was J.A. Wheeler who introduced the term wormhole [5], although his wormholes were at the Planck scale. We have to wait for M. S. Morris and K. S. Thorne [6] to see the subject of wormholes seriously considered by the scientific community. The violation of the null energy conditions is fundamental for the existence of a traversable wormholes. This means that the matter threading the wormhole's throat has to be "*exotic*". Classical matter satisfies the usual energy conditions, while Casimir energy on a fixed background has the correct properties to substitute the exotic matter. Usually one considers some matter or gauge fields which contribute to the Casimir energy necessary to the traversability of the wormholes, nevertheless nothing forbids to use the Casimir energy of the graviton on a background of a traversable wormhole. In this way, one can think that the quantum fluctuations of the gravitational field of a traversable wormhole are the same ones which are responsible to sustain traversability. Note that in [7], the ZPE was used as an indicator for a topology change without a Gravity's Rainbow scheme, while in [8], it has been shown that a topology change is a ZPE consequence induced by Gravity's Rainbow. In this contribution we would like to probe ZPE with the help of Gravity's Rainbow and an equation of state.

19.2 Gravity's Rainbow and the Equation of State

In Schwarzschild coordinates, the traversable wormhole metric can be cast into the form

$$\mathrm{d}s^2 = -\exp\left(-2\phi\left(r\right)\right)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{1-\frac{b(r)}{r}} + r^2\mathrm{d}\Omega^2. \tag{19.3}$$

where $\phi\left(r\right)$ is called the redshift function, while $b\left(r\right)$ is called the shape function and where $\mathrm{d}\Omega^2 = \mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2$ is the line element of the unit sphere. Using the Einstein field equation

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}, \tag{19.4}$$

in an orthonormal reference frame, we obtain the following set of equations

$$\rho\left(r\right)=\frac{1}{8\pi G}\frac{b'}{r^2}, \tag{19.5}$$

$$p_r\left(r\right)=\frac{1}{8\pi G}\left[\frac{2}{r}\left(1-\frac{b\left(r\right)}{r}\right)\phi'-\frac{b}{r^3}\right], \tag{19.6}$$

$$p_t\left(r\right)=\frac{1}{8\pi G}\left(1-\frac{b\left(r\right)}{r}\right)\left[\phi''+\phi'\left(\phi'+\frac{1}{r}\right)\right]-\frac{b'r-b}{2r^2}\left(\phi'+\frac{1}{r}\right), \tag{19.7}$$

in which $\rho\left(r\right)$ is the energy density, $p_r\left(r\right)$ is the radial pressure, and $p_t\left(r\right)$ is the lateral pressure. Using the conservation of the stress-energy tensor, in the same orthonormal reference frame, we get

$$p_r'=\frac{2}{r}\left(p_t-p_r\right)-\left(\rho+p_r\right)\phi'. \tag{19.8}$$

When Gravity's Rainbow comes into play, the line element (19.3) becomes [3]

$$ds^2=-\exp\left(-2\phi\left(r\right)\right)dt^2g_1^2\left(E/E_P\right)+\frac{dr^2}{\left(1-\frac{b(r)}{r}\right)g_2^2\left(E/E_P\right)}+\frac{r^2}{g_2^2\left(E/E_P\right)}d\Omega^2 \tag{19.9}$$

and the Einstein's Field Equations (19.5), (19.6) and (19.7) can be rearranged to give

$$b'=\frac{8\pi G\rho\left(r\right)r^2}{g_2^2\left(E/E_P\right)}, \tag{19.10}$$

$$\phi'=\frac{b+8\pi Gp_rr^3/g_2^2\left(E/E_P\right)}{2r^2\left(1-\frac{b(r)}{r}\right)}. \tag{19.11}$$

Now, we introduce the equation of state $p_r=\omega\rho$, and using (19.10), (19.11) becomes

$$\begin{aligned}\phi'&=\frac{b+8\pi G\left(\omega g_2^2\left(E/E_P\right)b'\left(r\right)/\left(8\pi Gr^2\right)\right)r^3/g_2^2\left(E/E_P\right)}{2r^2\left(1-\frac{b(r)}{r}\right)}\\&=\frac{b+\omega b'r}{2r^2\left(1-\frac{b(r)}{r}\right)}.\end{aligned} \tag{19.12}$$

It is immediate to see that the equation related the redshift is unchanged and can be set to a constant with respect to the radial distance if

$$b+\omega b'r=0. \tag{19.13}$$

The integration of this simple equation leads to

$$b\left(r\right)=r_{0}\left(\frac{r_{0}}{r}\right)^{\frac{1}{\omega}}, \tag{19.14}$$

where we have used the condition $b\left(r_{t}\right)=r_{t}$. In this situation, the line element (19.9) becomes

$$ds^{2}=-\frac{A}{g_{1}^{2}\left(E/E_{P}\right)}dt^{2}+\frac{dr^{2}}{1-\left(\frac{r_{0}}{r}\right)^{1+\frac{1}{\omega}}g_{2}^{2}\left(E/E_{P}\right)}+\frac{r^{2}}{g_{2}^{2}\left(E/E_{P}\right)}d\Omega^{2}, \tag{19.15}$$

where A is a constant coming from $\phi'=0$ which can be set to one without loss of generality. The parameter ω is restricted by the following conditions

$$b'\left(r_{0}\right)<1;\qquad \frac{b\left(r\right)}{r}\underset{r\rightarrow+\infty}{\rightarrow}0\qquad\Longrightarrow\qquad\begin{cases}\omega>0\\ \omega<-1\end{cases}. \tag{19.16}$$

19.3 Self-sustained Traversable Wormholes, Gravity's Rainbow and Phantom Energy

In this section we shall consider the formalism outlined in detail in [9, 10], where the graviton one loop contribution to a classical energy in a wormhole background is used. A traversable wormhole is said to be "*self sustained*" if

$$H_{\Sigma}^{(0)}=-E^{TT}, \tag{19.17}$$

where E^{TT} is the total regularized graviton one loop energy and $H_{\Sigma}^{(0)}$ is the classical term. When we deal with spherically symmetric line element, the classical Hamiltonian reduces to

$$\begin{aligned}H_{\Sigma}^{(0)}&=\int_{\Sigma}d^{3}x\left[\left(16\pi G\right)G_{ijkl}\pi^{ij}\pi^{kl}-\frac{\sqrt{g}}{16\pi G}R\right]\\&=-\frac{1}{16\pi G}\int_{\Sigma}d^{3}x\sqrt{g}\,R=-\frac{1}{2G}\int_{r_{0}}^{\infty}\frac{dr\,r^{2}}{\sqrt{1-b(r)/r}}\frac{b'(r)}{r^{2}g_{2}\left(E/E_{P}\right)},\end{aligned} \tag{19.18}$$

where we have used the explicit expression of the scalar curvature in three dimensions in terms of the shape function. G_{ijkl} is the super-metric and π^{ij} is the super-momentum. Note that, in this context, the kinetic term disappears. Note also that boundary terms become important when one compares different configurations like Wormholes and Dark Stars [7] or Wormholes and Gravastars [11]. With the help of (19.13), the classical energy becomes

$$H_{\Sigma}^{(0)} = \frac{1}{2G} \int_{r_0}^{\infty} \frac{\mathrm{d}r\, r^2}{\sqrt{1 - b(r)/r}} \frac{b(r)}{r^3 g_2 \left(E/E_P\right) \omega} \tag{19.19}$$

and following [12], the self-sustained equation (19.17) becomes

$$-\frac{b(r)}{2Gr^3 g_2 \left(E/E_P\right) \omega} = \frac{2}{3\pi^2} \left(I_1 + I_2\right) , \tag{19.20}$$

where the r.h.s. of (19.20) is represented by

$$I_1 = \int_{E^*}^{\infty} E \frac{g_1 \left(E/E_P\right)}{g_2^2 \left(E/E_P\right)} \frac{\mathrm{d}}{\mathrm{d}E} \left(\frac{E^2}{g_2^2 \left(E/E_P\right)} - m_1^2 \left(r\right) \right)^{\frac{3}{2}} \mathrm{d}E \tag{19.21}$$

and

$$I_2 = \int_{E^*}^{\infty} E \frac{g_1 \left(E/E_P\right)}{g_2^2 \left(E/E_P\right)} \frac{\mathrm{d}}{\mathrm{d}E} \left(\frac{E^2}{g_2^2 \left(E/E_P\right)} - m_2^2 \left(r\right) \right)^{\frac{3}{2}} \mathrm{d}E , \tag{19.22}$$

respectively. E^* is the value which annihilates the argument of the root while $m_1^2 \left(r\right)$ and $m_2^2 \left(r\right)$ are two r-dependent effective masses. Of course, I_1 and I_2 are finite for appropriate choices of the Rainbow's functions $g_1 \left(E/E_P\right)$ and $g_2 \left(E/E_P\right)$. With the help of the EoS, one finds

$$\begin{cases} m_1^2 \left(r\right) = \frac{6}{r^2} \left(1 - \frac{b(r)}{r}\right) + \frac{3}{2r^3\omega} b\left(r\right) \left(\omega + 1\right) \\ m_2^2 \left(r\right) = \frac{6}{r^2} \left(1 - \frac{b(r)}{r}\right) + \frac{3}{2r^3\omega} b\left(r\right) \left(\frac{1}{3} - \omega\right) \end{cases} \tag{19.23}$$

and on the throat, the effective masses reduce to

$$\begin{array}{ll} m_1^2 \left(r_0\right) = \frac{3}{2r_0^2\omega} \left(\omega + 1\right) & \begin{cases} > 0 & \text{when} \quad \omega > 0 \quad \text{or} \quad \omega < -1 \\ < 0 & \text{when} \quad -1 < \omega < 0 \end{cases} \\ m_2^2 \left(r_0\right) = \frac{3}{2r_0^2\omega} \left(\frac{1}{3} - \omega\right) & \begin{cases} > 0 & \text{when} \quad 1/3 > \omega > 0 \\ < 0 & \text{when} \quad \omega > 1/3 \quad \text{or} \quad \omega < 0 \end{cases} \end{array} . \tag{19.24}$$

However, to have values of ω compatible with the traversability condition, only the cases with $\omega > 0$ and $\omega < -1$ are allowed. It is easy to see that if we assume

$$g_1 \left(E/E_P\right) = 1 \qquad g_2 \left(E/E_P\right) = \begin{cases} 1 & \text{when} \quad E < E_P \\ E/E_P & \text{when} \quad E > E_P \end{cases} , \tag{19.25}$$

(19.20) becomes, close to the throat,

$$-\frac{1}{2Gr_0^2\omega}=\frac{2}{\pi^2}\left(\int_{\sqrt{m_1^2(r)}}^{E_P}E^2\sqrt{E^2-m_1^2\left(r_0\right)}\mathrm{d}E+\int_{\sqrt{m_2^2(r)}}^{E_P}E^2\sqrt{E^2-m_2^2\left(r_0\right)}\mathrm{d}E\right), \tag{19.26}$$

where $m_1^2\left(r_0\right)$ and $m_2^2\left(r_0\right)$ have been defined in (19.24). Since the r.h.s. is certainly positive, in order to have real solutions compatible with asymptotic flatness, we need to impose $\omega<-1$, that it means that we are in the Phantom regime. With this choice, the effective masses (19.24) become, on the throat

$$m_1^2\left(r_0\right)=\frac{3}{2r_0^2\omega}\left(\omega+1\right)\qquad m_2^2\left(r_0\right)=-\frac{3}{2r_0^2\omega}\left(\frac{1}{3}-\omega\right) \tag{19.27}$$

and (19.26) simplifies into

$$1=-\frac{4r_0^2\omega}{\pi^2E_P^2}\left(\int_{\sqrt{m_1^2(r_0)}}^{E_P}E^2\sqrt{E^2-\frac{3}{2r_0^2\omega}\left(\omega+1\right)}\mathrm{d}E+\int_0^{E_P}E^2\sqrt{E^2+\frac{3}{2r_0^2}\left|\frac{1}{3\omega}-1\right|}\mathrm{d}E\right) \tag{19.28}$$

The solution can be easily computed numerically and we find

$$\begin{aligned}-1\geq\omega\geq-4.5\\ 2.038\geq x\geq1.083.\end{aligned} \tag{19.29}$$

Therefore we can conclude that a wormhole which is traversable in principle, but not in practice, can be produced joining Gravity's Rainbow and phantom energy. Of course, the result is strongly dependent on the rainbow's functions which, nonetheless must be chosen in such a way to give finite results for the one loop integrals (19.21) and (19.22).

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Chapter 20
Cosmology via Metric-Independent Volume-Form Dynamics

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Abstract The method of non-Riemannian volume-forms (metric-independent covariant integration measure densities on the spacetime manifold) is applied to construct a unified model of dynamical dark energy plus dark matter as a dust fluid resulting from a hidden Noether symmetry of the pertinent scalar field Lagrangian. Canonical Hamiltonian treatment and Wheeler-DeWitt quantization of the latter model are briefly discussed.

20.1 Introduction

Alternative spacetime volume-forms (generally-covariant integration measure densities) independent on the Riemannian metric on the pertinent spacetime manifold have profound impact in any field theory models with general coordinate reparametrization invariance, such as general relativity and its extensions, strings and (higher-dimensional) membranes [10, 11, 13, 14].

The principal idea is to replace or employ alongside the standard Riemannian integration density given by $\sqrt{-g}$ (square root of the determinant $g = \det \|g_{\mu\nu}\|$ of the Riemannian metric $g_{\mu\nu}$) one or more non-Riemannian (metric-independent) covariant integration measure densities defined in terms of dual field-strengths $\Phi(B)$ of auxiliary maximal rank antisymmetric tensor gauge fields $B_{\mu\nu\lambda}$:

$$\Phi(B) = \frac{1}{3!}\varepsilon^{\mu\nu\kappa\lambda}\partial_\mu B_{\nu\kappa\lambda} \ , \tag{20.1}$$

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P. Nicolini et al. (eds.), *2nd Karl Schwarzschild Meeting on Gravitational Physics*, Springer Proceedings in Physics 208,
https://doi.org/10.1007/978-3-319-94256-8_20

The corresponding non-Riemannian-modified-measure gravity-matter models were called "two-measure (gravity) theories" and the associated auxiliary tensor gauge fields $B_{\mu\nu\lambda}$ – "measure gauge fields".

The auxiliary "measure" gauge fields trigger a number of physically interesting phenomena:

- The equations of motion w.r.t. $B_{\mu\nu\lambda}$ produce dynamical constraints involving *arbitrary integration constants*, where one of the latter *always* acquires the meaning of a *dynamically generated cosmological constant*.
- Employing the canonical Hamiltonian formalism for Dirac-constrained systems we find that $B_{\mu\nu\lambda}$ are in fact almost pure gauge degrees of freedom except for the above mentioned arbitrary integration constants which are identified with the conserved Dirac-constrained canonical momenta conjugated to the "magnetic" components (B_{ijk}) of the "measure" gauge fields.
- Upon applying the non-Riemannian volume-form formalism to minimal $N = 1$ supergravity the dynamically generated cosmological constant triggers spontaneous supersymmetry breaking and mass generation for the gravitino (supersymmetric Brout–Englert–Higgs effect) [16]. Applying the same formalism to anti-de Sitter supergravity allows to produce simultaneously a very large physical gravitino mass and a very small *positive* observable cosmological constant [16] in accordance with modern cosmological scenarios for slowly expanding universe of the present epoch [19–21].
- Employing two independent non-Riemannian volume-forms like (20.1) in generalized gravity-gauge+scalar-field models [12], thanks to the appearance of several arbitrary integration constants through the equations of motion w.r.t. the "measure" gauge fields, we obtain in the physical"Einstein-frame" a remarkable effective scalar potential with two infinitely large flat regions (for large negative and large positive values of the scalar field φ) with vastly different scales appropriate for a unified description of both the early and late universe' evolution. Another remarkable feature is the existence of a stable initial phase of *non-singular* universe creation preceding the inflationary phase – stable "emergent universe" without "Big-Bang" [12].

As a specific illustration of the usefulness of the non-Riemannian volume-form method and extending the study in [1, 17] we discuss a modified gravity+single-scalar-field model where the scalar Lagrangian couples symmetrically both to the standard Riemannian volume-form given by $\sqrt{-g}$ as well as to another non-Riemannian volume-form (20.1). The pertinent scalar field dynamics provides a unified description of both dark energy via dynamical generation of a cosmological constant, and dark matter as a "dust" fluid with geodesic flow as a result of a hidden Noether symmetry. Further, we briefly consider the canonical Hamiltonian treatment and the Wheeler–DeWitt quantization of the above unified dark energy plus dust fluid dark matter model.

20.2 Dark Energy and Dust Fluid Dark Matter Via Non-riemannian Volume-Form Dynamics

We will consider the following non-conventional gravity+scalar-field action – a particular case of the general class of the "two-measure" gravity-matter theories [10, 11, 13] (for simplicity we use units with the Newton constant $G_N = 1/16\pi$):

$$S = \int d^4x \sqrt{-g}\, R + \int d^4x \big(\sqrt{-g} + \Phi(B)\big) L(\varphi, X) \,. \tag{20.2}$$

Here $\Phi(B)$ is as in (20.1) and $L(\varphi, X)$ is general-coordinate invariant Lagrangian of a single scalar field $\varphi(x)$ of a generic "k-essence" form [2, 8] (i.e., a nonlinear (in general) function of the scalar kinetic term X): $L(\varphi, X) = \sum_{n=1}^{N} A_n(\varphi) X^n - V(\varphi)$, $X \equiv -\frac{1}{2} g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi$. The energy-monentum tensor corresponding to (20.2) reads:

$$T_{\mu\nu} = g_{\mu\nu}\, L(\varphi, X) + \Big(1 + \frac{\Phi(B)}{\sqrt{-g}}\Big) \frac{\partial L}{\partial X}\, \partial_\mu \varphi\, \partial_\nu \varphi \,. \tag{20.3}$$

The essential new feature is the dynamical constraint on the scalar Lagrangian, which results from the equation of motion w.r.t. "measure" gauge field $B_{\mu\nu\lambda}$:

$$\partial_\mu L(\varphi, X) = 0 \quad \longrightarrow \quad L(\varphi, X) = -2M = \text{const} \,, \tag{20.4}$$

where M is an *arbitrary integration constant*. We will take $M > 0$ in view of its interpretation as a *dynamically generated cosmological constant* (see (20.7) below).

A remarkable property of the scalar field action in (20.2) is the presence of a hidden Noether symmetry of the latter under the nonlinear transformations:

$$\delta_\varepsilon \varphi = \varepsilon \sqrt{X}, \quad \delta_\varepsilon g_{\mu\nu} = 0, \quad \delta_\varepsilon B_{\mu\nu\lambda} = -\varepsilon \frac{1}{2\sqrt{X}} \varepsilon_{\mu\nu\lambda\kappa} g^{\kappa\rho} \partial_\rho\, \varphi \big(\Phi(B) + \sqrt{-g}\big) \,. \tag{20.5}$$

The standard Noether procedure yields the conserved current:

$$\nabla_\mu J^\mu = 0, \quad J^\mu \equiv \Big(1 + \frac{\Phi(B)}{\sqrt{-g}}\Big) \sqrt{2X} g^{\mu\nu} \partial_\nu \varphi \frac{\partial L}{\partial X} \,. \tag{20.6}$$

Let us stress that the existence of the hidden symmetry (20.5) of the action (20.2) *does not* depend on the specific form of the scalar field Lagrangian.

Now, $T_{\mu\nu}$ (20.3) and J^μ (20.6) can be rewritten in a relativistic hydrodynamical form (taking into account (20.4)):

$$T_{\mu\nu} = \rho_0 u_\mu u_\nu - 2M g_{\mu\nu}, \quad J^\mu = \rho_0 u^\mu, \tag{20.7}$$

where:

$$\rho_0 \equiv \Big(1 + \frac{\Phi(B)}{\sqrt{-g}}\Big) 2X \frac{\partial L}{\partial X}, \quad u_\mu \equiv \frac{\partial_\mu \varphi}{\sqrt{2X}} \quad (u^\mu u_\mu = -1) \; . \tag{20.8}$$

For the pressure p and energy density ρ we obtain:

$$p = -2M = \text{const}, \quad \rho = \rho_0 - p = 2M + \Big(1 + \frac{\Phi(B)}{\sqrt{-g}}\Big) 2X \frac{\partial L}{\partial X} \; , \tag{20.9}$$

wherefrom indeed the integration constant M appears as *dynamically generated cosmological constant*. Moreover the covariant energy-momentum conservation $\nabla^\nu T_{\mu\nu} = 0$, due to the constancy of the pressure (first (20.9)), actually implies *both* the conservation of the Noether current J^μ (20.6) as well as the *geodesic flow* equation: $u_\nu \nabla^\nu u_\mu = 0$.

The above results lead to the following interpretation in accordance with the standard Λ-CDM model (see e.g. [9]). The energy-momentum tensor (20.7) consists of two parts:

- Dark energy part given by the second cosmological constant term in $T_{\mu\nu}$ (20.7), which arises due to the dynamical constraint on the scalar field Lagrangian (20.4) with $p_{\rm DE} = -2M$, $\rho_{\rm DE} = 2M$ (cf. (20.9)).
- Dark matter part given by the first term in (20.7) (cf. also (20.9)) with $p_{\rm DM} = 0$, $\rho_{\rm DM} = \rho_0$ (ρ_0 as in (20.8)). The latter describe a dust fluid with dust "particle number" conservation (20.6) and flowing along geodesics.

The idea of unified description of dark energy and dark matter is the subject of numerous earlier papers exploiting a variety of different approaches. Among them are generalized Chaplygin gas models [5, 22], "mimetic" dark matter models [6, 7], constant pressure ansatz models [3] etc.

20.3 Canonical Hamiltonian Formalism and Wheller-DeWitt Equation

For a systematic canonical Hamiltonian treatment of gravity-matter models based on metric-independent volume-forms we refer to [15] and specifically to the second reference therein for the full Hamiltonian treatment of the present model (20.2). Here, for simplicity, we will consider a reduction of (20.2) where the spacetime metric is taken of the Friedmann–Lemaitre–Robinson–Walker (FLRW) class:

$$ds^2 = -N^2(t)dt^2 + a^2(t)\Big[\frac{dr^2}{1 - Kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\Big] \; , \tag{20.10}$$

and where φ and the "measure" gauge field B are taken to depend only on t. The reduced action resulting from (20.2) reads (taking the standard form of the scalar

Lagrangian):

$$S = 6\int \mathrm{d}t\, Na^3\Big[-\frac{1}{N^2}\Big(\frac{\dot{a}}{a}\Big)^2 + \frac{K}{a^2}\Big] + \int \mathrm{d}t\big(\partial_t B + Na^3\big)\Big(\frac{1}{2N^2}\,\dot{\varphi}^2 - V(\varphi)\Big)\,. \tag{20.11}$$

The equation of motion w.r.t. B produces the dynamical constraint (reduced form of (20.4)) with explicit solution for $\varphi(t)$:

$$\dot{\varphi}^2 = 2\big(V(\varphi) - 2M\big) \quad\longrightarrow\quad \int_{\varphi(0)}^{\varphi(t)} \frac{\mathrm{d}\varphi}{\sqrt{2\big(V(\varphi)-2M\big)}} = \pm t\,. \tag{20.12}$$

The hidden "dust" Noether symmetry (cf. (20.5) and (20.6)) of the reduced action (20.11) now takes the form:

$$\delta_\varepsilon \varphi = \varepsilon\frac{\dot{\varphi}}{N}\;\;,\;\; \delta_\varepsilon B = \varepsilon\frac{1}{N}\big(\partial_t B + Na^3\big)\;\;,\;\; \delta_\varepsilon a = 0\;\;,\;\; \frac{\mathrm{d}}{\mathrm{d}t}\Big[\Big(Na^3 + \partial_t B\Big)\frac{\dot{\varphi}^2}{N^3}\Big] = 0\,. \tag{20.13}$$

The canonical Hamiltonian treatment a'la Dirac of the reduced action (20.11) yields the following Dirac-constrained Hamiltonian (N appearing as a Lagrange multiplier of the first class constraint in the brackets):

$$\mathscr{H}_{\mathrm{total}} = N\Big[-\frac{p_a^2}{24a} - 6Ka - \pi_B a^3 + \sqrt{2\big(V(\varphi)+\pi_B\big)}\,p_\varphi\Big]\,, \tag{20.14}$$

where p_a and π_B are the canonically conjugated momenta of a and B, respectively.

The quantum Wheeler-DeWitt equation corresponding to (20.14) is significantly simplified upon changing variables as:

$$a \to \widetilde{a} = \frac{4}{\sqrt{3}}\,a^{3/2}, \quad \varphi \to \widetilde{\varphi} = \int \frac{\mathrm{d}\varphi}{\sqrt{2\big(V(\varphi)-2M\big)}}\,, \tag{20.15}$$

where from (20.12) we find that the new scalar field coordinate $\widetilde{\varphi}$ will have the meaning of a (cosmic) time. Since B turns out to be a cyclic variable in (20.14) the quantized canonical momentum $\widehat{\pi}_B = -i\delta/\delta B$ is immediately diagonalized whose eigenvalues are denoted by $\pi_{\mathscr{B}} = -2M$, so that M will have the meaning of a dynamically generated cosmological constant. Further, we notice that the quantized form of the last term in (20.14), which is the Hamiltonian expression for the conserved "dust" Noether symmetry charge (20.13), will simplify to $\sqrt{2\big(V(\varphi)+\pi_B\big)}\,\big(-i\frac{\mathrm{d}}{\mathrm{d}\varphi}\big) = -i\mathrm{d}/\mathrm{d}\widetilde{\varphi}$ and is straightforwardly diagonalized with eigenvalues $\mathscr{E}$. Accordingly, the total Wheeler-DeWitt wave function will have the form $\psi(a,\varphi,B) = \psi_{\mathrm{grav}}(\widetilde{a})\,e^{i\mathscr{E}\widetilde{\varphi} - i\,2MB}$

(with $\widetilde{a}$ and $\widetilde{\varphi}$ as in (20.15)), and the Wheeler-DeWitt equation reduces to "energy" eigenvalue Schrödinger equation for the gravitational part of the total wave function:

$$\left[-\frac{1}{2}\frac{\partial^2}{\partial\widetilde{a}^2} - \frac{3}{8}M\widetilde{a}^2 + 6K\left(\frac{\sqrt{3}}{4}\widetilde{a}\right)^{2/3} - \mathscr{E}\right]\psi_{\text{grav}}(\widetilde{a}) = 0 \tag{20.16}$$

In the special case of zero spacial curvature $K = 0$ in the FLRW metric (20.10), (20.16) reduces to the energy eigenvalue Schrödinger equation for the *inverted* harmonic oscillator [4] with negative frequency squared $\omega^2 \equiv -\frac{3}{4}M$ (the dynamically generated cosmological constant M must be positive).

In particular, the inverted oscillator was applied in [18] to study the quantum mechanical dynamics of the scalar field in the so called "new inflationary" scenario. Since the energy eigenvalue spectrum of the inverted harmonic oscillator is continuous ($\mathscr{E} \in (-\infty, +\infty)$) and the corresponding energy eigenfunctions are not square-integrable, its application in the context of cosmology [18] requires employment of wave-packets instead of energy eigenfunctions.

Acknowledgements E.G. thanks Frankfurt Institute for Advanced Studies (FIAS) for hospitality. We gratefully acknowledge support of our collaboration through the academic exchange agreement between the Ben-Gurion University in Beer-Sheva, Israel, and the Bulgarian Academy of Sciences. S.P. and E.N. have received partial support from European COST actions MP-1210 and MP-1405, respectively, as well from Bulgarian National Science Fund Grant DFNI-T02/6.

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Chapter 21
Size Scaling of Self Gravitating Polymers and Strings

S. Kawamoto and T. Matsuo

Abstract A typical configuration of a long free fundamental string is described as a free random walk. With self-gravitational interaction, the configuration contracts and eventually the size becomes comparable to the Schwarzschild radius of a black hole of the same energy, where the string configuration is identified with the corresponding black hole. We consider the size change of a long string at a fixed large excited level by use of tools developed in polymer physics. We introduce a contact self-repulsive interaction as well as Newtonian gravitational interaction and find that the size exhibits interesting scaling behaviors, which are summarized in diagrams.

21.1 Introduction

A century ago, the first exact (nontrivial) solution of general theory of relativity (GR) was found by K. Schwarzschild just after GR was presented [1]. Since then, the solution and its siblings, black hole geometries, have been stimulating physicists and mathematicians, and have also attracted interests of general public for its mysterious description as "a space of no return" from which even light cannot escape. Black holes are not of merely theoretical interest as there have been accumulating observational supports for existence. It has turned out that black holes have thermodynamic properties; they have temperature and emit radiation. A black hole also possesses entropy that is proportional to its surface area, which implies that the degrees of freedom are distributed on a lower dimensional space than the black hole occupies; the reason behind this dimensional reduction is not clear yet, but it serves a key behind the holographic principle. On the other hand, the evaporation process of a black hole presents a puzzle; if an object of a pure state collapses into a black hole, it may evolve

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P. Nicolini et al. (eds.), *2nd Karl Schwarzschild Meeting on Gravitational Physics*, Springer Proceedings in Physics 208,
https://doi.org/10.1007/978-3-319-94256-8_21

into a mixed state of Hawking radiation in later time, which contradicts to unitarity quantum mechanics. There is another possibility that the final state may be a pure state, a massive remnant or highly entangled and unitarity remains intact. However, a consistent model of remnants appears difficult. If the final state is entangled pure radiation, it may enable us to duplicate the initial pure state that is contradict to linearity of quantum mechanics, or such entanglement would lead to highly excited degrees of freedom at the horizon, known as a "firewall," with which equivalence principle may be questioned.

These issues may be resolved through further understanding of quantum mechanics and GR, but it is also possible that final resolution calls for quantum gravity. Since string theory is expected to be a consistent quantum gravity, it is interesting to analyze the black hole formation and evaporation in terms of string dynamics. The profile of a highly excited free string is described by a random walk model [2, 3]. It thus occupies a large volume, but may be folded into a compact volume due to self-interaction to form a microscopic massive object. Susskind has argued that at a critical value of string coupling the description of the system of a folded string may be replaced by a corresponding black hole, and vice versa [4, 5]. Horowitz and Polchinski analyzed the profile of interacting strings by use of thermal scalar model to clarify how the size changes [6]. There are also some related works [7–12]. We revisit this problem by use of another physical system that is also described by interacting random walks — a polymer. Realistic polymers are described by self-avoiding, not free, walks, since monomers cannot not occupy the same location. This excluded volume effect can also be implemented as a contact repulsive interaction in free walks. We introduce this repulsive interaction since it has been also speculated that there emerges such interaction in a high density regime near a black hole [13]. In this article, we give a brief description of the size scaling of long strings at the presence of Newtonian self-gravitational and contract repulsive interactions, based on [14].

21.2 The Size Scaling of Self-Interacting Long Strings

In this section, we introduce an effective Hamiltonian that describes an interacting random walk, called Edwards Hamiltonian [15–17]. The size is evaluated with the help of two different approximation schemes and is found to exhibit interesting scaling behavior. We also observe that at certain critical couplings the configuration is enclosed by Schwarzschild radius and may collapse into a black hole. This size scaling is summarized in a "phase diagrams" in Fig. 21.1, which are parametrized by the magnitudes of two interaction.

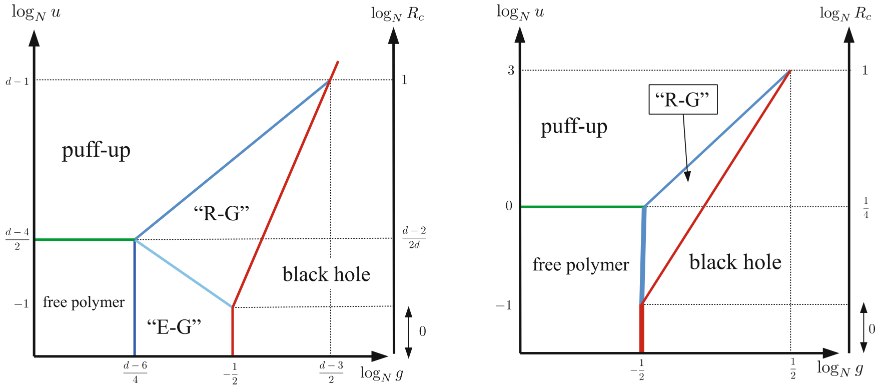

Fig. 21.1 The size scaling in $2 < \mathrm{d} < 4$ (left) and $\mathrm{d} = 4$ (right). The horizontal and the vertical axes denote $\log_N g$ and $\log_N u$ respectively. Namely, the numbers shown here represent the power of N. On the right, the size of the corresponding black hole for fixed u is also shown

21.2.1 Edwards Hamiltonian and the Evaluation of the Size

We consider the following Edwards Hamiltonian in $\mathrm{d}(> 2)$ dimensions,

$$\beta H = \frac{\mathrm{d}}{2\ell^2} \int_0^N \mathrm{d}\sigma \left(\frac{\partial \mathbf{R}}{\partial \sigma} \right)^2 + V \,. \tag{21.1}$$

The potential term V consists of long-range Newton interaction as well as a point-like repelling force expressed by the delta function,

$$V = \int_0^N \mathrm{d}\sigma \int_0^N \mathrm{d}\sigma' \left[- \frac{g^2 \ell^{\mathrm{d}-2}}{|\mathbf{R}(\sigma) - \mathbf{R}(\sigma')|^{\mathrm{d}-2}} + u\ell^{\mathrm{d}} \delta^{(\mathrm{d})}(\mathbf{R}(\sigma) - \mathbf{R}(\sigma')) \right] . \tag{21.2}$$

g^2 and u are dimensionless coupling constants, and are assumed to be positive; namely they correspond to attractive and repulsive interactions respectively. N is the number of monomers and ℓ is the (Kuhn) length of the bond between monomers, which will be identified with the string scale ℓ_s. With $V = 0$, the size is given by the free random walk as $R_0 = \ell\sqrt{N}$ and by comparing this with the size of a free string [2] the excited level of the string is identified with N^2. The effective temperature β^{-1} is the order of the string scale $1/\ell_s$, but its explicit value is not important; it just sets the scale of the analysis with which the coupling constants are measured.

The purpose of this article is to summarize the size scaling of a long string with respect to a large number of monomers N in accordance with the changes of N dependence in the coupling constants g and u. The size here stands for the root mean square of the end-to-end distance, which is evaluated by use of Edwards Hamiltonian as

$$R(g,u)^2 = \langle (\mathbf{R}(N) - \mathbf{R}(0))^2 \rangle = \frac{1}{Z} \int \mathscr{D}\mathbf{R}(N)\, (\mathbf{R}(N) - \mathbf{R}(0))^2 e^{-\beta H}\,, \quad (21.3)$$

where the partition function $Z = \int \mathscr{D}\mathbf{R}(N)\, e^{-\beta H}$ and we can fix, and will fix from now on, the position of the one end $\mathbf{R}(0)$ at the origin by use of translational invariance. We are not capable of evaluating this path-integral analytically and then introduce two approximation methods; variational principle and a uniform expansion model. These two methods turn out to be valid in the complementary situations.

In the variational principle, we consider the following harmonic Hamiltonian with a variational parameter q,

$$\beta H_q = \frac{\mathrm{d}}{2\ell^2} \int_0^N \mathrm{d}\sigma \left(\frac{\partial \mathbf{R}}{\partial \sigma} \right)^2 + \frac{\mathrm{d}q^2}{2\ell^2} \int_0^N \mathrm{d}\sigma\, \mathbf{R}(\sigma)^2\,, \quad (21.4)$$

and rewrite (21.1) as $\beta H = \beta H_q + V_q$ where $V_q = V - \frac{\mathrm{d}q^2}{2\ell^2} \int_0^N \mathrm{d}\sigma\, \mathbf{R}(\sigma)^2$, and treat V_q as perturbation. The approximated size is calculated as $R(q) = \sqrt{\langle \mathbf{R}(N)^2 \rangle_q}$ where the subscript for the bracket indicates that the expectation value is taken with respect to βH_q. Since the harmonic Hamiltonian is quadratic, the expectation value is immediately evaluated,

$$R(q) = \frac{\ell}{\sqrt{q}} \sqrt{\tanh qN} \simeq \begin{cases} \ell\sqrt{N} & qN \ll 1 \\ \ell q^{-1/2} & qN \geq O(1) \end{cases}. \quad (21.5)$$

The free energy $\beta F = -\ln Z$ satisfies the inequality $\beta F \leq \beta F_q + \langle V_q \rangle_q$ thanks to convexity of logarithm. F_q is the free energy of the harmonic Hamiltonian H_q. This inequality is spelled out as

$$\beta F \leq qN - N^2 \left(g^2 q^{\frac{\mathrm{d}-2}{2}} - u q^{\frac{\mathrm{d}}{2}} \right)\,, \quad (21.6)$$

where we have dropped $O(1)$ positive numerical coefficients since we are only interested in N dependence of R. The optimal value of q, denoted as q_0, is chosen so that it minimizes the right hand side. We then obtain the approximated size $R(q_0)$ which becomes a function of g and u through q_0. As seen in (21.5), the approximated size is bounded from above by the free walk size $R_0 = \ell\sqrt{N}$ for small values of q_0, and decreases like ${q_0}^{-1/2}$ as q_0 increases from $q_0 \simeq N^{-1}$. Hence, this approximation is viable if the attractive interactions are dominant and the configuration gets smaller compared to the free size.

The other approximation we employ is called the uniform expansion model (UEM), in which the change of the size is encoded in the change of the length of the bond, $\ell' = a\ell$, but the configuration is assumed to remain to be free one.

In this case the size rescaling factor a is a parameter of the approximation, and we use the following free Edwards Hamiltonian with the bond length $\ell' = a\ell$,

$$\beta H' = \frac{\mathrm{d}}{2a^2\ell^2} \int_0^N \mathrm{d}\sigma \left(\frac{\partial \mathbf{R}}{\partial \sigma} \right)^2 , \tag{21.7}$$

to evaluate the size R as

$$\begin{aligned} R^2 &= \frac{\int (\mathbf{R}(N))^2 e^{-\beta H}}{\int e^{-\beta H}} = \frac{\langle (\mathbf{R}(N))^2 e^{-\beta(H-H')} \rangle'}{\langle e^{-\beta(H-H')} \rangle'} \\ &\simeq \langle (\mathbf{R}(N))^2 \rangle' \big(1 + \langle \beta(H - H') \rangle'\big) - \langle (\mathbf{R}(N))^2 \beta(H - H') \rangle' , \end{aligned} \tag{21.8}$$

where $\langle \cdots \rangle'$ denotes the expectation value with respect $\beta H'$ and we have taken the first order in $H - H'$. It is straightforward to evaluate the expectation values as $\beta H'$ is Gaussian, and we find $R^2 \simeq N\ell'^2 + N\ell^2 a^{2-d} f(g, u; a)$ where

$$f(g, u; a) = a^{\mathrm{d}} - a^{\mathrm{d}+2} + uN^{\frac{4-\mathrm{d}}{2}} - g^2 a^2 N^{\frac{6-\mathrm{d}}{2}} , \tag{21.9}$$

and $O(1)$ positive numerical coefficients are omitted again. The requirement of UEM is that the size is given by the free walk size of the bond length ℓ', namely $R = \ell'\sqrt{N}$. Then the consistency condition of UEM is $f(g, u; a) = 0$. This condition fixes a, and the approximated size is obtained by use of this value as $R = \ell'\sqrt{N} = a\ell\sqrt{N}$. Note that again R becomes a function of g and u through a. Detailed analysis shows that UEM provides consistent size scaling if the repulsive interaction is important. This turns out to cover the complementary regions in coupling space to the previous variational calculation.

21.2.2 The Results and the Phase Diagram

We now determine the parameters q and a from the conditions (21.6) and $f(g, u; a) = 0$ with (21.9). As we are only interested in N-dependence of R, we assume the N dependence of the couplings, $g \sim N^{\xi}$ and $u \sim N^{\zeta}$, and solve the conditions. Both in (21.6) and (21.9), the terms independent of g and u correspond to the entropic force; diffusion and elasticity. The solutions will depend on which term is dominant in large N. For example, in (21.6), if u term is dominant over g term, the only possible solution is $q_0 = 0$ and we have a free configuration (recall that g and u are assumed positive). In this manner, we can determine the following four regions in which the different pair of interactions are dominant and balance. In the dimension $2 < \mathrm{d} < 4$, the results are summarized in the following list;

- Free polymer region: Here, entropic diffusion and elasticity balance, and gravity and repulsive interactions are negligible. This configuration is that of a free polymer, and the size scales as $R_0 \simeq \ell\sqrt{N}$.

- Puff-up region: Entropic elasticity and repulsive force balance. Newton gravity is a subleading effect, and the repulsive force is somehow strong, $u \geq N^{\frac{d-4}{2}}$. The size scales as $R \simeq \ell(uN^3)^{\frac{1}{d+2}}$, which is larger than R_0.
- "E–G" region: Entropic diffusion and gravity balance. The gravitational coupling is required to be larger than a critical value, $g \sim N^{\frac{d-6}{4}}$. The configuration shrinks and the size is $R \simeq \ell(g^2 N)^{\frac{1}{d-4}}$.
- "R–G" region: Repulsive force and gravity balance. This region is realized if both interactions are rather strong, and the size is given by $R \simeq \ell\sqrt{u}/g$.
- Black hole region: For large g, the Schwarzschild radius of the configuration, $R_s \simeq \ell(g^2 N)^{\frac{1}{d-2}}$, eventually exceeds the size R and we come into a "black hole" region.

The more details of analysis are presented in [14]. These behaviors are summarized in a "phase diagram" which is given by the left diagram in Fig. 21.1. The two axes are given by $\log_N g$ and $\log_N u$, and the boundaries of different regions turn out to be straight lines in this log–log plot. If u is fixed and g increases, the size R decreases and eventually coincides with the Schwarzschild radius R_s; the size at this corresponding point R_c is also shown in the figure. We observe that if the repulsive force is stronger than a critical value, $u \geq N^{-1}$, the corresponding black hole size R_c is no longer microscopic; it scales as N with a positive power. In $2 < \mathrm{d} < 4$, one can check that the size changes smoothly (in N dependence) as it goes across the borders.

As d approaches to 4, the "R–G" region eventually disappears and we obtain the right figure of Fig. 21.1. The size dependence on the couplings remains unchanged (apply $d = 4$ in the above relations). However, in this case, the size of the configuration may jump when it crosses the borderline on the right of the "free polymer" region (denoted as a slightly thick line).

21.3 Conclusion

In this article, we have summarized the size behavior of a long string under the effect of two interactions, Newton gravity and a contract repulsive interaction, by use the technique of polymer physics and two different approximation schemes. We find that the configuration exhibits a rather rich scaling behaviors with respect to the strength of the interactions, and the scaling behaviors are summarized in phase diagrams in Fig. 21.1. It is intriguing to see that the long string may collapse into a black hole of a macroscopic size if it exhibits sufficiently strong repulsive nature. Though the origin of such repulsive nature of strings is not clear yet, it would emerge in high density regime nonperturbatively [13], and it may enable us to study macroscopic black holes by using fundamental strings.

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Chapter 22
The Hot and Dense QCD Equation of State in Heavy Ion Collisions and Neutron Star Mergers

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Abstract The underlying open questions in the fields of general relativistic astrophysics and elementary particle and nuclear physics are strongly connected and their results are interdependent. Although the physical systems are quite different, the properties of a merged binary system of two neutron stars and the properties of the hot and dense matter created in high energy heavy ion collisions, strongly depend on the equation of state of fundamental elementary matter. Neutron star mergers represent optimal astrophysical laboratories to investigate the QCD phase structure using a spectrogram of the post-merger phase of the emitted gravitational waves. These studies can be supplemented by observations from heavy ion collisions to possibly reach a conclusive picture on the QCD phase structure at high density and temperature.

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P. Nicolini et al. (eds.), *2nd Karl Schwarzschild Meeting on Gravitational Physics*, Springer Proceedings in Physics 208,
https://doi.org/10.1007/978-3-319-94256-8_22

22.1 Introduction

Gravitational waves (GWs) have been recently observed from a pair of merging black holes (BHs) by the LIGO detectors [1] and GWs emitted from merging neutron star (NS) binaries are on the verge of their first detection. The main difference between GWs originating from a merger of two BHs or NSs is the possibility of the existence of a post-merger phase after the collisions of the two objects. The GWs produced by a merger of NSs are by far more interesting, as the equation of state (EOS) of elementary matter might be deduced by a frequency analysis of the GW [2–4]. This is insofar interesting, as the EOS of quantum chromo dynamics (QCD) until now is mainly investigated by high energy heavy ion collisions and only coarse constraints are coming from astrophysical observations, like the observed maximum mass in neutron stars, i.e., $2.01 \pm 0.04\, M_{\odot}$ [5]. We will discuss how one can create a similar state of hot and dense nuclear matter in two seemingly different 'experimental' setups, namely the mergers of two neutron stars and relativistic heavy ion collisions. Similarities and differences in the composition of the matter created are discussed. By studying the properties of this QCD matter in a single consistent approach we may finally address one of the most relevant challenges of high-energy nuclear theory. This is to determine the properties and phase structure of QCD at large densities and temperature.

22.2 Numerical General Relativity of Neutron Star Mergers

The basic equations which are used to describe the dynamics of neutron star mergers,

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \;=\; 8\pi\, T_{\mu\nu}\,, \quad \nabla_\mu T^{\mu\nu} \;=\; 0\,, \quad \nabla_\mu\left(\rho\, u^\mu\right) \;=\; 0 \quad , \tag{22.1}$$

include the equations of relativistic fluid dynamics, where the ideal-fluid energy-momentum tensor $T_{\mu\nu} = (e + p)\; u_\mu u_\nu + p\;\, g_{\mu\nu}$ introduces the pressure through the EOS of the underlying neutron star matter, $u^\mu = \mathrm{d}x^\mu/\mathrm{d}\tau$ describes the four velocity of the star's fluid which is defined as the derivative of the coordinates $x^\mu = (t, x, y, z)$ by the proper time τ. Here ∇_μ is the covariant derivative. These equations are complemented by Einsteins' equations of general relativity where $R_{\mu\nu}$ is the Ricci tensor, which contains first and second derivatives of the space-time metric $g_{\mu\nu}$. The Einstein equation (first equation in (22.1)), a highly non-linear differential equation, describes how matter moves in a curved space-time and formulates in which way the amount of energy-momentum curves the space-time structure.

In a similar way the dynamics of relativistic collisions of heavy ions can also be described by the equations of relativistic fluid dynamics, with the difference that gravitation does not play a role for the small systems created here. A common advantage for the description of both systems, NS mergers and heavy ion collisions, is that the EOS of dense and hot nuclear matter can be readily introduced in the equations of fluid dynamics. Therefore the EOS plays an integral part in both the description of neutron star mergers and heavy ion collisions.

The results of 3+1-dimensional numerical simulations of merging neutron star binaries in full general relativity [6] show that the emitted GWs of the merger and post-merger phase are strongly determined by the high density region of the equation of state. The underlying general-relativistic hydrodynamical evolution of the produced hypermassive neutron star depends on the hadronic and quark matter properties, i.e. the EOS, which also enters in the description of heavy ion collisions.

An even higher compression and heating of the neutron star matter can be achieved in a head on merger of two NS. Due to the considerable radial acceleration of the matter at the shock fronts of the colliding neutron star matter slabs, densities of several times the central density of the separated NS may be achieved. These velocities are close to the velocities and densities reached in relativistic heavy ion collisions at the GSI and FAIR accelerators, but below those at RHIC and the LHC.

The EOS used within most NS merger frameworks is usually composed of a cold nuclear-physics part and a thermal component (for examples see [3, 6, 7]). Within such a model, the hadronic phase is connected to deconfined quark matter above a certain transition rest-mass density ρ_{trans}. Alternatively an EOS of purely hadronic temperature dependent matter is used (see e.g. [8]). Such models are well established for the $T = 0$ (zero temperature) properties of dense nuclear matter. However, it is expected that QCD thermodynamics at finite temperature will quickly be dominated by additional degrees of freedom not present at vanishing temperature, i.e. mesons and/or gluons. In the description of neutron star mergers and relativistic heavy ion collisions a new class of EOS models is therefore necessary which also consistently describes QCD thermodynamics at finite temperature.

In the following we will introduce an EOS that can be used to describe symmetric and asymmetric nuclear matter at high densities and temperatures and explain how it is used to connect NS mergers and heavy ion collisions.

22.3 The Hot and Dense QCD Equation of State for Heavy Ion Collisions and Neutron Star Mergers

In heavy ion experiments at particle colliders, heavy nuclei are accelerated to relativistic velocities. As they collide, they create a small system (of several fm in size and a lifetime of approximately 20 fm/c) which is expected to have a Temperature of $T \geq 80\,\mathrm{MeV}$ and densities several times the nuclear ground state density. It is therefore very intriguing to study QCD matter at similar temperatures and densities in two rather different 'experimental' setups, in neutron star mergers and heavy ion collisions. By combining the findings from both observations one may be able to deduce information on the properties of the QCD matter at high densities and finally on the phase structure of QCD. The properties of the equation of state of QCD are the link connecting the neutron star mergers and relativistic nuclear collisions. Consequently the goal of such studies has to be to find a description for the EOS that is able to describe neutron star merger and nuclear collision observables and therefore establish the connection. From this description we can understand common features and differences of the systems created in heavy ion collisions and neutron star mergers.

First one has to establish common features and differences of the systems created. For that we estimated the expected maximal compression reached in nuclear collisions at different colliding beam energies. Since the very early stage of a nuclear collision is a very rapid and violent process, expected to take place out of thermal equilibrium, estimating the maximal compression is no unambiguous task. We compare the energy and net baryon densities reached, as function of the colliding beam energy per nucleon pair $\sqrt{s_{NN}}$ in Fig. 22.1. Different methods, not depending on the EOS, to estimate these densities are used, but give results of similar magnitude. The dashed lines follow from a simple geometric overlap model where one assumes that the total energy and baryon number of the colliding nuclei is completely stopped in a volume which is equal to the Lorentz contracted volume of a singe nucleus. The expected densities then can be written as:

$$\rho_{\text{ini}} = 2\, \gamma_{\text{c.m.}}\, \rho_0 \quad \text{and} \quad \varepsilon_{\text{ini}} = 2\, m_N\, \rho_0\, \gamma_{\text{c.m.}}^2 \tag{22.2}$$

where ρ is the net-baryon density, ε is the energy density, $\gamma_{\text{c.m.}}$ is the Lorentz gamma of the nuclei in the center of mass (c.m.) frame of the collision and m_N is the nucleon mass. The densities from the geometric overlap model serve as a lower bound of the expected densities since this simple approach does not take into account the additional compression which occurs as the two nuclei penetrate each other.
To get a more realistic estimate for the initial compression we also show results where a microscopic transport model is used to simulate the initial non-equilibrium compression stage (grey band in Fig. 22.1). The Ultrarelativistic Quantum Molecular Dynamics Model (UrQMD) is used in its cascade mode as well as a setup that includes nuclear interactions via potentials [9, 10]. To obtain a smooth density distribution from the microscopic model we run a large number of events and average densities over this event-ensemble. The so obtained values of the densities are generally larger, by up to a factor of 2, than the values of the overlap model. We find that is we want to study systems that have densities of approximately 4 times the nuclear ground state density $\rho \approx 0.6\, fm^3$ we have to study systems created at beam energies of $\sqrt{s_{\text{NN}}} \approx 2.5 - 3.0$ GeV. This is the energy region of the current SIS18 accelerator at GSI, as can be seen in Fig. 22.1.

As in the case of neutron star mergers the spatial density distribution in nuclear collisions is far from uniform. While in the center of the collision zone very high densities and temperatures can be obtained, steep spatial gradient of the densities are observed. To illustrate this Fig. 22.2 show contour plots of the net-baryon density and the corresponding temperatures for collisions of Au+Au nuclei at a fixed target beam energy of 1.5 A GeV, as expected for the SIS18 accelerator. This snap-shot of the densities as taken at a time $t = 15\, fm/c$, a time where one expects the system to be at least partially in local equilibrium. Again the bulk of the system reaches densities ranging from 1–4 time nuclear ground state density and temperatures from 50–100 MeV.

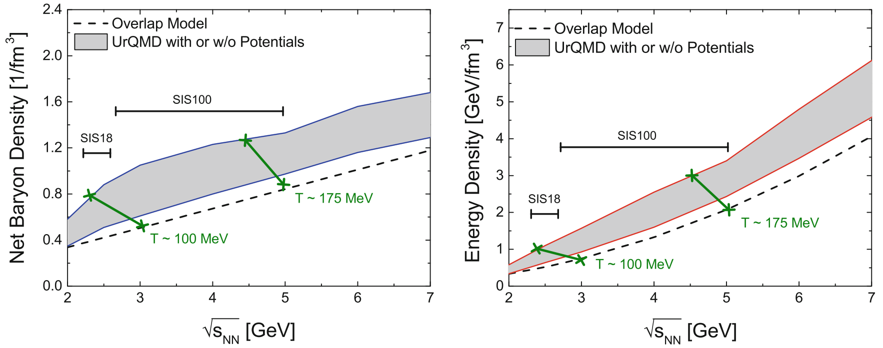

Fig. 22.1 Left: Largest net-baryon densities achieved in central collisions of Au+Au nuclei at different colliding beam energies. Right: Largest energy density achieved in central collisions of Au+Au nuclei at different colliding beam energies. For both figures we compare results from an overlap model (dashed black line) with results where we used the UrQMD model to estimate the initial compression. The green lines with crosses indicate the beam energies where we expect the maximal temperature to exceed 100 or 175 MeV. The temperatures are calculated using the QχP model described in the text

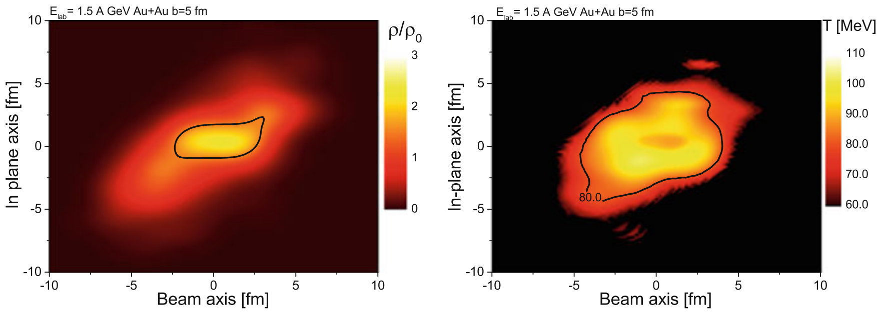

Fig. 22.2 Left: Net baryon density contour in the reaction plane of a non-central (b = 5 fm) nuclear collision of Au+Au nuclei at a beam energy of $E_{\mathrm{lab}} = 1.5$ A GeV. The contour where the density exceeds two times nuclear ground state density is highlighted (black dashed line). The densities where calculated using the UrQMD transport model. Right: Same as left but for the temperature. The temperature has been calculated from the density and energy density using the QχP model for the equation of state

In order to determine for example the temperature of the system at given densities, one requires knowledge on the effective degrees of freedom of the system, encoded in the equation of state. Depending on the EOS used, the Temperatures reached in these relativistic collisions and neutron star mergers may vary significantly. It is consequently most important to employ an EOS that entails a realistic set of degrees of freedom as well as interactions. In the following we will present a model for such an EOS which can be employed to describe the matter produced in neutron star mergers as well as heavy ion collisions, thus an EOS which is able to link the properties of collision events in drastically different environments.

22.3.1 The QχP Model

The model we employed is the so called Quark-Hadron Chiral Parity Doublet Model (QχP) [11, 12]. In this approach, an explicit mass term for baryons in the Lagrangian is possible, which preserves chiral symmetry. Here, the signature for chiral symmetry restoration is the degeneracy of the usual baryons and their respective negative-parity partner states. In the model approach, positive and negative parity states of the SU(3)$_f$ baryons are grouped in doublets $N = (N^+, N^-)$ as discussed in [13, 14].

Taking into account the scalar and vector condensates in mean-field approximation, the resulting Lagrangian includes the scalar meson interaction, driving the spontaneous breaking of the chiral symmetry, is expressed in terms of SU(3) invariants $I_2 = (\sigma^2 + \zeta^2),\ I_4 = -(\sigma^4/2 + \zeta^4)$ and $I_6 = (\sigma^6 + 4\zeta^6)$ as:

$$V = V_0 + \frac{1}{2}k_0 I_2 - k_1 I_2^2 - k_2 I_4 + k_6 I_6\ , \tag{22.3}$$

where V_0 is fixed by demanding a vanishing potential in the vacuum. The quark and gluonic degrees of freedom are introduced as done in the PNJL approach [15, 16]. This model uses the Polyakov loop Φ as the order parameter for de-confinement. To suppress hadrons in the deconfined phase we also introduced a simple excluded volume for the hadrons. The various parameters of this model are fixed by demanding a reasonable description of nuclear ground state properties like the saturation density, binding energy and symmetry energy. Furthermore if this model is extended to finite temperature and vanishing chemical potentials it gives a reasonable qualitative description of lattice QCD thermodynamics. A more detailed description of this model, can be found in [11, 12].

This model is highly qualified to study the properties of matter at high densities and intermediate temperatures, as expected in nuclear collisions as well as neutron star mergers. A straight forward way of consistently connecting the features of the EOS with the maximally achievable compression of a relativistic collision is by employing the so called Rankine–Hugoniot–Taub-Adiabat [17]. The Taub-Adiabat is essentially a shock wave solution of two colliding infinite slaps of matter. If the EoS, i.e. the connection between pressure, energy density and baryon density is know (as p(ε,ρ)), then one can calculate the maximum compression in a collision by solving the following Taub-equation:

$$(\rho_0\, X_0)^2 - (\rho\, X)^2 - (p_0 - p)(X_0 + X) = 0 \tag{22.4}$$

with $X = (\varepsilon + p)/\rho^2$, the generalized volume. For simplicity we assume $p_0 = 0$. One can furthermore connect the center of mass gamma factor $\gamma_{\text{c.m.}}$ of the colliding slabs to the densities created using $\gamma_{\text{c.m.}}^2 = \left(\frac{\varepsilon\ \rho_0}{\rho\ \varepsilon_0}\right)^2$. The resulting beam energy dependence of the net-baryon density and temperatures reached is shown in Fig. 22.3 (left) for two different scenarios:

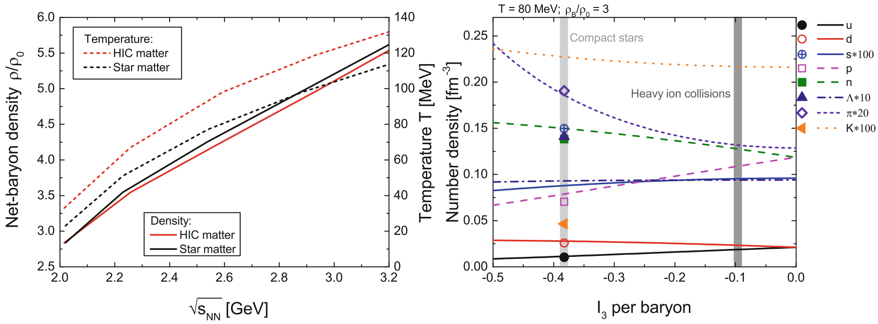

Fig. 22.3 Left: Largest net-baryon density (solid lines) and Temperatures (dashed lines) achieved in collisions of heavy ions and compact stars at a given center of mass beam energy $\sqrt{s_{\rm NN}} = 2 \cdot \gamma_{\rm c.m.} \cdot m_N$. To calculate the densities and temperatures we used the Taub adiabat (see text) with the QχP EoS. Due to the different properties of the EoS as function of iso-spin the temperatures in heavy ion collisions are larger and densities slightly smaller, at the same relative velocities. Right: Number densities of different hadronic and free quark species (rescaled for visibility) as function of the iso-spin per baryon at a fixed temperature T $=$ 80 MeV and net-baryon density $\rho = 3\rho_0$. The lines correspond to matter with conserved net strangeness, as expected for heavy ion collisions while the symbols represent results where the matter is on β-equilibrium (as expected for neutron star matter

1. The EOS for heavy ion collisions, i.e. with conserved strangeness and no beta-equilibrium
2. The EOS for compact stars, i.e. in beta-equilibrium

Figure 22.3 therefore presents, for different accelerator energies, the compressions achieved, using the relativistic one dimensional hydrodynamic Rankine–Hugoniot–Taub-Adiabat with a realistic EOS for hot and dense nuclear matter. This figure highlights the difference in the compression in collisions of projectiles with iso-spin symmetric matter (heavy ion collisions) and asymmetric matter (NS mergers).

To point out similarities and differences in the chemical composition of the systems created in these collisions we show in Fig. 22.3 (right) the number densities of different hadronic species at a fixed temperature and net-baryon density, as function of the iso-spin per baryon of the system. The composition of the systems created in collisions of Au+Au nuclei (where the iso-spin per baryon is -0.1) and of neutron star matter is quite different, as expected. The matter in neutron stars not only has a an iso-spin per baryon of -0.38 but also, according to beta equilibrated strangeness, a significantly different composition of strange particles.

22.4 Conclusions

In this article we show that the properties of elementary matter at high temperatures ($T \approx 100$ MeV) and densities ($\rho \approx 3\,\rho_0$) can be studied in two different physical scenarios [18]. High energy heavy ion collision experiments try do determine the phase structure of the iso-spin symmetric QCD equation of state, and the knowledge of the iso-spin asymmetric QCD EOS is needed in a general relativistic computer simulation of binary neutron star mergers. These two different fields of physics, namely elementary particle physics and astrophysics, combine when two neutron stars collide. It is therefore possible to study the properties of dense QCD for systems of different size, time-scales and chemical composition, which will eventually lead to an understanding of the properties of this elementary form of matter.

Acknowledgements Support comes from "NewCompStar", COST Action MP1304, from the LOEWE-Program in HIC for FAIR, and from the European Union's Horizon 2020 Research and Innovation Programme (Grant 671698) (call FETHPC-1-2014, project ExaHyPE). MH gratefully acknowledges support from the Frankfurt Institute for Advanced Studies (FIAS) and the Goethe University Frankfurt, while HS acknowledges the Judah M. Eisenberg laureatus Professur endowment. The simulations were performed on LOEWE at CSC-Frankfurt and on Hazelhen at HLRS in Stuttgart.

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