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Atlantis Studies in Variational Geometry
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Introduction to Global Variational Geometry

Atlantis Studies in Variational Geometry

Volume 1

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ISSN 2214-0700 ISSN 2214-0719 (electronic)
ISBN 978-94-6239-072-0 ISBN 978-94-6239-073-7 (eBook)
DOI 10.2991/978-94-6239-073-7

Library of Congress Control Number: 2014945965

Printed on acid-free paper

Preface

The *global variational geometry* as introduced in this book is a branch of mathematics, devoted to extremal problems on the frontiers of differential geometry, global analysis, the calculus of variations, and mathematical physics. Its subject is, generally speaking, a geometric structure consisting of a smooth manifold endowed with a differential form.

More specifically, by a *variational structure*, or a *Lagrange structure*, we mean in this book a pair (Y, ρ), where Y is a *smooth fibered manifold* over an *n-dimensional* base manifold X and ρ a *differential n-form*, defined on the r-jet prolongation J^rY of Y. The forms ρ, satisfying a *horizontality condition*, are called the *Lagrangians*. The *variational functional*, associated with (Y, ρ), is the real-valued function $\Gamma_\Omega(\pi) \ni \gamma \rightarrow \rho_\Omega(\gamma) = \int J^r\gamma^*\rho \in \mathbf{R}$, where $\Gamma_\Omega(\pi)$ is the set of sections of Y over a compact set $\Omega \subset X$, $J^r\gamma$ is the r-jet prolongation of a section γ, and $J^r\gamma^*\rho$ is an n-form on X, the pull-back of ρ by $J^r\gamma$.

Over the past few decades the subject has developed to a self-contained theory of extremals of *integral variational functionals* for sections of fibered manifolds, invariance theory under transformations of underlying geometric structures, and differential equations related to them. The *variational methods* for the study of these functionals extended the corresponding notions of global analysis such as differentiation and integration theory on manifolds. Innovations appeared in the developments of *topological methods* needed for a deeper understanding of the global character of variational concepts such as equations for extremals and conservation laws. It has also become clear that the higher order variational functionals could hardly be studied without innovations in the multi-linear algebra, namely in the decomposition theory of tensors and differential forms by the trace operation.

The resulting theory differs in many aspects from the classical approach to variational problems: The underlying *Euclidean spaces*, are replaced by *smooth manifolds* and *fibered spaces*, the classical *Lagrange functions* and their variations are replaced by *Lagrange differential forms* and their *Lie derivatives*, etc. Within the classical setting, a (first order) variational structure is a pair (Y, λ), where $Y = J^1(\mathbf{R}^n \times \mathbf{R}^m)$ is the 1-jet prolongation of the product $\mathbf{R}^n \times \mathbf{R}^m$ of Euclidean spaces, and in the canonical coordinates, $\lambda = Ldx^1 \wedge dx^2 \wedge \cdots \wedge dx^n$, where $L : \mathbf{R}^n \times \mathbf{R}^m \times \mathbf{R}^{nm} \to \mathbf{R}$ is a Lagrange function, depending on n independent variables, m dependent variables, and nm partial derivatives of dependent variables.

Basic geometric ideas allowing us to globalize the classical calculus of variations come from the concepts of E. Cartan [C] in the calculus of variations of simple integrals, and especially from the work of Lepage (see e.g. [Le]). Further developments after Cartan and Lepage have led to a deeper understanding of the structure and geometric nature of general variational procedures and their compatibility with manifold structures. Main contributors to the global theory are Dedecker [D] (geometric approach to the calculus of variations, regularity), Garcia [G] (Poincare-Cartan form, invariant geometric operations, connections), Goldschmidt and Sternberg [GS] (Cartan form, vector-valued Euler–Lagrange form, Hamilton theory, Hamilton–Jacobi equation), Krupka [K13], [K1] (Lepage forms, higher order variational functionals, infinitesimal first variation formula, Euler–Lagrange form, invariance), and Trautman [Tr1, Tr2] (invariance of Lagrange systems, Noether's theory).

This book covers the subjects that are considered as basic in the classical monographs on the (local) calculus of variations on Euclidean spaces: variational functionals and their variations, the (first) variation formula, extremals and the Euler–Lagrange equations, invariance and conservation laws. We study these topics within the framework of much broader underlying structures, smooth manifolds. This requires, in particular, a systematic use of analysis and topology of manifolds. In addition, new questions appear in this framework such as for instance *global existence* of the notions, constructed in charts. We also study global properties of the Euler–Lagrange mapping; to this purpose two chapters devoted to sheaves and the variational sequence theory are included. It is however obvious that these themes do not reflect the foundations of the global variational theory completely. Further comprehensive expositions including applications, based on modern geometric methods in the calculus of variations on manifolds, can be found in the monographs Giachetta, Mangiarotti and Sardanashvily [GMS1], [GMS2], De Leon and Rodrigues [LR], Mangiarotti and Modugno [MM], and Mei Fengxiang and Wu Huibin [MW]. For orientation in recent research in these fields we refer to Krupka and Saunders [KS].

The text of the book requires a solid background in topology, multi-linear algebra, and differential and integral calculus on manifolds; to this purpose we recommend the monograph Lee [L]. Essentials of the classical and modern calculus of variations can be found e.g. in Gelfand and Fomin [GF], Jost and Li-Jost [JL], and in the handbook Krupka and Saunders [KS], where differential forms are considered. For the theory of jets, natural bundles and applications we refer to original works of Ehresmann [E] and to the books Kolar, Michor and Slovak [KMS], Krupka and Janyska [KJ], and Saunders [S]. We also need an elementary sheaf theory; our exposition extends a chapter of the book Wells [We]. For reference, some theorems and formulas are collected in the Appendix. We should especially mention the section devoted to the *trace decomposition theory* on real vector spaces, which is needed for the decomposition of differential forms on jet manifolds (Krupka [K15]); although the trace decomposition is an elementary topic, it is difficult to find an adequate reference in classical and contemporary algebraic literature.

Chapter 1 covers fundamentals of fibered manifolds and their jet prolongations. The usual topics related to the jet structure, such as the horizontalization morphism, jet prolongations of sections and morphisms of fibered manifolds, and prolongations of vector fields are introduced. It should be pointed out that the vector fields and their jet prolongations represent a geometric, coordinate-free construction, replacing in the global variational theory the classical "variations of functions", and "induced variations" of their derivatives.

Chapter 2 studies differential forms on the jet prolongations of fibered manifolds. The *contact forms* are introduced, generating a *differential ideal* of the exterior algebra, and the corresponding decompositions of forms are studied. It is also shown that the *trace operation*, acting on the components of forms, leads to a decomposition related to the exterior derivative of forms. The meaning of the structure theorems for the global variational theory, explained in the subsequent chapters, consists in their variational interpretation; in different situations the decompositions lead to the Lagrangian forms, the source forms, the Helmholtz forms, etc.

Chapter 3 is devoted to the *formal divergence equations* on jet manifolds, a specific topic that needs independent exposition. It is proved that the integrability of these equations is equivalent with the vanishing of the Euler–Lagrange operator.

The objective of Chaps. 4–6 is to study the behaviour of the variational functional $\Gamma_\Omega(\pi) \ni \gamma \to \rho_\Omega(\gamma) = \int J^r\gamma^*\rho \in \mathbf{R}$ with respect to the variable γ. But in general, the domain of definition $\Gamma_\Omega(\pi)$ has *no* natural algebraic and topological structures; this fact prevents an immediate application of the methods of the differentiation theory in topological vector spaces, based on the concept of the derivative of a mapping. However, even when *no* topology on $\Gamma_\Omega(\pi)$ has been introduced, the *geometric*, or *variational* method to investigate the functional ρ_Ω can still be used: we can always vary (deform) each section $\gamma \in \Gamma_\Omega(\pi)$ within the set $\Gamma_\Omega(\pi)$, and study the induced variations (deformations) of the value $\rho_\Omega(\gamma)$.

The key notions in Chap. 4 are the *variational derivative*, *Lepage form*, the *first variation formula*, *Euler–Lagrange form*, *trivial Lagrangian*, *source form, Vainberg–Tonti Lagrangian*, and the *inverse problem of the calculus of variations* and the *Helmholtz expressions*.

The exposition begins with the description of variations of sections of the fibered manifold *Y*, considered as vector fields, and the *induced variations* of the variational functional $\int J^r\gamma^*\rho$. It turns out in this geometric setting that the induced variations are naturally characterized by the *Lie derivative* of ρ. An immediate consequence of this observation is that one can study the functional ρ_Ω by means of the differential calculus of forms and vector fields on the underlying jet manifold.

Next we introduce the fundamental concept of the global variational theory on fibered manifolds, a *Lepage form*. We prove that to any variational structure (Y, ρ) there always exists an *n*-form Θ_ρ with the following two properties: first, the form Θ_ρ defines the same integral variational functional as the form ρ, that is, the identity

$J^r\gamma^*\rho = J^r\gamma^*\Theta_\rho$ holds for all sections γ of the fibered manifold Y, and second, the exterior derivative $d\Theta_\rho$ defines *equations for the extremals*, thus, γ is an extremal if and only if $d\Theta_\rho$ vanishes along $J^r\gamma$. Any form Θ_ρ is called a *Lepage equivalent* of the form ρ.

As a basic consequence of the existence of Lepage equivalents we derive a geometric, coordinate-free analog of the classical (integral) first variation formula – the *infinitesimal first variation formula*, which is essentially the Lie derivative formula for the form Θ_ρ with respect to the vector fields defining the induced variations. The *infinitesimal first variation formula* becomes a main tool for further investigation of extremals and symmetries of the functional. It should also be noted that the geometric structure of the formula admits immediate extensions to *second* and *higher* variations.

We may say that these two properties defining Θ_ρ explain the meaning of the *first* and *second Lepage congruences*, considered by Lepage and Dedecker in their study of the classical variational calculus for submanifolds (cf. Dedecker [D]).

The exterior derivative $d\Theta_\rho$ splits in two terms, one of them, characterizing extremals, is a (globally well-defined) differential form, the *Euler–Lagrange form*; its components in a fibered chart are the well-known *Euler–Lagrange expressions*. The corresponding system of partial differential equations, *Euler–Lagrange equations*, are then related to each fibered chart. Solving these equations requires their analysis in any concrete case from the local and global viewpoints.

Next we study in Chap. 4 the structure of the *Euler–Lagrange mapping*, assigning to a Lagrangian its Euler–Lagrange form. Since the Euler–Lagrange mapping is a morphism of Abelian groups of differential forms on the underlying jet spaces, its basic characteristics include descriptions of its *kernel* and its *image*. We describe these spaces by their *local* properties.

The *kernel* consists of *variationally trivial* Lagrangians – the Lagrangians whose Euler–Lagrange forms vanish identically. These Lagrangians are characterized in terms of the exterior derivative operator d; their local structure corresponds with the classical *divergence expressions*. The *global structure* depends on the topology of the underlying fibered manifold Y, and is studied in Chap. 8.

The problem of how to characterize the *image* of the Euler–Lagrange mapping is known as the *inverse problem of the calculus of variations*. Its simple coordinate version for systems of partial differential equations consists in searching for conditions when the given equations coincide with the Euler–Lagrange equations of some Lagrangian. On a fibered manifold, the inverse problem is formulated for a *source form*, defined on J^rY; it is required that the *components* of the source form coincide with the Euler–Lagrange expressions of a Lagrangian. We find the obstructions for variationality of source forms by means of the Lagrangians of Vainberg–Tonti type, constructed by a fibered homotopy operator, and used for the first time by Vainberg [V]. The resulting theorem gives the necessary and sufficient local variationality conditions in terms of the *Helmholtz expression* (cf. Anderson and Duchamp [AD] and Krupka [K8, K11]).

Chapter 5 is devoted to variational structures whose Lagrangians, or Euler–Lagrange forms, admit some invariance transformations. The *invariance transformations* are defined naturally as the transformations preserving a given differential form; this immediately leads to criteria for a vector field to be the *generator* of these transformations. Then we prove a generalization of the *Noether's theorem* for a given variational structure (Y, ρ), relating the generators of invariance transformations of ρ with the existence of *conservation laws* for the solutions of the system of Euler–Lagrange equations. The theory extends the well-known classical results on invariance and conservation laws originally formulated for multiple-integral variational problems in Euclidean spaces (Noether [N]).

It should be noted that the invariance theorems for variational structures as stated in this book become comparatively simple (compare with Olver [O1], where a complete classical approach is given). The reason can be found in the fundamental concepts of the theory of variational structures – differential forms, for which invariance theorems are formulated. To explain the basic ideas, consider a manifold Y of dimension p endowed with a differential n-form ρ. Then for any vector field ξ on Y, the *Lie derivative* $\partial_\xi \rho$ can be expressed by the *Cartan's formula* $\partial_\xi \rho = i_\xi d\rho + d i_\xi \rho$, where i_ξ is the *contraction* of ρ by the vector field by ξ and d is the *exterior derivative*. Then for any mapping $f : X \to Y$, where X is a manifold of dimension n, the Lie derivative satisfies $f^* \partial_\xi \rho = f^* i_\xi d\rho + df^* i_\xi \rho$. Thus, if ρ is *invariant* with respect to ξ, that is, $\partial_\xi \rho = 0$, we have $f^* i_\xi d\rho + df^* i_\xi \rho = 0$. If in addition f satisfies the equation $f^* i_\xi d\rho = 0$, then f necessarily satisfies the *conservation law equation* $df^* i_\xi \rho = 0$ (Noether's theorem). Similar conservation law theorems for variational structures on jet manifolds are proved along the same lines.

In Chap. 6 we consider a few examples of *natural* variational structures as introduced in Krupka [K10] (for natural variational principles on Riemannian manifolds see Anderson [A1]). Main purpose is to establish basic (global) structures and find the corresponding Lepage forms. The *Hilbert variational functional* for the metric fields on a manifold (Hilbert [H]) and a variational functional for connections are briefly discussed. The approach should be compared with the standard formulation of the variational principles of the general relativity and other field theories. Clearly, these examples as well as many others whose role are variational principles of physics need a more complex and more detailed study.

As mentioned above, the theory of variational structures gives rise to the *Euler–Lagrange mapping*, which assigns to an n-form λ, a *Lagrangian*, an $(n+1)$-form E_λ, the *Euler–Lagrange form* associated with λ. Its definition results from the properties of the exterior derivative operator d, an appropriate canonical decomposition of underlying spaces of forms, and from the concept of a Lepage form (cf. Krupka [K1]). On this basis we easily come to the basic observation that the Euler–Lagrange mapping can be included in a differential sequence of Abelian sheaves as one of its arrows. We proceed to introduce the sequence and the associated complex of global sections, and to study on this basis *global properties* of the Euler–Lagrange mapping.

To this purpose we first explain in Chap. 7 elements of the sheaf theory (see e.g. Wells [We]). Attension is paid to those theorems, which are needed for the variational structures; complete proofs of these theorems are included. In particular, the formulation and proof of the *abstract De Rham theorem* is given.

The *variational geometry* is devoted to geometric, coordinate-independent properties of ρ_Ω. In particular, the geometric problems include the study of *critical points* (or *extremals*) of the variational functionals; their *maxima* and *minima*, where a topology on $\Gamma_\Omega(\pi)$ is needed, are not considered. Many other typical geometric problems are connected with various kinds of symmetries of the variational functionals and the corresponding equations for the extremals. The problem of restricting a given functional defined, say, on a Euclidean space, to a submanifold (the *constraint submanifold*) is obviously included in this framework.

It should be pointed out that the geometric variational theory completely covers the problems, related with the variational principles in *physical field theory* and *geometric mechanics*, where concrete underlying geometric structures and variational functionals are considered.

Chapter 8 is devoted to the *variational sequence* of order r for a fibered manifold Y. Its construction has *no* a priori relations with the theory of variational structures. The sequence is established on the observation that the *De Rham sequence* of differential forms on the r-jet prolongation J^rY has a remarkable *subsequence*, defined by the *contact forms*; the variational sequence is then defined to be the *quotient sheaf sequence* of the De Rham sheaf sequence (see Krupka [K19]).

With the obvious definition of the quotient groups, we denote the variational sequence as $0 \to \mathbf{R}_Y \to \Omega_0^r \to \Omega_1^r/\Theta_1^r \to \Omega_2^r/\Theta_2^r \to \Omega_3^r/\Theta_3^r \to \cdots$. Its properties relevant to the calculus of variations can be divided into two parts:

(a) *Local properties*, represented by theorems on the structure of the classes of forms in the quotient sequence and morphisms between these quotient groups:

- the *classes* $[\rho]$ *of n-forms* $\rho \in \Omega_n^r$, where n is the dimension of the base X of the fibered manifold Y, can canonically be identified with *Lagrangians* for the fibered manifold Y,
- the classes $[\rho]$ of $(n+1)$-forms $\rho \in \Omega_{n+1}^r$ can canonically be identified with the *source forms*,
- the quotient morphism $E_n : \Omega_n^r/\Theta_n^r \to \Omega_{n+1}^r/\Theta_{n+1}^r$ is exactly the *Euler–Lagrange mapping* of the calculus of variations,
- the quotient morphism $E_{n+1} : \Omega_{n+1}^r/\Theta_{n+1}^r \to \Omega_{n+2}^r/\Theta_{n+2}^r$ is exactly the *Helmholtz mapping* of the calculus of variations.

All these classes and morphisms are described *explicitly* in fibered charts; their expressions coincide with the corresponding expressions given in Chap. 4. Thus, the variational sequence allows us to *rediscover* basic variational concepts from abstract structure constructions on the jet manifolds J^rY.

(b) *Global properties*, represented by the theorem on the cohomology of the *complex of global sections* of the variational sequence; this implies, on the basis of the De Rham theorem that:

- there exists an isomorphism between the cohomology groups of the complex of global sections and the De Rham cohomology groups,
- the obstructions for global variational triviality of Lagrangians lie in the cohomology group $H^n Y$, where $n = \dim X$,
- the obstructions for global variationality of source forms lie in the cohomology group $H^{n+1} Y$.

We also provide a list of manifolds Y and its cohomology groups, which allows us to decide whether local variational triviality of a Lagrangian, resp. local variationality of a source form, necessarily implies its global triviality, resp. global variationality.

This book originated from my research in global variational geometry and from numerous courses and lectures at different universities and international summer schools. Its first five chapters, essentially extending original notes, have been written during my stay at Beijing Institute of Technology under a key programme of National Science Foundation of China (grant No. 10932002). I am deeply indebted to BIT for the excellent conditions and fruitful scientific atmosphere during my work at the School of Mathematics. Especially I would like to thank Prof. Donghua Shi for generous collaboration and kind hospitality, and to Prof. Huafei Sun and Prof. Yong-xin Guo for fruitful discussions and support.

I also highly appreciate research conditions, created for me by Prof. Michal Lenc, head of the Department or Theoretical Physics and Astrophysics, while working on the manuscript at my *Alma Mater* Masaryk University in Brno. Without his personal support this work could hardly be completed.

It remains for me to acknowledge the help I have received in preparing the manuscript of this book. I am especially indebted to Zhang Chen Xu and Kong Xin Lei from BIT who read very carefully a large part of the text, pointed out mistakes and suggested improvements.

Levoca, May 2014 D. Krupka

References

[A1] I. Anderson, Natural variational principles on Riemannian manifolds, Annals of Mathematics 120 (1984) 329-370

[AD] I. Anderson, T. Duchamp, On the existence of global variational principles, Am. J. Math. 102 (1980) 781-867

[C] E. Cartan, *Lecons sur les Invariants Intégraux*, Hermann, Paris, 1922

[D] P. Dedecker, On the generalization of symplectic geometry to multiple integrals in the calculus of variations, in: *Lecture Notes in Math.* 570, Springer, Berlin, 1977, 395-456

[E] C. Ehresmann, Les prolongements d'une variete différentiable I. –V., C. R. Acad. Sci. Paris 223 (1951) 598-600, 777-779, 1081-1083; 234 (1952) 1028-1030, 1424-1425

[G] P.L. Garcia, The Poincare-Cartan invariant in the calculus of variations, Symposia Mathematica 14 (1974) 219-246

[GF] I.M. Gelfand, S.V. Fomin, *Calculus of Variations*, Prentice Hall, New Jerseey, 1967

[GMS1] G. Giachetta, L. Mangiarotti, G. Sardanashvily, *Advanced Classical Field Theory*, World Scientific, 2009

[GMS2] G. Giachetta, L. Mangiarotti, G. Sardanashvily, *New Lagrangian and Hamiltonian Methods in Field Theory*, World Scientific, 1997

[GS] H. Goldschmidt, S. Sternberg, The Hamilton-Cartan formalism in the calculus of variations, Ann. Inst. H. Poincare 23 (1973) 203-267

[H] P. Havas, The range of applicability of the Lagrange formalism. I, Nuovo Cimento 5 (1957) 363-383

[JL] J. Jost, X. Li-Jost, *Calculus of Variations*, Cambridge Univ. Press, Cambridge, 1998

[K1] D. Krupka, A geometric theory of ordinary first order variational problems in fibered manifolds, I. Critical sections, II. Invariance, J. Math. Anal. Appl. 49 (1975) 180-206, 469-476

[K8] D. Krupka, Lepagean forms in higher order variational theory, in: *Modern Developments in Analytical Mechanics*, Proc. IUTAM-ISIMM Sympos., Turin, June 1982, Academy of Sciences of Turin, 1983, 197-238

[K10] D. Krupka, Natural Lagrange structures, in: *Semester on Differential Geometry*, 1979, Banach Center, Warsaw, Banach Center Publications, 12, 1984, 185-210

[K11] D. Krupka, On the local structure of the Euler–Lagrange mapping of the calculus of variations, in: O. Kowalski, Ed., *Differential Geometry and its Applications*, Proc. Conf., N. Mesto na Morave, Czechoslovakia, Sept. 1980; Charles University, Prague, 1981, 181-188; arXiv:math-ph/0203034, 2002

[K13] D. Krupka, *Some Geometric Aspects of Variational Problems in Fibered Manifolds*, Folia Fac. Sci. Nat. UJEP Brunensis, Physica 14, Brno, Czech Republic, 1973, 65 pp.; arXiv: math-ph/0110005, 2001

[K15] D. Krupka, Trace decompositions of tensor spaces, Linear and Multilinear Algebra 54 (2006) 235-263

[K19] D. Krupka, Variational sequences on finite-order jet spaces, Proc. Conf., World Scientific, 1990, 236-254

[KJ] D. Krupka, J. Janyska, *Lectures on Differential Invariants*, J.E. Purkyne University, Faculty of Science, Brno, Czechoslovakia, 1990

[KMS] I. Kolar, P. Michor, J. Slovak, *Natural Operations in Differential Geometry*, Springer-Verlag, Berlin, 1993

[KS] D. Krupka, D. Saunders, Eds., *Handbook of Global Analysis*, Elsevier, 2008

[K-S] Y. Kosmann-Schwarzbach, *The Noether Theorems*, Springer, 2011

[L] J.M. Lee, *Introduction to Smooth Manifolds*, Graduate Texts in Math. 218, Springer, 2006

[Le] Th.H.J. Lepage, Sur les champs géodésiques du calcul des variations, I, II, Bull. Acad. Roy. Belg. 22 (1936), 716–729, 1036-1046

[LR] M. De Leon, P.R. Rodrigues, *Generalized Classical Mechanics and Field Theory: A geometric approach of Lagrangian and Hamiltonian formalisms involving higher order derivatives*, Elsevier, 2011

[MM] L. Mangiarotti, M. Modugno, Some results of the calculus of variations on jet spaces, Annales de l'Institut Henri Poincaré (A) Physique théorique (1983) 29-43

[MW] Mei Fengxiang, Wu Huibin, *Dynamics of Constrained Mechanical Systems*, Beijing Institute of Technology Press, 2009

[N] E. Noether, Invariante Variationsprobleme, Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse (1918) 235-257

[O1] P.J. Olver, *Applications of Lie Groups to Differential Equations*, Springer-Verlag, New York, 1998

[S] D.J. Saunders, *The Geometry of Jet Bundles*, Cambridge Univ. Press, 1989

[V] M.M. Vainberg, *Variational Methods for the Study of Nonlinear Operators* (in Russian), Gostekhizdat, Moscow, 1956; English translation: Holden-Day, San Francisco, 1964

[Tr1] A. Trautman, Invariance of Lagrangian systems, in: *General Relativity*, Papers in Honour of J.L. Synge, Oxford, Clarendon Press, 1972, 85-99

[Tr2] A. Trautman, Noether equations and conservation laws, Commun. Math. Phys. 6 (1967) 248-261

[We] R.O. Wells, *Differential Analysis on Complex Manifolds*, Springer-Verlag, New York, 1980

Contents

Chapter 1
Jet Prolongations of Fibered Manifolds

This chapter introduces fibered manifolds and their jet prolongations. First, we recall properties of differentiable mappings of constant rank and introduce, with the help of rank, the notion of a fibered manifold. Then, we define automorphisms of fibered manifolds as the mappings preserving their fibered structure. The r-jets of sections of a fibered manifold Y, with a fixed positive integer r, constitute a new fibered manifold, the *r-jet prolongation* J^rY of Y; we describe the structure of J^rY and a canonical construction of automorphisms of J^rY from automorphisms of the fibered manifold Y, their *r-jet prolongation*. The prolongation procedure immediately extends, via flows, to vector fields. For this background material, we refer to Krupka [K17], Lee [L], and Saunders [S].

These concepts are prerequisites for the geometric definition of *variations of sections* of a fibered manifold, extending the corresponding notion used in the classical multiple-integral variational theory on Euclidean spaces to smooth fibered manifolds.

1.1 The Rank Theorem

Recall that the *rank* of a linear mapping $u: E \to F$ of vector spaces is defined to be the dimension of its image space, rank $u = \dim \operatorname{Im} u$. This definition applies to tangent mappings of differentiable mappings of smooth manifolds. Let $f: X \to Y$ be a C^r mapping of smooth manifolds, where $r \geq 1$. We define the *rank* of f *at a point* $x \in X$ to be the rank of the tangent mapping $T_x f: T_x X \to T_{f(x)} Y$. We denote

$$\operatorname{rank}_x f = \dim \operatorname{Im} T_x f. \tag{1}$$

The function $x \to \operatorname{rank}_x f$, defined on X, is the *rank function*.

Elementary examples of real-valued functions f of one real variable show that the rank function is not, in general, locally constant. Our main objective in this section is to study differentiable mappings whose rank function *is* locally constant.

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Atlantis Studies in Variational Geometry 1, DOI 10.2991/978-94-6239-073-7_1

First, we prove a manifold version of the constant rank theorem, a fundamental tool for a classification of differentiable mappings. The proof is based on the rank theorem in Euclidean spaces (see Appendix 3) and a standard use of charts on a smooth manifold.

Theorem 1 (Rank theorem) *Let* X *and* Y *be two manifolds,* $n = \dim X, m = \dim Y$, *and let* q *be a positive integer such that* $q \le \min(n, m)$. *Let* $W \subset X$ *be an open set, and let* $f: W \to Y$ *be a* C^r *mapping. The following conditions are equivalent*:

(1) *f has constant rank on W equal to q.*

(2) *To every point $x_0 \in W$, there exists a chart (U, φ), $\varphi = (x^i)$ at x_0, an open rectangle $P \subset \mathbf{R}^n$ with center 0 such that $\varphi(U) = P$, $\varphi(x_0) = 0$, a chart (V, ψ), $\psi = (y^\sigma)$, at $y_0 = f(x_0)$, such that $f(U) \subset V$, and an open rectangle $Q \subset \mathbf{R}^m$ with center 0 such that $\psi(V) = Q$, $\psi(y_0) = 0$, and*

$$y^\sigma \circ f = \begin{cases} x^\sigma, & \sigma = 1, 2, \ldots, q, \\ 0, & \sigma = q+1, q+2, \ldots, m. \end{cases} \tag{2}$$

Proof

1. Suppose that f has constant rank on W equal to q. We choose a chart $(\bar{U}, \bar{\varphi})$, $\bar{\varphi} = (\bar{x}^i)$, at x_0, and a chart $(\bar{V}, \bar{\psi})$, $\bar{\psi} = (\bar{y}^\sigma)$, at y_0, and set $g = \bar{\psi} f \bar{\varphi}^{-1}$; g is a C^r mapping from $\bar{\varphi}(\bar{U}) \subset \mathbf{R}^n$ into $\bar{\psi}(\bar{V}) \subset \mathbf{R}^m$. Since for every tangent vector $\xi \in T_x X$ expressed as

$$\xi = \bar{\xi}^i \left(\frac{\partial}{\partial \bar{x}^i} \right)_x, \tag{3}$$

we have

$$T_x f \cdot \xi = D_i(\bar{y}^\sigma f \bar{\varphi}^{-1})(\bar{\varphi}(x)) \bar{\xi}^i \left(\frac{\partial}{\partial \bar{y}^\sigma} \right)_{f(x)}, \tag{4}$$

the rank of f at x is rank $T_x f = \operatorname{rank} D_i(\bar{y}^\sigma f \bar{\varphi}^{-1})(\bar{\varphi}(x)$. Consequently, the rank of f is constant on the open set $\bar{\varphi}(\bar{U}) \subset \mathbf{R}^n$ and is equal to q. Shrinking $\bar{U}$ to a neighborhood U of x_0 and $\bar{V}$ to a neighborhood V of y_0 if necessary, we may suppose that there exists an open rectangle $P \subset \mathbf{R}^n$ with center 0, a diffeomorphism $\alpha: \bar{\varphi}(U) \to P$, an open rectangle $Q \subset \mathbf{R}^m$ with center 0, and a diffeomorphism $\beta: \bar{\psi}(V) \to Q$, such that in the canonical coordinates z^i on P and w^σ on Q, $\beta g \alpha^{-1}(z^1, z^2, \ldots, z^n) = (z^1, z^2, \ldots, z^q, 0, 0, \ldots, 0)$. We set $\varphi = \alpha \bar{\varphi}$, $\varphi = (x^i)$, and

$\psi = \beta\bar{\psi}$, $\psi = (y^\sigma)$. Then, (U, φ) and (V, ψ) are charts on the manifolds X and Y, respectively. In these charts, the mapping $\psi f \varphi^{-1}$ can be expressed as $\psi f \varphi^{-1} = \beta\bar{\psi} f \bar{\varphi}^{-1}\alpha^{-1} = \beta g \alpha^{-1}$; thus, for every point $x \in U$

$$\begin{aligned} \psi f(x) &= \psi f \varphi^{-1}\varphi(x) = \beta g \alpha^{-1}\varphi(x) \\ &= \beta g \alpha^{-1}(x^1(x), x^2(x), \ldots, x^n(x)) \\ &= (x^1(x), x^2(x), \ldots, x^q(x), 0, 0, \ldots, 0). \end{aligned} \tag{5}$$

In components,

$$y^\sigma \circ f(x) = \begin{cases} x^\sigma(x), & \sigma = 1, 2, \ldots, q, \\ 0, & \sigma = q+1, q+2, \ldots, m, \end{cases} \tag{6}$$

proving (2).

2. Conversely, suppose that on a neighborhood of $x_0 \in W$, the mapping f is expressed by (2). Then, rank $T_{x_0}f =$ rank $D_i(y^\sigma f \varphi^{-1})(\varphi(x_0)) = q$. □

Let $f\colon X \to Y$ be a C^r mapping, and let $x_0 \in X$ be a point. We say that f is a *constant rank mapping* at x_0, if there exists a neighborhood W of x_0 such that the rank function $x \to \operatorname{rank}_x f$ is constant on W. Then, the charts (U, φ) and (V, ψ) in which the mapping f has an expression (2) are said to be *adapted* to f at x_0, or just *f-adapted.* A C^r mapping f that is a constant rank mapping at every point is called a C^r mapping of *locally constant rank.*

A C^r mapping $f\colon W \to Y$ such that the tangent mapping $T_{x_0}f$ is *injective* is called an *immersion at* x_0. From the definition of the rank, it is immediate that f is an immersion at x_0 if and only if $\operatorname{rank}_{x_0} f = n \le m$. If f is an immersion at every point of the set W, we say that f is an *immersion*.

From the rank theorem, we get the following criterion.

Theorem 2 (Immersions) *Let X and Y be two manifolds, $n = \dim X$, $m = \dim Y \ge n$. Let $f\colon X \to Y$ be a C^r mapping, $x_0 \in X$ a point, and let $y_0 = f(x_0)$. The following two conditions are equivalent*:

(1) *f is an immersion at x_0.*

(2) *There exists a chart (U, φ), $\varphi = (x^i)$ at x_0, an open rectangle $P \subset \mathbf{R}^n$ with center 0 such that $\varphi(U) = P$ and $\varphi(x_0) = 0$, a chart (V, ψ), $\psi = (y^\sigma)$ at $y_0 = f(x_0)$, and an open rectangle $Q \subset \mathbf{R}^m$ with center 0 such that $\psi(V) = Q$ and $\psi(y_0) = 0$, such that in these charts, f is expressed by*

$$y^\sigma \circ f = \begin{cases} x^\sigma, & \sigma = 1, 2, \ldots, n, \\ 0, & \sigma = n+1, n+2, \ldots, m. \end{cases} \tag{7}$$

Proof The matrix of the linear operator $T_{x_0}f$ in some charts (U,φ), $\varphi = (x^i)$, at x_0 and (V,ψ), $\psi = (y^\sigma)$, at y_0 is formed by partial derivatives $D_i(y^\sigma f\varphi^{-1})(\varphi(x_0))$ and is of dimension $n \times m$. If rank $T_{x_0}f = n$ at x_0, then rank $T_x f = n$ on a neighborhood of x_0, by continuity of the determinant function. Equivalence of conditions (1) and (2) is now an immediate consequence of Theorem 1. □

Let $f\colon X \to Y$ be an immersion, let $x_0 \in X$ be a point, and let (U,φ) and (V,ψ) be the charts from Theorem 2, (2). Shrinking P and Q if necessary, we may suppose without loss of generality that the rectangle Q is of the form $Q = P \times R$, where $R$is an open rectangle in $\mathbf{R}^{m-n}$. Then, the chart expression $\psi f\varphi^{-1}\colon P \to P \times R$ of the immersion f in these charts is the mapping $(x^1,x^2,\ldots,x^n) \to (x^1,x^2,\ldots,x^n,0,0,\ldots,0)$. The charts (U,φ), (V,ψ) with these properties are said to be *adapted* to the immersion f at x_0.

Example 1 (Sections) Let $s \geq r$, let $f\colon X \to Y$ be a surjective mapping of smooth manifolds. By a C^r *section*, or simply a *section* of *f*, we mean a C^r mapping $\gamma\colon Y \to X$ such that

$$f \circ \gamma = \mathrm{id}_Y. \tag{8}$$

Every section is an immersion. Indeed, $T_{\gamma(y)}f \circ T_y\gamma = \mathrm{id}_{T_yY}$ at any point $y \in Y$. Thus, for any two tangent vectors $\xi_1, \xi_2 \in T_yY$ satisfying the condition $T_y\gamma \cdot \xi_1 = T_y\gamma \cdot \xi_2$, we have $T_{\gamma(y)}f \circ T_y\gamma \cdot \xi_1 = T_{\gamma(y)}f \circ T_y\gamma \cdot \xi_2$. From this condition, we conclude that $\xi_1 = \xi_2$.

A C^r mapping $f\colon W \to Y$ such that the tangent mapping $T_{x_0}f$ is surjective, is called a *submersion at* x_0. From the definition of the rank, it is immediate that f is a submersion at x_0 if and only if $\mathrm{rank}_{x_0} = m \leq n$. A *submersion* $f\colon W \to Y$ is a C^r mapping that is a submersion at every point $x \in W$.

Theorem 3 (Submersions) *Let X and Y be manifolds, let* $n = \dim X$, $m = \dim Y$. *Let* $f\colon X \to Y$ *be a* C^r *mapping,* x_0 *a point of* X, $y_0 = f(x_0)$. *The following conditions are equivalent*:

(1) *f is a submersion at x_0.*
(2) *There exists a chart (U,φ), $\varphi = (x^i)$, at x_0, an open rectangle $P \subset \mathbf{R}^n$ with center 0 such that $\varphi(U) = P$, $\varphi(x_0) = 0$, a chart (V,ψ), $\psi = (y^\sigma)$, at $y_0 = f(x_0)$, and an open rectangle $Q \subset \mathbf{R}^m$ with center 0 such that $\psi(V) = Q$, $\psi(y_0) = 0$, and*

$$y^\sigma \circ f = x^\sigma, \quad \sigma = 1,2,\ldots,m. \tag{9}$$

(3) *There exists a neighborhood V of y_0 and a C^r section $\gamma\colon V \to Y$ such that $\gamma(y_0) = x_0$.*

Proof

1. Suppose that f is a submersion at x_0. Then, rank $T_x f = m$ on a neighborhood of x_0, and equivalence of conditions (1) and (2) follows from Theorem 1.
2. Suppose that condition (2) is satisfied. Consider the chart expression $\psi f \varphi^{-1}$: $P \to Q$ of the submersion f that is equal to the Cartesian projection $(x^1, x^2, \ldots, x^m, x^{m+1}, x^{m+1}, \ldots, x^n) \to (x^1, x^2, \ldots, x^m)$. $\psi f \varphi^{-1}$ admits a C^r section δ. Since $\psi f \varphi^{-1} \circ \delta = \mathrm{id}_Q$, hence $f \varphi^{-1} \circ \delta = \psi^{-1}$. Setting $\gamma = \varphi^{-1} \delta \psi$, we have $f\gamma = f\varphi^{-1}\delta\psi = \psi^{-1}\psi = \mathrm{id}_V$ proving that γ is a section of f. This proves (3).
3. If f admits a C^r section γ defined on a neighborhood V of a point y, then $f \circ \gamma = \mathrm{id}_V$ and $T_y(f \circ \gamma) = T_x f \circ T_y \gamma = T_y \mathrm{id}_V = \mathrm{id}_{T_y Y}$, where $x = \gamma(y)$. Thus, $T_{x_0} f$ must be surjective, proving (1). □

Let f be a C^r submersion, $x_0 \in X$ a point, and let (U, φ) and (V, ψ) be the charts from Theorem 3, (2). Shrinking P and Q if necessary, we may suppose that the rectangle P is of the form $P = Q \times R$, where R is an open rectangle in $\mathbf{R}^{n-m}$. Then, the chart expression (9) of the submersion f is the mapping $(x^1, x^2, \ldots, x^m, x^{m+1}, x^{m+1}, \ldots, x^n) \to (x^1, x^2, \ldots, x^m)$. The charts (U, φ), (V, ψ) with these properties are said to be *adapted* to the submersion f at x_0.

Corollary 1 *A submersion is an open mapping.*

Proof In adapted charts, a submersion is expressed as a Cartesian projection that is an open mapping. Corollary 1 now follows from the definition of the manifold topology in which the charts are homeomorphisms. □

Corollary 2 *Let $f\colon X \to Y$ be a submersion, (U, φ) a chart on X and (V, ψ) a chart on Y. If (U, φ) and (V, ψ) are adapted to f at a point $x_0 \in X$, and $V = f(U)$, then the chart (V, ψ) is uniquely determined by (U, φ).*

Proof This is an immediate consequence of the definition of adapted charts and of Corollary 1. □

Example 2 (Cartesian projections) Cartesian projections of the Cartesian product of C^∞ manifolds X and Y, $\mathrm{pr}_1\colon X \times Y \to X$ and $\mathrm{pr}_2\colon X \times Y \to Y$, are C^∞ submersions. Indeed, let us verify for instance the rank condition for the projection pr_1. If $(x, y) \in X \times Y$ is a point and (U, φ), $\varphi = (x^i)$ (resp. (V, ψ), $\psi = (y^\sigma)$) is a chart at x (resp. y), we have on the chart neighborhood $U \times V \subset X \times Y$, $(x, y) = \psi^{-1}\psi(x, y) = \psi^{-1}(x^1, x^2, \ldots, x^n, y^1, y^2, \ldots, y^m)$ and $\mathrm{pr}_1(x, y) = x = \varphi^{-1}\varphi(x) = \varphi^{-1}(x^1, x^2, \ldots, x^n)$. Then, for all vectors $\xi \in T_x X$ and $\zeta \in T_y Y$, expressed as

$$\xi = \xi^i \left(\frac{\partial}{\partial x^i}\right)_x, \quad \zeta = \zeta^i \left(\frac{\partial}{\partial x^i}\right)_y, \tag{10}$$

equations of the projection pr_1 yield

$$T_{(x,y)}\mathrm{pr}_1 \cdot (\xi, \zeta) = \frac{\partial (x^i \circ \mathrm{pr}_1)}{\partial x^k} \xi^k \frac{\partial}{\partial x^i} + \frac{\partial (x^i \circ \mathrm{pr}_1)}{\partial y^\sigma} \zeta^\sigma \frac{\partial}{\partial y^\sigma} = \xi. \tag{11}$$

In particular, $T_{(x,y)}\mathrm{pr}_1$ is surjective so pr_1 is a surjective submersion.

Example 3 The tangent bundle projection is a surjective submersion. All tensor bundle projections are surjective submersions.

With the help of Corollary 1, submersions at a point can be characterized as follows.

Corollary 3 *Let X and Y be manifolds,* $n = \dim X$, $m = \dim Y \leq n$. *A* C^r *mapping* $f\colon X \to Y$ *is a submersion at a point* $x_0 \in X$ *if and only if there exists a neighborhood* U *of* x_0*, an open rectangle* $R \subset \mathbf{R}^{n-m}$*, and a diffeomorphism* $\chi\colon U \to f(U) \times \mathbf{R}^{n-m}$ *such that* $\mathrm{pr}_1 \circ \chi = f$.

Proof

1. Suppose f is a submersion at x_0, and choose some adapted charts (U, φ), $\varphi = (x^i)$, at x_0 and (V, ψ), $\psi = (y^\sigma)$ at y_0. Every point $x \in U$ has the coordinates $(x^1(x), x^2(x), \ldots, x^m(x), x^{m+1}(x), x^{m+2}(x), \ldots, x^n(x))$. We define a mapping $\chi\colon U \to Y \times \mathbf{R}^{n-m}$ by

$$\chi(x) = (f(x), x^{m+1}(x), x^{m+2}(x), \ldots, x^n(x)). \tag{12}$$

   Then, $\mathrm{pr}_1 \circ \chi = f$, and from Corollary 1, $f(U)$ is an open set in Y. It remains to show that χ is a diffeomorphism. We easily find the chart expression of the mapping χ with respect to the chart (U, φ) and the chart $(V \times \mathbf{R}^{n-m}, \eta)$, $\eta = (y^1, y^2, \ldots y^m, t^1, t^2, \ldots, t^{n-m})$, on $Y \times \mathbf{R}^{n-m}$, where t^k are the canonical coordinates on $\mathbf{R}^{n-m}$. We have for every $x \in U$, $y^\sigma \chi(x) = y^\sigma f(x) = x^\sigma(x)$, $1 \leq \sigma \leq m$, and $t^k \chi(x) = x^{m+k}(x)$, $1 \leq k \leq n - m$, that is,

$$\begin{aligned} y^i \chi &= x^i, \qquad & i &= 1, 2, \ldots, m, \\ t^k \chi &= x^{m+k}, \qquad & k &= 1, 2, \ldots, n - m, \end{aligned} \tag{13}$$

   that is, $\eta \circ \chi = \varphi$. Thus, $\chi = \eta^{-1} \varphi$ is a diffeomorphism.

2. Conversely, if $\mathrm{pr}_1 \circ \chi = f$, we have $T_{x_0} f = T_{\chi(x_0)} \mathrm{pr}_1 \circ T_{x_0} \chi$, and since χ is by hypothesis a diffeomorphism, $\operatorname{rank} T_{x_0} f = \operatorname{rank} T_{\chi(x_0)} \mathrm{pr}_1$. But the rank of the projection pr_1 is m (Example 2). □

1.2 Fibered Manifolds

By a *fibered manifold structure* on a C^∞ manifold Y, we mean a C^∞ manifold X together with a surjective submersion $\pi\colon Y \to X$ of class C^∞. A manifold Y endowed with a fibered manifold structure is called a *fibered manifold* of class C^∞, or just a *fibered manifold*. X is the *base*, and π is the *projection* of the fibered manifold Y.

According to Sect. 1.1, Theorem 3 and Corollary 2, any manifold, endowed with a fibered manifold structure, admits the charts with some specific properties. Let Y be a fibered manifold with base X and projection π, $\dim X = n$, and $\dim Y = n + m$. By hypothesis, to every point $y \in Y$, there exists a chart at y, (V, ψ), $\psi = (u^i, y^\sigma)$, where $1 \le i \le n$, $1 \le \sigma \le m$, with the following properties:

(a) There exists a chart (U, φ), $\varphi = (x^i)$, at $x = \pi(y)$, where $1 \le i \le n$, in which the projection π is expressed by the equation $x^i \circ \pi = u^i$.
(b) $U = \pi(V)$.

The chart (V, ψ) with these properties is called a *fibered chart* on Y. The chart (U, φ) is defined uniquely and is said to be *associated* with (V, ψ). Having in mind this correspondence, we usually write x^i instead of u^i and denote a fibered chart as (V, ψ), $\psi = (x^i, y^\sigma)$.

Lemma 1 *Every fibered manifold has an atlas consisting of fibered charts.*

Proof An immediate consequence of the definition of a submersion. □

A C^r *section* of the fibered manifold Y, defined on an open set $W \subset X$, is by definition a C^r section γ: $W \to Y$ of its projection π (cf. Sect. 1.1, Example 1). In terms of a fibered chart (V, ψ), $\psi = (x^i, y^\sigma)$, and the associated chart (U, φ), $\varphi = (x^i)$, such that $U \subset W$ and $\gamma(U) \subset V$, γ has equations of the form

$$x^i \circ \gamma = x^i, \quad y^\sigma \circ \gamma = f^\sigma, \tag{14}$$

where f^σ are real C^r functions, defined on U.

Let Y_1 (resp. Y_2) be a fibered manifold with base X_1 (resp. X_2) and projection π_1 (resp. π_2). A C^r mapping α: $W \to Y_2$, where W is an open set in Y_1, is called a C^r *morphism* of the fibered manifold Y_1 into Y_2, if there exists a C^r mapping α_0: $W_0 \to X_2$ where $W_0 = \pi_1(W_1)$, such that

$$\pi_2 \circ \alpha = \alpha_0 \circ \pi_1. \tag{15}$$

Note that W_0 is always an open set in X_1 (Sect. 1.1, Corollary 1). If α_0 exists, it is unique and is called the *projection* of α. We also say that α is a morphism *over* α_0. A morphism of fibered manifolds α: $Y_1 \to Y_2$ that is a diffeomorphism is called an *isomorphism*; the projection of an isomorphism of fibered manifolds is a diffeomorphism of their bases.

If the fibered manifolds Y_1 and Y_2 coincide, $Y_1 = Y_2 = Y$, then a morphism α: $W \to Y$ is also called an *automorphism* of Y.

We find the expression of a morphism of fibered manifolds in fibered charts. Consider a fibered chart (V_1, ψ_1), $\psi_1 = (x_1^i, y_1^\sigma)$, on Y_1 and a fibered chart (V_2, ψ_2), $\psi_2 = (x_2^p, y_2^\tau)$, on Y_2 such that $\alpha(V_1) \subset V_2$. We have the commutative diagram

$$\begin{array}{ccc} V_1 & \xrightarrow{\alpha} & V_2 \\ \downarrow & & \downarrow \\ \pi_1(V_1) & \xrightarrow{\alpha_0} & \pi_2(V_2) \end{array} \tag{16}$$

expressing condition (15). In terms of the charts, we can write

$$\begin{aligned} \alpha_0\pi_1 &= \varphi_2^{-1} \circ \varphi_2\alpha_0\varphi_1^{-1} \circ \varphi_1\pi_1\psi_1^{-1} \circ \psi_1, \\ \pi_2\alpha &= \varphi_2^{-1} \circ \varphi_2\pi_2\psi_2^{-1} \circ \psi_2\alpha\psi_1^{-1} \circ \psi_1, \end{aligned} \tag{17}$$

so the commutativity yields

$$\varphi_2\alpha_0\varphi_1^{-1} \circ \varphi_1\pi_1\psi_1^{-1} = \varphi_2\pi_2\psi_2^{-1} \circ \psi_2\alpha\psi_1^{-1}. \tag{18}$$

But in our fibered charts, $\varphi_1\pi_1\psi_1^{-1}$ is the Cartesian projection $(x_1^i, y_1^\sigma) \to (x_1^i)$, and $\varphi_2\pi_2\psi_2^{-1}$ is the Cartesian projection $(x_2^p, y_2^\tau) \to (x_2^p)$. Consequently, writing in components

$$\begin{aligned} \varphi_2\alpha_0\varphi_1^{-1} \circ \varphi_1\pi_1\psi_1^{-1}(x_1^i, y_1^\sigma) &= \varphi_2\alpha_0\varphi_1^{-1}(x_1^i) = (x_2^p\alpha_0\varphi_1^{-1}(x_1^i)), \\ \varphi_2\pi_2\psi_2^{-1} \circ \psi_2\alpha\psi_1^{-1}(x_1^i, y_1^\sigma) &= \varphi_2\pi_2\psi_2^{-1}(x_2^p\alpha\psi_1^{-1}(x_1^i, y_1^\sigma), y_2^\tau\alpha\psi_1^{-1}(x_1^i, y_1^\sigma)) \\ &= (x_2^p\alpha\psi_1^{-1}(x_1^i, y_1^\sigma)), \end{aligned} \tag{19}$$

we see that condition (18) implies $x_2^p\alpha_0\varphi_1^{-1}(x_1^i) = x_2^p\alpha\psi_1^{-1}(x_1^i, y_1^\sigma)$. This shows that the right-hand side expression is independent of the coordinates y_1^σ. Therefore, we conclude that the equations of the morphism α in fibered charts are always of the form

$$x_2^p = f^p(x_1^i), \quad y_2^\tau = F^\tau(x_1^i, y_1^\sigma). \tag{20}$$

Let Y be a fibered manifold with base X and projection π. If Ξ is a tangent vector to Y at a point $y \in Y$, then the tangent vector ξ to X at $x = \pi(y) \in X$, defined by

$$T_y\pi \cdot \Xi = \xi, \tag{21}$$

is called the π-*projection*, or simply the *projection* of Ξ. By definition of the submersion, the tangent mapping of the projection π at a point y, $T_y\pi$: $T_yY \to T_{\pi(x)}X$, is surjective.

A tangent vector Ξ at a point $y \in Y$ is said to be π-*vertical,* if

$$T_y\pi \cdot \Xi = 0. \tag{22}$$

The vector subspace of T_yY consisted of π-vertical vectors, is denoted by VT_yY. If Ξ is expressed in a fibered chart (V, ψ), $\psi = (x^i, y^\sigma)$, by

$$\Xi = \xi^i \left(\frac{\partial}{\partial x^i} \right)_y + \Xi^\sigma \left(\frac{\partial}{\partial y^\sigma} \right)_y, \tag{23}$$

then by (21)

$$\xi = \xi^i \left(\frac{\partial}{\partial x^i} \right)_x = 0. \tag{24}$$

Thus, Ξ is π-vertical if and only if

$$\Xi = \Xi^\sigma \left(\frac{\partial}{\partial y^\sigma} \right)_y. \tag{25}$$

If in particular, $\dim Y = n + m$ and $\dim X = n$, then $\dim VT_yY = m$.

The subset VTY of the tangent bundle TY, defined by

$$VTY = \bigcup_{y \in Y} VT_yY, \tag{26}$$

is a vector subbundle of TY.

The projection $\pi\colon Y \to X$ induces a vector bundle morphism $T\pi\colon TY \to TX$; from the definition of a fibered manifold, it follows that the image is $\operatorname{Im} T\pi = TX$. The vector subbundle $VTY = \operatorname{Ker} T\pi$ of the vector bundle TY is called the *vertical subbundle* over Y.

Let ρ be a differential k-form, defined on an open set W in Y. We say that ρ is π-*horizontal*, or just *horizontal*, if it vanishes whenever one of its vector arguments is a π-vertical vector.

We describe the chart expressions of π-horizontal forms.

Lemma 2 *The form ρ is π-horizontal if and only if in any fibered chart (V, ψ), $\psi = (x^i, y^\sigma)$, it has an expression*

$$\rho = \frac{1}{k!} \rho_{i_1 i_2 \ldots i_k} dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_k}. \tag{27}$$

Proof Choose a point $y \in V$ and express the form $\rho(y)$ as

$$\rho(y) = \frac{1}{k!}\rho_{i_1 i_2 \dots i_k}(y) dx^{i_1}(y) \wedge dx^{i_2}(y) \wedge \cdots \wedge dx^{i_k}(y) + dy^1(y) \wedge \rho_1(y) + dy^2(y) \wedge \rho_2(y) + \cdots + dy^m(y) \wedge \rho_m(y), \tag{28}$$

where the forms $\rho_1(y)$, $\rho_2(y)$, …, $\rho_m(y)$ do not contain $dy^1(y)$, the forms $\rho_2(y)$, $\rho_3(y)$, …, $\rho_m(y)$ do not contain $dy^1(y)$ and $dy^2(y)$, etc. Suppose that ρ is π-horizontal. Then contracting the form $\rho(y)$ by the vertical vector $(\partial/\partial y^1)_y$, we get $i_{(\partial/\partial y^1)_y}\rho(y) = \rho_1(y) = 0$. Contracting $\rho(y)$ by the vertical vector $(\partial/\partial y^2)_y$, we get $i_{(\partial/\partial y^2)_y}\rho(y) = \rho_2(y) = 0$, etc., clearly, this proves formula (27). □

Example 4 The first Cartesian projection pr_1 of the product of Euclidean spaces $\mathbf{R}^n \times \mathbf{R}^m$ onto $\mathbf{R}^n$, restricted to the product of open sets $U \times V$, where $U \subset \mathbf{R}^n$ and $V \subset \mathbf{R}^m$, is a fibered manifold over U. The restriction of pr_1 to *any* open set $W \subset \mathbf{R}^n \times \mathbf{R}^m$ is a fibered manifold over $\mathrm{pr}_1(W) \subset \mathbf{R}^n$.

Example 5 Moebius band is a fibered manifold over the circle.

A form ρ, defined on an open set W in Y, is said to be *π-projectable*, or just *projectable*, if there exists a form ρ_0, defined on the set $\pi(W)$, such that

$$\rho = \pi^* \rho_0. \tag{29}$$

If the form ρ_0 exists, it is unique and is called the *π-projection*, of just the *projection* of ρ.

Convention Formula (29) shows that a π-projectable form can canonically be identified with its π-projection. Thus, to simplify the notation, we sometimes denote a π-projectable form $\pi^*\rho_0$ by its π-projection ρ_0.

1.3 The Contact of Differentiable Mappings

Let X and Y be two smooth manifolds, $n = \dim X$, and $m = \dim Y$. Let $x \in X$ be a point, $f_1 \colon W \to Y$ and $f_2 \colon W \to Y$ two mappings, defined on a neighborhood W of x. We say that f_1, f_2 have the *contact of order* 0 at x, if

$$f_1(x) = f_2(x). \tag{30}$$

Suppose that f_1 and f_2 are of class C^r, where r is a positive integer. We say that f_1, f_2 have the *contact of order* r at x, if they have the contact of order 0, and there exists a chart (U, φ), $\varphi = (x^i)$, at x and a chart (V, ψ), $\psi = (y^\sigma)$, at $f_1(x)$ such that $U \subset W$, $f_1(U)$, $f_2(U) \subset V$, and

$$D^k(\psi f_1 \varphi^{-1})(\varphi(x)) = D^k(\psi f_2 \varphi^{-1})(\varphi(x)) \tag{31}$$

for all $k \le r$. These definitions immediately extend to C^∞ mappings f_1, f_2; in this case f_1, f_2 are said to have the *contact of order* ∞ at x, if they have the contact of order r for every r.

Writing in components $\psi f_1 \varphi^{-1} = y^\sigma f_1 \varphi^{-1}$, $\psi f_2 \varphi^{-1} = y^\sigma f_2 \varphi^{-1}$, we see at once that f_1 and f_2 have contact of order r if and only if $f_1(x) = f_2(x)$ and

$$D_{i_1} D_{i_2} \ldots D_{i_k}(y^\sigma f_1 \varphi^{-1})(\varphi(x)) = D_{i_1} D_{i_2} \ldots D_{i_k}(y^\sigma f_2 \varphi^{-1})(\varphi(x)) \tag{32}$$

for all $k = 1, 2, \ldots, r$, all σ and all $i_1, i_2, \ldots, i_k$ such that $1 \le \sigma \le m$ and $1 \le i_1 \le i_2 \le \cdots \le i_k \le n$.

We claim that if f_1, f_2 have contact of order r at a point x, then for any chart $(\bar{U}, \bar{\varphi})$, $\bar{\varphi} = (\bar{x}^i)$, at x and any chart $(\bar{V}, \bar{\psi})$, $\bar{\psi} = (\bar{y}^\sigma)$, at $f_1(x)$,

$$D^k(\bar{\psi} f_1 \bar{\varphi}^{-1})(\bar{\varphi}(x)) = D^k(\bar{\psi} f_2 \bar{\varphi}^{-1})(\bar{\varphi}(x)) \tag{33}$$

for all $k = 1, 2, \ldots, r$. We can verify this formula by means of the chain rule for derivatives of mappings of Euclidean spaces. Using the charts (U, φ), (V, ψ), we express the derivative

$$\begin{aligned} &D_{i_1} D_{i_2} \ldots D_{i_k}(\bar{y}^\sigma f_1 \bar{\varphi}^{-1})(\bar{\varphi}(x)) \\ &\quad = D_{i_1} D_{i_2} \ldots D_{i_k}(\bar{y}^\sigma \psi^{-1} \circ \psi f_1 \varphi^{-1} \circ \varphi \bar{\varphi}^{-1})(\bar{\varphi}(x)) \end{aligned} \tag{34}$$

as a polynomial in the variables $D_{j_1}(y^\nu f_1 \varphi^{-1})(\varphi(x))$, $D_{j_1} D_{j_2}(y^\nu f_1 \varphi^{-1})(\varphi(x))$, $\ldots$, $D_{j_1} D_{j_2} \ldots D_{j_k}(y^\nu f_1 \varphi^{-1})(\varphi(x))$. The derivative $D_{i_1} D_{i_2} \ldots D_{i_k}(\bar{y}^\sigma f_2 \bar{\varphi}^{-1})(\bar{\varphi}(x))$ is expressed by the same polynomial in the variables $D_{j_1}(y^\nu f_2 \varphi^{-1})(\varphi(x))$ $D_{j_1} D_{j_2}(y^\nu f_2 \varphi^{-1})(\varphi(x))$, $\ldots$, $D_{j_1} D_{j_2} \ldots D_{j_k}(y^\nu f_2 \varphi^{-1})(\varphi(x))$. Clearly, equality (33) now follows from (32).

Fix two points $x \in X$, $y \in Y$ and denote by $C^r_{(x,y)}(X, Y)$ the set of C^r mappings $f \colon W \to Y$, where W is a neighborhood of x and $f(x) = y$. The binary relation "*f, g have the contact of order r at x*" on $C^r_{(x,y)}(X, Y)$ is obviously *reflexive*, *transitive*, and *symmetric*, so is an equivalence relation. Equivalence classes of this equivalence relation are called *r-jets* with *source x* and *target y*. The r-jet whose representative is a mapping $f \in C^r_{(x,y)}(X, Y)$ is called the *r-jet of f at the point x* and is denoted by $J^r_x f$. If there is no danger of misunderstanding, we call an r-jet with source x and target y an *r-jet*, or just a *jet*. The set of r-jets with source $x \in X$ and target $y \in Y$ is denoted by $J^r_{(x,y)}(X, Y)$.

Let $f \in C^r_{(x,y)}(X, Y)$ be a mapping, $f \colon W \to Y$, let U be a neighborhood of x and V a neighborhood of y. Assigning to f the restriction of f to the set $f^{-1}(V) \cap U \cap W$, we get a *bijection* $J^r_x f \to J^r_x(f|_{f^{-1}(V) \cap U \cap W})$ of the set $J^r_{(x,y)}(X, Y)$ onto $J^r_{(x,y)}(U, V)$.

Let X, Y, and Z be three smooth manifolds. Two r-jets $A \in J^r_{(x,u)}(X,Y)$, $A = J^r_x f$, and $B \in J^r_{(y,z)}(Y,Z)$, $B = J^r_y g$ are said to be *composable*, if they have representatives which are composable (as mappings), i.e., if $u = y$; this equality means that the target of A coincides with the source of B. In this case, the composite $g \circ f$ of any representatives of A and B is a mapping of class C^r defined on a neighborhood of x. It is easily seen that the r-jet $J^r_x(g \circ f)$ is independent of the representatives of the r-jets A and B. If $\bar{f}$ and $\bar{g}$ are such that $J^r_x f = J^r_x \bar{f}$ and $J^r_x g = J^r_x \bar{g}$, then for any charts (U, φ), $\varphi = (x^i)$ at x, (V, ψ), $\psi = (y^\sigma)$, at $y = f(x)$, and (W, η), $\eta = (z^p)$, at $z = g(y)$, the derivatives $D_{i_1} D_{i_2} \ldots D_{i_k}(z^p g f \varphi^{-1})(\varphi(x))$ are expressible in the form

$$D_{i_1} D_{i_2} \ldots D_{i_k}(z^p g f \varphi^{-1})(\varphi(x)) = D_{i_1} D_{i_2} \ldots D_{i_k}(z^p g \psi^{-1} \circ \psi f \varphi^{-1})(\varphi(x)) \tag{35}$$

for all $k = 1, 2, \ldots, r$. By the chain rule for mappings of Euclidean spaces, expressions (35) are polynomial in the variables $D_{v_1} D_{v_2} \ldots D_{v_q}(z^p g \psi^{-1})(\psi(y))$ and $D_{i_1} D_{i_2} \ldots D_{i_m}(y^v f \varphi^{-1})(\varphi(x))$, where $m, q \le k$. The same polynomials in the derivatives $D_{v_1} D_{v_2} \ldots D_{v_q}(z^p \bar{g} \psi^{-1})(\psi(y))$, $D_{i_1} D_{i_2} \ldots D_{i_m}(y^v \bar{f} \varphi^{-1})(\varphi(x))$ are obtained when expressing $D_{i_1} D_{i_2} \ldots D_{i_k}(z^p \bar{g} \bar{f} \varphi^{-1})(\varphi(x))$ by means of the chain rule. Now since by definition

$$\begin{aligned} D_{i_1} D_{i_2} \ldots D_{i_m}(y^v f \varphi^{-1})(\varphi(x)) &= D_{i_1} D_{i_2} \ldots D_{i_m}(y^v \bar{f} \varphi^{-1})(\varphi(x)), \\ D_{v_1} D_{v_2} \ldots D_{v_q}(z^p g \psi^{-1})(\psi(y)) &= D_{v_1} D_{v_2} \ldots D_{v_q}(z^p \bar{g} \psi^{-1})(\psi(y)), \end{aligned} \tag{36}$$

we have

$$D_{i_1} D_{i_2} \ldots D_{i_k}(z^p g f \varphi^{-1})(\varphi(x)) = D_{i_1} D_{i_2} \ldots D_{i_k}(z^p \bar{g} \bar{f} \varphi^{-1})(\varphi(x)). \tag{37}$$

This proves that the r-jet $J^r_x(g \circ f)$ is independent of the choice of A and B.

If X, Y, and Z are three manifolds and $A \in J^r_{(x,y)}(X,Y)$, $A = J^r_x f$, and $B \in J^r_{(y,z)}(Y,Z)$, $B = J^r_y g$ are composable r-jets, we define

$$B \circ A = J^r_x(g \circ f), \tag{38}$$

or, explicitly, $J^r_x g \circ J^r_x f = J^r_x(g \circ f)$. The r-jet $B \circ A$ is called the *composite* of A and B, and the mapping $(A, B) \to B \circ A$ of $J^r_{(x,f(x))}(X,Y) \times J^r_{(y,g(y))}(Y,Z)$ into $J^r_{(x,z)}(X,Z)$, where $z = g(y)$, is the *composition* of r-jets.

A chart on X at the point x and a chart on Y at the point y induce a chart on the set $J^r_{(x,y)}(X,Y)$. Let (U, φ), $\varphi = (x^i)$ (resp. (V, ψ), $\psi = (x^i, y^\sigma)$), be a chart on X (resp. Y). We assign to any r-jet $J^r_x f \in J^r_{(x,y)}(X,Y)$ the numbers

$$z^\sigma_{j_1 j_2 \ldots j_k}(J^r_x \gamma) = D_{j_1} D_{j_2} \ldots D_{j_k}(y^\sigma f \varphi^{-1})(\varphi(x)), \quad 1 \le k \le r. \tag{39}$$

Then, the collection of functions $\chi^r = (x^i, y^\sigma, y^\sigma_{j_1}, y^\sigma_{j_1 j_2}, \ldots, y^\sigma_{j_1 j_2 \ldots j_r})$, such that

$$1 \le j_1 \le j_2 \le \cdots \le j_k \le n, \quad 1 \le \sigma \le m, \tag{40}$$

is a bijection of the set $J^r_{(x,y)}(X,Y)$ and the Euclidean space $\mathbf{R}^N$ of dimension

$$N = n + m\left(1 + n + \binom{n+1}{2} + \binom{n+2}{3} + \cdots + \binom{n+r-1}{r}\right). \tag{41}$$

Thus, the pair $(J^r_{(x,y)}(X,Y), \chi^r)$ is a (global) chart on $J^r_{(x,y)}(X,Y)$. This chart is said to be *associated* with the charts (U,φ) and (V,ψ).

Lemma 3

(a) *The associated charts* $(J^r_{(x,y)}(X,Y), \chi^r)$*, such that the charts* (U,φ) *and* (V,ψ) *belong to smooth structures on X and Y, form a smooth atlas on* $J^r_{(x,y)}(X,Y)$*. With this atlas,* $J^r_{(x,y)}(X,Y)$ *is a smooth manifold of dimension N.*

(b) *The composition of jets*

$$J^r_{(x,y)}(X,Y) \times J^r_{(y,z)}(Y,Z) \ni (A,B) \to B \circ A \in J^r_{(x,z)}(X,Z) \tag{42}$$

is smooth.

Proof

1. It is enough to prove that the transformation equations between the associated charts are of class C^∞. However, this follows from (34).
2. (b) is an immediate consequence of Formula (35). □

1.4 Jet Prolongations of Fibered Manifolds

In this section, we apply the concept of contact of differentiable mappings (Sect. 1.3) to C^r *sections* of fibered manifolds. We introduce the smooth manifold structure on the sets of jets of sections and establish the coordinate transformation formulas.

Let Y be a fibered manifold with base X and projection π, let $n = \dim X$ and $m = \dim Y - n$. We denote by $J^r Y$, where $r \ge 0$ is any integer, the set of r-jets $J^r_x\gamma$ of C^r *sections* γ of Y with source $x \in X$ and target $y = \gamma(x) \in Y$; if $r = 0$, then $J^0 Y = Y$. Note that the representatives of an r-jet $J^r_x\gamma$ are C^r sections $\gamma\colon W \to Y$, where W is an open set in X; the condition that γ is a section,

$$\pi \circ \gamma = \mathrm{id}_W \tag{43}$$

implies that the target $y = \gamma(x)$ of the r-jet $J^r_x\gamma$ belongs to the fiber $\pi^{-1}(x) \subset Y$ over the source point x. For any s such that $0 \le s \le r$, we have surjective mappings $\pi^{r,s}\colon J^rY \to J^sY$ and $\pi^r\colon J^rY \to X$, defined by the conditions

$$\pi^{r,s}(J^r_x\gamma) = J^s_x\gamma, \quad \pi^r(J^r_x\gamma) = x. \tag{44}$$

These mappings are called the *canonical jet projections*.

The smooth structure of the fibered manifold Y induces a smooth structure on the set J^rY. This is based on a canonical construction that assigns to any fibered chart on Y a chart on J^rY. Let (V,ψ), $\psi = (x^i, y^\sigma)$, be a fibered chart on Y, and let (U,φ), $\varphi = (x^i)$, be the associated chart on X. We set $V^r = (\pi^{r,0})^{-1}(V)$ and introduce, for all values of the indices, a family of functions $x^i, y^\sigma, y^\sigma_{j_1j_2\ldots j_k}$, defined on V^r, by

$$\begin{aligned} x^i(J^r_x\gamma) &= x^i(x), \\ y^\sigma(J^r_x\gamma) &= y^\sigma(\gamma(x)), \\ y^\sigma_{j_1j_2\ldots j_k}(J^r_x\gamma) &= D_{j_1}D_{j_2}\ldots D_{j_k}(y^\sigma\gamma\varphi^{-1})(\varphi(x)), \quad 1 \le k \le r. \end{aligned} \tag{45}$$

Then, the collection of functions $\psi^r = (x^i, y^\sigma, y^\sigma_{j_1}, y^\sigma_{j_1j_2}, \ldots, y^\sigma_{j_1j_2\ldots j_r})$, where the indices satisfy

$$1 \le i \le n, \quad 1 \le \sigma \le m, \quad 1 \le j_1 \le j_2 \le \cdots \le j_k \le n, \quad k = 2,3,\ldots,r, \tag{46}$$

is a bijection of the set V^r onto an open subset of the Euclidean space $\mathbf{R}^N$ of dimension

$$N = n + m\left(1 + n + \binom{n+1}{2} + \binom{n+2}{3} + \cdots + \binom{n+r-1}{r}\right). \tag{47}$$

The pair (V^r, ψ^r), $\psi^r = (x^i, y^\sigma, y^\sigma_{j_1}, y^\sigma_{j_1j_2}, \ldots, y^\sigma_{j_1j_2\ldots j_r})$, is a chart on the set J^rY, which is said to be *associated* with the fibered chart (V,ψ), $\psi = (x^i, y^\sigma)$.

Lemma 4 (Smooth structure on the set J^rY) *The set of associated charts* (V^r, ψ^r), $\psi^r = (x^i, y^\sigma, y^\sigma_{j_1}, y^\sigma_{j_1j_2}, \ldots, y^\sigma_{j_1j_2\ldots j_r})$, *such that the fibered charts* (V,ψ) *constitute an atlas on Y, is an atlas on* J^rY.

Proof Let $\mathscr{A}$ be an atlas on Y whose elements are fibered charts (Sect. 1.2, Lemma 1). One can easily check that $\mathscr{A}$ defines a topology on J^rY by requiring that for any fibered chart (V,ψ) from $\mathscr{A}$, the mapping $\psi^r\colon V^r \to \psi^r(V^r) \subset \mathbf{R}^N$ is a homeomorphism; we consider the set J^rY with this topology.

It is clear that the associated charts with fibered charts from $\mathscr{A}$ cover the set J^rY. Thus, to prove Lemma 4, it remains to check that the corresponding coordinate transformations are smooth. □

Suppose we have two fibered charts on Y, (V,ψ), $\psi=(x^i,y^\sigma)$, and $(\bar{V},\bar{\psi})$, $\bar{\psi}=(\bar{x}^i,\bar{y}^\sigma)$, such that $V\cap\bar{V}\neq\emptyset$. Consider the associated charts (V^r,ψ^r), $(\bar{V}^r,\bar{\psi}^r)$, and an element $J^r_x\gamma\in V^r\cap\bar{V}^r$. Let the coordinate transformation $\bar{\psi}\psi^{-1}$ be expressed by the equations

$$\bar{x}^i=f^i(x^j),\quad \bar{y}^\sigma=g^\sigma(x^j,y^\nu). \tag{48}$$

Note that the functions f^i and g^σ in formula (48) are defined by the formulas $\bar{x}^i(x)=\bar{x}^i\varphi^{-1}(\varphi(x))=f^i(\varphi(x))$ and $\bar{y}^\sigma(y)=\bar{y}^\sigma\psi^{-1}(\psi(y))=g^\sigma(\psi(y))$. We have

$$\begin{aligned}
\bar{x}^i(J^r_x\gamma)&=\bar{x}^i(x)=\bar{x}^i\varphi^{-1}(\varphi(x))=\bar{x}^i\varphi^{-1}(\varphi(J^r_x\gamma)),\\
\bar{y}^\sigma(J^r_x\gamma)&=\bar{y}^\sigma(\gamma(x))=(\bar{y}^\sigma\psi^{-1}\circ\psi)(\gamma(x))=\bar{y}^\sigma\psi^{-1}(\psi(J^r_x\gamma)),\\
\bar{y}^\sigma_{j_1j_2\ldots j_k}(J^r_x\gamma)&=D_{j_1}D_{j_2}\ldots D_{j_k}(\bar{y}^\sigma\gamma\bar{\varphi}^{-1})(\bar{\varphi}(x))\\
&=D_{j_1}D_{j_2}\ldots D_{j_k}(\bar{y}^\sigma\psi^{-1}\circ\psi\gamma\varphi^{-1}\circ\varphi\bar{\varphi}^{-1})(\bar{\varphi}(x)).
\end{aligned} \tag{49}$$

From the chain rule, it is now obvious that the left-hand sides, the coordinates of the r-jet $J^r_x\gamma$ in the chart $(\bar{V}^r,\bar{\psi}^r)$, depend smoothly on the coordinates of $J^r_x\gamma$ in the chart (V^r,ψ^r).

From now on, the set J^rY is always considered with the smooth structure, defined by Lemma 4, and is called the *r-jet prolongation* of the fibered manifold Y.

Lemma 5 *Each of the canonical jet projections (44) is smooth and defines a fibered manifold structure on the manifold J^rY.*

Proof Indeed, in the associated charts, each of the canonical jet projections is expressed as a Cartesian projection, which is smooth. □

Every C^r section $\gamma\colon W\to Y$, where W is an open set in X, defines a mapping

$$W\ni x\to J^r\gamma(x)=J^r_x\gamma\in J^rY, \tag{50}$$

called the *r-jet prolongation* of γ.

Example 6 (Coordinate transformations on J^2Y) Consider two fibered charts on a fibered manifold Y, (V,ψ), $\psi=(x^i,y^\sigma)$, and $(\bar{V},\bar{\psi})$, $\bar{\psi}=(\bar{x}^i,\bar{y}^\sigma)$, such that $V\cap\bar{V}\neq\emptyset$. Suppose that the corresponding transformation equations are expressed as

$$\bar{x}^i=\bar{x}^i(x^j),\quad \bar{y}^\sigma=\bar{y}^\sigma(x^j,y^\nu). \tag{51}$$

Then, the induced coordinate transformation on J^2Y is expressed by the equations

$$\begin{aligned}
\bar{x}^i &= \bar{x}^i(x^j),\\
\bar{y}^\sigma &= \bar{y}^\sigma(x^j, y^\nu),\\
\bar{y}^\sigma_{j_1} &= \left(\frac{\partial \bar{y}^\sigma}{\partial x^l} + \frac{\partial \bar{y}^\sigma}{\partial y^\nu} y^\nu_l\right)\frac{\partial x^l}{\partial \bar{x}^{j_1}},\\
\bar{y}^\sigma_{j_1 j_2} &= \left(\frac{\partial^2 \bar{y}^\sigma}{\partial x^l \partial x^m} + \frac{\partial^2 \bar{y}^\sigma}{\partial x^l \partial y^\mu} y^\mu_m + \frac{\partial^2 \bar{y}^\sigma}{\partial x^m \partial y^\nu} y^\nu_l + \frac{\partial^2 \bar{y}^\sigma}{\partial y^\mu \partial y^\nu} y^\nu_l y^\mu_m \right.\\
&\quad \left. + \frac{\partial \bar{y}^\sigma}{\partial y^\nu} y^\nu_{lm}\right)\frac{\partial x^m}{\partial \bar{x}^{j_2}}\frac{\partial x^l}{\partial \bar{x}^{j_1}} + \left(\frac{\partial \bar{y}^\sigma}{\partial x^l} + \frac{\partial \bar{y}^\sigma}{\partial y^\nu} y^\nu_l\right)\frac{\partial^2 x^l}{\partial \bar{x}^{j_1}\partial \bar{x}^{j_2}}.
\end{aligned} \tag{52}$$

To derive these equations, we use the chain rule for partial derivative operators. Let $J^2_x\gamma \in V^2 \cap \bar{V}^2$. The 2-jet $J^2_x\gamma$ has the coordinates

$$\begin{aligned}
x^i(J^2_x\gamma) &= x^i(x),\\
y^\sigma(J^2_x\gamma) &= y^\sigma(\gamma(x)),\\
y^\sigma_{j_1}(J^r_x\gamma) &= D_{j_1}(y^\sigma\gamma\varphi^{-1})(\varphi(x)),\\
y^\sigma_{j_1 j_2}(J^r_x\gamma) &= D_{j_1}D_{j_2}(y^\sigma\gamma\varphi^{-1})(\varphi(x)),
\end{aligned} \tag{53}$$

and analogous formulas arise for the chart $(\bar{V}, \bar{\psi})$. Then, by the chain rule

$$\begin{aligned}
D_{j_1}(\bar{y}^\sigma\gamma\bar{\varphi}^{-1})(\bar{\varphi}(x)) &= D_{j_1}(\bar{y}^\sigma\psi^{-1}\circ\psi\gamma\varphi^{-1}\circ\varphi\bar{\varphi}^{-1})(\bar{\varphi}(x))\\
&= D_k(\bar{y}^\sigma\psi^{-1})(\psi\gamma(x))D_l(x^k\gamma\varphi^{-1})(\varphi\bar{\varphi}^{-1}(\bar{\varphi}(x))D_{j_1}(x^l\bar{\varphi}^{-1})(\bar{\varphi}(x))\\
&\quad + D_\nu(\bar{y}^\sigma\psi^{-1}(\psi\gamma(x))D_l(y^\nu\gamma\varphi^{-1})(\varphi\bar{\varphi}^{-1}(\bar{\varphi}(x))D_{j_1}(x^l\bar{\varphi}^{-1})(\bar{\varphi}(x))\\
&= D_k(\bar{y}^\sigma\psi^{-1})(\psi\gamma(x))\delta^k_l D_{j_1}(x^l\bar{\varphi}^{-1})(\bar{\varphi}(x))\\
&\quad + D_\nu(\bar{y}^\sigma\psi^{-1})(\psi\gamma(x))D_l(y^\nu\gamma\bar{\varphi}^{-1})(\varphi(x))D_{j_1}(x^l\bar{\varphi}^{-1})(\bar{\varphi}(x))\\
&= (D_l(\bar{y}^\sigma\psi^{-1})(\psi\gamma(x)) + D_\nu(\bar{y}^\sigma\psi^{-1})(\psi\gamma(x))D_l(y^\nu\gamma\bar{\varphi}^{-1})(\varphi(x)))\\
&\quad \cdot D_{j_1}(x^l\bar{\varphi}^{-1})(\bar{\varphi}(x)),
\end{aligned} \tag{54}$$

which proves the third one of equations (52). To prove the fourth equation, we differentiate (54) again and apply the chain rule. We can also derive the fourth equation by differentiating the third one.

Consider a morphism $\alpha\colon W \to Y$ of a fibered manifold Y with projection π. The projection $\alpha_0\colon \pi(W) \to X$ of the morphism α is a unique morphism of smooth manifolds such that

$$\pi \circ \alpha = \alpha_0 \circ \pi. \tag{55}$$

Suppose that α_0 is a diffeomorphism of the open subsets $\pi(W)$ and $U_0 = \alpha_0(\pi(W))$ in X. Then, for any section γ of Y, defined on $\pi(W)$, formula $\gamma' = \alpha\gamma\alpha_0^{-1}$ defines a section of Y over U_0: indeed, since γ is a section, then $\pi\gamma' = \pi\alpha\gamma\alpha_0^{-1} = \alpha_0\pi\gamma\alpha_0^{-1} = \mathrm{i}d_{U_0}$. In this sense, α transforms sections γ of Y into sections $\alpha\gamma\alpha_0^{-1}$ of Y. In particular, setting for every r-jet $J^r_x\gamma \in W^r$

$$J^r\alpha(J^r_x\gamma) = J^r_{\alpha_0}\alpha\gamma\alpha_0^{-1} \tag{56}$$

we get a mapping $J^r\alpha$: $W^r \to J^rY$. This mapping is differentiable and satisfies, for all integers s such that $0 \le s \le r$,

$$\pi^{r,s} \circ J^r\alpha = J^s\alpha \circ \pi^{r,s}, \quad \pi^r \circ J^r\alpha = \alpha_0 \circ \pi^r. \tag{57}$$

These formulas show that the *mapping* $J^r\alpha$ is a morphisms of the r-jet prolongation J^rY of the fibered manifold Y over J^sY for all s such that $0 \le s \le r$, and over X. $J^r\alpha$ is called the *r-jet prolongation* of the morphism $J^r\alpha$ of Y. Note that $J^r\alpha$ is *not* defined for morphisms α whose projections are *not* diffeomorphisms.

1.5 The Horizontalization

Let Y be a fibered manifold with base X and projection π, $\dim X = n$ and $\dim Y = n + m$. For any open set $W \subset Y$, we denote by W^r the open set $(\pi^{r,0})^{-1}(W)$ in the r-jet prolongation J^rY of Y. We show that the fibered manifold structure on Y induces a vector bundle morphism between the tangent bundles $T^{r+1}Y$ and T^rY and study the decomposition of tangent vectors, associated with this mapping.

Let $J^{r+1}_x\gamma$ be a point of the manifold $J^{r+1}Y$. We assign to any tangent vector ξ of $J^{r+1}Y$ at the point $J^{r+1}_x\gamma$ a tangent vector of J^rY at the point $\pi^{r+1,r}(J^{r+1}_x\gamma) = J^r_x\gamma$ by

$$h\xi = T_xJ^r\gamma \circ T\pi^{r+1} \cdot \xi. \tag{58}$$

We get a vector bundle morphism h: $TJ^{r+1}Y \to TJ^rY$ over the jet projection $\pi^{r+1,r}$, called the *π-horizontalization*, or simply the *horizontalization*. Sometimes we call $h\xi$ the *horizontal component* of ξ (note, however, that ξ and $h\xi$ do not belong to the same vector space). Using a complementary construction, one can also assign to every tangent vector $\xi \in TJ^{r+1}Y$ at the point $J^{r+1}_x\gamma \in J^{r+1}Y$ a tangent vector $p\xi \in TJ^rY$ at $J^r_x\gamma$ by the decomposition

$$T\pi^{r+1,r} \cdot \xi = h\xi + p\xi. \tag{59}$$

$p\xi$ is called the *contact component* of the vector ξ.

Lemma 6 *The horizontal and contact components satisfy*

$$T\pi^r \cdot h\xi = T\pi^{r+1} \cdot \xi, \quad T\pi^r \cdot p\xi = 0. \tag{60}$$

Proof The first property follows from (58). Then, however,

$$\begin{aligned} T\pi^r \cdot p\xi &= T\pi^r \cdot T\pi^{r+1,r} \cdot \xi - T\pi^r \cdot h\xi = T\pi^{r+1} \cdot \xi - T\pi^r \cdot h\xi \\ &= T\pi^{r+1} \cdot \xi - T\pi^r \cdot T_x J^r\gamma \circ T\pi^{r+1} \cdot \xi = 0. \end{aligned} \tag{61}$$

□

Remark 1 If $h\xi = 0$, then necessarily $T\pi^{r+1} \cdot \xi = 0$ so ξ is π^{r+1}-vertical. This observation explains why $h\xi$ is called the *horizontal component* of ξ.

One can easily find the chart expressions for the vectors $h\xi$ and $p\xi$. If in a fibered chart (V, ψ), $\psi = (x^i, y^\sigma)$, ξ has an expression

$$\xi = \xi^i \left(\frac{\partial}{\partial x^i} \right)_{J_x^{r+1}\gamma} + \sum_{k=0}^{r+1} \sum_{j_1 \le j_2 \le \cdots \le j_k} \Xi^\sigma_{j_1 j_2 \ldots j_k} \left(\frac{\partial}{\partial y^\sigma_{j_1 j_2 \ldots j_k}} \right)_{J_x^{r+1}\gamma}, \tag{62}$$

then

$$h\xi = \xi^i \left(\left(\frac{\partial}{\partial x^i} \right)_{J_x^r\gamma} + \sum_{k=0}^{r} \sum_{j_1 \le j_2 \le \cdots \le j_k} y^\sigma_{j_1 j_2 \ldots j_k i} \left(\frac{\partial}{\partial y^\sigma_{j_1 j_2 \ldots j_k}} \right)_{J_x^r\gamma} \right), \tag{63}$$

and

$$p\xi = \sum_{k=0}^{r} \sum_{j_1 \le j_2 \le \cdots \le j_k} \left(\Xi^\sigma_{j_1 j_2 \ldots j_k} - y^\sigma_{j_1 j_2 \ldots j_k i} \xi^i \right) \left(\frac{\partial}{\partial y^\sigma_{j_1 j_2 \ldots j_k}} \right)_{J_x^r\gamma}. \tag{64}$$

Note that the conditions $h\xi = 0$ and $p\xi = 0$ do *not* imply $\xi = 0$; they are equivalent to the condition that ξ be $\pi^{r+1,r}$-vertical,

$$\xi = \sum_{j_1 \le j_2 \le \cdots \le j_{r+1}} \Xi^\sigma_{j_1 j_2 \ldots j_{r+1}} \left(\frac{\partial}{\partial y^\sigma_{j_1 j_2 \ldots j_{r+1}}} \right)_{J_x^{r+1}\gamma}. \tag{65}$$

The structure of the chart expression (63) can also be characterized by means of the vector fields d_i along the projection $\pi^{r+1,r}$, defined on V^{r+1} by

$$d_i = \left(\frac{\partial}{\partial x^i} \right)_{J_x^r\gamma} + \sum_{k=0}^{r} \sum_{j_1 \le j_2 \le \cdots \le j_k} y^\sigma_{j_1 j_2 \ldots j_k i} \left(\frac{\partial}{\partial y^\sigma_{j_1 j_2 \ldots j_k}} \right)_{J_x^r\gamma}. \tag{66}$$

d_i is called the i-th *formal derivative operator* (relative to the fibered chart (V, ψ)). Note that these vector fields are closely connected with the tangent mapping of the functions $f\colon J^rY \to \mathbf{R}$, composed with the prolongations $J^r\gamma$ of sections γ of Y. Namely, if (V, ψ), $\psi = (x^i, y^\sigma)$ is a fibered chart, $x \in \pi(U)$ a point and γ a section defined on U, then for every tangent vector $\xi_0 \in T_xX$, expressed as $\xi_0 = \xi_0^i(\partial/\partial x^i)_x$,

$$T_x(f \circ J^r\gamma) \cdot \xi_0 = \left(\frac{\partial(f \circ J^r\gamma \circ \varphi^{-1})}{\partial x^k}\right)_x \xi^k. \tag{67}$$

For each i such that $1 \le i \le n$, the formula

$$d_i f(J_x^{r+1}\gamma) = \left(\frac{\partial(f \circ J^r\gamma \circ \varphi^{-1})}{\partial x^k}\right)_x \tag{68}$$

defines a function $d_i f\colon V^{r+1} \to \mathbf{R}$, called the i-th *formal derivative* of the function f (relative to the given fibered chart). In the chart,

$$d_i f = \xi^i \frac{\partial f}{\partial x^i} + \sum_{k=0}^{r} \sum_{j_1 \le j_2 \le \cdots \le j_k} y^\sigma_{j_1 j_2 \ldots j_k i} \frac{\partial f}{\partial y^\sigma_{j_1 j_2 \ldots j_k}}. \tag{69}$$

Remark 2 Canonically extending the partial derivatives $\partial/\partial y^\sigma_{j_1 j_2 \ldots j_k}$ to *all* sequences $j_1, j_2, \ldots, j_k$, the formal derivative d_i can be expressed as

$$d_i = \frac{\partial}{\partial x^i} + \sum_{k=0}^{r} y^\sigma_{j_1 j_2 \ldots j_k i} \frac{\partial}{\partial y^\sigma_{j_1 j_2 \ldots j_k}} \tag{70}$$

(see Appendix 2).

Remark 3 In general, decomposition (59) of tangent vectors does *not* hold for vector fields. However, if ξ is a π^{r+1}-*vertical* vector field on W^{r+1}, then $h\xi$ is the zero vector field on W^r and condition (59) reduces to the $\pi^{r+1,r}$-projectability equation

$$T\pi^{r+1,r} \cdot \xi = \xi_0 \circ \pi^{r+1,r} \tag{71}$$

for the $\pi^{r+1,r}$-projection ξ_0 of ξ. Thus, $p\xi(J_x^{r+1}\gamma) = \xi_0(J_x^r\gamma)$.

1.6 Jet Prolongations of Automorphisms of Fibered Manifolds

Let (V, ψ), $\psi = (x^i, y^\sigma)$, be a chart, and let $f\colon V^r \to \mathbf{R}$ be a differentiable function. We set for every i, $1 \leq i \leq n$,

$$d_i f = \frac{\partial f}{\partial x^i} + \sum_{0 \leq k \leq r} \frac{\partial f}{\partial y^\sigma_{j_1 j_2 \ldots j_k}} y^\sigma_{j_1 j_2 \ldots j_k i}. \tag{72}$$

In this formula, the function $d_i f\colon V^{r+1} \to \mathbf{R}$ is the i-th *formal derivative* of the function f (Sect. 1.5). A notable formula

$$d_i y^\sigma_{j_1 j_2 \ldots j_k} = y^\sigma_{j_1 j_2 \ldots j_k i} \tag{73}$$

says that d_i may be treated as a mapping, acting on jet coordinates of the given chart.

Let r be a positive integer. Consider an open set W in the fibered manifold Y and a C^r automorphism $\alpha\colon W \to Y$ with projection $\alpha_0\colon W_0 \to X$, defined on an open set $W_0 = \pi(W)$. In this section, we suppose that the projection α_0 is a C^r *diffeomorphism*.

Every section $\gamma\colon W_0 \to Y$ defines the mapping $\alpha\gamma\alpha_0^{-1} = \alpha \circ \gamma \circ \alpha_0^{-1}$; it is easily seen that this mapping is a section of Y over the open set $\alpha_0(W_0) \subset X$: indeed, using properties of morphisms and sections of fibered manifolds, we get $\pi \circ \alpha\gamma\alpha_0^{-1} = \alpha_0 \circ \pi \circ \gamma \circ \alpha_0^{-1} = \alpha_0 \circ \alpha_0^{-1} = \mathrm{i}d_{W_0}$. Then, however, the r-jets of the section $x \to \alpha\gamma\alpha_0^{-1}(x)$ are defined and are elements of the set $J^r Y$. An r-jet $J^r_{\alpha_0(x)}\alpha\gamma\alpha_0^{-1}$ can be decomposed as $J^r_{\gamma(x)}\alpha \circ J^r_x\gamma \circ J^r_{\alpha_0(x)}\alpha_0^{-1}$, so it is independent of the choice of the representative γ and depends on the r-jet $J^r_x\gamma$ only. We set for every $J^r_x\gamma \in W^r = (\pi^{r,0})^{-1}(W)$

$$J^r\alpha(J^r_x\gamma) = J^r_{\alpha_0(x)}\alpha\gamma\alpha_0^{-1}. \tag{74}$$

This formula defines a mapping $J^r\alpha\colon W^r \to J^r Y$, called the *r-jet prolongation*, or just *prolongation* of the C^r automorphism α.

Note an immediate consequence of the definition (74). Given a C^r *section* $\gamma\colon W_0 \to Y$, then we have $J^r\alpha \circ J^r\gamma = J^r\alpha\gamma\alpha_0^{-1} \circ \alpha_0$ so the r-jet prolongation $J^r\alpha\gamma\alpha_0^{-1}$ of the section $\alpha\gamma\alpha_0^{-1}$ satisfies

$$J^r\alpha\gamma\alpha_0^{-1} = J^r\alpha \circ J^r\gamma \circ \alpha_0^{-1} \tag{75}$$

on the set $\alpha_0(W_0)$. In particular, this formula shows that the r-jet prolongations of automorphisms carry sections of Y into sections of $J^r Y$ (over X).

We find the chart expression of the mapping $J^r\alpha$.

Lemma 7 *Suppose that in two fibered charts on Y,* (V,ψ), $\psi=(x^i,y^\sigma)$, *and* $(\bar{V},\bar{\psi})$, $\bar{\psi}=(\bar{x}^i,\bar{y}^\sigma)$, *on Y such that* $\alpha(V)\subset\bar{V}$, *the* C^r *automorphism* α *is expressed by equations*

$$\bar{x}^i\circ\alpha(y)=f^i(x^j(x)),\quad \bar{y}^\sigma\circ\alpha(y)=F^\sigma(x^j(x),y^\nu(y)). \tag{76}$$

Then for every point $J^r_x\gamma\in V^r$, *the transformed point* $J^r\alpha(J^r_x\gamma)$ *has the coordinates*

$$\begin{aligned}\bar{x}^i\circ J^r\alpha(J^r_x\gamma)&=f^i(x^j(x)),\\ \bar{y}^\sigma\circ J^r\alpha(J^r_x\gamma)&=F^\sigma(x^j(x),y^\nu(\gamma(x))),\end{aligned} \tag{77}$$

$$\bar{y}^\sigma_{j_1j_2\ldots j_k}\circ J^r\alpha(J^r_x\gamma)=D_{j_1}D_{j_2}\ldots D_{j_k}(\bar{y}^\sigma\alpha\psi^{-1}\circ\psi\gamma\varphi^{-1}\circ\varphi\alpha_0^{-1}\bar{\varphi}^{-1})(\bar{\varphi}(\alpha_0(x))),$$
$$1\le k\le r.$$

Proof We have

$$\begin{aligned}\bar{x}^i\circ J^r\alpha(J^r_x\gamma)&=\bar{x}^i\circ\alpha_0(x)\\ &=\bar{x}^i\alpha_0\varphi^{-1}(\varphi(x))=f^i(x^j(x)),\\ \bar{y}^\sigma\circ J^r\alpha(J^r_x\gamma)&=\bar{y}^\sigma\circ\alpha(\gamma(x))=\bar{y}^\sigma\alpha\psi^{-1}(\psi(\gamma(x)))\\ &=F^\sigma(x^j(x),y^\nu(\gamma(x))),\end{aligned} \tag{78}$$

and by definition

$$\begin{aligned}\bar{y}^\sigma_{j_1j_2\ldots j_k}\circ J^r\alpha(J^s_x\gamma)&=\bar{y}^\sigma_{j_1j_2\ldots j_k}(J^s_{\alpha_0(x)}\alpha\gamma\alpha_0^{-1})\\ &=D_{j_1}D_{j_2}\ldots D_{j_k}(\bar{y}^\sigma\circ\alpha\gamma\alpha_0^{-1}\bar{\varphi}^{-1})(\bar{\varphi}(\alpha_0(x)))\\ &=D_{j_1}D_{j_2}\ldots D_{j_k}(\bar{y}^\sigma\alpha\psi^{-1}\circ\psi\gamma\varphi^{-1}\circ\varphi\alpha_0^{-1}\bar{\varphi}^{-1})(\bar{\varphi}(\alpha_0(x))).\end{aligned} \tag{79}$$

□

Formulas (77) contains partial derivatives of the functions f^i and F^σ, and also partial derivatives of the functions g^k, representing the chart expression $\varphi\alpha_0^{-1}\bar{\varphi}^{-1}$ of the inverse diffeomorphism α_0^{-1}. These functions are defined by

$$x^k\circ\alpha_0^{-1}(x')=g^k(\bar{x}^l(x')). \tag{80}$$

To obtain explicit dependence of the coordinates $\bar{y}^\sigma_{j_1j_2\ldots j_k}(J^r\alpha(J^r_x\gamma))$ on the coordinates of the r-jet $J^r_x\gamma$, we have to use the chain rule k times, which leads to polynomial dependence of the jet coordinates $\bar{y}^\sigma_{j_1j_2\ldots j_k}(J^r\alpha(J^r_x\gamma))$ on the jet coordinates $y^\nu_{i_1}(J^r_x\gamma)$, $y^\nu_{i_1i_2}(J^r_x\gamma)$, $\ldots$, $y^\nu_{i_1i_2\ldots i_k}(J^r_x\gamma)$. This shows, in particular, that if α is of class C^r, then $J^r\alpha$ is of class C^0; if α is of class C^s, where $s\ge r$, then $J^r\alpha$ is of class C^{s-r}.

Equations (77) can be viewed as the recurrence formulas for the chart expression of the mapping $J^r\alpha$. Writing

$$\bar{y}^{\sigma}_{j_1 j_2 \ldots j_{k-1}} \circ J^r\alpha(J^r_x\gamma) = (\bar{y}^{\sigma}_{j_1 j_2 \ldots j_{k-1}} \circ J^r\alpha \circ J^r\gamma \circ \varphi^{-1} \circ \varphi\alpha_0^{-1}\bar{\varphi}^{-1})(\bar{\varphi}(\alpha_0(x))), \quad (81)$$

we have

$$\begin{aligned} \bar{y}^{\sigma}_{j_1 j_2 \ldots j_k} \circ J^r\alpha(J^r_x\gamma) &= D_{j_k}(\bar{y}^{\sigma}_{j_1 j_2 \ldots j_{k-1}} \circ J^r\alpha \circ J^r\gamma \circ \varphi^{-1} \circ \varphi\alpha_0^{-1}\bar{\varphi}^{-1})(\bar{\varphi}(\alpha_0(x))) \\ &= D_l(\bar{y}^{\sigma}_{j_1 j_2 \ldots j_{k-1}} \circ J^r\alpha \circ J^r\gamma \circ \varphi^{-1})(\varphi(x)) D_{j_k}(x^l\alpha_0^{-1}\bar{\varphi}^{-1})(\bar{\varphi}(\alpha_0(x))). \end{aligned} \quad (82)$$

Thus, if we already have the functions $\bar{y}^{\sigma}_{j_1 j_2 \ldots j_{k-1}} \circ J^r\alpha$, then the functions $\bar{y}^{\sigma}_{j_1 j_2 \ldots j_k} \circ J^r\alpha$ are determined by (77).

We derive explicit expressions for the second-jet prolongation $J^2\alpha$.

Example 7 (2-jet prolongation of an automorphism) Let $r = 2$. We have from (76)

$$\begin{aligned} \bar{y}^{\sigma}_{j_1} \circ J^2\alpha(J^r_x\gamma) &= D_{j_1}(\bar{y}^{\sigma}\alpha\psi^{-1} \circ \psi\gamma\varphi^{-1} \circ \varphi\alpha_0^{-1}\bar{\varphi}^{-1})(\bar{\varphi}(\alpha_0(x))) \\ &= D_k(\bar{y}^{\sigma}\alpha\psi^{-1})(\psi\gamma(x)) D_l(x^k\gamma\varphi^{-1})(\varphi(x)) D_{j_1}(x^l\alpha_0^{-1}\bar{\varphi}^{-1})(\bar{\varphi}(\alpha_0(x))) \\ &= D_k(\bar{y}^{\sigma}\alpha\psi^{-1})(\psi\gamma(x)) \delta^k_l D_{j_1}(x^l\alpha_0^{-1}\bar{\varphi}^{-1})(\bar{\varphi}(\alpha_0(x))) \\ &\quad + D_\lambda(\bar{y}^{\sigma}\alpha\psi^{-1})(\psi\gamma(x)) y^{\lambda}_l(J^r_x\gamma) D_{j_1}(x^l\alpha_0^{-1}\bar{\varphi}^{-1})(\bar{\varphi}(\alpha_0(x))) \\ &= (D_l(\bar{y}^{\sigma}\alpha\psi^{-1})(\psi\gamma(x)) + D_\lambda(\bar{y}^{\sigma}\alpha\psi^{-1})(\psi\gamma(x)) y^{\lambda}_l(J^s_x\gamma)) \\ &\quad \cdot D_{j_1}(x^l\alpha_0^{-1}\bar{\varphi}^{-1})(\bar{\varphi}(\alpha_0(x))), \end{aligned} \quad (83)$$

or, in a different notation,

$$\bar{y}^{\sigma}_{j_1} \circ J^2\alpha(J^r_x\gamma) = d_l F^{\sigma}(J^r_x\gamma) \left(\frac{\partial g^l}{\partial \bar{x}^{j_1}}\right)_{\bar{\varphi}(\alpha_0(x))}, \quad (84)$$

where d_l denotes the formal derivative operator. Differentiating (83) or (84) again, we get the following equations for the 2-jet prolongation $J^2\alpha$ of α:

$$\begin{aligned} &\bar{x}^i = f^i(x^i), \quad \bar{y}^{\sigma} = F^{\sigma}(x^i, y^{\nu}), \quad \bar{y}^{\sigma}_{j_1} = d_{k_1}F^{\sigma} \cdot \frac{\partial g^{k_1}}{\partial \bar{x}^{j_1}}, \\ &\bar{y}^{\sigma}_{j_1 j_2} = d_{k_1}d_{k_2}F^{\sigma} \cdot \frac{\partial g^{k_1}}{\partial \bar{x}^{j_1}} \frac{\partial g^{k_2}}{\partial \bar{x}^{j_2}} + d_{k_1}F^{\sigma} \cdot \frac{\partial^2 g^{k_1}}{\partial \bar{x}^{j_1} \partial \bar{x}^{j_2}}. \end{aligned} \quad (85)$$

We can easily prove the following statements.

Lemma 8

(a) *For any s such that* $0 \le s \le r$,

$$\pi^r \circ J^r\alpha = \alpha_0 \circ \pi^r, \quad \pi^{r,s} \circ J^r\alpha = J^s\alpha \circ \pi^{r,s}. \tag{86}$$

(b) *If two* C^r *automorphisms* α *and* β *of the fibered manifold Y are composable, then* $J^r\alpha$ *and* $J^r\beta$ *are composable and*

$$J^r\alpha \circ J^r\beta = J^r(\alpha \circ \beta). \tag{87}$$

Proof All these assertions are easy consequences of definitions. □

Formula (86) shows that $J^r\alpha$ is an C^r automorphism of the r-jet prolongation J^rY of the fibered manifold Y, and also C^r automorphisms of J^rY over J^sY.

1.7 Jet Prolongations of Vector Fields

Let Y be a fibered manifold with base X and projection π. Our aim in this section is to extend the theory of jet prolongations of automorphisms of a fibered manifold Y to local flows of vector fields, defined on Y.

Let Ξ be a C^r vector field on Y, let $y_0 \in Y$ be a point, and consider a local flow α^Ξ: $(-\varepsilon, \varepsilon) \times V \to Y$ of Ξ at y_0 (see Appendix 4). As usual, define the mappings α_t^Ξ and α_y^Ξ by

$$\alpha^\Xi(t, y) = \alpha_t^\Xi(y) = \alpha_y^\Xi(t). \tag{88}$$

Then for any point $y \in V$, the mapping $t \to \alpha_y^\Xi(t)$ is an integral curve of Ξ passing through y at $t = 0$, i.e.,

$$T_t\alpha_y^\Xi = \Xi(\alpha_y^\Xi(t)), \quad \alpha_y^\Xi(0) = y. \tag{89}$$

Moreover, shrinking the domain of definition $(-\varepsilon, \varepsilon) \times V$ of α^Ξ to a subset $(-\kappa, \kappa) \times W \subset (-\varepsilon, \varepsilon) \times V$, where W is a neighborhood of the point y_0, we have

$$\alpha^\Xi(s + t, y) = \alpha^\Xi(s, \alpha^\Xi(t, y)), \quad \alpha^\Xi(-t, \alpha^\Xi(t, y)) = y \tag{90}$$

for all $(s, t) \in (-\kappa, \kappa)$ and $y \in W$ or, which is the same,

$$\alpha_{s+t}^\Xi(y) = \alpha_s^\Xi(\alpha_t^\Xi(y)), \quad \alpha_{-t}^\Xi\alpha_t^\Xi(y) = y. \tag{91}$$

Note that the second formula implies

$$(\alpha_t^{\Xi})^{-1} = \alpha_{-t}^{\Xi}. \tag{92}$$

In the following lemma, we study properties of flows of a *π-projectable* vector field.

Lemma 9 *Let Ξ be a C^r vector field on Y. The following two conditions are equivalent*:

(1) *The local* 1*-parameter groups of* Ξ *consist of C^r automorphisms of the fibered manifold Y.*
(2) Ξ *is π-projectable.*

Proof

1. Choose $y_0 \in Y$ be a point and let $x_0 = \pi(y_0)$. Choose a local flow α^{Ξ}: $(-\varepsilon, \varepsilon) \times V \to Y$ at y_0 and suppose that the mappings α_t^{Ξ}: $V \to Y$ are C^r automorphisms of Y. Then for each t, there exists a unique C^r mapping α_t: $U \to X$, where $U = \pi(V)$ is an open set, such that

$$\pi \circ \alpha_t^{\Xi} = \alpha_t \circ \pi \tag{93}$$

on V. Setting $\alpha(t, x) = \alpha_t(x)$, we get a mapping α: $(-\varepsilon, \varepsilon) \times U \to X$. It is easily seen that this mapping is of class C^r. Indeed, there exists a C^r section γ: $U \to Y$ such that $\gamma(x_0) = y_0$ (Sect. 1.1, Theorem 3); using this section, we can write $\alpha(t, x) = \alpha_t(x) = \pi \circ \alpha_t^{\Xi} \circ \gamma(x) = \pi \circ \alpha^{\Xi}(t, \gamma(x))$, so α can be expressed as the composite of C^r mappings. Since α satisfies $\alpha(0, x) = x$, setting

$$\xi(x) = T_0\alpha_x \cdot 1 \tag{94}$$

we get a C^{r-1} vector field on U.
On the other hand, Formula (93) implies $\pi \circ \alpha^{\Xi}(t, y) = \alpha(t, \pi(y))$, that is, $\pi \circ \alpha_y^{\Xi} = \alpha_{\pi(y)}$. Then from (89), $T_t(\pi \circ \alpha_y^{\Xi}) = T_{\alpha_y^{\Xi}(t)}\pi \cdot \Xi(\alpha_y^{\Xi}(t)) = T_t\alpha_{\pi(y)}$ and we have at the point $t = 0$

$$T_0\alpha_{\pi(y)} = T_y\pi \cdot \Xi(y). \tag{95}$$

Combining (94) and (95),

$$\xi(\pi(y)) = T_y\pi \cdot \Xi(y). \tag{96}$$

π-projectability of Ξ (on Y) now follows from the uniqueness of the π-projection.

2. Suppose that Ξ is π-projectable and denote by ξ its π-projection. Then

$$T_y\pi \cdot \Xi(y) = \xi(\pi(y)) \tag{97}$$

at every point y of the fibered manifold Y. The local flow α^Ξ satisfies Eq. (89) $T_t\alpha_y^\Xi = \Xi(\alpha_y^\Xi(t))$. Applying the tangent mapping $T\pi$ to both sides, we get

$$T_t(\pi \circ \alpha_y^\Xi) = T_{\alpha_y^\Xi(t)}\pi \cdot \Xi(\alpha_y^\Xi(t)) = \xi(\pi(\alpha_y^\Xi(t))). \tag{98}$$

This equality means that the curve $t \to \pi(\alpha_y^\Xi(t)) = \alpha_{\pi(y)}^\xi(t)$ is an integral curve of the vector field ξ. Thus, denoting by α^ξ the local flow of ξ at the point $x_0 = \pi(y_0)$, we have

$$\pi(\alpha^\Xi(t,y)) = \alpha^\xi(t,\pi(y)) \tag{99}$$

as required. □

Let Ξ be a π-projectable C^r vector field on Y, ξ its π-projection. Let α_t^Ξ (resp. α_t^ξ) be a local 1-parameter group of Ξ (resp. ξ). Since the mappings α_t^ξ are C^r diffeomorphisms, for each t, the C^r automorphism α_t^Ξ can be prolonged to the jet prolongation J^sY of Y, for any s, $0 \le s \le r$. The prolonged mapping is an automorphism of the fibered manifold J^sY over X, defined by

$$J^s\alpha_t^\Xi(J_x^r\gamma) = J^s_{\alpha_t^\xi(x)}\alpha_t^\Xi\gamma\alpha_{-t}^\xi, \tag{100}$$

the *s-jet prolongation* of α_t^Ξ.

It is easily seen that there exists a unique C^s vector field on J^sY whose integral curves are exactly the curves $t \to J^s\alpha_t^\Xi(J_x^r\gamma)$. This vector field is defined by

$$J^s\Xi(J_x^r\gamma) = \left(\frac{d}{dt}J^s\alpha_t^\Xi(J_x^r\gamma)\right)_0, \tag{101}$$

and is called the *r-jet prolongation* of the vector field Ξ. It follows from the definition that the vector field $J^s\Xi$ is π^s-projectable (resp. $\pi^{s,k}$-projectable for every k, $0 \le k \le s$) and its π^k-projection (resp. $\pi^{s,k}$-projection) is ξ (resp. $J^k\Xi$).

The following lemma explains the local structure of the jet prolongations of projectable vector fields (Krupka [K13]); its proof is based on the chain rule.

Lemma 10 *Let* Ξ *be a* π*-projectable vector field on* Y*, expressed in a fibered chart* (V,ψ)*,* $\psi=(x^i,y^\sigma)$*, by*

$$\Xi=\xi^i\frac{\partial}{\partial x^i}+\Xi^\sigma\frac{\partial}{\partial y^\sigma}. \tag{102}$$

Then, $J^s\Xi$ *is expressed in the associated chart* (V^s,ψ^s) *by*

$$J^s\Xi=\xi^i\frac{\partial}{\partial x^i}+\Xi^\sigma\frac{\partial}{\partial y^\sigma}+\sum_{k=1}^{s}\sum_{j_1\le j_2\le\dots\le j_k}\Xi^\sigma_{j_1j_2\dots j_k}\frac{\partial}{\partial y^\sigma_{j_1j_2\dots j_k}}, \tag{103}$$

where the components $\Xi^\sigma_{j_1j_2\dots j_k}$ *are determined by the recurrence formula*

$$\Xi^\sigma_{j_1j_2\dots j_k}=d_{j_k}\Xi^\sigma_{j_1j_2\dots j_{k-1}}-y^\sigma_{j_1j_2\dots j_{k-1}i}\frac{\partial\xi^i}{\partial x^{j_k}}. \tag{104}$$

Proof For all sufficiently small t, we can express the local 1-parameter group of Ξ in one chart only. Equations of the C^r automorphism α^Ξ_t are expressed as

$$x^i\circ\alpha^\Xi_t(y)=x^i\alpha^\xi_t(x),\quad y^\sigma\circ\alpha^\Xi_t(y)=y^\sigma\alpha^\Xi_t(y). \tag{105}$$

From these equations, we obtain the components of the vector field Ξ in the form

$$\xi^i(y)=\left(\frac{dx^i\alpha^\xi_t(x)}{dt}\right)_0,\quad \Xi^\sigma(y)=\left(\frac{dy^\sigma\alpha^\Xi_t(y)}{dt}\right)_0. \tag{106}$$

To determine the components of $J^s\Xi$, we use Lemma 9. The 1-parameter group of $J^s\Xi$ has the equations

$$\begin{aligned}
x^i\circ J^r\alpha^\Xi_t(y)&=x^i\alpha^\xi_t(x),\\
y^\sigma\circ J^r\alpha^\Xi_t(y)&=y^\sigma\alpha^\Xi_t(y),\\
y^\sigma_{j_1j_2\dots j_k}\circ J^r\alpha^\Xi_t(J^r_x\gamma)&=D_{j_1}D_{j_2}\dots D_{j_k}(y^\sigma\alpha^\Xi_t\psi^{-1}\circ\psi\gamma\varphi^{-1}\circ\varphi\alpha^\xi_{-t}\varphi^{-1})(\varphi(\alpha^\xi_t(x))),\\
&\quad 1\le k\le s,
\end{aligned} \tag{107}$$

so by (106), it is sufficient to determine $\Xi^\sigma_{j_1j_2\dots j_k}$. By definition,

$$\Xi^\sigma_{j_1j_2\dots j_k}(J^r_x\gamma)=\left(\frac{d}{dt}(y^\sigma_{j_1j_2\dots j_k}\circ J^r\alpha^\Xi_t)(J^r_x\gamma)\right)_0. \tag{108}$$

But

$$\begin{aligned} & y^{\sigma}_{j_1 j_2 \ldots j_{k-1}} \circ J^r \alpha^{\Xi}_t (J^r_x \gamma) \\ & \quad = D_{j_1} D_{j_2} \ldots D_{j_{k-1}} (y^{\sigma} \alpha^{\Xi}_t \psi^{-1} \circ \psi \gamma \varphi^{-1} \circ \varphi \alpha^{\xi}_{-t} \varphi^{-1})(\varphi(\alpha^{\xi}_t(x))) \\ & \quad = y^{\sigma}_{j_1 j_2 \ldots j_{k-1}} \circ J^r \alpha^{\Xi}_t \circ J^r \gamma \circ \alpha^{\xi}_{-t} \varphi^{-1}(\varphi(\alpha^{\xi}_t(x))), \end{aligned} \tag{109}$$

thus,

$$\begin{aligned} & y^{\sigma}_{j_1 j_2 \ldots j_k} \circ J^r \alpha^{\Xi}_t (J^r_x \gamma) \\ & \quad = D_{j_k} (y^{\sigma}_{j_1 j_2 \ldots j_{k-1}} \circ J^r \alpha^{\Xi}_t \circ J^r \gamma \circ \varphi^{-1} \circ \varphi \alpha^{\xi}_{-t} \varphi^{-1})(\varphi(\alpha^{\xi}_t(x))) \\ & \quad = D_l (y^{\sigma}_{j_1 j_2 \ldots j_{k-1}} \circ J^r \alpha^{\Xi}_t \circ J^r \gamma \circ \varphi^{-1})(\varphi(x)) D_{j_k} (x^l \alpha^{\xi}_{-t} \varphi^{-1})(\varphi(\alpha^{\xi}_t(x))). \end{aligned} \tag{110}$$

To obtain $\Xi^{\sigma}_{j_1 j_2 \ldots j_k}(J^r_x \gamma)$ (108), we differentiate the function

$$(t, \varphi(x)) \to y^{\sigma}_{j_1 j_2 \ldots j_{k-1}} \circ J^r \alpha^{\Xi}_t (J^r_x \gamma) = (y^{\sigma}_{j_1 j_2 \ldots j_{k-1}} \circ J^r \alpha^{\Xi}_t \circ J^r \gamma \circ \varphi^{-1})(\varphi(x)) \tag{111}$$

with respect to t and x^l. Since the partial derivatives commute, we can first differentiate with respect to t at $t = 0$. We get the expression $\Xi^{\sigma}_{j_1 j_2 \ldots j_{k-1}}(J^r_x \gamma)$. Subsequent differentiation yields

$$D_l (\Xi^{\sigma}_{j_1 j_2 \ldots j_{k-1}} \circ J^r \gamma \circ \varphi^{-1})(\varphi(x)) = d_l \Xi^{\sigma}_{j_1 j_2 \ldots j_{k-1}}(J^r_x \gamma), \tag{112}$$

where d_l is the formal derivative operator.

We should also differentiate expression $D_{j_k}(x^l \alpha^{\xi}_{-t} \varphi^{-1})(\varphi(\alpha^{\xi}_t(x)))$ with respect to t. We have the identity $D_l (x^k \alpha^{\xi}_{-t} \varphi^{-1} \circ \varphi \alpha^{\xi}_t \varphi^{-1})(\varphi(x)) = \delta^k_l$, that is,

$$D_j (x^k \alpha^{\xi}_{-t} \varphi^{-1})(\varphi \alpha^{\xi}_t(x)) D_l (x^j \alpha^{\xi}_t \varphi^{-1})(\varphi(x)) = \delta^k_l. \tag{113}$$

From this formula,

$$\begin{aligned} & \frac{d}{dt} D_j (x^k \alpha^{\xi}_{-t} \varphi^{-1})(\varphi \alpha^{\xi}_t(x)) \cdot D_l (x^j \alpha^{\xi}_t \varphi^{-1})(\varphi(x)) \\ & \quad + D_j (x^k \alpha^{\xi}_{-t} \varphi^{-1})(\varphi \alpha^{\xi}_t(x)) \cdot \frac{d}{dt} D_l (x^j \alpha^{\xi}_t \varphi^{-1})(\varphi(x)) \\ & = 0 \end{aligned} \tag{114}$$

thus, at $t = 0$,

$$\left(\frac{d}{dt} D_j (x^k \alpha^{\xi}_{-t} \varphi^{-1})(\varphi \alpha^{\xi}_t(x)) \right)_0 \cdot \delta^j_l + \delta^k_j D_l \xi^j(\varphi(x)) = 0, \tag{115}$$

hence,

$$\left(\frac{d}{dt}D_l(x^k\alpha^{\xi}_{-t}\varphi^{-1})(\varphi\alpha^{\xi}_t(x))\right)_0 = -D_l\xi^k(\varphi(x)). \tag{116}$$

Now we can complete the differentiation of Formula (110) at $t = 0$. We have, using (112) and (116)

$$\begin{aligned}\Xi^{\sigma}_{j_1 j_2 \ldots j_k}(J^r_x\gamma) &= \left(\frac{d}{dt}(y^{\sigma}_{j_1 j_2 \ldots j_k}\circ J^r\alpha^{\Xi}_t)(J^r_x\gamma)\right)_0\\ &= \left(\frac{d}{dt}D_l(y^{\sigma}_{j_1 j_2 \ldots j_{k-1}}\circ J^r\alpha^{\Xi}_t\circ J^r\gamma\circ\varphi^{-1})(\varphi(x))\right)_0\delta^l_{j_k}\\ &\quad + D_l(y^{\sigma}_{j_1 j_2 \ldots j_{k-1}}\circ J^r\gamma\circ\varphi^{-1})(\varphi(x))\left(\frac{d}{dt}D_{j_k}(x^l\alpha^{\xi}_{-t}\varphi^{-1})(\varphi(\alpha^{\xi}_t(x)))\right)_0\\ &= d_l\Xi^{\sigma}_{j_1 j_2 \ldots j_{k-1}}(J^{r+1}_x\gamma)\delta^l_{j_k}\\ &\quad - D_l(y^{\sigma}_{j_1 j_2 \ldots j_{k-1}}\circ J^r\gamma\circ\varphi^{-1})(\varphi(x))D_{j_k}\xi^l(\varphi(x))\\ &= d_{j_k}\Xi^{\sigma}_{j_1 j_2 \ldots j_{k-1}}(J^r_x\gamma) - y^{\sigma}_{j_1 j_2 \ldots j_{k-1} l}(J^r_x\gamma)D_{j_k}\xi^l(\varphi(x)),\end{aligned} \tag{117}$$

which coincides with (104). □

Example 8 (2-jet prolongation of a vector field) Let a π-projectable vector field Ξ be expressed by

$$\Xi = \xi^i\frac{\partial}{\partial x^i} + \Xi^{\sigma}\frac{\partial}{\partial y^{\sigma}}. \tag{118}$$

We can calculate the components of the second-jet prolongation $J^2\Xi$ from Lemma 10. We get

$$J^2\Xi = \xi^i\frac{\partial}{\partial x^i} + \Xi^{\sigma}\frac{\partial}{\partial y^{\sigma}} + \Xi^{\sigma}_j\frac{\partial}{\partial y^{\sigma}_j} + \sum_{j\le k}\Xi^{\sigma}_{jk}\frac{\partial}{\partial y^{\sigma}_{jk}}, \tag{119}$$

we get

$$\begin{aligned}\Xi^{\sigma}_j &= d_j\Xi^{\sigma} - y^{\sigma}_i\frac{\partial\xi^i}{\partial x^j},\\ \Xi^{\sigma}_{jk} &= d_jd_k\Xi^{\sigma} - y^{\sigma}_{ij}\frac{\partial\xi^i}{\partial x^k} - y^{\sigma}_{ik}\frac{\partial\xi^i}{\partial x^j} - y^{\sigma}_i\frac{\partial^2\xi^i}{\partial x^j\partial x^k}.\end{aligned} \tag{120}$$

In the following lemma, we study the *Lie bracket* of r-jet prolongations of projectable vector fields.

Lemma 11 *For any two π-projectable vector fields Ξ and Z, the Lie bracket* $[\Xi, Z]$ *is also π-projectable, and*

$$J^r[\Xi, Z] = [J^r\Xi, J^r Z]. \tag{121}$$

Proof

1. First, we prove (121) for $r = 1$. Suppose that in a fibered chart

$$\Xi = \xi^i \frac{\partial}{\partial x^i} + \Xi^\sigma \frac{\partial}{\partial y^\sigma}, \quad Z = \zeta^k \frac{\partial}{\partial x^k} + Z^\nu \frac{\partial}{\partial y^\nu}. \tag{122}$$

Then

$$J^1\Xi = \xi^i \frac{\partial}{\partial x^i} + \Xi^\sigma \frac{\partial}{\partial y^\sigma} + \Xi^\sigma_j \frac{\partial}{\partial y^\sigma_j}, \quad J^1 Z = \zeta^i \frac{\partial}{\partial x^i} + Z^\sigma \frac{\partial}{\partial y^\sigma} + Z^\sigma_j \frac{\partial}{\partial y^\sigma_j}, \tag{123}$$

where

$$\Xi^\sigma_j = d_j \Xi^\sigma - y^\sigma_i \frac{\partial \xi^i}{\partial x^j}, \quad Z^\sigma_j = d_j Z^\sigma - y^\sigma_i \frac{\partial \zeta^i}{\partial x^j}, \tag{124}$$

and

$$\begin{aligned}
[J^1\Xi, J^1 Z] = {} & \left(\frac{\partial \zeta^i}{\partial x^l}\xi^l - \frac{\partial \xi^i}{\partial x^l}\zeta^l\right)\frac{\partial}{\partial x^i} \\
& + \left(\frac{\partial Z^\sigma}{\partial x^l}\xi^l + \frac{\partial Z^\sigma}{\partial y^\nu}\Xi^\nu - \frac{\partial \Xi^\sigma}{\partial x^l}\zeta^l - \frac{\partial \Xi^\sigma}{\partial y^\nu}Z^\nu\right)\frac{\partial}{\partial y^\sigma} \\
& + \left(\frac{\partial Z^\sigma_j}{\partial x^l}\xi^l + \frac{\partial Z^\sigma_j}{\partial y^\nu}\Xi^\nu + \frac{\partial Z^\sigma_j}{\partial y^\nu_l}\Xi^\nu_l - \frac{\partial \Xi^\sigma_j}{\partial x^l}\zeta^l - \frac{\partial \Xi^\sigma_j}{\partial y^\nu}Z^\nu - \frac{\partial \Xi^\sigma_j}{\partial y^\nu_l}Z^\nu_l\right)\frac{\partial}{\partial y^\sigma_j}.
\end{aligned} \tag{125}$$

On the other hand, denoting $\Theta = [\Xi, Z]$, we have

$$\Theta = \vartheta^i \frac{\partial}{\partial x^i} + \Theta^\sigma \frac{\partial}{\partial y^\sigma}, \tag{126}$$

where

$$\begin{aligned}\vartheta^i &= \frac{\partial \zeta^i}{\partial x^s}\xi^s - \frac{\partial \xi^i}{\partial x^s}\zeta^s,\\ \Theta^\sigma &= \frac{\partial Z^\sigma}{\partial x^s}\xi^s + \frac{\partial Z^\sigma}{\partial y^\nu}\Xi^\nu - \frac{\partial \Xi^\sigma}{\partial x^s}\zeta^s - \frac{\partial \Xi^\sigma}{\partial y^\nu}Z^\nu,\end{aligned} \tag{127}$$

and

$$J^1\Theta = \Theta^i\frac{\partial}{\partial x^i} + \Theta^\sigma\frac{\partial}{\partial y^\sigma} + \Theta^\sigma_j\frac{\partial}{\partial y^\sigma_j}, \tag{128}$$

where

$$\Theta^\sigma_j = d_j\Theta^\sigma - y^\sigma_i\frac{\partial \theta^i}{\partial x^j}. \tag{129}$$

Comparing formulas (121) and (129), we see that to prove our assertion for $r = 1$, it is sufficient to show that

$$\begin{aligned}&d_j\left(\frac{\partial Z^\sigma}{\partial x^s}\xi^s + \frac{\partial Z^\sigma}{\partial y^\nu}\Xi^\nu - \frac{\partial \Xi^\sigma}{\partial x^s}\zeta^s - \frac{\partial \Xi^\sigma}{\partial y^\nu}Z^\nu\right)\\ &\quad - y^\sigma_i\frac{\partial}{\partial x^j}\left(\frac{\partial \zeta^i}{\partial x^s}\xi^s - \frac{\partial \xi^i}{\partial x^s}\zeta^s\right)\\ &= \frac{\partial Z^\sigma_j}{\partial x^l}\xi^l + \frac{\partial Z^\sigma_j}{\partial y^\nu}\Xi^\nu + \frac{\partial Z^\sigma_j}{\partial y^\nu_l}\Xi^\nu_l - \frac{\partial \Xi^\sigma_j}{\partial x^l}\zeta^l - \frac{\partial \Xi^\sigma_j}{\partial y^\nu}Z^\nu - \frac{\partial \Xi^\sigma_j}{\partial y^\nu_l}Z^\nu_l.\end{aligned} \tag{130}$$

We shall consider the left- and right-hand sides of this formula separately. The left-hand side can be expressed as

$$\begin{aligned}&d_j\frac{\partial Z^\sigma}{\partial x^s}\xi^s + \frac{\partial Z^\sigma}{\partial x^s}\frac{\partial \xi^s}{\partial x^j} + d_j\frac{\partial Z^\sigma}{\partial y^\nu}\Xi^\nu + \frac{\partial Z^\sigma}{\partial y^\nu}d_j\Xi^\nu\\ &\quad - d_j\frac{\partial \Xi^\sigma}{\partial x^s}\zeta^s - \frac{\partial \Xi^\sigma}{\partial x^s}\frac{\partial \zeta^s}{\partial x^j} - d_j\frac{\partial \Xi^\sigma}{\partial y^\nu}Z^\nu - \frac{\partial \Xi^\sigma}{\partial y^\nu}d_jZ^\nu\\ &\quad - y^\sigma_i\left(\frac{\partial^2 \zeta^i}{\partial x^j\partial x^s}\xi^s + \frac{\partial \zeta^i}{\partial x^s}\frac{\partial \xi^s}{\partial x^j} - \frac{\partial^2 \xi^i}{\partial x^j\partial x^s}\zeta^s - \frac{\partial \xi^i}{\partial x^s}\frac{\partial \zeta^s}{\partial x^j}\right).\end{aligned} \tag{131}$$

The right-hand side of (130) is

$$\begin{aligned}&\left(d_j\frac{\partial Z^\sigma}{\partial x^l}-y_i^\sigma\frac{\partial^2\zeta^i}{\partial x^l\partial x^j}\right)\xi^l+d_j\frac{\partial Z^\sigma}{\partial y^\nu}\Xi^\nu+\frac{\partial}{\partial y_l^\nu}\left(d_jZ^\sigma-y_i^\sigma\frac{\partial\zeta^i}{\partial x^j}\right)\Xi_l^\nu\\&\quad-\left(d_j\frac{\partial\Xi^\sigma}{\partial x^l}-y_i^\sigma\frac{\partial^2\xi^i}{\partial x^l\partial x^j}\right)\zeta^l-d_j\frac{\partial\Xi^\sigma}{\partial y^\nu}Z^\nu-\frac{\partial}{\partial y_l^\nu}\left(d_j\Xi^\sigma-y_i^\sigma\frac{\partial\xi^i}{\partial x^j}\right)Z_l^\nu\\&=\left(d_j\frac{\partial Z^\sigma}{\partial x^l}-y_i^\sigma\frac{\partial^2\zeta^i}{\partial x^l\partial x^j}\right)\xi^l+d_j\frac{\partial Z^\sigma}{\partial y^\nu}\Xi^\nu+\left(d_j\Xi^\nu-y_i^\nu\frac{\partial\xi^i}{\partial x^j}\right)\frac{\partial Z^\sigma}{\partial y^\nu}\\&\quad-\left(d_l\Xi^\sigma-y_i^\sigma\frac{\partial\xi^i}{\partial x^l}\right)\frac{\partial\zeta^l}{\partial x^j}\\&\quad-\left(d_j\frac{\partial\Xi^\sigma}{\partial x^l}-y_i^\sigma\frac{\partial^2\xi^i}{\partial x^l\partial x^j}\right)\zeta^l-d_j\frac{\partial\Xi^\sigma}{\partial y^\nu}Z^\nu-\left(d_jZ^\nu-y_i^\nu\frac{\partial\zeta^i}{\partial x^j}\right)\frac{\partial\Xi^\sigma}{\partial y^\nu}\\&\quad+\left(d_lZ^\sigma-y_i^\sigma\frac{\partial\zeta^i}{\partial x^l}\right)\frac{\partial\xi^l}{\partial x^j}.\end{aligned}\tag{132}$$

In this formula,

$$\begin{aligned}d_lZ^\sigma\frac{\partial\xi^l}{\partial x^j}-y_i^\nu\frac{\partial\xi^i}{\partial x^j}\frac{\partial Z^\sigma}{\partial y^\nu}&=\frac{\partial Z^\sigma}{\partial x^l}\frac{\partial\xi^l}{\partial x^j}+\frac{\partial Z^\sigma}{\partial y^\nu}y_l^\nu\frac{\partial\xi^l}{\partial x^j}-y_i^\nu\frac{\partial\xi^i}{\partial x^j}\frac{\partial Z^\sigma}{\partial y^\nu}\\&=\frac{\partial Z^\sigma}{\partial x^l}\frac{\partial\xi^l}{\partial x^j},\end{aligned}\tag{133}$$

and

$$\begin{aligned}-d_l\Xi^\sigma\frac{\partial\zeta^l}{\partial x^j}+y_i^\nu\frac{\partial\zeta^i}{\partial x^j}\frac{\partial\Xi^\sigma}{\partial y^\nu}&=-\frac{\partial\Xi^\sigma}{\partial x^l}\frac{\partial\zeta^l}{\partial x^j}-\frac{\partial\Xi^\sigma}{\partial y^\nu}y_l^\nu\frac{\partial\zeta^l}{\partial x^j}+y_i^\nu\frac{\partial\zeta^i}{\partial x^j}\frac{\partial\Xi^\sigma}{\partial y^\nu}\\&=-\frac{\partial\Xi^\sigma}{\partial x^l}\frac{\partial\zeta^l}{\partial x^j},\end{aligned}\tag{134}$$

thus,

$$\begin{aligned}&\frac{\partial Z_j^\sigma}{\partial x^l}\xi^l+\frac{\partial Z_j^\sigma}{\partial y^\nu}\Xi^\nu+\frac{\partial Z_j^\sigma}{\partial y_l^\nu}\Xi_l^\nu-\frac{\partial\Xi_j^\sigma}{\partial x^l}\zeta^l-\frac{\partial\Xi_j^\sigma}{\partial y^\nu}Z^\nu-\frac{\partial\Xi_j^\sigma}{\partial y_l^\nu}Z_l^\nu\\&\qquad=\left(d_j\frac{\partial Z^\sigma}{\partial x^l}-y_i^\sigma\frac{\partial^2\zeta^i}{\partial x^l\partial x^j}\right)\xi^l+d_j\frac{\partial Z^\sigma}{\partial y^\nu}\Xi^\nu+d_j\Xi^\nu\frac{\partial Z^\sigma}{\partial y^\nu}+y_i^\sigma\frac{\partial\xi^i}{\partial x^l}\frac{\partial\zeta^l}{\partial x^j}\\&\qquad\quad-\left(d_j\frac{\partial\Xi^\sigma}{\partial x^l}-y_i^\sigma\frac{\partial^2\xi^i}{\partial x^l\partial x^j}\right)\zeta^l-d_j\frac{\partial\Xi^\sigma}{\partial y^\nu}Z^\nu-d_jZ^\nu\frac{\partial\Xi^\sigma}{\partial y^\nu}-y_i^\sigma\frac{\partial\zeta^i}{\partial x^l}\frac{\partial\xi^l}{\partial x^j}.\\&\qquad\quad+\frac{\partial Z^\sigma}{\partial x^l}\frac{\partial\xi^l}{\partial x^j}-\frac{\partial\Xi^\sigma}{\partial x^l}\frac{\partial\zeta^l}{\partial x^j}.\end{aligned}\tag{135}$$

This is, however, exactly expression (130), proving (121) for $r=1$.

2. In this part of the proof, we consider the r-jet prolongation $J^{r-1}Y$ as a fibered manifold with base X and projection $\pi^{r-1}: J^{r-1}Y \to X$, and the 1-jet prolongation of this fibered manifold, $J^1 J^{r-1}Y$. $J^r Y$ can be embedded in $J^1 J^{r-1}Y$ by the *canonical injection*

$$J^r Y \ni J^r_x \gamma \to \iota(J^r_x \gamma) = J^1_x J^{r-1}\gamma \in J^1 J^{r-1} Y. \tag{136}$$

Obviously, ι is compatible with jet prolongations of automorphisms α of Y in the sense that

$$\iota \circ J^r \alpha = (J^1 J^{r-1}\alpha) \circ \iota. \tag{137}$$

Indeed, we have for any point $J^r_x \gamma$ from the domain of $J^r \alpha$

$$\iota(J^r \alpha(J^r_x \gamma)) = \iota(J^r_{\alpha_0(x)} \alpha\gamma\alpha_0^{-1}) = J^1_{\alpha_0(x)}(J^{r-1}\alpha\gamma\alpha_0^{-1}), \tag{138}$$

and also

$$J^1 J^{r-1}\alpha(\iota(J^r_x \gamma)) = J^1 J^{r-1}\alpha(J^1_x J^{r-1}\gamma) = J^1_{\alpha_0(x)}(J^{r-1}\alpha \circ J^{r-1}\gamma \circ \alpha_0^{-1}). \tag{139}$$

Thus, (138) follows from the definition of the 1-jet prolongation of a fibered automorphism (Sect. 1.4, (101)).
Then, however, applying (139) to local 1-parameter groups of a π-projectable vector field Ξ, we get ι-compatibility of $J^1 J^{r-1}\Xi$ and $J^r \Xi$,

$$J^1 J^{r-1}\Xi \circ \iota = T\iota \cdot J^r \Xi. \tag{140}$$

Since for any two π-projectable vector fields Ξ and Z the vector fields $J^1 J^{r-1}\Xi$ $J^r\Xi$ and $J^1 J^{r-1}\mathrm{Z}$ and $J^r \mathrm{Z}$ are ι-compatible, the corresponding Lie brackets are also ι-compatible and we have

$$[J^1 J^{r-1}\Xi, J^1 J^{r-1}\mathrm{Z}] \circ \iota = T\iota \cdot [J^r \Xi, J^r \mathrm{Z}]. \tag{141}$$

3. Using Part 1 of this proof, we now express the vector field on the left-hand side of (141) in a different way. First note that

$$[J^1 J^{r-1}\Xi, J^1 J^{r-1}\mathrm{Z}] = J^1 [J^{r-1}\Xi, J^{r-1}\mathrm{Z}]. \tag{142}$$

But we may suppose that $[J^{r-1}\Xi, J^{r-1}\mathrm{Z}] = J^{r-1}[\Xi, \mathrm{Z}]$ (induction hypothesis), thus $[J^1 J^{r-1}\Xi, J^1 J^{r-1}\mathrm{Z}] = J^1 J^{r-1}[\Xi, \mathrm{Z}]$. Restricting both sides by ι and applying (137),

$$[J^1 J^{r-1}\Xi, J^1 J^{r-1}\mathrm{Z}] \circ \iota = J^1 J^{r-1}[\Xi, \mathrm{Z}] \circ \iota = T\iota \cdot J^r [\Xi, \mathrm{Z}]. \tag{143}$$

Now from (142) and (144), we conclude that $T\iota \cdot ([J^r\Xi, J^rZ] - J^r[\Xi, Z]) = 0$. This implies, however, $[J^r\Xi, J^rZ] - J^r[\Xi, Z] = 0$ because $T\iota$ is at every point injective.

This completes the proof of formula (121). □

Remark 4 (Equations of the canonical injection) We find the chart expression of the canonical injection $\iota \colon J^rY \to J^1J^{r-1}Y$ (136) in a fibered chart (V, ψ), $\psi = (x^i, y^\sigma)$, on Y and the induced fibered chart on J^rY. We also have a fibered chart on $J^1J^{r-1}Y$, induced by the fibered chart (V^{r-1}, ψ^{r-1}), $\psi = (x^i, y^\sigma, y^\sigma_{j_1}, y^\sigma_{j_1j_2}, \ldots, y^\sigma_{j_1j_2\ldots j_{r-1}})$, on $J^{r-1}Y$. We denote the fibered chart on $J^1J^{r-1}Y$ by (W, Ψ), where the coordinate functions are denoted as

$$\Psi = (x^i, y^\sigma, y^\sigma_{j_1}, y^\sigma_{j_1j_2}, \ldots, y^\sigma_{j_1j_2\ldots j_{r-1}}, y^\sigma_{,k}, y^\sigma_{j_1,k}, y^\sigma_{j_1j_2,k}, \ldots, y^\sigma_{j_1j_2\ldots j_{r-1},k}). \tag{144}$$

Then by definition,

$$\begin{aligned} y^\sigma_{j_1j_2\ldots j_s,k} \circ \iota(J^r_x\gamma) &= D_k(y^\sigma_{j_1j_2\ldots j_s} \circ J^{r-1}\gamma \circ \varphi^{-1})(\varphi(x)) \\ &= D_kD_{j_1}D_{j_2}\ldots D_{j_s}(y^\sigma\gamma\varphi^{-1})(\varphi(x)) = y^\sigma_{j_1j_2\ldots j_sk}(J^r_x\gamma) \end{aligned} \tag{145}$$

for all $s = 1, 2, \ldots, r-1$, so the canonical injection ι is expressed by the equations

$$\begin{aligned} &x^i \circ \iota = x^i, \quad y^\sigma \circ \iota = y^\sigma, \quad y^\sigma_{j_1j_2\ldots j_s} \circ \iota = y^\sigma_{j_1j_2\ldots j_s}, \quad 1 \le s \le r-1, \\ &y^\sigma_{j_1j_2\ldots j_s,k} \circ \iota = y^\sigma_{j_1j_2\ldots j_sk}, \quad 1 \le s \le r-1. \end{aligned} \tag{146}$$

References

[K13] D. Krupka, *Some Geometric Aspects of Variational Problems in Fibered Manifolds*, Folia Fac. Sci. Nat. UJEP Brunensis, Physica 14, Brno, Czech Republic, 1973, 65 pp.; arXiv. math-ph/0110005, 2001

[L] J.M. Lee, *Introduction to Smooth Manifolds*, Graduate Texts in Math. 218, Springer, 2006

[S] D.J. Saunders, *The Geometry of Jet Bundles*, Cambridge Univ. Press, 1989

[K17] D. Krupka, Variational principles for energy-momentum tensors, Rep. Math. Phys. 49 (2002) 259-268

Chapter 2
Differential Forms on Jet Prolongations of Fibered Manifolds

In this chapter, we present a decomposition theory of differential forms on jet prolongations of fibered manifolds; the tools inducing the decompositions are the algebraic trace decomposition theory and the canonical jet projections. Of particular interest is the structure of the *contact forms*, annihilating integrable sections of the jet prolongations. We also study decompositions of forms defined by fibered homotopy operators and state the corresponding fibered Poincare-Volterra lemma.

The theory of differential forms explained in this chapter has been developed along the lines indicated in the approach of Lepage and Dedecker to the calculus of variations (see Dedecker [D], Goldschmidt and Sternberg [GS] and Krupka [K13]). The exposition extends the theory explained in the handbook chapter Krupka [K4].

Throughout, Y is a smooth fibered manifold with base X and projection π, $n = \dim X$, $n + m = \dim Y$. $J^r Y$ is the *r-jet prolongation* of Y, and $\pi^r \colon J^r Y \to X$, $\pi^r \colon J^r Y \to X$ are the canonical jet projections. For any open set $W \subset Y$, $\Omega^r_q W$ denotes the module of q-forms on the open set $W^r = (\pi^{r,0})^{-1}(W)$ in $J^r Y$, and $\Omega^r W$ is the exterior algebra of differential forms on the set W^r. We say that a form η is *generated* by a finite family of forms μ_κ, if η is expressible as $\eta = \eta^\kappa \wedge \mu_\kappa$ for some forms η^κ; note that in this terminology, we do not require μ_κ to be 1-forms, or k-forms for a fixed integer k.

2.1 The Contact Ideal

We introduced in Sect. 1.5 a vector bundle homomorphism h between the tangent bundles $TJ^{r+1}Y$ and $TJ^r Y$ over the canonical jet projection $\pi^{r+1,r} \colon J^{r+1}Y \to J^r Y$, the *horizontalization*. In this section, the associated *dual* mapping between the modules of 1-forms $\Omega^r_1 W$ and $\Omega^{r+1}_1 W$ is studied. We show, in particular, that this mapping allows us to associate with any fibered chart (V, ψ) on Y and any function, defined on V^r, its *formal* (or *total*) *partial derivatives* in a geometric way and a specific basis of 1-forms on V^r, termed the *contact basis*. Then, we introduce by means of the contact

D. Krupka, *Introduction to Global Variational Geometry*,
Atlantis Studies in Variational Geometry 1, DOI 10.2991/978-94-6239-073-7_2

basis a differential ideal in the exterior algebra $\Omega^r W$, characterizing the structure of forms on jet prolongations of fibered manifolds, the *contact ideal*.

Recall that the horizontalization h is defined by the formula

$$h\xi = T_x J^r \gamma \circ T\pi^{r+1} \cdot \xi, \tag{1}$$

where ξ is a tangent vector to the manifold $J^{r+1}Y$ at a point $J^{r+1}_x \gamma$. The mapping h makes the following diagram

$$\begin{array}{ccc} TJ^{r+1}Y & \xrightarrow{h} & TJ^r Y \\ \downarrow & & \downarrow \\ J^{r+1}Y & \xrightarrow{\pi^{r+1,r}} & J^r Y \end{array} \tag{2}$$

commutative and induces a decomposition of the projections of the tangent vectors $T\pi^{r+1,r} \cdot \xi$,

$$T\pi^{r+1,r} \cdot \xi = h\xi + p\xi. \tag{3}$$

$h\xi$ (resp. $p\xi$) is the *horizontal* (resp. *contact*) *component* of the vector ξ. Note, however, that the terminology is *not* standard: The vectors ξ and $h\xi$ do not belong to the same vector space. The horizontal and contact components satisfy

$$T\pi^r \cdot h\xi = T\pi^{r+1} \cdot \xi, \quad T\pi^r \cdot p\xi = 0. \tag{4}$$

The horizontalization h induces a mapping of modules of linear differential forms as follows. Let $J^{r+1}_x \gamma \in J^{r+1}Y$. We set for any differential 1-form ρ on W^r and any vector ξ from the tangent space $TJ^{r+1}Y$ at $J^{r+1}_x \gamma$

$$h\rho(J^{r+1}_x \gamma) \cdot \xi = \rho(J^r_x \gamma) \cdot h\xi. \tag{5}$$

The mapping $\Omega^r_1 W \ni \rho \to h\rho \in \Omega^{r+1}_1 W$ is called the π-*horizontalization* or just the *horizontalization* (of differential forms).

Clearly, the form $h\rho$ vanishes on π^{r+1}-vertical vectors so it is π^{r+1}-horizontal; $h\rho$ is sometimes called the *horizontal component* of ρ.

The mapping h is linear over the ring of functions $\Omega^r_0 W$ along the jet projection $\pi^{r+1,r}$ in the sense that

$$h(\rho_1 + \rho_2) = h\rho_1 + h\rho_2 \quad h(f\rho) = (f \circ \pi^{r+1,r})h\rho \tag{6}$$

for all $\rho_1, \rho_2, \rho \in \Omega^r_1 W$ and $f \in \Omega^r_0 W$.

If in the fibered chart (V, ψ), $\psi = (x^i, y^\sigma)$, a 1-form ρ is expressed by

$$\rho = A_i dx^i + \sum_{0 \le k \le r} \sum_{j_1 \le j_2 \le \cdots \le j_k} B_\sigma^{j_1 j_2 \ldots j_k} dy^\sigma_{j_1 j_2 \ldots j_k}, \tag{7}$$

then we have from (5) at any point $J_x^{r+1}\gamma \in V^{r+1}$

$$\begin{aligned} h\rho(J_x^{r+1}\gamma) \cdot \xi &= A_i(J_x^r\gamma) dx^i (J_x^r\gamma) \cdot h\xi \\ &\quad + \sum_{0 \le k \le r} \sum_{j_1 \le j_2 \le \cdots \le j_k} B_\sigma^{j_1 j_2 \ldots j_k}(J_x^r\gamma) dy^\sigma_{j_1 j_2 \ldots j_k}(J_x^r\gamma) \cdot h\xi \\ &= \left(A_i(J_x^r\gamma) + \sum_{0 \le k \le r} \sum_{j_1 \le j_2 \le \cdots \le j_k} B_\sigma^{j_1 j_2 \ldots j_k}(J_x^r\gamma) y^\sigma_{j_1 j_2 \ldots j_k i} \right) \xi^i, \end{aligned} \tag{8}$$

thus,

$$h\rho = \left(A_i + \sum_{0 \le k \le r} \sum_{j_1 \le j_2 \le \cdots \le j_k} B_\sigma^{j_1 j_2 \ldots j_k} y^\sigma_{j_1 j_2 \ldots j_k i} \right) dx^i. \tag{9}$$

In particular, for any function $f \colon W^r \to \mathbf{R}$

$$hdf = d_i f \cdot dx^i, \tag{10}$$

where

$$d_i f = \frac{\partial f}{\partial x^i} + \sum_{j_1 \le j_2 \le \cdots \le j_k} \frac{\partial f}{\partial y^\sigma_{j_1 j_2 \ldots j_k}} y^\sigma_{j_1 j_2 \ldots j_k i}. \tag{11}$$

The function $d_i f \colon V^{r+1} \to \mathbf{R}$ is the *i-th formal derivative* of f with respect to the fibered chart (V, ψ). From (10), it follows that $d_i f$ are the components of an invariant object, the *horizontal component hdf* of the exterior derivative of f. Note that formal derivatives $d_i f$ have already been introduced in Sect. 1.5.

The following lemma summarizes basic rules for computations with the horizontalization and formal derivatives. We denote by $\bar{d}_i$ the formal derivative operator with respect to a fibered chart $(\bar{V}, \bar{\psi})$, $\bar{\psi} = (\bar{x}^i, \bar{y}^\sigma)$.

Lemma 1 *Let (V, ψ), $\psi = (x^i, y^\sigma)$, be a fibered chart on Y.*

(a) *The horizontalization h satisfies*

$$\begin{aligned} hdy^\sigma = y^\sigma_i dx^i, \quad hdy^\sigma_{j_1} = y^\sigma_{j_1 i} dx^i, \quad hdy^\sigma_{j_1 j_2} = y^\sigma_{j_1 j_2 i} dx^i, \\ \ldots, \; hdy^\sigma_{j_1 j_2 \ldots j_r} = y^\sigma_{j_1 j_2 \ldots j_r i} dx^i. \end{aligned} \tag{12}$$

(b) *The i-th formal derivative of the coordinate function* $y^{\nu}_{j_1 j_2 \dots j_k}$ *is given by*

$$d_i y^{\nu}_{j_1 j_2 \dots j_k} = y^{\nu}_{j_1 j_2 \dots j_k i}. \tag{13}$$

(c) *If* $(\overline{V}, \overline{\psi})$, $\overline{\psi} = (\overline{x}^i, \overline{y}^\sigma)$, *is another chart on* Y *such that* $V \cap \overline{V} \neq \emptyset$, *then for every function* $f \colon V^r \cap \overline{V}^r \to \mathbf{R}$,

$$\overline{d}_i f = d_j f \cdot \frac{\partial x^j}{\partial \overline{x}^i}. \tag{14}$$

(d) *For any two functions* $f, \; g \colon V^r \to \mathbf{R}$,

$$d_i (f \cdot g) = g \cdot d_i f + f \cdot d_i g. \tag{15}$$

(e) *For every function* $f \colon V^r \to \mathbf{R}$ *and every section* $\gamma \colon U \to V \subset Y$,

$$d_i f \circ J^{r+1}\gamma = \frac{\partial (f \circ J^r \gamma)}{\partial x^i}. \tag{16}$$

Remark 1 By (13), $\overline{y}^\sigma_{j_1 j_2 \dots j_k} = \overline{d}_{j_k} \overline{y}^\sigma_{j_1 j_2 \dots j_{k-1}}$. Thus, applying (14) to coordinates, we obtain the following *prolongation formula* for coordinate transformations in jet prolongations of fibered manifolds

$$\overline{y}^\sigma_{j_1 j_2 \dots j_k} = d_i \overline{y}^\sigma_{j_1 j_2 \dots j_{k-1}} \cdot \frac{\partial x^i}{\partial \overline{x}^{j_k}}. \tag{17}$$

Remark 2 If two functions $f, g \colon V^r \to \mathbf{R}$ coincide along a section $J^r\gamma$, that is, $f \circ J^r \gamma = g \circ J^r \gamma$, then their formal derivatives coincide along the $(r+1)$-prolongation $J^{r+1}\gamma$,

$$d_i f \circ J^{r+1}\gamma = d_i g \circ J^{r+1}\gamma. \tag{18}$$

This is an immediate consequence of formula (16).

Now, we study properties of 1-forms, belonging to the kernel of the horizontalization $\Omega^r_1 W \ni \rho \to h\rho \in \Omega^{r+1}_1 W$. We say that a 1-form $\rho \in \Omega^r_1 W$ is *contact*, if

$$h\rho = 0. \tag{19}$$

It is easy to find the chart expression of a contact 1-form. Writing ρ as in (7), condition (19) yields

$$A_i + \sum_{0 \leq k \leq r} \sum_{j_1 \leq j_2 \leq \dots \leq j_k} B^{j_1 j_2 \dots j_k}_\sigma y^\sigma_{j_1 j_2 \dots j_k i} = 0, \tag{20}$$

or, equivalently,

$$B_\sigma^{j_1 j_2 \dots j_r} = 0, \quad A_i = -\sum_{0 \le k \le r-1} \sum_{j_1 \le j_2 \le \dots \le j_k} B_\sigma^{j_1 j_2 \dots j_k} y^\sigma_{j_1 j_2 \dots j_k i}. \tag{21}$$

Thus, setting for all k, $0 \le k \le r-1$,

$$\omega^\sigma_{j_1 j_2 \dots j_k} = dy^\sigma_{j_1 j_2 \dots j_k} - y^\sigma_{j_1 j_2 \dots j_k j} dx^j, \tag{22}$$

we see that ρ has the chart expression

$$\rho = \sum_{0 \le k \le r-1} \sum_{j_1 \le j_2 \le \dots \le j_k} B_\sigma^{j_1 j_2 \dots j_k} \omega^\sigma_{j_1 j_2 \dots j_k}. \tag{23}$$

This formula shows that any contact 1-form is expressible as a linear combination of the forms $\omega^\sigma_{j_1 j_2 \dots j_k}$.

The following two theorems summarize properties of the forms $\omega^\sigma_{j_1 j_2 \dots j_k}$.

Theorem 1

(a) *For any fibered chart (V, ψ), $\psi = (x^i, y^\sigma)$, the forms*

$$dx^i, \quad \omega^\sigma_{j_1 j_2 \dots j_k}, \quad dy^\sigma_{l_1 l_2 \dots l_{r-1} l_r}, \tag{24}$$

such that $1 \le i \le n$, $1 \le \sigma \le m$, $1 \le k \le r-1$, $1 \le j_1 \le j_2 \le \dots \le j_k \le n$, and $1 \le l_1 \le l_2 \le \dots \le l_r \le n$, constitute a basis of linear forms on the set V^r.

(b) *If (V, ψ), $\psi = (x^i, y^\sigma)$, and $(\overline{V}, \overline{\psi})$, $\overline{\psi} = (\overline{x}^i, \overline{y}^\sigma)$, are two fibered charts such that $V \cap \overline{V} \ne \emptyset$, then*

$$\omega^\lambda_{p_1 p_2 \dots p_k} = \sum_{0 \le m \le k} \sum_{j_1 \le j_2 \le \dots \le j_k} \frac{\partial y^\lambda_{p_1 p_2 \dots p_k}}{\partial \overline{y}^\tau_{j_1 j_2 \dots j_m}} \overline{\omega}^\tau_{j_1 j_2 \dots j_m}. \tag{25}$$

(c) *Let (V, ψ), $\psi = (x^i, y^\sigma)$, and $(\overline{V}, \overline{\psi})$, $\overline{\psi} = (\overline{x}^i, \overline{y}^\sigma)$, be two fibered charts and α an automorphism of Y, defined on V and such that $\alpha(V) \subset \overline{V}$. Then*

$$J^r \alpha^* \overline{\omega}^\sigma_{j_1 j_2 \dots j_k} = \sum_{i < i_2 < \dots < i_p} \frac{\partial(\overline{y}^\sigma_{j_1 j_2 \dots j_k} \circ J^r \alpha)}{\partial y^\nu_{i_1 i_2 \dots i_p}} \omega^\nu_{i_1 i_2 \dots i_p}. \tag{26}$$

Proof

(a) Clearly, from formula (22), we conclude that the forms (24) are expressible as linear combinations of the forms of the canonical basis dx^i, $dy^\sigma_{j_1 j_2 \dots j_k}$, $dy^\sigma_{l_1 l_2 \dots l_{r-1} l_r}$.

(b) Consider two charts (V,ψ), $\psi=(x^i,y^\sigma)$, and $(\overline{V},\overline{\psi})$, $\overline{\psi}=(\overline{x}^i,\overline{y}^\sigma)$, such that $V\cap\overline{V}\neq\emptyset$. For any function f, defined on V^r,

$$
\begin{aligned}
(\pi^{r+1,r})^*df &= hdf+pdf=d_if\cdot dx^i+\sum_{0\le k\le r}\sum_{l_1\le l_2\le\cdots\le l_k}\frac{\partial f}{\partial y^\nu_{l_1l_2\ldots l_k}}\omega^\nu_{l_1l_2\ldots l_k}\\
&=\overline{d}_pf\cdot d\overline{x}^p+\sum_{0\le k\le r}\sum_{j_1\le j_2\le\cdots\le j_k}\frac{\partial f}{\partial\overline{y}^\tau_{j_1j_2\ldots j_m}}\overline{\omega}^\tau_{j_1j_2\ldots j_m}\\
&=\overline{d}_pf\frac{\partial\overline{x}^p}{\partial x^i}dx^i+\sum_{0\le k\le r}\sum_{j_1\le j_2\le\cdots\le j_k}\sum_{l_1\le l_2\le\cdots\le l_k}\frac{\partial f}{\partial y^\nu_{l_1l_2\ldots l_k}}\frac{\partial y^\nu_{l_1l_2\ldots l_k}}{\partial\overline{y}^\tau_{j_1j_2\ldots j_m}}\overline{\omega}^\tau_{j_1j_2\ldots j_m}.
\end{aligned}
\tag{27}
$$

Setting $f=y^\lambda_{p_1p_2\cdots p_k}$, where $p_1\le p_2\le\cdots\le p_k$, and using (17), we get (25).

(c) By definition,

$$J^r\alpha^*\overline{\omega}^\sigma_{j_1j_2\ldots j_k}=d(\overline{y}^\sigma_{j_1j_2\ldots j_k}\circ J^r\alpha)-(\overline{y}^\sigma_{j_1j_2\ldots j_kl}\circ J^r\alpha)d(\overline{x}^l\circ J^r\alpha). \tag{28}$$

Denote by α_0 the π-projection of α. Since from Sect. 1.6, (80)

$$
\begin{aligned}
&\overline{y}^\sigma_{j_1j_2\ldots j_kl}\circ J^r\alpha(J^r_x\gamma)\\
&\qquad=\frac{\partial(\overline{y}^\sigma_{j_1j_2\ldots j_k}\circ J^r\alpha\circ J^r\gamma\circ\varphi^{-1})}{\partial x^s}\frac{\partial(x^s\alpha_0^{-1}\overline{\varphi}^{-1})}{\partial\overline{x}^l},
\end{aligned}
\tag{29}
$$

then

$$
\begin{aligned}
J^r\alpha^*\overline{\omega}^\sigma_{j_1j_2\ldots j_k} &= \frac{\partial(\overline{y}^\sigma_{j_1j_2\ldots j_k}\circ J^r\alpha)}{\partial x^p}dx^p+\sum_{i<i_2<\cdots<i_p}\frac{\partial(\overline{y}^\sigma_{j_1j_2\ldots j_k}\circ J^r\alpha)}{\partial y^\nu_{i_1i_2\ldots i_p}}dy^\nu_{i_1i_2\ldots i_p}\\
&\quad-\frac{\partial(\overline{y}^\sigma_{j_1j_2\ldots j_k}\circ J^r\alpha\circ J^r\gamma\circ\varphi^{-1})}{\partial x^s}\frac{\partial(x^s\alpha_0^{-1}\overline{\varphi}^{-1})}{\partial\overline{x}^l}\frac{\partial(\overline{x}^l\circ J^r\alpha)}{\partial x^p}dx^p\\
&=\frac{\partial(\overline{y}^\sigma_{j_1j_2\ldots j_k}\circ J^r\alpha)}{\partial x^p}dx^p+\sum_{i<i_2<\cdots<i_p}\frac{\partial(\overline{y}^\sigma_{j_1j_2\ldots j_k}\circ J^r\alpha)}{\partial y^\nu_{i_1i_2\ldots i_p}}\omega^\nu_{i_1i_2\ldots i_p}\\
&\quad+\sum_{i<i_2<\cdots<i_p}\frac{\partial(\overline{y}^\sigma_{j_1j_2\ldots j_k}\circ J^r\alpha)}{\partial y^\nu_{i_1i_2\ldots i_p}}y^\nu_{i_1i_2\ldots i_ps}dx^s\\
&\quad-\frac{\partial(\overline{y}^\sigma_{j_1j_2\ldots j_k}\circ J^r\alpha\circ J^r\gamma\circ\varphi^{-1})}{\partial x^s}dx^s\\
&=\sum_{i<i_2<\cdots<i_p}\frac{\partial(\overline{y}^\sigma_{j_1j_2\ldots j_k}\circ J^r\alpha)}{\partial y^\nu_{i_1i_2\ldots i_p}}\omega^\nu_{i_1i_2\ldots i_p}.
\end{aligned}
\tag{30}
$$

These conditions mean that the section δ is of the form $\delta=J^r(\pi^{r,0}\circ\delta)$ as required. □

The basis of 1-forms (24) on V^r is usually called the *contact basis*.

The following observations show that the contact forms $\omega^\sigma_{j_1 j_2 \dots j_k}$, defined by a fibered atlas on Y, define a (global) module of 1-forms and an ideal of the exterior algebra $\Omega^r W$ (for elementary definitions, see Appendix 7).

Corollary 1 *The contact* 1*-forms* $\omega^\sigma_{j_1 j_2 \dots j_k}$ *locally generate a submodule of the module* $\Omega^r_1 W$.

Corollary 2 *The contact* 1*-forms* $\omega^\sigma_{j_1 j_2 \dots j_k}$ *locally generate an ideal of the exterior algebra* $\Omega^r W$. *This ideal is not closed under the exterior derivative operator.*

Proof Existence of the ideal is ensured by the transformation properties of the contact 1-forms $\omega^\sigma_{j_1 j_2 \dots j_k}$ (Theorem 1, (b)). It remains to show that the ideal contains a form, which is *not* generated by the forms $\omega^\sigma_{j_1 j_2 \dots j_k}$. If ρ is a contact 1-form expressed as

$$\rho = \sum_{0 \le k \le r-1} \sum_{j_1 \le j_2 \le \dots \le j_k} B^{j_1 j_2 \dots j_k}_\sigma \omega^\sigma_{j_1 j_2 \dots j_k}, \tag{31}$$

then

$$d\rho = \sum_{0 \le k \le r-1} \sum_{j_1 \le j_2 \le \dots \le j_k} \left(dB^{j_1 j_2 \dots j_k}_\sigma \wedge \omega^\sigma_{j_1 j_2 \dots j_k} + B^{j_1 j_2 \dots j_k}_\sigma d\omega^\sigma_{j_1 j_2 \dots j_k} \right). \tag{32}$$

But in this expression,

$$d\omega^\sigma_{j_1 j_2 \dots j_k} = \begin{cases} -\omega^\sigma_{j_1 j_2 \dots j_k l} \wedge dx^l, & 0 \le k \le r-2, \\ -dy^\sigma_{j_1 j_2 \dots j_{r-1} l} \wedge dx^l, & k = r-1, \end{cases} \tag{33}$$

thus, $d\omega^\sigma_{j_1 j_2 \dots j_{r-1}}$ and in general the form ρ are *not* generated by the contact forms $\omega^\sigma_{j_1 j_2 \dots j_k}$. □

The ideal of the exterior algebra $\Omega^r W$, locally generated by the 1-forms $\omega^\sigma_{j_1 j_2 \dots j_k}$, where $0 \le k \le r-1$, is denoted by $\Theta^r_0 W$. The 1-forms $\omega^\sigma_{j_1 j_2 \dots j_k}$, where $0 \le k \le r-1$, and 2-forms $d\omega^\sigma_{j_1 j_2 \dots j_{r-1}}$ locally generate an ideal $\Theta^r W$ of the exterior algebra $\Omega^r W$, *closed* under the exterior derivative operator, that is, a *differential ideal*. This ideal is called the *contact ideal* of the exterior algebra $\Omega^r W$, and its elements are called *contact forms*. We denote

$$\Theta^r_q W = \Omega^r_q W \cap \Theta^r W. \tag{34}$$

The set $\Theta^r_q W$ of contact q-forms is a submodule of the module $\Omega^r_q W$, called the *contact submodule*.

Since the exterior derivative of a contact form is again a contact form, we have the sequence

$$0 \to \Theta_1^r W \xrightarrow{d} \Theta_2^r W \xrightarrow{d} \cdots \xrightarrow{d} \Theta_n^r W, \tag{35}$$

where the arrows denote the exterior derivative operator. If ρ is a contact form, $\rho \in \Theta_q^r W$, and f is a function on W^r, $f \in \Theta_0^r W$, then the formula

$$d(f\rho) = df \wedge \rho + fd\rho \tag{36}$$

shows that the form $d(f\rho)$ is again a contact form; however, the exterior derivative in (36) is *not* a homomorphism of $\Theta_0^r W$-modules. Restricting the multiplication in (36) to *constant* functions f, that is, to *real numbers*, the exterior derivative in (36) becomes a morphism of vector spaces.

Another consequence of Theorem 1 is concerned with sections of the fibered manifold $J^r Y$ over the base X. We say that a section δ of $J^r Y$, defined on an open set in X, is *holonomic*, or *integrable*, if there exists a section γ of Y such that

$$\delta = J^r \gamma. \tag{37}$$

Obviously, if γ exists, then applying the projection $\pi^{r.0}$ to both sides, we get $\pi^{r.0} \circ \delta = \gamma$; thus, if γ exists, it is unique and is determined by

$$\gamma = \pi^{r.0} \circ \delta. \tag{38}$$

Theorem 2 *A section* δ: $U \to J^r Y$ *is holonomic if and only if for any fibered chart* (V, ψ), $\psi = (x^i, y^\sigma)$, *such that the set* $\pi(V)$ *lies in the domain of definition of* δ,

$$\delta^* \omega^\sigma_{i_1 i_2 \ldots i_k} = 0 \tag{39}$$

for all σ, k, *and* $i_1, i_2, \ldots, i_k$ *such that* $1 \le \sigma \le m$, $0 \le k \le r-1$, *and* $1 \le i_1 \le i_2 \le \cdots \le i_k \le n$.

Proof By definition,

$$\begin{aligned} \delta^* \omega^\sigma_{i_1 i_2 \ldots i_k} &= d(y^\sigma_{i_1 i_2 \ldots i_k} \circ \delta) - (y^\sigma_{i_1 i_2 \ldots i_k l} \circ \delta) dx^l \\ &= \left(\frac{\partial (y^\sigma_{i_1 i_2 \ldots i_k} \circ \delta)}{\partial x^l} - y^\sigma_{i_1 i_2 \ldots i_k l} \circ \delta \right) dx^l. \end{aligned} \tag{40}$$

Thus, condition (39) is equivalent to the conditions

$$\frac{\partial (y^\sigma_{i_1 i_2 \ldots i_k} \circ \delta)}{\partial x^l} - y^\sigma_{i_1 i_2 \ldots i_k l} \circ \delta = 0 \tag{41}$$

that can also be written as

$$\begin{aligned}
&\frac{\partial(y^\sigma\circ\delta)}{\partial x^l} - y^\sigma_l\circ\delta = 0,\\
&\frac{\partial(y^\sigma_{i_1}\circ\delta)}{\partial x^l} - y^\sigma_{i_1 l}\circ\delta = \frac{\partial^2(y^\sigma\circ\delta)}{\partial x^{i_1}\partial x^l} - y^\sigma_{i_1 l}\circ\delta = 0,\\
&\dots\\
&\frac{\partial(y^\sigma_{i_1 i_2\dots i_{r-1}}\circ\delta)}{\partial x^l} - y^\sigma_{i_1 i_2\dots i_{r-1} l}\circ\delta = \frac{\partial^{k+1}(y^\sigma\circ\delta)}{\partial x^{i_1}\partial x^{i_2}\dots\partial x^{i_{r-1}}\partial x^l} - y^\sigma_{i_1 i_2\dots i_{r-1} l}\circ\delta = 0.
\end{aligned} \tag{42}$$

These conditions mean that the section δ is of the form $\delta = J^r(\pi^{r,0}\circ\delta)$ as required. □

2.2 The Trace Decomposition

Main objective in this section is the application of the trace decomposition theory of tensor spaces to differential forms defined on the r-jet prolongation J^rY of a fibered manifold Y. We decompose the components of a form, expressed in a fibered chart, by the trace operation (see Appendix 9); the resulting decomposition of differential forms will be referred to as the *trace decomposition*.

In order to study the structure of the components of a form $\rho\in\Omega^r_qW$ for *general* r, it will be convenient to introduce a *multi-index notation*. We also need a convention on the alternation and symmetrization of tensor components in a given set of indices.

Convention 1 (**Multi-indices**) We introduce a multi-index I as an ordered k-tuple $I=(i_1i_2\dots i_k)$, where k = 1, 2,…, r and the entries are indices such that $1\le i_1,i_2,\dots,i_k\le n$. The number k is the *length* of I and is denoted by $|I|$. If j is any integer such that $1\le j\le n$, we denote by Ij the multi-index $Ij=(i_1i_2\dots i_kj)$. In this notation, the *contact basis* of 1-forms, introduced in Sect. 2.1, Theorem 1, (a), is sometimes denoted as $(dx^i,\omega^\sigma_J,dy^\sigma_I)$, where the multi-indices satisfy $0\le\ |J|\ \le r-1$ and $|I|\ = r$; it is understood, however, that the basis includes only linearly independent 1-forms ω^σ_J, where the multi-indices $I=(i_1i_2\dots i_k)$ satisfy $i_1\le i_2\le\dots\le i_k$.

Convention 2 (**Alternation, symmetrization**) We introduce the symbol $\mathrm{Alt}(i_1i_2\dots i_k)$ to denote *alternation* in the indices $i_1,i_2,\dots,i_k$. If $U=U_{i_1i_2\dots i_k}$ is a collection of real numbers, we denote by $U_{i_1i_2\dots i_k}\,\mathrm{Alt}(i_1i_2\dots i_k)$ the *skew-symmetric* component of U. Analogously, $\mathrm{Sym}(i_1i_2\dots i_k)$ denotes *symmetrization* in the indices $i_1,i_2,\dots,i_k$, and the symbol $U_{i_1i_2\dots i_k}\,\mathrm{Sym}(i_1i_2\dots i_k)$ means the symmetric component of U. The operators Alt and Sym are understood as *projectors* (the coefficient $1/k!$ is included).

Note that there exists a close relationship between the trace operation on the one hand and the exterior derivative operator on the other hand. For instance, decomposing in a fibered chart the 2-form $dy^{\sigma}_{Jj} \wedge dx^k$ by the trace operation, we get

$$dy^{\sigma}_{Jj} \wedge dx^k = \frac{1}{n}\delta^k_j dy^{\sigma}_{Js} \wedge dx^s + dy^{\sigma}_{Jj} \wedge dx^k - \frac{1}{n}\delta^k_j dy^{\sigma}_{Js} \wedge dx^s, \tag{43}$$

where the summand, representing the *Kronecker component* of $dy^{\sigma}_{Jj} \wedge dx^k$, coincides, up to a constant factor, with the *exterior derivative* $d\omega^{\sigma}_J$, and is therefore a contact form:

$$\frac{1}{n}\delta^k_j dy^{\sigma}_{Js} \wedge dx^s = -\frac{1}{n} d\omega^{\sigma}_J. \tag{44}$$

The complementary summand in the decomposition (43), represented by the second and the third terms, is *traceless* in the indices j and k. We wish to use this observation to generalize decomposition (43) to any q-forms on J^rY.

First, we apply the trace decomposition theorem (Appendix 9, Theorem 1) to q-forms of a specific type, not containing the contact forms ω^{ν}_J.

Lemma 2 *Let (V, ψ), $\psi = (x^i, y^{\sigma})$, be a fibered chart on Y. Let μ be a q-form on V^r such that*

$$\begin{aligned}\mu = {} & A_{i_1 i_2 \ldots i_q} dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_q} \\ & + B^{I_1}_{\sigma_1 i_2 i_3 \ldots i_q} dy^{\sigma_1}_{I_1} \wedge dx^{i_2} \wedge dx^{i_3} \wedge \cdots \wedge dx^{i_q} \\ & + B^{I_1 I_2}_{\sigma_1 \sigma_2 i_3 i_4 \ldots i_q} dy^{\sigma_1}_{I_1} \wedge dy^{\sigma_2}_{I_2} \wedge dx^{i_3} \wedge dx^{i_4} \wedge \cdots \wedge dx^{i_q} \\ & + \cdots + B^{I_1 I_2 \cdots I_{q-1}}_{\sigma_1 \sigma_2 \cdots \sigma_{q-1} i_q} dy^{\sigma_1}_{I_1} \wedge dy^{\sigma_2}_{I_2} \wedge \cdots \wedge dy^{\sigma_{q-1}}_{I_{q-1}} \wedge dx^{i_q} \\ & + A^{I_1 I_2 \cdots I_q}_{\sigma_1 \sigma_2 \cdots \sigma_q} dy^{\sigma_1}_{I_1} \wedge dy^{\sigma_2}_{I_2} \wedge \cdots \wedge dy^{\sigma_q}_{I_q},\end{aligned} \tag{45}$$

where the multi-indices satisfy $|I_1|, |I_2|, \ldots, |I_{q-1}| = r$. Then, μ has a decomposition

$$\mu = \mu_0 + \mu', \tag{46}$$

satisfying the following conditions:

(a) *μ_0 is generated by the forms $d\omega^{\sigma}_J$, where $|J| = r - 1$, that is,*

$$\mu_0 = \sum_{|J| = r-1} d\omega^{\sigma}_J \wedge \Phi^J_{\sigma}, \tag{47}$$

for some $(q-2)$-forms Φ^J_{σ}.

(b) *μ' has an expression*

$$\begin{aligned}\mu' = {} & A_{i_1 i_2 \dots i_q} dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_q} \\ & + A^{I_1}_{\sigma_1\, i_2 i_3 \dots i_q} dy^{\sigma_1}_{I_1} \wedge dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_q} \\ & + A^{I_1 I_2}_{\sigma_1 \sigma_2 i_2 i_3 \dots i_q} dy^{\sigma_1}_{I_1} \wedge dy^{\sigma_2}_{I_2} \wedge dx^{i_3} \wedge dx^{i_4} \wedge \cdots \wedge dx^{i_q} \\ & + \cdots + A^{I_1 I_2}_{\sigma_1 \sigma_2} {}^{\cdots}_{\cdots} {}^{I_{q-1}}_{\sigma_{q-1}\, i_q} dy^{\sigma_1}_{I_1} \wedge dy^{\sigma_2}_{I_2} \wedge \cdots \wedge dy^{\sigma_{q-1}}_{I_{q-1}} \wedge dx^{i_q} \\ & + A^{I_1 I_2}_{\sigma_1 \sigma_2} {}^{\cdots}_{\cdots} {}^{I_q}_{\sigma_q} dy^{\sigma_1}_{I_1} \wedge dy^{\sigma_2}_{I_2} \wedge \cdots \wedge dy^{\sigma_q}_{I_q},\end{aligned} \tag{48}$$

where $A^{I_1}_{\sigma_1 i_2 i_3 \dots i_q}, A^{I_1 I_2}_{\sigma_1 \sigma_2 i_2 i_3 \dots i_q}, \dots, A^{I_1 I_2}_{\sigma_1 \sigma_2} {}^{\cdots}_{\cdots} {}^{I_{q-1}}_{\sigma_{q-1}\, i_q}$ are traceless components of the coefficients $B^{I_1}_{\sigma_1 i_2 i_3 \dots i_q}, B^{I_1 I_2}_{\sigma_1 \sigma_2 i_2 i_3 \dots i_q}, \dots, B^{I_1 I_2}_{\sigma_1 \sigma_2} {}^{\cdots}_{\cdots} {}^{I_{q-1}}_{\sigma_{q-1}\, i_q}$.

Proof Applying the trace decomposition theorem (Appendix 9) to the coefficients $B^{I_1}_{\sigma_1 i_2 i_3 \dots i_q}, B^{I_1 I_2}_{\sigma_1 \sigma_2 i_2 i_3 \dots i_q}, \dots, B^{I_1 I_2}_{\sigma_1 \sigma_2} {}^{\cdots}_{\cdots} {}^{I_{q-1}}_{\sigma_{q-1}\, i_q}$ in (45), we get

$$\begin{aligned} & B^{I_1}_{\sigma_1 i_2 i_3 \dots i_q} = A^{I_1}_{\sigma_1\, i_2 i_3 \dots i_q} + C^{I_1}_{\sigma_1\, i_2 i_3 \dots i_q}, \\ & B^{I_1 I_2}_{\sigma_1 \sigma_2 i_3 i_4 \dots i_q} = A^{I_1 I_2}_{\sigma_1 \sigma_2 i_3 i_4 \dots i_q} + C^{I_1 I_2}_{\sigma_1 \sigma_2 i_3 i_4 \dots i_q}, \\ & \dots \\ & B^{I_1 I_2}_{\sigma_1 \sigma_2} {}^{\cdots}_{\cdots} {}^{I_{q-2}}_{\sigma_{q-2}\, i_{q-1} i_q} = A^{I_1 I_2}_{\sigma_1 \sigma_2} {}^{\cdots}_{\cdots} {}^{I_{q-2}}_{\sigma_{q-2}\, i_{q-1} i_q} + C^{I_1 I_2}_{\sigma_1 \sigma_2} {}^{\cdots}_{\cdots} {}^{I_{q-2}}_{\sigma_{q-2} i_{q-1} i_q}, \\ & B^{I_1 I_2}_{\sigma_1 \sigma_2} {}^{\cdots}_{\cdots} {}^{I_{q-2}}_{\sigma_{q-2} i_{q-1} i_q} = A^{I_1 I_2}_{\sigma_1 \sigma_2} {}^{\cdots}_{\cdots} {}^{I_{q-2}}_{\sigma_{q-2} i_{q-1} i_q} + C^{I_1 I_2}_{\sigma_1 \sigma_2} {}^{\cdots}_{\cdots} {}^{I_{q-2}}_{\sigma_{q-2} i_{q-1} i_q}, \end{aligned} \tag{49}$$

where the systems $A^{I_1}_{\sigma_1 i_2 i_3 \dots i_q}, A^{I_1 I_2}_{\sigma_1 \sigma_2 i_2 i_3 \dots i_q}, \dots, A^{I_1 I_2}_{\sigma_1 \sigma_2} {}^{\cdots}_{\cdots} {}^{I_{q-1}}_{\sigma_{q-1}\, i_q}$ are traceless and $C^{I_1}_{\sigma_1 i_2 i_3 \dots i_q}, C^{I_1 I_2}_{\sigma_1 \sigma_2 i_2 i_3 \dots i_q}, \dots, C^{I_1 I_2}_{\sigma_1 \sigma_2} {}^{\cdots}_{\cdots} {}^{I_{q-1}}_{\sigma_{q-1}\, i_q}$ are of Kronecker type. Thus, writing the multi-index I_l as $I_l = J_l j_l$, we have

$$\begin{aligned} & C^{I_1}_{\sigma_1 i_2 i_3 \dots i_q} = \delta^{j_1}_{i_2} D^{J_1}_{\sigma_1 i_3 i_4 \dots i_q} \quad \mathrm{Alt}(i_2 i_3 i_4 \dots i_q) \quad \mathrm{Sym}(J_1 j_1), \\ & C^{I_1 I_2}_{\sigma_1 \sigma_2 i_3 i_4 \dots i_q} = \delta^{j_1}_{i_3} D^{J_1 I_2}_{\sigma_1 \sigma_2 i_4 i_5 \dots i_q} \quad \mathrm{Alt}(i_3 i_4 i_5 \dots i_q) \quad \mathrm{Sym}(J_1 j_1) \quad \mathrm{Sym}(J_2 j_2), \\ & \dots \\ & C^{I_1 I_2}_{\sigma_1 \sigma_2} {}^{\cdots}_{\cdots} {}^{I_{q-2}}_{\sigma_{q-2} i_{q-1} i_q} = \delta^{j_1}_{i_{q-1}} D^{J_1 I_2 I_3}_{\sigma_1 \sigma_2 \sigma_3} {}^{\cdots}_{\cdots} {}^{I_{q-2}}_{\sigma_{q-2} i_q} \quad \mathrm{Alt}(i_{q-1} i_q) \quad \mathrm{Sym}(J_1 j_1) \\ & \quad \mathrm{Sym}(J_2 j_2) \quad \dots \quad \mathrm{Sym}(J_{q-2} j_{q-2}), \\ & C^{I_1 I_2}_{\sigma_1 \sigma_2} {}^{\cdots}_{\cdots} {}^{I_{q-1}}_{\sigma_{q-1} i_q} = \delta^{j_1}_{i_q} D^{J_1 I_2 I_3}_{\sigma_1 \sigma_2 \sigma_3} {}^{\cdots}_{\cdots} {}^{I_{q-1}}_{\sigma_{q-1}} \quad \mathrm{Sym}(J_1 j_1) \quad \mathrm{Sym}(J_2 j_2) \\ & \quad \dots \quad \mathrm{Sym}(J_{q-2} j_{q-2}). \end{aligned} \tag{50}$$

Then

$$\begin{aligned}\mu = {} & A_{i_1 i_2 \dots i_q} dx^{i_1} \wedge dx^{i_2} \wedge \dots \wedge dx^{i_q} \\ & + A^{I_1}_{\sigma_1 i_2 i_3 \dots i_q} dy^{\sigma_1}_{I_1} \wedge dx^{i_2} \wedge dx^{i_3} \wedge \dots \wedge dx^{i_q} \\ & + A^{I_1 I_2}_{\sigma_1 \sigma_2 i_3 i_4 \dots i_q} dy^{\sigma_1}_{I_1} \wedge dy^{\sigma_2}_{I_2} \wedge dx^{i_3} \wedge dx^{i_4} \wedge \dots \wedge dx^{i_q} \\ & + \dots + A^{I_1 I_2 \dots I_{q-1}}_{\sigma_1 \sigma_2 \dots \sigma_{q-1} i_q} dy^{\sigma_1}_{I_1} \wedge dy^{\sigma_2}_{I_2} \wedge \dots \wedge dy^{\sigma_{q-1}}_{I_{q-1}} \wedge dx^{i_q} \\ & + A^{I_1 I_2 \dots I_q}_{\sigma_1 \sigma_2 \dots \sigma_q} dy^{\sigma_1}_{I_1} \wedge dy^{\sigma_2}_{I_2} \wedge \dots \wedge dy^{\sigma_q}_{I_q} \\ & + \delta^{j_1}_{i_2} D^{J_1}_{\sigma_1 i_3 i_4 \dots i_q} dy^{\sigma_1}_{J_1 j_1} \wedge dx^{i_2} \wedge dx^{i_3} \wedge \dots \wedge dx^{i_q} \\ & + \delta^{j_1}_{i_3} D^{J_1 I_2}_{\sigma_1 \sigma_2 i_4 i_5 \dots i_q} dy^{\sigma_1}_{J_1 j_1} \wedge dy^{\sigma_2}_{I_2} \wedge dx^{i_3} \wedge dx^{i_4} \wedge \dots \wedge dx^{i_q} \\ & + \dots + \delta^{j_1}_{i_q} D^{J_1 I_2 I_3 \dots I_{q-1}}_{\sigma_1 \sigma_2 \sigma_3 \dots \sigma_{q-1}} dy^{\sigma_1}_{J_1 j_1} \wedge dy^{\sigma_2}_{I_2} \wedge \dots \wedge dy^{\sigma_{q-1}}_{I_{q-1}} \wedge dx^{i_q},\end{aligned} \tag{51}$$

and now our assertion follows from the formula (44). □

The following theorem generalizes Lemma 2 to arbitrary forms on open sets in the r-jet prolongation $J^r Y$.

Theorem 3 (The trace decomposition theorem) *Let q be any positive integer, and let $\rho \in \Omega^r_q W$ be a q-form. Let (V, ψ), $\psi = (x^i, y^\sigma)$, be a fibered chart on Y, such that $V \subset W$. Then, ρ has on V^r an expression*

$$\rho = \rho_0 + \rho', \tag{52}$$

with the following properties:

(a) *ρ_0 is generated by the 1-forms ω^σ_J with $0 \le |J| \le r-1$ and 2-forms $d\omega^\sigma_I$ where $|I| = r-1$.*
(b) *ρ' has an expression*

$$\begin{aligned}\rho' = {} & A_{i_1 i_2 \dots i_q} dx^{i_1} \wedge dx^{i_2} \wedge \dots \wedge dx^{i_q} \\ & + A^{I_1}_{\sigma_1 i_2 i_3 \dots i_q} dy^{\sigma_1}_{I_1} \wedge dx^{i_2} \wedge dx^{i_3} \wedge \dots \wedge dx^{i_q} \\ & + A^{I_1 I_2}_{\sigma_1 \sigma_2 i_2 i_3 \dots i_q} dy^{\sigma_1}_{I_1} \wedge dy^{\sigma_2}_{I_2} \wedge dx^{i_3} \wedge dx^{i_4} \wedge \dots \wedge dx^{i_q} \\ & + \dots + A^{I_1 I_2 \dots I_{q-1}}_{\sigma_1 \sigma_2 \dots \sigma_{q-1} i_q} dy^{\sigma_1}_{I_1} \wedge dy^{\sigma_2}_{I_2} \wedge \dots \wedge dy^{\sigma_{q-1}}_{I_{q-1}} \wedge dx^{i_q} \\ & + A^{I_1 I_2 \dots I_q}_{\sigma_1 \sigma_2 \dots \sigma_q} dy^{\sigma_1}_{I_1} \wedge dy^{\sigma_2}_{I_2} \wedge \dots \wedge dy^{\sigma_q}_{I_q},\end{aligned} \tag{53}$$

where $|I_1|, |I_2|, \dots, |I_{q-1}| = r$ and all coefficients $A^{I_1}_{\sigma_1 i_2 i_3 \dots i_q}, A^{I_1 I_2}_{\sigma_1 \sigma_2 i_2 i_3 \dots i_q}, \dots, A^{I_1 I_2 \dots I_{q-1}}_{\sigma_1 \sigma_2 \dots \sigma_{q-1} i_2 i_3 \dots i_q}$ are traceless.

Proof To prove Theorem 3, we express ρ in the contact basis. Then, $\rho = \rho_1 + \mu$, where ρ_1 is generated by contact 1-forms ω_J^σ, $0 \leq |J| \leq r-1$, and μ does not contain any factor ω_J^σ. Thus, μ has an expression (45) and can be decomposed as in Lemma 2, (46). Using this decomposition, we get the formula (52). □

Theorem 3 is the *trace decomposition theorem* for differential forms; formula (52) is referred to as the *trace decomposition formula*. The form ρ_0 in this decomposition (43) is contact and is called the *contact component* of ρ; the form ρ' is the *traceless component* of ρ with respect to the fibered chart (V, ψ).

Lemma 3 *Let $\rho \in \Omega_q^r W$ be a q-form, and let (V, ψ), $\psi = (x^i, y^\sigma)$, and $(\overline{V}, \overline{\psi})$, $\overline{\psi} = (\overline{x}^i, \overline{y}^\sigma)$, be two fibered charts such that $V \cap \overline{V} \neq \emptyset$. Suppose that we have the trace decomposition of the form ρ with respect to (V, ψ) and $(\overline{V}, \overline{\psi})$, respectively,*

$$\rho = \rho_0 + \rho' = \overline{\rho}_0 + \overline{\rho'}. \tag{54}$$

Then, the traceless components satisfy

$$\rho' = \overline{\rho'} + \overline{\eta}, \tag{55}$$

where $\overline{\eta}$ is a contact form on the intersection $V \cap \overline{V}$.

Proof Lemma 3 can be easily verified by a direct calculation. Consider for instance the term $A^I_{\sigma\, i_2 i_3 \ldots i_q} dy^\sigma_{I_1} \wedge dx^{i_2} \wedge dx^{i_3} \wedge \cdots \wedge dx^{i_q}$ in formula (53), and the transformation equation is

$$\frac{\partial y^\sigma_{i_1 i_2 \ldots i_r}}{\partial \overline{y}^\nu_{j_1 j_2 \ldots j_r}} = \frac{\partial y^\sigma}{\partial \overline{y}^\nu} \frac{\partial \overline{x}^{j_1}}{\partial x^{i_1}} \frac{\partial \overline{x}^{j_2}}{\partial x^{i_2}} \cdots \frac{\partial \overline{x}^{j_r}}{\partial x^{i_r}} \quad \mathrm{Sym}(j_1 j_2 \ldots j_r). \tag{56}$$

Denote $\overline{\omega}^\nu_{j_1 j_2 \ldots j_k} = d\overline{y}^\nu_{j_1 j_2 \ldots j_k} - \overline{y}^\nu_{j_1 j_2 \ldots j_k l} d\overline{x}^l$. Then, we have

$$\begin{aligned} &A_\sigma^{i_1 i_2 \ldots i_r}{}_{s_2 s_3 \ldots s_q} dy^\sigma_{i_1 i_2 \ldots i_r} \wedge dx^{s_2} \wedge dx^{s_3} \wedge \cdots \wedge dx^{s_q} \\ &\quad = A_\sigma^{i_1 i_2 \ldots i_r}{}_{s_2 s_3 \ldots s_q} \frac{\partial x^{s_2}}{\partial \overline{x}^{l_2}} \frac{\partial x^{s_3}}{\partial \overline{x}^{l_3}} \cdots \frac{\partial x^{s_q}}{\partial \overline{x}^{l_q}} \cdot \left(\left(\frac{\partial y^\sigma_{i_1 i_2 \ldots i_r}}{\partial \overline{x}^p} + \sum_{0 \leq k \leq r-1} \frac{\partial y^\sigma_{i_1 i_2 \ldots i_r}}{\partial \overline{y}^\nu_{j_1 j_2 \ldots j_k}} y^\nu_{j_1 j_2 \ldots j_k p} \right) d\overline{x}^p \right. \\ &\qquad \left. + \sum_{0 \leq k \leq r-1} \frac{\partial y^\sigma_{i_1 i_2 \ldots i_r}}{\partial \overline{y}^\nu_{j_1 j_2 \ldots j_k}} \overline{\omega}^\nu_{j_1 j_2 \ldots j_k} + \frac{\partial y^\sigma_{i_1 i_2 \ldots i_r}}{\partial \overline{y}^\nu_{j_1 j_2 \ldots j_r}} d\overline{y}^\nu_{j_1 j_2 \ldots j_r} \right) \wedge d\overline{x}^{l_2} \wedge d\overline{x}^{l_3} \wedge \cdots \wedge d\overline{x}^{l_q}. \end{aligned} \tag{57}$$

Consequently, the last summand in (57) implies

$$\overline{A}_\nu^{j_1 j_2 \ldots j_r}{}_{l_2 l_3 \ldots l_q} = A_\sigma^{i_1 i_2 \ldots i_r}{}_{s_2 s_3 \ldots s_q} \frac{\partial x^{s_2}}{\partial \overline{x}^{l_2}} \frac{\partial x^{s_3}}{\partial \overline{x}^{l_3}} \cdots \frac{\partial x^{s_q}}{\partial \overline{x}^{l_q}} \frac{\partial y^\sigma_{i_1 i_2 \ldots i_r}}{\partial \overline{y}^\nu_{j_1 j_2 \ldots j_r}}. \tag{58}$$

Substituting from (56) in this formula, we see that the trace of $\overline{A}^{j_1 j_2 \ldots j_r}_{\nu\, l_2 l_3 \ldots l_q}$ vanishes if and only if the same is true for the trace of $A^{i_1 i_2 \ldots i_r}_{\sigma\, s_2 s_3 \ldots s_q}$. Thus, the decomposition (55) is valid for the summand (56). The same applies to any other summand. □

Following Theorem 3, we can write the q-form ρ in the contact basis as $\rho = \rho_1 + \rho_2 + \rho'$, where ρ_1 is generated by the forms ω^σ_J, $0 \le |J| \le r-1$, ρ_2 is generated by $d\omega^\sigma_I$, $|I| = r-1$, and does not contain any factor ω^σ_J, and the form ρ' is traceless. Thus,

$$\rho_1 = \sum_{0 \le |J| \le r-1} \omega^\sigma_J \wedge \Phi^J_\sigma, \quad \rho_2 = \sum_{|I| = r-1} d\omega^\sigma_I \wedge \Psi^I_\sigma \tag{59}$$

for some forms Φ^J_σ and Ψ^I_σ. Then,

$$\rho = \omega^\sigma_J \wedge \Phi^J_\sigma + \omega^\sigma_I \wedge d\Psi^I_\sigma + d(\omega^\sigma_I \wedge \Psi^I_\sigma) + \rho'. \tag{60}$$

Setting

$$P\rho = \omega^\sigma_J \wedge \Phi^J_\sigma + \omega^\sigma_I \wedge d\Psi^I_\sigma, \quad Q\rho = \omega^\sigma_I \wedge \Psi^I_\sigma, \quad R\rho = \rho', \tag{61}$$

we get the following version of Theorem 3.

Theorem 4 *Let q be arbitrary, and let $\rho \in \Omega^r_q W$ be a q-form. Let (V, ψ), $\psi = (x^i, y^\sigma)$, be a fibered chart on Y such that $V \subset W$. Then, ρ can be expressed on V^r as*

$$\rho = P\rho + dQ\rho + R\rho. \tag{62}$$

Proof This is an immediate consequence of definitions and Theorem 3. □

In the following two examples, we discuss the trace decomposition formula and the transformation equations for the *traceless* components of some differential forms on 1-jet prolongation of the fibered manifold Y. The aim is to illustrate the decomposition methods for lower-degree differential forms.

Example 1 We find the trace decomposition of a 3-form μ, written in a fibered chart (V, ψ), $\psi = (x^i, y^\sigma)$, as

$$\begin{aligned} \mu = {} & A_{ijk} dx^i \wedge dx^j \wedge dx^k + B^p_{\sigma\, jk} dy^\sigma_p \wedge dx^j \wedge dx^k \\ & + B^{pq}_{\sigma\nu k} dy^\sigma_p \wedge dy^\nu_q \wedge dx^k + A^{pqr}_{\sigma\nu\tau} dy^\sigma_p \wedge dy^\nu_q \wedge dy^\tau_r. \end{aligned} \tag{63}$$

Decomposing $B^p_{\sigma jk}$, we have $B^p_{\sigma jk} = A^p_{\sigma jk} + \delta^p_j C_{\sigma k} + \delta^p_k D_{\sigma j}$, where $A^p_{\sigma jk}$ is traceless. Then, the condition $B^p_{\sigma jk} = -B^p_{\sigma kj}$ yields

$$\begin{aligned} B^p_{\sigma pk} &= \delta^p_p C_{\sigma k} + \delta^p_k D_{\sigma p} = nC_{\sigma k} + D_{\sigma k} \\ &= -B^p_{\sigma kp} = -\delta^p_k C_{\sigma p} - \delta^p_p D_{\sigma k} = -C_{\sigma k} - nD_{\sigma k}, \end{aligned} \tag{64}$$

and hence, $C_{\sigma k} = -D_{\sigma k}$. Thus,

$$B^p_{\sigma jk} = A^p_{\sigma jk} + \delta^p_j C_{\sigma k} - \delta^p_k C_{\sigma j}. \tag{65}$$

Decomposing $B^{pq}_{\sigma\nu k}$, we have $B^{pq}_{\sigma\nu k} = A^{pq}_{\sigma\nu k} + \delta^p_k C^q_{\sigma\nu} + \delta^q_k D^p_{\sigma\nu}$. Now, the condition $B^{pq}_{\sigma\nu k} = -B^{qp}_{\nu\sigma k}$ yields

$$\begin{aligned} B^{pq}_{\sigma\nu p} &= \delta^p_p C^q_{\sigma\nu} + \delta^q_p D^p_{\sigma\nu} = nC^q_{\sigma\nu} + D^q_{\sigma\nu} \\ &= -B^{qp}_{\nu\sigma p} = -\delta^q_p C^p_{\nu\sigma} - \delta^p_p D^q_{\nu\sigma} = -C^q_{\nu\sigma} - nD^q_{\nu\sigma}, \end{aligned} \tag{66}$$

and hence, $nC^q_{\sigma\nu} + C^q_{\nu\sigma} = -nD^q_{\nu\sigma} - D^q_{\sigma\nu}$. It can be easily verified that this condition implies

$$C^q_{\sigma\nu} = -D^q_{\nu\sigma}. \tag{67}$$

Indeed, symmetrization and alternation yield

$$nC^q_{\sigma\nu} + C^q_{\nu\sigma} + nC^q_{\nu\sigma} + C^q_{\sigma\nu} = -nD^q_{\nu\sigma} - D^q_{\sigma\nu} - nD^q_{\sigma\nu} - D^q_{\nu\sigma} \tag{68}$$

and

$$nC^q_{\sigma\nu} + C^q_{\nu\sigma} - nC^q_{\nu\sigma} - C^q_{\sigma\nu} = -nD^q_{\nu\sigma} - D^q_{\sigma\nu} + nD^q_{\sigma\nu} + D^q_{\nu\sigma}, \tag{69}$$

hence, $C^q_{\sigma\nu} + C^q_{\nu\sigma} = -D^q_{\nu\sigma} - D^q_{\sigma\nu}$ and $C^q_{\sigma\nu} - C^q_{\sigma\nu} = -D^q_{\nu\sigma} + D^q_{\sigma\nu}$. These equations already imply (47). Thus,

$$B^{pq}_{\sigma\nu k} = A^{pq}_{\sigma\nu k} + \delta^p_k C^q_{\sigma\nu} - \delta^q_k C^p_{\nu\sigma}. \tag{70}$$

Summarizing (65) and (70), we get

$$\begin{aligned} \mu = {} & A_{ijk} dx^i \wedge dx^j \wedge dx^k + A^p_{\sigma jk} dy^\sigma_p \wedge dx^j \wedge dx^k + A^p_{\sigma\nu k} dy^\sigma_p \wedge dy^\nu_q \wedge dx^k \\ & + \delta^p_j C_{\sigma k} dy^\sigma_p \wedge dx^j \wedge dx^k - \delta^p_k C_{\sigma j} dy^\sigma_p \wedge dx^j \wedge dx^k \\ & + \delta^p_k C^q_{\sigma\nu} dy^\sigma_p \wedge dy^\nu_q \wedge dx^k - \delta^q_k C^p_{\nu\sigma} dy^\sigma_p \wedge dy^\nu_q \wedge dx^k \\ & + A^{pqr}_{\sigma\nu\tau} dy^\sigma_p \wedge dy^\nu_q \wedge dy^\tau_r \end{aligned}$$

$$\begin{aligned}
&= A_{ijk}dx^i \wedge dx^j \wedge dx^k + A^p_{\sigma jk}dy^\sigma_p \wedge dx^j \wedge dx^k \\
&\quad + A^{pq}_{\sigma\nu k}dy^\sigma_p \wedge dy^\nu_q \wedge dx^k + A^{pqr}_{\sigma\nu\tau}dy^\sigma_p \wedge dy^\nu_q \wedge dy^\tau_r \\
&\quad + C_{\sigma k}dy^\sigma_p \wedge dx^p \wedge dx^k - C_{\sigma j}dy^\sigma_p \wedge dx^j \wedge dx^p \\
&\quad + C^q_{\sigma\nu}dy^\sigma_p \wedge dy^\nu_q \wedge dx^p - C^p_{\nu\sigma}dy^\sigma_p \wedge dy^\nu_q \wedge dx^q \\
&= A_{ijk}dx^i \wedge dx^j \wedge dx^k + A^p_{\sigma jk}dy^\sigma_p \wedge dx^j \wedge dx^k \\
&\quad + A^{pq}_{\sigma\nu k}dy^\sigma_p \wedge dy^\nu_q \wedge dx^k + A^{pqr}_{\sigma\nu\tau}dy^\sigma_p \wedge dy^\nu_q \wedge dy^\tau_r \\
&\quad - 2C_{\sigma k}d\omega^\sigma \wedge dx^k + 2C^p_{\sigma\nu}d\omega^\sigma \wedge dy^\nu_p.
\end{aligned} \tag{71}$$

Thus, applying formula (51) to any 3-form ρ on V^1, we get the decomposition

$$\rho = \rho_1 + \rho_2 + \rho', \tag{72}$$

where ρ_1 is generated by ω^σ, that is, $\rho_1 = \omega^\sigma \wedge \Phi_\sigma$, ρ_2 is generated by the contact 2-forms $d\omega^\sigma$, $\rho_2 = d\omega^\sigma \wedge \Psi_\sigma$, where the 1-forms Ψ_σ do not contain any factor ω^ν, and ρ' is traceless.

Example 2 (Transformation properties) Consider a 2-form on the 1-jet prolongation J^1Y, expressed in two fibered charts (V, ψ), $\psi = (x^i, y^\sigma)$, and $(\overline{V}, \overline{\psi})$, $\overline{\psi} = (\overline{x}^i, \overline{y}^\sigma)$, as

$$\rho = \rho_1 + \rho_2 + \rho' = \overline{\rho}_1 + \overline{\rho}_2 + \overline{\rho'}, \tag{73}$$

where according to Theorem 3,

$$\begin{aligned}
&\rho_1 = \omega^\sigma \wedge P_\sigma, \quad \rho_2 = Q_\sigma d\omega^\sigma, \\
&\rho' = A_{ij}dx^i \wedge dx^j + A^i_{\nu j}dy^\nu_i \wedge dx^j + A^{ij}_{\nu\tau}dy^\nu_i \wedge dy^\tau_j,
\end{aligned} \tag{74}$$

and

$$\begin{aligned}
&\overline{\rho}_1 = \overline{\omega}^\sigma \wedge \overline{P}_\sigma, \quad \overline{\rho}_2 = \overline{Q}_\sigma d\overline{\omega}^\sigma, \\
&\overline{\rho'} = \overline{A}_{ij}d\overline{x}^i \wedge d\overline{x}^j + \overline{A}^i_{\nu j}d\overline{y}^\nu_i \wedge d\overline{x}^j + \overline{A}^{il}_{\nu\tau}d\overline{y}^\nu_i \wedge d\overline{y}^\tau_l.
\end{aligned} \tag{75}$$

We want to determine transformation formulas for the traceless components $A^{ij}_{\nu\tau}$, $A^i_{\nu j}$, and A_{ij}. Transformation equations are of the form

$$\overline{x}^i = \overline{x}^i(x^j), \quad \overline{y}^\sigma = \overline{y}^\sigma(x^j, y^\nu), \quad \overline{y}^\sigma_j = \left(\frac{\partial \overline{y}^\sigma}{\partial x^l} + \frac{\partial \overline{y}^\sigma}{\partial y^\nu}y^\nu_l\right)\frac{\partial x^l}{\partial \overline{x}^j}, \tag{76}$$

and imply

$$d\overline{y}_i^\nu = \left(\frac{\partial \overline{y}_i^\nu}{\partial x^p} + \frac{\partial \overline{y}_i^\nu}{\partial y^\kappa} y_p^\kappa\right) dx^p + \frac{\partial \overline{y}_i^\nu}{\partial y^\kappa}\omega^\kappa + \frac{\partial \overline{y}^\nu}{\partial y^\kappa}\frac{\partial x^s}{\partial \overline{x}^i} dy_s^\kappa. \tag{77}$$

Then, a direct calculation yields

$$\begin{aligned}
\overline{A}_{\nu\tau}^{i\,l} d\overline{y}_i^\nu \wedge d\overline{y}_l^\tau = {} & \overline{A}_{\nu\tau}^{i\,l}\left(\frac{\partial \overline{y}_i^\nu}{\partial x^p} + \frac{\partial \overline{y}_i^\nu}{\partial y^\kappa} y_p^\kappa\right)\left(\frac{\partial \overline{y}_l^\tau}{\partial x^q} + \frac{\partial \overline{y}_l^\tau}{\partial y^\lambda} y_q^\lambda\right) dx^p \wedge dx^q \\
& + \overline{A}_{\nu\tau}^{i\,l}\left(\frac{\partial \overline{y}_i^\nu}{\partial x^p} + \frac{\partial \overline{y}_i^\nu}{\partial y^\kappa} y_p^\kappa\right)\frac{\partial \overline{y}_l^\tau}{\partial y^\lambda} dx^p \wedge \omega^\lambda \\
& + \overline{A}_{\nu\tau}^{il}\frac{\partial \overline{y}_i^\nu}{\partial y^\kappa}\left(\frac{\partial \overline{y}_l^\tau}{\partial x^p} + \frac{\partial \overline{y}_l^\tau}{\partial y^\lambda} y_q^\lambda\right)\omega^\kappa \wedge dx^q \\
& + \overline{A}_{\nu\tau}^{i\,l}\left(\frac{\partial \overline{y}_i^\nu}{\partial x^p} + \frac{\partial \overline{y}_i^\nu}{\partial y^\kappa} y_p^\kappa\right)\frac{\partial \overline{y}^\tau}{\partial y^\lambda}\frac{\partial x^j}{\partial \overline{x}^l} dx^p \wedge dy_j^\lambda \\
& + \overline{A}_{\nu\tau}^{i\,l}\frac{\partial \overline{y}^\nu}{\partial y^\kappa}\frac{\partial x^s}{\partial \overline{x}^i}\left(\frac{\partial \overline{y}_l^\tau}{\partial x^p} + \frac{\partial \overline{y}_l^\tau}{\partial y^\lambda} y_q^\lambda\right) dy_s^\kappa \wedge dx^q + \overline{A}_{\nu\tau}^{i\,l}\frac{\partial \overline{y}_i^\nu}{\partial y^\kappa}\frac{\partial \overline{y}_l^\tau}{\partial y^\lambda}\omega^\kappa \wedge \omega^\lambda \\
& + \overline{A}_{\nu\tau}^{i\,l}\frac{\partial \overline{y}_i^\nu}{\partial y^\kappa}\frac{\partial \overline{y}^\tau}{\partial y^\lambda}\frac{\partial x^j}{\partial \overline{x}^l}\omega^\kappa \wedge dy_j^\lambda + \overline{A}_{\nu\tau}^{i\,l}\frac{\partial \overline{y}^\nu}{\partial y^\kappa}\frac{\partial x^s}{\partial \overline{x}^i}\frac{\partial \overline{y}_l^\tau}{\partial y^\lambda} dy_s^\kappa \wedge \omega^\lambda \\
& + \overline{A}_{\nu\tau}^{i\,l}\frac{\partial \overline{y}^\nu}{\partial y^\kappa}\frac{\partial x^s}{\partial \overline{x}^i}\frac{\partial \overline{y}^\tau}{\partial y^\lambda}\frac{\partial x^j}{\partial \overline{x}^l} dy_s^\kappa \wedge dy_j^\lambda.
\end{aligned} \tag{78}$$

Similarly,

$$\begin{aligned}
\overline{A}_{\nu j}^i d\overline{y}_i^\nu \wedge d\overline{x}^j = {} & \overline{A}_{\nu j}^i \frac{\partial \overline{x}^j}{\partial x^l}\left(\frac{\partial \overline{y}_i^\nu}{\partial x^p} + \frac{\partial \overline{y}_i^\nu}{\partial y^\kappa} y_p^\kappa\right) dx^p \wedge dx^l \\
& + \overline{A}_{\nu j}^i \frac{\partial \overline{x}^j}{\partial x^l}\frac{\partial \overline{y}_i^\nu}{\partial y^\kappa}\omega^\kappa \wedge dx^l + \overline{A}_{\nu j}^i \frac{\partial \overline{x}^j}{\partial x^l}\frac{\partial \overline{y}^\nu}{\partial y^\kappa}\frac{\partial x^s}{\partial \overline{x}^i} dy_s^\kappa \wedge dx^l,
\end{aligned} \tag{79}$$

and

$$\overline{A}_{ij} d\overline{x}^i \wedge d\overline{x}^j = \overline{A}_{ij}\frac{\partial \overline{x}^i}{\partial x^p}\frac{\partial \overline{x}^j}{\partial x^l} dx^p \wedge dx^l. \tag{80}$$

To determine the traceless components $A^{ij}_{\nu\tau}$, $A^{i}_{\nu j}$, and A_{ij} from the formulas (78)–(80), respectively, we need the terms not containing ω^{τ}; we get

$$\begin{aligned}
&\bar{A}^{i\,l}_{\nu\tau}\left(\frac{\partial\bar{y}^{\nu}_{i}}{\partial x^{p}}+\frac{\partial\bar{y}^{\nu}_{i}}{\partial y^{\kappa}}y^{\kappa}_{p}\right)\left(\frac{\partial\bar{y}^{\tau}_{l}}{\partial x^{q}}+\frac{\partial\bar{y}^{\tau}_{l}}{\partial y^{\lambda}}y^{\lambda}_{q}\right)dx^{p}\wedge dx^{q}\\
&\quad+\bar{A}^{i\,l}_{\nu\tau}\left(\frac{\partial\bar{y}^{\nu}_{i}}{\partial x^{p}}+\frac{\partial\bar{y}^{\nu}_{i}}{\partial y^{\kappa}}y^{\kappa}_{p}\right)\frac{\partial\bar{y}^{\tau}}{\partial y^{\lambda}}\frac{\partial x^{j}}{\partial\bar{x}^{l}}dx^{p}\wedge dy^{\lambda}_{j}\\
&\quad+\bar{A}^{i\,l}_{\nu\tau}\frac{\partial\bar{y}^{\nu}}{\partial y^{\kappa}}\frac{\partial x^{s}}{\partial\bar{x}^{i}}\left(\frac{\partial\bar{y}^{\tau}_{l}}{\partial x^{p}}+\frac{\partial\bar{y}^{\tau}_{l}}{\partial y^{\lambda}}y^{\lambda}_{q}\right)dy^{\kappa}_{s}\wedge dx^{q}\\
&\quad+\bar{A}^{i\,l}_{\nu\tau}\frac{\partial\bar{y}^{\nu}}{\partial y^{\kappa}}\frac{\partial x^{s}}{\partial\bar{x}^{i}}\frac{\partial\bar{y}^{\tau}}{\partial y^{\lambda}}\frac{\partial x^{j}}{\partial\bar{x}^{l}}dy^{\kappa}_{s}\wedge dy^{\lambda}_{j}\\
&\quad+\bar{A}^{i}_{\nu j}\frac{\partial\bar{x}^{j}}{\partial x^{l}}\left(\frac{\partial\bar{y}^{\nu}_{i}}{\partial x^{p}}+\frac{\partial\bar{y}^{\nu}_{i}}{\partial y^{\kappa}}y^{\kappa}_{p}\right)dx^{p}\wedge dx^{l}\\
&\quad+\bar{A}^{i}_{\nu j}\frac{\partial\bar{x}^{j}}{\partial x^{l}}\frac{\partial\bar{y}^{\nu}}{\partial y^{\kappa}}\frac{\partial x^{s}}{\partial\bar{x}^{i}}dy^{\kappa}_{s}\wedge dx^{l}\\
&\quad+\bar{A}_{ij}\frac{\partial\bar{x}^{i}}{\partial x^{p}}\frac{\partial\bar{x}^{j}}{\partial x^{l}}dx^{p}\wedge dx^{l}.
\end{aligned}\tag{81}$$

Now, it is immediate that

$$\begin{aligned}
A_{pq}&=\bar{A}^{i\,l}_{\nu\tau}\left(\frac{\partial\bar{y}^{\nu}_{i}}{\partial x^{p}}+\frac{\partial\bar{y}^{\nu}_{i}}{\partial y^{\kappa}}y^{\kappa}_{p}\right)\left(\frac{\partial\bar{y}^{\tau}_{l}}{\partial x^{q}}+\frac{\partial\bar{y}^{\tau}_{l}}{\partial y^{\lambda}}y^{\lambda}_{q}\right)\\
&\quad+\frac{1}{2}\bar{A}^{i}_{\nu j}\left(\frac{\partial\bar{x}^{j}}{\partial x^{q}}\left(\frac{\partial\bar{y}^{\nu}_{i}}{\partial x^{p}}+\frac{\partial\bar{y}^{\nu}_{i}}{\partial y^{\kappa}}y^{\kappa}_{p}\right)-\frac{\partial\bar{x}^{j}}{\partial x^{p}}\left(\frac{\partial\bar{y}^{\nu}_{i}}{\partial x^{q}}+\frac{\partial\bar{y}^{\nu}_{i}}{\partial y^{\kappa}}y^{\kappa}_{q}\right)\right)\\
&\quad+\bar{A}_{ij}\frac{\partial\bar{x}^{i}}{\partial x^{p}}\frac{\partial\bar{x}^{j}}{\partial x^{q}}
\end{aligned}\tag{82}$$

and

$$A^{sj}_{\kappa\lambda}=\frac{1}{2}\bar{A}^{i\,l}_{\nu\tau}\left(\frac{\partial\bar{y}^{\nu}}{\partial y^{\kappa}}\frac{\partial x^{s}}{\partial\bar{x}^{i}}\frac{\partial\bar{y}^{\tau}}{\partial y^{\lambda}}\frac{\partial x^{j}}{\partial\bar{x}^{l}}-\frac{\partial\bar{y}^{\nu}}{\partial y^{\lambda}}\frac{\partial x^{j}}{\partial\bar{x}^{i}}\frac{\partial\bar{y}^{\tau}}{\partial y^{\kappa}}\frac{\partial x^{s}}{\partial\bar{x}^{l}}\right).\tag{83}$$

The remaining terms should determine $A^{s}_{\kappa q}$ as the traceless component of the expression

$$\begin{aligned}
&-\bar{A}^{i\,l}_{\nu\tau}\left(\frac{\partial\bar{y}^{\nu}_{i}}{\partial x^{q}}+\frac{\partial\bar{y}^{\nu}_{i}}{\partial y^{\lambda}}y^{\lambda}_{q}\right)\frac{\partial\bar{y}^{\tau}}{\partial y^{\kappa}}\frac{\partial x^{s}}{\partial\bar{x}^{l}}+\bar{A}^{i\,l}_{\nu\tau}\frac{\partial\bar{y}^{\nu}}{\partial y^{\kappa}}\frac{\partial x^{s}}{\partial\bar{x}^{i}}\left(\frac{\partial\bar{y}^{\tau}_{l}}{\partial x^{q}}+\frac{\partial\bar{y}^{\tau}_{l}}{\partial y^{\lambda}}y^{\lambda}_{q}\right)\\
&\quad+\bar{A}^{i}_{\nu j}\frac{\partial\bar{x}^{j}}{\partial x^{q}}\frac{\partial\bar{y}^{\nu}}{\partial y^{\kappa}}\frac{\partial x^{s}}{\partial\bar{x}^{i}}.
\end{aligned}\tag{84}$$

Recall that the traceless component W^i_k of a general system P^i_k, indexed with one contravariant and one covariant index, is defined by

$$W^s_q = P^s_q - \frac{1}{n}\delta^s_q P, \tag{85}$$

where $P = P^j_j$ is the trace of P^i_k. To apply this definition, we first calculate the trace of (84) in s and q. We get

$$\begin{aligned} &-\overline{A}^{i\,l}_{\nu\tau}\left(\frac{\partial \overline{y}^\nu_i}{\partial x^s} + \frac{\partial \overline{y}^\nu_i}{\partial y^\lambda} y^\lambda_s\right)\frac{\partial \overline{y}^\tau}{\partial y^\kappa}\frac{\partial x^s}{\partial \overline{x}^l} + \overline{A}^{i\,l}_{\nu\tau}\frac{\partial \overline{y}^\nu}{\partial y^\kappa}\frac{\partial x^s}{\partial \overline{x}^i}\left(\frac{\partial \overline{y}^\tau_l}{\partial x^s} + \frac{\partial \overline{y}^\tau_l}{\partial y^\lambda} y^\lambda_s\right) \\ &+\overline{A}^i_{\nu j}\frac{\partial \overline{x}^j}{\partial x^s}\frac{\partial \overline{y}^\nu}{\partial y^\kappa}\frac{\partial x^s}{\partial \overline{x}^i}. \end{aligned} \tag{86}$$

Now, we can determine the traceless component of (84). Since the resulting expression must be equal to $A^s_{\kappa q}$, we get the transformation formula

$$\begin{aligned} A^s_{\kappa q} = {}& \overline{A}^i_{\nu j}\frac{\partial \overline{x}^j}{\partial x^q}\frac{\partial \overline{y}^\nu}{\partial y^\kappa}\frac{\partial x^s}{\partial \overline{x}^i} \\ &-\overline{A}^{i\,l}_{\nu\tau}\left(\frac{\partial \overline{y}^\nu_i}{\partial x^q} + \frac{\partial \overline{y}^\nu_i}{\partial y^\lambda} y^\lambda_q\right)\frac{\partial \overline{y}^\tau}{\partial y^\kappa}\frac{\partial x^s}{\partial \overline{x}^l} + \overline{A}^{i\,l}_{\nu\tau}\frac{\partial \overline{y}^\nu}{\partial y^\kappa}\frac{\partial x^s}{\partial \overline{x}^i}\left(\frac{\partial \overline{y}^\tau_l}{\partial x^q} + \frac{\partial \overline{y}^\tau_l}{\partial y^\lambda} y^\lambda_q\right) \\ &+\frac{1}{n}\delta^s_q\overline{A}^{i\,l}_{\nu\tau}\left(\left(\frac{\partial \overline{y}^\nu_i}{\partial x^m} + \frac{\partial \overline{y}^\nu_i}{\partial y^\lambda} y^\lambda_m\right)\frac{\partial \overline{y}^\tau}{\partial y^\kappa}\frac{\partial x^m}{\partial \overline{x}^l} - \frac{\partial \overline{y}^\nu}{\partial y^\kappa}\frac{\partial x^m}{\partial \overline{x}^i}\left(\frac{\partial \overline{y}^\tau_l}{\partial x^m} + \frac{\partial \overline{y}^\tau_l}{\partial y^\lambda} y^\lambda_m\right)\right) \end{aligned} \tag{87}$$

as desired. It is straightforward to verify that the expression on the right-hand side is traceless. This completes Example 2.

2.3 The Horizontalization

We extend the horizontalization $\Omega^r_1 W \ni \rho \to h\rho \in \Omega^{r+1}_1 W$, introduced in Sect. 2.1, to a morphism h: $\Omega^r W \to \Omega^{r+1} W$ of exterior algebras.

Let $\rho \in \Omega^r_q W$ be a q-form, where $q \geq 1$, $J^{r+1}_x\gamma \in W^{r+1}$ a point. Consider the pullback $(\pi^{r+1,r})^*\rho$ and the value $(\pi^{r+1,r})^*\rho(J^{r+1}_x\gamma)(\xi_1, \xi_2, \ldots, \xi_q)$ on any tangent vectors ξ_1, ξ_2, …, ξ_q of $J^{r+1}Y$ at the point $J^{r+1}_x\gamma$. Decompose each of these vectors into the horizontal and contact components,

$$T\pi^{r+1} \cdot \xi_l = h\xi_l + p\xi_l, \tag{88}$$

and set

$$h\rho(J_x^{r+1}\gamma)(\xi_1, \xi_2, \ldots, \xi_q) = \rho(J_x^r\gamma)(h\xi_1, h\xi_2, \ldots, h\xi_q). \tag{89}$$

This formula defines a q-form $h\rho \in \Omega_q^{r+1}W$. This definition can be extended to 0-forms (functions); we set for any function f: $W^r \to \mathbf{R}$

$$hf = (\pi^{r+1,r})^*f. \tag{90}$$

It follows from the properties of the decomposition (88) that the value $h\rho(J_x^{r+1}\gamma)(\xi_1, \xi_2, \ldots, \xi_q)$ vanishes whenever at least one of the vectors ξ_1, ξ_2, …, ξ_q is π^{r+1}-vertical (cf. Sect. 1.5). Thus, the q-form $h\rho$ is π^{r+1}*-horizontal*. In particular, $h\rho = 0$ whenever $q \geq n+1$. Sometimes $h\rho$ is called the *horizontal component* of ρ.

Formulas (89) and (90) define a mapping h: $\Omega^r W \to \Omega^{r+1}W$ of exterior algebras, called the *horizontalization*. The mapping h satisfies

$$h(\rho_1 + \rho_1) = h\rho_1 + h\rho_1, \quad h(f\rho) = (\pi^{r+1,r})^*f \cdot h\rho \tag{91}$$

for all q-forms ρ_1, ρ_1, and ρ and all functions f. In particular, restricting these formulas to *constant* functions f, we see that the horizontalization h is *linear* over the field of real numbers.

Theorem 5 *The mapping h: $\Omega^r W \to \Omega^{r+1}W$ is a morphism of exterior algebras.*

Proof This assertion is a straightforward consequence of the definition of exterior product and formula (89) for the horizontal component of a form ρ. Indeed,

$$\begin{aligned} h(\rho \wedge \eta)&(J_x^{r+1}\gamma)(\xi_1, \xi_2, \ldots, \xi_q, \xi_{p+1}, \xi_{p+2}, \ldots, \xi_{p+q}) \\ &= (\rho \wedge \eta)(J_x^r\gamma)(h\xi_1, h\xi_2, \ldots, h\xi_p, h\xi_{p+1}, h\xi_{p+2}, \ldots, h\xi_{p+q}) \\ &= \sum_\tau \operatorname{sgn}\tau \cdot \rho(J_x^r\gamma)(h\xi_{\tau(1)}, h\xi_{\tau(2)}, \ldots, h\xi_{\tau(p)}) \\ &\quad \cdot \eta(J_x^r\gamma)(h\xi_{\tau(p+1)}, h\xi_{\tau(p+2)}, \ldots, h\xi_{\tau(p+q)}) \\ &= \sum_\tau \operatorname{sgn}\tau \cdot h\rho(J_x^r\gamma)(\xi_{\tau(1)}, \xi_{\tau(2)}, \ldots, \xi_{\tau(p)}) \\ &\quad \cdot h\eta(J_x^r\gamma)(\xi_{\tau(p+1)}, \xi_{\tau(p+2)}, \ldots, \xi_{\tau(p+q)}) \\ &= (h\rho(J_x^{r+1}\gamma) \wedge h\eta(J_x^{r+1}\gamma))(\xi_1, \xi_2, \ldots, \xi_q, \xi_{p+1}, \xi_{p+2}, \ldots, \xi_{p+q}) \end{aligned} \tag{92}$$

(summation through all permutations τ of the set $\{1,2,\ldots,p,p+1,\ldots,p+q\}$ such that $\tau(1)<\tau(2)<\cdots<\tau(p)$ and $\tau(p+1)<\tau(p+2)<\cdots<\tau(p+q)$). This means, however, that

$$h(\rho\wedge\eta)=h\rho\wedge h\eta. \tag{93}$$

□

The following theorem shows that the horizontalization is completely determined by its action on functions and their exterior derivatives.

Theorem 6 *Let W be an open set in the fibered manifold Y. Then, the horizontalization* $\Omega^r W\ni\rho\to h\rho\in\Omega^{r+1}W$ *is a unique* **R**-*linear, exterior-product-preserving mapping such that for any function* $f\colon W^r\to\mathbf{R}$, *and any fibered chart* (V,ψ), $\psi=(y^\sigma)$, *with* $V\subset W$,

$$hf=f\circ\pi^{r+1,r},\quad hdf=d_if\cdot dx^i, \tag{94}$$

where

$$d_if=\frac{\partial f}{\partial x^i}+\sum_{j_1\le j_2\le\cdots\le j_k}\frac{\partial f}{\partial y^\sigma_{j_1j_2\ldots j_k}}y^\sigma_{j_1j_2\ldots j_ki}. \tag{95}$$

Proof The proof that h, defined by (89) and (90), has the desired properties (94) and (95), is standard. To prove uniqueness, note that (94) and (95) imply

$$hdx^i=dx^i,\quad hdy^\sigma_{j_1j_2\ldots j_k}=y^\sigma_{j_1j_2\ldots j_ki}dx^i. \tag{96}$$

It remains to check that any two mappings h_1 and h_2 satisfying the assumptions of Theorem 6 that agree on functions and their exterior derivatives coincide. □

We determine the kernel and the image of the horizontalization h. The following are elementary consequences of the definition.

Lemma 4

(a) *A function f satisfies* $hf=0$ *if and only if* $f=0$.
(b) *If* $q\ge n+1$, *then every q-form* $\rho\in\Omega^r_qW$ *satisfies* $h\rho=0$.
(c) *Let* $1\le q\le n$, *and let* $\rho\in\Omega^r_qW$ *be a form. Then,* $h\rho=0$ *if and only if*

$$J^r\gamma^*\rho=0 \tag{97}$$

for every C^r *section* γ *of Y defined on an open subset of W.*
(d) *If* $h\rho=0$, *then also the exterior derivative* $hd\rho=0$.

Proof

(a) This is a mere restatement of the definition.
(b) This is an immediate consequence of the definition.
(c) Choose a section γ of Y, a point x from the domain of definition of γ and any tangent vectors $\zeta_1, \zeta_2, \ldots, \zeta_q$ of X at x. Then,

$$\begin{aligned} & J^r\gamma^*\rho(x)(\zeta_1, \zeta_2, \ldots, \zeta_q) \\ & \quad = \rho(J^r_x\gamma)(T_xJ^r\gamma \cdot \zeta_1, T_xJ^r\gamma \cdot \zeta_2, \ldots, T_xJ^r\gamma \cdot \zeta_q). \end{aligned} \tag{98}$$

Since $T\pi^{r+1}$ is surjective, there exist tangent vectors ξ_l to $J^{r+1}Y$ at $J^{r+1}_x\gamma$, such that $\zeta_l = T\pi^{r+1} \cdot \xi_l$. For these tangent vectors,

$$\begin{aligned} & J^r\gamma^*\rho(x)(\zeta_1, \zeta_2, \ldots, \zeta_q) \\ & \quad = \rho(J^r_x\gamma)(T_xJ^r\gamma \cdot T\pi^{r+1} \cdot \xi_1, T_xJ^r\gamma \cdot T\pi^{r+1} \cdot \xi_2, \ldots, T_xJ^r\gamma \cdot T\pi^{r+1} \cdot \xi_q). \end{aligned} \tag{99}$$

But $h\xi = T_xJ^r\gamma \circ T\pi^{r+1} \cdot \xi$, and hence,

$$\begin{aligned} J^r\gamma^*\rho(x)(\zeta_1, \zeta_2, \ldots, \zeta_q) & = \rho(J^r_x\gamma)(h\xi_1, h\xi_2, \ldots, h\xi_q) \\ & = h\rho(J^{r+1}_x\gamma)(\xi_1, \xi_2, \ldots, \xi_q). \end{aligned} \tag{100}$$

This correspondence already proves assertion (a).
(d) This assertion (d) follows from (c). □

We are now in a position to complete the description of the kernel of the horizontalization h for q-forms such that $1 \le q \le n$.

Theorem 7 *Let $W \subset Y$ be an open set, $\rho \in \Omega^r_q W$ a form, and let (V, ψ), $\psi = (x^i, y^\sigma)$, be a fibered chart such that $V \subset W$.*

(a) *Let $q = 1$. Then, ρ satisfies $h\rho = 0$ if and only if its chart expression is of the form*

$$\rho = \sum_{0 \le |J| \le r-1} \Phi^J_\sigma \omega^\sigma_J \tag{101}$$

for some functions $\Phi^\sigma_J \colon V^r \to \mathbf{R}$.

(b) *Let $2 \le q \le n$. Then, ρ satisfies $h\rho = 0$ if and only if its chart expression is of the form*

$$\rho = \sum_{0 \le |J| \le r-1} \omega^\sigma_J \wedge \Phi^J_\sigma + \sum_{|I| = r-1} d\omega^\sigma_I \wedge \Psi^I_\sigma, \tag{102}$$

where Φ^J_σ (resp. Ψ^I_σ) are some $(q-1)$-forms (resp. $(q-2)$-forms) on V^r.

Proof Suppose that we have a contact q-form ρ on W^r, where $1 \le q \le n$. Write as in Sect. 2.2, Theorem 3, $\rho = \rho_0 + \rho'$, where ρ_0 is contact and ρ' is traceless. But the horizontalization h preserves exterior product and $h\rho = 0$, so we get $h\rho' = 0$ because ρ_0 is generated by the contact forms ω_J^σ, $d\omega_I^\sigma$, which satisfy $h\omega_J^\sigma = 0$ and $hd\omega_I^\sigma = 0$. Now, using formula $hdy_I^\sigma = y_{Ii}^\sigma dx^i$, we get, expressing ρ' as in Sect. 2.2, (53)

$$\begin{aligned} h\rho' = (&A_{i_1 i_2 \ldots i_q} + A^{I_1}_{\sigma_1 i_2 i_3 \ldots i_q} y^{\sigma_1}_{I_1 i_1} + A^{I_1 I_2}_{\sigma_1 \sigma_2 i_3 i_4 \ldots i_q} y^{\sigma_1}_{I_1 i_1} y^{\sigma_3}_{I_2 i_2} \\ &+ \cdots + A^{I_1 I_2}_{\sigma_1 \sigma_2} \cdots {}^{I_{q-1}}_{\sigma_{q-1} i_q} y^{\sigma_1}_{I_1 i_1} y^{\sigma_3}_{I_2 i_2} \cdots y^{\sigma_{q-1}}_{I_{q-1} i_{q-1}} \\ &+ A^{I_1 I_2}_{\sigma_1 \sigma_2} \cdots {}^{I_q}_{\sigma_q} y^{\sigma_1}_{I_1 i_1} y^{\sigma_3}_{I_2 i_2} \cdots y^{\sigma_q}_{I_q i_q}) dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_q}, \end{aligned} \tag{103}$$

where $|I_1|, |I_2|, \ldots, |I_{q-1}| = r$ and the coefficients $A^{I_1}_{\sigma_1 i_2 i_3 \ldots i_q}$, $A^{I_1 I_2}_{\sigma_1 \sigma_2 i_3 i_4 \ldots i_q}$, $\ldots$, $A^{I_1 I_2}_{\sigma_1 \sigma_2} \cdots {}^{I_{q-1}}_{\sigma_{q-1}\, i_q}$ are traceless. Then,

$$\begin{aligned} &A_{i_1 i_2 \ldots i_q} + A^{I_1}_{\sigma_1 i_2 i_3 \ldots i_q} y^{\sigma_1}_{I_1 i_1} + A^{I_1 I_2}_{\sigma_1 \sigma_2 i_3 i_4 \ldots i_q} y^{\sigma_1}_{I_1 i_1} y^{\sigma_3}_{I_2 i_2} \\ &\quad + \cdots + A^{I_1 I_2}_{\sigma_1 \sigma_2} \cdots {}^{I_{q-1}}_{\sigma_{q-1}}{}_{i_q} y^{\sigma_1}_{I_1 i_1} y^{\sigma_3}_{I_2 i_2} \cdots y^{\sigma_{q-1}}_{I_{q-1} i_{q-1}} \\ &\quad + A^{I_1 I_2}_{\sigma_1 \sigma_2} \cdots {}^{I_q}_{\sigma_q} y^{\sigma_1}_{I_1 i_1} y^{\sigma_3}_{I_2 i_2} \cdots y^{\sigma_q}_{I_q i_q} = 0 \quad \mathrm{Alt}(i_1 i_2 \ldots i_q). \end{aligned} \tag{104}$$

But the expressions on the left-hand sides of these equations are polynomial in the variables y^ν_K with $|K| = r + 1$, so the corresponding homogeneous components in (104) must vanish separately. Then, we have $A_{i_1 i_2 \ldots i_q} = 0$, $A^{I_1 I_2}_{\sigma_1 \sigma_2} \cdots {}^{I_q}_{\sigma_q} = 0$, and

$$\begin{aligned} &A^{I_1}_{\sigma_1 i_2 i_3 \ldots i_q} \delta^{l_1}_{i_1} = 0 \quad \mathrm{Alt}(i_1 i_2 \ldots i_q) \quad \mathrm{Sym}(I_1 l_1), \\ &A^{I_1 I_2}_{\sigma_1 \sigma_2 i_3 i_4 \ldots i_q} \delta^{l_1}_{i_1} \delta^{l_2}_{i_2} = 0 \quad \mathrm{Alt}(i_1 i_2 \ldots i_q) \quad \mathrm{Sym}(I_1 l_1) \quad \mathrm{Sym}(I_2 l_2), \\ &\ldots \\ &A^{I_1 I_2}_{\sigma_1 \sigma_2} \cdots {}^{I_{q-1}}_{\sigma_{q-1} i_q} \delta^{l_1}_{i_1} \delta^{l_2}_{i_2} \cdots \delta^{l_{q-1}}_{i_{q-1}} = 0 \quad \mathrm{Alt}(i_1 i_2 \ldots i_q) \quad \mathrm{Sym}(I_1 l_1) \\ &\mathrm{Sym}(I_2 l_2) \quad \ldots \mathrm{Sym}(I_{q-1} l_{q-1}). \end{aligned} \tag{105}$$

However, since the coefficients $A^{I_1}_{\sigma_1\, i_2 i_3 \ldots i_q}, A^{I_1 I_2}_{\sigma_1 \sigma_2 i_3 i_4 \ldots i_q}, \ldots, A^{I_1 I_2}_{\sigma_1 \sigma_2} \cdots {}^{I_{q-1}}_{\sigma_{q-1}\, i_q}$ are traceless, they must vanish identically (see Appendix 9, Theorem 4). Thus, we have in (103)

$$\begin{aligned} &A_{i_1 i_2 \ldots i_q} = 0, \quad A^{I_1}_{\sigma_1 i_2 i_3 \ldots i_q} = 0, \quad A^{I_1 I_2}_{\sigma_1 \sigma_2 i_3 i_4 \ldots i_q} = 0, \\ &\ldots, \quad A^{I_1 I_2}_{\sigma_1 \sigma_2} \cdots {}^{I_{q-1}}_{\sigma_{q-1} i_q} = 0, \quad A^{I_1 I_2}_{\sigma_1 \sigma_2} \cdots {}^{I_q}_{\sigma_q} = 0 \end{aligned} \tag{106}$$

and hence, $h\rho' = 0$. Thus $\rho = \rho_0$, and to close the proof, we just write this result for $q = 1$ and $q > 1$ separately. □

Corollary 1 *If* $0 \le q \le n$, *then a* q*-form belongs to the kernel of the horizontalization* h *if and only if it is a contact form.*

Corollary 2 *Let* $W \subset Y$ *be an open set,* $\rho \in \Omega_q^r W$ *a* q*-form such that* $2 \le q \le n$, *and let* (V, ψ), $\psi = (x^i, y^\sigma)$, *be a fibered chart such that* $V \subset W$. *Then, the form* ρ *satisfies the condition* $h\rho = 0$ *if and only if its chart expression is of the form*

$$\rho = \sum_{0 \le |J| \le r-1} \omega_J^\sigma \wedge \Phi_\sigma^J + \sum_{|I|=r-1} d(\omega_I^\sigma \wedge \Psi_\sigma^I), \tag{107}$$

where Φ_σ^J *are* $(q-1)$*-forms and* Ψ_σ^I *are* $(q-2)$*-forms) on* V^r, *which do not contain* ω_J^σ, $0 \le |J| \le r-1$.

Proof We write (102) as

$$\rho = \sum_{0 \le |J| \le r-1} \omega_J^\sigma \wedge \Phi_\sigma^J - \sum_{|I|=r-1} \omega_I^\sigma \wedge d\Psi_\sigma^I + \sum_{0 \le |I| \le r-1} d(\omega_I^\sigma \wedge \Psi_\sigma^I). \tag{108}$$

□

The *image* of the horizontalization h is characterized as follows.

Lemma 5 *Let* $\rho \in \Omega_q^r W$ *be a form.*

(a) *If* $q = 0$, *then* $h\rho = (\pi^{r+1,r})^* \rho$.
(b) *If* $1 \le q \le n$, *then*

$$h\rho = h\rho'. \tag{109}$$

(c) *If* $q \ge n+1$, *then* $h\rho = h\rho' = 0$.

Proof This assertion is an immediate consequence of the definition of the horizontalization h. □

2.4 The Canonical Decomposition

Beside the horizontalization of q-forms $\Omega_q^r W$, introduced in Sects. 2.1 and 2.3, the vector bundle morphism h: $TJ^{r+1}Y \to TJ^r Y$ also induces a decomposition of the modules of q-forms $\Omega_q^r W$. Let $\rho \in \Omega_q^r W$ be a q-form, where $q \ge 1$, $J_x^{r+1}\gamma \in W^{r+1}$ a point. Consider the pullback $(\pi^{r+1,r})^* \rho$ and the value $(\pi^{r+1,r})^* \rho(J_x^{r+1}\gamma)$ $(\xi_1, \xi_2, \ldots, \xi_q)$ on any tangent vectors ξ_1, ξ_2, ..., ξ_q of $J^{r+1}Y$ at the point $J_x^{r+1}\gamma$. Write for each l,

$$T\pi^{r+1} \cdot \xi_l = h\xi_l + p\xi_l, \tag{110}$$

and substitute these vectors in the pullback $(\pi^{r+1,r})^*\rho$. We get

$$\begin{aligned}&(\pi^{r+1,r})^*\rho(J_x^{r+1}\gamma)(\xi_1,\xi_2,\ldots,\xi_q)\\&\quad=\rho(J_x^r\gamma)(h\xi_1+p\xi_1,h\xi_2+p\xi_2,\ldots,h\xi_q+p\xi_q).\end{aligned} \tag{111}$$

We study in this section, for each $k=0,1,2,\ldots,q$, the summands on the right-hand side, homogeneous of degree k in the contact components $p\xi_l$ of the vectors ξ_l, and describe the corresponding decomposition of the form $(\pi^{r+1,r})^*\rho$. Using properties of ρ, we set

$$\begin{aligned}&p_k\rho(J_x^{r+1}\gamma)(\xi_1,\xi_2,\ldots,\xi_q)\\&\quad=\sum\varepsilon^{j_1j_2\ldots j_kj_{k+1}\ldots j_q}\rho(J_x^r\gamma)(p\xi_{j_1},p\xi_{j_2},\ldots,p\xi_{j_k},h\xi_{j_{k+1}},h\xi_{j_{k+2}},\ldots,h\xi_{j_q}),\end{aligned} \tag{112}$$

where the summation is understood through all sequences $j_1<j_2<\cdots<j_k$ and $j_{k+1}<j_{k+2}<\cdots<j_q$. Equivalently, $p_k\rho(J_x^{r+1}\gamma)$ can also be defined by

$$\begin{aligned}&p_k\rho(J_x^{r+1}\gamma)(\xi_1,\xi_2,\ldots,\xi_q)\\&\quad=\frac{1}{k!(q-k)!}\varepsilon^{j_1j_2\ldots j_kj_{k+1}\ldots j_q}\rho(J_x^r\gamma)(p\xi_{j_1},p\xi_{j_2},\ldots,p\xi_{j_k},h\xi_{j_{k+1}},\ldots,h\xi_{j_q})\end{aligned} \tag{113}$$

(summation through *all* values of the indices $j_1,j_2,\ldots,j_k,j_{k+1},\ldots,j_q$).

Note that if $k=0$, then $p_0\rho$ coincides with the *horizontal component* of ρ, defined in Sect. 2.1, (5),

$$p_0\rho=h\rho. \tag{114}$$

We also introduce the notation

$$p\rho=p_1\rho+p_2\rho+\cdots+p_q\rho. \tag{115}$$

These definitions can be extended to 0-forms (functions). Since for a function $f\colon W^r\to\mathbf{R}$, hf was defined to be $(\pi^{r+1,r})^*f$, we set

$$pf=0. \tag{116}$$

With this notation, any q-form $\rho\in\Omega_q^rW$, where $q\geq0$, can be expressed as $(\pi^{r+1,r})^*\rho=h\rho+p\rho$, or

$$(\pi^{r+1,r})^*\rho=h\rho+p_1\rho+p_2\rho+\cdots+p_q\rho. \tag{117}$$

This formula will be referred to as the *canonical decomposition* of the form ρ (however, the decomposition concerns rather the pullback $(\pi^{r+1,r})^*\rho$ than ρ itself).

Lemma 6 *Let $q \geq 1$, and let $\rho \in \Omega_q^r W$ be a q-form. In any fibered chart (V, ψ), $\psi = (x^i, y^\sigma)$, such that $V \subset W$, $p_k\rho$ has a chart expression*

$$p_k\rho = \sum_{0 \leq |J_1|, |J_2|, \ldots, |J_k| \leq r} P^{J_1 J_2 \ldots J_k}_{\sigma_1 \sigma_2 \ldots \sigma_k\, i_{k+1} i_{k+2} \ldots i_q} \omega^{\sigma_1}_{J_1} \wedge \omega^{\sigma_2}_{J_2} \wedge \cdots \wedge \omega^{\sigma_k}_{J_k} \wedge dx^{i_{k+1}} \wedge dx^{i_{k+2}} \wedge \cdots \wedge dx^{i_q}, \tag{118}$$

where the components $P^{J_1 J_2 \ldots J_k}_{\sigma_1 \sigma_2 \ldots \sigma_k\, i_{k+1} i_{k+2} \ldots i_q}$ are real-valued functions on the set $V^r \subset W^r$.

Proof We express the pullback $(\pi^{r+1,r})^*\rho$ in the contact basis on W^{r+1}. Write in a fibered chart

$$\rho = dx^i \wedge \Phi_i + \sum_{0 < |J| < r-1} \omega^\sigma_J \wedge \Psi^J_\sigma + \sum_{|I| = r} dy^\sigma_I \wedge \Theta^I_\sigma \tag{119}$$

for some $(q-1)$-forms Φ_i, Ψ^J_σ, and Θ^I_σ. But $dy^\sigma_I = \omega^\sigma_I + y^\sigma_{Ii} dx^i$, and hence,

$$\begin{aligned}(\pi^{r+1,r})^*\rho &= dx^i \wedge \left((\pi^{r+1,r})^*\Phi_i + \sum_{|I|=r} y^\sigma_{Ii} (\pi^{r+1,r})^*\Theta^I_\sigma \right) \\ &\quad + \sum_{0 < |J| < r-1} \omega^\sigma_J \wedge (\pi^{r+1,r})^*\Psi^J_\sigma + \sum_{|I|=r} \omega^\sigma_I \wedge (\pi^{r+1,r})^*\Theta^I_\sigma.\end{aligned} \tag{120}$$

Thus, the pullback $(\pi^{r+1,r})^*\rho$ is generated by the form dx^i, ω^σ_J, where $0 < |J| < r - 1$ and ω^σ_I, $|I| = r$. The same decomposition can be applied to the $(q-1)$-forms Φ_i, Ψ^J_σ, and Θ^I_σ. Consequently, $(\pi^{r+1,r})^*\rho$ has an expression

$$(\pi^{r+1,r})^*\rho = \rho_0 + \rho_1 + \rho_2 + \cdots + \rho_q, \tag{121}$$

where

$$\begin{aligned}\rho_0 &= A_{i_1 i_2 \ldots i_q} dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_q}, \\ \rho_k &= \sum_{0 \leq |J_1|, |J_2|, \ldots, |J_k| \leq r} B^{J_1 J_2 \ldots J_k}_{\sigma_1 \sigma_2 \ldots \sigma_k\, i_{k+1} i_{k+2} \ldots i_q} \omega^{\sigma_1}_{J_1} \wedge \omega^{\sigma_2}_{J_2} \wedge \cdots \wedge \omega^{\sigma_k}_{J_k} \\ &\quad \wedge dx^{i_{k+1}} \wedge dx^{i_{k+2}} \wedge \cdots \wedge dx^{i_q}, \quad 1 \leq k \leq q-1, \\ \rho_q &= \sum_{0 \leq |J_1|, |J_2|, \ldots, |J_q| \leq r} B^{J_1 J_2 \ldots J_q}_{\sigma_1 \sigma_2 \ldots \sigma_q i_q} \omega^{\sigma_1}_{J_1} \wedge \omega^{\sigma_3}_{J_2} \wedge \cdots \wedge \omega^{\sigma_q}_{J_q}.\end{aligned} \tag{122}$$

Theorem 1, Sect. 2.1, implies that the decomposition (121) is invariant.

We prove that $\rho_k = p_k\rho$. It is sufficient to determine the chart expression of $p_k\rho$. Let ξ be a tangent vector,

$$\xi = \xi^i \left(\frac{\partial}{\partial x^i}\right)_{J_x^{r+1}\gamma} + \sum_{k=0}^{r+1} \sum_{j_1 \le j_2 \le \cdots \le j_k} \Xi^{\sigma}_{j_1 j_2 \ldots j_k} \left(\frac{\partial}{\partial y^{\sigma}_{j_1 j_2 \ldots j_k}}\right)_{J_x^{r+1}\gamma}. \tag{123}$$

From Sect. 1.5, (62)

$$h\xi = \xi^i \left(\left(\frac{\partial}{\partial x^i}\right)_{J_x^r\gamma} + \sum_{k=0}^{r} \sum_{j_1 \le j_2 \le \cdots \le j_k} y^{\sigma}_{j_1 j_2 \ldots j_k i} \left(\frac{\partial}{\partial y^{\sigma}_{j_1 j_2 \ldots j_k}}\right)_{J_x^r\gamma} \right), \tag{124}$$

and

$$p\xi = \sum_{k=0}^{r} \sum_{j_1 \le j_2 \le \cdots \le j_k} (\Xi^{\sigma}_{j_1 j_2 \ldots j_k} - y^{\sigma}_{j_1 j_2 \ldots j_k i}\xi^i) \left(\frac{\partial}{\partial y^{\sigma}_{j_1 j_2 \ldots j_k}}\right)_{J_x^r\gamma}. \tag{125}$$

If $h\xi = 0$, then $\xi^i = 0$, and we have

$$p\xi = \sum_{k=0}^{r} \sum_{j_1 \le j_2 \le \cdots \le j_k} \Xi^{\sigma}_{j_1 j_2 \ldots j_k} \left(\frac{\partial}{\partial y^{\sigma}_{j_1 j_2 \ldots j_k}}\right)_{J_x^r\gamma}. \tag{126}$$

If $p\xi = 0$, then $\Xi^{\sigma}_{j_1 j_2 \ldots j_k} = y^{\sigma}_{j_1 j_2 \ldots j_k i}\xi^i$, and hence,

$$h\xi = \xi^i \left(\left(\frac{\partial}{\partial x^i}\right)_{J_x^r\gamma} + \sum_{k=0}^{r} \sum_{j_1 \le j_2 \le \cdots \le j_k} y^{\sigma}_{j_1 j_2 \ldots j_k i} \left(\frac{\partial}{\partial y^{\sigma}_{j_1 j_2 \ldots j_k}}\right)_{J_x^r\gamma} \right). \tag{127}$$

We substitute from these formulas to expression (112). Consider the expression $p_k\rho(J_x^{r+1}\gamma)(\xi_1, \xi_2, \ldots, \xi_q)$ for $\xi_1, \xi_2, \ldots, \xi_q$ such that $h\xi_1 = 0$, $h\xi_2 = 0$, …, $h\xi_k = 0$ and $p\xi_{k+1} = 0$, $p\xi_2 = 0$, …, $p\xi_q = 0$. Then, (112) reduces to

$$\begin{aligned} &p_k\rho(J_x^{r+1}\gamma)(\xi_1, \xi_2, \ldots, \xi_q) \\ &\quad = \rho(J_x^r\gamma)(p\xi_1, p\xi_2, \ldots, p\xi_k, h\xi_{k+1}, h\xi_{k+2}, \ldots, h\xi_q). \end{aligned} \tag{128}$$

Writing

$$\begin{aligned} p\xi_l &= \sum_{k=0}^{r} \sum_{j_1 \le j_2 \le \cdots \le j_k} {}^{(l)}\Xi^{\sigma}_{j_1 j_2 \ldots j_k} \left(\frac{\partial}{\partial y^{\sigma}_{j_1 j_2 \ldots j_k}}\right)_{J^r_x\gamma}, \quad 1 \le l \le k, \\ h\xi_l &= {}^{(l)}\xi^i \left(\left(\frac{\partial}{\partial x^i}\right)_{J^r_x\gamma} + \sum_{k=0}^{r} \sum_{j_1 \le j_2 \le \cdots \le j_k} y^{\sigma}_{j_1 j_2 \ldots j_k i} \left(\frac{\partial}{\partial y^{\sigma}_{j_1 j_2 \ldots j_k}}\right)_{J^r_x\gamma}\right), \\ &\quad k+1 \le l \le q, \end{aligned} \tag{129}$$

with l indexing the vectors ξ_l, and substituting into (128), we get

$$\begin{aligned} &p_k\rho(J^{r+1}_x\gamma)(\xi_1, \xi_2, \ldots, \xi_k, \xi_{k+1}, \xi_{k+2}, \ldots, \xi_q), \\ &= C^{I_1 I_2 \ldots I_k}_{\sigma_1 \sigma_2 \ldots \sigma_k\, i_{k+1} i_{k+2} \ldots i_q} {}^1\Xi^{\sigma_1}_{I_1} {}^2\Xi^{\sigma_2}_{I_2} \ldots {}^k\Xi^{\sigma_k}_{I_k} {}^{k+1}\xi^{i_{k+1}} {}^{k+2}\xi^{i_{k+2}} \ldots {}^q\xi^{i_q}. \end{aligned} \tag{130}$$

But

$${}^l\Xi^{\sigma}_I = \omega^{\sigma}_I(J^{r+1}_x\gamma) \cdot \xi_l, \quad {}^l\xi^i = dx^i(J^{r+1}_x\gamma) \cdot \xi_l \tag{131}$$

Therefore, $p_k\rho(J^{r+1}_x\gamma)$ must be of the form (118). □

Formula (118) implies that for any $k \ge 1$, the form $p_k\rho$ is contact; $p_k\rho$ is called the *k-contact component* of the form ρ.

If $(\pi^{r+1,r})^*\rho = p_k\rho$ or, equivalently, if $p_j\rho = 0$ for all $j \ne k$, then we say that ρ is *k-contact*, and k is the *degree of contactness* of ρ. The degree of contactness of the q-form $\rho = 0$ is equal to k for every $k = 0, 1, 2, \ldots, q$. We say that ρ is of *degree of contactness* $\ge k$, if $p_0\rho = 0$, $p_1\rho = 0$, …, $p_{k-1}\rho = 0$. If $k = 0$, then the 0-contact form $p_0\rho = h\rho$ is $\pi^{r+1,r}$*-horizontal*. The mapping $\Omega^r_q W \ni \rho \to h\rho \in \Omega^{r+1}_q W$ is called the *horizontalization.*

The following observation is immediate.

Lemma 7 *If $q - k > n$, then*

$$\begin{aligned} &h\rho = 0, \\ &p_1\rho = 0, \quad p_2\rho = 0, \quad \ldots, \quad p_{q-n-1} = 0. \end{aligned} \tag{132}$$

Proof Expression $\rho(J^r_x\gamma)(p\xi_{j_1}, p\xi_{j_2}, \ldots, p\xi_{j_k}, h\xi_{j_{k+1}}, h\xi_{j_{k+2}}, \ldots, h\xi_{j_q})$ in (113) is a $(q-k)$-linear function of vectors $\zeta_{j_{k+1}} = T\pi^{r+1} \cdot \xi_{j_{k+1}}, \zeta_{j_{k+2}} = T\pi^{r+1} \cdot \xi_{j_{k+2}}, \ldots, \zeta_{j_q} = T\pi^{r+1} \cdot \xi_{j_q}$, belonging to the tangent space T_xX. Consequently, if $q - k > n = \dim X$, then the skew symmetry of the form $p_k\rho(J^{r+1}_x\gamma$ implies $p_k\rho(J^{r+1}_x\gamma)(\xi_1, \xi_2, \ldots, \xi_q) = 0$. □

To complete the local description of the decomposition (117), we express the components $P^{J_1 J_2 \dots J_k}_{\sigma_1 \sigma_2 \dots \sigma_k i_{k+1} i_{k+2} \dots i_q}$ (118) of the k-contact components $p_k \rho$ in terms of the components of ρ.

Lemma 8 *Let W be an open set in Y, q an integer, $\eta \in \Omega^r_q W$ a form, and let (V, ψ), $\psi = (x^i, y^\sigma)$, be a fibered chart on Y such that $V \subset W$. Assume that η has on V^r a chart expression*

$$\eta = \sum_{s=0}^{q} \frac{1}{s!(q-s)!} A^{I_1 I_2 \dots I_s}_{\sigma_1 \sigma_2 \dots \sigma_s\, i_{s+1} i_{s+2} \dots i_q} dy^{\sigma_1}_{I_1} \wedge dy^{\sigma_2}_{I_2} \wedge \dots \wedge dy^{\sigma_s}_{I_s} \wedge dx^{i_{s+1}} \wedge dx^{i_{s+2}} \wedge \dots \wedge dx^{i_q}, \tag{133}$$

with multi-indices $I_1, I_2, \dots, I_s$ of length r. Then, the k-contact component $p_k \eta$ of η has on V^{r+1} a chart expression

$$p_k \eta = \frac{1}{k!(q-k)!} B^{I_1 I_1 \dots I_k}_{\sigma_1 \sigma_1 \dots \sigma_k\, i_{k+1} i_{k+2} \dots i_q} \omega^{\sigma_1}_{I_1} \wedge \omega^{\sigma_2}_{I_2} \wedge \dots \wedge \omega^{\sigma_k}_{I_k} \wedge dx^{i_{k+1}} \wedge dx^{i_{k+2}} \wedge \dots \wedge dx^{i_q}, \tag{134}$$

where

$$B^{I_1 I_2 \dots I_k}_{\sigma_1 \sigma_2 \dots \sigma_k i_{k+1} i_{k+2} \dots i_q} = \sum_{s=k}^{q} \binom{q-k}{q-s} A^{I_1 I_2 \dots I_k\, I_{k+1}\, I_{k+2} \dots I_s}_{\sigma_1 \sigma_2 \dots \sigma_k\, \sigma_{k+1}\, \sigma_{k+2} \dots \sigma_s\, i_{s+1} i_{s+2} \dots i_q} y^{\sigma_{k+1}}_{I_{k+1} i_{k+1}} y^{\sigma_{k+2}}_{I_{k+2} i_{k+2}} \dots y^{\sigma_s}_{I_s i_s} \operatorname{Alt}(i_{k+1} i_{k+2} \dots i_s i_{s+1} \dots i_q). \tag{135}$$

Proof To derive the formula (134), we pullback the form η to V^{r+1} and express the form $(\pi^{r+1,r})^* \Psi$ in terms of the contact basis; in the multi-index notation, the transformation equations are

$$dx^i = dx^i, \quad dy^\sigma_I = \omega^\sigma_I + y^\sigma_{Ii} dx^i, \quad |I| = r \tag{136}$$

(Sect. 2.1, Theorem 1, (a)). Thus, we set in (133) $dy^{\sigma_l}_{I_l} = \omega^{\sigma_l}_{I_l} + y^{\sigma_l}_{I_l i_l} dx^{i_l}$ and consider the terms in (133) such that $s \geq 1$. Then, the pullback of the form $dy^{\sigma_1}_{I_1} \wedge dy^{\sigma_2}_{I_2} \wedge \dots \wedge dy^{\sigma_s}_{I_s}$ by $\pi^{r+1,r}$ is equal to

$$(\omega^{\sigma_1}_{I_1} + y^{\sigma_1}_{I_1 i_1} dx^{i_1}) \wedge (\omega^{\sigma_2}_{I_2} + y^{\sigma_2}_{I_2 i_2} dx^{i_2}) \wedge \dots \wedge (\omega^{\sigma_s}_{I_s} + y^{\sigma_s}_{I_s i_s} dx^{i_s}). \tag{137}$$

Collecting together all terms homogeneous of degree k in the contact 1-forms $\omega^{\sigma_l}_{I_l}$, we get $\binom{s}{k}$ summands with exactly k entries the contact 1-forms $\omega^{\sigma_l}_{I_l}$. Thus, using symmetry properties of the components $A^{I_1 I_1 \dots I_s}_{\sigma_1 \sigma_1 \dots \sigma_s i_{s+1} i_{s+2} \dots i_q}$ in (133) and interchanging multi-indices, we get the terms containing k entries $\omega^{\sigma_l}_{I_l}$, for fixed s and each $k = 1, 2, \ldots, s$,

$$\frac{1}{s!(q-s)!}\binom{s}{k}A^{I_1 I_2}_{\sigma_1\sigma_2}\cdots{}^{I_s}_{\sigma_s i_{s+1} i_{s+2}\ldots i_q} y^{\sigma_{k+1}}_{I_{k+1}i_{k+1}} y^{\sigma_{k+2}}_{I_{k+2}i_{k+2}}\cdots y^{\sigma_s}_{I_s i_s}\omega^{\sigma_1}_{I_1}\wedge\omega^{\sigma_2}_{I_2}\wedge\cdots\wedge\omega^{\sigma_k}_{I_k}$$
$$\wedge\, dx^{i_{k+1}}\wedge dx^{i_{k+2}}\wedge\cdots\wedge dx^{i_s}\wedge dx^{i_{s+1}}\wedge dx^{i_{s+2}}\wedge\cdots\wedge dx^{i_q}. \tag{138}$$

Writing the factor as

$$\frac{1}{s!(q-s)!}\binom{s}{k} = \frac{1}{k!(q-k)!}\binom{q-k}{q-s}, \tag{139}$$

we can express (138) as

$$\frac{1}{k!(q-k)!}\binom{q-k}{q-s}A^{I_1 I_2}_{\sigma_1\sigma_2}\cdots{}^{I_s}_{\sigma_s\, i_{s+1} i_{s+2}\ldots i_q} y^{\sigma_{k+1}}_{I_{k+1}i_{k+1}} y^{\sigma_{k+2}}_{I_{k+2}i_{k+2}}\cdots y^{\sigma_s}_{I_s i_s}\omega^{\sigma_1}_{I_1}\wedge\omega^{\sigma_2}_{I_2}$$
$$\wedge\cdots\wedge\omega^{\sigma_k}_{I_k}\wedge dx^{i_{k+1}}\wedge dx^{i_{k+2}}\wedge\cdots\wedge dx^{i_s}\wedge dx^{i_{s+1}}\wedge dx^{i_{s+2}}\wedge\cdots\wedge dx^{i_q}. \tag{140}$$

Formula (138) is valid for each $s = 1, 2, \ldots, q$ and each $k = 1, 2, \ldots, s$ and includes summation through all these terms to get expression (133). The summation through the pairs (s, k) is given by the table

s	1	2	3	…	$q-1$	q
k	1	1,2	1,2,3	…	1,2,3,…,$q-1$	1,2,3,…,q

(141)

It will be convenient to pass to the summation over the same written in the opposite order. The summation through the pairs (k, s) is expressed by the table

k	1	2	3	…	$q-1$	q
s	1,2,3,…,q	2,3,…,q	3,4,…,q	…	$q-1$,q	q

(142)

Now, we can substitute from (140) back to (133). We have, with multi-indices of length r,

$$\eta = \frac{1}{q!}A_{i_1 i_2\ldots i_q}dx^{i_1}\wedge dx^{i_2}\wedge\cdots\wedge dx^{i_q}$$
$$+\sum_{s=1}^{q}\sum_{k=1}^{s}\frac{1}{k!(q-k)!}\binom{q-k}{q-s}A^{I_1 I_2}_{\sigma_1\sigma_2}\cdots{}^{I_s}_{\sigma_s\, i_{s+1} i_{s+2}\ldots i_q} y^{\sigma_{k+1}}_{I_{k+1}i_{k+1}} y^{\sigma_{k+2}}_{I_{k+2}i_{k+2}}\cdots y^{\sigma_s}_{I_s i_s}$$
$$\cdot\,\omega^{\sigma_1}_{I_1}\wedge\omega^{\sigma_2}_{I_2}\wedge\cdots\wedge\omega^{\sigma_k}_{I_k}\wedge dx^{i_{k+1}}\wedge dx^{i_{k+2}}\wedge\cdots\wedge dx^{i_s}\wedge dx^{i_{s+1}}\wedge\cdots\wedge dx^{i_q} \tag{143}$$

hence,

$$p_k\eta = \frac{1}{q!}A_{i_1 i_2 \ldots i_q} dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_q}$$
$$+ \sum_{k=1}^{q} \frac{1}{k!(q-k)!} \left(\sum_{s=k}^{q} \binom{q-k}{q-s} A^{I_1 I_2}_{\sigma_1 \sigma_2} \cdots^{I_s}_{\sigma_s \, i_{s+1} i_{s+2} \ldots i_q} y^{\sigma_{k+1}}_{I_{k+1} i_{k+1}} y^{\sigma_{k+2}}_{I_{k+2} i_{k+2}} \cdots y^{\sigma_s}_{I_s i_s} \right)$$
$$\cdot\, \omega^{\sigma_1}_{I_1} \wedge \omega^{\sigma_2}_{I_2} \wedge \cdots \wedge \omega^{\sigma_k}_{I_k} \wedge dx^{i_{k+1}} \wedge dx^{i_{k+2}} \wedge \cdots \wedge dx^{i_q}. \tag{144}$$

This proves the formulas (134) and (135). □

Remark 5 Formulas (133) and (134) are *not* invariant; the transformation properties of the components are determined in Sect. 2.1, Theorem 1, (b).

Lemma 8 can now be easily extended to general q-forms. It is sufficient to consider the case of q-forms generated by p-forms $\omega^{\nu_1}_{J_1} \wedge \omega^{\nu_2}_{J_2} \wedge \cdots \wedge \omega^{\nu_p}_{J_p}$ with fixed p, $1 \le p \le q - p$. The proof then consists in a formal application of Lemma 8.

Theorem 8 *Let W be an open set in Y, q a positive integer, and $\rho \in \Omega^r_q W$ a q-form, and let (V, ψ), $\psi = (x^i, y^\sigma)$, be a fibered chart on Y such that $V \subset W$. Assume that ρ has on V^r a chart expression*

$$\rho = \sum_{s=0}^{q-p} \frac{1}{s!(q-p-s)!} A^{J_1 J_2}_{\nu_1 \nu_2} \cdots^{J_p I_1 I_2}_{\nu_p \sigma_1 \sigma_2} \cdots^{I_s}_{\sigma_s \, i_{s+1} i_{s+2} \ldots i_{q-p}} \omega^{\nu_1}_{J_1} \wedge \omega^{\nu_2}_{J_2} \wedge \cdots \wedge \omega^{\nu_p}_{J_p}$$
$$\wedge\, dy^{\sigma_1}_{I_1} \wedge dy^{\sigma_2}_{I_2} \wedge \cdots \wedge dy^{\sigma_s}_{I_s} \wedge dx^{i_{s+1}} \wedge dx^{i_{s+2}} \wedge \cdots \wedge dx^{i_{q-p}}, \tag{145}$$

with multi-indices $J_1, J_2, \ldots, J_p$ of length $r-1$ and multi-indices $I_1, I_2, \ldots, I_s$ of length r. Then, the k-contact component $p_k\rho$ of ρ has on V^{r+1} the chart expression

$$p_k\rho = \frac{1}{(k-p)!(q-p-k)!} B^{J_1 J_2}_{\nu_1 \nu_2} \cdots^{J_p I_1 I_1}_{\nu_p \sigma_1 \sigma_1} \cdots^{I_{k-p}}_{\sigma_{k-p} \, i_{k-p+1} i_{k-p+2} \ldots i_{q-p}} \omega^{\nu_1}_{J_1} \wedge \omega^{\nu_2}_{J_2}$$
$$\wedge \cdots \wedge \omega^{\nu_p}_{J_p} \wedge \omega^{\sigma_1}_{I_1} \wedge \omega^{\sigma_2}_{I_2} \wedge \cdots \wedge \omega^{\sigma_{k-p}}_{I_{k-p}} \wedge dx^{i_{k-p+1}} \wedge dx^{i_{k-p+2}} \wedge \cdots \wedge dx^{i_{q-p}}, \tag{146}$$

where

$$B^{J_1 J_2}_{\nu_1 \nu_2} \cdots^{J_p I_1 I_1}_{\nu_p \sigma_1 \sigma_1} \cdots^{I_{k-p}}_{\sigma_{k-p} \, i_{k-p+1} i_{k-p+2} \ldots i_{q-p}}$$
$$= \sum_{s=k-p}^{q-p} \binom{q-k}{q-p-s} A^{J_1 J_2}_{\nu_1 \nu_2} \cdots^{J_p I_1 I_2}_{\nu_p \sigma_1 \sigma_2} \cdots^{I_{k-p} I_{k-p+1} I_{k-p+2}}_{\sigma_{k-p} \sigma_{k-p+1} \sigma_{k-p+2}} \cdots^{I_s}_{\sigma_s i_{s+1} i_{s+2} \ldots i_{q-p}} \tag{147}$$
$$\cdot\, y^{\sigma_{k-p+1}}_{I_{k-p+1} i_{k-p+1}} y^{\sigma_{k-p+2}}_{I_{k-p+2} i_{k-p+2}} \cdots y^{\sigma_s}_{I_s i_s} \quad \mathrm{Alt}(i_{k-p+1} i_{k-p+2} \ldots i_s i_{s+1} \ldots i_{q-p}).$$

Proof ρ can be expressed as

$$\rho = \omega^{\nu_1}_{J_1} \wedge \omega^{\nu_2}_{J_2} \wedge \cdots \wedge \omega^{\nu_p}_{J_p} \wedge \eta^{J_1 J_2 \dots J_p}_{\nu_1 \nu_2 \dots \nu_p}, \tag{148}$$

where

$$\begin{aligned} \eta^{J_1 J_2 \dots J_p}_{\nu_1 \nu_2 \dots \nu_p} = \sum_{s=0}^{q-p} \frac{1}{s!(q-p-s)!} & A^{J_1 J_2 \dots J_p I_1 I_2 \dots I_s}_{\nu_1 \nu_2 \dots \nu_p \sigma_1 \sigma_2 \dots \sigma_s i_{s+1} i_{s+2} \dots i_{q-p}} \\ & \wedge dy^{\sigma_1}_{I_1} \wedge dy^{\sigma_2}_{I_2} \wedge \cdots \wedge dy^{\sigma_s}_{I_s} \wedge dx^{i_{s+1}} \wedge dx^{i_{s+2}} \wedge \cdots \wedge dx^{i_{q-p}}. \end{aligned} \tag{149}$$

We can apply to $\eta^{J_1 J_2 \dots J_p}_{\nu_1 \nu_2 \dots \nu_p}$ formula (134). Replacing q with $q-p$ and k with $k-p$,

$$\begin{aligned} p_{k-p} \eta^{J_1 J_2 \dots J_p}_{\nu_1 \nu_2 \dots \nu_p} = & \frac{1}{(k-p)!(q-p-k)!} B^{J_1 J_2 \dots J_p I_1 I_1 \dots I_{k-p}}_{\nu_1 \nu_2 \dots \nu_p \sigma_1 \sigma_1 \dots \sigma_{k-p} i_{k-p+1} i_{k-p+2} \dots i_{q-p}} \\ & \cdot \omega^{\sigma_1}_{I_1} \wedge \omega^{\sigma_2}_{I_2} \wedge \cdots \wedge \omega^{\sigma_{k-p}}_{I_{k-p}} \wedge dx^{i_{k-p+1}} \wedge dx^{i_{k-p+2}} \wedge \cdots \wedge dx^{i_{q-p}}, \end{aligned} \tag{150}$$

where

$$\begin{aligned} & B^{J_1 J_2 \dots J_p I_1 I_1 \dots I_{k-p}}_{\nu_1 \nu_2 \dots \nu_p \sigma_1 \sigma_1 \dots \sigma_{k-p} \, i_{k-p+1} i_{k-p+2} \dots i_{q-p}} \\ & \quad = \sum_{s=k-p}^{q-p} \binom{q-k}{q-p-s} A^{J_1 J_2 \dots J_p I_1 I_2 \dots I_{k-p} I_{k-p+1} I_{k-p+2} \dots I_s}_{\nu_1 \nu_2 \dots \nu_p \sigma_1 \sigma_2 \dots \sigma_{k-p} \sigma_{k-p+1} \sigma_{k-p+2} \dots \sigma_s \, i_{s+1} i_{s+2} \dots i_{q-p}} \\ & \qquad \cdot y^{\sigma_{k-p+1}}_{I_{k-p+1} i_{k-p+1}} y^{\sigma_{k-p+2}}_{I_{k-p+2} i_{k-p+2}} \dots y^{\sigma_s}_{I_s i_s} \quad \mathrm{Alt}(i_{k-p+1} i_{k-p+2} \dots i_s i_{s+1} \dots i_{q-p}). \end{aligned} \tag{151}$$

□

The following two corollaries are immediate consequences of Theorem 8 and Sect. 2.1, Theorem 1. The first one shows that the operators p_k behave like *projector operators* in linear algebra. The second one is a consequence of the identity $d(\pi^{r+1,r})^* \rho = (\pi^{r+1,r})^* d\rho$ for the exterior derivative operator, the canonical decomposition of forms on jet manifolds, applied to both sides, as well as the formula

$$d\omega^\nu_J = -\omega^\nu_{Jj} \wedge dx^j. \tag{152}$$

Corollary 1 *For any k and l,*

$$p_k p_l \rho = \begin{cases} (\pi^{r+2,r+1})^* p_k \rho, & k = l, \\ 0, & k \neq l. \end{cases} \tag{153}$$

Corollary 2 *For every $k \geq 1$,*

$$(\pi^{r+2,r+1})^* p_k \rho = p_k d p_{k-1} \rho + p_k d_k \rho. \tag{154}$$

Remark 6 According to Sect. 2.3, Theorem 5, the horizontalization $h\colon \Omega^r W \to \Omega^{r+1} W$ is a morphism of exterior algebras. On the other hand, if k is a *positive* integer, then the mapping $p_k\colon \Omega^r W \to \Omega^{r+1} W$ satisfies

$$p_k(\rho + \eta) = p_k\rho + p_k\eta, \quad p_k(f\rho) = (f \circ \pi^{r+1,r})p_k\rho \tag{155}$$

for all ρ, η, and f. However, $p_k\colon \Omega^r W \to \Omega^{r+1} W$ are *not* morphisms of exterior algebras.

2.5 Contact Components and Geometric Operations

In this section, we summarize some properties of the contact components and the differential-geometric operations acting on forms, such as the wedge product $\wedge$, the contraction i_ζ of a form by a vector ζ, and the Lie derivative ∂_ξ by a vector field ξ.

Theorem 9 *Let W be an open set in Y.*

(a) *For any two forms ρ and η on $W^r \subset J^r Y$,*

$$p_k(\rho \wedge \eta) = \sum_{i+j=k} p_k\rho \wedge p_k\eta. \tag{156}$$

(b) *For any form ρ and any π^{r+1}-vertical, $\pi^{r+1,r}$-projectable vector field Ξ on W^{r+1}, with $\pi^{r+1,r}$-projection ξ,*

$$i_\Xi p_k\rho = p_{k-1} i_\xi \rho. \tag{157}$$

(c) *For any form ρ and any automorphism α of Y, defined on W,*

$$p_k(J^r\alpha^*\rho) = J^{r+1}\alpha^* p_k\rho. \tag{158}$$

(d) *For any form ρ and any π-projectable vector field on Y on W*

$$p_k(\partial_{J^r\Xi}\rho) = \partial_{J^{r+1}\Xi} p_k\rho. \tag{159}$$

Proof

(a) The exterior product $(\pi^{r+1,r})^*(\rho \wedge \eta)$ commutes with the pullback, so we have $(\pi^{r+1,r})^*(\rho \wedge \eta) = (\pi^{r+1,r})^*\rho \wedge (\pi^{r+1,r})^*\eta$. Applying the trace decomposition formula (Sect. 2.2, Theorem 3) to $(\pi^{r+1,r})^*\rho$ and $(\pi^{r+1,r})^*\eta$, and comparing the k-contact components on both sides, we obtain formula (156).

(b) To prove formula (157), we use the definition of the k-contact component of a form (Sect. 2.4, (112)) and the identity $p\Xi(J_x^{r+1}\gamma) = \xi(J_x^r\gamma)$ (Sect. 1.5, Remark 2). Set $\xi_1 = \Xi(J_x^{r+1}\gamma)$. Then, $h\xi_1 = 0$ and $p\xi_1 = \xi(J_x^r\gamma)$. By definition,

$$\begin{aligned} & i_\Xi p_k\rho(J_x^{r+1}\gamma)(\xi_2,\xi_3,\ldots,\xi_q) \\ & \quad = p_k\rho(J_x^{r+1}\gamma)(\Xi(J_x^{r+1}\gamma),\xi_2,\xi_3,\ldots,\xi_q) \\ & \quad = p_k\rho(J_x^{r+1}\gamma)(\xi_1,\xi_2,\xi_3,\ldots,\xi_q) \\ & \quad = \sum \varepsilon^{j_1j_2\ldots j_kj_{k+1}\ldots j_q}\rho(J_x^r\gamma)(p\xi_{j_1},p\xi_{j_2},\ldots,p\xi_{j_k},h\xi_{j_{k+1}},h\xi_{j_{k+2}},\ldots,h\xi_{j_q}) \end{aligned} \tag{160}$$

with summation through the sequences $j_1 < j_2 < \cdots < j_k$, $j_{k+1} < j_{k+2} < \cdots < j_q$ (Sect. 2.4, (112)). On the other hand,

$$\begin{aligned} & p_{k-1}i_\xi\rho(J_x^{r+1}\gamma)(\xi_2,\xi_3,\ldots,\xi_q) \\ & \quad = \sum \varepsilon^{i_2i_3\ldots i_ki_{k+1}\ldots j_q} i_\xi\rho(J_x^r\gamma)(p\xi_{i_2},p\xi_{i_3},\ldots,p\xi_{i_k},h\xi_{i_{k+1}},h\xi_{i_{k+2}},\ldots,h\xi_{i_q}) \\ & \quad = \sum \varepsilon^{i_2i_3\ldots i_ki_{k+1}\ldots j_q}\rho(J_x^r\gamma)(p\xi_1,p\xi_{i_2},p\xi_{i_3},\ldots,p\xi_{i_k},h\xi_{i_{k+1}},h\xi_{i_{k+2}},\ldots,h\xi_{i_q}) \end{aligned} \tag{161}$$

(summation through $i_2 < i_3 < \cdots < i_k$, $i_{k+1} < i_{k+2} < \cdots < i_q$). Since $h\xi_1 = 0$, the summation in (161) can be extended to the sequences $1 < i_2 < i_3 < \cdots < i_k$ and $1 < i_{k+1} < i_{k+2} < \cdots < i_q$, and therefore, (161) coincides with (160).

(c) Formula (158) follows from the commutativity of the r-jet prolongation of automorphisms of the fibered manifold Y and the canonical jet projections, $(\pi^{r+1,r})^*J^r\alpha^*\rho = J^{r-1}\alpha^*(\pi^{r+1,r})^*\rho$, and from the property of the contact 1-forms $\omega^\nu_{i_1i_2\ldots i_p}$

$$J^r\alpha^*\overline{\omega}^\sigma_{j_1j_2\ldots j_k} = \sum_{i<i_2<\cdots<i_p} \frac{\partial(\bar{y}^\sigma_{j_1j_2\ldots j_k}\circ J^r\alpha)}{\partial y^\nu_{i_1i_2\ldots i_p}}\omega^\nu_{i_1i_2\ldots i_p} \tag{162}$$

(Sect. 2.1, Theorem 1, (c)).

(d) Formula (159) is an immediate consequence of (162). □

Remark 7 If $k = 0$, (156) reduces to the condition $h(\rho\wedge\eta) = h(\rho)\wedge h(\eta)$, stating that h is a homomorphism of exterior algebras (Sect. 2.3, Theorem 5).

2.6 Strongly Contact Forms

Let $\rho \in \Omega^r_q W$ be a q-form such that $n+1 \le q \le \dim J^rY$. Since $h\rho = 0$ and also $p_1\rho = 0, p_2\rho = 0, \ldots, p_{q-n-1}\rho = 0$ (Sect. 2.4, Theorem 8), ρ is always *contact*, and its canonical decomposition has the form

$$(\pi^{r+1,r})^*\rho = p_{q-n}\rho + p_{q-n+1}\rho + \cdots + p_q\rho. \tag{163}$$

We introduce by induction a class of q-forms, imposing a condition on the contact component $p_{q-n}\rho$. If $q = n+1$, then we say that ρ is *strongly contact*, if for every point $y_0 \in W$ there exist a fibered chart (V, ψ), $\psi = (x^i, y^\sigma)$, at y_0 and a contact n-form τ, defined on V^r, such that

$$p_1(\rho - d\tau) = 0. \tag{164}$$

If $q > n+1$, then we say that ρ is *strongly contact*, if for every $y_0 \in W$ there exist (V, ψ), $\psi = (x^i, y^\sigma)$, at y_0 and a strongly contact n-form τ, defined on V^r, such that

$$p_{q-n}(\rho - d\tau) = 0. \tag{165}$$

Lemma 9 *The following conditions are equivalent:*

(a) *ρ is strongly contact.*
(b) *There exist a q-form η and a $(q-1)$-form τ such that*

$$\rho = \eta + d\tau, \quad p_{q-n}\eta = 0, \quad p_{q-n-1}\tau = 0. \tag{166}$$

Proof If ρ is strongly contact and we have τ such that (165) holds, then we set $\eta = \rho - d\tau$. The converse is obvious. □

In view of part (b) of Lemma 9, to study the properties of strongly contact forms, we need the chart expressions of the q-forms $p_{q-n}\rho$ and $p_{q-n-1}\tau = 0$. We also need, in particular, the chart expressions of the forms ρ whose $(q-n)$-contact component vanishes,

$$p_{q-n}\rho = 0. \tag{167}$$

To this purpose, we use the contact basis. The formulas as well as the proof the subsequent theorem are based on the complete trace decomposition theory and are technically tedious because we cannot avoid extensive index notation. We write

$$\begin{aligned}\rho = \sum A^{J_1 J_2}_{\nu_1 \nu_2} \cdots {}^{J_p I_{p+1} I_{p+2}}_{\nu_p \sigma_{p+1} \sigma_{p+2}} \cdots {}^{I_{p+s}}_{\sigma_{p+s}\; i_{p+s+1} i_{p+s+2} \ldots i_q}\, \omega^{\nu_1}_{J_1} \wedge \omega^{\nu_2}_{J_2} \wedge \cdots \wedge \omega^{\nu_p}_{J_p} \\ \wedge dy^{\sigma_{p+1}}_{I_{p+1}} \wedge dy^{\sigma_{p+2}}_{I_{p+2}} \wedge \cdots \wedge dy^{\sigma_{p+s}}_{I_{p+s}} \wedge dx^{i_{p+s+1}} \wedge dx^{i_{p+s+2}} \wedge \cdots \wedge dx^{i_q},\end{aligned} \tag{168}$$

where summation is taking place through the multi-indices $J_1, J_2, \ldots, J_p$ of length less or equal to $r-1$ and the multi-indices $I_{p+1}, I_{p+2}, \ldots, I_{p+s}$ of length equal to r.

Applying the trace decomposition theorem (Appendix 9, Theorem 1) as many times as necessary, we can write

$$\begin{aligned}\rho = \sum B^{J_1 J_2}_{\nu_1 \nu_2} \cdots {}^{J_l K_{l+1} K_{l+2}}_{\nu_l \kappa_{l+1} \kappa_{l+2}} \cdots {}^{K_{l+p} I_{l+p+1} I_{l+p+2}}_{\kappa_{l+p} \sigma_{l+p+1} \sigma_{l+p+2}} \cdots {}^{I_{l+p+s}}_{\sigma_{l+p+s} i_{l+p+s+1} i_{l+p+s+2} \dots i_Q} \\ \cdot \omega^{\nu_1}_{J_1} \wedge \omega^{\nu_2}_{J_2} \wedge \cdots \wedge \omega^{\nu_l}_{J_l} \wedge d\omega^{\kappa_{l+1}}_{K_{l+1}} \wedge d\omega^{\kappa_{l+2}}_{K_{l+2}} \wedge \cdots \wedge d\omega^{\kappa_{l+p}}_{K_{l+p}} \\ \wedge dy^{\sigma_{l+p+1}}_{I_{l+p+1}} \wedge dy^{\sigma_{l+p+2}}_{I_{l+p+2}} \wedge \cdots \wedge dy^{\sigma_{l+p+s}}_{I_{l+p+s}} \\ \wedge dx^{i_{l+p+s+1}} \wedge dx^{i_{l+p+s+2}} \wedge \cdots \wedge dx^{i_Q},\end{aligned} \tag{169}$$

where

$$\begin{aligned} &0 \le |J_1|,\ |J_2|, \dots,\ |J_l|\ \le r-1, \\ &|K_{l+1}|, |K_{l+2}|, \dots, |K_{l+p}| = r-1, \\ &|I_{l+p+1}|, |I_{l+p+2}|, \dots, |I_{l+p+s}| = r, \end{aligned} \tag{170}$$

and the coefficients are *traceless*. The number Q in (169) is *not* the degree of ρ; it is related to the degree q by $l + 2p + s + Q - l - p - s = q$, that is,

$$p + Q = q. \tag{171}$$

Theorem 10 *Let $W \subset Y$ be an open set, q an integer such that $n + 1 \le q \le \dim J^r Y$, and $\eta \in \Omega^r_q W$ a form, and let (V, ψ), $\psi = (x^i, y^\sigma)$, be a fibered chart such that $V \subset W$. Then, $p_{q-n}\eta = 0$ if and only if*

$$\begin{aligned}\eta = \sum_{q-n+1 \le l+p} \omega^{\sigma_1}_{J_1} \wedge \omega^{\sigma_2}_{J_2} \wedge \cdots \wedge \omega^{\sigma_l}_{J_l} \wedge d\omega^{\nu_1}_{I_1} \wedge d\omega^{\nu_2}_{I_2} \wedge \cdots \wedge d\omega^{\nu_p}_{I_p} \\ \wedge \Phi^{J_1 J_2}_{\sigma_1 \sigma_2} \cdots {}^{J_l I_1 I_2}_{\sigma_l \nu_1 \nu_2} \cdots {}^{I_p}_{\nu_p},\end{aligned} \tag{172}$$

where $\Phi^{J_1 J_2}_{\sigma_1 \sigma_2} \cdots {}^{J_l I_1 I_2}_{\sigma_l \nu_1 \nu_2} \cdots {}^{I_p}_{\nu_p}$ are some $(q - l - 2p)$-forms on V^r and the multi-indices satisfy $0 \le |J_1|, |J_2|, \dots, |J_l| \le r - 1$, $|I_1|, |I_2|, \dots, |I_p| = r - 1$.

Proof Expression (169) for η can be written as V^{r+1}, where

$$\begin{aligned}\eta_0 = \sum_{l+p \ge q-n} B^{J_1 J_2}_{\nu_{11} \nu_2} \cdots {}^{J_l K_{l+1} K_{l+2}}_{\nu_l \kappa_{l+1} \kappa_{l+2}} \cdots {}^{K_{l+p} I_{l+p+1} I_{l+p+2}}_{\kappa_{l+p} \sigma_{l+p+1} \sigma_{l+p+2}} \cdots {}^{I_{l+p+s}}_{\sigma_{l+p+s} i_{l+p+s+1} i_{l+p+s+2} \dots i_Q} \\ \cdot \omega^{\nu_1}_{J_1} \wedge \omega^{\nu_2}_{J_2} \wedge \cdots \wedge \omega^{\nu_l}_{J_l} \wedge d\omega^{\kappa_{l+1}}_{K_{l+1}} \wedge d\omega^{K_{l+2}}_{\kappa_{l+2}} \wedge \cdots \wedge d\omega^{K_{l+p}}_{\kappa_{l+p}} \\ \wedge dy^{\sigma_{l+p+1}}_{I_{l+p+1}} \wedge dy^{\sigma_{l+p+2}}_{I_{l+p+2}} \wedge \cdots \wedge dy^{\sigma_{l+p+s}}_{I_{l+p+s}} \wedge dx^{i_{l+p+s+1}} \wedge dx^{i_{l+p+s+2}} \wedge \cdots \wedge dx^{i_Q}\end{aligned} \tag{173}$$

and

$$\eta_1 = \sum_{l+p<q-n} B^{J_1 J_2 \ldots J_l K_{l+1} K_{l+2} \ldots K_{l+p} I_{l+p+1} I_{l+p+2} \ldots I_{l+p+s}}_{\nu_1 \nu_2 \ldots \nu_l \kappa_{l+1} \kappa_{l+2} \ldots \kappa_{l+p} \sigma_{l+p+1} \sigma_{l+p+2} \ldots \sigma_{l+p+s} i_{l+p+s+1} i_{l+p+s+2} \ldots i_Q}$$
$$\cdot \omega^{\nu_1}_{J_1} \wedge \omega^{\nu_2}_{J_2} \wedge \cdots \wedge \omega^{\nu_l}_{J_l} \wedge d\omega^{\kappa_{l+1}}_{K_{l+1}} \wedge d\omega^{K_{l+2}}_{\kappa_{l+2}} \wedge \cdots \wedge d\omega^{K_{l+p}}_{\kappa_{l+p}}$$
$$\wedge dy^{\sigma_{l+p+1}}_{I_{l+p+1}} \wedge dy^{\sigma_{l+p+2}}_{I_{l+p+2}} \wedge \cdots \wedge dy^{\sigma_{l+p+s}}_{I_{l+p+s}} \wedge dx^{i_{l+p+s+1}} \wedge dx^{i_{l+p+s+2}} \wedge \cdots \wedge dx^{i_Q}. \tag{174}$$

We want to show that the condition $p_{q-n}\eta = 0$ implies $\eta_1 = 0$.

To determine $p_{q-n}\eta_1$, we need the pullback $(\pi^{r+1,r})^*\eta_1$; this can be obtained by replacing dy^σ_I with

$$dy^\sigma_I = \omega^\sigma_I + y^\sigma_{Ii} dx^i. \tag{175}$$

Then, the corresponding expressions on the right-hand side of the formula (174) arise by substitution

$$\begin{aligned} & dy^{\sigma_{l+p+1}}_{I_{l+p+1}} \wedge dy^{\sigma_{l+p+2}}_{I_{l+p+2}} \wedge \cdots \wedge dy^{\sigma_{l+p+s}}_{I_{l+p+s}} \\ & \quad = \left(\omega^{\sigma_{l+p+1}}_{I_{l+p+1}} + y^{\sigma_{l+p+1}}_{I_{l+p+1} i_{l+p+1}} dx^{i_{l+p+1}}\right) \wedge \left(\omega^{\sigma_{l+p+2}}_{I_{l+p+2}} + y^{\sigma_{l+p+2}}_{I_{l+p+2} i_{l+p+2}} dx^{i_{l+p+2}}\right) \\ & \qquad \wedge \cdots \wedge \left(\omega^{\sigma_{l+p+s}}_{I_{l+p+s}} + y^{\sigma_{l+p+s}}_{I_{l+p+s} i_{l+p+s}} dx^{i_{l+p+s}}\right). \end{aligned} \tag{176}$$

Computing the right-hand side, we obtain

$$\begin{aligned} & dy^{\sigma_{l+p+1}}_{I_{l+p+1}} \wedge dy^{\sigma_{l+p+2}}_{I_{l+p+2}} \wedge \cdots \wedge dy^{\sigma_{l+p+s}}_{I_{l+p+s}} = \omega^{\sigma_{l+p+1}}_{I_{l+p+1}} \wedge \omega^{\sigma_{l+p+2}}_{I_{l+p+2}} \wedge \cdots \wedge \omega^{\sigma_{l+p+s}}_{I_{l+p+s}} \\ & \quad + s y^{\sigma_{l+p+s}}_{I_{l+p+s} i_{l+p+s}} \omega^{\sigma_{l+p+1}}_{I_{l+p+1}} \wedge \omega^{\sigma_{l+p+2}}_{I_{l+p+2}} \wedge \cdots \wedge \omega^{\sigma_{l+p+s-1}}_{I_{l+p+s-1}} \wedge dx^{i_{l+p+s}} \\ & \quad + \binom{s}{2} y^{\sigma_{l+p+s-1}}_{I_{l+p+s-1} i_{l+p+s-1}} y^{\sigma_{l+p+s}}_{I_{l+p+s} i_{l+p+s}} \omega^{\sigma_{l+p+1}}_{I_{l+p+1}} \wedge \omega^{\sigma_{l+p+2}}_{I_{l+p+2}} \\ & \quad \wedge \cdots \wedge \omega^{\sigma_{l+p+s-2}}_{l+p+s-2} \wedge dx^{i_{l+p+s-1}} \wedge dx^{i_{l+p+s}} \\ & \quad + \cdots + s y^{\sigma_{l+p+2}}_{I_{l+p+2} i_{l+p+2}} \ldots y^{\sigma_{l+p+s-1}}_{I_{l+p+s-1} i_{l+p+s-1}} y^{\sigma_{l+p+s}}_{I_{l+p+s} i_{l+p+s}} \omega^{\sigma_{l+p+1}}_{I_{l+p+1}} \\ & \quad \wedge dx^{i_{l+p+2}} \wedge \cdots \wedge dx^{i_{l+p+s-1}} \wedge dx^{i_{l+p+s}} \\ & \quad + y^{\sigma_{l+p+1}}_{I_{l+p+1} i_{l+p+1}} \ldots y^{\sigma_{l+p+s-1}}_{I_{l+p+s-1} i_{l+p+s-1}} y^{\sigma_{l+p+s}}_{I_{l+p+s} i_{l+p+s}} dx^{i_{l+p+1}} \wedge \cdots \wedge dx^{i_{l+p+s-1}} \wedge dx^{i_{l+p+s}}. \end{aligned} \tag{177}$$

Now, consider a fixed summand in expression (174), with given l, p, and s,

$$\begin{aligned} & B^{J_1 J_2 \ldots J_l K_{l+1} K_{l+2} \ldots K_{k+p} I_{l+p+1} I_{l+p+2} \ldots I_{l+p+s}}_{\nu_1 \nu_2 \ldots \nu_l \kappa_{l+1} \kappa_{l+2} \ldots \kappa_{l+p} \sigma_{l+p+1} \sigma_{l+p+2} \ldots \sigma_{l+p+s} i_{l+p+s+1} i_{l+p+s+2} \ldots i_Q} \\ & \quad \cdot \omega^{\nu_1}_{J_1} \wedge \omega^{\nu_2}_{J_2} \wedge \cdots \wedge \omega^{\nu_l}_{J_l} \wedge d\omega^{\kappa_{l+1}}_{K_{l+1}} \wedge d\omega^{\kappa_{l+2}}_{K_{l+2}} \wedge \cdots \wedge d\omega^{\kappa_{l+p}}_{K_{l+p}} \\ & \quad \wedge \omega^{\sigma_{l+p+1}}_{I_{l+p+1}} \wedge \omega^{\sigma_{l+p+2}}_{I_{l+p+2}} \wedge \cdots \wedge \omega^{\sigma_{l+p+s}}_{I_{l+p+s}} \wedge dx^{i_{l+p+s+1}} \wedge dx^{i_{l+p+s+2}} \wedge \cdots \wedge dx^{i_Q}. \end{aligned} \tag{178}$$

Using (178), we get the terms

$$
\begin{aligned}
& sB^{J_1 J_2}_{\nu_{11} \nu_2} \cdots {}^{J_l K_{l+1} K_{l+2}}_{\nu_l \kappa_{l+1} \kappa_{l+2}} \cdots {}^{K_{k+p} I_{l+p+1} I_{l+p+2}}_{\kappa_{l+p} \sigma_{l+p+1} \sigma_{l+p+2}} \cdots {}^{I_{l+p+s}}_{\sigma_{l+p+s} i_{l+p+s+1} i_{l+p+s+2} \dots i_Q} \\
&\quad \cdot y^{\sigma_{l+p+s}}_{I_{l+p+s} i_{l+p+s}} \omega^{\nu_1}_{J_1} \wedge \omega^{\nu_2}_{J_2} \wedge \cdots \wedge \omega^{\nu_l}_{J_l} \wedge d\omega^{\kappa_{l+1}}_{K_{l+1}} \wedge d\omega^{\kappa_{l+2}}_{K_{l+2}} \wedge \cdots \wedge d\omega^{\kappa_{l+p}}_{K_{l+p}} \\
&\quad \wedge \omega^{\sigma_{l+p+1}}_{I_{l+p+1}} \wedge \omega^{\sigma_{l+p+2}}_{I_{l+p+2}} \wedge \cdots \wedge \omega^{\sigma_{l+p+s-1}}_{I_{l+p+s-1}} \\
&\quad \wedge dx^{i_{l+p+s}} \wedge dx^{i_{l+p+s+1}} \wedge dx^{i_{l+p+s+2}} \wedge \cdots \wedge dx^{i_Q}, \\
&\quad \binom{S}{2} B^{J_1 J_2}_{\nu_{11} \nu_2} \cdots {}^{J_l K_{l+1} K_{l+2}}_{\nu_l \kappa_{l+1} \kappa_{l+2}} \cdots {}^{K_{k+p} I_{l+p+1} I_{l+p+2}}_{\kappa_{l+p} \sigma_{l+p+1} \sigma_{l+p+2}} \cdots {}^{I_{l+p+s}}_{\sigma_{l+p+s}\, i_{l+p+s+1} i_{l+p+s+2} \dots i_Q} \\
&\quad \cdot y^{\sigma_{l+p+s-1}}_{I_{l+p+s-1} i_{l+p+s-1}} y^{\sigma_{l+p+s}}_{I_{l+p+s} i_{l+p+s}} \omega^{\nu_1}_{J_1} \wedge \omega^{\nu_2}_{J_2} \wedge \cdots \wedge \omega^{\nu_l}_{J_l} \\
&\quad \wedge d\omega^{\kappa_{l+1}}_{K_{l+1}} \wedge d\omega^{\kappa_{l+2}}_{K_{l+2}} \wedge \cdots \wedge d\omega^{\kappa_{l+p}}_{K_{l+p}} \wedge \omega^{\sigma_{l+p+1}}_{I_{l+p+1}} \wedge \omega^{\sigma_{l+p+2}}_{I_{l+p+2}} \\
&\quad \wedge \cdots \wedge \omega^{\sigma_{l+p+s-2}}_{l+p+s-2} \wedge dx^{i_{l+p+s-1}} \wedge dx^{i_{l+p+s}} \wedge dx^{i_{l+p+s+1}} \wedge dx^{i_{l+p+s+2}} \wedge \cdots \wedge dx^{i_Q}, \\
&\quad \dots \\
&\quad sB^{J_1 J_2}_{\nu_{11} \nu_2} \cdots {}^{J_l K_{l+1} K_{l+2}}_{\nu_l \kappa_{l+1} \kappa_{l+2}} \cdots {}^{K_{k+p} I_{l+p+1} I_{l+p+2}}_{\kappa_{l+p} \sigma_{l+p+1} \sigma_{l+p+2}} \cdots {}^{I_{l+p+s}}_{\sigma_{l+p+s}\, i_{l+p+s+1} i_{l+p+s+2} \dots i_Q} \\
&\quad \cdot y^{\sigma_{l+p+2}}_{I_{l+p+2} i_{l+p+2}} \dots y^{\sigma_{l+p+s-1}}_{I_{l+p+s-1} i_{l+p+s-1}} y^{\sigma_{l+p+s}}_{I_{l+p+s} i_{l+p+s}} \omega^{\nu_1}_{J_1} \wedge \omega^{\nu_2}_{J_2} \wedge \cdots \wedge \omega^{\nu_l}_{J_l} \\
&\quad \wedge d\omega^{\kappa_{l+1}}_{K_{l+1}} \wedge d\omega^{\kappa_{l+2}}_{K_{l+2}} \wedge \cdots \wedge d\omega^{\kappa_{l+p}}_{K_{l+p}} \wedge \omega^{\sigma_{l+p+1}}_{I_{l+p+1}} \\
&\quad \wedge dx^{i_{l+p+2}} \wedge \cdots \wedge dx^{i_{l+p+s-1}} \wedge dx^{i_{l+p+s}} \wedge dx^{i_{l+p+s+1}} \wedge dx^{i_{l+p+s+2}} \wedge \cdots \wedge dx^{i_Q},
\end{aligned}
\tag{179}
$$

and

$$
\begin{aligned}
& B^{J_1 J_2}_{\nu_{11} \nu_2} \cdots {}^{J_l K_{l+1} K_{l+2}}_{\nu_l \kappa_{l+1} \kappa_{l+2}} \cdots {}^{K_{k+p} I_{l+p+1} I_{l+p+2}}_{\kappa_{l+p} \sigma_{l+p+1} \sigma_{l+p+2}} \cdots {}^{I_{l+p+s}}_{\sigma_{l+p+s} i_{l+p+s+1} i_{l+p+s+2} \dots i_Q} \\
&\quad \cdot y^{\sigma_{l+p+1}}_{I_{l+p+1} i_{l+p+1}} \dots y^{\sigma_{l+p+s-1}}_{I_{l+p+s-1} i_{l+p+s-1}} y^{\sigma_{l+p+s}}_{I_{l+p+s} i_{l+p+s}} \omega^{\nu_1}_{J_1} \wedge \omega^{\nu_2}_{J_2} \wedge \cdots \wedge \omega^{\nu_l}_{J_l} \\
&\quad \wedge d\omega^{\kappa_{l+1}}_{K_{l+1}} \wedge d\omega^{\kappa_{l+2}}_{K_{l+2}} \wedge \cdots \wedge d\omega^{\kappa_{l+p}}_{K_{l+p}} \wedge dx^{i_{l+p+1}} \wedge \cdots \wedge dx^{i_{l+p+s-1}} \wedge dx^{i_{l+p+s}} \\
&\quad \wedge dx^{i_{l+p+s+1}} \wedge dx^{i_{l+p+s+2}} \wedge \cdots \wedge dx^{i_Q}.
\end{aligned}
\tag{180}
$$

We see that the degrees of contactness of these terms are

$$
l + p + s > l + p + s - 1 > l + p + s - 2 > \cdots > l + p + 1 > l + p, \tag{181}
$$

respectively. Clearly, since we consider the terms where $l + p < q - n$, (180) does not contribute to $p_{q-n}\eta_1$. We claim that among the terms (178), there is one whose degree of contactness is $q - n$. Suppose the opposite; then $l + p + s < q - n$, but this is not possible, because the term satisfying this inequality would contain more than n factors dx^i.

Thus, the condition $p_1\eta_1 = 0$ applies to one of the expressions (179) and states that the coefficient in this expression vanishes. But the components of η_1 are traceless, and we have already seen that this is only possible when they also vanish.

This implies in turn that the forms on the left of (179) all vanish, which proves that $\eta_1 = 0$. The proof is complete. □

Corollary 1 *Let $W \subset Y$ be an open set, q an integer such that $n+1 \le q \le \dim J^r Y$, and $\eta \in \Omega_q^r W$ a form, and let (V,ψ), $\psi = (x^i, y^\sigma)$, be a fibered chart such that $V \subset W$. Then, $p_{q-n}\eta = 0$ if and only if*

$$\eta = \eta_0 + d\mu, \tag{182}$$

where η_0 and μ are ω_J^σ-generated, $0 \le |I| \le r-1$, such that $p_{q-n}\eta_0 = 0$ and $p_{q-n-1}\mu = 0$.

Proof Write in Theorem 10 $\eta = \eta_0 + \eta'$, where η_0 includes *all* ω_J^σ-generated terms, defined by the condition $l \ge 1$, and

$$\begin{aligned}\eta' &= \sum_{q-n+1 \le p} d\omega_{I_1}^{\nu_1} \wedge d\omega_{I_2}^{\nu_2} \wedge \dots \wedge d\omega_{I_p}^{\nu_p} \wedge \Phi_{\sigma_1\sigma_2}^{J_1 J_2} \dots {}_{\sigma_l \nu_1 \nu_2}^{J_l I_1 I_2} \dots {}_{\nu_p}^{I_p} \\ &= \sum_{q-n+1 \le p} d(\omega_{I_1}^{\nu_1} \wedge d\omega_{I_2}^{\nu_2} \wedge \dots \wedge d\omega_{I_p}^{\nu_p} \wedge \Phi_{\sigma_1\sigma_2}^{J_1 J_2} \dots {}_{\sigma_l \nu_1 \nu_2}^{J_l I_1 I_2} \dots {}_{\nu_p}^{I_p}) \\ &\quad + \sum_{q-n+1 \le p} \omega_{I_1}^{\nu_1} \wedge d\omega_{I_2}^{\nu_2} \wedge \dots \wedge d\omega_{I_p}^{\nu_p} \wedge d(\Phi_{\sigma_1\sigma_2}^{J_1 J_2} \dots {}_{\sigma_l \nu_1 \nu_2}^{J_l I_1 I_2} \dots {}_{\nu_p}^{I_p}).\end{aligned} \tag{183}$$

Thus, η can also be written as $\eta = \eta_0 + d\mu$, where η_0 is ω_J^σ-generated, and μ is also ω_J^σ-generated and contains p contact factors ω_J^σ and $d\omega_I^\nu$; in particular, $p_{q-n-1}\mu = 0$. □

Remark 8 Note that the summation in Theorem 10 through the pairs (l,p) can also be defined by the inequality $q-n+1-p \le l \le q-2p$, where the range of p is given by the conditions $p = 0,1,2,\dots$ and $q-2p \ge 0$.

Lemma 10

(a) *If ρ is a strongly contact form such that $q \ge n+2$, then for any π-vertical vector field Ξ, the form $i_{J^r\Xi}\rho$ is strongly contact.*
(b) *The exterior derivative of a strongly contact form is strongly contact.*

Proof

(a) We have $i_{J^r\Xi}\rho = i_{J^r\Xi}\eta + i_{J^r\Xi}d\tau = i_{J^r\Xi}\eta + \partial_{J^r\Xi}\tau - di_{J^r\Xi}\tau$. But by Sect. 2.5, Theorem 9 $p_{q-n-1}(i_{J^r\Xi}\eta + \partial_{J^r\Xi}\tau) = i_{J^{r+1}\Xi}p_{q-n}\eta + \partial_{J^{r+1}\Xi}p_{q-n-1}\tau$ and $p_{q-n-2}i_{J^r\Xi}\tau = i_{J^{r+1}\Xi}p_{q-n-1}\tau$; however, these expressions vanish because ρ is strongly contact. Now, we apply Lemma 9.
(b) Let the form ρ be strongly contact. Then, from (166), $d\rho = d\eta$, where $p_{q-n}\eta = 0$. We want to show that to any point y_0 from the domain of definition of ρ, there exists a fibered chart (V,ψ), $\psi = (x^i, y^\sigma)$, at y_0 and a q-form τ, defined on V^r, such that $p_{q+1-n}(d\rho - d\tau) = 0$ and $p_{q-n}\tau = 0$. Taking $\tau = \eta$, we get the result.

For $n+1 \le q \le \dim J^r Y$, strongly contact forms constitute an *Abelian subgroup* $\Theta^r_q W$ of the Abelian group of q-forms $\Omega^r_q W$; they do not form a submodule of $\Omega^r_q W$. It follows from Lemma 10, (b) that the subgroups $\Theta^r_q W$ together with the exterior derivative operator define a sequence

$$\Theta^r_n W \to \Theta^r_{n+1} W \to \cdots \to \Theta^r_M W \to 0. \tag{184}$$

The number M labeling the last nonzero term in this sequence is

$$M = m\binom{n+r-1}{n} + 2n - 1. \tag{185}$$

□

Remark 9 If $n+1 \le q \le \dim J^r Y$, then by Lemma 1, the canonical decomposition of a contact form $\rho \in \Theta^r_q W$ is

$$(\pi^{r+1,r})^* \rho = p_{q-n} d\tau + p_{q-n+1}\rho + p_{q-n+2}\rho + \cdots + p_q \rho. \tag{186}$$

Remark 10 It is easily seen that the definition of a contact q-form $\rho \in \Omega^r_q W$ for $1 \le q \le n$ agrees with (165). Indeed, if $1 \le q \le n$, we have for any contact form $\rho' \in \Theta^r_{q-1} W$, $h(\rho - d\rho') = h\rho$ as $(\pi^{r+1})^* h d\rho' = hdh\rho' = 0$ (Corollary 2). Thus, if $h\rho = 0$, then $h(\rho - d\rho') = 0$ for any $\rho' \in \Theta^r_{q-1} W$.

2.7 Fibered Homotopy Operators on Jet Prolongations of Fibered Manifolds

In this section, we introduce the fibered homotopy operators for differential forms on jet prolongations of fibered manifolds. We study their relations with the canonical decomposition of forms and the exactness problem for contact and strongly contact forms. The general theory of fibered homotopy operators is summarized in Appendix 6.

The relevant underlying structure we need is a trivial fibered manifold $W = U \times V$, where U is an open set in $\mathbf{R}^n$ and V an open ball in $\mathbf{R}^m$ with center at the origin; the projection is the first Cartesian projection of $U \times V$ onto U, denoted by π. The r-jet prolongation $J^r W$ is also denoted by W^r. By definition

$$W^r = U \times V \times L(\mathbf{R}^n, \mathbf{R}^m) \times L^2_{\text{sym}}(\mathbf{R}^n, \mathbf{R}^m) \times \cdots \times L^r_{\text{sym}}(\mathbf{R}^n, \mathbf{R}^m), \tag{187}$$

where $L^k_{\text{sym}}(\mathbf{R}^n, \mathbf{R}^m)$ is the vector space of k-linear symmetric mappings from $\mathbf{R}^n$ to $\mathbf{R}^m$. The canonical coordinates on W are denoted by (x^i, y^σ), and the associated coordinates on W^r are $(x^i, y^\sigma, y^\sigma_{j_1}, y^\sigma_{j_1 j_2}, \ldots, y^\sigma_{j_1 j_2 \ldots j_r})$. Any Cartesian projections

$\pi^{r,s}$: $W^r \to W^s$, with $0 \leq s < r$, define in an obvious way a homotopy $\chi^{r,s}$ and the *fibered homotopy operator* $I^{r,s}$ (see Appendix 6, (27)), so the Volterra-Poincare lemma holds in these cases.

In this section, we consider the fibered homotopy operator $I = I^{r,0}$. Recall that the homotopy $\chi = \chi^{r,s}$ is a mapping from $[0,1] \times W^r$ to W^r, defined by

$$\chi(s, (x^i, y^\sigma, y^\sigma_{j_1}, y^\sigma_{j_1 j_2}, \ldots, y^\sigma_{j_1 j_2 \ldots j_r})) = (x^i, sy^\sigma, sy^\sigma_{j_1}, sy^\sigma_{j_1 j_2}, \ldots, sy^\sigma_{j_1 j_2 \ldots j_r}). \tag{188}$$

It is immediately verified that the pullback by χ satisfies

$$\begin{aligned} \chi^* dx^i = dx^i, \quad \chi^* dy^\sigma_{j_1 j_2 \ldots j_k} &= y^\sigma_{j_1 j_2 \ldots j_k} ds + s dy^\sigma_{j_1 j_2 \ldots j_k}, \\ \chi^* \omega^\sigma_{j_1 j_2 \ldots j_k} = y^\sigma_{j_1 j_2 \ldots j_k} ds + s\omega^\sigma_{j_1 j_2 \ldots j_k}. \end{aligned} \tag{189}$$

In accordance with the general theory, these formulas lead to explicit description of the operator I. For any q-form ρ on W^r, $\chi^*\rho$ has a unique decomposition

$$\chi^*\rho = ds \wedge \rho^{(0)}(s) + \rho'(s) \tag{190}$$

such that the $(q-1)$-form $\rho^{(0)}(s)$ and the q-form $\rho'(s)$ do not contain ds. Then,

$$I\rho = \int_0^1 \rho^{(0)}(s), \tag{191}$$

where the expression on the right-hand side denotes the integration of the coefficients in the form $\rho^{(0)}(s)$ over s from 0 to 1.

The following is a version of a general theorem on fibered homotopy operators on fibered manifolds. ζ stands for the *zero section* of W^r over U.

Theorem 11

(a) *For every differentiable function* f: $W^r \to \mathbf{R}$,

$$f = Idf + (\pi^r)^*\zeta^* f. \tag{192}$$

(b) *Let* $q \geq 1$. *Then, for every differential* q*-form* ρ *on* W^r,

$$\rho = Id\rho + dI\rho + (\pi^r)^*\zeta^*\rho. \tag{193}$$

Proof Slight modification of Theorem 1, Appendix 6. □

Theorem 12 *Let ρ be a contact q-form on W^r.*

(a) *The contact components of ρ satisfy*

$$Ih\rho = 0, \quad Ip_k\rho = p_{k-1}I\rho, \quad 1 \le k \le q. \tag{194}$$

(b) *If ρ is strongly contact, then $I\rho$ is strongly contact.*

Proof

(a) Expressing the forms ρ and $(\pi^{r+1,r})^*\rho$ in the basis of 1-forms (dx^i, dy^σ_J), $0 \le |J| \le r$, we have

$$(\pi^{r+1,r})^* I\rho = I(\pi^{r+1,r})^*\rho. \tag{195}$$

The canonical decomposition of the form ρ yields

$$(\pi^{r+1,r})^* I\rho = I(\pi^{r+1,r})^*\rho = I\left(\sum_{0 \le l \le q} p_l\rho\right) = \sum_{0 \le l \le q} Ip_l\rho. \tag{196}$$

But by (191), $Ip_l\rho$ is $(l-1)$-contact; thus, applying p_k to both sides of (195) and comparing k-contact components, we get (194).

(b) Let $q \ge n+1$ and suppose we have a strongly contact q-form ρ on W^r. Then, $\rho = \eta + d\tau$ for some q-form η and $(q-1)$-form τ such that $p_{q-n}\eta = 0$ and $p_{q-n-1}\tau = 0$; hence, $I\rho = I\eta + Id\tau = I\eta + \tau - dI\tau - \tau_0$, where τ_0 is a $(q-1)$-form on U. If $q > n+1$, then always $\tau_0 = 0$. If $q = n+1$, then always $d\tau_0 = 0$, and we may replace τ with $\tau - \tau_0$; then, $I\rho = I\eta + \tau - dI\tau$. The $(q-1)$-form $I\eta + \tau$ satisfies

$$p_{q-n-1}(I\eta + \tau) = Ip_{q-n}\eta + p_{q-n-1}\tau = p_{q-n-1}\tau = 0. \tag{197}$$

If $q \ge n+2$, then $q-n-2 \ge 0$ and $p_{q-n-2}I\tau = Ip_{q-n-1}\tau = 0$; consequently, $I\rho$ is strongly contact. If $q = n+1$, then from (195), $h\tau = 0$ as required. □

Corollary 1 (The fibered Volterra–Poincare lemma) *If $d\rho = 0$, then there exists a $(q-1)$-form η such that $\rho = d\eta$.*

The following two theorems extend the fibered Volterra-Poincare lemma to contact and strongly contact forms. Their proofs are based on the trace decomposition theorem (Sect. 2.2, Theorem 3), Appendix 9, Theorem 4, and on the fibered Volterra-Poincare lemma.

Theorem 13 *Let $1 \le q \le n$ and let ρ be a contact q-form such that $d\rho = 0$. Then $\rho = d\eta$ for some contact $(q-1)$-form η.*

Proof

1. Let ρ be a contact 1-form, expressed as

$$\rho = \sum_{0 \le |J| \le r-1} \Phi_\nu^J \omega_J^\nu. \tag{198}$$

Then,

$$d\rho = \sum_{0 \le |J| \le r-1} (d\Phi_\nu^J \wedge \omega_J^\nu - \Phi_\nu^J dy_{Jj}^\nu \wedge dx^j). \tag{199}$$

Condition $d\rho = 0$ implies, for $|J| = r - 1$, $\Phi_\nu^J \delta_j^k = 0$ $\operatorname{Sym}(Jk)$, and the trace operation yields, up to the factor $(n + r - 1)/r$,

$$\Phi_\nu^J = 0. \tag{200}$$

Thus, ρ must be of the form

$$\rho = \sum_{0 \le |J| \le r-2} \Phi_\nu^J \omega_J^\nu. \tag{201}$$

Repeating the same procedure, we get $\rho = 0$.

2. Let $2 \le q \le n$. We show in several steps that if ρ is a contact q-form such that $d\rho = 0$, then there exist a contact q-form τ and a contact $(q - 1)$-form κ such that

$$\rho = \tau + d\kappa, \quad p_1\tau = 0. \tag{202}$$

First, we find a decomposition

$$\rho = \rho_0 + \tau_0 + d\kappa_0, \tag{203}$$

with the following properties:

(a) ρ_0 is generated by the forms ω_J^σ such that $0 \le |J| \le r - 1$,

$$\rho_0 = \sum_{0 \le |J| \le r-2} \omega_J^\sigma \wedge \Phi_\sigma^J + \sum_{|J|=r-1} \omega_J^\sigma \wedge \Delta_\sigma^J, \tag{204}$$

where the $(q - 1)$-forms Δ_σ^J are traceless.

(b) τ_0 is generated by $\omega_J^\sigma \wedge \omega_I^\nu$ and $\omega_J^\sigma \wedge d\omega_L^\nu$, where $|J| = r - 1$, $0 \le |I| \le r - 1$, $|L| = r - 1$.

(c) κ_0 is a contact $(q - 1)$-form.

Expressing ρ as in Sect. 2.3, Corollary 2, we have

$$\rho = \sum_{0 \le |J| \le r-2} \omega_J^\sigma \wedge \Phi_\sigma^J + \sum_{|J|=r-1} \omega_J^\sigma \wedge \Phi_\sigma^J + d\kappa_0, \tag{205}$$

where κ_0 is a contact $(q-1)$-form. Decompose the $(q-1)$-forms Φ_ν^J, indexed with multi-indices J of length $r-1$, by the trace operation. We get a decomposition

$$\Phi_\nu^J = \Delta_\nu^J + Z_\nu^J, \tag{206}$$

where the expression Δ_ν^J is the traceless and Z_ν^J is the contact component. Then,

$$\rho = \sum_{0 \le |J| \le r-2} \omega_J^\sigma \wedge \Phi_\sigma^J + \sum_{|J|=r-1} \omega_J^\sigma \wedge \Delta_\sigma^J + \sum_{|J|=r-1} \omega_J^\sigma \wedge Z_\sigma^J + d\kappa_0. \tag{207}$$

Setting

$$\begin{aligned} \rho_0 &= \sum_{0 \le |J| \le r-2} \omega_J^\sigma \wedge \Phi_\sigma^J + \sum_{|J|=r-1} \omega_J^\sigma \wedge \Delta_\sigma^J, \\ \tau_0 &= \sum_{|J|=r-1} \omega_J^\sigma \wedge Z_\sigma^J, \end{aligned} \tag{208}$$

we get (203).

Second, we show that ρ has a decomposition

$$\rho = \rho_1 + \tau_1 + d\kappa_1 \tag{209}$$

with the following properties:

(a) The form ρ_1 is generated by the contact forms ω_J^σ, such that $0 \le |J| \le r-2$, that is,

$$\rho_1 = \sum_{0 \le |J| \le r-3} \omega_J^\sigma \wedge \Phi_\sigma^J + \sum_{|J|=r-2} \omega_J^\sigma \wedge \Delta_\sigma^J, \tag{210}$$

where the $(q-1)$-forms Δ_σ^J are traceless.

(b) τ_1 is generated by $\omega_J^\sigma \wedge \omega_I^\nu$ and $\omega_J^\sigma \wedge d\omega_L^\nu$, where $|J| = r-1$, $0 \le |I| \le r-1$, $|L| = r-1$.

(c) κ_1 is a contact $(q-1)$-form.

Indeed, we apply condition $d\rho = 0$ to expression (203). We have, since $d\omega^\sigma_J = -dy^\sigma_{Jj} \wedge dx^j$,

$$\sum_{0 \le |J| \le r-2} d(\omega^\sigma_J \wedge \Phi^J_\sigma) - \sum_{|J|=r-1} (dy^\sigma_{Jj} \wedge dx^j \wedge \Delta^J_\sigma + \omega^\sigma_J \wedge d\Delta^J_\sigma) + d\tau_0 = 0. \tag{211}$$

But the terms $dy^\nu_{Jj} \wedge dx^j \wedge \Delta^J_\nu$ in this expression do not contain any form ω^ν_J or $d\omega^\nu_J$ and must vanish separately. Thus,

$$\sum_{|J|=r-1} dy^\nu_{Jj} \wedge dx^j \wedge \Delta^J_\nu = 0. \tag{212}$$

The 1-contact component gives

$$\sum_{|J|=r-1} \omega^\nu_{Jj} \wedge h(dx^j \wedge \Delta^J_\nu) = 0 \tag{213}$$

hence

$$h(dx^j \wedge \Delta^J_\nu) = 0 \quad \mathrm{Sym}(Jj). \tag{214}$$

The traceless form Δ^J_ν can be expressed as

$$\begin{aligned} \Delta^J_\nu = {} & A^J_{\nu i_2 i_3 \ldots i_q} dx^{i_2} \wedge dx^{i_3} \wedge \cdots \wedge dx^{i_q} \\ & + A^{JI_2}_{\nu\sigma_2 i_3 i_4 \ldots i_q} dy^{\sigma_2}_{I_2} \wedge dx^{i_3} \wedge dx^{i_4} \wedge \cdots \wedge dx^{i_q} \\ & + A^{JI_2I_3}_{\nu\sigma_2\sigma_3 i_4 i_5 \ldots i_q} dy^{\sigma_2}_{I_2} \wedge dy^{\sigma_3}_{I_3} \wedge dx^{i_4} \wedge dx^{i_5} \wedge \cdots \wedge dx^{i_q} \\ & + \cdots + A^{JI_2I_3 \ldots I_{q-1}}_{\nu\sigma_2\sigma_3 \ldots \sigma_{q-1} i_q} dy^{\sigma_2}_{I_2} \wedge dy^{\sigma_3}_{I_3} \wedge \cdots \wedge dy^{\sigma_{q-1}}_{I_{q-1}} \wedge dx^{i_q} \\ & + A^{JI_2I_3 \ldots I_q}_{\nu\sigma_2\sigma_3 \ldots \sigma_q} dy^{\sigma_2}_{I_2} \wedge dy^{\sigma_3}_{I_3} \wedge \cdots \wedge dy^{\sigma_q}_{I_q}, \end{aligned} \tag{215}$$

where the multi-indices $I_2, I_3, \ldots, I_q$ satisfy $|I_2|, |I_3|, \ldots, |I_q| = r$ and all coefficients $A^{JI_2}_{\nu\sigma_2 i_3 i_4 \ldots i_q}, A^{JI_2I_3}_{\nu\sigma_2\sigma_3 i_4 i_5 \ldots i_q}, \ldots, A^{JI_2I_3 \ldots I_{q-1}}_{\nu\sigma_2\sigma_3 \ldots \sigma_{q-1} i_q}$ are traceless in the indices $i_3, i_4, \ldots, i_q$ and the multi-indices $I_2, I_3, \ldots, I_{q-1}$. Then, Eq. (214) reads

$$\begin{aligned} (A^J_{\nu i_2 i_3 \ldots i_q} & + A^{JI_2}_{\nu\sigma_2 i_3 i_4 \ldots i_q} y^{\sigma_2}_{I_2 i_2} + A^{JI_2I_3}_{\nu\sigma_2\sigma_3 i_4 i_5 \ldots i_q} y^{\sigma_2}_{I_2 i_2} y^{\sigma_3}_{I_3 i_3} \\ & + \cdots + A^{JI_2I_3 \ldots I_{q-1}}_{\nu\sigma_2\sigma_3 \ldots \sigma_{q-1} i_q} y^{\sigma_2}_{I_2 i_2} y^{\sigma_3}_{I_3 i_3} \ldots y^{\sigma_{q-1}}_{I_{q-1} i_{q-1}} \\ & + A^{JI_2I_3 \ldots I_q}_{\nu\sigma_2\sigma_3 \ldots \sigma_q} y^{\sigma_2}_{I_2 i_2} y^{\sigma_3}_{I_3 i_3} \ldots y^{\sigma_q}_{I_q i_q}) \\ & \cdot \delta^l_{i_1} dx^{i_1} \wedge dx^{i_2} \wedge dx^{i_3} \wedge \cdots \wedge dx^{i_q} = 0 \quad \mathrm{Sym}(Jj). \end{aligned} \tag{216}$$

Setting

$$\begin{aligned}
&B^{Jl}_{\nu i_1 i_2 i_3 \dots i_q} = A^{J}_{\nu i_2 i_3 \dots i_q}\delta^l_{i_1} \quad \mathrm{Sym}(Jl) \quad \mathrm{Alt}(i_1 i_2 i_3 \dots i_q),\\
&B^{JlI_2}_{\nu\sigma_2 i_1 i_3 i_4 \dots i_q} = A^{JI_2}_{\sigma_2 i_3 i_4 \dots i_q \nu}\delta^l_{i_1} \quad \mathrm{Sym}(Jl) \quad \mathrm{Alt}(i_1 i_3 i_4 \dots i_q),\\
&B^{JlI_2I_3}_{\nu\sigma_2\sigma_3 i_1 i_4 i_5 \dots i_q} = A^{JI_2I_3}_{\nu\sigma_2\sigma_3 i_4 i_5 \dots i_q}\delta^l_{i_1} \quad \mathrm{Sym}(Jl) \quad \mathrm{Alt}(i_1 i_4 i_5 \dots i_q),\\
&\dots\\
&B^{JI_2I_3 \dots I_{q-1}}_{\nu\sigma_2\sigma_3 \dots \sigma_{q-1} i_1 i_q} = A^{JI_2I_3 \dots I_{q-1}}_{\nu\sigma_2\sigma_3 \dots \sigma_{q-1} i_q}\delta^l_{i_1} \quad \mathrm{Sym}(Jl) \quad \mathrm{Alt}(i_1 i_q),\\
&B^{JlI_2I_3 \dots I_q}_{\nu\sigma_2\sigma_3 \dots \sigma_q i_1} = A^{JI_2I_3 \dots I_q}_{\nu\sigma_2\sigma_3 \dots \sigma_q}\delta^l_{i_1} \quad \mathrm{Sym}(Jl),
\end{aligned} \tag{217}$$

we get the system

$$\begin{aligned}
&B^{Jl}_{\nu i_1 i_2 i_3 \dots i_q} = 0,\\
&B^{JlI_2}_{\nu\sigma_2 i_1 i_3 i_4 \dots i_q}\delta^{j_2}_{i_2} = 0 \quad \mathrm{Sym}(I_2 j_2) \quad \mathrm{Alt}(i_1 i_2 i_3 \dots i_q),\\
&B^{JlI_2I_3}_{\nu\sigma_2\sigma_3 i_1 i_4 i_5 \dots i_q}\delta^{j_2}_{i_2}\delta^{j_3}_{i_3} = 0 \quad \mathrm{Sym}(I_2 j_2) \quad \mathrm{Sym}(I_3 j_3) \quad \mathrm{Alt}(i_1 i_2 i_3 \dots i_q),\\
&\dots\\
&B^{JlI_2I_3 \dots I_{q-1}}_{\nu\sigma_2\sigma_3 \dots \sigma_{q-1} i_1 i_q}\delta^{j_2}_{i_2}\delta^{j_3}_{i_3}\dots\delta^{j_{q-1}}_{i_{q-1}} = 0 \quad \mathrm{Sym}(I_2 j_2) \quad \mathrm{Sym}(I_3 j_3)\\
&\quad \dots \quad \mathrm{Sym}(I_{q-1} j_{q-1}) \quad \mathrm{Alt}(i_1 i_2 i_3 \dots i_q),\\
&B^{JlI_2I_3 \dots I_q}_{\nu\sigma_2\sigma_3 \dots \sigma_q i_1}\delta^{j_2}_{i_2}\delta^{j_3}_{i_3}\dots\delta^{j_q}_{i_q} = 0 \quad \mathrm{Sym}(I_2 j_2) \quad \mathrm{Sym}(I_3 j_3)\\
&\quad \dots \quad \mathrm{Sym}(I_q j_q) \quad \mathrm{Alt}(i_1 i_2 i_3 \dots i_q).
\end{aligned} \tag{218}$$

Since the unknown functions, $B^{Jl\,I_2}_{\nu\,\sigma_2\,i_1 i_3 i_4 \dots i_q}$, $B^{JI_2 lI_3}_{\nu\sigma_2\sigma_3 i_1 i_4 i_5 \dots i_q}$, …, $B^{Jl\,I_2\,I_3 \dots I_{q-1}}_{\nu\,\sigma_2\,\sigma_3 \dots \sigma_{q-1}\,i_1 i_q}$, $B^{Jl\,I_2\,I_3 \dots I_q}_{\nu\,\sigma_2\,\sigma_3 \dots \sigma_q\,i_1}$, are traceless, for each fixed multi-index $I = Jl$ and each index ν, this system has only the trivial solution (see Appendix 9), and we have from (217)

$$\begin{aligned}
&A^{J}_{\nu i_2 i_3 \dots i_q}\delta^l_{i_1} = 0 \quad \mathrm{Sym}(Jl) \quad \mathrm{Alt}(i_1 i_2 i_3 \dots i_q),\\
&A^{JI_2}_{\nu\sigma_{2 i_3 i_4 \dots i_q}}\delta^l_{i_1} = 0 \quad \mathrm{Sym}(Jl) \quad \mathrm{Alt}(i_1 i_3 i_4 \dots i_q),\\
&A^{JI_2I_3}_{\nu\sigma_2\sigma_3 i_4 i_5 \dots i_q}\delta^l_{i_1} = 0 \quad \mathrm{Sym}(Jl) \quad \mathrm{Alt}(i_1 i_4 i_5 \dots i_q),\\
&\dots\\
&A^{JI_2I_3 \dots I_{q-1}}_{\nu\sigma_2\sigma_3 \dots \sigma_{q-1} i_q}\delta^l_{i_1} = 0 \quad \mathrm{Sym}(Jl) \quad \mathrm{Alt}(i_1 i_q),\\
&A^{JI_2I_3 \dots I_q}_{\sigma_2\nu\sigma_3 \dots \sigma_q}\delta^l_{i_1} = 0 \quad \mathrm{Sym}(Jl).
\end{aligned} \tag{219}$$

The solutions of this system are of *Kronecker type*; we have, denoting the multi-index J as $J = Kk$,

$$\begin{aligned}
&A^{Kk}_{\nu i_2 i_3 \dots i_q} = C^{K}_{\nu i_3 i_4 \dots i_q}\delta^k_{i_2} \quad \mathrm{Sym}(Kk) \quad \mathrm{Alt}(i_2 i_3 i_4 \dots i_q),\\
&A^{KkI_2}_{\nu\sigma_2 i_3 i_4 \dots i_q} = C^{KI_2}_{\nu\sigma_2 i_4 i_5 \dots i_q}\delta^k_{i_3} \quad \mathrm{Sym}(Kk) \quad \mathrm{Alt}(i_3 i_4 i_5 \dots i_q),\\
&A^{KkI_2I_3}_{\nu\sigma_2\sigma_3 i_4 i_5 \dots i_q} = C^{KI_2I_3}_{\nu\sigma_2\sigma_3 i_5 i_6 \dots i_q}\delta^k_{i_4} \quad \mathrm{Sym}(Kk) \quad \mathrm{Alt}(i_4 i_5 i_6 \dots i_q),\\
&\dots\\
&A^{KkI_2I_3 \dots I_{q-1}}_{\nu\sigma_2\sigma_3 \dots \sigma_{q-1} i_q} = C^{KI_2I_3 \dots I_{q-1}}_{\nu\sigma_2\sigma_3 \dots \sigma_{q-1}}\delta^k_{i_q} \quad \mathrm{Sym}(Jl),\\
&A^{KkI_2I_3 \dots I_q}_{\nu\sigma_2\sigma_3 \dots \sigma_q} = 0.
\end{aligned} \tag{220}$$

Consequently,

$$\begin{aligned}\sum_{|J|=r-1} \omega_J^\sigma \wedge \Delta_\nu^J &= \omega_{Kk}^\sigma \wedge (C_{\nu i_3 i_4 \ldots i_q}^K \delta_{i_2}^k dx^{i_2} \wedge dx^{i_3} \wedge \ldots \wedge dx^{i_q} \\ &\quad + C_{\nu\,\sigma_2 i_4 i_5 \ldots i_q}^{K\,I_2} \delta_{i_3}^k dy_{I_2}^{\sigma_2} \wedge dx^{i_3} \wedge dx^{i_4} \wedge \cdots \wedge dx^{i_q} \\ &\quad + C_{\nu\sigma_2\sigma_3 i_5 i_6 \ldots i_q}^{KI_2I_3} \delta_{i_4}^k dy_{I_2}^{\sigma_2} \wedge dy_{I_3}^{\sigma_3} \wedge dx^{i_4} \wedge dx^{i_5} \wedge \cdots \wedge dx^{i_q} \\ &\quad + \cdots + C_{\nu\sigma_2\sigma_3 \ldots \sigma_{q-1}}^{KI_2I_3 \ldots I_{q-1}} \delta_{i_q}^k dy_{I_2}^{\sigma_2} \wedge dy_{I_3}^{\sigma_3} \wedge \cdots \wedge dy_{I_{q-1}}^{\sigma_{q-1}} \wedge dx^{i_q}) \\ &= d\omega_K^\sigma \wedge (-C_{\nu i_3 i_4 \ldots i_q}^K dx^{i_3} \wedge dx^{i_4} \wedge \cdots \wedge dx^{i_q} \\ &\quad + C_{\nu\sigma_2 i_4 i_5 \ldots i_q}^{KI_2} dy_{I_2}^{\sigma_2} \wedge dx^{i_4} \wedge dx^{i_5} \wedge \cdots \wedge dx^{i_q} \\ &\quad - C_{\nu\sigma_2\sigma_3 i_5 i_6 \ldots i_q}^{KI_2I_3} dy_{I_2}^{\sigma_2} \wedge dy_{I_3}^{\sigma_3} \wedge dx^{i_5} \wedge dx^{i_6} \wedge \cdots \wedge dx^{i_q} \\ &\quad + \cdots + (-1)^{q-1} C_{\nu\sigma_2\sigma_3 \ldots \sigma_{q-1}}^{KI_2I_3 \ldots I_{q-1}} dy_{I_2}^{\sigma_2} \wedge dy_{I_3}^{\sigma_3} \wedge \cdots \wedge dy_{I_{q-1}}^{\sigma_{q-1}}).\end{aligned} \tag{221}$$

This expression splits in two terms,

$$\begin{aligned}& d(\omega_K^\sigma \wedge (-C_{\nu i_3 i_4 \ldots i_q}^K dx^{i_3} \wedge dx^{i_4} \wedge \cdots \wedge dx^{i_q} \\ &\quad + C_{\nu\sigma_2 i_4 i_5 \ldots i_q}^{KI_2} dy_{I_2}^{\sigma_2} \wedge dx^{i_4} \wedge dx^{i_5} \wedge \cdots \wedge dx^{i_q} \\ &\quad - C_{\nu\sigma_2\sigma_3 i_5 i_6 \ldots i_q}^{KI_2I_3} dy_{I_2}^{\sigma_2} \wedge dy_{I_3}^{\sigma_3} \wedge dx^{i_5} \wedge dx^{i_6} \wedge \cdots \wedge dx^{i_q} \\ &\quad + \cdots + (-1)^{q-1} C_{\nu\sigma_2\sigma_3 \ldots \sigma_{q-1}}^{KI_2I_3 \ldots I_{q-1}} dy_{I_2}^{\sigma_2} \wedge dy_{I_3}^{\sigma_3} \wedge \cdots \wedge dy_{I_{q-1}}^{\sigma_{q-1}})),\end{aligned} \tag{222}$$

and

$$\begin{aligned}& -\omega_K^\sigma \wedge d(-C_{\nu i_3 i_4 \ldots i_q}^K dx^{i_3} \wedge dx^{i_4} \wedge \cdots \wedge dx^{i_q} \\ &\quad + C_{\nu\sigma_2 i_4 i_5 \ldots i_q}^{KI_2} dy_{I_2}^{\sigma_2} \wedge dx^{i_4} \wedge dx^{i_5} \wedge \cdots \wedge dx^{i_q} \\ &\quad - C_{\nu\sigma_2\sigma_3 i_5 i_6 \ldots i_q}^{KI_2I_3} dy_{I_2}^{\sigma_2} \wedge dy_{I_3}^{\sigma_3} \wedge dx^{i_5} \wedge dx^{i_6} \wedge \cdots \wedge dx^{i_q} \\ &\quad + \cdots + (-1)^{q-1} C_{\nu\sigma_2\sigma_3 \ldots \sigma_{q-1}}^{KI_2I_3 \ldots I_{q-1}} dy_{I_2}^{\sigma_2} \wedge dy_{I_3}^{\sigma_3} \wedge \cdots \wedge dy_{I_{q-1}}^{\sigma_{q-1}}),\end{aligned} \tag{223}$$

which can be distributed to the terms $d\kappa_0$ and ρ_0 in the decomposition (207). Therefore, ρ can be written as

$$\begin{aligned}\rho &= \sum_{0 \le |J| \le r-2} \omega_J^\sigma \wedge \Phi_\sigma^J + \sum_{|J|=r-1} \omega_J^\sigma \wedge \Delta_\sigma^J + \sum_{|J|=r-1} \omega_J^\sigma \wedge Z_\sigma^J + d\kappa_0 \\ &= \sum_{0 \le |J| \le r-2} \omega_J^\sigma \wedge \tilde{\Phi}_\sigma^J + \sum_{|J|=r-1} \omega_J^\sigma \wedge Z_\sigma^J + d\kappa_1 \\ &= \sum_{0 \le |J| \le r-3} \omega_J^\sigma \wedge \tilde{\Phi}_\sigma^J + \sum_{|J|=r-2} \omega_J^\sigma \wedge \tilde{\Phi}_\sigma^J + \sum_{|J|=r-1} \omega_J^\sigma \wedge Z_\sigma^J + d\kappa_1\end{aligned}$$

$$
\begin{aligned}
&= \sum_{0 \le |J| \le r-3} \omega_J^\sigma \wedge \tilde{\Phi}_\sigma^J + \sum_{|J| = r-2} \omega_J^\sigma \wedge \tilde{\Phi}_\sigma^J + \sum_{|J| = r-1} \omega_J^\sigma \wedge \mathrm{Z}_\sigma^J + d\kappa_1 \\
&= \sum_{0 \le |J| \le r-3} \omega_J^\sigma \wedge \tilde{\Phi}_\sigma^J + \sum_{|J| = r-2} \omega_J^\sigma \wedge \Delta_\sigma^J + \sum_{|J| = r-2} \omega_J^\sigma \wedge \mathrm{Z}_\sigma^J \\
&\quad + \sum_{|J| = r-1} \omega_J^\sigma \wedge \mathrm{Z}_\sigma^J + d\kappa_1
\end{aligned} \tag{224}
$$

where we use the trace decomposition $\tilde{\Phi}_\sigma^J = \Delta_\sigma^J + \mathrm{Z}_\sigma^J$ for $|J| = r - 1$. Summarizing and replacing for simplicity of notation $\tilde{\Phi}_\sigma^J$ with Φ_σ^J, we get the decomposition (209).

Third, we construct as in the second step the decompositions

$$
\begin{aligned}
&\rho_0 = \sum_{0 \le |J| \le r-2} \omega_J^\sigma \wedge \Phi_\sigma^J + \sum_{|J|=r-1} \omega_J^\sigma \wedge \Delta_\sigma^J, \\
&\rho_1 = \sum_{0 \le |J| \le r-3} \omega_J^\sigma \wedge \Phi_\sigma^J + \sum_{|J|=r-2} \omega_J^\sigma \wedge \Delta_\sigma^J, \\
&\ldots \\
&\rho_{r-2} = \omega^\sigma \wedge \Phi_\sigma + \sum_j \omega_j^\sigma \wedge \Delta_\sigma^j, \\
&\rho_{r-1} = \omega^\sigma \wedge \Delta_\sigma,
\end{aligned} \tag{225}
$$

and

$$
\begin{aligned}
&\rho = \rho_0 + \tau_0 + d\kappa_0 = \rho_1 + \tau_1 + d\kappa_1 = \rho_2 + \tau_2 + d\kappa_2 \\
&\ldots = \rho_{r-2} + \tau_{r-2} + d\kappa_{r-2} = \rho_{r-1} + \tau_{r-1} + d\kappa_{r-1}.
\end{aligned} \tag{226}
$$

Note, however, the different meaning of the symbols Φ_σ^J and Δ_σ^J in the lines of expressions (225), which are defined in the construction.

Finally, we show that ρ has a decomposition

$$
\rho = \tau_{r-1} + d\kappa_{r-1}, \tag{227}
$$

where τ_{r-1} is generated by the contact forms $\omega_J^\sigma \wedge \omega_I^\nu$ and $\omega_J^\sigma \wedge d\omega_L^\nu$, $|J| = r - 1$, $0 \le |I| \le r - 1$, $|L| = r - 1$ and κ_{r-1} is a contact $(q - 1)$-form. It is sufficient to show that in the decomposition $\rho = \rho_{r-1} + \tau_{r-1} + d\kappa_{r-1}$ (226), the form ρ_{r-1} vanishes. Condition $d\rho = 0$ implies

$$
d\omega^\sigma \wedge \Delta_\sigma - \omega^\sigma \wedge d\Delta_\sigma + d\tau_{r-1} = 0. \tag{228}
$$

The 1-contact component yields $-\omega_l^\sigma \wedge dx^l \wedge h\Delta_\sigma - \omega^\sigma \wedge hd\Delta_\sigma = 0$; hence,

$$
h(dx^l \wedge \Delta_\sigma) = 0. \tag{229}
$$

Writing the traceless form Δ_ν as

$$\begin{aligned}\Delta_\nu = {} & A_{\nu i_2 i_3 \ldots i_q} dx^{i_2} \wedge dx^{i_3} \wedge \cdots \wedge dx^{i_q} \\ & + A^{I_2}_{\nu\sigma_2 i_3 i_4 \ldots i_q} dy^{\sigma_2}_{I_2} \wedge dx^{i_3} \wedge dx^{i_4} \wedge \cdots \wedge dx^{i_q} \\ & + A^{I_2 I_3}_{\nu\sigma_2\sigma_3 i_4 i_5 \ldots i_q} dy^{\sigma_2}_{I_2} \wedge dy^{\sigma_3}_{I_3} \wedge dx^{i_4} \wedge dx^{i_5} \wedge \cdots \wedge dx^{i_q} \\ & + \cdots + A^{I_2 I_3 \ldots I_{q-1}}_{\nu\sigma_2\sigma_3 \ldots \sigma_{q-1} i_q} dy^{\sigma_2}_{I_2} \wedge dy^{\sigma_3}_{I_3} \wedge \cdots \wedge dy^{\sigma_{q-1}}_{I_{q-1}} \wedge dx^{i_q} \\ & + A^{I_2 I_3 \ldots I_q}_{\nu\sigma_2\sigma_3 \ldots \sigma_q} dy^{\sigma_2}_{I_2} \wedge dy^{\sigma_3}_{I_3} \wedge \cdots \wedge dy^{\sigma_q}_{I_q},\end{aligned} \tag{230}$$

we have

$$\begin{aligned} h(dx^l \wedge \Delta_\nu) = {} & \Big(A_{\nu i_2 i_3 \ldots i_q} + A^{I_2}_{\nu\sigma_2 i_3 i_4 \ldots i_q} y^{\sigma_2}_{I_2 i_2} + A^{I_2 I_3}_{\nu\sigma_2\sigma_3 i_4 i_5 \ldots i_q} y^{\sigma_2}_{I_2 i_2} y^{\sigma_3}_{I_3 i_3} \\ & + \cdots + A^{I_2 I_3 \ldots I_{q-1}}_{\nu\sigma_2\sigma_3 \ldots \sigma_{q-1} i_q} y^{\sigma_2}_{I_2 i_2} y^{\sigma_3}_{I_3 i_3} \ldots y^{\sigma_{q-1}}_{I_{q-1} i_{q-1}} + A^{I_2 I_3 \ldots I_q}_{\nu\sigma_2\sigma_3 \ldots \sigma_q} y^{\sigma_2}_{I_2 i_2} y^{\sigma_3}_{I_3 i_3} \ldots y^{\sigma_q}_{I_q i_q} \Big) \\ & \cdot dx^l \wedge dx^{i_2} \wedge dx^{i_3} \wedge \cdots \wedge dx^{i_q} = 0, \end{aligned} \tag{231}$$

which implies, because the coefficients are traceless,

$$\begin{aligned} & A_{\nu i_2 i_3 \ldots i_q} = 0, \quad A^{I_2}_{\nu\sigma_2 i_3 i_4 \ldots i_q} = 0, \quad A^{I_2 I_3}_{\nu\sigma_2\sigma_3 i_4 i_5 \ldots i_q} = 0, \\ & \ldots \quad A^{I_2 I_3 \ldots I_{q-1}}_{\nu\sigma_2\sigma_3 \ldots \sigma_{q-1} i_q} = 0, \quad A^{I_2 I_3 \ldots I_q}_{\nu\sigma_2\sigma_3 \ldots \sigma_q} = 0. \end{aligned} \tag{232}$$

Consequently, $\rho_{r-1} = 0$ proving (227).

3. To conclude the proof, we apply the contact homotopy decomposition to the form τ_{r-1} (Theorem 11). We have $\tau_{r-1} = Id\tau_{r-1} + dI\tau_{r-1}$. But $d\tau_{r-1} = 0$, and thus, $\tau_{r-1} = dI\tau_{r-1}$, and since the order of contactness of τ_{r-1} is ≥ 2, we have $hI\tau_{r-1} = Ihp_1\tau_{r-1} = 0$, so $I\tau_{r-1}$ is contact. Then, however,

$$\rho = Id\tau_{r-1} + dI\tau_{r-1} + d\kappa_{r-1} = d(I\tau_{r-1} + d\kappa_{r-1}). \tag{233}$$

Setting $\eta = I\tau_{r-1} + d\kappa_{r-1}$, we complete the proof. □

Theorem 14 *If ρ is strongly contact and $d\rho = 0$, then there exists a strongly contact $(q-1)$-form η such that $\rho = d\eta$.*

Proof We express ρ as $\rho = Id\rho + dI\rho$. But by hypothesis $d\rho = 0$, thus setting $\eta = I\rho$, we have $\rho = d\eta$; now, our assertion follows from Theorem 12, (b). □

Remark 11 The concept of a strongly contact form, used in Theorem 14, has been introduced by means of the exterior derivative d and the pullback operation by the canonical jet projection $\pi^{r+1,r}$: $J^{r+1}Y \to J^rY$. The decompositions of the forms on J^rY, related to this concept, represent a basic tool in the higher-order variational theory on the jet spaces J^rY. A broader concept of a strongly contact form is considered in Chap. 8.

References

[D] P. Dedecker, On the generalization of symplectic geometry to multiple integrals in the calculus of variations, in: *Lecture notes in Math.* 570, Springer, Berlin, 1977, 395-456

[GS] H. Goldschmidt, S. Sternberg, The Hamilton-Cartan formalism in the calculus of variations, Ann. Inst. H. Poincare 23 (1973) 203-267

[K13] D. Krupka, *Some Geometric Aspects of Variational Problems in Fibered Manifolds*, Folia Fac. Sci. Nat. UJEP Brunensis, Physica 14, Brno, Czech Republic, 1973, 65 pp.; arXiv:math-ph/0110005

[K4] D. Krupka, Global variational theory in fibred spaces, in: D. Krupka, D. Saunders, Eds., *Handbook of Global Analysis*, Elsevier, 2008, 773-836

Chapter 3
Formal Divergence Equations

In this chapter, we introduce formal divergence equations on Euclidean spaces and study their basic properties. These partial differential equations naturally appear in the variational geometry on fibered manifolds, but also have a broader meaning related to differential equations, conservation laws, and integration of forms on manifolds with boundary. A formal divergence equation is not always integrable; we show that the obstructions are connected with the *Euler–Lagrange expressions* known from the higher-order variational theory of multiple integrals. If a solution exists, then it defines a solution of the associated "ordinary" divergence equation along any section of the underlying fibered manifold. The notable fact is that the solutions of formal divergence equations of order r are in one–one correspondence with a class of differential forms on the $(r-1)$-st jet prolongation of the underlying fibered manifold, defined by the exterior derivative operator.

The chapter extends the theory explained in Krupka [K14].

3.1 Formal Divergence Equations

Let $U \subset \mathbf{R}^n$ be an open set, let $V \subset \mathbf{R}^m$ be an open ball with center $0 \in \mathbf{R}^m$, and denote $W = U \times V$. We consider W as a fibered manifold over U with the first Cartesian projection π: $W \to U$. As before, we denote by W^r the r-jet prolongation of W. The set W^r can explicitly be expressed as the Cartesian product

$$W^r = U \times V \times L(\mathbf{R}^n, \mathbf{R}^m) \times L^2_{\mathrm{sym}}(\mathbf{R}^n, \mathbf{R}^m) \times \cdots \times L^r_{\mathrm{sym}}(\mathbf{R}^n, \mathbf{R}^m), \tag{1}$$

where $L^k_{\mathrm{sym}}(\mathbf{R}^n, \mathbf{R}^m)$ is the vector space of k-linear, symmetric mappings from $\mathbf{R}^n$ to $\mathbf{R}^m$. The Cartesian coordinates on W, and the associated jet coordinates on W^r, are denoted by (x^i, y^σ) and $(x^i, y^\sigma, y^\sigma_{j_1}, y^\sigma_{j_1 j_2}, \ldots, y^\sigma_{j_1 j_2 \ldots j_r})$, respectively.

Let $s \geq 1$ and let f: $W^s \to \mathbf{R}$ be a function. In this section, we study the differential equation

D. Krupka, *Introduction to Global Variational Geometry*,
Atlantis Studies in Variational Geometry 1, DOI 10.2991/978-94-6239-073-7_3

$$d_i g^i = f \tag{2}$$

for a collection $g = g^i$ of differentiable functions g^i: $W^r \to \mathbf{R}$, where $r \geq s$, and

$$d_i g^i = \frac{\partial g^i}{\partial x^i} + \sum_{0 \leq k \leq s} \sum_{1 \leq i \leq n} \sum_{j_1 \leq j_2 \leq \cdots \leq j_k} \frac{\partial g^i}{\partial y^\sigma_{j_1 j_2 \ldots j_k}} y^\sigma_{j_1 j_2 \ldots j_k i} \tag{3}$$

is the *formal divergence* of the collection g^i. Equation (2) is the *formal divergence equation*, and g^i is its *solution of order r*. Clearly, a solution of order r is also a solution of order $r + 1$. Our aim will be to find all solutions of order s, defined on the same domain as the function f.

In expression (3), we differentiate with respect to independent variables $y^\sigma_{j_1 j_2 \ldots j_k}$, where $j_1 \leq j_2 \leq \cdots \leq j_k$. However, it will be convenient to find another expression for the formal divergence with no restriction to the summation indices. According to Appendix 2,

$$\sum_i \sum_{j_1 \leq j_2 \leq \cdots \leq j_k} \frac{\partial g^i}{\partial y^\sigma_{j_1 j_2 \ldots j_k}} y^\sigma_{j_1 j_2 \ldots j_k i} = \frac{\partial g^i}{\partial y^\sigma_{i_1 i_2 \ldots i_k}} y^\sigma_{i_1 i_2 \ldots i_k i}, \tag{4}$$

where $y^\sigma_{i_1 i_2 \ldots i_k}$ on the right side stands for the canonical extension of the variables $y^\sigma_{i_1 i_2 \ldots i_k}$, $j_1 \leq j_2 \leq \cdots \leq j_k$ to all values of the subscripts. With this convention, the formal derivative (3) can be expressed as

$$d_i g^i = \frac{\partial g^i}{\partial x^i} + \frac{\partial g^i}{\partial y^\sigma} y^\sigma_i + \frac{\partial g^i}{\partial y^\sigma_{j_1}} y^\sigma_{j_1 i} + \frac{\partial g^i}{\partial y^\sigma_{j_1 j_2}} y^\sigma_{j_1 j_2 i} + \cdots + \frac{\partial g^i}{\partial y^\sigma_{j_1 j_2 \ldots j_s}} y^\sigma_{j_1 j_2 \ldots j_s i}. \tag{5}$$

From expression (5), we immediately see that every solution g^i, defined on the set W^r such that $r \geq s$, satisfies the system of partial differential equations

$$\frac{\partial g^i}{\partial y^\sigma_{j_1 j_2 \ldots j_r}} + \frac{\partial g^{j_1}}{\partial y^\sigma_{i j_2 j_3 \ldots j_r}} + \frac{\partial g^{j_2}}{\partial y^\sigma_{j_1 i j_3 j_4 \ldots j_r}} + \cdots + \frac{\partial g^{j_{r-1}}}{\partial y^\sigma_{j_1 j_2 \ldots j_{r-2} i j_r}} + \frac{\partial g^{j_r}}{\partial y^\sigma_{j_1 j_2 \ldots j_{r-1} i}} = 0. \tag{6}$$

Our first aim will be to find solutions of this system.

The proof of the following lemma is based on the Young decomposition theory of tensor spaces.

Lemma 1

(a) *Every solution* $g = g^i$ *of the system* (6) *is a polynomial function of the variables* $y^\sigma_{j_1 j_2 \ldots j_s}$.

(b) *If the system* (6) *has a solution* $g = g^i$ *of order* $r \geq s$, *then it also has a solution of order s.*

Proof

(a) To prove Lemma 1, it is convenient to use multi-indices of the form $J = (j_1, j_2, \ldots, j_r)$. First, we show that condition (6) implies that the expression

$$\frac{\partial^n g^i}{\partial y^{\sigma_1}_{J_1} \partial y^{\sigma_2}_{J_2} \ldots \partial y^{\sigma_n}_{J_n}} \tag{7}$$

vanishes for all $\sigma_1, \sigma_2, \ldots, \sigma_n$ and $J_1, J_2, \ldots, J_n$. This expression is indexed with $nr + 1$ indices q_l, where $l = 1, 2, \ldots, n, n + 1, n + 2, \ldots, nr, nr + 1$ and $1 \leq q_l \leq n$ (entries of the multi-indices and the index i). The (unique) cycle decomposition of the number $nr + 1$ includes exactly one scheme, namely the scheme $(r + 1, r, \ldots, r)$ (one row with $r + 1$ boxes, $n - 1$ rows with r boxes). The corresponding Young diagrams as well as (non-trivial) Young projectors are then necessarily of the form

$$\begin{array}{|c|c|} \hline J_1 & i \\ \hline J_2 & \multicolumn{1}{c}{} \\ \cline{1-1} J_3 & \multicolumn{1}{c}{} \\ \cline{1-1} \ldots & \multicolumn{1}{c}{} \\ \cline{1-1} J_n & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \tag{8}$$

The first row represents symmetrization in the entries of the multi-index J_1 and the index i. But according to (6), these Young symmetrizers annihilate (7), so the Young decomposition yields

$$\frac{\partial^n g^i}{\partial y^{\sigma_1}_{J_1} \partial y^{\sigma_2}_{J_2} \ldots \partial y^{\sigma_n}_{J_n}} = 0. \tag{9}$$

Consequently, g^i is polynomial in the variables y^σ_J.

(b) Consider the formal divergence Eq. (2) with the right-hand side $f = f\left(x^i, y^\sigma, y^\sigma_{j_1}, y^\sigma_{j_1 j_2}, \ldots, y^\sigma_{j_1 j_2 \ldots j_s}\right)$, and its solution $g = g^i$ of order $r \geq s + 1$. Then

$$\frac{\partial g^i}{\partial x^i} + \frac{\partial g^i}{\partial y^\sigma} y^\sigma_i + \frac{\partial g^i}{\partial y^\sigma_{j_1}} y^\sigma_{j_1 i} + \frac{\partial g^i}{\partial y^\sigma_{j_1 j_2}} y^\sigma_{j_1 j_2 i} + \cdots + \frac{\partial g^i}{\partial y^\sigma_{j_1 j_2 \ldots j_r}} y^\sigma_{j_1 j_2 \ldots j_r i} = f, \tag{10}$$

and condition (6) is satisfied. Then by the first part of this proof

$$g^i = g^i_0 + g^i_1 + g^i_2 + \cdots + g^i_{n-1}, \tag{11}$$

where g_p^i is a homogeneous polynomial of degree p in the variables $y^\sigma_{j_1 j_2 \ldots j_r}$. Substituting from (11) into (10), we get, because f does not depend on $y^\sigma_{j_1 j_2 \ldots j_r}$,

$$\frac{\partial g_0^i}{\partial x^i} + \frac{\partial g_0^i}{\partial y^\sigma} y_i^\sigma + \frac{\partial g_0^i}{\partial y_{j_1}^\sigma} y^\sigma_{j_1 i} + \frac{\partial g_0^i}{\partial y^\sigma_{j_1 j_2}} y^\sigma_{j_1 j_2 i} + \cdots + \frac{\partial g_0^i}{\partial y^\sigma_{j_1 j_2 \ldots j_{r-1}}} y^\sigma_{j_1 j_2 \ldots j_{r-1} i} = f. \quad (12)$$

Repeating this procedure, we get some functions $h = h^i$, defined on V^s, satisfying

$$\frac{\partial h^i}{\partial x^i} + \frac{\partial h^i}{\partial y^\sigma} y_i^\sigma + \frac{\partial h^i}{\partial y_{j_1}^\sigma} y^\sigma_{j_1 i} + \frac{\partial h^i}{\partial y^\sigma_{j_1 j_2}} y^\sigma_{j_1 j_2 i} + \cdots + \frac{\partial h^i}{\partial y^\sigma_{j_1 j_2 \ldots j_s}} y^\sigma_{j_1 j_2 \ldots j_s i} = f. \quad (13)$$

h^i is a solution of order s. □

Remark 1 If $g = g^i$ is a solution of order r of the formal divergence Eq. (2), then Eq. (6) represent restrictions to the *coefficients* of the polynomials g^i.

Remark 2 Every solution of the homogeneous formal divergence equation

$$d_i g^i = 0 \quad (14)$$

is defined on U. Indeed, according to Lemma 1, if (14) has a solution, then this solution is defined on V; thus

$$\frac{\partial g^i}{\partial x^i} + \frac{\partial g^i}{\partial y^\sigma} y_i^\sigma = 0, \quad (15)$$

hence $(\partial g^i / \partial y^\sigma) = 0$ and g^i depends on x^i only.

Let $s \geq 1$ and let f: $W^s \to \mathbf{R}$ be a differentiable function. Sometimes, it is useful to divide the formal derivative $d_i f$ of the function f in two terms; by the ith *cut formal derivative* of f, we mean the function $d'_i f$: $W^s \to \mathbf{R}$ defined by

$$d'_i f = \frac{\partial f}{\partial x^i} + \frac{\partial f}{\partial y^\sigma} y_i^\sigma + \frac{\partial f}{\partial y_{j_1}^\sigma} y^\sigma_{j_1 i} + \frac{\partial f}{\partial y^\sigma_{j_1 j_2}} y^\sigma_{j_1 j_2 i} + \cdots + \frac{\partial f}{\partial y^\sigma_{j_1 j_2 \ldots j_{s-1}}} y^\sigma_{j_1 j_2 \ldots j_{s-1} i}. \quad (16)$$

The i-th formal derivative, which is defined on W^{s+1}, is then expressed as

$$d_i f = d'_i f + \frac{\partial f}{\partial y^\sigma_{j_1 j_2 \ldots j_s}} y^\sigma_{j_1 j_2 \ldots j_s i}. \quad (17)$$

The following assertion is a restatement of the definition of a solution of the formal divergence equation (17), Sect. 3.1.

Lemma 2 *Let* $f\colon W^s \to \mathbf{R}$ *and* $g^i\colon W^s \to \mathbf{R}$ *be differentiable functions. The following conditions are equivalent:*

(a) *The functions* g^i *satisfy the formal divergence equation.*
(b) *The functions* g^i *satisfy the system*

$$d'_i g^i = f \tag{18}$$

and

$$\frac{\partial g^i}{\partial y^\sigma_{j_1 j_2 \ldots j_s}} + \frac{\partial g^{j_1}}{\partial y^\sigma_{i j_2 j_3 \ldots j_s}} + \frac{\partial g^{j_2}}{\partial y^\sigma_{j_1 i j_3 j_4 \ldots j_s}} + \cdots + \frac{\partial g^{j_s}}{\partial y^\sigma_{j_1 j_2 \ldots j_{s-1} i}} = 0. \tag{19}$$

Proof Immediate. □

3.2 Integrability of Formal Divergence Equations

We introduce the concepts, responsible for integrability of the formal divergence equation, and prove the integrability theorem.

To any function $f\colon W^s \to \mathbf{R}$, we assign an n-form λ_f and an $(n+1)$-form E_f on W^s, by

$$\lambda_f = f\omega_0, \tag{20}$$

and

$$E_f = E_\sigma(f)\omega^\sigma \wedge \omega_0, \tag{21}$$

where the components $E_\sigma(f)$ are defined by

$$E_\sigma(f) = \frac{\partial f}{\partial y^\sigma} + \sum_{k=1}^{s} (-1)^k d_{p_1} d_{p_2} \ldots d_{p_k} \frac{\partial f}{\partial y^\sigma_{p_1 p_2 \ldots p_k}}. \tag{22}$$

We call λ_f the *Lagrange form*, or the *Lagrangian*, and E_f the *Euler–Lagrange form*, associated with f. The components $E_\sigma(f)$ are called the *Euler–Lagrange expressions*.

In the following lemma, we use the *horizontalization morphism* h and the 1-contact homomorphism p_1, acting on modules of differential forms on the r-jet prolongation $W^r = J^r W$ of the fibered manifold W (see Chap. 2).

Lemma 3 *For any function* $f\colon W^s \to \mathbf{R}$*, there exists an n-form* Θ_f*, defined on* W^{2s-1}*, such that*

(a) $h\Theta_f = \lambda_f$.
(b) *The form* $p_1 d\Theta_f$ *is* ω^σ*-generated.*

Proof We search for Θ_f of the form

$$\Theta_f = f\omega_0 + \left(f^i_\sigma \omega^\sigma + f^{ij_1}_\sigma \omega^\sigma_{j_1} + f^{ij_1j_2}_\sigma \omega^\sigma_{j_1j_2} + \cdots + f^{ij_1j_2\cdots j_{s-1}}_\sigma \omega^\sigma_{j_1j_2\cdots j_{s-1}}\right) \wedge \omega_i, \quad (23)$$

where the coefficients $f^{ij_1j_2\cdots j_k}_\sigma$ are supposed to be *symmetric* in the superscripts $i, j_1, j_2, \ldots, j_k$. Then condition (a) is obviously satisfied. Computing $p_1 d\Theta$, we have

$$\begin{aligned} p_1 d\Theta_f &= df \wedge \omega_0 + \big(hdf^i_\sigma \wedge \omega^\sigma + f^i_\sigma d\omega^\sigma + hdf^{ij_1}_\sigma \wedge \omega^\sigma_{j_1} \\ &\quad + f^{ij_1}_\sigma d\omega^\sigma_{j_1} + hdf^{ij_1j_2}_\sigma \wedge \omega^\sigma_{j_1j_2} + f^{ij_1j_2}_\sigma d\omega^\sigma_{j_1j_2} \\ &\quad + \cdots + hdf^{ij_1j_2\cdots j_{s-1}}_\sigma \wedge \omega^\sigma_{j_1j_2\cdots j_{s-1}} + f^{ij_1j_2\cdots j_{s-1}}_\sigma d\omega^\sigma_{j_1j_2\cdots j_{s-1}}\big) \wedge \omega_i \\ &= \left(\frac{\partial f}{\partial y^\sigma}\omega^\sigma + \frac{\partial f}{\partial y^\sigma_{j_1}}\omega^\sigma_{j_1} + \frac{\partial f}{\partial y^\sigma_{j_1j_2}}\omega^\sigma_{j_1j_2} + \cdots + \frac{\partial f}{\partial y^\sigma_{j_1j_2\cdots j_s}}\omega^\sigma_{j_1j_2\cdots j_s}\right) \wedge \omega_0 \\ &\quad + \big(d_k f^i_\sigma dx^k \wedge \omega^\sigma + d_k f^{ij_1}_\sigma dx^k \wedge \omega^\sigma_{j_1} + d_k f^{ij_1j_2}_\sigma dx^k \wedge \omega^\sigma_{j_1j_2} \\ &\quad + \cdots + d_k f^{ij_1j_2\cdots j_{s-1}}_\sigma dx^k \wedge \omega^\sigma_{j_1j_2\cdots j_{s-1}}\big) \wedge \omega_i \\ &\quad - \big(f^i_\sigma \omega^\sigma_k \wedge dx^k + f^{ij_1}_\sigma \omega^\sigma_{j_1k} \wedge dx^k + f^{ij_1j_2}_\sigma \omega^\sigma_{j_1j_2k} \wedge dx^k \\ &\quad + \cdots + f^i_\sigma \omega^\sigma_{j_1j_2\cdots j_{s-1}k} \wedge dx^k\big) \wedge \omega_i. \end{aligned} \quad (24)$$

This expression can also be written as

$$\begin{aligned} p_1 d\Theta_f &= \left(\frac{\partial f}{\partial y^\sigma} - d_i f^i_\sigma\right)\omega^\sigma \wedge \omega_0 + \left(\frac{\partial f}{\partial y^\sigma_{j_1}} - d_i f^{ij_1}_\sigma - f^{j_1}_\sigma\right)\omega^\sigma_{j_1} \wedge \omega_0 \\ &\quad + \left(\frac{\partial f}{\partial y^\sigma_{j_1j_2}} - d_i f^{ij_1j_2}_\sigma - f^{j_2j_1}{}_\sigma\right)\omega^\sigma_{j_1j_2} \wedge \omega_0 \\ &\quad + \cdots + \left(\frac{\partial f}{\partial y^\sigma_{j_1j_2\cdots j_{s-1}}} - d_i f^{ij_1j_2\cdots j_{s-1}}_\sigma - f^{j_{r-1}j_1j_2\cdots j_{s-2}}{}_\sigma\right)\omega^\sigma_{j_1j_2\cdots j_{s-1}} \wedge \omega_0 \\ &\quad + \left(\frac{\partial f}{\partial y^\sigma_{j_1j_2\cdots j_s}} - f^{j_rj_1j_2\cdots j_{s-1}}_\sigma\right)\omega^\sigma_{j_1j_2\cdots j_{s-1}j_s} \wedge \omega_0. \end{aligned} \quad (25)$$

But we can choose f^i_σ, $f^{ij_1}_\sigma$, $f^{ij_1j_2}_\sigma$, $\ldots f^{ij_1j_2\cdots j_{s-1}}_\sigma$ from the conditions

$$\begin{aligned}
f_\sigma^{j_s j_1 j_2 \ldots j_{s-1}} &= \frac{\partial f}{\partial y^\sigma_{j_1 j_2 \ldots j_s}},\\
f_\sigma^{j_{r-1} j_1 j_2 \ldots j_{s-2}} &= \frac{\partial f}{\partial y^\sigma_{j_1 j_2 \ldots j_{s-1}}} - d_i f_\sigma^{i j_1 j_2 \ldots j_{s-1}} = \frac{\partial f}{\partial y^\sigma_{j_1 j_2 \ldots j_{s-1}}} - d_{i_1} \frac{\partial f}{\partial y^\sigma_{j_1 j_2 \ldots j_{s-1} i_1}},\\
&\ldots\\
f_\sigma^{j_2 j_1} &= \frac{\partial f}{\partial y^\sigma_{j_1 j_2}} - d_i f_\sigma^{i j_1 j_2} = \frac{\partial f}{\partial y^\sigma_{j_1 j_2}} - d_{i_1} \frac{\partial f}{\partial y^\sigma_{j_1 j_2 i_1}} + d_{i_1} d_{i_2} \frac{\partial f}{\partial y^\sigma_{j_1 j_2 i_1 i_2}}\\
&\quad - \cdots + (-1)^{r-2} d_{i_1} d_{i_2} \ldots d_{i_{s-2}} \frac{\partial f}{\partial y^\sigma_{j_1 j_2 i_1 i_2 \ldots i_{s-2}}},\\
f_\sigma^{j_1} &= \frac{\partial f}{\partial y^\sigma_{j_1}} - d_i f_\sigma^{i j_1} = \frac{\partial f}{\partial y^\sigma_{j_1}} - d_{i_1} \frac{\partial f}{\partial y^\sigma_{j_1 i_1}} + d_{i_1} d_{i_2} \frac{\partial f}{\partial y^\sigma_{j_1 i_1 i_2}}\\
&\quad - \cdots + (-1)^{s-1} d_{i_1} d_{i_2} \ldots d_{i_{s-1}} \frac{\partial f}{\partial y^\sigma_{j_1 i_1 i_2 \ldots i_{s-1}}},
\end{aligned} \tag{26}$$

and for this choice, the form $p_1 d\Theta$ is ω^σ-generated, proving (b).

Using formulas (23) and (26), we see that the form $\Theta = \Theta_f$, constructed in the proof, has the expression

$$\Theta_f = f\omega_0 + \sum_{k=0}^{s} \left(\sum_{l=0}^{s-k-1} (-1)^l d_{p_1} d_{p_2} \ldots d_{p_l} \frac{\partial f}{\partial y^\sigma_{j_1 j_2 \ldots j_k p_1 p_2 \ldots p_l i}} \right) \omega^\sigma_{j_1 j_2 \ldots j_k} \wedge \omega_i. \tag{27}$$

This form obeys properties (a) and (b) of Lemma 3. We call Θ_f the *principal Lepage equivalent* of the function f or of the Lagrange form λ_f. Computing $p_1 d\Theta_f$, we get the *Euler–Lagrange form*, associated with f,

$$p_1 d\Theta_f = E_f. \tag{28}$$

□

Now we are in a position to study integrability of the formal divergence equation; the proof includes the construction of the solutions.

Theorem 1 *Let* $f\colon W^s \to \mathbf{R}$ *be a function. The following two conditions are equivalent*:

(a) *The formal divergence equation* $d_i g^i = f$ *has a solution defined on the set* W^s.
(b) *The Euler–Lagrange form, associated with* f, *vanishes,*

$$E_f = 0. \tag{29}$$

Proof

1. Suppose that condition (a) is satisfied and the formal divergence equation has a solution $g = g^i$, defined on W^s. Differentiating the function $d_i g^i$, we get the formulas

$$\frac{\partial d_i g^i}{\partial y^\sigma} = d_i \frac{\partial g^i}{\partial y^\sigma}, \tag{30}$$

and for every $k = 1, 2, \ldots, s$,

$$\begin{aligned} \frac{\partial d_i g^i}{\partial y^\sigma_{i_1 i_2 \ldots i_k}} &= d_i \frac{\partial g^i}{\partial y^\sigma_{i_1 i_2 \ldots i_k}} \\ &\quad + \frac{1}{k}\left(\frac{\partial g^{i_1}}{\partial y^\sigma_{i_2 i_3 \ldots i_k}} + \frac{\partial g^{i_2}}{\partial y^\sigma_{i_1 i_3 \ldots i_k}} + \frac{\partial g^{i_3}}{\partial y^\sigma_{i_2 i_1 i_4 \ldots i_k}} + \cdots + \frac{\partial g^{i_k}}{\partial y^\sigma_{i_2 i_3 \ldots i_{k-1}}}\right). \end{aligned} \tag{31}$$

Using these formulas, we can compute the Euler–Lagrange expressions $E_\sigma(f) = E_\sigma(d_i g^i)$ in several steps. First, we have

$$\begin{aligned} E_\sigma(d_i g^i) &= d_{i_1}\left(\frac{\partial g^{i_1}}{\partial y^\sigma} - \frac{\partial d_i g^i}{\partial y^\sigma_{i_1}} + d_{i_2}\frac{\partial d_i g^i}{\partial y^\sigma_{i_1 i_2}} - \cdots + (-1)^s d_{i_2} d_{i_3} \ldots d_{i_s} \frac{\partial d_i g^i}{\partial y^\sigma_{i_1 i_2 \ldots i_s}}\right) \\ &= d_{i_1} d_{i_2}\left(-\frac{\partial g^{i_2}}{\partial y^\sigma_{i_1}} + \frac{\partial d_i g^i}{\partial y^\sigma_{i_1 i_2}} - d_{i_3}\frac{\partial d_i g^i}{\partial y^\sigma_{i_1 i_2 i_3}} + \cdots + (-1)^s d_{i_3} d_{i_4} \ldots d_{i_s} \frac{\partial d_i g^i}{\partial y^\sigma_{i_1 i_2 \ldots i_s}}\right). \end{aligned} \tag{32}$$

Second, using symmetrization,

$$\begin{aligned} E_\sigma(d_i g^i) &= d_{i_1} d_{i_2}\left(-\frac{\partial g^{i_2}}{\partial y^\sigma_{i_1}} + d_i \frac{\partial g^i}{\partial y^\sigma_{i_1 i_2}} + \frac{1}{2}\left(\frac{\partial g^{i_1}}{\partial y^\sigma_{i_2}} + \frac{\partial g^{i_2}}{\partial y^\sigma_{i_1}}\right)\right. \\ &\quad \left. - d_{i_3}\frac{\partial d_s g^s}{\partial y^\sigma_{i_1 i_2 i_3}} + \cdots + (-1)^r d_{i_3} d_{i_4} \ldots d_{i_r} \frac{\partial d_s g^s}{\partial y^\sigma_{i_1 i_2 \ldots i_r}}\right) \\ &= d_{i_1} d_{i_2} d_{i_3}\left(\frac{\partial g^{i_3}}{\partial y^\sigma_{i_1 i_2}} - \frac{\partial d_s g^s}{\partial y^\sigma_{i_1 i_2 i_3}} + \cdots + (-1)^r d_{i_4} d_{i_5} \ldots d_{i_r} \frac{\partial d_s g^s}{\partial y^\sigma_{i_1 i_2 \ldots i_s}}\right). \end{aligned} \tag{33}$$

We continue this process and obtain after $s - 1$ steps

$$E_\sigma(d_i g^i) = (-1)^s d_{i_1} d_{i_2} \ldots d_{i_{s-1}} d_{i_s} d_i \frac{\partial g^i}{\partial y^\sigma_{i_1 i_2 \ldots i_s}}. \tag{34}$$

But since f is defined on W^s, the solution g^i necessarily satisfies

$$\frac{\partial g^{i_1}}{\partial y^\sigma_{i_2 i_3 \ldots i_{s+1}}} + \frac{\partial g^{i_2}}{\partial y^\sigma_{i_1 i_3 i_4 \ldots i_{s+1}}} + \frac{\partial g^{i_3}}{\partial y^\sigma_{i_2 i_1 i_4 i_5 \ldots i_{s+1}}} + \cdots + \frac{\partial g^{i_{s+1}}}{\partial y^\sigma_{i_2 i_3 \ldots i_{s-1} i_r i_1}} = 0, \qquad (35)$$

proving that $E_\sigma(d_i g^i) = 0$.

2. Suppose that $E_\sigma(f) = 0$. We want to show that there exist functions $g^i\colon V^s \to \mathbf{R}$ such that $f = d_i g^i$. Let I be the fibered homotopy operator for differential forms on V^{2s}, associated with the projection $\pi^{2s}\colon V \to U$ (Sect. 2.7). We have

$$\Theta_f = Id\Theta_f + dI\Theta_f + \Theta_0 = Ip_1 d\Theta_f + Ip_2 d\Theta_f + dI\Theta_f + \Theta_0, \qquad (36)$$

where Θ_0 is an n-form, projectable on U. In this formula, $p_1 d\Theta_f = 0$ by hypothesis, $Ip_2 d\Theta_f$ is 1-contact, and since $d\Theta_0 = 0$ identically, we have $\Theta_0 = d\vartheta_0$ for some ϑ_0 (on U). Moreover $h\Theta_f = hd(I\Theta_f + \vartheta_0) = f\omega_0$. Defining functions g^i on V^{2s} by the condition

$$h(I\Theta_f + \vartheta_0) = g^i \omega_i, \qquad (37)$$

we see we have constructed a solution of the formal divergence equation. Indeed, from (35), $hd(I\Theta_f + \vartheta_0) = hdh(I\Theta_f + \vartheta_0) = d_i g^i \cdot \omega_0 = f\omega_0$. Then, however, we may choose g^i to be defined on W^s as required (Sect. 3.1, Lemma 1). □

If the formal divergence equation has a solution, then this solution is unique, up to a system of functions $g^i = g^i(x^j)$, such that $(\partial g^i/\partial x^i) = 0$.

Remark 3 If a formal divergence equation $d_i g^i = f$ has a solution g^i, defined on the set W^s, then any other solution is given as $g^i + h^i$, where h^i are functions on U such that $\partial h^i/\partial x^i = 0$ (see Sect. 3.1, Remark 2).

Condition $E_\lambda = 0$ (28) is called the *integrability condition* for the formal divergence equation. In terms of differential equations, this condition can equivalently be written as

$$E_\sigma(f) = 0. \qquad (38)$$

3.3 Projectable Extensions of Differential Forms

Denote

$$\omega_0 = dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n, \qquad (39)$$

and $\omega_i = i_{\partial/\partial x^i}\omega_0$, that is,

$$\omega_i = \frac{1}{(n-1)!}\varepsilon_{ij_2j_3\ldots j_n} dx^{j_2} \wedge dx^{j_3} \wedge \cdots \wedge dx^{j_n}. \tag{40}$$

Consider a π^s-horizontal $(n-1)$-form η on W^s, expressed as

$$\eta = g^i \omega_i = \frac{1}{(n-1)!} h_{j_2j_3\ldots j_n} dx^{j_2} \wedge dx^{j_3} \wedge \cdots \wedge dx^{j_n}. \tag{41}$$

Note that from expression (40), the components of the form η satisfy the transformation formulas

$$h_{j_2j_3\ldots j_n} = \varepsilon_{ij_2j_3\ldots j_n} g^i, \quad g^k = \frac{1}{(n-1)!}\varepsilon^{kj_2j_3\ldots j_n} h_{j_2j_3\ldots j_n}. \tag{42}$$

In the following lemma, we derive a formula for the derivatives of the functions $h_{j_2j_3\ldots j_n}$ and g^k; to this purpose, a straightforward calculation is needed. Denote by Alt and Sym the *alternation* and *symmetrization* in the corresponding indices.

Lemma 4 *The functions g^i and $h_{j_1j_2\ldots j_{n-1}}$ satisfy*

$$\begin{aligned} &\frac{1}{r+1}\varepsilon_{il_2l_3\ldots l_n}\left(\frac{\partial g^i}{\partial y^\sigma_{k_1k_2\ldots k_s}} + \frac{\partial g^{k_1}}{\partial y^\sigma_{ik_2k_3\ldots k_s}} + \frac{\partial g^{k_2}}{\partial y^\sigma_{k_1ik_3k_4\ldots k_s}} + \cdots + \frac{\partial g^{k_s}}{\partial y^\sigma_{k_1k_2\ldots k_{s-1}i}}\right) \\ &\quad = \frac{\partial h_{l_2l_3\ldots l_n}}{\partial y^\sigma_{k_1k_2\ldots k_s}} - \frac{s(n-1)}{s+1}\frac{\partial h_{il_3l_4\ldots l_n}}{\partial y^\sigma_{ik_2k_3\ldots k_s}}\delta^{k_1}_{l_2} \quad \mathrm{Sym}(k_1k_2\ldots k_s) \quad \mathrm{Alt}(l_2l_3\ldots l_n). \end{aligned} \tag{43}$$

Proof Formula (43) is an immediate consequence of equation (42). Differentiating we get

$$\frac{\partial g^i}{\partial y^\sigma_{k_1k_2\ldots k_s}} = \frac{1}{(n-1)!}\varepsilon^{ij_2j_3\ldots j_n}\frac{\partial h_{j_2j_3\ldots j_n}}{\partial y^\sigma_{k_1k_2\ldots k_s}}, \tag{44}$$

hence

$$\begin{aligned} &\frac{1}{s+1}\varepsilon_{il_2l_3\ldots l_n}\left(\frac{\partial g^i}{\partial y^\sigma_{k_1k_2\ldots k_s}} + \frac{\partial g^{k_1}}{\partial y^\sigma_{ik_2k_3\ldots k_s}} + \frac{\partial g^{k_2}}{\partial y^\sigma_{k_1ik_3k_4\ldots k_s}} + \cdots + \frac{\partial g^{k_s}}{\partial y^\sigma_{k_1k_2\ldots k_{s-1}i}}\right) \\ &\quad = \frac{1}{s+1}\frac{1}{(n-1)!}\varepsilon_{il_2l_3\ldots l_n}\varepsilon^{ij_2j_3\ldots j_n}\frac{\partial h_{j_2j_3\ldots j_n}}{\partial y^\sigma_{k_1k_2\ldots k_s}} \\ &\quad\quad + \frac{1}{s+1}\frac{1}{(n-1)!}\varepsilon_{il_2l_3\ldots l_n}\varepsilon^{k_1j_2j_3\ldots j_n}\frac{\partial h_{j_2j_3\ldots j_n}}{\partial y^\sigma_{ik_2k_3\ldots k_s}} \end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{s+1}\frac{1}{(n-1)!}\varepsilon_{il_2l_3\ldots l_n}\varepsilon^{k_2j_2j_3\ldots j_n}\frac{\partial h_{j_2j_3\ldots j_n}}{\partial y^\sigma_{k_1ik_3k_4\ldots k_s}}\\
&+\cdots+\frac{1}{s+1}\frac{1}{(n-1)!}\varepsilon_{il_2l_3\ldots l_n}\varepsilon^{k_sj_2j_3\ldots j_n}\frac{\partial h_{j_2j_3\ldots j_n}}{\partial y^\sigma_{k_1k_2\ldots k_{s-1}i}}\\
&=\frac{1}{s+1}\frac{\partial h_{l_2l_3\ldots l_n}}{\partial y^\sigma_{k_1k_2\ldots k_s}}+\frac{1}{s+1}\frac{n!}{(n-1)!}\left(\delta^{k_1}_i\delta^{j_2}_{l_2}\delta^{j_3}_{l_3}\ldots\delta^{j_n}_{l_n}\frac{\partial h_{j_2j_3\ldots j_n}}{\partial y^\sigma_{ik_2k_3\ldots k_s}}\right.\\
&\left.+\delta^{k_2}_i\delta^{j_2}_{l_2}\delta^{j_3}_{l_3}\ldots\delta^{j_n}_{l_n}\frac{\partial h_{j_2j_3\ldots j_n}}{\partial y^\sigma_{k_1ik_3k_4\ldots k_s}}+\cdots+\delta^{k_s}_i\delta^{j_2}_{l_2}\delta^{j_3}_{l_3}\ldots\delta^{j_n}_{l_n}\frac{\partial h_{j_2j_3\ldots j_n}}{\partial y^\sigma_{k_1k_2\ldots k_{s-1}i}}\right)\\
&\quad\mathrm{Alt}(il_2l_3\ldots l_n).
\end{aligned}
\tag{45}
$$

We calculate the alternations $\mathrm{Alt}(il_2l_3\ldots l_n)$ of the summands in the parentheses in two steps. Consider the first summand. Alternating in the indices $(l_2l_3\ldots l_n)$ and then in $(il_2l_3\ldots l_n)$, we get

$$
\begin{aligned}
&\delta^{k_1}_i\delta^{j_2}_{l_2}\delta^{j_3}_{l_3}\ldots\delta^{j_n}_{l_n}\frac{\partial h_{j_2j_3\ldots j_n}}{\partial y^\sigma_{ik_2k_3\ldots k_s}}\quad\mathrm{Alt}(il_2l_3\ldots l_n)\\
&=\frac{1}{n}\left(\delta^{k_1}_i\frac{\partial h_{l_2l_3\ldots l_n}}{\partial y^\sigma_{ik_2k_3\ldots k_s}}-\delta^{k_1}_{l_2}\frac{\partial h_{il_3l_4\ldots l_n}}{\partial y^\sigma_{ik_2k_3\ldots k_s}}-\delta^{k_1}_{l_3}\frac{\partial h_{l_2il_4l_5\ldots l_n}}{\partial y^\sigma_{ik_2k_3\ldots k_s}}-\cdots-\delta^{k_1}_{l_n}\frac{\partial h_{l_2l_3\ldots l_{n-1}i}}{\partial y^\sigma_{ik_2k_3\ldots k_s}}\right)\\
&=\frac{1}{n}\left(\frac{\partial h_{l_2l_3\ldots l_n}}{\partial y^\sigma_{k_1k_2k_3\ldots k_s}}-\delta^{k_1}_{l_2}\frac{\partial h_{il_3l_4\ldots l_n}}{\partial y^\sigma_{ik_2k_3\ldots k_s}}-\delta^{k_1}_{l_3}\frac{\partial h_{l_2il_4l_5\ldots l_n}}{\partial y^\sigma_{ik_2k_3\ldots k_s}}-\cdots-\delta^{k_1}_{l_n}\frac{\partial h_{l_2l_3\ldots l_{n-1}i}}{\partial y^\sigma_{ik_2k_3\ldots k_s}}\right),
\end{aligned}
\tag{46}
$$

and similarly for the remaining terms. Altogether

$$
\begin{aligned}
&\frac{1}{s+1}\varepsilon_{il_2l_3\ldots l_n}\left(\frac{\partial g^i}{\partial y^\sigma_{k_1k_2\ldots k_s}}+\frac{\partial g^{k_1}}{\partial y^\sigma_{ik_2k_3\ldots k_s}}+\frac{\partial g^{k_2}}{\partial y^\sigma_{k_1ik_3k_4\ldots k_s}}+\cdots+\frac{\partial g^{k_s}}{\partial y^\sigma_{k_1k_2\ldots k_{s-1}i}}\right)\\
&=\frac{1}{s+1}\left(\frac{\partial h_{l_2l_3\ldots l_n}}{\partial y^\sigma_{k_1k_2\ldots k_s}}+\frac{\partial h_{l_2l_3\ldots l_n}}{\partial y^\sigma_{k_1k_2k_3\ldots k_s}}-\delta^{k_1}_{l_2}\frac{\partial h_{il_3l_4\ldots l_n}}{\partial y^\sigma_{ik_2k_3\ldots k_s}}-\delta^{k_1}_{l_3}\frac{\partial h_{l_2il_4l_5\ldots l_n}}{\partial y^\sigma_{ik_2k_3\ldots k_s}}\right.\\
&-\cdots-\delta^{k_1}_{l_n}\frac{\partial h_{l_2l_3\ldots l_{n-1}i}}{\partial y^\sigma_{ik_2k_3\ldots k_s}}+\frac{\partial h_{l_2l_3\ldots l_n}}{\partial y^\sigma_{k_1k_2k_3\ldots k_s}}-\delta^{k_1}_{l_2}\frac{\partial h_{il_3l_4\ldots l_n}}{\partial y^\sigma_{ik_2k_3\ldots k_s}}-\delta^{k_1}_{l_3}\frac{\partial h_{l_2il_4l_5\ldots l_n}}{\partial y^\sigma_{ik_2k_3\ldots k_s}}\\
&-\cdots-\delta^{k_1}_{l_n}\frac{\partial h_{l_2l_3\ldots l_{n-1}i}}{\partial y^\sigma_{ik_2k_3\ldots k_s}}+\frac{\partial h_{l_2l_3\ldots l_n}}{\partial y^\sigma_{k_1k_2k_3\ldots k_s}}-\delta^{k_2}_{l_2}\frac{\partial h_{il_3l_4\ldots l_n}}{\partial y^\sigma_{ik_1k_3\ldots k_s}}-\delta^{k_2}_{l_3}\frac{\partial h_{l_2il_4l_5\ldots l_n}}{\partial y^\sigma_{ik_1k_3\ldots k_s}}\\
&-\cdots-\delta^{k_2}_{l_n}\frac{\partial h_{l_2l_3\ldots l_{n-1}i}}{\partial y^\sigma_{ik_1k_3\ldots k_s}}+\cdots+\frac{\partial h_{l_2l_3\ldots l_n}}{\partial y^\sigma_{k_1k_2k_3\ldots k_s}}\\
&\left.-\delta^{k_s}_{l_2}\frac{\partial h_{il_3l_4\ldots l_n}}{\partial y^\sigma_{ik_2k_3\ldots k_{s-1}k_1}}-\delta^{k_s}_{l_3}\frac{\partial h_{l_2il_4l_5\ldots l_n}}{\partial y^\sigma_{ik_2k_3\ldots k_{s-1}k_1}}-\cdots-\delta^{k_s}_{l_n}\frac{\partial h_{l_2l_3\ldots l_{n-1}i}}{\partial y^\sigma_{ik_2k_3\ldots k_{s-1}k_1}}\right)
\end{aligned}
$$

$$= \frac{\partial h_{l_2 l_3 \ldots l_n}}{\partial y^{\sigma}_{k_1 k_2 \ldots k_s}} - \frac{1}{s+1}\left(\delta^{k_1}_{l_2}\frac{\partial h_{i l_3 l_4 \ldots l_n}}{\partial y^{\sigma}_{i k_2 k_3 \ldots k_s}} + \delta^{k_1}_{l_3}\frac{\partial h_{l_2 i l_4 l_5 \ldots l_n}}{\partial y^{\sigma}_{i k_2 k_3 \ldots k_s}} + \cdots + \delta^{k_1}_{l_n}\frac{\partial h_{l_2 l_3 \ldots l_{n-1} i}}{\partial y^{\sigma}_{i k_2 k_3 \ldots k_s}}\right.$$
$$+ \delta^{k_2}_{l_2}\frac{\partial h_{i l_3 l_4 \ldots l_n}}{\partial y^{\sigma}_{i k_1 k_3 \ldots k_s}} + \delta^{k_2}_{l_3}\frac{\partial h_{l_2 i l_4 l_5 \ldots l_n}}{\partial y^{\sigma}_{i k_1 k_3 \ldots k_s}} + \cdots + \delta^{k_2}_{l_n}\frac{\partial h_{l_2 l_3 \ldots l_{n-1} i}}{\partial y^{\sigma}_{i k_1 k_3 \ldots k_s}} \tag{47}$$
$$\left. + \cdots + \delta^{k_s}_{l_2}\frac{\partial h_{i l_3 l_4 \ldots l_n}}{\partial y^{\sigma}_{i k_2 k_3 \ldots k_{s-1} k_1}} + \delta^{k_s}_{l_3}\frac{\partial h_{l_2 i l_4 l_5 \ldots l_n}}{\partial y^{\sigma}_{i k_2 k_3 \ldots k_{s-1} k_1}} + \cdots + \delta^{k_s}_{l_n}\frac{\partial h_{l_2 l_3 \ldots l_{n-1} i}}{\partial y^{\sigma}_{i k_2 k_3 \ldots k_{s-1} k_1}}\right),$$

and, with the help of alternations and symmetrizations,

$$\frac{1}{s+1}\varepsilon_{i l_2 l_3 \ldots l_n}\left(\frac{\partial g^{i}}{\partial y^{\sigma}_{k_1 k_2 \ldots k_s}} + \frac{\partial g^{k_1}}{\partial y^{\sigma}_{i k_2 k_3 \ldots k_s}} + \frac{\partial g^{k_2}}{\partial y^{\sigma}_{k_1 i k_3 k_4 \ldots k_s}} + \cdots + \frac{\partial g^{k_s}}{\partial y^{\sigma}_{k_1 k_2 \ldots k_{s-1} i}}\right)$$
$$= \frac{\partial h_{l_2 l_3 \ldots l_n}}{\partial y^{\sigma}_{k_1 k_2 \ldots k_s}} - \frac{n-1}{s+1}\frac{1}{n-1}\left(\delta^{k_1}_{l_2}\frac{\partial h_{i l_3 l_4 \ldots l_n}}{{}^{\sigma}_{i k_2 k_3 \ldots k_s}} + \delta^{k_1}_{l_3}\frac{\partial h_{l_2 i l_4 l_5 \ldots l_n}}{\partial y^{\sigma}_{i k_2 k_3 \ldots k_s}} + \cdots + \delta^{k_1}_{l_n}\frac{\partial h_{l_2 l_3 \ldots l_{n-1} i}}{\partial y^{\sigma}_{i k_2 k_3 \ldots k_s}}\right.$$
$$+ \delta^{k_2}_{l_2}\frac{\partial h_{i l_3 l_4 \ldots l_n}}{{}^{\sigma}_{i k_1 k_3 \ldots k_s}} + \delta^{k_2}_{l_3}\frac{\partial h_{l_2 i l_4 l_5 \ldots l_n}}{\partial y^{\sigma}_{i k_1 k_3 \ldots k_s}} + \cdots + \delta^{k_2}_{l_n}\frac{{}_{l_2 l_3 \ldots l_{n-1} i}}{\partial y^{\sigma}_{i k_1 k_3 \ldots k_s}}$$
$$\left. + \cdots + \delta^{k_s}_{l_2}\frac{\partial h_{i l_3 l_4 n}}{\partial y^{\sigma}_{i k_2 k_3 \ldots k_{s-1} k_1}} + \delta^{k_s}_{l_3}\frac{{}_{l_2 i l_4 l_5 \ldots l_n}}{\partial y^{\sigma}_{i k_2 k_3 \ldots k_{s-1} k_1}} + \cdots + \delta^{k_s}_{l_n}\frac{\partial h_{l_2 l_3 n-1 i}}{\partial y^{\sigma}_{i k_2 k_3 \ldots k_{s-1} k_1}}\right)$$
$$= \frac{{}_{l_2 l_3 \ldots l_n}}{\partial y^{\sigma}_{k_1 k_2 s}} - \frac{n-1}{s+1}\delta^{k_1}_{l_2}\frac{\partial h_{i l_3 l_4 \ldots l_n}}{{}^{\sigma}_{i k_2 k_3 \ldots k_s}} - \frac{n-1}{s+1}\delta^{k_2}_{l_2}\frac{\partial h_{i l_3 l_4 \ldots l_n}}{\partial y^{\sigma}_{i k_1 k_3 \ldots k_s}} - \cdots - \frac{n-1}{s+1}\delta^{k_s}_{l_2}\frac{\partial h_{i l_3 l_4 \ldots l_n}}{\partial y^{\sigma}_{i k_2 k_3 \ldots k_{s-1} k_1}}$$
$$\mathrm{Alt}(l_2 l_3 \ldots l_n)$$
$$= \frac{\partial h_{l_2 l_3 \ldots l_n}}{{}^{\sigma}_{k_1 k_2 \ldots k_s}} - \frac{s(n-1)}{s+1}\delta^{k_1}_{l_2}\frac{\partial h_{i l_3 l_4 \ldots l_n}}{\partial y^{\sigma}_{i k_2 k_3 \ldots k_s}}\mathrm{Alt}(l_2 l_3 \ldots l_n)\,\mathrm{Sym}(k_1 k_2 \ldots k_s). \tag{48}$$

□

Let η be a π^s-horizontal form η, defined on W^s. A form μ on W^{s-1} is said to be a $\pi^{s,s-1}$-*projectable extension* of η, if η is equal to the horizontal components of μ,

$$\eta = h\mu. \tag{49}$$

Our objective now will be to find conditions for η ensuring that μ does exist. Let η be expressed in two bases of $(n-1)$-forms by formula (41).

Theorem 2 *The following two conditions are equivalent*:

(a) *η has a $\pi^{s,s-1}$-projectable extension.*
(b) *The components g^i satisfy*

$$\frac{\partial g^{i}}{\partial y^{\sigma}_{j_1 j_2 \ldots j_s}} + \frac{\partial g^{j_1}}{\partial y^{\sigma}_{i j_2 j_3 \ldots j_s}} + \frac{\partial g^{j_2}}{\partial y^{\sigma}_{j_1 i j_3 j_4 \ldots j_s}} + \cdots + \frac{\partial g^{j_s}}{\partial y^{\sigma}_{j_1 j_2 \ldots j_{s-1} i}} = 0. \tag{50}$$

(c) *The components* $h_{i_1 i_2 \ldots i_{n-1}}$ *satisfy*

$$\frac{\partial h_{l_2 l_3 \ldots l_n}}{\partial y^\sigma_{k_1 k_2 \ldots k_s}} - \frac{r(n-1)}{r+1} \frac{\partial h_{i l_3 l_4 \ldots l_n}}{\partial y^\sigma_{i k_2 k_3 \ldots k_s}} \delta^{k_1}_{l_2} = 0 \quad \mathrm{Sym}(k_1 k_2 \ldots k_s)\ \mathrm{Alt}(l_2 l_3 \ldots l_n). \quad (51)$$

Proof

1. To show that (a) implies (b), suppose that we have an $(n-1)$-form μ, defined on W^{s-1}, such that $\eta = h\mu$. Then $hd\eta = d_i g^i \cdot \omega_0$, which is a form on W^{s+1}. But $(\pi^{s,s-1})^* d\mu = d(\pi^{s,s-1})^* \mu$ hence $hd\eta = hdh\mu = hd\mu$, so $hd\eta$ is $\pi^{s+1,s}$-projectable (with projection $hd\mu$). But

$$\begin{aligned} hd\eta &= d_i g^i \cdot \omega_0 \\ &= \left(\frac{\partial g^i}{\partial x^i} + \frac{\partial g^i}{\partial y^\sigma} y^\sigma_i + \frac{\partial g^i}{\partial y^\sigma_{j_1}} y^\sigma_{j_1 i} + \frac{\partial g^i}{\partial y^\sigma_{j_1 j_2}} y^\sigma_{j_1 j_2 i} + \cdots + \frac{\partial g^i}{\partial y^\sigma_{j_1 j_2 \ldots j_s}} y^\sigma_{j_1 j_2 \ldots j_s i} \right) \omega_0, \end{aligned} \quad (52)$$

so $\pi^{s+1,s}$-projectability implies (50).

2. (c) follows from (b) by Lemma 4.
3. Now we prove that condition (c) implies (a). Write η as in (3),

$$\eta = \frac{1}{(n-1)!} h_{j_2 j_3 \ldots j_n} dx^{j_2} \wedge dx^{j_3} \wedge \cdots \wedge dx^{j_n}. \quad (53)$$

By Lemma 1, Sect. 3.1, and formula (42), the functions $h_{j_2 j_3 \ldots j_n}$ are polynomial in the variables y^σ_{Jj}, where J is a multi-index of length $s-1$. Thus,

$$\begin{aligned} h_{i_1 i_2 \ldots i_{n-1}} &= B_{i_1 i_2 \ldots i_{n-1}} + B^{J_1 k_1}_{\sigma_1}{}_{i_1 i_2 \ldots i_{n-1}} y^{\sigma_1}_{J_1 k_1} + B^{J_1 k_1}_{\sigma_1}{}^{J_2 k_2}_{\sigma_2}{}_{i_1 i_2 \ldots i_{n-1}} y^{\sigma_1}_{J_1 k_1} y^{\sigma_2}_{J_2 k_2} \\ &\quad + \cdots + B^{J_1 k_1}_{\sigma_1}{}^{J_2 k_2}_{\sigma_2} \cdots {}^{J_{n-2} k_{n-2}}_{\sigma_{n-2}}{}_{i_1 i_2 \ldots i_{n-1}} y^{\sigma_1}_{J_1 k_1} y^{\sigma_2}_{J_2 k_2} \cdots y^{\sigma_{n-2}}_{J_{n-2} k_{n-2}} \\ &\quad + B^{J_1 k_1}_{\sigma_1}{}^{J_2 k_2}_{\sigma_2} \cdots {}^{J_{n-2} k_{n-2}}_{\sigma_{n-2}}{}^{J_{n-1} k_{n-1}}_{\sigma_{n-1}}{}_{i_1 i_2 \ldots i_{n-1}} y^{\sigma_1}_{J_1 k_1} y^{\sigma_2}_{J_2 k_2} \cdots y^{\sigma_{n-2}}_{J_{n-2} k_{n-2}} y^{\sigma_{n-1}}_{J_{n-1} k_{n-1}}. \end{aligned} \quad (54)$$

The coefficients in this expression are supposed to be symmetric in the multi-indices ${}^{Jk}_{\sigma}$, ${}^{Lj}_{\nu}$. By hypothesis, the polynomials (54) satisfy condition (51)

$$\frac{\partial h_{i_2 i_3 \ldots i_n}}{\partial y^\sigma_{Jk}} - \frac{r(n-1)}{r+1} \frac{\partial h_{l i_3 i_4 \ldots i_n}}{\partial y^\sigma_{Jl}} \delta^k_{i_2} = 0 \quad \mathrm{Sym}(Jk)\ \mathrm{Alt}(i_2 i_3 \ldots i_n), \quad (55)$$

which reduces to some conditions for the coefficients. To find these conditions, we compute

$$\frac{\partial h_{i_1 i_2 \ldots i_{n-1}}}{\partial y^{\sigma}_{Jk}} = B_{\sigma}^{Jk}{}_{i_1 i_2 \ldots i_{n-1}} + 2B_{\sigma}^{Jk}{}_{\sigma_2}^{J_2 k_2}{}_{i_1 i_2 \ldots i_{n-1}} y^{\sigma_2}_{J_2 k_2}$$
$$+ \cdots + (n-2) B_{\sigma}^{Jk}{}_{\sigma_2}^{J_2 k_2} \ldots {}_{\sigma_{n-2}}^{J_{n-2} k_{n-2}}{}_{i_1 i_2 \ldots i_{n-1}} y^{\sigma_2}_{J_2 k_2} y^{\sigma_3}_{J_3 k_3} \ldots y^{\sigma_{n-2}}_{J_{n-2} k_{n-2}}$$
$$+ (n-1) B_{\sigma}^{Jk}{}_{\sigma_2}^{J_2 k_2} \ldots {}_{\sigma_{n-2}}^{J_{n-2} k_{n-2}}{}_{\sigma_{n-1}}^{J_{n-1} k_{n-1}}{}_{i_1 i_2 \ldots i_{n-1}} y^{\sigma_2}_{J_2 k_2} y^{\sigma_3}_{J_3 k_3} \ldots y^{\sigma_{n-2}}_{J_{n-2} k_{n-2}} y^{\sigma_{n-1}}_{J_{n-1} k_{n-1}}, \tag{56}$$

and

$$\frac{\partial h_{l i_2 i_3 \ldots i_{n-1}}}{\partial y^{\sigma}_{Jl}} = B_{\sigma}^{Jl}{}_{l i_2 i_3 \ldots i_{n-1}} + 2B_{\sigma}^{Jl}{}_{\sigma_2}^{J_2 k_2}{}_{l i_2 i_3 \ldots i_{n-1}} y^{\sigma_2}_{J_2 k_2}$$
$$+ \cdots + (n-2) B_{\sigma}^{Jl}{}_{\sigma_2}^{J_2 k_2} \ldots {}_{\sigma_{n-2}}^{J_{n-2} k_{n-2}}{}_{l i_2 i_3 \ldots i_{n-1}} y^{\sigma_2}_{J_2 k_2} y^{\sigma_3}_{J_3 k_3} \ldots y^{\sigma_{n-2}}_{J_{n-2} k_{n-2}}$$
$$+ (n-1) B_{\sigma}^{Jl}{}_{\sigma_2}^{J_2 k_2} \ldots {}_{\sigma_{n-2}}^{J_{n-2} k_{n-2}}{}_{\sigma_{n-1}}^{J_{n-1} k_{n-1}}{}_{l i_2 i_3 \ldots i_{n-1}} y^{\sigma_2}_{J_2 k_2} y^{\sigma_3}_{J_3 k_3} \ldots y^{\sigma_{n-2}}_{J_{n-2} k_{n-2}} y^{\sigma_{n-1}}_{J_{n-1} k_{n-1}}, \tag{57}$$

from which we have, changing the index notation,

$$\frac{\partial h_{l i_3 i_4 \ldots i_n}}{\partial y^{\sigma}_{Jl}} \delta^k_{i_2} = B_{\sigma}^{Jl}{}_{l i_3 i_4 \ldots i_n} \delta^k_{i_2} + 2B_{\sigma}^{Jl}{}_{\sigma_2}^{J_2 k_2}{}_{l i_3 i_4 \ldots i_n} \delta^k_{i_2} y^{\sigma_2}_{J_2 k_2}$$
$$+ \cdots + (n-2) B_{\sigma}^{Jl}{}_{\sigma_2}^{J_2 k_2} \ldots {}_{\sigma_{n-2}}^{J_{n-2} k_{n-2}}{}_{l i_3 i_4 \ldots i_n} \delta^k_{i_2} y^{\sigma_2}_{J_2 k_2} y^{\sigma_3}_{J_3 k_3} \ldots y^{\sigma_{n-2}}_{J_{n-2} k_{n-2}}$$
$$+ (n-1) B_{\sigma}^{Jl}{}_{\sigma_2}^{J_2 k_2} \ldots {}_{\sigma_{n-2}}^{J_{n-2} k_{n-2}}{}_{\sigma_{n-1}}^{J_{n-1} k_{n-1}}{}_{l i_3 i_4 \ldots i_n} \delta^k_{i_2} y^{\sigma_2}_{J_2 k_2} y^{\sigma_3}_{J_3 k_3} \ldots y^{\sigma_{n-2}}_{J_{n-2} k_{n-2}} y^{\sigma_{n-1}}_{J_{n-1} k_{n-1}}$$
$$\mathrm{Sym}(Jk) \quad \mathrm{Alt}(i_2 i_3 \ldots i_n). \tag{58}$$

Thus, comparing the coefficients in (58) and (56), condition (55) yields

$$B_{\sigma}^{Jk}{}_{i_1 i_2 \ldots i_{n-1}} = \frac{s(n-1)}{s+1} B_{\sigma}^{Jl}{}_{l i_2 i_3 \ldots i_{n-1}} \delta^k_{i_1}$$
$$B_{\sigma}^{Jk}{}_{\sigma_2}^{J_2 k_2}{}_{i_1 i_2 \ldots i_{n-1}} = \frac{s(n-1)}{s+1} B_{\sigma}^{Jl}{}_{\sigma_2}^{J_2 k_2}{}_{l i_2 i_3 \ldots i_{n-1}} \delta^k_{i_1}$$
$$\ldots$$
$$B_{\sigma}^{Jk}{}_{\sigma_2}^{J_2 k_2}{}_{\sigma_3}^{J_3 k_3} \ldots {}_{\sigma_{n-2}}^{J_{n-2} k_{n-2}}{}_{i_1 i_2 \ldots i_{n-1}} = \frac{s(n-1)}{s+1} B_{\sigma}^{Jl}{}_{\sigma_2}^{J_2 k_2}{}_{\sigma_3}^{J_3 k_3} \ldots {}_{\sigma_{n-2}}^{J_{n-2} k_{n-2}}{}_{l i_2 i_3 \ldots i_{n-1}} \delta^k_{i_1},$$
$$B_{\sigma}^{Jk}{}_{\sigma_2}^{J_2 k_2}{}_{\sigma_3}^{J_3 k_3} \ldots {}_{\sigma_{n-1}}^{J_{n-1} k_{n-1}}{}_{i_1 i_2 \ldots i_{n-1}} = \frac{s(n-1)}{s+1} B_{\sigma}^{Jl}{}_{\sigma_2}^{J_2 k_2}{}_{\sigma_3}^{J_3 k_3} \ldots {}_{\sigma_{n-1}}^{J_{n-1} k_{n-1}}{}_{l i_2 i_3 \ldots i_{n-1}} \delta^k_{i_1},$$
$$\mathrm{Sym}(Jk) \ \mathrm{Alt}(i_1 i_2 \ldots i_{n-1}). \tag{59}$$

On the other hand, any $(n-1)$-form μ on W^{s-1} can be expressed as

$$\mu = \mu_0 + \omega_I^{\nu} \wedge \Phi_{\nu}^{I} + d\omega_I^{\nu} \wedge \Psi_{\nu}^{I}, \tag{60}$$

where

$$\begin{aligned}\mu_0 = {} & A_{i_1 i_2 \ldots i_{n-1}} dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_{n-1}} \\ & + A^{J_1}_{\sigma_1}{}_{i_2 i_3 \ldots i_{n-1}} dy^{\sigma_1}_{J_1} \wedge dx^{i_2} \wedge dx^{i_3} \wedge \cdots \wedge dx^{i_{n-1}} \\ & + A^{J_1}_{\sigma_1}{}^{J_2}_{\sigma_2}{}_{i_3 i_4 \ldots i_{n-1}} dy^{\sigma_1}_{J_1} \wedge dy^{\sigma_2}_{J_2} \wedge dx^{i_3} \wedge dx^{i_4} \wedge \cdots \wedge dx^{i_{n-1}} \\ & + \cdots + A^{J_1}_{\sigma_1}{}^{J_2}_{\sigma_2} \cdots {}^{j_{n-2}}_{\sigma_{n-2}}{}_{i_{n-1}} dy^{\sigma_1}_{J_1} \wedge dy^{\sigma_2}_{J_2} \wedge \cdots \wedge dy^{\sigma_{n-2}}_{J_{n-2}} \wedge dx^{i_{n-1}} \\ & + A^{J_1}_{\sigma_1}{}^{J_2}_{\sigma_2} \cdots {}^{J_{n-1}}_{\sigma_{n-1}} dy^{\sigma_1}_{J_1} \wedge dy^{\sigma_2}_{J_2} \wedge \cdots \wedge dy^{\sigma_{n-1}}_{J_{n-1}},\end{aligned} \tag{61}$$

and the coefficients are *traceless* (Sect. 2.2, Theorem 3). Then, $h\mu = h\mu_0$ because h is an exterior algebra homomorphism, annihilating the contact forms ω^{ν}, and

$$\begin{aligned}h\mu = {} & \Big(A_{i_1 i_2 \ldots i_{n-1}} + A^{J_1}_{\sigma_1}{}_{i_2 i_3 \ldots i_{n-1}} y^{\sigma_1}_{J_1 i_1} + A^{J_1 J_2}_{\sigma_1 \sigma_2}{}_{i_3 i_4 \ldots i_{n-1}} y^{\sigma_1}_{J_1 i_1} y^{\sigma_2}_{J_2 i_2} \\ & + \cdots + A^{J_1 J_2}_{\sigma_1 \sigma_2} \cdots {}^{J_{n-2}}_{\sigma_{n-2}}{}_{i_{n-1}} y^{\sigma_1}_{J_1 i_1} y^{\sigma_2}_{J_2 i_2} \cdots y^{\sigma_{n-2}}_{J_{n-2} i_{n-2}} \\ & + A^{J_1 J_2}_{\sigma_1 \sigma_2} \cdots {}^{j_{n-1}}_{\sigma_{n-1}} y^{\sigma_1}_{J_1 i_1} y^{\sigma_2}_{J_2 j_2} \cdots y^{\sigma_{n-1}}_{J_{n-1} i_{n-1}} \Big) dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_{n-1}}.\end{aligned} \tag{62}$$

Now comparing the coefficients in (62) and (54), we see that the equation $h\mu = \eta$ for $\pi^{s,s-1}$-projectable extensions of the form η is equivalent with the system

$$\begin{aligned} B_{i_1 i_2 \ldots i_{n-1}} &= A_{i_1 i_2 \ldots i_{n-1}}, \\ B^{J_1 k_1}_{\sigma_1}{}_{i_1 i_2 \ldots i_{n-1}} &= A^{J_1}_{\sigma_1}{}_{i_2 i_3 \ldots i_{n-1}} \delta^{k_1}_{i_1} \quad \mathrm{Sym}(J_1 k_1) \quad \mathrm{Alt}(i_1 i_2 \ldots i_{n-1}), \\ B^{J_1 k_1}_{\sigma_1}{}^{j_2 k_2}_{\sigma_2}{}_{i_1 i_2 \ldots i_{n-1}} &= A^{J_1 j_2}_{\sigma_1 \sigma_2}{}_{i_3 i_4 \ldots i_{n-1}} \delta^{k_1}_{i_1} \delta^{k_2}_{i_2} \quad \mathrm{Sym}(J_1 k_1) \quad \mathrm{Sym}(J_2 k_2) \\ & \quad \mathrm{Alt}(i_1 i_2 \ldots i_{n-1}), \\ & \ldots \\ B^{J_1 k_1}_{\sigma_1}{}^{J_2 k_2}_{\sigma_2} \cdots {}^{J_{n-2} k_{n-2}}_{\sigma_{n-2}}{}_{i_1 i_2 \ldots i_{n-1}} &= A^{J_1 J_2}_{\sigma_1 \sigma_2} \cdots {}^{J_{n-2}}_{\sigma_{n-2}}{}_{i_{n-1}} \delta^{k_1}_{i_1} \delta^{k_2}_{i_2} \cdots \delta^{k_{n-2}}_{i_{n-2}} \quad \mathrm{Sym}(J_1 k_1) \\ & \quad \mathrm{Sym}(J_2 k_2) \ldots \mathrm{Sym}(J_{n-2} k_{n-2}) \quad \mathrm{Alt}(i_1 i_2 \ldots i_{n-1}), \\ B^{J_1 k_1}_{\sigma_1}{}^{J_2 k_2}_{\sigma_2} \cdots {}^{J_{n-2} k_{n-2}}_{\sigma_{n-2}}{}^{J_{n-1} k_{n-1}}_{\sigma_{n-1}}{}_{i_1 i_2 \ldots i_{n-1}} &= A^{J_1 J_2}_{\sigma_1 \sigma_2} \cdots {}^{J_{n-2}}_{\sigma_{n-2}}{}^{j_{n-1}}_{\sigma_{n-1}} \delta^{k_1}_{i_1} \delta^{k_2}_{i_2} \cdots \delta^{k_{n-2}}_{i_{n-2}} \delta^{k_{n-1}}_{i_{n-1}} \\ & \quad \mathrm{Sym}(j_1 k_1) \mathrm{Sym}(j_2 k_2) \ldots \mathrm{Sym}(j_{n-1} k_{n-1}) \quad \mathrm{Alt}(i_1 i_2 \ldots i_{n-1}) \end{aligned} \tag{63}$$

for unknown functions $A_{i_1 i_2 \ldots i_{n-1}}$, $A^{J_1}_{\sigma_1}{}_{i_2 i_3 \ldots i_{n-1}}$, $A^{J_1 J_2}_{\sigma_1 \sigma_2}{}_{i_3 i_4 \ldots i_{n-1}}$, $\ldots$, $A^{J_1 J_2}_{\sigma_1 \sigma_2} \cdots {}^{J_{n-2}}_{\sigma_{n-2}}{}_{i_{n-1}}$, and $A^{J_1 J_2}_{\sigma_1 \sigma_2} \cdots {}^{J_{n-2}}_{\sigma_{n-2}}{}^{J_{n-1}}_{\sigma_{n-1}}$.

We can now solve this system with the help of the trace decomposition theory, namely with the trace decomposition formula of the symmetric-alternating tensors; in what follows we use the notation of Appendix 8 and Appendix 9.

We consider each of equations (63) separately. The second equation is

$$B^{Jk}_{\sigma\ \ i_1 i_2 \ldots i_{n-1}} = A^{J}_{\sigma\ \ i_2 i_3 \ldots i_{n-1}} \delta^{k}_{i_1} \quad \mathrm{Sym}(Jk)\ \mathrm{Alt}(i_1 i_2 \ldots i_{n-1}). \tag{64}$$

Denoting $B = B^{J_1 k_1}_{\sigma_1\ \ i_1 i_2 \ldots i_{n-1}}$ and $A = A^{J}_{\sigma\ \ i_2 i_3 \ldots i_{n-1}} \delta^{k}_{i_1}$, this equation can also be written as $B = \mathbf{q}\tilde{A}$ where $\tilde{A} = \tilde{A}^{J}_{\sigma\ \ i_2 i_3 \ldots i_{n-1}}$ is defined by

$$A^{J}_{\sigma\ \ i_2 i_3 \ldots i_{n-1}} = \frac{s(n-1)}{s+1} \tilde{A}^{J}_{\sigma\ \ i_2 i_3 \ldots i_{n-1}}. \tag{65}$$

But B satisfies the first condition (59), which can also be written as $B = \mathbf{q}\mathrm{tr}B$. Consequently, the trace decomposition formula yields $\tilde{A} = \mathrm{tr}\mathbf{q}\tilde{A} + \mathbf{q}\mathrm{tr}\tilde{A} = \mathrm{tr}B$ because $\tilde{A}$ is traceless; thus, we get a solution

$$A = \frac{s(n-1)}{s+1} \tilde{A} = \frac{s(n-1)}{s+1} \mathrm{tr}B. \tag{66}$$

Next equation (63) is

$$\begin{aligned} B^{J_1 k_1\ \ J_2 k_2}_{\sigma_1\ \ \ \ \sigma_2\ \ \ \ i_1 i_2 \ldots i_{n-1}} = A^{J_1 J_2}_{\sigma_1 \sigma_2\ \ i_3 i_4 \ldots i_{n-1}} \delta^{k_1}_{i_1} \delta^{k_2}_{i_2} \quad \mathrm{Sym}(J_1 k_1)\ \mathrm{Sym}(J_2 k_2) \\ \mathrm{Alt}(i_1 i_2 \ldots i_{n-1}). \end{aligned} \tag{67}$$

This equation can be understood as a condition for the trace decomposition of the tensor $B = B^{J_1 k_1\ \ J_2 k_2}_{\sigma_1\ \ \ \ \sigma_2\ \ \ \ i_1 i_2 \ldots i_{n-1}}$(Appendix 9). According to conditions (59)

$$\begin{aligned} B^{J_1 k_1\ \ J_2 k_2}_{\sigma_1\ \ \ \ \sigma_2\ \ \ \ i_1 i_2 \ldots i_{n-1}} = \frac{s(n-1)}{s+1} B^{J_1 l\ \ J_2 k_2}_{\sigma_1\ \ \ \ \sigma_2\ \ \ \ l i_2 i_3 \ldots i_{n-1}} \delta^{k_1}_{i_1} \quad \mathrm{Sym}(J_1 k_1) \\ \mathrm{Alt}(i_1 i_2 \ldots i_{n-1}). \end{aligned} \tag{68}$$

Analogously

$$B^{J_1 k_1\ \ J_2 k_2}_{\sigma_1\ \ \ \ \sigma_2\ \ \ \ i_1 i_2 \ldots i_{n-1}} = \frac{s(n-1)}{s+1} B^{J_1 k_1\ \ J_2 l}_{\sigma_1\ \ \ \ \sigma_2\ \ \ \ l i_2 i_3 \ldots i_{n-1}} \delta^{k_2}_{i_1} \quad \mathrm{Sym}(J_2 k_2)\ \mathrm{Alt}(i_1 i_2 \ldots i_{n-1}). \tag{69}$$

These conditions mean that B is a Kronecker tensor whose summands contain exactly one factor of the form δ^{α}_{i}, where α runs through $J_1 k_1$ and i through the set $\{i_1, i_2, \ldots, i_{n-1}\}$, and exactly one factor δ^{β}_{i}, where β runs through $J_2 k_2$ and i through $\{i_1, i_2, \ldots, i_{n-1}\}$; thus, B must be a linear combination of the terms of the form $\delta^{j_1}_{i} \delta^{j_2}_{l}$, $\delta^{j_1}_{i} \delta^{k_2}_{l}$, $\delta^{k_1}_{i} \delta^{j_2}_{l}$, $\delta^{k_1}_{i} \delta^{k_2}_{l}$. From the complete trace decomposition theorem, it now follows that the coefficients at these Kronecker tensors can be chosen traceless. This shows, however, that equation (67) has a solution $A^{J_1\ \ J_2}_{\sigma_1\ \ \sigma_2\ \ i_3 i_4 \ldots i_{n-1}}$.

To complete the construction of the $\pi^{s,s-1}$-projectable extension μ of the form η, we proceed in the same way. □

A remarkable property of solutions of the formal divergence equation is obtained when we combine Theorem 1 and Theorem 2: We show that the solutions can also be described as projectable extensions of forms on W^s.

Theorem 3 *Let $f\colon W^s \to \mathbf{R}$ be a function, let $g = g^i$ be a system of functions, defined on W^s, and let $\eta = g^i \omega_i$. Then, the following conditions are equivalent:*

(a) *The system g^i is a solution of the formal divergence equation*

$$d_i g^i = f. \tag{70}$$

(b) *There exists a projectable extension μ of the form η such that*

$$hd\mu = f\omega_0. \tag{71}$$

Proof

1. If the functions g^i solve the formal divergence equation $d_i g^i = f$, then condition (50) is satisfied and η has a projectable extension μ (Theorem 2). Then $\eta = h\mu$, hence

$$(\pi^{s+1,s})^* hd\mu = hdh\mu = hd\eta = d_i g^i \cdot \omega_0 = f\omega_0, \tag{72}$$

proving (71).

2. Conversely, suppose that $g^i \omega_i = h\mu$. Then, a direct calculation yields $hd\mu = hdh\mu = d_1 g^i \cdot \omega_0$, hence (70) follows from (71). □

Reference

[K14] D. Krupka, The total divergence equation, Lobachevskii Journal of Mathematics 23 (2006) 71-93

Chapter 4
Variational Structures

In this chapter, a complete treatment of the foundations of the calculus of variations on fibered manifolds is presented. The aim is to study higher-order integral variational functionals of the form $\gamma \to \int J^r\gamma^*\rho$, depending on sections γ of a fibered manifold Y, where ρ is a general differential form on the jet manifold J^rY and $J^r\gamma$ is the r-jet prolongation γ. The *horizontal* forms ρ are the *Lagrangians*.

In Sects. 4.1–4.7 we consider *variations* (deformations) of sections of Y as *vector fields*, permuting the set of sections, and the *prolongations* of these vector fields to the jet manifolds J^rY. The variations are applied to the functionals in a geometric way by means of the *Lepage forms* (Krupka [K13, K1]). The main idea can be introduced by means of the *Cartan's formula* for the Lie derivative of a differential form η on a manifold Z, $\partial_\xi\eta = i_\xi d\eta + di_\xi\eta$, where i_ξ is the contraction by a vector field ξ and d is the exterior derivative operator. For any manifold X and any mapping $f\colon X \to Z$, the Lie derivative satisfies $f^*\partial_\xi\eta = f^*i_\xi d\eta + df^*i_\xi\eta$. Replacing Z with the r-jet prolongation J^rY and η with ρ, we prove that the form ρ in the variational functional $\gamma \to \int J^r\gamma^*\rho$ may be chosen in such a way that the Cartan's formula for ρ becomes a geometric version of the classical *first variation formula*. These forms are the *Lepage forms*; a structure theorem we prove implies that for different underlying manifold structures and order of their jet prolongations, this concept generalizes the well-known *Cartan form* in classical mechanics (Carton [C]), the *Poincaré-Cartan forms* in the first-order field theory (Garcia [G]), the so-called *fundamental forms* (Betounes, Krupka [B, K2, K13]) and [K5], the *second-order generalization of the Poincaré-Cartan form* [K13], the *Carathéodory form* (Crampin, Saunders [CS]), and the *Hilbert form* in Finsler geometry (Crampin, Saunders, Krupka [CS, K7]). For survey research, we refer to Krupka et al. [KKS1, KKS2] and [K5].

The first variation formula, expressed by means of a Lepage form ρ, leads to the concept of the *Euler–Lagrange form*, a global differential form, defined by means of the exterior derivative $d\rho$ (cf. Krupka [K13] and also Goldschmidt and Sternberg [GS], where the Euler–Lagrange form is interpreted as a vector-valued form). The coordinate components of the Euler–Lagrange form coincide with the *Euler–Lagrange expressions* of the classical variational calculus, and its classical analogue is simply the collection of the Euler–Lagrange expression. The corresponding *Euler–Lagrange*

D. Krupka, *Introduction to Global Variational Geometry*,
Atlantis Studies in Variational Geometry 1, DOI 10.2991/978-94-6239-073-7_4

equations for *extremals* of a variational functional are then related to each fibered chart and should be analyzed in any concrete case from local and global viewpoints.

The first variation formula also gives rise to the *Euler–Lagrange mapping*, assigning to a Lagrangian its Euler–Lagrange form. The domain and image of this mapping are some Abelian groups of differential forms. A complete treatment of the local theory is presented in Sects. 4.9–4.11, using the fibered homotopy operator as the basic tool. First the *Vainberg–Tonti formula*, allowing us to assign a Lagrangian to *any* source form, is considered (Tonti, Vainberg [To, V]) and is extended to the higher-order variational theory (Krupka [K8, K16]). The theorem on the Euler–Lagrange equations of the *Vainberg–Tonti Lagrangian*, proved in Sect. 4.9, determining the *image* of the Euler–Lagrange mapping in terms of the (local variationality) *Helmholtz* conditions, is a basic instrument for the *local inverse variational problem*, treated in Sects. 4.10 and 4.11 (Anderson, Duchamp, Krupka [AD, K11]).

Specific research directions in the variational geometry have been developed for several decades. Different aspects of the local inverse problem are given extensive investigation in Anderson and Thompson [AT], Zenkov (Ed.) [Z], Bucataru [Bu], Crampin [Cr], Krupka and Saunders [KS], Krupková and Prince [KrP], Olver [O2], Sarlet et al. [SCM], Urban and Krupka [UK2], and many others. Remarks on the history of the inverse problem can be found in Havas [H]; original sources are Helmholtz [He] (the inverse problem for systems of second-order ordinary differential equations), Sonin [So] and Douglas [Do] (for variational integrating factors).

The theorem on the kernel of the Euler–Lagrange mapping is proved in Sect. 4.10 on the basis of the formal divergence equations (Chap. 3) and the approach initiated in Krupka [K12], Krupka and Musilová [KM].

Our basic notation in this chapter follows Chaps. 2 and 3: Y is a fixed fibered manifold with orientable base manifold X and projection π, and $\dim X = n$, $\dim Y = n + m$. J^rY is the r-jet prolongation of Y, $\pi^{r,s}$ and π^r are the canonical jet projections. For any set $W \subset Y$, we denote $W^r = (\pi^{r,0})^{-1}(W)$. $\Omega^r_q W$ is the module of q-forms defined on W^r. Sometimes, when no misunderstanding may possibly arise, to simplify formulas we do not distinguish between the differential forms ρ, defined on the base manifold X of a fibered manifold $\pi^s\colon J^sY \to X$ and its canonical lifting $(\pi^s)^*\rho$ to the jet manifold J^sY. Similarly, the Lie derivative $\partial_{J^r\Xi}\rho$ and contraction $i_{J^r\Xi}\rho$ are sometimes denoted simply by $\partial_\Xi \rho$ and $i_\Xi \rho$.

Since the subject of this chapter is the *higher-order* calculus of variations, some proofs of our statements include extensive coordinate calculations; in order not to make difficult the understanding, we prefer to present them as complete as possible.

4.1 Variational Structures on Fibered Manifolds

By a *variational structure,* we shall mean a pair (Y, ρ), where Y is a fibered manifold over an n-dimensional manifold X with projection π and ρ is an n-form on the r-jet prolongation J^rY.

Suppose that we have a variational structure (Y, ρ). Let Ω be a compact, n-dimensional submanifold of X with boundary (a *piece* of X). Denote by $\Gamma_\Omega(\pi)$ the set of differentiable sections of π over Ω (of a fixed order of differentiability). Then for any section $\gamma \in \Gamma_\Omega(\pi)$ of Y, the pullback $J^r\gamma^*\rho$ by the r-jet prolongation $J^r\gamma$ is an n-form on a neighborhood of the piece Ω. Integrating the n-form $J^r\gamma^*\rho$ on Ω, we get a function $\Gamma_\Omega(\pi) \ni \gamma \to \rho_\Omega(\gamma) \in \mathbf{R}$, defined by

$$\rho_\Omega(\gamma) = \int_\Omega J^r\gamma^*\rho. \tag{1}$$

ρ_Ω is called the *variational functional*, associated with (Y, ρ) (over Ω). The variational functional of the form (1) is referred to as the *integral variational functional, associated* with ρ.

If W is an open set in Y, considered as a fibered manifold with projection $\pi|_W$, then restricting the n-form ρ to $W^r \subset J^rY$ we get a variational structure (W, ρ). The corresponding variational functional is the restriction of the variational functional (1) to the set $\Gamma_\Omega(\pi|_W) \subset \Gamma_\Omega(\pi)$. Elements of this set are sections whose values lie in W.

On the other hand, *any* n-form ρ on the set W^r defines a variational structure (W, ρ). The corresponding variational functional is given by

$$\Gamma_\Omega(\pi|_W) \ni \gamma \to \rho_\Omega(\gamma) = \int_\Omega J^r\gamma^*\rho \in \mathbf{R}. \tag{2}$$

If $W = Y$, then $\Gamma_\Omega(\pi|_W) = \Gamma_\Omega(\pi)$ and formula (2) reduces to (1).

Let W be an open set in Y. For every r, we denote by $\Omega^r_{n,X}W$ the submodule of the module of q-forms $\Omega^r_n W$, consisting of π^r-*horizontal* forms. Elements of the set $\Omega^r_{n,X}W$ are called *Lagrangians* (of order r) for the fibered manifold Y.

Let $\rho \in \Omega^r_n W$. There exists a unique Lagrangian $\lambda_\rho \in \Omega^{r+1}_{n,X} W$ such that

$$J^{r+1}\gamma^*\lambda_\rho = J^r\gamma^*\rho \tag{3}$$

for all sections γ of Y. The n-form λ_ρ can alternatively be defined by the first canonical decomposition the form ρ (Sect. 2.4)

$$(\pi^{r+1,r})^*\rho = h\rho + p_1\rho + p_2\rho + \cdots + p_n\rho \tag{4}$$

as the *horizontal component* of ρ,

$$\lambda_\rho = h\rho. \tag{5}$$

λ_ρ is a Lagrangian, said to be *associated with* ρ. Property (3) says that the variational functional ρ_Ω can also be expressed as

$$\rho_\Omega(\gamma) = \int_\Omega J^{r+1}\gamma^* \lambda_\rho. \tag{6}$$

We give the chart expressions of ρ and $h\rho$ in a fibered chart (V,ψ), $\psi = (x^i, y^\sigma)$, on Y (or, more exactly, in the associated charts on J^rY and $J^{r+1}Y$). Recall that in multi-index notation, the *contact basis* of 1-forms on V^r (and analogously on V^{r+1}) is defined to be the basis $(dx^i, \omega^\sigma_J, dy^\sigma_I)$, where the multi-indices satisfy $0 \le |J| \le r-1$, $|I| = r$, and

$$\omega^\sigma_J = dy^\sigma_J - y^\sigma_{Jj} dx^j. \tag{7}$$

We also associate with the given chart the n-form

$$\omega_0 = dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n \tag{8}$$

(considered on $U = \pi(V) \subset X$, and also on V^r), sometimes called the *local volume form*, associated with (V, ψ).

According to the trace decomposition theorem (Sect. 2.2, Theorem 3), ρ has an expression

$$\rho = \sum_{0 \le |J| \le r-1} \omega^\sigma_J \wedge \Phi^J_\sigma + \sum_{|J|=r-1} d\omega^\sigma_J \wedge \Psi^J_\sigma + \rho_0, \tag{9}$$

where

$$\begin{aligned}
\rho_0 = {} & A_{i_1 i_2 \ldots i_n} dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_n} \\
& + A^{J_1}_{\sigma_1}{}_{i_2 i_3 \ldots i_n} dy^{\sigma_1}_{J_1} \wedge dx^{i_2} \wedge dx^{i_3} \wedge \cdots \wedge dx^{i_n} \\
& + A^{J_1}_{\sigma_1}{}^{J_2}_{\sigma_2}{}_{i_3 i_4 \ldots i_n} dy^{\sigma_1}_{J_1} \wedge dy^{\sigma_2}_{J_2} \wedge dx^{i_3} \wedge dx^{i_4} \wedge \cdots \wedge dx^{i_n} \\
& + \cdots + A^{J_1}_{\sigma_1}{}^{J_2}_{\sigma_2} \cdots {}^{J_{n-1}}_{\sigma_{n-1}}{}_{i_n} dy^{\sigma_1}_{J_1} \wedge dy^{\sigma_2}_{J_2} \wedge \cdots \wedge dy^{\sigma_{n-1}}_{J_{n-1}} \wedge dx^{i_n} \\
& + A^{J_1}_{\sigma_1}{}^{J_2}_{\sigma_2} \cdots {}^{J_n}_{\sigma_n} dy^{\sigma_1}_{J_1} \wedge dy^{\sigma_2}_{J_2} \wedge \cdots \wedge dy^{\sigma_n}_{J_n},
\end{aligned} \tag{10}$$

and the coefficients $A^{J_1}_{\sigma_1}{}^{J_2}_{\sigma_2} \cdots {}^{J_s}_{\sigma_s}{}_{i_{s+1} i_{s+2} \ldots i_n}$ are *traceless*. Then, $h\rho = h\rho_0$ because h is an exterior algebra homomorphism, annihilating the contact forms ω^σ_J and $d\omega^\sigma_J$. Thus,

$$\begin{aligned}
\lambda_\rho = {} & (A_{i_1 i_2 \ldots i_n} + A^{J_1}_{\sigma_1}{}_{i_2 i_3 \ldots i_n} y^{\sigma_1}_{J_1 i_1} + A^{J_1}_{\sigma_1}{}^{J_2}_{\sigma_2}{}_{i_3 i_4 \ldots i_n} y^{\sigma_1}_{J_1 i_1} y^{\sigma_2}_{J_2 i_2} \\
& + \cdots + A^{J_1}_{\sigma_1}{}^{J_2}_{\sigma_2} \cdots {}^{J_{n-1}}_{\sigma_{n-1}}{}_{i_n} y^{\sigma_1}_{J_1 i_1} y^{\sigma_2}_{J_2 i_2} \cdots y^{\sigma_{n-1}}_{J_{n-1} i_{n-1}} + A^{J_1}_{\sigma_1}{}^{J_2}_{\sigma_2} \cdots {}^{J_n}_{\sigma_n} y^{\sigma_1}_{J_1 i_1} y^{\sigma_2}_{J_2 j_2} \cdots y^{\sigma_n}_{J_n i_n}) \\
& \cdot dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_n}.
\end{aligned} \tag{11}$$

Using the local volume form (8), we also write

$$\lambda_\rho = \mathscr{L}\omega_0, \tag{12}$$

where

$$\begin{aligned}\mathscr{L} = \varepsilon^{i_1 i_2 \ldots i_n}(&A_{i_1 i_2 \ldots i_n} + A^{J_1}_{\sigma_1}{}_{i_2 i_3 \ldots i_n} y^{\sigma_1}_{J_1 i_1} + A^{J_1}_{\sigma_1}{}^{J_2}_{\sigma_2}{}_{i_3 i_4 \ldots i_n} y^{\sigma_1}_{J_1 i_1} y^{\sigma_2}_{J_2 i_2}\\ &+ \cdots + A^{J_1}_{\sigma_1}{}^{J_2}_{\sigma_2} \ldots {}^{J_{n-1}}_{\sigma_{n-1}}{}_{i_n} y^{\sigma_1}_{J_1 i_1} y^{\sigma_2}_{J_2 i_2} \ldots y^{\sigma_{n-1}}_{J_{n-1} i_{n-1}} + A^{J_1}_{\sigma_1}{}^{J_2}_{\sigma_2} \ldots {}^{J_n}_{\sigma_n} y^{\sigma_1}_{J_1 i_1} y^{\sigma_2}_{J_2 j_2} \ldots y^{\sigma_n}_{J_n i_n}).\end{aligned} \tag{13}$$

$\mathscr{L}$ is a function on V^{r+1} called the *Lagrange function*, associated with ρ (or with the Lagrangian λ_ρ).

Remark 1 Sometimes, the integration domain Ω in the variational functional ρ_Ω is not fixed, but is arbitrary. Then, formula (2) defines a *family* of variational functionals labeled by Ω. This situation usually appears in variational principles in physics.

Remark 2 Orientability of the base X of the fibered manifold Y is not an essential assumption; replacing differential forms by *twisted base differential forms*, one can also develop the variational theory for *non-orientable* bases X [K10]. Variational functionals, defined on fibered manifolds over non-orientable bases, may appear in the general relativity theory and field theory, and in the variational theory for submanifolds.

Remark 3 (The structure of Lagrange functions) Formulas (12) and (13) describe the *general structure* of the Lagrangians, associated with the class of variational functionals (2). The Lagrange functions $\mathscr{L}$ that appear in chart descriptions of the Lagrangians are multilinear, symmetric functions of the variables y^σ_I, where $|I| = r + 1$.

Remark 4 (Lagrangians) Let ρ be an n-form belonging to the submodule $\Omega^r_{n,X} W \subset \Omega^r_n W$ of π^r-*horizontal forms*, expressed as

$$\rho = \frac{1}{n!} A_{i_1 i_2 \ldots i_n} dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_n}. \tag{14}$$

Then, since $dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_n} = \varepsilon^{i_1 i_2 \ldots i_n}\omega_0$, one can equivalently write

$$\rho = \mathscr{L}\omega_0, \tag{15}$$

where the Lagrange function $\mathscr{L}$ is given by

$$\mathscr{L} = \frac{1}{n!} A_{i_1 i_2 \ldots i_n} \varepsilon^{i_1 i_2 \ldots i_n}. \tag{16}$$

The following lemma describes all n-forms $\rho \in \Omega_n^r W$, whose associated Lagrangians belong to the module $\Omega_n^r W$, that is, are of order r.

Lemma 1 *For a form $\rho \in \Omega_n^r W$, the following two conditions are equivalent:*

1. *The Lagrangian λ_ρ is defined on W^r.*
2. *In any fibered chart $(V, \psi), \psi = (x^i, y^\sigma)$, on Y, ρ has an expression*

$$\rho = \mathscr{L}\omega_0 + \sum_{0 \le |J| \le r-1} \omega_J^\sigma \wedge \Phi_\sigma^J + \sum_{|J|=r-1} d\omega_J^\sigma \wedge \Psi_\sigma^J \tag{17}$$

for some function $\mathscr{L}$ and some forms Φ_σ^J and Ψ_σ^J.

Proof This follows from (5) and (13). □

4.2 Variational Derivatives

Let U be an open subset of X, γ: $U \to Y$ a section, and let Ξ be a π-projectable vector field on an open set $W \subset Y$ such that $\gamma(U) \subset W$. If α_t is the local 1-parameter group of Ξ, and $\alpha_{(0)t}$ its π-projection, then

$$\gamma_t = \alpha_t \gamma \alpha_{(0)t}^{-1} \tag{18}$$

is a 1-parameter family of *sections* of Y, depending differentiably on the parameter t: Indeed, since $\pi\alpha_t = \alpha_{(0)t}\pi$, we have

$$\pi\gamma_t(x) = \pi\alpha_t\gamma\alpha_{(0)t}^{-1}(x) = \alpha_{(0)t}\pi\gamma\alpha_{(0)t}^{-1}(x) = \alpha_{(0)t}\alpha_{(0)t}^{-1}(x) = x \tag{19}$$

on the domain of γ_t, so γ_t is a section for each t. The family γ_t is called the *variation*, or *deformation*, of the section γ, *induced* by the vector field Ξ.

Recall that a *vector field along* γ is a mapping Ξ: $U \to TY$ such that $\Xi(x) \in T_{\gamma(x)}Y$ for every point $x \in U$. Given Ξ, formula

$$\xi = T\pi \cdot \Xi \tag{20}$$

then defines a vector field ξ on U, called the *π-projection* of Ξ.

The following theorem says that every vector field along a section γ can be extended to a π-projectable vector field, defined on a neighborhood of the image of γ in Y. Moreover, the r-jet prolongation of the extended vector field, considered along $J^r\gamma$, is independent of the extension.

Theorem 1 *Let γ be a section of Y defined on an open set $U \subset X$, let Ξ be a vector field along γ.*

(a) *There exists a π-projectable vector field $\tilde{\Xi}$, defined on a neighborhood of the set $\gamma(U)$, such that for each $x \in U$*

$$\tilde{\Xi}(\gamma(x)) = \Xi(\gamma(x)). \tag{21}$$

(b) *Any two π-projectable vector fields Ξ_1, Ξ_2, defined on a neighborhood of $\gamma(U)$, such that $\Xi_1(\gamma(x)) = \Xi_2(\gamma(x))$ for all $x \in U$, satisfy*

$$J^r\Xi_1(J^r_x\gamma) = J^r\Xi_2(J^r_x\gamma). \tag{22}$$

Proof

(a) Choose $x_0 \in U$ and a fibered chart $(V_0, \psi_0), \psi_0 = (x_0^i, y_0^\sigma)$, at the point $\gamma(x_0) \in Y$, such that $\pi(V_0) \subset U$ and $\gamma(\pi(V_0)) \subset V_0$. Ξ has in this chart an expression

$$\Xi(\gamma(x)) = \xi^i(x)\left(\frac{\partial}{\partial x^i}\right)_{\gamma(x)} + \Xi^\sigma(x)\left(\frac{\partial}{\partial y^\sigma}\right)_{\gamma(x)} \tag{23}$$

on $\pi(V_0)$. Set for any $y \in V_0, \tilde{\xi}^i(y) = \xi^i(\pi(y)), \tilde{\Xi}^\sigma(y) = \Xi^\sigma(\pi(y))$ and define a vector field $\tilde{\Xi}$ on V_0 by

$$\tilde{\Xi} = \tilde{\xi}^i\frac{\partial}{\partial x^i} + \tilde{\Xi}^\sigma\frac{\partial}{\partial y^\sigma}. \tag{24}$$

The vector field $\tilde{\Xi}$ satisfies $\tilde{\Xi}(\gamma(x)) = \Xi(\gamma(x))$ on $\pi(V_0)$.
Applying this construction to every point of the domain of definition U of Ξ, we may suppose that we have families of fibered charts $(V_\iota, \psi_\iota), \psi_\iota = (x_\iota^i, y_\iota^\sigma)$, and vector fields $\tilde{\Xi}_\iota$, where ι runs through an index set I, such that $\pi(V_\iota) \subset U, \gamma(\pi(V_\iota)) \subset V_\iota$ for every $\iota \in I, \tilde{\Xi}_\iota$ is defined on V_ι, and $\tilde{\Xi}_\iota(\gamma(x)) = \tilde{\Xi}(\gamma(x))$ for all $\pi(V_\iota)$.
Let $\{\chi_\iota\}_{\iota \in I}$ be a partition of unity, subordinate to the covering $\{V_\iota\}_{\iota \in I}$ of the set $\gamma(U) \subset Y$. Setting

$$\tilde{\Xi} = \sum_{\iota \in I} \chi_\iota \tilde{\Xi}_\iota, \tag{25}$$

we get a vector field on the open set $V = \cup V_\iota$. For any $x \in U$, the point $\gamma(x)$ belongs to some of the sets V_ι; thus, $\gamma(U) \subset V$. The value of $\tilde{\Xi}(\gamma(x))$ is

$$\tilde{\Xi}(\gamma(x)) = \sum_{\iota \in I} \chi_\iota(\gamma(x)) \tilde{\Xi}_\iota(\gamma(x)) = \left(\sum_{\iota \in I} \chi_\iota(\gamma(x)) \right) \Xi(\gamma(x)) \\ = \Xi(\gamma(x)) \tag{26}$$

because $\{\chi_\iota\}_{\iota \in I}$ is a partition of unity.

(b) It is sufficient to verify equality (22) in a chart. Suppose that

$$\Xi_1 = \xi^i \frac{\partial}{\partial x^i} + \Xi^\sigma \frac{\partial}{\partial y^\sigma}, \quad \Xi_2 = \zeta^i \frac{\partial}{\partial x^i} + Z^\sigma \frac{\partial}{\partial y^\sigma} \tag{27}$$

and

$$\xi^i = \zeta^i, \quad \Xi^\sigma \circ \gamma = Z^\sigma \circ \gamma. \tag{28}$$

Then from the formulas,

$$\Xi^\sigma_{j_1 j_2 \ldots j_k} = d_{j_k} \Xi^\sigma_{j_1 j_2 \ldots j_{k-1}} - y^\sigma_{j_1 j_2 \ldots j_{k-1} i} \frac{\partial \xi^i}{\partial x^{j_k}}, \\ Z^\sigma_{j_1 j_2 \ldots j_k} = d_{j_k} Z^\sigma_{j_1 j_2 \ldots j_{k-1}} - y^\sigma_{j_1 j_2 \ldots j_{k-1} i} \frac{\partial \zeta^i}{\partial x^{j_k}} \tag{29}$$

for the components of $J^r\Xi_1$ and $J^r\Xi_2$ (Sect. 1.7, Lemma 10), and from the formal derivative formula (28), Sect. 2.1, we observe that the left-hand sides in (29) are polynomials in the variables $y^\sigma_{j_1 j_2 \ldots j_s}$, $1 \le s \le r$. Therefore, condition (28) applies to the coefficients of these polynomials, and we get $\Xi^\sigma_{j_1 j_2 \ldots j_k} \circ J^r\gamma = Z^\sigma_{j_1 j_2 \ldots j_k} \circ J^r\gamma$. □

A π-projectable vector field $\tilde{\Xi}$, satisfying condition (a) of Theorem 1, is called a *π-projectable extension* of Ξ. Using (b) and *any* π-projectable extension $\tilde{\Xi}$, we may define, for the given section γ,

$$J^r\Xi(J^r_x\gamma) = J^r\tilde{\Xi}(J^r_x\gamma). \tag{30}$$

Then, $J^r\Xi$ is a vector field along the r-jet prolongation $J^r\gamma$ of γ; we call this vector field the *r-jet prolongation* of the vector field (along γ) Ξ.

Variations ("deformations") of sections induce the corresponding variations ("deformations") of the variational functionals. Let $\rho \in \Omega^r_n W$ be a form, $\Omega \subset \pi(W)$ a piece of X. Choose a section $\gamma \in \Gamma_\Omega(\pi|_W)$ and a π-projectable vector field Ξ on W, and consider the variation (1) of γ, induced by Ξ. Since the domain of γ_t contains Ω for all sufficiently small t, the value of the variational functional $\Gamma_\Omega(\pi|_W) \ni \gamma \to \rho_\Omega(\gamma) \in \mathbf{R}$ at γ_t is defined, and we get a real-valued function, defined on a neighborhood $(-\varepsilon, \varepsilon)$ of the point $0 \in \mathbf{R}$,

$$(-\varepsilon,\varepsilon) \ni t \to \rho_{\alpha_{(0)t}(\Omega)}(\alpha_t \gamma \alpha_{(0)t}^{-1}) = \int\limits_{\alpha_{(0)t}(\Omega)} J^r(\alpha_t \gamma \alpha_{(0)t}^{-1})^* \rho \in \mathbf{R}. \tag{31}$$

It is easily seen that this function is differentiable. Since

$$J^r(\alpha_t \gamma \alpha_{(0)t}^{-1})^* \rho = (\alpha_{(0)t}^{-1})^* (J^r \gamma)^* (J^r \alpha_t)^* \rho, \tag{32}$$

where $J^r \alpha_t$ is the local 1-parameter group of the r-jet prolongation $J^r \Xi$ of the vector field Ξ, we have, using properties of the pullback operation and the theorem on transformation of the integration domain,

$$\int\limits_{\alpha_{(0)t}(\Omega)} (J^r(\alpha_t \gamma \alpha_{(0)t}^{-1}))^* \rho = \int\limits_{\Omega} J^r \gamma^* (J^r \alpha_t)^* \rho. \tag{33}$$

Thus, since the piece Ω is compact, differentiability of the function (31) follows from the theorem on differentiation of an integral, depending upon a parameter.

Differentiating (31) at $t = 0$ one obtains, using (33) and the definition of the *Lie derivative*,

$$\left(\frac{d}{dt} \rho_{\Omega}(\alpha_t \gamma \alpha_{(0)t}^{-1}) \right)_0 = \int\limits_{\Omega} J^r \gamma^* \partial_{J^r \Xi} \rho. \tag{34}$$

Note that this expression can be written, in the notation introduced by formula (19), Sect. 4.1, as

$$(\partial_{J^r \Xi} \rho)_{\Omega}(\gamma) = \int\limits_{\Omega} J^r \gamma^* \partial_{J^r \Xi} \rho. \tag{35}$$

The number (35) is called the *variation* of the integral variational functional ρ_{Ω} *at the point* γ*, induced by the vector field* Ξ.

This formula shows that the function $\Gamma_{\Omega}(\pi|_W) \ni \gamma \to (\partial_{J^r \Xi} \lambda)_{\Omega}(\gamma) \in \mathbf{R}$ is the variational functional (over Ω), defined by the form $\partial_{J^r \Xi} \rho$. We call this functional the *variational derivative*, or the *first variation* of the variational functional ρ_{Ω} by the vector field Ξ.

Formula (35) admits a direct generalization. If Z is another π-projectable vector field on W, then the *second variational derivative*, or the *second variation*, of the variational functional ρ_{Ω} by the vector fields Ξ and Z, is the mapping $\Gamma_{\Omega}(\pi|_W) \ni \gamma \to (\partial_{J^r Z} \partial_{J^r \Xi} \rho)_{\Omega}(\gamma) \in \mathbf{R}$, defined by

$$(\partial_{J^r Z} \partial_{J^r \Xi} \rho)_{\Omega}(\gamma) = \int\limits_{\Omega} J^r \gamma^* \partial_{J^r Z} \partial_{J^r \Xi} \rho. \tag{36}$$

It is now obvious how *higher-order variational derivatives* are defined: one should simply apply the Lie derivative (with respect to different vector fields) several times.

A section $\gamma \in \Gamma_\Omega(\pi|_W)$ is called a *stable point* of the variational functional λ_Ω with respect to its variation Ξ, if

$$(\partial_{J^r\Xi}\rho)_\Omega(\gamma) = 0. \tag{37}$$

In practice, one usually requires that a section be a stable point with respect to a *family* of its variations, defined by the problem considered.

Formula (35) can also be expressed in terms of the Lagrangian $\lambda_\rho = h\rho$, the horizontal component of ρ. Since for any π-projectable vector field Ξ, the Lie derivative by its r-jet prolongation $J^r\Xi$ commutes with the horizontalization,

$$h\partial_{J^r\Xi}\rho = \partial_{J^r\Xi}h\rho \tag{38}$$

(see Sect. 2.5, Theorem 9, (d)), the first variation of the integral variational functional ρ_Ω at a point $\gamma \in \Gamma_\Omega(\pi|_W)$, induced by the vector field Ξ, can be written as

$$(\partial_{J^r\Xi}\rho)_\Omega(\gamma) = \int_\Omega J^{r+1}\gamma^*\partial_{J^{r+1}\Xi}\lambda_\rho. \tag{39}$$

4.3 Lepage Forms

In this section, we introduce a class of n-forms ρ on the r-jet prolongation J^rY of the fibered manifold Y, defining variational structures (W, ρ) by imposing certain conditions on the exterior derivative $d\rho$. Properties of these forms determine the structure of the Lie derivatives $\partial_{J^r\Xi}\rho$, where Ξ is a π-projectable vector field on Y, and of the integrands of the variational functionals $\gamma \to (\partial_{J^r\Xi}\rho)_\Omega(\gamma)$ (35). Roughly speaking, we study those forms ρ for which the well-known Cartan's formula $\partial_{J^r\Xi}\rho = i_{J^r\Xi}d\rho + di_{J^r\Xi}\rho$ of the calculus of forms becomes an *infinitesimal analogue* of the integral first variation formula, known from the classical calculus of variations on Euclidean spaces.

First, we summarize some useful notation related with a chart (U, φ), $\varphi = (x^i)$, on an n-dimensional manifold X. Denote

$$\begin{aligned}
\omega_0 &= \frac{1}{n!}\varepsilon_{i_1 i_2 \dots i_n} dx^{i_1} \wedge dx^{i_2} \wedge \dots \wedge dx^{i_n},\\
\omega_{k_1} &= \frac{1}{1!(n-1)!}\varepsilon_{k_1 i_2 i_3 \dots i_n} dx^{i_2} \wedge dx^{i_3} \wedge \dots \wedge dx^{i_n},\\
\omega_{k_1 k_2} &= \frac{1}{2!(n-2)!}\varepsilon_{k_1 k_2 i_3 i_4 \dots i_n} dx^{i_3} \wedge dx^{i_4} \wedge \dots \wedge dx^{i_n}.
\end{aligned} \tag{40}$$

The inverse transformation formulas are

$$\begin{aligned} dx^{l_1} \wedge dx^{l_2} \wedge \cdots \wedge dx^{l_n} &= \varepsilon^{l_1 l_2 \ldots l_n} \omega_0, \\ dx^{l_2} \wedge dx^{l_3} \wedge \cdots \wedge dx^{l_n} &= \varepsilon^{k_1 l_2 l_3 \ldots l_n} \omega_{k_1}, \\ dx^{l_3} \wedge dx^{l_4} \wedge \cdots \wedge dx^{i_n} &= \varepsilon^{k_1 k_2 l_3 l_4 \ldots l_n} \omega_{k_1 k_2} \end{aligned} \tag{41}$$

(cf. Appendix 10). Also note that ω_{jk} can be written as

$$\begin{aligned} \omega_{jk} &= i_{\partial/\partial x^j} i_{\partial/\partial x^k} \omega_0 \\ &= (-1)^{j+k} dx^1 \wedge dx^2 \wedge \cdots \wedge dx^{j-1} \wedge dx^{j+1} \wedge \cdots \wedge dx^{k-1} \wedge \cdots \wedge dx^n, \end{aligned} \tag{42}$$

whenever $j < k$. Then,

$$dx^l \wedge \omega_{jk} = \delta^l_j \omega_k - \delta^l_k \omega_j, \tag{43}$$

which is an immediate consequence of definitions: since we have the identity $\omega_k = (-1)^{k-1} dx^l \wedge dx^1 \wedge dx^2 \wedge \cdots \wedge dx^{k-1} \wedge dx^{k+1} \wedge \cdots \wedge dx$, then

$$i_{\partial/\partial x^j}(dx^l \wedge \omega_k) = \begin{cases} \delta^l_k i_{\partial/\partial x^j} \omega_0 = \delta^l_k \omega_j, \\ \delta^l_j \omega_k - dx^l \wedge i_{\partial/\partial x^j} \omega_k = \delta^l_j \omega_k - dx^l \wedge \omega_{jk}. \end{cases} \tag{44}$$

We prove three lemmas characterizing the structure of n-forms on the r-jet prolongation $J^r Y$.

Lemma 2 *An n-form ρ on $W^r \subset J^r Y$ has in a fibered chart $(V, \psi), \psi = (x^i, y^\sigma)$, an expression*

$$\rho = \rho_0 + \tilde{\rho} + d\eta \tag{45}$$

with the following properties:

(a) *The n-form ρ_0 is generated by the contact forms ω^σ_J, where $0 \leq |J| \leq r-1$, that is,*

$$\rho_0 = \sum_{0 \leq |J| \leq r-1} \omega^\sigma_J \wedge \Phi^J_\sigma, \tag{46}$$

where

$$\Phi^J_\sigma = \Phi^J_{\sigma\ (1)} + \Phi^J_{\sigma\ (2)} + \tilde{\Phi}^J_\sigma, \tag{47}$$

the forms $\Phi^J_{\sigma\ (1)}$ are generated by the contact forms $\omega^\sigma_J, 0 \leq |J| \leq r-1, \Phi^J_{\sigma(2)}$ are generated by $d\omega^\sigma_I$ with $|I| = r-1$, and

$$\begin{aligned}\tilde{\Phi}^J_\sigma &= \tilde{\Phi}^J_{\sigma\ i_1 i_2 \ldots i_{n-1}} dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_{n-1}} \\ &+ \tilde{\Phi}^{J\ I_1}_{\sigma\ \sigma_1\ i_2 i_3 \ldots i_{n-1}} dy^{\sigma_1}_{I_1} \wedge dx^{i_2} \wedge dx^{i_3} \wedge \cdots \wedge dx^{i_{n-1}} \\ &+ \tilde{\Phi}^{J\ I_1\ I_2}_{\sigma\ \sigma_1\ \sigma_2\ i_3 i_4 \ldots i_{n-1}} dy^{\sigma_1}_{I_1} \wedge dy^{\sigma_2}_{I_2} \wedge dx^{i_3} \wedge dx^{i_4} \wedge \cdots \wedge dx^{i_{n-1}} \\ &+ \cdots + \tilde{\Phi}^{J\ I_1\ I_2 \ldots I_{n-2}}_{\sigma\ \sigma_1\ \sigma_2 \ldots \sigma_{n-2}\ i_{n-1}} dy^{\sigma_1}_{I_1} \wedge dy^{\sigma_2}_{I_2} \wedge \cdots \wedge dy^{\sigma_{n-2}}_{I_{n-2}} \wedge dx^{i_{n-1}} \\ &+ \tilde{\Phi}^{J\ I_1\ I_2 \ldots I_{n-1}}_{\sigma\ \sigma_1\ \sigma_2 \ldots \sigma_{n-1}} dy^{\sigma_1}_{I_1} \wedge dy^{\sigma_2}_{I_2} \wedge \cdots \wedge dy^{\sigma_{n-1}}_{I_{n-1}},\end{aligned} \tag{48}$$

where the multi-indices are of length $|I_1|, |I_2|, \ldots, |I_{n-1}| = r$ *and all the coefficients* $\tilde{\Phi}^{J\ I_1}_{\sigma\ \sigma_1\ i_2 i_3 \ldots i_{n-1}}, \tilde{\Phi}^{J\ I_1\ I_2}_{\sigma\ \sigma_1\ \sigma_2\ i_3 i_4 \ldots i_{n-1}}, \ldots, \tilde{\Phi}^{J\ I_1\ I_2 \ldots I_{n-2}}_{\sigma\ \sigma_1\ \sigma_2 \ldots \sigma_{n-2}\ i_{n-1}}$ *are traceless.*

(b) η *is a contact* $(n-1)$*-form such that*

$$\eta = \sum_{|I|=r-1} \omega^\sigma_I \wedge \Psi^I_\sigma, \tag{49}$$

where the forms Ψ^I_σ *do not contain any exterior factor* ω^σ_J *such that* $0 \le |J| \le r-1$.

(c) $\tilde{\rho}$ *has an expression*

$$\begin{aligned}\tilde{\rho} &= A_{i_1 i_2 \ldots i_n} dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_n} \\ &+ A^{I_1}_{\sigma_1\ i_2 i_3 \ldots i_n} dy^{\sigma_1}_{I_1} \wedge dx^{i_2} \wedge dx^{i_3} \wedge \cdots \wedge dx^{i_n} \\ &+ A^{I_1\ I_2}_{\sigma_1\ \sigma_2\ i_3 i_4 \ldots i_n} dy^{\sigma_1}_{I_1} \wedge dy^{\sigma_2}_{I_2} \wedge dx^{i_3} \wedge dx^{i_4} \wedge \cdots \wedge dx^{i_n} \\ &+ \cdots + A^{I_1\ I_2 \ldots I_{n-1}}_{\sigma_1\ \sigma_2 \ldots \sigma_{n-1}\ i_n} dy^{\sigma_1}_{I_1} \wedge dy^{\sigma_2}_{I_2} \wedge \cdots \wedge dy^{\sigma_{n-1}}_{I_{n-1}} \wedge dx^{i_n} \\ &+ A^{I_1\ I_2 \ldots I_n}_{\sigma_1\ \sigma_2 \ldots \sigma_n} dy^{\sigma_1}_{I_1} \wedge dy^{\sigma_2}_{I_2} \wedge \cdots \wedge dy^{\sigma_n}_{I_n},\end{aligned} \tag{50}$$

where $|I_1|, |I_2|, \ldots, |I_n| = r$ *and all the coefficients* $A^{I_1}_{\sigma_1\ i_2 i_3 \ldots i_n}, A^{I_1\ I_2}_{\sigma_1\ \sigma_2\ i_3 i_4 \ldots i_n}, \ldots, A^{I_1\ I_2 \ldots I_{n-1}}_{\sigma_1\ \sigma_2 \ldots \sigma_{n-1}\ i_n}$ *are traceless.*

Proof From the trace decomposition theorem (Sect. 2.2, Theorem 3), ρ can be written as

$$\rho = \rho_{(1)} + \rho_{(2)} + \tilde{\rho}, \tag{51}$$

where $\rho_{(1)}$ includes all ω^σ_J-generated terms, where $0 \le |J| \le r-1$, $\rho_{(2)}$ includes all $d\omega^\sigma_I$-generated terms with $|J| = r-1$ (and does not contain any exterior factor ω^σ_J), and $\tilde{\rho}$ is expressed by (50). Then,

$$\rho_{(2)} = \sum_{|I|=r-1} d\omega^\sigma_I \wedge \Psi^I_\sigma = d\left(\sum_{|I|=r-1} \omega^\sigma_I \wedge \Psi^I_\sigma\right) - \sum_{|I|=r-1} \omega^\sigma_I \wedge d\Psi^I_\sigma, \tag{52}$$

so we get

$$\begin{aligned}\rho &= \rho_{(1)} - \sum_{|I|=r-1} \omega_I^\sigma \wedge d\Psi_\sigma^I + d\left(\sum_{|I|=r-1} \omega_I^\sigma \wedge \Psi_\sigma^I\right) + \tilde{\rho} \\ &= \rho_0 + d\left(\sum_{|I|=r-1} \omega_I^\sigma \wedge \Psi_\sigma^I\right) + \tilde{\rho},\end{aligned} \tag{53}$$

proving Lemma 2. □

Our next aim will be to find the chart expression for the horizontal and 1-contact components of the n-form

$$\tau = \rho_0 + \tilde{\rho} \tag{54}$$

from Lemma 2.

Lemma 3 *Suppose that* τ *has an expression* (46) *and* (50).

(a) *The horizontal component* $h\tau$ *is given by*

$$\begin{aligned}h\tau = (&A_{i_1 i_2 \ldots i_n} + A_{\sigma_1}^{I_1}{}_{i_2 i_3 \ldots i_n} y_{I_1 i_1}^{\sigma_1} + A_{\sigma_1}^{I_1}{}_{\sigma_2}^{I_2}{}_{i_3 i_4 \ldots i_n} y_{I_1 i_1}^{\sigma_1} y_{I_2 i_2}^{\sigma_2} \\ &+ \cdots + A_{\sigma_1}^{I_1}{}_{\sigma_2}^{I_2} \cdots {}_{\sigma_{n-1}}^{I_{n-1}}{}_{i_n} y_{I_1 i_1}^{\sigma_1} y_{I_2 i_2}^{\sigma_2} \cdots y_{I_{n-1} i_{n-1}}^{\sigma_{n-1}} \\ &+ A_{\sigma_1}^{I_1}{}_{\sigma_2}^{I_2} \cdots {}_{\sigma_n}^{I_n} y_{I_1 i_1}^{\sigma_1} y_{I_2 i_2}^{\sigma_2} \cdots y_{I_n i_n}^{\sigma_n}) dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_n}.\end{aligned} \tag{55}$$

(b) *The* 1-*contact component* $p_1\tau$ *is given by*

$$\begin{aligned}p_1\tau = \sum_{0 \le |J| \le r-1} &(\tilde{\Phi}_\sigma^J{}_{i_2 i_3 \ldots i_n} + \tilde{\Phi}_\sigma^J{}_{\sigma_2}^{I_2}{}_{i_3 i_4 \ldots i_n} y_{I_2 i_2}^{\sigma_2} + \tilde{\Phi}_\sigma^J{}_{\sigma_2}^{I_2}{}_{\sigma_3}^{I_3}{}_{i_4 i_5 \ldots i_n} y_{I_2 i_2}^{\sigma_2} y_{I_3 i_3}^{\sigma_3} \\ &+ \cdots + \tilde{\Phi}_\sigma^J{}_{\sigma_2}^{I_2}{}_{\sigma_3}^{I_3} \cdots {}_{\sigma_{n-1}}^{I_{n-1}}{}_{i_n} y_{I_2 i_2}^{\sigma_2} y_{I_3 i_3}^{\sigma_3} \cdots y_{I_{n-1} i_{n-1}}^{\sigma_{n-1}} \\ &+ \tilde{\Phi}_\sigma^J{}_{\sigma_2}^{I_2}{}_{\sigma_3}^{I_3} \cdots {}_{\sigma_n}^{I_n} y_{I_2 i_2}^{\sigma_2} y_{I_3 i_3}^{\sigma_3} \cdots y_{I_n i_n}^{\sigma_n}) \omega_J^\sigma \wedge dx^{i_2} \wedge dx^{i_3} \wedge \cdots \wedge dx^{i_n} \\ &+ \sum_{|I|=r} (A_\sigma^I{}_{i_2 i_3 \ldots i_n} + 2A_{\sigma_1}^I{}_{\sigma_2}^{I_2}{}_{i_3 i_4 \ldots i_n} y_{I_2 i_2}^{\sigma_2} + 3A_\sigma^I{}_{\sigma_2}^{I_2}{}_{\sigma_3}^{I_3}{}_{i_4 i_5 \ldots i_n} y_{I_2 i_2}^{\sigma_2} y_{I_3 i_3}^{\sigma_3} \\ &+ \cdots + (n-1) A_\sigma^I{}_{\sigma_2}^{I_2} \cdots {}_{\sigma_{n-1}}^{I_{n-1}}{}_{i_n} y_{I_2 i_2}^{\sigma_2} y_{I_3 i_3}^{\sigma_3} \cdots y_{I_{n-1} i_{n-1}}^{\sigma_{n-1}} \\ &+ nA_\sigma^I{}_{\sigma_2}^{I_2} \cdots {}_{\sigma_n}^{I_n} y_{I_2 i_2}^{\sigma_2} y_{I_3 i_3}^{\sigma_3} \cdots y_{I_n i_n}^{\sigma_n}) \omega_I^\sigma \wedge dx^{i_2} \wedge dx^{i_3} \wedge \cdots \wedge dx^{i_n}.\end{aligned} \tag{56}$$

Proof

(a) Clearly, $h\tau = h\tilde{\rho}$ and (55) follows.
(b) The form $p_1\tau$ is given by

$$p_1\tau = \sum_{0 \le |J| \le r-1} \omega^\sigma_J \wedge h\Phi^J_\sigma + p_1\tilde{\rho}. \tag{57}$$

Then,

$$\begin{aligned} h\tilde{\Phi}^J_\sigma &= (\tilde{\Phi}^J_{\sigma\ i_1 i_2 \ldots i_{n-1}} + \tilde{\Phi}^{J\ I_1}_{\sigma\ \sigma_1\ i_2 i_3 \ldots i_{n-1}} y^{\sigma_1}_{I_1 i_1} + \tilde{\Phi}^{J\ I_1\ I_2}_{\sigma\ \sigma_1\ \sigma_2\ i_3 i_4 \ldots i_{n-1}} y^{\sigma_1}_{I_1 i_1} y^{\sigma_2}_{I_2 i_2} \\ &\quad + \cdots + \tilde{\Phi}^{J\ I_1\ I_2 \ldots I_{n-2}}_{\sigma\ \sigma_1\ \sigma_2 \ldots \sigma_{n-2}\ i_{n-1}} y^{\sigma_1}_{I_1 i_1} y^{\sigma_2}_{I_2 i_2} \ldots y^{\sigma_{n-2}}_{I_{n-2} i_{n-2}} \\ &\quad + \tilde{\Phi}^{J\ I_1\ I_2 \ldots I_{n-1}}_{\sigma\ \sigma_1\ \sigma_2 \ldots \sigma_{n-1}} y^{\sigma_1}_{I_1 i_1} y^{\sigma_2}_{I_2 i_2} \ldots y^{\sigma_{n-1}}_{I_{n-1} i_{n-1}}) dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_{n-1}} \\ &= (\tilde{\Phi}^J_{\sigma\ i_2 i_3 \ldots i_n} + \tilde{\Phi}^{J\ I_2}_{\sigma\ \sigma_2\ i_3 i_4 \ldots i_n} y^{\sigma_2}_{I_2 i_2} + \tilde{\Phi}^{J\ I_2\ I_3}_{\sigma\ \sigma_2\ \sigma_3\ i_4 i_5 \ldots i_n} y^{\sigma_2}_{I_2 i_2} y^{\sigma_3}_{I_3 i_3} \\ &\quad + \cdots + \tilde{\Phi}^{J\ I_2\ I_3 \ldots I_{n-1}}_{\sigma\ \sigma_2\ \sigma_3 \ldots \sigma_{n-1}\ i_n} y^{\sigma_2}_{I_2 i_2} y^{\sigma_3}_{I_3 i_3} \ldots y^{\sigma_{n-1}}_{I_{n-1} i_{n-1}} \\ &\quad + \tilde{\Phi}^{J\ I_2\ I_3 \ldots I_n}_{\sigma\ \sigma_2\ \sigma_3 \ldots \sigma_n} y^{\sigma_2}_{I_2 i_2} y^{\sigma_3}_{I_3 i_3} \ldots y^{\sigma_n}_{I_n i_n}) dx^{i_2} \wedge dx^{i_3} \wedge \cdots \wedge dx^{i_n}, \end{aligned} \tag{58}$$

and

$$\begin{aligned} p_1\tilde{\rho} &= (A^{I_1}_{\sigma_1\ i_2 i_3 \ldots i_n} + 2A^{I_1\ I_2}_{\sigma_1\ \sigma_2\ i_3 i_4 \ldots i_n} y^{\sigma_2}_{I_2 i_2} + 3A^{I_1\ I_2\ I_3}_{\sigma_1\ \sigma_2\ \sigma_3\ i_4 i_5 \ldots i_n} y^{\sigma_2}_{I_2 i_2} y^{\sigma_3}_{I_3 i_3} \\ &\quad + \cdots + (n-1) A^{I_1\ I_2 \ldots I_{n-1}}_{\sigma_1\ \sigma_2 \ldots \sigma_{n-1}\ i_n} y^{\sigma_2}_{I_2 i_2} y^{\sigma_3}_{I_3 i_3} \ldots y^{\sigma_{n-1}}_{I_{n-1} i_{n-1}} \\ &\quad + nA^{I_1\ I_2 \ldots I_n}_{\sigma_1\ \sigma_2 \ldots \sigma_n} y^{\sigma_2}_{I_2 i_2} y^{\sigma_3}_{I_3 i_3} \ldots y^{\sigma_n}_{I_n i_n}) \omega^{\sigma_1}_{I_1} \wedge dx^{i_2} \wedge dx^{i_3} \wedge \cdots \wedge dx^{i_n} \\ &= \sum_{|I|=r} (A^I_{\sigma\ i_2 i_3 \ldots i_n} + 2A^{I\ I_2}_{\sigma_1\ \sigma_2\ i_3 i_4 \ldots i_n} y^{\sigma_2}_{I_2 i_2} + 3A^{I\ I_2\ I_3}_{\sigma\ \sigma_2\ \sigma_3\ i_4 i_5 \ldots i_n} y^{\sigma_2}_{I_2 i_2} y^{\sigma_3}_{I_3 i_3} \\ &\quad + \cdots + (n-1) A^{I\ I_2 \ldots I_{n-1}}_{\sigma\ \sigma_2 \ldots \sigma_{n-1}\ i_n} y^{\sigma_2}_{I_2 i_2} y^{\sigma_3}_{I_3 i_3} \ldots y^{\sigma_{n-1}}_{I_{n-1} i_{n-1}} \\ &\quad + nA^{I\ I_2 \ldots I_n}_{\sigma\ \sigma_2 \ldots \sigma_n} y^{\sigma_2}_{I_2 i_2} y^{\sigma_3}_{I_3 i_3} \ldots y^{\sigma_n}_{I_n i_n}) \omega^\sigma_I \wedge dx^{i_2} \wedge dx^{i_3} \wedge \cdots \wedge dx^{i_n} \end{aligned} \tag{59}$$

(56) now follows from (58) and (59). □

Now we find the chart expression for the pullback $(\pi^{r+1,r})^*\rho$. According to Lemma 2,

$$(\pi^{r+1,r})^*\rho = h\tilde{\rho} + p_1(\rho_0 + \tilde{\rho}) + d\eta + \mu, \tag{60}$$

where $h\tilde{\rho} = h\tau$ and $p_1\rho_0 + p_1\tilde{\rho}$ are given by Lemma 3, and the order of contactness of μ is ≥2 . We define f_0 and $f^{J\ i}_\sigma$ by the formulas

$$h\tilde{\rho} = f_0\omega_0, \quad p_1(\rho_0 + \tilde{\rho}) = \sum_{0 \le |J| \le r} f^{J\ i}_\sigma \omega^\sigma_J \wedge \omega_i. \tag{61}$$

Explicitly,

$$\begin{aligned} f_0 &= \varepsilon^{i_1 i_2 \ldots i_n} (A_{i_1 i_2 \ldots i_n} + A^{I_1}_{\sigma_1\ i_2 i_3 \ldots i_n} y^{\sigma_1}_{I_1 i_1} + A^{I_1\ I_2}_{\sigma_1\ \sigma_2\ i_3 i_4 \ldots i_n} y^{\sigma_1}_{I_1 i_1} y^{\sigma_2}_{I_2 i_2} \\ &\quad + \cdots + A^{I_1\ I_2 \ldots I_{n-1}}_{\sigma_1\ \sigma_2 \ldots \sigma_{n-1}\ i_n} y^{\sigma_1}_{I_1 i_1} y^{\sigma_2}_{I_2 i_2} \ldots y^{\sigma_{n-1}}_{I_{n-1} i_{n-1}} + A^{I_1\ I_2 \ldots I_n}_{\sigma_1\ \sigma_2 \ldots \sigma_n} y^{\sigma_1}_{I_1 i_1} y^{\sigma_2}_{I_2 i_2} \ldots y^{\sigma_n}_{I_n i_n}), \end{aligned} \tag{62}$$

and, since $\varepsilon^{ii_2i_3\ldots i_n}\omega_i = dx^{i_2}\wedge dx^{i_3}\wedge\cdots\wedge dx^{i_n}$,

$$\begin{aligned} f_\sigma^{J\ i} = \varepsilon^{ii_2i_3\ldots i_n}(&\tilde{\Phi}_{\sigma\ i_2i_3\ldots i_n}^{J} + \tilde{\Phi}_{\sigma\ \sigma_2\ i_3i_4\ldots i_n}^{J\ I_2} y_{I_2i_2}^{\sigma_2} + \tilde{\Phi}_{\sigma\ \sigma_2\ \sigma_3\ i_4i_5\ldots i_n}^{J\ I_2\ I_3} y_{I_2i_2}^{\sigma_2} y_{I_3i_3}^{\sigma_3} \\ &+ \cdots + \tilde{\Phi}_{\sigma\ \sigma_2\ \sigma_3\ldots\sigma_{n-1}\ i_n}^{J\ I_2\ I_3\ldots I_{n-1}} y_{I_2i_2}^{\sigma_2} y_{I_3i_3}^{\sigma_3}\ldots y_{I_{n-1}i_{n-1}}^{\sigma_{n-1}} \\ &+ \tilde{\Phi}_{\sigma\ \sigma_2\ \sigma_3\ldots\sigma_n}^{J\ I_2\ I_3\ldots I_n} y_{I_2i_2}^{\sigma_2} y_{I_3i_3}^{\sigma_3}\ldots y_{I_ni_n}^{\sigma_n}), \end{aligned} \tag{63}$$

and

$$\begin{aligned} f_\sigma^{I\ i} = \varepsilon^{ii_2i_3\ldots i_n}(&A_{\sigma\ i_2i_3\ldots i_n}^{I} + 2A_{\sigma\ \sigma_2\ i_3i_4\ldots i_n}^{I\ I_2} y_{I_2i_2}^{\sigma_2} + 3A_{\sigma\ \sigma_2\ \sigma_3\ i_4i_5\ldots i_n}^{I\ I_2\ I_3} y_{I_2i_2}^{\sigma_2} y_{I_3i_3}^{\sigma_3} \\ &+ \cdots + (n-1)A_{\sigma\ \sigma_2\ldots\sigma_{n-1}\ i_n}^{I\ I_2\ldots I_{n-1}} y_{I_2i_2}^{\sigma_2} y_{I_3i_3}^{\sigma_3}\ldots y_{I_{n-1}i_{n-1}}^{\sigma_{n-1}} \\ &+ nA_{\sigma\ \sigma_2\ldots\sigma_n}^{I\ I_2\ldots I_n} y_{I_2i_2}^{\sigma_2} y_{I_3i_3}^{\sigma_3}\ldots y_{I_ni_n}^{\sigma_n}), \end{aligned} \tag{64}$$

where $0\le |J| \le r-1$ and $|I| = r$.

We further decompose the forms $f_\sigma^{J\ i}\omega_J^\sigma\wedge\omega_i$.

Lemma 4 *For $k\ge 1$, the forms $\omega_{j_1j_2\ldots j_k}^\sigma\wedge\omega_i$ can be decomposed as*

$$\begin{aligned} \omega_{l_1l_2\ldots l_k}^\sigma\wedge\omega_i = \frac{1}{k+1}(&\omega_{l_1l_2\ldots l_k}^\sigma\wedge\omega_i \\ &+ \omega_{il_2l_3\ldots l_k}^\sigma\wedge\omega_{l_1} + \omega_{l_1il_3l_4\ldots l_k}^\sigma\wedge\omega_{l_2} + \cdots + \omega_{l_1l_2\ldots l_{k-1}i}^\sigma\wedge\omega_{l_k}) \\ &+ \frac{1}{k+1}((\omega_{l_1l_2\ldots l_k}^\sigma\wedge\omega_i - \omega_{il_2l_3\ldots l_k}^\sigma\wedge\omega_{l_1}) + (\omega_{l_1l_2\ldots l_k}^\sigma\wedge\omega_i - \omega_{l_1il_3l_4\ldots l_k}^\sigma\wedge\omega_{l_2}) \\ &+ \cdots + (\omega_{l_1l_2\ldots l_k}^\sigma\wedge\omega_i - \omega_{l_1l_2\ldots l_{k-1}i}^\sigma\wedge\omega_{l_k})). \end{aligned} \tag{65}$$

The forms $\omega_{l_1l_2\ldots l_k}^\sigma\wedge\omega_i - \omega_{l_1l_2\ldots l_{p-1}il_{p+1}\ldots l_{k-1}l_k}^\sigma\wedge\omega_{l_p}$ are closed and can be expressed as

$$\omega_{l_1l_2\ldots l_k}^\sigma\wedge\omega_i - \omega_{l_1l_2\ldots l_{p-1}il_{p+1}\ldots l_{k-1}l_k}^\sigma\wedge\omega_{l_p} = d(\omega_{l_1l_2\ldots l_{p-1}l_{p+1}\ldots l_{k-1}l_k}^\sigma\wedge\omega_{il_p}). \tag{66}$$

Proof Indeed, from (43)

$$\begin{aligned} &d\omega_{l_1l_2\ldots l_{p-1}l_{p+1}\ldots l_{k-1}l_k}^\sigma\wedge\omega_{l_pi} \\ &\quad= -\omega_{l_1l_2\ldots l_{p-1}l_{p+1}\ldots l_{k-1}l_kj}^\sigma\wedge dx^j\wedge\omega_{l_pi} \\ &\quad= -\omega_{l_1l_2\ldots l_{p-1}l_{p+1}\ldots l_{k-1}l_kj}^\sigma\wedge dx^j\wedge\omega_{l_pi} \\ &\quad= \omega_{l_1l_2\ldots l_{p-1}l_{p+1}\ldots l_{k-1}l_kj}^\sigma\wedge(\delta_i^j\omega_{l_p} - \delta_{l_p}^j\omega_i) \\ &\quad= -\omega_{l_1l_2\ldots l_{p-1}l_{p+1}\ldots l_{k-1}l_kl_p}^\sigma\wedge\omega_i + \omega_{l_1l_2\ldots l_{p-1}l_{p+1}\ldots l_{k-1}l_ki}^\sigma\wedge\omega_{l_p}. \end{aligned} \tag{67}$$

□

Now we are in a position to prove the following theorem on the structure of n-forms on W^r.

Theorem 2 *Let $\rho \in \Omega_n^r W$. For every fibered chart $(V,\psi), \psi = (x^i, y^\sigma)$, the pull-back $(\pi^{r+1,r})^*\rho$ has an expression*

$$(\pi^{r+1,r})^*\rho = f_0\omega_0 + \sum_{0 \le |J| \le r} P_\sigma^{J\ i}\omega_J^\sigma \wedge \omega_i + d\eta + \mu, \tag{68}$$

where the components $P_\sigma^{J\ i}$ are symmetric in the superscripts, η is a 1-contact form, and μ is a contact form whose order of contactness is ≥2. The functions $P_\sigma^{I\ i}$ such that $|I|\ = r$ satisfy

$$P_\sigma^{I\ i} = \frac{\partial f_0}{\partial y_{Ii}^\sigma}. \tag{69}$$

The forms $f_0\omega_0, \sum P_\sigma^{J\ i}\omega_J^\sigma \wedge \omega_i$ and μ in this decomposition are unique.

Proof We use formulas (60) and (61) and apply Lemma 4 to the forms $f_\sigma^{J\ i}\omega_J^\sigma \wedge \omega_i$. Write with explicit index notation $f_\sigma^{J\ i} = P_\sigma^{j_1 j_2 \ldots j_k\ i}$. We have the decomposition

$$f_\sigma^{j_1 j_2 \ldots j_k\ i} = P_\sigma^{j_1 j_2 \ldots j_k\ i} + Q_\sigma^{j_1 j_2 \ldots j_k\ i}, \tag{70}$$

where $P_\sigma^{j_1 j_2 \ldots j_k\ i} = f_\sigma^{j_1 j_2 \ldots j_k\ i}$ $\mathrm{Sym}(j_1 j_2 \ldots j_k i)$ is the symmetric component, and $Q_\sigma^{j_1 j_2 \ldots j_k\ i}$ is the complementary component of the system $f_\sigma^{j_1 j_2 \ldots j_k\ i}$. We have, for each k, $1 \le k \le r$,

$$\begin{aligned} f_\sigma^{j_1 j_2 \ldots j_k\ i}\omega_{j_1 j_2 \ldots j_k}^\sigma \wedge \omega_i &= P_\sigma^{j_1 j_2 \ldots j_k\ i}\omega_{j_1 j_2 \ldots j_k}^\sigma \wedge \omega_i - \frac{1}{k+1} Q_\sigma^{j_1 j_2 \ldots j_k\ i} d(\omega_{j_2 j_3 \ldots j_k}^\sigma \wedge \omega_{j_1 i} \\ &\quad + \omega_{j_1 j_3 j_4 \ldots j_k}^\sigma \wedge \omega_{j_2 i} + \cdots + \omega_{j_1 j_2 \ldots j_{k-1}}^\sigma \wedge \omega_{j_k i}) \\ &= P_\sigma^{j_1 j_2 \ldots j_k\ i}\omega_{j_1 j_2 \ldots j_k}^\sigma \wedge \omega_i - \frac{1}{k+1} d(Q_\sigma^{j_1 j_2 \ldots j_k\ i}(\omega_{j_2 j_3 \ldots j_k}^\sigma \wedge \omega_{j_1 i} \\ &\quad + \omega_{j_1 j_3 j_4 \ldots j_k}^\sigma \wedge \omega_{j_2 i} + \cdots + \omega_{j_1 j_2 \ldots j_{k-1}}^\sigma \wedge \omega_{j_k i})) \\ &\quad + \frac{1}{k+1} dQ_\sigma^{j_1 j_2 \ldots j_k\ i} \wedge (\omega_{j_2 j_3 \ldots j_k}^\sigma \wedge \omega_{j_1 i} + \omega_{j_1 j_3 j_4 \ldots j_k}^\sigma \wedge \omega_{j_2 i} \\ &\quad + \cdots + \omega_{j_1 j_2 \ldots j_{k-1}}^\sigma \wedge \omega_{j_k i}). \end{aligned} \tag{71}$$

The exterior derivative $dQ_\sigma^{j_1 j_2 \ldots j_k\ i}$, when lifted to the set V^{r+2}, can be decomposed as

$$\begin{aligned} (\pi^{r+2,r+1})^* dQ_\sigma^{j_1 j_2 \ldots j_k\ i} &= hdQ_\sigma^{j_1 j_2 \ldots j_k\ i} + pdQ_\sigma^{j_1 j_2 \ldots j_k\ i} \\ &= d_p Q_\sigma^{j_1 j_2 \ldots j_k\ i} dx^p + pdQ_\sigma^{j_1 j_2 \ldots j_k\ i}. \end{aligned} \tag{72}$$

Substituting from (72) back to (71), we get 1-contact and a 2-contact summands. The 1-contact summands are equal to

$$\begin{aligned}
&hdQ_\sigma^{j_1j_2\ldots j_k\ i} \wedge (\omega^\sigma_{j_2j_3\ldots j_k} \wedge \omega_{j_1i} + \omega^\sigma_{j_1j_3j_4\ldots j_k} \wedge \omega_{j_2i} + \cdots + \omega^\sigma_{j_1j_2\ldots j_{k-1}} \wedge \omega_{j_ki}) \\
&\quad = -d_pQ_\sigma^{j_1j_2\ldots j_k\ i}(\omega^\sigma_{j_2j_3\ldots j_k} \wedge dx^p \wedge \omega_{j_1i} + \omega^\sigma_{j_1j_3j_4\ldots j_k} \wedge dx^p \wedge \omega_{j_2i} \\
&\qquad + \cdots + \omega^\sigma_{j_1j_2\ldots j_{k-1}} \wedge dx^p \wedge \omega_{j_ki}) \\
&\quad = -d_pQ_\sigma^{j_1j_2\ldots j_k\ i}(\omega^\sigma_{j_2j_3\ldots j_k} \wedge (\delta^p_{j_1}\omega_i - \delta^p_i\omega_{j_1}) \\
&\qquad + \omega^\sigma_{j_1j_3j_4\ldots j_k} \wedge (\delta^p_{j_2}\omega_i - \delta^p_i\omega_{j_2}) + \cdots + \omega^\sigma_{j_1j_2\ldots j_{k-1}} \wedge (\delta^p_{j_k}\omega_i - \delta^p_i\omega_{j_k})) \qquad (73) \\
&\quad = -(d_pQ_\sigma^{pj_2j_3\ldots j_k\ i}\omega^\sigma_{j_2j_3\ldots j_k} + d_pQ_\sigma^{j_1pj_3j_4\ldots j_k\ i}\omega^\sigma_{j_1j_3j_4\ldots j_k} \\
&\qquad + \cdots + d_pQ_\sigma^{j_1j_2\ldots j_{k-1}p\ i}\omega^\sigma_{j_1j_2\ldots j_{k-1}})\omega_i + d_pQ_\sigma^{j_1j_2\ldots j_k\ p}(\omega^\sigma_{j_2j_3\ldots j_k} \wedge \omega_{j_1} \\
&\qquad + \omega^\sigma_{j_1j_3j_4\ldots j_k} \wedge \omega_{j_2} + \cdots + \omega^\sigma_{j_1j_2\ldots j_{k-1}} \wedge \omega_{j_k}) \\
&\quad = -kd_p(Q_\sigma^{pj_2j_3\ldots j_k\ i} - Q_\sigma^{ij_2j_3\ldots j_k\ p})\omega^\sigma_{j_2j_3\ldots j_k} \wedge \omega_i.
\end{aligned}$$

Note that from the definition of the functions $Q_\sigma^{pj_2j_3\ldots j_k\ i}$ and from formula (63), we easily see that this form is $\pi^{r+2,r+1}$-projectable. Thus, returning to (71), we have on V^{r+1}

$$\begin{aligned}
f_\sigma^{j_1j_2\ldots j_k\ i}\omega^\sigma_{j_1j_2\ldots j_k} \wedge \omega_i = P_\sigma^{j_1j_2\ldots j_k\ i}&\omega^\sigma_{j_1j_2\ldots j_k} \wedge \omega_i \\
&- \frac{k}{k+1}d_p(Q_\sigma^{pj_2j_3\ldots j_k\ i} - Q_\sigma^{ij_2j_3\ldots j_k\ p})\omega^\sigma_{j_2j_3\ldots j_k} \wedge \omega_i \\
&- \frac{1}{k+1}d(Q_\sigma^{j_1j_2\ldots j_k\ i}(\omega^\sigma_{j_2j_3\ldots j_k} \wedge \omega_{j_1i} + \omega^\sigma_{j_1j_3j_4\ldots j_k} \wedge \omega_{j_2i} \\
&+ \cdots + \omega^\sigma_{j_1j_2\ldots j_{k-1}} \wedge \omega_{j_ki})) \\
&+ \frac{1}{k+1}pdQ_\sigma^{j_1j_2\ldots j_k\ i} \wedge (\omega^\sigma_{j_2j_3\ldots j_k} \wedge \omega_{j_1i} + \omega^\sigma_{j_1j_3j_4\ldots j_k} \wedge \omega_{j_2i} \\
&+ \cdots + \omega^\sigma_{j_1j_2\ldots j_{k-1}} \wedge \omega_{j_ki}). \qquad (74)
\end{aligned}$$

This sum replaces $f_\sigma^{J\ i}\omega^\sigma_J \wedge \omega_i$, where $|J| = k$, with the symmetrized term $P_\sigma^{J\ i}\omega^\sigma_J \wedge \omega_i$, a term $d_p(Q_\sigma^{pj_2j_3\ldots j_k\ i} - Q_\sigma^{ij_2j_3\ldots j_k\ p})\omega^\sigma_{j_2j_3\ldots j_k} \wedge \omega_i$ containing $\omega^\sigma_J \wedge \omega_i$ with $|J| = k-1$, a closed form, and a 2-contact term.

Using these expressions in (60), written as

$$(\pi^{r+1,r})^*\rho = f_0\omega_0 + \sum_{0 \le |J| \le r} f_\sigma^{J\ i}\omega^\sigma_J \wedge \omega_i + d\eta + \mu, \qquad (75)$$

we can redefine the coefficients and get

$$(\pi^{r+1,r})^*\rho = f_0\omega_0 + \sum_{0 \le |J| \le r-1} f_\sigma^{J\ i} \omega_J^\sigma \wedge \omega_i + \sum_{|J| \le r} P_\sigma^{J\ i} \omega_J^\sigma \wedge \omega_i + d\eta + \mu. \quad (76)$$

After r steps, we get (68).

To prove (69), we differentiate (62) and compare the result with (64).

It remains to prove uniqueness of the decomposition (68). Supposing that $(\pi^{r+1,r})^*\rho = 0$, we immediately obtain $f_0\omega_0 = 0$ and $\mu = 0$; hence,

$$\sum_{0 \le |J| \le r} P_\sigma^{J\ i} \omega_J^\sigma \wedge \omega_i + d\eta = 0. \quad (77)$$

Differentiating (77) and taking into account the 1-contact component of the resulting $(n + 1)$-form,

$$\begin{aligned} &\sum_{0 \le |J| \le r} p_1(dP_\sigma^{J\ i} \wedge \omega_J^\sigma \wedge \omega_i - P_\sigma^{J\ i} \omega_{Ji}^\sigma \wedge \omega_0) \\ &= -\sum_{0 \le |J| \le r} (d_i P_\sigma^{J\ i} \wedge \omega_J^\sigma - P_\sigma^{J\ i} \omega_{Ji}^\sigma) \wedge \omega_0 = 0, \end{aligned} \quad (78)$$

which is only possible when $P_\sigma^{J\ i} = 0$ because $P_\sigma^{J\ i}$ are symmetric in the superscripts. □

In the following lemma, we consider vector fields on any fibered manifold Y with base X and projection π.

Lemma 5 *Let ξ be a vector field on X. There exists a π-projectable vector field $\tilde{\xi}$ on Y whose π-projection is ξ.*

Proof We can construct $\tilde{\xi}$ by means of an atlas on Y, consisting of fibered charts, and a subordinate partition of unity (cf. Theorem 1, Sect. 4.2). □

Now we study properties of differential n-forms ρ, defined on $W^r \subset J^rY$, which play a key role in global variational geometry. To this purpose, we write the decomposition formula (68) as

$$(\pi^{r+1,r})^*\rho = f_0\omega_0 + P_\sigma^{\ i}\omega^\sigma \wedge \omega_i + \sum_{k=1}^{r} P_\sigma^{j_1 j_2 \ldots j_k\ i} \omega_{j_1 j_2 \ldots j_k}^\sigma \wedge \omega_i + d\eta + \mu, \quad (79)$$

where

$$P_\sigma^{j_1 j_2 \ldots j_r\ i} = \frac{\partial f_0}{\partial y_{j_1 j_2 \ldots j_r i}^\sigma}. \quad (80)$$

Lemma 6 *Let $\rho \in \Omega_n^r W$. The following three conditions are equivalent*:

(a) *$p_1 d\rho$ is a $\pi^{r+1,0}$-horizontal $(n+1)$-form.*
(b) *For each $\pi^{r,0}$-vertical vector field ξ on W^r,*

$$hi_\xi d\rho = 0. \tag{81}$$

(c) *The pullback $(\pi^{r+1,r})^*\rho$ has the chart expression (79), such that the coefficients satisfy*

$$\frac{\partial f_0}{\partial y^\sigma_{j_1 j_2 \ldots j_k}} - d_i P_\sigma^{j_1 j_2 \ldots j_k\ i} - P_\sigma^{j_1 j_2 \ldots j_{k-1}\ j_k} = 0, \quad k = 1, 2, \ldots, r. \tag{82}$$

(d) *$p_1 d\rho$ belongs to the ideal on the exterior algebra on W^{r+1}, locally generated by the forms ω^σ.*

Proof

1. Let Ξ be a vector field on W^r, $\tilde{\Xi}$ a vector field on W^{r+1} such that $T\pi^{r+1,r} \cdot \tilde{\Xi} = \Xi \circ \pi^{r+1,r}$ (Lemma 5). Then, $i_{\tilde{\Xi}}(\pi^{s+1,s})^* d\rho = (\pi^{s+1,s})^* i_\Xi d\rho$, and the forms on both sides can canonically be decomposed into their contact components. We have

$$i_{\tilde{\Xi}} p_1 d\rho + i_{\tilde{\Xi}} p_2 d\rho + \cdots + i_{\tilde{\Xi}} p_{n+1} d\rho = hi_\Xi d\rho + p_1 i_\Xi d\rho + \cdots + p_n i_\Xi d\rho. \tag{83}$$

Comparing the horizontal components on both sides, we get

$$hi_{\tilde{\Xi}} p_1 d\rho = (\pi^{r+2,r+1})^* hi_\Xi d\rho. \tag{84}$$

Let $p_1 d\rho$ be $\pi^{r+1,0}$-horizontal. Then if Ξ is $\pi^{r,0}$-vertical, $\tilde{\Xi}$ is $\pi^{r+1,0}$-vertical, and we get $hi_{\tilde{\Xi}} p_1 d\rho = (\pi^{r+2,r+1})^* hi_\Xi d\rho = 0$, which implies, by injectivity of the mapping $(\pi^{r+2,r+1})^*$ that $hi_\Xi d\rho = 0$.
Conversely, let $hi_\Xi d\rho = 0$ for each $\pi^{r,0}$-vertical vector field ξ. Then by (84), $hi_{\tilde{\Xi}} p_1 d\rho = i_{\tilde{\Xi}} p_1 d\rho = 0$ for all $\pi^{r+1,r}$-projectable, $\pi^{r+1,0}$-vertical vector fields $\tilde{\Xi}$. If in a fibered chart,

$$\tilde{\Xi} = \sum_{k=1}^{r} \Xi^\sigma_{j_1 j_2 \ldots j_k} \frac{\partial}{\partial y^\sigma_{j_1 j_2 \ldots j_k}} \tag{85}$$

and

$$p_1 d\rho = \sum_{k=0}^{r} A_\sigma^{j_1 j_2 \ldots j_k} \omega^\sigma_{j_1 j_2 \ldots j_k} \wedge \omega_0, \tag{86}$$

then we get

$$A_{\sigma}^{j_1 j_2 \dots j_k} = 0, \quad 1 \le k \le r, \tag{87}$$

proving $\pi^{r+1,0}$-horizontality of $p_1 d\rho$. This proves that conditions (a) and (b) are equivalent.

2. Express $(\pi^{r+1,r})^*\rho$ in a fibered chart by (79). Then,

$$\begin{aligned} p_1 d\rho = & \left(\frac{\partial f_0}{\partial y^{\sigma}} - d_i P_{\sigma}{}^{i}\right)\omega^{\sigma} \wedge \omega_0 \\ & + \sum_{k=1}^{r}\left(\frac{\partial f_0}{\partial y^{\sigma}_{j_1 j_2 \dots j_k}} - d_i P_{\sigma}^{j_1 j_2 \dots j_k\ i} - P_{\sigma}^{j_1 j_2 \dots j_{k-1}\ j_k}\right)\omega^{\sigma}_{j_1 j_2 \dots j_k} \wedge \omega_0 \\ & + \left(\frac{\partial f_0}{\partial y^{\sigma}_{j_1 j_2 \dots j_{r+1}}} - P_{\sigma}^{j_1 j_2 \dots j_r\ j_{r+1}}\right)\omega^{\sigma}_{j_1 j_2 \dots j_r j_{r+1}} \wedge \omega_0 \end{aligned} \tag{88}$$

Formula (88) proves equivalence of conditions (a) and (c).

3. Conditions (a) and (d) are obviously equivalent. □

Any form $\rho \in \Omega_n^r W$ such that the 1-contact form $p_1 d\rho$ is $\pi^{r+1,0}$-horizontal, is called a *Lepage form*. Lepage forms may equivalently be defined by any of the equivalent conditions of Lemma 6.

Remark 5 (Existence of Lepage forms) It is easily seen that the system (82) has always a solution, and the solution is unique. Indeed,

$$\begin{aligned} P_{\sigma}^{j_1 j_2 \dots j_{k-1}\ j_k} & = \frac{\partial f_0}{\partial y^{\sigma}_{j_1 j_2 \dots j_k}} - d_{i_1} P_{\sigma}^{j_1 j_2 \dots j_k\ i_1} \\ & = \frac{\partial f_0}{\partial y^{\sigma}_{j_1 j_2 \dots j_k}} - d_{i_1}\left(\frac{\partial f_0}{\partial y^{\sigma}_{j_1 j_2 \dots j_k i_1}} - d_{i_2} P_{\sigma}^{j_1 j_2 \dots j_k i_1\ i_2}\right) \\ & = \frac{\partial f_0}{\partial y^{\sigma}_{j_1 j_2 \dots j_k}} - d_{i_1}\frac{\partial f_0}{\partial y^{\sigma}_{j_1 j_2 \dots j_k i_1}} + d_{i_1} d_{i_2} P_{\sigma}^{j_1 j_2 \dots j_{k-1} i_1\ i_2} \\ & = \frac{\partial f_0}{\partial y^{\sigma}_{j_1 j_2 \dots j_k}} - d_{i_1}\frac{\partial f_0}{\partial y^{\sigma}_{j_1 j_2 \dots j_k i_1}} + d_{i_1} d_{i_2}\left(\frac{\partial f_0}{\partial y^{\sigma}_{j_1 j_2 \dots j_{k-1} i_1 i_2}} - d_{i_3} P_{\sigma}^{j_1 j_2 \dots j_{k-1} i_1 i_2\ i_3}\right) \\ & = \ldots = \sum_{l=0}^{r+1-k} (-1)^l d_{i_1} d_{i_2} \ldots d_{i_l} \frac{\partial f_0}{\partial y^{\sigma}_{j_1 j_2 \dots j_k i_1 i_2 \dots i_l}}, \end{aligned} \tag{89}$$

so the coefficients $P^{j_1}_{\sigma}, P^{j_1 j_2 \ldots j_{k-1}\, j_k}_{\sigma}$ are completely determined by the function f_0. In particular, Lepage forms always exist over fibered coordinate neighborhoods. One can also interpret this result in such a way that to any form $\rho \in \Omega^r_n W$ and any fibered chart $(V, \psi), \psi = (x^i, y^\sigma)$, on W, one can always assign a Lepage form, belonging to the module $\Omega^{r+1}_n V$. Note that we have already considered conditions (82) in connection with the integrability condition for formal differential equations (cf. Sect. 3.2, Lemma 3).

Theorem 3 *A form $\rho \in \Omega^r_n W$ is a Lepage form if and only if for every fibered chart $(V, \psi), \psi = (x^i, y^\sigma)$, on Y such that $V \subset W$, $(\pi^{r+1,r})^*\rho$ has an expression*

$$(\pi^{r+1,r})^*\rho = \Theta + d\eta + \mu, \tag{90}$$

where

$$\Theta = f_0\omega_0 + \sum_{k=0}^{r}\left(\sum_{l=0}^{r-k}(-1)^l d_{p_1} d_{p_2} \ldots d_{p_l}\frac{\partial f_0}{\partial y^\sigma_{j_1 j_2 \ldots j_k p_1 p_2 \ldots p_l i}}\right)\omega^\sigma_{j_1 j_2 \ldots j_k} \wedge \omega_i, \tag{91}$$

f_0 is a function, defined by the chart expression $h\rho = f_0\omega_0$, η is a 1-contact form, and μ is a contact form whose order of contactness is ≥ 2.

Proof Suppose we have a Lepage form ρ expressed by (79) where conditions (82) are satisfied, and consider conditions (59). Then repeating (89), we get formula (91). The converse follows from (88) and (79). □

The n-form Θ defined by (91) is sometimes called the *principal component* of the Lepage form ρ with respect to the fibered chart (V, ψ). Note that Θ depends only on the Lagrangian $h\rho = \lambda_\rho$ associated with ρ; the forms Θ constructed this way are defined only locally, but their horizontal components define a global form.

Remark 6 Equation (82) include conditions ensuring that the order of the functions $P^{j_1 j_2 \ldots j_k\, i}_{\sigma}$ does not exceed the order of f_0. We obtained these conditions using polynomiality of the expression on the left-hand side in the jet variables $y^\sigma_{j_1 j_2 \ldots j_k}, k > r + 1$. Similarly, when Θ is expressed by (91), the order restrictions apply to f_0 since the coefficients at $\omega^\sigma_{j_1 j_2 \ldots j_k} \wedge \omega_i$ should be of order $\leq r + 1$.

4.4 Euler–Lagrange Forms

We defined in Sect. 4.3 a Lepage form $\rho \in \Omega^r_n W$ by a condition on the exterior derivative $\rho \in \Omega^r_n W$, derived from the fibered manifold structure on Y. Namely, we required that the 1-contact component $p_1 d\rho$ should belong to the ideal of forms, defined on W^{r+1}, generated in any fibered chart $(V, \psi), \psi = (x^i, y^\sigma)$, by the contact 1-forms ω^σ. Now we study properties of the exterior derivative $d\rho$. We express a Lepage form ρ as in formula (89), Sect. 4.3.

Theorem 4 *If $\rho \in \Omega_n^r W$ is a Lepage form, then the form $(\pi^{r+1,r})^* d\rho$ has an expression*

$$(\pi^{r+1,r})^* d\rho = E + F, \tag{92}$$

where E is a 1-contact, $(\pi^{r+1,0})$-horizontal $(n+1)$-form, and F is a form whose order of contactness is ≥2. E is unique and has the chart expression

$$E = \left(\frac{\partial f_0}{\partial y^\sigma} - \sum_{l=1}^{r+1} (-1)^{l-1} d_{p_1} d_{p_2} \ldots d_{p_l} \frac{\partial f_0}{\partial y^\sigma_{p_1 p_2 \ldots p_l}} \right) \omega^\sigma \wedge \omega_0. \tag{93}$$

Proof For any $\rho, E = p_1 d\rho$ and $F = p_2 d\rho + p_3 d\rho + \cdots + p_{n+1} d\rho$. But for a Lepage form ρ,

$$E = p_1 d\Theta = \left(\frac{\partial f_0}{\partial y^\sigma} - d_i P^i_\sigma \right) \omega^\sigma \wedge \omega_0, \tag{94}$$

where by Sect. 4.3, (89),

$$P_\sigma{}^i = \sum_{l=0}^{s} (-1)^l d_{p_1} d_{p_2} \ldots d_{p_l} \frac{\partial f_0}{\partial y^\sigma_{p_1 p_2 \ldots p_l i}}. \tag{95}$$

This proves formula (93). □

Note that similarly as the form Θ, E depends only on the Lagrangian $\lambda_\rho = f_0 \omega_0$, associated with Θ. The $(n+1)$-form E is called the *Euler–Lagrange form*, associated with the Lepage form ρ, or with the Lagrangian $\lambda_\rho = f_0 \omega_0$. The components of E

$$E_\sigma(f_0) = \frac{\partial f_0}{\partial y^\sigma} - \sum_{l=1}^{r+1} (-1)^{l-1} d_{p_1} d_{p_2} \ldots d_{p_l} \frac{\partial f_0}{\partial y^\sigma_{p_1 p_2 \ldots p_l}} \tag{96}$$

are called the *Euler–Lagrange expressions* of the function f_0, or of the Lagrangian λ_ρ (in the given fibered chart).

4.5 Lepage Equivalents and the Euler–Lagrange Mapping

Our aim now will be to study Lepage forms with fixed horizontal components – the Lagrangians. As before, denote by $\Omega^r_{n,X} W$ the submodule of the module $\Omega^r_n W$, formed by π^r*-horizontal n-forms* (*Lagrangians of order r* for Y). Clearly, the set $\Omega^r_{n,X} W$ contains the Lagrangians λ_η, associated with the n-forms $\eta \in \Omega^{r-1}_n W$, defined on W^{r-1}.

The following is an existence theorem of Lepage forms whose horizontal component is given.

Theorem 5 *To any Lagrangian $\lambda \in \Omega^r_{n,X}W$, there exists an integer $s \leq 2r-1$ and a Lepage form $\rho \in \Omega^s_n W$ of order of contactness ≤1 such that*

$$h\rho = \lambda. \tag{97}$$

Proof We show that the theorem is true for $s = 2r-1$. Choose an atlas $\{(V_\iota, \psi_\iota)\}$ on Y, consisting of fibered charts $(V_\iota, \psi_\iota), \psi_\iota = (x^i_\iota, y^\sigma_\iota)$, and a partition of unity $\{\chi_\iota\}$, subordinate to the covering $\{V_\iota\}$ of the fibered manifold Y. The functions χ_ι define (global) Lagrangians $\chi_\iota \lambda \in \Omega^r_{n,X}W$. We have in the chart (V_ι, ψ_ι)

$$\lambda = \mathscr{L}_\iota \omega_{0,\iota}, \tag{98}$$

where $\omega_{0,\iota} = dx^1_\iota \wedge dx^2_\iota \wedge \cdots \wedge dx^n_\iota$. Then, we set for each ι

$$\begin{aligned}\Theta_\iota &= \chi_\iota \mathscr{L}_\iota \omega_{0,\iota} \\ &\quad + \sum_{k=0}^{r-1} \left(\sum_{l=0}^{r-1-k} (-1)^l d_{p_1} d_{p_2} \ldots d_{p_l} \frac{\partial(\chi_\iota \mathscr{L}_\iota)}{\partial y_{(\iota)}{}^{\sigma}_{j_1 j_2 \ldots j_k p_1 p_2 \ldots p_l i}} \right) \omega^\sigma_{j_1 j_2 \ldots j_k,\iota} \wedge \omega_{0,\iota},\end{aligned} \tag{99}$$

where $\omega^\sigma_{j_1 j_2 \ldots j_k,\iota} = dy^\sigma_{j_1 j_2 \ldots j_k,\iota} - y^\sigma_{j_1 j_2 \ldots j_k l,\iota} dx^l_\iota$. Thus, Θ_ι is the principal Lepage equivalent of the Lagrangian $\lambda = \mathscr{L}_\iota \omega_{0,\iota}$. Since the family $\{\chi_\iota\}$ is locally finite, the family $\{\Theta_\iota\}$ is also locally finite; thus, the sum $\rho = \sum \Theta_\iota$ is defined. Then, we have $p_1 d\rho = \sum p_1 d\Theta_\iota$; thus, ρ is a Lepage form, because each of the forms Θ_ι is Lepage. It remains to show that $h\rho = \lambda$. We have $h\rho = \sum h\Theta_\iota = \sum \chi_\iota \mathscr{L}_\iota \omega_{0,\iota}$. To compute this expression, choose a fibered chart $(V, \psi), \psi = (x^i, y^\sigma)$, such that the intersection $V \cap V_\iota$ is non-void for only finitely many indices ι. Using this chart, we have $\lambda = \mathscr{L}_\iota \omega_{0,\iota} = \mathscr{L}\omega_0$ on $V \cap V_\iota$ and, since

$$\omega_{0,\iota} = \det\left(\frac{\partial x^i_\iota}{\partial x^j}\right) \cdot \omega_0, \tag{100}$$

then,

$$\mathscr{L}_\iota \det\left(\frac{\partial x^i_\iota}{\partial x^j}\right) = \mathscr{L}. \tag{101}$$

Consequently,

$$h\rho = \sum \chi_\iota \mathscr{L}_\iota \omega_{0,\iota} = \sum \chi_\iota \mathscr{L}_\iota \det\left(\frac{\partial x^i_\iota}{\partial x^j}\right) \cdot \omega_0 = (\sum \chi_\iota) \mathscr{L}\omega_0 = \mathscr{L}\omega_0 \tag{102}$$

because $\sum \chi_\iota = 1$.

Let $\lambda \in \Omega^r_{n,X}W$ be a Lagrangian. A Lepage form $\rho \in \Omega^s_n W$ such that $h\rho = \lambda$ (possibly up to a canonical jet projection) is called a *Lepage equivalent* of λ.

If λ is expressed in a fibered chart (V,ψ), $\psi=(x^i,y^\sigma)$, as

$$\lambda=\mathscr{L}\omega_0, \tag{103}$$

then the form

$$\Theta_{\mathscr{L}}=\mathscr{L}\omega_0+\sum_{k=0}^{r-1}\left(\sum_{l=0}^{r-1-k}(-1)^l d_{p_1}d_{p_2}\ldots d_{p_l}\frac{\partial\mathscr{L}}{\partial y^\sigma_{j_1j_2\ldots j_kp_1p_2\ldots p_li}}\right)\omega^\sigma_{j_1j_2\ldots j_k}\wedge\omega_i \tag{104}$$

is called the *principal Lepage equivalent* of λ for the fibered chart (V,ψ). This form is in general defined on the set $V^{2r-1}\subset W^{2r-1}$. □

Remark 7 The Lepage equivalent constructed in the proof of Theorem 5 is $\pi^{2r-1,r-1}$-horizontal, and its order of contactness is ≤1.

Remark 8 Theorem 5 says that the class of variational functionals, associated with the variational structures (W,ρ), introduced in Sect. 4.1, remains the same when we restrict ourselves to *Lepage* forms ρ. Thus, from now on, we may suppose without loss of generality that the variational functionals

$$\Gamma_\Omega(\pi|_W)\ni\gamma\to\rho_\Omega(\gamma)=\int_\Omega J^r\gamma^*\rho\in\mathbf{R} \tag{105}$$

are defined by Lepage forms.

We give two basic examples of Lepage equivalents of Lagrangians.

Example 1 (Lepage forms of order 1) If $\lambda=\mathscr{L}\omega_0$ is a Lagrangian of order 1, then its principal Lepage equivalent is given by

$$\Theta_\lambda=\mathscr{L}\omega_0+\frac{\partial\mathscr{L}}{\partial y^\sigma_i}\omega^\sigma\wedge\omega_i. \tag{106}$$

The form (106) is called, due to Garcia [G], the *Poincare-Cartan form*. Its invariance with respect to transformations of fibered charts can be proved by a direct calculation (see Example 2).

Example 2 (Lepage forms of order 2) The principal Lepage equivalent of a second-order Lagrangian $\lambda=\mathscr{L}\omega_0$ is given by

$$\Theta_{\mathscr{L}}=\mathscr{L}\omega_0+\left(\frac{\partial\mathscr{L}}{\partial y^\sigma_i}-d_j\frac{\partial\mathscr{L}}{\partial y^\sigma_{ij}}\right)\omega^\sigma\wedge\omega_i+\frac{\partial\mathscr{L}}{\partial y^\sigma_{ij}}\omega^\sigma_j\wedge\omega_i \tag{107}$$

(Krupka [K13]). We show that in this case, $\Theta_{\mathscr{L}}$ is invariant with respect to all transformations of fibered coordinates. It is sufficient to show that $\Theta_{\mathscr{L}}$ can be introduced in a unique way by invariant conditions. We define a form Θ on W^3 by the following three conditions:

(a) Θ is a Lepage form, that is, $p_1 d\Theta$ is $\pi^{3,0}$-horizontal.
(b) The horizontal component of Θ coincides with the given Lagrangian λ; this condition reads $h\Theta = \lambda$.

To state the third condition, we assign to any fibered chart $(V, \psi), \psi = (x^i, y^\sigma)$, the contact forms $\omega_j^\sigma \wedge \omega_i$. One can easily derive the transformation properties of these forms. For any other fibered chart $(\bar{V}, \bar{\psi}), \bar{\psi} = (\bar{x}^i, \bar{y}^\sigma)$, the local volume elements satisfy on the intersection $V \cap \bar{V}$

$$\omega_0 = \det\left(\frac{\partial x^p}{\partial \bar{x}^p}\right)\bar{\omega}_0. \tag{108}$$

Using this formula, we get

$$\omega_i = i_{\partial/\partial x^i}\omega_0 = \frac{\partial \bar{x}^l}{\partial x^i}\det\left(\frac{\partial x^p}{\partial \bar{x}^q}\right) \cdot i_{\partial/\partial \bar{x}^l}\bar{\omega}_0 = \frac{\partial \bar{x}^l}{\partial x^i}\det\left(\frac{\partial x^p}{\partial \bar{x}^q}\right) \cdot \bar{\omega}_l. \tag{109}$$

On the other hand, we know that

$$\omega_j^\sigma = \frac{\partial y_j^\sigma}{\partial \bar{y}^\tau}\bar{\omega}^\tau + \frac{\partial y_j^\sigma}{\partial \bar{y}_j^\tau}\bar{\omega}_j^\tau = \frac{\partial y_j^\sigma}{\partial \bar{y}^\tau}\bar{\omega}^\tau + \frac{\partial y^\sigma}{\partial \bar{y}^\tau}\frac{\partial \bar{x}^l}{\partial x^j}\bar{\omega}_l^\tau \tag{110}$$

(Sect. 2.1, Theorem 1, Sect. 1.4, Example 5). These formulas imply

$$\begin{aligned}\omega_j^\sigma \wedge \omega_i = \det\left(\frac{\partial x^p}{\partial \bar{x}^q}\right)\frac{\partial y_j^\sigma}{\partial \bar{y}^\tau}\frac{\partial \bar{x}^l}{\partial x^i}\bar{\omega}^\tau \wedge \bar{\omega}_l \\ + \det\left(\frac{\partial x^p}{\partial \bar{x}^q}\right)\frac{\partial y^\sigma}{\partial \bar{y}^\tau}\frac{\partial \bar{x}^l}{\partial x^i}\frac{\partial \bar{x}^k}{\partial x^j}\bar{\omega}_k^\tau \wedge \bar{\omega}_l.\end{aligned} \tag{111}$$

In particular, the forms $\omega_i^\sigma \wedge \omega_j + \omega_j^\sigma \wedge \omega_i$ locally generate a submodule of the module $\Omega_n^3(W^3)$. For the purpose of this example, we denote this submodule by $\Theta_{n,1}^3(W^3)$. Now we require, in addition to conditions (a) and (b),

(c) $\Theta \in \Theta_{n,1}^3(W^3)$.

Conditions (a), (b), and (c) uniquely define an n-form on W^3, and this n-form is obviously the form $\Theta_{\mathscr{L}}$ (107). Consequently, the principal Lepage equivalent $\Theta_{\mathscr{L}}$ of a second-order Lagrangian λ is globally well-defined. We usually write Θ_λ instead of $\Theta_{\mathscr{L}}$.

Choosing for any Lagrangian $\lambda \in \Omega^r_{n,X}W$ a Lepage equivalent ρ of λ, we can construct the Euler–Lagrange form E associated with ρ (93); this $(n + 1)$-form depends on λ only. We denote this form by E_λ and call it the *Euler–Lagrange form, associated with* λ. Clearly, E_λ may be defined by (local) principal Lepage equivalents $\Theta_{\mathscr{L}}$. Denoting by $\Omega^{2r-1}_{n+1,Y}W$ the module of $\pi^{2r-1,0}$-*horizontal* $(n + 1)$-*forms* on W^{2r-1}, we get the mapping

$$\Omega^r_{n,X}W \ni \lambda \to E_\lambda \in \Omega^r_{n+1,Y}W \tag{112}$$

called the *Euler–Lagrange mapping*.

Remark 9 We can summarize basic motivations and properties of the Lepage forms by means of their relationship to the Euler–Lagrange forms. Denote by $\mathrm{Lep}^r_n W$ the vector subspace of the real vector space $\Omega^r_n W$, whose elements are Lepage forms. Taking into account properties of the exterior derivative of a Lepage form, we see that the Euler–Lagrange mapping makes the following diagram commutative:

$$\begin{array}{ccc} \mathrm{Lep}^r_n W & \xrightarrow{\ h\ } & \Omega^{r+1}_{n,X}W \\ \downarrow d & & \downarrow E \\ \Omega^{r+1}_{n+1}W & \xrightarrow{\ p_1\ } & \Omega^{2(r+1)}_{n,Y}W \end{array} \tag{113}$$

Basic motivation for the notion of a Lepage form is the construction of this diagram. Its commutativity demonstrates the relationship of the Euler–Lagrange mapping and the exterior derivative of differential forms, just in the spirit of the work of Lepage [Le]. Equation (113) shows that the Euler–Lagrange form has its origin in the exterior derivative operator.

The following theorem describes the behavior of the Euler–Lagrange mapping under automorphisms of the underlying fibered manifold; it says that transformed Lagrangians have transformed Euler–Lagrange forms.

Theorem 6 *For each Lagrangian* $\lambda \in \Omega^r_{n,X}W$ *and each automorphism* α *of* Y

$$J^{2r}\alpha^* E_\lambda = E_{J^{2r}\alpha^*\lambda}. \tag{114}$$

Proof To prove (114), we apply Theorem 4 of Sect. 4.4 to Lepage equivalents. Let $\rho_\lambda \in \Omega^s_n W$ be any Lepage equivalent of λ. Then,

$$(\pi^{s+1,s})^* d\rho = E_\lambda + F_\lambda. \tag{115}$$

It is easily seen that the pullback $J^s\alpha^*\rho$ is a Lepage form whose Lagrangian is $hJ^s\alpha^*\rho = J^{s+1}\alpha^* h\rho = J^{s+1}\alpha^*\lambda$. Then from standard commutativity of the pullback and the exterior derivative, we have

$$(\pi^{s+1,s})^*dJ^s\alpha^*\rho = (\pi^{s+1,s})^*J^s\alpha^*d\rho = J^{s+1}\alpha^*(\pi^{s+1,s})^*d\rho, \tag{116}$$

from which we conclude that $J^{s+1}\alpha^*E_\lambda + J^{s+1}\alpha^*F_\lambda = E_{J^s\alpha^*\lambda} + F_{J^s\alpha^*\lambda}$. Theorem 6 now follows from the uniqueness of the 1-contact component of these forms. □

4.6 The First Variation Formula

Suppose that we have a variational structure (W, ρ), where W is an open set in a fibered manifold Y with n-dimensional base X, and ρ is a *Lepage form* on the set $W^r \subset J^rY$. Recall that for any piece Ω of X, and any open set $W \subset Y$, the Lepage form ρ defines the variational functional $\Gamma_W(\pi|_W) \ni \gamma \to \rho_\Omega(\gamma) \in \mathbf{R}$ by

$$\rho_\Omega(\gamma) = \int_\Omega J^r\gamma^*\rho \tag{117}$$

(Equation 2). The *first variation* of ρ_Ω by a π-projectable vector field Ξ is the variational functional $\Gamma_\Omega(\pi|_W) \ni \gamma \to (\partial_{J^r\Xi}\rho)_\Omega(\gamma) \in \mathbf{R}$, where

$$(\partial_{J^r\Xi}\rho)_\Omega(\gamma) = \int_\Omega J^r\gamma^*\partial_{J^r\Xi}\rho \tag{118}$$

(Equation 31). As before, denote by λ_ρ the *horizontal component* of the n-form ρ, that is the *Lagrangian*, associated with ρ. For Lepage forms, the following theorem on the structure of the integrand in the first variation (118) is just a restatement of definitions.

Theorem 7 *Let $\rho \in \Omega_n^rW$ be a Lepage form, Ξ a π-projectable vector field on W.*

(a) *The Lie derivative $\partial_{J^r\Xi}\rho$ can be expressed as*

$$\partial_{J^r\Xi}\rho = i_{J^r\Xi}d\rho + di_{J^r\Xi}\rho. \tag{119}$$

(b) *If Ξ is π-vertical, then*

$$\partial_{J^{r+1}\Xi}\lambda_\rho = i_{J^{r+1}\Xi}E_{\lambda_\rho} + hdi_{J^r\Xi}\rho. \tag{120}$$

(c) *For any section γ of Y with values in W,*

$$J^r\gamma^*\partial_{J^r\Xi}\rho = J^{r+1}\gamma^*i_{J^{r+1}\Xi}E_{\lambda_\rho} + dJ^r\gamma^*i_{J^r\Xi}\rho \tag{121}$$

(d) *For every piece* Ω *of* X *and every section* γ *of* Y *defined on* Ω,

$$\int_{\Omega} J^r\gamma^* \partial_{J^r\Xi}\rho = \int_{\Omega} J^{r+1}\gamma^* i_{J^{r+1}\Xi} E_\lambda + \int_{\partial\Omega} J^{r+1}\gamma^* i_{J^{r+1}\Xi}\rho. \tag{122}$$

Proof

(a) This is a standard Lie derivative formula.
(b) If Ξ π-vertical, then since $h\partial_{J^r\Xi}\rho = \partial_{J^r\Xi}h\rho$, we have from (119) $h\partial_{J^r\Xi}\rho = i_{J^r\Xi}p_1 d\rho + hdi_{J^r\Xi}\rho$, but $p_1 d\rho = E_{\lambda_\rho}$ because ρ is a Lepage form.
(c) Formula (120) can be proved by a straightforward calculation:

$$\begin{aligned} J^r\gamma^*\partial_{J^r\Xi}\rho &= J^r\gamma^* i_{J^r\Xi} d\rho + J^r\gamma^* di_{J^r\Xi}\rho \\ &= J^{r+1}\gamma^* h i_{J^r\Xi} d\rho + J^r\gamma^* di_{J^r\Xi}\rho \\ &= J^{r+1}\gamma^* i_{J^{r+1}\Xi} p_1 d\rho + J^{r+1}\gamma^* i_{J^r\Xi} p_2 d\rho + J^r\gamma^* di_{J^r\Xi}\rho \\ &= J^{r+1}\gamma^* i_{J^{r+1}\Xi} E_{\lambda_\rho} + J^r\gamma^* di_{J^r\Xi}\rho. \end{aligned} \tag{123}$$

(d) Integrating (121) and using the Stokes' theorem on integration of closed $(n-1)$-forms on pieces of n-dimensional manifolds, we get (122). □

Any of the formulas (119–121) is called, in the context of the variational theory on fibered manifolds, the *infinitesimal first variation formula*; (122) is the *integral first variation formula*.

Remark 10 Note that the infinitesimal first variation formulas in Theorem 7 have no analogue in the classical formulation of the calculus of variations. These formulas are based on the concept of a (global) Lepage form as well as on the use of (invariant) geometric operations such as the Lie derivative, exterior derivative, and contraction of a form by a vector field, describing the variation procedure.

Remark 11 Theorem 7 can be used to obtain the corresponding formulas for higher variational derivatives (see Sect. 4.2).

4.7 Extremals

Let $U \subset X$ be an open set, γ: $U \to W$ a section, and let Ξ: $U \to TY$ be a vector field along the section γ; in our standard notation, γ is an element of the set $\Gamma_\Omega(\pi|_W)$. The *support* of the vector field Ξ is defined to be the set $supp\Xi = \mathrm{cl}\{x \in U | \Xi(x) \neq 0\}$ (cl means *closure*). We know that each differentiable vector field Ξ along γ can be differentiably prolonged to a π-projectable vector field $\tilde{\Xi}$ defined on a neighborhood of the set $\gamma(U)$ in W (Sect. 4.2, Theorem 1). $\tilde{\Xi}$ satisfies

$$\tilde{\Xi} \circ \gamma = \Xi. \tag{124}$$

This property of vector fields along sections will be used in the definition of extremal sections, which can be introduced as follows.

Consider a Lepage form $\rho \in \Omega_n^r W$, and fix a piece Ω of X. We shall say that a section $\gamma \in \Gamma_\Omega(\pi|_U)$ is an *extremal* of the variational functional $\Gamma_\Omega(\pi|_W) \ni \gamma \to \rho_\Omega(\gamma) \in \mathbf{R}$ on Ω, if for all π-projectable vector fields Ξ, such that $supp(\Xi \circ \gamma) \subset \Omega$,

$$\int_\Omega J^r\gamma^* \partial_{J^r\Xi}\rho = 0. \tag{125}$$

Condition (125) can also be expressed as $(\partial_{J^r\Xi}\rho)_\Omega(\gamma) = 0$. γ is called an *extremal of the Lagrange structure* (W, ρ), or simply an *extremal*, if it is an extremal of the variational functional ρ_Ω for *every* Ω in the domain of definition of γ.

In this sense, the extremals can also be defined as those sections γ for which the values $\rho_\Omega(\gamma)$ of the variational functional ρ_Ω are not sensitive to small compact deformations of γ.

In the following necessary and sufficient conditions for a section to be an extremal, we use the *Euler–Lagrange form* E_{λ_ρ}, associated with the Lagrangian $\lambda_\rho = h\rho$, written in a fibered chart as

$$E_{\lambda_\rho} = E_\sigma(\mathscr{L})\omega^\sigma \wedge \omega_0, \tag{126}$$

where the components $E_\sigma(\mathscr{L})$ are the *Euler–Lagrange expressions* (Sect. 4.4). Explicitly, if $h\rho = \mathscr{L}\omega_0$, then

$$E_\sigma(\mathscr{L}) = \frac{\partial \mathscr{L}}{\partial y^\sigma} - \sum_{l=1}^{r+1} (-1)^{l-1} d_{p_1} d_{p_2} \ldots d_{p_l} \frac{\partial \mathscr{L}}{\partial y^\sigma_{p_1 p_2 \ldots p_l}}. \tag{127}$$

Theorem 8 *Let $\rho \in \Omega_n^r W$ be a Lepage form. Let γ: $U \to W$ a section, and $\Omega \subset U$ be a piece of X. The following conditions are equivalent*:

(a) *γ is an extremal on Ω.*
(b) *For every π-vertical vector field Ξ defined on a neighborhood of $\gamma(U)$, such that $supp(\Xi \circ \gamma) \subset \Omega$,*

$$J^r\gamma^* i_{J^r\Xi} d\rho = 0. \tag{128}$$

(c) *The Euler–Lagrange form associated with the Lagrangian $\lambda_\rho = h\rho$ vanishes along $J^{r+1}\gamma$, i.e.,*

$$E_{\lambda_\rho} \circ J^{r+1}\gamma = 0. \tag{129}$$

(d) *For every fibered chart* $(V, \psi), \psi = (x^i, y^\sigma)$, *such that* $\pi(V) \subset U$ *and* $\gamma(\pi(V)) \subset V, \gamma$ *satisfies the system of partial differential equations*

$$E_\sigma(\mathscr{L}_\rho) \circ J^{r+1}\gamma = 0, \quad 1 \leq \sigma \leq m. \tag{130}$$

Proof

1. We show that (a) implies (b). By Theorem 7, (d), for any piece Ω of X and any π-vertical vector field Ξ such that $supp(\Xi \circ \gamma) \subset \Omega$,

$$\int_\Omega J^r\gamma^* \partial_{J^r\Xi}\rho = \int_\Omega J^r\gamma^* i_{J^r\Xi} d\rho, \tag{131}$$

because the vector field $J^r\Xi$ vanishes along the boundary $\partial\Omega$. Then,

$$\int_\Omega J^r\gamma^* i_{J^r\Xi} d\rho = \int_\Omega J^{r+1}\gamma^* (\pi^{r+1,r})^* i_{J^r\Xi} d\rho = \int_\Omega J^{r+1}\gamma^* i_{J^{r+1}\Xi} p_1 d\rho, \tag{132}$$

where $p_1 d\rho = E_{h\rho}$ is the Euler–Lagrange form.

If Ω is contained in a coordinate neighborhood, the support $supp(\Xi \circ \gamma) \subset \Omega$ lies in the same coordinate neighborhood. Writing $\Xi = \Xi^\sigma \cdot \partial/\partial y^\sigma$ and $p_1 d\rho = E_\sigma(\mathscr{L}_\rho)\omega^\sigma \wedge \omega_0$ then $i_{J^{r+1}\Xi} p_1 d\rho = E_\sigma(\mathscr{L}_\rho)\Xi^\sigma\omega_0$ and

$$J^r\gamma^* i_{J^r\Xi} d\rho = (E_\sigma(\mathscr{L}_\rho) \circ J^{r+1}\gamma) \cdot (\Xi^\sigma \circ \gamma) \cdot \omega_0. \tag{133}$$

Now supposing that $J^r\gamma^* i_{J^r\Xi} d\rho \neq 0$ for some π-vertical vector field Ξ, the first variation formula

$$\int_\Omega J^r\gamma^* i_{J^r\Xi} d\rho = \int_\Omega (E_\sigma(\mathscr{L}_\rho) \circ J^{r+1}\gamma) \cdot (\Xi^\sigma \circ \gamma) \cdot \omega_0 \tag{134}$$

would give us a contradiction

$$\int_\Omega J^3\gamma^* \partial_{J^r\Xi}\rho \neq 0. \tag{135}$$

Thus, (a) implies (b).

2. (c) is an immediate consequence of condition (b). Indeed, we can write for Ξ π-vertical

$$J^r\gamma^* i_{J^r\Xi} d\rho = (\pi^{r+1,r} \circ J^{r+1}\gamma)^* i_{J^r\Xi} d\rho = J^{r+1}\gamma^* (\pi^{r+1,r})^* i_{J^r\Xi} d\rho$$
$$= J^{r+1}\gamma^* i_{J^{r+1}\Xi} (\pi^{r+1,r})^* d\rho = J^{r+1}\gamma^* i_{J^{r+1}\Xi} p_1 d\rho = J^{r+1}\gamma^* i_{J^{r+1}\Xi} E_{\lambda_\rho}. \tag{136}$$

3. (d) is just a restatement of (b) for the components of the form E_{λ_ρ}.
4. We apply Theorem 7, (d). □

Equation (130) are called the *Euler–Lagrange equations*; these equations are indeed related to the chosen fibered chart $(V, \psi), \psi = (x^i, y^\sigma)$. However, since the Euler–Lagrange expressions are components of a (global) differential form, the Euler–Lagrange form, the solutions are independent of fibered charts.

If a Lagrangian $\lambda \in \Omega^r_{n,X} W$ is given and ρ is a Lepage equivalent of λ of order $s = 2r - 1$ (Sect. 4.5, Theorem 5), then the Euler–Lagrange equations are of order $\leq 2r$.

Remark 12 For a fixed fibered chart $(V, \psi), \psi = (x^i, y^\sigma)$, the Euler–Lagrange equations represent a system of partial differential equations of order $r + 1$ for unknown functions $(x^i) \to \gamma^\sigma(x^i)$, where $1 \leq i \leq n$ and $1 \leq \sigma \leq m$. This fact is due to the origin of the Lagrange function $\mathscr{L}$ that comes from a Lepage form, which is of order r. If we start with a given Lagrangian of order r, then the Euler–Lagrange equations are of order $2r$. To get an extremal γ on a piece $\Omega \subset X$, we have to solve this system for every fibered chart $(V_\iota, \psi_\iota), \psi = (x^i_\iota, y^\sigma_\iota)$, from a collection of fibered charts, such that the sets $\pi(V_\iota)$ cover Ω; then, the solutions $(x^i_\iota) \to \gamma^\sigma_\iota(x^i_\iota)$ should be used to find a section γ such that $\gamma^\sigma_\iota = y^\sigma_\iota \gamma \varphi^{-1}_\iota$ for all indices ι.

Remark 13 Properties of nonlinear equations (130) depend on the form ρ; their *global* structure can also be understood by means of condition (128). This condition says that a section γ is an extremal if and only if its r-jet prolongation is an *integral mapping* of an ideal of forms generated by the family of n-forms $i_{J^3\Xi} d\rho$. Using fibered chart formulas, one can find explicit expressions for local generators of the ideal.

4.8 Trivial Lagrangians

Consider the Euler–Lagrange mapping, assigning to a Lagrangian its Euler–Lagrange form (112)

$$\Omega^r_{n,X} W \ni \lambda \to E_\lambda \in \Omega^{2r}_{n+1,Y} W. \tag{137}$$

The domain and the range of this mapping have the structure of Abelian groups (and real vector spaces), and the Euler–Lagrange mapping is a homomorphism of these Abelian groups. The purpose of this section is to describe the *kernel* of the Euler–Lagrange mapping. Elements of the kernel are the Lagrangians $\lambda \in \Omega^r_{n,X} W$ such that

$$E_\lambda = 0. \tag{138}$$

These Lagrangians are called (*variationally*) *trivial*, or *null*.

Trivial Lagrangians can locally be characterized as formal divergences or some closed forms.

Theorem 9 *Let $\lambda \in \Omega^r_{n,X} W$ be a Lagrangian. The following conditions are equivalent*:

(a) *λ is variationally trivial.*

(b) *For any fibered chart $(V, \psi), \psi = (x^i, y^\sigma)$, there exist functions $g^i \colon V^r \to \mathbf{R}$, such that on $V^r, \lambda = \mathscr{L}\omega_0$, where*

$$\mathscr{L} = d_i g^i. \tag{139}$$

(c) *For every fibered chart $(V, \psi), \psi = (x^i, y^\sigma)$, such that $V \subset W$, there exists an $(n-1)$-form $\mu \in \Omega^{r-1}_{n-1} V$ such that on V^r*

$$\lambda = hd\mu. \tag{140}$$

Proof

1. We show that (a) is equivalent with (b). Suppose that we have a variationally trivial Lagrangian $\lambda \in \Omega^r_n W$. Write for any fibered chart $(V, \psi), \psi = (x^i, y^\sigma)$, $\lambda = \mathscr{L}\omega_0$. Since by hypothesis, the Euler–Lagrange expressions $E_\sigma(\mathscr{L})$ vanish, consequently, by Sect. 3.2, Theorem 1, $\mathscr{L} = d_i g^i$ for some functions g^i on V^r. The converse follows from the same Theorem.
2. Equivalence of (a) and (c) follows from Sect. 3.3, Theorem 3. □

In general, Theorem 9 does not ensure existence of a *globally defined* form μ or $d\mu$. However, for first-order Lagrangians local triviality already induces global variationality.

Corollary 1 *A first-order Lagrange form $\lambda \in \Omega^1_{n,X} W$ is variationally trivial if and only if there exists an n-form $\eta \in \Omega^0_n W$ such that*

$$\lambda = h\eta \tag{141}$$

and

$$d\eta = 0. \tag{142}$$

Proof By Theorem 9, for any two points $y_1, y_2 \in W$ there exist two $(n-1)$-forms $\mu_1, \mu_2 \in Y$, defined on a neighborhood of y_1 and y_2, such that $hd\mu_1 = \lambda$ and $hd\mu_2 = \lambda$, respectively. Then, $hd(\mu_1 - \mu_2) = 0$ on the intersection of the corresponding neighborhoods in W^1. But the horizontalization h, considered on forms on $J^0Y = Y$, is injective. Consequently, condition $hd(\mu_1 - \mu_2) = 0$ implies $d(\mu_1 - \mu_2) = 0$, so there exists an n-form $\eta \in \Omega_n^0 W$ whose restriction agrees with $d\mu_1$ and $d\mu_2$. Clearly, $d\eta = 0$. □

4.9 Source Forms and the Vainberg–Tonti Lagrangians

A 1-contact $(n + 1)$-form $\varepsilon \in \Omega^s_{n+1,Y}W$, where s is a nonnegative integer, is called a *source form* (Takens [T]). From this definition it follows that ε has in a fibered chart $(V, \psi), \psi = (x^i, y^\sigma)$, an expression

$$\varepsilon = \varepsilon_\sigma \omega^\sigma \wedge \omega_0, \tag{143}$$

where the components ε_σ depend on the jet coordinates $x^i, y^\sigma, y^\sigma_{j_1}, y^\sigma_{j_1 j_2}, \ldots, y^\sigma_{j_1 j_2 \ldots j_s}$. Clearly, every Euler–Lagrange form E_λ is a source form; thus, the set of source forms contains the Euler–Lagrange forms as a subset.

We assign to any source form ε a family of Lagrangians as follows. Let ε be defined on W^s, and let $(V, \psi), \psi = (x^i, y^\sigma)$, be a fibered chart on Y, such that $V \subset W$, and the set $\psi(V)$ is star-shaped. Denote by I the fibered homotopy operator on V^s (Sect. 2.7). Then, $I\varepsilon$ is a π^s-horizontal form, that is, a *Lagrangian* for Y, defined on V^s. This Lagrangian, denoted

$$\lambda_\varepsilon = I\varepsilon, \tag{144}$$

is called the *Vainberg–Tonti Lagrangian*, associated with the source form ε (and the fibered chart (V, ψ)) (cf. [To, V]).

Recall that $I\varepsilon$ is defined by the fibered homotopy χ_s: $[0, 1] \times V^s \to V^s$, where $\chi_s(t, (x^i, y^\sigma, y^\sigma_{j_1}, y^\sigma_{j_1 j_2}, \ldots, y^\sigma_{j_1 j_2 \ldots j_s})) = (x^i, ty^\sigma, ty^\sigma_{j_1}, ty^\sigma_{j_1 j_2}, \ldots, ty^\sigma_{j_1 j_2 \ldots j_s})$. Since χ_s satisfies $\chi_s^* \varepsilon = (\varepsilon_\sigma \circ \chi_s)(t\omega^\sigma + y^\sigma dt) \wedge \omega_0$, we have, integrating the coefficient in this expression at dt,

$$\lambda_\varepsilon = \mathscr{L}_\varepsilon \omega_0, \tag{145}$$

where

$$\mathscr{L}_\varepsilon = y^\sigma \int_0^1 \varepsilon_\sigma \circ \chi_s \cdot dt, \tag{146}$$

or, which is the same,

$$\mathscr{L}_{\varepsilon}(x^i, y^\sigma, y^\sigma_{j_1}, \ldots, y^\sigma_{j_1 j_2 \ldots j_s}) = y^\sigma \int_0^1 \varepsilon_\sigma(x^i, ty^\nu, ty^\nu_{j_1}, \ldots, ty^\nu_{j_1 j_2 \ldots j_s}) dt. \tag{147}$$

We can find the chart expression for the Euler–Lagrange form E_{λ_ε} of the Vainberg–Tonti Lagrangian λ_ε; recall that

$$E_{\lambda_\varepsilon} = E_\sigma(\mathscr{L}_\varepsilon)\omega^\sigma \wedge \omega_0, \tag{148}$$

where

$$E_\sigma(\mathscr{L}_\varepsilon) = \sum_{l=0}^{s} (-1)^l d_{p_1} d_{p_2} \ldots d_{p_l} \frac{\partial \mathscr{L}_\varepsilon}{\partial y^\sigma_{p_1 p_2 \ldots p_l}}. \tag{149}$$

To this purpose, we derive two formulas for the formal derivative operator d_i. The formulas are completely parallel with the well-known classical Leibniz rules for partial derivatives of the product of functions.

Lemma 7

(a) *For every function f on* V^p

$$d_i(f \circ \chi_p) = d_i f \circ \chi_{p+1}. \tag{150}$$

(b) *For every function f on* V^s *and a collection of functions* $g^{p_1 p_2 \ldots p_k}$ *on* V^s, *symmetric in the superscripts,*

$$d_{p_1} d_{p_2} \ldots d_{p_k}(f \cdot g^{p_1 p_2 \ldots p_k}) = \sum_{i=0}^{k} \binom{k}{i} d_{p_1} d_{p_2} \ldots d_{p_i} f \cdot d_{p_{i+1}} d_{p_{i+2}} \ldots d_{p_k} g^{p_1 p_2 \ldots p_i p_{i+1} p_{i+2} \ldots p_k}. \tag{151}$$

Proof

(a) Formula (150) is an easy consequence of definitions.
(b) The proof is standard. We have

$$\begin{aligned} d_{p_1}(f \cdot g^{p_1}) &= d_{p_1}f \cdot g^{p_1} + f \cdot d_{p_1}g^{p_1} \\ &= \binom{1}{0} d_{p_1}f \cdot g^{p_1} + \binom{1}{1} f \cdot d_{p_1}g^{p_1}, \\ d_{p_1}d_{p_2}(f \cdot g^{p_1p_2}) &= d_{p_2}(d_{p_1}f \cdot g^{p_1p_2} + f \cdot d_{p_1}g^{p_1p_2}) \\ &= d_{p_2}d_{p_1}f \cdot g^{p_1p_2} + d_{p_1}f \cdot d_{p_2}g^{p_1p_2} + d_{p_2}f \cdot d_{p_1}g^{p_1p_2} + f \cdot d_{p_1}d_{p_1}g^{p_1p_2} \\ &= \binom{2}{0} d_{p_2}d_{p_1}f \cdot g^{p_1p_2} + \binom{2}{1} d_{p_1}f \cdot d_{p_2}g^{p_1p_2} + \binom{2}{2} f \cdot d_{p_1}d_{p_1}g^{p_1p_2}. \end{aligned} \tag{152}$$

Then, supposing that

$$d_{p_1}d_{p_2}\ldots d_{p_{k-1}}(f \cdot g^{p_1p_2\ldots p_{k-1}}) = \sum_{i=0}^{k-1} \binom{k-1}{i} d_{p_1}d_{p_2}\ldots d_{p_i}f \cdot d_{p_{i+1}}d_{p_{i+2}}\ldots d_{p_{k-1}}g^{p_1p_2\ldots p_ip_{i+1}p_{i+2}\ldots p_{k-1}}, \tag{153}$$

we have (150)

$$\begin{aligned} d_{p_1}d_{p_2}\ldots d_{p_{k-1}}d_{p_k}(f \cdot g^{p_1p_2\ldots p_{k-1}p_k}) = {} & f \cdot d_{p_1}d_{p_2}\ldots d_{p_{k-1}}d_{p_k}g^{p_1p_2\ldots p_{k-1}p_k} \\ & + \left(\binom{k-1}{0} + \binom{k-1}{1}\right) d_{p_1}f \cdot d_{p_2}d_{p_3}\ldots d_{p_k}g^{p_1p_2\ldots p_{k-1}p_k} \\ & + \left(\binom{k-1}{1} + \binom{k-1}{2}\right) d_{p_1}d_{p_2}f \cdot d_{p_3}d_{p_4}\ldots d_{p_k}g^{p_1p_2\ldots p_{k-1}p_k} \\ & + \cdots + \left(\binom{k-1}{k-2} + \binom{k-1}{k-1}\right) d_{p_1}d_{p_2}\ldots d_{p_{k-1}} \cdot d_{p_k}g^{p_1p_2\ldots p_{k-1}p_k} \\ & + \binom{k-1}{k-1} d_{p_k}d_{p_1}d_{p_2}\ldots d_{p_{k-1}} \cdot g^{p_1p_2\ldots p_{k-1}p_k} \end{aligned} \tag{154}$$

and

$$\binom{k-1}{p} + \binom{k-1}{p+1} = \binom{k}{p+1}; \tag{155}$$

thus,

$$\begin{aligned} d_{p_1}d_{p_2}\ldots d_{p_{k-1}}d_{p_k}(f \cdot g^{p_1p_2\ldots p_{k-1}p_k}) = {} & \binom{k}{0} f \cdot d_{p_1}d_{p_2}\ldots d_{p_{k-1}}d_{p_k}g^{p_1p_2\ldots p_{k-1}p_k} \\ & + \binom{k}{1} d_{p_1}f \cdot d_{p_2}d_{p_3}\ldots d_{p_k}g^{p_1p_2\ldots p_{k-1}p_k} \\ & + \binom{k}{2} d_{p_1}d_{p_2}f \cdot d_{p_3}d_{p_4}\ldots d_{p_k}g^{p_1p_2\ldots p_{k-1}p_k} \\ & + \cdots + \binom{k}{k-1} d_{p_1}d_{p_2}\ldots d_{p_{k-1}} \cdot d_{p_k}g^{p_1p_2\ldots p_{k-1}p_k} \\ & + \binom{k}{k} d_{p_k}d_{p_1}d_{p_2}\ldots d_{p_{k-1}} \cdot g^{p_1p_2\ldots p_{k-1}p_k}. \end{aligned} \tag{156}$$

which is formula (150). □

The Vainberg–Tonti Lagrangian $\lambda_\varepsilon = \mathscr{L}_\varepsilon \omega_0$ allows us to assign to *any* source form $\varepsilon = \varepsilon_\sigma \omega^\sigma \wedge \omega_0$ a variational functional and the corresponding Euler–Lagrange form of this functional, with the Euler–Lagrange expressions $E_\sigma(\mathscr{L}_\varepsilon)$. We shall determine the functions $E_\sigma(\mathscr{L}_\varepsilon)$ and compare them with the components ε_σ of the source form.

Theorem 10 *The Euler–Lagrange expressions of the Vainberg–Tonti Lagrangian λ_ε of a source form $\varepsilon = \varepsilon_\sigma \omega^\sigma \wedge \omega_0$ are*

$$E_\sigma(\mathscr{L}_\varepsilon) = \varepsilon_\sigma - \sum_{k=0}^{s} y^\nu_{q_1 q_2 \ldots q_k} \int_0^1 H_\sigma{}^{q_1 q_2 \ldots q_k}_\nu (\varepsilon) \circ \chi_{2s} \cdot t dt, \tag{157}$$

where for every $k = 0, 1, 2, \ldots, s$

$$\begin{aligned} H^{q_1 q_2 \ldots q_k}_{\sigma\nu}(\varepsilon) = {} & \frac{\partial \varepsilon_\sigma}{\partial y^\nu_{q_1 q_2 \ldots q_k}} - (-1)^k \frac{\partial \varepsilon_\nu}{\partial y^\sigma_{q_1 q_2 \ldots q_k}} \\ & - \sum_{l=k+1}^{s} (-1)^l \binom{l}{k} d_{p_{k+1}} d_{p_{k+2}} \ldots d_{p_l} \frac{\partial \varepsilon_\nu}{\partial y^\sigma_{q_1 q_2 \ldots q_k p_{k+1} p_{k+2} \ldots p_l}}. \end{aligned} \tag{158}$$

Proof We find a formula for the difference $\varepsilon_\sigma - E_\sigma(\mathscr{L}_\varepsilon)$. To simplify the formulas, we denote the homotopy χ_{s+l-i} simply by χ. Calculating the derivatives, we have

$$\frac{\partial \mathscr{L}_\varepsilon}{\partial y^\sigma} = \int_0^1 \varepsilon_\sigma \circ \chi \cdot dt + y^\nu \int_0^1 \frac{\partial \varepsilon_\nu}{\partial y^\sigma} \circ \chi \cdot t dt, \tag{159}$$

and, by Lemma 7, (150) and (151), for every l, $1 \le l \le s$,

$$\begin{aligned} & d_{p_l} \ldots d_{p_2} d_{p_1} \frac{\partial \mathscr{L}_\varepsilon}{\partial y^\sigma_{p_1 p_2 \ldots p_l}} \\ & \quad = d_{p_l} \ldots d_{p_2} d_{p_1} \left(y^\nu \int_0^1 \frac{\partial \varepsilon_\nu}{\partial y^\sigma_{p_1 p_2 \ldots p_l}} \circ \chi \cdot t dt \right) \\ & \quad = \sum_{i=0}^{l} \binom{l}{i} d_{p_1} d_{p_2} \ldots d_{p_i} y^\nu \cdot d_{p_{i+1}} d_{p_{i+2}} \ldots d_{p_l} \int_0^1 \frac{\partial \varepsilon_\nu}{\partial y^\sigma_{p_1 p_2 \ldots p_i p_{i+1} p_{i+2} \ldots p_l}} \circ \chi \cdot t dt \\ & \quad = \sum_{i=0}^{l} \binom{l}{i} y^\nu_{p_1 p_2 \ldots p_i} \int_0^1 d_{p_{i+1}} d_{p_{i+2}} \ldots d_{p_l} \frac{\partial \varepsilon_\nu}{\partial y^\sigma_{p_1 p_2 \ldots p_i p_{i+1} p_{i+2} \ldots p_l}} \circ \chi \cdot t dt. \end{aligned} \tag{160}$$

Then by (159) and (160),

$$E_\sigma(\mathscr{L}_\varepsilon) = \int_0^1 \varepsilon_\sigma \circ \chi \cdot dt + y^\nu \int_0^1 \frac{\partial \varepsilon_\nu}{\partial y^\sigma} \circ \chi \cdot tdt$$
$$+ \sum_{l=1}^{s} (-1)^l \sum_{i=0}^{l} \binom{l}{i} y^\nu_{p_1 p_2 \ldots p_i} \int_0^1 d_{p_{i+1}} d_{p_{i+2}} \ldots d_{p_l} \frac{\partial \varepsilon_\nu}{\partial y^\sigma_{p_1 p_2 \ldots p_i p_{i+1} p_{i+2} \ldots p_l}} \circ \chi \cdot tdt. \tag{161}$$

On the other hand,

$$\begin{aligned}
\varepsilon_\sigma &= \int_0^1 \frac{d}{dt}(\varepsilon_\sigma \circ \chi \cdot t)dt \\
&= \int_0^1 \frac{d(\varepsilon_\sigma \circ \chi)}{dt} \cdot tdt + \int_0^1 \varepsilon_\sigma \circ \chi \cdot dt \qquad (162) \\
&= \sum_{i=0}^{s} \int_0^1 \frac{\partial \varepsilon_\sigma}{\partial y^\nu_{p_1 p_2 \ldots p_i}} \circ \chi \cdot y^\nu_{p_1 p_2 \ldots p_i} \cdot tdt + \int_0^1 \varepsilon_\sigma \circ \chi \cdot dt,
\end{aligned}$$

hence,

$$\begin{aligned}
\varepsilon_\sigma - E_\sigma(\mathscr{L}_\varepsilon) &= \sum_{i=0}^{s} \int_0^1 \frac{\partial \varepsilon_\sigma}{\partial y^\nu_{p_1 p_2 \ldots p_i}} \circ \chi \cdot y^\nu_{p_1 p_2 \ldots p_i} \cdot tdt - y^\nu \int_0^1 \frac{\partial \varepsilon_\nu}{\partial y^\sigma} \circ \chi \cdot tdt \\
&\quad - \sum_{l=1}^{s} (-1)^l \sum_{i=0}^{l} \binom{l}{i} y^\nu_{p_1 p_2 \ldots p_i} \int_0^1 d_{p_{i+1}} d_{p_{i+2}} \ldots d_{p_l} \frac{\partial \varepsilon_\nu}{\partial y^\sigma_{p_1 p_2 \ldots p_i p_{i+1} p_{i+2} \ldots p_l}} \circ \chi \cdot tdt \\
&= \int_0^1 \frac{\partial \varepsilon_\sigma}{\partial y^\nu} \circ \chi \cdot y^\nu \cdot tdt - y^\nu \int_0^1 \frac{\partial \varepsilon_\nu}{\partial y^\sigma} \circ \chi \cdot tdt \\
&\quad - \sum_{l=1}^{s} (-1)^l \binom{l}{0} y^\nu \cdot \int_0^1 d_{p_1} d_{p_2} \ldots d_{p_l} \frac{\partial \varepsilon_\nu}{\partial y^\sigma_{p_1 p_2 \ldots p_l}} \circ \chi \cdot tdt \\
&\quad + \sum_{i=1}^{s} \int_0^1 \frac{\partial \varepsilon_\sigma}{\partial y^\nu_{p_1 p_2 \ldots p_i}} \circ \chi \cdot y^\nu_{p_1 p_2 \ldots p_i} \cdot tdt \\
&\quad - \sum_{l=1}^{s} (-1)^l \sum_{i=1}^{l} \binom{l}{i} y^\nu_{p_1 p_2 \ldots p_i} \int_0^1 d_{p_{i+1}} d_{p_{i+2}} \ldots d_{p_l} \frac{\partial \varepsilon_\nu}{\partial y^\sigma_{p_1 p_2 \ldots p_i p_{i+1} p_{i+2} \ldots p_l}} \circ \chi \cdot tdt.
\end{aligned} \tag{163}$$

We change summation in the double sum, replacing the summation through the pairs (l, i) with the summation through (i, l). Summation through (l, i) can be expressed by the scheme

$$\begin{array}{l} (1,1) \\ (2,1),(2,2) \\ (3,1),(3,2),(3,3) \\ \ldots \\ (s,1),(s,2),(s,3),\ldots,(s-1,s),(s,s) \end{array} \tag{164}$$

Then, it is easily seen that the same summation, but represented by the pairs, (i, l), is expressed by the scheme

$$\begin{array}{l} (1,1),(1,2),(1,3),\ldots,(1,s-1),(1,s) \\ (2,2),(2,3),\ldots,(2,s-1),(2,s) \\ \ldots \\ (s-1,s-1),(s-1,s) \\ (s,s) \end{array} \tag{165}$$

Consider the double sum in (163). The summation through (i, l) now becomes,

$$\begin{aligned} &\sum_{l=1}^{s}(-1)^{l}\sum_{i=1}^{l}\binom{l}{i}y^{\nu}_{p_1p_2\ldots p_i}\int_0^1 d_{p_{i+1}}d_{p_{i+2}}\ldots d_{p_l}\frac{\partial\varepsilon_\nu}{\partial y^{\sigma}_{p_1p_2\ldots p_ip_{i+1}p_{i+2}\ldots p_l}}\circ\chi\cdot tdt \\ &\quad=\sum_{i=1}^{s}(-1)^{l}\sum_{l=i}^{s}\binom{l}{i}y^{\nu}_{p_1p_2\ldots p_i}\int_0^1 d_{p_{i+1}}d_{p_{i+2}}\ldots d_{p_l}\frac{\partial\varepsilon_\nu}{\partial y^{\sigma}_{p_1p_2\ldots p_ip_{i+1}p_{i+2}\ldots p_l}}\circ\chi\cdot tdt \\ &\quad=\sum_{i=1}^{s}(-1)^{i}y^{\nu}_{p_1p_2\ldots p_i}\int_0^1\frac{\partial\varepsilon_\nu}{\partial y^{\sigma}_{p_1p_2\ldots p_l}}\circ\chi\cdot tdt \\ &\quad+\sum_{i=1}^{s}(-1)^{l}\sum_{l=i+1}^{s}\binom{l}{i}y^{\nu}_{p_1p_2\ldots p_i}\int_0^1 d_{p_{i+1}}d_{p_{i+2}}\ldots d_{p_l}\frac{\partial\varepsilon_\nu}{\partial y^{\sigma}_{p_1p_2\ldots p_ip_{i+1}p_{i+2}\ldots p_l}}\circ\chi\cdot tdt. \end{aligned} \tag{166}$$

Returning to (163), we get,

$$\begin{aligned} \varepsilon_\sigma-E_\sigma(\mathscr{L}_\varepsilon)&=\int_0^1\frac{\partial\varepsilon_\sigma}{\partial y^\nu}\circ\chi\cdot y^\nu\cdot tdt-y^\nu\int_0^1\frac{\partial\varepsilon_\nu}{\partial y^\sigma}\circ\chi\cdot tdt \\ &\quad-\sum_{l=1}^{s}(-1)^{l}y^{\nu}\cdot\int_0^1 d_{p_1}d_{p_2}\ldots d_{p_l}\frac{\partial\varepsilon_\nu}{\partial y^{\sigma}_{p_1p_2\ldots p_l}}\circ\chi\cdot tdt \end{aligned}$$

$$+\sum_{i=1}^{s}\int_0^1 \frac{\partial \varepsilon_\sigma}{\partial y^\nu_{p_1p_2\ldots p_i}} \circ \chi \cdot y^\nu_{p_1p_2\ldots p_i} \cdot tdt$$
$$-\sum_{i=1}^{s}(-1)^i y^\nu_{p_1p_2\ldots p_i} \cdot \int_0^1 \frac{\partial \varepsilon_\nu}{\partial y^\sigma_{p_1p_2\ldots p_l}} \circ \chi \cdot tdt$$
$$-\sum_{i=1}^{s}\sum_{l=i+1}^{s}(-1)^l \binom{l}{i} y^\nu_{p_1p_2\ldots p_i} \int_0^1 d_{p_{i+1}} d_{p_{i+2}} \ldots d_{p_l} \frac{\partial \varepsilon_\nu}{\partial y^\sigma_{p_1p_2\ldots p_i p_{i+1} p_{i+2}\ldots p_l}} \circ \chi \cdot tdt$$
$$= y^\nu \int_0^1 \left(\frac{\partial \varepsilon_\sigma}{\partial y^\nu} - \frac{\partial \varepsilon_\nu}{\partial y^\sigma} - \sum_{l=1}^{s}(-1)^l d_{p_1} d_{p_2} \ldots d_{p_l} \frac{\partial \varepsilon_\nu}{\partial y^\sigma_{p_1p_2\ldots p_l}} \right) \circ \chi \cdot tdt$$
$$+ y^\nu_{p_1p_2\ldots p_i} \sum_{i=1}^{s} \int_0^1 \left(\frac{\partial \varepsilon_\sigma}{\partial y^\nu_{p_1p_2\ldots p_i}} - (-1)^i \frac{\partial \varepsilon_\nu}{\partial y^\sigma_{p_1p_2\ldots p_l}} \right.$$
$$\left. - \sum_{l=i+1}^{s}(-1)^l \binom{l}{i} \cdot d_{p_{i+1}} d_{p_{i+2}} \ldots d_{p_l} \frac{\partial \varepsilon_\nu}{\partial y^\sigma_{p_1p_2\ldots p_i p_{i+1} p_{i+2}\ldots p_l}} \right) \circ \chi \cdot tdt. \tag{167}$$

This formula proves Theorem 10. □

The functions $H^{q_1q_2\ldots q_k}_{\sigma\nu}(\varepsilon)$ (158) are called the *Helmholtz expressions*, associated with the source form ε.

It will be instructive to write up the Helmholtz expressions for lower-order source forms.

Remark 14 The Helmholtz expressions for the source forms of order $s = 3$ with components ε_σ are

$$\begin{aligned}
H^{ijk}_{\sigma\nu}(\varepsilon) &= \frac{\partial \varepsilon_\sigma}{\partial y^\nu_{ijk}} + \frac{\partial \varepsilon_\nu}{\partial y^\sigma_{ijk}},\\
H^{ij}_{\sigma\nu}(\varepsilon) &= \frac{\partial \varepsilon_\sigma}{\partial y^\nu_{ij}} - \frac{\partial \varepsilon_\nu}{\partial y^\sigma_{ij}} + 3d_k \frac{\partial \varepsilon_\nu}{\partial y^\sigma_{ijk}},\\
H^{i}_{\sigma\nu}(\varepsilon) &= \frac{\partial \varepsilon_\sigma}{\partial y^\nu_{i}} + \frac{\partial \varepsilon_\nu}{\partial y^\sigma_{i}} - 2d_j \frac{\partial \varepsilon_\nu}{\partial y^\sigma_{ij}} + 3d_j d_k \frac{\partial \varepsilon_\nu}{\partial y^\sigma_{ijk}},\\
H_{\sigma\nu}(\varepsilon) &= \frac{\partial \varepsilon_\sigma}{\partial y^\nu} - \frac{\partial \varepsilon_\nu}{\partial y^\sigma} + d_i \frac{\partial \varepsilon_\nu}{\partial y^\sigma_{i}} - d_i d_j \frac{\partial \varepsilon_\nu}{\partial y^\sigma_{ij}} + d_i d_j d_k \frac{\partial \varepsilon_\nu}{\partial y^\sigma_{ijk}}.
\end{aligned} \tag{168}$$

Remark 15 Theorem 10 describes the difference between the given source form and the Euler–Lagrange form of the Vainberg–Tonti Lagrangian; we see, in particular, that responsibility for the difference lies on the properties of the source form and is characterized by the Helmholtz expressions.

Lemma 8 *Let $\lambda = \mathscr{L}\omega_0$ be a Lagrangian, and let Θ_λ be its principal Lepage equivalent. Then the Vainberg–Tonti Lagrangian of the Euler–Lagrange form $E_\lambda = E_\sigma(\mathscr{L})\omega^\sigma \wedge \omega_0$,*

$$\lambda_{E_\lambda} = IE_\lambda, \tag{169}$$

satisfies

$$\lambda_{E_\lambda} = \lambda - hd(I\Theta_\lambda + \mu_0). \tag{170}$$

Proof Using the fibered homotopy operator I, we can express the principal Lepage equivalent Θ_λ of λ as $\Theta_\lambda = Id\Theta_\lambda + dI\Theta_\lambda + \Theta_0$. Then, the horizontal component is

$$\begin{aligned} h\Theta_\lambda &= hId\Theta_\lambda + hdI\Theta_\lambda + h\Theta_0 = hIp_1 d\Theta_\lambda + hd(I\Theta_\lambda + \mu_0) \\ &= IE_\lambda + hd(I\Theta_\lambda + \mu_0) \end{aligned} \tag{171}$$

for some $(n-1)$-form μ_0 on X such that $\Theta = d\mu_0$, where $\lambda = h\Theta_\lambda$, and IE_λ is the Vainberg–Tonti Lagrangian. □

Note that, in particular, formula (170) shows that the Vainberg–Tonti Lagrangian differs from the given Lagrangian λ by the term $hd(I\Theta_\lambda + \mu_0)$ that belongs to the kernel of the Euler–Lagrange mapping. This demonstrates that the Euler–Lagrange forms of λ and the Vainberg–Tonti Lagrangian λ_{E_λ} coincide.

Remark 16 (Euler–Lagrange source forms) Using homotopies and properties of formal divergence expressions (Chap. 3), we can give an elementary proof of Lemma 8, based on direct calculations. Namely, we prove that the Vainberg–Tonti Lagrangian of a source form $\varepsilon = E_\sigma(\mathscr{L})\omega^\sigma \wedge \omega_0$, which *is* the Euler–Lagrange form of a Lagrangian $\lambda = \mathscr{L}\omega_0$, is given by

$$y^\sigma \int_0^1 E_\sigma(\mathscr{L}) \circ \chi \cdot dt = \mathscr{L} + d_i \Psi^i. \tag{172}$$

First note that for any family of functions g^i on V^s, the formal divergence $d_i g^i$ satisfies the integral homotopy formula

$$\int_0^1 d_i g^i \circ \chi \cdot dt = d_i \int_0^1 g^i \circ \chi \cdot dt. \tag{173}$$

Indeed, we have

$$\begin{aligned} d_i(g^i \circ \chi) &= \frac{\partial (g^i \circ \chi)}{\partial x^i} + \sum_{l=0}^{s} \frac{\partial (g^i \circ \chi)}{\partial y^\sigma_{p_1 p_2 \ldots p_l}} y^\sigma_{p_1 p_2 \ldots p_l k} \\ &= \left(\frac{\partial g^i}{\partial x^i} + \sum_{l=0}^{s} \frac{\partial g^i}{\partial y^\sigma_{p_1 p_2 \ldots p_l}} y^\sigma_{p_1 p_2 \ldots p_l k} \right) \circ \chi, \end{aligned} \tag{174}$$

and formula (173) arises by integration.

Consider the Euler–Lagrange expressions $E_\sigma(\mathscr{L})$ of a Lagrangian of order r expressed as $\lambda = \mathscr{L}\omega_0$,

$$\begin{aligned} E_\sigma(\mathscr{L}) &= \frac{\partial \mathscr{L}}{\partial y^\sigma} - \sum_{l=1}^{r} (-1)^{l-1} d_{p_1} d_{p_2} \ldots d_{p_l} \frac{\partial \mathscr{L}}{\partial y^\sigma_{p_1 p_2 \ldots p_l}} \\ &= \frac{\partial \mathscr{L}}{\partial y^\sigma} - d_{p_1} \frac{\partial \mathscr{L}}{\partial y^\sigma_{p_1}} + d_{p_1} d_{p_2} \frac{\partial \mathscr{L}}{\partial y^\sigma_{p_1 p_2}} - \cdots + (-1)^r d_{p_1} d_{p_2} \ldots d_{p_r} \frac{\partial \mathscr{L}}{\partial y^\sigma_{p_1 p_2 \ldots p_r}}, \end{aligned} \tag{175}$$

and set

$$\begin{aligned} \Phi^{i_1}_\sigma &= \frac{\partial \mathscr{L}}{\partial y^\sigma_{i_1}} - d_{p_2} \frac{\partial \mathscr{L}}{\partial y^\sigma_{i_1 p_2}} + d_{p_2} d_{p_3} \frac{\partial \mathscr{L}}{\partial y^\sigma_{i_1 p_2 p_3}} \\ &\qquad - \cdots + (-1)^{r-1} d_{p_2} d_{p_3} \ldots d_{p_r} \frac{\partial \mathscr{L}}{\partial y^\sigma_{i_1 p_2 p_3 \ldots p_r}}, \\ \Phi^{i_1 i_2}_\sigma &= \frac{\partial \mathscr{L}}{\partial y^\sigma_{i_1 i_2}} - d_{p_3} \frac{\partial \mathscr{L}}{\partial y^\sigma_{i_1 i_2 p_3}} - d_{p_3} d_{p_4} \frac{\partial \mathscr{L}}{\partial y^\sigma_{i_1 i_2 p_3 p_4}} \\ &\qquad - \cdots + (-1)^{r-1} d_{p_3} d_{p_4} \ldots d_{p_r} \frac{\partial \mathscr{L}}{\partial y^\sigma_{i_1 i_2 p_3 p_4 \ldots p_r}}, \\ &\ldots \\ \Phi^{i_1 i_2 \ldots i_k}_\sigma &= \frac{\partial \mathscr{L}}{\partial y^\sigma_{i_1 i_2 \ldots i_k}} - d_{p_{k+1}} \frac{\partial \mathscr{L}}{\partial y^\sigma_{i_1 i_2 \ldots i_k p_{k+1}}} - d_{p_{k+1}} d_{p_{k+2}} \frac{\partial \mathscr{L}}{\partial y^\sigma_{i_1 i_2 \ldots i_k p_{k+1} p_{k+2}}} \\ &\qquad - \cdots + (-1)^{r-1} d_{p_{k+1}} d_{p_{k+2}} \ldots d_{p_r} \frac{\partial \mathscr{L}}{\partial y^\sigma_{i_1 i_2 \ldots i_k p_{k+1} p_{k+2} \ldots p_r}}, \\ &\ldots \\ \Phi^{i_1 i_2 \ldots i_{r-1}}_\sigma &= \frac{\partial \mathscr{L}}{\partial y^\sigma_{i_1 i_2 \ldots i_{r-1}}} - d_{p_r} \frac{\partial \mathscr{L}}{\partial y^\sigma_{i_1 i_2 \ldots i_{r-1} p_r}}, \\ \Phi^{i_1 i_2 \ldots i_r}_\sigma &= \frac{\partial \mathscr{L}}{\partial y^\sigma_{i_1 i_2 \ldots i_r}}. \end{aligned} \tag{176}$$

It is immediately seen that these functions, entering the Euler–Lagrange expression $E_\sigma(\mathscr{L})$ (175), satisfy the recurrence formula

$$\Phi_\sigma^{i_1 i_2 \ldots i_k} = \frac{\partial \mathscr{L}}{\partial y^\sigma_{i_1 i_2 \ldots i_k}} - d_{p_{k+1}} \Phi_\sigma^{i_1 i_2 \ldots i_k p_{k+1}}. \tag{177}$$

Using properties of the homotopy χ,

$$\frac{d\mathscr{L} \circ \chi}{dt} = \frac{\partial \mathscr{L}}{\partial y^\sigma} \circ \chi \cdot y^\sigma + \sum_{l=1}^{r} \frac{\partial \mathscr{L}}{\partial y^\sigma_{p_1 p_2 \ldots p_l}} \circ \chi \cdot y^\sigma_{p_1 p_2 \ldots p_l k}. \tag{178}$$

Hence, denoting $\mathscr{L}_0(x^i, y^\sigma, y^\sigma_{j_1}, y^\sigma_{j_1 j_2}, \ldots, y^\sigma_{j_1 j_2 \ldots j_r}) = \mathscr{L}(x^i, 0, 0, 0, \ldots, 0)$, we get for the Vainberg–Tonti Lagrangian

$$\begin{aligned}
y^\sigma \int_0^1 E_\sigma(\mathscr{L}) \circ \chi \cdot dt &= y^\sigma \int_0^1 \frac{\partial \mathscr{L}}{\partial y^\sigma} \circ \chi \cdot dt - y^\sigma \int_0^1 d_i \Phi^i_\sigma \circ \chi \cdot dt \\
&= \int_0^1 \left(\frac{\mathrm{d}\mathscr{L} \circ \chi}{dt} - \sum_{l=1}^{r} \frac{\partial \mathscr{L}}{\partial y^\sigma_{p_1 p_2 \ldots p_l}} \circ \chi \cdot y^\sigma_{p_1 p_2 \ldots p_l k} \right) dt - y^\sigma \int_0^1 d_i \Phi^i_\sigma \circ \chi \cdot dt \\
&= \mathscr{L} - \mathscr{L}_0 - \sum_{l=1}^{r} y^\sigma_{p_1 p_2 \ldots p_l} \int_0^1 \frac{\partial \mathscr{L}}{\partial y^\sigma_{p_1 p_2 \ldots p_l}} \circ \chi \cdot dt - y^\sigma d_i \int_0^1 \Phi^i_\sigma \circ \chi \cdot dt \\
&= \mathscr{L} - \mathscr{L}_0 - \sum_{l=1}^{r} y^\sigma_{p_1 p_2 \ldots p_l} \int_0^1 \frac{\partial \mathscr{L}}{\partial y^\sigma_{p_1 p_2 \ldots p_l}} \circ \chi \cdot dt \\
&\quad + y^\sigma_i \int_0^1 \Phi^i_\sigma \circ \chi \cdot dt - d_i \left(y^\sigma \int_0^1 \Phi^i_\sigma \circ \chi \cdot dt \right) \\
&\approx \mathscr{L} - \mathscr{L}_0 + y^\sigma_i \int_0^1 \left(\Phi^i_\sigma - \frac{\partial \mathscr{L}}{\partial y^\sigma_i} \right) \circ \chi \cdot dt \\
&\quad - \sum_{l=2}^{r} y^\sigma_{p_1 p_2 \ldots p_l} \int_0^1 \frac{\partial \mathscr{L}}{\partial y^\sigma_{p_1 p_2 \ldots p_l}} \circ \chi \cdot dt.
\end{aligned} \tag{179}$$

The symbol $\approx$, replacing the equality sign =, means that we have omitted a formal divergence expression, annihilating the Euler–Lagrange expressions of the Vainberg–Tonti Lagrangian.

In formula (179),

$$\begin{aligned}
& y_i^\sigma \int_0^1 \left(\Phi_\sigma^i - \frac{\partial \mathscr{L}}{\partial y_i^\sigma} \right) \circ \chi \cdot dt - y_{p_1 p_2}^\sigma \int_0^1 \frac{\partial \mathscr{L}}{\partial y_{p_1 p_2}^\sigma} \circ \chi \cdot dt \\
& \quad = -y_i^\sigma \int_0^1 d_p \Phi_\sigma^{ip} \circ \chi \cdot dt - y_{p_1 p_2}^\sigma \int_0^1 \frac{\partial \mathscr{L}}{\partial y_{p_1 p_2}^\sigma} \circ \chi \cdot dt \\
& \quad = -y_i^\sigma d_p \int_0^1 \Phi_\sigma^{ip} \circ \chi \cdot dt - y_{p_1 p_2}^\sigma \int_0^1 \frac{\partial \mathscr{L}}{\partial y_{p_1 p_2}^\sigma} \circ \chi \cdot dt \\
& \quad = -d_p \left(y_i^\sigma \int_0^1 \Phi_\sigma^{ip} \circ \chi \cdot dt \right) + y_{ip}^\sigma \int_0^1 \Phi_\sigma^{ip} \circ \chi \cdot dt - y_{p_1 p_2}^\sigma \int_0^1 \frac{\partial \mathscr{L}}{\partial y_{p_1 p_2}^\sigma} \circ \chi \cdot dt \\
& \quad \approx y_{p_1 p_2}^\sigma \int_0^1 \left(\Phi_\sigma^{p_1 p_2} - \frac{\partial \mathscr{L}}{\partial y_{p_1 p_2}^\sigma} \right) \circ \chi \cdot dt
\end{aligned} \tag{180}$$

thus,

$$\begin{aligned}
y^\sigma \int_0^1 E_\sigma(\mathscr{L}) \circ \chi \cdot dt & \approx \mathscr{L} - \mathscr{L}_0 + y_i^\sigma \int_0^1 \left(\Phi_\sigma^i - \frac{\partial \mathscr{L}}{\partial y_i^\sigma} \right) \circ \chi \cdot dt \\
& \quad - y_{p_1 p_2}^\sigma \int_0^1 \frac{\partial \mathscr{L}}{\partial y_{p_1 p_2}^\sigma} \circ \chi \cdot dt - \sum_{l=3}^r y_{p_1 p_2 \ldots p_l}^\sigma \int_0^1 \frac{\partial \mathscr{L}}{\partial y_{p_1 p_2 \ldots p_l}^\sigma} \circ \chi \cdot dt \\
& \approx \mathscr{L} - \mathscr{L}_0 + y_{p_1 p_2}^\sigma \int_0^1 \left(\Phi_\sigma^{p_1 p_2} - \frac{\partial \mathscr{L}}{\partial y_{p_1 p_2}^\sigma} \right) \circ \chi \cdot dt \\
& \quad - \sum_{l=3}^r y_{p_1 p_2 \ldots p_l}^\sigma \int_0^1 \frac{\partial \mathscr{L}}{\partial y_{p_1 p_2 \ldots p_l}^\sigma} \circ \chi \cdot dt.
\end{aligned} \tag{181}$$

Repeating these decompositions, we finally get the terms

$$\begin{aligned}
& y^{\sigma}_{p_1p_2\ldots p_{r-1}} \int_0^1 \left(\Phi^{p_1p_2\ldots p_{r-1}}_{\sigma} - \frac{\partial \mathscr{L}}{\partial y^{\sigma}_{p_1p_2\ldots p_{r-1}}} \right) \circ \chi \cdot dt - y^{\sigma}_{p_1p_2\ldots p_r} \int_0^1 \frac{\partial \mathscr{L}}{\partial y^{\sigma}_{p_1p_2\ldots p_r}} \circ \chi \cdot dt \\
&= -y^{\sigma}_{p_1p_2\ldots p_{r-1}} d_{p_r} \int_0^1 \Phi^{p_1p_2\ldots p_r}_{\sigma} \circ \chi \cdot dt - y^{\sigma}_{p_1p_2\ldots p_r} \int_0^1 \frac{\partial \mathscr{L}}{\partial y^{\sigma}_{p_1p_2\ldots p_r}} \circ \chi \cdot dt \\
&= -d_{p_r} \left(y^{\sigma}_{p_1p_2\ldots p_{r-1}} \int_0^1 \Phi^{p_1p_2\ldots p_r}_{\sigma} \circ \chi \cdot dt \right) + y^{\sigma}_{p_1p_2\ldots p_r} \int_0^1 \Phi^{p_1p_2\ldots p_r}_{\sigma} \circ \chi \cdot dt \qquad (182) \\
&\quad - y^{\sigma}_{p_1p_2\ldots p_r} \int_0^1 \frac{\partial \mathscr{L}}{\partial y^{\sigma}_{p_1p_2\ldots p_r}} \circ \chi \cdot dt \\
&= -d_{p_r} \left(y^{\sigma}_{p_1p_2\ldots p_{r-1}} \int_0^1 \Phi^{p_1p_2\ldots p_r}_{\sigma} \circ \chi \cdot dt \right).
\end{aligned}$$

Since $\mathscr{L}_0$ is always, as a function of x^i only, of the formal divergence type, this proves that

$$y^{\sigma} \int_0^1 E_{\sigma}(\mathscr{L}) \circ \chi \cdot dt \approx \mathscr{L}, \tag{183}$$

proving formula (172). □

4.10 The Inverse Problem of the Calculus of Variations

Our objective in this section is to study the *image* of the Euler–Lagrange mapping $\Omega^r_{n,X} W \ni \lambda \to E_\lambda \in \Omega^r_{n+1,Y} W$, considered as a subset of the set of source forms $\varepsilon \in \Omega^s_{n+1,Y} W$ (Sect. 4.9). The problem is to find a criterion for a source form to belong to the subset of the Euler–Lagrange forms.

First we show that the image of the Euler–Lagrange mapping is closed under the Lie derivative with respect to projectable vector fields.

Theorem 11 (Invariance of the image) *Let $\lambda \in \Omega^r_{n,X} W$. Then for any π-projectable vector field Ξ on W the Lie derivative $\partial_{J^r\Xi}\lambda$ belongs to the module $\Omega^r_{n,X} W$ and*

$$\partial_{J^{2r}\Xi} E_\lambda = E_{\partial_{J^r\Xi}\lambda}. \tag{184}$$

Proof Since $\lambda \in \Omega^r_{n,X}W$, then $\partial_{J^r\Xi}\lambda \in \Omega^r_{n,X}W$. If ρ_λ is a Lepage equivalent of λ, and $\rho_{\partial_{J^r\Xi}\lambda}$ is a Lepage equivalent of the Lagrangian $\partial_{J^r\Xi}\lambda$, both defined on the set W^s, then, with the notation of Sect. 4.3, Theorem 3, $\rho_\lambda = \Theta_\lambda + d\eta + \mu$, $\rho_{\partial_{J^r\Xi}\lambda} = \Theta_{\partial_{J^r\Xi}\lambda} + d\eta' + \mu'$, and

$$\partial_{J^s\Xi}\rho_\lambda = \partial_{J^s\Xi}\Theta_\lambda + d\partial_{J^s\Xi}\eta + \partial_{J^s\Xi}\mu. \tag{185}$$

The form $\partial_{J^s\Xi}\rho_\lambda$ has the horizontal component $h\partial_{J^s\Xi}\rho_\lambda = \partial_{J^{s+1}\Xi}h\rho_\lambda = \partial_{J^r\Xi}\lambda$ and is a Lepage form, because $p_1 d\partial_{J^s\Xi}\rho_\lambda = p_1 d\partial_{J^s\Xi}\Theta_\lambda = p_1\partial_{J^s\Xi}d\Theta_\lambda$ and the Lie derivative $\partial_{J^s\Xi}$ preserves contact forms (Sect. 2.5, Theorem 9). Thus, the forms $\rho_{\partial_{J^r\Xi}\lambda}$ and $\partial_{J^s\Xi}\rho_\lambda$ are both Lepage forms and have the same Lagrangians. Consequently, their Euler–Lagrange forms agree, $\partial_{J^{2r}\Xi}E_\lambda = E_{\partial_{J^r\Xi}\lambda}$. □

Rephrasing formula (184), we see that the Lie derivative of an Euler–Lagrange form by a vector field $J^{2r}\Xi$, where Ξ is a π-projectable vector field, permutes the set of Euler–Lagrange forms; the corresponding Lagrangians are also related by the Lie derivative operation.

Consider a source form $\varepsilon \in \Omega^s_{n+1,Y}W$. We say that ε is *variational*, if

$$\varepsilon = E_\lambda \tag{186}$$

for some Lagrangian $\lambda \in \Omega^r_{n,X}W$. ε is said to be *locally variational*, if there exists an atlas on Y, consisting of fibered charts, such that for each chart (V,ψ), $\psi = (x^i, y^\sigma)$, from this atlas, the restriction of ε to V^s is variational.

The *inverse problem* of the calculus of variations, or the *variationality problem* for source forms, consists in finding conditions under which there exists a Lagrangian λ, satisfying equation (186); if these conditions are satisfied, then the problem is to find *all* Lagrangians for the source form ε. The *local inverse problem*, or *local variationality problem*, for a source form ε consists in finding existence (integrability) conditions and solutions $\mathscr{L}$ of the system of partial differential equations

$$\varepsilon_\sigma = \frac{\partial\mathscr{L}}{\partial y^\sigma} + \sum_{l=1}^{r}(-1)^l d_{p_1}d_{p_2}\ldots d_{p_l}\frac{\partial\mathscr{L}}{\partial y^\sigma_{p_1p_2\ldots p_l}} \tag{187}$$

with given functions $\varepsilon_\sigma = \varepsilon_\sigma(x^i, y^\sigma, y^\sigma_{j_1}, y^\sigma_{j_1j_2}, \ldots, y^\sigma_{j_1j_2\ldots j_s})$ on the left-hand side (cf. Sect. 4.4, Theorem 4).

Let r be a fixed positive integer. We shall characterize the subspace of the vector space of source forms, which is in general larger than the image of the Euler–Lagrange mapping, namely the subspace of *locally variational forms* [K11]. Our next theorem states the relationship between the exterior derivative operator and the concept of variationality. It also indicates the meaning of *Lepage forms* for the inverse problem.

Theorem 12 (Local variationality of source forms) *Let* $\varepsilon \in \Omega^{s}_{n+1,Y}W$ *be a source form. The following two conditions are equivalent*:

(a) ε *is locally variational.*
(b) *For every point* $y \in W$ *there exist an integer* r, *a fibered chart* $(V,\psi), \psi = (x^i, y^\sigma)$, *at* y *and a form* $F \in \Omega^{r}_{n+1}V$ *of order of contactness* 2 *such that on* V^r

$$d(\varepsilon + F) = 0. \tag{188}$$

Proof

1. Suppose that ε is locally variational, and choose a fibered chart $(V,\psi), \psi = (x^i, y^\sigma)$, such that ε is variational on V; then $\varepsilon = E_\lambda$ for some Lagrangian $\lambda \in \Omega^{r}_{n,X}V$. Let Θ_λ denote the principal Lepage equivalent of λ and set $F = p_2 d\Theta_\lambda$. Then, $d(\varepsilon + F) = dd\Theta_\lambda = 0$.
2. Conversely, if for some fibered chart $(V,\psi), \psi = (x^i, y^\sigma)$, condition $d(\varepsilon + F) = 0$ holds on V^s, then $\varepsilon + F = d\rho$ for some ρ. ρ is obviously a Lepage form; hence, $\varepsilon = p_1 d\rho$, so ε is a locally variational form whose Lagrangian is $h\rho$. □

Remark 17 Theorem 12 indicates possible *geometric interpretation* of the exterior derivative dε. Namely, formula (188) says that the variationality condition means that the *class* of dε modulo (n + 2)-forms whose order of contactness is greater than 1 vanishes if and only if ε is locally variational. Developing this point of view to q-forms of *any* degree q leads to an idea to characterize the Euler–Lagrange mapping as a morphism in a suitable sheaf sequence of classes of forms (a "variational sequence").

Properties of the form F in Theorem 1 can be further specified. Namely, for a given Lagrangian λ of order r, F can be determined from the exterior derivative of the principal Lepage equivalent Θ_λ (104) and is $\pi^{2r-1,s-1}$-horizontal.

The following lemma is needed in the proof of another theorem on the local inverse problem of the calculus of variations.

Lemma 9 *Let* U *be an open set in* $\mathbf{R}^n$ *such that for each point* $x_0 = (x_0^1, x_0^2, \ldots, x_0^n)$ *the segment* $\{(tx_0^1, tx_0^2, \ldots, tx_0^n) | t \in [0,1]\}$ *belongs to* U. *Let* $f\colon U \to \mathbf{R}$ *be a function such that*

$$\int_0^1 F(tx_0^1, tx_0^2, \ldots, tx_0^n)dt = 0 \tag{189}$$

for all points $(x_0^1, x_0^2, \ldots, x_0^n) \in U$. *Then,* F = 0.

Proof If (189) is true, then for any $s \in [0,1], (sx_0^1, sx_0^2, \ldots, sx_0^n) \in U$, thus,

$$\int_0^1 F(tsx_0^1, tsx_0^2, \ldots, tsx_0^n)dt = 0. \tag{190}$$

Differentiating with respect to s

$$\int_0^1 \left(\frac{\partial F}{\partial x^k}\right)_{tsx_0} tsx_0^k dt = 0, \tag{191}$$

so at $s = 1$

$$\int_0^1 \left(\frac{\partial F}{\partial x^k}\right)_{tx_0} x_0^k t dt = 0. \tag{192}$$

On the other hand,

$$\begin{aligned} \frac{d}{dt}(tF(tx_0^1, tx_0^2, \ldots, tx_0^n)) &= F(tx_0^1, tx_0^2, \ldots, tx_0^n) + t\frac{d}{dt}F(tx_0^1, tx_0^2, \ldots, tx_0^n) \\ &= F(tx_0^1, tx_0^2, \ldots, tx_0^n) + \left(\frac{\partial F}{\partial x^k}\right)_{tx_0} x_0^k t. \end{aligned} \tag{193}$$

Integrating we have

$$\begin{aligned} F(x_0^1, x_0^2, \ldots, x_0^n) &= \int_0^1 F(tx_0^1, tx_0^2, \ldots, tx_0^n)dt + \int_0^1 \left(\frac{\partial F}{\partial x^k}\right)_{tx_0} x_0^k t dt \\ &= 0. \end{aligned} \tag{194}$$

□

Consider now the local inverse problem of the calculus of variations. We wish to find integrability conditions for the system of partial differential equations (187) and describe all solutions $\mathscr{L}$ of this system in an explicit form. To characterize locally variational forms, we need the *Helmholtz expressions* $H_{\sigma}{}_{\nu}^{q_1 q_2 \ldots q_k}(\varepsilon)$ (Sect. 4.9, (158) and Remark 14). Recall that

$$\begin{aligned} H_{\sigma\nu}^{q_1 q_2 \ldots q_k}(\varepsilon) &= \frac{\partial \varepsilon_\sigma}{\partial y^\nu_{q_1 q_2 \ldots q_k}} - (-1)^k \frac{\partial \varepsilon_\nu}{\partial y^\sigma_{q_1 q_2 \ldots q_k}} \\ &\quad - \sum_{l=k+1}^{s} (-1)^l \binom{l}{k} d_{p_{k+1}} d_{p_{k+2}} \ldots d_{p_l} \frac{\partial \varepsilon_\nu}{\partial y^\sigma_{q_1 q_2 \ldots q_k p_{k+1} p_{k+2} \ldots p_l}}, \end{aligned} \tag{195}$$

where $k = 0, 1, 2, \ldots, s$, and s is the order or ε.

Theorem 13 *Let V be an open star-shaped set in the Euclidean space* $\mathbf{R}^m$, *and let* $\varepsilon_\sigma\colon V^s \to \mathbf{R}$ *be differentiable functions. The following two conditions are equivalent*:

(a) *Equation*

$$\varepsilon_\sigma = \frac{\partial \mathscr{L}}{\partial y^\sigma} + \sum_{l=1}^{s} (-1)^l d_{p_1} d_{p_2} \ldots d_{p_l} \frac{\partial \mathscr{L}}{\partial y^\sigma_{p_1 p_2 \ldots p_l}} \tag{196}$$

has a solution $\mathscr{L}\colon V^s \to \mathbf{R}$.

(b) *For all* $k = 0, 1, 2, \ldots, s$, *the function* ε_σ *satisfies*

$$H^{q_1 q_2 \ldots q_k}_{\sigma \nu}(\varepsilon) = 0 \tag{197}$$

Proof

1. Suppose that the system (196) has a solution $\mathscr{L}$, defined on the set V^r. Then, ε is the Euler–Lagrange form $E_\sigma(\mathscr{L})\omega^\sigma \wedge \omega_0$ of the Lagrangian $\lambda = \mathscr{L}\omega_0$; we may suppose without loss of generality that the Helmholtz expressions (195) are of order $s = 2r$. Since the Lagrangian λ and the Vainberg–Tonti Lagrangian have the same Euler–Lagrange form (Sect. 4.9, Lemma 8), the Helmholtz expressions satisfy

$$\int_0^1 \sum_{k=0}^{2r} (y^\nu_{q_1 q_2 \ldots q_k} H^{q_1 q_2 \ldots q_k}_{\sigma \nu}(\varepsilon)) \circ \chi \cdot dt = 0 \tag{198}$$

(Sect. 4.9, Theorem 10); hence, from Lemma 9,

$$\sum_{k=0}^{2r} y^\nu_{q_1 q_2 \ldots q_k} H^{q_1 q_2 \ldots q_k}_{\sigma \nu}(\varepsilon) = 0. \tag{199}$$

Since by hypothesis ε is variational, that is, $\varepsilon = E_\lambda$ for some Lagrangian λ, then for any π-projectable vector field Ξ, $\partial_{J^{2r}\Xi}\varepsilon = \partial_{J^{2r}\Xi} E_\lambda = E_{\partial_{J^{2r}\Xi}\lambda}$ (Theorem 11); hence, the form $\partial_{J^{2r}\Xi}\varepsilon$ is also variational. Thus, the Helmholtz expressions satisfy for all projectable vector fields Ξ,

$$\sum_{k=0}^{2r} y^\nu_{q_1 q_2 \ldots q_k} H^{q_1 q_2 \ldots q_k}_{\sigma \nu}(\partial_{J^{2r}\Xi}\varepsilon) = 0 \tag{200}$$

We shall show that this condition implies $H^{q_1 q_2 \ldots q_k}_{\sigma \nu}(\varepsilon) = 0$.

Consider condition (200) for different choices of the vector field Ξ. It is sufficient to consider π-*vertical* vector fields, whose components do not depend on y^τ, that is,

$$\Xi = \Xi^{\sigma} \frac{\partial}{\partial y^{\sigma}}, \tag{201}$$

where $\Xi^{\sigma} = \Xi^{\sigma}(x^k)$. Then, the components of the r-jet prolongation $J^r\Xi$ are

$$\Xi^{\sigma}_{j_1 j_2 \ldots j_k} = \frac{\partial^k \Xi^{\sigma}}{\partial x^{j_1} \partial x^{j_2} \ldots \partial x^{j_k}}. \tag{202}$$

Writing $\varepsilon = \varepsilon_{\sigma}\omega^{\sigma} \wedge \omega_0$ and using properties of the vector field Ξ, the Lie derivative $\partial_{J^{2r}\Xi}\varepsilon$, standing in (200), is given by

$$\partial_{J^{2r}\Xi}\varepsilon = \partial_{J^{2s}\Xi}\varepsilon_{\sigma} \cdot \omega^{\sigma} \wedge \omega_0 = \sum_{k=0}^{2r} \frac{\partial \varepsilon_{\sigma}}{\partial y^{\kappa}_{j_1 j_2 \ldots j_k}} \Xi^{\kappa}_{j_1 j_2 \ldots j_k} \cdot \omega^{\sigma} \wedge \omega_0. \tag{203}$$

We denote

$$\varepsilon' = \partial_{J^{2r}\Xi}\varepsilon, \quad \varepsilon'_{\sigma} = \sum_{k=0}^{2r} \frac{\partial \varepsilon_{\sigma}}{\partial y^{\kappa}_{j_1 j_2 \ldots j_k}} \Xi^{\kappa}_{j_1 j_2 \ldots j_k}. \tag{204}$$

Choose the vector field Ξ in the form

$$\Xi = \frac{\partial}{\partial y^{\tau}}, \tag{205}$$

where τ is any fixed integer. In components,

$$\Xi^{\sigma} = \begin{cases} 1, & \sigma = \tau, \\ 0, & \sigma \neq \tau. \end{cases} \tag{206}$$

Then, the r-jet prolongation $J^r\Xi$ has the components $\Xi^{\sigma}_{j_1 j_2 \ldots j_r} = 0$, and the expression

$$J^r\Xi = \frac{\partial}{\partial y^{\tau}}. \tag{207}$$

The Lie derivative (203) yields

$$\varepsilon' = \frac{\partial \varepsilon_{\sigma}}{\partial y^{\kappa}} \Xi^{\kappa} \omega^{\sigma} \wedge \omega_0 = \frac{\partial \varepsilon_{\sigma}}{\partial y^{\tau}} \omega^{\sigma} \wedge \omega_0. \tag{208}$$

Thus, for the vector field (205),

$$\varepsilon'_\sigma = \frac{\partial \varepsilon_\sigma}{\partial y^\tau}. \tag{209}$$

The Helmholtz expression $H_\sigma{}^{q_1 q_2 \ldots q_k}_{\nu}(\varepsilon')$ for the source form ε' can be written as

$$\begin{aligned} H_{\sigma\nu}^{q_1 q_2 \ldots q_k}(\varepsilon') &= \frac{\partial}{\partial y^\nu_{q_1 q_2 \ldots q_k}} \frac{\partial \varepsilon_\sigma}{\partial y^\tau} - (-1)^k \frac{\partial}{\partial y^\sigma_{q_1 q_2 \ldots q_k}} \frac{\partial \varepsilon_\nu}{\partial y^\tau} \\ &\quad - \sum_{l=k+1}^{2r} (-1)^l \binom{l}{k} d_{p_{k+1}} d_{p_{k+2}} \ldots d_{p_l} \frac{\partial}{\partial y^\sigma_{q_1 q_2 \ldots q_k p_{k+1} p_{k+2} \ldots p_l}} \frac{\partial \varepsilon_\nu}{\partial y^\tau} \\ &= \frac{\partial H_{\sigma\nu}^{q_1 q_2 \ldots q_k}(\varepsilon)}{\partial y^\tau}, \end{aligned} \tag{210}$$

because the differential operators $\partial/\partial y^\tau$ and d_k commute. Condition (200) now implies

$$\begin{aligned} \sum_{k=0}^{2r} y^\nu_{q_1 q_2 \ldots q_k} H_{\sigma\nu}^{q_1 q_2 \ldots q_k}(\partial_{J^{2r}\Xi}\varepsilon) &= \sum_{k=0}^{2r} y^\nu_{q_1 q_2 \ldots q_k} \frac{\partial H_{\sigma\nu}^{q_1 q_2 \ldots q_k}(\varepsilon)}{\partial y^\tau} \\ &= \sum_{k=0}^{2r} \frac{\partial (y^\nu_{q_1 q_2 \ldots q_k} H_{\sigma\nu}^{q_1 q_2 \ldots q_k}(\varepsilon))}{\partial y^\tau} - H_{\sigma\tau}(\varepsilon) \\ &= -H_{\sigma\tau}(\varepsilon) = 0. \end{aligned} \tag{211}$$

Consequently, (200) reduces to

$$\sum_{k=1}^{2r} y^\nu_{q_1 q_2 \ldots q_k} H_{\sigma\nu}^{q_1 q_2 \ldots q_k}(\varepsilon) = 0. \tag{212}$$

Then by Theorem 11,

$$\sum_{k=1}^{2r} y^\nu_{q_1 q_2 \ldots q_k} H_{\sigma\nu}^{q_1 q_2 \ldots q_k}(\partial_{J^{2r}\Xi}\varepsilon) = 0. \tag{213}$$

Now consider equation (200) for the vector field

$$\Xi = x^i \frac{\partial}{\partial y^\tau}, \tag{214}$$

where i and τ are fixed integers. In components,

$$\Xi^\sigma = \begin{cases} x^i, & \sigma = \tau, \\ 0, & \sigma \neq \tau. \end{cases} \tag{215}$$

Then, the r-jet prolongation $J^r\Xi$ has the components

$$\Xi_j^\sigma = d_j\Xi^\sigma = \begin{cases} 1, & \sigma = \tau, \quad j = i, \\ 0, & \sigma = \tau, \quad j \neq i, \\ 0, & \sigma \neq \tau, \end{cases} \quad \Xi^\sigma_{j_1 j_2 \ldots j_k} = 0, \quad k \geq 2, \tag{216}$$

hence,

$$J^r\Xi = x^i \frac{\partial}{\partial y^\tau} + \frac{\partial}{\partial y_i^\tau}. \tag{217}$$

The Lie derivative (203) yields

$$\varepsilon' = \left(\frac{\partial \varepsilon_\sigma}{\partial y^\kappa}\Xi^\kappa + \frac{\partial \varepsilon_\sigma}{\partial y_j^\kappa}\Xi^\kappa_{j_k}\right)\omega^\sigma \wedge \omega_0 = \left(\frac{\partial \varepsilon_\sigma}{\partial y^\tau}x^i + \frac{\partial \varepsilon_\sigma}{\partial y_i^\tau}\right)\omega^\sigma \wedge \omega_0. \tag{218}$$

Consequently, using the vector field (214),

$$\varepsilon'_\sigma = \frac{\partial \varepsilon_\sigma}{\partial y^\tau}x^i + \frac{\partial \varepsilon_\sigma}{\partial y_i^\tau}. \tag{219}$$

The Helmholtz expressions for ε'_σ become

$$\begin{aligned} H_{\sigma\nu}^{q_1 q_2 \ldots q_k}(\varepsilon') = {} & \frac{\partial}{\partial y^\nu_{q_1 q_2 \ldots q_k}}\left(\frac{\partial \varepsilon_\sigma}{\partial y^\tau}x^i + \frac{\partial \varepsilon_\sigma}{\partial y_i^\tau}\right) - (-1)^k \frac{\partial}{\partial y^\sigma_{q_1 q_2 \ldots q_k}}\left(\frac{\partial \varepsilon_\nu}{\partial y^\tau}x^i + \frac{\partial \varepsilon_\nu}{\partial y_i^\tau}\right) \\ & - \sum_{l=k+1}^{2r} (-1)^l \binom{l}{k} d_{p_{k+1}} d_{p_{k+2}} \ldots d_{p_l} \frac{\partial}{\partial y^\sigma_{q_1 q_2 \ldots q_k p_{k+1} p_{k+2} \ldots p_l}}\left(\frac{\partial \varepsilon_\nu}{\partial y^\tau}x^i + \frac{\partial \varepsilon_\nu}{\partial y_i^\tau}\right) \\ = {} & \frac{\partial}{\partial y^\tau}\left(x^i\left(\frac{\partial \varepsilon_\sigma}{\partial y^\nu_{q_1 q_2 \ldots q_k}} - (-1)^k \frac{\partial \varepsilon_\nu}{\partial y^\sigma_{q_1 q_2 \ldots q_k}}\right)\right. \\ & \left. - \sum_{l=k+1}^{2r} (-1)^l \binom{l}{k} d_{p_{k+1}} d_{p_{k+2}} \ldots d_{p_l}\left(x^i \frac{\partial \varepsilon_\nu}{\partial y^\sigma_{q_1 q_2 \ldots q_k p_{k+1} p_{k+2} \ldots p_l}}\right)\right) \\ & + \frac{\partial}{\partial y^\nu_{q_1 q_2 \ldots q_k}}\frac{\partial \varepsilon_\sigma}{\partial y_i^\tau} - (-1)^k \frac{\partial}{\partial y^\sigma_{q_1 q_2 \ldots q_k}}\frac{\partial \varepsilon_\nu}{\partial y_i^\tau} \\ & - \sum_{l=k+1}^{2r} (-1)^l \binom{l}{k} d_{p_{k+1}} d_{p_{k+2}} \ldots d_{p_l} \frac{\partial}{\partial y^\sigma_{q_1 q_2 \ldots q_k p_{k+1} p_{k+2} \ldots p_l}}\frac{\partial \varepsilon_\nu}{\partial y_i^\tau}. \end{aligned} \tag{220}$$

In this expression,

$$d_{p_{k+1}}d_{p_{k+2}}\dots d_{p_l}\left(x^i\frac{\partial\varepsilon_\nu}{\partial y^\sigma_{q_1q_2\dots q_kp_{k+1}p_{k+2}\dots p_l}}\right)$$
$$= d_{p_{k+2}}d_{p_{k+3}}\dots d_{p_l}\frac{\partial\varepsilon_\nu}{\partial y^\sigma_{q_1q_2\dots q_kip_{k+2}p_{k+3}\dots p_l}}$$
$$+ d_{p_{k+2}}d_{p_{k+3}}\dots d_{p_l}\left(x^i d_{p_{k+1}}\frac{\partial\varepsilon_\nu}{\partial y^\sigma_{q_1q_2\dots q_kp_{k+1}p_{k+2}\dots p_l}}\right). \tag{221}$$

Note that for any function f, the formal derivative satisfies

$$d_p\frac{\partial f}{\partial y^\tau_i} = \frac{\partial d_pf}{\partial y^\tau_i} - \frac{\partial f}{\partial y^\tau}\delta^i_p. \tag{222}$$

Applying this rule, we find

$$d_{p_{k+1}}d_{p_{k+2}}\dots d_{p_l}\frac{\partial}{\partial y^\tau_i}\frac{\partial\varepsilon_\nu}{\partial y^\sigma_{q_1q_2\dots q_kp_{k+1}p_{k+2}\dots p_l}}$$
$$= \dots = \frac{\partial}{\partial y^\tau_i}d_{p_{k+3}}d_{p_{k+4}}\dots d_{p_l}d_{p_{k+2}}d_{p_{k+1}}\frac{\partial\varepsilon_\nu}{\partial y^\sigma_{q_1q_2\dots q_kp_{k+1}p_{k+2}\dots p_l}}$$
$$-(l-k)\frac{\partial}{\partial y^\tau}d_{p_{k+2}}d_{p_{k+3}}\dots d_{p_l}\frac{\partial\varepsilon_\nu}{\partial y^\sigma_{q_1q_2\dots q_kip_{k+2}p_{k+3}\dots p_l}}. \tag{223}$$

Returning to (220)

$$H^{q_1q_2\dots q_k}_{\sigma\nu}(\varepsilon') = \frac{\partial}{\partial y^\tau}\left(x^i\left(\frac{\partial\varepsilon_\sigma}{\partial y^\nu_{q_1q_2\dots q_k}} - (-1)^k\frac{\partial\varepsilon_\nu}{\partial y^\sigma_{q_1q_2\dots q_k}}\right)\right.$$
$$-\left(\sum_{l=k+1}^{2r}(-1)^l\binom{l}{k}(l-k)d_{p_{k+2}}d_{p_{k+3}}\dots d_{p_l}\frac{\partial\varepsilon_\nu}{\partial y^\sigma_{q_1q_2\dots q_kip_{k+2}p_{k+3}\dots p_l}}\right.$$
$$\left.\left.+x^i d_{p_{k+1}}d_{p_{k+2}}\dots d_{p_l}\frac{\partial\varepsilon_\nu}{\partial y^\sigma_{q_1q_2\dots q_kp_{k+1}p_{k+2}\dots p_l}}\right)\right)$$
$$+\frac{\partial}{\partial y^\tau_i}\left(\frac{\partial\varepsilon_\sigma}{\partial y^\nu_{q_1q_2\dots q_k}} - (-1)^k\frac{\partial\varepsilon_\nu}{\partial y^\sigma_{q_1q_2\dots q_k}}\right)$$
$$-\sum_{l=k+1}^{2r}(-1)^l\binom{l}{k}\left(\frac{\partial}{\partial y^\tau_i}d_{p_{k+1}}d_{p_{k+2}}d_{p_{k+3}}\dots d_{p_l}\frac{\partial\varepsilon_\nu}{\partial y^\sigma_{q_1q_2\dots q_kp_{k+1}p_{k+2}\dots p_l}}\right.$$
$$\left.-(l-k)\frac{\partial}{\partial y^\tau}d_{p_{k+2}}d_{p_{k+3}}\dots d_{p_l}\frac{\partial\varepsilon_\nu}{\partial y^\sigma_{q_1q_2\dots q_kip_{k+2}p_{k+3}\dots p_l}}\right). \tag{224}$$

Therefore,

$$H_{\sigma\nu}^{q_1 q_2 \ldots q_k}(\varepsilon') = x^i \frac{\partial H_{\sigma\nu}^{q_1 q_2 \ldots q_k}(\varepsilon)}{\partial y^\tau} + \frac{\partial H_{\sigma\nu}^{q_1 q_2 \ldots q_k}(\varepsilon)}{\partial y_i^\tau}. \tag{225}$$

Now (200) is expressed as

$$\begin{aligned} \sum_{k=1}^{2r} y_{q_1 q_2 \ldots q_k}^\nu \left(x^i \frac{\partial H_{\sigma\nu}^{q_1 q_2 \ldots q_k}(\varepsilon)}{\partial y^\tau} + \frac{\partial H_{\sigma\nu}^{q_1 q_2 \ldots q_k}(\varepsilon)}{\partial y_i^\tau} \right) \\ = x^i \frac{\partial}{\partial y^\tau} \sum_{k=1}^{2r} y_{q_1 q_2 \ldots q_k}^\nu H_{\sigma\nu}^{q_1 q_2 \ldots q_k}(\varepsilon) \\ + \sum_{k=1}^{2r} \frac{\partial}{\partial y_i^\tau} (y_{q_1 q_2 \ldots q_k}^\nu H_{\sigma\nu}^{q_1 q_2 \ldots q_k}(\varepsilon)) - H_{\sigma\tau}^i(\varepsilon) = -H_{\sigma\tau}^i(\varepsilon) = 0. \end{aligned} \tag{226}$$

The proof can be completed by induction. To this purpose, one should assume that $H_{\sigma\nu} = 0, H_{\sigma\nu}^{q_1} = 0, H_{\sigma\nu}^{q_1 q_2} = 0, \ldots, H_{\sigma\nu}^{q_1 q_2 \ldots q_p} = 0$ for some p (induction hypothesis). Then, conditions (212) and (213) are replaced with

$$\sum_{k=p+1}^{2r} y_{q_1 q_2 \ldots q_k}^\nu H_{\sigma\nu}^{q_1 q_2 \ldots q_k}(\varepsilon) = 0 \tag{227}$$

and

$$\sum_{k=p+1}^{2r} y_{q_1 q_2 \ldots q_k}^\nu H_{\sigma\nu}^{q_1 q_2 \ldots q_k}(\partial_{J^{2r}\Xi}\varepsilon) = 0, \tag{228}$$

where the vector fields Ξ are of the form

$$\Xi^\sigma = \begin{cases} x^{k_1} x^{k_2} \ldots x^{k_p}, & \sigma = \tau, \\ 0, & \sigma \neq \tau. \end{cases} \tag{229}$$

2. We prove that (b) implies (a). Suppose that a system of functions ε_σ satisfies conditions (197) and denotes by $\varepsilon = \varepsilon_\sigma \omega^\sigma \wedge \omega_0$ the corresponding source form. Then, the Euler–Lagrange expressions of the Vainberg–Tonti Lagrangian $\lambda_\varepsilon = \mathscr{L}_\varepsilon \omega_0$,

$$E_\sigma(\mathscr{L}_\varepsilon) = \varepsilon_\sigma - \sum_{k=0}^{s} \int (y_{q_1 q_2 \ldots q_k}^\nu H_{\sigma\nu}^{q_1 q_2 \ldots q_k}(\varepsilon)) \circ \chi \cdot dt, \tag{230}$$

reduce to ε_σ (Sect. 4.9, Theorem 10). Thus, $\lambda_\varepsilon = \mathscr{L}_\varepsilon \omega_0$. □

Remark 18 One can easily prove condition

$$H_{\sigma\nu}(\varepsilon) = 0 \tag{231}$$

in Theorem 13 by means of the *integrability criterion* for formal divergence equations (Sect. 3.2, Theorem 1). Consider the inverse problem equation

$$\begin{aligned}\varepsilon_\sigma = \frac{\partial \mathscr{L}}{\partial y^\sigma} - d_{p_1}\frac{\partial \mathscr{L}}{\partial y^\sigma_{p_1}} + d_{p_1}d_{p_2}\frac{\partial \mathscr{L}}{\partial y^\sigma_{p_1p_2}} \\ - \cdots + (-1)^{r-1}d_{p_1}d_{p_2}\ldots d_{p_{r-1}}\frac{\partial \mathscr{L}}{\partial y^\sigma_{p_1p_2\ldots p_{r-1}}} + (-1)^r d_{p_1}d_{p_2}\ldots d_{p_r}\frac{\partial \mathscr{L}}{\partial y^\sigma_{p_1p_2\ldots p_r}}\end{aligned} \tag{232}$$

and suppose it has a solution $\mathscr{L}$. Denoting

$$\begin{aligned}\Phi^{p_1}_\sigma = \frac{\partial \mathscr{L}}{\partial y^\sigma_{p_1}} + d_{p_2}\frac{\partial \mathscr{L}}{\partial y^\sigma_{p_1p_2}} - \cdots + (-1)^{r-1}d_{p_2}d_{p_3}\ldots d_{p_{r-1}}\frac{\partial \mathscr{L}}{\partial y^\sigma_{p_1p_2\ldots p_{r-1}}} \\ + (-1)^r d_{p_2}d_{p_3}\ldots d_{p_r}\frac{\partial \mathscr{L}}{\partial y^\sigma_{p_1p_2\ldots p_r}},\end{aligned} \tag{233}$$

we get the formal divergence equation

$$\varepsilon_\sigma - \frac{\partial \mathscr{L}}{\partial y^\sigma} = -d_{p_1}\Phi^{p_1}_\sigma. \tag{234}$$

Since by hypothesis there exists a solution, the integrability condition for this equation is satisfied, that is,

$$E_\tau\left(\varepsilon_\sigma - \frac{\partial \mathscr{L}}{\partial y^\sigma}\right) = 0. \tag{235}$$

Explicitly, since the derivative d_i and the partial derivative $\partial/\partial y^\tau$ commute,

$$\begin{aligned}E_\tau\left(\varepsilon_\sigma - \frac{\partial \mathscr{L}}{\partial y^\sigma}\right) = \frac{\partial \varepsilon_\sigma}{\partial y^\tau} - d_{p_1}\frac{\partial \varepsilon_\sigma}{\partial y^\tau_{p_1}} + d_{p_1}d_{p_2}\frac{\partial \varepsilon_\sigma}{\partial y^\tau_{p_1p_2}} \\ - \cdots + (-1)^{r-1}d_{p_1}d_{p_2}\ldots d_{p_{r-1}}\frac{\partial \varepsilon_\sigma}{\partial y^\tau_{p_1p_2\ldots p_{r-1}}} \\ + (-1)^r d_{p_1}d_{p_2}\ldots d_{p_r}\frac{\partial \varepsilon_\sigma}{\partial y^\tau_{p_1p_2\ldots p_r}} - \frac{\partial \varepsilon_\tau}{\partial y^\sigma} = 0.\end{aligned} \tag{236}$$

Comparing this formula with (195), we get exactly $H_{\sigma\ \tau}(\varepsilon) = 0$.

We end this section with two remarks on the inverse problem for systems of differential equations.

Remark 19 (Variationality of differential equations) The concept of local variationality can be applied to the systems of partial differential equations. Fixing the functions ε_σ, we sometimes say, without aspiration to rigor, that the *system of partial differential equations*

$$\varepsilon_\sigma(x^i, y^\tau, y^\tau_{j_1}, y^\tau_{j_1 j_2}, \ldots, y^\tau_{j_1 j_2 \ldots j_s}) = 0 \tag{237}$$

is *variational* and its left-hand sides coincide with the Euler–Lagrange equations of some Lagrangian. It is clear, however, that this concept is not well-defined; indeed, setting $\varepsilon'_\nu = \Phi^\nu_\sigma \varepsilon_\nu$ with any functions Φ^ν_σ such that $\det \Phi^\nu_\sigma \neq 0$, we get two equivalent systems $\varepsilon_\sigma = 0$ and $\varepsilon'_\nu = 0$, but it may happen that the first one is variational and the second one is not. If (188) is *not* variational and there exists Φ^ν_σ such that the equivalent system $\Phi^\nu_\sigma \varepsilon_\nu = 0$ *is* variational, we say that Φ^ν_σ are *variational integrators* for the system (188). It should be noted, however, that this terminology is also used in a different context of differential equations, expressed in a *contravariant* form.

Remark 20 (Sonin, Helmholtz, and Douglas) The inverse problem of the calculus of variations was first considered in 1886 for *one* second-order ordinary differential equation by Sonin (see Sonin [So]; for this reference, the author is indebted to V.D. Skarzhinski). He proved that *every* second-order equation has a Lagrangian. It should be pointed out that in this paper the *variational multiplier*, in contemporary terminology, was used as a natural factor ensuring *covariance* of the considered equation. The variationality of *systems* of second-order ordinary differential equations, expressed in the covariant form, was studied by Helmholtz in 1887 and subsequently by many followers (Helmholtz [He]; see also Havas [H], where further references can be found). The systems of second-order ordinary differential equations, solved with respect to the second derivatives, were considered by Douglas in 1940 with the techniques of variational multipliers (see Douglas [Do], and e.g., Anderson and Thompson [AT], Bucataru [Bu], Crampin [Cr], Sarlet et al. [SCM]).

4.11 Local Variationality of Second-Order Source Forms

In this section, we shall primarily be concerned with the *second-order* source forms and *second-order* systems of partial differential equations. The aim is to present a solution of the inverse problem of the calculus of variations for this class of source forms entirely by means of the theory of Lepage forms (Sect. 4.10, Theorem 12) and elementary integration theory of exterior differential systems.

Suppose we are given a second-order source form ε on $W^2 \subset J^2 Y$, expressed in a fibered chart $(V, \psi), \psi = (x^i, y^\sigma)$, as

$$\varepsilon = \varepsilon_\sigma \omega^\sigma \wedge \omega_0. \tag{238}$$

Consider the system of partial differential equations

$$\frac{\partial \mathscr{L}}{\partial y^\sigma} - d_p \frac{\partial \mathscr{L}}{\partial y^\sigma_p} + d_p d_q \frac{\partial \mathscr{L}}{\partial y^\sigma_{pq}} = \varepsilon_\sigma \tag{239}$$

for an unknown Lagrangian $\mathscr{L}$ of order 2. Clearly the left-hand sides of these equations are exactly the Euler–Lagrange expressions $E_\sigma(\mathscr{L})$ of the Lagrangian $\mathscr{L}$. The problem we consider is twofold: (a) to find the *variationality* (*integrability) conditions* for ε, ensuring existence of a solution $\mathscr{L}$, and (b) to find all solutions provided the integrability conditions are satisfied.

The following theorem, following from the theory of the Vainberg–Tonti Lagrangians, states that a second-order variational source form always admits a *first-order* Lagrangian; it seems that this extension of the well-known statement of the calculus of variations of simple integrals to the general multiple-integral problems is new. Note that the result restricts the class of locally variational forms to the source forms, depending on the second derivative variables *linearly*.

Theorem 14 *If a second-order source form* ε, *defined on* $W^2 \subset J^2Y$, *is locally variational, then for every point* $y \in W$ *there exists a fibered chart* $(V, \psi), \psi = (x^i, y^\sigma)$, *at* y *and a first-order Lagrangian* $\lambda_0 = \mathscr{L}_0\omega_0$, *defined on* V^1, *such that*

$$E_\sigma(\mathscr{L}_0) = \varepsilon_\sigma. \tag{240}$$

Proof If ε is variational, then by hypothesis the form $\varepsilon_\sigma \omega^\sigma \wedge \omega_i$ has a *second-order* Lagrangian $\lambda = \mathscr{L}\omega_0$ (the Vainberg–Tonti Lagrangian). The Euler–Lagrange form associated with λ is then given by

$$E_\lambda = E_\sigma(\mathscr{L})\omega^\sigma \wedge \omega_0, \tag{241}$$

where

$$\varepsilon_\sigma = E_\sigma(\mathscr{L}) = \frac{\partial \mathscr{L}}{\partial y^\sigma} - d_i \frac{\partial \mathscr{L}}{\partial y^\sigma_i} + d_i d_j \frac{\partial \mathscr{L}}{\partial y^\sigma_{ij}}. \tag{242}$$

One can find an explicit formula for the Euler–Lagrange expression (242); this expression does not depend on y^σ_{ijk} and y^σ_{ijkl}. Introducing the *cut formal derivative* of a function $f = f(x^i, y^\sigma, y^\sigma_j, y^\sigma_{jk})$ as the function

$$d'_j f = \frac{\partial f}{\partial x^j} + \frac{\partial f}{\partial y^\sigma} y^\sigma_j + \frac{\partial f}{\partial y^\sigma_i} y^\sigma_{ij} \tag{243}$$

(see Sect. 3.1), we easily find

$$\begin{aligned}E_\sigma(\mathscr{L}) &= \frac{\partial\mathscr{L}}{\partial y^\sigma} - d'_i\frac{\partial\mathscr{L}}{\partial y_i^\sigma} + d'_i d'_j\frac{\partial\mathscr{L}}{\partial y_{ij}^\sigma} + 2d'_i\frac{\partial^2\mathscr{L}}{\partial y_{ij}^\sigma\partial y_{kl}^\nu}y_{klj}^\nu \\ &\quad + \left(\frac{\partial^2\mathscr{L}}{\partial y_j^\nu\partial y_{kl}^\sigma} - \frac{\partial^2\mathscr{L}}{\partial y_j^\sigma\partial y_{kl}^\nu}\right)y_{klj}^\nu + \frac{\partial^3\mathscr{L}}{\partial y_{ij}^\sigma\partial y_{kl}^\nu\partial y_{pq}^\tau}y_{pqi}^\tau y_{klj}^\nu \\ &\quad + \frac{\partial^2\mathscr{L}}{\partial y_{ij}^\sigma\partial y_{kl}^\nu}y_{klij}^\nu.\end{aligned} \tag{244}$$

However, this function does not depend on y_{klj}^ν and y_{klij}^ν. Hence, $\mathscr{L}$ must satisfy, among others,

$$\frac{\partial^2\mathscr{L}}{\partial y_{ij}^\sigma\,\partial y_{kl}^\nu} = 0 \quad \mathrm{Sym}(klij). \tag{245}$$

But this condition implies

$$\frac{\partial^2\mathscr{L}}{\partial y_{ij}^\sigma\,\partial y_{kl}^\sigma} + \frac{\partial^2\mathscr{L}}{\partial y_{il}^\sigma\,\partial y_{jk}^\sigma} + \frac{\partial^2\mathscr{L}}{\partial y_{ik}^\sigma\,\partial y_{jl}^\sigma} = 0. \tag{246}$$

Then, for any two fixed indices i, j,

$$\frac{\partial^2\mathscr{L}}{\partial y_{ij}^\sigma\partial y_{ii}^\sigma} = 0, \quad \frac{\partial^2\mathscr{L}}{\partial y_{ij}^\sigma\partial y_{ii}^\sigma} = 0, \quad \frac{\partial^2\mathscr{L}}{\partial y_{ij}^\sigma\partial y_{ij}^\sigma} + 2\frac{\partial^2\mathscr{L}}{\partial y_{ii}^\sigma\partial y_{jj}^\sigma} = 0, \tag{247}$$

hence, differentiating,

$$\frac{\partial^3\mathscr{L}}{\partial y_{ij}^\sigma\partial y_{ij}^\sigma\partial y_{ij}^\sigma} = 0, \quad \frac{\partial^3\mathscr{L}}{\partial y_{ij}^\sigma\partial y_{ii}^\sigma\partial y_{jj}^\sigma} = 0. \tag{248}$$

In particular, $\mathscr{L}$ must be a polynomial function of y_{jj}^σ, quadratic in each of the variables y_{jj}^σ. We can write

$$\mathscr{L} = \mathscr{L}_0 + \sum_{p\geq 1}\mathscr{L}_p, \tag{249}$$

where $\mathscr{L}_0 = \mathscr{L}_0(x^k, y^\sigma, y_j^\sigma)$ is a function independent of y_{ij}^ν and $\mathscr{L}_p$ is a homogeneous polynomial of degree p,

$$\mathscr{L}_p = P_{\sigma_1\ \sigma_2\ \ldots\ \sigma_p}^{i_1j_1\ i_2j_2\ \ldots\ i_pj_p} y_{i_1j_1}^{\sigma_1}y_{i_2j_2}^{\sigma_2}\ldots y_{i_pj_p}^{\sigma_p}. \tag{250}$$

Substituting from this formula into (244),

$$E_\sigma(\mathscr{L}) = \frac{\partial \mathscr{L}_0}{\partial y^\sigma} - d'_i \frac{\partial \mathscr{L}_0}{\partial y^\sigma_i} + \sum_{p \geq 1} \left(\frac{\partial \mathscr{L}_p}{\partial y^\sigma} - d'_i \frac{\partial \mathscr{L}_p}{\partial y^\sigma_i} + d'_i d'_j \frac{\partial \mathscr{L}_p}{\partial y^\sigma_{ij}} + 2d'_i \frac{\partial^2 \mathscr{L}_p}{\partial y^\sigma_{ij} \partial y^\nu_{kl}} y^\nu_{klj} + \left(\frac{\partial^2 \mathscr{L}_p}{\partial y^\nu_j \partial y^\sigma_{kl}} - \frac{\partial^2 \mathscr{L}_p}{\partial y^\sigma_j \partial y^\nu_{kl}} \right) y^\nu_{klj} + \frac{\partial^3 \mathscr{L}_p}{\partial y^\sigma_{ij} \partial y^\nu_{kl} \partial y^\tau_{pq}} y^\tau_{pqi} y^\nu_{klj} + \frac{\partial^2 \mathscr{L}_p}{\partial y^\sigma_{ij} \partial y^\nu_{kl}} y^\nu_{klij} \right). \tag{251}$$

But the left-hand side does not depend on y^ν_{ijk} and y^ν_{ijkl}, so setting $y^\nu_{ijk} = 0$ and $y^\nu_{ijkl} = 0$, we get

$$E_\sigma(\mathscr{L}) = E_\sigma(\mathscr{L}_0) = \frac{\partial \mathscr{L}_0}{\partial y^\sigma} - d'_i \frac{\partial \mathscr{L}_0}{\partial y^\sigma_i}. \tag{252}$$

Replacing the cut formal derivative d'_i with d_i, this formula shows that the Euler–Lagrange expressions $E_\sigma(\mathscr{L}_0)$ of the first-order Lagrangian $\lambda_0 = \mathscr{L}_0\omega_0$ coincide with the components ε_σ of the source form ε. This proves Theorem 14.□

Corollary 1 *Suppose that a second-order source form $\varepsilon = \varepsilon_\sigma \omega^\sigma \wedge \omega_i$ is variational. Then, the components ε_σ depend linearly on the second derivative variables y^ν_{ij}, that is,*

$$\varepsilon_\sigma = A_\sigma + B^{ij}_{\sigma\nu} y^\nu_{ij}, \tag{253}$$

where the functions $B^{ij}_{\sigma\nu}$ do not depend on the variables y^ν_{ij}.

Now we wish to find a criterion for a second-order source form ε (1) to be locally variational. As a main tool in the proof, we use the concept of a Lepage form and the basic theorem on locally variational source forms (Sect. 4.10, Theorem 12).

Theorem 15 (Local variationality of source forms) *Let $\varepsilon \in \Omega^2_{n+1,Y} W$ be a source form. The following two conditions are equivalent*:

(a) *ε is locally variational.*
(b) *For every point $y \in W$ there exist an integer r and a fibered chart $(V, \psi), \psi = (x^i, y^\sigma)$, at y, such that $\varepsilon = \varepsilon_\sigma \omega^\sigma \wedge \omega_0$, and the components ε_σ satisfy*

$$\frac{\partial \varepsilon_\sigma}{\partial y^\nu_{ij}} - \frac{\partial \varepsilon_\nu}{\partial y^\sigma_{ij}} = 0, \quad \frac{\partial \varepsilon_\sigma}{\partial y^\nu_j} + \frac{\partial \varepsilon_\nu}{\partial y^\sigma_j} - d_i \left(\frac{\partial \varepsilon_\sigma}{\partial y^\nu_{ij}} + \frac{\partial \varepsilon_\nu}{\partial y^\sigma_{ij}} \right) = 0,$$
$$\frac{\partial \varepsilon_\sigma}{\partial y^\nu} - \frac{\partial \varepsilon_\nu}{\partial y^\sigma} - \frac{1}{2} d_j \left(\frac{\partial \varepsilon_\sigma}{\partial y^\nu_j} - \frac{\partial \varepsilon_\nu}{\partial y^\sigma_j} \right) = 0. \tag{254}$$

(c) *For every point* $y \in W$ *there exist an integer* r *and a fibered chart* $(V, \psi), \psi = (x^i, y^\sigma)$, *at* y *and a form* $F \in \Omega^r_{n+1} V$ *of order of contactness* ≤2 *such that on* V^r

$$d(\varepsilon + F) = 0. \tag{255}$$

Proof

1. If (a) holds, then (b) is obtained by a direct calculation. Indeed, suppose that $\varepsilon_\sigma = E_\sigma(\mathscr{L})$ are the Euler–Lagrange expressions of a first-order Lagrangian $\lambda = \mathscr{L}\omega_0$; then

$$E_\sigma(\mathscr{L}) = \frac{\partial \mathscr{L}}{\partial y^\sigma} - \frac{\partial^2 \mathscr{L}}{\partial x^i \partial y^\sigma_i} - \frac{\partial^2 \mathscr{L}}{\partial y^\tau \partial y^\sigma_i} y^\tau_i - \frac{\partial^2 \mathscr{L}}{\partial y^\tau_i \partial y^\sigma_j} y^\tau_{ij}. \tag{256}$$

Differentiating we have

$$\begin{aligned}
\frac{\partial \varepsilon_\sigma}{\partial y^\nu_{pq}} &= -\frac{1}{2}\left(\frac{\partial^2 \mathscr{L}}{\partial y^\tau_p \partial y^\sigma_q} + \frac{\partial^2 \mathscr{L}}{\partial y^\tau_q \partial y^\sigma_p}\right),\\
\frac{\partial \varepsilon_\sigma}{\partial y^\nu_q} &= \frac{\partial^2 \mathscr{L}}{\partial y^\sigma \partial y^\nu_q} - d_s \frac{\partial^2 \mathscr{L}}{\partial y^\nu_q \partial y^\sigma_s} - \frac{\partial^2 \mathscr{L}}{\partial y^\nu \partial y^\sigma_q},\\
\frac{\partial \varepsilon_\sigma}{\partial y^\nu} &= \frac{\partial^2 \mathscr{L}}{\partial y^\nu \partial y^\sigma} - d_s \frac{\partial^2 \mathscr{L}}{\partial y^\nu \partial y^\sigma_s},
\end{aligned} \tag{257}$$

from which we get

$$\begin{aligned}
\frac{\partial \varepsilon_\sigma}{\partial y^\nu_{pq}} - \frac{\partial \varepsilon_\nu}{\partial y^\sigma_{pq}} = &-\frac{1}{2}\left(\frac{\partial^2 \mathscr{L}}{\partial y^\nu_p \partial y^\sigma_q} + \frac{\partial^2 \mathscr{L}}{\partial y^\nu_q \partial y^\sigma_p}\right)\\
&+\frac{1}{2}\left(\frac{\partial^2 \mathscr{L}}{\partial y^\sigma_p \partial y^\nu_q} + \frac{\partial^2 \mathscr{L}}{\partial y^\sigma_q \partial y^\nu_p}\right) = 0,
\end{aligned} \tag{258}$$

and

$$\begin{aligned}
\frac{\partial \varepsilon_\sigma}{\partial y^\nu_q} + \frac{\partial \varepsilon_\nu}{\partial y^\sigma_q} - 2 d_p \frac{\partial \varepsilon_\nu}{\partial y^\sigma_{qp}} = &\frac{\partial^2 \mathscr{L}}{\partial y^\sigma \partial y^\nu_q} - d_s \frac{\partial^2 \mathscr{L}}{\partial y^\nu_q \partial y^\sigma_s} - \frac{\partial^2 \mathscr{L}}{\partial y^\nu \partial y^\sigma_q} + \frac{\partial^2 \mathscr{L}}{\partial y^\nu \partial y^\sigma_q}\\
&- d_s \frac{\partial^2 \mathscr{L}}{\partial y^\sigma_q \partial y^\nu_s} - \frac{\partial^2 \mathscr{L}}{\partial y^\sigma \partial y^\nu_q} + d_p\left(\frac{\partial^2 \mathscr{L}}{\partial y^\sigma_p \partial y^\nu_q} + \frac{\partial^2 \mathscr{L}}{\partial y^\sigma_q \partial y^\nu_p}\right) = 0,
\end{aligned} \tag{259}$$

and

$$\begin{aligned}
&\frac{\partial \varepsilon_\sigma}{\partial y^\nu} - \frac{\partial \varepsilon_\nu}{\partial y^\sigma} + d_p \frac{\partial \varepsilon_\nu}{\partial y^\sigma_p} - d_p d_q \frac{\partial \varepsilon_\nu}{\partial y^\sigma_{pq}} \\
&\qquad = \frac{\partial^2 \mathscr{L}}{\partial y^\nu \partial y^\sigma} - d_s \frac{\partial^2 \mathscr{L}}{\partial y^\nu \partial y^\sigma_s} - \frac{\partial^2 \mathscr{L}}{\partial y^\nu \partial y^\sigma} + d_s \frac{\partial^2 \mathscr{L}}{\partial y^\sigma \partial y^\nu_s} \\
&\qquad\quad + d_q \frac{\partial^2 \mathscr{L}}{\partial y^\nu \partial y^\sigma_q} - d_q d_s \frac{\partial^2 \mathscr{L}}{\partial y^\sigma_q \partial y^\nu_s} - d_q \frac{\partial^2 \mathscr{L}}{\partial y^\sigma \partial y^\nu_q} \\
&\qquad\quad + \frac{1}{2} d_p d_q \left(\frac{\partial^2 \mathscr{L}}{\partial y^\sigma_p \partial y^\nu_q} + \frac{\partial^2 \mathscr{L}}{\partial y^\sigma_q \partial y^\nu_p} \right) = 0.
\end{aligned} \tag{260}$$

2. Suppose that the components $\varepsilon_\sigma = E_\sigma(\mathscr{L})$ of the Euler–Lagrange expressions of $\lambda = \mathscr{L}\omega_0$ satisfy condition (b). Setting

$$F = -\left(\frac{1}{4} \left(\frac{\partial \varepsilon_\sigma}{\partial y^\nu_i} - \frac{\partial \varepsilon_\nu}{\partial y^\sigma_i} \right) \omega^\nu + \frac{\partial \varepsilon_\sigma}{\partial y^\nu_{ij}} \omega^\nu_j \right) \wedge \omega^\sigma \wedge \omega_i, \tag{261}$$

we get by a straightforward calculation, using the canonical decomposition of forms into their horizontal and contact components and the identities $d\omega^\nu = -\omega^\nu_l \wedge dx^l, d\omega^\nu_j = -\omega^\nu_{jl} \wedge dx^l$, and $dx^l \wedge \omega_i = \delta^l_i \omega_0$,

$$\begin{aligned}
dF &= -\frac{1}{4} d \left(\frac{\partial \varepsilon_\sigma}{\partial y^\nu_i} - \frac{\partial \varepsilon_\nu}{\partial y^\sigma_i} \right) \wedge \omega^\nu \wedge \omega^\sigma \wedge \omega_i \\
&\quad - \frac{1}{4} \left(\frac{\partial \varepsilon_\sigma}{\partial y^\nu_i} - \frac{\partial \varepsilon_\nu}{\partial y^\sigma_i} \right) d\omega^\nu \wedge \omega^\sigma \wedge \omega_i \\
&\quad - d \frac{\partial \varepsilon_\sigma}{\partial y^\nu_{ij}} \wedge \omega^\nu_j \wedge \omega^\sigma \wedge \omega_i - \frac{\partial \varepsilon_\sigma}{\partial y^\nu_{ij}} d\omega^\nu_j \wedge \omega^\sigma \wedge \omega_i \\
&\quad + \left(\frac{1}{4} \left(\frac{\partial \varepsilon_\sigma}{\partial y^\nu_i} - \frac{\partial \varepsilon_\nu}{\partial y^\sigma_i} \right) \omega^\nu + \frac{\partial \varepsilon_\sigma}{\partial y^\nu_{ij}} \omega^\nu_j \right) \wedge d\omega^\sigma \wedge \omega_i \\
&= -\frac{1}{4} d_i \left(\frac{\partial \varepsilon_\sigma}{\partial y^\nu_i} - \frac{\partial \varepsilon_\nu}{\partial y^\sigma_i} \right) \omega^\nu \wedge \omega^\sigma \wedge \omega_0 \\
&\quad + \left(\frac{1}{2} \left(\frac{\partial \varepsilon_\nu}{\partial y^\sigma_i} - \frac{\partial \varepsilon_\sigma}{\partial y^\nu_i} \right) - d_j \frac{\partial \varepsilon_\sigma}{\partial y^\nu_{ij}} \right) \omega^\nu_i \wedge \omega^\sigma \wedge \omega_0 \\
&\quad - \frac{\partial \varepsilon_\sigma}{\partial y^\nu_{ij}} \omega^\nu_{ij} \wedge \omega^\sigma \wedge \omega_0 - \frac{\partial \varepsilon_\sigma}{\partial y^\nu_{ij}} \omega^\nu_j \wedge \omega^\sigma_i \wedge \omega_0 \\
&\quad - \frac{1}{4} pd \left(\frac{\partial \varepsilon_\sigma}{\partial y^\nu_i} - \frac{\partial \varepsilon_\nu}{\partial y^\sigma_i} \right) \wedge \omega^\nu \wedge \omega^\sigma \wedge \omega_i - pd \frac{\partial \varepsilon_\sigma}{\partial y^\nu_{ij}} \wedge \omega^\nu_j \wedge \omega^\sigma \wedge \omega_i.
\end{aligned} \tag{262}$$

Consequently, since

$$d\varepsilon = \left(\frac{\partial \varepsilon_\sigma}{\partial y^\nu} \omega^\nu + \frac{\partial \varepsilon_\sigma}{\partial y^\nu_i} \omega^\nu_i + \frac{\partial \varepsilon_\sigma}{\partial y^\nu_{ij}} \omega^\nu_{ij} \right) \wedge \omega^\sigma \wedge \omega_0, \tag{263}$$

the exterior derivative $d(\varepsilon + F)$ is expressed as

$$\begin{aligned} d(\varepsilon + F) = {} & \frac{1}{4} \left(\frac{\partial \varepsilon_\sigma}{\partial y^\nu} - \frac{\partial \varepsilon_\nu}{\partial y^\sigma} - \frac{1}{2} d_i \left(\frac{\partial \varepsilon_\sigma}{\partial y^\nu_i} - \frac{\partial \varepsilon_\nu}{\partial y^\sigma_i} \right) \right) \omega^\nu \wedge \omega^\sigma \wedge \omega_0 \\ & + \left(\frac{1}{2} \left(\frac{\partial \varepsilon_\sigma}{\partial y^\nu_i} + \frac{\partial \varepsilon_\nu}{\partial y^\sigma_i} \right) - d_j \frac{\partial \varepsilon_\sigma}{\partial y^\nu_{ij}} \right) \omega^\nu_i \wedge \omega^\sigma \wedge \omega_0 \\ & - \frac{1}{2} \left(\frac{\partial \varepsilon_\sigma}{\partial y^\nu_{ij}} - \frac{\partial \varepsilon_\nu}{\partial y^\sigma_{ij}} \right) \omega^\nu_j \wedge \omega^\sigma_i \wedge \omega_0 \\ & - \frac{1}{4} pd \left(\frac{\partial \varepsilon_\sigma}{\partial y^\nu_i} - \frac{\partial \varepsilon_\nu}{\partial y^\sigma_i} \right) \wedge \omega^\nu \wedge \omega^\sigma \wedge \omega_i - pd \frac{\partial \varepsilon_\sigma}{\partial y^\nu_{ij}} \wedge \omega^\nu_j \wedge \omega^\sigma \wedge \omega_i. \end{aligned} \tag{264}$$

Thus, by hypothesis (b),

$$\begin{aligned} d(\varepsilon + F) = {} & -\frac{1}{4} pd \left(\frac{\partial \varepsilon_\sigma}{\partial y^\nu_i} - \frac{\partial \varepsilon_\nu}{\partial y^\sigma_i} \right) \wedge \omega^\nu \wedge \omega^\sigma \wedge \omega_i \\ & - pd \frac{\partial \varepsilon_\sigma}{\partial y^\nu_{ij}} \wedge \omega^\nu_j \wedge \omega^\sigma \wedge \omega_i. \end{aligned} \tag{265}$$

3. Suppose that the functions ε_σ satisfy condition (b). Substituting from (254) to dε, we have

$$\begin{aligned} d\varepsilon = {} & \left(\frac{1}{4} d_j \left(\frac{\partial \varepsilon_\sigma}{\partial y^\nu_j} - \frac{\partial \varepsilon_\nu}{\partial y^\sigma_j} \right) \omega^\nu + \frac{1}{2} d_j \left(\frac{\partial \varepsilon_\sigma}{\partial y^\nu_{ij}} + \frac{\partial \varepsilon_\nu}{\partial y^\sigma_{ij}} \right) \omega^\nu_i \right. \\ & \left. + \frac{1}{2} \left(\frac{\partial \varepsilon_\sigma}{\partial y^\nu_i} - \frac{\partial \varepsilon_\nu}{\partial y^\sigma_i} \right) \omega^\nu_i + \frac{1}{2} \left(\frac{\partial \varepsilon_\sigma}{\partial y^\nu_{ij}} + \frac{\partial \varepsilon_\nu}{\partial y^\sigma_{ij}} \right) \omega^\nu_{ij} \right) \wedge \omega^\sigma \wedge \omega_0 \\ = {} & \left(\frac{1}{4} d_j \left(\frac{\partial \varepsilon_\sigma}{\partial y^\nu_j} - \frac{\partial \varepsilon_\nu}{\partial y^\sigma_j} \right) \omega^\nu + \frac{1}{2} \left(d_j \left(\frac{\partial \varepsilon_\sigma}{\partial y^\nu_{ij}} + \frac{\partial \varepsilon_\nu}{\partial y^\sigma_{ij}} \right) + \frac{\partial \varepsilon_\sigma}{\partial y^\nu_i} - \frac{\partial \varepsilon_\nu}{\partial y^\sigma_i} \right) \omega^\nu_i \right. \\ & \left. + \frac{1}{2} \left(\frac{\partial \varepsilon_\sigma}{\partial y^\nu_{ij}} + \frac{\partial \varepsilon_\nu}{\partial y^\sigma_{ij}} \right) \omega^\nu_{ij} \right) \wedge \omega^\sigma \wedge \omega_0. \end{aligned} \tag{266}$$

On the other hand, we can recognize in formula (266) some terms in the form of an exterior derivative. Observe that

$$
\begin{aligned}
p_2 d\left(\left(\frac{\partial\varepsilon_\sigma}{\partial y^\nu_{ij}}+\frac{\partial\varepsilon_\nu}{\partial y^\sigma_{ij}}\right)\omega^\nu_i\wedge\omega^\sigma\wedge\omega_j\right)
&= d_j\left(\frac{\partial\varepsilon_\sigma}{\partial y^\nu_{ij}}+\frac{\partial\varepsilon_\nu}{\partial y^\sigma_{ij}}\right)\omega^\nu_i\wedge\omega^\sigma\wedge\omega_0\\
&\quad+\left(\frac{\partial\varepsilon_\sigma}{\partial y^\nu_{ij}}+\frac{\partial\varepsilon_\nu}{\partial y^\sigma_{ij}}\right)d\omega^\nu_i\wedge\omega^\sigma\wedge\omega_j-\left(\frac{\partial\varepsilon_\sigma}{\partial y^\nu_{ij}}+\frac{\partial\varepsilon_\nu}{\partial y^\sigma_{ij}}\right)\omega^\nu_i\wedge d\omega^\sigma\wedge\omega_j\\
&= d_j\left(\frac{\partial\varepsilon_\sigma}{\partial y^\nu_{ij}}+\frac{\partial\varepsilon_\nu}{\partial y^\sigma_{ij}}\right)\omega^\nu_i\wedge\omega^\sigma\wedge\omega_0\\
&= \left(d_j\left(\frac{\partial\varepsilon_\sigma}{\partial y^\nu_{ij}}+\frac{\partial\varepsilon_\nu}{\partial y^\sigma_{ij}}\right)\omega^\nu_i+\left(\frac{\partial\varepsilon_\sigma}{\partial y^\nu_{ij}}+\frac{\partial\varepsilon_\nu}{\partial y^\sigma_{ij}}\right)\omega^\nu_{ij}\right)\wedge\omega^\sigma\wedge\omega_0,
\end{aligned}
\tag{267}
$$

and

$$
\begin{aligned}
p_2 d\left(\left(\frac{\partial\varepsilon_\sigma}{\partial y^\nu_i}-\frac{\partial\varepsilon_\nu}{\partial y^\sigma_i}\right)\omega^\nu\wedge\omega^\sigma\wedge\omega_i\right) = \Bigg(d_i&\left(\frac{\partial\varepsilon_\sigma}{\partial y^\nu_i}-\frac{\partial\varepsilon_\nu}{\partial y^\sigma_i}\right)\omega^\nu\\
&+2\left(\frac{\partial\varepsilon_\sigma}{\partial y^\nu_i}-\frac{\partial\varepsilon_\nu}{\partial y^\sigma_i}\right)\omega^\nu_i)\wedge\omega^\sigma\wedge\omega_0.
\end{aligned}
\tag{268}
$$

Thus, $d\varepsilon$ is expressible as

$$
\begin{aligned}
d\varepsilon = {}&\frac{1}{4}p_2 d\left(\left(\frac{\partial\varepsilon_\sigma}{\partial y^\nu_i}-\frac{\partial\varepsilon_\nu}{\partial y^\sigma_i}\right)\omega^\nu\wedge\omega^\sigma\wedge\omega_i\right)\\
&+\frac{1}{2}p_2 d\left(\left(\frac{\partial\varepsilon_\sigma}{\partial y^\nu_{ij}}+\frac{\partial\varepsilon_\nu}{\partial y^\sigma_{ij}}\right)\omega^\nu_i\wedge\omega^\sigma\wedge\omega_j\right).
\end{aligned}
\tag{269}
$$

Setting

$$
F=-\frac{1}{2}\left(\frac{1}{2}\left(\frac{\partial\varepsilon_\sigma}{\partial y^\nu_i}-\frac{\partial\varepsilon_\nu}{\partial y^\sigma_i}\right)\omega^\nu-\left(\frac{\partial\varepsilon_\sigma}{\partial y^\nu_{ij}}+\frac{\partial\varepsilon_\nu}{\partial y^\sigma_{ij}}\right)\omega^\nu_i\right)\wedge\omega^\sigma\wedge\omega_j \tag{270}
$$

and $\rho=\varepsilon+F$ we get assertion (c).

4. To show that condition (c) implies (a), we can repeat the proof of Theorem 12 for source forms of order 2. Suppose that for some fibered chart (V,ψ), $\psi=(x^i,y^\sigma)$, on the fibered manifold Y condition $d(\varepsilon+F)=0$ holds on V^2. Integrating we get $\varepsilon+F=d\eta$ for some n-form η. But since $\varepsilon=p_1 d\eta$, the form η is a Lepage form, therefore, so ε must be a locally variational form whose Lagrangian is $h\eta$. □

Remark 21 In the proof of Theorem 15, we have assigned to a second-order source form $\varepsilon = \varepsilon_\sigma \omega^\sigma \wedge \omega_0$ the form $\rho = \varepsilon + F$, defined by the requirement $d\rho = 0$. The solution

$$\begin{aligned} \rho &= \varepsilon_\sigma \omega^\sigma \wedge \omega_0 \\ &\quad - \frac{1}{2}\left(\frac{1}{2}\left(\frac{\partial \varepsilon_\sigma}{\partial y_i^\nu} - \frac{\partial \varepsilon_\nu}{\partial y_i^\sigma}\right)\omega^\nu - \left(\frac{\partial \varepsilon_\sigma}{\partial y_{ij}^\nu} + \frac{\partial \varepsilon_\nu}{\partial y_{ij}^\sigma}\right)\omega_i^\nu\right) \wedge \omega^\sigma \wedge \omega_j \end{aligned} \tag{271}$$

extends the source form by a form of order of contactness ≥2. This construction, involving the exterior derivative operator, is closely related to the variationality of the form ε and can be considered as a motivation for possible generalizations of the geometric theory of Lepage differential n-forms to $(n + 1)$-forms and differential forms of higher degree (cf. [KKS2]). This notable construction also indicates the possibility to interpret a source forms as a *class* of forms modulo contact forms; this idea has been developed by the theory of variational sequences (cf. [K19] and Chap. 8).

Theorem 16 (First-order Lepage forms) *Let $\rho \in \Omega_n^1 V$ be an n-form. The following two conditions are equivalent*:

(a) *$\rho \in \Omega_n^1 V$ is a Lepage form.*
(b) *There exists a first-order Lagrangian $\lambda \in \Omega_{n,X}^1 V$, an n-form κ of order of contactness ≥2 and a contact $(n - 1)$-form τ, such that*

$$\rho = \Theta_\lambda + \kappa + d\tau. \tag{272}$$

Proof

1. Let (V, ψ), $\psi = (x^i, y^\sigma)$ be a fibered chart on Y, and let ρ be a first-order Lepage form, defined on the set V^1. Then, the form $\varepsilon = p_1 d\rho$ is a *second-order* Euler–Lagrange form, defined on V^2, associated with the *second-order* Lagrangian $h\rho$ – the *horizontal component* of ρ. On the other hand, it follows from Theorem 14 that ε has a *first-order* Lagrangian λ; denoting by Θ_λ the principal Lepage equivalent of λ, we have $\varepsilon = p_1 d\Theta_\lambda$; hence,

$$p_1 d\rho = p_1 d\Theta_\lambda. \tag{273}$$

Consequently, $p_1 d(\rho - \Theta_\lambda) = 0$ and by the theorem on the kernel of the Euler–Lagrange mapping (Sect. 4.8, Theorem 9, (c)), there exists an $(n - 1)$-form μ, defined on V^1, such that $h(\rho - \Theta_\lambda) = hd\mu$; hence,

$$\rho - \Theta_\lambda = \eta + d\mu \tag{274}$$

for some contact form η such that $p_1 d\eta = 0$. Therefore, η satisfies two conditions

$$h\eta = 0, \quad p_1 d\eta = 0. \tag{275}$$

The first one implies that $\eta = \omega^\sigma \wedge \Phi_\sigma + d\omega^\sigma \wedge \Psi_\sigma$ for some forms Φ_σ and Ψ_σ (Sect. 2.3, Theorem 7, (b)). We can also write

$$\eta = \omega^\sigma \wedge (\Phi_\sigma + d\Psi_\sigma) + d(\omega^\sigma \wedge \Psi_\sigma) \tag{276}$$

for some forms Φ_σ and Ψ_σ. Setting $\tau_\sigma = \Phi_\sigma + d\Psi_\sigma$, the second condition (275) implies

$$p_1 d\eta = -\omega_l^\sigma \wedge dx^l \wedge h\tau_\sigma - \omega^\sigma \wedge hd\tau_\sigma = 0. \tag{277}$$

We want to show that this condition implies $h\tau_\sigma = 0$. Indeed, for any $\pi^{2,0}$-vertical vector field

$$\Xi = \Xi_i^\sigma \frac{\partial}{\partial y_i^\sigma} + \Xi_{ij}^\sigma \frac{\partial}{\partial y_{ij}^\sigma} \tag{278}$$

condition (278) yields $\Xi_l^\sigma dx^l \wedge h\tau_\sigma = 0$. Writing $h\tau_\sigma = A_\sigma^i \omega_i$, this condition implies $\Xi_l^\sigma A_\sigma^i dx^l \wedge \omega_i = \Xi_l^\sigma A_\sigma^l \omega_0 = 0$; hence, $A_\sigma^l = 0$. Thus, $h\tau_\sigma = 0$. Substituting from this result to (277) and (275), we see that assertion (a) implies (b).

2. The converse is obvious. □

References

[AD] I. Anderson, T. Duchamp, On the existence of global variational principles, Am. J. Math. 102 (1980) 781-867

[AT] I. Anderson, G. Thompson, *The inverse problem of the calculus of variations for ordinary differential equations*, Mem. Amer. Math. Soc. 98, 1992, 1-110

[B] D. Betounes, Extension of the classical Cartan form, Phys. Rev. D29 (1984) 599-606

[Bu] I. Bucataru, A setting for higher order differential equation fields and higher order Lagrange and Finsler spaces, Journal of Geometric Mechanics 5 (2013) 257-279

[C] E. Cartan, *Lecons sur les Invariants Intégraux*, Hermann, Paris, 1922

[Cr] M. Crampin, On the inverse problem for sprays, Publ. Math. Debrecen 70, 2007, 319-335

[CS] M. Crampin, D.J. Saunders, The Hilbert-Carathéodory form and Poincaré-Cartan forms for higher-order multiple-integral variational problems, Houston J. Math. 30 (2004) 657-689

[Do] J. Douglas, Solution of the inverse problem of the calculus of variations, Transactions AMS 50 (1941) 71-128

[G] P.L. Garcia, The Poincare-Cartan invariant in the calculus of variations, Symposia Mathematica 14 (1974) 219-246

[GS] H. Goldschmidt, S. Sternberg, The Hamilton-Cartan formalism in the calculus of variations, Ann. Inst. H. Poincare 23 (1973) 203-267

[H] P. Havas, The range of applicability of the Lagrange formalism. I, Nuovo Cimento 5 (1957) 363-383

[He] H. von Helmholtz, Ueber die physikalische Bedeutung des Princips der kleinsten Wirkung, Journal fur die reine und angewandte Mathematik 100 (1887) 137-166, 213-222

[K1] D. Krupka, A geometric theory of ordinary first order variational problems in fibered manifolds, I. Critical sections, II. Invariance, J. Math. Anal. Appl. 49 (1975) 180-206, 469-476

[K2] D. Krupka, A map associated to the Lepagean forms of the calculus of variations in fibered manifolds, Czech. Math. J. 27 (1977) 114-118

[K5] D. Krupka, *Introduction to Global Variational Geometry*, Chap. 1-5, Beijing, 2011, Lepage Inst. Archive, No.1, 2012

[K7] D. Krupka, Lepage forms in Kawaguchi spaces and the Hilbert form, paper in honor of Professor Lajos Tamassy, Publ. Math. Debrecen 84 (2014), 147-164; DOI: 10.5486/PMD.2014.5791

[K8] D. Krupka, Lepagean forms in higher order variational theory, in: *Modern Developments in Analytical Mechanics*, Proc. IUTAM-ISIMM Sympos., Turin, June 1982, Academy of Sciences of Turin, 1983, 197-238

[K10] D. Krupka, Natural Lagrange structures, in: *Semester on Differential Geometry*, 1979, Banach Center, Warsaw, Banach Center Publications, 12, 1984, 185-210

[K11] D. Krupka, On the local structure of the Euler-Lagrange mapping of the calculus of variations, in: O. Kowalski, Ed., *Differential Geometry and its Applications*, Proc. Conf., N. Mesto na Morave, Czechoslovakia, Sept. 1980; Charles University, Prague, 1981, 181-188; arXiv:math-ph/0203034

[K12] D. Krupka, On the structure of the Euler mapping, Arch. Math. (Brno) 10 (1974) 55–61

[K13] D. Krupka, *Some Geometric Aspects of Variational Problems in Fibered Manifolds*, Folia Fac. Sci. Nat. UJEP Brunensis, Physica 14, Brno, Czech Republic, 1973, 65 pp.; arXiv: math-ph/0110005

[K16] D. Krupka, The Vainberg-Tonti Lagrangian and the Euler-Lagrange mapping, in: F. Cantrijn, B. Langerock, Eds., *Differential Geometric Methods in Mechanics and Field Theory*, Volume in Honor of W. Sarlet, Gent, Academia Press, 2007, 81-90

[K19] D. Krupka, Variational sequences on finite-order jet spaces, Proc. Conf., World Scientific, 1990, 236-254

[KKS1] D. Krupka, O. Krupková, D. Saunders, Cartan-Lepage forms in geometric mechanics, doi: 10.1016/j.ijnonlinmec.2011.09.002, Internat. J. of Non-linear Mechanics 47 (2011) 1154-1160

[KKS2] D. Krupka, O. Krupková, D. Saunders, The Cartan form and its generalisations in the calculus of variations, Int. J. Geom. Met. Mod. Phys. 7 (2010) 631-654

[KM] D. Krupka, J. Musilová, Trivial Lagrangians in field theory, Diff. Geom. Appl. 9 (1998) 293-305; 10 (1999) 303

[KS] D. Krupka, D. Saunders, Eds., *Handbook of Global Analysis*, Elsevier, 2008

[KrP] O. Krupková, G. Prince, Second order ordinary differential equations in jet bundles and the inverse problem of the calculus of variations, in *Handbook of Global Analysis*, Elsevier, 2008, 837-904

[Le] Th.H.J. Lepage, Sur les champs géodésiques du calcul des variations, I, II, Bull. Acad. Roy. Belg. 22 (1936), 716-729, 1036-1046

[O2] P.J. Olver, Equivalence and the Cartan form, Acta Appl. Math. 31 (1993) 99-136

[SCM] W. Sarlet, M. Crampin, E. Martinez, The integrability conditions in the inverse problem of the calculus of variations for second-order ordinary differential equations, Acta Appl. Math. 54 (1998) 233-273

[So] N.J. Sonin, About determining maximal and minimal properties of plane curves (in Russian), Warsawskye Universitetskye Izvestiya 1-2 (1886) 1-68; English translations: Lepage Inst. Archive No. 1, 2012

[To] E. Tonti, Variational formulation of nonlinear differential equations, I, II, Bull. Acad. Roy. Belg. C. Sci. 55 (1969) 137-165, 262-278

[UK2] Z. Urban, D. Krupka, The Helmholtz conditions for systems of second order homogeneous differential equations, Publ. Math. Debrecen 83 (1-2) (2013) 71-84

[V] M. M. Vainberg, Variational Methods for the Study of Nonlinear Operators (in Russian), Gostekhizdat, Moscow, 1956; English translation: Holden-Day, San Francisco, 1964

[T] F. Takens, A global version of the inverse problem of the calculus of variations, J. Differential Geometry 14 (1979) 543-562

[Z] D. Zenkov (Ed.), The Inverse Problem of the Calculus of Variations, Local and Global Theory and Applications, Atlantis Series in Global Variational Geometry, to appear

Chapter 5
Invariant Variational Structures

Let X be any manifold, W an open set in X, and let α: $W \to X$ be a smooth mapping. A differential form η, defined on the set $\alpha(W)$ in X, is said to be *invariant* with respect to α, if the transformed form $\alpha^*\eta$ coincides with η, that is, if $\alpha^*\eta = \eta$ on the set $W \cap \alpha(W)$; in this case, we also say that α is an *invariance transformation* of η. A vector field, whose local one-parameter group consists of invariance transformations of η, is called the *generator* of invariance transformations.

These definitions can naturally be extended to variational structures (Y, ρ) and to the integral variational functionals associated with them. Our objective in this chapter is to study invariance properties of the form ρ and other differential forms, associated with ρ, the Lagrangian λ, and the Euler–Lagrange form E_λ. The class of transformations we consider is formed by automorphisms of fibered manifolds and their jet prolongations. This part of the variational theory represents a notable extension of the classical coordinate concepts and methods to topologically non-trivial fibered manifolds that cannot be covered by a single chart. The geometric coordinate-free structure of the infinitesimal first variation formula leads in several consequences, such as the geometric invariance criteria of the Lagrangians and the Euler–Lagrange forms, a global theorem on the conservation law equations, and the relationship between extremals and conservation laws. Resuming that we can say that these results as a whole represent an extension of the classical Noether's theory to higher-order variational functionals on fibered manifolds (Noether [N]).

In this chapter, we basically follow Trautman's formulation of the invariance theory based on the geometric understanding of the topic (Trautman [Tr1, Tr2]). The concept of the *jet prolongation* of a vector field and its meaning for the geometric notion of a variation for invariance theory was discussed in Krupka [K6, K1]. The fundamentals of the invariance theory for differential equations and the calculus of variations in Euclidean spaces developed along the classical lines can be found in Olver [O1]; however, in this work, the Trautman's approach using geometric characteristics of the underlying transformations, such as the Lie derivatives, is not included. A complete treatment of the work of Noether on invariant variational principles is presented, also within the classical local framework, by Kosmann-Schwarzbach [K-S].

D. Krupka, *Introduction to Global Variational Geometry*,
Atlantis Studies in Variational Geometry 1, DOI 10.2991/978-94-6239-073-7_5

In this chapter, we follow our previous notations. Throughout, Y is a fixed fibered manifold with base X and projection π. We set $\dim X = n$, $\dim Y = n + m$. J^rY denotes the *r-jet prolongation* of Y, and $\pi^{r,s}$ and π^r are the canonical jet projections. For any set $W \subset Y$, the set $(\pi^{r,0})^{-1}(W)$ is denoted by W^r. $\Omega^r_q W$ denotes the module of q-forms defined on W^r, $\Omega^r_{q,Y} W$ is the submodule of $\pi^{r,0}$-horizontal forms, and $\Omega^r W$ is the exterior algebra of differential forms on W^r. We use the *horizontalization morphism* of exterior algebras $h: \Omega^r W \to \Omega^{r+1} W$. The r-jet prolongation of a morphism α of the fibered manifold Y is denoted by $J^r\alpha$. Analogously, the r-jet prolongation of a π-projectable vector field is denoted by $J^r\Xi$.

5.1 Invariant Differential Forms

We present in this section some elementary remarks on the invariance of differential forms on smooth manifolds under diffeomorphisms. We prove two standard lemmas that are permanently used in the theory of invariant variational structures.

Let X be a smooth manifold, W an open set in X and $\alpha: W \to X$ a diffeomorphism. Let ρ be a p-form on X. We say that ρ is *invariant with respect to* α, if its pull-back $\alpha^*\rho$ coincides with α,

$$\alpha^*\rho = \rho. \tag{1}$$

A diffeomorphism α, satisfying condition (1), is called the *invariance transformation* of ρ.

These definitions immediately transfer to vector fields. Let ξ be a vector field on X, α^ξ its flow, and α^ξ_t its local 1-parameter groups, defined by the condition $\alpha^\xi_t(x) = \alpha^\xi(t, x)$, where the points (t, x) belong to the domain of definition of α^ξ. We say that ξ is the *generator of invariance transformations* of ρ, if its local 1-parameter groups are invariance transformations of ρ, that is,

$$(\alpha^\xi_t)^*\rho(x) = \rho(x) \tag{2}$$

for all points (t, x) from the domain of α^ξ.

Lemma 1 *For every point* (t, x) *from the domain of definition of the flow of the vector field* ξ,

$$\frac{d}{dt}(\alpha^\xi_t)^*\rho(x) = ((\alpha^\xi_t)^*\partial_\xi\rho)(x). \tag{3}$$

Proof Let (t, x_0) be a point from the domain of α^ξ. Choose tangent vectors $\xi_1, \xi_2, \ldots, \xi_p \in T_{x_0}X$ and consider the value of the form $(\alpha^\xi_t)^*p(x_0)$ on these tangent vectors. This gives rise to a real-valued function $t \to ((\alpha^\xi_t)^*\rho)(x_0)(\xi_1, \xi_2, \ldots, \xi_p)$. Differentiating this function at a point t_0, we have

$$\begin{aligned}&\left(\frac{d}{dt}((\alpha_t^{\xi})^*\rho)(x_0)(\xi_1,\xi_2,\ldots,\xi_p)\right)_{t_0}\\&\quad=\left(\frac{d}{ds}((\alpha_{t_0+s}^{\xi})^*\rho)(x_0)(\xi_1,\xi_2,\ldots,\xi_p)\right)_0.\end{aligned}\tag{4}$$

But the flow satisfies the condition $\alpha_{t_0+s}^{\xi}=\alpha_s^{\xi}\circ\alpha_{t_0}^{\xi}$ so we have

$$\begin{aligned}&\left(\frac{d}{dt}((\alpha_t^{\xi})^*\rho)(x_0)(\xi_1,\xi_2,\ldots,\xi_p)\right)_{t_0}\\&=\left(\frac{d}{ds}((\alpha_{t_0}^{\xi})^*(\alpha_s^{\xi})^*\rho)(x_0)(\xi_1,\xi_2,\ldots,\xi_p)\right)_0\\&=\left(\frac{d}{ds}((\alpha_s^{\xi})^*\rho)(\alpha_{t_0}^{\xi}(x_0))(T\alpha_{t_0}^{\xi}\cdot\xi_1,T\alpha_{t_0}^{\xi}\cdot\xi_2,\ldots,T\alpha_{t_0}^{\xi}\cdot\xi_p)\right)_0\\&=\partial_{\xi}\rho(\alpha_{t_0}^{\xi}(x_0)(T\alpha_{t_0}^{\xi}\cdot\xi_1,T\alpha_{t_0}^{\xi}\cdot\xi_2,\ldots,T\alpha_{t_0}^{\xi}\cdot\xi_p)\\&=((\alpha_s^{\xi})^*\partial_{\xi}\rho)(x_0)(\xi_1,\xi_2,\ldots,\xi_p).\end{aligned}\tag{5}$$

This is formula (3). □

Lemma 2 (Invariance lemma) *Let ξ be a vector field on X, and let ρ be a p-form on X. The following two conditions are equivalent*:

(a) *ξ generates invariance transformations of ρ.*
(b) *The Lie derivative of ρ by ξ vanishes*,

$$\partial_{\xi}\rho=0.\tag{6}$$

Proof

1. If ξ generates invariance transformations of ρ, then we differentiate both sides of equation (2) with respect to t at $t = 0$ and obtain formula (6).
2. If condition (6) is satisfied, then by Lemma 1,

$$\frac{d}{dt}((\alpha_t^{\xi})^*\rho)(x)=0\tag{7}$$

on the domain of the flow α^{ξ}. Thus, the curve $t\to((\alpha_t^{\xi})^*\rho)(x)$ is independent of t, and since its domain is connected, its value is constant and must be equal to $((\alpha_0^{\xi})^*\rho)(x)=\rho(x)$. This proves condition (2). □

5.2 Invariant Lagrangians and Conservation Equations

Let W be an open set in Y, let λ be a Lagrangian of order r for Y, defined on $W^r \subset J^rY$. Consider an automorphism α: $W \to Y$ of Y, and its r-jet prolongation $J^r\alpha$: $W^r \to J^rY$. We say that α is an *invariance transformation* of λ if $J^r\alpha^*\lambda = \lambda$. The *generator* of invariance transformations of λ is a π-projectable vector field on Y whose local one-parameter group consists of invariance transformations of λ.

In the following lemma, we use fibered charts (V, ψ), $\psi = (x^i, y^\sigma)$, and our standard multi-index notation. Recall that the *contact* 1-forms ω_J^σ, locally generating the *contact ideal*, are the 1-forms, defined by the formula $\omega_J^\sigma = \mathrm{d}y_J^\sigma - y_{Jj}^\sigma\,\mathrm{d}x^j$ (Sect. 2.1, Theorem 1).

Lemma 3 *Suppose we have a vector field Z on J^rY. The following two conditions are equivalent*:

(a) *For every fibered chart (V, ψ), $\psi = (x^i, y^\sigma)$, on Y, every σ, and every multi-index J such that $0 \leq |J| \leq r - 1$, the form $\partial_Z\omega_J^\sigma$ is a contact form.*

(b) *There exists a π-projectable vector field Ξ such that $Z = J^r\Xi$.*

Proof Write $\omega_J^\tau = \mathrm{d}y_J^\tau - y_{Jj}^\tau\mathrm{d}x^j$ and

$$Z = \zeta^i\frac{\partial}{\partial x^i} + Z_I^\sigma\frac{\partial}{\partial y_I^\sigma}. \tag{8}$$

Then

$$\begin{aligned}\partial_Z\omega_J^\tau &= i_Zd\omega_J^\tau + di_Z\omega_J^\tau = -i_Z(dy_{Jj}^\tau \wedge dx^j) + di_Z(dy_J^\tau - y_{Jl}^\tau dx^l)\\ &= -Z_{Jj}^\tau dx^j + \zeta^j dy_{Jj}^\tau + d(Z_J^\tau - y_{Jl}^\tau\zeta^l)\\ &= -Z_{Jj}^\tau dx^j + \zeta^j dy_{Jj}^\tau + dZ_J^\tau - \zeta^l dy_{Jl}^\tau - y_{Jl}^\tau d\zeta^l\\ &= -Z_{Jj}^\tau dx^j + dZ_J^\tau - y_{Jl}^\tau d\zeta^l\\ &= (-Z_{Jj}^\tau + d_jZ_J^\tau - y_{Jl}^\tau d_j\zeta^l)dx^j + \frac{\partial Z_J^\tau}{\partial y_K^\lambda}\omega_K^\lambda,\end{aligned} \tag{9}$$

and our assertion follows from Sect. 1.7, Lemma 8. □

Lemma 4 *Let λ be a Lagrangian of order r for Y.*

(a) *A π-projectable vector field Ξ on Y generates invariance transformations of λ if and only if*

$$\partial_{J^r\Xi}\lambda = 0. \tag{10}$$

(b) *Generators of invariance transformations of λ constitute a subalgebra of the algebra of vector fields on J^rY.*

Proof

(a) This is a trivial consequence of definitions.
(b) Any two generators satisfy $[J^r\Xi_1, J^r\Xi_2] = J^r[\Xi_1, \Xi_2]$ (Sect. 1.7, Lemma 11). Then, however,

$$\partial_{J^r[\Xi_1,\Xi_2]}\lambda = \partial_{[J^r\Xi_1,J^r\Xi_2]}\lambda = \partial_{J^r\Xi_1}\partial_{J^r\Xi_2}\lambda - \partial_{J^r\Xi_2}\partial_{J^r\Xi_1}\lambda = 0. \tag{11}$$

□

We keep terminology used by Trautman [Tr1, Tr2] and call Eq. (10), the *Noether equation*. This equation represents a relation between the Lagrangian λ and the generator Ξ of invariance transformation. Given λ, we can use the Noether equation to determine the generators Ξ. On the other hand, given a Lie algebra of π-projectable vector fields Ξ, one can use the corresponding Noether equations *to determine* invariant Lagrangians λ.

Theorem 1 *Suppose that a Lagrangian λ is invariant with respect to a π-projectable vector field Ξ. Then for any Lepage equivalent ρ of λ*

$$hi_{J^r\Xi}d\rho + hdi_{J^r\Xi}\rho = 0, \tag{12}$$

or, which is the same,

$$J^r\gamma^* i_{J^r\Xi}d\rho + dJ^r\gamma^* i_{J^r\Xi}\rho = 0 \tag{13}$$

for every section γ of Y.

Proof From Sect. 4.6, Theorem 7,

$$h\partial_{J^r\Xi}\rho = \partial_{J^{r+1}\Xi}h\rho = \partial_{J^{r+1}\Xi}\lambda = hi_{J^r\Xi}d\rho + hdi_{J^r\Xi}\rho \tag{14}$$

which implies (12). □

Remark 1 According to Sect. 4.3, Theorem 3, condition (12) reduces locally to

$$hi_{J^r\Xi}d\Theta_\lambda + hdi_{J^r\Xi}\Theta_\lambda = 0, \tag{15}$$

where Θ_λ is the principal Lepage equivalent of the Lagrangian form λ.

By a *conserved current* for a section $\gamma \in \Gamma_\Omega(\pi)$, we mean any $(n-1)$-form $\eta \in \Omega^r_n W$ such that

$$dJ^s\gamma^*\eta = 0. \tag{16}$$

We call formula (16) the *conservation law equation*; it is also called a *conservation law* for the section γ.

The following assertion says that *extremals* of invariant Lagrangians satisfy, in addition to the Euler–Lagrange equations, also some other conditions, expressed by the *conservation law equations*.

Theorem 2 (First theorem of Emmy Noether) *Let $\lambda \in \Omega^r_{n,X} W$ be a Lagrangian, ρ a Lepage equivalent of λ defined on J^sY, and let γ be an extremal. Then for every generator Ξ of invariance transformations of λ*

$$dJ^s\gamma^* i_{J^s\Xi}\rho = 0. \tag{17}$$

Proof The proof is based on the first variation formula (Sect. 4.6, Theorem 7, (c)), and is trivial. Indeed, we have

$$J^r\gamma^* \partial_{J^r\Xi}\lambda = J^s\gamma^* i_{J^s\Xi} d\rho + dJ^s\gamma^* i_{J^s\Xi}\rho, \tag{18}$$

and since the left-hand side vanishes, by invariance, and the first summand on the right-hand side also vanishes, because γ is an extremal, we get formula (17) as required. □

Note that (global) condition (17) can also be written in a different way, by means of locally defined principal Lepage equivalents Θ_λ of the Lagrangian λ. From the structure theorem on Lepage forms, we know that, locally, $\rho = \Theta_\lambda + \mathrm{d}\nu + \mu$, where ν is a contact form, and μ is a contact form of order of contactness ≥2. Then $dJ^s\gamma^* i_{J^s\Xi}\rho = dJ^s\gamma^*(i_{J^s\Xi}\Theta_\lambda + i_{J^s\Xi} d\nu + i_{J^s\Xi}\mu)$. But the form $i_{J^s\Xi}\mu$ is contact; moreover, $i_{J^s\Xi} d\nu = \partial_{J^s\Xi}\nu - d i_{J^s\Xi}\nu$, from which we deduce that

$$J^s\gamma^* i_{J^s\Xi}\mu = 0, \quad dJ^s\gamma^* i_{J^s\Xi} d\nu = dJ^s\gamma^* \partial_{J^s\Xi}\nu - dJ^s\gamma^* d i_{J^s\Xi}\nu = 0. \tag{19}$$

Consequently, under the hypothesis of Theorem 1, condition

$$dJ^s\gamma^* i_{J^s\Xi}\Theta_\lambda = 0 \tag{20}$$

holds over coordinate neighborhoods of fibered charts on Y.

One can also use invariance of variational functionals in a different way. Namely, the infinitesimal first variation formula shows that the property of a Lagrangian to be invariant reduces the number of the Euler–Lagrange equations.

Theorem 3 *If λ is invariant, ρ is a Lepage equivalent of λ, and γ a section satisfying the conservation law equation*

$$dJ^r\gamma^* i_{J^r\Xi}\rho = 0, \tag{21}$$

then for any fibered chart (V, ψ), $\psi = (x^i, y^\sigma)$, the associated Euler–Lagrange expressions are linearly dependent along γ.

Proof The infinitesimal first variation formula gives

$$J^r\gamma^* i_{J^r\Xi} d\rho = J^r\gamma^* i_{J^r\Xi} p_1 d\rho = J^r\gamma^* i_{J^r\Xi} E_{h\rho} = 0. \tag{22}$$

Consequently, in the chart (V, ψ), $\psi = (x^i, y^\sigma)$, for the given vector field Ξ, the Euler–Lagrange expressions of the Lagrangian $\lambda = h\rho$ satisfy (22) and are linearly dependent along γ. □

Example (Conservation law equations) In the following example, we consider the product fibered manifold $Y = X \times \mathbf{R}^m$. Denote by y^σ the canonical coordinates on $\mathbf{R}^m$, and by x^i, y^σ some coordinates on Y. Consider the translation vector fields

$$\Xi_\tau = \frac{\partial}{\partial y^\tau}. \tag{23}$$

One can easily determine the r-jet prolongations of these vertical vector fields. We get

$$J^r\Xi_\tau = \frac{\partial}{\partial y^\tau}. \tag{24}$$

Invariance conditions for a Lagrangian $\lambda = \mathscr{L}\omega_0$ are $\partial_{J^r\Xi_\tau}\lambda = i_{J^r\Xi_\tau} d\lambda = 0$, that is,

$$\frac{\partial \mathscr{L}}{\partial y^\tau} = 0. \tag{25}$$

In classical variational calculus, condition (25) is sometimes called the *Routh condition*. The principal Lepage equivalent is

$$\Theta_{\mathscr{L}} = \mathscr{L}\omega_0 + \sum_{k=0}^{r-1}\left(\sum_{l=0}^{r-1-k}(-1)^l d_{p_1}d_{p_2}\ldots d_{p_l}\frac{\partial \mathscr{L}}{\partial y^\sigma_{j_1 j_2\ldots j_k p_1 p_2\ldots p_l i}}\right)\omega^\sigma_{j_1 j_2\ldots j_k}\wedge\omega_i, \tag{26}$$

and its contraction by $J^r\Xi_\tau$ is

$$i_{J^{r+1}\Xi}\Theta_{\lambda_\rho} = \sum_{l=0}^{r-1}(-1)^l d_{p_1}d_{p_2}\ldots d_{p_l}\frac{\partial \mathscr{L}}{\partial y^\tau_{p_1 p_2\ldots p_l i}}\omega_i. \tag{27}$$

Therefore, the invariance condition $J^{r+1}\gamma^* E_\tau(\lambda)\omega_0 + dJ^{r+1}\gamma^* i_{J^{r+1}\Xi}\Theta_{\lambda_\rho} = 0$ reduces to

$$E_\tau(\lambda) - \sum_{l=0}^{r-1}(-1)^l d_i d_{p_1}d_{p_2}\ldots d_{p_l}\frac{\partial \mathscr{L}}{\partial y^\tau_{p_1 p_2\ldots p_l i}} = 0. \tag{28}$$

In particular, if γ satisfies the conservation law equation

$$\sum_{l=0}^{r-1}(-1)^l d_i d_{p_1} d_{p_2}\dots d_{p_l}\frac{\partial\mathscr{L}}{\partial y^\tau_{p_1p_2\dots p_l i}}=0, \tag{29}$$

it also solves the Euler–Lagrange equation

$$E_\tau(\lambda)\circ J^{r+1}\gamma=0. \tag{30}$$

In particular, if λ is invariant with respect to all translation vector fields Ξ_τ, then the system of the Euler–Lagrange equations is equivalent with the system of the conservation law equations.

Remark 2 It should be pointed out that in general, the principal Lepage equivalent Θ_λ, considered as a form depending on the Lagrangian λ, does not satisfy the invariance condition $\partial_{J^r\Xi}\Theta_\lambda=\Theta_{\partial_{J^r\Xi}\lambda}$.

Remark 3 The geometric structure of the first Noether's theorem may be explained as follows. Let Y be any manifold, ρ a differential form on Y. If ξ is a vector field on Y. such that the Lie derivative $\partial_\xi\rho$ vanishes, $\partial_\xi\rho=0$, then by the *Cartan's formula*, ρ and ξ satisfy $i_\xi d\rho+di_\xi\rho=0$. Then for any mapping f: $X\to Y$ satisfying the "Euler–Lagrange equation" $f^*i_\xi d\rho=0$, the identity $f^*i_\xi d\rho+df^*i_\xi\rho=0$ yields the "conservation law equation" $df^*i_\xi\rho=0$.

Remark 4 (Invariance with respect to a Lie group action) The first theorem of Emmy Noether (Theorem 2) is concerned with variational integrals, invariant with respect to 1-parameter groups of automorphisms of underlying manifolds Y. Clearly, the same theorem applies to invariance with respect to group actions of (finite-dimensional) Lie groups G on Y. The corresponding conservation law equations $dJ^s\gamma^*i_{J^s\Xi}\rho=0$ (21) represent a system, in which the vector fields Ξ are fundamental vector fields, defined by the Lie algebra of G. Thus, we get the system of k equations on J^sY, where k is the dimension of G.

Remark 5 (Second theorem of Emmy Noether) Some variational functionals admit broad classes of invariance transformations that cannot be characterized as Lie group actions. These transformations depend rather on arbitrary functions than on finite number of real parameters. Consequences of invariance of this kind are known as the second Noether's theorem (cf. Olver [O1], where the systems possessing the second Noether's theorem are characterized as abnormal). However, also this type of invariance can sometimes be understood as invariance with respect to a (finite-dimensional) Lie group; namely, this situation arises when the given Lagrangian is a differential invariant (Krupka and Trautman [KT], Krupka [K10]; see also Chap. 6 of this book).

5.3 Invariant Euler–Lagrange Forms

Let α: $W \to Y$ be an automorphism of Y, and let ε be a source form on J^sY. We say that α is an *invariance transformation* of ε, if $J^s\alpha^*\varepsilon = \varepsilon$. The *generator* of invariance transformations of ε is a π-projectable vector field on Y whose local one-parameter group consists of invariance transformations of ε.

Lemma 5 (Noether–Bessel–Hagen equation) *Let ε be a source form of order s for Y.*

(a) *A π-projectable vector field Ξ on Y is the generator of invariance transformations of ε if and only if*

$$\partial_{J^r\Xi}\varepsilon = 0. \tag{31}$$

(b) *Generators of invariance transformations of ε constitute a subalgebra of the algebra of vector fields on J^rY.*

Proof The same as the proof of Lemma 4, Sect. 5.2. □

Equation (31) is a geometric version of what is known in the classical calculus of variations as the *Noether–Bessel–Hagen equation*.

Let λ be a Lagrangian of order r for Y, let α be any automorphism of Y, and let E_λ be the Euler–Lagrange form of λ. Using the identity

$$J^{2r}\alpha^*E_\lambda = E_{J^r\alpha^*\lambda} \tag{32}$$

(Section 4.5, Theorem 6), we easily obtain the following statement.

Lemma 6

(a) *Every invariance transformation of λ is an invariance transformation of the Euler–Lagrange form E_λ.*
(b) *For every invariance transformation α of E_λ, the Lagrangian $\lambda - J^r\alpha^*\lambda$ is variationally trivial.*

Proof

(a) This follows from (32): if $J^r\alpha^*\lambda = 0,$ *then* $J^{2r}\alpha^*E_\lambda = 0$.
(b) This is again an immediate consequence of (32): if $J^{2r}\alpha^*E_\lambda = 0$ then $E_{J^r\alpha^*\lambda} = 0$. □

We can generalize the Noether's theorem to invariance transformations of the Euler–Lagrange form. However, since the proof is based on the theorem on the kernel of the Euler–Lagrange mapping, the assertion we obtain is of local character. We denote by Θ_λ the principal Lepage equivalent of λ.

Theorem 4 *Let λ be a Lagrangian of order r, let γ be an extremal, and let Ξ be a generator of invariance transformations of the Euler–Lagrange form E_λ. Then for every point $y_0 \in Y$ there exists a fibered chart (V, ψ) at y_0 and an $(n - 1)$-form η, defined on V^{r-1}, such that on $\pi(V)$*

$$dJ^{2r}\gamma^*(i_{J^s\Xi}\Theta_\lambda + \eta) = 0. \tag{33}$$

Proof Under the hypothesis of Theorem 4, from Sect. 4.10, Theorem 1, from formula $\partial_{J^{2r}\Xi}E_\lambda = E_{\partial_{J^r\Xi}\lambda}$ we obtain $E_{\partial_{J^r\Xi}\lambda} = 0$. Thus, the Lagrangian $\partial_{J^r\Xi}\lambda$ belongs to the kernel of the Euler–Lagrange mapping, so it must be of the form $\partial_{J^r\Xi}\lambda = hd\eta$ over sufficiently small open sets V in Y such that (V, ψ) is a fibered chart (Sect. 4.8, Theorem 9). Then, however, from the infinitesimal first variation formula over V, expression

$$J^r\gamma^*\partial_{J^r\Xi}\lambda = J^{2r-1}\gamma^*i_{J^s\Xi}d\Theta_\lambda + dJ^{2r-1}\gamma^*i_{J^s\Xi}\Theta_\lambda, \tag{34}$$

reduces to

$$J^r\gamma^*hd\eta = dJ^s\gamma^*i_{J^s\Xi}\Theta_\lambda. \tag{35}$$

Since $J^r\gamma^*hd\eta = J^r\gamma^*d\eta = dJ^r\gamma^*\eta$, this proves formula (33). □

Remark 6 If $r = 1$, then the principal Lepage equivalent Θ_λ is globally well defined. Moreover, it follows from the properties of the Euler–Lagrange mapping that the form η may be taken as a globally defined form on Y.

5.4 Symmetries of Extremals and Jacobi Vector Fields

Let λ be a Lagrangian of order r for a fibered manifold Y, and let γ be an extremal of λ; thus, we suppose that γ satisfies the Euler–Lagrange equation

$$E_\lambda \circ J^{2r}\gamma = 0. \tag{36}$$

Consider an automorphism $\alpha: W \to Y$ of Y with projection α_0, and its r-jet prolongation $J^r\alpha: W^r \to J^rY$. We say that α is a *symmetry* of γ, if the section $\alpha\gamma\alpha_0^{-1}$ is also a solution of the Euler–Lagrange equations, that is,

$$E_\lambda \circ J^{2r}(\alpha\gamma\alpha_0^{-1}) = 0. \tag{37}$$

We say that a π-projectable vector field Ξ is the *generator of symmetries* of γ, or *generates symmetries* of γ, if its local one-parameter group consists of symmetries of γ.

We need a lemma on pushforward vector fields. Consider a vector field ξ and a diffeomorphism α: $W \to X$, defined on an open set $W \subset X$. By the *pushforward vector field* of ξ with respect to α, we mean the vector field $\xi^{(\alpha)}$ defined on W by

$$\xi^{(\alpha)}(x) = T_{\alpha^{-1}(x)}\alpha \cdot \xi(\alpha^{-1}(x)). \tag{38}$$

Lemma 7 *Let X be a manifold, W an open set in X, Z a vector field on X,* α: $W \to X$ *a diffeomorphism, and* ρ *a p-form. Then*

$$i_\xi \alpha^* \rho = \alpha^* i_{\xi^{(\alpha)}} \rho. \tag{39}$$

Proof We have, with standard notation,

$$\begin{aligned}
&(i_\xi \alpha^* \rho)(x)(\xi_1, \xi_2, \ldots, \xi_p) \\
&\quad = \rho(\alpha(x))(T_x\alpha \cdot \xi(x), T_x\alpha \cdot \xi_1, T_x\alpha \cdot \xi_2, \ldots, T_x\alpha \cdot \xi_p) \\
&\quad = \rho(\alpha(x))(\xi^{(\alpha)}(\alpha(x)), T_x\alpha \cdot \xi_1, T_x\alpha \cdot \xi_2, \ldots, T_x\alpha \cdot \xi_p) \\
&\quad = i_{\xi^{(\alpha)}(\alpha(x))}\rho(\alpha(x))(T_x\alpha \cdot \xi_1, T_x\alpha \cdot \xi_2, \ldots, T_x\alpha \cdot \xi_p) \\
&\quad = \alpha^*(i_{\xi^{(\alpha)}}\rho)(x)(\xi_1, \xi_2, \ldots, \xi_p).
\end{aligned} \tag{40}$$

This is exactly formula (39). □

The following theorem says that invariance transformations of the Euler–Lagrange form E_λ permute extremals of the variational structure (λ, Y) and give us examples of symmetries.

Theorem 5 *An invariance transformation of the Euler–Lagrange form* E_λ *is the symmetry of every extremal* γ.

Proof

1. Let α: $W \to Y$ be any automorphism of Y with projection α_0: $\pi(W) \to X$. Let Z be any π-projectable vector field with projection Z_0. We show that the pushforward vector field $Z^{(\alpha)} = T\alpha \cdot Z \circ \alpha^{-1}$ is π-projectable, with projection $Z_0^{(\alpha_0)} = T\alpha_0 \cdot Z_0 \circ \alpha_0^{-1}$. Indeed, for every $y \in \alpha(W)$

$$\begin{aligned}
T_y\pi \cdot Z^{(\alpha)}(y) &= T_y\pi \cdot T_{\alpha^{-1}(y)}\alpha \cdot Z(\alpha^{-1}(y)) = T_{\alpha^{-1}(y)}(\pi\alpha) \cdot Z(\alpha^{-1}(y)) \\
&= T_{\pi(\alpha^{-1}(y))}\alpha_0 \cdot T_{\alpha^{-1}(y)}\pi \cdot Z(\alpha^{-1}(y)) = T_{\alpha_0^{-1}\pi(y)}\alpha_0 \cdot Z_0(\pi\alpha^{-1}(y)) \\
&= T_{\alpha_0^{-1}\pi(y)}\alpha_0 \cdot Z_0(\alpha_0^{-1}\pi(y)) = Z^{(\alpha_0)}(\pi(y)),
\end{aligned} \tag{41}$$

proving that $Z^{(\alpha)}$ is projectable and its projection is $Z_0^{(\alpha_0)}$.

Let β_t denote the local 1-parameter group of Z, and let $\beta_{0,t}$ be its projection. Then since

$$\begin{aligned}\left(\frac{d}{dt}\alpha\beta_t\alpha^{-1}(y)\right)_0 &= T_{\alpha^{-1}(y)}\alpha\cdot\left(\frac{d}{dt}\beta_t\alpha^{-1}(y)\right)_0 \\ &= T_{\alpha^{-1}(y)}\alpha\cdot Z(\alpha^{-1}(y)) = Z^{(\alpha)}(y),\end{aligned} \tag{42}$$

$\alpha\beta_t\alpha^{-1}$ is the 1-parameter group of $Z^{(\alpha)}$. The 1-parameter group of the projection $Z_0^{(\alpha_0)}$ is defined by $\pi\alpha\beta_t\alpha^{-1} = \alpha\pi\beta_t\alpha^{-1} = \alpha\beta_{0,t}\pi\alpha^{-1} = \alpha\beta_{0,t}\alpha_0^{-1}\pi$ and is equal to $\alpha\beta_{0,t}\alpha_0^{-1}$.

Since $Z^{(\alpha)}$ is projectable, its s-jet prolongation $J^sZ^{(\alpha)}$ is defined. Since we know the 1-parameter groups of $Z^{(\alpha)}$, then $J^sZ^{(\alpha)}$ at a point $J^s_x\gamma$ is given by differentiation of the curve $t \to J^s_{\alpha_0\beta_{0,t}\alpha_0^{-1}(x)}(\alpha\beta_t\alpha^{-1})\gamma(\alpha_0\beta_{0,t}^{-1}\alpha_0^{-1})$ at $t = 0$,

$$J^sZ^{(\alpha)}(J^s_x\gamma) = \left(\frac{d}{dt}J^s_{\alpha_0\beta_{0,t}\alpha_0^{-1}(x)}(\alpha\beta_t\alpha^{-1})\gamma(\alpha_0\beta_{0,t}^{-1}\alpha_0^{-1})\right)_0. \tag{43}$$

It can be easily seen that the vector field $J^sZ^{(\alpha)}$ can be determined by

$$J^sZ^{(\alpha)} = TJ^s\alpha\cdot J^sZ\circ J^s\alpha^{-1}. \tag{44}$$

We determine the right-hand side at a point $J^s_x\gamma\in J^s\alpha(W^s)$. Using standard differentiations, we have

$$T_{J^s\alpha^{-1}(J^r_x\gamma)}J^s\alpha\cdot J^sZ(J^s\alpha^{-1}(J^r_x\gamma)) = \left(\frac{d}{dt}J^s\alpha(J^s\beta_t(J^s\alpha^{-1}(J^r_x\gamma)))\right)_0. \tag{45}$$

The curve $t\to J^s\alpha(J^s\beta_t(J^s\alpha^{-1}(J^r_x\gamma)))$ can be expressed from the definition of s-jet prolongation of a fibered automorphism (see Sect. 1.4). We have

$$\begin{aligned}J^s\alpha(J^s\beta_t(J^s\alpha^{-1}(J^r_x\gamma))) &= J^s\alpha(J^s\beta_t(J^s_{\alpha_0^{-1}(x)}\alpha^{-1}\gamma\alpha_0)) \\ &= J^s\alpha(J^s_{\beta_{0,t}\alpha_0^{-1}(x)}\beta_t\alpha^{-1}\gamma\alpha_0\beta_{0,t}^{-1})) \\ &= J^s_{\alpha_0\beta_{0,t}\alpha_0^{-1}(x)}(\alpha\beta_t\alpha^{-1})\gamma(\alpha_0\beta_{0,t}^{-1}\alpha_0^{-1}).\end{aligned} \tag{46}$$

Differentiating this curve, we get the vector field $J^sZ^{(\alpha)}$ (44).

2. Let W be the domain of α. We have by definition for every point $J^s_x\gamma\in W^r$, $J^s\alpha(J^s_x\gamma) = J^s_{\alpha_0(x)}\alpha\gamma\alpha_0^{-1}$. Then $(J^s\alpha\circ J^s\gamma)(x) = (J^s\alpha\gamma\alpha_0^{-1}\circ\alpha_0)(x)$, and we can write on the domain $\alpha_0(\pi(W))$ of the section $\alpha\gamma\alpha_0^{-1}$

$$J^s\alpha\circ J^s\gamma\circ\alpha_0^{-1} = J^s\alpha\gamma\alpha_0^{-1}. \tag{47}$$

Consider the Euler–Lagrange form E_λ, the n-form $i_{J^sZ}E_\lambda$ that appears in the first variation formula and its values along the section $J^s\alpha\gamma\alpha_0^{-1}$. We have

$$(J^s\alpha\gamma\alpha_0^{-1})^* i_{J^sZ}E_\lambda = (\alpha_0^{-1})^*(J^s\gamma)^*(J^s\alpha)^* i_{J^sZ}E_\lambda \tag{48}$$

on the domain $\alpha_0(\pi(W))$ of the section $\alpha\gamma\alpha_0^{-1}$. We can easily find an expression for the form $(J^s\alpha)^* i_{J^sZ}E_\lambda$ on W^r. Choose any tangent vectors $\Xi_1, \Xi_2,\ldots, \Xi_n$ at the point $J^s_x\gamma \in W^r$. Then

$$\begin{aligned}&((J^s\alpha)^* i_{J^sZ}E_\lambda)(J^s_x\gamma)(\Xi_1, \Xi_2, \ldots \Xi_n)\\ &\quad = E_\lambda(J^s\alpha(J^s_x\gamma))(J^sZ(J^s\alpha(J^s_x\gamma)), TJ^s\alpha\cdot\Xi_1, TJ^s\alpha\cdot\Xi_2, \ldots, TJ^s\alpha\cdot\Xi_n).\end{aligned} \tag{49}$$

Writing $J^sZ(J^s\alpha(J^s_x\gamma)) = TJ^s\alpha \cdot TJ^s\alpha^{-1} \cdot J^sZ(J^s\alpha(J^s_x\gamma))$, we get from (44)

$$T_{J^s\alpha(J^s_x\gamma)}J^s\alpha^{-1}\cdot J^sZ(J^s\alpha(J^s_x\gamma)) = J^sZ^{(\alpha^{-1})}(J^s_x\gamma) \tag{50}$$

and

$$\begin{aligned}&((J^s\alpha)^* i_{J^sZ}E_\lambda)(J^s_x\gamma)(\Xi_1, \Xi_2, \ldots \Xi_n)\\ &\quad = E_\lambda(J^s\alpha(J^s_x\gamma))(TJ^s\alpha\cdot J^sZ^{(\alpha^{-1})}(J^s_x\gamma), TJ^s\alpha\cdot\Xi_1, \ldots, TJ^s\alpha\cdot\Xi_n)\\ &\quad = (J^s\alpha)^*E_\lambda(J^s_x\gamma)(J^sZ^{(\alpha^{-1})}(J^s_x\gamma), \Xi_1, \Xi_2, \ldots, \Xi_n)\\ &\quad = i_{J^sZ^{(\alpha^{-1})}(J^s_x\gamma)}(J^s\alpha)^*E_\lambda(J^s_x\gamma)(\Xi_1, \Xi_2, \ldots, \Xi_n)\\ &\quad = (i_{J^sZ^{(\alpha^{-1})}}(J^s\alpha)^*E_\lambda(J^s_x\gamma))(\Xi_1, \Xi_2, \ldots, \Xi_n),\end{aligned} \tag{51}$$

or, which is the same,

$$J^s\alpha^* i_{J^sZ}E_\lambda = i_{J^sZ^{(\alpha^{-1})}}J^s\alpha^*E_\lambda. \tag{52}$$

3. Now we can show that if γ is an extremal, and α is an invariance transformation of E_λ, then for any Z

$$(J^s\alpha\gamma\alpha_0^{-1})^* i_{J^sZ}E_\lambda = 0. \tag{53}$$

Since by hypothesis, $(J^s\alpha) * E_\lambda = E_\lambda$, (52) implies $J^s\alpha^* i_{J^sZ}E_\lambda = i_{J^sZ^{(\alpha^{-1})}}E_\lambda$, thus, along $J^s\gamma$,

$$J^s\gamma^* J^s\alpha^* i_{J^sZ}E_\lambda = 0. \tag{54}$$

But the left-hand side is, from (47)

$$\begin{aligned} J^s\gamma^* J^s\alpha^* i_{J^sZ}E_\lambda &= (J^s\alpha \circ J^s\gamma)^* i_{J^sZ}E_\lambda \\ &= (J^s\alpha\gamma\alpha_0^{-1} \circ \alpha_0)^* i_{J^sZ}E_\lambda \\ &= \alpha_0^* J^s\alpha\gamma\alpha_0^{-1*} i_{J^sZ}E_\lambda, \end{aligned} \tag{55a}$$

proving (53) as well as Theorem 5. □

The following theorem describes properties of individual extremals.

Theorem 6 *Let λ be a Lagrangian of order r, let s be the order of the Euler–Lagrange form E_λ, and let γ be an extremal. Then* a *π-projectable vector field Ξ generates symmetries of γ if and only if*

$$E_{\partial_{J^r\Xi}\lambda} \circ J^s\gamma = 0. \tag{55b}$$

Proof

1. Suppose we have an extremal γ and a vector field Ξ generating symmetries of γ; we prove that condition (37) is satisfied. We proceed in several steps.
   Denote by α_t and $\alpha_{0,t}$, the 1-parameter group of Ξ and its projection, respectively. Using formulas (48) and (52) and invariance of the Euler–Lagrange mapping (Sect. 4.5, Theorem 6), we get

$$\begin{aligned} (J^s\alpha_t\gamma\alpha_{0,t}^{-1})^* i_{J^sZ}E_\lambda &= (\alpha_{0,t}^{-1})^*(J^s\gamma)^* i_{J^sZ^{(\alpha_{-t})}}(J^s\alpha_t^* E_\lambda) \\ &= (\alpha_{0,t}^{-1})^* J^s\gamma^* i_{J^sZ^{(\alpha_{-t})}} E_{J^s\alpha_t^*\lambda}. \end{aligned} \tag{56}$$

   Since the left-hand side vanishes by hypothesis, the right-hand side yields

$$J^s\gamma^* i_{J^sZ^{(\alpha_t)}} E_{J^r\alpha_t^*\lambda} = 0. \tag{57}$$

   We want to differentiate the form $i_{J^sZ^{(\alpha_{-t})}} E_{J^s\alpha_t^*\lambda}$ with respect to t at $t = 0$ and then consider the resulting form along the prolongation $J^s\gamma$ of the extremal γ. To perform differentiation, note that the derivative of $i_{J^sZ^{(\alpha_{-t})}} E_{J^s\alpha_t^*\lambda}$ at $t = 0$ is the Lie derivative of the form $i_{J^sZ}E_\lambda$ by the vector field J^sZ. Indeed, for every point $J^r_x\delta$ belonging to the domain of $J^s\alpha_t$ for sufficiently small t, and any tangent vectors $\Xi_1, \Xi_2, \ldots, \Xi_n$ at $J^r_x\delta$,

$$\begin{aligned} &(J^s\alpha_t^* i_{J^sZ}E_\lambda)(J^r_x\delta)(\Xi_1, \Xi_2, \ldots, \Xi_n) \\ &\quad = E_\lambda(J^s\alpha_t(J^r_x\delta))(J^sZ(J^s\alpha_t(J^r_x\delta)), TJ^s\alpha_t \cdot \Xi_1, \ldots, TJ^s\alpha_t \cdot \Xi_n). \end{aligned} \tag{58}$$

Substituting

$$\begin{aligned} J^sZ(J^s\alpha_t(J^r_x\delta)) &= TJ^s\alpha_t \cdot TJ^s\alpha_t^{-1} \cdot J^sZ(J^s\alpha_t(J^r_x\delta)) \\ &= TJ^s\alpha_t \cdot J^sZ^{(\alpha_{-t})}(J^r_x\delta) \end{aligned} \tag{59}$$

from (22), we have

$$\begin{aligned} &(J^s\alpha_t^* i_{J^sZ}E_\lambda)(J^r_x\delta)(\Xi_1, \Xi_2, \ldots, \Xi_n) \\ &\quad = J^s\alpha_t^* E_\lambda(J^r_x\delta)(J^sZ^{(\alpha_{-t})}(J^r_x\delta)), (\Xi_1, \Xi_2, \ldots, \Xi_n) \\ &\quad = i_{J^sZ^{(\alpha_{-t})}(J^r_x\delta)} E_{J^r\alpha_t^*\lambda}(J^r_x\delta)(\Xi_1, \Xi_2, \ldots, \Xi_n) \end{aligned} \tag{60}$$

hence

$$J^s\alpha_t^* i_{J^sZ}E_\lambda = i_{J^sZ^{(\alpha_{-t})}} E_{J^r\alpha_t^*\lambda}. \tag{61}$$

This formula proves that the derivative with respect to t at $t = 0$ of the right-hand side is exactly the Lie derivative of the form $i_{J^sZ}E_\lambda$ with respect to the vector field $J^r\Xi$.

Then, however, since

$$\frac{d}{dt} J^s\alpha_t^* i_{J^sZ}E_\lambda = J^s\alpha_t^* \partial_\Xi i_{J^sZ}E_\lambda = \frac{d}{dt} i_{J^sZ^{(\alpha_{-t})}} E_{J^r\alpha_t^*\lambda} \tag{62}$$

(Lemma 1), so we have along the extremal γ, from (57),

$$\begin{aligned} J^r\gamma^* J^s\alpha_t^* \partial_\Xi i_{J^sZ}E_\lambda &= J^r\gamma^* \frac{d}{dt} J^s\alpha_t^* i_{J^sZ}E_\lambda \\ &= J^r\gamma^* \frac{d}{dt} i_{J^sZ^{(\alpha_{-t})}} E_{J^r\alpha_t^*\lambda} \\ &= 0. \end{aligned} \tag{63}$$

On the other hand, using the Cartan's formula for the Lie derivative of a differential form (see Appendix 5, (44)), we have

$$\begin{aligned} \partial_{J^s\Xi} i_{J^sZ}E_\lambda &= i_{J^s\Xi} d i_{J^sZ}E_\lambda + d i_{J^s\Xi} i_{J^sZ}E_\lambda \\ &= i_{J^s\Xi}(\partial_{J^sZ}E_\lambda - i_{J^sZ}dE_\lambda) - d i_{J^sZ} i_{J^s\Xi}E_\lambda \\ &= i_{J^s\Xi}\partial_{J^sZ}E_\lambda - i_{J^s\Xi} i_{J^sZ}dE_\lambda - \partial_{J^sZ} i_{J^s\Xi}E_\lambda + i_{J^sZ} d i_{J^s\Xi}E_\lambda \\ &= i_{J^s\Xi}\partial_{J^sZ}E_\lambda - i_{J^s\Xi} i_{J^sZ}dE_\lambda - \partial_{J^sZ} i_{J^s\Xi}E_\lambda + i_{J^sZ}(E_{\partial_{J^r\Xi}\lambda} - i_{J^s\Xi}dE_\lambda) \\ &= i_{J^s\Xi}\partial_{J^sZ}E_\lambda - \partial_{J^sZ} i_{J^s\Xi}E_\lambda + i_{J^sZ}E_{\partial_{J^r\Xi}\lambda}, \end{aligned} \tag{64}$$

and from the Lie bracket formula

$$i_{[J^s Z, J^s \Xi]} E_\lambda = \partial_{J^s Z} i_{J^s \Xi} E_\lambda - i_{J^s \Xi} \partial_{J^s Z} E_\lambda \tag{65}$$

we get

$$\partial_{J^s \Xi} i_{J^s Z} E_\lambda = -i_{[J^s Z, J^s \Xi]} E_\lambda + i_{J^s Z} E_{\partial_{J^r \Xi} \lambda}. \tag{66}$$

Now, since γ is an extremal and Ξ generates symmetries of γ, we have $J^s\gamma^* i_{[J^s Z, J^s \Xi]} E_\lambda = 0$ and from equation (63), $J^s\gamma^* \partial_{J^s \Xi} i_{J^s Z} E_\lambda = 0$, thus, $J^s\gamma^* i_{J^s Z} E_{\partial_{J^r \Xi} \lambda} = 0$ as required.

2. Conversely, suppose that we have an extremal γ and a vector field Ξ such that condition $E_{\partial_{J^r \Xi} \lambda} \circ J^s\gamma = 0$ (27) holds. We want to show that Ξ generates symmetries of γ, that is,

$$\alpha^*_{0,t} J^s(\alpha_t \gamma \alpha^{-1}_{0,t})^* i_{J^s Z} E_\lambda = 0, \tag{67}$$

where α_t is the local 1-parameter group of Ξ and Z is any π-projectable vector field.
According to Sect. 4.10, Theorem 11, condition (37) implies

$$J^s\gamma^* i_{J^s Z} E_{\partial_{J^r \Xi} \lambda} = J^s\gamma^* i_{J^s Z} \partial_{J^s \Xi} E_\lambda = 0 \tag{68}$$

for all π-projectable vector fields Z. Thus, at any point $J^r_x\gamma$

$$i_{J^s Z(J^r_x \gamma)} \partial_{J^s \Xi} E_\lambda (J^r_x \gamma) = 0 \tag{69}$$

therefore, $\partial_{J^s \Xi} E_\lambda (J^r_x \gamma) = 0$ because the Euler–Lagrange form is 1-contact. Thus by Sect. 5.1, Lemma 2,

$$(J^s\alpha_t)^* E_\lambda (J^s_x \gamma) = E_\lambda (J^s_x \gamma). \tag{70}$$

Contracting the left-hand side by $J^s Z(J^r_x\gamma)$ and using Lemma 7,

$$\begin{aligned} J^r\gamma^* i_{J^s Z} (J^s\alpha_t)^* E_\lambda &= J^r\gamma^* (J^s\alpha_t)^* i_{J^s Z^{(\alpha_{-t})}} E_\lambda \\ &= (J^s\alpha_t \circ J^r\gamma)^* i_{J^s Z^{(\alpha_{-t})}} E_\lambda = (J^s\alpha_t \gamma \alpha^{-1}_{0,t} \circ \alpha_{0,t})^* i_{J^s Z^{(\alpha_{-t})}} E_\lambda \\ &= (\alpha_{0,t})^* (J^s\alpha_t \gamma \alpha^{-1}_{0,t})^* i_{J^s Z^{(\alpha_{-t})}} E_\lambda = \alpha^*_{0,t} J^s(\alpha_t \gamma \alpha^{-1}_{0,t})^* i_{J^s Z} E_\lambda. \end{aligned} \tag{71}$$

Since the contraction of the right-hand side vanishes, because γ is an extremal, we have $\alpha^*_{0,t} J^s(\alpha_t \gamma \alpha^{-1}_{0,t})^* i_{J^s Z} E_\lambda = 0$, proving (67). □

Remark 7 Properties of the systems of partial differential equations, described in this section, strongly rely on the variational origin of these systems. The structure of these equations, esp. their invariance properties, indicates possibilities of applying specific methods of solving these equations. Clearly, these specific topics need further research.

References

[K-S] Y. Kosmann-Schwarzbach, *The Noether Theorems*, Springer, 2011
[K1] D. Krupka, A geometric theory of ordinary first order variational problems in fibered manifolds, I. Critical sections, II. Invariance, J. Math. Anal. Appl. 49 (1975) 180-206, 469-476
[K6] D. Krupka, Lagrange theory in fibered manifolds, Rep. Math. Phys. 2 (1971) 121-133
[KT] D. Krupka, A. Trautman, General invariance of Lagrangian structures, Bull. Acad. Polon. Sci., Ser. Sci. Math. Astronom. Phys. 22 (1974) 207-211
[K10] D. Krupka, Natural Lagrange structures, in: *Semester on Differential Geometry*, 1979, Banach Center, Warsaw, Banach Center Publications 12, 1984, 185-210
[N] E. Noether, Invariante Variationsprobleme, Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse (1918) 235-257
[O1] P.J. Olver, *Applications of Lie Groups to Differential Equations*, Springer-Verlag, New York, 1998
[Tr1] A. Trautman, Invariance of Lagrangian systems, in: General Relativity, Papers in Honour of J.L. Synge, Oxford, Clarendon Press, 1972, 85-99
[Tr2] A. Trautman, Noether equations and conservation laws, Commun. Math. Phys. 6 (1967) 248-261

Chapter 6
Examples: Natural Lagrange Structures

Examples presented in this chapter include typical variational functionals that appear as variational principles in the theory of geometric and physical fields. We begin by the discussion of the well-known *Hilbert variational functional* for the metric fields, first considered in Hilbert [H] in 1915, whose Euler–Lagrange equations are the *Einstein vacuum equations*. We give a manifold interpretation of this functional and show that its *second-order* Lagrangian, the *formal scalar curvature*, possesses a global *first-order Lepage* equivalent. The Lagrangian used by Hilbert is an example of a *differential invariant* of a metric field (and its first and second derivatives). It should be pointed out, however, that the variational considerations as well as the resulting extremal equations are independent of the signature of underlying metric fields.

Our approach to the subject closely follows the preprint Krupka and Lenc [KL]. The theory of jets and differential invariants including applications is explained in Krupka and Janyska [KJ] (see also a general treatment by Kolar, Michor, Slovak [KMS]). Variational principles with similar invariance properties were studied by Anderson [A1] in connection with the inverse variational problem. More general classes of *natural bundles* and *natural Lagrangians* that are differential invariants of *any* collection of tensor fields, or *any* geometric object fields, were introduced in Krupka and Trautman [KT] and Krupka [K3, K10]. The claims in this chapter are *not* routine; the reader should provide a proof of them or consult the corresponding references.

For contemporary research in the theory of natural Lagrange structures, we refer to Ferraris et al. [FFPW], Patak and Krupka [PK], Palese and Winterroth [PW] and the references therein. Extensive literature on the classical invariant theory, related with the subject, can be found in Kolar et al. [KMS] and Krupka and Janyska [KJ]; however, this topic is outside the scope of this book. The variational functionals for submanifolds, whose underlying structures *differ* from jet prolongation of fibered manifolds, are not considered in this book (cf. Urban and Krupka [UK3]).

D. Krupka, *Introduction to Global Variational Geometry*,
Atlantis Studies in Variational Geometry 1, DOI 10.2991/978-94-6239-073-7_6

6.1 The Hilbert Variational Functional

The modern geometric interpretation of variational principles in physics requires the knowledge of the structure of underlying fibered spaces as well as adequate (intrinsic and also coordinate) methods of the calculus of variations on these spaces. In this example, we briefly consider the *Hilbert variational functional* for metric fields on a general n-dimensional manifold X, a well-known functional providing, for $n = 4$, the variational principle for the *Einstein vacuum equations* in the general relativity theory (Hilbert [H]). Note that the Hilbert variational principle does *not* restrict the topology of the underlying (*spacetime*) manifold X. If we require that the *topology* of spacetime should have its origin in *matter* and *physical fields*, then this principle should be completed with some other one.

In this example, we follow the preprint Krupka and Lenc [KL]; the topic certainly needs further investigations. Our assertions are formulated without proof, which can however be easily reconstructed by means of the general theory. Basic knowledge of the concepts of Riemannian (and pseudo-Riemannian) geometry is supposed.

Let X be an n-dimensional smooth manifold, T^0_2X the vector bundle of tensors of type (0,2) over X, and let τ: $T^0_2X \to X$ be the tensor bundle projection. T^0_2X contains the open set $\mathrm{Met}\,X$ of *symmetric, regular bilinear forms* on the tangent spaces at the points of X. Then, the restriction of the tensor bundle projection τ defines a *fibered manifold structure* on the set $\mathrm{Met}\,X$ over the manifold X; we call this fibered manifold the *bundle of metrics* over X. Its *sections* are *metric fields* on the manifold X. Integral variational functionals for the metric fields are defined by n-forms on the r-jet prolongations $J^r\mathrm{Met}\,X$ of the fibered manifold $\mathrm{Met}\,X$.

Any chart (U, φ), $\varphi = (x^i)$, on X induces a chart (V, ψ), $\psi = (x^i, g_{ij})$, on $\mathrm{Met}\,X$, where $V = \tau^{-1}(U)$ and g_{ij} are functions on V defined by the decomposition $g = g_{ij}dx^i \otimes dx^j$ of the bilinear forms; the *coordinate functions* g_{ij} entering the chart (V, ψ) satisfy $1 \le i \le j \le n$. The associated fibered charts on the r-jet prolongations $J^r\mathrm{Met}\,X$ are then defined in a standard way. In particular, if $r = 2$, then the associated chart is denoted by (V^2, ψ^2), $\psi^2 = (x^i, g_{ij}, g_{ij,k}, g_{ij,kl})$, where $i \le j$, $k \le l$, and $g_{ij,k} = d_k g_{ij}$, $g_{ij,kl} = d_k d_l g_{ij}$; d_k is the *formal derivative operator*. We denote

$$\begin{aligned} \omega_0 &= dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n, \\ \omega_k &= (-1)^{k-1} dx^1 \wedge dx^2 \wedge \cdots \wedge dx^{k-1} \wedge dx^{k+1} \wedge \cdots \wedge dx^n, \\ \omega_{ij} &= dg_{ij} - g_{ij,p}dx^p, \\ \omega_{ij,k} &= dg_{ij,k} - g_{ij,kp}dx^p. \end{aligned} \tag{1}$$

Then, the forms $dx^i, \omega_{ij}, \omega_{ij,k}, dg_{ij,kl}$ constitute the *contact basis* on the set V^2. We need some systems of functions on V^2. The functions

$$\Gamma^i_{jk} = \frac{1}{2} g^{im}(g_{mk,j} + g_{jm,k} - g_{jk,m}), \tag{2}$$

where g^{im} are elements of the *inverse matrix* of the matrix g_{ij}, which are called the *formal Christoffel symbols*; note that the derivative $g_{pj,k}$ can be reconstructed from Γ^i_{jk} by the formula $g_{pj,k} = g_{pi}\Gamma^i_{jk} + g_{ji}\Gamma^i_{pk}$. The expressions

$$R_{ik} = \Gamma^l_{ik,l} + \Gamma^l_{ik}\Gamma^m_{lm} - \Gamma^l_{il,k} - \Gamma^m_{il}\Gamma^l_{km}, \quad R = g^{ik}R_{ik}, \tag{3}$$

where $\Gamma^l_{ik,j}$ are the formal derivatives $d_j\Gamma^l_{ik}$, define the *formal Ricci tensor* with components R_{ik}, and a function $R\colon J^2\mathrm{Met}\,X \to \mathbf{R}$, the *formal scalar curvature*. Every metric field $U \ni x \to g(x) \in \mathrm{Met}\,X$, defined on an open set in X, can be prolonged to the section $U \ni x \to J^2g(x) \in J^2\mathrm{Met}\,X$ of the second jet prolongation $J^2\mathrm{Met}\,X$. Composing the second jet prolongation J^2g with the formal scalar curvature, we get a real-valued function on U, $x \to (R \circ J^2g)(x) = R(J^r_x g)$, the *scalar curvature of the metric* g, and a second-order Lagrangian

$$\lambda = R\sqrt{|\det g_{ij}|} \cdot \omega_0. \tag{4}$$

λ is called the *Hilbert Lagrangian*. The variational functional

$$\Gamma_\Omega(\tau) \ni g \to \lambda_\Omega(\gamma) = \int_\Omega J^2g^*\lambda \in \mathbf{R}, \tag{5}$$

where Ω is any compact set in the domain of definition of the section γ, which is the *Hilbert variational functional* for the metric fields on X.

We shall restate basic general theorems of the variational theory on fibered manifolds for this special case. It should be pointed out, however, that all these statements could also be proved *directly*, without reference to the general theory. Our first statement rephrases the existence theorem for Lepage equivalents of a given Lagrangian; we claim in addition that the (*second-order*) Hilbert Lagrangian possesses a *first-order* Lepage equivalent.

Recall that $\tau^{2,0}$ is the canonical jet projection of $J^2\mathrm{Met}\,X$ onto $\mathrm{Met}\,X$, expressed as the mapping $(x^i, g_{ij}, g_{ij,k}, g_{ij,kl}) \to (x^i, g_{ij})$, and denote

$$\mathscr{R} = R\sqrt{|\det g_{ij}|}. \tag{6}$$

$\mathscr{R}$ is the *component* of the Hilbert Lagrangian with respect to the chart on $J^2\mathrm{Met}\,X$, associated with the chart (U, φ), $\varphi = (x^i)$.

Theorem 1 (Existence of Lepage equivalents) *There exists an n-form* Θ_H *on the first jet prolongation* $J^1 \mathrm{Met}\, X$ *with the following properties*:

(a) $h\Theta_H = \lambda$.
(b) $p_1 d\Theta_H$ *is* $\tau^{2,0}$*-horizontal.*

To prove Theorem 1, we can use the principal Lepage equivalent of a second-order Lagrangian (Sect. 4.5, Example 2), which is now given by

$$\Theta_H = \mathscr{R}\omega_0 + \left(\left(\frac{\partial \mathscr{R}}{\partial g_{ij,k}} - d_l \frac{\partial \mathscr{R}}{\partial g_{ij,kl}} \right) \omega_{ij} + \frac{\partial \mathscr{R}}{\partial g_{ij,kl}} \omega_{ij,l} \right) \wedge \omega_k. \tag{7}$$

Substituting from (6), we get the *principal Lepage equivalent of the Hilbert Lagrangian*

$$\begin{aligned} \Theta_H = & \sqrt{|\det g_{rs}|} g^{ip} (\Gamma^j_{ip}\Gamma^k_{jk} - \Gamma^j_{ik}\Gamma^k_{jp})\omega_0 \\ & + \sqrt{|\det g_{rs}|} (g^{jp}g^{iq} - g^{pq}g^{ij})(dg_{pq,j} + \Gamma^k_{pq} dg_{jk}) \wedge \omega_i. \end{aligned} \tag{8}$$

One can also prove Theorem 1 by searching for Θ_H in the form

$$\Theta_H = \mathscr{R}\omega_0 + (f^{ijk}\omega_{ij} + f^{ijkl}\omega_{ij,l}) \wedge \omega_k, \tag{9}$$

with an invariant condition $f^{ijkl} = f^{ijlk}$. The following is another expression for Θ_H.

Theorem 2 *The form* Θ_H *satisfying conditions* (a) and (b) of Theorem 1 *has an expression*

$$\Theta_H = -\mathscr{H}\omega_0 + \mathscr{P}^{ij,k} dg_{ij} \wedge \omega_k + d\eta, \tag{10}$$

where

$$\begin{aligned} \mathscr{H} &= \sqrt{|\det g_{rs}|} \cdot g^{ij}(\Gamma^k_{ik}\Gamma^r_{jr} - \Gamma^k_{ij}\Gamma^r_{kr}), \\ \mathscr{P}^{ij,k} &= \frac{1}{2}\sqrt{|\det g_{rs}|}(-g^{ki}g^{sj}\Gamma^q_{qs} - g^{kj}g^{si}\Gamma^q_{qs} + g^{ks}g^{ij}\Gamma^q_{qs} \\ & \quad + g^{pi}g^{sj}\Gamma^k_{ps} + g^{pj}g^{si}\Gamma^k_{ps} - g^{ij}g^{ps}\Gamma^k_{ps}), \\ \eta &= \sqrt{|\det g_{rs}|}(g^{jl}\Gamma^k_{jl} - g^{kl}\Gamma^r_{rl})\omega_k. \end{aligned} \tag{11}$$

These explicit formulas show that the Lepage form Θ_H is of the first order. Since $h\Theta_H = \lambda$, the Hilbert variational functional (1) can also be treated as a *first-order* functional

$$\Gamma_\Omega(\tau) \ni g \to \lambda_H(\gamma) = \int_\Omega J^1 g^* \Theta_H \in \mathbf{R}. \tag{12}$$

Existence of the Lepage equivalent Θ_H has a few immediate consequences. The most important one is the form of the first variation formula (Sect. 4.6). Recall this formula for any τ-projectable vector field Ξ on the fibered manifold Met X, expressed by

$$\Xi = \xi^i \frac{\partial}{\partial x^i} + \Xi_{ij} \frac{\partial}{\partial g_{ij}}. \tag{13}$$

Then for every metric field g, defined on an open set in X, the Lie derivative $\partial_{J^1\Xi}\Theta_H$ is along $J^1 g$ expressed as

$$J^1 g^* \partial_{J^1\Xi}\Theta_H = J^1 g^* i_{J^1\Xi} d\Theta_H + dJ^1 g^* i_{J^1\Xi}\Theta_H. \tag{14}$$

This is the basic (global) *infinitesimal first variation formula* for the Hilbert Lagrangian, allowing us to study its *extremals* and *conservation law equations*. The horizontal components $hi_{J^1\Xi} d\Theta_H$ and $hdJ^1 g^* i_{J^1\Xi}\Theta_H$ corresponding with formula (14) are

$$hi_{J^1\Xi} d\Theta_H = \left(\frac{\partial \mathscr{R}}{\partial g_{ij}} - d_k \frac{\partial \mathscr{R}}{\partial g_{ij,k}} + d_k d_l \frac{\partial \mathscr{R}}{\partial g_{ij,kl}} \right) (\Xi_{ij} - g_{ij,p}\xi^p)\omega_0, \tag{15}$$

and

$$hdi_{J^1\Xi}\Theta_H = d_i w^i \cdot \omega_0, \tag{16}$$

where

$$w^i = \mathscr{R}\xi^i + \left(\frac{\partial \mathscr{R}}{\partial g_{kl,i}} + d_j \frac{\partial \mathscr{R}}{\partial g_{kl,ij}} \right) (\Xi_{kl} - g_{kl,p}\xi^p) + \frac{\partial \mathscr{R}}{\partial g_{kl,ij}} (\Xi_{klj} - g_{kl,jp}\xi^p). \tag{17}$$

Note that the horizontalization h in (15) and (16) characterizes the forms $i_{J^1\Xi} d\Theta_H$ and $di_{J^1\Xi}\Theta_H$ *along the* 1*-jet prolongations* $J^1 g$ of sections of the fibered manifold Met X. Expression (15) represents the *Euler–Lagrange term*, and (16) is the *boundary term*. Since from the definition of the r-jet prolongation of a vector field, the expression $\Xi_{klj} - g_{kl,jp}\xi^p$ can be expressed as

$$\begin{aligned} d_j(\Xi_{kl} - g_{kl,p}\xi^p) &= d_j \Xi_{kl} - g_{kl,pj}\xi^p - g_{kl,p} \frac{\partial \xi^p}{\partial x^j} \\ &= \Xi_{klj} - g_{kl,pj}\xi^p \end{aligned} \tag{18}$$

(see Sect. 1.7), we can also write formula (17) as

$$\begin{aligned} w^i &= \mathscr{R}\xi^i + \left(\frac{\partial \mathscr{R}}{\partial g_{kl,i}} + d_j \frac{\partial \mathscr{R}}{\partial g_{kl,ij}}\right)(\Xi_{kl} - g_{kl,p}\xi^p) + \frac{\partial \mathscr{R}}{\partial g_{kl,ij}} d_j(\Xi_{kl} - g_{kl,p}\xi^p) \\ &= \mathscr{R}\xi^i + \frac{\partial \mathscr{R}}{\partial g_{kl,i}}(\Xi_{kl} - g_{kl,p}\xi^p) + d_j\left(\frac{\partial \mathscr{R}}{\partial g_{kl,ij}}(\Xi_{kl} - g_{kl,p}\xi^p)\right). \end{aligned} \tag{19}$$

The Lapage equivalent Θ_H determines the Euler–Lagrange equations:

Theorem 3 (Euler–Lagrange expressions, Noether currents)

(a) *The Euler–Lagrange term in the first variation formula* (14) *has an expression*

$$hi_{J^1\Xi}d\Theta_H = \left(\frac{1}{2}g_{ij}R - R_{ij}\right)g^{ir}g^{js}(\Xi_{rs} - g_{rs,p}\xi^p)\sqrt{|\det g_{rs}|}\omega_0. \tag{20}$$

(b) *The boundary term is given by the expression*

$$\begin{aligned} w^i = \mathscr{R}\xi^i &+ \sqrt{|\det g_{rs}|}(g^{jl}g^{pi} - g^{pj}g^{li})\Gamma^k_{pj}(\Xi_{kl} - g_{kl,m}\xi^m) \\ &+ \sqrt{|\det g_{rs}|}(g^{kj}g^{il} - g^{ij}g^{kl})(\Xi_{klj} - g_{kl,jm}\xi^m). \end{aligned} \tag{21}$$

The $(n+1)$-form defined by expression (20), characterizing *extremals* of the Hilbert variational functionals, is the *Euler–Lagrange form*

$$E(\lambda) = p_1 d\Theta_H = \sqrt{|\det g_{rs}|}E_{ij}g^{ir}g^{js}\omega_{rs} \wedge \omega_0, \tag{22}$$

where E_{ij} is the *formal Einstein tensor*,

$$E_{ij} = \frac{1}{2}g_{ij}R - R_{ij}. \tag{23}$$

The corresponding Euler–Lagrange equations are the *Einstein equations*

$$E_{ij} \circ J^2 g = 0. \tag{24}$$

The $(n-1)$-form $i_{J^1\Xi}\Theta_H$ in (16) is the *Noether current* associated with the vector field Ξ.

A specific property of the Hilbert Lagrangian consists in its invariance under *all* diffeomorphisms of the fibered manifold $\mathrm{Met}\,X$, induced by diffeomorphisms of the underlying manifold X. Recall briefly the corresponding definitions (Krupka [K3]). Suppose we are given a diffeomorphism $\alpha\colon U \to \bar{U}$, where U and $\bar{U}$ are open subsets of X. First, we wish to show that α lifts to a diffeomorphism α_{Met} of the set $\tau^{-1}(U)$ into $\tau^{-1}(\bar{U})$ and find equations of α_{Met}. If U and $\bar{U}$ are domains of definition of two charts, (U, φ), $\varphi = (x^i)$, and $(\bar{U}, \bar{\varphi})$, $\bar{\varphi} = (\bar{x}^\sigma)$, then for any point $x \in U$, a metric $\bar{g}$ at the point $\alpha(x) \in \bar{U}$ is expressed as

$$\bar{g} = \bar{g}_{\sigma\nu} \cdot dy^\sigma(\alpha(x)) \otimes dy^\nu(\alpha(x)), \tag{25}$$

where $\bar{g}_{\sigma\nu}$ are real numbers. Then setting

$$\begin{aligned} T_2^0\alpha \cdot \bar{g} &= \bar{g}_{\sigma\nu}(\alpha^* dy^\sigma)(x) \otimes (\alpha^* dy^\nu)(x) \\ &= \bar{g}_{\sigma\nu} d(y^\sigma \circ \alpha)(x) \otimes (y^\nu \circ \alpha)(x) \\ &= \bar{g}_{\sigma\nu} \left(\frac{\partial(y^\sigma \alpha\varphi^{-1})}{\partial x^i}\right)_{\varphi(x)} \left(\frac{\partial(y^\nu \alpha\varphi^{-1})}{\partial x^j}\right)_{\varphi(x)} dx^i(x) \otimes dx^j(x), \end{aligned} \tag{26}$$

we get a metric $g = T_2^0\alpha \cdot \bar{g}$ at the point x. Thus, replacing α with α^{-1}, we get a diffeomorphism $\operatorname{Met}\alpha\colon \tau^{-1}(U) \to \tau^{-1}(\bar{U})$, defined in components as the correspondence

$$\begin{aligned} &x^i \to x^i\alpha\varphi^{-1}(\varphi(x)), \\ &g_{ij} \to \bar{g}_{\sigma\nu} = g_{ij}\left(\frac{\partial(x^i\alpha^{-1}\bar{\varphi}^{-1})}{\partial \bar{x}^\sigma}\right)_{\varphi(\alpha(x))} \left(\frac{\partial(x^j\alpha^{-1}\bar{\varphi}^{-1})}{\partial \bar{x}^\nu}\right)_{\varphi(\alpha(x))}. \end{aligned} \tag{27}$$

This construction can be adapted to the local 1-parameter group α_t of a vector field ξ on X. To this purpose, we may choose, for all sufficiently small t, $(\bar{U}, \bar{\varphi}) = (U, \varphi)$. Express ξ as

$$\xi = \xi^i \frac{\partial}{\partial x^i}. \tag{28}$$

Then, the mapping $\operatorname{Met}\alpha$ (27) is replaced with the mapping expressed as

$$\begin{aligned} &(t, x^i) \to x^i\alpha_t\varphi^{-1}(\varphi(x)) = x^i\alpha_t(x), \\ &(t, g_{ij}) \to \bar{g}_{rs} = g_{ij}\left(\frac{\partial(x^i\alpha_t^{-1}\varphi^{-1})}{\partial x^r}\right)_{\varphi(\alpha_t(x))} \left(\frac{\partial(x^j\alpha_t^{-1}\bar{\varphi}^{-1})}{\partial x^s}\right)_{\varphi(\alpha_t(x))}, \end{aligned} \tag{29}$$

representing the *canonical lift* $\operatorname{Met}\alpha_t$ of the flow α_t to the fibered manifold $\operatorname{Met}X$. The corresponding lift of the vector field ξ to the fibered manifold $\operatorname{Met}X$, denoted $\operatorname{Met}\xi$, is obtained by differentiating of the functions (29) at $t = 0$. Differentiating the mapping $(t, x^i) \to x^i\alpha_t(x)$ yields the component ξ^i of ξ. Since $\alpha_t^{-1} = \alpha_{-t}$ and $\alpha_0 = \mathrm{id}$, the second row in (29) yields the expression

$$g_{ij}\left(\frac{\partial}{\partial x^r}\left(\frac{d(x^i\alpha_{-t}\varphi^{-1})}{dt}\right)_0\right)_{\varphi(x)}\delta^j_s + g_{ij}\delta^i_r\left(\frac{\partial}{\partial x^s}\left(\frac{d(x^j\alpha_{-t}\bar{\varphi}^{-1})}{dt}\right)_0\right)_{\varphi(x)}$$
$$= -g_{is}\left(\frac{\partial\xi^i}{\partial x^r{}_0}\right)_{\varphi(x)} - g_{rj}\left(\frac{\partial\xi^j}{\partial x^s{}_0}\right)_{\varphi(x)}. \tag{30}$$

Thus, since the vector field $\mathrm{Met}\,\xi$ is determined by its flow, we have

$$\mathrm{Met}\,\xi = \xi^i\frac{\partial}{\partial x^i} - \left(g_{is}\frac{\partial\xi^i}{\partial x^r} + g_{ri}\frac{\partial\xi^i}{\partial x^s}\right)\frac{\partial}{\partial g_{rs}}. \tag{31}$$

The Hilbert Lagrangian λ is easily seen to be diffeomorphism invariant or, which is the same, a *differential invariant* (cf. Krupka and Janyska [KJ]; Kolar et al. [KMS]). This property can also be expressed in terms of Lie derivatives.

Theorem 4 *For every vector field ξ, defined on an open set in X,*

$$\partial_{J^2\mathrm{Met}\xi}\lambda = 0. \tag{32}$$

Combining Theorem 4 and the first variation formula (14), where $\Xi = \mathrm{Met}\,\xi$ we obtain the identity

$$J^1g^*i_{J^1\mathrm{Met}\,\xi}d\Theta_H + dJ^1g^*i_{J^1\mathrm{Met}\,\xi}\Theta_H = 0 \tag{33}$$

holding for all ξ and all γ. The meaning of this condition requires further analysis, given, for more general variational functionals, in subsequent sections.

6.2 Natural Lagrange Structures

The class of *natural Lagrange structures* represents a far-going generalization of the Hilbert variational principle, discussed in the previous example. The *Lagrangians* for these Lagrange structures are defined on natural bundles by an invariance condition with respect to diffeomorphisms of the underlying manifold, analogous to property $\partial_{J^2\mathrm{Met}\xi}\lambda = 0$, of the Hilbert Lagrangian λ (see Sect. 6.1, (32)). Conditions of this kind can be rephrased by saying that the Lagrangians should be *differential invariants* (Krupka and Janyska [KJ]); a specific feature of such a Lagrangian consists in its property to define a variational principle not only for one specific fibered manifold but rather for the *category* of locally isomorphic fibered manifolds. For the natural bundles and their generalizations – gauge natural bundles – we refer to Kolar et al. [KMS].

Our brief exposition follows the general theory explained in Chap. 4 and two papers on natural Lagrange structures (Krupka [K3, K10]). The relationship between natural Lagrangians and the inverse problem of the calculus of variations was studied by Anderson [A1].

By the rth *differential group* of the Euclidean space $\mathbf{R}^n$, we mean the group L_n^r of invertible r-jets with source and target at the origin $0 \in \mathbf{R}^n$. An element of the group L_n^r is an r-jet $J_0^r\alpha$, whose representative is a diffeomorphisms $\alpha\colon U \to V$, where U and V are neighborhoods of the origin and $\alpha(0) = 0$. The group operation $L_n^r \times L_n^r \ni (J_0^r\alpha, J_0^r\beta) \to J_0^r(\alpha \circ \beta) \in L_n^r$ is defined by the composition of jets. The *canonical* (global) *coordinates* $a^i_{j_1 j_2 \ldots j_k}$ on L_n^r are defined by the condition $a^i_{j_1 j_2 \ldots j_k}(J_0^r\alpha) = D_{j_1}D_{j_2}\ldots D_{j_k}\alpha^i(0)$, where $1 \le k \le r$, $1 \le j_1 \le j_2 \le \cdots \le j_k \le n$, and α^i are components of the diffeomorphism α. Since the group operation is polynomial, the differential group is a Lie group. Clearly, L_n^1 can be canonically identified with the general linear group $GL_n(\mathbf{R})$.

Let X be a smooth manifold of dimension n. By an *r-frame* at a point $x \in X$, we mean an invertible r-jet $J_0^r\zeta$ with source $0 \in \mathbf{R}^n$ and target x. The set of r-frames, denoted $\mathscr{F}^r X$, has a natural smooth structure and is endowed with the canonical jet projection $\pi^r\colon \mathscr{F}^r X \to X$: Every chart (U, φ), $\varphi = (x^i)$, on X induces a chart $((\pi^r)^{-1}(U), \varphi^r)$, $\varphi^r = (x^i, \zeta^i_{j_1 j_2 \ldots j_k})$, on $\mathscr{F}^r X$ by $\zeta^i_{j_1 j_2 \ldots j_k}(J_0^r\zeta) = D_{j_1}D_{j_2}\ldots D_{j_k}\zeta^i(0)$, where $1 \le k \le r$, $1 \le j_1 \le j_2 \le \cdots \le j_k \le n$, and ζ^i are the components of ζ in the chart (U, φ). The mapping $\mathscr{F}^r X \times L_n^r \ni (J_0^r\zeta, J_0^r\alpha) \to J_0^r(\zeta \circ \alpha) \in \mathscr{F}^r X$ defines on $\mathscr{F}^r X$ the structure of a (right) *principal fiber bundle* with structure group L_n^r. $\mathscr{F}^r X$ is called the *bundle of r-frames* over X. If $r = 1$, then $\mathscr{F}^1 X$ can be canonically identified with the bundle of linear frames $\mathscr{F}X$.

As an example, one can easily derive the equations, describing the structure of the principal L_n^2-bundle of 2-frames. The group multiplication in the differential group L_n^2 is given by

$$
\begin{aligned}
a^i_j(A \circ B) &= a^i_k(A)a^k_j(B),\\
a^i_{j_1 j_2}(A \circ B) &= a^i_{k_1 k_2}(A)a^{k_1}_{j_1}(B)a^{k_2}_{j_2}(B) + a^i_k(A)a^k_{j_1 j_2}(B),
\end{aligned}
\tag{34}
$$

where $A = J_0^2\alpha$, $B = J_0^2\beta$. The right action of L_n^2 on $\mathscr{F}^2 X$ is expressed by the formulas

$$
\begin{aligned}
\zeta^i_j(\zeta \circ A) &= \zeta^i_k(\zeta)a^k_j(A),\\
\zeta^i_{j_1 j_2}(\zeta \circ A) &= \zeta^i_{k_1 k_2}(\zeta)a^{k_1}_{j_1}(A)a^{k_2}_{j_2}(A) + \zeta^i_k(\zeta)a^k_{j_1 j_2}(A).
\end{aligned}
\tag{35}
$$

We need some categories:

(a) $\mathscr{D}_n$ – the category of diffeomorphisms of smooth, n-dimensional manifolds,
(b) $\mathscr{PB}_n(G)$ – the category of homomorphisms of principal fiber bundles with structure group G, whose projections are morphisms of $\mathscr{D}_n$,
(c) $\mathscr{FB}_n(G)$ – the category of homomorphisms of fiber bundles, associated with principal fiber bundles from $\mathscr{PB}_n(G)$.

Let $\tau\colon \mathscr{D}_n \to \mathscr{PB}_n(G)$ be a *lifting*, that is, a covariant functor, assigning to an object X of the category $\mathscr{D}_n$ an object τX of $\mathscr{PB}_n(G)$ and to a morphism $f\colon U \to V$

of $\mathscr{D}_n$ a morphism $\tau f\colon \tau U \to \tau V$ of $\mathscr{PB}_n(G)$. Let Q be a manifold, endowed with a left action of the Lie group G. For any manifold X belonging to $\mathscr{D}_n$, Q defines a fiber bundle $\tau_Q X$ with type fiber Q, associated with τX. $f\colon U \to V$ also defines a morphism $\tau_Q f\colon \tau_Q U \to \tau_Q V$ of the category $\mathscr{FB}_n(G)$. The correspondence $X \to \tau_Q X, f \to \tau_Q f$ is a covariant functor from $\mathscr{D}_n$ to $\mathscr{FB}_n(G)$, called the *Q-lifting* associated with the lifting τ. This lifting is denoted by τ_Q.

In many applications, Q is a space of tensors on the vector space $\mathbf{R}^n$. Then, Q is endowed with the *tensor action* $GL_n(\mathbf{R}) \times Q \ni (g,p) \to g \cdot p \in Q$. In this case, the Q-lifting τ_Q assigns to a smooth n-dimensional manifold X the tensor bundle $\tau_Q X$ of *tensors of type* Q over X and to a morphism $f\colon U \to V$ of $\mathscr{D}_n$ the corresponding morphism $\tau_Q f\colon \tau_Q U \to \tau_Q V$ of the category $\mathscr{FB}_n(GL_n(\mathbf{R}))$.

In the calculus of variations, we need the *jet prolongations* of these fiber bundles. Denote by $T^r_n Q$ the set of r-jets with source $0 \in \mathbf{R}^n$ and target in Q. $T^r_n Q$ is endowed with the action of the differential group L^{r+1}_n,

$$L^{r+1}_n \times T^r_n Q \ni (J^{r+1}_0 \alpha, J^r_0 \zeta) \to J^r_0((D\alpha \cdot \zeta) \circ \alpha^{-1}) \in T^r_n Q \tag{36}$$

(Krupka [K3]). Calculating this mapping in a chart, we easily find that formally, this jet formula represents *transformation properties* of the derivatives of a tensor field of type Q. The following interpretation is important for applications; namely, it possesses a tool how to construct *natural Lagrangians* for collections of tensor fields of a given type Q.

Lemma 1 *Let X be a smooth n-dimensional manifold.*

(a) *Formula (36) defines the structure of a fiber bundle with type fiber $T^r_n Q$, associated with the principal L^{r+1}_n-bundle $\mathscr{F}^{r+1}X$.*
(b) *The correspondence $X \to J^r \tau_Q X, f \to J^r f_Q X$ is a covariant functor from the category $\mathscr{D}_n$ to the category $\mathscr{FB}_n(L^{r+1}_n)$.*

The lifting $J^r \tau_Q$ is called the *r-jet prolongation* of the lifting τ_Q.

The notion of the r-jet prolongation can naturally be extended to *any* manifolds Q endowed with a left action of the general linear group $GL_n(\mathbf{R})$.

These notions represent the underlying general concepts of the theory of natural variational structures. Namely let X be an n-dimensional manifold (an object of the category $\mathscr{D}_n$), Q a manifold endowed with a left action of the general linear group $L^1_n = GL_n(\mathbf{R})$, $\tau_Q X$ the fiber bundle with base X and type fiber Q, associated with the bundle of frames $\mathscr{F}X$ (an object of the category $\mathscr{FB}_n(L^1_n)$), and let $J^r \tau_Q X$ be the r-jet prolongation of $\tau_Q X$ (an object of the category $\mathscr{FB}_n(L^{r+1}_n)$). Let $J^r \tau_Q \xi$ be the *lift* of a vector field ξ, defined on X, to the bundle $J^r \tau_Q X$ (an object of $\mathscr{FB}_n(L^{r+1}_n)$). We say that a Lagrangian λ defined on $J^r \tau_Q X$ is *natural*, if for all vector fields ξ,

$$\partial_{J^r \tau_Q \xi} \lambda = 0. \tag{37}$$

Now let (Y, λ) be a variational structure of order r, let X be the base of the fibered manifold Y, and suppose without loss of generality that the form λ is a *Lagrangian*. We shall say that the variational structure (Y, λ) is *natural*, if there exists a left L_n^1-manifold Q such that $Y = \tau_Q X$, and λ is a *natural Lagrangian* for this natural bundle.

Examples

1. The variational structure $(\mathrm{Met}X, \lambda)$, where λ is the Hilbert Lagrangian (Sect. 6.1).
2. The Lagrangian for a covector field and a metric field in the general relativity theory, representing interaction of the electromagnetic and gravitational fields in the general relativity theory. The corresponding natural Lagrange structure is the pair (Y, λ), where the fibered manifold Y is the fiber product $\mathrm{Met}\,X \oplus T^*X$ over a manifold X; its sections are the pairs of tensor fields (g, A), locally expressible as

$$g = g_{ij} dx^i \otimes dx^j, \quad A = A_i dx^i. \tag{38}$$

The Lagrangian is of the form $\lambda = \lambda_H + \lambda'$, where λ_H is the Hilbert Lagrangian and the term λ', describing the interaction of the *gravitational* and *electromagnetic* field, is defined by the *interaction Lagrangian*

$$\lambda' = g^{ij} g^{kl} (A_{i,k} - A_{k,i})(A_{j,l} - A_{l,j}) \sqrt{|\det g_{rs}|}\,\omega_0. \tag{39}$$

In this formula $A_{i,k} = d_k A_i$ are *formal derivatives*. The Euler–Lagrange equations consist of two systems, the *Maxwell equations* and the *Einstein equations* whose left-hand side is the Einstein tensor E_{ij} (23) and the right-hand side is the variational energy-momentum tensor of the electromagnetic field.
3. An example of a *gauge natural* variational structure is provided by the Hilbert–Young–Mills Lagrangian (see e.g., Patak and Krupka [PK]).

6.3 Connections

We give in this section an example of a first-order natural Lagrange structure $(\mathscr{C}X, \lambda_{\mathscr{C}})$, whose underlying fibered manifold is *not* a tensor bundle.

Consider the vector space $Q = \mathbf{R}^n \otimes (\mathbf{R}^n)^* \otimes (\mathbf{R}^n)^*$ of tensors of type $(1, 2)$ on the vector space $\mathbf{R}^n$, with the canonical coordinates Γ^i_{jk}. We shall refer to Γ^i_{jk} as the *formal Christoffel symbols*. Q is endowed with a *nonlinear* left action of the differential group L_n^2, defined in charts by

$$\bar{\Gamma}^i_{jk} = a^i_p (b^q_j b^r_k \Gamma^p_{qr} + b^p_{jk}), \tag{40}$$

where a^i_j, a^i_{jk} are the canonical coordinates on L^2_n, and b^i_j, b^i_{jk} are functions on L^2_n defined by the formulas $a^i_p b^p_j = \delta^i_j$, $a^i_{pq} b^p_j + a^i_p a^s_q b^p_{js} = 0$. Note that this action is defined by the *transformation equations* for the components of a connection. For *any* n-dimensional manifold X, the left action (40) defines in a standard way a fiber bundle over X with type fiber Q, associated with the principal L^2_n-bundle of 2 frames $\mathscr{F}^2X$, denoted $\mathscr{C}X = \mathscr{F}^2_Q X$. We call this fiber bundle the *connection bundle*. Its sections are *connection fields*, or *connections* on the underlying manifold X. One can also assign to any diffeomorphism α of n-dimensional manifolds its lifting $\mathscr{F}^2\alpha$, an isomorphism of the corresponding bundles of 2 frames, and the associated lifting $\mathscr{F}^2_Q\alpha$, an isomorphism of the corresponding fiber bundles with type fiber $\mathscr{C}\alpha = T^1_n Q$. Then, the correspondence $X \to \mathscr{C}X$, $\alpha \to \mathscr{C}\alpha$ is a Q-lifting, associated with the 2 frame lifting $\mathscr{F}^2$ from the category $\mathscr{D}_n$ to $\mathscr{F}\mathscr{B}_n(L^2_n)$.

The notion of the connection bundle was introduced in this way for the *symmetric* tensor product $Q = \mathbf{R}^n \otimes ((\mathbf{R}^n)^* \odot (\mathbf{R}^n)^*)$ in the paper Krupka [K9], with the aim to study differential invariants of symmetric linear connections. The formal Christoffel symbols entering formula (40) are in general *not* symmetric.

Now the q-lifting $X \to \mathscr{C}X$, $\alpha \to \mathscr{C}\alpha$ induces in a standard way its r-jet prolongation liftings $X \to J^r\mathscr{C}X$, $\alpha \to J^r\mathscr{C}\alpha$ from $\mathscr{D}_n$ to $\mathscr{F}\mathscr{B}_n(L^{r+2}_n)$. In this example, we need the case $r = 1$. If X is a fixed n-dimensional manifold with some local coordinates (x^i) are some local coordinates on X, then the associated fibered coordinates on $\mathscr{C}X$ are (x^i, Γ^i_{jk}), and the associated coordinates on $J^1\mathscr{C}X$ are $(x^i, \Gamma^i_{jk}, \Gamma^i_{jk,l})$, where the coordinate functions $\Gamma^i_{jk,l}$ are defined by the formal derivative operator as $\Gamma^i_{jk,l} = d_l\Gamma^i_{jk}$.

Using these coordinates, we set

$$R_{ik} = \Gamma^s_{ik,s} - \Gamma^s_{is,k} + \Gamma^s_{ik}\Gamma^m_{sm} - \Gamma^m_{is}\Gamma^s_{km} \tag{41}$$

and

$$\lambda_{\mathscr{C}} = \sqrt{|\det R_{ij}|} \cdot \omega_0. \tag{42}$$

The system of functions R_{ik} is called the *formal Ricci tensor*, and $\lambda_{\mathscr{C}}$ is a global horizontal n-form, defined on the fibered manifold $J^1\mathscr{C}X$. Formula (42) concludes the construction of a natural Lagrange structure $(\mathscr{C}X, \lambda_{\mathscr{C}})$.

We show that the principal Lepage equivalent of the Lagrangian $\lambda_{\mathscr{C}}$ is given by

$$\Theta_{\mathscr{C}} = \sqrt{|\det R_{ij}|}\left(\omega_0 + \frac{1}{2}(R^{jk}\delta^l_i - R^{jl}\delta^k_i)\omega^i_{jk} \wedge \omega_l\right), \tag{43}$$

where

$$\begin{aligned} \omega_0 &= dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n, \\ \omega_l &= dx^1 \wedge dx^2 \wedge \cdots \wedge dx^{l-1} \wedge dx^{l+1} \wedge \cdots \wedge dx^n, \\ \omega^i_{jk} &= d\Gamma^i_{jk} - \Gamma^i_{jk,s} dx^s. \end{aligned} \tag{44}$$

Denote for further calculations $\upsilon = \det R_{rs}$ and $C = \sqrt{|\upsilon|}$. We shall consider the open set in the fibered manifold $J^1\mathscr{C}X$ defined by the condition $\upsilon \neq 0$. Differentiating we have

$$\begin{aligned} \frac{\partial C}{\partial \Gamma^i_{jk}} &= \frac{1}{2\sqrt{|\upsilon|}} \operatorname{sgn}\upsilon \frac{\partial \upsilon}{\partial R_{pq}} \frac{\partial R_{pq}}{\partial \Gamma^i_{jk}} = \frac{1}{2\sqrt{|\upsilon|}} \operatorname{sgn}\upsilon \cdot \upsilon \cdot R^{pq} \frac{\partial R_{pq}}{\partial \Gamma^i_{jk}} \\ &= \frac{\sqrt{|\upsilon|}}{2} R^{pq} \cdot (\delta^s_i \delta^j_p \delta^k_q \Gamma^m_{sm} + \delta^m_i \delta^j_s \delta^k_m \Gamma^s_{pq} - \Gamma^s_{qm} \delta^m_i \delta^j_p \delta^k_s - \Gamma^m_{ps} \delta^s_i \delta^j_q \delta^k_m) \\ &= \frac{\sqrt{|\upsilon|}}{2} (R^{jk} \Gamma^m_{\text{im}} + \delta^k_i R^{pq} \Gamma^j_{pq} - R^{jq} \Gamma^k_{qi} - R^{pj} \Gamma^k_{pi}), \end{aligned} \tag{45}$$

and

$$\begin{aligned} \frac{\partial C}{\partial \Gamma^i_{jk,l}} &= \frac{\sqrt{|\upsilon|}}{2} R^{pq} \frac{\partial R_{pq}}{\partial \Gamma^i_{jk,l}} = \frac{\sqrt{|\upsilon|}}{2} R^{pq} (\delta^s_i \delta^j_p \delta^k_q \delta^l_s - \delta^s_i \delta^j_p \delta^k_s \delta^l_q) \\ &= \frac{\sqrt{|\upsilon|}}{2} (R^{jk} \delta^l_i - R^{jl} \delta^k_i). \end{aligned} \tag{46}$$

Hence, the principal Lepage equivalent is

$$\Theta_{\mathscr{C}} = C\omega_0 + \frac{\partial C}{\partial \Gamma^i_{jk,l}} \omega^i_{jk} \wedge \omega_l = \sqrt{|\rho|} \left(\omega_0 + \frac{1}{2} (R^{jk} \delta^l_i - R^{jl} \delta^k_i) \omega^i_{jk} \wedge \omega_l \right) \tag{47}$$

as required.

Formula (43) can be used for explicit description of the properties of the variational functional

$$\Gamma_\Omega(\tau_X) \ni \Gamma \to \int_\Omega J^1\Gamma^* \lambda_{\mathscr{C}} = \int_\Omega J^1\Gamma^* \Theta_{\mathscr{C}} \in \mathbf{R}, \tag{48}$$

for connections Γ on an n-dimensional manifold X; in this formula, τ_X is the projection of the fibered manifold $\mathscr{C}X$ onto X. In particular, we can determine the *Euler–Lagrange form* $p_1 d\Theta_{\mathscr{C}}$ for extremal connections and the corresponding Noether currents. We do not analyze the resulting formulas here.

Remark A fundamental notion of the differential geometry of connections on a manifold X is the curvature tensor. From the point of view of the *variational geometry*, this notion can be represented by the *formal curvature tensor*

$$R^l_{ikj} = \Gamma^l_{ik,j} - \Gamma^l_{ij,k} + \Gamma^l_{ik}\Gamma^m_{jm} - \Gamma^m_{ij}\Gamma^l_{km}, \tag{49}$$

defined on the 1-jet prolongation $J^1\mathscr{C}X$ of the bundle of connections $\mathscr{C}X$. Note that the formal Ricci tensor (41) represents the trace of the formal curvature tensor (49) in the indices l and j; one can also consider a different variational functional for connection fields whose Lagrangian is based on the trace of R^l_{ikj} in the indices l and i, $\lambda = \sqrt{|\det R^s_{skj}|}\omega_0$.

References

[A1] I. Anderson, Natural variational principles on Riemannian manifolds, Annals of Mathematics 120 (1984) 329-370

[FFPW] M. Ferraris, M. Francaviglia, M. Palese, E. Winterroth, Gauge-natural Noether currents and connection fields, Int. J. of Geom. Methods in Mod. Phys. 01/2011; 8(1); 1-9

[H] D. Hilbert, Die Grundlagen der Physik, Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse (1915) 325-407

[KMS] I. Kolar, P. Michor, J. Slovak, *Natural Operations in Differential Geometry*, Springer-Verlag, Berlin, 1993

[K3] D. Krupka, A setting for generally invariant Lagrangian structures in tensor bundles, Bull. Acad. Polon. Sci. Ser. Sci. Math. Astronom. Phys. 22 (1974) 967-972

[K9] D. Krupka, Local invariants of a linear connection, in: Differential Geometry, Colloq. Math. Soc. Janos Bolyai 31, North Holland, 1982, 349-369

[K10] D. Krupka, Natural Lagrange structures, in: *Semester on Differential Geometry*, 1979, Banach Center, Warsaw, Banach Center Publications, 12, 1984, 185-210

[KJ] D. Krupka, J. Janyska, *Lectures on Differential Invariants*, J.E. Purkyne University, Faculty of Science, Brno, Czechoslovakia, 1990

[KL] D. Krupka, M. Lenc, The Hilbert variational principle, Preprint 3/200GACR (201/00/0724), Masaryk University, Brno, 2002, 75 pp.

[KT] D. Krupka, A. Trautman, General invariance of Lagrangian structures, Bull. Acad. Polon. Sci., Ser. Sci. Math. Astronom. Phys. 22 (1974) 207-211

[PK] A. Patak, D. Krupka, Geometric structure of the Hilbert-Yang-Mills functional, Internat. J. Geom. Met. Mod. Phys. 5 (2008) 387-405

[PW] M. Palese, E. Winterroth, A variational perspective on classical Higgs fields in gauge-natural theories, Theoretical and Mathematical Physics 10/2011; 168(1)

[UK3] Z. Urban, D. Krupka, Foundations of higher-order variational theory on Grassmann fibrations, Internat. J. of Geom. Methods in Modern Physics, 11 (2014); doi:10.1142/S0219887814600238

[Z] D. Zenkov, *The Inverse Problem of the Calculus of variations, Local and Global Theory and Applications*, Atlantic Series in Global Variational Geometry, to appear

Chapter 7
Elementary Sheaf Theory

The purpose of this chapter is to explain selected topics of the sheaf theory over paracompact, Hausdorff topological spaces. The choice of questions we consider are predetermined by the global variational theory over (topologically nontrivial) fibered manifolds, namely by the problem how to characterize differences between the local and global properties of the Euler–Lagrange mapping, between *locally* and *globally trivial Lagrangians*, and *locally and globally variational source forms*. To this purpose, the central topic we follow is the *abstract De Rham theorem* and its consequences. In particular, in the context of this book, the cohomology of abstract sheaves should be compared with the cohomology of the associated complexes of global sections, and the cohomology of underlying smooth manifolds.

This chapter requires basic knowledge of the point-set topology; to help the reader some parts of the topology of local homeomorphisms have been included. Our treatment, intended for larger audience of readers who are not specialists in algebraic topology and sheaf theory, includes all proofs and also their technical details, and from this point of view is wider than similar advanced texts in specialized monograph literature.

The main reference covering the choice of material needed in this book is Wells [We]; for different aspects of the sheaf theory, especially the cohomology, we also refer to Bott and Tu [BT], Bredon [Br], Godement [Go], Lee [L], and Warner [W].

7.1 Sheaf Spaces

Recall that a continuous mapping $\sigma\colon S \to X$ of a topological space S into a topological space X is called a *local homeomorphism*, if every point $s \in S$ has a neighborhood V such that the set $\sigma(V)$ is open set in X and the restricted mapping $\sigma|_V$ is a homeomorphism of V onto $\sigma(V)$.

By a *sheaf space structure* on a topological space S, we mean a topological space X together with a *surjective* local homeomorphism $\sigma\colon S \to X$. The topological space S endowed with a sheaf space structure is called a *sheaf space* or an *étalé*

D. Krupka, *Introduction to Global Variational Geometry*,
Atlantis Studies in Variational Geometry 1, DOI 10.2991/978-94-6239-073-7_7

space. X is the *base space*, and σ is the *projection* of the sheaf space S. For every point $x \in X$, the set $S_x = \sigma^{-1}(x)$ is called the *fiber* over x. We denote a sheaf space by $\sigma\colon S \to X$ or just by S when no misunderstanding may possibly arise.

A mapping $\gamma\colon Y \to S$, where Y is a subset of X, is called a *section* of the topological space S over Y (or more precisely, a section of the projection σ), if $\gamma(x) \in S_x$ for all points $x \in Y$. Obviously, γ is a section if and only if

$$\sigma \circ \gamma = \mathrm{id}_Y. \tag{1}$$

If $Y = X$, γ is a *global section*. The set of sections (resp. *continuous* sections), defined on a set U, is denoted by $(\mathrm{Sec}\, S)U$ (resp. $(\mathrm{Sec}^{(c)}\, S)U$, and also $\Gamma(U, S)$). The union of the sets $(\mathrm{Sec}\, S)U$ (resp. $(\mathrm{Sec}^{(c)}\, S)U$) through $U \subset X$ is denoted by $\mathrm{Sec}\, S$ (resp. $\mathrm{Sec}^{(c)}\, S$).

Lemma 1

(a) *A local homeomorphism is an open mapping.*
(b) *The restriction of a local homeomorphism to a topological subspace is a local homeomorphism.*
(c) *The composition of two local homeomorphisms is a local homeomorphism.*

Proof

(a) Let $\sigma\colon S \to X$ be a local homeomorphism. Any open subset V of S is expressible as tue union $\cup V_\iota$, where V_ι is an open set such that $\sigma|_{V_\iota}$ is a homeomorphism. Then, the set $\sigma(V) = \cup\sigma(V_\iota)$ must be open as the union of open sets.
(b) Let $T \subset S$ be a subspace and $V \subset S$ an open set such that $\sigma|_V$ is a homeomorphism. Then, $V \cap T = V \cap (\sigma|_V)^{-1}(\sigma(T)) = (\sigma|_V)^{-1}(\sigma(V) \cap \sigma(T))$, and $\sigma(V \cap T) = \sigma(V) \cap \sigma(T)$. Thus, the image of the open set $\sigma(V \cap T) \subset T$ by $\sigma|_T$ is open in $\sigma(T)$. Since $\sigma|_T|_{V\cap T} = \sigma|_{V\cap T}$ is a continuous bijection and is an open mapping hence a homeomorphism, $\sigma|_T|_{V\cap T}$ is a homeomorphism.
(c) The proof is immediate. □

Lemma 2 *Let S be a sheaf space with base X and projection σ.*

(a) *To every point $s \in S$, there exists a neighborhood U of the point $x = \sigma(s)$ in X and a continuous section $\gamma\colon U \to S$ such that $\gamma(x) = s$.*
(b) *Let γ be a continuous section of S, defined on an open subset of X. Then, to every point x from the domain of γ and every neighborhood V of $\gamma(x)$ such that $\sigma|_V$ is a homeomorphism, there exists a neighborhood U of X such that $\gamma(U) \subset V$ and $\gamma|_U = (\sigma|_V)^{-1}|_U$.*
(c) *If U and V are open sets in X and $\gamma\colon U \to S$ and $\delta\colon V \to S$ are continuous sections, then the set $\{x \in U \cap V | \gamma(x) = \delta(x)\}$ is open.*
(d) *Every continuous section of S, defined on an open set in X, is an open mapping.*

Proof

(a) We choose a neighborhood V of s such that $\sigma|_V$ is a homeomorphism and set $U = \sigma(V)$, $\gamma = (\sigma|_V)^{-1}$.

(b) By continuity of γ, we choose a neighborhood U of x such that $\gamma(U) \subset V$, and apply the mapping $\gamma = (\sigma|_V)^{-1}$ to both sides of the identity $\sigma|_V \circ \gamma|_U = \mathrm{id}_U$. We get $\gamma|_U = (\sigma|_V)^{-1}$.

(c) We may suppose that $\{x \in U \cap V | \gamma(x) = \delta(x)\} \neq \emptyset$. Choose a point $x_0 \in U \cap V$, and a neighborhood W of the point $\gamma(x) = \delta(x)$ such that $\sigma(W)$ is open and $\sigma|_W$ is a homeomorphism. By condition (b), x_0 has a neighborhood U_0 such that $\gamma(U_0) \subset V$ and $\gamma|_{U_0} \subset (\sigma|_V)^{-1}|_{U_0}$. Analogously x_0 has a neighborhood of V_0 such that $\delta(V_0) \subset W$ and $\delta|_{V_0} \subset (\sigma|_W)^{-1}|_{V_0}$. Thus, $\gamma|_{U_0 \cap V_0} \subset (\sigma|_W)^{-1}|_{U_0 \cap V_0} = \delta|_{U_0 \cap V_0}$ proving (c).

(d) Let U be an open set in X, γ: $U \to S$ a continuous section. It is sufficient to show that the set $\gamma(U) \subset S$ is open. To every point $x \in U$, we assign a neighborhood $V_{\gamma(x)}$ of the point $\gamma(x)$ such that $\sigma(V_{\gamma(x)})$ is open and the mapping $\sigma|_{V_{\gamma(x)}}$ is a homeomorphism, and a neighborhood U_x of the point x such that $U_x \subset U$, $\gamma(U_x) \subset V_{\gamma(x)}$, and $\gamma|_{U_x} = (\sigma_{\gamma(x)})^{-1}|_{U_x}$ (see Part (b) of this lemma). Then since $(\sigma_{\gamma(x)})^{-1}$: $\sigma(V_{\gamma(x)}) \to V_{\gamma(x)} \subset S$ is a homeomorphism, $\gamma(U_x)$ is open in S, and we have $\gamma(U) = \gamma(\cup U_x) = \cup\gamma(U_x)$, which is an open set. □

Remark 1 Suppose that S is a *Hausdorff* space. Let γ: $U \to S$ and δ: $V \to S$ be two continuous sections, defined on open sets U and V in X, such that $U \cap V \neq \emptyset$ and $\gamma(x_0) = \delta(x_0)$ at a point $x_0 \in U \cap V$. Then, $\gamma = \delta$ on the connected component of $U \cap V$ containing x_0. Indeed, since S is Hausdorff, the set $U_0 = \{x \in U \cap V | \gamma(x) = \delta(x)\}$ is closed. Since by Lemma 2, (c) the set U_0 is open, it must be equal to the connected component of the point x_0. This remark shows that if a sheaf space S is Hausdorff, it satisfies the *principle of analytic continuation*. On the other hand, if the principle of analytic continuation is not valid, S cannot be Hausdorff.

Suppose that we have a *set* S, a topological space X, and a mapping σ: $S \to X$. Then, there exists at most one topology on S for which σ is a local homeomorphism. Indeed, if τ_1 and τ_2 are two such topologies, $s \in S$ a point, $V \in \tau_1$ and $W \in \tau_2$ its neighborhood: such that $\sigma|_V$ and $\sigma|_W$ are homeomorphisms, then $U = \sigma(V) \cap \sigma(W)$ is a neighborhood of the point, $x = \sigma(s)$ and $\sigma^{-1}(U)$ is a neighborhood of the point s both in τ_1 and τ_2. This implies, in particular, that the identity mapping id_S is a homeomorphism.

Let S be a sheaf space with base X and projection σ. Beside its own topology, the set S may be endowed with the *final topology*, associated with the family of continuous sections, defined on open subsets of X.

Lemma 3 *Let S be a sheaf space with base X and projection σ.*

(a) *The open sets* $V \subset S$ *such that* $\sigma|_V$ *is a homeomorphism form a basis of the topology of S.*
(b) *The topology of S coincides with the final topology, associated with the set* $\mathrm{Sec}^{(c)}\, S$ *of continuous sections of S.*
(c) *The topology induced on fibers of S is the discrete topology.*

Proof

(a) This is an immediate consequence of the definition of a local homeomorphism.
(b) If a subset W of S is an open set in the topology of S, then for every continuous section γ of S, $\gamma^{-1}(W)$ is an open subset of X hence by definition, W is open in the final topology. Conversely, let W be open in the final topology. For any section $\gamma: U \to S$, $\gamma(\gamma^{-1}(W)) \subset W \cap \gamma(U) \subset W$. If the section γ is continuous, then by the definition of the final topology, $\gamma^{-1}(W)$ is an open set; moreover, since γ is open in the topology of S (Lemma 2, (d)), the set $\gamma(\gamma^{-1}(W))$ is open in the topology of S. But by Lemma 2, (a), the sets $\gamma(\gamma^{-1}(W))$ cover W which implies that W is open in the topological space S.
(c) This assertion is evident.

Let $\sigma: S \to X$ and $\tau: T \to Y$ be two sheaf spaces. Recall that a mapping $f: S \to T$ is said to be *projectable*, if

$$\tau \circ f = f_0 \circ \sigma \tag{2}$$

for some mapping $f_0: X \to Y$. Obviously, the same can be expressed by saying that there exists f_0 such that the diagram

$$\begin{array}{ccc} S & \xrightarrow{f} & T \\ \downarrow \sigma & & \downarrow \tau \\ X & \xrightarrow{f_0} & Y \end{array} \tag{3}$$

commutes. If f_0 exists, it follows from condition (2) that it is unique. If f is continuous, then the mapping f_0 is also continuous since it is always expressible on open sets as $f_0 = \tau \circ f \circ \gamma$ for some continuous sections γ of the topological space S. □

A continuous projectable mapping $f: S \to T$ is called a *morphism* of the sheaf space S into the sheaf space T, or just a *sheaf space morphism*.

Lemma 4 *Let* $\sigma: S \to X$ *and* $\tau: T \to Y$ *be sheaf spaces,* $f: S \to T$ *a surjective mapping and* $f_0: X \to Y$ *its projection. Then, f is a local homeomorphism if and only if* f_0 *is a local homeomorphism.*

Proof Let $x \in X$ be a point, γ a continuous section of S defined on a neighborhood of x. Choose a neighborhood W of the point $f(\gamma(x))$ such that $\tau|_W$ is a homeomorphism, a neighborhood V of $\gamma(x)$ such that $f(V) \subset W$, and a neighborhood U of x such that $U \subset \sigma(V)$ and $\gamma|_U$ is a homeomorphism. Then $\tau|_W \circ f|_V \circ \gamma|_U = (\tau \circ f \circ \gamma)|_U$, and from condition (2), $(\tau \circ f \circ \gamma)|_U = (f_0 \circ \sigma \circ \gamma)|_U = f_0|_U$ proving Lemma 4. □

Denote by f_x the *restriction* of a mapping $f\colon S \to T$ to the fiber S_x over a point $x \in X$. If $X = Y$, we have the following assertion.

Corollary 1 *Let $\sigma\colon S \to X$ and $\tau\colon T \to X$ be two sheaf spaces, and let $f\colon S \to T$ be a projectable mapping whose projection is the identity mapping* id_X.

(a) *f is a local homeomorphism.*
(b) *f is injective (resp. surjective) if and only if f_x is injective (resp. surjective) for each $x \in X$.*

Proof

(a) This follows from Lemma 4.
(b) These assertions follow immediately from the definitions. □

Let $\sigma\colon S \to X$ and $\tau\colon T \to Y$ be two sheaf spaces. The Cartesian product $S \times T$ together with the mapping $\sigma \times \tau\colon S \times T \to X \times Y$ defined by the formula $(\sigma \times \tau)(s,t) = (\sigma(s), \tau(t))$ is a sheaf space, called the *product* of S and T. If $X = Y$, then we define a subset of the Cartesian product $S \times T$ by $S \times_X T = \{(s,t) \in S \times T | \sigma(t) = \tau(s)\}$, and a mapping $\sigma \times_X \tau\colon S \times_X T \to X$ by $(\sigma \times_X \tau)(s,t) = \sigma(s) = \tau(t)$. If we consider the set $S \times_X T$ with the induced topology, the mapping $\sigma \times_X \tau$ defines on $S \times_X T$ the structure of a sheaf space, called the *fiber product* of the sheaf spaces S and T.

Let $\sigma\colon S \to X$, $\sigma'\colon S' \to X$ and $\tau\colon T \to Y$, $\tau'\colon T' \to Y$ be sheaf spaces. Let $f\colon S \to T$ and $f'\colon S' \to T'$ be two projectable mappings over the same projection $f_0\colon X \to Y$. For every point (s,s') we define a mapping $f \times_X f'\colon S \times S' \to T \times T'$ by $(f \times_X f')(s,s') = (f(s), f'(s'))$. This gives rise to the following commutative diagram

$$\begin{array}{ccc} S \times_X S' & \xrightarrow{\iota} & S \times S' \\ \downarrow f \times_X f' & & \downarrow f \times f' \\ T \times_Y T' & \xrightarrow{\kappa} & T \times T' \end{array} \tag{4}$$

where the horizontal arrows denote the canonical inclusions. The mapping $f \times_X f'$ is called the *fiber product* of f and f'. It is easily seen that if f and f' are *continuous*, then the fiber product $f \times_X f'$ is also continuous: indeed, for any open set U in $T \times_Y T'$, there exists an open set V in $T \times T'$ such that $U = \kappa^{-1}(V)$; since

$$(f \times_X f')^{-1}(U) = (f \times_X f')^{-1}(\kappa^{-1}(V))$$
$$= (\kappa \circ (f \times_X f'))^{-1}(V) = ((f \times f') \circ \iota)^{-1}(V) \tag{5}$$

is an open set in $S \times_X S'$, the mapping $f \times_X f'$ must be continuous.

We give some examples of sheaf spaces; using these examples we also discuss properties of the topology of sheaf spaces.

Examples

1. Continuous global sections of a sheaf space need not necessarily exist. Consider for example the *real line* $\mathbf{R} = \mathbf{R}^1$ and the *unit circle* $S^1 = \{(x, y) \in \mathbf{R}^2 | x^2 + y^2 = 1\}$. The mapping $\sigma: \mathbf{R} \to S^1$, defined by the formula $\sigma(s) = (\cos 2\pi s, \sin 2\pi s)$ is a surjective local homeomorphism. It is easily seen that σ has *no* continuous global section. Suppose the opposite. Then, if γ is a continuous global section, $\gamma(S^1) \subset \mathbf{R}$ is a non-void compact and open set in $\mathbf{R}$ hence coincides with $\mathbf{R}$. However, this is a contradiction since $\mathbf{R}$ is non-compact.
2. Let $S^2 = \{(x, y) \in \mathbf{R}^3 | x^2 + y^2 + z^2 = 1\}$ be the unit sphere in $\mathbf{R}^3$, and consider an equivalence relation $\sim$ on S^2 "$(x, y, z) \sim (x', y', z')$ *if either* $(x, y, z) = (x', y', z')$ *or* $(x, y, z) \sim -(x', y', z')$." The quotient space $S^2/\sim$ is called the *real projective plane* and is denoted by RP^2. The quotient projection $\sigma: S^2 \to RP^2$ is a sheaf space. The set RP^2 can be identified with the set of straight lines in $\mathbf{R}^3$ passing through the origin.
3. A local homeomorphism admitting a *global* continuous section is *not* necessarily a homeomorphism: Define a subspace $S = \{(x, r) \in \mathbf{R}^2 | r = 0, 1\}$ of $\mathbf{R}^2$ and a mapping $\sigma: S \to \mathbf{R}$ by the condition $\sigma(x, r) = x$. Then, the mapping $\gamma: \mathbf{R} \to S$ defined by $\gamma(x, 0) = x$ is a global continuous section of S but σ is not a homeomorphism.
4. Consider the subspace $S = \{(x, r) \in \mathbf{R}^2 | r = -1, 1\}$ of $\mathbf{R}^2$, two points $a, b \in \mathbf{R}$ such that $a < b$, and a partitions of S defined by the subsets $\{(x, -1)\}$, $\{(x, 1)\}$ if $x \leq a$, $x \geq b$, and $\{(x, -1), (x, 1)\}$ if $a < x < b$ (one- and two-element subsets). Let $\sim$ be an equivalence relation on S defined by this partition and denote $X = X/\sim$. The quotient mapping of S onto X is a surjective local homeomorphism; the quotient space X is *not* Hausdorff. Further, assigning to each of the sets $\{(x, -1)\}$, $\{(x, 1)\}$, $\{\{x, -1\}, \{x, -1\}\}$ the point $x \in \mathbf{R}$, we obtain a local homeomorphim of X onto the real line $\mathbf{R}$.
5. The topological subspace S of $\mathbf{R}^3$, defined in a parametric form as $S = \{(x, y, z) \in \mathbf{R}^3 | x = \cos t, y = \sin t, z = t, t \in \mathbf{R}\}$ (the *helix*), together with the restriction of the Cartesian projection $\pi: \mathbf{R}^3 \to \mathbf{R}^2$ to S is a local homeomorphism of S onto the circle S^1 (Example 1). This example shows that for a general local homeomorphism $\sigma: S \to X$ the topology of S does *not* necessarily coincide with the *initial* topology of the topology of X by the mapping σ.

6. If σ: $S \to X$ is a sheaf space and Y is an open subset of X, then the restriction $\sigma|_{\sigma^{-1}(Y)}$: $\sigma^{-1}(Y) \to Y$ is a sheaf space.
7. The Cartesian projection π: $X \times Q \to X$, where X is a topological space and Q is a non-void set endowed with the *discrete* topology, is a sheaf space.
8. Using the notation of Example 1, we obtain a surjective local homeomorphism $\sigma \times \sigma$ of the real plane $\mathbf{R}^2$ onto the *torus* $S^1 \times S^1$.

7.2 Abelian Sheaf Spaces

An *Abelian sheaf space structure* on a topological space S consists of a sheaf space structure with base X and projection σ such that for every point $x \in X$ the fiber S_x over x is an Abelian group and the subtraction mapping $S \times_X S \ni (s, t) \to s - t \in S$ is continuous. A topological space S, endowed with an Abelian sheaf space structure is called an *Abelian sheaf space*. We usually denote an Abelian sheaf space σ: $S \to X$, or simply by S. Sometimes, when no misunderstanding may arise, we call an Abelian sheaf space just a *sheaf space*.

A *sheaf subspace* of the Abelian sheaf space S is an open set $T \subset S$ such that for every point $x \in X$, the intersection $T \cap S_x$ is a subgroup of the Abelian group S_x.

The Abelian sheaf space structure on a topological space S induces the Abelian group structure on sections of S. The *zero section* is the mapping θ: $X \to S$, assigning to a point $x \in X$ the neutral element of the Abelian group S_x. Clearly, θ is a *global continuous section* of S: If $x_0 \in X$ is a point and γ is any continuous section over a neighborhood U of x_0, then $\theta(x) = \gamma(x) - \gamma(x)$ on U, which implies that θ is expressible as the composition of two continuous mappings $U \ni x \to (\gamma(x), \gamma(x)) \in S \times_X S$ and $S \times_X S \ni (s, t) \to s - t \in S$. The open set $\theta(X)$ is called the *zero sheaf subspace* of S. For any two sections γ and δ, defined on the same set in X, one can naturally define the *sum* $\gamma + \delta$ and the *opposite* $-\gamma$ of the section γ. Thus, the set of sections over an open subset of X has an Abelian group structure. If the sections γ and δ are continuous, then $\gamma + \delta$ and $-\gamma$ are also continuous.

For any subspace Y of the base space X, the restriction of the projection σ to the set $\sigma^{-1}(Y)$ is a sheaf subspace of the Abelian sheaf space S with base Y, called the *restriction* of S to Y.

Remark 2 If a local homeomorphism admits an Abelian sheaf space structure, then it necessarily admits a continuous global section (the zero section). Conversely, local homeomorphisms, which do not admit a global continuous section, do not admit an Abelian sheaf space structure.

Examples

9. In this example we construct a sheaf space of Abelian groups, the *skyscraper sheaf space*, whose topology is *not* Hausdorff. Denote by $\mathbf{Z}$ the set of integers in the set of real numbers $\mathbf{R}$. Let X be a Hausdorff space, x_0 a point of X, and let

S be a subset of the Cartesian product $X \times \mathbf{Z}$, defined as $S = (X \backslash \{x_0\}) \times \{0\}) \cup (\{x_0\} \times \mathbf{Z})$. The subsets of S of the form $U \times \{x_0\}$, where U is an open set in X and $\{x_0\} \notin U$, and $((V \backslash \{x_0\} \times \{0\}) \cup \{(x_0, z)\}$, where V is open in X, $x_0 \in V$ and $z \in \mathbf{Z}$, is a basis for a topology on S. In this topology, the restriction of the first Cartesian projection is a local homeomorphism of S onto X. For any two different points $z_1, z_2 \in \mathbf{Z}$, every neighborhood of the point $(x_0, z_1) \in S$ (resp. $(x_0, z_2) \in S$) contains a neighborhood $((V_1 \backslash \{x_0\} \times \{0\}) \cup \{(x_0, z_1)\}$ of the point $(x_0, z_1) \in S$ (resp. $((V_2 \backslash \{x_0\} \times \{0\}) \cup \{(x_0, z_2)\}$ of $(x_0, z_2) \in S$), whose intersection is $((V_1 \cap V_2) \backslash \{x_0\}) \times \{0\}$. Assuming $(V_1 \cap V_2) \backslash \{x_0\} = \varnothing$, we get a neighborhood $V_1 \cap V_2$ of $\{x_0\}$ equal to $\{x_0\}$. Thus, if $\{x_0\}$ is *not* an isolated point, S is *not* Hausdorff.

10. The restriction of the Cartesian projection π: $\mathbf{R}^3 \to \mathbf{R}^2$ to the helix (Sect. 7.1, Example 5) is a surjective local homeomorphism of S onto the unit circle S^1. This local homeomorphism cannot be endowed with a sheaf structure because it does not admit a continuous global section.
11. Consider a topological space X and an Abelian group G with *discrete topology*. The Cartesian product $X \times G$, endowed with the product topology, and the first Cartesian projection is a sheaf space, called the *constant sheaf space* over *X with fiber G*. We usually denote this sheaf by G_X. If U is an open set in X and γ: $U \to G_X$ a continuous section, then the restriction of γ to any connected open subset V of U is constant, that is, of the form $V \ni x \to \gamma(x) = (x, g) \in G_X$ for some $g \in G$. Since the continuous image of a connected subspace is connected, the second Cartesian projection $\mathrm{pr}_2 \circ \gamma(V) \in G$ consists of a single point. In particular, every continuous section of a constant sheaf space is constant on connected components of the base, that is, *locally constant*.
12. The *trivial sheaf space of Abelian groups* over a topological space X is defined as X together with the identity homeomorphism id_X: $X \to X$, and trivial Abelian group structure on every fiber $\{x\} = {\mathrm{id}_X}^{-1}(x)$. Thus, the trivial sheaf space is the sheaf space 0_X.
13. Let T be a sheaf space of Abelian groups with base X and projection τ, and let R and S be two sheaf subspaces of T. For every point $x \in X$, $R_x + S_x$ is a subgroup of the Abelian group T_x. We set

$$R + S = \bigcup_{x \in X} (R_x + S_x). \tag{6}$$

$R + S$ is an open subset of T: if $t \in R + S$, then $t = r + s$, where $r \in R$ and $s \in S$, and because R (resp. S) is a sheaf subspace of T, r (resp. s) has a neighborhood U (resp. V) in R (resp. S) such that τ restricted to U (resp. V) is a homeomorphism. But both R and S are open in T. Thus, $U + V$ is open in T, proving that $R + S$ is open in T. Therefore, $R + S$ is a sheaf subspace of T. We call this subspace the *sum* of R and S.

Let S and T be two Abelian sheaf spaces over a topological space X, σ and τ the corresponding projections. A projectable continuous mapping $f: S \to T$ over the identity mapping id_X is called a *morphism of Abelian sheaf spaces*, if for every point $x \in X$ the restriction $f_x = f|_{\sigma^{-1}(x)}$ to the fiber over x is a morphism of Abelian groups. A morphism $f: S \to T$ of Abelian sheaf spaces such that both f and f^{-1} are bijections, is called an *isomorphism* of Abelian sheaf spaces. The mapping id_S is the *identity morphism* of S. To simplify terminology, we sometimes call morphisms of Abelian sheaf spaces just *morphisms of sheaf spaces*, of *sheaf space morphisms*.

The composite $f \circ g$ of two morphisms of Abelian sheaf spaces is again a morphism of Abelian sheaf spaces.

Consider a sheaf space morphism $f: S \to T$ and set

$$\mathrm{Ker}\, f = \{s \in S | f(s) = 0\}, \quad \mathrm{Im}\, f = f(S). \tag{7}$$

Obviously, these sets can be expressed as

$$\mathrm{Ker}\, f = \bigcup_{x \in X} \mathrm{Ker}\, f_x, \quad \mathrm{Im}\, f = \bigcup_{x \in X} \mathrm{Im}\, f_x. \tag{8}$$

Lemma 5 *Let S and T be two Abelian sheaf spaces over a topological space X with projections σ and τ, $f: S \to T$ a sheaf space morphism.*

(a) $\mathrm{Ker}\, f$ *is a sheaf subspace of S.*
(b) $\mathrm{Im}\, f = f(S)$ *is a sheaf subspace of T.*

Proof

(a) Since $\mathrm{Ker}\, f = f^{-1}(0(X))$, where $0(X)$ is the zero sheaf subspace of T, which is an open set in T, the set $\mathrm{Ker}\, f$ is open in S. Since $\sigma(\mathrm{Ker} f) = X$ and for each $x \in X$, $\mathrm{Ker} f \cap S_x$ is a subgroup of S_x, $\mathrm{Ker}\, f$ is a sheaf subspace of S.
(b) By Lemma 1, (b), the restriction of the projection τ to $f(S)$ is a local homeomorphism. The image of $\tau|_{f(S)}$ is given by $\tau(f(S)) = \sigma)S) = X$. For each point $x \in X$, the set $f(S) \cap T_x$ is a subgroup of T_x. The commutative diagram

$$\begin{array}{ccc} f(S) \times_X f(S) & \longrightarrow & T \times_X T \\ \downarrow & & \downarrow \\ f(S) & \longrightarrow & T \end{array} \tag{9}$$

in which the horizontal arrows are inclusions and the vertical arrows are subtractions (in fibers), shows that the subtractions $f(S) \times_X f(S) \to f(S)$ are continuous. □

The sheaf subspace $\mathrm{Ker}\, f$ (resp. $\mathrm{Im}\, f$) is called the *kernel* (resp. *image*) of the morphism of Abelian sheaf spaces $f: S \to T$.

Let $\sigma: S \to X$ be a sheaf space, T a sheaf subspace of S. Consider an equivalence relation on S "*$s_1 \sim s_2$ if $\sigma(s_1) = \sigma(s_2)$ and $s_1 - s_2 \in T$*." Let S/T be the quotient space (endowed with the quotient topology), and let ρ denote the quotient projection; if $[s]$ is the class of an element $s \in S$, then $\rho(s) = [s]$. Define a mapping $\tau: S/T \to X$ by $\tau([s]) = \sigma(s)$. Since ρ is surjective, τ is a unique mapping such that

$$\tau \circ \rho = \sigma. \tag{10}$$

Since the composite $\tau \circ \rho = \sigma$ is continuous, τ is also continuous.

Note that for every point $x \in X$ the fiber $\tau^{-1}(x) = (S/T)_x = S_x/T_x$ has the structure of an Abelian group. We wish to show that the quotient S/T has the structure of a sheaf space over X with projection τ, and ρ is a morphism of Abelian sheaf spaces.

It is easily seen that the quotient mapping is open. Let $V \subset S$ be an open set. To show that $\rho(V)$ is open in the quotient topology means to show that $V' = \rho^{-1}(\rho(V))$ is open in the topology of S. But $V' = V + (\sigma|_T)^{-1}(\sigma(V))$. Since through every point of T passes a continuous section, defined on an open subset of $\sigma(V)$, the set V' is expressible as a union of open sets arising as images of continuous sections (Lemma 2, (d)). Thus, ρ is open.

We show that ρ is a local homeomorphism. Clearly, if $s \in S$ is a point and V is its neighborhood such that $\sigma|_V$ is a bijection, then $\sigma|_V = \tau|_W \circ \rho|_V$, where $W = \rho(V)$; since $\rho|_V: V \to W$ is surjective, both $\tau|_W$ and $\rho|_V$ must be bijective. Hence, $(\sigma|_V)^{-1} \circ \tau|_W \circ \rho|_V = \mathrm{id}_V$. Thus, we have the identity $(\sigma|_V)^{-1} = (\rho|_V)^{-1} \circ (\tau|_W)^{-1}$ and $\rho|_V \circ (\sigma|_V)^{-1} \circ \tau|_W = \mathrm{id}_W$. But W is open since the quotient mapping ρ is open and $(\rho|_V)^{-1} = (\sigma|_V)^{-1} \circ \tau|_W$, which is a continuous mapping. This proves that $\rho|_V$ is a homeomorphism. Now it is easy to conclude that the mapping τ is a local homeomorphism: We take the sets W and V as above and write $\tau|_W = \sigma|_V \circ (\rho|_V)^{-1}$.

It remains to check that the subtraction in S/T is continuous. We have a commutative diagram

$$\begin{array}{ccc} S \times_X S & \xrightarrow{\varphi} & S \\ \downarrow \rho \times_X \rho & & \downarrow \rho \\ (S/T) \times_X (S/T) & \xrightarrow{\psi} & S/T \end{array} \tag{11}$$

in which φ denotes the mapping $(s_1, s_2) \to s_1 - s_2$ and ψ is the mapping $([s_1], [s_2]) \to [s_1 - s_2])$, and $\rho \times_X \rho$ is the fiber product. But ρ, φ and $\rho \times_X \rho$ are local homeomorphisms, so from Lemma 4 we conclude that ψ is also a local homeomorphism.

The Abelian sheaf space S/T is called the *quotient sheaf space* of the sheaf space S by T. The morphism of Abelian sheaf spaces $\rho: S \to S/T$ is the *quotient projection*.

7.3 Sections of Abelian Sheaf Spaces

Suppose that we have an Abelian sheaf space S with base X and projection σ. Consider the correspondence $U \to \mathrm{Sec}^{(c)}\ U$, denoted by $\mathrm{Sec}^{(c)}$, assigning to every non-empty open set U in X the Abelian group $\mathrm{Sec}^{(c)}\ U$ of continuous sections over U. We extend this correspondence to the whole topology of X by assigning to the empty set $\varnothing$ the trivial one-point Abelian group 0. To any open sets U, V in X such that $U \subset V$ we assign a group morphism s_{VU}: $(\mathrm{Sec}^{(c)}\ S)V \to (\mathrm{Sec}^{(c)}\ S)U$ defined by

$$s_{VU} \circ \gamma = \gamma|_U \tag{12}$$

(the *restriction* of the continuous section γ to the set U). We get a family $\{(\mathrm{Sec}^{(c)}S)U\}$, labeled by the set U, and a family $\{s_{VU}\}$, labeled by the sets U and V. s_{VU} are called *restriction mappings*, or *restrictions* of the Abelian sheaf space S.

We say that two continuous sections $\gamma, \delta \in (\mathrm{Sec}^{(c)}\ S)U$ *coincide locally*, if there exists an open covering $\{U_\iota\}_{\iota \in I}$ of U such that $s_{UU_\iota}(\gamma) = s_{UU_\iota}(\delta)$ for each ι from the indexing set I. A family $\{\gamma_\iota\}_{\iota \in I}$ of continuous sections $\gamma_\iota \in (\mathrm{Sec}^{(c)}\ S)U_\iota$ is said to be *compatible*, if $s_{U_\iota, U_\iota \cap U_\kappa}(\gamma_\iota) = s_{U_\kappa, U_\iota \cap U_\kappa}(\gamma_\kappa)$ for all indices $\iota, \kappa \in I$. We say that the family of sections $\{\gamma_\iota\}_{\iota \in I}$ *locally generates* a section $\gamma \in (\mathrm{Sec}^{(c)}\ S)U$, where $U = \cup U_\iota$, if $s_{UU_\iota}(\gamma) = \gamma_\iota$ for all $\iota \in I$; we also say that γ is *locally generated* by the family $\{\gamma_\iota\}_{\iota \in I}$. A family of continuous sections, locally generating a continuous section, is compatible.

The following are basic properties of the restriction mappings s_{VU} and the Abelian groups $(\mathrm{Sec}^{(c)}\ S)U$.

Lemma 6 *The correspondence* $\mathrm{Sec}^{(c)}\ S$ *has the following properties*:

(1) $(\mathrm{Sec}^{(c)}\ S)\varnothing = 0$.
(2) $s_{UU} = \mathrm{id}_U$ *for every open set U in X.*
(3) $s_{WU} = s_{VU} \circ s_{WV}$ *for all open sets U, V, W such that $U \subset V \subset W$.*
(4) *If two continuous sections γ and δ coincide locally, then $\gamma = \delta$.*
(5) *Every compatible family of continuous sections of S locally generates a continuous section of S.*

Proof (1) holds by definition, and assertions (2) and (3) are immediate. We prove condition (4). Let $\{U_\iota\}_{\iota \in I}$ be a family of open sets in X, $U = \cup U_\iota$, $\gamma_1, \gamma_2 \in (\mathrm{Sec}^{(c)}\ S)U$ two sections such that the restrictions satisfy $\gamma_1|_{U_\iota} = \gamma_2|_{U_\iota}$ for all ι. Let $x \in U$. Then by hypothesis, there exists an index ι such that $x \in U_\iota$; consequently, $\gamma_1(x) = \gamma_1|_{U_\iota}(x) = \gamma_2|_{U_\iota}(x) = \gamma_2(x)$, and since the point x is arbitrary, we have $\gamma_1 = \gamma_2$ proving (4). Now we prove condition (5). Let $\{\gamma_\iota\}_{\iota \in I}$ be a family such that $\gamma_\iota \in (\mathrm{Sec}^{(c)}\ S)U_\iota$ and $\gamma_\iota|_{U_\iota \cap U_\kappa} = \gamma_\kappa|_{U_\iota \cap U_\kappa}$ for all indices $\iota, \kappa \in I$. Let $x \in U$ be a point. Then, there exists an index ι such that $x \in U_\iota$; we choose ι and set $\gamma(x) = \gamma_\iota(x)$. If also $x \in U_\kappa$, then $\gamma_\iota|_{U_\iota \cap U_\kappa}(x) = \gamma_\kappa|_{U_\iota \cap U_\kappa}(x)$ hence $\gamma(x) = \gamma_\kappa(x)$, so the value $\gamma(x)$ is defined independently of the choice of the index ι. It follows from the definition that γ, defined in this way, is continuous on U_ι for every ι hence on U, thus, $\gamma \in (\mathrm{Sec}^{(c)}\, S)U$ proving (5). $\square$

The correspondence $\mathrm{Sec}^{(c)}\, S$, assigning to an open set $U \subset X$ the Abelian group $(\mathrm{Sec}^{(c)}\, S)U$, is called the *sheaf of continuous sections* of the Abelian sheaf space S, or just the *Abelian sheaf, associated* with S.

Let $\sigma\colon S \to X$ and $\tau\colon T \to X$ be two Abelian sheaf spaces over the same base space X, $f\colon S \to T$ a sheaf space morphism. Consider the associated Abelian sheaves $\mathrm{Sec}^{(c)}\, S$ and $\mathrm{Sec}^{(c)}\, T$, and denote by $\{s_{VU}\}$ and $\{t_{VU}\}$ the corresponding families of restrictions in these sheaves. If γ is a continuous section of S, $\gamma \in (\mathrm{Sec}^{(c)}\, S)U$, then $f \circ \gamma \in (\mathrm{Sec}^{(c)}\, T)U$. Setting

$$\mathsf{f}_U(\gamma) = f \circ \gamma, \tag{13}$$

we obtain an Abelian group morphism $\mathsf{f}_U\colon (\mathrm{Sec}^{(c)}\, S)U \to (\mathrm{Sec}^{(c)}\, T)U$. Obviously, for every pair of open sets $U, V \subset X$ such that $U \subset V$, the diagram

$$\begin{array}{ccc} (\mathrm{Sec}^{(c)}\, S)V & \xrightarrow{\mathsf{f}_V} & S \\ \downarrow s_{VU} & & \downarrow t_{VU} \\ (\mathrm{Sec}^{(c)}\, S)U & \xrightarrow{\mathsf{f}_U} & S/T \end{array} \tag{14}$$

commutes. The family $\mathsf{f} = \{\mathsf{f}_U\}$, labeled by U, is called the *Abelian sheaf morphism* of the sheaf $\mathrm{Sec}^{(c)}\, S$ into the sheaf $\mathrm{Sec}^{(c)}\, T$, *associated* with the Abelian sheaf space morphism $f : S \to T$. We usually denote the associated Abelian sheaf morphism by $\mathsf{f}\colon \mathrm{Sec}^{(c)}\, S \to \mathrm{Sec}^{(c)}\, T$.

Now we study the sheaves associated with a sheaf subspace of an Abelian sheaf space, and the sheaves associated with the kernel and the image of an Abelian sheaf space morphism. Recall that the kernel $\mathrm{Ker}\, f$ and the image $\mathrm{Im}\, f$ of a sheaf space morphism $f\colon S \to T$ is a sheaf subspace of S and T, respectively.

Lemma 7

(a) *S is a sheaf subspace of an Abelian sheaf space T if and only if the Abelian group $(\mathrm{Sec}^{(c)}\, S)U$ is a subgroup of $(\mathrm{Sec}^{(c)}\, T)U$ for every open set U in X.*

(b) *Let $\sigma\colon S \to X$ and $\tau\colon T \to X$ be two Abelian sheaf spaces, $f\colon S \to T$ an Abelian sheaf space morphism, and let $\gamma \in (\mathrm{Sec}^{(c)}\, S)U$. Then $\gamma \in (\mathrm{Sec}^{(c)} \mathrm{Ker} f)U$ if and only if $\mathsf{f}_U(\gamma) = 0$.*

(c) *Let $\sigma\colon S \to X$ and $\tau\colon T \to X$ be two Abelian sheaf spaces, let $f\colon S \to T$ be a sheaf space morphism, and let $\delta \in (\mathrm{Sec}^{(c)}\, T)U$ be a continuous section. Then $\delta \in (\mathrm{Sec}^{(c)} \mathrm{Im}\, f)U$ if and only if it is locally generated by a family of continuous sections $\{\mathsf{f}_{U_\iota}(\gamma_\iota)\}_{\iota\in I}$, where $\gamma_\iota \in (\mathrm{Sec}^{(c)}\, S)U_\iota$, and the family $\{U_\iota\}_{\iota\in I}$ is an open covering of U.*

Proof

(a) If S is a sheaf subspace of T, then S is open in the sheaf space T, and $S_x = S \cap T_x \subset T_x$ is a subgroup for every $x \in X$. If $\gamma \in (\mathrm{Sec}^{(c)}\, S)U$, then γ is continuous in T because S is open. Thus, $\gamma \in (\mathrm{Sec}^{(c)}\, T)U$, and $(\mathrm{Sec}^{(c)}\, S)U$ must be a subgroup of $(\mathrm{Sec}^{(c)}\, T)U$. Conversely, let $x \in X$, $s_1, s_2 \in S_x$, and let $\gamma_1, \gamma_2 \in (\mathrm{Sec}^{(c)}\, S)U_x$ be continuous sections defined on a neighborhood U_x of x such that $\gamma_1(x) = s_1$, $\gamma_2(x) = s_2$ (Lemma 2, (a)). The union of the sets U_x coincides with U which implies that U is open. Moreover since $\gamma_1 + \gamma_2 \in (\mathrm{Sec}^{(c)}\, S)U$ then $s_1 + s_2 = \gamma_1(x) + \gamma_2(x) = (\gamma_1 + \gamma_2)(x) \in S_x$.

(b) This is a trivial consequence of (13).

(c) Let $\delta \in (\mathrm{Sec}^{(c)}\mathrm{Im}\, f)U$, and let $x \in X$. Then $\delta(x) = f(\gamma_x(x))$ for some continuous section γ_x, defined on a neighborhood U_x of x such that $U_x \subset U$ (Lemma 2, (b)). We may assume, shrinking U_x if necessary, that both δ and γ_x are homeomorphisms on U_x. Then $s_{UU_x}(\delta) = f \circ \gamma_x = \mathsf{f}_{U_x}(\gamma_x)$, so the family $\{\mathsf{f}_{U_x}(\gamma_x)\}_{x \in U}$ locally generates δ. The converse is obvious. □

Remark 3 Lemma 7, assertion (c) does not assure that for a continuous section $\delta \in (\mathrm{Sec}^{(c)}\mathrm{Im}\, f)U$, there always exists a continuous section $\gamma \in (\mathrm{Sec}^{(c)}\, S)U$ such that $\delta = \mathsf{f}_U(\gamma)$.

In accordance with Lemma 7, (a), given a sheaf subspace S of an Abelian sheaf T, we define a *subsheaf* of the sheaf $\mathrm{Sec}^{(c)}\, T$ as the correspondence $U \to (\mathrm{Sec}^{(c)}\, S)U$, and write $\mathrm{Sec}^{(c)}\, S \subset \mathrm{Sec}^{(c)}\, T$. If $f\colon S \to T$ is a sheaf space morphism, then the *kernel* (resp. the *image*) of the sheaf morphism $\mathsf{f}\colon \mathrm{Sec}^{(c)}\, S \to \mathrm{Sec}^{(c)}\, T$ is defined to be the Abelian sheaf, associated with the sheaf space $\mathrm{Ker}\, f$ (resp. $\mathrm{Im}\, f$); that is, we set

$$\mathrm{Ker}\ \mathsf{f} = \mathrm{Sec}^{(c)}\mathrm{Ker} f, \quad \mathrm{Im}\ \mathsf{f} = \mathrm{Sec}^{(c)}\mathrm{Im}\, f. \tag{15}$$

7.4 Abelian Presheaves

We can use properties (1), (2), and (3) of the sets of sections of an Abelian sheaf space (Sect. 7.3, Lemma 6) to introduce the concept of an *Abelian presheaf*. Diagram (14) will then be used to define Abelian presheaf morphisms. Properties (4) and (5) will be required to define *complete presheaves*, that is, (abstract) *sheaves*.

Let X be a topological space, S a correspondence assigning to an open set $U \subset X$ an Abelian group $\mathsf{S}U$ and to every pair of open sets U, V such that $V \subset U$ an Abelian group morphism $s_{VU}\colon \mathsf{S}V \to \mathsf{S}U$. S is said to be an *Abelian presheaf*, or just a *presheaf*, if the following conditions are satisfied:

(1) $\mathsf{S}\emptyset = 0$.
(2) $s_{UU} = \mathrm{id}_U$ for every open set $U \subset X$.
(3) $s_{WU} = s_{VU} \circ s_{WV}$ for all open sets $U, V, W \subset X$ such that $U \subset V \subset W$.

The topological space X is called the *base* of the Abelian presheaf $\mathcal{S}$. Elements of the Abelian groups $\mathcal{S}U$ are called *sections* of $\mathcal{S}$ over U, and the Abelian group morphisms s_{VU} are *restriction morphisms*, or just *restrictions* of $\mathcal{S}$. If $\gamma \in \mathcal{S}V$ and $U \subset V$, then the section $s_{VU}(\gamma)$ is called the *restriction* of the section γ to U.

Let $\mathcal{S}$ be an Abelian presheaf with base X and restrictions $\{s_{VU}\}$. Let U be an open subset of X. We say that two sections $\gamma, \delta \in \mathcal{S}U$ *coincide locally*, if there exists an open covering $\{U_\iota\}_{\iota \in I}$ of U such that for every $\iota \in I$

$$s_{UU_\iota}(\gamma) = s_{UU_\iota}(\delta). \tag{16}$$

A family $\{\gamma_\iota\}_{\iota \in I}$ of sections of $\mathcal{S}$, where $\gamma_\iota \in \mathcal{S}U_\iota$, is said to be *compatible*, if the condition

$$s_{U_\iota, U_\iota \cap U_\kappa}(\gamma_\iota) = s_{U_\kappa, U_\iota \cap U_\kappa}(\gamma_\kappa) \tag{17}$$

holds for all $\iota, \kappa \in I$. We say that a family $\{\gamma_\iota\}_{\iota \in I}$ *locally generates* a section $\gamma \in \mathcal{S}U$, where $U = \cup U_\iota$, if

$$s_{UU_\iota}(\gamma) = \gamma_\iota \tag{18}$$

for all $\iota \in I$. A family of sections, locally generating a section, is always compatible.

A *complete Abelian presheaf*, or an *Abelian sheaf*, is a presheaf $\mathcal{S}$ satisfying, in addition to conditions (1), (2) and (3) from the definition of an Abelian presheaf, the following two conditions:

(4) Any two sections of $\mathcal{S}$ which coincide locally, coincide.
(5) Every compatible family of sections of $\mathcal{S}$ locally generates a section of $\mathcal{S}$.

If an Abelian presheaf $\mathcal{S}$ is complete, then any section, locally generated by a compatible family of sections, is unique. Indeed, if γ_1, γ_2 are two sections locally generated by a compatible family $\{\gamma_\iota\}_{\iota \in I}$, then according to (5), $s_{UU_\iota}(\gamma_1) = \gamma_\iota = s_{UU_\iota}(\gamma_2)$, and property (4) implies $\gamma_1 = \gamma_2$.

Let $\mathcal{S}$ (resp. $\mathcal{T}$) be an Abelian presheaf over X, $\{s_{UV}\}$ (resp. $\{t_{UV}\}$) the family of restrictions of $\mathcal{S}$ (resp. $\mathcal{T}$). Let $f = \{f_U\}$ be a family of Abelian group morphisms f_U: $\mathcal{S}U \to \mathcal{T}U$. f is said to be a *morphism of Abelian presheaves*, or simply a *presheaf morphism*, if for every pair of open sets U and V in X such that $U \subset V$, the diagram

$$\begin{array}{ccc} \mathcal{S}V & \xrightarrow{f_V} & \mathcal{T}V \\ \downarrow s_{VU} & & \downarrow t_{VU} \\ \mathcal{S}U & \xrightarrow{f_U} & \mathcal{T}U \end{array} \tag{19}$$

commutes. We also denote this presheaf morphism by f: $\mathcal{S} \to \mathcal{T}$.

A *subpresheaf* $\mathcal{S}$ of an Abelian presheaf $\mathcal{T}$ is a presheaf such that $\mathcal{S}U$ is a subgroup of $\mathcal{T}U$ for every open set U in X. If ι_U are the corresponding inclusions, then the presheaf morphism ι: $\mathcal{S} \to \mathcal{T}$, is called the *inclusion* of the subpresheaf $\mathcal{S}$ into $\mathcal{T}$.

The *composition* of presheaf morphisms is defined in an obvious way. If $g: R \to S$ and $f: S \to T$ are two presheaf morphism, where $g = \{g_U\}$ and $f = \{f_U\}$, then we define $g \circ f: R \to T$ to be the family $\{g_U \circ f_U\}$.

If S is an Abelian presheaf, then the family $\mathrm{id}_S = \{\mathrm{id}_{SU}\}$ is a presheaf morphism, called the *identity morphism* of id_S. If $f: S \to T$ and $g: T \to S$ (resp. $h: T \to S$) are two Abelian presheaf morphisms and $g \circ f = \mathrm{id}_S$ (resp. $f \circ h = \mathrm{id}_T$), we call g (resp. h) a *left inverse* (resp. *right inverse*) for f. If f has a left inverse g and a right inverse h, then $h = (g \circ f) \circ h = g \circ (f \circ h) = g$ hence the presheaf morphism $h = g$ is unique. It is called the *inverse* of f and is denoted f^{-1}. f is called a *presheaf isomorphism*, if it has the inverse.

An Abelian presheaf morphism $f = \{f_U\}$ is called *injective* (resp. *surjective*), if the group morphisms f_U are injective (resp. surjective).

Let $f: S \to T$ be an Abelian presheaf morphism, $f = \{f_U\}$. We define a presheaf $\mathrm{Ker}\, f$ (resp. $\mathrm{Im}\, f$) as the correspondence, assigning to every open set $U \subset X$ the Abelian group $\mathrm{Ker}\, f_U \subset SU$ (resp. $\mathrm{Im}\, f_U \subset TU$), and to every two open sets $U, V \subset X$, where $U \subset V$, the restriction $s_{VU}|_{\mathrm{Ker}\, f_V}: \mathrm{Ker}\, f_V \to SU$ (resp. $t_{VU}|_{\mathrm{Im}\, f_V}: \mathrm{Im}\, f_V \to TU$). $\mathrm{Ker}\, f$ (resp. $\mathrm{Im}\, f$) is a subpresheaf of S (resp. T) called the *kernel* (resp. *image*) of f.

Remark 4 If the family $\{U_\iota\}_{\iota \in I}$ consists of two disjoint sets U_1, U_2, then condition (2) $s_{U_1,\varnothing}(\gamma_\iota) = s_{U_2,\varnothing}(\gamma_\kappa)$ reduces to the identity $0 = 0$. Thus, property (5), used for the definition of a complete presheaf, implies that there should always exist an extension of γ_1 and γ_2 to $U_1 \cup U_2$. This observation can sometimes be used to easily check that a presheaf is *not* complete: It is sufficient to verify that in the considered Abelian presheaf such an extension does not exist.

Examples

14. By definition, the *sheaf of continuous sections* of an Abelian sheaf space, introduced in Sect. 7.3, is a sheaf.
15. Let S and T be Abelian sheaves with base X and let $f: S \to T$ be an Abelian presheaf morphism. It is easily seen that $\mathrm{Ker}\, f$ is a complete presheaf of S. Indeed, $\mathrm{Ker}\, f$ satisfies condition (4) from the definition of a sheaf. To investigate condition (5), denote by $\{s_{VU}\}$ (resp. $\{t_{VU}\}$) the family of restrictions of S (resp. T). Let $\{U_\iota\}_{\iota \in I}$ be a family of open sets in X, $U = \cup U_\iota$. Let $\{\gamma_\iota\}_{\iota \in I}$ be a family of sections such that $\gamma_\iota \in (\mathrm{Ker}\, f)U_\iota$ and $s_{U_\iota, U_\iota \cap U_\kappa}(\gamma_\iota) = s_{U_\kappa, U_\iota \cap U_\kappa}(\gamma_\kappa)$ for all $\iota, \kappa \in I$. Then by condition (5), there exists $\gamma \in SU$ such that $s_{UU_\iota}(\gamma) = \gamma_\iota$. Using this condition and the commutative diagram (19), we get $t_{UU_\iota}(f_U(\gamma)) = f_{U_\iota}(s_{UU_\iota}(\gamma)) = f_{U_\iota}(\gamma_\iota) = 0$. Since T is complete, condition (5) implies $f_U(\gamma) = 0$.
16. The *trivial sheaf* over a topological space X is a complete presheaf, assigning to each open set $U \subset X$ the Abelian group id_U, with the restrictions $s_{UV}(\mathrm{id}_U) = \mathrm{id}_V$. The trivial sheaf over X is denoted by 0_X.
17. Assume that we have an Abelian sheaf space S with base X and projection σ. Consider the correspondence $\mathrm{Sec}\, S$, assigning to an open set $U \subset X$ the Abelian

group $(\mathrm{Sec}\ S)U$ of *all*, not necessarily continuous, sections of the local homeomorphism σ, defined on U. To any open sets $U, V \subset X$ such that $U \subset V$ we assign the restriction mapping s_{VU} in a standard way; we get Abelian group morphisms s_{VU}: $(\mathrm{Sec}\ S)V \to (\mathrm{Sec}\ S)U$. In this way, we get an Abelian sheaf Sec S, called the sheaf of (*discontinuous*) *sections*, associated with the sheaf space S.

18. Let X be a topological space. Assign to every open set $U \subset X$ the Abelian group $C_{X,\mathbf{R}}U$ of continuous real-valued functions, defined on U, and to any open sets $U, V \subset X$ such that $U \subset V$, the restriction mapping defined as $C_{X,\mathbf{R}}V \ni f \to s_{VU}(f) = f|_U \in C_{X,\mathbf{R}}U$. This correspondence obviously satisfies the axioms (1)–(5) of a complete Abelian presheaf (Abelian sheaf). Indeed, axioms (1), (2), and (3) are satisfied trivially. To formally verify (4), suppose we have two continuous functions $f, g \in C_{X,\mathbf{R}}U$ such that
$$s_{UU_\iota}(f) = f|_{U_\iota} = s_{UU_\iota}(g) = g|_{U_\iota} \tag{20}$$
for some open covering $\{U_\iota\}_{\iota \in I}$ of U. Clearly, then for every point $x \in U$, $f(x) = g(x)$, so f and g coincide on U. To verify axiom (5), consider a compatible family of continuous functions $\{f_\iota\}_{\iota \in I}$, where f_ι is defined on U_ι. Setting $f(x) = f_\iota(x)$ whenever $x \in U_\iota$, we get a continuous function f, defined on $U = \cup U_\iota$. Thus, the presheaf $C_{X,\mathbf{R}}$, defined in this way, is complete. This complete Abelian presheaf is referred to as the *sheaf of continuous functions* on the topological space X.
19. Let X be a smooth manifold. Assign to every open set $U \subset X$ the Abelian group $C^r_{X,\mathbf{R}}U$ of real-valued functions of class C^r, defined on U, where $r = 0, 1, 2, \ldots, \infty$, and to any open sets $U, V \subset X$ such that $U \subset V$, the restriction mapping $C^r_{X,\mathbf{R}}V \ni f \to s_{VU}(f) = f|_U \in C^r_{X,\mathbf{R}}U$. This correspondence obviously satisfies the axioms (1)–(5) of a complete presheaf; we get a complete Abelian presheaf called the *sheaf of functions of class* C^r on X.
20. Let E be a smooth vector bundle over a manifold X with projection π. For any $r = 0, 1, 2, \ldots, \infty$, assign to every open set $U \subset X$ the Abelian group $\Gamma^r_U(\pi)$ of C^r-sections of E, defined on U, and to any open sets $U, V \subset X$, where $U \subset V$, the restrictions $\Gamma_V(\pi) \ni \gamma \to s_{VU}(\gamma) = \gamma|_U \in \Gamma_U(\pi)$. This correspondence obviously satisfies the axioms (1)–(5) of a complete Abelian presheaf, the *sheaf of sections of class* C^r of the vector bundle E.
21. We show in this example that the image of a complete Abelian presheaf by an Abelian presheaf morphism into a complete presheaf is not necessarily a complete subpresheaf. Consider the Abelian sheaf $C^\infty_{X,\mathbf{R}} = \Omega^0_X$ of smooth functions (0-forms) and the sheaf $\mathcal{T} = \Omega^1_X$ of smooth 1-forms over the smooth manifold $X = \mathbf{R}^2 \backslash \{(0,0)\}$. The exterior derivative d: $\Omega^0_X \to \Omega^1_X$ defines, for every open set $U \subset X$, a morphism of Abelian groups d: $\Omega^0_X U \to \Omega^1_X U$, and a presheaf morphism d: $\Omega^0_X \to \Omega^1_X$. We show that the image presheaf $\mathrm{Im}\, d \subset \Omega^1_X$

does not satisfy condition (5) of a complete presheaf, so consequently, it is not complete. Consider in the canonical coordinates x, y in $\mathbf{R}^2$, the 1-form

$$\omega = \frac{xdy - ydx}{x^2 + y^2}. \tag{21}$$

Let $\{U_\iota\}_{\iota \in I}$ be a covering of X by open balls. Then by the Volterra–Poincare lemma, $\omega = d\varphi_\iota$ on U_ι, where $\varphi_\iota \in \Omega^0_X U_\iota$, but there is no function $\varphi \in \Omega^0_X$ satisfying $\omega = d\varphi$ (see e.g., Schwartz [Sc]). Thus ω is locally expressible as the exterior derivative, but there is *no* global function φ such that $\omega = d\varphi$.

7.5 Sheaf Spaces Associated with Abelian Presheaves

We introduce in this section a correspondence, assigning to an Abelian presheaf an Abelian sheaf space, and to an Abelian presheaf morphism an Abelian sheaf space morphism, and study basic properties of this correspondence.

Let $\mathcal{S}$ be an Abelian presheaf with base X, $\{s_{VU}\}$ the family of its restriction mappings. For any point $x \in X$, consider the set of all pairs (U, γ), where U is a neighborhood of x and γ a section of $\mathcal{S}$, belonging to the Abelian group $\mathcal{S}U$. There is an equivalence relation on this set "$\gamma \sim \delta$, *if there exists a neighborhood* W *of* x *such that the restrictions of* γ *and* δ *to* W *coincide*." Indeed, the binary relation $\sim$ is obviously symmetric and reflexive. To show that it is transitive, consider three sections $\gamma_1 \in \mathcal{S}U_1$, $\gamma_2 \in \mathcal{S}U_2$, and $\gamma_3 \in \mathcal{S}U_3$, such that $\gamma_1 \sim \gamma_2$ and $\gamma_2 \sim \gamma_3$. Then by definition, there exist two neighborhoods V and W of the point x such that $V \subset U_1 \cap U_2$, $W \subset U_2 \cap U_3$ and $s_{U_1 V}(\gamma_1) = s_{U_2 V}(\gamma_2)$ and $s_{U_2 W}(\gamma_2) = s_{U_2 W}(\gamma_3)$. Then on $V \cap W$

$$\begin{aligned} s_{U_1, V\cap W}(\gamma_1) &= s_{V, V\cap W} \circ s_{U_1, V}(\gamma_1) = s_{V, V\cap W} \circ s_{U_2, V}(\gamma_2) = s_{U_2, V\cap W}(\gamma_2) \\ &= s_{W, V\cap W} \circ s_{U_2 W}(\gamma_2) = s_{W, V\cap W} \circ s_{U_3 W}(\gamma_3) = s_{U_3, V\cap W}(\gamma_3). \end{aligned} \tag{22}$$

The equivalence class of a section γ is called the *germ* of γ at the point x and is denoted by $[\gamma]_x$. Denote by $\mathcal{S}_x$ the quotient set and consider the set

$$\operatorname{Germ} \mathcal{S} = \bigcup_{x \in X} \mathcal{S}_x \tag{23}$$

Define a mapping σ: Germ $\mathcal{S} \to X$ by the equation

$$\sigma([\gamma]_x) = x. \tag{24}$$

We need a topology on the set Germ $\mathcal{S}$ and an Abelian group structure on each of the sets $\mathcal{S}_x$ defining on Germ $\mathcal{S}$ the structure of a sheaf space of Abelian groups with base X and projection σ. Let U be an open set in X, $\gamma \in SU$ a section. We define a mapping $\tilde{\gamma}$: $U \to \operatorname{Germ} \mathcal{S}$ by

$$\tilde{\gamma}(x) = [\gamma]_x. \tag{25}$$

The set Germ $\mathcal{S}$ will be considered with the *final topology*, associated with the family $\{\tilde{\gamma}\}$, where γ runs through the set of sections of the presheaf $\mathcal{S}$; this is the strongest topology on the set Germ $\mathcal{S}$ in which all the mappings $\tilde{\gamma}$ are continuous.

Note that if $\gamma \in \mathcal{S}U$ is a section then the set $\tilde{\gamma}(U)$ is open in Germ $\mathcal{S}$. Clearly, if $\delta \in \mathcal{S}V$ is another section, we have

$$\tilde{\delta}^{-1}\tilde{\gamma}(U) = \{x \in V | \tilde{\delta}(x) = \tilde{\gamma}(x)\} = \{x \in U \cap V | \tilde{\delta}(x) = \tilde{\gamma}(x)\}, \tag{26}$$

which is an open subset of $U \cap V$ formed by all points x such that $\delta = \gamma$ on a neighborhood of x. Now we apply the definition of the final topology to observe that $\tilde{\gamma}(U)$ is open.

It is easy to see that the mapping σ: Germ $\mathcal{S} \to X$ defined by (25) is a local homeomorphism. If $y \in$ Germ $\mathcal{S}$ is any germ at $x \in X$ and $\gamma \in \mathcal{S}U$ any representative of y, then $W = \tilde{\gamma}(U)$ is a neighborhood of y and

$$\sigma|_W \circ \tilde{\gamma} = \mathrm{id}_U, \quad \tilde{\gamma} \circ \sigma|_W = \mathrm{id}_W. \tag{27}$$

Every fiber $\mathcal{S}_x$ of σ has the structure of an Abelian group defined by

$$[\gamma]_x + [\delta]_x = [s_{UW}(\gamma) + s_{VW}(\delta)]_x, \tag{28}$$

where $\gamma \in \mathcal{S}U$, $\delta \in \mathcal{S}V$, and $W = U \cap V$. Clearly, this definition is correct, because the germ on the right-hand side is independent of the choice of the representatives γ and δ. Indeed, with obvious notation

$$\begin{aligned} [s_{U'W'}(\gamma') + s_{U'W'}(\delta')]_x &= [s_{W'W''}(s_{U'V'}(\gamma') + s_{U'V'}(\delta'))]_x \\ &= [s_{U'W''}(\gamma') + s_{U'W''}(\delta')]_x, \\ [s_{UW}(\gamma) + s_{VW}(\delta)]_x &= [s_{UW''}(\gamma) + s_{VW''}(\delta)]_x, \end{aligned} \tag{29}$$

where $W' = U' \cap V'$. Since one may choose the set W'' in such a way that $s_{UW''}(\gamma) = s_{U'W''}(\gamma)$ and $s_{VW''}(\delta) = s_{V'W''}(\delta')$, we have

$$[s_{UW}(\gamma) + s_{VW}(\delta)]_x = [s_{U'W''}(\gamma') + s_{V'W''}(\delta')]_x. \tag{30}$$

It remains to check that the mapping $(p, q) \to (p - q)$ of the fiber product Germ $\mathcal{S} \times_X$ Germ $\mathcal{S}$ into Germ $\mathcal{S}$ is continuous. Let (p_0, q_0) be an arbitrary point of the set Germ $\mathcal{S} \times_X$ Germ $\mathcal{S}$, where $p_0 = [\gamma]_x$, $q_0 = [\delta]_x$. We may assume without loss of generality that $\gamma, \delta \in \mathcal{S}W$, where W is a neighborhood of x. Then $p_0 - q_0 = [\gamma - \delta]_x$. If $\eta = \gamma - \delta$, then $\tilde{\eta}(W)$ is a neighborhood of the point $p_0 - q_0$.

The set $\tilde{\gamma}(W) + \tilde{\delta}(W) \subset \text{Germ}\,\mathcal{S} \times \text{Germ}\,\mathcal{S}$ is open, and the set $(\tilde{\gamma}(W) + \tilde{\delta}(W)) \cap (\text{Germ}\,\mathcal{S} \times_S \text{Germ}\,\mathcal{S})$ is open in the set $\text{Germ}\,\mathcal{S} \times_S \text{Germ}\,\mathcal{S}$. Since the image of $(\tilde{\gamma}(W) + \tilde{\delta}(W)) \cap (\text{Germ}\,\mathcal{S} \times_S \text{Germ}\,\mathcal{S})$ under the mapping $(p, q) \to (p - q)$ coincides with $\tilde{\eta}(W)$, this mapping is continuous at (p_0, q_0). This completes the construction of the Abelian sheaf space $\text{Germ}\,\mathcal{S}$ from a given presheaf $\mathcal{S}$.

We call $\text{Germ}\,\mathcal{S}$ the *Abelian sheaf space*, *associated* with the Abelian presheaf $\mathcal{S}$. The continuous section $\tilde{\gamma}$: $U \to \text{Germ}\,\mathcal{S}$ is said to be *associated with* the section $\gamma \in \mathcal{S}U$.

Let $\mathcal{S}$ (resp. $\mathcal{T}$) be an Abelian presheaf over a topological space X, $\{s_{UV}\}$ (resp. $\{t_{UV}\}$) the family of restrictions of $\mathcal{S}$ (resp. $\mathcal{T}$). Let $\boldsymbol{f} = \{\boldsymbol{f}_U\}$ be a *presheaf morphism* of the presheaf $\mathcal{S}$ into $\mathcal{T}$. Denote by σ: $\text{Germ}\,\mathcal{S} \to X$ and τ: $\text{Germ}\,\mathcal{T} \to X$ the corresponding sheaf spaces. We define a mapping f: $\text{Germ}\,\mathcal{S} \to \text{Germ}\,\mathcal{T}$ by the equation

$$f([\gamma]_x) = [\boldsymbol{f}_U(\gamma)]_x, \tag{31}$$

where $[\gamma]_x \in \text{Germ}\,\mathcal{S}$ and $\gamma \in \mathcal{S}U$ is any representative of the germ $[\gamma]_x$. It can be readily verified that the germ $[\boldsymbol{f}_U(\gamma)]_x$ is defined independently of the choice of the representative γ. Indeed, let $\delta \in \mathcal{S}V$ be such that $[\delta]_x = [\gamma]_x$. Then $s_{UW}(\gamma) = s_{VW}(\delta)$ for some neighborhood W of the point x. Applying the definition of the presheaf morphism, we obtain

$$t_{UW} \circ \boldsymbol{f}_U(\gamma) = \boldsymbol{f}_W \circ s_{UW}(\gamma) = \boldsymbol{f}_W \circ s_{VW}(\delta) = t_{VW} \circ \boldsymbol{f}_V(\delta), \tag{32}$$

hence $[\boldsymbol{f}_U(\gamma)]_x = [\boldsymbol{f}_V(\delta)]_x$.

We assert that the mapping f, defined by (31), is a sheaf space morphism. f obviously satisfies $\tau \circ f = \sigma$. Note that if $\gamma \in \mathcal{S}U$, then $\boldsymbol{f}_U(\gamma)$ is a section of $\mathcal{T}$; in particular, the mapping $x \to f([\gamma]_x) = f \circ \tilde{\gamma}(x) = [\boldsymbol{f}_U(\gamma)]_x$ of U into the set germ $\mathcal{T}$ is continuous (with respect to the final topology on $\text{Germ}\,\mathcal{T}$). This means, however, that $f \circ \tilde{\gamma}$ is continuous, and using the properties of the topology of the set $\text{Germ}\,\mathcal{S}$, we conclude that the mapping f is continuous. Finally, the restriction f_x of f to each fiber $(\text{Germ}\,\mathcal{S})_x$ is an Abelian group morphism. Summarizing, we see that all conditions for f to be an Abelian sheaf space morphism hold. f is said to be *associated* with the Abelian presheaf morphism $\boldsymbol{f}$.

Consider a sheaf space of Abelian groups S with base X and projection σ, the associated sheaf of Abelian groups $\text{Sec}^{(c)}\,S$, and the sheaf space $\text{Germ}\,\text{Sec}^{(c)}\,S$, associated with the sheaf $\text{Sec}^{(c)}\,S$. Let σ': $\text{Germ}\,\text{Sec}^{(c)}\,S \to X$ be the sheaf space projection. Let $s \in S$ be a point and V a neighborhood if s such that $\sigma|_V$ is a homeomorphism. Put $x = \sigma(s)$, $\gamma_s = (\sigma|_V)^{-1}$, and

$$\nu_S(s) = [\gamma_s(x)]. \tag{33}$$

This defines a mapping ν_S: $S \to \text{Germ}\,\text{Sec}^{(c)}\,S$ such that $\sigma' \circ \nu_S = \sigma$.

Lemma 8

(a) *Let* S *and* T *be two Abelian presheaves with base X,* $\mathsf{f}\colon \mathsf{S} \to \mathsf{T}$ *an Abelian presheaf morphism, and let* $f\colon \text{Germ } \mathsf{S} \to \text{Germ } \mathsf{T}$ *be the sheaf space morphism associated with* f*. Then for every point* $x \in X$

$$(\text{Germ Ker } \mathsf{f})_x = \text{Ker} f_x, \quad (\text{Germ Im } \mathsf{f})_x = \text{Im} f_x. \tag{34}$$

(b) *Let* $f\colon \text{Germ } \mathsf{R} \to \text{Germ } \mathsf{S}$ *(resp.* $g\colon \text{Germ } \mathsf{S} \to \text{Germ } \mathsf{T}$*) be the Abelian sheaf space morphism associated with an Abelian presheaf morphism* $\mathsf{f}\colon \mathsf{R} \to \mathsf{S}$ *(resp.* $\mathsf{g}\colon \mathsf{S} \to \mathsf{T}$*), and* $h\colon \text{Germ } \mathsf{R} \to \text{Germ } \mathsf{T}$ *the Abelian sheaf space morphism associated with the Abelian presheaf morphism* $\mathsf{h} = \mathsf{g} \circ \mathsf{f}$*. Then* $h = g \circ f$*.*

(c) *The mapping* $v_S\colon S \to \text{Germ Sec}^{(c)} S$ *is an Abelian sheaf space isomorphism.*

Proof

(a) Let $[\gamma]_x \in \text{Germ Ker} f$. Then $\gamma \in (\text{Ker} f)U$, where U is a neighborhood of x. Thus, the representative γ satisfies $\mathsf{f}_U(\gamma) = 0$ hence by (31), $f([\gamma]_x) = 0$ and $[\gamma]_x \in \text{Ker} f$. Conversely, assume that $[\gamma]_x \in \text{Ker} f$. Then by (31) $f([\gamma]_x) = [\mathsf{f}_V(\gamma)]_x = 0$. In particular, $\mathsf{f}_V(\gamma)$ is equivalent to the zero section, $t_{VU}(\mathsf{f}_V(\gamma)) = \mathsf{f}_V(s_{VU}(\gamma)) = 0$ for a neighborhood U of x such that $U \subset V$. Thus $[\gamma]_x = [s_{VU}(\gamma)]_x$, where $s_{VU}(\gamma) \in \text{Ker } \mathsf{f}_U$.

Let $[\delta]_x \in \text{Germ Im } \mathsf{f}$. Then for some neighborhood V of x, $\delta = \mathsf{f}_V(\gamma)$, where $\gamma \in \mathsf{S}U$. Thus by (31), $f([\gamma]_x) = [\mathsf{f}_U(\gamma)]_x = [\delta]_x$ which means that $[\delta]_x \in \text{Im} f_x$. Conversely, let $[\delta]_x \in \text{Im} f_x$. Then there exists $[\gamma]_x$ such that $f_x([\gamma]_x) = [\delta]_x$. Assume that $\gamma \in \mathsf{S}V$, $\delta \in \mathsf{T}V$. Then on a neighborhood U of x, $\mathsf{f}_U(s_{VU}(\gamma)) = t_{VU}(\delta)$ which implies $[\delta]_x = [t_{VU}(\delta)]_x = [\mathsf{f}_U s_{VU}(\gamma)]_x$, which is an element of the set Germ Im f_x.

(b) The proof is straightforward.

(c) We shall show that v_S is an Abelian sheaf space isomorphism. Let $[\gamma]_x \in \text{Germ Sec}^{(c)} S$ be a germ represented by a section $\gamma \in (\text{Sec}^{(c)} S)U$. Write $\tau_S([\gamma]_x) = \gamma(x)$. Clearly, the point $\gamma(x) \in S$ is defined independently of the choice of the representative γ. We have $\tau_S([\gamma]_x) = (\sigma_V)^{-1}(x)$, where V is a neighborhood of the point $\gamma(x) \in S$ such that the restriction $\sigma|_U$ is a homeomorphism. Since $v_S \circ \tau_S([\gamma]_x) = v_S((\sigma_V)^{-1}(x)) = [(\sigma_V)^{-1}]_x = [\gamma]_x$ and

$$\tau_S \circ v_S(s) = \tau_S([\gamma]_x) = \gamma_S(x) = s, \tag{35}$$

τ_S is the inverse of v_S.

We shall verify that v_S is continuous. Let $s \in S$ be a point, $x = \sigma(s)$, V a neighborhood of the point $v_S(s) \in \text{Germ Sec}^{(c)} S$. The point $v_S(s)$ has a neighborhood $\tilde{\gamma}_s(U)$, where $\gamma_s\colon U \to S$ is a section, defined on a neighborhood

U of x, and $\tilde{\gamma}_s(y) = [\gamma_s]_y$. Since $\tilde{\gamma}_s$ is continuous, we may suppose that $\tilde{\gamma}_s(U) \subset V$. But the set $\gamma_s(U)$ is a neighborhood of the point s, and $v_S(\gamma_s(U)) = \tilde{\gamma}_s(U) \subset V$, hence v_S is continuous at s.

Now we shall show that for every point $x \in X$ and any two points $s_1, s_2 \in S_x$, $v_S(s_1 + s_2) = v_S(s_1) + v_S(s_2)$. Let V_1 (resp. V_2) be a neighborhood of s_1 (resp. s_2) such that $\sigma|_{V_1}$ (resp. $\sigma|_{V_2}$) is a homeomorphism. One may suppose that $\sigma(V_1) = \sigma(V_2) = U$. Then $\gamma_{s_1}, \gamma_{s_2}, \gamma_{s_1+s_2} \in (\mathrm{Sec}^{(c)}\ S)U$ and by definition $[\gamma_{s_1}]_x + [\gamma_{s_2}]_x = [\gamma_{s_1} + \gamma_{s_2}]_x$, that is, $v_S(s_1) + v_S(s_2) = v_S(s_1 + s_2)$. This proves that the mapping v_S is an Abelian sheaf space morphism.

The mapping v_S is obviously injective and surjective hence bijective. The inverse mapping $(v_S)^{-1}$: Germ $\mathrm{Sec}^{(c)}\ S \to S$ is continuous by the properties of the final topology, since for every section $\gamma \in \mathrm{Sec}^{(c)}\ S$ the composite $(v_S)^{-1} \circ \tilde{\gamma} = \gamma$ is continuous. Summarizing, this proves that v_S is an Abelian sheaf space isomorphism. □

We call the Abelian sheaf space isomorphism $v_S\colon S \to$ Germ $\mathrm{Sec}^{(c)}\ S$ the *canonical isomorphism*.

7.6 Sheaves Associated with Abelian Presheaves

The concepts of an *Abelian sheaf associated with an Abelian sheaf space* and the *Abelian sheaf space associated with an Abelian presheaf* allow to assign to any Abelian presheaf S the sheaf $\mathrm{Sec}^{(c)}\mathrm{Germ}\,\mathsf{S}$, which is said to be *associated* with S. We study properties of this correspondence.

Let S be an Abelian presheaf over a topological space X, $\{s_{VU}\}$ the family of its restrictions. For every open set $U \subset X$ define a morphism of Abelian groups $\vartheta_U\colon \mathsf{S}U \to (\mathrm{Sec}^{(c)}\mathrm{Germ}\,\mathsf{S})U$ by

$$\vartheta_U(\gamma) = \tilde{\gamma}, \tag{36}$$

where $\tilde{\gamma}$ is a section of the sheaf Germ S, associated with γ (Sect. 7.5, (4)). The Abelian presheaf morphism $\vartheta_{\mathsf{S}} = \{\vartheta_U\}$ of S into $\mathrm{Sec}^{(c)}\mathrm{Germ}\,\mathsf{S}$ is said to be *canonical*. Since for every open sets $U, V \subset X$ such that $U \subset V$, and every point $x \in U$, $\vartheta_U(s_{VU}(\gamma))(x) = [s_{VU}(\gamma)]_x = [\gamma]_x = \tilde{\gamma}(x) = \vartheta_V(\gamma)|_U(x)$, ϑ_{S} commutes with the restrictions,

$$\vartheta_U \circ s_{VU}(\gamma) = \vartheta_U(\gamma)|_U. \tag{37}$$

Note that any section δ of the sheaf $\mathrm{Sec}^{(c)}\mathrm{Germ}\,\mathsf{S}$ is locally generated by a family of sections, generated by sections of S. To prove it, consider a continuous section $\delta \in (\mathrm{Sec}^{(c)}\mathrm{Germ}\,\mathsf{S})U$ and any point $x \in U$. By definition $\delta(x)$ is the germ of a section $\gamma_x \in \mathsf{S}U_x$, where U_x is a neighborhood of the point x in U. That is,

$\delta(x) = [\gamma_x]_x = \tilde{\gamma}_x(x)$. The projection σ: Germ $\mathcal{S} \to X$ of the sheaf space Germ $\mathcal{S}$ is a local homeomorphism and $\sigma \circ \delta = \mathrm{id}_U$. On the other hand, $\sigma \circ \tilde{\gamma}_x = \mathrm{id}_{U_x}$, and since the inverse mapping is unique,

$$\delta|_{U_x} = \tilde{\gamma}_x = \vartheta_{U_x}(\gamma_x). \tag{38}$$

Obviously, $U = \cup U_x$ and for any two points $x, y \in U$, $\delta|_{U_x} = \tilde{\gamma}_x$ hence

$$\delta|_{U_x \cap U_y} = \tilde{\gamma}_x|_{U_x \cap U_y} = \tilde{\gamma}_y|_{U_x \cap U_y}. \tag{39}$$

Thus $[\gamma_x]_z = [\gamma_y]_z$ for every $z \in U_x \cap U_y$. Therefore, every point $z \in U_x \cap U_y$ has a neighborhood W_z such that

$$s_{U_x W_z}(\gamma_x) = s_{U_y W_z}(\gamma_y). \tag{40}$$

In view of (38), we say that the continuous section $\delta \in (\mathrm{Sec}^{(c)}\mathrm{Germ}\,\mathcal{S})U$ is *locally generated* by the family of sections $\{\gamma_x\}_{x \in U}$ of $\mathcal{S}$.

Our aim now will be to find conditions ensuring that the canonical morphism $\vartheta_{\mathcal{S}}$: $\mathcal{S} \to \mathrm{Sec}^{(c)}\mathrm{Germ}\,\mathcal{S}$ is a presheaf isomorphism.

Theorem 1 *Let* $\mathcal{S}$ *be an Abelian presheaf. The following conditions are equivalent:*

(1) $\mathcal{S}$ *is complete.*

(2) *The canonical presheaf morphism* $\vartheta_{\mathcal{S}}$: $\mathcal{S} \to \mathrm{Sec}^{(c)}\mathrm{Germ}\,\mathcal{S}$ *is a presheaf isomorphism.*

Proof

1. Suppose that $\vartheta_{\mathcal{S}} = \{\vartheta_U\}$ is a presheaf isomorphism. Let $\{s_{UV}\}$ be the restrictions of the presheaf $\mathcal{S}$, $\{t_{UV}\}$ the restrictions of the sheaf $\mathrm{Sec}^{(c)}\mathrm{Germ}\,\mathcal{S}$. Let $\{U_\iota\}_{\iota \in I}$ be a family of open sets in X, $U = \cup U_\iota$, and γ, δ two sections from $\mathcal{S}U$ such that $s_{UU_\iota}(\gamma) = s_{UU_\iota}(\delta)$. Then by the definition of the presheaf morphism, $\vartheta_{U_\iota} \circ s_{UU_\iota}(\gamma) = t_{UU_\iota} \circ \vartheta_U(\gamma) = t_{UU_\iota} \circ \vartheta_U(\delta)$. Hence $\vartheta_U(\gamma) = \vartheta_U(\delta)$ and, since ϑ_U is a group isomorphism, $\gamma = \delta$. This means that the presheaf $\mathcal{S}$ satisfies condition (4) of the definition of a complete presheaf. Now suppose that a family $\{\gamma_\iota\}_{\iota \in I}$, where $\gamma_\iota \in \mathcal{S}U_\iota$, satisfies the condition $s_{U_\iota, U_\iota \cap U_\kappa}(\gamma_\iota) = s_{U_\iota, U_\iota \cap U_\kappa}(\gamma_\kappa)$ for all $\iota, \kappa \in I$. Then

$$\begin{aligned} \vartheta_{U_\iota \cap U_\kappa} \circ s_{U_\iota, U_\iota \cap U_\kappa}(\gamma_\iota) &= t_{U_\iota, U_\iota \cap U_\kappa}(\gamma_\iota) \circ \vartheta_{U_\iota}(\gamma_\iota) \\ &= t_{U_\kappa, U_\iota \cap U_\kappa}(\gamma_\kappa) \circ \vartheta_{U_\kappa}(\gamma_\kappa), \end{aligned} \tag{41}$$

so there must exist a section $\delta \in (\mathrm{Sec}^{(c)}\mathrm{Germ}\,\mathcal{S})U$, where $U = \cup U_\iota$, such that $t_{UU_\iota}(\delta) = \vartheta_{U_\iota}(\gamma_\iota)$ for all indices $\iota \in I$. If $\gamma \in \mathcal{S}U$ is such that $\delta = \vartheta_U(\gamma)$, we have $t_{UU_\iota} \circ \vartheta_U(\gamma) = \vartheta_{U_\iota} \circ s_{UU_\iota}(\gamma) = \vartheta_{U_\iota}(\gamma_\iota)$, hence $s_{UU_\iota}(\gamma) = \gamma_\iota$. Thus, condition (5) is also satisfied. This means, however, that $\mathcal{S}$ must be complete.

2. Conversely, suppose that the presheaf $\mathcal{S}$ is complete. We wish to show that there exists a presheaf morphism $\boldsymbol{f}$: $\mathrm{Sec}^{(c)}\mathrm{Germ}\,\mathcal{S} \to \mathcal{S}$, $\boldsymbol{f} = \{\boldsymbol{f}_U\}$, such that $\vartheta_{\mathcal{S}} \circ \boldsymbol{f} = \mathrm{id}_{\mathrm{Sec}^{(c)}\mathrm{Germ}\,\mathcal{S}}$ and $\boldsymbol{f} \circ \vartheta_{\mathcal{S}} = \mathrm{id}_{\mathcal{S}}$, that is,

$$\vartheta_U \circ \boldsymbol{f}_U = \mathrm{id}_{(\mathrm{Sec}^{(c)}\mathrm{Germ}\,\mathcal{S})\mathrm{U}}, \quad \boldsymbol{f}_U \circ \vartheta_U = \mathrm{id}_{\mathcal{S}U} \tag{42}$$

for all open sets $U \subset X$. Obviously, these equations have a solution $\boldsymbol{f}_U$ if and only if the mapping ϑ_U is bijective. Since we have already shown that ϑ_U is injective, it is sufficient to prove that it is surjective.
Let $\delta \in (\mathrm{Sec}^{(c)}\mathrm{Germ}\,\mathcal{S})U$ be a section, and let $x \in U$ be a point. Applying the definition of a presheaf (condition (3), Sect. 7.4) of to Eq. (38),

$$s_{U_x \cap U_y, W_z} \circ s_{U_x, U_x \cap U_y}(\gamma_x) = s_{U_x \cap U_y, W_z} \circ s_{U_x, U_x \cap U_y}(\gamma_y). \tag{43}$$

Covering $U_x \cap U_y$ by the sets W_z we get from condition (4) of the definition of a presheaf

$$s_{U_x, U_x \cap U_y}(\gamma_x) = s_{U_x, U_x \cap U_y}(\gamma_y). \tag{44}$$

Condition (5) now implies that there exists a section $\gamma \in \mathcal{S}U$ such that

$$s_{UU_x}(\gamma) = \gamma_x \tag{45}$$

for all $x \in U$. Therefore, the sections γ and γ_x belong to the same germ at every point of the set U_x. This means that $\tilde{\gamma}|_{U_x}) = \tilde{\gamma}_x$ and

$$\delta|_{U_x} = \tilde{\gamma}_x = \tilde{\gamma}|_{U_x}. \tag{46}$$

Since the presheaf of sections of the sheaf space Germ $\mathcal{S}$ is a sheaf (Lemma 6), we get $\delta = \tilde{\gamma}$ proving that the mapping ϑ_U is surjective.
Consequently, the mapping $\boldsymbol{f}_U$ exists and is given by the formula $\boldsymbol{f}_U = (\vartheta_U)^{-1}$. It remains to show that $t_{VU} \circ \boldsymbol{f}_V = \boldsymbol{f}_U \circ s_{VU}$ for any two open sets $U, V \subset X$ such that $U \subset V$, where t_{VU} are restrictions of the presheaf $\mathrm{Sec}^{(c)}\mathrm{Germ}\,\mathcal{S}$. Let $\delta \in (\mathrm{Sec}^{(c)}\mathrm{Germ}\,\mathcal{S})U$ be a section; then $\delta = \tilde{\gamma} = \vartheta_V(\gamma)$ for some section $\gamma \in \mathcal{S}V$. We have

$$s_{VU} \circ \boldsymbol{f}_V(\tilde{\gamma}) = s_{VU} \circ \boldsymbol{f}_V \circ \vartheta_V(\gamma) = s_{VU}(\gamma), \tag{47}$$

and

$$\boldsymbol{f}_U \circ t_{VU}(\tilde{\gamma}) = \boldsymbol{f}_U \circ t_{VU} \circ \vartheta_V(\gamma) = \boldsymbol{f}_U \circ \vartheta_U \circ s_{VU}(\gamma) = s_{VU}(\gamma), \tag{48}$$

proving the desired identity $t_{VU} \circ \boldsymbol{f}_V = \boldsymbol{f}_U \circ s_{VU}$. Now the proof is complete. □

Theorem 2 *Let S (resp. T) be an Abelian presheaf with restrictions $\{s_{UV}\}$ (resp. $\{t_{UV}\}$), let $\mathsf{f}\colon \mathsf{S} \to \mathsf{T}$ be an Abelian presheaf morphism. There exists a unique Abelian presheaf morphism g: $\mathrm{Sec}^{(c)}\mathrm{Germ}\,\mathsf{S} \to \mathrm{Sec}^{(c)}\mathrm{Germ}\,\mathsf{T}$ such that the diagram*

$$\begin{array}{ccc} \mathsf{S} & \xrightarrow{\mathsf{f}} & \mathsf{T} \\ \downarrow \vartheta_{\mathsf{S}} & & \downarrow \vartheta_{\mathsf{T}} \\ \mathrm{Sec}^{(c)}\mathrm{Germ}\,\mathsf{S} & \xrightarrow{\mathsf{g}} & \mathrm{Sec}^{(c)}\mathrm{Germ}\,\mathsf{T} \end{array} \tag{49}$$

commutes.

Proof f generates a sheaf space morphism f: Germ $\mathsf{S} \to$ Germ T by the formula $f([\gamma]_x) = [\mathsf{f}_U(\gamma)]_x$, where U is a neighborhood of x and $\gamma \in \mathsf{S}U$ is a representative of the germ $[\gamma]_x$. f defines a sheaf morphism g: $\mathrm{Sec}^{(c)}\mathrm{Germ}\,\mathsf{S} \to \mathrm{Sec}^{(c)}\mathrm{Germ}\,\mathsf{T}$, $\mathsf{g} = \{\mathsf{g}_U\}$ by

$$\mathsf{g}_U(\delta) = f \circ \delta, \tag{50}$$

where $\delta \in (\mathrm{Sec}^{(c)}\mathrm{Germ}\,\mathsf{S})U$. Note that condition (45), Sect. 7.5 can be expressed in the form $f(\vartheta_{\mathsf{S},U}(\gamma)(x)) = \vartheta_{\mathsf{T},U}(\mathsf{f}_U(\gamma))(x))$ or, equivalently, $f \circ \vartheta_{\mathsf{S},U}(\gamma) = \vartheta_{\mathsf{T},U} \circ \mathsf{f}_U(\gamma)$, which implies

$$\mathsf{g}_U(\vartheta_{\mathsf{S},U}(\gamma)) = f \circ \vartheta_{\mathsf{S},U}(\gamma) = \vartheta_{\mathsf{T},U} \circ \mathsf{f}_U(\gamma). \tag{51}$$

This proves existence and uniqueness of g. □

To describe the morphism g: $\mathrm{Sec}^{(c)}\mathrm{Germ}\,\mathsf{S} \to \mathrm{Sec}^{(c)}\mathrm{Germ}\,\mathsf{T}$ explicitly, choose a continuous section $\delta \in (\mathrm{Sec}^{(c)}\mathrm{Germ}\,\mathsf{S})U$. We have already seen that there exists a family $\{\gamma_x\}_{x \in U}$ of sections $\gamma_x \in \mathsf{T}U_x$, where U_x is a neighborhood of x in U, such that

$$\delta|_{U_x} = \vartheta_{\mathsf{S},U_x}(\gamma_x). \tag{52}$$

If $z \in U_x \cap U_y$, then $s_{U_xW_z}(\gamma_x) = s_{U_xW_z}(\gamma_y)$ on some neighborhood W_z of the point z in $U_x \cap U_y$. Obviously, on U_x

$$\mathsf{g}_U(\delta)|_{U_x} = \vartheta_{\mathsf{T},U_x}(\mathsf{f}_{U_x}(\gamma_x)), \tag{53}$$

because for every $y \in U_x$

$$\begin{aligned} \mathsf{g}_U(\delta)|_{U_x}(y) &= f(\delta(y)) = f(\vartheta_{\mathsf{S},U_x}(\gamma_x)(y)) = f([\gamma_x]_y) \\ &= [\mathsf{f}_{U_x}(\gamma_x)]_y = \vartheta_{\mathsf{T},U_x}(\mathsf{f}_{U_x}(\gamma_x))(y). \end{aligned} \tag{54}$$

Thus, if δ is locally generated by the family $\{\gamma_x\}_{x \in U}$, then $\mathsf{g}_U(\delta)$ is locally generated by the family $\{\mathsf{f}_{U_x}(\gamma_x)\}_{x \in U}$.

Note that if in diagram (49), $\mathcal{T}$ is a complete Abelian presheaf, then by Theorem 1, $\vartheta_{\mathcal{T}}$ is an Abelian presheaf isomorphism, so we have, with obvious conventions,

$$f = \vartheta_{\mathcal{T}}^{-1} \circ g \circ \vartheta_{\mathcal{S}}. \tag{55}$$

If $\mathcal{S}$ is a complete presheaf, then

$$g = \vartheta_{\mathcal{T}} \circ f \circ \vartheta_{\mathcal{S}}^{-1}. \tag{56}$$

Corollary 2 *If $\mathcal{S}$ is a subpresheaf of an Abelian presheaf $\mathcal{T}$, then the sheaf* $\mathrm{Sec}^{(c)}\mathrm{Germ}\,\mathcal{S}$ *is a subsheaf of* $\mathrm{Sec}^{(c)}\mathrm{Germ}\,\mathcal{T}$.

Corollary 3

(a) *Every complete Abelian presheaf is isomorphic with an Abelian sheaf, associated with an Abelian sheaf space.*
(b) *Every presheaf morphism of complete Abelian presheaves is expressible as a sheaf morphism, associated with a sheaf space morphism.*

Proof

(a) This follows from Theorem 1.
(b) If both $\mathcal{S}$ and $\mathcal{T}$ in Theorem 2 are complete presheaves, then formulas (55) and (56) establish a one-to-one correspondence between presheaf morphisms f of complete presheaves and sheaf morphisms g associated with sheaf space morphisms. □

Let f: $\mathcal{S} \to \mathcal{T}$ be an Abelian presheaf morphism, and suppose that the Abelian presheaf $\mathcal{T}$ is complete. Let f: Germ $\mathcal{S} \to$ Germ $\mathcal{T}$ be the associated morphism of sheaf spaces. Note that we have defined the image Im f as a subpresheaf of $\mathcal{T}$. On the other hand, we have also defined the image of the sheaf $\mathrm{Sec}^{(c)}\mathrm{Germ}\,\mathcal{S}$ by the sheaf morphism induced by f, which is equal to the subsheaf $\mathrm{Sec}^{(c)}\mathrm{Im}\,f$ of the Abelian sheaf $\mathrm{Sec}^{(c)}\mathrm{Germ}\,\mathcal{T}$. Obviously, we have $\mathrm{Im}\,f \subset \vartheta_{\mathcal{T}}^{-1}(\mathrm{Sec}^{(c)}\mathrm{Im}\,f)$, and $\vartheta_{\mathcal{T}}^{-1}(\mathrm{Sec}^{(c)}\mathrm{Im}\,f)$ is a complete subpresheaf of $\mathcal{T}$. To distinguish between $\mathrm{Im}\,f$ and $\vartheta_{\mathcal{T}}^{-1}(\mathrm{Sec}^{(c)}\mathrm{Im}\,f)$, we sometimes call $\vartheta_{\mathcal{T}}^{-1}(\mathrm{Sec}^{(c)}\mathrm{Im}\,f)$ the *complete image* of $\mathcal{S}$ by the presheaf morphism f, or the *complete subpresheaf*, generated by $\mathcal{S}$.

If $\mathcal{S}$ is a subpresheaf of the presheaf $\mathcal{T}$, then the canonical inclusion $\iota_{\mathcal{S}}$: $\mathcal{S} \to \mathcal{T}$ defines the image $\mathrm{Im}\,\iota_{\mathcal{S}}$ and the complete image $\vartheta_{\mathcal{T}}^{-1}(\mathrm{Sec}^{(c)}\mathrm{Im}\,\iota_{\mathcal{S}})$. If the presheaf $\mathcal{S}$ is complete, then the following three subpresheaves $\mathcal{S}$, $\mathrm{Im}\,\iota_{\mathcal{S}}$ and $\vartheta_{\mathcal{T}}^{-1}(\mathrm{Sec}^{(c)}\mathrm{Im}\,\iota_{\mathcal{S}})$ coincide.

Examples

22. Let X be a topological space, G a group. We set for each non-void open set $U \subset X$, $\mathcal{G}U = G$, and $\mathcal{G}\emptyset = 0$ (the neutral element of G). For any two open

sets $U, V \subset X$ such that $U \subset V$, we set s_{UV}: $\mathbf{G}U \to \mathbf{G}V$ to be the restriction of the identity mapping id_G. Then the family $\mathbf{G} = \{\mathbf{G}U\}$ is a presheaf over X, called the *constant presheaf*. $\mathbf{G}$ is *not* complete, because it does not satisfy condition (5), Sect. 7.4 of the definition of a complete presheaf. Indeed, if U and V are *disjoint* open sets in X, and $g \in \mathbf{G}U = G$, $h \in \mathbf{G}V = G$ are two *different* points, then there is *no* element in G equal to both g and h (cf. Sect. 7.4, Remark 4). It is easily seen that the sheaf space, associated with the presheaf $\mathbf{G}$, Germ $\mathbf{G}$, coincides with the *constant sheaf space* G_X (Sect. 7.2, Example 11).

Remark 5 One can define sheaves with *different* algebraic structures on the fibers than the Abelian group structure. Let σ: $S \to X$ be a local homeomorphism of topological spaces. Assume that for every point $x \in X$ the fiber S_x is a commutative ring with unity such that the subtraction $S \times_X S \ni (s_1, s_2) \to s_1 - s_2 \in S$ and multiplication $S \times_X S \ni (s_1, s_2) \to s_1 \cdot s_2 \in S$ are continuous. Then, S is called the *sheaf space of commutative rings with unity*. If τ: $T \to X$ is another local homeomorphism, such that the fibers T_x are modules over S_x and the mappings $T \times_X T \ni (t_1, t_2) \to t_1 - t_2 \in T$ and $S \times_X T \ni (s, t) \to s \cdot t \in S$ are continuous, then T is called a *sheaf space of S-modules*.

7.7 Sequences of Abelian Groups, Complexes

We summarize in this section elementary notions of the homological algebra of sequences of Abelian groups such as the complex, the connecting homorphism, and the long exact sequence.

A family $A^* = \{A^i, d^i\}_{i \in Z}$, of Abelian groups and their morphisms d^i: $A^i \to A^{i+1}$, indexed with the integers $i \in \mathbf{Z}$, is called a *sequence of Abelian groups*. The family of the group morphisms in this sequence is denoted by $\{d^i\}_{i \in Z}$. We usually write A^* in the form

$$\cdots \longrightarrow A^{i-1} \xrightarrow{d^{i-1}} A^i \xrightarrow{d^i} A^{i+1} \xrightarrow{d^{i+1}} \cdots \tag{57}$$

Note that the asterisk in the symbol A^* of the sequence refers to the position of indices in the sequence.

A sequence of Abelian groups may begin or end with an infinite string of trivial, one-element Abelian groups 0, and their trivial group morphisms. If $A^i = 0$ for all $i < 0$, then the sequence A^* is said to be *nonnegative*, and is written as $A^* = \{A^i, d^i\}_{i \in N}$, with indexing set the nonnegative integers, or

$$0 \longrightarrow A^0 \xrightarrow{d^0} A^1 \xrightarrow{d^1} A^2 \xrightarrow{d^2} A^2 \xrightarrow{d^3} \cdots \tag{58}$$

In this notation, the mapping $0 \to A^0$ is the *trivial* group morphism. If there exist the smallest and greatest integer r and s) such that $A^r \neq 0$ and $A^s \neq 0$, then the sequence A^* is said to be *finite*, and A^r (resp. A^s) is called its *first* (resp. *last*) element. In this case, we write A^* as

$$0 \longrightarrow A^r \xrightarrow{d^r} A^{r+1} \xrightarrow{d^{r+1}} \cdots \xrightarrow{d^{s-1}} A^s \xrightarrow{d^s} 0 \tag{59}$$

with trivial group morphisms $0 \to A^r$ and $A^s \to 0$. To simplify notation, we sometimes omit the indexing set and write just $A^* = \{A^i, d^i\}$, or $A^* = \{A^i, d\}$ for the sequence (59) when no misunderstanding may arise.

A sequence of Abelian groups $A^* = \{A^i, d^i\}$ is said to be *exact* at the term A^q, if $\operatorname{Ker} d^q = \operatorname{Im} d^{q-1}$. A^* is an *exact sequence*, if it is exact in *every* term. Exact sequence of the form

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0 \tag{60}$$

is called a *short exact sequence*.

The following are elementary properties of short exact sequences.

Lemma 9

(a) *The sequence* (60) *is exact at* C *if and only if the group morphism* g *is surjective.*
(b) *The sequence* (60) *is exact at* A *if and only if the* f *is injective.*
(c) *A sequence of Abelian groups*

$$0 \longrightarrow A \xrightarrow{\iota} B \xrightarrow{\pi} B/A \longrightarrow 0 \tag{61}$$

in which $A \subset B$, $\iota\colon A \to B$ *is inclusion and* $\pi\colon B \to B/A$ *is the quotient projection, is a short exact sequence.*
(d) *Suppose we have a diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A^0 & \xrightarrow{f^0} & A^1 & \xrightarrow{f^1} & A^2 & \longrightarrow & 0 \\ & & \downarrow \varphi^0 & & \downarrow \varphi^1 & & & & \\ 0 & \longrightarrow & B^0 & \xrightarrow{g^0} & B^1 & \xrightarrow{g^1} & B^2 & \longrightarrow & 0 \end{array} \tag{62}$$

where the horizontal sequences are short exact sequences of Abelian groups, φ^0 *and* φ^1 *are morphisms of Abelian groups, and the first square commutes,*

$$g^0 \circ \varphi^0 = \varphi^1 \circ g^1. \tag{63}$$

Then, there exists a unique morphism of Abelian groups $\varphi^2\colon A^2 \to B^2$ *such that the second square of the diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A^0 & \xrightarrow{f^0} & A^1 & \xrightarrow{f^1} & A^2 & \longrightarrow & 0 \\ & & \downarrow \varphi^0 & & \downarrow \varphi^1 & & \downarrow \varphi^2 & & \\ 0 & \longrightarrow & B^0 & \xrightarrow{g^0} & B^1 & \xrightarrow{g^1} & B^2 & \longrightarrow & 0 \end{array} \tag{64}$$

commutes.

(e) *Consider the exact sequence of Abelian groups* (60) *and the quotient projection* $\pi: B \to B/f(A)$. *There exists a unique group isomorphism* $\varphi: C \to B/f(A)$ *such that the diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \\ & & \downarrow \mathrm{id}_A & & \downarrow \mathrm{id}_B & & \downarrow \varphi & & \\ 0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{\pi} & B/f(A) & \longrightarrow & 0 \end{array} \tag{65}$$

commutes.

Proof

1. Assertions (a), (b), and (c) are immediate consequences of definitions.
2. Consider the diagram (62). We first construct a morphism of Abelian groups $\varphi^2: A^2 \to B^2$ and then prove its uniqueness. Let $a'' \in A^2$ be a point. We set

$$\varphi^2(a'') = g^1\varphi^1(a'), \tag{66}$$

where $a' \in A^1$ is any element such that $f^1(a') = a''$. We shall show that this equation defines a point $\varphi^2(a'') \in B^2$ independently of the choice of a'. Let $a'_1, a'_2 \in A^1$ be any two points such that $f^1(a'_1) = a''$ and $f^1(a'_2) = a''$. Then $f^1(a'_1 - a'_2) = 0$ hence $a'_1 - a'_2 = f^0(a)$ for some $a \in A^1$ (exactness of the first row). Then, however, $g^1(\varphi^1(a'_1)) = g^1(\varphi^1(a'_2)) + g^1(\varphi^1(f^0(a))) = g^1(\varphi^1(a'_2))$ because $g^1(\varphi^1(f^0(a))) = g^1(g^0(\varphi^0(a))) = 0$ (exactness of the second row). Therefore, formula (66) defines a mapping $\varphi^2: A^2 \to B^2$, and the same formula immediately implies that φ^2 satisfies the condition $\varphi^2 \circ f^1 = g^1 \circ \varphi^1$. This means that the second square of the diagram (62) commutes.
To show that the mapping φ^2 is a group morphism, take $a''_1, a''_2 \in A^2$ and $a'_1, a'_2 \in A^1$ such that $f^1(a'_1) = a''_1$ and $f^1(a'_2) = a''_2$. Then, we have $f^1(a'_1 + a'_2) = a''_1 + a''_2$, therefore

$$\varphi^2(a''_1 + a''_2) = g^1(\varphi^1(a'_1 + a'_2)) = \varphi^2(a''_1) + \varphi^2(a''_2) \tag{67}$$

since both g^1 and φ^1 are group morphisms. This proves existence of the group morphism φ^2. Its uniqueness follows from the surjectivity of f^1.
3. To prove (e) we combine (c) and (d). □

A sequence of Abelian groups $A* = \{A^i, d^i\}$ is called a *complex of Abelian groups*, or just a *complex*, if

$$d^{i+1} \circ d^i = 0 \tag{68}$$

for all i. The family of group morphisms $d* = \{d^i\}$ is called the *differential* of the complex $A*$. Condition (68) is equivalent to saying that the kernel $\operatorname{Ker} d^{i+1}$ and the image $\operatorname{Im} d^i$ satisfy $\operatorname{Im} d^i \subset \operatorname{Ker} d^{i+1}$. To simplify notation, we usually denote the Abelian group morphisms d^i by the same letter, d; condition (68) then reads $d \circ d = 0$.

Let $A* = \{A^i, d\}$ be a complex. For every index i, the complex $A*$ defines an Abelian group $H^i A*$, the ith *cohomology group* of $A*$, by

$$H^i A* = \operatorname{Ker} d^{i+1} / \operatorname{Im} d^i. \tag{69}$$

Elements of this group are called ith *cohomology classes* of the complex $A*$. Note that the complex is exact in the ith term if and only if the ith cohomology group $H^i A*$ is trivial.

If A is an Abelian group, then any exact sequence Abelian groups of the form

$$0 \longrightarrow A \xrightarrow{\varepsilon} B^0 \xrightarrow{d} B^1 \xrightarrow{d} B^2 \xrightarrow{d} \cdots \tag{70}$$

is called a *resolution* of A. A resolution (70) defines a nonnegative complex $B* = \{B^i, d\}$ as

$$0 \longrightarrow B^0 \xrightarrow{d} B^1 \xrightarrow{d} B^2 \xrightarrow{d} B^3 \xrightarrow{d} \cdots \tag{71}$$

such that

$$H^0 B* = A, \quad H^i B* = 0, \quad i \geq 1. \tag{72}$$

Using this complex, the resolution can also be expressed in a shortened form

$$0 \longrightarrow A \xrightarrow{\varepsilon} B* \tag{73}$$

Let $A* = \{A^i, d\}$ and $B* = \{B^i, d'\}$ be two complexes, and let $\Phi = \{\varphi^i\}$ be a family of Abelian group morphisms $\varphi^i \colon A^i \to B^i$. These complexes and group morphisms can be expressed by the diagram

$$\begin{array}{ccccccccc}
\ldots & \longrightarrow & A^{i-1} & \xrightarrow{d} & A^i & \xrightarrow{d} & A^{i+1} & \longrightarrow & \ldots \\
 & & \downarrow\varphi^{i-1} & & \downarrow\varphi^i & & \downarrow\varphi^{i+1} & & \\
\ldots & \longrightarrow & B^{i-1} & \xrightarrow{d'} & B^i & \xrightarrow{d'} & B^{i+1} & \longrightarrow & \ldots
\end{array} \tag{74}$$

If all squares in this diagram commute, that is,

$$\varphi^{i+1} \circ d = d' \circ \varphi^i, \tag{75}$$

then we say that Φ is a *morphism of the complex* A^* into B^*. Property (75) can also be expressed by writing $\Phi : A^* \to B^*$. The *composition* of two morphisms Φ and Ψ, defined in an obvious way, and is denoted by $\Psi \circ \Phi$.

As before, the asterisk in the following lemma denotes position of indices, labeling different elements of Abelian groups belonging to a complex.

Lemma 10 *Let* $A^* = \{A^i_j, d^i_j\}$ *and* $A_* = \{A^j_i, \delta^j_i\}$ *be two families of nonnegative complexes. Suppose that we have a commutative diagram*

$$\begin{array}{ccccccccccc}
 & & 0 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A^0_0 & \xrightarrow{d^0_0} & A^1_0 & \xrightarrow{d^1_0} & A^2_0 & \xrightarrow{d^2_0} & A^3_0 & \longrightarrow & \cdots \\
 & & \downarrow \delta^0_0 & & \downarrow \delta^1_0 & & \downarrow \delta^2_0 & & \downarrow \delta^3_0 & & \\
0 & \longrightarrow & A^0_1 & \xrightarrow{d^0_1} & A^1_1 & \xrightarrow{d^1_1} & A^2_1 & \xrightarrow{d^2_1} & A^3_1 & \longrightarrow & \cdots \\
 & & \downarrow \delta^0_1 & & \downarrow \delta^1_1 & & \downarrow \delta^2_1 & & \downarrow \delta^3_1 & & \\
0 & \longrightarrow & A^0_2 & \xrightarrow{d^0_2} & A^1_2 & \xrightarrow{d^1_2} & A^2_2 & \xrightarrow{d^2_2} & A^3_2 & \longrightarrow & \cdots \\
 & & \downarrow \delta^0_2 & & \downarrow \delta^1_2 & & \downarrow \delta^2_2 & & \downarrow \delta^3_2 & & \\
0 & \longrightarrow & A^0_3 & \xrightarrow{d^0_3} & A^1_3 & \xrightarrow{d^1_3} & A^2_3 & \xrightarrow{d^2_3} & A^3_3 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & &
\end{array} \tag{76}$$

such that all its rows (resp. columns) except possibly the first row (resp. column) are exact sequences of Abelian groups. Then for each $q \geq 0$, *the cohomology groups* $H^q A_0{}^*$ *and* $H^q A^0_*$ *are isomorphic.*

Proof Let $q = 0$ and let $[a] \in H^0 A^0_* = \operatorname{Ker} \delta^0_0$. Then $[a] = a$, $\delta^0_0(a) = 0$ hence $\delta^1_0 d^0_0(a) = d^0_1 \delta^0_0(a) = 0$ and injectivity of δ^1_0 implies $d^0_0(a) = 0$, that is, $a \in \operatorname{Ker} d^0_0 = H^0 A_0{}^*$. Thus, $H^0 A^0_* \subset H^0 A_0{}^*$. The opposite inclusion is obtained in the same way.

Consider the case $q \geq 1$. Let $[a] \in H^q A^0_* = \operatorname{Ker} \delta^0_q / \operatorname{Im} \delta^0_{q-1}$, and let a be a representative of $[a]$. Then $\delta^0_q(a) = 0$ hence $\delta^1_q d^0_q(a) = d^0_{q-1} \delta^0_q(a) = 0$, that is, $d^0_q(a) \in \operatorname{Ker} \delta^1_q = \operatorname{Im} \delta^1_{q-1}$, and for some $b_1 \in A^1_{q-1}$,

$$d^0_q(a) = \delta^1_{q-1}(b_1). \tag{77}$$

But $\delta^2_{q-1} d^1_{q-1}(b_1) = d^1_q \delta^1_{q-1}(b_1) = d^1_q d^0_q(a) = 0$ and $d^1_{q-1}(b_1) \in \operatorname{Ker} \delta^2_{q-1} = \operatorname{Im} \delta^2_{q-2}$. Thus, for some $b_2 = A^1_{q-1}$ we have $d^1_{q-1}(b_1) = \delta^2_{q-2}(b_2)$.

Suppose that for some k, $1 \le k \le q-2$, and A^k_{q-k}, there exists $b_k \in A^k_{q-k}$ such that $d^k_{q-1}(b_k) = \delta^{k+1}_{q-k-1}(b_{k+1})$. Then

$$\delta^{k+2}_{q-k-1} d^{k+1}_{q-k-1}(b_{k+1}) = d^{k+1}_{q-k}\delta^{k+1}_{q-k-1}(b_{k+1}) = d^{k+1}_{q-k} d^k_{q-k}(b_k) = 0 \tag{78}$$

hence $d^{k+1}_{q-k-1}(b_{k+1}) \in \operatorname{Ker} \delta^{k+2}_{q-k-1} = \operatorname{Im} \delta^{k+2}_{q-k-2}$. Thus for some $b_{k+2} \in A^{k+1}_{q-k-2}$,

$$d^{k+1}_{q-k-1}(b_{k+1}) = \delta^{k+2}_{q-k-2}(b_{k+2}). \tag{79}$$

The construction is described by the following part of diagram (76):

$$\begin{array}{cccccccc}
 & & & & b_{k+2} & A^{k+2}_{q-k-2} \\
 & & & & & \downarrow \delta^{k+2}_{q-k-2} \\
 & & b_{k+1} & A^{k+1}_{q-k-1} & \xrightarrow{d^{k+1}_{q-k-1}} & A^{k+2}_{q-k-1} \\
 & & & \downarrow \delta^{k+1}_{q-k-1} & & \downarrow \delta^{k+2}_{q-k-1} \\
b_k & A^k_{q-k} & \xrightarrow{d^k_{q-k}} & A^{k+1}_{q-k} & \xrightarrow{d^{k+1}_{q-k}} & A^{k+2}_{q-k} \\
 & \downarrow \delta^k_{q-k} & & \downarrow \delta^{k+1}_{q-k} \\
 & A^k_{q-k+1} & \xrightarrow{d^k_{q-k+1}} & A^{k+1}_{q-k+1}
\end{array} \tag{80}$$

For $k = q-2$, formula (79) gives $d^{q-1}_1(b_{q-1}) = \delta^q_0(b_q)$ hence

$$\delta^{q+1}_0 d^q_0(b_q) = d^q_1 \delta^q_0(b_q) = d^q_1 d^{q-1}_1(b_{q-1}) = 0, \tag{81}$$

and injectivity of δ^{q+1}_0 implies $d^q_0(b_q) = 0$ hence $b_q \in \operatorname{Ker} d^q_0$. Thus, to a representative a of a class $[a] \in H^q A^0_*$ we have constructed a sequence $(b_1, b_2, \ldots, b_q)$ such that $b_i \in A^i_{q-1}$ for each i, $b_q \in \operatorname{Ker} d^q_0$, and

$$d^0_q(a) = \delta^1_{q-1}(b_1), \quad d^k_{q-k}(b_k) = \delta^{k+1}_{q-k-1}(b_{k+1}). \tag{82}$$

Let a' be another representative of the class $[a]$, and let $(b'_1, b'_2, \ldots, b'_q)$ be another sequence satisfying condition (82),

$$d^0_q(a') = \delta^1_{q-1}(b'_1), \quad d^k_{q-k}(b'_k) = \delta^{k+1}_{q-k-1}(b'_{k+1}). \tag{83}$$

We set $a'' = a - a'$, $b''_i = b_i - b'_i$. We wish to show that $[b''_q] = 0$ hence $b''_q \in \operatorname{Im} d^{q-1}_0$. By definition $[a''] = 0$ hence $a'' \in \operatorname{Im} \delta^0_{q-1}$ and $a'' = \delta^0_{q-1}(c_1)$ for some $c_1 \in A^0_{q-1}$. But by (82) and (83), $\delta^1_{q-1}(b''_1) = d^0_q(\delta^0_{q-1}(c_1)) = \delta^1_{q-1} d^0_{q-1}(c_1)$, which implies $b''_1 - d^0_{q-1}(c_1) \in \text{Ker}\, \delta^1_{q-1} = \text{Im}\ \delta^1_{q-2}$, hence for some $c_2 \in A^1_{q-2}$,

$$b''_1 - d^0_{q-1}(c_1) = \delta^1_{q-2}(c_2). \tag{84}$$

Now suppose that for some $k \geq 1$ and some $c_k \in A^{k-1}_{q-k}$ there exists $c_{k+1} \in A^{k-1}_{q-k-1}$ such that $b''_k - d^{k-1}_{q-k}(c_k) = \delta^k_{q-k-1}(c_{k+1})$. Using (82), (83) and (84),

$$\begin{aligned} \delta^{k+1}_{q-k-1}(b''_{k+1}) &= d^k_{q-k}(b''_k) = d^k_{q-k}(\delta^k_{q-k-1}(c_{k+1}) + d^{k-1}_{q-k}(c_k)) \\ &= d^k_{q-k}\delta^k_{q-k-1}(c_{k+1}) = \delta^{k+1}_{q-k-1} d^k_{q-k-1}(c_{k+1}), \end{aligned} \tag{85}$$

so that $b''_{k+1} - d^k_{q-k-1}(c_{k+1}) \in \text{Ker}\, \delta^{k+1}_{q-k-1} = \text{Im}\ \delta^{k+1}_{q-k-2}$. Thus, for some element $c_{k+2} \in A^{k+1}_{q-k-2}$

$$b''_{k+1} - d^k_{q-k-1}(c_{k+1}) = \delta^{k+1}_{q-k-2}(c_{k+2}). \tag{86}$$

The derivation of this formula includes the following part of diagram (76) of Lemma 10:

$$\begin{array}{ccccccccc} & & & & & & c_{k+2} & A^{k+1}_{q-k-2} \\ & & & & & & & \downarrow \delta^{k+1}_{q-k-2} \\ & & & & c_{k+1} & A^k_{q-k-1} & \overset{d^k_{q-k}}{\longrightarrow} & A^{k+1}_{q-k-1} \\ & & & & & \downarrow \delta^k_{q-k-1} & & \downarrow \delta^{k+1}_{q-k-1} \\ c_{k+1} & A^{k-1}_{q-k} & \overset{d^{k-1}_{q-k}}{\longrightarrow} & & & A^k_{q-k} & \overset{d^k_{q-k}}{\longrightarrow} & A^{k+1}_{q-k} \\ & \downarrow \delta^{k-1}_{q-k} & & & & \downarrow \delta^k_{q-k} \\ & A^{k-1}_{q-k+1} & \overset{d^{k-1}_{q-k+1}}{\longrightarrow} & & & A^k_{q-k-1} \end{array} \tag{87}$$

If $k = q - 2$, formula (86) gives for some $c_q \in A^{q-1}_0$

$$b''_{q-1} - d^k_1(c_{q-1}) = \delta^{q-1}_0(c_q). \tag{88}$$

Then by (82), (83) and (88)

$$\begin{aligned} \delta^q_0(b''_q) &= d^{q-1}_1(b''_{q-1}) = d^{q-1}_1(\delta^{q-1}_0(c_q) + d^{q-2}_1(c_{q-1})) \\ &= d^{q-1}_1\delta^{q-1}_0(c_q) = \delta^q_0 d^{q-1}_0(c_q), \end{aligned} \tag{89}$$

that is, $b''_q - d^{q-1}_0(c_q) = 0$ because δ^q_0 is injective. Therefore, $b''_q \in \text{Im}\ d^{q-1}_0$.

Consequently, equation

$$f^q([a]) = [b_q] \tag{90}$$

defines a mapping f^q: $H^q A_*^0 \to H^q A_0{}^*$ which is a morphism of Abelian groups. In the same way, we define a morphism of Abelian groups f_q: $H^q A_0{}^* \to H^q A_*^0$, and it remains to verify that the morphism f_q is the inverse of f^q.

Let $[b] \in H^q A_0{}^*$ be a class, represented by an element b. There exists a sequence $(a_1, a_2, \ldots, a_q)$, where $a_i \in A_i^{q-1}$, such that

$$\delta_0^q(b) = d_1^{q-1}(a_1), \quad \delta_k^{q-k}(a_k) = d_{k+1}^{q-k-1}(a_{k+1}), \tag{91}$$

where $k = 1, 2, \ldots, q-1$. By definition,

$$f_q([b]) = [a_q]. \tag{92}$$

Let $[b] = [b_q]$, where $[b_q]$ is determined by (90). Taking $a_1 = b_{q-1}$, $a_2 = b_{q-2}$, ..., $a_{q-1} = b_1$, $a_q = a$ we get from (77) and (79) that (91) is satisfied. Consequently, $[a_q] = [a]$ proving that f_q is the inverse of f^q.

This completes the proof of Lemma 10. □

Now we consider three complexes $A^* = \{A^i, d^i\}$, $B^* = \{B^i, \delta^i\}$ and $C^* = \{C^i, \Delta^i\}$ and two morphisms of complexes Φ: $A^* \to B^*$, $\Phi = \{\varphi^i\}$, and Ψ: $B^* \to C^*$, $\Psi = \{\psi^i\}$ between them. The composition of these morphisms yields a morphism of complexes $\Psi \circ \Phi$: $A^* \to C^*$, defined by

$$(\Psi \circ \Phi)^q = \psi^q \circ \varphi^q. \tag{93}$$

We show that under some exactness hypothesis these morphisms induce an exact sequence of Abelian groups, formed by cohomology groups of these complexes. Note that the morphism Φ induces the diagrams

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \operatorname{Im} d^{i-1} & \longrightarrow & \operatorname{Ker} d^i & \longrightarrow & H^i A^* & \longrightarrow & 0 \\ & & \downarrow \varphi^i & & \downarrow \varphi^i & & \downarrow \varphi_i & & \\ 0 & \longrightarrow & \operatorname{Im} \delta^{i-1} & \longrightarrow & \operatorname{Ker} \delta^i & \longrightarrow & H^i B^* & \longrightarrow & 0 \end{array} \tag{94}$$

where the first two vertical arrows are the restrictions of the morphism φ^i to the subgroups of A^i, the mappings $\operatorname{Im} d^{i-1} \to \operatorname{Ker} d^i$ and $\operatorname{Im} \delta^{i-1} \to \operatorname{Ker} \delta^i$ are the canonical inclusions, and φ_i is the unique morphism of Abelian groups for which the second square in the diagram (94) commutes (Lemma 10, (e)).

The following statement is sometimes referred to as the *zig-zag lemma*. Its proof is based on the technique known as the *diagram chasing*.

Lemma 11 *Let* $A^* = \{A^i, d^i\}$, $B^* = \{B^i, \delta^i\}$ *and* $C^* = \{C^i, \Delta^i\}$ *be three non-negative complexes,* $\Phi\colon A^* \to B^*$, $\Phi = \{\varphi^i\}$, *and* $\Psi\colon B^* \to C^*$, $\Psi = \{\psi^i\}$ *morphisms of complexes. Suppose that we have a commutative diagram*

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A^0 & \xrightarrow{d^0} & A^1 & \xrightarrow{d^1} & A^2 & \xrightarrow{d^2} & \cdots \\
 & & \downarrow \varphi^0 & & \downarrow \varphi^1 & & \downarrow \varphi^2 & & \\
0 & \longrightarrow & B^0 & \xrightarrow{\delta^0} & B^1 & \xrightarrow{\delta^1} & B^2 & \xrightarrow{\delta^2} & \cdots \\
 & & \downarrow \psi^0 & & \downarrow \psi^1 & & \downarrow \psi^2 & & \\
0 & \longrightarrow & C^0 & \xrightarrow{\Delta^0} & C^1 & \xrightarrow{\Delta^1} & C^2 & \xrightarrow{\Delta^2} & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array} \tag{95}$$

with exact columns. Then for every $q \geq 0$, *there exists a morphism of sequences of Abelian groups* $\partial = \{\partial^q\}$, $\partial^q\colon H^qC^* \to H^{q+1}A^*$ *such that the sequence of Abelian groups*

$$\begin{array}{l}
0 \longrightarrow H^0A^* \xrightarrow{\varphi^0} H^0B^* \xrightarrow{\psi^0} H^0C^* \xrightarrow{\partial^0} H^1A^* \\
\xrightarrow{\varphi_1} H^1B^* \xrightarrow{\psi_1} H^1C^* \xrightarrow{\partial^1} H^2A^* \xrightarrow{\varphi^2}
\end{array} \tag{96}$$

is exact.

Proof

1. First, we construct the group morphisms $\partial^q\colon H^qC^* \to H^{q+1}A^*$. Consider the following commutative diagram

$$\begin{array}{ccccccc}
0 & & 0 & & 0 & & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
A^{q-1} & \xrightarrow{d^{q-1}} & A^q & \xrightarrow{d^q} & A^{q+1} & \xrightarrow{d^{q+1}} & A^{q+2} \\
\downarrow \varphi^{q-1} & & \downarrow \varphi^q & & \downarrow \varphi^{q+1} & & \downarrow \varphi^{q+2} \\
B^{q-1} & \xrightarrow{\delta^{q-1}} & B^q & \xrightarrow{\delta^q} & B^{q+1} & \xrightarrow{\delta^{q+1}} & B^{q+2} \\
\downarrow \psi^{q-1} & & \downarrow \psi^q & & \downarrow \psi^{q+1} & & \\
C^{q-1} & \xrightarrow{\Delta^{q-1}} & C^q & \xrightarrow{\Delta^q} & C^{q+1} & & \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & & 0 & & 0 & &
\end{array} \tag{97}$$

Let $[c] \in H^qC^* = \operatorname{Ker}\Delta^q/\operatorname{Im}\Delta^q$ be a class, represented by an element $c \in \operatorname{Ker}\Delta^q$. Since ψ^q is surjective, there exists an element $b \in B^q$ such that $\psi^q(b) = c$. But $\psi^{q+1}\delta^q(b) = \Delta^q\psi^q(b) = 0$ so that $\delta^q(b) \in \operatorname{Ker}\psi^{q+1}$ and by exactness of the third column, there exists an element $a \in A^{q+1}$ such that $\delta^q(b) = \varphi^{q+1}(a)$. Since $\varphi^{q+2}d^{q+1}(a) = \delta^{q+1}\varphi^{q+1}(a) = \delta^{q+1}\delta^q(a) = 0$, and since φ^{q+2} is injective, $d^{q+1}(a) = 0$ and $a \in \operatorname{Ker} d^{q+1}$. Thus, given $c \in \operatorname{Ker} \Delta^q$, there exists $b \in B^q$ and $a \in \operatorname{Ker} d^{q+1}$ such that

$$c = \psi^q(b), \quad \delta^q(b) = \varphi^{q+1}(a). \tag{98}$$

If c' is some other representative of the class $[c]$, then there exist $b' \in B^q$, $a' \in \operatorname{Ker} d^{q+1}$ and $d \in C^{q-1}$ such that

$$c' = \psi^q(b'), \quad \delta^q(b') = \varphi^{q+1}(a'), \quad c' = c - \Delta^{q-1}(d). \tag{99}$$

We show that $[a] = [a']$. We have $d = \psi^{q-1}(b_0)$ for some $b_0 \in B^{q-1}$ (by surjectivity of ψ^{q-1}). Thus, $\psi^q\delta^{q-1}(b_0) = \Delta^{q-1}\psi^{q-1}(b_0) = \Delta^{q-1}(d)$, and the third formula (97) gives $\psi^q(b' - b + \delta^{q-1}(b_0)) = 0$, that is, by exactness of the column, $b' - b + \delta^{q-1}(b_0) \in \operatorname{Im} \varphi^q$. Thus, $b' - b + \delta^{q-1}(b_0) = \varphi^q(a_0)$ for some $a_0 \in A^q$. But $\delta^q(b' - b + \delta^{q-1}(b_0)) = \delta^q\varphi^q(a_0) = \varphi^{q+1}d^q(a_0)$ by commutativity of the diagram (97). Applying (98) and (99) and the property $\delta^{q+1}\delta^q = 0$ of the complex B^* one obtains $\varphi^{q+1}(a') - \varphi^{q+1}(a) = \varphi^{q+1}d^q(a_0)$. Finally, injectivity of φ^{q+1} yields $a' - a = d^q(a_0)$. This proves that $[a] = [a']$.
Now since the class $[a]$ is defined independently of the choice of the representative c of the class $[c]$, we may define a mapping ∂^q of H^qC^* into $H^{q+1}A^*$ by the formula

$$\partial^q([c]) = [a]. \tag{100}$$

It is easily verified that this mapping is an Abelian group morphism. Let c_1 be a representative of a class $[c_1]$ in H^qC^*. There exists $b_1 \in B^q$ and $a_1 \in \operatorname{Ker} d^{q+1}$ such that $c_1 = \psi^q(b_1)$, $\delta^q(b_1) = \varphi^{q+1}(a_1)$. Similarly, let c_2 be a representative of a class $[c_2]$ in H^qC^*. There exist elements $b_2 \in B^q$ and $a_2 \in \operatorname{Ker} d^{q+1}$ such that $c_2 = \psi^q(b_2)$, $\delta^q(b_2) = \varphi^{q+1}(a_2)$. Then

$$c_1 + c_2 = \psi^q(b_1 + b_2), \quad \delta^q(b_1 + b_2) = \varphi^{q+1}(a_1 + a_2), \tag{101}$$

proving that ∂^q is a group morphism.

2. Now we prove exactness of the sequence of Abelian groups (96). We proceed in several steps.

   (a) Exactness at $H^0A^* = \operatorname{Ker} d^0$ is obvious: Since $H^0B^* = \operatorname{Ker} \delta^0$ and the commutativity of the left upper square in the diagram (95) implies $\varphi^0(\operatorname{Ker} d^0) \subset \operatorname{Ker} \delta^0$, exactness at H^0A^* follows from injectivity of φ^0.
   (b) We verify exactness at the term H^0B^*. Let $b \in H^0B^* = \operatorname{Ker} \delta^0$ and $b \in \operatorname{Ker} \psi^0$. Then $b = \varphi^0(a)$ for some $a \in A_0 = H^0A^*$, and we want to

show that $a \in \operatorname{Ker} d^0$. But $\varphi^1 d^0(a) = \delta^0\varphi^0(a) = \delta^0(b) = 0$ hence $d^0(a) = 0$ (injectivity of φ^1) and $a \in \operatorname{Ker} d^0 = H^0A^*$. Thus $\operatorname{Ker}\psi^0 = \operatorname{Im}\varphi^0$.

(c) We prove exactness at H^0C^*. Consider an element $c \in H^0C^*$ such that $c \in \operatorname{Ker}\Delta^0$, that is, $\partial^0 c = 0$. We want to show that $c = \psi^0(b)$ for some $b \in H^0B^* = \operatorname{Ker}\delta^0$. By definition, $\partial^0 c = [a]$, where $a \in \operatorname{Ker} d^1$ is an arbitrary point such that for some $b' \in B^0$, $c = \psi^0(b')$ and $\delta^0(b') = \varphi^1(a)$ (98). But $[a] = 0$ hence $a \in \operatorname{Im} d^0$ and $a = d^0(a')$ for some $a' \in A^0$. Consequently, $\delta^0(b') = \varphi^1 d^0(a') = \delta^0\varphi^0(a')$. We set $b = b' - \varphi^0(a')$. Then

$$\delta^0(b) = \delta^0(b') - \delta^0\varphi^0(a') = 0, \tag{102}$$

that is, $b \in \operatorname{Ker}\delta^0$. Moreover,

$$\psi^0(b) = \psi^0(b') - \psi^0\varphi^0(a') = \psi^0(b') = c, \tag{103}$$

thus $\operatorname{Ker}\delta^0 \subset \operatorname{Im}\psi^0$.

Conversely, if $c \in \operatorname{Im}\psi^0$, then $c = \psi^0(b)$ for some $b \in H^0B^* = \operatorname{Ker}\delta^0$, and $\partial^0(c) = [a]$, where $c = \psi^0(b')$ and $\delta^0(b') = \varphi^1(a)$ for some $b' \in B^0$, $a \in \operatorname{Ker} d^1$(98). But $\psi^0(b - b') = 0$ hence $b - b' = \varphi^0(a')$, where $a' \in A^0$. Now $\varphi^1 d^0(a') = \delta^0\varphi^0(a') = \delta^0(b - b') = -\delta^0(b') = -\varphi^1(a)$ that is, by injectivity, $d^0(a') = -a$. Hence $[a] = -[d^0(a')] = 0$ and we get $\operatorname{Im}\psi^0 \subset \operatorname{Ker}\partial^0$.

Summarizing, $\operatorname{Im}\psi^0 = \operatorname{Ker}\partial^0$ as required.

(d) We check exactness at H^qA^*, where $q > 0$. Let $[a] \in H^qA^*$ and $\varphi_q([a]) = 0$. Since $\varphi_q([a]) = [\varphi^q(a)] = 0$, we have $\varphi^q(a) \in \operatorname{Im}\delta^{q-1}$. Thus, there exists $b \in B^{q-1}$ such that $\delta^{q-1}(b) = \varphi^q(a)$. We set $c = \psi^{q-1}(b)$. Then by definition, $\partial([c]) = [a]$, therefore $\operatorname{Ker}\varphi_q \subset \operatorname{Im}\partial^{q-1}$.

Conversely, consider a class $[c] \in H^{q-1}C^*$. Then $\varphi_q\delta^{q-1}([c]) = \varphi_q([a])$, where $c = \psi^{q-1}(b)$, $\delta^{q-1}(b) = \varphi^q(a)$ for some $b \in B^{q-1}$, $a \in \operatorname{Ker} d^q$. But then $\varphi_q\delta^{q-1}([c]) = [\varphi^q(a)] = [\delta^{q-1}(b)] = 0$ since $H^qB^* = \operatorname{Ker}\delta^q/\operatorname{Im}\delta^{q-1}$.

(e) We prove exactness at H^qB^*, $q > 0$. Let $[b] \in H^qB^*$ be a class such that $\psi_q([b]) = [\psi^q(b)] = 0$. Then $\psi^q(b) \in \operatorname{Im}\Delta^{q-1}$ hence there exists $c \in C^{q-1}$ such that $\psi^q(b) = \Delta^{q-1}(c)$. But $c = \psi^{q-1}(b')$ for some $b' \in B^{q-1}$; applying Δ^{q-1} we have $\Delta^{q-1}\psi^{q-1}(b') = \psi^q\delta^{q-1}(b')$, that is, $\psi^q(b) = \psi^q\delta^{q-1}(b')$ hence $\psi^q(b - \delta^{q-1}(b')) = 0$ and $b - \delta^{q-1}(b') = \varphi^q(a)$ for some $a \in A^q$. Now $\varphi^{q+1}d^q(a) = \delta^q\varphi^q(a) = \delta^q(b - \psi^q\delta^{q-1}(b')) = 0$ because $\delta^q(a) = 0$, $\delta^q\delta^{q-1} = 0$. Hence $d^q(a) = 0$ and $a \in \operatorname{Ker} d^q$. Now

$$\varphi_q([a]) = [\varphi^q(a)] = [b - \delta^{q-1}(b')] = [b], \tag{104}$$

so we get the inclusion $\operatorname{Ker}\psi_q \subset \operatorname{Im}\varphi_q$.
The inverse inclusion follows from the equality $\psi_q \circ \varphi^q = 0$ and from the diagram (94), which implies

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \operatorname{Im} d^{q-1} & \longrightarrow & \operatorname{Ker} d^q & \longrightarrow & H^qA^* & \longrightarrow & 0 \\
 & & \downarrow \varphi^q & & \downarrow \varphi^q & & \downarrow \varphi_q & & \\
0 & \longrightarrow & \operatorname{Im} \delta^{q-1} & \longrightarrow & \operatorname{Ker} \delta^q & \longrightarrow & H^qB^* & \longrightarrow & 0 \\
 & & \downarrow \psi^q & & \downarrow \psi^q & & \downarrow \psi_q & & \\
0 & \longrightarrow & \operatorname{Im} \Delta^{q-1} & \longrightarrow & \operatorname{Ker} \delta^q & \longrightarrow & H^qC^* & \longrightarrow & 0
\end{array} \tag{105}$$

in which the group morphisms φ_q and ψ_q are unique, and the composition law $(\Psi \circ \Phi)^q = \psi^q \circ \varphi^q$ (93) holds.

(f) We prove exactness at H^qC^*, where $q > 0$. Let $[c] \in H^qC^*$ be a class such that $\partial^q([c]) = 0$. We want to show that there exists $[b] \in H^qB^*$ such that $[c] = \psi_q([b])$. Let c be a representative of $[c]$. By (98), there exist an element $b \in B^q$ and $a \in \operatorname{Ker} d^{q+1}$ such that $c = \psi^q(b)$, $\delta^q(b) = \varphi^{q+1}(a)$. From the condition $\partial^q([c]) = 0$, it follows that $[a] = 0$ hence $a \in \operatorname{Im} d^q$ and $a = d^q(a')$ for some $a' \in A^q$. Then $\delta^q(b) = \varphi^{q+1}d^q(a') = \delta^q\varphi^q(a')$ hence $b - \varphi^q(a') \in \operatorname{Ker}\delta^q$. Setting $b' = b - \varphi^q(a')$ we have $\delta^q(b') = 0$, $b' \in \operatorname{Ker}\delta^q$. Moreover, $\psi^q(b') = \psi^q(b - \varphi^q(a')) = \psi^q(b) = c$, therefore

$$\psi_q([b']) = [\psi^q(b')] = [c]. \tag{106}$$

This implies that $\operatorname{Ker}\partial^q \subset \operatorname{Im}\psi_q$.

Conversely, let $[c] \in \operatorname{Im}\psi_q$. Then $[c] = \psi_q([b]) = [\psi^q(b)]$ for some element $[b] \in H^qB^*$. Thus $\partial^q([c]) = [a]$, where $c = \psi^q(b')$, $\delta^q(b') = \varphi^{q+1}(a)$ for some $b' \in B^q$. But $\psi^q(b - b') = 0$ so that $b - b' = \varphi^q(a')$, where $a' \in A^q$. Now

$$\varphi^{q+1}d^q(a') = \delta^q\varphi^q(a') = \delta^q(b - b') = -\delta^q(b') \tag{107}$$

hence $\varphi^{q+1}(a) = -\varphi^{q+1}d^q(a')$, $\varphi^{q+1}(a + d^q(a')) = 0$, and $a + d^q(a') = 0$. Hence $[a] = -[d^q(a')] = 0$, therefore $\operatorname{Im}\psi_q \subset \operatorname{Ker}\partial^q$. This completes the proof. □

The exact sequence of Abelian groups (96) is referred to as the *long exact sequence*, associated with the morphisms of complexes $\Phi\colon A^* \to B^*$ and $\Psi\colon B^* \to C^*$. The family of Abelian group morphisms $\partial = \{\partial^q\}$, where

$\partial^q\colon H^qC^* \to H^{q+1}A^*$, is called the *connecting morphism*, associated to the morphisms Φ and Ψ.

The following two corollaries follow from the long exact sequence (96).

Corollary 4 *Suppose that in the commutative diagram of morphisms of Abelian groups*

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A^0 & \longrightarrow & A^1 & \longrightarrow & A^2 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & B^0 & \longrightarrow & B^1 & \longrightarrow & B^2 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & C^0 & \longrightarrow & C^1 & \longrightarrow & C^2 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array} \tag{108}$$

all columns are exact. Then if two rows are exact, the third row is also exact.

Corollary 5 *Let A^*, B^* and C^* be three nonnegative complexes, $\Phi\colon A^* \to B^*$ and $\Psi\colon B^* \to C^*$ morphisms of complexes. Suppose that the diagram (95) commutes and all its columns are exact. Then if any two of the complexes A^*, B^*, and C^* are exact, the third is also exact.*

Proof This follows from the long exact sequence (96). □

7.8 Exact Sequences of Abelian Sheaves

The concepts we have introduced for sequences of Abelian groups apply to sequences of Abelian sheaves. First, we briefly formulate the definitions and describe basic properties of exact sequences. Then, we study the *canonical resolution* of an Abelian sheaf, an exact sequence, relating properties of a sheaf with topological properties of its base space.

A family $\mathcal{S}^* = \{\mathcal{S}^i, f^i\}_{i\in \boldsymbol{Z}}$ of Abelian sheaves $\mathcal{S}^i$ over the same base, and their morphisms $f^i\colon \mathcal{S}^i \to \mathcal{S}^{i+1}$, indexed with the integers $i \in \boldsymbol{Z}$, is called a *sequence of Abelian sheaves*. The family of sheaf morphisms in this sequence is denoted by $\{f^i\}_{i\in \boldsymbol{Z}}$. The sequence $\mathcal{S}^*$ is called a *nonnegative*, if $\mathcal{S}^i = 0$ for all $i<0$. Then, the sequence $\mathcal{S}^*$ is usually written as $\mathcal{S}^* = \{\mathcal{S}^i, f^i\}_{i\in \boldsymbol{N}}$, with indexing set the nonnegative integers $\boldsymbol{N}$, or just as

$$0 \longrightarrow \mathcal{S}^0 \xrightarrow{f^0} \mathcal{S}^1 \xrightarrow{f^1} \mathcal{S}^2 \xrightarrow{f^2} \cdots \tag{109}$$

In this notation, the mapping $0 \to \mathcal{S}^0$ is the *trivial* sheaf morphism. If there exist the smallest and greatest integers r and s such that $\mathcal{S}^r \neq 0$ and $\mathcal{S}^s \neq 0$, then the sequence $\mathcal{S}*$ is said to be *finite*, and $\mathcal{S}^r$ (resp. $\mathcal{S}^s$) is called its *first* (resp. *last*) element. In this case, we write $\mathcal{S}*$ as

$$0 \longrightarrow \mathcal{S}^r \xrightarrow{f^r} \mathcal{S}^{r+1} \xrightarrow{f^{r+1}} \cdots \xrightarrow{f^{s-1}} \mathcal{S}^s \longrightarrow 0 \tag{110}$$

with trivial sheaf morphisms $0 \to \mathcal{S}^r$ and $\mathcal{S}^s \to 0$. To further simplify notation, we sometimes omit the indexing set and write just $\mathcal{S}* = \{\mathcal{S}^i, f^i\}$, or $\mathcal{S}* = \{\mathcal{S}^i, f^i\}$ instead of $\mathcal{S}* = \{\mathcal{S}^i, f^i\}_{i \in \boldsymbol{N}}$.

Let $\mathcal{S}* = \{\mathcal{S}^i, f^i\}$ be a family of sheaves of Abelian groups over a topological space X, $x \in X$ a point. Denote by $\mathcal{S}^p_x = (\mathrm{Germ}\, \mathcal{S}^p)_x$ the *fiber* of the sheaf space $\mathrm{Germ}\, \mathcal{S}^p$ over x, and by $f^p_x \colon \mathcal{S}^p_x \to \mathcal{S}^{p+1}_x$ the restriction to the fiber of the morphism $f^i \colon \mathcal{S}^i \to \mathcal{S}^{i+1}$. Restricting all the sheaf morphisms to the fibers $\mathcal{S}^p_x$ we get a sequence of Abelian groups

$$0 \longrightarrow \mathcal{S}^0_x \xrightarrow{f^0_x} \mathcal{S}^1_x \xrightarrow{f^1_x} \mathcal{S}^2_x \xrightarrow{f^2_x} \cdots \tag{111}$$

This sequence is called the *restriction* of the sequence (109) to the point x.

The sequence $\mathcal{S}*$ (109) is said to be *exact* at the term $\mathcal{S}^q$ *over* x, if the restricted sequence (111) is exact as the sequence of Abelian groups, that is, if $\mathrm{Ker}\, f^q_x = \mathrm{Im}\, f^{q-1}_x$. $\mathcal{S}*$ is said to be *exact* at the term $\mathcal{S}^q$ if it is exact at x for every $x \in X$. We say that $\mathcal{S}*$ is an *exact sequence*, if it is exact in every term $\mathcal{S}^q$.

Let $\mathcal{S}$ be an Abelian sheaf. A sequence of Abelian sheaves $\mathcal{S}* = \{\mathcal{S}^i, f^i\}$, such that

$$f^{q+1} \circ f^q = 0 \tag{112}$$

for all q is called a *differential sequence*. An exact sequence is a differential sequence.

An exact sequence of the form

$$0 \longrightarrow \mathcal{S} \xrightarrow{\varepsilon} \mathcal{T}^0 \xrightarrow{f^0} \mathcal{T}^1 \xrightarrow{f^1} \mathcal{T}^2 \xrightarrow{f^2} \cdots \tag{113}$$

is called a *resolution* of $\mathcal{S}$. The resolution defines a nonnegative differential sequence $\mathcal{T}* = \{\mathcal{T}^i, f^i\}$. To shorten notation, we sometimes write the sequence (113) as

$$0 \longrightarrow \mathcal{S} \xrightarrow{f} \mathcal{T}*, \tag{114}$$

the mappings being understood.

An exact sequence of the form

$$0 \longrightarrow \mathcal{R} \xrightarrow{f} \mathcal{S} \xrightarrow{g} \mathcal{T} \longrightarrow 0 \tag{115}$$

where $0 \to \mathcal{R}$ and $\mathcal{T} \to 0$ are trivial sheaf morphisms, is called a *short exact sequence*.

Let Y be a subspace of the topological space X. Denote by $\mathcal{S}_Y$ the restriction of the Abelian sheaf $\mathcal{S}$ to Y and by f^i_Y the restriction of the sheaf morphism f^i: $\mathcal{S}^i \to \mathcal{S}^{i+1}$ to Y. We obtain a sequence of sheaves

$$0 \longrightarrow \mathcal{S}^0_Y \xrightarrow{f^0_Y} \mathcal{S}^1_Y \xrightarrow{f^1_Y} \mathcal{S}^2_Y \xrightarrow{f^2_Y} \cdots \tag{116}$$

called the *restriction* of the sequence $\mathcal{S}* = \{\mathcal{S}^i, f^i\}$ to the subspace Y.

The following are elementary properties of exact sequences.

Lemma 12

(a) *A sequence of Abelian sheaves $\mathcal{S}* = \{\mathcal{S}^i, f^i\}$ is exact at $\mathcal{S}^q$ if and only if* $\operatorname{Ker} f^q = \operatorname{Im} f^{q-1}$.
(b) *If a sequence of Abelian sheaves $\mathcal{S}* = \{\mathcal{S}^i, f^i\}$ over a topological space X is exact at the term $\mathcal{S}^q$, then its restriction to a subspace $Y \subset X$ is exact at $\mathcal{S}^q_Y$.*
(c) *A sequence of sheaves of the form* (115) *is exact at $\mathcal{T}$ if and only if the sheaf morphism g is surjective.*
(d) *A sequence of sheaves of the form* (115) *is exact at $\mathcal{R}$ if and only if the sheaf morphism f is injective.*
(e) *A sequence of Abelian sheaves*

$$0 \longrightarrow \mathcal{R} \xrightarrow{\imath} \mathcal{S} \xrightarrow{\pi} \mathcal{S}/\imath(\mathcal{R}) \longrightarrow 0 \tag{117}$$

where $\mathcal{R} \subset \mathcal{S}$ is a subsheaf, $\imath$: $\mathcal{R} \to \mathcal{S}$ its inclusion, $\mathcal{S}/\imath(\mathcal{R})$ the quotient sheaf and π: $\mathcal{S} \to \mathcal{S}/\imath(\mathcal{R})$ the quotient projection, is a short exact sequence.
(f) *Suppose we have a diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{R}_0 & \xrightarrow{f^0} & \mathcal{R}_1 & \xrightarrow{f^1} & \mathcal{R}_2 & \longrightarrow & 0 \\ & & \downarrow \varphi_0 & & \downarrow \varphi_1 & & & & \\ 0 & \longrightarrow & \mathcal{S}_0 & \xrightarrow{g^0} & \mathcal{S}_1 & \xrightarrow{g^0} & \mathcal{S}_2 & \longrightarrow & 0 \end{array} \tag{118}$$

such that the horizontal sequences are short exact sequences of sheaves, φ_0 and φ_1 are sheaf morphisms and

$$g^0 \circ \varphi^0 = \varphi^1 \circ f^0. \tag{119}$$

Then there exists a unique Abelian sheaf morphism φ_2: $\mathcal{R}^2 \to \mathcal{S}^2$ *such that the second square of the diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{R}_0 & \xrightarrow{f^0} & \mathcal{R}_1 & \xrightarrow{f^1} & \mathcal{R}_2 & \longrightarrow & 0 \\ & & \downarrow \varphi^0 & & \downarrow \varphi^1 & & \downarrow \varphi^2 & & \\ 0 & \longrightarrow & \mathcal{S}_0 & \xrightarrow{g^0} & \mathcal{S}_1 & \xrightarrow{g^1} & \mathcal{S}_2 & \longrightarrow & 0 \end{array} \tag{120}$$

commutes.

(g) *Consider the exact sequence of Abelian sheaves* (115), *the quotient sheaf* $\mathcal{S}/f(\mathcal{R})$ *and the quotient projection* π: $\mathcal{S} \to \mathcal{S}/f(\mathcal{R})$. *There exists a unique sheaf isomorphism* φ: $\mathcal{T} \to \mathcal{S}/f(\mathcal{R})$ *such that*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{R} & \xrightarrow{f^0} & \mathcal{S} & \xrightarrow{f^1} & \mathcal{T} & \longrightarrow & 0 \\ & & \downarrow \mathrm{id}_{\mathcal{R}} & & \downarrow \mathrm{id}_{\mathcal{S}} & & \downarrow \varphi & & \\ 0 & \longrightarrow & \mathcal{R} & \xrightarrow{g^0} & \mathcal{S} & \xrightarrow{\pi} & \mathcal{S}/f(\mathcal{R}) & \longrightarrow & 0 \end{array} \tag{121}$$

commutes.

Proof

1. We prove assertion (a). Suppose that $\mathcal{S}^*$ is exact at $\mathcal{S}^q$. Then by definition $\mathrm{Ker}\, f_x^q = \mathrm{Im}\, f_x^{q-1}$ for every x, where f_x^q is the restriction of the sheaf space morphism f^q: Germ $\mathcal{S}^q \to$ Germ $\mathcal{S}^{q+1}$, associated with f^q, to x. Thus

$$\mathrm{Ker} f^q = \bigcup_{x \in X} \mathrm{Ker} f_x^q = \bigcup_{x \in X} \mathrm{Im}\, f_x^{q-1} = \mathrm{Im}\, f^q. \tag{122}$$

Then $\mathrm{Ker}\, f^q = \mathrm{Sec}^{(c)} \mathrm{Ker}\, f^q = \mathrm{Sec}^{(c)} \mathrm{Im}\, f^{q-1} = \mathrm{Im}\, f^{q-1}$ as required. The converse is obvious.
2. Assertions (b), (c), (d), and (e) of Lemma 12 are immediate consequences of definitions.
3. To prove (f) we apply (b) and Lemma 9, (d).
4. To prove (g) we apply (b) and Lemma 9, (e). □

A sequence of Abelian sheaves (109) over a topological space X induces, for every open set U in X, the Abelian groups $\mathcal{S}^i U$ of continuous sections and their morphisms ${f_U}^i$: $\mathcal{S}^i U \to \mathcal{S}^{i+1} U$. We usually denote these morphisms by the same letters, f^i. The sequence of Abelian groups is then denoted by

$$0 \longrightarrow \mathcal{S}^0 U \xrightarrow{f^0} \mathcal{S}^1 U \xrightarrow{f^1} \mathcal{S}^2 U \xrightarrow{f^2} \cdots \tag{123}$$

and is said to be *induced* by the sequence of sheaves (109). In particular, if $U = X$, the sequence of Abelian groups

$$0 \longrightarrow \mathcal{S}^0 X \xrightarrow{f^0} \mathcal{S}^1 X \xrightarrow{f^1} \mathcal{S}^2 X \xrightarrow{f^2} \cdots \tag{124}$$

is referred to as the *sequence of global sections*, associated with the sequence of Abelian sheaves (109).

Exactness of the sequence (109) does not imply exactness of (123). This is demonstrated by the following example.

7.9 Cohomology Groups of a Sheaf

In this section, we construct a resolution of an Abelian sheaf, known as the *canonical*, or *Godement resolution* (Godement [G]). We also introduce canonical morphisms of the canonical resolutions, and study properties of the corresponding diagrams.

Consider the sheaf space Germ $\mathcal{S}$, associated with $\mathcal{S}$ and the sheaf of (not necessarily continuous) sections of the sheaf space Germ $\mathcal{S}$, denoted by

$$\mathcal{C}^0\mathcal{S} = \mathrm{SecGerm}\,\mathcal{S} \tag{125}$$

(cf. Sect. 7.4, Example 17). We have the *canonical injective sheaf morphism* ι: $\mathrm{Sec}^{(c)}\mathrm{Germ}\,\mathcal{S} \to \mathcal{C}^0\mathcal{S}$. Since $\mathrm{Sec}^{(c)}\mathrm{Germ}\,\mathcal{S}$ is canonically isomorphic with the Abelian sheaf $\mathcal{S}$, setting

$$\mathcal{D}^1\mathcal{S} = \mathcal{C}^0\mathcal{S}/\mathrm{Im}\,\iota \tag{126}$$

we get an exact sequence of sheaves

$$0 \to \mathcal{S} \xrightarrow{\iota} \mathcal{C}^0\mathcal{S} \to \mathcal{D}^1\mathcal{S} \to 0. \tag{127}$$

The same construction can be repeated for the sheaf $\mathcal{D}^1\mathcal{S}$. Replacing $\mathcal{S}$ with $\mathcal{D}^1\mathcal{S}$, we have the Abelian sheaf of (*discontinuous*) sections of the sheaf space Germ $\mathcal{D}^1\mathcal{S}$, $\mathcal{C}^0\mathcal{D}^1\mathcal{S} = \mathrm{SecGerm}\,\mathcal{D}^1\mathcal{S}$, the Abelian sheaf of *continuous sections* $\mathrm{Sec}^{(c)}\mathrm{Germ}\,\mathcal{D}^1\mathcal{S}$, canonically isomorphic with the sheaf $\mathcal{D}^1\mathcal{S}$, and the canonical sheaf morphism of continuous sections into discontinuous sections, ι^1: $\mathrm{Sec}^{(c)}\mathrm{Germ}\,\mathcal{D}^1\mathcal{S} \to \mathrm{SecGerm}\,\mathcal{D}^1\mathcal{S}$. Setting $\mathcal{D}^1(\mathcal{D}^1\mathcal{S}) = \mathcal{C}^0(\mathcal{D}^1\mathcal{S})/\mathrm{Im}\,\iota^1$ we get an exact sequence

$$0 \longrightarrow D^1 S \overset{\iota^1}{\longrightarrow} C^0 D^1 S \longrightarrow D^1 D^1 S \longrightarrow 0. \tag{128}$$

Combining these two constructions

$$\begin{array}{ccccccccccc} & & & & 0 & & & & & & \\ & & & & \downarrow & & & & & & \\ & & & & S & & & & & & \\ & & & & \downarrow & & & & & & \\ 0 & \longrightarrow & S & \longrightarrow & C^0 S & \longrightarrow & D^1 S & \longrightarrow & 0 & & \\ & & & & \downarrow & & & & & & \\ & & 0 & \longrightarrow & D^1 S & \longrightarrow & C^1 S & \longrightarrow & D^1 D^1 S & \longrightarrow & 0 \\ & & & & \downarrow & & & & & & \\ & & & & 0 & & & & & & \end{array} \tag{129}$$

Similarly we get, with obvious notation, the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & S & \longrightarrow & C^0 S & & & & \\ & & & & \downarrow & & & & \\ & & 0 & \longrightarrow & D^1 S & \longrightarrow & C^1 S & & \\ & & & & \downarrow & & \downarrow & & \\ & & & & 0 & \longrightarrow & D^2 S & \longrightarrow & C^2 S \\ & & & & & & \downarrow & & \downarrow \\ & & & & & & 0 & \longrightarrow & D^3 S & \longrightarrow & C^2 S \end{array} \tag{130}$$

etc. This diagram gives rise to the sheaf morphisms $c^p \colon C^p S \to C^{p+1} S$, for every $p \geq 0$. We get a sequence of sheaves of Abelian groups

$$0 \longrightarrow S \overset{\iota}{\longrightarrow} C^0 S \overset{c^0}{\longrightarrow} C^1 S \overset{c^1}{\longrightarrow} C^2 S \overset{c^2}{\longrightarrow} \cdots \tag{131}$$

Lemma 13 *The sequence of sheaves of Abelian group* (131) *is a resolution of the sheaf* S.

Proof We want to verify exactness. Since ι is injective, the sequence is exact at S. To check exactness at the term $C^0 S$, we use the diagram (131), where the sheaf morphism $g \colon C^0 S \to D^1 S$ is the quotient morphism and $h \colon D^1 S \to C^1 S$ is an inclusion. Let $a \in \operatorname{Im} \iota$. Evidently $a \in \operatorname{Ker} c^0$ since $c^0 = h \circ g$ and $a \in \operatorname{Ker} g$. Conversely, let $a \in \operatorname{Ker} c^0$. Then $h(g(a)) = 0$ and since h is injective, $g(a) = 0$ and $a \in \operatorname{Ker} h$ hence $a \in \operatorname{Im} \iota$. Exactness at $C^q S$ can be proved in the same way. □

The resolution (131) of the Abelian sheaf S is called the *canonical resolution*. Setting $C^* S = \{C^i S, c^i\}$, we can write the sequence (131) as

$$0 \longrightarrow \mathcal{S} \xrightarrow{\iota} \mathcal{C}^*\mathcal{S}. \tag{132}$$

The Abelian sheaves $\mathcal{C}^p\mathcal{S}$, where $p \geq 0$, in the sequence (132), have some specific properties, namely, they belong to the class of soft sheaves. A sheaf of Abelian groups $\mathcal{S}$ over a topological space X is said to be *soft* if any section of the associated sheaf space Germ $\mathcal{S}$, defined on a closed subset $Y \subset X$, can be prolonged to a global section of $\mathcal{S}$.

Lemma 14 *The sheaves $\mathcal{C}^p\mathcal{S}$, where $p \geq 0$, are soft.*

Proof It is sufficient to show that the sheaf $\mathcal{C}^0\mathcal{S} = \text{SecGerm}\,\mathcal{S}$. is soft; the same proof applies to $\mathcal{C}^p\mathcal{S}$, where $p > 0$. Let $Y \subset X$ be a closed subset, $\delta \in \mathcal{C}^0\mathcal{S}$ any section of Germ $\mathcal{S}$, defined on Y. By definition, $\delta(x)$, where x is a point of Y, is the germ of a (not necessarily continuous) section $\gamma \in \mathcal{S}U$, where U is a neighborhood of x in X; thus $\delta(x) = [\gamma]_x$. Consider a family of (not necessarily continuous) sections $\gamma_x \in \mathcal{S}U_x$ such that $\delta(x) = [\gamma_x]_x$ for all points $x \in Y$, and set

$$\tilde{\delta}(x) = \begin{cases} [\gamma_x]_x, & x \in Y, \\ 0, & x \notin Y. \end{cases} \tag{133}$$

Then $\tilde{\delta}$ is a global section of the sheaf space Germ $\mathcal{S}$. Here, 0 is the germ of the zero section, defined on the open set $X \backslash Y \subset X$.

Let $v: \mathcal{S} \to \mathcal{T}$ be a morphism of Abelian sheaves over a topological space X. We shall construct a family of sheaf morphisms $v^p: \mathcal{C}^p\mathcal{S} \to \mathcal{C}^p\mathcal{T}$, $p \geq 0$, between the canonical resolutions $0 \to \mathcal{S} \to \mathcal{C}^*\mathcal{S}$ and $0 \to \mathcal{T} \to \mathcal{C}^*\mathcal{T}$ of these sheaves, such that the diagram

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & \mathcal{S} & \xrightarrow{\iota_\mathcal{S}} & \mathcal{C}^0\mathcal{S} & \longrightarrow & \mathcal{C}^1\mathcal{S} & \longrightarrow & \mathcal{C}^2\mathcal{S} & \longrightarrow \\ & & \downarrow v & & \downarrow v^0 & & \downarrow v^1 & & \downarrow v^2 & \\ 0 & \longrightarrow & \mathcal{T} & \xrightarrow{\iota_\mathcal{T}} & \mathcal{C}^0\mathcal{T} & \longrightarrow & \mathcal{C}^1\mathcal{T} & \longrightarrow & \mathcal{C}^2\mathcal{T} & \longrightarrow \end{array} \tag{134}$$

commutes.

Let $S = \text{Germ}\,\mathcal{S}$ and $T = \text{Germ}\,\mathcal{T}$ be the associated sheaf spaces, σ and τ the corresponding sheaf space projections, and let $\tilde{v}: S \to T$ be the associated sheaf space morphism. Recall that $\tilde{v}$ is defined as the mapping $S \ni [\gamma]_x \to \tilde{v}([\gamma]_x) = [v_U(\gamma)]_x \in T$, where $\gamma \in \mathcal{S}U$ is a representative of the germ $[\gamma]_x$ (Sect. 7.5, (31)). We shall consider the Abelian sheaves $\mathcal{S}$ and $\mathcal{T}$ as the sheaves of continuous sections of the sheaf spaces S and T. Then $\mathcal{C}^0\mathcal{S}$ and $\mathcal{C}^0\mathcal{T}$ are the corresponding Abelian sheaves of *discontinuous* sections. We set for any section $\delta: U \to S$

$$v^0(\delta) = \tilde{v} \circ \delta. \tag{135}$$

This formula defines the first square in the diagram (134). If δ is a continuous section of $\mathcal{C}^0\mathcal{S}$, we have $v^0(\imath_{\mathcal{S}}(\gamma)) = \tilde{v} \circ \imath_{\mathcal{S}}(\gamma)$

$$\begin{aligned}(v^0\imath_{\mathcal{S}}(\gamma))(x) &= \tilde{v}(\imath_{\mathcal{S}}(\gamma)(x)) = \tilde{v}(\gamma(x)) = \tilde{v}([\gamma]_x) \\ &= [v_U(\gamma)]_x = v \circ \gamma(x) = \imath_{\mathcal{T}}(v \circ \gamma)(x),\end{aligned} \tag{136}$$

proving the commutativity.

Consider the next squares in the diagram (134)

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{S} & \longrightarrow & \mathcal{C}^0\mathcal{S} & \longrightarrow & \mathcal{D}^1\mathcal{S} & \longrightarrow & 0 \\ & & \downarrow v & & \downarrow v^0 & & \downarrow \bar{v}^1 & & \\ 0 & \longrightarrow & \mathcal{T} & \longrightarrow & \mathcal{C}^0\mathcal{T} & \longrightarrow & \mathcal{D}^1\mathcal{T} & \longrightarrow & 0 \end{array} \tag{137}$$

defining $\bar{v}^1$ (Lemma 12, (f)). If we replace $\mathcal{S}$ (resp. $\mathcal{T}$) with $\mathcal{D}^i\mathcal{S}$ (resp. $\mathcal{D}^i\mathcal{T}$), where $i \geq 1$, we get the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{D}^i\mathcal{S} & \longrightarrow & \mathcal{C}^i\mathcal{S} & \longrightarrow & \mathcal{D}^{i+1}\mathcal{S} & \longrightarrow & 0 \\ & & \downarrow \bar{v}^i & & \downarrow v^i & & \downarrow \bar{v}^{i+1} & & \\ 0 & \longrightarrow & \mathcal{D}^i\mathcal{T} & \longrightarrow & \mathcal{C}^i\mathcal{T} & \longrightarrow & \mathcal{D}^{i+1}\mathcal{T} & \longrightarrow & 0 \end{array} \tag{138}$$

We show that the ith square also commutes. Combining (130) and (138) and using a suitable temporary notation, we get the commutative diagrams

$$\begin{array}{ccc} \mathcal{C}^{i-1}\mathcal{S} & \xrightarrow{a} & \mathcal{D}^i\mathcal{S} \\ \downarrow v^{i-1} & & \downarrow \bar{v}^i \\ \mathcal{C}^{i-1}\mathcal{T} & \xrightarrow{b} & \mathcal{D}^i\mathcal{T} \end{array} \quad \begin{array}{ccc} \mathcal{C}^{i-1}\mathcal{S} & \xrightarrow{g} & \mathcal{C}^i\mathcal{S} \\ \downarrow v^{i-1} & & \downarrow \bar{v}^i \\ \mathcal{C}^{i-1}\mathcal{T} & \xrightarrow{h} & \mathcal{C}^i\mathcal{T} \end{array} \quad \begin{array}{ccc} \mathcal{D}^i\mathcal{S} & \xrightarrow{b} & \mathcal{C}^i\mathcal{S} \\ \downarrow \bar{v}^{i-1} & & \downarrow v^i \\ \mathcal{D}^i\mathcal{T} & \xrightarrow{d} & \mathcal{C}^i\mathcal{T} \end{array} \tag{139}$$

Combining these diagrams with (126), we obtain

$$g = b \circ a, \quad d \circ \bar{v}^i = v^i \circ b, \quad \bar{v}^i \circ a = c \circ v^{i-1}, \quad h = d \circ c, \tag{140}$$

which implies $v^i \circ g = v^i \circ b \circ a = d \circ \bar{v}^i \circ a = d \circ c \circ v^{i-1} = h \circ v^{i-1}$. Since $i \geq 1$, this proves commutativity of all squares in the diagram (134). □

The family of sheaf morphisms $\{v, v^0, v^1, v^2, \ldots\}$ is called the *canonical morphism* of the canonical resolutions $0 \to \mathcal{S} \to \mathcal{C}^*\mathcal{S}$ and $0 \to \mathcal{T} \to \mathcal{C}^*\mathcal{T}$, associated with the Abelian sheaf morphism v: $\mathcal{S} \to \mathcal{T}$.

Elementary properties of the canonical resolutions are formulated in the following lemma.

Lemma 15

(a) *The canonical resolution of a trivial Abelian sheaf* 0_X *over a topological space* X *consists of the trivial sheaves* $\mathcal{C}^p 0_X = 0_X$.
(b) *The canonical resolution associated with the identity sheaf morphism* $\mathrm{id}_{\mathcal{S}}$ *is the identity morphism* $\{\mathrm{id}_{\mathcal{S}}, \mathrm{id}_{\mathcal{C}^0\mathcal{S}}, \mathrm{id}_{\mathcal{C}^1\mathcal{S}}, \mathrm{id}_{\mathcal{C}^2\mathcal{S}}, \ldots\}$.
(c) *If the Abelian sheaf morphism* $\nu\colon \mathcal{S} \to \mathcal{T}$ *is injective (resp. surjective), then each* $\nu^p\colon \mathcal{S}^p \to \mathcal{T}^p$ *is injective (resp. surjective).*
(d) *Let* $\mathcal{R}$, $\mathcal{S}$, *and* $\mathcal{T}$ *be three Abelian sheaves with base* X, $\mu\colon \mathcal{R} \to \mathcal{S}$, $\nu\colon \mathcal{S} \to \mathcal{T}$ *two Abelian sheaf morphisms, and* $\eta = \nu \circ \mu$. *Then, the diagram*

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & \mathcal{R} & \longrightarrow & \mathcal{C}^0\mathcal{R} & \longrightarrow & \mathcal{C}^1\mathcal{R} & \longrightarrow & \mathcal{C}^2\mathcal{R} & \longrightarrow \\ & & \downarrow \mu & & \downarrow \mu^0 & & \downarrow \mu^1 & & \downarrow \mu^2 & \\ 0 & \longrightarrow & \mathcal{S} & \longrightarrow & \mathcal{C}^0\mathcal{S} & \longrightarrow & \mathcal{C}^1\mathcal{S} & \longrightarrow & \mathcal{C}^2\mathcal{S} & \longrightarrow \\ & & \downarrow \nu & & \downarrow \nu^0 & & \downarrow \nu^1 & & \downarrow \nu^2 & \\ 0 & \longrightarrow & \mathcal{T} & \longrightarrow & \mathcal{C}^0\mathcal{T} & \longrightarrow & \mathcal{C}^1\mathcal{T} & \longrightarrow & \mathcal{C}^2\mathcal{T} & \longrightarrow \end{array} \tag{141}$$

satisfies, for every $p \geq 0$,

$$\eta^p = \nu^p \circ \mu^p. \tag{142}$$

(e) *Suppose that the first column of the diagram*

$$\begin{array}{ccccccccccc} & & 0 & & 0 & & 0 & & 0 & \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ 0 & \longrightarrow & \mathcal{S} & \longrightarrow & \mathcal{C}^0\mathcal{S} & \longrightarrow & \mathcal{C}^1\mathcal{S} & \longrightarrow & \mathcal{C}^2\mathcal{S} & \longrightarrow \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ 0 & \longrightarrow & \mathcal{S}^0 & \longrightarrow & \mathcal{C}^0\mathcal{S}^0 & \longrightarrow & \mathcal{C}^1\mathcal{S}^0 & \longrightarrow & \mathcal{C}^2\mathcal{S}^0 & \longrightarrow \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ 0 & \longrightarrow & \mathcal{S}^1 & \longrightarrow & \mathcal{C}^0\mathcal{S}^1 & \longrightarrow & \mathcal{C}^1\mathcal{S}^1 & \longrightarrow & \mathcal{C}^2\mathcal{S}^1 & \longrightarrow \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ 0 & \longrightarrow & \mathcal{S}^2 & \longrightarrow & \mathcal{C}^0\mathcal{S}^2 & \longrightarrow & \mathcal{C}^1\mathcal{S}^2 & \longrightarrow & \mathcal{C}^2\mathcal{S}^2 & \longrightarrow \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \end{array} \tag{143}$$

consists of the resolution

$$0 \longrightarrow \mathcal{S} \xrightarrow{\varepsilon} \mathcal{S}^0 \xrightarrow{f^0} \mathcal{S}^1 \xrightarrow{f^1} \mathcal{S}^2 \xrightarrow{f^2} \tag{144}$$

of the sheaf $\mathcal{S}$, *the rows are formed by the canonical resolutions, and the columns are the canonical morphisms of the canonical resolutions. Then this diagram commutes, and all its columns are exact.*

Proof

(a) This follows from formulas (126)–(128).
(b) We set in (134) $\mathcal{S} = \mathcal{T}$, $\nu = \mathrm{id}_{\mathcal{S}}$. Then, $\nu^0\colon \mathcal{C}^0\mathcal{S} \to \mathcal{C}^0\mathcal{S}$ satisfies

$$\nu^0 = \mathrm{id}_{C^0 S} \tag{145}$$

and (137) implies

$$\bar{\nu}^1 = \mathrm{id}_{D^1 S} \tag{146}$$

hence $\nu^1 = \mathrm{id}_{C^1 S}$ and by induction $\nu^i = \mathrm{id}_{C^i S}$ for all $i \geq 1$.

(c) This follows from (135).

(d) Denote by $\tilde{\mu}$ ($\tilde{\nu}$, resp. $\tilde{\eta}$) the sheaf space morphism associated with μ (ν, resp. η). Since $\eta = \nu \circ \mu$, we have $\tilde{\eta} = \tilde{\nu} \circ \tilde{\mu}$ (Sect. 7.7, Lemma 9, (b)). Thus, using (135) we get for every section $\delta\colon U \to \mathrm{Germ}\, S$, $\eta_0(\delta) = \tilde{\eta} \circ \delta = \tilde{\nu} \circ \tilde{\mu} \circ \delta = \tilde{\nu} \circ \mu_0(\delta) = \nu_0(\mu_0(\delta))$ proving (d) for $p = 0$. Repeating this procedure, we get $\eta^i = \nu^i \circ \mu^i$ for all $i \geq 1$.

(e) Commutativity is ensured by diagram (134). We want to prove exactness of the pth column of the diagram (143). Consider the second column

$$0 \longrightarrow C^0 S \xrightarrow{\varepsilon^0} C^0 S^0 \xrightarrow{f^{00}} C^0 S^1 \xrightarrow{f^{10}} C^0 S^2 \xrightarrow{f^{20}} \tag{147}$$

Exactness at the term $C^0 S$ follows from the injectivity of ε^0 (see (c)). Now let $\delta\colon U \to \mathrm{Germ}\, C^0 S^0$ be a section such that $f^{00}(\delta) = \tilde{f}^0 \circ \delta = 0$. Then if $\delta(x) = [\gamma_x]_x$ for some continuous section $\gamma_x\colon U_x \to \mathrm{Germ}\, C^0 S^0$, we have $\tilde{f}^0([\gamma_x]_x) = 0$ and $[\gamma_x]_x \in \mathrm{Ker}\, \tilde{f}^0 = \mathrm{Im}\, \tilde{\varepsilon}_x$. Therefore, δ is a section of $\mathrm{Im}\, \varepsilon$, proving exactness at $C^0 S^0$. Continuing in the same way, we get exactness of the first column. Exactness in the next columns can be proved by induction. □

Corollary 6 *Suppose that we have a commutative diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & R & \longrightarrow & S & \longrightarrow & T & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & R' & \longrightarrow & S' & \longrightarrow & T' & \longrightarrow & 0 \end{array} \tag{148}$$

with exact rows. Then for every $i \geq 0$, the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C^i R & \longrightarrow & C^i S & \longrightarrow & C^i T & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & C^i R' & \longrightarrow & C^i S' & \longrightarrow & C^i T' & \longrightarrow & 0 \end{array} \tag{149}$$

commutes and has exact rows.

Proof To prove commutativity of the diagram (149), we use commutativity of the square

$$\begin{array}{ccc} \mathcal{R} & \stackrel{h}{\longrightarrow} & \mathcal{S} \\ \downarrow \mu & & \downarrow \nu \\ \mathcal{R}' & \stackrel{k}{\longrightarrow} & \mathcal{S}' \end{array} \tag{150}$$

in (148) and formulas (126–128). Exactness of the rows follows from Lemma 15, (e). □

Corollary 7 *For any isomorphism of Abelian sheaves* $f\colon \mathcal{R} \to \mathcal{S}$ *the sheaf morphisms* $f^p\colon \mathcal{C}^p\mathcal{R} \to \mathcal{C}^p\mathcal{S}$ *are isomorphisms.*

Proof This follows from Lemma 15, (b) and (d). □

Let $\mathcal{S}$ be an Abelian sheaf over a topological space X. Consider the canonical resolution of $\mathcal{S}$

$$0 \longrightarrow \mathcal{S} \stackrel{\iota}{\longrightarrow} \mathcal{C}^0\mathcal{S} \stackrel{c^0}{\longrightarrow} \mathcal{C}^1\mathcal{S} \stackrel{c^1}{\longrightarrow} \mathcal{C}^2\mathcal{S} \stackrel{c^2}{\longrightarrow} \tag{151}$$

Taking *global sections* of every term we obtain a complex of Abelian groups

$$\begin{aligned} &0 \longrightarrow \mathcal{S}X \stackrel{\iota}{\longrightarrow} (\mathcal{C}^0\mathcal{S})X \stackrel{c^0}{\longrightarrow} (\mathcal{C}^1\mathcal{S})X \\ &\stackrel{c^1}{\longrightarrow} (\mathcal{C}^2\mathcal{S})X \stackrel{c^2}{\longrightarrow} \cdots \end{aligned} \tag{152}$$

where the induced Abelian group morphisms in this diagram are denoted by the same letters as in the sequence (151). Denote by $(\mathcal{C}^*\mathcal{S})X$ the nonnegative complex

$$0 \longrightarrow \mathcal{C}^0\mathcal{S} \stackrel{c^0}{\longrightarrow} \mathcal{C}^1\mathcal{S} \stackrel{c^1}{\longrightarrow} \mathcal{C}^2\mathcal{S} \stackrel{c^2}{\longrightarrow} \cdots \tag{153}$$

Then (152) can also be written as

$$0 \longrightarrow \mathcal{S}X \stackrel{\iota}{\longrightarrow} (\mathcal{C}^*\mathcal{S})X. \tag{154}$$

We set for every $p \geq 0$

$$H^p(X, \mathcal{S}) = H^p((\mathcal{C}^*\mathcal{S})X). \tag{155}$$

The Abelian group $H^p(X, \mathcal{S}) = H^p((\mathcal{C}^*\mathcal{S})X)$ is called the pth *cohomology group* of the topological space X *with coefficients* in the sheaf $\mathcal{S}$.

Lemma 16 *Let $\mathcal{S}$ be an Abelian sheaf over a topological space X. The complex of Abelian groups (152) is exact at the terms $\mathcal{S}X$ and $(\mathcal{C}^0\mathcal{S})X$.*

Proof Let $\gamma \in \mathcal{S}X$ and let $\imath(\gamma) = 0$. Then by definition $\imath(\gamma(x)) = 0$ for all $x \in X$. Since the canonical resolution (151) is exact at $\mathcal{S}$ we have $\gamma(x) = 0$ for every x hence $\gamma = 0$. Thus, the complex (152) is exact at $\mathcal{S}X$.

We prove exactness at $(\mathcal{C}^0\mathcal{S})X$. Only inclusion $\operatorname{Ker} c^0 \subset \operatorname{Im} \imath$ needs proof. Let $\gamma \in (\mathcal{C}^0\mathcal{S})X$ and let $c^0(\gamma) = 0$. Then $c^0(\gamma)(x) = 0$ for every point $x \in X$. But (151) is exact at the term $\mathcal{C}^0\mathcal{S}$ hence to each $x \in X$ there exists a unique germ $s_x \in \mathcal{S}_x$ such that $\imath(s_x) = \gamma(x) = 0$, and we have a mapping $X \ni x \to \delta(x) = s_x \in \mathcal{S}$ satisfying $\imath \circ \delta = \gamma$. We want to show that this mapping is continuous. Let $x_0 \in X$ be a point. There exists a neighborhood V (resp. W, resp. U) of the point $\delta(x_0)$ (resp. $\imath(\delta(x_0))$, resp. x_0) such that $\imath|_V\colon V \to W$ (resp. $\gamma|_U\colon U \to W$) is a homeomorphism. Then the composition $(\imath|_V)^{-1} \circ \gamma|_U\colon U \to V$ satisfies, for each $x \in U$,

$$\imath((\imath|_V)^{-1} \circ \gamma|_U(x)) = \gamma(x) = \imath(\delta(x)). \tag{156}$$

Since $\delta(x), (\imath|_V)^{-1} \circ \gamma|_U(x) \in \mathcal{S}_x$ and the restriction of $\imath$ to the fiber $\mathcal{S}_x$ is injective, we have $\delta(x) = (\imath|_V)^{-1} \circ \gamma|_U(x)$, which shows that the mapping δ is continuous at x_0. Consequently, $\operatorname{Ker} c^0 \subset \operatorname{Im} \imath$. □

Corollary 8 *For any Abelian sheaf $\mathcal{S}$ with base X, $H^0(X, \mathcal{S}) = \mathcal{S}X$.*

Let $\mathcal{S}$ and $\mathcal{T}$ be Abelian sheaves over a topological space X, $\nu\colon \mathcal{S} \to \mathcal{T}$ a morphism of Abelian sheaves, and let $\{\nu, \nu^0, \nu^1, \nu^2, \ldots\}$ be the canonical morphism of the canonical resolutions of these sheaves. This morphism induces a comutative diagram of Abelian groups of global sections

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{S}X & \xrightarrow{\imath_{\mathcal{S}}} & (\mathcal{C}^0\mathcal{S})X & \longrightarrow & (\mathcal{C}^1\mathcal{S})X & \longrightarrow & (\mathcal{C}^2\mathcal{S})X & \longrightarrow \\ & & \downarrow \nu & & \downarrow \nu^0 & & \downarrow \nu^1 & & \downarrow \nu^2 & \\ 0 & \longrightarrow & \mathcal{T}X & \xrightarrow{\imath_{\mathcal{T}}} & (\mathcal{C}^0\mathcal{T})X & \longrightarrow & (\mathcal{C}^1\mathcal{T})X & \longrightarrow & (\mathcal{C}^2\mathcal{T})X & \longrightarrow \end{array} \tag{157}$$

and a commutative diagram of nonnegative complexes of global sections

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (\mathcal{C}^0\mathcal{S})X & \longrightarrow & (\mathcal{C}^1\mathcal{S})X & \longrightarrow & (\mathcal{C}^2\mathcal{S})X & \longrightarrow & \cdots \\ & & \downarrow \nu^0 & & \downarrow \nu^1 & & \downarrow \nu^2 & & \\ 0 & \longrightarrow & (\mathcal{C}^0\mathcal{T})X & \longrightarrow & (\mathcal{C}^1\mathcal{T})X & \longrightarrow & (\mathcal{C}^2\mathcal{T})X & \longrightarrow & \cdots \end{array} \tag{158}$$

with obvious notation for the morphisms. Applying standard definitions we obtain, passing to the quotients, the induced group morphisms of cohomology groups $\nu_q\colon H^q(X, \mathcal{S}) \to H^q(X, \mathcal{T})$, $q \geq 0$.

If $\mu\colon \mathcal{T} \to \mathcal{P}$ is some other Abelian sheaf morphism and the family $\{\mu, \mu^0, \mu^1, \mu^2, \ldots\}$ is the morphism of the corresponding canonical resolutions,

$\mu_q\colon H^q(X, \mathcal{T}) \to H^q(X, \mathcal{P})$, we have for every $q \geq 0$, an Abelian group morphism $(\mu \circ \nu)_q\colon H^q(X, \mathcal{S}) \to H^q(X, \mathcal{P})$. Using Lemma 12, (f) and Lemma 15, (d)

$$\mu^q \circ \nu^q = (\mu \circ \nu)^q. \tag{159}$$

Corollary 9 *If $\nu\colon \mathcal{S} \to \mathcal{T}$ is an isomorphism of Abelian sheaves, then $\nu_q\colon H^q(X, \mathcal{S}) \to H^q(X, \mathcal{T})$ is an Abelian group isomorphism for every $q \geq 0$.*

7.10 Sheaves over Paracompact Hausdorff Spaces

All sheaves considered in this section are *Abelian sheaves* over topological spaces whose topology is *Hausdorff* and *paracompact*.

Recall that an Abelian sheaf $\mathcal{S}$ with base X can be considered as the sheaf of continuous sections of the corresponding Abelian sheaf space $S = \text{Germ}\,\mathcal{S}$, defined on open subsets of X. Every morphism $\mathsf{f}\colon \mathcal{S} \to \mathcal{T}$ of Abelian sheaves can be considered as a morphism of Abelian sheaf spaces $f\colon S \to T$.

A *soft sheaf* is by definition a sheaf $\mathcal{S}$ with base X such that every continuous section of $\mathcal{S}$, defined on a closed subset of X can be prolonged to a global section. The proof of the following theorem on short exact sequences of soft sheaves is based on the *Zorn's lemma*.

Theorem 3 *Let X be a paracompact Hausdorff space, and let*

$$0 \longrightarrow \mathcal{R} \xrightarrow{\ f\ } \mathcal{S} \xrightarrow{\ g\ } \mathcal{T} \longrightarrow 0 \tag{160}$$

be a short exact sequence of sheaves over X. If $\mathcal{R}$ is a soft sheaf, then the sequence of Abelian groups of global sections

$$0 \longrightarrow \mathcal{R}X \xrightarrow{\ f_X\ } \mathcal{S}X \xrightarrow{\ g_X\ } \mathcal{T}X \longrightarrow 0 \tag{161}$$

is exact.

Proof

1. We prove exactness at $\mathcal{R}X$. If $\gamma \in \mathcal{R}X$ and $\mathsf{f}_X(\gamma) = 0$, then $f(\tilde{\gamma}(x)) = 0$, then for every point $x \in X$ we get, by injectivity of f, $\tilde{\gamma}(x) = 0$. Thus, the germ $\tilde{\gamma}(x)$ can be represented at every point by the zero section hence $\gamma = 0$.
2. We prove exactness of the sequence (161) at $\mathcal{S}X$. Let $\gamma \in \text{Ker}\,\mathsf{g}_X$. Then $\text{Ker}\,\mathsf{g}_X(\gamma) = 0$ hence $g \circ \tilde{\gamma}(x) = 0$ for all $x \in X$. Since the sequence (160) is exact at $\mathcal{S}$, to every point $x \in X$ there exists an element $\delta(x) \in \mathcal{R}$ such that $f(\delta(x)) = \gamma(x)$ and, since the morphism f is injective, this point is unique. Since $\sigma \circ f = \rho$, where σ (resp. σ) is the projection of S (resp. T), we have $\rho \circ \delta = \sigma \circ f \circ \delta = \sigma \circ \gamma = \text{id}_X$ showing that δ is a global section of $\mathcal{R}$. To show that δ

is continuous, observe that $f \circ \delta = \gamma$ is continuous; then the continuity of δ follows from the property of f to be a local homeomorphism.

3. We show that the mapping g_X is surjective. Let $\gamma \in TX$ be a global section of T. Since the sequence of Abelian sheaves (160) is exact at T, to each point $x \in X$ there exists a neighborhood U_x and a continuous section $\beta_x \in SU_x$ such that $g_{U_x}(\beta_x) = \gamma|_{U_x}$. Thus, in a different notation, there exists an open covering $\{U_\iota\}_{\iota \in I}$ of X, such that for each $\iota \in I$ there exists $\beta_\iota \in SU_\iota$ with the property

$$g_{U_\iota}(\beta_\iota) = \gamma|_{U_\iota}. \tag{162}$$

Since X is paracompact and Hausdorff, there exists a locally finite open covering $\{V_\iota\}_{\iota \in I}$ of X such that $\mathrm{Cl}V_\iota \subset U_\iota$ (Cl denotes the *closure*). The sets $K_\iota = \mathrm{Cl}V_\iota$ are closed and form a closed covering $\{K_\iota\}_{\iota \in I}$ of X. Thus, to every $\iota \in I$ we have assigned a pair (K_ι, β_ι), where $\beta_\iota \in SU_\iota$. Consider the non-empty set K of pairs (K, β), where $K = \cup K_\kappa$ is the union of some sets belonging to the family $\{K_\iota\}_{\iota \in I}$, and β is a section of S defined on the open set $U = \cup U_\kappa$. K becomes a *partially ordered set*, defined by the order relation "$(K, \beta) \leq (K', \beta')$ *if* $K \subset K'$ *and* $\beta'|_U = \beta$."
We show that any linearly ordered family of subsets of the set K has an upper bound. Let $\{(K_\lambda, \beta_\lambda)\}_{\lambda \in L}$ be a linearly ordered family of subsets of K, $K_\lambda \subset U_\lambda$. Denote $K = \cup K_\lambda$; then $K \subset U = \cup U_\lambda$. The family $\{\beta_\lambda\}_{\lambda \in L}$ is a compatible family of sections of the sheaf S. But every compatible family of sections of S locally generates a section of S (Sect. 7.4, condition (5)); thus, there exists a section $\beta \in SU$ such that $\beta|_{U_\lambda} = \beta$ for each $\lambda \in L$. Then, the pair (K, β) is the *upper bound* of the linearly ordered family $\{(K_\lambda, \beta_\lambda)\}_{\lambda \in L}$.
This shows that the set K satisfies the assumptions of the *Zorn's lemma*, therefore, it has a maximal element (K_0, β_0). It remains to show that $K_0 = X$. Suppose the opposite; then there exists a point $x \in X$ such that $x \notin K_0$, and since $K = \cup K_\iota = X$, there must exist an index $\iota \in I$ such that $K_\iota \not\subset K_0$. On $K_\iota \cap K_0$, $g \circ (\beta_0 - \beta_\iota) = \gamma_0 - \gamma_\iota = 0$. But the sequence (161) is exact at SX hence $f(\delta) = \beta_0 - \beta_\iota$ for some $\delta \in R(K_\iota \cap K_0)$. Since R is soft, δ can be prolonged to a section $\bar{\delta}$ over X; then $\delta = \bar{\delta}|_{K_\iota \cap K_0}$. We define a section $\bar{\beta}$ over $K_\iota \cup K_0$ by the conditions

$$\bar{\beta}|_{K_0} = \beta_0, \quad \bar{\beta}|_{K_\iota} = \beta_\iota + f(\delta). \tag{163}$$

Clearly, the $\bar{\beta}$ is defined correctly since on $K_\iota \cap K_0$

$$\beta_0|_{K_\iota \cap K_0} = (\beta_\iota + f(\delta))|_{K_\iota \cap K_0} = (\beta_\iota + \beta_0 - \beta_\iota)|_{K_\iota \cap K_0}. \tag{164}$$

Consequently, the pair $(K_\iota \cup K_0, \bar{\beta})$ belongs to the set K. But this pair satisfies $(K_0, \beta_0) \leq (K_\iota \cup K_0, \bar{\beta})$, which contradicts maximality of the pair (K_0, β_0) unless $K_0 = X$. □

Corollary 10 *If the Abelian sheaves* $\mathcal{R}$ *and* $\mathcal{S}$ *in the short exact sequence* (160) *are soft, then also the Abelian sheaf* $\mathcal{T}$ *is soft.*

Proof Let K be a closed set in the base X, and consider the restriction of the exact sequence (160) to K. The restricted sequence is also exact. Then by Theorem 3, the corresponding sequence of Abelian group (161) over K is exact. Choose a section $\gamma \in \mathcal{T}K$. There exists $\delta \in \mathcal{S}K$ such that $g_K(\delta) = \gamma$. If $\tilde{\delta}$ is an extension of δ to X, then $g_X(\tilde{\delta}) = g \circ \tilde{\delta}$ is the extension of γ to X. □

Corollary 11 *Let X be a paracompact Hausdorff space and let*

$$0 \longrightarrow \mathcal{S}_0 \xrightarrow{f_0} \mathcal{S}_1 \xrightarrow{f_1} \mathcal{S}_2 \xrightarrow{f_2} \cdots \tag{165}$$

be an exact sequence of Abelian sheaves over X. If each of the sheaves $\mathcal{S}_0$, $\mathcal{S}_1$, $\mathcal{S}_2$, … *is soft, then the induced sequence of Abelian groups*

$$0 \longrightarrow \mathcal{S}_0 X \longrightarrow \mathcal{S}_1 X \longrightarrow \mathcal{S}_2 X \longrightarrow \cdots \tag{166}$$

is exact.

Proof The sequence (165) is exact if and only if for each $i = 1, 2, 3, \ldots$ the sequence

$$0 \longrightarrow \operatorname{Ker} f_i \longrightarrow \mathcal{S}_i \xrightarrow{f_i} \operatorname{Ker} f_{i+1} \longrightarrow 0 \tag{167}$$

is exact. Since by hypothesis $\operatorname{Ker} f_1 = \mathcal{S}_0$ and $\mathcal{S}_1$ are soft sheaves, the sheaf $\operatorname{Ker} f_2$ is also soft (Corollary 10). Since the sheaf $\mathcal{S}_1$ is soft, the sheaf $\operatorname{Ker} f_3$ must also be soft, according to Corollary 10, etc. Therefore, for all i, the sequence of global sections

$$0 \longrightarrow (\operatorname{Ker} f_i)X \longrightarrow \mathcal{S}_i X \xrightarrow{f_i} (\operatorname{Ker} f_{i+1})X \longrightarrow 0 \tag{168}$$

is exact, by Theorem 3. Now it is immediate that the sequence (166) must be exact. □

Corollary 12 *If* $\mathcal{S}$ *is a soft sheaf over a paracompact Hausdorff space X, then* $H^q(X, \mathcal{S}) = 0$ *for all* $q \geq 1$.

Proof Consider the canonical resolution of $\mathcal{S}$,

$$0 \longrightarrow \mathcal{S} \xrightarrow{\iota} \mathcal{C}^0\mathcal{S} \xrightarrow{c^0} \mathcal{C}^1\mathcal{S} \xrightarrow{c^1} \mathcal{C}^2\mathcal{S} \xrightarrow{c^2} \tag{169}$$

Since all the sheaves $\mathcal{C}^i\mathcal{S}$ are soft (Sect. 7.8, Lemma 14), the associated sequence of global sections

$$0 \longrightarrow (\mathcal{C}^9\mathcal{S})X \xrightarrow{c^0} (\mathcal{C}^1\mathcal{S})X \xrightarrow{c^1} (\mathcal{C}^2\mathcal{S})X \xrightarrow{c^2} \cdots \tag{170}$$

is exact (Corollary 11). Now, Corollary 12 follows from the definition of a cohomology group. □

Examples

22. Let G be an Abelian group, X *connected Hausdorff space*, and $S = X \times G$ the constant sheaf space (Sect. 7.2, Example 11). We show that the constant sheaf $\mathrm{Sec}^{(c)}S$ is *not* soft. Let x and y be two different points of the base X. Consider the closed subset $Y = \{x\} \cup \{y\}$ of X and the section γ of S defined on Y by $\gamma(x) = g,\ \gamma(y) = h,$ where g and h are two distinct point of G. If U is a neighborhood of x and V is a neighborhood of y such that $U \cap V = \emptyset$, then we have a section $\tilde{\gamma}$: $U \cup V \to S$, equal to g on U and h on V. The restriction of $\tilde{\gamma}$ to Y is equal to γ; in particular, $\tilde{\gamma}$ is continuous. But since X is connected, $\tilde{\gamma}$ cannot be prolonged to a global continuous section of S.
23. If X is a *normal space*, then every continuous, real-valued function defined on a closed subspace of X, can be prolonged to a globally defined continuous function (*Tietze theorem*). Consequently, the sheaf $C_{X,\mathbf{R}}$ is soft (cf. Sect. 7.4, Example 18).
24. We shall show that the sheaf of modules $\mathcal{S}$ over a soft sheaf of commutative rings with unity $\mathcal{R}$ is soft. Let X be the base of $\mathcal{R}$ (and $\mathcal{S}$), K a closed subset of X, and let $\gamma \in \mathrm{Sec}^{(c)}S$ be a continuous section, defined on K. Then by definition γ can be prolonged to a continuous section, also denoted by γ, defined on a neighborhood U of K. Define a continuous section $\rho \in \mathrm{Sec}^{(c)}(K \cup (X \backslash U)))$ by

$$\rho(x) = \begin{cases} 1, & x \in K, \\ 0, & x \in X \backslash U. \end{cases} \tag{171}$$

Since $\mathcal{R}$ is soft, there exists a section $\tilde{\rho} \in \mathrm{Sec}^{(c)}X$ prolonging ρ to X. We define $\tilde{\gamma}(x) = \tilde{\rho}(x) \cdot \gamma(x)$; $\tilde{\gamma}$ is the desired prolongation of γ.
25. The sum of two soft subsheaves of a sheaf is a soft subsheaf (cf. Sect. 7.2, Example 13).

Let $\mathcal{S}$ be an Abelian sheaf over a topological space X, η: $\mathcal{S} \to \mathcal{S}$ a sheaf morphism. We define the *support* of η to be a closed subspace of X

$$\mathrm{supp}\ \eta = \mathrm{cl}\{x \in X | \eta(x) \neq 0\}. \tag{172}$$

Let $\{U_\iota\}_{\iota \in I}$ be a locally finite open covering of the paracompact Hausdorff space X, $\mathcal{S}$ an Abelian sheaf with base X. By a *sheaf partition of unity* for $\mathcal{S}$, subordinate to $\{U_\iota\}_{\iota \in I}$ we mean any family $\{\chi_\iota\}_{\iota \in I}$ of sheaf morphisms χ_ι: $\mathcal{S} \to \mathcal{S}$ over X with the following two properties:

(1) supp $\chi_\iota \subset U_\iota$ for every $\iota \in I$.
(2) For every point $x \in X$

$$\sum_{\iota \in I} \chi_\iota(x) = x. \tag{173}$$

Note that the sum on the left-hand side of formula (173) is well defined, because for every fixed point x the summation is taking place through only a *finitely many* indices ι from the indexing set I.

An Abelian sheaf $\mathcal{S}$ is said to be *fine*, if to every locally finite open covering $\{U_\iota\}_{\iota\in I}$ of X there exists a sheaf partition of unity $\{\chi_\iota\}_{\iota\in I}$ subordinate to $\{U_\iota\}_{\iota\in I}$.

Theorem 4 *Every fine Abelian sheaf over a paracompact Hausdorff space is soft.*

Proof Let $\mathcal{S}$ be an Abelian sheaf over a paracompact Hausdorff space X, $S = \text{Germ}\,\mathcal{S}$, and let σ be the projection of S. Let Y be a closed subspace of X, γ a continuous section, defined on Y. To every point $x \in Y$, there exists a neighborhood U_x of x and a continuous section γ_x: $U_x \to S$ such that $\gamma(x) = \gamma_x$. Shrinking γ_x to $U_x \cap Y$ we get a continuous section of the restriction of S to $U_x \cap Y$. Shrinking U_x if necessary we may assume without loss of generality that $\gamma_x|_{U_x\cap Y} = \gamma|_{U_x\cap Y}$. The sets U_x together with the set $X\backslash Y$ cover X. Since X is paracompact, there exists a locally finite refinement $\{V_\iota\}_{\iota\in I}$ of this covering. If for some $\iota \in I$, $V_\iota \cap Y \neq \emptyset$, then there exists a continuous section γ_ι: $V_\iota \to S$ such that $\gamma_\iota|_{V_\iota\cap Y} = \gamma|_{V_\iota\cap Y}$; if $V_\iota \cap Y = \emptyset$, we set $\gamma_\iota = 0$. In this way, we assign to each of the sets V_ι a continuous section γ_ι: $V_\iota \to S$.

Let $\{\eta_\iota\}_{\iota\in I}$ be a partition of unity subordinate to the covering $\{V_\iota\}_{\iota\in I}$. Set for all $\iota \in I$

$$\delta_\iota(x) = \begin{cases} \eta_\iota(\gamma_\iota(x)), & x \in V_\iota, \\ 0, & x \in X\backslash V_\iota, \end{cases} \tag{174}$$

where 0 denotes the neutral element of the Abelian group S_x. We get a mapping δ_ι: $X \to S$ satisfying the condition $\sigma \circ \delta_\iota = \text{id}_X$. This mapping is obviously continuous on the set V_ι, and also on a neighborhood $X\backslash\text{supp}\eta_\iota$ of the closed set $X\backslash V_\iota$. We set $\delta = \sum \delta_\iota$. Then, δ is a global continuous section of the sheaf space S. Then for every point $x \in X$,

$$\delta(x) = \sum_{V_\kappa \ni x} \eta_\kappa(\gamma_\kappa(x)) = \sum_\kappa \eta_\kappa \gamma(x) = \left(\sum_\kappa \eta_\kappa\right)\gamma(x) = \gamma(x). \tag{175}$$

Therefore, $\delta|_Y = \gamma$.

Examples

26. The Abelian sheaf $C_{X,\mathbf{R}}$ of continuous real-valued functions on a paracompact Hausdorff space X is fine. Indeed, any locally finite open covering $\{U_\iota\}_{\iota\in I}$ of X, and any subordinate partition of unity $\{\chi_\iota\}_{\iota\in I}$, define a sheaf partition of unity as the family of sheaf morphisms $f \to \chi_\iota f$. The Abelian sheaf $C_{X,\mathbf{R}}$ can also be considered as a *sheaf of commutative rings with unity*.
27. Let $\mathcal{S}$ be a sheaf of $C_{X,\mathbf{R}}$-modules over a paracompact Hausdorff space X, let S be the associated sheaf space, with projection $\sigma\colon S \to X$. Every continuous function $f\colon X \to \mathbf{R}$ defines an Abelian sheaf morphism of the sheaf space S by

$$f_S(s) = f(\sigma(s)) \cdot s. \tag{176}$$

If $\{U_\iota\}_{\iota\in I}$ is an open covering of X, and $\{\chi_\iota\}_{\iota\in I}$ a partition of unity on X, subordinate to $\{U_\iota\}_{\iota\in I}$, then formula (176) applies to the functions from the family of functions $\{\chi_\iota\}_{\iota\in I}$; the corresponding family of sheaf morphisms $\{\chi_{\iota,S}\}_{\iota\in I}$ is then a sheaf partition of unity on S. Consequently, the Abelian sheaf $\mathcal{S}$ is fine.
28. The Abelian sheaves $C^r_{X,\mathbf{R}}$ of r times continuously differentiable functions on a smooth manifold X, where $r = 0, 1, 2, \ldots, \infty$, are fine (cf. Example 26), and can also be considered as *sheaves of commutative rings with unity*.
29. Every sheaf of modules over a fine sheaf of commutative rings with unity is fine.

Let us consider a short exact sequence of Abelian sheaves over a paracompact Hausdorff manifold X

$$0 \longrightarrow \mathcal{R} \xrightarrow{f} \mathcal{S} \xrightarrow{g} \mathcal{T} \longrightarrow 0, \tag{177}$$

and the commutative diagram of the canonical resolutions

$$\begin{array}{ccccccccccc}
 & & 0 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{R} & \longrightarrow & \mathcal{C}^0\mathcal{R} & \longrightarrow & \mathcal{C}^1\mathcal{R} & \longrightarrow & \mathcal{C}^2\mathcal{R} & \longrightarrow & \\
 & & \downarrow f & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{S} & \longrightarrow & \mathcal{C}^0\mathcal{S} & \longrightarrow & \mathcal{C}^1\mathcal{S} & \longrightarrow & \mathcal{C}^2\mathcal{S} & \longrightarrow & \\
 & & \downarrow g & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{T} & \longrightarrow & \mathcal{C}^0\mathcal{T} & \longrightarrow & \mathcal{C}^1\mathcal{T} & \longrightarrow & \mathcal{C}^2\mathcal{T} & \longrightarrow & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & & 0 & &
\end{array} \tag{178}$$

This diagram induces the commutative diagram of global sections

$$\begin{array}{ccccccccccc} & & 0 & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & RX & \longrightarrow & (C^0R)X & \longrightarrow & (C^1R)X & \longrightarrow & (C^2R)X & \longrightarrow & \\ & & \downarrow f & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & SX & \longrightarrow & (C^0S)X & \longrightarrow & (C^1S)X & \longrightarrow & (C^2S)X & \longrightarrow & \\ & & \downarrow g & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & TX & \longrightarrow & (C^0T)X & \longrightarrow & (C^1T)X & \longrightarrow & (C^2T)X & \longrightarrow & \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & 0 & & \end{array} \tag{179}$$

All the sheaves C^iR, C^iS, and C^iT in (178) are soft (Sect. 7.9, Lemma 14). Applying Corollary 11, we see that the columns are exact. Therefore, by Lemma 11, we get the *long exact sequence*

$$\begin{aligned} &0 \longrightarrow H^0(X,R) \xrightarrow{f} H^0(X,S) \xrightarrow{g} H^0(X,T) \xrightarrow{\partial^0} \\ &H^1(X,R) \longrightarrow H^1(X,S) \longrightarrow H^1(X,T) \xrightarrow{\partial^1} H^2(X,R) \longrightarrow ..., \end{aligned} \tag{180}$$

where the family $(\partial^0, \partial^1, \partial^2, \ldots)$ is the *connected morphism*.

The long exact sequence can be applied to commutative diagrams of short exact sequences.

Lemma 17 *Let X be a paracompact Hausdorff space. Suppose that we have a commutative diagram of Abelian sheaves over X*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & R & \xrightarrow{f} & S & \xrightarrow{g} & T & \longrightarrow & 0 \\ & & \downarrow h & & \downarrow k & & \downarrow j & & \\ 0 & \longrightarrow & \bar{R} & \xrightarrow{\bar{f}} & \bar{S} & \xrightarrow{\bar{g}} & \bar{T} & \longrightarrow & 0 \end{array} \tag{181}$$

whose rows are exact. Then the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^0(X,R) & \xrightarrow{f} & H^0(X,S) & \xrightarrow{g} & H^0(X,T) & \xrightarrow{\partial^0} & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & H^0(X,\bar{R}) & \xrightarrow{\bar{f}} & H^0(X,\bar{S}) & \xrightarrow{\bar{g}} & H^0(X,\bar{T}) & \xrightarrow{\partial^0} & \\ & H^1(X,R) & \longrightarrow & H^1(X,S) & \longrightarrow & H^1(X,T) & \xrightarrow{\partial^1} & H^2(X,R) & \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ & H^1(X,\bar{R}) & \longrightarrow & H^1(X,\bar{S}) & \longrightarrow & H^1(X,\bar{T}) & \xrightarrow{\partial^1} & H^2(X,\bar{R}) & \end{array} \tag{182}$$

where the first (resp. the second) row is the long exact sequence associated with the first (resp. the second) row in (181)*, commutes.*

Proof It is enough to prove commutativity of the squares in (182) containing the group morphisms ∂^i. Commutativity of the other squares is an immediate consequence of the diagrams (181) and Sect. 7.9, (151).

Consider the square

$$\begin{array}{ccc} H^0(X, \mathsf{T}) & \xrightarrow{\partial^0} & H^1(X, \mathsf{R}) \\ \downarrow & & \downarrow \\ H^0(X, \bar{\mathsf{T}}) & \xrightarrow{\partial^0} & H^1(X, \bar{\mathsf{R}}) \end{array} \tag{183}$$

For the purpose of this proof denote by $\varepsilon_{\mathsf{R}}\colon \mathsf{R} \to C^0\mathsf{R}$ and $c^i_{\mathsf{R}}\colon C^i\mathsf{R} \to C^{i+1}\mathsf{R}$ the corresponding sheaf morphisms in the canonical resolution of the sheaf R, $0 \to \mathsf{R} \to C^0\mathsf{R} \to C^1\mathsf{R} \to C^2\mathsf{R} \to \cdots$, and introduce analogous notation for the sheaves S and T. Let $c \in H^0(X, \mathsf{T}) = \operatorname{Ker} c^0_{\mathsf{T}}$. There exist an element $b \in (C^0\mathsf{S})X$ and $a \in \operatorname{Ker} c^1_{\mathsf{R}}$ such that $c \in g^0(b)$, $c^0_S(b) = \mathsf{f}^1(a)$, and by definition

$$\begin{aligned} \partial^0(c) &= [a], \\ \mathsf{h}^1\partial^0(c) &= \mathsf{h}^1([a]) = [\mathsf{h}^1(a)]. \end{aligned} \tag{184}$$

We set

$$\bar{b} = \mathsf{k}_0(b), \quad \bar{a} = \mathsf{h}^1(a). \tag{185}$$

Then, we get by immediate calculations $\bar{\mathsf{g}}^0(b') = \bar{\mathsf{g}}^0\mathsf{k}^0(b) = \mathsf{j}^0\mathsf{g}^0(b) = \mathsf{j}^0(c)$, $\bar{\mathsf{f}}^1(\bar{a}) = \bar{\mathsf{f}}^1(\mathsf{h}^1(a)) = \mathsf{k}^1\mathsf{f}^1(a)$, and $c^0_{\mathsf{S}'}(\bar{b}) = c^0_{\mathsf{S}'}\mathsf{k}^0(b) = \mathsf{k}^0c^0_{\mathsf{S}}(b) = \mathsf{k}^1\mathsf{f}^1(a)$. Hence $\bar{b}$ and $\bar{a}$ satisfy

$$\mathsf{j}^0(c) = \bar{\mathsf{g}}(\bar{b}), \quad c^0_{\bar{\mathsf{S}}}(\bar{b}) = \bar{\mathsf{f}}^1(\bar{a}). \tag{186}$$

Consequently,

$$\partial^0\bar{\mathsf{j}}^0(c) = a' = \mathsf{h}^1\partial^0(c) \tag{187}$$

proving commutativity of (183).

Commutativity of the square

$$\begin{array}{ccc} H^q(X, \mathsf{T}) & \xrightarrow{\partial^q} & H^{q+1}(X, \mathsf{R}) \\ \downarrow & & \downarrow \\ H^q(X, \bar{\mathsf{T}}) & \xrightarrow{\partial^q} & H^{q+1}(X, \bar{\mathsf{R}}) \end{array} \tag{188}$$

can be proved in the same way. Let $[c] \in H^q(X, \mathcal{T}) = \operatorname{Ker} c^q_{\mathcal{T}} / \operatorname{Im} c^{q-1}_{\mathcal{T}}$. There exist elements $b \in (C^q S)X$ and $a \in \operatorname{Ker} c^{q+1}_{\mathcal{R}}$ such that

$$c = \mathcal{g}^q(b), \quad c^q_{\mathcal{S}}(b) = \mathcal{f}^{q+1}(a), \tag{189}$$

and by definition

$$\begin{aligned} \partial^q([c]) &= [a], \\ \mathcal{h}^{q+1}\partial^q([c]) &= \mathcal{h}^{q+1}([a]) = [\mathcal{h}^{q+1}(a)]. \end{aligned} \tag{190}$$

We denote

$$\bar{b} = \mathcal{k}^k(b), \quad \bar{a} = \mathcal{h}^{q+1}(a). \tag{191}$$

Then

$$\begin{aligned} \bar{\mathcal{g}}^q(\bar{b}) &= \bar{\mathcal{g}}^q \mathcal{k}^q(b) = \mathcal{j}^q \mathcal{g}^q(b) = \mathcal{j}^q(c), \\ \bar{\mathcal{f}}^{q+1}(\bar{a}) &= \bar{\mathcal{f}}^{q+1}\mathcal{h}^{q+1}(a) = \mathcal{k}^{q+1}\mathcal{f}^{q+1}(a), \\ c^q_{\bar{\mathcal{S}}}(\bar{b}) &= c^q_{\bar{\mathcal{S}}}\mathcal{k}^q(b) = \mathcal{k}^{q+1}c^q_{\mathcal{S}}(b) = \mathcal{k}^{q+1}\mathcal{f}^{q+1}(a), \end{aligned} \tag{192}$$

so that

$$c^q_{\bar{\mathcal{S}}}(\bar{b}) = \bar{\mathcal{f}}^{q+1}(\bar{a}). \tag{193}$$

Now using the definition of ∂^q we get

$$\begin{aligned} \partial^q \mathcal{j}^q([c]) &= \partial^q([\mathcal{j}^q(c)]) = [a'] \\ &= [\mathcal{h}^{q+1}(a)] = \mathcal{h}^{q+1}\partial^q([c]), \end{aligned} \tag{194}$$

which proves commutativity of the square (188). □

An Abelian sheaf $\mathcal{S}$ over a topological space X is said to be *acyclic*, if $H^q(X, \mathcal{S}) = 0$ for all $q \geq 1$. A resolution of $\mathcal{S}$

$$0 \longrightarrow \mathcal{S} \longrightarrow \mathcal{S}^0 \longrightarrow \mathcal{S}^1 \longrightarrow \mathcal{S}^2 \longrightarrow \cdots \tag{195}$$

is said to be *acyclic*, if each of the sheaves $\mathcal{S}^i$, where $i \geq 1$, is acyclic.

Lemma 18 *Let $\mathcal{S}$ be an Abelian sheaf over a paracompact Hausdorff space X.*

(a) *If $\mathcal{S}$ is soft, it is acyclic.*
(b) *The canonical resolution of $\mathcal{S}$ is acyclic.*

Proof

(a) This follows from Corollary 12.

(c) We want to show that each of the sheaves $\mathcal{C}^p\mathcal{S}$, where $p \geq 0$, is acyclic. But we have already shown that these sheaves are soft (Sect. 7.9, Lemma 14)); since by hypothesis the base X of $\mathcal{S}$ is paracompact and Hausdorff, they are acyclic by part (a) of this lemma. □

Denote by $\mathcal{T}^*X$ the complex $0 \to \mathcal{T}^0X \to \mathcal{T}^1X \to \mathcal{T}^2X \to \cdots$, and let $H^q(\mathcal{T}^*X)$ be the qth cohomology group of this complex.

Theorem 5 (Abstract De Rham theorem) *Let $\mathcal{S}$ be an Abelian sheaf over a paracompact Hausdorff manifold X, let*

$$0 \longrightarrow \mathcal{S} \longrightarrow \mathcal{T}^0 \longrightarrow \mathcal{T}^1 \longrightarrow \mathcal{T}^2 \longrightarrow \cdots \tag{196}$$

be a resolution of $\mathcal{S}$. If this resolution is acyclic, then for every $q \geq 0$ the cohomology groups $H^q(X, \mathcal{S})$ and $H^q(\mathcal{T}^*X)$ are isomorphic.

Proof Let us consider the following commutative diagram of Abelian sheaves

$$\begin{array}{ccccccccccc}
 & & 0 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{S} & \longrightarrow & \mathcal{C}^0\mathcal{S} & \longrightarrow & \mathcal{C}^1\mathcal{S} & \longrightarrow & \mathcal{C}^2\mathcal{S} & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{T}^0 & \longrightarrow & \mathcal{C}^0\mathcal{T}^0 & \longrightarrow & \mathcal{C}^1\mathcal{T}^0 & \longrightarrow & \mathcal{C}^2\mathcal{T}^0 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{T}^1 & \longrightarrow & \mathcal{C}^0\mathcal{T}^1 & \longrightarrow & \mathcal{C}^1\mathcal{T}^1 & \longrightarrow & \mathcal{C}^2\mathcal{T}^1 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{T}^2 & \longrightarrow & \mathcal{C}^0\mathcal{T}^1 & \longrightarrow & \mathcal{C}^1\mathcal{T}^1 & \longrightarrow & \mathcal{C}^2\mathcal{T}^1 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & &
\end{array} \tag{197}$$

with exact rows and columns, and the associated diagram of global sections

$$\begin{array}{ccccccccccc}
 & & 0 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{S}X & \longrightarrow & (\mathcal{C}^0\mathcal{S})X & \longrightarrow & (\mathcal{C}^1\mathcal{S})X & \longrightarrow & (\mathcal{C}^2\mathcal{S})X & \longrightarrow & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{T}^0X & \longrightarrow & (\mathcal{C}^0\mathcal{T}^0)X & \longrightarrow & (\mathcal{C}^1\mathcal{T}^0)X & \longrightarrow & (\mathcal{C}^2\mathcal{T}^0)X & \longrightarrow & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{T}^1X & \longrightarrow & (\mathcal{C}^0\mathcal{T}^1)X & \longrightarrow & (\mathcal{C}^1\mathcal{T}^1)X & \longrightarrow & (\mathcal{C}^2\mathcal{T}^1)X & \longrightarrow & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{T}^2X & \longrightarrow & (\mathcal{C}^0\mathcal{T}^1)X & \longrightarrow & (\mathcal{C}^1\mathcal{T}^1)X & \longrightarrow & (\mathcal{C}^2\mathcal{T}^1)X & \longrightarrow & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & &
\end{array} \tag{198}$$

By Sect. 7.9, Corollary 6 and Corollary 7, every column in this diagram except possibly the first one, are exact. We shall show that each row, except possibly the first row, is exact.

Consider the k-th row

$$0 \longrightarrow \mathcal{T}^k X \longrightarrow (\mathcal{C}^0 \mathcal{T}^k)X \longrightarrow (\mathcal{C}^1 \mathcal{T}^k)X \longrightarrow (\mathcal{C}^2 \mathcal{T}^k)X \longrightarrow \tag{199}$$

This sequence is exact at the first and the second terms (Sect. 7.9, Lemma 16). Since the sheaf $\mathcal{T}^k$ is acyclic, we have for each $q \geq 1$,

$$H^q(X, \mathcal{T}^k) = 0, \tag{200}$$

which means that the sequence (199) is exact everywhere. In particular, the diagram (199) is exact everywhere except possibly the first column and the first row. Now, we apply (Sect. 7.7, Lemma 10). □

Corollary 13 *For any two acyclic resolutions of an Abelian sheaf $\mathcal{S}$ over a paracompact Hausdorff space X, expressed by the diagram*

$$\begin{array}{ccccccccc} & & & & \mathcal{R}^0 & \longrightarrow & \mathcal{R}^1 & \longrightarrow & \mathcal{R}^2 & \longrightarrow & \cdots \\ & & & \nearrow & & & & & & & \\ 0 & \longrightarrow & \mathcal{S} & & & & & & & & \\ & & & \searrow & & & & & & & \\ & & & & \mathcal{T}^0 & \longrightarrow & \mathcal{T}^1 & \longrightarrow & \mathcal{T}^2 & \longrightarrow & \cdots \end{array} \tag{201}$$

the cohomology groups of the complexes of global sections $H^q(\mathcal{R}^*X)$ and $H^q(\mathcal{T}^*X)$ are isomorphic.

Proof Indeed, according to Theorem 5, $H^q(\mathcal{R}^*X)$ and $H^q(\mathcal{T}^*X)$ are isomorphic with the cohomology group $H^q(X, \mathcal{S})$. □

Examples

30. Any sheaf $\mathcal{S}$ of C^r-sections of a smooth vector bundle over a smooth paracompact Hausdorff manifold X admits multiplication by functions of class C^r and is therefore fine. Consequently, $\mathcal{S}$ is soft (Theorem 4) and acyclic (Lemma 18).

Remark 6 Consider an n-dimensional smooth manifold X, the constant sheaf $\mathbf{R}$ and the sheaves of p-forms Ω^p of class C^∞ on X. The exterior derivative of differential forms d: $\Omega^p \to \Omega^{p+1}$ defines a differential sequence

$$0 \longrightarrow \mathbf{R} \longrightarrow \Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d} \Omega^2 \longrightarrow \tag{202}$$

where the mapping $\mathbf{R} \to \Omega^0$ is the canonical inclusion. It follows from the Volterra–Poincare lemma that this sequence is *exact*, therefore, it is a resolution of the constant sheaf $\mathbf{R}$. Since the sheaves Ω^p are fine they are soft (Example 29, Example 30) and acyclic (Lemma 18). Thus, the resolution (202) is acyclic; in particular, according to the abstract De Rham theorem, the cohomology groups $H^q(\Omega^* X)$ of the complex of global sections

$$0 \longrightarrow \Omega^0 X \xrightarrow{d} \Omega^1 X \xrightarrow{d} \Omega^2 X \xrightarrow{d} \cdots \tag{203}$$

coincide with the cohomology groups $H^q(X, \mathbf{R})$. The sequence (202) is called the *De Rham sequence* (of sheaves); (203) is the *De Rham sequence* of differential forms on X, and the groups $H^q(\Omega^* X)$, usually denoted just by $H^q X$, are the *De Rham cohomology groups* of X. Note that according to Corollary 13, for *any* acyclic resolution of the constant sheaf $\mathbf{R}$ on X,

$$0 \longrightarrow \mathbf{R} \longrightarrow \mathcal{S}^*, \tag{204}$$

the cohomology groups $H^q(\mathcal{S}^* X)$ *coincide* (that is, are isomorphic) with the De Rham cohomology groups $H^q X$,

$$H^q(\mathcal{S}^* X) = H^q X. \tag{205}$$

References

[BT] R. Bott, L.V. Tu, *Differential Forms and Algebraic Topology*, Springer-Verlag, New York, 1982

[Br] G.E. Bredon, *Sheaf Theory*, Springer-Verlag, New York, 1997

[G] P.L. Garcia, The Poincare-Cartan invariant in the calculus of variations, Symposia Mathematica 14 (1974) 219-246

[Go] R. Godement, *Topologie Algébrique et Théorie des Faisceaux*, Hermann, Paris, 1958

[L] J.M. Lee, *Introduction to Smooth Manifolds*, Graduate Texts in Math. 218, Springer, 2006

[Sc] L. Schwartz, *Analyse Mathematique* II, Hermann, Paris, 1967

[W] F.W. Warner, *Foundations of Differentiable Manifolds and Lie Groups*, Springer-Verlag, New York, 1983

[We] R.O. Wells, *Differential Analysis on Complex Manifolds*, Springer-Verlag, New York, 1980

Chapter 8
Variational Sequences

We introduced in Chap. 4 the *Euler–Lagrange mapping* of the calculus of variations as an **R**-linear mapping, assigning to a Lagrangian λ, defined on the r-jet prolongation J^rY of a fibered manifold Y, its Euler–Lagrange form E_λ. Local properties of this mapping are determined by the *components* of the Euler–Lagrange form, the Euler–Lagrange expressions of the Lagrangian λ. In this chapter, we construct an exact sequence of Abelian sheaves, the *variational sequence*, such that one of its sheaf morphisms coincides with the Euler–Lagrange mapping. Existence of the sequence provides a possibility to study basic global characteristics of the Euler–Lagrange mapping in terms of the cohomology groups of the corresponding *complex of global sections* and the underlying manifold Y. In particular, for variational purposes, the structure of the *kernel* and the *image* of the Euler–Lagrange mapping $\lambda \to E_\lambda$ is considered.

The variational sequence is defined by means of the exterior derivative operator, acting on differential forms on jet spaces. Recall that for *any* smooth, paracompact, Hausdorff manifold X the following facts have already been stated in Chap. 7:

(a) The set of real-valued functions, defined on open subsets of X, with standard restrictions, is a sheaf; the sets of *continuous*, C^k*-differentiable* and *smooth* functions are also sheaves.
(b) More generally, the set of *differentiable* k-forms on open subsets of X, with standard restrictions, is a sheaf.
(c) The set of *closed* differentiable k-forms, defined on open subsets of X, with standard restrictions, is a sheaf.
(d) An *exact* form ρ on an open set $U \subset X$ is a form such that there exists a form η, defined on U, such that $\rho = d\eta$; the exact forms constitute a presheaf but *not* a sheaf: if $\{U_\iota\}_{\iota \in I}$ is an open covering of an open set $U \subset X$, such that $\rho|_{U_\iota} = d\eta_\iota$ for each $\iota \in I$, then in general, there is no η such that $\rho = d\eta$.

This chapter treats the foundations of the variational sequence theory. The approach, which we have followed, is due to the original papers Krupka [K18, K19]. Main innovations consist in the use of *variational projectors* (also called the *interior*

D. Krupka, *Introduction to Global Variational Geometry*,
Atlantis Studies in Variational Geometry 1, DOI 10.2991/978-94-6239-073-7_8

Euler–Lagrange operators, see Anderson [A2], Krupka and Sedenková – Volná [KSe], Volná and Urban [VU]). The idea to apply sheaves comes from Takens [T].

A number of important topics have not been included. For recent research in the structure of the variational sequence, its relations with topology, symmetries and differential equations, and possible extensions to Grassmann fibrations and submanifold theory, we refer to Zenkov [Z], Brajercik and Krupka [BK], Francaviglia et al. [FPW], Grigore [Gr], Krupka [K16, K17], Krbek and Musilova [KM], Pommaret [Po], Urban and Krupka [UK1], Vitolo [Vit] and Zenkov [Z] (see also the handbook Krupka and Saunders [KS], where further references can be found).

Note that the variational sequence theory does *not* follow the approach to the "formal calculus of variations" based on a *variational bicomplex theory* on *infinite* jet prolongations of fibered manifolds, although some technical aspects of these two theories appear to be parallel (Anderson [A2]; Anderson and Duchamp [AD]; Dedecker and Tulczyjew [DT]; Olver [O1]; Saunders [S]; Takens [T]; Urban and Krupka [UK1]; Vinogradov et al. [VKL] and others). In particular, the finite-order sequence can never be considered as a "subsequence" of the bicomplex. The results, however, and require a deeper comparison. It seems for instance that the infinite jet structure of the bicomplex theory is a serious obstacle for obtaining local and global characteristics of the "variational" morphisms within this theory; although a main motivation was to study these morphisms, no *explicit* (or at least *effective*) formulas say for the inverse problem of the calculus of variations and Helmholtz morphism have been derived yet.

As before, Y denotes in this chapter a smooth fibered manifold with n-dimensional base X and projection π, and $n + m = \dim Y$. J^rY is its *r-jet prolongation* and π^r: $J^rY \to X$, $\pi^{r,s}$: $J^rY \to J^sY$ are the canonical jet projections. For any open set $W \subset Y$, $\Omega^r_q W$ is the module of q-forms on the set $W^r = (\pi^{r,0})^{-1}(W)$, and $\Omega^r W$ is the exterior algebra of forms on W^r. The horizontalization morphism of the exterior algebra $\Omega^r W$ into $\Omega^{r+1} W$ is denoted by h. If Ξ is a π-projectable vector field and $J^r\Xi$ its r-jet prolongation, then to simplify notation, we sometimes denote the contraction $i_{J^r\Xi}\rho$, and the Lie derivative $\partial_{J^r\Xi}\rho$ of a form ρ, just by $i_\Xi\rho$, or $\partial_\Xi\rho$.

8.1 The Contact Sequence

We saw in Sect. 7.10, Remark 6, that the exterior differential forms on a finite-dimensional smooth manifold X together with the exterior derivative morphism constitute a resolution of the constant sheaf $\mathbf{R}$ over X, the *De Rham resolution*. In this section, we provide analogous construction for differential forms on the r-jet prolongation J^rY of a fibered manifold Y over X. We use the fibered structure of Y to construct a slightly modified version of the De Rham resolution, in which the underlying topological space is the manifold Y itself instead of J^rY.

Following our previous notation (Chaps. 4 and 7), consider a smooth fibered manifold Y with base X and projection π. For any open set W in Y, denote by $\Omega^r_0 W$

the Abelian group of real-valued functions of class C^r (0-forms), defined on the open set $W^r \subset J^r Y$; one can also consider $\Omega_0^r W$ with its algebraic structure of a commutative ring with unity. Next, let $q \geq 1$, and denote by $\Omega_q^r W$ the Abelian group of q-forms of class C^r, defined on $W^r \subset J^r Y$. This way we get, for every non-negative integer q, a correspondence $W \to \Omega_q^r W$, assigning to an open set $W \subset Y$ the Abelian group of q-forms on W^r. One can easily verify that this correspondence defines a *sheaf structure* on the family $\{\Omega_q^r W\}$, labeled by the open sets W. Indeed, to any two open sets W_1 and W_2 in Y such that $W_2 \subset W_1$, and any $\rho \in \Omega_q^r W_1$, the restrictions $\Omega_q^r W_1 \ni \rho \to \rho|_{W_2} \in \Omega_q^r W_2$ define an *Abelian presheaf structure* on $\{\Omega_q^r W\}$. Since this presheaf is obviously *complete*, it has the *Abelian sheaf structure* (Sect. 7.4); with this structure, the family $\{\Omega_q^r W\}$ will be referred to as the *sheaf of q-forms of order r* over Y, and will be denoted by Ω_q^r.

The exterior derivative operator d defines, for each $W \subset Y$, a sequence of Abelian groups

$$\begin{aligned} 0 \longrightarrow \mathbf{R} \longrightarrow \Omega_0^r W \xrightarrow{d} \Omega_1^r W \xrightarrow{d} \Omega_2^r W \xrightarrow{d} \cdots \xrightarrow{d} \Omega_n^r W \\ \xrightarrow{d} \Omega_{n+1}^r W \xrightarrow{d} \cdots \xrightarrow{d} \Omega_M^r W \to 0, \end{aligned} \tag{1}$$

and an exact sequence of Abelian sheaves

$$\begin{aligned} 0 \longrightarrow \mathbf{R} \longrightarrow \Omega_0^r \xrightarrow{d} \Omega_1^r \xrightarrow{d} \Omega_2^r \xrightarrow{d} \cdots \xrightarrow{d} \Omega_n^r \\ \xrightarrow{d} \Omega_{n+1}^r \xrightarrow{d} \cdots \xrightarrow{d} \Omega_M^r \to 0. \end{aligned} \tag{2}$$

We call this sequence the *De Rham (sheaf) sequence* over $J^r Y$. We now construct a subsequence of the De Rham sequence. First, recall the notion of a *contact form* and introduce the notion of a *strongly contact form*, a (higher-order) analogy of a similar concept introduced in Sect. 8.2.

Let W be an open set in the fibered manifold Y. Recall that the *horizontalisation* h: $\Omega^r W \to \Omega^{r+1} W$ is a morphism of exterior algebras, which assigns to a q-form $\rho \in \Omega_q^r W$, $q \geq 1$, a π^{r+1}-horizontal q-form $h\rho \in \Omega_q^{r+1} W$ by the formula

$$h\rho(J_x^{r+1}\gamma)(\xi_1, \xi_2, \ldots, \xi_q) = \rho(J_x^r\gamma)(h\xi_1, h\xi_2, \ldots, h\xi_q), \tag{3}$$

where $J_x^{r+1}\gamma \in W^{r+1}$ is any point and $\xi_1, \xi_2, \ldots, \xi_q$ are any tangent vectors of $J^{r+1}Y$ at this point. If f is a function, then

$$hf = (\pi^{r+1,r})^* f. \tag{4}$$

One can equivalently introduce h as a morphism, defined in a fibered chart (V, ψ), $\psi = (x^i, y^\sigma)$, by the equations

$$hf = f \circ \pi^{r+1,r}, \quad hdx^i = dx^i, \quad hdy^\sigma_{j_1 j_2 \ldots j_k} = y^\sigma_{j_1 j_2 \ldots j_k\, i} dx^i, \tag{5}$$

where f is any function on V^r and $0 \le k \le r$. A form $\rho \in \Omega^r_q W$ such that

$$h\rho = 0 \tag{6}$$

is said to be *contact*. Clearly, every q-form ρ such that $q \ge n+1$ is contact, and the 1-forms

$$\omega^\sigma_{j_1 j_2 \ldots j_l} = dy^\sigma_{j_1 j_2 \ldots j_l} - y^\sigma_{j_1 j_2 \ldots j_l\, i} dx^i, \quad 0 \le l \le r-1, \tag{7}$$

defined on the open set $V^r \subset J^r Y$ are examples of contact 1-forms. The collection of 1-forms $\{dx^i, \omega^\sigma_{j_1 j_2 \ldots j_k}, dy^\sigma_{l_1 l_2 \ldots l_{r-1} l_r}\}$, where $1 \le i \le n$, $1 \le \sigma \le m$, $1 \le k \le r-1$, $1 \le j_1 \le j_2 \le \cdots \le j_k \le n$, and $1 \le l_1 \le l_2 \le \cdots \le l_r \le n$, constitutes a basis of linear forms on the set V^r, called the *contact basis* (Sect. 2.1, Theorem 1). The exterior derivative df, or more precisely, $(\pi^{r+1,r})^* df$, can be decomposed as $(\pi^{r+1,r})^* df = hdf + pdf$, where pdf is a contact 1-form, called the *contact component* of f. Any form $\rho \in \Omega^r_q W$, of more precisely $(\pi^{r+1,r})^* \rho$, has the *canonical decomposition* $(\pi^{r+1,r})^* \rho = h\rho + p_1\rho + p_2\rho + \cdots + p_q\rho$, where $h\rho$ is π^{r+1}-horizontal and $p_k\rho$ is *k-contact*; this condition can equivalently be expressed by saying that the chart expression of $p_k\rho$ is generated by the product of k exterior factors $\omega^\sigma_{j_1 j_2 \ldots j_p}$, where $0 \le p \le r$.

The 1-forms $\omega^\sigma_{j_1 j_2 \ldots j_k}$ and 2-forms $d\omega^\sigma_{j_1 j_2 \ldots j_{r-1}}$ locally generate the *contact ideal* $\Theta^r W$ of the exterior algebra $\Omega^r W$, which is *closed* under the exterior derivative operator d; its elements are called *contact forms*. The *contact q-forms* are elements of the *contact submodules* $\Omega^r_q W \cap \Theta^r W$. We need these submodules for $q \le n$; denote

$$\Theta^r_q W = \Omega^r_q W \cap \Theta^r W, \quad q \le n. \tag{8}$$

The 1-forms $\omega^\sigma_{j_1 j_2 \ldots j_k}$, where $0 \le k \le r-1$, determined by a fibered atlas on Y, locally generate a (global) module of 1-forms, and an ideal $\Theta^r_0 W$ of the exterior algebra $\Omega^r W$ (for definitions see Appendix 7). Clearly, the contact ideal contains $\Theta^r_0 W$ as a subset.

Since the contact ideal is closed under the exterior derivative, we have the sequence of Abelian groups

$$0 \longrightarrow \Theta^r_1 W \xrightarrow{d} \Theta^r_2 W \xrightarrow{d} \cdots \xrightarrow{d} \Theta^r_n W. \tag{9}$$

If $\rho \in \Theta^r_q W$ is a contact form and f is a function on W^r, then the formula

$$d(f\rho) = df \wedge \rho + fd\rho \tag{10}$$

shows that the form $d(f\rho)$ is again a contact form. Thus, the mapping $\rho \to d(f\rho)$ is a morphism of Abelian groups; however, the exterior derivative in the sequence (9) is *not* a homomorphism of modules. Restricting the multiplication to *constant* functions *f*, that is, to *real numbers*, (9) can be considered as a sequence of real vector spaces.

Consider now the sets of q-forms $\Omega_q^r W$ such that $n+1 \leq q \leq \dim J^r Y$. Denote $q = n+k$. If $\rho \in \Omega_{n+k}^r W$, then $h\rho = 0$, and also $p_1\rho = 0$, $p_2\rho = 0, \ldots, p_{k-1}\rho = 0$ identically (cf. Sect. 2.4, Theorem 8), thus ρ is always contact, and its canonical decomposition has the form

$$(\pi^{r+1,r})^*\rho = p_k\rho + p_{k+1}\rho + \cdots + p_{k+n}\rho. \tag{11}$$

To introduce the notion of a strongly contact form, it is convenient to proceed in two steps. First, we slightly modify the definition given in Sect. 2.6 and introduce the class of strongly contact forms as follows. We say that an $(n+1)$-form $\rho \in \Omega_{n+1}^r W$ is *strongly contact*, if for every point $J_x^r\gamma \in V^r$, there exists an integer $s \geq r$, a fibered chart (V, ψ), $\psi = (x^i, y^\sigma)$, at $\gamma(x) \in V$ and a contact n-form $\eta \in \Theta_n^s V$ such that

$$p_1((\pi^{s,r})^*\rho - d\eta) = 0. \tag{12}$$

Second, if $\rho \in \Omega_{n+k}^r W$ where $k \geq 2$, we say that ρ is *strongly contact*, if for every point $J_x^r\gamma \in V^r$, there exists $s \geq r$, a fibered chart (V, ψ), $\psi = (x^i, y^\sigma)$, at $\gamma(x) \in V$ and a strongly contact $(n+k-1)$-form $\eta \in \Omega_{n+k}^s V$ such that

$$p_k((\pi^{s,r})^*\rho - d\eta) = 0. \tag{13}$$

Lemma 1 *Let $\rho \in \Omega_{n+k}^r W$. The following conditions are equivalent:*

(a) *ρ is strongly contact.*
(b) *There exists an integer $s \geq r$ and an $(n+k-1)$-form $\eta \in \Omega_{n+k}^s V$ such that*

$$(\pi^{s,r})^*\rho = \mu + d\eta, \quad p_k\mu = 0, \quad p_{k-1}\eta = 0. \tag{14}$$

Proof If ρ is strongly contact, then $(\pi^{s,r})^*\rho - d\eta = \mu$ for some form μ on V^s such that $p_k\mu = 0$. Then $(\pi^{s,r})^*\rho = \mu + d\eta$ proving (14). The converse is obvious. □

Lemma 2

(a) *Every form $\rho \in \Omega_{n+k}^r W$ such that $p_k\rho = 0$, is strongly contact.*

(b) *Exterior derivative of a contact n-form is strongly contact. Exterior derivative of a strongly contact form is strongly contact.*
(c) *Let* Ξ *be a* π*-vertical vector field,* $\rho \in \Omega^r_{n+k} W$ *a strongly contact form. If* $k \geq 2$*, then the* $(n+k-1)$*-form* $i_\Xi \rho$ *is strongly contact.*

Proof

(a) Obvious.
(b) We use the identity $p_{k+1}(d\rho - d\rho) = 0$.
(c) This follows from Lemma 9 and Sect. 2.5, Theorem 9. Indeed, for every π-vertical vector field Ξ

$$\begin{aligned} & i_\Xi p_k((\pi^{s,r})^* \rho - d\eta) \\ & = p_{k-1}(i_\Xi (\pi^{s,r})^* \rho - i_\Xi d\eta) \\ & = p_{k-1}(i_\Xi (\pi^{s,r})^* \rho - \partial_\Xi \eta) \\ & = p_{k-1}(i_\Xi (\pi^{s,r})^* \rho + d i_\Xi \eta) = 0. \end{aligned} \tag{15}$$

But $p_{k-2} i_\Xi \eta = i_\Xi p_{k-1} \eta = 0$ proving (c). □

Remark 1 It follows from Lemma 1 that the canonical decomposition of a strongly contact form $\rho \in \Theta^r_{n+k} W$ is

$$\begin{aligned} (\pi^{s,r})^* \rho & = p_k d\tau + p_{k+1}\rho + p_{k+2}\rho + \cdots + p_{n+k}\rho \\ & = d\tau + p_{k+1}(\rho - d\tau) + p_{k+2}(\rho - d\tau) + \cdots + p_{n+k}(\rho - d\tau), \end{aligned} \tag{16}$$

where the forms on the right-hand side are considered as canonically lifted to the set $V^s \subset J^s Y$.

Remark 2 One can formally extend the definition of a strongly contact form to the q-forms $\rho \in \Omega^r_q W$ such that $1 \leq q \leq n$. Indeed, we have for any contact form $\rho' \in \Theta^r_{q-1} W$, $h(\rho - d\rho') = h\rho$; thus, if $h\rho = 0$, then we have $h(\rho - d\rho') = 0$ for any $\rho' \in \Theta^r_{q-1} W$.

Remark 3 The definition of a strongly contact form, given above, has its natural origin in the theory of systems of partial differential equations for mappings of n independent variables, defined by *differential forms of degree* $n + k > n$: Such differential equations can equivalently be described by systems of *n-forms* arising by contraction of $(n + k)$-forms with k vector fields. For an ad hoc construction in this context, similar to the concept of a strongly contact form, see the *differential systems with independence condition* in Bryant et al. [Bry].

Remark 4 The definition of a strongly contact form is closely related to the concept of a Lepage form (Sect. 4.3).

Strongly contact $(n+k)$-forms on W^r constitute a *subgroup* $\Theta_q^r W$ of the Abelian group $\Omega_q^r W$; they do not form a submodule of $\Omega_q^r W$. The Abelian groups $\Theta_q^r W$ together with the exterior derivative d form a sequence

$$\Theta_n^r W \xrightarrow{d} \Theta_{n+1}^r W \xrightarrow{d} \cdots \xrightarrow{d} \Theta_M^r W \longrightarrow 0. \tag{17}$$

The index M of the last nonzero term in this sequence is

$$M = m\binom{n+r-1}{n} + 2n - 1. \tag{18}$$

If η is a contact n-form, then η is automatically a strongly contact form. Thus, sequences (9) and (17) can be glued together. We get a sequence

$$\begin{aligned} 0 \longrightarrow \Theta_1^r W \xrightarrow{d} \Theta_2^r W \xrightarrow{d} \cdots \xrightarrow{d} \Theta_n^r W \\ \xrightarrow{d} \Theta_{n+1}^r W \xrightarrow{d} \cdots \xrightarrow{d} \Theta_M^r W \longrightarrow 0. \end{aligned} \tag{19}$$

The families of Abelian groups $\{\Theta_q^r W\}$, where W runs through open subsets of the fibered manifold Y, induce Abelian sheaves, and the sequences (19) induce a sequence of Abelian sheaves. Indeed, consider for any integer q such that $1 \le q \le M$ the family of Abelian groups $\Theta_q^r = \{\Theta_q^r W\}$. Any two open sets $W_1, W_2 \subset Y$ such that $W_2 \subset W_1$ define a morphism of Abelian groups $\Theta_q^r W_1 \ni \rho \to \rho|_{W_2} \in \Theta_q^r W_2$, the *restriction* of a form, defined on the open set $W_1^r \subset J^r Y$, to the open set $W_2^r \subset W_1^r$. Clearly, Θ_q^r with these restriction morphisms forms an Abelian presheaf over Y. The restriction morphisms obviously satisfy the axioms of an Abelian sheaf (Sect. 7.4). Thus, the presheaf Θ_q^r has the structure of an Abelian sheaf.

If $1 \le q \le n$ (resp. $n+1 \le q \le M$), this sheaf is called the *sheaf of contact* (resp. *strongly contact*) *q-forms of order r* on Y.

Remark 5 The sheaf Θ_q^r, defined over the fibered manifold Y, *differs* from the sheaf of q-forms over the r-jet prolongation $J^r Y$ of Y; Θ_q^r can be characterized as the *direct image* of the sheaf of q-forms of order r over $J^r Y$ by the jet projection $\pi^{r,0}\colon J^r Y \to Y$. Our construction, for the forms of degree $q \le n$, is the same as an analogous construction in Anderson and Duchamp [AD].

The sequences (19) induce the *sequence of Abelian sheaves*

$$\begin{aligned} 0 \longrightarrow \Theta_1^r \xrightarrow{d} \Theta_2^r \xrightarrow{d} \cdots \xrightarrow{d} \Theta_n^r \xrightarrow{d} \Theta_{n+1}^r \\ \xrightarrow{d} \cdots \xrightarrow{d} \Theta_M^r \longrightarrow 0. \end{aligned} \tag{20}$$

The following basic observation shows that the De Rham sequence can be factored through the sequence (20).

Lemma 3 *The sequence of Abelian sheaves* (20) *is an exact subsequence of the De Rham sequence* (2).

Proof

1. To prove exactness of the sequence (20) at the term Θ_q^r, where $1 \leq q \leq n$, it is sufficient to consider differential forms defined on the chart neighborhood of a fibered chart (V, ψ), $\psi = (x^i, y^\sigma)$, on Y. However, for these differential forms, the statement already follows from Sect. 2.7, Theorem 13.
2. Exactness at the terms Θ_q^r, where $n + 1 \leq q \leq M$, follows from Sect. 2.7, Theorem 14. □

The sequence (19) will be referred to as the *contact sequence*, or the *contact subsequence* of the De Rham sequence.

We show that the sheaves Θ_q^r in the contact subsequence are all soft. To describe the structure of these sheaves Θ_q^r such that $n + 1 \leq q \leq M$, note that any q-form ρ on the r-jet prolongation $J^r Y$ identically satisfies

$$h\rho = 0, \quad p_1\rho = 0, \quad p_2\rho = 0\,, \ldots, \; p_{q-n-1}\rho = 0 \tag{21}$$

(Sect. 2.4, Theorem 8). We denote by $\Omega_{q(c)}^r W$ the submodule of the module of q-form $\Omega_q^r W$ defined by the condition

$$p_{q-n}\rho = 0. \tag{22}$$

This condition states that the submodule $\Omega_{q(c)}^r W$ consists of the forms whose order of contactness is $\geq q - n + 1$. The family of the modules $\Omega_{q(c)}^r W$ defines the sheaf of modules

$$\Omega_{q(c)}^r = \{\Omega_{q(c)}^r W\}. \tag{23}$$

Clearly, $\Omega_{q(c)}^r$ is a soft sheaf.

Lemma 4 *For every integer q such that $1 \leq q \leq M$ the sheaf Θ_q^r is soft.*

Proof

1. If $1 \leq q \leq n$, then the sheaf Θ_q^r admits multiplication by functions so it is fine; then, however, according to Sect. 7.1, Theorem 4, the sheaf Θ_q^r is soft.
2. Consider the contact subsequence (20) and the short exact sequence

$$0 \longrightarrow \Theta_1^r \xrightarrow{d} \Theta_2^r \xrightarrow{d} d\Theta_2^r \longrightarrow 0, \tag{24}$$

where $d\Theta_2^r$ denotes the image sheaf, $d\Theta_2^r = \operatorname{Im} d$. Since the sheaves Θ_1^r and Θ_2^r are soft, the sheaf $d\Theta_2^r$ is also soft (Sect. 7.10, Corollary 1). Similarly, assign to the sequence

$$0 \longrightarrow \Theta_1^r \xrightarrow{d} \Theta_2^r \xrightarrow{d} \Theta_3^r \xrightarrow{d} d\Theta_3^r \longrightarrow 0 \tag{25}$$

the short exact sequence

$$0 \longrightarrow \operatorname{Ker} d \longrightarrow \Theta_3^r \xrightarrow{d} d\Theta_3^r \longrightarrow 0. \tag{26}$$

Using exactness of (25) at Θ_3^r, we have $\operatorname{Ker} d = d\Theta_2^r$, so the sheaf $\operatorname{Ker} d$ in (26) is soft. Consequently, the sheaf $d\Theta_3^r$ is also soft. Continuing this way, we assign to the sequence

$$0 \longrightarrow \Theta_1^r \xrightarrow{d} \Theta_2^r \xrightarrow{d} \cdots \xrightarrow{d} \Theta_n^r \xrightarrow{d} d\Theta_n^r \longrightarrow 0 \tag{27}$$

the short exact sequence

$$0 \longrightarrow \operatorname{Ker} d \longrightarrow \Theta_n^r \xrightarrow{d} d\Theta_n^r \longrightarrow 0 \tag{28}$$

and since $\operatorname{Ker} d = d\Theta_{n-1}^r$ and this sheaf is soft, the sheaf $d\Theta_n^r$ is also soft. Now consider the sheaf Θ_{n+1}^r. Note that by definition, we have a sheaf morphism, expressed (by means of representatives of the germs) as

$$\Theta_n^r \times_Y \Omega_{n+1(c)}^r \ni (\tau, \mu) \to \mu + d\tau \in \Omega_{n+2}^r, \tag{29}$$

where $\Theta_n^r \times_Y \Omega_{n+1(c)}^r$ is the fiber product of the sheaves Θ_n^r and $\Omega_{n+1(c)}^r$. The sheaf Θ_{n+1}^r can be regarded as the *image sheaf* of this morphism; its *kernel* consists of the pairs $(\tau, -d\tau) \in \Theta_n^r \times_Y d\Theta_n^r$. We get a short exact sequence

$$0 \longrightarrow \Theta_n^r \times_Y d\Theta_n^r \longrightarrow \Theta_n^r \times_X \Omega_{n+1(c)}^r \xrightarrow{d} \Theta_{n+1}^r \longrightarrow 0. \tag{30}$$

The sheaves $\Theta_n^r \times_Y d\Theta_n^r$ and $\Theta_n^r \times_X \Omega_{n+1(c)}^r$ in this sequence are fiber products of soft sheaves Θ_n^r, $d\Theta_n^r$, and $\Omega_{n+1(c)}^r$, and are therefore soft; hence, the sheaf Θ_{n+1}^r is also soft.

Extending this construction to any of the sheaves Θ_q^r in the variational sequence (20), where $q \geq n+1$, we complete the proof. □

8.2 The Variational Sequence

Consider the De Rham sequence (32), and its contact subsequence (19), Sect. 8.1. Using Sect. 8.1, Lemma 3, we get a commutative diagram

$$\begin{array}{ccccccccccccc}
 & & & & & & 0 & & 0 & & 0 & & \\
 & & & & & & \downarrow & & \downarrow & & \downarrow & & \\
 & & & & 0 & \longrightarrow & \Theta_1^r & \xrightarrow{d} & \Theta_2^r & \xrightarrow{d} & \Theta_3^r & \xrightarrow{d} & \dots \\
 & & & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbf{R}_Y & \longrightarrow & \Omega_0^r & \xrightarrow{d} & \Omega_1^r & \xrightarrow{d} & \Omega_2^r & \xrightarrow{d} & \Omega_3^r & \xrightarrow{d} & \dots
\end{array} \tag{31}$$

in which $\mathbf{R}_Y \to \Omega_0^r$ is the canonical inclusion and the vertical arrows represent canonical inclusions of subsheaves. Passing to the quotient sheaves and quotient sheaf morphisms, this diagram induces a commutative diagram, written in two parts as

$$\begin{array}{ccccccccccc}
 & & & & & & 0 & & 0 & & \\
 & & & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & \longrightarrow & \Theta_1^r & \xrightarrow{d} & \Theta_2^r & \xrightarrow{d} & \dots \\
 & & & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbf{R}_Y & \longrightarrow & \Omega_0^r & \xrightarrow{d} & \Omega_1^r & \xrightarrow{d} & \Omega_2^r & \xrightarrow{d} & \dots \\
 & & & & & \searrow & \downarrow & & \downarrow & & \\
 & & & & & & \Omega_1^r/\Theta_1^r & \longrightarrow & \Omega_2^r/\Theta_2^r & \longrightarrow & \dots \\
 & & & & & & \downarrow & & \downarrow & & \\
 & & & & & & 0 & & 0 & &
\end{array} \tag{32}$$

$$\begin{array}{ccccccccccc}
 & & 0 & & & & & & & & \\
 & & \downarrow & & & & & & & & \\
\dots & \longrightarrow & \Theta_M^r & \longrightarrow & 0 & & & & & & \\
 & & \downarrow & & \downarrow & & & & & & \\
\dots & \xrightarrow{d} & \Omega_M^r & \xrightarrow{d} & \Omega_M^r & \xrightarrow{d} & \dots & \xrightarrow{d} & \Omega_N^r & \longrightarrow & 0 \\
 & & \downarrow & \nearrow & & & & & & & \\
\dots & \longrightarrow & \Omega_M^r/\Theta_M^r & & & & & & & & \\
 & & \downarrow & & & & & & & & \\
 & & 0 & & & & & & & &
\end{array}$$

The quotient sequence of Abelian sheaves, defined by this diagram,

$$0 \longrightarrow \mathbf{R}_Y \longrightarrow \Omega_0^r \longrightarrow \Omega_1^r/\Theta_1^r \longrightarrow \Omega_2^r/\Theta_2^r \longrightarrow \Omega_3^r/\Theta_3^r \longrightarrow \tag{33}$$

is called the (rth order) *variational sequence* over the fibered manifold Y. Since the De Rham sequence and its contact subsequence are exact, it can be easily verified that the quotient sequence is also exact (see also Sect. 7.7, Corollary 2). Thus, the variational sequence is a *resolution* of the constant sheaf $\mathbf{R}_Y$ over Y. We call the Abelian group morphisms in (33) the *Euler–Lagrange morphisms* and denote them by E_j: $\Omega^r_j/\Theta^r_j \longrightarrow \Omega^r_{j+1}/\Theta^r_{j+1}$, or just by E. The variational sequence is also denoted by

$$0 \longrightarrow \mathbf{R}_Y \longrightarrow \mathrm{Var}^r_Y. \tag{34}$$

Consider the complex of global sections

$$0 \longrightarrow \Omega^r_0 Y \longrightarrow (\Omega^r_1/\Theta^r_1)Y \longrightarrow (\Omega^r_2/\Theta^r_2)Y \longrightarrow (\Omega^r_3/\Theta^r_3)Y \longrightarrow \tag{35}$$

associated with the variational sequence (34), its cohomology groups $H^k(\mathrm{Var}^r_Y Y)$, and the cohomology groups of the fibered manifold Y with coefficients in the constant sheaf $\mathbf{R}_Y$; by the De Rham theorem, we identify these cohomology groups with the *De Rham cohomology groups*; thus, $H^k Y = H^k(Y, \mathbf{R}_Y)$ (Sect. 7.10, Remark 6). We are now going to establish two theorems, representing central results of this chapter, namely the tools for the study of the global variational functionals, considered in Chaps. 4 and 5 of this book.

Theorem 1 *The variational sequence* $0 \to \mathbf{R}_Y \to \mathrm{Var}^r_Y$ *is an acyclic resolution of the constant sheaf* $\mathbf{R}_Y$.

Proof Since the sheaves Ω^r_k and Θ^r_k are soft (Sect. 8.1, Lemma 4), the quotient sheaves Ω^r_k/Θ^r_k are also soft (Sect. 7.9, Corollary 1). Then, however, the sheaves Ω^r_k/Θ^r_k are acyclic, so the resolution $0 \to \mathbf{R}_Y \to \mathrm{Var}^r_Y$ is acyclic (Sect. 7.10, Lemma 18). □

Theorem 2 *The cohomology groups* $H^k(\mathrm{Var}^r_Y Y)$ *of the complex of global sections and the De Rham cohomology groups* $H^k Y$ *of the manifold* Y *are isomorphic.*

Proof This follows from Sect. 7.10, Theorem 5 (see also Corollary 13 and Remark 6). □

Remark 6 The cohomology groups $H^k(Y, \mathbf{R}_Y)$ have been constructed by means of the topology of the underlying fibered manifold Y. On the other hand, it follows from Theorem 2 that the same cohomology groups characterize properties of the complex of global sections associated with the variational sequence. In this sense, Theorem 2 clarifies the relationship between existence of global sections of the quotient Abelian groups and topological properties of Y.

8.3 Variational Projectors

In this section, we consider the columns of the diagram (33), Sect. 8.2, defining the variational sequence of order r over the fibered manifold Y. The main goal is to show that the *classes of forms* – elements of the quotient groups Ω^r_k/Θ^r_k – can be represented as *global differential forms*, defined on the s-jet prolongation $J^s Y$ for some s. Basic idea for constructing this representation leans on the definition of the quotient space, which is defined up to a canonical isomorphism. We shall construct an Abelian group of forms Φ_k^r and a group morphism $\mathscr{I}_k^r\colon \Omega_k^r \to \Phi_k^r$ such that $\mathrm{Ker}\mathscr{I}_k^r = \Theta_k^r$; then, the quotient sheaf Ω_k^r/Θ_k^r becomes canonically isomorphic with the image $\mathrm{Im}\,\mathscr{I}_k^r \subset \Phi_k^r$, according to the diagram

$$\begin{array}{ccccc} & & \Theta_k^r & & \\ & & \downarrow & & \\ & & \Omega_k^r & & \\ & \swarrow & & \searrow & \\ \Omega_k^r/\Theta_k^r & & \longleftrightarrow & & \mathrm{Im}\,\mathscr{I}_k^r \end{array} \tag{36}$$

Let $k \geq 1$, let W be an open set in Y, and let η be a k-contact $(n+k)$-form η, defined on the open set W^{r+1} in J^rY. In a fibered chart (V,ψ), $\psi = (x^i, y^\sigma)$, on Y, η has an expression

$$\eta = \sum_{0 \leq k \leq r} \Phi_\sigma^{j_1 j_2 \ldots j_k} \wedge \omega_{j_1 j_2 \ldots j_k}^\sigma \wedge \omega_0, \tag{37}$$

where $\Phi_\sigma^{j_1 j_2 \ldots j_k}$ are some $(k-1)$-contact $(k-1)$-forms. In this section, we construct a decomposition of the canonical lift $(\pi^{2r+1,r+1})^*\eta$ of η to W^{2r+1}; to this purpose, we use the property

$$\omega_{j_1 j_2 \ldots j_k}^\sigma \wedge \omega_0 = -d(\omega_{j_1 j_2 \ldots j_{k-1}}^\sigma \wedge \omega_{j_k}) \tag{38}$$

of the contact 1-forms $\omega_{j_1 j_2 \ldots j_k}^\sigma$. Although the decomposition will be constructed by means of fibered charts, it will be independent of the chosen charts.

First, consider the decomposition of $(n+1)$-forms, defined on the set W^{r+1}; the idea will be to identify in a form a summand, which is an *exact* form. The proof of the following theorem is based on the algebraic trace decomposition theory explained in Appendix 9.

Theorem 3 *Let η be a 1-contact $\pi^{r+1,r}$-horizontal $(n+1)$-form on W^{r+1}, expressed in a fibered chart (V,ψ), $\psi = (x^i, y^\sigma)$, by*

$$\eta = \sum_{0 \leq |J| \leq r} A_\sigma^J \omega_J^\sigma \wedge \omega_0. \tag{39}$$

(a) *There exist a 1-contact ω^σ-generated $(n+1)$-form $I_1\eta$ on V^{2r+1}, a 1-contact n-form $J_1\eta$ and a 2-contact $(n+1)$-form $K_1\eta$, defined on V^{2r+1}, such that*

$$(\pi^{2r+1,r+1})^*\eta = I_1\eta - dJ_1\eta + K_1\eta, \tag{40}$$

where

$$I_1\eta = \left(A_\sigma + \sum_{1\le s\le r} (-1)^s d_{j_1} d_{j_2}\ldots d_{j_s} A_\sigma^{j_1 j_2 \ldots j_s}\right)\omega^\sigma \wedge \omega_0,$$
$$J_1\eta = \sum_{1\le s\le r}\sum_{0\le k\le r-1} (-1)^k d_{j_{s-k+1}} d_{j_{s-k+2}}\ldots d_{j_s} A_\sigma^{i_1 i_2 \ldots i_{s-k} j_{s-k+1} j_{s-k+2}\ldots j_s}\omega^\sigma_{i_1 i_2\ldots i_{s-k-1}}\wedge\omega_{i_{s-k}},$$
$$K_1\eta = \sum_{1\le s\le r}\sum_{0\le k\le s-1} (-1)^{k+1} pd(d_{j_{s-k+1}} d_{j_{s-k+2}}\ldots d_{j_s} A_\sigma^{i_1 i_2 \ldots i_{s-k} j_{s-k+1} j_{s-k+2}\ldots j_s})$$
$$\wedge\,\omega^\sigma_{i_1 i_2\ldots i_{s-k-1}}\wedge\omega_{i_{s-k}}. \tag{41}$$

(b) *Suppose that we have a decomposition*

$$(\pi^{2r+1,r+1})^*\eta = \eta_0 - d\eta_1 + \eta_2 \tag{42}$$

such that η_0 *is* 1*-contact and* ω^σ*-generated,* η_1 *is* 1*-contact, and* η_2 *is a* 2*-contact form. Then*

$$\eta_0 = I_1\eta, \quad d\eta_1 = dJ_1\eta, \quad \eta_2 = K_1\eta. \tag{43}$$

Proof

(a) Write expression (39) as

$$\begin{aligned}\eta &= \sum_{0\le |J|\le r} A_\sigma^J \omega_J^\sigma\wedge\omega_0\\ &= A_\sigma\omega^\sigma\wedge\omega_0 + \sum_{1\le|J|\le r} A_\sigma^J\omega_J^\sigma\wedge\omega_0,\end{aligned} \tag{44}$$

and consider a summand $A_\sigma^J\omega_J^\sigma\wedge\omega_0$, where $|J| = s\ge 1$. Then, in the standard index notation

$$\begin{aligned}A_\sigma^J\omega_J^\sigma\wedge\omega_0 &= -d(A_\sigma^{i_1 i_2\ldots i_s}\omega^\sigma_{i_1 i_2\ldots i_{s-1}}\wedge\omega_{i_s}) + dA_\sigma^{i_1 i_2\ldots i_s}\omega^\sigma_{i_1 i_2\ldots i_{s-1}}\wedge\omega_{i_s}\\
&= hdA_\sigma^{i_1 i_2\ldots i_s}\wedge\omega^\sigma_{i_1 i_2\ldots i_{s-1}}\wedge\omega_{i_s} + pdA_\sigma^{i_1 i_2\ldots i_s}\wedge\omega^\sigma_{i_1 i_2\ldots i_{s-1}}\wedge\omega_{i_s}\\
&\quad - d(A_\sigma^{i_1 i_2\ldots i_s}\omega^\sigma_{i_1 i_2\ldots i_{s-1}}\wedge\omega_{i_s})\\
&= d(d_{j_s}A_\sigma^{i_1 i_2\ldots i_{s-1} j_s}\omega^\sigma_{i_1 i_2\ldots i_{s-2}}\wedge\omega_{i_{s-1}}) + d_{j_{s-1}}d_{j_s}A_\sigma^{i_1 i_2\ldots i_{s-2} j_{s-1} j_s}\wedge\omega^\sigma_{i_1 i_2\ldots i_{s-2}}\wedge\omega_0\\
&\quad - pd(d_{j_s}A_\sigma^{i_1 i_2\ldots i_{s-1} j_s})\omega^\sigma_{i_1 i_2\ldots i_{s-2}}\wedge\omega_{i_{s-1}}\\
&\quad + pdA_\sigma^{i_1 i_2\ldots i_s}\wedge\omega^\sigma_{i_1 i_2\ldots i_{s-1}}\wedge\omega_{i_s} - d(A_\sigma^{i_1 i_2\ldots i_s}\omega^\sigma_{i_1 i_2\ldots i_{s-1}}\wedge\omega_{i_s})\\
&= d_{j_{s-1}}d_{j_s}A_\sigma^{i_1 i_2\ldots i_{s-2} j_{s-1} j_s}\wedge\omega^\sigma_{i_1 i_2\ldots i_{s-2}}\wedge\omega_0\\
&\quad - pd(d_{j_s}A_\sigma^{i_1 i_2\ldots i_{s-1} j_s})\omega^\sigma_{i_1 i_2\ldots i_{s-2}}\wedge\omega_{i_{s-1}} + pdA_\sigma^{i_1 i_2\ldots i_s}\wedge\omega^\sigma_{i_1 i_2\ldots i_{s-1}}\wedge\omega_{i_s}\\
&\quad + d(A_\sigma^{i_1 i_2\ldots i_{s-1} j_s}\omega^\sigma_{i_1 i_2\ldots i_{s-2}}\wedge\omega_{i_{s-1}} - A_\sigma^{i_1 i_2\ldots i_s}\omega^\sigma_{i_1 i_2\ldots i_{s-1}}\wedge\omega_{i_s}).\end{aligned} \tag{45}$$

Further calculations yield

$$\begin{aligned} A_\sigma^{i_1 i_2 \dots i_s} \omega^\sigma_{i_1 i_2 \dots i_s} \wedge \omega_0 &= (-1)^s d_{j_1} d_{j_2} \dots d_{j_s} A_\sigma^{j_1 j_2 \dots j_s} \omega^\sigma \wedge \omega_0 \\ &- \sum_{0 \le k \le s-1} (-1)^k pd(d_{j_{s-k+1}} d_{j_{s-k+2}} \dots d_{j_s} A_\sigma^{i_1 i_2 \dots i_{s-k} j_{s-k+1} j_{s-k+2} \dots j_s}) \wedge \omega^\sigma_{i_1 i_2 \dots i_{s-k-1}} \wedge \omega_{i_{s-k}} \\ &- d\left(\sum_{0 \le k \le s-1} (-1)^k d_{j_{s-k+1}} d_{j_{s-k+2}} \dots d_{j_s} A_\sigma^{i_1 i_2 \dots i_{s-k} j_{s-k+1} j_{s-k+2} \dots j_s} \omega^\sigma_{i_1 i_2 \dots i_{s-k-1}} \wedge \omega_{i_{s-k}} \right). \end{aligned} \tag{46}$$

These formulas prove statement (a).

(b) To prove (b), suppose that $\eta_0 - d\eta_1 + \eta_2 = 0$, where η_0 is 1-contact and ω^σ-generated, η_1 is 1-contact, and η_2 is a 2-contact form; we want to show that this condition implies $\eta_0 = 0$, $\eta_2 = 0$; indeed, these conditions will also prove that $d\eta_1 = 0$. The forms η_0 and η_1 can be expressed in the form

$$\eta_0 = A_\sigma \omega^\sigma \wedge \omega_0, \quad \eta_1 = B_\sigma{}^i \omega^\sigma \wedge \omega_i + \sum_{1 \le k \le 2r} B_\sigma^{j_1 j_2 \dots j_k \, i} \omega^\sigma_{j_1 j_2 \dots j_k} \wedge \omega_i. \tag{47}$$

If $k \ge 1$, then $B_\sigma^{j_1 j_2 \dots j_k \, i}$ can be decomposed as

$$\begin{aligned} B_\sigma^{j_1 j_2 \dots j_k \ i} &= \tilde{B}_\sigma^{j_1 j_2 \dots j_k \ i} \\ &+ \frac{1}{k+1}(B_\sigma^{j_1 j_2 \dots j_k \ i} - B_\sigma^{i j_2 j_3 \dots j_k \ j_1}) + \frac{1}{k+1}(B_\sigma^{j_1 j_2 \dots j_k \ i} - B_\sigma^{j_1 i j_3 j_4 \dots j_k \ j_2}) \\ &+ \dots + \frac{1}{k+1}(B_\sigma^{j_1 j_2 \dots j_k \ i} \dots B_\sigma^{j_1 j_2 \dots j_{k-1} i \ j_k}), \end{aligned} \tag{48}$$

where $\tilde{B}_\sigma^{j_1 j_2 \dots j_k \, i}$ is the symmetric component,

$$\tilde{B}_\sigma^{j_1 j_2 \dots j_k \ i} = \frac{1}{k+1}(B_\sigma^{j_1 j_2 \dots j_k \ i} + B_\sigma^{i j_2 j_3 \dots j_k j_1} + B_\sigma^{j_1 i j_3 j_4 \dots j_k \, j_2} + \dots + B_\sigma^{j_1 j_2 \dots j_{k-1} \ i j_k}). \tag{49}$$

Now calculating $p_1 d\eta_1$, we have

$$\begin{aligned} p_1 d\eta_1 &= -d_i B_\sigma{}^i \omega^\sigma \wedge \omega_0 - B_\sigma{}^i \omega^\sigma_i \wedge \omega_0 \\ &- \sum_{1 \le k \le 2r} d_i B_\sigma^{j_1 j_2 \dots j_k \ i} \omega^\sigma_{j_1 j_2 \dots j_k} \wedge \omega_0 - \sum_{1 \le k \le 2r} B_\sigma^{j_1 j_2 \dots j_k \ i} \omega^\sigma_{j_1 j_2 \dots j_k i} \wedge \omega_0 \\ &= -d_i B_\sigma{}^i \omega^\sigma \wedge \omega_0 - (B_\sigma{}^{j_1} + d_i B_\sigma^{j_1 \ i}) \omega^\sigma_{j_1} \wedge \omega_0 \\ &- \sum_{2 \le k \le 2r} (d_i B^{j_1 j_2 \dots j_k \ i} + B_\sigma^{j_1 j_2 \dots j_{k-1} j_k}) \omega^\sigma_{j_1 j_2 \dots j_{k-1}} \wedge \omega_0 \\ &- B_\sigma^{j_1 j_2 \dots j_{2r} i} \omega^\sigma_{j_1 j_2 \dots j_{2r} \, i} \wedge \omega_0. \end{aligned} \tag{50}$$

Equation $\eta_0 - d\eta_1 + \eta_2 = 0$ implies $\eta_0 - p_1 d\eta_1 = 0$ hence

$$\begin{aligned}
&(A_\sigma - d_{j_1}B_\sigma^{j_1})\omega^\sigma \wedge \omega_0 - (d_i B_\sigma^{j_1 i} + B_\sigma^{j_1})\omega_{j_1}^\sigma \wedge \omega_0 \\
&\quad - (d_i B_\sigma^{j_1 j_2 i} + B_\sigma^{j_1 j_2})\omega_{j_1 j_2}^\sigma \wedge \omega_0 - (d_i B_\sigma^{j_1 j_2 i} + B_\sigma^{j_1 j_2})\omega_{j_1 j_2}^\sigma \wedge \omega_0 \\
&\quad - \cdots - (d_i B_\sigma^{j_1 j_2 \ldots j_{2r-1}\, i} + B_\sigma^{j_1 j_2 \ldots j_{2r-2}\, j_{2r-1}})\omega_{j_1 j_2 \ldots j_{2r-1}}^\sigma \wedge \omega_0 \\
&\quad - (d_i B_\sigma^{j_1 j_2 \ldots j_{2r}\, i} + B_\sigma^{j_1 j_2 \ldots j_{2r-1}\, j_{2r}})\omega_{j_1 j_2 \ldots j_{2r}}^\sigma \wedge \omega_0 \\
&\quad - B_\sigma^{j_1 j_2 \ldots j_{2r}\, j_{2r+1}}\omega_{j_1 j_2 \ldots j_{2r}\, j_{2r+1}}^\sigma \wedge \omega_0 = 0,
\end{aligned} \tag{51}$$

therefore, the components $B_\sigma^{j_1 j_2 \ldots j_k\, i}$ satisfy

$$\begin{aligned}
&\tilde{B}_\sigma^{j_1 j_2 \ldots j_{2r}\, j_{2r+1}} = 0, \\
&\tilde{B}_\sigma^{j_1 j_2 \ldots j_{2r-1}\, j_{2r}} = -d_i B_\sigma^{j_1 j_2 \ldots j_{2r}\, i}, \\
&\ldots \\
&B_\sigma^{j_1} = -d_i B_\sigma^{j_1 i}, \\
&B_\sigma^{j_1} = -d_i B_\sigma^{j_1 i},
\end{aligned} \tag{52}$$

and $A_\sigma = d_{j_1} B_\sigma^{j_1}$. Consequently,

$$\begin{aligned}
A_\sigma &= d_{j_1} B_\sigma^{j_1} = -d_{j_1} d_{j_2} B_\sigma^{j_1 j_2} = -d_{j_1} d_{j_2} \tilde{B}_\sigma^{j_1 j_2} = d_{j_1} d_{j_2} d_{j_3} B_\sigma^{j_1 j_2 j_3} \\
&= d_{j_1} d_{j_2} d_{j_3} \tilde{B}_\sigma^{j_1 j_2 j_3} = \cdots = (-1)^{k-1} d_{j_1} d_{j_2} \ldots d_{j_{k-1}} d_{j_k} B_\sigma^{j_1 j_2 \ldots j_{k-1}\, j_k} \\
&= (-1)^{k-1} d_{j_1} d_{j_2} \ldots d_{j_{k-1}}\; d_{j_k} \tilde{B}_\sigma^{j_1 j_2 \ldots j_{k-1}\, j_k} \\
&= \cdots = (-1)^{2r-1} d_{j_1} d_{j_2} \ldots d_{j_{2r-1}} d_{j_{2r}} B_\sigma^{j_1 j_2 \ldots j_{2r-1}\, j_{2r}} \\
&= (-1)^{2r-1} d_{j_1} d_{j_2} \ldots d_{j_{2r-1}}\; d_{j_{2r}} \tilde{B}_\sigma^{j_1 j_2 \ldots j_{2r-1}\, j_{2r}} \\
&= (-1)^{2r} d_{j_1} d_{j_2} \ldots d_{j_{2r}} d_{j_{2r+1}} B_\sigma^{j_1 j_2 \ldots j_{2r}\, j_{2r+1}} \\
&= (-1)^{2r} d_{j_1} d_{j_2} \ldots d_{j_{2r}} d_{j_{2r+1}} \tilde{B}_\sigma^{j_1 j_2 \ldots j_{2r}\, j_{2r+1}} \\
&= 0,
\end{aligned} \tag{53}$$

proving that $A_\sigma = 0$; hence, $\eta_0 = 0$.
Substituting from this identity to Eq. (52),

$$\begin{aligned}
&\tilde{B}_\sigma^{j_1 j_2 \ldots j_{2r}\, j_{2r+1}} = 0, \quad d_i B_\sigma^{j_1 j_2 \ldots j_{2r}\, i} = -\tilde{B}_\sigma^{j_1 j_2 \ldots j_{2r-1}\, j_{2r}}, \\
&d_i B_\sigma^{j_1 j_2 \ldots j_{2r-1}\, i} = -\tilde{B}_\sigma^{j_1 j_2 \ldots j_{2r-2}\, j_{2r-1}}, \ldots, d_i B_\sigma^{j_1 j_2\, i} = -\tilde{B}_\sigma^{j_1 j_2}, \\
&d_i B_\sigma^{j_1 i} = -B_\sigma^{j_1}, \quad d_{j_1} B_\sigma^{j_1} = 0.
\end{aligned} \tag{54}$$

Then, by Sect. 3.1, Remark 2 and Sect. 3.2, Theorem 1, the functions $B_\sigma^{j_1}, B_\sigma^{j_1 i}, B_\sigma^{j_1 j_2 i}, \ldots, B_\sigma^{j_1 j_2 \ldots j_{2r-1}\, i}, B_\sigma^{j_1 j_2 \ldots j_{2r}\, i}$ depend on the variable x^i only. Then, formula (47) implies $p_2 d\eta_1 = 0$; hence, from equation $\eta_0 - d\eta_1 + \eta_2 = 0$, $\eta_2 = 0$. This proves (b). □

Note that for any n-form ρ on W^r, the 1-contact component $p_1\rho$ is an n-form on the set W^{r+1}, and since $p_1 d\rho = p_1 dh\rho + p_1 dp_1\rho = dh\rho + p_1 dp_1\rho$, the 1-contact $(n+1)$-form $p_1 dp_1\rho$ is also defined on W^{r+1}. Therefore, the form $I_1 p_1 dp_1\rho$ is defined and is an $(n+1)$-form on W^{2r+1}.

Corollary 1 *The form $I_1 p_1 dp_1\rho$ vanishes identically,*

$$I_1 p_1 dp_1\rho = 0. \tag{55}$$

Proof We have the identity

$$\begin{aligned}
&(\pi^{2r+1,r+1})^* p_1 dp_1\rho \\
&\quad = (\pi^{2r+1,r+1})^*(dp_1\rho - p_2 dp_1\rho - p_3 dp_1\rho - \cdots - p_{n+1} dp_1\rho) \\
&\quad = d(\pi^{2r+1,r+1})^* p_1\rho - p_2(\pi^{2r,r+1})^* dp_1\rho
\end{aligned} \tag{56}$$

because $p_3 dp_1\rho = 0$, $p_4 dp_1\rho = 0$, $\ldots$, $p_{n+1} dp_1\rho = 0$. Comparing this formula with decomposition (5) and using the uniqueness of the component $I_1 p_1 dp_1\rho$ (Theorem 3, (b)), we get identity (55). □

Remark 7 If $p_2 d\eta_1$ is ω^σ-generated, then $p_2 d\eta_1 = 0$ (see the proof of Theorem 3).

Remark 8 Part (b) of Theorem 3 can alternatively be proved by means of the properties of Lepage forms. Note that the uniqueness condition $\eta_0 - d\eta_1 + \eta_2 = 0$ implies that $\eta_0 = p_1 d\eta_1$; this means, however, that η_1 is a *Lepage form* whose Lagrangian $h\eta_1 = 0$ is the *zero Lagrangian*. Using Sect. 4.3, Theorem 3, we get $\eta_1 = d\kappa + \mu$, where the form κ is 1-contact and the form μ is of order of contactness ≥ 2. Then, however, $d\eta_1 = d\mu$, which is a form of order or contactness ≥ 2. Equation $\eta_0 - d\eta_1 + \eta_2 = 0$ now implies that $\eta_0 = 0$ because η_0 is 1-contact (and $-d\mu + \eta_2$ is of order of contactness ≥ 2).

Next, consider $(n+k)$-forms on W^{r+1} for arbitrary $k \geq 1$. The following result generalizes Theorem 3.

Theorem 4 *Let $k \geq 1$, let η be a k-contact, $\pi^{r+1,r}$-horizontal $(n+k)$-form on W^{r+1}, expressed in a fibered chart (V, ψ), $\psi = (x^i, y^\sigma)$, by*

$$\eta = \sum_{0 \leq k \leq r} \Phi_\sigma^{j_1 j_2 \ldots j_k} \wedge \omega^\sigma_{j_1 j_2 \ldots j_k} \wedge \omega_0. \tag{57}$$

There exist k-contact ω^σ-generated k-form $I_k\eta$ on V^{2r+1}, a $(k-1)$-contact $(n+k-1)$-form $J_k\eta$ and an $(k+1)$-contact $(n+k)$-form $K_k\eta$, defined on V^{2r+1}, such that

$$(\pi^{2r+1,r+1})^*\eta = I_k\eta - dJ_k\eta + K_k\eta. \tag{58}$$

(b) *Suppose that we have a decomposition*

$$(\pi^{2r+1,r+1})^*\eta = \eta_0 - d\eta_1 + \eta_2 \tag{59}$$

such that η_0 *is* 1*-contact and* ω^σ*-generated,* η_1 *is* 1*-contact, and* η_2 *is a* 2*-contact form. Then,*

$$\eta_0 = I_k\eta. \tag{60}$$

Proof

(a) Let $k \geq 1$, let W be an open set in Y, and let η be a k-contact, $(n+k)$-form, defined on some open set W^{r+1}. In a fibered chart (V,ψ), $\psi = (x^i, y^\sigma)$, on Y, η has a unique decomposition

$$\eta = \eta_0 + \eta_1 + \eta_2 + \cdots + \eta_r, \tag{61}$$

where η_0 is the ω^σ-generated component, η_1 includes all $\omega^\sigma_{j_1}$-generated terms, which do not contain any factor ω^σ, η_2 includes all $\omega^\sigma_{j_1j_2}$-generated terms, which do not contain any factors ω^σ, $\omega^\sigma_{j_1}$, etc.; finally, η_r consists of $\omega^\sigma_{j_1j_2\ldots j_r}$-generated terms which do not include any factors ω^σ, $\omega^\sigma_{j_1}$, $\omega^\sigma_{j_1j_2}$, …, $\omega^\sigma_{j_1j_2\ldots j_{r-1}}$. η_r has an expression

$$\eta_r = \Psi_\sigma^{j_1j_2\ldots j_r} \wedge \omega^\sigma_{j_1j_2\ldots j_r} \wedge \omega_0 \tag{62}$$

for some $(k-1)$-contact $(k-1)$-forms $\Psi_\sigma^{j_1j_2\ldots j_r}$, which do not include any factors ω^σ, $\omega^\sigma_{j_1}$, $\omega^\sigma_{j_1j_2}$, …, $\omega^\sigma_{j_1j_2\ldots j_{r-1}}$. Then, by (38),

$$\begin{aligned}
\eta_r &= -\Psi_\sigma^{j_1j_2\ldots j_r} \wedge d(\omega^\sigma_{j_1j_2\ldots j_{r-1}} \wedge \omega_{j_r}) \\
&= (-1)^k d(\Psi_\sigma^{j_1j_2\ldots j_r} \wedge \omega^\sigma_{j_1j_2\ldots j_{r-1}} \wedge \omega_{j_r}) - (-1)^k d\Psi_\sigma^{j_1j_2\ldots j_r} \wedge \omega^\sigma_{j_1j_2\ldots j_{r-1}} \wedge \omega_{j_r} \\
&= -(-1)^k p_{k-1} d\Psi_\sigma^{j_1j_2\ldots j_r} \wedge \omega^\sigma_{j_1j_2\ldots j_{r-1}} \wedge \omega_{j_r} \\
&\quad + (-1)^k d(\Psi_\sigma^{j_1j_2\ldots j_r} \wedge \omega^\sigma_{j_1j_2\ldots j_{r-1}} \wedge \omega_{j_r}) \\
&\quad - (-1)^k p_k d\Psi_\sigma^{j_1j_2\ldots j_r} \wedge \omega^\sigma_{j_1j_2\ldots j_{r-1}} \wedge \omega_{j_r}.
\end{aligned} \tag{63}$$

The term $p_{k-1} d\Psi_\sigma^{j_1j_2\ldots j_r} \wedge \omega^\sigma_{j_1j_2\ldots j_{r-1}} \wedge \omega_{j_r}$ in this expression is k-contact (and therefore contains the factor $dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n$) and is generated by the forms $\omega^\sigma_{j_1j_2\ldots j_{r-1}}$. Thus, from the definition of the $(k-1)$-component $p_{k-1} d\Psi_\sigma^{j_1j_2\ldots j_r}$, it follows that the form $p_{k-1} d\Psi_\sigma^{j_1j_2\ldots j_r} \wedge \omega^\sigma_{j_1j_2\ldots j_{r-1}} \wedge \omega_{j_r}$ contains the exterior factors $\omega^\nu_{l_1l_2\ldots l_{r-1}}$, $\omega^\nu_{l_1l_2\ldots l_r}$ and $\omega^\nu_{l_1l_2\ldots l_r i_1}$ only. Decomposition (61) now reads

$$\begin{aligned}(\pi^{2r+1,r+1})^*\eta = \eta_0 + \eta_1 + \eta_2 + \cdots + \eta_{r-2} + \tilde{\eta}_{r-1} \\ + (-1)^k d(\Psi_\sigma^{j_1 j_2 \ldots j_r} \wedge \omega_{j_1 j_2 \ldots j_{r-1}}^\sigma \wedge \omega_{j_r}) \\ - (-1)^k p_k d\Psi_\sigma^{j_1 j_2 \ldots j_r} \wedge \omega_{j_1 j_2 \ldots j_{r-1}}^\sigma \wedge \omega_{j_r},\end{aligned} \tag{64}$$

where $\tilde{\eta}_{r-1}$ can be written as

$$\begin{aligned}\tilde{\eta}_{r-1} &= \eta_{r-1} - (-1)^k p_{k-1} d\Psi_\sigma^{j_1 j_2 \ldots j_r} \wedge \omega_{j_1 j_2 \ldots j_{r-1}}^\sigma \wedge \omega_{j_r} \\ &= \Psi_\sigma^{j_1 j_2 \ldots j_{r-1}} \wedge \omega_{j_1 j_2 \ldots j_{r-1}}^\sigma \wedge \omega_0.\end{aligned} \tag{65}$$

Then, however,

$$\begin{aligned}\tilde{\eta}_{r-1} &= \Psi_\sigma^{j_1 j_2 \ldots j_{r-1}} \wedge d(\omega_{j_1 j_2 \ldots j_{r-2}}^\sigma \wedge \omega_{j_{r-1}}) \\ &= -(-1)^k p_{k-1} d\Psi_\sigma^{j_1 j_2 \ldots j_{r-1}} \wedge \omega_{j_1 j_2 \ldots j_{r-2}}^\sigma \wedge \omega_{j_{r-1}} \\ &\quad + (-1)^k d(\Psi_\sigma^{j_1 j_2 \ldots j_{r-1}} \wedge \omega_{j_1 j_2 \ldots j_{r-2}}^\sigma \wedge \omega_{j_{r-1}}) \\ &\quad - (-1)^k p_k d\Psi_\sigma^{j_1 j_2 \ldots j_{r-1}} \wedge \omega_{j_1 j_2 \ldots j_{r-2}}^\sigma \wedge \omega_{j_{r-1}}.\end{aligned} \tag{66}$$

The term $p_{k-1} d\Psi_\sigma^{j_1 j_2 \ldots j_{r-1}} \wedge \omega_{j_1 j_2 \ldots j_{r-2}}^\sigma \wedge \omega_{j_{r-1}}$ in this expression is k-contact, contains the factor ω_0, and is generated by the forms $\omega_{j_1 j_2 \ldots j_{r-1}}^\sigma$. From the definition of the $(k-1)$-component $p_{k-1} d\Psi_\sigma^{j_1 j_2 \ldots j_r}$, it follows that this term contains the exterior factors $\omega_{l_1 l_2 \ldots l_{r-2}}^\nu$, $\omega_{l_1 l_2 \ldots l_{r-1}}^\nu$, $\omega_{l_1 l_2 \ldots l_r}^\nu$, $\omega_{l_1 l_2 \ldots l_r i_1 i_2}^\nu$ only. The decomposition (61) (or (64)) now reads

$$\begin{aligned}(\pi^{2r+1,r+1})^*\eta = \eta_0 + \eta_2 + \eta_3 + \cdots + \eta_{r-3} + \tilde{\eta}_{r-2} \\ + (-1)^k d(\Psi_\sigma^{j_1 j_2 \ldots j_{r-1}} \wedge \omega_{j_1 j_2 \ldots j_{r-2}}^\sigma \wedge \omega_{j_{r-1}}) \\ - (-1)^k p_k d\Psi_\sigma^{j_1 j_2 \ldots j_{r-1}} \wedge \omega_{j_1 j_2 \ldots j_{r-2}}^\sigma \wedge \omega_{j_{r-1}} \\ + (-1)^k (d(\Psi_\sigma^{j_1 j_2 \ldots j_r} \wedge \omega_{j_1 j_2 \ldots j_{r-1}}^\sigma \wedge \omega_{j_r}) \\ - p_k d\Psi_\sigma^{j_1 j_2 \ldots j_r} \wedge \omega_{j_1 j_2 \ldots j_{r-1}}^\sigma \wedge \omega_{j_r}),\end{aligned} \tag{67}$$

where

$$\begin{aligned}\tilde{\eta}_{r-2} &= \eta_{r-2} - (-1)^k p_{k-1} d\Psi_\sigma^{j_1 j_2 \ldots j_{r-1}} \wedge \omega_{j_1 j_2 \ldots j_{r-2}}^\sigma \wedge \omega_{j_{r-1}} \\ &= \Psi_\sigma^{j_1 j_2 \ldots j_2} \wedge \omega_{j_1 j_2 \ldots j_{r-2}}^\sigma \wedge \omega_0.\end{aligned} \tag{68}$$

Continuing in the same way, we get after $r-1$ steps

$$\begin{aligned}(\pi^{2r+1,r+1})^*\eta = \eta_0 + \tilde{\eta}_1 &+ (-1)^k d(\Psi_\sigma^{j_1 j_2} \wedge \omega_{j_1}^\sigma \wedge \omega_{j_2}) \\ &- (-1)^k p_k d\Psi_\sigma^{j_1 j_2} \wedge \omega_{j_1 j_2}^\sigma \wedge \omega_{j_2} \\ &+ \cdots + (-1)^k d(\Psi_\sigma^{j_1 j_2 \ldots j_{r-1}} \wedge \omega_{j_1 j_2 \ldots j_{r-2}}^\sigma \wedge \omega_{j_{r-1}}) \\ &- (-1)^k p_k d\Psi_\sigma^{j_1 j_2 \ldots j_{r-1}} \wedge \omega_{j_1 j_2 \ldots j_{r-2}}^\sigma \wedge \omega_{j_{r-1}} \\ &+ (-1)^k d(\Psi_\sigma^{j_1 j_2 \ldots j_r} \wedge \omega_{j_1 j_2 \ldots j_{r-1}}^\sigma \wedge \omega_{j_r}) \\ &- (-1)^k p_k d\Psi_\sigma^{j_1 j_2 \ldots j_r} \wedge \omega_{j_1 j_2 \ldots j_{r-1}}^\sigma \wedge \omega_{j_r},\end{aligned} \tag{69}$$

where

$$\begin{aligned}\tilde{\eta}_1 &= \eta_1 - (-1)^k p_{k-1} d\Psi_\sigma^{j_1 j_2} \wedge \omega_{j_1}^\sigma \wedge \omega_{j_2} \\ &= \Psi_\sigma^{j_1} \wedge \omega_{j_1}^\sigma \wedge \omega_0.\end{aligned} \tag{70}$$

The form $\tilde{\eta}_1$ contains $\omega_{j_1}^\sigma, \omega_{j_1 j_2}^\sigma, \ldots, \omega_{j_1 j_2 \ldots j_r}^\sigma, \omega_{j_1 j_2 \ldots j_r i_1}^\sigma, \ldots, \omega_{j_1 j_2 \ldots j_r i_1 i_2 \ldots i_{r-1}}^\sigma$ but no factor ω^σ. Then,

$$\begin{aligned}\tilde{\eta}_1 &= -\Psi_\sigma^{j_1} \wedge d(\omega^\sigma \wedge \omega_{j_1}) \\ &= (-1)^k d(\Psi_\sigma^{j_1} \wedge \omega^\sigma \wedge \omega_{j_1}) - (-1)^k d\Psi_\sigma^{j_1} \wedge \omega^\sigma \wedge \omega_{j_1} \\ &= -(-1)^k p_{k-1} d\Psi_\sigma^{j_1} \wedge \omega^\sigma \wedge \omega_{j_1} \\ &\quad + (-1)^k d(\Psi_\sigma^{j_1} \wedge \omega^\sigma \wedge \omega_{j_1}) - (-1)^k p_k d\Psi_\sigma^{j_1} \wedge \omega^\sigma \wedge \omega_{j_1},\end{aligned} \tag{71}$$

and

$$\begin{aligned}&(\pi^{2r+1,r+1})^*\eta \\ &= \eta_0 - (-1)^k p_{k-1} d\Psi_\sigma^{j_1} \wedge \omega^\sigma \wedge \omega_{j_1} \\ &- (-1)^{k-1} d(\Psi_\sigma^{j_1} \wedge \omega^\sigma \wedge \omega_{j_1} + \Psi_\sigma^{j_1 j_2} \wedge \omega_{j_1}^\sigma \wedge \omega_{j_2} \\ &+ \cdots + \Psi_\sigma^{j_1 j_2 \ldots j_{r-1}} \wedge \omega_{j_1 j_2 \ldots j_{r-2}}^\sigma \wedge \omega_{j_{r-1}} + \Psi_\sigma^{j_1 j_2 \ldots j_r} \wedge \omega_{j_1 j_2 \ldots j_{r-1}}^\sigma \wedge \omega_{j_r}) \\ &- (-1)^k (p_k d\Psi_\sigma^{j_1} \wedge \omega^\sigma \wedge \omega_{j_1} + p_k d\Psi_\sigma^{j_1 j_2} \wedge \omega_{j_1 j_2}^\sigma \wedge \omega_{j_2} \\ &+ \cdots + p_k d\Psi_\sigma^{j_1 j_2 \ldots j_{r-1}} \wedge \omega_{j_1 j_2 \ldots j_{r-2}}^\sigma \wedge \omega_{j_{r-1}} \\ &+ p_k d\Psi_\sigma^{j_1 j_2 \ldots j_r} \wedge \omega_{j_1 j_2 \ldots j_{r-1}}^\sigma \wedge \omega_{j_r}).\end{aligned} \tag{72}$$

Summarizing

$$(\pi^{2r+1,r+1})^*\eta = I_k\eta - dJ_k\eta + K_k\eta, \tag{73}$$

where

$$\begin{aligned} I_k\eta &= \eta_0 - (-1)^k p_{k-1} d\Psi^{j_1}_\sigma \wedge \omega^\sigma \wedge \omega_{j_1}, \\ J_k\eta &= (-1)^{k-1}(\Psi^{j_1}_\sigma \wedge \omega^\sigma \wedge \omega_{j_1} + \Psi^{j_1 j_2}_\sigma \wedge \omega^\sigma_{j_1} \wedge \omega_{j_2} \\ &\quad + \dots + \Psi^{j_1 j_2 \dots j_{r-1}}_\sigma \wedge \omega^\sigma_{j_1 j_2 \dots j_{r-2}} \wedge \omega_{j_{r-1}} \\ &\quad + \Psi^{j_1 j_2 \dots j_r}_\sigma \wedge \omega^\sigma_{j_1 j_2 \dots j_{r-1}} \wedge \omega_{j_r}), \\ K_k\eta &= (-1)^{k-1} p_{k+1}(d\Psi^{j_1}_\sigma \wedge \omega^\sigma \wedge \omega_{j_1} + d\Psi^{j_1 j_2}_\sigma \wedge \omega^\sigma_{j_1 j_2} \wedge \omega_{j_2} \\ &\quad + \dots + d\Psi^{j_1 j_2 \dots j_{r-1}}_\sigma \wedge \omega^\sigma_{j_1 j_2 \dots j_{r-2}} \wedge \omega_{j_{r-1}} \\ &\quad + d\Psi^{j_1 j_2 \dots j_r}_\sigma \wedge \omega^\sigma_{j_1 j_2 \dots j_{r-1}} \wedge \omega_{j_r}). \end{aligned} \tag{74}$$

(b) To prove uniqueness of the component $I_k\eta$, we adapt to the decomposition (73) a classical integration approach. It is sufficient to consider the case when

$$I_k\eta - dJ_k\eta + K_k\eta = 0, \tag{75}$$

and to prove that $I_k\eta = 0$. Choose π-vertical vector fields $\Xi_1, \Xi_2, \dots, \Xi_k$ on Y and consider the pullback of this n-form by the r-jet prolongation of a section γ of Y, $J^{2r+1}\gamma^* i_{\Xi_k} \dots i_{\Xi_2} i_{\Xi_1} I_k\eta$. Clearly, the pullback $J^{2r+1}\gamma^*$ annihilates contact n-forms. Since the Lie derivative of a contact form by a π-vertical vector field is a contact form (Sect. 2.5, Theorem 9, (d)), hence

$$\begin{aligned} &J^{2r+1}\gamma^* i_{\Xi_k} \dots i_{\Xi_2} i_{\Xi_1} I_k\eta \\ &= J^{2r+1}\gamma^* i_{\Xi_k} \dots i_{\Xi_2} i_{\Xi_1} dJ_k\eta + J^{2r+1}\gamma^* i_{\Xi_k} \dots i_{\Xi_2} i_{\Xi_1} K_k\eta \\ &= J^{2r+1}\gamma^* i_{\Xi_k} \dots i_{\Xi_2} i_{\Xi_2}(\partial_{\Xi_1} J_k\eta - d i_{\Xi_1} J_k\eta) + J^{2r+1}\gamma^* i_{\Xi_k} \dots i_{\Xi_2} i_{\Xi_1} K_k\eta \\ &= -J^{2r+1}\gamma^* i_{\Xi_k} \dots i_{\Xi_3} i_{\Xi_2} d i_{\Xi_1} J_k\eta \end{aligned} \tag{76}$$

because the forms $i_{\Xi_k} \dots i_{\Xi_2} i_{\Xi_2} \partial_{\Xi_1} J_k\eta$ and $i_{\Xi_k} \dots i_{\Xi_2} i_{\Xi_1} K_k\eta$ are contact. Repeating this step,

$$\begin{aligned} &J^{2r+1}\gamma^* i_{\Xi_k} \dots i_{\Xi_2} i_{\Xi_1} I_k\eta \\ &= -J^{2r+1}\gamma^* i_{\Xi_k} \dots i_{\Xi_4} i_{\Xi_3} \partial_{\Xi_2} i_{\Xi_1} J_k\eta + J^{2r+1}\gamma^* i_{\Xi_k} \dots i_{\Xi_4} i_{\Xi_3} d i_{\Xi_2} i_{\Xi_1} J_k\eta \\ &= J^{2r+1}\gamma^* i_{\Xi_k} \dots i_{\Xi_4} i_{\Xi_3} d i_{\Xi_2} i_{\Xi_1} J_k\eta \\ &= \dots = (-1)^p J^{2r+1}\gamma^* i_{\Xi_k} \dots i_{\Xi_{p+2}} i_{\Xi_{p+1}} d i_{\Xi_p} i_{\Xi_{p-1}} \dots i_{\Xi_2} i_{\Xi_1} J_k\eta \\ &= \dots = (-1)^k J^{2r+1}\gamma^* d i_{\Xi_k} i_{\Xi_{k-1}} \dots i_{\Xi_2} i_{\Xi_1} J_k\eta \\ &= (-1)^k d J^{2r+1}\gamma^* i_{\Xi_k} i_{\Xi_{k-1}} \dots i_{\Xi_2} i_{\Xi_1} J_k\eta. \end{aligned} \tag{77}$$

Thus, integrating over an arbitrary piece $\Omega \subset X$ with boundary $\partial\Omega$,

$$\begin{aligned}\int_{\Omega} J^{2r+1}\gamma^* i_{\Xi_k} \ldots i_{\Xi_2} i_{\Xi_1} I_k \eta &= (-1)^k \int_{\Omega} dJ^{2r+1}\gamma^* i_{\Xi_k} i_{\Xi_{k-1}} \ldots i_{\Xi_2} i_{\Xi_1} J_k \eta \\ &= (-1)^k \int_{\partial\Omega} J^{2r+1}\gamma^* i_{\Xi_k} i_{\Xi_{k-1}} \ldots i_{\Xi_2} i_{\Xi_1} J_k \eta.\end{aligned} \tag{78}$$

This identity holds for all π-vertical vector fields $\Xi_1, \Xi_2, \ldots, \Xi_k$, but on the other hand, the right-hand side depends on their values along the boundary $\partial\Omega$ only. Replace the vector field Ξ_1 with $f\Xi_1$, where f is a function, defined on a neighborhood of Ω, vanishing along $\partial\Omega$. Then, we get

$$\int_{\Omega} J^{2r+1}\gamma^* i_{\Xi_k} \ldots i_{\Xi_2} i_{f\Xi_1} I_k \eta = (-1)^k \int_{\Omega} f J^{2r+1}\gamma^* i_{\Xi_k} \ldots i_{\Xi_2} i_{\Xi_1} I_k \eta = 0. \tag{79}$$

Since the function f is arbitrary in the interior of the piece Ω, this is only possible when the integrand satisfies $J^{2r+1}\gamma^* i_{\Xi_k} \ldots i_{\Xi_2} i_{\Xi_1} I_k \eta = 0$. Finally, the section γ is also arbitrary; since through every point of the domain of definition of the form $i_{\Xi_k} \ldots i_{\Xi_2} i_{\Xi_1} I_k \eta$ passes the $(2r+1)$-jet prolongation $J^{2r+1}\gamma$ of γ; therefore,

$$I_k \eta = 0. \tag{80}$$

This proves that the form $I_k \eta$ in formula (73) is defined uniquely by the assumptions of Theorem 4. □

Corollary 1 *Extends to arbitrary forms as follows.*

Corollary 2 *For any integer $k \geq 1$ and any $(n+k-1)$-form ρ on W^r the form $I_k p_k dp_k \rho$ vanishes,*

$$I_k p_k dp_k \rho = 0. \tag{81}$$

Proof Using the canonical decomposition of the form $dp_1\rho$, we get the identity

$$\begin{aligned}&(\pi^{2r+1,r+1})^* p_k dp_k \rho \\ &\quad = (\pi^{2r+1,r+1})^* (dp_k \rho - p_{k+1} dp_k \rho - p_{k+2} dp_k \rho - \cdots - p_{k+n} dp_k \rho) \\ &\quad = d(\pi^{2r+1,r+1})^* p_k \rho - p_{k+1} (\pi^{2r,r+1})^* dp_k \rho\end{aligned} \tag{82}$$

because the components satisfy the conditions $p_{k+1} dp_k \rho = 0$, $p_{k+2} dp_k \rho = 0, \ldots,$ $p_{k+n} dp_k \rho = 0$. Comparing this formula with decomposition (58) and using the uniqueness of the component $I_1 p_1 dp_1 \rho$ (Theorem 4, (b)), we get identity (81). □

Our next aim is to determine an explicit formula for the component $I_k\eta$ of a form η by a geometric construction; the result will be proved on a successive application of Theorem 3.

Theorem 5

(a) *Let η be a 2-contact, $\pi^{r+1,r}$-horizontal $(n+2)$-form on the set W^{r+1}. Then for any π-vertical vector fields Ξ_1 and Ξ_2*

$$i_{\Xi_2} i_{\Xi_1} I_2\eta = \frac{1}{2}(i_{\Xi_2} I_1 i_{\Xi_1}\eta - i_{\Xi_1} I_1 i_{\Xi_2}\eta). \tag{83}$$

(b) *Let $k \geq 2$ and let η be a k-contact, $\pi^{r+1,r}$-horizontal $(n+k)$-form defined on W^{r+1}. Then for any π-vertical vector fields Ξ_1, Ξ_2. ..., Ξ_k*

$$\begin{aligned} i_{\Xi_k} \ldots i_{\Xi_2} i_{\Xi_1} I_k\eta = \frac{1}{k}(& i_{\Xi_k} i_{\Xi_{k-1}} \ldots i_{\Xi_2} I_{k-1} i_{\Xi_1}\eta - i_{\Xi_k} i_{\Xi_{k-1}} \ldots i_{\Xi_3} i_{\Xi_1} I_{k-1} i_{\Xi_2}\eta \\ & - i_{\Xi_k} i_{\Xi_{k-1}} \ldots i_{\Xi_4} i_{\Xi_2} i_{\Xi_1} I_{k-1} i_{\Xi_3}\eta - \cdots - i_{\Xi_{k-1}} \ldots i_{\Xi_2} i_{\Xi_1} I_{k-1} i_{\Xi_k}\eta). \end{aligned} \tag{84}$$

Proof

(a) From the decompositions

$$(\pi^{2r+1,r+1})^* i_{\Xi_1}\eta = \begin{cases} i_{\Xi_1} I_2\eta - i_{\Xi_1} dJ_2\eta + i_{\Xi_1} K_2\eta \\ I_1 i_{\Xi_1}\eta - dJ_1 i_{\Xi_1}\eta + K_1 i_{\Xi_1}\eta \end{cases} \tag{85}$$

it follows that

$$\begin{aligned} i_{\Xi_2} i_{\Xi_1} I_2\eta - \frac{1}{2}(i_{\Xi_2} I_1 i_{\Xi_1}\eta - i_{\Xi_1} I_1 i_{\Xi_2}\eta) &= i_{\Xi_2} i_{\Xi_1} dJ_2\eta - i_{\Xi_2} i_{\Xi_1} K_2\eta \\ &+ \frac{1}{2}(-i_{\Xi_2} dJ_1 i_{\Xi_1}\eta + i_{\Xi_2} K_1 i_{\Xi_1}\eta - i_{\Xi_1} dJ_2 i_{\Xi_1}\eta + i_{\Xi_1} K_1 i_{\Xi_2}\eta). \end{aligned} \tag{86}$$

Using the properties of the Lie derivative operator (see Appendix 5), we can write

$$\begin{aligned} & i_{\Xi_2} i_{\Xi_1} dJ_2\eta + \frac{1}{2}(-i_{\Xi_2} dJ_1 i_{\Xi_1}\eta - i_{\Xi_1} dJ_2 i_{\Xi_1}\eta) \\ & = i_{\Xi_2} \partial_{\Xi_1} J_2\eta - i_{\Xi_2} d i_{\Xi_1} J_2\eta \\ & \quad + \frac{1}{2}(-\partial_{\Xi_2} J_1 i_{\Xi_1}\eta + d i_{\Xi_2} J_1 i_{\Xi_1}\eta - \partial_{\Xi_1} J_2 i_{\Xi_1}\eta + d i_{\Xi_1} J_2 i_{\Xi_1}\eta) \\ & = i_{\Xi_2} \partial_{\Xi_1} J_2\eta - \partial_{\Xi_2} i_{\Xi_1} J_2\eta - d i_{\Xi_2} i_{\Xi_1} J_2\eta \\ & \quad + \frac{1}{2}(-\partial_{\Xi_2} J_1 i_{\Xi_1}\eta + d i_{\Xi_2} J_1 i_{\Xi_1}\eta - \partial_{\Xi_1} J_2 i_{\Xi_1}\eta + d i_{\Xi_1} J_2 i_{\Xi_1}\eta), \end{aligned} \tag{87}$$

thus

$$\begin{aligned} & i_{\Xi_2} i_{\Xi_1} I_2\eta - \frac{1}{2}(i_{\Xi_2} I_1 i_{\Xi_1}\eta - i_{\Xi_1} I_1 i_{\Xi_2}\eta) \\ & \quad = i_{\Xi_2}\partial_{\Xi_1} J_2\eta - \partial_{\Xi_2} i_{\Xi_1} J_2\eta - d i_{\Xi_2} i_{\Xi_1} J_2\eta \\ & \qquad + \frac{1}{2}(-\partial_{\Xi_2} J_1 i_{\Xi_1}\eta + d i_{\Xi_2} J_1 i_{\Xi_1}\eta - \partial_{\Xi_1} J_2 i_{\Xi_1}\eta + d i_{\Xi_1} J_2 i_{\Xi_1}\eta) \\ & \qquad - i_{\Xi_2} i_{\Xi_1} K_2\eta + \frac{1}{2}(i_{\Xi_2} K_1 i_{\Xi_1}\eta + i_{\Xi_1} K_1 i_{\Xi_2}\eta). \end{aligned} \tag{88}$$

Now integrating

$$\begin{aligned} & \int_\Omega J^{2r+1}\gamma^*\left(i_{\Xi_2} i_{\Xi_1} I_2\eta - \frac{1}{2}(i_{\Xi_2} I_1 i_{\Xi_1}\eta - i_{\Xi_1} I_1 i_{\Xi_2}\eta)\right) \\ & \quad = \int_{\partial\Omega} J^{2r+1}\gamma^*\left(-i_{\Xi_2} i_{\Xi_1} J_2\eta + \frac{1}{2}(i_{\Xi_2} J_1 i_{\Xi_1}\eta + i_{\Xi_1} J_2 i_{\Xi_1}\eta)\right). \end{aligned} \tag{89}$$

To conclude that this condition implies

$$i_{\Xi_2} i_{\Xi_1} I_2\eta - \frac{1}{2}(i_{\Xi_2} I_1 i_{\Xi_1}\eta - i_{\Xi_1} I_1 i_{\Xi_2}\eta) = 0 \tag{90}$$

we proceed as in the proof of Theorem 4.

(b) To complete the proof, we apply elementary induction. □

According to Theorem 5, formula (59) defines a mapping I_k from the Abelian group of k-contact $(n+k)$-forms on W^{r+1} to $\pi^{2r+1,0}$-horizontal $(n+k)$-forms on W^{r+1}. I_k is clearly a morphism of Abelian groups.

Theorem 6

(a) *Condition $I_k\eta = 0$ is satisfied if and only if η is a strongly contact form.*
(b) *The mapping I_k satisfies*

$$I_k \circ I_k = I_k \tag{91}$$

Proof

(a) This follows from Theorem 4, (b).
(b) To prove (b), write $(\pi^{2r+1,r+1})^*\eta = I_k\eta - dJ_k\eta + K_k\eta$. Then

$$\begin{aligned} & (\pi^{2(2r+1+,2r+2})^*(\pi^{2r+1,r+1})^*\eta \\ & \quad = I_k(\pi^{2r+1,r+1})^*\eta - dJ_k(\pi^{2r+1,r+1})^*\eta + K_k(\pi^{2r+1,r+1})^*\eta \end{aligned} \tag{92}$$

and from the properties of the pullback operation

$$\begin{aligned}(\pi^{2(2r+1+,2r+2)})^*(\pi^{2r+1,r+1})^*\eta \\ &= (\pi^{2(2r+1+,2r+2)})^*(I_k\eta - dJ_k\eta + K_k\eta) \\ &= (\pi^{2(2r+1+,2r+2)})^*I_k\eta - d(\pi^{2(2r+1+,2r+2)})^*J_k\eta \\ &\quad + (\pi^{2(2r+1+,2r+2)})^*K_k\eta \\ &= I_kI_k\eta - dJ_kI_k\eta + K_kI_k\eta \\ &\quad - d(\pi^{2(2r+1+,2r+2)})^*J_k\eta + (\pi^{2(2r+1+,2r+2)})^*K_k\eta.\end{aligned} \tag{93}$$

Comparing (92) with (93) and using the uniqueness of these decompositions (Theorem 4 (b)), we get formula (91). □

Remark 9 Property (a) characterizes the *kernel* of the mapping I_k. Its *image* consists of the k-contact, ω^σ-generated $(n+k)$-forms ε on W^{2r+1} for which the equation

$$\varepsilon = I_k\eta \tag{94}$$

has a solution η. The corresponding *integrability conditions*, which should be satisfied by ε, are determined by the structure of the mapping I_k and can be studied by means of the formal divergence equations (Chap. 3).

Remark 10 The uniqueness of the component $I_k\eta$ in the decomposition (59) means that the pullback of the vector space of k-contact $(n+k)$-forms on W^{r+1} is isomorphic with the direct sum of two subspaces of the vector space of k-contact $(n+k)$-forms on W^{2r+1}, one of which is the subspace of strongly contact forms.

We conclude this section by extending the decomposition (59), defined for k-contact $(n+k)$-forms on W^{r+1}, to any forms $\rho \in \Omega^r_{n+k}W$. Substituting in formula (56) $\eta = p_k\rho$, we get

$$\begin{aligned}&(\pi^{2r+1,r})^*\rho \\ &= (\pi^{2r+1,r+1})^*p_k\rho + (\pi^{2r+1,r+1})^*(p_{k+1}\rho + p_{k+2}\rho + \cdots + p_{k+n}\rho) \\ &= (\pi^{2r+1,r+1})^*(I_kp_k\rho - dJ_kp_k\rho + K_kp_k\rho) \\ &\quad + (\pi^{2r+1,r+1})^*(p_{k+1}\rho + p_{k+2}\rho + \cdots + p_{k+n}\rho) \\ &= (\pi^{2r+1,r+1})^*I_kp_k\rho - d(\pi^{2r+1,r+1})^*J_kp_k\rho \\ &\quad + (\pi^{2r+1,r+1})^*K_kp_k\rho + (\pi^{2r+1,r+1})^*(p_{k+1}\rho + p_{k+2}\rho + \cdots + p_{k+n}\rho).\end{aligned} \tag{95}$$

Therefore, setting

$$\begin{aligned}\mathscr{I}_k\rho &= (\pi^{2r+1,r+1})^* I_k p_k \rho,\\ \mathscr{J}_k\rho &= (\pi^{2r+1,r+1})^* J_k p_k \rho,\\ \mathscr{K}_k\rho &= (\pi^{2r+1,r+1})^* K_k p_k \rho + (\pi^{2r+1,r+1})^*(p_{k+1}\rho + p_{k+2}\rho + \cdots + p_{k+n}\rho)\end{aligned} \tag{96}$$

we get the decomposition

$$(\pi^{2r+1,r})^*\rho = \mathscr{I}_k\rho - d\mathscr{J}_k\rho + \mathscr{K}_k\rho. \tag{97}$$

According to Theorem 4, this formula defines a mapping $\rho \rightarrow \mathscr{I}_k\rho$ of the Abelian group of $\Omega^r_{n+k}W$ of $(n+k)$-forms, defined on W^r, into the Abelian group $\Omega^{2r+1}_{n+k}W$ of $(n+k)$-forms on W^{2r+1}.

The following theorem summarizes elementary properties of the mapping $\Omega^r_{n+k}W \ni \rho \rightarrow \mathscr{I}_k\rho \in \Omega^{2r+1}_{n+k}W$. As before, to simplify notation, we omit obvious pullback operations on differential forms with respect to the canonical jet projections $\pi^{r,s}\colon J^rY \rightarrow J^sY$.

Theorem 7

(a) *The mapping $\rho \rightarrow \mathscr{I}_k\rho$ of the Abelian group $\Omega^r_{n+k}W$ into $\Omega^{2r+1}_{n+k}W$ is a morphism of Abelian groups.*
(b) *The kernel of the mapping $\mathscr{I}$ is the Abelian group of strongly contact forms $\Theta^r_{n+k}W$, and its image is isomorphic with the quotient group $\Omega^r_{n+k}W/\Theta^r_{n+k}W$.*
(c) *For every $\rho \in \Omega^r_{n+k}W$ the mapping $\mathscr{I}$ satisfies*

$$\mathscr{I}_k\mathscr{I}_k\rho = \mathscr{I}_k\rho. \tag{98}$$

Proof

(a) Obvious.
(b) If $\mathscr{I}_k\rho = 0$, then by Lemma 3, ρ is *strongly contact*, thus ρ belongs to the Abelian group $\Theta^r_{n+k}W$.
(c) Applying the pullback operation to both sides of formula (97) and using the properties $\mathscr{J}_k\mathscr{I}_k\rho = 0$ and $\mathscr{K}_k\mathscr{I}_k\rho = 0$ of the mappings $\mathscr{I}_k$, $\mathscr{J}_k$ and $\mathscr{K}_k$,

$$\begin{aligned}&(\pi^{2(2r+1)+1,2r+1})^*(\pi^{2r+1,r})^*\rho\\ &\quad = \mathscr{I}_k(\pi^{2r+1,r})^*\rho - d\mathscr{J}_k(\pi^{2r+1,r})^*\rho + \mathscr{K}_k(\pi^{2r+1,r})^*\rho,\end{aligned} \tag{99}$$

and

$$\begin{aligned}
&(\pi^{2(2r+1)+1,2r+1})^*(\pi^{2r+1,r})^*\rho \\
&\quad = (\pi^{2(2r+1)+1,2r+1})^*\mathscr{I}_k\rho - d(\pi^{2(2r+1)+1,2r+1})^*\mathscr{I}_k\rho \\
&\qquad + (\pi^{2(2r+1)+1,2r+1})^*\mathscr{K}_k\rho \\
&\quad = \mathscr{I}_k\mathscr{I}_k\rho - d\mathscr{I}_k\mathscr{I}_k\rho + \mathscr{K}_k\mathscr{I}_k\rho - d(\pi^{2(2r+1)+1,2r+1})^*\mathscr{I}_k\rho \\
&\qquad + (\pi^{2(2r+1)+1,2r+1})^*\mathscr{K}_k\rho \\
&\quad = \mathscr{I}_k\mathscr{I}_k\rho - d(\pi^{2(2r+1)+1,2r+1})^*\mathscr{I}_k\rho + (\pi^{2(2r+1)+1,2r+1})^*\mathscr{K}_k\rho.
\end{aligned} \tag{100}$$

Comparing these formulas and using the uniqueness of the decompositions, we get assertion (c). □

We call the Abelian group morphism $\Omega^r_{n+k}W \ni \rho \to \mathscr{I}_k\rho \in \Omega^{2r+1}_{n+k}W$ the kth *variational projector*. To simplify notation, we sometimes write just $\mathscr{I}$ instead of $\mathscr{I}_k$.

8.4 The Euler–Lagrange Morphisms

Consider the variational sequence (33), Sect. 8.2

$$0 \longrightarrow \mathbf{R}_Y \longrightarrow \Omega^r_0 \longrightarrow \Omega^r_1/\Theta^r_1 \longrightarrow \Omega^r_2/\Theta^r_2 \longrightarrow \Omega^r_3/\Theta^r_3 \longrightarrow \tag{101}$$

Note that by definition of the horizontalization morphism h: $\Omega^r_pW \to \Omega^{r+1}_pW$, the equivalence relation on the Abelian group Ω^r_pW associated with the subgroup of contact forms $\Theta^r_pW \subset \Omega^r_pW$ coincides with the equivalence relation defined by h. Similarly, Part Theorem 7, Sect. 8.3, shows that for each $k \geq 1$, the equivalence relation on the Abelian group $\Omega^r_{n+k}W$, associated with the subgroup $\Theta^r_{n+k}W \subset \Omega^r_{n+k}W$ of strongly contact forms, coincides with the equivalence relation induced by the *variational projectors* $\mathscr{I}_k$. Thus, the diagram, defining the variational sequence, can be expressed as

$$\begin{array}{ccccccccccccc}
 & & & & & & 0 & & 0 & & & & 0 \\
 & & & & & & \downarrow & & \downarrow & & & & \downarrow \\
 & & & & 0 & \longrightarrow & \Theta^r_1 & \xrightarrow{d} & \Theta^r_2 & \xrightarrow{d} & \cdots & \xrightarrow{d} & \Theta^r_n \\
 & & & & \downarrow & & \downarrow & & \downarrow & & & & \downarrow \\
0 & \longrightarrow & \mathbf{R}_Y & \longrightarrow & \Omega^r_0 & \xrightarrow{d} & \Omega^r_1 & \xrightarrow{d} & \Omega^r_2 & \xrightarrow{d} & \cdots & \xrightarrow{d} & \Omega^r_n \\
 & & & & & \searrow & \downarrow h & & \downarrow h & & & & \downarrow h \\
 & & & & & & h^r_1 & \longrightarrow & h^r_2 & \longrightarrow & \cdots & \longrightarrow & h^r_n \\
 & & & & & & \downarrow & & \downarrow & & & & \downarrow \\
 & & & & & & 0 & & 0 & & & & 0
\end{array} \tag{102}$$

$$\begin{array}{ccccccccccc}
 & 0 & & 0 & & & & 0 & & & \\
 & \downarrow & & \downarrow & & & & \downarrow & & & \\
\longrightarrow & \Theta^r_{n+1} & \longrightarrow & \Theta^r_{n+2} & \longrightarrow & \cdots & \longrightarrow & \Theta^r_M & \longrightarrow & 0 & \\
 & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & \\
\longrightarrow & \Omega^r_{n+1} & \xrightarrow{d} & \Omega^r_{n+2} & \xrightarrow{d} & \cdots & \xrightarrow{d} & \Omega^r_M & \xrightarrow{d} & \Omega^r_{M+1} & \xrightarrow{d} \\
 & \downarrow h & & \downarrow h & & & & \downarrow h & \nearrow & & \\
\longrightarrow & \mathscr{I}^r_{n+1} & \longrightarrow & \mathscr{I}^r_{n+2} & \longrightarrow & \cdots & \longrightarrow & \mathscr{I}^r_M & & & \\
 & \downarrow & & \downarrow & & & & \downarrow & & & \\
 & 0 & & 0 & & & & 0 & & &
\end{array}$$

The corresponding representation of the variational sequence (101) is

$$\begin{aligned} &0 \longrightarrow \mathbf{R}_Y \longrightarrow \Omega^r_0 \xrightarrow{E_0} h^r_1 \xrightarrow{E_1} h^r_2 \xrightarrow{E_2} \cdots \xrightarrow{E_{n-1}} h^r_n \\ &\xrightarrow{E_n} \mathscr{I}^r_1 \xrightarrow{E_{n+1}} \mathscr{I}^r_2 \xrightarrow{E_{n+2}} \cdots \end{aligned} \tag{103}$$

The Abelian group morphisms E_k in this sequence will be called the *Euler–Lagrange morphisms*. Our task in this section will be to determine the structure of the morphisms E_k. The formulas we derive establish *explicit* correspondence between the morphisms E_k and basic concepts of the calculus of variations on fibered manifolds such as the Euler–Lagrange mapping and the Helmholtz mappings. The following two theorems give us a way to calculate the chart expressions of these morphisms E_k.

Theorem 8 *The Euler–Lagrange morphisms in the variational sequence* (103) *can be expressed as*

$$E_k h\rho = \begin{cases} hd\rho, & \rho \in \Omega^r_k W, \quad 0 \le k \le n-1, \\ I_1 dh\rho, & \rho \in \Omega^r_n W, \quad k = n, \end{cases} \tag{104}$$

and

$$E_{n+k}\mathscr{I}_k\rho = I_{k+1} dp_k\rho, \quad \rho \in \Omega^r_{n+k} W, \quad k \ge 1. \tag{105}$$

Proof If $\rho \in \Omega^r_n W$, then $E_n h\rho = \mathscr{I}_1 d\rho = I_1 p_1 d\rho = I_1 p_1 dh\rho + I_1 p_1 dp_1\rho$. Thus, by Sect. 8.3, Corollary 1,

$$E_n h\rho = I_1 dh\rho. \tag{106}$$

If $\rho \in \Omega^r_{n+k} W$, where $k \ge 1$, then

$$\begin{aligned} E_{n+k}\mathscr{I}_k\rho &= \mathscr{I}_{k+1} d\rho = I_{k+1} p_{k+1} d\rho \\ &= I_{k+1} p_{k+1} dp_k\rho + I_{k+1} p_{k+1} dp_{k+1}\rho \end{aligned} \tag{107}$$

hence, by Corollary 2, (81)

$$E_{n+k}\mathscr{I}_k\rho = I_{k+1}dp_k\rho. \tag{108}$$

□

Theorem 9 *Let* (V,ψ), $\psi = (x^i, y^\sigma)$, *be a fibered chart on* Y.

(a) *If* $f \in \Omega_0^r V$, *then*

$$E_0 f = d_i f \cdot dx^i. \tag{109}$$

(b) *Let* $1 \le k \le n-1$ *and let* $h\rho \in \mathscr{I}_j^r V$ *be a class. Then if* $h\rho$ *is expressed by*

$$h\rho = \rho_{i_1 i_2 \dots i_k} dx^{i_1} \wedge dx^{i_2} \wedge \dots \wedge dx^{i_k}, \tag{110}$$

then the image $E_k h\rho$ *is given by*

$$E_k h\rho = d_{i_0}\rho_{i_1 i_2 \dots i_k} \wedge dx^{i_0} \wedge dx^{i_1} \wedge dx^{i_2} \wedge \dots \wedge dx^{i_k}. \tag{111}$$

Proof We prove assertion (b). According to the trace decomposition theorem (Sect. 2.2, Theorem 3), a form $\rho \in \Omega_k^r V$ has an expression

$$\rho = \sum_{0 \le |J| \le r-1} \omega_J^\sigma \wedge \Phi_\sigma^J + \sum_{|I| = r-1} d(\omega_J^\sigma \wedge \Psi_\sigma^J) + \rho_0, \tag{112}$$

where ρ_0 is the traceless component of ρ and Φ_σ^J, Ψ_σ^J are some forms. Since the morphism h annihilates the contact forms ω_J^σ and $d\omega_I^\sigma$, ρ_0 has an expression

$$\begin{aligned} \rho_0 = {} & A_{i_1 i_2 \dots i_k} dx^{i_1} \wedge dx^{i_2} \wedge \dots \wedge dx^{i_k} \\ & + A_{\sigma_1 \; i_2 i_3 \dots i_k}^{J_1} dy_{J_1}^{\sigma_1} \wedge dx^{i_2} \wedge dx^{i_3} \wedge \dots \wedge dx^{i_k} \\ & + A_{\sigma_1 \sigma_2 \; i_3 i_4 \dots i_k}^{J_1 J_2} dy_{J_1}^{\sigma_1} \wedge dy_{J_2}^{\sigma_2} \wedge dx^{i_3} \wedge dx^{i_4} \wedge \dots \wedge dx^{i_k} \\ & + \dots + A_{\sigma_1 \, \sigma_2 \dots \sigma_{k-1} i_k}^{J_1 \, J_2 \dots J_{k-1}} dy_{J_1}^{\sigma_1} \wedge dy_{J_2}^{\sigma_2} \wedge \dots \wedge dy_{J_{j-1}}^{\sigma_{j-1}} \wedge dx^{i_k} \\ & + A_{\sigma_1 \, \sigma_2 \dots \sigma_k}^{J_1 \, J_2 \dots J_k} dy_{J_1}^{\sigma_1} \wedge dy_{J_2}^{\sigma_2} \wedge \dots \wedge dy_{J_k}^{\sigma_k}, \end{aligned} \tag{113}$$

where the coefficients $A_{\sigma_1 \sigma_2 \dots \sigma_s i_{s+1} i_{s+2} \dots i_k}^{J_1 J_2 \dots J_s}$ are *traceless*. Thus, any class $h\rho$ is expressed as the k-form

$$h\rho = \rho_{i_1 i_2 \dots i_k} dx^{i_1} \wedge dx^{i_2} \wedge \dots \wedge dx^{i_k}, \tag{114}$$

where

$$\begin{aligned}\rho_{i_1 i_2 \ldots i_k} = {} & (A_{i_1 i_2 \ldots i_k} + A^{J_1}_{\sigma_1\ i_2 i_3 \ldots i_k} y^{\sigma_1}_{J_1 i_1} + A^{J_1 J_2}_{\sigma_1 \sigma_2\ i_3 i_4 \ldots i_k} y^{\sigma_1}_{J_1 i_1} y^{\sigma_2}_{J_2 i_2} \\ & + \cdots + A^{J_1 J_2 \ldots J_{k-1}}_{\sigma_1 \sigma_2 \ldots \sigma_{k-1} i_k} y^{\sigma_1}_{J_1 i_1} y^{\sigma_2}_{J_2 i_2} \cdots y^{\sigma_{k-1}}_{J_{k-1} i_{k-1}} + A^{J_1 J_2 \ldots J_k}_{\sigma_1 \sigma_2 \ldots \sigma_k} y^{\sigma_1}_{J_1 i_1} y^{\sigma_2}_{J_2 j_2} \cdots y^{\sigma_k}_{J_k i_k}) \\ & \mathrm{Alt}(i_1 i_2 \ldots i_k).\end{aligned} \tag{115}$$

The class $hd\rho$ of $d\rho$ is then given by

$$hd\rho = d_{i_0} \rho_{i_1 i_2 \ldots i_k} \wedge dx^{i_0} \wedge dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_k}. \tag{116}$$

Clearly, $hd\rho$ is defined on V^{r+1}. □

Remark 11 If $k = n - 1$, then since $\varepsilon^{l i_1 i_2 \ldots i_{n-1}} \omega_l = dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_{n-1}}$, the class $h\rho = \rho_{i_1 i_2 \ldots i_{n-1}} dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_{n-1}}$ (110) can be written as $h\rho = \rho^l \omega_l$. Then, the image $E_{n-1} h\rho$ is expressed as

$$\begin{aligned}E_{n-1} h\rho &= d_{i_0} \rho_{i_1 i_2 \ldots i_{n-1}} \wedge dx^{i_0} \wedge dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_{n-1}} \\ &= d_{i_0} \rho_{i_1 i_2 \ldots i_{n-1}} \varepsilon^{i_0 i_1 i_2 \ldots i_{n-1}} \wedge \omega_0 = d_i \rho^i \cdot \omega_0 \\ &= hdh\rho,\end{aligned} \tag{117}$$

where $d_i \rho^i$ is the *formal divergence* of the family ρ^l. Thus, the Euler–Lagrange morphism E_{n-1} can also be expressed in short as $E_{n-1} = hd$.

Now, we study the Euler–Lagrange morphisms E_{n+k} for $k \geq 0$. We derive explicit formulas for $k = 0, 1$; in subsequent sections, these formulas will be compared with basic variational concepts, which appeared already in the previous sections devoted to the calculus of variations.

In order to study the morphism E_n, we find the chart expression of the class $h\rho$ of a form $\rho \in \Omega^r_n V$. According to the trace decomposition theorem (Sect. 2.2, Theorem 3), ρ has an expression

$$\rho = \sum_{0 \leq |J| \leq r-1} \omega^\sigma_J \wedge \Phi^J_\sigma + \sum_{|I| = r-1} d\omega^\sigma_I \wedge \Psi^I_\sigma + \rho_0, \tag{118}$$

where ρ_0 is the traceless component of ρ. Clearly, $h\rho = h\rho_0$. But ρ_0 has an expression

$$\begin{aligned}\rho_0 &= A_{i_1 i_2 \ldots i_n} dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_n} \\ &\quad + A^{J_1}_{\sigma_1 i_2 i_3 \ldots i_n} dy^{\sigma_1}_{J_1} \wedge dx^{i_2} \wedge dx^{i_3} \wedge \cdots \wedge dx^{i_n} \\ &\quad + A^{J_1 J_2}_{\sigma_1 \sigma_2 i_3 i_4 \ldots i_n} dy^{\sigma_1}_{J_1} \wedge dy^{\sigma_2}_{J_2} \wedge dx^{i_3} \wedge dx^{i_4} \wedge \cdots \wedge dx^{i_n} \\ &\quad + \cdots + A^{J_1 J_2 \ldots J_{n-1}}_{\sigma_1 \sigma_2 \ldots \sigma_{n-1} i_n} dy^{\sigma_1}_{J_1} \wedge dy^{\sigma_2}_{J_2} \wedge \cdots \wedge dy^{\sigma_{n-1}}_{J_{n-1}} \wedge dx^{i_n} \\ &\quad + A^{J_1 J_2 \ldots J_n}_{\sigma_1 \sigma_2 \ldots \sigma_n} dy^{\sigma_1}_{J_1} \wedge dy^{\sigma_2}_{J_2} \wedge \cdots \wedge dy^{\sigma_n}_{J_n},\end{aligned} \tag{119}$$

where the summation indices satisfy $|J_1| = |J_2| = \cdots = |J_{n+1}| = r$, and the coefficients $A^{J_1 J_2 \ldots J_s}_{\sigma_1 \sigma_2 \ldots \sigma_s\, i_{s+1} i_{s+2} \ldots i_n}$ are *traceless*. Thus, any class $h\rho$ can be expressed as the n-form

$$h\rho = \rho_{i_1 i_2 \ldots i_n} dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_n}, \tag{120}$$

where

$$\begin{aligned}\rho_{i_1 i_2 \ldots i_n} &= A_{i_1 i_2 \ldots i_n} + A^{J_1}_{\sigma_1\, i_2 i_3 \ldots i_n} y^{\sigma_1}_{J_1 i_1} + A^{J_1 J_2}_{\sigma_1 \sigma_2\, i_3 i_4 \ldots i_n} y^{\sigma_1}_{J_1 i_1} y^{\sigma_2}_{J_2 i_2} \\ &\quad + \cdots + A^{J_1 J_2 \ldots J_{n-1}}_{\sigma_1 \sigma_2 \ldots \sigma_{n-1} i_n} y^{\sigma_1}_{J_1 i_1} y^{\sigma_2}_{J_2 i_2} \ldots y^{\sigma_{n-1}}_{J_{n-1} i_{n-1}} + A^{J_1 J_2 \ldots J_n}_{\sigma_1 \sigma_2 \ldots \sigma_n} y^{\sigma_1}_{J_1 i_1} y^{\sigma_2}_{J_2 j_2} \ldots y^{\sigma_n}_{J_n i_n} \\ &\quad \operatorname{Alt}(i_1 i_2 \ldots i_n).\end{aligned} \tag{121}$$

Thus, $h\rho$ can also be characterized as

$$h\rho = \mathscr{L}\omega_0, \tag{122}$$

where $\mathscr{L} = \varepsilon^{i_1 i_2 \ldots i_n} \rho_{i_1 i_2 \ldots i_n}$ (Sect. 4.1, (112)).

Remark 12 In variational terminology, the class $\lambda = h\rho$ is the *Lagrangian*, associated with the n-form ρ, that is, an element of the module $\Omega^{r+1}_{n,X} V$ of π^{r+1}-horizontal forms, defined on $V^{r+1} \subset J^{r+1} Y$. The function $\mathscr{L}$, characterizing the class $h\rho$ locally, is the *Lagrange function*, associated with $h\rho$ (and with the given fibered chart, cf. Sect. 4.1).

We can now prove the following theorem.

Theorem 10 *If the class $h\rho$ of an n-form $\rho \in \Omega^r_n V$ is expressed as*

$$h\rho = \mathscr{L}\omega_0, \tag{123}$$

then

$$E_n h\rho = \left(\frac{\partial \mathscr{L}}{\partial y^\sigma} + \sum_{1 \le s \le r} (-1)^s d_{j_1} d_{j_2} \ldots d_{j_s} \frac{\partial \mathscr{L}}{\partial y^\sigma_{j_1 j_2 \ldots j_s}} \right) \omega^\sigma \wedge \omega_0. \tag{124}$$

Proof The class $E_n h\rho$ is defined to be $E_n h\rho = \mathscr{I}_1 d\rho = I_1 dh\rho$ (Theorem 8, (104)). Since

$$dh\rho = d\mathscr{L} \wedge \omega_0 = \sum_{0 \le |J| \le r+1} \frac{\partial \mathscr{L}}{\partial y_J^\nu} \omega_J^\nu \wedge \omega_0, \tag{125}$$

we have

$$I_1 dh\rho = \left(\frac{\partial \mathscr{L}}{\partial y^\sigma} + \sum_{1 \le s \le r} (-1)^s d_{j_1} d_{j_2} \ldots d_{j_s} \frac{\partial \mathscr{L}}{\partial y^\sigma_{j_1 j_2 \ldots j_s}} \right) \omega^\sigma \wedge \omega_0 \tag{126}$$

(Sect. 8.3, Theorem 3). □

Now we find the chart expression of the class $\mathscr{I}_1\rho$ of a form $\rho \in \Omega^r_{n+1} V$. Writing $p_1\rho$ as

$$p_1\rho = \sum_{0 \le s \le r} A_\sigma^{j_1 j_2 \ldots j_s} \omega^\sigma_{j_1 j_2 \ldots j_s} \wedge \omega_0, \tag{127}$$

we get, according to Sect. 8.3, Theorem 3,

$$\mathscr{I}_1\rho = I_1 p_1 \rho = \varepsilon_\sigma \omega^\sigma \wedge \omega_0, \tag{128}$$

where

$$\varepsilon_\sigma = A_\sigma + \sum_{1 \le s \le r} (-1)^s d_{j_1} d_{j_2} \ldots d_{j_s} A_\sigma^{j_1 j_2 \ldots j_s}. \tag{129}$$

Remark 13 According to formula (128), the class $\varepsilon = \mathscr{I}_1\rho$ of a form $\rho \in \Omega^r_{n+1} V$ is an element of the Abelian group $\Omega^{2r+1}_{n+1,Y} V$ of $\pi^{2r+1,0}$-horizontal forms, defined on the set $V^{2r+1} \subset J^{2r+1} Y$; in the variational theory, elements of the Abelian groups $\Omega^{2r+1}_{n+1,Y} V$ are the *source forms* on the fibered manifold Y (cf. Sect. 4.9).

Theorem 11 *If the class $\mathscr{I}_1\rho$ of an $(n+1)$-form $\rho \in \Omega^r_{n+1} V$ is expressed as*

$$\mathscr{I}_1\rho = \varepsilon_\sigma \omega^\sigma \wedge \omega_0, \tag{130}$$

then

$$E_{n+1} \mathscr{I}_1 \rho = \frac{1}{2} \sum_{0 \le k \le r} H_{\sigma\ \nu}^{\ j_1 j_2 \ldots j_k} (\varepsilon) \omega^\nu_{j_1 j_2 \ldots j_k} \wedge \omega^\sigma \wedge \omega_0, \tag{131}$$

where

$$H_{\sigma\nu}^{j_1 j_2 \ldots j_k}(\varepsilon) = \frac{\partial \varepsilon_\sigma}{\partial y^\nu_{j_1 j_2 \ldots j_k}} - (-1)^k \frac{\partial \varepsilon_\nu}{\partial y^\sigma_{j_1 j_2 \ldots j_k}} - \sum_{l=k+1}^{s} (-1)^l \binom{l}{k} d_{p_{k+1}} d_{p_{k+2}} \ldots d_{p_l} \frac{\partial \varepsilon_\nu}{\partial y^\sigma_{j_1 j_2 \ldots j_k p_{k+1} p_{k+2} \ldots p_l}}. \tag{132}$$

Proof The image $E_{n+1}\mathscr{I}_1\rho$ is defined by the equation $E_{n+1}\mathscr{I}_1\rho = \mathscr{I}_2 d\rho$. However, if $\mathscr{I}_1\rho$ is defined on V^s, then $\mathscr{I}_1\mathscr{I}_1\rho = (\pi^{2s+1,s})^*\mathscr{I}_1\rho$ (Sect. 8.3, Theorem 7); thus, the image can also be calculated from the equation

$$E_{n+1}\mathscr{I}_1\mathscr{I}_1\rho = E_{n+1}(\pi^{2s+1,s})^*\mathscr{I}_1\rho = (\pi^{2s+1,s})^* E_{n+1}\mathscr{I}_1\rho = (\pi^{2s+1,s})^*\mathscr{I}_2 d\rho. \tag{133}$$

□

We apply this formula to the representation (130) of the class of ρ. Setting $\mathscr{I}_1\rho = \varepsilon$, we have

$$E_{n+1}\mathscr{I}_1\varepsilon = \mathscr{I}_2 d\varepsilon = I_2 p_2 d\varepsilon = I_2 d\varepsilon. \tag{134}$$

This expression can be easily determined by means of the mapping I_2, defined by

$$i_{\Xi_2} i_{\Xi_1} I_2 d\varepsilon = \frac{1}{2}(i_{\Xi_2} I_1 i_{\Xi_1} d\varepsilon - i_{\Xi_1} I_1 i_{\Xi_2} d\varepsilon), \tag{135}$$

where Ξ_1 and Ξ_2 are any π-vertical vector fields (Sect. 8.3, Theorem 5). From this expression, we conclude that

$$I_2 d\varepsilon = \frac{1}{2} \sum_{0 \le k \le r} H_{\sigma\,\nu}^{j_1 j_2 \ldots j_k}(\varepsilon) \omega^\nu_{j_1 j_2 \ldots j_k} \wedge \omega^\sigma \wedge \omega_0, \tag{136}$$

where the components $H_{\sigma\,\nu}^{j_1 j_2 \ldots j_k}(\varepsilon)$ are given by (132).

Consider the variational sequence (103). Theorem 10 shows that the morphism E_n in this Abelian sheaf sequence is exactly the *Euler–Lagrange mapping* of the calculus of variations (cf. Sect. 4.5). The mappings E_{n-1} and E_{n+1} also admit a direct variational interpretation (Remark 11, Theorem 11). In the subsequent sections, we consider the part of the variational sequence Var^r_Y including E_n,

$$\cdots \longrightarrow h^r_{n-1} \xrightarrow{E_{n-1}} h^r_n \xrightarrow{E_n} \mathscr{I}^r_1 \xrightarrow{E_{n+1}} \mathscr{I}^r_2 \longrightarrow \cdots \tag{137}$$

and the corresponding part of the associated *complex of global sections* $\mathrm{Var}^r_Y Y$ (Sect. 8.2, (35)),

$$\cdots \longrightarrow h^r_{n-1}Y \xrightarrow{E_{n-1}} h^r_n Y \xrightarrow{E_n} \mathscr{I}^r_1 Y \xrightarrow{E_{n+1}} \mathscr{I}^r_2 Y \longrightarrow \cdots \tag{138}$$

Since by Sect. 8.2, Theorem 2, the cohomology groups $H^k(\mathrm{Var}^r_Y Y)$ of (138) and the cohomology groups $H^k(Y, \mathbf{R}_Y)$ are isomorphic, this fact allows us to complete the properties of the *kernel* and the *image* of the Euler–Lagrange mapping by their *global* characteristics. The results bind together properties of the *variationally trivial Lagrangians*, and *variational source forms* with the *topology* of the underlying fibered manifold Y in terms of its (De Rham) cohomology groups.

Remark 14 In general, to determine the De Rham cohomology groups of a smooth manifold of a smooth fibered manifold is a hard problem; for basic theory of the DeRham cohomology, we refer to Lee [L] and Warner [W]; in simple cases, one can apply the *Künneth theorem* (Bott and Tu [BT]).

The following are well-known standard examples of manifolds and their cohomology groups:

(a) *Euclidean spaces* $\mathbf{R}^n$: $H^k\mathbf{R}^n = 0$ for all $k \geq 1$.
(b) *Spheres* S^n:

$$H^k S^n = \begin{cases} \mathbf{R}, & k = 0, n, \\ 0, & 0 < k < n. \end{cases} \tag{139}$$

(c) *Punctured Euclidean spaces* (complements of one-point sets $\{x\}$ in $\mathbf{R}^n$), *complements of closed balls* $B \subset \mathbf{R}^n$:

$$H^k(\mathbf{R}^n \backslash \{x\}) = H^k(\mathbf{R}^n \backslash B) = H^k S^{n-1}. \tag{140}$$

(d) Tori $T^k = S^1 \times S^1 \times \cdots \times S^1$ (k factors S^1):

$$H^k T^n = \mathbf{R}^{\binom{n}{k}}. \tag{141}$$

(e) *Möbius band*:

$$H^k M = H^k S^1. \tag{142}$$

(f) $H^0(X \times Y) = \mathbf{R}$
(g) *Cartesian products* (*Künneth theorem*), $k \geq 0$:

$$H^k(X \times Y) = \bigoplus_{r+s=k} H^r X \otimes H^s Y \tag{143}$$

(h) *Disjoint unions* (M_1, M_2 disjoint):

$$H^k(M_1 \cup M_2) = H^k M_1 \oplus H^k M_2. \tag{144}$$

8.5 Variationally Trivial Lagrangians

Let W be an open set in Y. Recall that a Lagrangian $\lambda \in h_n^r W$ is called *variationally trivial*, if its Euler–Lagrange form vanishes,

$$E_n \lambda = 0. \tag{145}$$

This condition can be considered as an *equation* for the unknown n-form λ. Our main objective in this section is to summarize previous local results on the solutions of this equation and to complete these results by a theorem on global solutions.

The mapping E_n is the Euler–Lagrange morphism in the complex of global sections

$$\cdots \longrightarrow h_{n-1}^r W \xrightarrow{E_{n-1}} h_n^r W \xrightarrow{E_n} \mathscr{I}_1^r W \xrightarrow{E_{n+1}} \mathscr{I}_2^r W \longrightarrow \cdots \tag{146}$$

and equation (145) has the meaning of the *integrability condition* for the corresponding equation for an unknown $(n-1)$-form η,

$$\lambda = E_{n-1}\eta. \tag{147}$$

Thus, since $E_{n-1}\eta$ is defined to be $hd\eta$, equation (147) can also be written as

$$\lambda = hd\eta. \tag{148}$$

Integrability condition (145), representing exactness of the sheaf variational sequence, ensures existence of *local* solutions, defined on chart neighborhoods in the set W. According to Theorem 9, Sect. 4.8, the following conditions are equivalent:

(a) λ is variationally trivial.
(b) For any fibered chart (V, ψ), $\psi = (x^i, y^\sigma)$, such that $V \subset W$, there exist functions $g^i \colon V^r \to \mathbf{R}$, such that on V^r, λ is expressible as $\lambda = \mathscr{L}\omega_0$, where

$$\mathscr{L} = d_i g^i. \tag{149}$$

(c) For every fibered chart (V, ψ), $\psi = (x^i, y^\sigma)$, such that $V \subset W$, there exists an $(n-1)$-form $\mu \in \Omega^{r-1}_{n-1} V$ such that on V^r

$$\lambda = hd\mu. \tag{150}$$

A question still remains open, namely, under what conditions there exists a solution μ, defined *globally* over W or, in other words, when a given Lagrangian, locally expressible as "divergence," can be expressed as a "divergence" globally. The following theorem is an immediate consequence of the properties of the complex of global sections ((138), Sect. 8.4).

Theorem 12 *Let Y be a fibered manifold over an n-dimensional manifold X, such that $H^n Y = 0$. Let λ be a π^r*-horizontal *Lagrangian. Then the following conditions are equivalent:*

(a) *λ is variationally trivial.*
(b) *There exists an $(n-1)$-form $\mu \in \Omega^{r-1}_{n-1} Y$ such that on $J^r Y$*

$$\lambda = hd\mu. \tag{151}$$

Proof

1. We show that (a) implies (b). In view of Sect. 4.8, Theorem 9, only existence of μ, defined globally on $J^r Y$, needs proof. But by Sect. 8.2, Theorem 2, the cohomology groups $H^k(\mathrm{Var}^r_Y Y)$ are isomorphic with the De Rham cohomology groups $H^k(Y, \mathbf{R}_Y)$; thus, condition $H^n Y = 0$ implies $H^n(\mathrm{Var}^r_Y Y) = 0$ proving existence of μ.
2. The converse is obvious. □

On analogy with the De Rham sequence, a variationally trivial Lagrangian can also be called *variationally closed*. A variationally closed Lagrangian $\lambda \in h^r_n W$ is called *variationally exact*, if $\lambda = hd\mu$ for some $\mu \in h^r_{n-1} W$. Theorem 12 then says that if $H^n Y = 0$, then every variationally closed Lagrangian is variationally exact.

In the following examples, we refer to the cohomology groups given in Sect. 8.4, Remark 13.

Examples (Obstructions for variational triviality)

1. If the fibered manifold Y is the Cartesian product $\mathbf{R}^n \times \mathbf{R}^m$, endowed with the first canonical projection, then every variationally trivial Lagrangian on Y is variationally exact.

2. Let $Y = S^3$, and consider S^3 as a fibered manifold over S^2 (the *Hopf fibration*). Then, $H^3 S^3 = \mathbf{R} \neq 0$; therefore, a variationally trivial Lagrangian on $J^r S^3$ need not be closed.
3. If $Y = \mathbf{R}^n \times Q$; then, the Künneth theorem yields $H^n(\mathbf{R}^n \times Q) = H^n Q$. Thus, if $H^n Q = 0$, then variational triviality always implies variational exactness. If for example Q is an n-sphere S^n, punctured Euclidean space $\mathbf{R}^{n+1} \backslash \{0\}$, or the k-torus T^k, then variational triviality does not imply variational exactness.

8.6 Global Inverse Problem of the Calculus of Variations

Let W be an open set in Y. Recall that a source form $\varepsilon \in \mathscr{I}_1^r W$ is said to be *variational*, if there exists a Lagrangian $\lambda \in h_n^r W$ such that its Euler–Lagrange form $E_n \lambda$ coincides with ε,

$$\varepsilon = E_n \lambda. \tag{152}$$

ε is said to be *locally variational*, if there exists an atlas on Y, consisting of fibered charts, such that for each chart (V, ψ), $\psi = (x^i, y^\sigma)$, from this atlas, the restriction of ε to V^s is variational.

The mapping E_n in formula (152) is the *Euler–Lagrange morphism* in the complex of global sections

$$\cdots \longrightarrow h_{n-1}^r W \xrightarrow{E_{n-1}} h_n^r W \xrightarrow{E_n} \mathscr{I}_1^r W \xrightarrow{E_{n+1}} \mathscr{I}_2^r W \longrightarrow \cdots \tag{153}$$

which determines the *integrability condition* for equation (152),

$$E_{n+1} \varepsilon = 0. \tag{154}$$

The problem to determine conditions ensuring existence of the Lagrangian λ, and to determine λ as a function of the source form ε, is the *inverse problem of the calculus of variations*.

If the source form ε is expressed in the form

$$\varepsilon = \varepsilon_\sigma \omega^\sigma \wedge \omega_0, \tag{155}$$

then equation (152) is expressed as a system of partial differential equations

$$\varepsilon_\sigma = \frac{\partial \mathscr{L}}{\partial y^\sigma} + \sum_{1 \le s \le r} (-1)^s d_{j_1} d_{j_2} \ldots d_{j_s} \frac{\partial \mathscr{L}}{\partial y^\sigma_{j_1 j_2 \ldots j_s}}, \quad 1 \le \sigma \le m, \tag{156}$$

for an unknown function $\mathscr{L} = \mathscr{L}(x^i, y^\sigma, y^\sigma_{j_1}, y^\sigma_{j_1 j_2}, \ldots, y^\sigma_{j_1 j_2 \ldots j_r})$. Integrability condition (154) is then of the form

$$E_{n+1}\varepsilon = \frac{1}{2}\sum_{0 \le k \le r} H_{\sigma\ \nu}^{\ j_1 j_2 \ldots j_k}(\varepsilon)\omega^\nu_{j_1 j_2 \ldots j_k} \wedge \omega^\sigma \wedge \omega_0 = 0, \tag{157}$$

where $H_{\sigma\ \nu}^{\ j_1 j_2 \ldots j_k}(\varepsilon)$ are the *Helmholtz expressions* (Sect. 8.4, Theorem 11); thus, if s is the *order* of the functions ε_σ, the integrability condition reads

$$\begin{gathered} \frac{\partial \varepsilon_\sigma}{\partial y^\nu_{j_1 j_2 \ldots j_k}} - (-1)^k \frac{\partial \varepsilon_\nu}{\partial y^\sigma_{j_1 j_2 \ldots j_k}} \\ - \sum_{l=k+1}^{s} (-1)^l \binom{l}{k} d_{p_{k+1}} d_{p_{k+2}} \ldots d_{p_l} \frac{\partial \varepsilon_\nu}{\partial y^\sigma_{j_1 j_2 \ldots j_k p_{k+1} p_{k+2} \ldots p_l}} = 0, \\ 1 \le \sigma, \nu \le m, \quad 0 \le k \le s, \quad 1 \le j_1, j_2, \ldots, j_k \le n. \end{gathered} \tag{158}$$

Integrability condition (158) ensures existence of local solutions λ of equation (152), or which is the same solutions $\mathscr{L}$ of the system (5); solutions are given explicitly by the *Vainberg–Tonti Lagrangians*

$$\lambda_\varepsilon = \mathscr{L}_\varepsilon \omega_0, \tag{159}$$

where

$$\begin{aligned} &\mathscr{L}_\varepsilon(x^i, y^\sigma, y^\sigma_{j_1}, y^\sigma_{j_1 j_2}, \ldots, y^\sigma_{j_1 j_2 \ldots j_s}) \\ &\quad = y^\sigma \int_0^1 \varepsilon_\sigma(x^i, ty^\nu, ty^\nu_{j_1}, ty^\nu_{j_1 j_2}, \ldots, ty^\nu_{j_1 j_2 \ldots j_s}) dt \end{aligned} \tag{160}$$

(Sect. 4.9, (3), Sect. 4.10, Theorem 12 and Theorem 13, Sect. 8.4, Theorem 11).

In this section, we complete these results by a theorem ensuring existence of *global* solutions of equation (152), where the open set $W \subset Y$ coincides with the fibered manifold Y. The following result completes properties of the source forms by establishing a topological condition ensuring that local variationality implies (global) variationality.

Theorem 13 *Let Y be a fibered manifold with n-dimensional base X, such that $H^{n+1}Y = 0$. Let $\varepsilon \in \mathscr{F}^r_1 W$ be a source form. Then the following conditions are equivalent:*

(a) *ε is locally variational.*
(b) *ε is variational.*

Proof This assertion is an immediate consequence of the existence of an isomorphism between the cohomology groups $H^k(\mathrm{Var}^r_Y Y)$ and the De Rham cohomology groups $H^k(Y, \mathbf{R}_Y)$ (Sect. 8.2, Theorem 2); thus, condition $H^{n+1}Y = 0$ implies $H^{n+1}(\mathrm{Var}^r_Y Y) = 0$ as required. □

Remark 14 The meaning of Theorem 13 can be rephrased as follows. First, it states that in order to ensure that a given source form ε is *locally variational*, one should verify that its components satisfy the *Helmholtz conditions* (158), and second, if in addition the $(n+1)$-st cohomology group $H^{n+1}Y$ of the underlying fibered manifold vanishes, then ε is automatically variational.

Examples (Obstructions for global variationality)

4. If $Y = \mathbf{R} \times M$, where M is the Möbius band, then $H^2 Y = 0$; hence, local variationality always implies variationality.
5. If $Y = S^1 \times M$, where S^1 is the circle and M is the Möbius band, then $H^2 Y = H^2(S^1 \times M) = H^1 S^1 \oplus H^1 M = \mathbf{R} \oplus \mathbf{R} = \mathbf{R}^2$. Thus, in general, local variationality does not imply variationality.
6. If the 3-sphere S^3 is considered as a fibered manifold over S^2 (Hopf fibration), then since $H^3 S^3 = \mathbf{R} \neq 0$, local variationality does not necessarily imply global variationality.
7. If $k \geq l$, then the k-torus T^k can be fibered over the l-torus T^l by means of the Cartesian projection. Since $H^{l+1} T^k \neq 0$, we have obstructions against global variationality.

References

[A2] I. Anderson, *The variational bicomplex*, preprint, Utah State University, 1989, 289 pp.

[AD] I. Anderson, T. Duchamp, On the existence of global variational principles, Am. J. Math. 102 (1980) 781-867

[BK] J. Brajercik, D. Krupka, Cohomology and local variational principles, Proc. of the XVth International Workshop on Geometry and Physics (Puerto de la Cruz, Tenerife, Canary Islands, September 11-16, 2006, Publ. de la RSME, (2007) 119-124

[Bry] R.L. Bryant, S.S. Chern, R.B. Gardner, H.J. Goldschmidt, P.A. Griffiths, *Exterior Differential Systems*, Mathematical Sciences Research Institute Publications 18, Springer-Verlag, New York, 1991

[BT] R. Bott, L.V. Tu, *Differential Forms and Algebraic Topology*, Springer-Verlag, New York, 1982

[DT] P. Dedecker, W. Tulczyjew, Spectral sequences and the inverse problem of the calculus of variations, Internat. Colloq., Aix-en-Provence, 1979; in: *Differential-Geometric Methods in Mathematical Physics*, Lecture Notes in Math. 826 Springer, Berlin, 1980, 498-503

[FPW] M. Francaviglia, M. Palese, E. Winterroth, Cohomological obstructions in locally variational field theory, Journal of Physics: Conference Series 474 (2013) 012017, doi:10.1088/1742-6596/474/1/012017

[Gr] D.R. Grigore, Lagrangian formalism on Grassmann manifolds, in: D. Krupka, D. Saunders, Eds., *Handbook of Global Analysis*, Elsevier, 2008, 327-373

[K16] D. Krupka, The Vainberg-Tonti Lagrangian and the Euler–Lagrange mapping, in: F. Cantrijn, B. Langerock, Eds., *Differential Geometric Methods in Mechanics and Field Theory*, Volume in Honor of W. Sarlet, Gent, Academia Press, 2007, 81-90

[K17] D. Krupka, Variational principles for energy-momentum tensors, Rep. Math. Phys. 49 (2002) 259-268

[K18] D. Krupka, Variational sequences in mechanics, Calc. Var. 5 (1997) 557-583

[K19] D. Krupka, Variational sequences on finite-order jet spaces, Proc. Conf., World Scientific, 1990, 236-254

[KrM] M. Krbek, J. Musilová, Representation of the variational sequence by differential forms, Acta Appl. Math. 88 (2005) 177-199

[KS] D. Krupka, D. Saunders, Eds., *Handbook of Global Analysis*, Elsevier, 2008

[KSe] D. Krupka, J. Sedenková, Variational sequences and Lepage forms, in: *Diff. Geom. Appl.*, Proc. Conf., Charles University, Prague, Czech Republic, 2005, 617-627

[L] J.M. Lee, *Introduction to Smooth Manifolds*, Graduate Texts in Math. 218, Springer, 2006

[O1] P.J. Olver, *Applications of Lie Groups to Differential Equations*, Springer-Verlag, New York, 1998

[Po] J.F. Pommaret, Spencer sequence and variational sequence, Acta Appl. Math. 41 (1995) 285-296

[S] D.J. Saunders, *The Geometry of Jet Bundles*, Cambridge Univ. Press, 1989

[T] F. Takens, A global version of the inverse problem of the calculus of variations, J. Differential Geometry 14 (1979) 543-562

[UK1] Z. Urban, D. Krupka, Variational sequences in mechanics on Grassmann fibrations, Acta Appl. Math. 112 (2010) 225-249

[VKL] A.M. Vinogradov, I.S. Krasilschik, V.V. Lychagin, *Introduction to the Geometry of Non-linear Differential Equations* (in Russian) Nauka, Moscow, 1986

[Vit] R. Vitolo, Finite order Lagrangian bicomplexes, Math. Soc. Cambridge Phil. Soc. 125 (1999) 321-333

[VU] J. Volna, Z. Urban, The interior Euler-Lagrange operator in field theory, Math. Slovaca, to appear

[W] F.W. Warner, *Foundations of Differentiable Manifolds and Lie Groups*, Springer-Verlag, New York, 1983

[Z] D. Zenkov (Ed.), The Inverse Problem of the Calculus of Variations, Local and Global Theory and Applications, Atlantis Series in Global Variational Geometry, to appear

Appendix
Analysis on Euclidean Spaces and Smooth Manifolds

In this appendix, we summarize for the reference essential notions and theorems of differentiation and integration theory on Euclidean spaces as needed in this book. Main coordinate formulas of the calculus of vector fields and differential forms on smooth manifolds are also given. We also included elementary concepts from multilinear algebra and the trace decomposition theory over a real vector space.

A.1 Jets of Mappings of Euclidean Spaces

Let $L(\mathbf{R}^n, \mathbf{R}^m)$ be the vector space of *linear* mappings of $\mathbf{R}^n$ into $\mathbf{R}^m$, $L^k(\mathbf{R}^n, \mathbf{R}^m)$ the vector space of k-linear mappings of the Cartesian product $\mathbf{R}^n \times \mathbf{R}^n \times \cdots \times \mathbf{R}^n$ (k factors) into $\mathbf{R}^m$, and let $L^k_{\mathrm{sym}}(\mathbf{R}^n, \mathbf{R}^m)$ be the vector space of k-linear *symmetric* mappings from of $\mathbf{R}^n \times \mathbf{R}^n \times \cdots \times \mathbf{R}^n$ (k factors) into $\mathbf{R}^m$. Let $U \subset \mathbf{R}^n$ and $V \subset \mathbf{R}^m$ be open sets, and denote

$$J^r(U,V) = U \times V \times L(\mathbf{R}^n, \mathbf{R}^m) \times L^2_{\mathrm{sym}}(\mathbf{R}^n, \mathbf{R}^m) \times \cdots \times L^r_{\mathrm{sym}}(\mathbf{R}^n, \mathbf{R}^m). \qquad (1)$$

$J^r(U,V)$ is an open set in the Euclidean vector space

$$\mathbf{R}^n \times \mathbf{R}^m \times L(\mathbf{R}^n, \mathbf{R}^m) \times L^2_{\mathrm{sym}}(\mathbf{R}^n, \mathbf{R}^m) \times \cdots \times L^r_{\mathrm{sym}}(\mathbf{R}^n, \mathbf{R}^m). \qquad (2)$$

Using the canonical bases of the vector spaces $\mathbf{R}^n$ and $\mathbf{R}^m$, this vector space can be identified with the Euclidean vector space $\mathbf{R}^N$ of dimension

$$N = n + m\left(1 + n + \binom{n+1}{2} + \binom{n+2}{3} + \cdots + \binom{n+r-1}{r}\right). \qquad (3)$$

The set $J^r(U,V)$ can be identified with collections of real numbers $P = (x^i, y^\sigma, y^\sigma_{j_1}, y^\sigma_{j_1 j_2}, \ldots, y^\sigma_{j_1 j_2 \ldots j_r})$, $1 \le i, j_1, j_2, \ldots, j_r \le n$, $1 \le \sigma \le m$, such that the systems $y^\sigma_{j_1 j_2 \cdots j_k}$ are *symmetric* in the subscripts. We call P an *r-jet*; the point $x \in U$, $x = x^i$ is called the *source* of P and the point $y \in V$, $y = y^\sigma$, *is called the* target of P.

We set for every point $P \in J^r(U,V)$, $P = (x^i, y^\sigma, y^\sigma_{j_1}, y^\sigma_{j_1 j_2}, \ldots, y^\sigma_{j_1 j_2 \ldots j_r})$,

$$x^i = x^i(P), \quad y^\sigma = y^\sigma(P), \quad y^\sigma_{j_1 j_2 \cdots j_k} = y^\sigma_{j_1 j_2 \cdots j_k}(P), \quad 1 \le k \le r. \qquad (4)$$

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Atlantis Studies in Variational Geometry 1, DOI 10.2991/978-94-6239-073-7

Then, by abuse of language, x^i, y^σ, and $y^\sigma_{j_1 j_2 \cdots j_k}$, denote both the components of P and also real-valued functions on $J^r(U,V)$. Restricting ourselves to independent functions, we get a global chart, the *canonical chart* $(x^i, y^\sigma, y^\sigma_{j_1}, y^\sigma_{j_1 j_2}, \ldots, y^\sigma_{j_1 j_2 \ldots j_r})$, $j_1 \le j_2 \le \cdots \le j_k$, defining the *canonical smooth manifold structure* on $J^r(U,V)$; elements of this chart are the *canonical coordinates* on $J^r(U,V)$. The set $J^r(U,V)$, endowed with its canonical smooth manifold structure, is called *the manifold of r-jets* (with *source in U* and *target in V*).

We sometimes express without notice an element $P \in J^r(U,V)$ as a collection of real numbers $P = (x^i, y^\sigma, y^\sigma_{j_1}, y^\sigma_{j_1 j_2}, \ldots, y^\sigma_{j_1 j_2 \ldots j_r})$, subject to the condition $j_1 \le j_2 \le \cdots \le j_k$.

We show that the r-jets can be viewed as classes of mappings, transferring the source of an r-jet to its target. Given an r-jet $P = J^r(U,V)$, $P = (x^i, y^\sigma, y^\sigma_{j_1}, y^\sigma_{j_1 j_2}, \ldots, y^\sigma_{j_1 j_2 \ldots j_r})$, one can always find a mapping $f = f^\sigma$, defined on a neighborhood of the source $x \in U$, such that $f(x) = y$, whose derivatives satisfy

$$\begin{gathered} D_{i_1} f^\sigma(x^i(P)) = y^\sigma_{i_1}(P), \quad D_{i_1} D_{i_2} f^\sigma(x^i(P)) = y^\sigma_{i_1 i_2}(P), \\ \ldots, D_{i_1} D_{i_2} \ldots D_{i_r} f^\sigma(x^i(P)) = y^\sigma_{j_1 j_2 \cdots j_r}(P). \end{gathered} \tag{5}$$

Indeed, one can choose for the components of f the polynomials

$$\begin{aligned} f^\sigma(t^j) = {} & y^\sigma + \frac{1}{1!} y^\sigma_{j_1}(t^{j_1} - x^{j_1}) + \frac{1}{2!} y^\sigma_{j_1 j_2}(t^{j_1} - x^{j_1})(t^{j_2} - x^{j_2}) \\ & + \cdots + \frac{1}{r!} y^\sigma_{j_1 j_2 \cdots j_r}(t^{j_1} - x^{j_1})(t^{j_2} - x^{j_2}) \cdots (t^{j_r} - x^{j_r}). \end{aligned} \tag{6}$$

Any mapping f, satisfying conditions (5), is called a *representative* of the r-jet P. Using representatives, we usually denote $P = J^r_x f$.

A.2 Summation Conventions

This section contains some remarks to the summation conventions used in this book. We distinguish essentially three different cases:

(a) Summations through pairs of indices, one in contravariant and one in covariant position (the Einstein summation convention). In this case, the summation symbol is not explicitly designated.
(b) Summations through more indices or multi-indices. In this case, we usually omit the summation symbols for summations, which are evident.
(c) Summations of expressions through variables, labeled with non-decreasing sequences of integers. In this Appendix, we discuss the corresponding conventions in more detail.

Let k be a positive integer, let $L^k\mathbf{R}^n$ be the vector space of collections of real numbers $u = u_{i_1 i_2 \ldots i_k}$, where $1 \le i_1, i_2, \ldots, i_k \le n$, and $J^k\mathbf{R}^n$ the vector space of collections of real numbers $v = v_{i_1 i_2 \ldots i_k}$, where $1 \le i_1 \le i_2 \le \cdots \le i_k \le n$. We introduce two mappings $\iota\colon J^k\mathbf{R}^n \to L^k\mathbf{R}^n$ and $\kappa\colon L^k\mathbf{R}^n \to J^k\mathbf{R}^n$ as follows. Choose a vector $v \in J^k\mathbf{R}^n$, $v = v_{i_1 i_2 \ldots i_k}$, where $1 \le i_1 \le i_2 \le \cdots \le i_k \le n$, and set for *any* sequence of the indices $j_1, j_2, \ldots, j_k$, not necessarily a non-decreasing one,

$$v_{j_1 j_2 \ldots j_k} = v_{j_{\tau(1)} j_{\tau(2)} \ldots j_{\tau(k)}}, \tag{7}$$

where τ is any permutation of the set $\{1, 2, \ldots, k\}$, such that the subscripts satisfy $j_{\tau(1)} \le j_{\tau(2)} \le \cdots \le j_{\tau(k)}$. Then set

$$\iota(v) = v_{j_1 j_2 \ldots j_k}. \tag{8}$$

The vector $\iota(v)$ is *symmetric* in all subscripts and is called the *canonical extension* of v to $L^k\mathbf{R}^n$; the mapping ι is *the canonical extension* (by symmetry). If $u \in L^k\mathbf{R}^n$, $u = u_{i_1 i_2 \ldots i_k}$, set

$$\kappa(u) = v_{j_1 j_2 \ldots j_k} = \frac{1}{k!} \sum_{\nu} u_{j_{\nu(1)} j_{\nu(2)} \ldots j_{\nu(k)}}, \tag{9}$$

whenever $j_1 \le j_2 \le \cdots \le j_k$; κ is called the *symmetrization*. For any function $f\colon J^k\mathbf{R}^n \to \mathbf{R}$, the function $f \circ \kappa\colon L^k\mathbf{R}^n \to \mathbf{R}$ is called the *canonical extension* of f. When no misunderstanding may possibly arise, we write just f instead of $f \circ \kappa$. Clearly, definitions (8) and (9) imply

$$\kappa \circ \iota = \mathrm{id}_{J^k\mathbf{R}^n}. \tag{10}$$

Note that in the finite-dimensional Euclidean vector space $\mathbf{R}^N$, the points of $\mathbf{R}^N$ are canonically identified with the canonical coordinates of these point. In what follows we shall consider the symbols $u_{i_1 i_2 \ldots i_k}$ and $v_{i_1 i_2 \ldots i_k}$ both as the points of $\mathbf{R}^N$ as well as the *canonical coordinates* on the vector spaces $L^k\mathbf{R}^n$ and $J^k\mathbf{R}^n$, respectively.

Denote

$$N(j_1 j_2 \ldots j_k) = \frac{N_1! N_2! \ldots N_n!}{k!}, \tag{11}$$

where N_l is the number of occurrences of the index $l = 1, 2, \ldots, n$ in the k-tuple $(j_1, j_2, \ldots, j_k)$. The following lemma states two formulas how to express a linear form, whose variables are indexed with non-decreasing sequences; these formulas are based on simple algebraic relations.

Let

$$\Phi = \sum_{i_1 \le i_2 \le \cdots \le i_k} A^{i_1 i_2 \ldots i_k} v_{i_1 i_2 \ldots i_k} \tag{12}$$

be a linear form on $J^k\mathbf{R}^n$.

Lemma 1 *A linear form* Φ (12) *on* $J^k\mathbf{R}^n$ *can be expressed as*

$$\Phi = B^{j_1 j_2 \ldots j_k} v_{j_1 j_2 \ldots j_k}, \tag{13}$$

where

$$B^{j_1 j_2 \ldots j_k} = \frac{1}{N(j_1 j_2 \ldots j_k)} A^{i_1 i_2 \ldots i_k}. \tag{14}$$

Proof Supposing that $B^{j_1 j_2 \ldots j_k}$ and $v_{j_1 j_2 \ldots j_k}$ are symmetric, we have

$$\begin{aligned} B^{j_1 j_2 \ldots j_k} v_{j_1 j_2 \ldots j_k} &= \sum_{j_1, j_2, \ldots, j_k} \sum_{\kappa} \frac{1}{k!} B^{j_{\kappa(1)} j_{\kappa(2)} \ldots j_{\kappa(k)}} v_{j_{\kappa(1)} j_{\kappa(2)} \ldots j_{\kappa(k)}} \\ &= \sum_{j_1 \le j_2 \le \cdots \le j_k} \frac{1}{k!} N_1! N_2! \ldots N_n! B^{j_1 j_2 \ldots j_k} v_{j_1 j_2 \ldots j_k} \\ &= \sum_{j_1 \le j_2 \le \cdots \le j_k} N(i_1 i_2 \ldots i_k) B^{j_1 j_2 \ldots j_k} v_{j_1 j_2 \ldots j_k}. \end{aligned} \tag{15}$$

If this expression equals Φ, we get (14). □

Lemma 1 can be applied to linear forms df, where $f\colon J^k\mathbf{R}^n \to \mathbf{R}$ is a function. df is defined by

$$df(v) \cdot \Xi = \sum_{i_1 \le i_2 \le \cdots \le i_k} \left(\frac{\partial f}{\partial v_{i_1 i_2 \ldots i_k}} \right)_v \Xi_{i_1 i_2 \ldots i_k}, \tag{16}$$

where

$$\Xi = \sum_{i_1 \le i_2 \le \cdots \le i_k} \Xi_{i_1 i_2 \ldots i_k} \frac{\partial}{\partial v_{i_1 i_2 \ldots i_k}} \tag{17}$$

is a tangent vector. But the chain rule yields $T_v f \cdot \Xi = T_{\iota(v)}(f \circ \kappa) \circ T_v \iota \cdot \Xi$, so we have the following assertion.

Lemma 2 *The linear form df (16) can be expressed as*

$$df(v) \cdot \Xi = \left(\frac{\partial (f \circ \kappa)}{\partial u_{j_1 j_2 \ldots j_k}} \right)_{\iota(v)} \Xi_{j_1 j_2 \ldots j_k}. \tag{18}$$

Proof Using formula (10), we get from (16)

$$\begin{aligned} df(v) \cdot \Xi &= \sum_{i_1 \le i_2 \le \cdots \le i_k} \left(\frac{\partial (f \circ \kappa \circ \iota)}{\partial v_{i_1 i_2 \ldots i_k}} \right)_v \Xi_{i_1 i_2 \ldots i_k} \\ &= \sum_{i_1 \le i_2 \le \cdots \le i_k} \sum_{j_1, j_2, \ldots, j_k} \left(\frac{\partial (f \circ \kappa)}{\partial u_{j_1 j_2 \ldots j_k}} \right)_{\iota(v)} \left(\frac{\partial (u_{j_1 j_2 \ldots j_k} \circ \iota)}{\partial v_{i_1 i_2 \ldots i_k}} \right)_v \Xi_{i_1 i_2 \ldots i_k} \\ &= \sum_{j_1, j_2, \ldots, j_k} \left(\frac{\partial (f \circ \kappa)}{\partial u_{j_1 j_2 \ldots j_k}} \right)_{\iota(v)} \sum_{i_1 \le i_2 \le \cdots \le i_k} \left(\frac{\partial (u_{j_1 j_2 \ldots j_k} \circ \iota)}{\partial v_{i_1 i_2 \ldots i_k}} \right)_v \Xi_{i_1 i_2 \ldots i_k}. \end{aligned} \tag{19}$$

But writing

$$T_v \imath \cdot \Xi = \Xi_{j_1 j_2 \ldots j_k} \frac{\partial}{\partial u_{j_1 j_2 \ldots j_k}}, \tag{20}$$

we see that $T_v \imath$ extends the components $\Xi_{i_1 i_2 \ldots i_k}$, $i_1 \leq i_2 \leq \cdots \leq i_k$ by the index symmetry,

$$\Xi_{j_1 j_2 \ldots j_k} = \sum_{i_1 \leq i_2 \leq \cdots \leq i_k} \left(\frac{\partial (u_{j_1 j_2 \ldots j_k} \circ \imath)}{\partial v_{i_1 i_2 \ldots i_k}} \right)_v \Xi_{i_1 i_2 \ldots i_k}. \tag{21}$$

Thus, using the symmetric components (21), one can also express the exterior derivative d*f* (19) as in (18). □

Corollary 1 *Let* $f : J^k \mathbf{R}^n \to \mathbf{R}$ *be a function,* $v \in J^k \mathbf{R}^n$ *a point, and let* $\Xi = \Xi_{j_1 j_2 \ldots j_k}$, *where* $1 \leq j_1 \leq j_2 \leq \cdots \leq j_k \leq n$, *be the components of a tangent vector of* $J^k \mathbf{R}^n$ *at the point* v. *Then the derivatives of the functions* f *and* $f \circ \kappa$ *satisfy*

$$\sum_{j_1 \leq j_2 \leq \cdots \leq j_k} \left(\frac{\partial f}{\partial v_{j_1 j_2 \ldots j_k}} \right)_v \Xi_{j_1 j_2 \ldots j_k} = \left(\frac{\partial (f \circ \kappa)}{\partial u_{i_1 i_2 \ldots i_k}} \right)_{\imath(v)} \Xi_{i_1 i_2 \ldots i_k}. \tag{22}$$

Proof (22) follows from (16) and (18). □

Corollary 2

(a) *Partial derivatives of the functions* f *and* $f \circ \kappa$ *satisfy the condition*

$$\frac{\partial (f \circ \kappa)}{\partial u_{j_1 j_2 \ldots j_k}} = N(j_1 j_2 \ldots j_k) \frac{\partial f}{\partial v_{j_{\lambda(1)} j_{\lambda(2)} \ldots j_{\lambda(k)}}} \circ \kappa, \tag{23}$$

where λ *is any permutation of the index set* $\{1, 2, \ldots, k\}$, *such that* $j_{\lambda(1)} \leq j_{\lambda(2)} \leq \cdots \leq j_{\lambda(k)}$, *and*

$$\frac{\partial f}{\partial v_{i_1 i_2 \ldots i_k}} = \frac{1}{N(i_1 i_2 \ldots i_k)} \frac{\partial (f \circ \kappa)}{\partial u_{i_{\tau(1)} i_{\tau(2)} \ldots i_{\tau(k)}}} \circ \imath \tag{24}$$

for any permutation τ.

(b) *For any permutation* $l_{\tau(1)}, l_{\tau(2)}, \ldots, l_{\tau(k)}$ *of the indices* $l_1, l_2, \ldots, l_k$, *the derivatives of the function* $f \circ \kappa$ *satisfy*

$$\frac{\partial (f \circ \kappa)}{\partial u_{l_{\tau(1)} l_{\tau(2)} \ldots l_{\tau(k)}}} = \frac{\partial (f \circ \kappa)}{\partial u_{l_1 l_2 \ldots l_k}}. \tag{25}$$

Proof

(a) From the chain rule, we have for any $(j_1, j_2, \ldots, j_k)$

$$\frac{\partial(f \circ \kappa)}{\partial u_{j_1 j_2 \ldots j_k}} = \sum_{i_1 \le i_2 \le \cdots \le i_k} \left(\frac{\partial f}{\partial v_{i_1 i_2 \ldots i_k}} \circ \kappa \right) \frac{\partial (v_{i_1 i_2 \ldots i_k} \circ \kappa)}{\partial u_{j_1 j_2 \ldots j_k}}. \tag{26}$$

But from Eq. (9), there is exactly one nonzero term on the right-hand side, namely the term in which $(i_1 i_2 \ldots i_k) = (j_{\lambda(1)} j_{\lambda(2)} \ldots j_{\lambda(k)})$, such that $j_{\lambda(1)} \le j_{\lambda(2)} \le \cdots \le j_{\lambda(k)}$ for some permutation λ. Then

$$\frac{\partial(f \circ \kappa)}{\partial u_{j_1 j_2 \ldots j_k}} = \frac{\partial f}{\partial v_{j_{\lambda(1)} j_{\lambda(2)} \ldots j_{\lambda(k)}}} \circ \kappa \cdot \frac{\partial (v_{j_{\lambda(1)} j_{\lambda(2)} \ldots j_{\lambda(k)}} \circ \kappa)}{\partial u_{j_1 j_2 \ldots j_k}}, \tag{27}$$

where by (9)

$$v_{j_{\lambda(1)} j_{\lambda(2)} \ldots j_{\lambda(k)}} \circ \kappa = \frac{1}{k!} \sum_{\tau} u_{j_{\tau(1)} j_{\tau(2)} \ldots j_{\tau(k)}}. \tag{28}$$

Differentiating (24), we get

$$\begin{aligned} \frac{\partial (v_{j_{\lambda(1)} j_{\lambda(2)} \ldots j_{\lambda(k)}} \circ \kappa)}{\partial u_{j_1 j_2 \ldots j_k}} &= \frac{1}{k!} \sum_{\tau} \frac{\partial u_{j_{\tau(1)} j_{\tau(2)} \ldots j_{\tau(k)}}}{\partial u_{j_1 j_2 \ldots j_k}} \\ &= \frac{N_1! N_2! \ldots N_n!}{k!}. \end{aligned} \tag{29}$$

Substituting from (29) back to (27), we have

$$\frac{\partial(f \circ \kappa)}{\partial u_{j_1 j_2 \ldots j_k}} = N(j_1 j_2 \ldots j_k) \frac{\partial f}{\partial v_{j_{\lambda(1)} j_{\lambda(2)} \ldots j_{\lambda(k)}}} \circ \kappa. \tag{30}$$

Conversely, given a k-tuple of indices $(i_1, i_2, \ldots, i_k)$ such that $1 \le i_1 \le i_2 \le \cdots \le i_k \le n$, we get from (30) and (10)

$$\frac{\partial f}{\partial v_{i_1 i_2 \ldots i_k}} = \frac{1}{N(i_1 i_2 \ldots i_k)} \frac{\partial(f \circ \kappa)}{\partial u_{i_{\tau(1)} i_{\tau(2)} \ldots i_{\tau(k)}}} \circ \iota \tag{31}$$

for any permutation τ. Formulas (30) and (31) prove Corollary 2.

(b) Formula (25) follows from (23). □

Remark Formula (22) can also be used, with obvious simplification, in the form

$$\sum_{j_1 \le j_2 \le \cdots \le j_k} \frac{\partial f}{\partial u_{j_1 j_2 \ldots j_k}} \Xi_{j_1 j_2 \ldots j_k} = \frac{\partial f}{\partial u_{i_1 i_2 \ldots i_k}} \Xi_{i_1 i_2 \ldots i_k}. \tag{32}$$

A.3 The Rank Theorem

In the following two basic theorems of analysis of real-valued functions on finite-dimensional Euclidean spaces, we denote by x^i and y^σ the canonical coordinates on the Euclidean spaces $\mathbf{R}^n$ and $\mathbf{R}^m$, respectively.

Theorem 1 (The Rank theorem) *Let W be an open set in $\mathbf{R}^n$, and let $f: W \to \mathbf{R}^m$ be a C^r-mapping. Let $q \le \min(m, n)$ be a positive integer. The following conditions are equivalent:*

(1) *The mapping f has constant rank* $\operatorname{rank} Df(x) = q$ *on W.*
(2) *For every point $x_0 \in W$ there exist a neighborhood U of x_0 in W, an open rectangle $P \subset \mathbf{R}^n$ with center 0, a C^r diffeomorphism $\varphi: U \to P$ such that $\varphi(x_0) = 0$, a neighborhood V of $f(x_0)$ such that $f(U) \subset V$, an open rectangle $Q \subset \mathbf{R}^m$ with center 0, and a C^r diffeomorphism $\psi: V \to Q$ such that $\psi(f(x_0)) = 0$, and on P,*

$$\psi f \varphi^{-1}(x^1, x^2, \ldots, x^q, x^{q+1}, x^{q+2}, \ldots, x^n) = (x^1, x^2, \ldots, x^q, 0, 0, \ldots, 0). \tag{33}$$

Formula (33) can be expressed in terms of *equations* of the mapping $\psi f \varphi^{-1}$, which are of the form

$$y^\sigma \circ f = \begin{cases} x^\sigma, & 1 \le \sigma \le q, \\ 0, & q + 1 \le \sigma \le m. \end{cases} \tag{34}$$

In particular, if $q = n \le m$, then $\psi f \varphi^{-1}$ is the restriction of the canonical injection $(x^1, x^2, \ldots, x^n) \to (x^1, x^2, \ldots, x^n, 0, 0, \ldots, 0)$ of Euclidean spaces; if $q = n = m$, $\psi f \varphi^{-1}$ is the restriction of the identity mapping of $\mathbf{R}^n$; if the dimensions n and m satisfy $n > m$, then $\psi f \varphi^{-1}$ is the restriction of the Cartesian projection $(x^1, x^2, \ldots, x^m, x^{m+1}, x^{m+2}, \ldots, x^n) \to (x^1, x^2, \ldots, x^m)$ of Euclidean spaces.

The following is an immediate consequence of Theorem 1.

Theorem 2 (The Inverse function theorem) *Let $W \subset \mathbf{R}^n$ be an open set, and let $f: W \to \mathbf{R}^n$ be a C^r-mapping. Suppose that* $\det Df(x_0) \neq 0$ *at a point $x_0 \in W$. Then there exists a neighborhood U of x_0 in W and a neighborhood V of $f(x_0)$ in $\mathbf{R}^n$ such that $f(U) = V$ and the restriction $f|_U: U \to V$ is a C^r-diffeomorphism.*

A.4 Local Flows of Vector Fields

In this book, the symbol $T_x f$ denotes the tangent mapping of a mapping f at a point x. Sometimes, we also use another notation, which may simplify calculations and resulting formulas. If $t \to \zeta(t)$ is a *curve* in a manifold, then its tangent vector at a point t_0 is denoted by either of the symbols

$$T_{t_0}\zeta \cdot 1, \quad \left(\frac{d\zeta}{dt}\right)_{t_0}. \tag{35}$$

The tangent vector field is denoted by

$$T_t\zeta \cdot 1 = \frac{d\zeta}{dt}. \tag{36}$$

Note, however, that sometimes the symbol $d\zeta/dt$ may cause notational problems when using the chain rule.

The following is a well-known result of the theory of integral curves of vector fields on smooth manifolds.

Theorem (The local flow theorem) *Let $r \geq 1$ and let ξ be a C^r vector field on a smooth manifold X.*

(a) *For every point $x_0 \in X$ there exists an open interval J containing the point $0 \in \mathbf{R}$, a neighborhood V of x_0, and a unique C^r mapping $\alpha: J \times V \to X$ such that for every point $x \in V$, $\alpha(0, x) = x$ and the mapping $J \ni t \to \alpha_x(t) = \alpha(t, x) \in X$ satisfies*

$$T_t\alpha_x = \xi(\alpha_x(t)). \tag{37}$$

(b) *There exist a subinterval K of J with center 0 and a neighborhood W of x_0 in V such that*

$$\alpha(s + t, x) = \alpha(s, \alpha(t, x)), \quad \alpha(-t, \alpha(t, x)) = x \tag{38}$$

for all points $(s, t) \in K$ and $x \in W$. For every $t \in K$, the mapping $W \ni x \to \alpha(t, x) \in X$ is a C^k diffeomorphism.

Condition (37) means that $t \to \alpha_x(t)$ is an *integral curve* of the vector field ξ, and the mapping $(t, x) \to \alpha_x(t) = \alpha(t, x)$ is a *local flow* of ξ at the point x_0; we also say that α is a *local flow* of ξ *on the set V*. Equation (37) can also be written as

$$\frac{d\alpha_x}{dt} = \xi(\alpha_x(t)). \tag{39}$$

A.5 Calculus on Manifolds

In this Appendix, we give a list of basic rules and coordinate formulas of the calculus of differential forms and vector fields on smooth manifolds.

We use the following notation:

Tf	tangent mapping of a differentiable mapping f
$f^*\eta$	pull-back of a differential form η by f
$[\xi, \zeta]$	Lie bracket of vector fields ξ and ζ

d	exterior derivative of a differential form
$i_\xi \eta$	contraction of a differential form η by a vector field ξ
$\partial_\xi \eta$	Lie derivative of a differential form η by a vector field ξ

Theorem 1 (The pull-back of a differential form) *Let X, Y and Z be smooth manifolds.*

(a) *For any differentiable mapping $f: X \to Y$, any p-form η and any q-form ρ on Y*

$$f^*(\eta \wedge \rho) = f^*\eta \wedge f^*\rho. \tag{40}$$

(b) *Let $f: X \to Y$ and $g: Y \to Z$ be differentiable mappings. Then for any p-form μ on Z*

$$f^*g^*\mu = (g \circ f)^*\mu. \tag{41}$$

Theorem 2 (Exterior derivative) *Let X and Y be smooth manifolds.*

(a) *For any p-form η and q-form ρ on X*

$$d(\eta \wedge \rho) = d\eta \wedge \rho + (-1)^p \eta \wedge d\rho. \tag{42}$$

(b) *For every p-form η on X*

$$d(d\eta) = 0. \tag{43}$$

(c) *For any differentiable mapping $f: X \to Y$ and any p-form η on Y*

$$df^*\eta = f^*d\eta. \tag{44}$$

Theorem 3 (Contraction of forms by a vector field) *Let X and Y be smooth manifolds.*

(a) *Let η be a p-form on X, and let ξ and ζ be two vector fields on X. Then*

$$i_\zeta i_\xi \eta = -i_\xi i_\zeta \eta. \tag{45}$$

(b) *Let η be a p-form, ρ a q-form, and let ζ be a vector field on X. Then*

$$i_\zeta(\eta \wedge \rho) = i_\zeta \eta \wedge \rho + (-1)^p \eta \wedge i_\zeta \rho. \tag{46}$$

(c) *Let $f: X \to Y$ be a differentiable mapping, η a p-form on Y, and let ξ be a vector field on X, ζ a vector field on Y. Suppose that ξ and $\zeta$are f-related. Then*

$$f^*i_\zeta \eta = i_\xi f^*\eta. \tag{47}$$

Theorem 4 (Lie derivative)

(a) *Let X be a smooth manifold, η a p-form, ρ a q-form, and let ξ and ζ be vector fields on X. Then*

$$\partial_\xi \eta = i_\xi d\eta + d i_\xi \eta, \tag{48}$$

$$\partial_\xi d\eta = d\partial_\xi \eta, \tag{49}$$

$$\partial_\xi (\eta \wedge \rho) = \partial_\xi \eta \wedge \rho + \eta \wedge \partial_\xi \rho, \tag{50}$$

$$i_{[\xi,\zeta]}\eta = \partial_\xi i_\zeta \eta - i_\zeta \partial_\xi \eta, \tag{51}$$

$$\partial_{[\xi,\zeta]}\eta = \partial_\xi \partial_\zeta \eta - \partial_\zeta \partial_\xi \eta. \tag{52}$$

(b) *Let $f \colon X \to Y$ be a differentiable mapping of smooth manifolds, let ξ be a vector field on X, and ζ be a vector field on Y. Suppose that ξ and ζ are f-compatible. Then for any p-form η on Y*

$$f^* \partial_\zeta \eta = \partial_\xi f^* \eta. \tag{53}$$

Theorem 5 *Let X and Y be smooth manifolds, $f \colon X \to Y$ a C^1 mapping. Let (U, φ), $\varphi = (x^i)$, be a chart on X, and (V, ψ), $\psi = (y^\sigma)$, a chart on Y, such that $f(U) \subset V$.*

(a) *For any point $x \in U$ and any tangent vector $\xi \in T_x X$ at the point x, expressed as*

$$\xi = \xi^k \left(\frac{\partial}{\partial x^k} \right)_x, \tag{54}$$

the image $Tf \cdot \xi$ is

$$Tf \cdot \xi = \left(\frac{\partial (y^\sigma f \varphi^{-1})}{\partial x^i} \right)_{\varphi(x)} \xi^i \left(\frac{\partial}{\partial y^\sigma} \right)_{f(x)}. \tag{55}$$

(b) *The pull-back $f^*\eta$ of a differential p-form η on Y, expressed as*

$$\eta = \frac{1}{p!} \eta_{i_1 i_2 \dots i_p} dx^{i_1} \wedge dx^{i_2} \wedge \dots \wedge dx^{i_p}, \tag{56}$$

is given by

$$\begin{aligned} f^*\eta = \frac{1}{p!} & \frac{\partial (y^{\sigma_1} f \varphi^{-1})}{\partial x^{i_1}} \frac{\partial (y^{\sigma_2} f \varphi^{-1})}{\partial x^{i_2}} \dots \frac{\partial (y^{\sigma_p} f \varphi^{-1})}{\partial x^{i_p}} \\ & \cdot (\eta_{\sigma_1 \sigma_2 \dots \sigma_p} \circ f) \cdot dx^{i_1} \wedge dx^{i_2} \wedge \dots \wedge dx^{i_p}. \end{aligned} \tag{57}$$

Theorem 6 *Let* (U, φ), $\varphi = (x^i)$, *be a chart on* X.

(a) *For any two vector fields* ξ *and* ζ *on* X, *expressed by*

$$\xi = \xi^i \frac{\partial}{\partial x^i}, \quad \zeta = \zeta^i \frac{\partial}{\partial x^i}, \tag{58}$$

the Lie bracket $[\xi, \zeta]$ *is expressed by*

$$[\xi, \zeta] = \left(\frac{\partial \zeta^i}{\partial x^l} \xi^l - \frac{\partial \xi^i}{\partial x^l} \zeta^l \right) \frac{\partial}{\partial x^i}. \tag{59}$$

(b) *The exterior derivative* df *of a function* $f \colon X \to \mathbf{R}$ *is expressed by*

$$df = \frac{\partial f}{\partial x^k} dx^k. \tag{60}$$

The exterior derivative $d\eta$ *of a p-form* η (56) *has the chart expression*

$$d\eta = \frac{1}{p!} d\eta_{i_1 i_2 \ldots i_p} \wedge dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_p}, \tag{61}$$

where the exterior derivative $d\eta_{i_1 i_2 \ldots i_p}$ *is determined by formula* (60).

(c) *The contraction* $i_\xi \eta$ *of the form* η (56) *by a vector field* ξ (58) *has the chart expression*

$$i_\xi \eta = \frac{1}{(k-1)!} \eta_{s i_1 i_2 \ldots i_{k-1}} \xi^s dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_{k-1}}. \tag{62}$$

(d) *The Lie derivative* $\partial_\xi \eta$ *of the form* η (56) *by a vector field* ξ (58) *has the chart expression*

$$\begin{aligned} \partial_\xi \eta = \frac{1}{p!} \Bigg(& \frac{\partial \xi^s}{\partial x^{i_1}} \eta_{s i_2 i_3 \ldots i_p} - \frac{\partial \xi^s}{\partial x^{i_2}} \eta_{s i_1 i_3 i_4 \ldots i_p} + \frac{\partial \xi^s}{\partial x^{i_3}} \eta_{s i_1 i_2 i_4 i_5 \ldots i_p} \\ & - \cdots + (-1)^{p-1} \frac{\partial \xi^s}{\partial x^{i_p}} \eta_{s i_1 i_2 \ldots i_{p-1}} + \frac{\partial \eta_{i_1 i_2 \ldots i_p}}{\partial x^k} \xi^k \Bigg) dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_p}. \end{aligned} \tag{63}$$

A.6 Fibered Homotopy Operators

In this section, we study differential forms, defined on open star-shaped sets U in an Euclidean space $\mathbf{R}^n$ and on trivial fibered manifolds $U \times V$, where V is an open star-shaped set in $\mathbf{R}^m$. Our aim will be to investigate properties of the exterior derivative operator d on U and on $U \times V$.

First, we consider a differential k-form ρ, where $k \geq 1$, defined on an open star-shaped set $U \subset \mathbf{R}^n$ with center at the origin $0 \in \mathbf{R}^n$. We shall study the equation

$$d\eta = \rho \tag{64}$$

for an unknown $(k-1)$-form η on V. Denote by x^i the canonical coordinates on U. Define a mapping $\chi : [0,1] \times V \to V$ as the restriction of the image of the mapping $(s, x^1, x^2, \ldots, x^n) = (sx^1, sx^2, \ldots, sx^n)$ from $\mathbf{R} \times \mathbf{R}^n$ to $\mathbf{R}^n$ to the open set V; thus, in short

$$\chi(s, x^i) = (sx^i). \tag{65}$$

Then

$$\chi^* dx^i = x^i ds + s dx^i. \tag{66}$$

The pull-back $\chi^*\rho$ is a k-form on a neighborhood of the set $[0,1] \times V$. Obviously, there exists a unique decomposition

$$\chi^*\rho = ds \wedge \rho^{(0)}(s) + \rho'(s), \tag{67}$$

such that the k-forms $\rho^{(0)}(s)$ and $\rho'(s)$ do not contain ds. Note that by formula (66), $\rho'(s)$ arises from ρ by replacing each factor dx^i with sdx^i, and by replacing each coefficient f with $f \circ \chi$. Thus, $\rho'(s)$ obeys

$$\rho'(1) = \rho, \quad \rho'(0) = 0. \tag{68}$$

We set

$$I\rho = \int_0^1 \rho^{(0)}(s), \tag{69}$$

where the expression on the right-hand side means integration of the coefficients in the form $\rho^{(0)}(s)$ over s from 0 to 1.

Lemma 1 *Let U be an open ball in $\mathbf{R}^n$ with center* 0.

(a) *For every differentiable function $f: U \to \mathbf{R}$,*

$$f = Idf + f(0). \tag{70}$$

(b) *Suppose that $k \geq 1$. Then for any differential k-form ρ on U,*

$$\rho = Id\rho + dI\rho. \tag{71}$$

Proof

1. If f is a function, then $df = (\partial f / \partial x^i) dx^i$, and we have by (66) $\chi^* df = ((\partial f / \partial x^i) \circ \chi) \cdot (x^i ds + s dx^i)$. Consequently,

$$Idf = x^i \int_0^1 \left(\frac{\partial f}{\partial x^i} \circ \chi \right) ds. \tag{72}$$

Now (70) follows from the identity

$$f - f(0) = (f \circ \chi)|_{s=1} - (f \circ \chi)|_{s=0} = \int_0^1 \frac{d(f \circ \chi)}{ds} ds$$
$$= x^i \int_0^1 \left(\frac{\partial f}{\partial x^i} \circ \chi \right) ds. \tag{73}$$

2. Let $k = 1$. Then ρ has an expression $\rho = B_i dx^i$, and the pull-back $\chi^* \rho$ is given by $\chi^* \rho = x^i (B_i \circ \chi) ds + (B_i \circ \chi) s dx^i$. Differentiating we get

$$\chi^* d\rho = d\chi^* \rho = ds \wedge \left(-d(x^i (B_i \circ \chi) + \frac{\partial((B_i \circ \chi)s)}{\partial s} dy^i \right)$$
$$+ s \frac{\partial(B_i \circ \chi)}{\partial x^j} dx^j \wedge dx^i, \tag{74}$$

hence

$$I\rho = x^i \int_0^1 B_i \circ \chi \cdot ds. \tag{75}$$

Thus,

$$Id\rho = \int_0^1 \left(\frac{\partial((B_i \circ \chi)s)}{\partial s} - \frac{\partial(x^j \cdot B_j \circ \chi)}{\partial x^i} \right) ds \cdot dx^i, \tag{76}$$

and

$$dI\rho = \int_0^1 \frac{\partial(x^j \cdot B_j \circ \chi)}{\partial x^i} ds \cdot dx^i. \tag{77}$$

Consequently,

$$Id\rho + dI\rho = \int_0^1 \left(\frac{\partial((B_i \circ \chi)s)}{\partial s} \right) ds \cdot dx^i$$
$$= ((B_i \circ \chi \cdot s)|_{s=1} - (B_i \circ \chi \cdot s)|_{s=0}) dx^i = \rho. \tag{78}$$

3. Let $k \geq 2$. Write ρ in the form

$$\rho = dx^i \wedge \Psi_i, \tag{79}$$

and define differential forms $\Psi_i^{(0)}(s)$ and $\Psi'_i(s)$ by

$$\chi^*\Psi_i = ds \wedge \Psi_i^{(0)}(s) + \Psi'_i(s). \tag{80}$$

Then

$$\begin{aligned}\chi^*\rho &= (sdx^i + x^i ds) \wedge (ds \wedge \Psi_i^{(0)}(s) + \Psi'_i(s)) \\ &= ds \wedge (-sdx^i \wedge \Psi_i^{(0)}(s) + y^\sigma \Psi'_i(s)) + sdy^i \wedge \Psi'_i(s).\end{aligned} \tag{81}$$

Thus,

$$I\rho = \int_0^1 (-sdx^i \wedge \Psi_i^{(0)}(s) + x^i \Psi'_i(s)). \tag{82}$$

To determine $Id\rho$, we compute $\chi^* d\rho$. Property $\chi^* d\rho = d\chi^*\rho$ of the pull-back yields

$$\begin{aligned}\chi^* d\rho = -ds \wedge (&sdx^i \wedge d\Psi_i^{(0)}(s) + dx^i \wedge \Psi'_i(s) \\ &+ x^i d\Psi'_i(s)) - dx^i \wedge d(s\Psi'_i(s))) \\ = ds \wedge \Big(&-sdx^i \wedge d\Psi_i^{(0)}(s) - dx^i \wedge \Psi'_i(s) \\ &-x^i d\Psi'_i(s) + dx^i \wedge \frac{\partial(s\Psi'_i(s))}{\partial s}\Big) - dx^i \wedge dx^j \wedge \frac{\partial(s\Psi'_i(s))}{\partial x^j},\end{aligned} \tag{83}$$

where $\partial\eta(s)/\partial s$ denotes the form, arising from $\eta(s)$ by differentiation with respect to s, followed by multiplication by ds. Now by (83) and (69),

$$\begin{aligned}Id\rho = -dx^i \wedge \int_0^1 sd\Psi_i^{(0)}(s) - dx^i \wedge \int_0^1 \Psi'_i(s) \\ - x^i \int_0^1 d\Psi'_i(s) + dx^i \wedge \int_0^1 \frac{\partial(s\Psi'_i(s))}{\partial s}.\end{aligned} \tag{84}$$

It is important to notice that the exterior derivatives $d\Psi_\sigma^{(0)}(s)$, and $d\Psi'_\sigma(s)$ have the meaning of the derivatives with respect to x^i (the terms containing ds are canceled; see the definition of I (67), (69)).
Now, we easily get

$$Id\rho + dI\rho = dx^i \wedge \int_0^1 \frac{\partial(s\Psi'_i(s))}{\partial s}. \tag{85}$$

Remembering that the integral symbol denotes integration of *coefficients* in the corresponding forms with respect to the parameter s from 0 to 1, and using (68), one obtains

$$\begin{aligned} Id\rho + dI\rho &= dx^i \wedge (1 \cdot \Psi_i'(1) - 0 \cdot \Psi_i'(0)) \\ &= dx^i \wedge \Psi_i'(1) = dx^i \wedge \Psi_i = \rho, \end{aligned} \tag{86}$$

as desired. □

As an immediate consequence, we get the following statement.

Lemma 2 (The Volterra–Poincare lemma) *Let U be an open ball in $\mathbf{R}^n$ with center* 0, *ρ a differential k-form on U, where $k \geq 1$. The following two conditions are equivalent:*

(a) *There exists a form η on U such that*

$$d\eta = \rho. \tag{87}$$

(b) *ρ satisfies*

$$d\rho = 0. \tag{88}$$

Proof If $d\eta = \rho$ for some η, we have $d\rho = dd\eta = 0$. Conversely, if $d\rho = 0$, we take $\eta = I\rho$ in Lemma 1. □

Condition (88) is sometimes called *integrability condition* for the differential equation (87).

Now we consider a different kind of differential equations, reducing to (64) for differential forms of sufficiently high degree. Let U be an open set U in $\mathbf{R}^n$, and V an open ball V in $\mathbf{R}^m$ with center at the origin. Denote by k the first Cartesian projection of $U \times V$ onto U. Suppose we are given ρ on $U \times V$, where k is a positive integer. Our objective will be to study the equation

$$d\eta + \pi^* \eta_0 = \rho \tag{89}$$

for the unknowns a $(k-1)$-form η on $U \times V$, and a k-form η_0 on U.

Let (x^i, y^σ), where $1 \leq i \leq n$, $1 \leq \sigma \leq m$, be the canonical coordinates on $U \times V$, and $\zeta \colon U \to U \times V$ be the *zero section* of $U \times V$. Consider the mapping $(s, (x^1, x^2, \ldots, x^n, y^1, y^2, \ldots, y^m)) \to (x^1, x^2, \ldots, x^n, sy^1, sy^2, \ldots, sy^m)$ of $\mathbf{R} \times \mathbf{R}^n \times \mathbf{R}^m$ with values in $\mathbf{R}^n \times \mathbf{R}^m$. Restricting the range of this mapping to $U \times V$, we define a mapping $\chi \colon [0,1] \times U \times V \to U \times V$ by

$$\chi(s, (x^i, y^\sigma)) = (x^i, sy^\sigma). \tag{90}$$

Then

$$\chi^* dx^i = dx^i, \quad \chi^* dy^\sigma = y^\sigma ds + s dy^\sigma. \tag{91}$$

Consider the pull-back $\chi^* \rho$, which is a k-form on a neighborhood of the set $[0, 1] \times U \times V$. There exists a unique decomposition

$$\chi^* \rho = ds \wedge \rho^{(0)}(s) + \rho'(s) \tag{92}$$

such that the k-forms $\rho^{(0)}(s)$ and $\rho'(s)$ do not contain ds. Note that by (91), $\rho'(s)$ arises from ρ by replacing each factor dy^σ with sdy^σ, and by replacing each coefficient f with $f \circ \chi$; the factors dx^i remain unchanged. Thus, $\rho'(s)$ obeys

$$\rho'(1) = \rho, \quad \rho'(0) = \pi^* \zeta^* \rho. \tag{93}$$

We define

$$I\rho = \int_0^1 \rho^{(0)}(s), \tag{94}$$

where the expression on the right-hand side means integration of the coefficients in the form $\rho^{(0)}(s)$ over s from 0 to 1.

Theorem 1 *Let* $U \subset \mathbf{R}^n$ *be an open set, and let* $V \subset \mathbf{R}^m$ *be an open ball with center* 0.

(a) *For every differentiable function* $f: U \times V \to \mathbf{R}$,

$$f = Idf + \pi^* \zeta^* f. \tag{95}$$

(b) *Let* $k \geq 1$. *Then for every differential k-form* ρ *on the Cartesian product* $U \times V$,

$$\rho = Id\rho + dI\rho + \pi^* \zeta^* \rho. \tag{96}$$

Proof

1. We have

$$df = \frac{\partial f}{\partial x^i} dx^i + \frac{\partial f}{\partial y^\sigma} dy^\sigma, \tag{97}$$

and by (91)

$$\chi^* f = \left(\frac{\partial f}{\partial x^i} \circ \chi \right) dx^i + \left(\frac{\partial f}{\partial y^\sigma} \circ \chi \right) (y^\sigma ds + sdy^\sigma). \tag{98}$$

Now the identity

$$\begin{aligned} f - \pi^*\zeta^* f &= f \circ \chi|_{s=1} - f \circ \chi|_{s=0} \\ &= \int_0^1 \frac{d(f \circ \chi)}{ds} ds = y^\sigma \int_0^1 \left(\frac{\partial f}{\partial y^\sigma} \circ \chi \right) ds = Idf, \end{aligned} \tag{99}$$

which follows from (94), gives the result.

2. Let $k = 1$. Then ρ has an expression $\rho = A_i dx^i + B_\sigma dy^\sigma$, thus

$$\begin{aligned} \chi^*\rho &= (A_i \circ \chi) dx^i + (B_\sigma \circ \chi)(s dy^\sigma + y^\sigma ds) \\ &= y^\sigma (B_\sigma \circ \chi) ds + (A_i \circ \chi) dx^i + (B_\sigma \circ \chi) s dy^\sigma, \end{aligned} \tag{100}$$

and

$$\begin{aligned} \chi^* d\rho &= d\chi^*\rho \\ &= ds \wedge \left(-d(y^\sigma (B_\sigma \circ \chi)) + \frac{\partial (A_i \circ \chi)}{\partial s} dx^i + \frac{\partial ((B_\sigma \circ \chi) s)}{\partial s} dy^\sigma \right) \\ &\quad + \left(\frac{\partial (A_i \circ \chi)}{\partial x^j} dx^j + \frac{\partial (A_i \circ \chi)}{\partial y^\nu} dy^\nu \right) \wedge dx^i \\ &\quad + s \left(\frac{\partial (B_\sigma \circ \chi)}{\partial x^j} dx^j + \frac{\partial (B_\sigma \circ \chi)}{\partial y^\nu} dy^\nu \right) \wedge dy^\sigma, \end{aligned} \tag{101}$$

hence

$$I\rho = y^\sigma \int_0^1 B_\sigma \circ \chi \cdot ds, \tag{102}$$

and

$$\begin{aligned} Id\rho &= \int_0^1 \left(\frac{\partial (A_i \circ \chi)}{\partial s} - \frac{\partial (y^\nu \cdot B_\nu \circ \chi)}{\partial x^i} \right) ds \cdot dx^i \\ &\quad + \int_0^1 \left(\frac{\partial ((B_\sigma \circ \chi) s)}{\partial s} - \frac{\partial (y^\nu \cdot B_\nu \circ \chi)}{\partial y^\sigma} \right) ds \cdot dy^\sigma. \end{aligned} \tag{103}$$

We also get

$$dI\rho = y^\sigma \int_0^1 \frac{\partial (B_\sigma \circ \chi)}{\partial x^i} ds \cdot dx^i + \int_0^1 \frac{\partial (y^\nu \cdot B_\nu \circ \chi)}{\partial y^\sigma} ds \cdot dy^\sigma, \tag{104}$$

consequently,

$$\begin{aligned} Id\rho + dI\rho &= A_i \circ \chi|_{s=1} - A_i \circ \chi|_{s=0} + (B_\sigma \circ \chi \cdot s)|_{s=1} - (B_\sigma \circ \chi \cdot s)|_{s=0} \\ &= \rho - \pi^*\zeta^*\rho. \end{aligned} \tag{105}$$

Let $k \geq 2$. Write ρ in the form $\rho = dx^i \wedge \Phi_i + dy^\sigma \wedge \Psi_\sigma$, and define differential forms $\Phi_i^{(0)}(s)$, $\Phi'_i(s)$, $\Psi_\sigma^{(0)}(s)$ by

$$\begin{aligned} \chi^*\Phi_i &= ds \wedge \Phi_i^{(0)}(s) + \Phi'_i(s), \\ \chi^*\Psi_\sigma &= ds \wedge \Psi_\sigma^{(0)}(s) + \Psi'_\sigma(s). \end{aligned} \tag{106}$$

Then

$$\begin{aligned} \chi^*\rho = ds \wedge (&-dx^i \wedge \Phi_i^{(0)}(s) - sdy^\sigma \Psi_\sigma^{(0)}(s) + y^\sigma \Psi'_\sigma(s)) \\ &+ dx^i \wedge \Phi'_i(s) + sdy^\sigma + sy^\sigma \Psi'_\sigma(s)). \end{aligned} \tag{107}$$

Thus,

$$I\rho = -dx^i \wedge \int_0^1 \Phi_i^{(0)}(s) - dy^\sigma \wedge \int_0^1 (s\Psi_\sigma^{(0)}(s) + y^\sigma \Psi'_\sigma(s))ds. \tag{108}$$

To determine $Id\rho$, we compute $\chi^* d\rho$. We get

$$\begin{aligned} \chi^* d\rho &= d\chi^*\rho \\ &= -ds \wedge (dx^i \wedge d\Phi_i^{(0)}(s)) + sdy^\sigma \wedge d\Psi_\sigma^{(0)}(s) + dy^\sigma \wedge \Psi'_\sigma(s) \\ &\quad + y^\sigma d\Psi'_\sigma(s) - dx^i \wedge d\Phi'_i(s) - dy^\sigma \wedge d(s\Psi'_\sigma(s)) \\ &= ds \wedge \left(-dx^i \wedge d\Phi_i^{(0)}(s) + dx^i \wedge \frac{\partial \Phi'_i(s)}{\partial s} - sdy^\sigma \wedge d\Psi_\sigma^{(0)}(s) \right. \\ &\quad \left. - dy^\sigma \wedge \Psi'_\sigma(s) - y^\sigma d\Psi'_\sigma(s) + dy^\sigma \wedge \frac{\partial (s\Psi'_\sigma(s))}{\partial s} \right) \\ &\quad - dx^i \wedge \left(dx^j \wedge \frac{\partial \Phi'_i(s)}{\partial x^j} + dy^\nu \wedge \frac{\partial \Phi'_i(s)}{\partial y^\nu} \right) \\ &\quad - dy^\sigma \wedge \left(dx^j \wedge \frac{\partial (s\Psi'_\sigma(s))}{\partial x^j} + dy^\nu \wedge \frac{\partial (s\Psi'_\sigma(s))}{\partial y^\nu} \right), \end{aligned} \tag{109}$$

where $\partial\eta(s)/\partial s$ denotes the form, arising by differentiation of $\eta(s)$ with respect to s, followed by multiplication by ds. Now by (108) and (93),

$$\begin{aligned} Id\rho = -dx^i \wedge \int_0^1 d\Phi_i^{(0)}(s) - dy^\sigma \wedge \int_0^1 sd\Psi_\sigma^{(0)}(s) - dy^\sigma \wedge \int_0^1 \Psi'_\sigma(s) \\ - y^\sigma \int_0^1 d\Psi'_\sigma(s) + dx^i \wedge \int_0^1 \frac{\partial\Phi'_i(s)}{\partial s} + dy^\sigma \wedge \int_0^1 \frac{\partial(s\Psi'_\sigma(s))}{\partial s}. \end{aligned} \tag{110}$$

Note that the expressions $d\Phi_i^{(0)}(s)$, $d\Psi_\sigma^{(0)}(s)$, and $d\Psi'_\sigma(s)$ have the meaning of the exterior derivatives with respect to x^i, y^σ (the terms containing ds are canceled; see the definition of I (93), (94)).
Now

$$Id\rho + dI\rho = dx^i \wedge \int_0^1 \frac{\partial\Phi'_i(s)}{\partial s} + dy^\sigma \wedge \int_0^1 \frac{\partial(s\Psi'_\sigma(s))}{\partial s}, \tag{111}$$

and using formula (93),

$$\begin{aligned} Id\rho + dI\rho &= dx^i \wedge (\Phi'_i(1) - \Phi'_i(0)) + dy^\sigma \wedge (1 \cdot \Psi'_\sigma(1) - 0 \cdot \Psi'_\sigma(0)) \\ &= dx^i \wedge \Phi'_i(1) + dy^\sigma \wedge \Psi'_\sigma(1) - dx^i \wedge \Phi'_i(0) \\ &= dx^i \wedge \Phi_i + dy^\sigma \wedge \Psi_\sigma - dx^i \wedge \pi^*\zeta^*\Phi_i \\ &= \rho - \pi^*\zeta^*\rho. \end{aligned} \tag{112}$$

□

As a consequence, we have the following statement.

Theorem 2 (The fibered Volterra–Poincare lemma) *Let $U \subset \mathbf{R}^n$ be an open set, $V \subset \mathbf{R}^m$ an open ball with center 0. Let $k \geq 1$ and let ρ be a differential k-form on $U \times V$. The following two conditions are equivalent:*

(a) *There exist a $(k-1)$-form η on $U \times V$ and a k-form η_0 on U such that*

$$d\eta + \pi^*\eta_0 = \rho. \tag{113}$$

(b) *The form $d\rho$ is π-projectable and its π-projection is $d\eta_0$.*

Proof Suppose we have some forms η and η_0 satisfying condition (a). Then $d\rho = d\pi^*\eta_0 = \pi^*d\eta_0$ proving (b). □

Conversely, if $d\rho$ is π-projectable, then by the definition of I, $Id\rho = 0$, and then by Theorem 1, $\rho = Id\rho + dI\rho + \pi^*\zeta^*\rho = d\eta + \pi^*\eta_0$ proving (a).

We also get two assertions on *projectability* of forms, and non-uniqueness of solutions of equation (89).

Corollary 1 *Let $U \subset \mathbf{R}^n$ be an open set, $V \subset \mathbf{R}^m$ an open ball with center the origin* 0*, ρ a differential form on $U \times V$. The following two conditions are equivalent:*

1. *There exists a form η on U such that $\rho = \pi^*\eta$.*
2. *$Id\rho + dI\rho = 0$.*

Proof This follows from Theorem 1. □

Corollary 2 *Suppose that the form $d\rho$ is π-projectable. Let (η, η_0) and $(\tilde{\eta}, \tilde{\eta}_0)$ be two solutions of* equation (89). *Then there exist a $(p-1)$-form τ on $U \times V$ and a $(p-1)$-form χ on U such that*

$$\tilde{\eta} = \eta + \pi^*\chi + d\tau, \quad \tilde{\eta}_0 = \eta_0 - d\chi. \tag{114}$$

Proof By hypothesis,

$$d\eta + \pi^*\eta_0 = \rho, \quad d\tilde{\eta} + \pi^*\tilde{\eta}_0 = \rho. \tag{115}$$

These equations imply $d\eta + \pi^*\eta_0 = d\tilde{\eta} + \pi^*\tilde{\eta}_0$ hence $\pi^*d\eta_0 = \pi^*d\tilde{\eta}_0$. But for any section δ of the projection π,

$$\delta^*\pi^*d\eta_0 = d\eta_0 = \delta^*\pi^*d\tilde{\eta}_0 = d\tilde{\eta}_0. \tag{116}$$

Thus, by the Volterra–Poincaré lemma, $\tilde{\eta}_0 - \eta_0 = d\chi$ for some χ. Then, however, $d\eta + \pi^*\eta_0 = d\tilde{\eta} + \pi^*(\eta_0 + d\chi)$ and

$$d(\eta - \tilde{\eta} - \pi^*\chi) = 0. \tag{117}$$

Applying the Volterra–Poincaré lemma again, we get (114). □

Remark 1 (The Volterra–Poincare lemma on manifolds) Let X be an n-dimensional manifold. Every point $x \in X$ has a neighborhood U such that the decomposition of forms, given in Theorem 1, is defined on U. Indeed, if (U, φ) is a chart at x such that $\varphi(U)$ is an open ball with center $0 \in \mathbf{R}^n$, then formulas $\rho = \varphi^*\mu$ and $(\varphi^{-1})^*\rho = \mu$ establish a bijective correspondence between forms on U and $\varphi(U)$, commuting with the exterior derivative d. In general, this correspondence does not provide a construction of solutions of differential equations (64) and (89), defined *globally* on X.

Remark 2 For k-forms ρ such that $k = n$, always $d\eta_0 = 0$ hence $\eta_0 = d\tau$ and equation $d\eta + \pi^*\eta_0 = \rho$ (89) reduces to $d\eta = \rho$ (64). The same is true for $k > n$ because in this case $\eta_0 = 0$.

Turning back to the definition of the fibered homotopy operator I (94), we have the following explicit assertion.

Lemma 3 *Let ρ be a differential k-form on the product of open sets $U \times V$, considered as a fibered manifold over U, expressed in the canonical coordinates (x^i, y^σ) on $U \times V$ as*

$$\rho = \frac{1}{p!} A_{\sigma_1\sigma_2\ldots\sigma_p\, i_1 i_2 \ldots i_q} dy^{\sigma_1} \wedge dy^{\sigma_2} \wedge \cdots \wedge dy^{\sigma_p} \wedge dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_p}, \tag{118}$$

where $k = p + q$. Then the fibered homotopy operator I is given by

$$\begin{aligned} I\rho = y^\sigma \int_0^1 A_{\sigma\sigma_1\sigma_2\ldots\sigma_{p-1}\, i_1 i_2 \ldots i_q}(x^j, sy^\nu) s^{p-1} ds \cdot dy^{\sigma_1} \wedge dy^{\sigma_2} \wedge \cdots \wedge dy^{\sigma_{p-1}} \\ \wedge dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_p}. \end{aligned} \tag{119}$$

I satisfies

$$I^2 \rho = 0. \tag{120}$$

Proof The homotopy $(x^i, y^\sigma) \to \chi(s, (x^i, y^\sigma)) = (x^i, sy^\sigma)$ yields

$$\begin{aligned} \chi^* \rho = \frac{1}{p!} (p y^\sigma (A_{\sigma\sigma_1\sigma_2\ldots\sigma_{p-1}\, i_1 i_2 \ldots i_q} \circ \chi) s^{p-1} ds \wedge dy^{\sigma_1} \wedge dy^{\sigma_2} \wedge \cdots \wedge dy^{\sigma_{p-1}} \\ + (A_{\sigma_1\sigma_2\ldots\sigma_p\, i_1 i_2 \ldots i_q} \circ \chi) s^p dy^{\sigma_1} \wedge dy^{\sigma_2} \wedge \cdots \wedge dy^{\sigma_p}) \\ \wedge dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_p} \end{aligned} \tag{121}$$

which implies that

$$\begin{aligned} I\rho &= (A_{\sigma_1\sigma_2\ldots\sigma_p\, i_1 i_2 \ldots i_q} \circ \chi) \chi^* (dy^{\sigma_1} \wedge dy^{\sigma_2} \wedge \cdots \wedge dy^{\sigma_p}) \\ &= y^\sigma \int_0^1 (A_{\sigma\sigma_1\sigma_2\ldots\sigma_{p-1}\, i_1 i_2 \ldots i_q} \circ \chi) s^{p-1} ds \cdot dy^{\sigma_1} \wedge dy^{\sigma_2} \wedge \cdots \wedge dy^{\sigma_{p-1}} \\ &\quad \wedge dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_p}. \end{aligned} \tag{122}$$

Identity (120) is now an immediate consequence of formula (119). □

A.7 Differential Ideals

For basic concepts of the theory of differential ideals and related topics, we refer to Bryant et al. [Br].

Let X be an n-dimensional smooth manifold. We denote by $\Lambda^p TX$ the bundle of alternating p-forms over X; in this notation, $\Lambda^1 TX = T^*X$ is the *cotangent bundle* of X. Sections of the bundle $\Lambda^p TX$, *differential p-forms* on X, form a *module* over the ring of functions, denoted by $\Omega_p X$. The direct sum

$$\Omega X = \Omega_0 X \oplus \Omega_1 X \oplus \Omega_2 X \oplus \cdots \oplus \Omega_n X \tag{123}$$

together with the exterior multiplication of forms is the *exterior algebra* of X. We usually consider elements of $\Omega_p X$ as elements of ΩX. The multiplication $\wedge$ in ΩX is *associative* and *distributive*, but *not* commutative; instead we have for any $\eta \in \Omega_p X$ and $\rho \in \Omega_q X$,

$$\eta \wedge \rho = (-1)^{pq} \rho \wedge \eta. \tag{124}$$

A subset $\Theta \subset \Omega X$ is called an *ideal*, if the following two conditions are satisfied:

(a) Θ is a subgroup of the additive group of ΩX.
(b) If $\eta \in \Theta$ and $\rho \in \Omega X$ then $\eta \wedge \rho \in \Theta$.

An ideal $\Theta \subset \Omega X$ is called a *differential ideal*, if for any $\eta \in \Theta$ also $d\eta \in \Theta$; thus, a differential ideal is an ideal *closed* under exterior derivative operation.

Any non-empty set $\theta \subset \Omega X$ generates a subgroup Θ_θ of the additive group of ΩX, formed by (finite) sums

$$\mu = \sum \eta_k \wedge \rho_k, \tag{125}$$

where $\eta_k \in \theta$ and $\rho_k \in \Omega X$. Θ_θ is an ideal, which is a subset of *any* ideal containing θ; it is said to be *generated* by the set θ (or by the *generators* $\eta \in \theta$). If the set θ is *finite*, we say that Θ_θ is *finitely generated.*

Let $\mathscr{V}X$ denote the module of vector fields on X. We denote

$$\mathscr{A}(\Theta) = \{\xi \in \mathscr{V}X | i_\xi \eta \subset \Theta, \eta \in \Theta\}. \tag{126}$$

This set, the *Cauchy characteristic space* of Θ, has the structure of a subgroup of the additive group of $\mathscr{V}X$. The annihilator

$$\mathscr{C}(\Theta) = \{\mu \in \Omega X | i_\xi \mu = 0, \xi \in \mathscr{A}(\Theta)\} \tag{127}$$

is the *retracting subspace* of Θ.

A.8 The Levi-Civita Symbol

We introduce in this appendix a real-valued function, defined on the symmetric group $\tau \in S_n$, the *Levi-Civita symbol*, playing an essential role in algebraic computations with skew-symmetric expressions. We also derive basic computation formulas for the Levi-Civita symbol, needed in this book.

Any permutation $\tau \in S_n$ can be written as the composition of transpositions τ_k, that is $\tau = \tau_M \circ \tau_{M-1} \circ \cdots \circ \tau_2 \circ \tau_1$. This decomposition of τ is not unique, but the number $\operatorname{sgn} \tau = (-1)^M$, the *sign* of the permutation τ, is independent of the choice of the decomposition. If $\operatorname{sgn} \tau = 1$ (resp. $\operatorname{sgn} \tau = -1$), the permutation τ is called *even* (resp. *odd*). The function $S_k \ni \tau \to \operatorname{sgn} \tau \in \{1, -1\}$ is sometimes called the *sign function*. As an immediate consequence of the definition, we have

$$\operatorname{sgn}(\nu \cdot \tau) = \operatorname{sgn} \nu \cdot \operatorname{sgn} \tau \tag{128}$$

for all permutations $\nu, \tau \in S_r$.

The *sign function* $\tau \to \operatorname{sgn} \tau$ can be considered as a function on the set of *distinct* n-tuples $(i_1, i_2, \ldots, i_n)$ of integers, such that $1 \le i_1, i_2, \ldots, i_n \le n$. We define the *Levi-Civita,* or *permutation symbol* $\varepsilon_{i_1 i_2 \ldots i_n}$ setting $\varepsilon_{i_1 i_2 \ldots i_n} = 1$ if the n-tuple $(i_1, i_2, \ldots, i_n)$ is an even permutation of $(1, 2, \ldots, n)$, $\varepsilon_{i_1 i_2 \ldots i_n} = -1$ if $(i_1, i_2, \ldots, i_n)$ is an odd permutation of $(1, 2, \ldots, n)$, and $\varepsilon_{i_1 i_2 \ldots i_n} = 0$ whenever at least two of the indices coincide. Clearly,

$$\varepsilon_{i_1 i_2 \ldots i_n} = \sum_{\tau \in S_n} \operatorname{sgn} \tau \cdot \delta^1_{i_{\tau(1)}} \delta^2_{i_{\tau(2)}} \ldots \delta^n_{i_{\tau(n)}}. \tag{129}$$

Sometimes it is convenient to express this formula in a different form, without explicit mentioning the permutations τ. To this purpose, we introduce the *alternation operation* in the indices $(i_1, i_2, \ldots, i_n)$, denoted $\operatorname{Alt}(i_1 i_2 \ldots i_n)$, by

$$\frac{1}{n!} \sum_{\tau \in S_n} \operatorname{sgn} \tau \cdot \delta^1_{i_{\tau(1)}} \delta^2_{i_{\tau(2)}} \ldots \delta^n_{i_{\tau(n)}} = \delta^1_{i_1} \delta^2_{i_2} \ldots \delta^n_{i_n} \quad \operatorname{Alt}(i_1 i_2 \ldots i_n). \tag{130}$$

It is understood in this formula that the operator $\operatorname{Alt}(i_1 i_2 \ldots i_n)$ is applied to the right-hand side expression, and represents explicit expression on the left-hand side. From (130) we get, in particular,

$$\varepsilon_{i_1 i_2 \ldots i_n} = n! \delta^1_{i_{\tau(1)}} \delta^2_{i_{\tau(2)}} \ldots \delta^n_{i_{\tau(n)}} \quad \operatorname{Alt}(i_1 i_2 \ldots i_n). \tag{131}$$

Formula (131) indicates that the Levi-Civita symbols $\varepsilon_{i_1 i_2 \ldots i_n}$ and $\varepsilon^{i_1 i_2 \ldots i_n}$ can be expressed by means of determinants. We have

$$\varepsilon_{i_1 i_2 \ldots i_n} = \begin{vmatrix} \delta^1_{i_1} & \delta^2_{i_1} & \ldots & \delta^n_{i_1} \\ \delta^1_{i_2} & \delta^2_{i_2} & \ldots & \delta^n_{i_2} \\ \ldots \\ \delta^1_{i_n} & \delta^2_{i_n} & \ldots & \delta^n_{i_n} \end{vmatrix}, \quad \varepsilon^{i_1 i_2 \ldots i_n} = \begin{vmatrix} \delta^{i_1}_1 & \delta^{i_2}_1 & \ldots & \delta^{i_n}_1 \\ \delta^{i_1}_2 & \delta^{i_2}_2 & \ldots & \delta^{i_n}_2 \\ \ldots \\ \delta^{i_1}_n & \delta^{i_2}_n & \ldots & \delta^{i_n}_n \end{vmatrix}. \tag{132}$$

Clearly, multiplying these determinants, we get

$$\begin{aligned} \varepsilon_{i_1 i_2 \ldots i_n} \varepsilon^{j_1 j_2 \ldots j_n} &= \begin{vmatrix} \delta^1_{i_1} & \delta^2_{i_1} & \ldots & \delta^n_{i_1} \\ \delta^1_{i_2} & \delta^2_{i_2} & \ldots & \delta^n_{i_2} \\ \ldots \\ \delta^1_{i_n} & \delta^2_{i_n} & \ldots & \delta^n_{i_n} \end{vmatrix} \begin{vmatrix} \delta^{j_1}_1 & \delta^{j_2}_1 & \ldots & \delta^{j_n}_1 \\ \delta^{j_1}_2 & \delta^{j_2}_2 & \ldots & \delta^{j_n}_2 \\ \ldots \\ \delta^{j_1}_n & \delta^{j_2}_n & \ldots & \delta^{j_n}_n \end{vmatrix} \\ &= \begin{vmatrix} \delta^{j_1}_{i_1} & \delta^{j_2}_{i_1} & \ldots & \delta^{j_n}_{i_1} \\ \delta^{j_1}_{i_2} & \delta^{j_2}_{i_2} & \ldots & \delta^{j_n}_{i_2} \\ \ldots \\ \delta^{j_1}_{i_n} & \delta^{j_2}_{i_n} & \ldots & \delta^{j_n}_{i_n} \end{vmatrix} = n! \delta^{j_1}_{i_1} \delta^{j_2}_{i_2} \ldots \delta^{j_n}_{i_n} \quad \operatorname{Alt}(i_1 i_2 \ldots i_n). \end{aligned} \tag{133}$$

Lemma 1

(a) *For every k such that* $1 \leq k \leq n$,

$$\varepsilon_{i_1 i_2 \ldots i_k s_{k+1} s_{k+2} \ldots s_n} \varepsilon^{j_1 j_2 \ldots j_k s_{k+1} s_{k+2} \ldots s_n} = k!(n-k)! \delta^{j_1}_{i_1} \delta^{j_2}_{i_2} \ldots \delta^{j_k}_{i_k} \quad \mathrm{Alt}(i_1 i_2 \ldots i_k). \tag{134}$$

(b) *For every k such that* $0 \leq k \leq s \leq n$,

$$\frac{1}{s!} \binom{n-k}{n-s} \varepsilon_{j_1 j_2 \ldots j_k j_{k+1} j_{k+2} \ldots j_s i_{s+1} i_{s+2} \ldots i_n} \delta^{l_{k+1}}_{i_{k+1}} \delta^{l_{k+2}}_{i_{k+2}} \ldots \delta^{l_s}_{i_s} \quad \mathrm{Alt}(i_{k+1} i_{k+2} \ldots i_s i_{s+1} i_{s+2} \ldots i_n)$$
$$= \frac{1}{k!(s-k)!} \delta^{l_{k+1}}_{j_{k+1}} \delta^{l_{k+2}}_{j_{k+2}} \ldots \delta^{l_s}_{j_s} \varepsilon_{j_1 j_2 \ldots j_k i_{k+1} i_{k+2} \ldots i_s i_{s+1} i_{s+2} \ldots i_n}$$
$$\mathrm{Alt}(j_1 j_2 \ldots j_k j_{k+1} j_{k+2} \ldots j_s). \tag{135}$$

Proof

1. Setting

$$\Delta^{j_1 j_2 \ldots j_l}_{i_1 i_2 \ldots i_l} = \delta^{j_1}_{i_1} \delta^{j_2}_{i_2} \ldots \delta^{j_l}_{i_l} \quad \mathrm{Alt}(i_1 i_2 \ldots i_l), \tag{136}$$

we have

$$\begin{aligned} \Delta^{j_1 j_2 \ldots j_l}_{i_1 i_2 \ldots i_l} &= \delta^{j_1}_{i_1} \delta^{j_2}_{i_2} \ldots \delta^{j_{l-1}}_{i_{l-1}} \delta^{j_l}_{i_l} \quad \mathrm{Alt}(i_1 i_2 \ldots i_{l-1}) \quad \mathrm{Alt}(i_1 i_2 \ldots i_l) \\ &= \Delta^{j_1 j_2 \ldots j_{l-1}}_{i_1 i_2 \ldots i_{l-1}} \delta^{j_l}_{i_l} \quad \mathrm{Alt}(i_1 i_2 \ldots i_l) \\ &= \frac{1}{l} \left(\Delta^{j_1 j_2 \ldots j_{l-1}}_{i_1 i_2 \ldots i_{l-1}} \delta^{j_l}_{i_l} - \Delta^{j_1 j_2 \ldots j_{l-1}}_{i_l i_2 i_3 \ldots i_{l-1}} \delta^{j_l}_{i_1} - \Delta^{j_1 j_2 \ldots j_{l-1}}_{i_1 i_l i_3 i_4 \ldots i_{l-1}} \delta^{j_l}_{i_2} \right. \\ &\quad \left. - \cdots - \Delta^{j_1 j_2 \ldots j_{l-1}}_{i_1 i_2 \ldots i_{l-3} i_l i_{l-1}} \delta^{j_l}_{i_{l-2}} - \Delta^{j_1 j_2 \ldots j_{l-1}}_{i_1 i_2 \ldots i_{l-3} i_{l-2} i_l} \delta^{j_l}_{i_{l-1}} \right). \end{aligned} \tag{137}$$

Note that contracting this expression, we obtain

$$\Delta^{i_1 i_2 \ldots i_{l-1} s}_{i_1 i_2 \ldots i_{l-1} s} = \frac{n-l+1}{l} \Delta^{i_1 i_2 \ldots i_{l-1}}_{i_1 i_2 \ldots i_{l-1}}. \tag{138}$$

Now formula (133) can be written in the form

$$\varepsilon_{i_1 i_2 \ldots i_n} \varepsilon^{j_1 j_2 \ldots j_n} = n! \Delta^{j_1 j_2 \ldots j_n}_{i_1 i_2 \ldots i_n}. \tag{139}$$

Contracting (139) in one pair of indices, we get

$$\varepsilon_{i_1 i_2 \ldots i_{n-1} s} \varepsilon^{j_1 j_2 \ldots j_{n-1} s} = n! \Delta^{j_1 j_2 \ldots j_{n-1} s}_{i_1 i_2 \ldots i_{n-1} s} = (n-1)! 1! \Delta^{j_1 j_2 \ldots j_{n-1}}_{i_1 i_2 \ldots i_{n-1}}, \tag{140}$$

proving (134) for $k = 1$. After $n - k$ contractions, we obtain

$$\Delta^{j_1 j_2 \ldots j_k s_{k+1} s_{k+2} \ldots s_n}_{i_1 i_2 \ldots i_k s_{k+1} s_{k+2} \ldots s_n} = \frac{1}{(n-n+k)!}\Delta^{j_1 j_2 \ldots j_k}_{i_1 i_2 \ldots i_k} = \frac{1}{k!}\Delta^{j_1 j_2 \ldots j_k}_{i_1 i_2 \ldots i_k}, \tag{141}$$

which leads to (134).

2. To prove formula (135), consider the tensors

$$\begin{aligned}&\binom{n-k}{n-s}\frac{1}{s!}\varepsilon_{j_1 j_2 \ldots j_k j_{k+1} j_{k+2} \ldots j_s i_{s+1} i_{s+2} \ldots i_n}\delta^{l_{k+1}}_{i_{k+1}}\delta^{l_{k+2}}_{i_{k+2}}\ldots\delta^{l_s}_{i_s}\\ &\quad \mathrm{Alt}(i_{k+1} i_{k+2} \ldots i_s i_{s+1} i_{s+2} \ldots i_n)\end{aligned} \tag{142}$$

and

$$\begin{aligned}&\frac{1}{k!(s-k)!}\varepsilon_{j_1 j_2 \ldots j_k i_{k+1} i_{k+2} \ldots i_s i_{s+1} i_{s+2} \ldots i_n}\delta^{l_{k+1}}_{j_{k+1}}\delta^{l_{k+2}}_{j_{k+2}}\ldots\delta^{l_s}_{j_s}\\ &\quad \mathrm{Alt}(j_1 j_2 \ldots j_k j_{k+1} j_{k+2} \ldots j_s).\end{aligned} \tag{143}$$

Suppose that the component (142) is different from 0. Then

(a) the set $\{i_{k+1}, i_{k+2}, \ldots, i_s, i_{s+1}, i_{s+2}, \ldots, i_n\}$ consists of distinct elements,
(b) the set $\{j_1, j_2, \ldots, j_k, j_{k+1}, j_{k+2}, \ldots, j_s\}$ consists of distinct elements,
(c) the set $\{l_{k+1}, l_{k+2}, \ldots, l_s\}$ satisfies

$$\begin{aligned}&\{i_{k+1}, i_{k+2}, \ldots, i_s, i_{s+1}, i_{s+2}, \ldots, i_n\} \cap \{j_1, j_2, \ldots, j_k, j_{k+1}, j_{k+2}, \ldots, j_s\}\\ &\quad = \{l_{k+1}, l_{k+2}, \ldots, l_s\}.\end{aligned} \tag{144}$$

Take $j_{k+1} = l_{k+1}$, $j_{k+2} = l_{k+2}$, $\ldots$, $j_s = l_s$. Then (142) reduces to

$$\begin{aligned}&\binom{n-k}{n-s}\frac{1}{s!}\varepsilon_{j_1 j_2 \ldots j_k l_{k+1} l_{k+2} \ldots {}_s i_{s+1} i_{s+2} \ldots i_n}\delta^{l_{k+1}}_{i_{k+1}}\delta^{l_{k+2}}_{i_{k+2}}\ldots\delta^{l_s}_{i_s}\\ &\quad \mathrm{Alt}(i_{k+1} i_{k+2} \ldots i_s i_{s+1} i_{s+2} \ldots i_n).\end{aligned} \tag{145}$$

There exist exactly one $(s-k)$-tuple in the set $i_{k+1}, i_{k+2}, \ldots, i_s, i_{s+1}, i_{s+2}, \ldots, i_n$, say $i_{k+1}, i_{k+2}, \ldots, i_s$ such that $\delta^{l_{k+1}}_{i_{k+1}}\delta^{l_{k+2}}_{i_{k+2}}\ldots\ \delta^{l_s}_{i_s} = 1$. Then

$$i_{k+1} = j_{k+1} = l_{k+1}, \quad i_{k+2} = j_{k+2} = l_{k+2}, \ldots, i_s = j_s = l_s, \tag{146}$$

and (146) gives the expression

$$\frac{(n-s)!}{(n-k)!}\binom{n-k}{n-s}\frac{1}{s!}\varepsilon_{j_1 j_2 \ldots j_k l_{k+1} l_{k+2} \ldots {}_s i_{s+1} i_{s+2} \ldots i_n} = \frac{1}{s!(s-k)!}\varepsilon_{j_1 j_2 \ldots j_k l_{k+1} l_{k+2} \ldots {}_s i_{s+1} i_{s+2} \ldots i_n}. \tag{147}$$

Compute now (143) for the same indices, satisfying conditions (147). We get

$$\begin{aligned}&\frac{1}{k!(s-k)!}\frac{k!}{s!}\delta^{l_{k+1}}_{j_{k+1}}\delta^{l_{k+2}}_{j_{k+2}}\ldots\delta^{l_s}_{j_s}\varepsilon_{j_1j_2\ldots j_kl_{k+1}l_{k+2}\ldots l_si_{s+1}i_{s+2}\ldots i_n}\\&\quad=\frac{1}{s!(s-k)!}\varepsilon_{j_1j_2\ldots j_kl_{k+1}l_{k+2}\ldots l_si_{s+1}i_{s+2}\ldots i_n}.\end{aligned}\tag{148}$$

This shows that if the component (142) is different from 0, then also the component (143) is different from 0, and is equal to (142).
Conversely, if (143) is different from 0, then

$$\begin{aligned}&\frac{1}{k!(s-k)!}\delta^{l_{k+1}}_{j_{k+1}}\delta^{l_{k+2}}_{j_{k+2}}\ldots\delta^{l_s}_{j_s}\varepsilon_{j_1j_2\ldots j_ki_{k+1}i_{k+2}\ldots i_si_{s+1}i_{s+2}\ldots i_n}\\&\quad\mathrm{Alt}(j_1j_2\ldots j_kj_{k+1}j_{k+2}\ldots j_s),\end{aligned}\tag{149}$$

we obtain again conditions (a), (b), and (c). □

Corollary 1 *If* $k = n$, (134) *coincides with* (133). *If* $k = 0$, *we have*

$$\varepsilon_{s_1s_2\ldots s_n}\varepsilon^{s_1s_2\ldots s_n} = n!.\tag{150}$$

Corollary 2 (Bases of forms) *Let X be an n-dimensional smooth manifold, and let* (U,φ), $\varphi = (x^i)$, *be a chart on X. Then the forms*

$$\begin{aligned}\omega_0 &= \frac{1}{n!}\varepsilon_{i_1i_2\ldots i_n}dx^{i_1}\wedge dx^{i_2}\wedge\cdots\wedge dx^{i_n},\\ \omega_{k_1k_2\ldots k_p} &= \frac{1}{(n-p)!}\varepsilon_{k_1k_2\ldots k_{p-1}k_pi_{p+1}i_{p+2}\ldots i_n}dx^{i_{p+1}}\wedge dx^{i_{p+2}}\wedge\cdots\wedge dx^{i_n},\\ &\quad 1\le p\le n-1,\end{aligned}\tag{151}$$

define bases of n-forms, $(n-1)$*-forms, ...,* 2*-forms, and* 1*-forms, respectively. The inverse transformation formulas are*

$$\begin{aligned}&\varepsilon^{l_1l_2\ldots l_n}\omega_0 = dx^{l_1}\wedge dx^{l_2}\wedge\cdots\wedge dx^{l_n},\\ &\varepsilon^{k_1k_2\ldots k_pl_{p+1}l_{p+2}\ldots l_n}\omega_{k_1k_2\ldots k_p} = dx^{l_{p+1}}\wedge dx^{l_{p+2}}\wedge\cdots\wedge dx^{l_n},\\ &\quad 1\le p\le n\end{aligned}\tag{152}$$

Proof Immediate: The forms (151) are defined by

$$\begin{aligned}&\omega_0 = dx^1\wedge dx^2\wedge\cdots\wedge dx^n,\quad \omega_{k_1} = i_{\partial/\partial x^{k_1}}\omega_0,\quad \omega_{k_1k_2} = i_{\partial/\partial x^{k_2}}\omega_{k_1},\\ &\quad\ldots,\quad \omega_{k_1k_2\ldots k_p} = i_{\partial/\partial x^{k_p}}\omega_{k_1k_2\ldots k_{p-1}},\quad\ldots,\quad \omega_{k_1k_2\ldots k_{n-1}} = i_{\partial/\partial x^{k_{n-1}}}\omega_{k_1k_2\ldots k_{n-2}},\end{aligned}\tag{153}$$

and are linearly independent. □

A.9 The Trace Decomposition

This appendix is devoted to specific algebraic methods, used in the decomposition theory of differential forms on jet manifolds. To this purpose we present elementary trace decomposition formulas and their proofs (Krupka [K15]).

Beside the usual index notation, we also use multi-indices of the form $I = (i_1 i_2 \ldots i_k)$, where r and n are positive integers, $k = 0, 1, 2, \ldots, r$, and $1 \le i_1, i_2, \ldots, i_k \le n$. The number k is called the *length* of I and is denoted by $|I|$. For any index j, such that $1 \le j \le n$, we denote by Ij the multi-index $(i_1 i_2 \ldots i_k j)$. The symbol $\mathrm{Alt}(i_1 i_2 \ldots i_k)$ (respectively, $\mathrm{Sym}(i_1 i_2 \ldots i_k)$) denotes *alternation* (respectively, *symmetrisation*) in the indices i_1, i_2, …, i_k.

Let E be an n-dimensional vector space, E^* its dual vector space, and let r and s be two non-negative integers; suppose that at least one of these integers is non-zero. Then by a *tensor of type* (r, s) over E, we mean a multilinear mapping $U\colon E^* \times E^* \times \cdots \times E^* \times E \times E \times \cdots \times E \to \mathbf{R}$ (r factors E^*, s factors E); r (respectively, s) is called the *contravariant* (respectively, *covariant*) *degree* of U. A tensor of type $(r, 0)$ (respectively, $(0, s)$) is called *contravariant* (*covariant*) of degree r (respectively, s). The set of tensors of type (r, s) considered with its natural real vector space structure is called the *tensor space of type* (r, s) over E, and is denoted by $T^r_s E$.

Let e_i be a basis of the vector space E, e^i the dual basis of E^*. The tensors $\mathrm{e}_{j_1} \otimes \mathrm{e}_{j_2} \otimes \cdots \otimes \mathrm{e}_{j_r} \otimes \mathrm{e}^{i_1} \otimes \mathrm{e}^{i_2} \otimes \cdots \otimes \mathrm{e}^{i_s}$, $1 \le j_1, j_2, \ldots, j_r, i_1, i_2, \ldots, i_s \le n$, form a *basis* of the vector space $T^r_s E$. Each tensor $U \in T^r_s E$ has a unique expression

$$U = U^{j_1 j_2 \ldots j_r}{}_{i_1 i_2 \ldots i_s} \mathrm{e}_{j_1} \otimes \mathrm{e}_{j_2} \otimes \cdots \otimes \mathrm{e}_{j_r} \otimes \mathrm{e}^{i_1} \otimes \mathrm{e}^{i_2} \otimes \cdots \otimes \mathrm{e}^{i_s}, \tag{154}$$

where the numbers $U^{i_1 i_2 \ldots i_r}{}_{i_1 i_2 \ldots i_s}$ are the *components* of U in the basis e_i.

Remark 1 If a basis of the vector space E is fixed, it is sometimes convenient to denote the tensors simply by their components; in this case, a tensor U of type (r, s) over E is usually written as

$$U = U^{j_1 j_2 \ldots j_r}{}_{i_1 i_2 \ldots i_s}. \tag{155}$$

Remark 2 The *canonical basis* of the vector space $E = \mathbf{R}^n$ consists of the vectors $\mathrm{e}_1 = (1, 0, 0, \ldots, 0)$, $\mathrm{e}_2 = (0, 1, 0, 0, \ldots, 0)$, …, $\mathrm{e}_n = (0, 0, \ldots, 0, 1)$. The basis of the tensor space $T^r_s \mathbf{R}^n$ associated with $(\mathrm{e}_1, \mathrm{e}_2, \ldots, \mathrm{e}_n)$ is also called *canonical*. A tensor $U \in T^r_s \mathbf{R}^n$ can be expressed either by formula (154) or by (155); these formulas define the *canonical identification* of the vector space $T^r_s \mathbf{R}^n$ with the vector space $\mathbf{R}^N$ of the collections $U = U^{j_1 j_2 \ldots j_r}{}_{i_1 i_2 \ldots i_s}$, where $N = \dim T^r_s \mathbf{R}^n = n^{rs}$.

Remark 3 The *transformation equations* for the associated bases in $T^r_s E$ are easily derived from the transformation equations for bases of the vector space E. Suppose, we have two bases e_i and $\bar{\mathrm{e}}_i$ of E. Let $\bar{\mathrm{e}}_i = A^p_i \mathrm{e}_p$ and $\bar{\mathrm{e}}^i = B^i_p \mathrm{e}^p$ be the corresponding transformation equations. Then

$$A_i^q B_p^i = \delta_p^q, \tag{156}$$

where δ_q^p is the *Kronecker symbol*, $\delta_p^p = 1$ and $\delta_q^p = 0$ if $p \neq q$, and

$$\begin{aligned}&\bar{\mathrm{e}}_{j_1} \otimes \bar{\mathrm{e}}_{j_2} \otimes \cdots \otimes \bar{\mathrm{e}}_{j_r} \otimes \bar{\mathrm{e}}^{i_1} \otimes \bar{\mathrm{e}}^{i_2} \otimes \cdots \otimes \bar{\mathrm{e}}^{i_s}\\ &\quad = A_{j_1}^{p_1} A_{j_2}^{p_2} \ldots A_{j_r}^{p_r} B_{q_1}^{i_1} B_{q_2}^{i_2} \ldots B_{q_s}^{i_s} e_{p_1} \otimes e_{p_2} \otimes \cdots \otimes e_{p_r} \otimes e^{q_1} \otimes e^{q_2} \otimes \cdots \otimes e^{q_s}.\end{aligned} \tag{157}$$

Expressing a tensor $U \in T_s^r E$ as in (154), we have

$$\begin{aligned} U &= \bar{U}^{j_1 j_2 \ldots j_r}{}_{i_1 i_2 \ldots i_s} \bar{\mathrm{e}}_{j_1} \otimes \bar{\mathrm{e}}_{j_2} \otimes \cdots \otimes \bar{\mathrm{e}}_{j_r} \otimes \bar{\mathrm{e}}^{i_1} \otimes \bar{\mathrm{e}}^{i_2} \otimes \cdots \otimes \bar{\mathrm{e}}^{i_s} \\ &= U^{p_1 p_2 \ldots p_r}{}_{q_1 q_2 \ldots q_s} \mathrm{e}_{p_1} \otimes \mathrm{e}_{p_2} \otimes \cdots \otimes \mathrm{e}_{p_r} \otimes \mathrm{e}^{q_1} \otimes \mathrm{e}^{q_2} \otimes \cdots \otimes \mathrm{e}^{q_s}. \end{aligned} \tag{158}$$

Clearly, then

$$U^{p_1 p_2 \ldots p_r}{}_{q_1 q_2 \ldots q_s} = A_{j_1}^{p_1} A_{j_2}^{p_2} \ldots A_{j_r}^{p_r} B_{q_1}^{i_1} B_{q_2}^{i_2} \ldots B_{q_s}^{i_s} \bar{U}^{j_1 j_2 \ldots j_r}{}_{i_1 i_2 \ldots i_s}. \tag{159}$$

The *Kronecker tensor* over E is a $(1,1)$-tensor δ, defined in any basis of E as

$$\delta = \mathrm{e}_i \otimes \mathrm{e}^i. \tag{160}$$

It is immediately seen that the tensor δ does not depend on the choice of the basis e_i. We can also write $\delta = \delta_j^i \mathrm{e}_i \otimes \mathrm{e}^j$, where δ_j^i is the *Kronecker symbol* (Remark 3).

This definition can be extended to tensors of type (r,s) for any positive integers r and s. Let α and β be integers such that $1 \leq \alpha \leq r$, $1 \leq \beta \leq s$, and let e_i be a basis of E. We introduce a linear mapping $\iota_\beta^\alpha : T_{s-1}^{r-1} E \to T_s^r E$ as follows. For every $V \in T_{s-1}^{r-1} E$,

$$V = V^{j_1 j_2 \ldots j_{r-1}}{}_{i_1 i_2 \ldots i_{s-1}} \mathrm{e}_{j_1} \otimes \mathrm{e}_{j_2} \otimes \cdots \otimes \mathrm{e}_{j_{r-1}} \otimes \mathrm{e}^{i_1} \otimes \mathrm{e}^{i_2} \otimes \cdots \otimes \mathrm{e}^{i_{s-1}}, \tag{161}$$

define a tensor $\iota_\beta^\alpha V \in T_s^r E$ by

$$\iota_\beta^\alpha V = W^{j_1 j_2 \ldots j_r}{}_{i_1 i_2 \ldots i_s} \mathrm{e}_{j_1} \otimes \mathrm{e}_{j_2} \otimes \cdots \otimes \mathrm{e}_{j_r} \otimes \mathrm{e}^{i_1} \otimes \mathrm{e}^{i_2} \otimes \cdots \otimes \mathrm{e}^{i_s}, \tag{162}$$

where

$$W^{j_1 j_2 \ldots j_{\alpha-1} j_\alpha j_{\alpha+1} \ldots j_r}{}_{i_1 i_2 \ldots i_{\beta-1} i_\beta i_{\beta+1} \ldots i_s} = \delta_{i_\beta}^{j_\alpha} V^{j_1 j_2 \ldots j_{\alpha-1} j_{\alpha+1} \ldots j_r}{}_{i_1 i_2 \ldots i_{\beta-1} i_{\beta+1} \ldots i_s}. \tag{163}$$

Thus,

$$\begin{aligned} \iota_\beta^\alpha V = V^{j_1 j_2 \ldots j_{r-1}}{}_{i_1 i_2 \ldots i_{s-1}} &\mathrm{e}_{j_1} \otimes \mathrm{e}_{j_2} \otimes \cdots \otimes \mathrm{e}_{j_{\alpha-1}} \otimes \mathrm{e}_s \otimes \mathrm{e}_{j_{\alpha+1}} \otimes \cdots \otimes \mathrm{e}_{j_r} \\ &\otimes \mathrm{e}^{i_1} \otimes \mathrm{e}^{i_2} \otimes \cdots \otimes \mathrm{e}^{i_{\beta-1}} \otimes \mathrm{e}^s \otimes \mathrm{e}^{i_{\beta+1}} \otimes \cdots \otimes \mathrm{e}^{i_s} \end{aligned} \tag{164}$$

(summation through s on the right-hand side). It is easily verified that this tensor is independent of the choice of e_i.

The mapping ι_β^α defined by formulas (152), (163) is the (α,β)-*canonical injection*. A tensor $U \in T_s^r E$, belonging to the vector subspace generated by the subspaces $\iota^{\alpha}_{\beta}(T^{r-1}_{s-1}E) \subset T^r_s E$, where $1 \le \alpha \le r$ and $1 \le \beta \le s$, is called a *Kronecker tensor*, or a tensor of *Kronecker type*.

A tensor $V \in T^r_s E$, $V = V^{k_1 k_2 \ldots k_r}{}_{l_1 l_2 \ldots l_s}$ is a Kronecker tensor if and only if there exist some tensors $V^{(p)}_{(q)} \in T^{r-1}_{s-1}E$, $V^{(p)}_{(q)} = V^{(p) k_1 k_2 \ldots k_{r-1}}_{(q)}{}_{l_1 l_2 \ldots l_{s-1}}$, where the indices satisfy $1 \le p \le r$, $1 \le q \le s$, such that $V^{k_1 k_2 \ldots k_r}{}_{l_1 l_2 \ldots l_s}$ can be expressed in the form

$$\begin{aligned} V^{j_1 j_2 \ldots j_r}{}_{l_1 l_2 \ldots l_s} = {} & \delta^{j_1}_{l_1} V^{(1) j_2 j_3 \ldots j_r}_{(1)}{}_{l_2 l_3 \ldots l_s} + \delta^{j_1}_{l_2} V^{(1) j_2 j_3 \ldots j_r}_{(2)}{}_{l_1 l_3 \ldots l_s} + \cdots + \delta^{j_1}_{l_s} V^{(1) j_2 j_3 \ldots j_r}_{(s)}{}_{l_1 l_2 \ldots l_{s-1}} \\ & + \delta^{j_2}_{l_1} V^{(2) j_1 j_3 \ldots j_r}_{(1)}{}_{l_2 l_3 \ldots l_s} + \delta^{j_2}_{l_2} V^{(2) j_1 j_3 \ldots j_r}_{(2)}{}_{l_1 l_3 \ldots l_s} + \cdots + \delta^{j_2}_{l_s} V^{(2) j_1 j_3 \ldots j_r}_{(s)}{}_{l_1 l_2 \ldots l_{s-1}} \\ & + \ldots \\ & + \delta^{j_r}_{l_1} V^{(r) j_1 j_2 \ldots j_{r-1}}_{(1)}{}_{l_2 l_3 \ldots l_s} + \delta^{j_r}_{l_2} V^{(r) j_1 j_2 \ldots j_{r-1}}_{(2)}{}_{l_1 l_3 \ldots l_s} + \cdots + \delta^{j_r}_{l_s} V^{(r) j_1 j_2 \ldots j_{r-1}}_{(s)}{}_{l_1 l_2 \ldots l_{s-1}}. \end{aligned} \tag{165}$$

A tensor $U \in T^r_s E$ expressed as in (154), is said to be *traceless*, if its traces are all zero,

$$\begin{aligned} & U^{s l_1 l_2 \ldots l_{r-1}}{}_{s j_1 j_2 \ldots j_{s-1}} = 0, \quad U^{l_1 s l_2 \ldots l_{r-1}}{}_{s j_1 j_2 \ldots j_{s-1}} = 0, \quad \ldots, \quad U^{l_1 l_2 \ldots l_{r-1} s}{}_{s j_1 j_2 \ldots j_{s-1}} = 0, \\ & U^{s l_1 l_2 \ldots l_{r-1}}{}_{j_1 s j_2 \ldots j_{s-1}} = 0, \quad U^{l_1 s l_2 \ldots l_{r-1}}{}_{j_1 s j_2 \ldots j_{s-1}} = 0, \quad \ldots, \quad U^{l_1 l_2 \ldots l_{r-1} s}{}_{j_1 s j_2 \ldots j_{s-1}} = 0, \\ & \ldots \\ & U^{s l_1 l_2 \ldots l_{r-1}}{}_{j_1 j_2 \ldots j_{s-1} s} = 0, \quad U^{l_1 s l_2 \ldots l_{r-1}}{}_{j_1 j_2 \ldots j_{s-1} s} = 0, \quad \ldots, \quad U^{l_1 l_2 \ldots l_{r-1} s}{}_{j_1 j_2 \ldots j_{s-1} s} = 0. \end{aligned} \tag{166}$$

To prove a theorem of the decomposition of the tensor space $T^r_s E$ by the trace operation, recall that every scalar product g on the vector space E induces a scalar product on $T^r_s E$ as follows. Let g be expressed in a basis as

$$g(\xi, \zeta) = g_{ij} \xi^i \zeta^j, \tag{167}$$

where $\xi = \xi^i$, $\zeta = \zeta^i$ are any vectors from E. Let $U, V \in T^r_s E$ be any tensors, $U = U^{j_1 j_2 \ldots j_r}{}_{i_1 i_2 \ldots i_s}$, $V = V^{i_1 i_2 \ldots i_r}{}_{j_1 j_2 \ldots j_s}$. We define a bilinear form on $T^r_s E$, denoted by the same letter, g, by

$$g(U, V) = g_{j_1 k_1} g_{j_2 k_2} \ldots g_{j_r k_r} g^{i_1 l_1} g^{i_2 l_2} \ldots g^{i_s l_s} U^{j_1 j_2 \ldots j_r}{}_{i_1 i_2 \ldots i_s} V^{k_1 k_2 \ldots k_r}{}_{l_1 l_2 \ldots l_s l_1 l_2 \ldots l_s}. \tag{168}$$

Lemma 1 *Formula* (168) *defines a scalar product on the tensor space* $T^r_s E$.

Proof Only positive definiteness of the bilinear form (168) needs proof. If we choose a basis of E such that $g_{jk} = \delta_{jk}$, then $g(U, V)$(168) has an expression

$$g(U, V) = \sum_{k_1, k_2, \ldots, k_r} \sum_{l_1, l_2, \ldots, l_s} U^{j_1 j_2 \ldots j_r}{}_{l_1 l_2 \ldots l_s} V^{j_1 j_2 \ldots j_r}{}_{l_1 l_2 \ldots l_s}. \tag{169}$$

Obviously, this is the Euclidean scalar product, which is positive definite. □

Theorem 1 (The trace decomposition theorem) *The vector space T^r_sE is the direct sum of its vector subspaces of traceless and Kronecker tensors.*

Proof We want to show that any tensor $W \in T^r_sE$ has a unique decomposition of the form $W = U + V$, where U is traceless and V is of Kronecker type. To prove existence of the decomposition, consider a scalar product g (169) on T^r_sE. It is immediately seen that the orthogonal complement of the subspace of Kronecker tensors coincides with the subspace of traceless tensors. Indeed, if $U \in T^r_sE$, $U = U^{i_1 i_2 \dots i_r}{}_{j_1 j_2 \dots j_s}$, then calculating the scalar product $g(U, V)$ for any tensor $V \in T^r_sE$, $V = V^{k_1 k_2 \dots k_r}{}_{l_1 l_2 \dots l_s}$, satisfying condition (165), the condition

$$g(U, V) = 0 \tag{170}$$

implies that U must be traceless. The uniqueness of the direct sum follows from the orthogonality of subspaces of traceless and Kronecker tensors in T^r_sE with respect to the scalar product g. □

Theorem 1 states that every tensor $W \in T^r_sE$, $W = W^{i_1 i_2 \dots i_r}{}_{l_1 l_2 \dots l_s}$ is expressible in the form

$$\begin{aligned} W^{i_1 i_2 \dots i_r}{}_{l_1 l_2 \dots l_s} = {} & U^{i_1 i_2 \dots i_r}{}_{l_1 l_2 \dots l_s} \\ & + \delta^{i_1}_{l_1} V^{(1) i_2 i_3 \dots i_r}_{(1)}{}_{l_2 l_3 \dots l_s} + \delta^{i_1}_{l_2} V^{(1) i_2 i_3 \dots i_r}_{(2)}{}_{l_1 l_3 \dots l_s} + \dots + \delta^{i_1}_{l_s} V^{(1) i_2 i_3 \dots i_r}_{(s)}{}_{l_1 l_2 \dots l_{s-1}} \\ & + \delta^{i_2}_{l_1} V^{(2) i_1 i_3 \dots i_r}_{(1)}{}_{l_2 l_3 \dots l_s} + \delta^{i_2}_{l_2} V^{(2) i_1 i_3 \dots i_r}_{(2)}{}_{l_1 l_3 \dots l_s} + \dots + \delta^{i_2}_{l_s} V^{(2) i_1 i_3 \dots i_r}_{(s)}{}_{l_1 l_2 \dots l_{s-1}} \\ & + \dots \\ & + \delta^{i_r}_{l_1} V^{(r) i_1 i_2 \dots i_{r-1}}_{(1)}{}_{l_2 l_3 \dots l_s} + \delta^{i_r}_{l_2} V^{(r) i_1 i_2 \dots i_{r-1}}_{(2)}{}_{l_1 l_3 \dots l_s} + \dots + \delta^{k_r}_{l_s} V^{(r) i_1 i_2 \dots i_{r-1}}_{(s)}{}_{l_1 l_2 \dots l_{s-1}}, \end{aligned} \tag{171}$$

where $U = U^{i_1 i_2 \dots i_r}{}_{l_1 l_2 \dots l_s}$ is a uniquely defined traceless tensor, and for every p and q such that $1 \le p \le r$, $1 \le q \le s$, the tensor $V^{(p)}_{(q)} = V^{(p) i_1 i_2 \dots i_{r-1}}_{(q)}{}_{l_1 l_2 \dots l_{s-1}}$ belongs to the tensor space $T^{r-1}_{s-1}E$.

Remark 4 The traceless component $U^{i_1 i_2 \dots i_r}{}_{l_1 l_2 \dots l_s}$ and the complementary Kronecker component of the tensor W in (171) are determined uniquely. However, this does not imply, in general, that the tensors $V^{(p)}_{(q)}$ are unique. If the contravariant and covariant degrees satisfy $r + s \le n + 1$, then the tensors $V^{(p)}_{(q)}$ may not be unique.

Formula (171) is called the *trace decomposition formula*.

Denote by E^r_s the vector subspace of tensors $U = U^{j_1 j_2 \dots j_r}{}_{i_1 i_2 \dots i_s}$ in the tensor space T^r_sE, *symmetric* in the superscripts and *skew-symmetric* in the subscripts; sometimes these tensors are called *symmetric-skew-symmetric*. We wish to find the trace decomposition formula for the tensors, belonging to the tensor space E^r_s. Set

$$\mathrm{tr} U = U^{k j_1 j_2 \dots j_{r-1}}{}_{k i_1 i_2 \dots i_{s-1}}, \tag{172}$$

and

$$\mathbf{q}U = \frac{(r+1)(s+1)}{n+r-s}\delta^{j_1}_{i_1}U^{j_2j_3\ldots j_{r+1}}{}_{i_2i_3\ldots i_{s+1}} \quad \mathrm{Alt}(i_1i_2\ldots i_{s+1}) \\ \mathrm{Sym}(j_1j_2\ldots j_{r+1}). \tag{173}$$

These formulas define two linear mappings tr: $E^r_s \to E^{r-1}_{s-1}$ and $\mathbf{q}$: $E^r_s \to E^{r+1}_{s+1}$.

Theorem 2

(a) *Any tensor* $U \in E^r_s$ *has a decomposition*

$$U = \mathrm{tr}\mathbf{q}U + \mathbf{q}\mathrm{tr}U. \tag{174}$$

(b) *The mappings* tr *and* $\mathbf{q}$ *satisfy*

$$\mathrm{tr}\mathrm{tr}U = 0, \quad \mathbf{qq}U = 0. \tag{175}$$

Proof

(a) Using (173) we have, with obvious notation,

$$\mathbf{q}\ U = \frac{r+1}{n+r-s}(\delta^{j_1}_{i_1}U^{j_2j_3\ldots j_{r+1}}{}_{i_2i_3\ldots i_{s+1}} - \delta^{j_1}_{i_2}U^{j_2j_3\ldots j_{r+1}}{}_{i_1i_3i_4\ldots i_{s+1}} \\ - \delta^{j_1}_{i_3}U^{j_2j_3\ldots j_{r+1}}{}_{i_2i_1i_4i_5\ldots i_{s+1}} - \cdots - \delta^{j_1}_{i_{s+1}}U^{j_2j_3\ldots j_{r+1}}{}_{i_2i_3\ldots i_si_1}) \\ \mathrm{Sym}(j_1j_2\ldots j_{r+1}). \tag{176}$$

Thus,

$$\mathrm{tr}\mathbf{q}U = \frac{1}{n+r-s}(\delta^k_kU^{j_2j_3\ldots j_{r+1}}{}_{i_2i_3\ldots i_{s+1}} - \delta^k_{i_2}U^{j_2j_3\ldots j_{r+1}}{}_{ki_3i_4\ldots i_{s+1}} \\ - \delta^k_{i_3}U^{j_2j_3\ldots j_{r+1}}{}_{i_2ki_4i_5\ldots i_{s+1}} - \cdots - \delta^k_{i_{s+1}}U^{j_2j_3\ldots j_{r+1}}{}_{i_2i_3\ldots i_sk} \\ + \delta^{j_2}_kU^{kj_3j_4\ldots j_{r+1}}{}_{i_2i_3\ldots i_{s+1}} - \delta^{j_2}_{i_2}U^{kj_3j_4\ldots j_{r+1}}{}_{ki_3i_4\ldots i_{s+1}} \\ - \delta^{j_2}_{i_3}U^{kj_3j_4\ldots j_{r+1}}{}_{i_2ki_4i_5\ldots i_{s+1}} - \cdots - \delta^{j_2}_{i_{s+1}}U^{kj_3j_4\ldots j_{r+1}}{}_{i_2i_3\ldots i_sk} \\ + \delta^{j_3}_kU^{j_2kj_4j_5\ldots j_{r+1}}{}_{i_2i_3\ldots i_{s+1}} - \delta^{j_3}_{i_2}U^{j_2kj_4j_5\ldots j_{r+1}}{}_{ki_3i_4\ldots i_{s+1}} \\ - \delta^{j_3}_{i_3}U^{j_2kj_4j_5\ldots j_{r+1}}{}_{i_2ki_4i_5\ldots i_{s+1}} - \cdots - \delta^{j_3}_{i_{s+1}}U^{j_2kj_4j_5\ldots j_{r+1}}{}_{i_2i_3\ldots i_sk} \\ + \cdots + \delta^{j_{r+1}}_kU^{j_2j_3\ldots j_rk}{}_{i_2i_3\ldots i_{s+1}} - \delta^{j_{r+1}}_{i_2}U^{j_2j_3\ldots j_rk}{}_{ki_3i_4\ldots i_{s+1}} \\ - \delta^{j_{r+1}}_{i_3}U^{j_2j_3\ldots j_rk}{}_{i_2ki_4i_5\ldots i_{s+1}} - \cdots - \delta^{j_{r+1}}_{i_{s+1}}U^{j_2j_3\ldots j_rk}{}_{i_2i_3\ldots i_sk}). \tag{177}$$

Computing the traces, we get

$$
\begin{aligned}
\operatorname{tr}\mathbf{q}U = \frac{1}{n+r-s}\big(& nU^{j_2j_3\ldots j_{r+1}}{}_{i_2i_3\ldots i_{s+1}} - U^{j_2j_3\ldots j_{r+1}}{}_{i_2i_3i_4\ldots i_{s+1}} - U^{j_2j_3\ldots j_{r+1}}{}_{i_2i_3i_4i_5\ldots i_{s+1}} \\
& - \cdots - U^{j_2j_3\ldots j_{r+1}}{}_{i_2i_3\ldots i_s i_{s+1}} + U^{j_2j_3j_4\ldots j_{r+1}}{}_{i_2i_3\ldots i_{s+1}} - \delta^{j_2}_{i_2} U^{kj_3j_4\ldots j_{r+1}}{}_{ki_3i_4\ldots i_{s+1}} \\
& - \delta^{j_2}_{i_3} U^{kj_3j_4\ldots j_{r+1}}{}_{i_2ki_4i_5\ldots i_{s+1}} - \cdots - \delta^{j_2}_{i_{s+1}} U^{kj_3j_4\ldots j_{r+1}}{}_{i_2i_3\ldots i_s k} \\
& + U^{j_2j_3j_4j_5\ldots j_{r+1}}{}_{i_2i_3\ldots i_{s+1}} - \delta^{j_3}_{i_2} U^{j_2kj_4j_5\ldots j_{r+1}}{}_{ki_3i_4\ldots i_{s+1}} \\
& - \delta^{j_3}_{i_3} U^{j_2kj_4j_5\ldots j_{r+1}}{}_{i_2ki_4i_5\ldots i_{s+1}} - \cdots - \delta^{j_3}_{i_{s+1}} U^{j_2kj_4j_5\ldots j_{r+1}}{}_{i_2i_3\ldots i_s k} \\
& + \cdots + U^{j_2j_3\ldots j_r j_{r+1}}{}_{i_2i_3\ldots i_{s+1}} - \delta^{j_{r+1}}_{i_2} U^{j_2j_3\ldots j_r k}{}_{ki_3i_4\ldots i_{s+1}} \\
& - \delta^{j_{r+1}}_{i_3} U^{j_2j_3\ldots j_r k}{}_{i_2ki_4i_5\ldots i_{s+1}} - \cdots - \delta^{j_{r+1}}_{i_{s+1}} U^{j_2j_3\ldots j_r k}{}_{i_2i_3\ldots i_s k}\big).
\end{aligned}
\tag{178}
$$

Further straightforward calculations yield

$$
\begin{aligned}
\operatorname{tr}\mathbf{q}U = U^{j_2j_3\ldots j_{r+1}}{}_{i_2i_3\ldots i_{s+1}} - \frac{rs}{n+r-s}\delta^{j_2}_{i_2} U^{kj_3j_4\ldots j_{r+1}}{}_{ki_3i_4\ldots i_{s+1}} \\
\operatorname{Sym}(j_2j_3\ldots j_{r+1}) \quad \operatorname{Alt}(i_2i_3\ldots i_{s+1}).
\end{aligned}
\tag{179}
$$

But by (172), the second term is exactly $\mathbf{q}\operatorname{tr}u$, proving (174).

(b) Formulas (175) are immediate. □

Formula (174) is the *trace decomposition formula* for tensors $U \in E^r_s$.

The following assertion is a consequence of Theorem 2. It states, in particular, that the decomposition (174) of a tensor $U \in E^r_s$ is unique.

Theorem 3 *Let* $U \in E^r_s$.

(a) *Equation* $\mathbf{q}V + \operatorname{tr}W = U$ *for unknown tensors* $V \in E^{r-1}_{s-1}$ *and* $W \in E^{r+1}_{s+1}$ *has a unique solution such that* $\operatorname{tr}V = 0$, $\mathbf{q}W = 0$. *This solution is given by* $V = \operatorname{tr}U$, $W = \mathbf{q}U$.

(b) *Equation* $\mathbf{q}X = U$ *has a solution* $X \in E^{r-1}_{s-1}$ *if and only if* $\mathbf{q}\,U = 0$. *If this condition is satisfied, then* $X = \operatorname{tr}U$ *is a solution. Any other solution is of the form* $X' = X + \mathbf{q}Y$ *for some tensor* $Y \in E^{r-1}_{s-2}$.

Proof

(a) If $\mathbf{q}V + \operatorname{tr}W = U$, $\operatorname{tr}V = 0$ then $V = \operatorname{tr}\mathbf{q}V = \operatorname{tr}U$ because $\operatorname{tr}\operatorname{tr}W = 0$; if $\mathbf{q}W = 0$, then $W = \mathbf{q}\operatorname{tr}W = \mathbf{q}(U - \mathbf{q}V) = \mathbf{q}U$.

(b) If equation $\mathbf{q}X = U$ has a solution U, then necessarily $\mathbf{q}U = 0$. Conversely, if $\mathbf{q}U = 0$, then $U = \mathbf{q}\operatorname{tr}U$ and $X = \operatorname{tr}U$ solves equation $\mathbf{q}X = U$. Clearly, the tensors $X' = X + \mathbf{q}Y$, where $Y \in E^{r-1}_{s-2}$ also solve this equation. □

Example 1 We find the trace decomposition formula (174) for $r = 1$. Writing $U = U^{j_1}_{i_1i_2\ldots i_s}$, we have $\operatorname{tr}U = U^k_{ki_1i_2\ldots i_{s-1}}$ and

$$\mathbf{q}\mathrm{tr}U = \frac{1}{n+1-s}(\delta^{j_1}_{i_1}U^k{}_{ki_2i_3\ldots i_s} + \delta^{j_1}_{i_2}U^k{}_{i_1ki_3i_4\ldots i_s} + \cdots + \delta^{j_1}_{i_s}U^k{}_{i_1i_2\ldots i_{s-1}k}). \tag{180}$$

Analogously

$$\begin{aligned}\mathbf{q}U &= \frac{2(s+1)}{n+1-s}\delta^{j_1}_{i_1}U^{j_2}{}_{i_2i_3\ldots i_{s+1}} \quad \mathrm{Alt}(i_1i_2\ldots i_{s+1}) \quad \mathrm{Sym}(j_1j_2)\\ &= \frac{1}{n+1-s}(\delta^{j_1}_{i_1}U^{j_2}{}_{i_2i_3\ldots i_{s+1}} - \delta^{j_1}_{i_2}U^{j_2}{}_{i_1i_3i_4\ldots i_{s+1}} - \cdots - \delta^{j_1}_{i_{s+1}}U^{j_2}{}_{i_2i_3\ldots i_si_1}\\ &\quad + \delta^{j_2}_{i_1}U^{j_1}{}_{i_2i_3\ldots i_{s+1}} - \delta^{j_2}_{i_2}U^{j_1}{}_{i_1i_3i_4\ldots i_{s+1}} - \cdots - \delta^{j_2}_{i_{s+1}}U^{j_1}{}_{i_2i_3\ldots i_si_1})\end{aligned} \tag{181}$$

hence

$$\begin{aligned}\mathrm{tr}\mathbf{q}U &= \frac{1}{n+1-s}(nU^{j_2}{}_{i_2i_3\ldots i_{s+1}} - (s-1)U^{j_2}{}_{i_2i_3i_4\ldots i_{s+1}}\\ &\quad - \frac{1}{n+1-s}(\delta^{j_2}_{i_2}U^k{}_{ki_3i_4\ldots i_{s+1}} + \delta^{j_2}_{i_3}U^k{}_{i_2ki_4i_5\ldots i_{s+1}} + \cdots + \delta^{j_2}_{i_{s+1}}U^k{}_{i_2i_3\ldots i_sk})\\ &= U^{j_2}{}_{i_2i_3\ldots i_{s+1}} - \mathbf{q}\mathrm{tr}U.\end{aligned} \tag{182}$$

Formulas (181) and (183) yield $U = \mathrm{tr}\mathbf{q}U + \mathbf{q}\mathrm{tr}U$. In particular, if $r = 1$ and $s = n$, then $U = U^j_{i_1i_2\ldots i_n}$, $\mathrm{tr}U = U^s_{si_1i_2\ldots i_{n-1}}$ and $\mathbf{q}U = 0$. Thus,

$$\begin{aligned}U &= n\delta^j_{i_1}U^s{}_{si_2i_3\ldots i_n} \quad \mathrm{Alt}(i_1i_2\ldots i_n)\\ &= \delta^j_{i_1}U^s{}_{si_2i_3\ldots i_n} + \delta^j_{i_2}U^s{}_{i_1si_3i_4\ldots i_n} + \cdots + \delta^j_{i_n}U^s{}_{i_1i_2\ldots i_{n-1}s}.\end{aligned} \tag{183}$$

Example 2 We determine decomposition (174) for $r = 2$ and $s = n-1$, and find explicit expressions for the traceless and Kronecker components $\mathrm{tr}\mathbf{q}\,U$ and $\mathbf{q}\,\mathrm{tr}U$ of the tensor U. Writing $U = U^{j_1j_2}{}_{i_1i_2\ldots i_{n-1}}$ and using the proof of Theorem 2, we have

$$\begin{aligned}\mathrm{tr}\mathbf{q}U = U^{j_2j_3}{}_{i_2i_3\ldots i_n} - \frac{1}{3}(&\delta^{j_2}_{i_2}U^{kj_3}{}_{ki_3i_4\ldots i_n} + \delta^{j_2}_{i_3}U^{kj_3}{}_{i_2ki_4i_5\ldots i_n} + \cdots + \delta^{j_2}_{i_n}U^{kj_3}{}_{i_2i_3\ldots i_{n-1}k}\\ &+ \delta^{j_3}_{i_2}U^{j_2k}{}_{ki_3i_4i_5\ldots i_n} + \delta^{j_3}_{i_3}U^{j_2k}{}_{i_2ki_4i_5\ldots i_n} + \cdots + \delta^{j_3}_{i_n}U^{j_2k}{}_{i_2i_3\ldots i_{n-1}k})\end{aligned} \tag{184}$$

and

$$\begin{aligned}\mathbf{q}\mathrm{tr}U &= \frac{1}{3}(\delta^{j_2}_{i_2}U^{kj_3}{}_{ki_3i_4\ldots i_n} + \delta^{j_2}_{i_3}U^{kj_3}{}_{i_2ki_4i_5\ldots i_n} + \cdots + \delta^{j_2}_{i_n}U^{kj_3}{}_{i_2i_3\ldots i_{n-1}k}\\ &\quad + \delta^{j_3}_{i_2}U^{j_2k}{}_{ki_3i_4i_5\ldots i_n} + \delta^{j_3}_{i_3}U^{j_2k}{}_{i_2ki_4i_5\ldots i_n} + \cdots + \delta^{j_3}_{i_n}U^{j_2k}{}_{i_2i_3\ldots i_{n-1}k})\\ &= \frac{1}{3}(\delta^{j_2}_{i_2}U^{kj_3}{}_{ki_3i_4\ldots i_n} - \delta^{j_2}_{i_3}U^{kj_3}{}_{ki_2i_4i_5\ldots i_n} - \cdots - \delta^{j_2}_{i_n}U^{kj_3}{}_{ki_3\ldots i_{n-1}i_2}\\ &\quad + \delta^{j_3}_{i_2}U^{kj_2}{}_{ki_3i_4i_5\ldots i_n} - \delta^{j_3}_{i_3}U^{kj_2}{}_{ki_2i_4i_5\ldots i_n} - \cdots - \delta^{j_3}_{i_n}U^{kj_2}{}_{ki_3\ldots i_{n-1}i_2})\\ &= \frac{2(n-1)}{3}\delta^{j_2}_{i_2}U^{kj_3}{}_{ki_3i_4\ldots i_n} \quad \mathrm{Sym}(j_2j_3) \quad \mathrm{Alt}(i_2i_3\ldots i_n).\end{aligned} \tag{185}$$

Let s and j be positive integers such that $j \le s \le n$. Consider the vector space of tensors $X = X^{I_1 I_2 \ldots I_j}{}_{i_{j+1} i_{j+2} \ldots i_s}$, indexed with multi-indices I_1, I_2, …, I_j of length r and indices $i_{j+1}, i_{j+2}, \ldots, i_s$, such that $1 \le i_{j+1}, i_{j+2}, \ldots i_s, \le n$, *symmetric* in the superscripts entering each of the multi-indices, and skew-*symmetric* in the subscripts. Our objective will be to solve the system of homogeneous equations

$$\begin{gathered} \delta^{p_1}_{p_1} \delta^{p_2}_{p_2} \ldots \delta^{p_j}_{p_j} X^{I_1 I_2 \ldots I_j}{}_{i_{j+1} i_{j+2} \ldots i_s} = 0 \quad \mathrm{Alt}(p_1 p_2 \ldots p_j i_{j+1} i_{j+2} \ldots i_s) \\ \mathrm{Sym}(I_1 p_1) \quad \mathrm{Sym}(I_2 p_2) \quad \ldots \quad \mathrm{Sym}(I_j p_j) \end{gathered} \tag{186}$$

for an unknown tensor X. In this formula, the alternation operation is applied to the subscripts, and the symmetrizations to the superscripts, and then the summations through double indices are provided.

In the proof of the following theorem, we want to distinguish between two groups of indices in the expression $\delta^{p_1}_{i_1} \delta^{p_2}_{i_2} \ldots \delta^{p_j}_{i_j} X^{I_1 I_2 \ldots I_j}{}_{i_{j+1} i_{j+2} \ldots i_s}$; the indices labeling the tensor $X^{I_1 I_2 \ldots I_j}{}_{i_{j+1} i_{j+2} \ldots i_s}$ will be called *interior* (the complementary indices, labeling the Kronecker tensors, are called *exterior*).

Theorem 4 *Let q and j be positive integers such that $1 \le j \le s \le n$. Let $X = X^{I_1 I_2 \ldots I_j}{}_{i_{j+1} i_{j+2} \ldots i_q}$ be a tensor, indexed with multi-indices I_1, I_2, …, I_j of length r and indices i_{j+1}, i_{j+2}, …, i_s, such that $1 \le i_{j+1}, i_{j+2}, \ldots, i_s \le n$, symmetric in the superscripts entering each of the multi-indices, and skew-symmetric in the subscripts. Then X satisfies* equation (186) *if and only if it is a Kronecker tensor.*

Proof Suppose we have a tensor $X^{I_1 I_2 \ldots I_j}{}_{i_{j+1} i_{j+2} \ldots i_s}$, satisfying equation (187). We want to show that X is a Kronecker tensor. Consider a fixed component $X^{I_1 I_2 \ldots I_j}{}_{i_{j+1} i_{j+2} \ldots i_s}$. Choose p_1, p_2, …, p_j and i_1, i_2, …, i_j such that the s-tuples $(p_1, p_2, \ldots, p_j, i_{j+1}, i_{j+2}, \ldots, i_s)$ and $(i_1, i_2, \ldots, i_j, i_{j+1}, i_{j+2}, \ldots, i_s)$ consist of mutually different indices, and consider expression

$$\begin{gathered} \delta^{p_1}_{i_1} \delta^{p_2}_{i_2} \ldots \delta^{p_j}_{i_j} X^{I_1 I_2 \ldots I_j}{}_{i_{j+1} i_{j+2} \ldots i_s} \quad \mathrm{Alt}(i_1 i_2 \ldots i_j i_{j+1} \ldots i_s) \\ \mathrm{Sym}(I_1 p_1) \quad \mathrm{Sym}(I_2 p_2) \quad \ldots \quad \mathrm{Sym}(I_j p_j). \end{gathered} \tag{187}$$

The summations in (187) are defined by the alternation $\mathrm{Alt}(i_1 i_2 \ldots i_j i_{j+1} \ldots i_s)$ and the symmetrizations $\mathrm{Sym}(I_1 p_1), \mathrm{Sym}(I_2 p_2), \ldots, \mathrm{Sym}(I_j p_j)$. We divide the summands in four groups according to the positions of the indices p_1, p_2, …, p_j and i_1, i_2, …, i_j.

(a) None of the indices p_1, p_2, …, p_j and i_1, i_2, …, i_j is interior.
(b) None of the indices p_1, p_2, …, p_j is interior, at least one of the indices i_1, i_2, …, i_j is interior.
(c) At least one of the indices p_1, p_2, …, p_j is interior, none of the indices i_1, i_2, …, i_j is interior.
(d) At least one of the indices p_1, p_2, …, p_j is interior, and at least one of the indices i_1, i_2, …, i_j is interior.

Equation (187) involves expressions (187) such that $i_1 = p_1$, $i_2 = p_2$, …, $i_q = p_q$. For this choice of indices, the terms (a) become

$$\begin{gathered}\delta^{p_1}_{p_1}\delta^{p_2}_{p_2}\ldots\delta^{p_q}_{p_q}X^{I_1I_2\ldots I_q}{}_{i_{q+1}i_{q+2}\ldots i_s} \quad \mathrm{Alt}(p_1p_2\ldots p_qi_{q+1}i_{q+2}\ldots i_s)\\ \mathrm{Sym}(I_1p_1) \quad \mathrm{Sym}(I_2p_2) \quad \ldots \quad \mathrm{Sym}(I_qp_q)\end{gathered} \tag{188}$$

(no summation through p_1, p_2, …, p_q). Expressions (b) and (c) vanish identically because the indices $(i_1, i_2, \ldots, i_q, i_{q+1}, i_{q+2}, \ldots, i_s)$ are mutually different and $X^{I_1I_2\ldots I_q}{}_{i_{q+1}i_{q+2}\ldots i_s}$ is skew-symmetric in the subscripts. The terms in (d) are of Kronecker type, each summand is a multiple of the Kronecker symbol δ^{α}_{β}, where $\alpha \notin \{p_1, p_2, \ldots, p_q\}$ and $\beta \in \{i_{q+1}i_{q+2}\ldots i_s\}$.

Thus, (187) is the sum of the terms (a) and (d). But the left-hand side of Eq. (186) is determined from (187) by the trace operation in $i_1 = p_1$, $i_2 = p_2$, …, $i_q = p_q$. The terms entering (a) lead to an expression of the form cX, where c is a non-zero constant, namely to the expression

$$\begin{aligned}&\frac{j!}{s!((r+1)!)^q}\delta^{p_1}_{p_1}\delta^{p_2}_{p_2}\ldots\delta^{p_j}_{p_j}X^{I_1I_2\ldots I_j}{}_{i_{j+1}i_{j+2}\ldots i_s} \quad \mathrm{Alt}(p_1p_2\ldots p_j)\\ &= \frac{1}{s!((r+1)!)^j}\det\delta^{p_i}_{p_l}\cdot X^{I_1I_2\ldots I_j}{}_{i_{j+1}i_{j+2}\ldots i_s}.\end{aligned} \tag{189}$$

Since the contraction of the terms (d) in $i_1 = p_1$, $i_2 = p_2$, …, $i_q = p_q$ does not influence the factors δ^{α}_{β}, (d) leads to a Kronecker tensor.

Corollary 1 *Assume that in addition to the assumptions of* Theorem 4, *the tensor* $X = X^{I_1I_2\ldots I_j}{}_{i_{j+1}i_{j+2}\ldots i_s}$ *is traceless. Then*

$$X^{I_1I_2\ldots I_j}{}_{i_{j+1}i_{j+2}\ldots i_s} = 0. \tag{190}$$

Proof This follows from Theorem 4, and from the orthogonality of traceless and Kronecker tensors. □

Example 3 For tensors of lower degrees equations (186) can be solved directly by means of the decomposition of the unknown tensor X. Consider for example the system

$$\delta^{p_1}_{p_1}\delta^{p_2}_{p_2}X^{i_1i_2}{}_{i_3} = 0 \quad \mathrm{Alt}(p_1p_2i_3) \quad \mathrm{Sym}(i_1p_1) \quad \mathrm{Sym}(i_2p_2) \tag{191}$$

for a *traceless* tensor $X = X^{i_1i_2}{}_k$. The decomposition of the left-hand side is

$$\begin{aligned}
&\delta^{p_1}_{p_1}\delta^{p_2}_{p_2}X^{i_1i_2}{}_{i_3} + \delta^{i_1}_{p_1}\delta^{p_2}_{p_2}X^{p_1i_2}{}_{i_3} + \delta^{p_1}_{p_1}\delta^{i_2}_{p_2}X^{i_1p_2}{}_{i_3} + \delta^{i_1}_{p_1}\delta^{i_2}_{p_2}X^{p_1p_2}{}_{i_3}\\
&\quad - \delta^{p_1}_{p_2}\delta^{p_2}_{p_1}X^{i_1i_2}{}_{i_3} - \delta^{i_1}_{p_2}\delta^{p_2}_{p_1}X^{p_1i_2}{}_{i_3} - \delta^{p_1}_{p_2}\delta^{i_2}_{p_1}X^{i_1p_2}{}_{i_3} - \delta^{i_1}_{p_2}\delta^{i_2}_{p_1}X^{p_1p_2}{}_{i_3}\\
&\quad - \delta^{p_1}_{i_3}\delta^{p_2}_{p_2}X^{i_1i_2}{}_{p_1} - \delta^{i_1}_{i_3}\delta^{p_2}_{p_2}X^{p_1i_2}{}_{p_1} - \delta^{p_1}_{i_3}\delta^{i_2}_{p_2}X^{i_1p_2}{}_{p_1} - \delta^{i_1}_{i_3}\delta^{i_2}_{p_2}X^{p_1p_2}{}_{p_1}\\
&\quad + \delta^{p_1}_{p_2}\delta^{p_2}_{i_3}X^{i_1i_2}{}_{p_1} + \delta^{i_1}_{p_2}\delta^{p_2}_{i_3}X^{p_1i_2}{}_{p_1} + \delta^{p_1}_{p_2}\delta^{i_2}_{i_3}X^{i_1p_2}{}_{p_1} + \delta^{i_1}_{p_2}\delta^{i_2}_{i_3}X^{p_1p_2}{}_{p_1}\\
&\quad - \delta^{p_1}_{p_1}\delta^{p_2}_{i_3}X^{i_1i_2}{}_{p_2} - \delta^{i_1}_{p_1}\delta^{p_2}_{i_3}X^{p_1i_2}{}_{p_2} - \delta^{p_1}_{p_1}\delta^{i_2}_{i_3}X^{i_1p_2}{}_{p_2} - \delta^{i_1}_{p_1}\delta^{i_2}_{i_3}X^{p_1p_2}{}_{p_2}\\
&\quad + \delta^{p_1}_{i_3}\delta^{p_2}_{p_1}X^{i_1i_2}{}_{p_2} + \delta^{i_1}_{i_3}\delta^{p_2}_{p_1}X^{p_1i_2}{}_{p_2} + \delta^{p_1}_{i_3}\delta^{i_2}_{p_1}X^{i_1p_2}{}_{p_2} + \delta^{i_1}_{i_3}\delta^{i_2}_{p_1}X^{p_1p_2}{}_{p_2}.
\end{aligned} \tag{192}$$

Contraction in p_1 and p_2 gives the expression

$$\begin{aligned}
&n^2X^{i_1i_2}{}_{i_3} + nX^{i_1i_2}{}_{i_3} + nX^{i_1i_2}{}_{i_3} + X^{i_1i_2}{}_{i_3} - nX^{i_1i_2}{}_{i_3}\\
&\quad - X^{i_1i_2}{}_{i_3} - X^{i_1i_2}{}_{i_3} - X^{i_2i_1}{}_{i_3} - X^{i_1i_2}{}_{i_3} - X^{i_1i_2}{}_{i_3}\\
&\quad + X^{i_1i_2}{}_{i_3} - nX^{i_1i_2}{}_{i_3} - X^{i_1i_2}{}_{i_3} + X^{i_1i_2}{}_{i_3}\\
&= (n^2-2)X^{i_1i_2}{}_{i_3} - X^{i_2i_1}{}_{i_3}.
\end{aligned} \tag{193}$$

Since this expression should vanish, we get $(n^2-2)X^{i_1i_2}{}_{i_3} - X^{i_2i_1}{}_{i_3} = 0$ which is only possible when $X^{i_1i_2}{}_{i_3} = 0$.

A.10 Bases of Forms

We summarize for reference some useful formulas for the bases of differential forms on an n-dimensional manifold X.

Lemma 1 (Bases of forms) *Let X be an n-dimensional smooth manifold, and let (U,φ), $\varphi = (x^i)$, be a chart on X. Then the forms*

$$\omega_0 = \frac{1}{n!}\varepsilon_{i_1i_2\ldots i_n}dx^{i_1}\wedge dx^{i_2}\wedge\cdots\wedge dx^{i_n} \tag{194}$$

and

$$\omega_{k_1k_2\ldots k_p} = \frac{1}{(n-p)!}\varepsilon_{k_1k_2\ldots k_pi_{p+1}i_{p+2}\ldots i_n}dx^{i_{p+1}}\wedge dx^{i_{p+2}}\wedge\cdots\wedge dx^{i_n},\quad 1\le p\le n-1, \tag{195}$$

define bases of n-forms and $(n-p)$-forms on U. The transformation formulas to the canonical bases are

$$\varepsilon^{k_1k_2\ldots k_pl_{p+1}l_{p+2}\ldots l_n}\omega_{k_1k_2\ldots k_p} = dx^{l_{p+1}}\wedge dx^{l_{p+2}}\wedge\cdots\wedge dx^{l_n}. \tag{196}$$

Proof See Appendix A.8. □

The *Jacobian determinant* of a transformation $\bar{x}^p = \bar{x}^p(x^1, x^2, \ldots, x^n)$, $\det(\partial\bar{x}^p/\partial x^p)$, has the following basic properties:

Lemma 2 (Jacobians)

(a) *The local volume forms on X are on intersections of the charts are related by the formula*

$$\bar{\omega}_0 = \det\left(\frac{\partial\bar{x}^p}{\partial x^p}\right)\omega_0. \tag{197}$$

(b) *The derivative of the Jacobian satisfies*

$$\frac{\partial}{\partial\bar{x}^m}\det\left(\frac{\partial x^r}{\partial\bar{x}^s}\right) = \det\left(\frac{\partial x^r}{\partial\bar{x}^s}\right)\cdot\frac{\partial^2 x^p}{\partial\bar{x}^m\partial\bar{x}^q}\frac{\partial\bar{x}^q}{\partial x^p}. \tag{198}$$

(c) *The $(n-1)$-forms ω_k and $\bar{\omega}_i$* obey the transformation formulas

$$\bar{\omega}_i = \frac{\partial x^k}{\partial\bar{x}^i}\det\frac{\partial\bar{x}^r}{\partial x^s}\cdot\omega_k. \tag{199}$$

Proof (b) To verify formula (198), consider any regular matrix a and its inverse a^{-1},

$$a = \begin{pmatrix} a_1^1 & a_2^1 & a_n^1 \\ a_1^2 & a_2^2 & a_n^2 \\ a_1^n & a_2^n & a_n^n \end{pmatrix}, \quad a^{-1} = \begin{pmatrix} b_1^1 & b_2^1 & b_n^1 \\ b_1^2 & b_2^2 & b_n^2 \\ b_1^n & b_2^n & b_2^n \end{pmatrix}, \tag{200}$$

and compute the derivative $\partial\det a/\partial a_q^p$. Multilinearity and the Laplace decomposition with respect to the *s-th* row of the determinant of a yields $\det a = a_1^s A_1^s + a_2^s A_2^s + \cdots + a_n^s A_n^s$, with algebraic complements A_k^s. Thus

$$\frac{\partial\det a}{\partial a_q^p} = A_q^p. \tag{201}$$

But a is regular, so the inverse matrix satisfies

$$\begin{pmatrix} b_1^1 & b_2^1 & \ldots & b_n^1 \\ b_1^2 & b_2^2 & \ldots & b_n^2 \\ \ldots & & & \\ b_1^n & b_2^n & \ldots & b_n^n \end{pmatrix} = \frac{1}{(\det a)^n}\begin{pmatrix} A_1^1 & A_1^2 & \ldots & A_1^n \\ A_2^1 & A_2^2 & \ldots & A_2^n \\ \ldots & & & \\ A_n^1 & A_n^2 & \ldots & A_n^n \end{pmatrix}, \tag{202}$$

hence $A_q^p = \det a\cdot b_p^q$ and we conclude that

$$\frac{\partial\det a}{\partial a_q^p} = \det a\cdot b_p^q. \tag{203}$$

Now substituting

$$a_s^r = \frac{\partial x^r}{\partial \bar{x}^s}, \quad b_s^r = \frac{\partial \bar{x}^r}{\partial x^s}, \tag{204}$$

we get

$$\frac{\partial}{\partial \bar{x}^m} \det\left(\frac{\partial x^r}{\partial \bar{x}^s}\right) = \sum_{p,q} \frac{\partial \det a}{\partial a_q^p} \frac{\partial a_q^p}{\partial \bar{x}^m} = \det\left(\frac{\partial x^r}{\partial \bar{x}^s}\right) \cdot \frac{\partial^2 x^p}{\partial \bar{x}^m \partial \bar{x}^q} \frac{\partial \bar{x}^q}{\partial x^p}. \tag{205}$$

(c) Using the transformation properties of the forms ω_0 and $\bar{\omega}_0$ (formula (197)),

$$\bar{\omega}_i = i_{\partial/\partial \bar{x}^i} \bar{\omega}_0 = \frac{\partial x^k}{\partial \bar{x}^i} \det \frac{\partial \bar{x}}{\partial x} \cdot i_{\partial/\partial x^k} \omega_0 = \frac{\partial x^k}{\partial \bar{x}^i} \det \frac{\partial \bar{x}}{\partial x} \cdot \omega_k. \tag{206}$$

□

Remark (Different bases) Sometimes it is convenient to consider bases of forms, differing from the forms (195) by a constant factor. If we set

$$\omega_{k_1 k_2 \ldots k_p} = \frac{1}{p!(n-p)!} \varepsilon_{k_1 k_2 \ldots k_p i_{p+1} i_{p+2} \ldots i_n} dx^{i_{p+1}} \wedge dx^{i_{p+2}} \wedge \cdots \wedge dx^{i_n}. \tag{207}$$

then for example

$$\begin{aligned} dx^l \wedge \omega_{k_1 k_2} &= \frac{1}{2!(n-2)!} \varepsilon_{k_1 k_2 i_3 i_4 \ldots i_n} dx^l \wedge dx^{i_3} \wedge dx^{i_4} \wedge \cdots \wedge dx^{i_n}, \\ &= \frac{1}{2!(n-2)!} \varepsilon_{k_1 k_2 i_3 i_4 \ldots i_n} \varepsilon^{p l i_3 i_4 \ldots i_n} \omega_p \\ &= \frac{2!(n-2)!}{2!(n-2)!} \frac{1}{2} (\delta_{k_1}^{p_1} \delta_{k_2}^l - \delta_{k_2}^{p_1} \delta_{k_1}^l) \omega_{p_1} \\ &= \frac{1}{2} (\delta_{k_2}^l \omega_{k_1} - \delta_{k_1}^l \omega_{k_2}), \end{aligned} \tag{208}$$

etc. (cf. Appendix A.8).

Bibliography

[A1] I. Anderson, Natural variational principles on Riemannian manifolds, Annals of Mathematics 120 (1984) 329-370

[A2] I. Anderson, *The variational bicomplex*, preprint, Utah State University, 1989, 289 pp.

[AD] I. Anderson, T. Duchamp, On the existence of global variational principles, Am. J. Math. 102 (1980) 781-867

[AT] I. Anderson, G. Thompson, *The inverse problem of the calculus of variations for ordinary differential equations,* Mem. Amer. Math. Soc. 98, 1992, 1-110

[B] D. Betounes, Extension of the classical Cartan form, Phys. Rev. D29 (1984) 599-606

[BT] R. Bott, L.V. Tu, *Differential Forms and Algebraic Topology*, Springer-Verlag, New York, 1982

[BK] J. Brajercik, D. Krupka, Cohomology and local variational principles, Proc. of the XVth International Workshop on Geometry and Physics (Puerto de la Cruz, Tenerife, Canary Islands, September 11-16, 2006, Publ. de la RSME, 2007) 119-124

[Br] G.E. Bredon, *Sheaf Theory*, Springer-Verlag, New York, 1997

[Bu] I. Bucataru, A setting for higher order differential equation fields and higher order Lagrange and Finsler spaces, Journal of Geometric Mechanics 5 (2013) 257-279

[Bry] R.L. Bryant, S.S. Chern, R.B. Gardner, H.J. Goldschmidt, P.A. Griffiths, *Exterior Differential Systems*, Mathematical Sciences Research Institute Publications 18, Springer-Verlag, New York, 1991

[C] E. Cartan, *Lecons sur les Invariants Intégraux*, Hermann, Paris, 1922

[Cr] M. Crampin, On the inverse problem for sprays, Publ. Math. Debrecen 70, 2007, 319-335

[CS] M. Crampin, D.J. Saunders, The Hilbert-Carathéodory form and Poincaré-Cartan forms for higher-order multiple-integral variational problems, Houston J. Math. 30 (2004) 657-689

D. Krupka, *Introduction to Global Variational Geometry*,
Atlantis Studies in Variational Geometry 1, DOI 10.2991/978-94-6239-073-7

[D] P. Dedecker, On the generalization of symplectic geometry to multiple integrals in the calculus of variations, in: *Lecture Notes in Math.* 570, Springer, Berlin, 1977, 395-456

[DT] P. Dedecker, W. Tulczyjew, Spectral sequences and the inverse problem of the calculus of variations, Internat. Colloq., Aix-en-Provence,1979; in: *Differential-Geometric Methods in Mathematical Physics*, Lecture Notes in Math. 826 Springer, Berlin, 1980, 498-503

[Do] J. Douglas, Solution of the inverse problem of the calculus of variations, Transactions AMS 50 (1941) 71-128

[E] C. Ehresmann, Les prolongements d'une variete différentiable I. - V., C. R. Acad. Sci. Paris 223 (1951) 598–600, 777-779, 1081-1083; 234 (1952) 1028-1030, 1424-1425

[FFPW] M. Ferraris, M. Francaviglia, M. Palese, E. Winterroth, Gauge-natural Noether currents and connection fields, Int. J. of Geom. Methods in Mod. Phys. 01/2011; 8(1); 1-9

[FPW] M. Ferraris, M. Palese, E. Winterroth, Local variational problems and conservation laws, Diff. Geom. Appl. 29 (2011), Suppl. 1, S80-S85

[G] P.L. Garcia, The Poincare-Cartan invariant in the calculus of variations, Symposia Mathematica 14 (1974) 219-246

[GF] I.M. Gelfand, S.V. Fomin, *Calculus of Variations*, Prentice Hall, New Jerseey, 1967

[GMS1] G. Giachetta, L. Mangiarotti, G. Sardanashvily, *Advanced Classical Field Theory*, World Scientific, 2009

[GMS2] G. Giachetta, L. Mangiarotti, G. Sardanashvily, *New Lagrangian and Hamiltonian Methods in Field Theory*, World Scientific, 1997

[Go] R. Godement, *Topologie Algébrique et Théorie des Faisceaux*, Hermann, Paris, 1958

[GS] H. Goldschmidt, S. Sternberg, The Hamilton-Cartan formalism in the calculus of variations, Ann. Inst. H. Poincare 23 (1973) 203-267

[Gr] D.R. Grigore, Lagrangian formalism on Grassmann manifolds, in: D. Krupka, D. Saunders, Eds., *Handbook of Global Analysis*, Elsevier, 2008, 327-373

[H] P. Havas, The range of applicability of the Lagrange formalism. I, Nuovo Cimento 5 (1957) 363-383

[He] H. von Helmholtz, Ueber die physikalische Bedeutung des Princips der kleinsten Wirkung, Journal fur die reine und angewandte Mathematik 100 (1887) 137-166, 213-222

[H] D. Hilbert, Die Grundlagen der Physik, Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse (1915) 325-407

[31] M. Horak, I. Kolar, On the higher order Poincare-Cartan forms, Czechoslovak Math. J. (1983) 467-475

[JL] J. Jost, X. Li-Jost, *Calculus of Variations*, Cambridge Univ. Press, Cambridge, 1998

[KMS] I. Kolar, P. Michor, J. Slovak, *Natural Operations in Differential Geometry*, Springer-Verlag, Berlin, 1993

[K-S] Y. Kosmann-Schwarzbach, *The Noether Theorems*, Springer, 2011

[K1] D. Krupka, A geometric theory of ordinary first order variational problems in fibered manifolds, I. Critical sections, II. Invariance, J. Math. Anal. Appl. 49 (1975) 180-206, 469-476

[K2] D. Krupka, A map associated to the Lepagean forms of the calculus of variations in fibered manifolds, Czech. Math. J. 27 (1977) 114-118

[K3] D. Krupka, A setting for generally invariant Lagrangian structures in tensor bundles, Bull. Acad. Polon. Sci. Ser. Sci. Math. Astronom. Phys. 22 (1974) 967-972

[K4] D. Krupka, Global variational theory in fibred spaces, in: D. Krupka, D. Saunders, Eds., *Handbook of Global Analysis*, Elsevier, 2008, 773-836

[K5] D. Krupka, *Introduction to Global Variational Geometry*, Beijing, 2011, Chapters 1 - 5, Lepage Inst. Archive, No. 1, 2012

[K6] D. Krupka, Lagrange theory in fibered manifolds, Rep. Math. Phys. 2 (1971) 121-133

[K7] D. Krupka, Lepage forms in Kawaguchi spaces and the Hilbert form, paper in honor of Professor Lajos Tamassy, Publ. Math. Debrecen 84 (2014), 147-164; DOI:10.5486/PMD.2014.5791

[K8] D. Krupka, Lepagean forms in higher order variational theory, in: *Modern Developments in Analytical Mechanics*, Proc. IUTAM-ISIMM Sympos., Turin, June 1982, Academy of Sciences of Turin, 1983, 197-238

[K9] D. Krupka, Local invariants of a linear connection, in: Differential Geometry, Colloq. Math. Soc. Janos Bolyai 31, North Holland, 1982, 349-369

[K10] D. Krupka, Natural Lagrange structures, in: *Semester on Differential Geometry*, 1979, Banach Center, Warsaw, Banach Center Publications 12, 1984, 185-210

[K11] D. Krupka, On the local structure of the Euler-Lagrange mapping of the calculus of variations, in: O. Kowalski, Ed., *Differential Geometry and its Applications*, Proc. Conf., N. Mesto na Morave, Czechoslovakia, Sept. 1980; Charles University, Prague, 1981, 181-188; arXiv:math-ph/0203034, 2002

[K12] D. Krupka, On the structure of the Euler mapping, Arch. Math. (Brno) 10 (1974) 55-61

[K13] D. Krupka, *Some Geometric Aspects of Variational Problems in Fibered Manifolds*, Folia Fac. Sci. Nat. UJEP Brunensis, Physica 14, Brno, Czech Republic, 1973, 65 pp.; arXiv:math-ph/0110005, 2001

[K14] D. Krupka, The total divergence equation, Lobachevskii Journal of Mathematics 23 (2006) 71-93

[K15] D. Krupka, Trace decompositions of tensor spaces, Linear and Multilinear Algebra 54 (2006) 235-263

[K16] D. Krupka, The Vainberg-Tonti Lagrangian and the Euler-Lagrange mapping, in: F. Cantrijn, B. Langerock, Eds., *Differential Geometric Methods in Mechanics and Field Theory*, Volume in Honor of W. Sarlet, Gent, Academia Press, 2007, 81-90

[K17] D. Krupka, Variational principles for energy-momentum tensors, Rep. Math. Phys. 49 (2002) 259-268

[K18] D. Krupka, Variational sequences in mechanics, Calc. Var. 5 (1997) 557-583

[K19] D. Krupka, Variational sequences on finite-order jet spaces, Proc. Conf., World Scientific, 1990, 236-254

[KJ] D. Krupka, J. Janyska, *Lectures on Differential Invariants*, J.E. Purkyne University, Faculty of Science, Brno, Czechoslovakia, 1990

[KKS1] D. Krupka, O. Krupková, D. Saunders, Cartan-Lepage forms in geometric mechanics, doi: 10.1016/j.ijnonlinmec.2011.09.002, Internat. J. of Non-linear Mechanics 47 (2011) 1154-1160

[KKS2] D. Krupka, O. Krupková, D. Saunders, The Cartan form and its generalisations in the calculus of variations, Int. J. Geom. Met. Mod. Phys. 7 (2010) 631-654

[KL] D. Krupka, M. Lenc, The Hilbert variational principle, Preprint 3/200GACR (201/00/0724), Masaryk University, Brno, 2002, 75 pp

[KM] D. Krupka, J. Musilová, Trivial Lagrangians in field theory, Diff. Geom. Appl. 9 (1998) 293-305; 10 (1999) 303

[KS] D. Krupka, D. Saunders, Eds., *Handbook of Global Analysis*, Elsevier, 2008

[KSe] D. Krupka, J. Sedenková, Variational sequences and Lepage forms, in: *Diff. Geom. Appl.*, Proc. Conf., Charles University, Prague, Czech Republic, 2005, 617-627

[KT] D. Krupka, A. Trautman, General invariance of Lagrangian structures, Bull. Acad. Polon. Sci., Ser. Sci. Math. Astronom. Phys. 22 (1974) 207-211

[KrM] M. Krbek, J. Musilová, Representation of the variational sequence by differential forms, Acta Appl. Math. 88 (2005), 177-199

[KrP] O. Krupková, G. Prince, Second order ordinary differential equations in jet bundles and the inverse problem of the calculus of variations, in *Handbook of Global Analysis*, Elsevier, 2008, 837-904

[L] J.M. Lee, *Introduction to Smooth Manifolds*, Graduate Texts in Math. 218, Springer, 2006

[LR] M. De Leon, P.R. Rodrigues, *Generalized Classical Mechanics and Field Theory: A geometric approach of Lagrangian and Hamiltonian formalisms involving higher order derivatives*, Elsevier, 2011

[Le] Th.H.J. Lepage, Sur les champs géodésiques du calcul des variations, I, II, Bull. Acad. Roy. Belg. 22 (1936), 716-729, 1036-1046

[MM] L. Mangiarotti, M. Modugno, Some results of the calculus of variations on jet spaces, Annales de l'Institut Henri Poincaré (A) Physique théorique (1983) 29-43

[MW] Mei Fengxiang, Wu Huibin, *Dynamics of Constrained Mechanical Systems*, Beijing Institute of Technology Press, 2009

[N] E. Noether, Invariante Variationsprobleme, Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse (1918) 235-257

[O1] P.J. Olver, *Applications of Lie Groups to Differential Equations*, Springer-Verlag, New York, 1998

[O2] P.J. Olver, Equivalence and the Cartan form, Acta Appl. Math. 31 (1993) 99-136

[PK] A. Patak, D. Krupka, Geometric structure of the Hilbert-Yang-Mills functional, Internat. J. Geom. Met. Mod. Phys. 5 (2008) 387-405

[PW] M. Palese, E. Winterroth, A variational perspective on classical Higgs fields in gauge-natural theories, Theoretical and Mathematical Physics 10/2011; 168(1)

[Po] J.F. Pommaret, Spencer sequence and variational sequence, Acta Appl. Math. 41 (1995) 285-296

[SCM] W. Sarlet, M. Crampin, E. Martinez, The integrability conditions in the inverse problem of the calculus of variations for second-order ordinary differential equations, Acta Appl. Math. 54 (1998) 233-273

[S] D.J. Saunders, *The Geometry of Jet Bundles*, Cambridge Univ. Press, 1989

[Sc] L. Schwartz, *Analyse Mathematique* II, Hermann, Paris, 1967

[So] N.J. Sonin, About determining maximal and minimal properties of plane curves (in Russian), Warsawskye Universitetskye Izvestiya 1-2 (1886) 1-68; English translation, Lepage Inst. Archive, No. 1, 2012

[T] F. Takens, A global version of the inverse problem of the calculus of variations, J. Differential Geometry 14 (1979) 543-562

[To] E. Tonti, Variational formulation of nonlinear differential equations, I, II, Bull. Acad. Roy. Belg. C. Sci. 55 ((1969) 137-165, 262-278

[Tr1] A. Trautman, Invariance of Lagrangian systems, in: General Relativity, Papers in Honour of J.L. Synge, Oxford, Clarendon Press, 1972, 85-99

[Tr2] A. Trautman, Noether equations and conservation laws, Commun. Math. Phys. 6 (1967) 248-261

[UK1] Z. Urban, D. Krupka, Variational sequences in mechanics on Grassmann fibrations, Acta Appl. Math. 112 (2010) 225-249

[UK2] Z. Urban, D. Krupka, The Helmholtz conditions for systems of second order homogeneous differential equations, Publ. Math. Debrecen 83 (1-2) (2013) 71-84

[UK3] Z. Urban, D. Krupka, Foundations of higher-order variational theory on Grassmann fibrations, Internat. J. of Geom. Methods in Modern Physics 11 (2014); doi:10.1142/S0219887814600238

[V] M. M. Vainberg, *Variational Methods for the Study of Nonlinear Operators*, (in Russian), Gostekhizdat, Moscow, 1956; English translation: Holden-Day, San Francisco, 1964

[VKL] A.M. Vinogradov, I.S. Krasilschik, V.V. Lychagin, *Introduction to the Geometry of Non-linear Differential Equations* (in Russian) Nauka, Moscow, 1986

[Vit] R. Vitolo, Finite order Lagrangian bicomplexes, Math. Soc. Cambridge Phil. Soc. 125 (1999) 321-333

[VU] J. Volna, Z. Urban, The interior Euler-Lagrange operator in field theory, Math. Slovaca, to appear

[W] F.W. Warner, *Foundations of Differentiable Manifolds and Lie Groups*, Springer-Verlag, New York, 1983

[We] R.O. Wells, *Differential Analysis on Complex Manifolds*, Springer-Verlag, New York, 1980

[Z] D. Zenkov (Ed.), *The Inverse Problem of the Calculus of Variations, Local and Global Theory and Applications,* Atlantis Series in Global Variatonal Geometry, to appear

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D. Krupka, *Introduction to Global Variational Geometry*,
Atlantis Studies in Variational Geometry 1, DOI 10.2991/978-94-6239-073-7

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