Richard H. Cushman
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Global Aspects of Classical Integrable Systems

Second Edition

Birkhäuser

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Foreword

This book gives a complete global geometric description of the motion of the two dimensional harmonic oscillator, the Kepler problem, the Euler top, the spherical pendulum and the Lagrange top. These classical integrable Hamiltonian systems one sees treated in almost every physics book on classical mechanics. So why is this book necessary? The answer is that the standard treatments are not complete. For instance in physics books one cannot see the monodromy in the spherical pendulum from its explicit solution in terms of elliptic functions nor can one read off from the explicit solution the fact that a tennis racket makes a near half twist when it is tossed so as to spin nearly about its intermediate axis. Modern mathematics books on mechanics do not use the symplectic geometric tools they develop to treat the qualitative features of these problems either. One reason for this is that their basic tool for removing symmetries of Hamiltonian systems, called regular reduction, is not general enough to handle removal of the symmetries which occur in the spherical pendulum or in the Lagrange top. For these symmetries one needs singular reduction. Another reason is that the obstructions to making local action angle coordinates global such as monodromy were not known when these works were written.

The point of view adopted in this book is to start with a somewhat unfamiliar abstract mathematical model of the physical system such as the study of the geodesic flow of a left invariant metric on the three dimensional rotation group. Using the symplectic geometric formulation of Hamiltonian mechanics we then show that the equations of motion agree with those found by more traditional methods for a well known physical system, namely, the force free rigid body or Euler top. This justifies our mathematical model. We do not try to build our model from fundamental physical principles. We have not written a book on mechanics or Hamiltonian particle dynamics. We only discuss five special integrable systems, which is a very small sample of the rich variety of general Hamiltonian systems. Moreover the behavior of the solutions of these integrable systems is much more regular than the nearly unpredictable motion of a general Hamiltonian system such as the three body problem.

Our main goal is to understand the global geometric features of our model integrable systems. The main tool we use is reduction to remove the symmetries and to obtain a system with one degree of freedom. This allows us to determine the range and the topology of every fiber of the energy momentum mapping of the system. The topology of a fiber corresponding to a singular value of the energy momentum mapping is of great interest. Physically, these motions are simpler than the general motion and therefore are easier to study experimentally. Mathematically, these fibers contain a relative equilibrium of the system, that is, a motion which is also an orbit of the symmetry group. For instance, in the spherical pendulum the relative equilibria are circular orbits on the 2-sphere which lie in a plane parallel to and below the equator. Other examples are the regular precession and sleeping motions of the Lagrange top. Finally, to complete the qualitative picture, we describe how the fibers of the energy momentum map fit together. Sometimes this involves showing that the monodromy of certain torus bundles are nontrivial. That this phenomenon happens in the spherical pendulum and the Lagrange top was not known until the 1980s.

This book is written from a bottom up approach with examples being given prominence over theory. The examples are treated in a uniform way. First the mathematical model is described and then the equations of motion are derived. Next the symmetries and corresponding integrals are obtained and it is shown that the given problem is Liouville integrable. Finally, the geometry of the level sets of the energy momentum map, which gives a complete geometric description of the motion, are obtained by first using reduction to remove the symmetries and then reconstructing the geometry from the geometry of the reduced system. This program may seem to be excessively lengthy. There are two reasons why we have followed it. First, our procedure gives complete answers, whereas short cut ones do not. Second, in carrying out our program the reader sees enough detail in the text to be able to understand the arguments without having to look at the theory. The theory given in chapters VI through XI is what the authors feel is the minimum necessary to justify all the unproved assertions in the examples.

This book was not written to be read in a sequential fashion. We strongly encourage the reader to browse.

Introduction

The mathematical pendulum

We begin by looking at the mathematical pendulum.

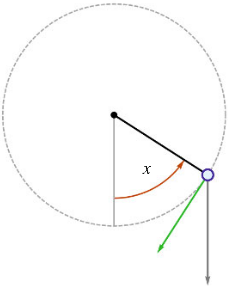

Figure 0.1. The mathematical pendulum.

Let $T^*\mathbf{R}$ be the *cotangent bundle* of $\mathbf{R}$, which we identify with $\mathbf{R}^2$ and give coordinates (x, y). The canonical symplectic form $\omega = \mathrm{d}x \wedge \mathrm{d}y$ on $T^*\mathbf{R}$ is the element of oriented area on $\mathbf{R}^2$. Consider the *Hamiltonian system* $(H, T^*\mathbf{R}, \omega)$ with *Hamiltonian*

$$H : T^*\mathbf{R} \to \mathbf{R} : (x,y) \mapsto \tfrac{1}{2}y^2 - \cos x.$$

$\triangleright$ The following argument shows that the *Hamiltonian vector field* X_H on $T^*\mathbf{R}$ corresponding to the Hamiltonian H is

$$X_H(x,y) = \dot{x}\frac{\partial}{\partial x} + \dot{y}\frac{\partial}{\partial y} = y\frac{\partial}{\partial x} - \sin x\frac{\partial}{\partial y}. \tag{1}$$

(0.1) **Proof**: By definition of Hamiltonian vector field, see appendix A §3,

$$\mathrm{d}H(p)z_p = \omega(p)(X_H(p), z_p) \tag{2}$$

for every $z_p = (v_p, w_p)$ in the tangent space $T_p(T^*\mathbf{R})$ to $T^*\mathbf{R}$ at p. Let $X_H(p) = \big(X(p), Y(p)\big)$. Now $\mathrm{d}H(p)z_p = \frac{\partial H}{\partial x}v_p + \frac{\partial H}{\partial y}w_p$. Moreover, $\omega(p)(X_H(p), z_p)$ is the oriented area spanned by the parallelogram with sides $X_H(p)$ and z_p, that is,

$$\omega(p)(X_H(p), z_p) = \det\begin{pmatrix} X(p) & v_p \\ Y(p) & w_p \end{pmatrix} = X(p)w_p - Y(p)v_p.$$

Therefore (2) is equivalent to

$$\frac{\partial H}{\partial x}v_p + \frac{\partial H}{\partial y}w_p = -Y(p)v_p + X(p)w_p \tag{3}$$

for every $(v_p, w_p) \in \mathbf{R}^2$. In (3) choose $(v_p, w_p) = (1,0)$. Then $X(p) = \frac{\partial H}{\partial y} = y$. Next choose $(v_p, w_p) = (0,1)$. Then $Y(p) = -\frac{\partial H}{\partial x} = -\sin x$. □

Note that (1) may be written as the second order differential equation

$$\ddot{x} = -\frac{\mathrm{d}}{\mathrm{d}x}(-\cos x) = -\sin x. \tag{4}$$

By Newton's second law of motion, an integral curve of (1) describes the motion of a particle of unit mass under a force coming from the *potential* $V : \mathbf{R} \to \mathbf{R} : x \mapsto -\cos x$.

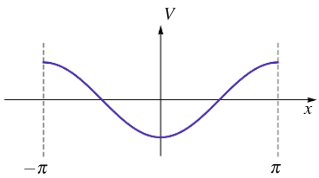

Figure 0.2. The graph of the potential $V(x) = -\cos x$.

Thus H is the total energy of the particle, namely the sum of the *kinetic* and potential ▷ energy. We now show that H is a Morse function on $T^*\mathbf{R}$.

(0.2) **Proof**: The point $p = (x,y)$ is a critical point of H if and only if $X_H(p) = 0$, that is, if and only if

$$0 = \frac{\partial H}{\partial y} = \sin x \qquad \text{and} \qquad 0 = \frac{\partial H}{\partial x} = y.$$

Thus $\{p = (n\pi, 0) \in \mathbf{R}^2 | n \in \mathbf{Z}\}$ is the set of *critical points* of H. The corresponding *critical value* of H is -1 if n is even or 1 if n is odd. Since the Hessian of H at p is

$$D^2H(p) = \left.\begin{pmatrix} \dfrac{\partial^2 H}{\partial x^2} & \dfrac{\partial^2 H}{\partial x \partial y} \\ \dfrac{\partial^2 H}{\partial y \partial x} & \dfrac{\partial^2 H}{\partial y^2} \end{pmatrix}\right|_p = \begin{pmatrix} \cos n\pi & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} (-1)^n & 0 \\ 0 & 1 \end{pmatrix},$$

H is a Morse function, because $D^2H(p)$ is nondegenerate. □

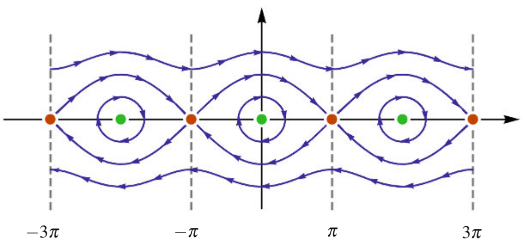

Figure 0.3. The level sets of $H(x,y) = \frac{1}{2}y^2 - \cos x$.

When $n = 2k$ the Morse index of $D^2H(n\pi, 0)$ is zero, and so the critical points $(2k\pi, 0)$ are relative minima of H; whereas when $n = 2k+1$ the *Morse index* of $D^2H(n\pi, 0)$ is one, and so the critical points $(2k\pi,0)$ are saddle points of H, see figure .3. Using the Morse lemma, see appendix F §1, there is a neighborhood U_k of $(2k\pi,0)$ in the open strip $((2k-1)\pi,(2k+1)\pi)\times\mathbf{R}$ such that for h slightly greater than -1 the *level set* $H^{-1}(h)\cap U_k$ is diffeomorphic to a circle. Since H has no critical values in the interval $(-1,1)$, by the Morse isotopy lemma, see appendix F §3, we deduce that for every $h\in(-1,1)$ the level set $H^{-1}(h)\cap U_k$ is diffeomorphic to a circle. Thus for $h\in(-1,1)$ the whole level set $H^{-1}(h)$ is diffeomorphic to a countable disjoint union of circles. If $h\ge 1$, then $H^{-1}(h)$ is the union of the graphs of two smooth functions $y_{\pm}=\pm\sqrt{2(h+\cos x)}$. The graphs of $y_{\pm}$ are disjoint if $h>1$. On the other hand, if $h=1$, then the graphs of $y_{\pm}=\pm 2\cos\frac{1}{2}x$ intersect only at the points $((2k+1)\pi,0)$. There they intersect transversely as can be seen by applying the Morse lemma at the points $((2k+1)\pi,0)$. Thus we have obtained a picture of the level curves of H as given in figure .3.

To simplify the topology of the level sets of H, we make use of the fact that H is invariant under the *translation symmetry*

$$\mathbf{Z}\times T^*\mathbf{R}\to T^*\mathbf{R}:\big(n,(x,y)\big)\mapsto(x+2n\pi,y). \tag{5}$$

Thus H induces a function $\widetilde{H}$ on the space of orbits $T^*\mathbf{R}/2\pi\mathbf{Z}$. Concretely, this orbit space is identified with the cotangent bundle T^*S^1 of the circle S^1. Here S^1 is

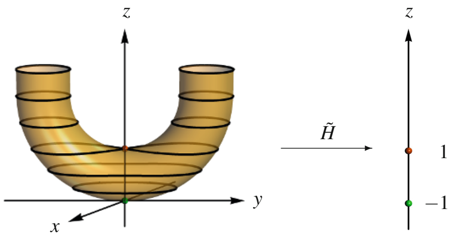

Figure 0.4. The graph of $\widetilde{H}(x,y)=\frac{1}{2}y^2-\cos x$ with $(x,y)\in T^*S^1$.

thought of as the orbit space $\mathbf{R}/2\pi\mathbf{Z}$ of the real numbers modulo 2π. Geometrically, T^*S^1 is the cylinder $S^1\times\mathbf{R}$ which is obtained from $\mathbf{R}^2$ by cutting along the vertical lines $x=0$ and $x=2\pi$ and then pasting the edges together. Applying this process to figure .3 gives figure .4 which depicts the level sets of $\widetilde{H}$ and hence the orbits of the induced Hamiltonian vector field $X_{\widetilde{H}}$. A short argument using Newton's second law shows that the second order differential equation

$$\ddot{x}=-\sin x \qquad x \bmod 2\pi$$

describes the motion of a particle of mass one on the unit circle under the influence of a constant vertical downward unit force, see figure .1.

From figure .4 we see that the topological circle, defined by the component of the level set $\widetilde{H}^{-1}(h)$ $(h>1)$ lying in the upper half cylinder, is very different from the topological circle defined by the level set $\widetilde{H}^{-1}(h)$ $(-1<h<1)$. The first circle is not contractible in TS^1 to a point whereas the second circle is. Hence it is impossible to continuously deform the first circle into the second one. This difference in the topological disposition of the two circles corresponds to the physical fact that for small energy the particle oscillates about the bottom of the circle, while for large energy the particle loops over the top of the circle.

Exercises

1. Let (x,y) be canonical coordinates on $T^*\mathbf{R} = \mathbf{R}^2$ with symplectic form $\omega = \mathrm{d}x \wedge \mathrm{d}y$. Suppose that the Hamiltonian $H : T^*\mathbf{R} \to \mathbf{R}$ is a sum of kinetic and potential energy, that is, $H(x,y) = \frac{1}{2}y^2 + V(x)$, where $V : \mathbf{R} \to \mathbf{R}$.

   a) Find a potential function V such that the zero level set of H is connected, compact and has one singular point which is a cusp.

   b) Construct a polynomial Hamiltonian on $T^*\mathbf{R}$ whose zero level set is an n-leaf clover.

   c) Show that there is *no* smooth Hamiltonian which is the sum of kinetic and potential energy which has a 3-leaf clover as a level set.

   d) For smooth V with countable many isolated critical points give a topological characterization of the critical level sets of H.

2. Construct a Hamiltonian function on S^2 which is a Morse function with two critical points. Draw its level sets. Construct a vector field on S^2 with only one equilibrium point and sketch its orbits. Show that this vector field is not Hamiltonian.

3. a) On $\mathbf{R}^2$ consider the action $\cdot$ of $\mathbf{Z}^2$ defined by $(n,m)\cdot(x,y) = (x+n, y+m)$. The orbit space $\mathbf{R}^2/\mathbf{Z}^2$ is a two dimensional torus T^2, which may be modeled by a square with the opposite sides identified. The symplectic form $\Omega = \mathrm{d}x \wedge \mathrm{d}y$ on $\mathbf{R}^2$ induces a symplectic form $\widetilde{\Omega}$ on T^2. Show that the vector field $\widetilde{X}$ on T^2 induced by the Hamiltonian vector field $X = \frac{\partial}{\partial x} + \frac{\partial}{\partial y}$ on $\mathbf{R}^2$ is *not* Hamiltonian on $(T^2, \widetilde{\Omega})$.

   b) Sketch the orbits of a Hamiltonian vector field on $(T^2, \widetilde{\Omega})$ where the Hamiltonian is a Morse function with the fewest number of critical points.

   c) Construct a vector field X on T^2 with *two* equilibrium points.

   d)* Show that a smooth function on T^2 must have at least three critical points. Find a smooth function on T^2 with exactly three critical points. Sketch its level sets.

   e) Deduce that the vector field X constructed in c) is not Hamiltonian.

4. Let M be a compact connected orientable smooth two dimensional manifold with volume form Ω. In what follows we show that every integral curve of a Hamiltonian vector field X_H of a one degree of freedom Hamiltonian system (H, M, Ω) is either an equilibrium point, a periodic orbit, or is asymptotic to an equilibrium point as $t \to \pm\infty$. For $m \in M$ let $\gamma : \mathbf{R} \to M : t \mapsto \varphi_t^H(m)$ be the integral curve of X_H through m. The *ω-limit set* $\omega(\gamma)$of γ is the closure of the set $\bigcap_{T>0} \{ \varphi_t^H(m) \big| t \geq T \}$.

   a) Show that $\omega(\gamma)$ is nonempty.

b) If γ is an equilibrium point or a periodic orbit of X_H, show that $\omega(\gamma) = \gamma$. Is the converse true?

c)* If γ is not a periodic orbit of X_H, show that $\omega(\gamma)$ is a critical point of H, that is, an equilibrium point of X_H.

4. (Period energy relation for the mathematical pendulum.) When $|h| < 1$ show that the period of an integral curve of the mathematical pendulum which starts at $(x_+, 0)$ where $0 < x_+ = x_+(h) < \pi$ and $h + \cos x_+ = 0$, is given by

$$\tau(h) = 2\int_{-x_+}^{x_+} \frac{1}{\sqrt{2(h+\cos x)}}\, dx.$$

Show that $\tau = 4K(\sqrt{(h+1)/2})$, where K is the complete elliptic integral of the first kind, see the exercises of chapter 1. Deduce that

a) $\tau(-1) = 2\pi$, $\tau(1) = \infty$ and $\tau'(-1) = \pi/4$.

b) τ is a real analytic function on $(-1, 1)$.

c)* $\tau' > 0$ on $(-1, 1)$. Hint: show that τ satisfies a differential equation.

5. a) Suppose that a particle moves on the graph of $y = f(x)$ under the influence of gravity and that the origin is a stable equilibrium point. Determine the shape of the graph of f so that the period of oscillation of the particle about the origin is a constant independent of the energy.

b)* Show that a) is equivalent to the fact the derivative of the area enclosed by a level set of the Hamiltonian of the particle with respect to the Hamiltonian itself is a constant. Hint: see appendix D §1.

6. (Reduction of discrete symmetry of mathematical pendulum.)

a) (Discrete symmetry.) Show that

$$\zeta : S^1 \times \mathbf{R} \to S^1 \times \mathbf{R} : (x, y) \mapsto (-x, -y). \tag{6}$$

generates a $\mathbf{Z}_2$-*symmetry* of the mathematical pendulum. Show that the fundamental domain $\mathscr{D}$ of the $\mathbf{Z}_2$-action on T^*S^1 generated by ζ is the piece of the cylinder in figure .4, which lies in the half space $y \ge 0$ with the points $(\pm x, 0)$ on the circle $\partial\mathscr{D} = T^*S^1 \cap \{y = 0\}$ identified. Deduce that the orbit space $P = T^*S^1/\mathbf{Z}_2$ is homeomorphic to a cone on S^1 with vertex at the $\mathbf{Z}_2$-orbit corresponding to the point $(0,0) \in T^*S^1$.

b) Show that the algebra of real analytic invariant functions of the abelian group $\mathbf{Z}_2$ generated by ζ is generated by

$$\tau_1 = \cos x, \qquad \tau_2 = y\sin x, \qquad \tau_3 = \tfrac{1}{2}y^2 - \cos x \tag{7}$$

subject to the relation

$$C(\tau) = \tfrac{1}{2}\tau_2^2 - (\tau_3 + \tau_1)(1 - \tau_1^2) = 0, \quad |\tau_1| \le 1 \;\&\; \tau_3 \ge -1, \tag{8}$$

which defines the orbit space P. Draw a picture of the semialgebraic variety P.

c) (Reduced Poisson bracket.) In order to have dynamics on the $\mathbf{Z}_2$-reduced space P we first need a Poisson bracket $\{\,,\,\}_{\mathbf{R}^3}$ on $C^\infty(\mathbf{R}^3)$. A calculation shows that

$$\begin{aligned}
\{\tau_1,\tau_2\} &= \tau_1^2 - 1 = \frac{\partial C}{\partial \tau_3} \\
\{\tau_2,\tau_3\} &= 2\tau_1(\tau_3+\tau_1) + \tau_1^2 - 1 = \frac{\partial C}{\partial \tau_1} \\
\{\tau_3,\tau_1\} &= \tau_2 = \frac{\partial C}{\partial \tau_2}
\end{aligned}$$

Then for every F, $G \in C^\infty(\mathbf{R}^3)$ we have $\{F,G\} = \sum_{i,j} \frac{\partial F}{\partial \tau_i}\frac{\partial G}{\partial \tau_j}\{\tau_i,\tau_j\}$. We say that a function f on P is smooth if there is a smooth function F on $\mathbf{R}^3$ such that $f = F|P$. Let $C^\infty(P)$ be the space of smooth functions on P. Then $(P, C^\infty(P))$ is a *differential space*, which is *subcartesian* because P is a semialgebraic variety. On $C^\infty(P)$ we define a *Poisson bracket* $\{\,,\,\}_P$ as follows. Suppose that $f,g \in C^\infty(P)$. Then there are $F,G \in C^\infty(\mathbf{R}^3)$ such that $f = F|P$ and $g = G|P$. Let $\{f,g\}_P = \{F,G\}_{\mathbf{R}^3}|P$. Because the defining function C (8) of the orbit space P is a *Casimir* in the Poisson algebra $\mathscr{A} = (C^\infty(\mathbf{R}^3), \{\,,\,\}_{\mathbf{R}^3}, \cdot)$, the collection $\mathscr{I}$ of all smooth functions on $\mathbf{R}^3$, which vanish identically on P, is a *Poisson ideal* in $\mathscr{A}$. Consequently, the Poisson bracket $\{\,,\,\}_P$ is well defined. So $\mathscr{B} = \big(C^\infty(P) = C^\infty(\mathbf{R}^3/\mathscr{I}), \{\,,\,\}_P, \cdot\big)$ is a Poisson algebra.

d) (Reduced dynamics.) Consider the derivation $-\mathrm{ad}_H$ on the Poisson algebra $\mathscr{A}$. This derivation gives rise to the $\mathbf{Z}_2$-reduced Hamiltonian vector field X_H on the subcartesian differential space $(P, C^\infty(P))$ associated to the $\mathbf{Z}_2$-*reduced Hamiltonian* $H : P \to \mathbf{R} : \tau \mapsto \tau_3$. On $\mathbf{R}^3$ the integral curves of $-\mathrm{ad}_H$ satisfy

$$\begin{aligned}
\dot\tau_1 &= \{\tau_1, H\}_P = \{\tau_1,\tau_3\}_P = -\tau_2 \\
\dot\tau_2 &= \{\tau_2, H\}_P = \{\tau_2,\tau_3\}_P = 2\tau_1(\tau_3+\tau_1) + \tau_1^2 - 1 \\
\dot\tau_3 &= \{\tau_1, H\}_P = \{\tau_1,\tau_3\}_P = 0.
\end{aligned}$$

The equality $\dot\tau_3 = 0$ shows that H is an integral of X_H. Check that C (8) is also an integral of X_H. A calculation shows that $-\mathrm{ad}_H$ leaves $C^{-1}(0)$, $\{\tau_3+\tau_1 = 0\}$, and $\{\tau_1 = \pm 1\}$ invariant. Thus the *reduced space* P is invariant under the flow of $-\mathrm{ad}_H$. Consequently, the *reduced Hamiltonian vector field* X_H on P is $-\mathrm{ad}_H|P$. Because the Hamiltonian vector field $X_{\widetilde{H}}$ of the mathematical pendulum is complete, the reduced vector field X_H is complete. Its flow φ_t^H is a 1-parameter group of diffeomorphisms of P. In fact, for $p \in H^{-1}(e)$ the closure of the integral curve $\{\varphi_t^H(p) \in P \,\big|\, t \in \mathbf{R}\}$ is a connected component of the level set $H^{-1}(e)$, since a level set of the reduced Hamiltonian H is compact.

Part I. Examples

Chapter I

The harmonic oscillator

1 Hamilton's equations and S^1 symmetry

Physically, the *harmonic oscillator* in the plane is described by a particle of unit mass acted upon by two linear springs of unit spring constant: one spring acts in the x_1-direction and the other in the x_2-direction. Mathematically, the *configuration space* of the harmonic oscillator is Euclidean 2-space. In other words, the space of positions of the particle is $\mathbf{R}^2$ with coordinates $x = (x_1, x_2)$ and *Euclidean inner product* $(\, , \,)$ where $(x, x') = x_1 x_1' + x_2 x_2'$. The space of all positions and momenta of the particle is the *cotangent bundle* $T^*\mathbf{R}^2$ of $\mathbf{R}^2$. This phase space has coordinates (x, y) and a canonical *symplectic form* $\omega = \mathrm{d}x_1 \wedge \mathrm{d}y_1 + \mathrm{d}x_2 \wedge \mathrm{d}y_2$. The *Hamiltonian* function $H : T^*\mathbf{R}^2 \to \mathbf{R}$ of the harmonic oscillator is the sum of *kinetic energy* $\frac{1}{2}(y, y)$ and *potential energy* $\frac{1}{2}(x, x)$. Letting $z = (x, y) \in \mathbf{R}^4$,

$$H(z) = \tfrac{1}{2}(y, y) + \tfrac{1}{2}(x, x) = \tfrac{1}{2}\langle z, z\rangle. \tag{1}$$

Here $\langle \, , \, \rangle$ is the Euclidean inner product on $\mathbf{R}^4$, which we have identified with $T^*\mathbf{R}^2$.

The motion of the harmonic oscillator is described by Hamilton's equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix} \begin{pmatrix} D_1 H(x, y) \\ D_2 H(x, y) \end{pmatrix} = \begin{pmatrix} y \\ -x \end{pmatrix}.$$

Here I_2 is the 2×2 identity matrix. Since the *Hamiltonian vector field* $X_H(x, y) = (y, -x)$ is linear, the flow of the linear vector field is

$$\varphi^H : \mathbf{R} \times \mathbf{R}^4 \to \mathbf{R}^4 : (t, z) \mapsto (\exp tX_H)\, z = \begin{pmatrix} (\cos t) I_2 & (\sin t) I_2 \\ -(\sin t) I_2 & (\cos t) I_2 \end{pmatrix} z. \tag{2}$$

Given any initial condition $z \in T^*\mathbf{R}^2$, the *integral curve* of X_H through the point z is $t \mapsto \varphi^H_t(z)$. Thus from a quantitative point of view, we know everything about the vector field X_H, because we have an explicit formula (2) for every integral curve. On the other hand, from a qualitative point of view, the explicit formula is *very* unsatisfactory. For instance, we do not know if the integral curves lie on a lower dimensional invariant manifold or

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how they fit together. In the rest of this chapter we will describe the *global qualitative* features of the invariant manifolds of X_H.

Claim: The h-level set $H^{-1}(h)$ of the Hamiltonian H (1) is an invariant manifold of the vector field of the harmonic oscillator.

(1.1) **Proof**: Since

$$\dot{H} = L_{X_H} H = \langle \dot{x}, x\rangle + \langle \dot{y}, y\rangle = \langle y, x\rangle - \langle x, y\rangle = 0,$$

H is constant on the integral curves of X_H, that is, H is an integral (= conserved quantity) of X_H, see chapter VII §3. In particular the h-level set

$$H^{-1}(h) = \big\{ z \in \mathbf{R}^4 \,\big|\, \tfrac{1}{2}\langle z, z\rangle = h \big\},$$

which is diffeomorphic to a 3-sphere S^3 when $h > 0$, a point when $h = 0$, and is empty when $h < 0$, is a smooth *invariant manifold* of X_H. In other words, every integral curve of X_H with initial condition in $H^{-1}(h)$ lies in $H^{-1}(h)$ for all time. □

The rotational symmetry of the potential energy $\frac{1}{2}\langle x, x\rangle$ gives rise to another conserved quantity, namely, the *angular momentum*

$$L : T^*\mathbf{R}^2 \to \mathbf{R} : (x, y) \mapsto x_1 y_2 - x_2 y_1. \tag{3}$$

To see this, consider the $S^1 = \mathbf{R}/2\pi\mathbf{Z}$-action on $\mathbf{R}^2$

$$\psi : S^1 \times \mathbf{R}^2 \to \mathbf{R}^2 : (s, x) \mapsto R_s x = \begin{pmatrix} \cos s & -\sin s \\ \sin s & \cos s \end{pmatrix} x.$$

ψ_s is a counterclockwise rotation through an angle s about the origin. The infinitesimal generator of the action ψ is the vector field

$$X(x) = \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0} \psi_s(x) = (-x_2, x_1).$$

ψ lifts to an S^1 action Ψ on $T^*\mathbf{R}^2$ defined by

$$\Psi : S^1 \times T^*\mathbf{R}^2 \to T^*\mathbf{R}^2 : \big(s, (x, y)\big) \mapsto (R_s x, R_s y).$$

Ψ preserves the *canonical* 1-*form* $\theta = y_1\,\mathrm{d}x_1 + y_2\,\mathrm{d}x_2 = (y, \mathrm{d}x)$ on $T^*\mathbf{R}^2$, since

$$\Psi_s^*\theta = (R_s y, \mathrm{d}R_s x) = (R_s y, R_s\,\mathrm{d}x) = (y, \mathrm{d}x) = \theta.$$

Therefore Ψ_s is a symplectic mapping, that is, $\Psi_s^*\omega = \omega$, since

$$\omega = -\,\mathrm{d}\theta = -\,\mathrm{d}\Psi_s^*\theta = -\Psi_s^*\,\mathrm{d}\theta = \Psi_s^*\omega.$$

The infinitesimal generator of the action Ψ is the vector field

$$Y(x, y) = \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0} \Psi_s(x, y) = (-x_2, x_1, -y_2, y_1) = \big(D_2 L(x, y), -D_1 L(x, y)\big).$$

Thus Y is a Hamiltonian vector field X_L corresponding to the angular momentum L (3). That the lifted S^1 action Ψ preserves the canonical 1-form and is the flow of the Hamiltonian vector field corresponding to the Hamiltonian $L = X \lrcorner\, \theta$ is a particular case of a more general set of results, see chapter VII §3. Since

$$(\Psi_s^* H)(x,y) = \tfrac{1}{2}(R_s y, R_s y) + \tfrac{1}{2}(R_s x, R_s x) = H(x,y),$$

H is constant on the integral curves of X_L. Therefore L is constant on the integral curves of X_H, that is, L is an integral of X_H, see chapter VI §4. This implies that the ℓ-level set of L,

$$L^{-1}(\ell) = \big\{(x,y) \in T^*\mathbf{R}^2 \big| x_1y_2 - x_2y_1 = \ell\big\},$$

is an invariant manifold of X_H. When $\ell \neq 0$ the level set $L^{-1}(\ell)$ is diffeomorphic to $S^1 \times \mathbf{R}^2$, while when $\ell = 0$ the level set $L^{-1}(0)$ is homeomorphic but *not* diffeomorphic to $S^1 \times \mathbf{R}^2$ as it is a cone on $S^1 \times S^1$ together with its vertex at the origin.

2 S^1 energy momentum mapping

In order to organize the qualitative information about the harmonic oscillator which can be obtained from the integrals of energy and angular momentum, define the S^1 *energy momentum mapping*

$$\mathscr{EM} : T^*\mathbf{R}^2 \to \underline{\mathbf{R}}^2 : (x,y) \mapsto \big(H(x,y), L(x,y)\big) = \big(\tfrac{1}{2}(y_1^2+y_2^2+x_1^2+x_2^2), x_1y_2 - x_2y_1\big).$$

Because H and L are polynomial integrals of X_H, the *fiber* $\mathscr{EM}^{-1}(h,\ell)$ is an invariant manifold of X_H which is a real algebraic variety. Other geometric properties of $\mathscr{EM}$ correspond to qualitative properties of X_H. To describe such global geometric properties of $\mathscr{EM}$, we shall

1. Find the critical points, critical values and range of $\mathscr{EM}$.
2. Find the topological type of *every* fiber of the energy momentum mapping. This determines the *bifurcation set* of $\mathscr{EM}$, the set of values (h,ℓ) where the *topological type of the fiber* changes.
3. Analyze how the fibers of constant angular momentum *foliate* a given energy level set.

▷ We begin by finding the critical points and corresponding critical values of the energy momentum map. A point $z = (x,y) \in T^*\mathbf{R}^2$ is a critical point of $\mathscr{EM}$ if and only if the derivative of $\mathscr{EM}$ at z is not surjective, that is,

$$D\mathscr{EM}(z) = \begin{pmatrix} DH(z) \\ DL(z) \end{pmatrix} = \begin{pmatrix} x_1 & x_2 & y_1 & y_2 \\ y_2 & -y_1 & -x_2 & x_1 \end{pmatrix} \tag{4}$$

has rank less than two. There are two cases to be considered.

CASE I: $\operatorname{rank} D\mathscr{EM}(z) = 0$. This can only happen if $DH(z) = DL(z) = 0$. Then $z = 0$ is the critical point and $\mathscr{EM}(0) = (0,0)$ is the corresponding critical value of $\mathscr{EM}$.

CASE II. rank$D\mathcal{EM}(z) = 1$. This occurs if and only if $DH(z)$ and $DL(z)$ are linearly dependent and are not both zero. From (4) it follows that $DH(z) = 0$ if and only if $DL(z) = 0$. Therefore we may suppose that $DH(z) \neq 0$, that is, $z \neq 0$. Thus for some $\mu \in \mathbf{R}$

$$0 = DL(z) - \mu\, DH(z) = (y_2, -y_1, -x_2, x_1) - \mu\,(x_1, x_2, y_1, y_2). \tag{5}$$

If $\mu = 0$, then $z = 0$, which is a contradiction. Therefore $\mu \neq 0$. Composing the linear mapping $(x_1, x_2, y_1, y_2) \mapsto \mu^{-1}(y_2, -y_1, -x_2, x_1)$ with itself gives $z = \mu^{-2} z$. Thus $\mu^2 = 1$, since $z \neq 0$. Consequently, the solutions of (5) define two punctured 2-planes

$$\Pi^*_+ = \big\{(x_1, x_2, -x_2, x_1) \in T^*\mathbf{R}^2 \big| (x_1, x_2) \in \big(\mathbf{R}^2 \setminus \{(0,0)\}\big)\big\}$$

and

$$\Pi^*_- = \big\{(x_1, x_2, x_2, -x_1) \in T^*\mathbf{R}^2 \big| (x_1, x_2) \in \big(\mathbf{R}^2 \setminus \{(0,0)\}\big)\big\}.$$

On Π^*_+ the corresponding set of critical values of $\mathcal{EM}$ is the diagonal ray $\{(h,h) \in \mathbf{R}^2 \,|\, h > 0\}$ since $0 < \ell = L|\Pi^*_+ = x_1^2 + x_2^2 = H|\Pi^*_+ = h$; while on Π^*_- the corresponding critical values of $\mathcal{EM}$ is the antidiagonal ray $\{(h,-h) \in \mathbf{R}^2 \,|\, h > 0\}$. Therefore the critical fiber $\mathcal{EM}^{-1}(h,h)$, $h > 0$ is the circle

$$S^1_{+,h} = H^{-1}(h) \cap \Pi^*_+ = \big\{(x_1, x_2, -x_2, x_1) \in T^*\mathbf{R}^2 \big| x_1^2 + x_2^2 = h,\ h > 0\big\}; \tag{6}$$

while the critical fiber $\mathcal{EM}^{-1}(h,-h)$, $h > 0$ is the circle

$$S^1_{-,h} = H^{-1}(h) \cap \Pi^*_- = \big\{(x_1, x_2, x_2, -x_1) \in T^*\mathbf{R}^2 \big| x_1^2 + x_2^2 = h,\ h > 0\big\}. \tag{7}$$

Note that the image of $S^1_{\pm,h}$ under the bundle projection $\pi : T^*\mathbf{R}^2 \to \mathbf{R}^2 : (x,y) \mapsto x$ is the circle $x_1^2 + x_2^2 = h$, which is positively oriented when the sign is $+$ and negatively otherwise. Another way to interpret the critical circles $S^1_{\pm,h}$ is to note that solving (5) subject to the condition that $z \in H^{-1}(h)$, $h > 0$ is equivalent to finding the critical points
▷ of L on $H^{-1}(h)$. Thus L has two critical manifolds $S^1_{\pm,h}$ on $H^{-1}(h)$. The manifolds $S^1_{\pm,h}$ are nondegenerate of Morse index 2 for $S^1_{+,h}$ and 0 for $S^1_{-,h}$.

(2.1) **Proof**: To show that $S^1_{+,h}$ is a nondegenerate critical manifold of $L|H^{-1}(h)$ we must verify that at every $p = (x_1, x_2, -x_2, x_1) \in S^1_{+,h}$, the Hessian of $L|H^{-1}(h)$ when restricted to a 2-plane $N_p S^1_{+,h}$ normal to $S^1_{+,h}$ in $T_p H^{-1}(h)$ has Morse index 2. From the fact that p is a critical point of $L|H^{-1}(h)$ with Lagrange multiplier $\mu = 1$, it follows that the Hessian of $L|H^{-1}(h)$ at p is Q_+, which equals

$$D^2\big(L|H^{-1}(h)\big)(p) = \big(D^2L(p) - D^2H(p)\big)\,|\,T_pH^{-1}(h) = \left.\begin{pmatrix} -1 & 0 & 0 & 1 \\ 0 & -1 & -1 & 0 \\ 0 & -1 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}\right|_{\ker DH(p)}$$

see chapter XI §2. Since $S^1_{+,h}$ is an *orbit* of X_H, we see that $T_p S^1_{+,h}$ is spanned by the vector $X_H(p) = (-x_2, x_1, -x_1, -x_2)$. As $\ker DH(p)$ is spanned by the linearly independent vectors

$$X_H = (-x_2, x_1, -x_1, -x_2),\ f_1 = (x_2, -x_1, -x_1, -x_2),\ f_2 = (x_1, x_2, x_2, -x_1),$$

a normal 2-plane $N_pS^1_{+,h}$ is spanned by the vectors $\{f_1, f_2\}$. A calculation shows that the matrix of Q_+ with respect to the basis $\{f_1, f_2\}$ is $-2I_2$. Therefore $Q_+|N_pS^1_{+,h}$ is nondegenerate with Morse index 2. A similar calculation shows that at $p \in S^1_{-,h}$ the Hessian $Q_- = D^2\big(L|H^{-1}(h)\big)(p)$ restricted to $N_pS^1_{-,h}$ is equal to $2I_2$. Thus $Q_-|N_pS^1_{-,h}$ is nondegenerate with Morse index 0. □

Consequently, on $S^1_{+,h}$ the function L assumes its maximum value h, while on $S^1_{-,h}$ the function L assumes its minimum value $-h$. Therefore, the closed wedge $\{(h,\ell) \in \mathbf{R}^2 \,\big|\, 0 \le |\ell| \le h\}$ is the image of the energy momentum mapping $\mathscr{EM}$.

▷ To find the topology of a fiber of $\mathscr{EM}$ corresponding to a regular value, we simultaneously diagonalize the quadratic forms defining the energy and angular momentum by a linear *symplectic coordinate change*. Consider the linear change of coordinates on $\mathbf{R}^4$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix} = P \begin{pmatrix} \xi \\ \eta \end{pmatrix}, \tag{8}$$

where $A = \frac{1}{\sqrt{2}}\left(\begin{smallmatrix} 0 & 0 \\ 1 & -1 \end{smallmatrix}\right)$ and $B = \frac{1}{\sqrt{2}}\left(\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix}\right)$. Since $A^tA + B^tB = I_2$ and $A^tB = B^tA$, the matrix P is *symplectic* and orthogonal, that is, $P^*\omega = \omega$ and $\langle Pz, Pw\rangle = \langle z, w\rangle$ for every $z, w \in \mathbf{R}^4$. With respect to the (ξ, η) coordinates, the Hamiltonian H becomes

$$\widetilde{H}(\xi,\eta) = (P^*H)(\xi,\eta) = (H \circ P)(\xi,\eta) \,=\, \tfrac{1}{2}(\eta_1^2 + \eta_2^2 + \xi_1^2 + \xi_2^2).$$

Because P is symplectic, the Hamiltonian vector field $X_{\widetilde{H}}$ corresponding to $\widetilde{H}$ is $P^{-1}X_HP$, that is,

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} \eta \\ -\xi \end{pmatrix}.$$

Moreover the angular momentum L becomes

$$\widetilde{L} = (P^*L)(\xi,\eta) = \tfrac{1}{2}(\eta_2^2 - \eta_1^2 + \xi_2^2 - \xi_1^2).$$

Since P is symplectic, $\widetilde{L}$ is an integral of $X_{\widetilde{H}}$. Therefore the fiber of $\widetilde{\mathscr{EM}} = \mathscr{EM} \circ P$ at (h,ℓ) is the set of $(\xi,\eta) \in \mathbf{R}^4$ which satisfy

$$\begin{aligned} \tfrac{1}{2}(\eta_1^2 + \eta_2^2 + \xi_1^2 + \xi_2^2) \,&=\, h \,=\, \widetilde{H}(\xi,\eta) \\ \tfrac{1}{2}(\eta_2^2 - \eta_1^2 + \xi_2^2 - \xi_1^2) \,&=\, \ell \,=\, \widetilde{L}(\xi,\eta). \end{aligned} \tag{9}$$

This implies

$$\begin{aligned} \eta_1^2 + \xi_1^2 \quad &= h - \ell \\ \eta_2^2 + \xi_2^2 \quad &= h + \ell. \end{aligned} \tag{10}$$

Therefore when $0 \le |\ell| < h$, that is, when (h,ℓ) is a regular value of $\widetilde{\mathscr{EM}}$, each of the equations in (10) defines a circle. Hence $\widetilde{\mathscr{EM}}^{-1}(h,\ell)$ is a 2-torus $\widetilde{T}^2_{h,\ell}$. Since P is a diffeomorphism, we find that $\mathscr{EM}^{-1}(h,\ell)$ is a 2-torus when (h,ℓ) is a regular value. □

We now describe geometrically how the orbits of $X_{\widetilde{H}}|\widetilde{T}^2_{h,\ell}$ foliate the 2-torus $\widetilde{T}^2_{h,\ell}$. Observe

▷ that the flow $\varphi_t^{\widetilde{H}}$ of $X_{\widetilde{H}}$ defines a free proper action of $S^1 = \mathbf{R}/2\pi\mathbf{Z}$ on the 2-torus $\widetilde{T}^2_{h,\ell}$.

(2.2) **Proof**: Since S^1 is compact, we need only show that at every point $p \in \widetilde{T}^2_{h,\ell}$ the isotropy group $\{t \in S^1 \,|\, \varphi_t^{\widetilde{H}}(p) = p\}$ is the identity element of S^1. For $(\xi,\eta) \neq 0$ and $t \in \mathbf{R}$

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} = \varphi_t^{\widetilde{H}} \begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} (\cos t)I_2 & (\sin t)I_2 \\ -(\sin t)I_2 & (\cos t)I_2 \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix}$$

implies that $t = 2n\pi$ for every $n \in \mathbf{Z}$. These values of t correspond to the identity element in S^1 under the orbit mapping $\mathbf{R} \rightarrow \mathbf{R}/2\pi\mathbf{Z}$. □

This implies the orbit space $\widetilde{T}^2_{h,\ell}/S^1$ is a smooth one dimensional manifold, see chapter VII ((2.9)).

Claim: The orbit space $\widetilde{T}^2_{h,\ell}/S^1$ is diffeomorphic to S^1.

(2.3) **Proof**: Since the S^1-action defined by the flow of $X_{\widetilde{H}}$ is free and proper, we know from results proved in chapter VII ((2.12)) that $\widetilde{T}^2_{h,\ell}$ is the total space of a smooth principal bundle with base a smooth one dimensional manifold with no boundary. Because the bundle projection map ρ is smooth, compactness and connectedness of $\widetilde{T}^2_{h,\ell}$ implies that the base is a compact connected one dimensional smooth manifold with no boundary. This implies that the base is a circle C. □

▷ We now show that a fiber $\mathscr{C} = \rho^{-1}(p)$ where $p \in C$ of the principal bundle ρ is a global *cross section* for the flow of $X_{\widetilde{H}}$ on $\widetilde{T}^2_{h,\ell}$. Suppose that $(\xi,\eta) \in \mathscr{C}$. After time 2π the integral curve of $X_{\widetilde{H}}$ through (ξ,η) intersects $\mathscr{C}$ for the first positive time at $\varphi_{2\pi}^{\widetilde{H}}(\xi,\eta)$, which is in fact (ξ,η). □

Since the image under the bundle projection ρ of the integral curve through (ξ,η) of $X_{\widetilde{H}-\widetilde{L}}$ parameterizes C and crosses p for the first positive time at 2π, we find that

$$\varphi_{2\pi}^{\widetilde{H}-\widetilde{L}}(\xi,\eta) = \varphi_{2\pi}^{\widetilde{H}}(\xi,\eta).$$

In other words, the integral curve of $X_{\widetilde{H}}$ through a point on $\mathscr{C}$ winds once around C as its projection winds once around the circle $\{(0,\xi_2,\sqrt{h-\ell},\eta_2) \in \widetilde{T}^2_{h,\ell} \,\big|\, \xi_2^2 + \eta_2^2 = h + \ell\}$. Therefore the rotation number of the flow of $X_{\widetilde{H}}$ on $\widetilde{T}^2_{h,\ell}$ is 1. Applying the linear symplectic coordinate change P^{-1} with P given by (8), we find that $P^{-1}\mathscr{C}$ is a global cross section for the flow of X_H on $T^2_{h,\ell}$ and that every integral curve of $X_H|T^2_{h,\ell}$ has rotation number 1.

The information we have obtained so far about the level sets of the energy momentum mapping of the harmonic oscillator is summarized in the *bifurcation diagram* figure 2.1. The set of regular values of $\mathscr{EM}$ is the open wedge $\mathscr{T}$ in the (h,ℓ) plane defined by $0 \le |\ell| < h$, since the critical values of $\mathscr{EM}$ are the two rays $\{(h,\pm h) \in \mathbf{R}_{>0} \times \mathbf{R} \,|\, h > 0\}$. Because $\mathscr{T}$ is *simply connected*, the energy momentum mapping $\mathscr{EM}$ defines a *trivial* smooth fibration over $\mathscr{T}$ with fiber T^2, that is, $\mathscr{EM}^{-1}(\mathscr{T})$ is diffeomorphic to $\mathscr{T} \times T^2$, see chapter VIII §2.

To complete the qualitative analysis of the energy momentum mapping we need only understand how an energy surface $H^{-1}(h)$, $h > 0$, which is diffeomorphic to the 3-sphere

S^3, is built up from the fibers of the energy momentum mapping $\mathcal{EM}^{-1}(h,\ell)$ as ℓ varies over $[-h,h]$. Recall that $\mathcal{EM}^{-1}(h,\ell)$ is a circle when $|\ell| = h$ and is a 2-torus T^2 when $|\ell| < h$. This problem will be solved in the next two sections by showing the the S^1 momentum mapping $\mathcal{EM}$ has an extension to a $\mathrm{U}(2)$-momentum mapping whose restriction to $H^{-1}(h)$ is the Hopf fibration.

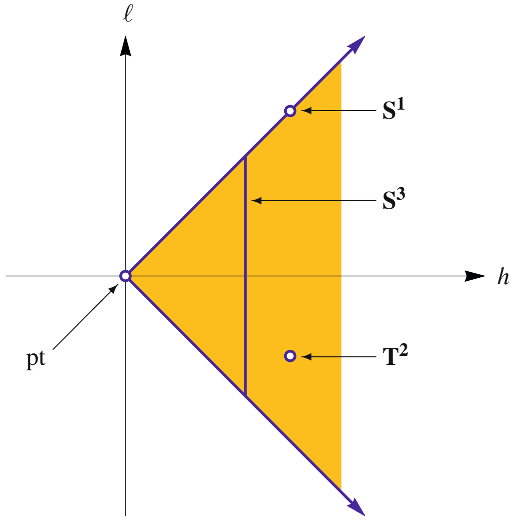

Figure 2.1. The bifurcation diagram. The image of $\mathcal{EM}$ is shaded. The topological type of its fibers or union of fibers is as indicated.

3 U(2)-momentum mapping

As with the construction of the S^1 energy momentum mapping, we begin our construction of the $\mathrm{U}(2)$-momentum mapping by looking for additional integrals of the harmonic oscillator.

We start by looking for quadratic ones. Suppose that F is a homogeneous real quadratic function on $\mathbf{R}^4$. Then F is an integral of X_H if and only if $0 = L_{X_H}F = \{F,H\}$, where $\{\,,\}$ is the *standard Poisson bracket* on $C^\infty(\mathbf{R}^4)$, see chapter VI §4. From $[X_F,X_H] = -X_{\{F,H\}}$, it follows that $[X_F,X_H] = 0$. Conversely, if $[X_F,X_H] = 0$, then $X_{\{F,H\}} = 0$. Thus the function $\{F,H\} = (\frac{\partial F}{\partial x}, \frac{\partial H}{\partial y}) - (\frac{\partial F}{\partial y}, \frac{\partial H}{\partial x})$ is constant. But F is a homogeneous real quadratic function on $\mathbf{R}^4$. Hence $\frac{\partial F}{\partial x} = \frac{\partial F}{\partial y} = 0$ at the origin. Therefore $\{F,H\} = 0$, that is, F is an integral of X_H. This proves

Claim: The homogeneous real quadratic function $F : \mathbf{R}^4 \to \mathbf{R}$ is an integral of the harmonic oscillator vector field X_H if and only if $[X_F,X_H] = 0$.

Since F is a homogeneous quadratic function, the Hamiltonian vector field X_F is linear. Thus F is a homogeneous quadratic integral of X_H if and only if the matrices X_F and X_H commute. Now every homogeneous real quadratic function F on $\mathbf{R}^4$ is given by a 4×4 symmetric matrix, which can be written as $\begin{pmatrix} -B & A^t \\ A & C \end{pmatrix}$, where A is a 2×2 real matrix and B and C are 2×2 real symmetric matrices. Therefore the Hamiltonian vector field X_F corresponding to F has integral curves which satisfy

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= \frac{\partial F}{\partial y} = Ax + Cy \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= -\frac{\partial F}{\partial x} = Bx - A^t y. \end{aligned}$$

So X_F is the 4×4 infinitesimally symplectic matrix $\begin{pmatrix} A & C \\ B & -A^t \end{pmatrix}$, where $B = B^t$ and $C = C^t$, that is, $X_F \in \mathrm{sp}(4, \mathbf{R})$, see chapter VII §5.1 example 2. The infinitesimally symplectic matrix corresponding to the harmonic oscillator vector field X_H is $\begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix}$. A calculation shows that X_F and X_H commute if and only if $X_F = \begin{pmatrix} A & -B \\ B & A \end{pmatrix}$, where $A = -A^t$ and $B = B^t$.

Claim: The set of all 4×4 infinitesimally symplectic matrices which commute with the infinitesimally symplectic matrix X_H corresponding to the harmonic oscillator vector field is isomorphic to the Lie algebra $\mathrm{u}(2)$ of the Lie group $\mathrm{U}(2)$ of 2×2 unitary matrices.

(3.1) **Proof**: Define a mapping

$$\vartheta : \mathrm{u}(2) \to \mathrm{sp}(4, \mathbf{R}) : A + iB \to \begin{pmatrix} A & -B \\ B & A \end{pmatrix}.$$

Since $A + iB \in \mathrm{u}(2)$, it follows that $A^t - iB^t = (\overline{A + iB})^t = -(A + iB)$. Hence $A = -A^t$ and $B = B^t$, which implies that $\mathscr{A} \in \mathrm{sp}(4, \mathbf{R})$. Thus the image of the mapping ϑ is contained in $\mathrm{sp}(4, \mathbf{R})$. Clearly ϑ is bijective on its image and linear. □

From now on we will consider $\mathrm{u}(2)$ to be a subspace of $\mathrm{sp}(4, \mathbf{R})$. Let

$$E_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad E_4 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.$$

Then $\{E_1, E_2, E_3, E_4\}$ form a basis for $\mathrm{u}(2)$. The quadratic Hamiltonian function corresponding to the linear Hamiltonian vector field E_i is $W_i(z) = \frac{1}{2}\omega(E_i z, z)$. Here $z = (x, y)$ and the matrix of the canonical symplectic form ω on $\mathbf{R}^4$ is $-E_4$. This establishes the

Claim: The functions

$$\begin{aligned} W_1(x,y) &= x_1 x_2 + y_1 y_2 \\ W_2(x,y) &= x_1 y_2 - x_2 y_1 = L(x,y) \\ W_3(x,y) &= \tfrac{1}{2}(y_1^2 + x_1^2 - y_2^2 - x_2^2) \\ W_4(x,y) &= \tfrac{1}{2}(y_1^2 + x_1^2 + y_2^2 + x_2^2) = H(x,y) \end{aligned} \tag{11}$$

are a basis for the vector space of all the quadratic integrals of the harmonic oscillator vector field.

From the claim we see that the functions W_1, W_2, W_3, W_4 are integrals of the harmonic oscillator vector field. Moreover, these integrals satisfy the relation

$$W_1^2 + W_2^2 = \widetilde{W}_3 \widetilde{W}_4 \tag{12}$$

with

$$\widetilde{W}_3 = W_4 - W_3 = y_1^2 + x_1^2 \ge 0 \text{ and } \widetilde{W}_4 = W_4 + W_3 = y_2^2 + x_2^2 \ge 0.$$

We now give a geometric interpretation of these integrals. Let $\gamma : \mathbf{R} \to \mathbf{R}^2 : t \mapsto \big(x_1(t), x_2(t)\big)$ be the projection onto configuration space of an integral curve of X_H of energy $h > 0$ starting at (x^0, y^0). From (2) we see that $\gamma(t)$ passes through x^0 at time $t = 0$ and is given by

$$\big(x_1(t), x_2(t)\big) = (x_1^0 \cos t + y_1^0 \sin t, x_2^0 \cos t + y_2^0 \sin t). \tag{13}$$

Since the integral curves of X_H of positive energy are closed, it follows that γ is a planar ▷ closed curve, called a *Lissajous figure*, which we now show is an ellipse.

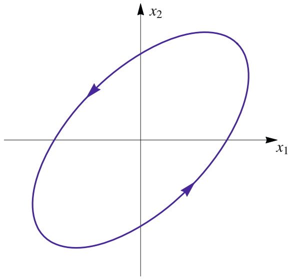

Figure 3.1. A Lissajous curve.

(3.2) **Proof**: First we note that the initial condition (x^0, y^0) determines the values of the integrals, namely,

$$W_1(x^0, y^0) = \widetilde{w}_1,\ W_2(x^0, y^0) = \widetilde{w}_2,\ \widetilde{W}_3(x^0, y^0) = \widetilde{w}_3,\ \widetilde{W}_4(x^0, y^0) = \widetilde{w}_4.$$

Therefore using the definition of $\widetilde{W}_3$ and $\widetilde{W}_4$ we get

$$\big(\widetilde{w}_3 - x_1(t)^2\big)\big(\widetilde{w}_4 - x_2(t)^2\big) = \big(y_1(t) y_2(t)\big)^2 = \big(\widetilde{w}_1 - x_1(t) x_2(t)\big)^2,$$

which upon simplification is

$$\widetilde{w}_4 x_1(t)^2 - 2\widetilde{w}_2 x_1(t) x_2(t) + \widetilde{w}_3 x_2(t)^2 = \widetilde{w}_3 \widetilde{w}_4 - \widetilde{w}_1^2 = \widetilde{w}_2^2. \tag{14}$$

Since $\widetilde{w}_3 + \widetilde{w}_4 = h > 0$ and $\widetilde{w}_3 \widetilde{w}_4 - \widetilde{w}_1^2 = \widetilde{w}_2^2 \ge 0$, if $\widetilde{w}_2 > 0$ then the quadratic form

$$Q(x_1, x_2) = \widetilde{w}_4 x_1^2 - 2\widetilde{w}_2 x_1 x_2 + \widetilde{w}_3 x_2^2 = \widetilde{w}_2^2 \tag{15}$$

is positive definite and hence the curve (13) is an ellipse $\mathscr{E}$.

We now describe the geometry of the ellipse $\mathscr{E}$ more precisely. In complex coordinates $\zeta = x_1 + i x_2$ the quadratic form $Q = \widetilde{w}_2^2$ becomes

$$4\mathscr{Q}(\zeta, \overline{\zeta}) = (\widetilde{w}_4 - \widetilde{w}_3 + 2i\widetilde{w}_1)\zeta^2 + 2(\widetilde{w}_4 + \widetilde{w}_3)\,\zeta\overline{\zeta} + (\widetilde{w}_4 - \widetilde{w}_3 - 2i\widetilde{w}_1)\overline{\zeta}^2 = 4\widetilde{w}_2^2. \tag{16}$$

Apply the rotation $\zeta = z\mathrm{e}^{i\vartheta}$. Then (16) becomes

$$(\widetilde{w}_4 - \widetilde{w}_3 + 2i\widetilde{w}_1)\mathrm{e}^{2i\vartheta} z^2 + 2(\widetilde{w}_4 + \widetilde{w}_3) z\bar{z} + (\widetilde{w}_4 - \widetilde{w}_3 - 2i\widetilde{w}_1)\mathrm{e}^{-2i\vartheta} \bar{z}^2 = 4\widetilde{w}_2^2, \qquad (17)$$

which is diagonal if the angle ϑ is chosen so that $(\widetilde{w}_4 - \widetilde{w}_3 + 2i\widetilde{w}_1)\mathrm{e}^{2i\vartheta}$ is real and, say, positive. So set

$$2\vartheta = -\tan^{-1}\frac{2\widetilde{w}_1}{\widetilde{w}_4 - \widetilde{w}_3}. \qquad (18)$$

Then the symmetry axis of $\mathscr{E}$ lies along the line in $\mathbf{R}^2$, which passes through the origin and subtends an angle 2ϑ with the positive x_1-axis. After performing the rotation $\mathrm{e}^{i\vartheta}$, where ϑ satisfies (18), equation (17) becomes

$$Az^2 + 2Bz\bar{z} + A\bar{z}^2 = 4\widetilde{w}_2^2, \qquad (19)$$

where $A = |\widetilde{w}_4 - \widetilde{w}_3 + 2i\widetilde{w}_1| \ge 0$ and $B = \widetilde{w}_4 + \widetilde{w}_3 > 0$. In real coordinates $\binom{x_1}{x_2} = \begin{pmatrix} \cos\vartheta & -\sin\vartheta \\ \sin\vartheta & \cos\vartheta \end{pmatrix}\binom{\xi_1}{\xi_2}$ equation (19) becomes

$$2(A+B)\xi_1^2 + 2(B-A)\xi_2^2 = 4\widetilde{w}_2^2. \qquad (20)$$

Since

$$B^2 - A^2 = (\widetilde{w}_3 + \widetilde{w}_4)^2 - (\widetilde{w}_4 - \widetilde{w}_3)^2 - 4\widetilde{w}_1^2 = 4(\widetilde{w}_4\widetilde{w}_3 - \widetilde{w}_1^2) = 4\widetilde{w}_2^2 > 0,$$

it follows that $B - A > 0$. Thus (20) is an equation for the ellipse $\mathscr{E}$, which in standard form is $\left(\frac{\xi_1}{b}\right)^2 + \left(\frac{\xi_2}{a}\right)^2 = 1$, where $a = \frac{\sqrt{2}w_2}{\sqrt{B-A}}$ is the major semi-axis a and $b = \frac{\sqrt{2}w_2}{\sqrt{A+B}}$ is its minor semi-axis. Consequently, the eccentricity e of $\mathscr{E}$ is

$$e = \sqrt{1 - \frac{b^2}{a^2}} = \sqrt{1 - \frac{B-A}{A+B}} = \sqrt{\frac{2A}{A+B}}.$$

Now suppose that $\widetilde{w}_2 = 0$. Then the quadratic form Q (15) factors into the product of the linear factors $\widetilde{w}_4 x_1 - \widetilde{w}_1 x_2$ and $-\widetilde{w}_1 x_1 + \widetilde{w}_3 x_2$. Hence $Q = 0$ defines two lines given by

$$\widetilde{w}_4 x_1 - \widetilde{w}_1 x_2 = 0 \quad \text{or} \quad -\widetilde{w}_1 x_1 + \widetilde{w}_3 x_2 = 0. \qquad (21)$$

Because x^0 lies on γ, it satisfies exactly one of the equations in (21). Since the energy is fixed, we have

$$x_1^2 + x_2^2 \le x_1^2 + y_1^2 + x_2^2 + y_2^2 = 2h. \qquad (22)$$

Therefore γ is a line segment, which lies inside the disc in configuration space defined by equation (22). □

We now return to the problem of finding an extension of the S^1-momentum map of section 2. Following the construction of the S^1-momentum mapping, we look for a group acting on $T^*\mathbf{R}^2$ which properly contains S^1 and has Hamiltonian vector fields for its infinitesimal generators. In contrast to the S^1 case, the action on $T^*\mathbf{R}^2$ we find will *not* be a *lift of an action* on the configuration space $\mathbf{R}^2$. This is reflected in the observation that some of the new integrals are not linear in the momenta, see chapter VII §5. Consider the *linear action* of the *unitary group*

$$\mathrm{U}(2) = \left\{ \mathrm{U} = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \in \mathrm{Gl}_4(\mathbf{R}) \,\middle|\, \begin{array}{c} a^t a + b^t b = I_2 \;\&\; a^t b = b^t a \\ a, b \in \mathrm{gl}_2(\mathbf{R}) \end{array} \right\}$$

on $\mathbf{R}^4 = T^*\mathbf{R}^2$ defined by $\Psi : \mathrm{U}(2) \times \mathbf{R}^4 \to \mathbf{R}^4 : \big(U,(x,y)\big) \mapsto U\begin{pmatrix} x \\ y \end{pmatrix}$. Since

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}^t \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix} \begin{pmatrix} a & -b \\ b & a \end{pmatrix} = \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix},$$

Ψ_U is a linear symplectic map, that is, $\Psi_U^*\omega = \omega$. For $u = \left(\begin{smallmatrix} A & -B \\ B & A \end{smallmatrix}\right) \in \mathrm{u}(2)$, the infinitesimal generator Y^u corresponding to u is the vector field

$$Y^u(x,y) = \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0} \Psi_{\exp su}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} A & -B \\ B & A \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

with flow $\psi_t^u : \mathbf{R}^4 \to \mathbf{R}^4 : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \exp tu \begin{pmatrix} x \\ y \end{pmatrix}$. Since the matrix Y^u is infinitesimally symplectic, the vector field Y^u is linear Hamiltonian with Hamiltonian function

$$J^u : \mathbf{R}^4 \to \mathbf{R} : (x,y) \mapsto \tfrac{1}{2}\omega\big(u(x,y)^t, (x,y)^t\big). \tag{23}$$

For fixed $(x,y) \in \mathbf{R}^4$, the function $u \mapsto J^u(x,y)$ is linear. Therefore by duality it makes sense to define the mapping $J : T^*\mathbf{R}^2 \to \mathrm{u}(2)^*$ by setting $J(x,y)u = J^u(x,y)$.

▷ An important property of J is that it intertwines the linear action of $\mathrm{U}(2)$ on $\mathbf{R}^4$ with the coadjoint action of $\mathrm{U}(2)$ on $\mathrm{u}(2)^*$, the dual of the Lie algebra $\mathrm{u}(2)$.

(3.3) **Proof**: This is a consequence of the calculation

$$\begin{aligned} J(U(x,y)^t)u &= J^u\big(U(x,y)^t\big) = \tfrac{1}{2}\omega\big(uU(x,y)^t, U(x,y)^t\big) \\ &= \tfrac{1}{2}\omega\big(U^{-1}uU(x,y)^t, (x,y)^t\big), \quad \text{since } U \text{ is symplectic} \\ &= J(x,y)\big(U^{-1}uU\big) = J(x,y)(\mathrm{Ad}_{U^{-1}}u) \\ &= \big(\mathrm{Ad}^t_{U^{-1}}J(x,y)\big)u. \end{aligned}$$

□

Therefore J is the U(2)-*momentum mapping* corresponding to the linear action Ψ, see chapter VII §5. If we identify $\mathrm{u}(2)^*$ with $\mathrm{u}(2)$ using the *Killing metric* k defined by $k(u,v) = \frac{1}{2}\mathrm{tr}\, u\overline{v}^t$, then the momentum mapping intertwines the linear action of $\mathrm{U}(2)$ on $\mathbf{R}^4$ with the adjoint action of $\mathrm{U}(2)$ on its Lie algebra $\mathrm{u}(2)$. Identifying $\mathrm{u}(2)$ with $\mathbf{R}^4$ by choosing the basis $\{E_i\}$, the U(2)-momentum mapping J becomes the mapping

$$J : \mathbf{R}^4 \to \mathbf{R}^4 : (x,y) \mapsto \big(W_1(x,y), W_2(x,y), W_3(x,y), W_4(x,y)\big).$$

▷ Thus the components of the U(2)-momentum mapping J are quadratic integrals of the harmonic oscillator.

(3.4) **Proof**: This is just the content of the equations $W_i(z) = \frac{1}{2}\omega(E_i z, z)$ for $i = 1, \ldots, 4$. An alternative argument starts by observing that every element $U \in \mathrm{U}(2)$ when written as

$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ is also an element of $\mathrm{SO}(4)$, the group of all orientation preserving linear *isometries* of $(\mathbf{R}^4, \langle\, , \rangle)$. Consequently, Ψ_U preserves H and so $0 = L_{Y^u}H = -L_{X_H}J^u$. $\square$

Now consider the 2-torus

$$T^2 = \left\{ \begin{pmatrix} ac & -ad & -bc & bd \\ ad & ac & -bd & -bc \\ bc & -bd & ac & -ad \\ bd & bc & ad & ac \end{pmatrix} \in \mathrm{U}(2) \;\middle|\; \begin{matrix} a^2+b^2=1 \\ c^2+d^2=1 \end{matrix} \right\},$$

which is an abelian subgroup of $\mathrm{U}(2)$. Restricting the $\mathrm{U}(2)$ action on $\mathbf{R}^4$ to a T^2-action gives rise to a momentum mapping $j : T^*\mathbf{R}^2 \to (t^2)^* = \mathbf{R}^2$ where $(t^2)^*$ is the dual of the Lie algebra t^2 of T^2 and $j(x,y) = \big(W_4(x,y), W_1(x,y)\big)$. In other words, j is the S^1 energy momentum map $\mathscr{E}\mathscr{M}$ studied in section 2. Therefore the $\mathrm{U}(2)$-momentum mapping J is a proper extension of $\mathscr{E}\mathscr{M}$.

4 The Hopf fibration

In this section we study the qualitative properties of the *Hopf mapping*

$$\mathscr{H} : \mathbf{R}^4 \to \mathbf{R}^4 : (x,y) \mapsto \big(w_1(x,y), w_2(x,y), w_3(x,y), w_4(x,y)\big), \tag{24}$$

where

$$\begin{aligned} w_1 &= W_1 = x_1x_2 + y_1y_2 & \qquad w_2 &= W_2 = x_1y_2 - x_2y_1 \\ w_3 &= W_3 = \tfrac{1}{2}(y_1^2 + x_1^2 - y_2^2 - x_2^2) & \qquad w_4 &= W_4 = \tfrac{1}{2}(y_1^2 + x_1^2 + y_2^2 + x_2^2). \end{aligned}$$

The *Hopf variables* w_i, $i = 1, \ldots, 4$ satisfy the relation

$$C(w_1, w_2, w_3, w_4) = w_1^2 + w_2^2 + w_3^2 - w_4^2 = 0, \quad w_4 \ge 0. \tag{25}$$

Therefore the image of the Hopf map is contained in the *semialgebraic variety* $\mathscr{C}$ defined by (25). Topologically $\mathscr{C}$ is a cone on S^2 with vertex at 0. To show that $\mathscr{C}$ is the image of the Hopf map, it suffices to verify that $D\mathscr{H}(x,y) : T_{(x,y)}\mathbf{R}^4 \to T_{\mathscr{H}(x,y)}\mathscr{C}$ is surjective for all $(x,y) \ne (0,0)$, since $\mathscr{H}(0) = 0$ and $\mathscr{C} \setminus \{0\}$ is a smooth three dimensional manifold.

(4.1) **Proof**: When $(x,y) \ne (0,0)$ the derivative

$$D\mathscr{H}(x,y) = \begin{pmatrix} x_2 & x_1 & y_2 & y_1 \\ y_2 & -y_1 & -x_2 & x_1 \\ x_1 & -x_2 & y_1 & -y_2 \\ x_1 & x_2 & y_1 & y_2 \end{pmatrix}$$

has rank ≥ 3 because its first three rows are nonzero and pairwise orthogonal. It has rank ≤ 3, since

$$\operatorname{im} D\mathscr{H}(x,y) \subseteq T_{\mathscr{H}(x,y)}C = \ker DC(\mathscr{H}(x,y)) = \ker(w_1, w_2, w_3, w_4)$$

and (w_1, w_2, w_3, w_4) is nonzero. $\square$

Restricting the Hopf mapping $\mathscr{H}$ to the 3-sphere

$$H^{-1}(h) = S^3_{\sqrt{2h}} = \{(x,y) \in \mathbf{R}^4 \,|\, y_1^2 + y_2^2 + x_1^2 + x_2^2 = 2h,\ h > 0\}$$

and using (25) gives the mapping

$$\mathscr{F} : S^3_{\sqrt{2h}} \subseteq \mathbf{R}^4 \to S^2_h \subseteq \mathbf{R}^3 : (x,y) \mapsto \big(w_1(x,y), w_2(x,y), w_3(x,y)\big). \tag{26}$$

Here S^2_h is the 2-sphere $\{w \in \mathbf{R}^3 \,|\, w_1^2 + w_2^2 + w_3^2 = h^2\}$. The map $\mathscr{F}$ is called the *Hopf fibration*. From a topological point of view the Hopf fibration is quite nontrivial and will require quite a bit of work to be understood.

Claim: The Hopf fibration $\mathscr{F}$ has the following properties:

I. $\mathscr{F}$ is a proper *submersion*.

II. For every $w \in S^2_h$ the fiber $\mathscr{F}^{-1}(w)$ is a *great circle* on $S^3_{\sqrt{2h}}$ contained in the 2-plane Π^w, see (27).

III. For every $w, w' \in S^2_h$ with $w \neq w'$, the circles $\mathscr{F}^{-1}(w)$ and $\mathscr{F}^{-1}(w')$ are linked in $S^3_{\sqrt{2h}}$ with linking number 1, see (29).

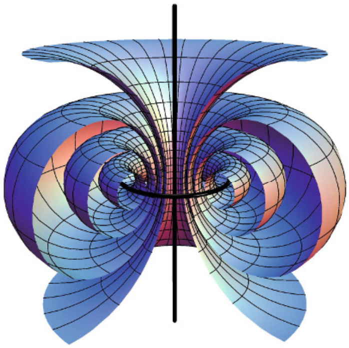

Figure 4.1. Visualization of the Hopf fibration.

(4.2) **Proof**: I. Consider the mapping

$$F : \mathbf{R}^4 \to \mathbf{R}^3 : (x,y) \mapsto (w_1, w_2, w_3) = \big(x_1x_2 + y_1y_2, x_1y_2 - x_2y_1, \tfrac{1}{2}(y_1^2 + x_1^2 - y_2^2 - x_2^2)\big).$$

Clearly F is smooth and has derivative

$$DF(x,y) = \begin{pmatrix} x_2 & x_1 & y_2 & y_1 \\ y_2 & -y_1 & -x_2 & x_1 \\ x_1 & -x_2 & y_1 & -y_2 \end{pmatrix}.$$

Since $\mathscr{F} = F|S^3_{\sqrt{2h}}$, the Hopf fibration $\mathscr{F}$ is smooth. When $(x,y) \neq (0,0)$ the rows of $DF(x,y)$ are nonzero and pairwise orthogonal. Hence $D\mathscr{F}(x,y) = DF(x,y)|T_{(x,y)}S^3_{\sqrt{2h}}$ is a surjective linear map from $T_{(x,y)}S^3_{\sqrt{2h}}$ to $T_{\mathscr{F}(x,y)}S^2_h$ for every $(x,y) \in S^3_{\sqrt{2h}}$. Thus $\mathscr{F}$ is a submersion. Moreover, $\mathscr{F}$ is a proper map, because its domain is compact.

II. The following argument shows that every fiber of $\mathscr{F}$ is a great circle on $S^3_{\sqrt{2h}}$. Let Π^w be the 2-plane in $\mathbf{R}^4$ defined by

$$\begin{aligned} \pi_1 \begin{pmatrix} x \\ y \end{pmatrix} &= (-w_1, h+w_3, w_2, 0) \begin{pmatrix} x \\ y \end{pmatrix} &= 0 \\ \pi_2 \begin{pmatrix} x \\ y \end{pmatrix} &= (-w_2, 0, -w_1, h+w_3) \begin{pmatrix} x \\ y \end{pmatrix} &= 0, \end{aligned} \tag{27}$$

when $w \in S^2_h \setminus \{(0,0,-h)\}$, and the 2-plane $\{(0,x_2,0,y_2) \in \mathbf{R}^4 \,\big|\, (x_2,y_2) \in \mathbf{R}^2\}$, when $w = (0,0,-h)$. To see that equation (27) defines a 2-plane we argue as follows. We know that the covectors π_1 and π_2 are linearly dependent if and only if $0 = \pi_1 \wedge \pi_2$, that is, when all the 2×2 minors of

$$\begin{pmatrix} -w_1 & h+w_3 & w_2 & 0 \\ -w_2 & w_1 & 0 & h+w_3 \end{pmatrix}$$

vanish. In other words, $w_1 = w_2 = 0$ and $w_3 = -h$. But this is excluded by hypothesis. Therefore (27) defines a 2-plane, when $w \in S^2_h \setminus \{(0,0,-h)\}$.

We now show that $\mathscr{F}^{-1}(w) \subseteq \Pi^w \cap S^3_{\sqrt{2h}}$. Suppose that $w \in S^2_h \setminus \{(0,0,-h)\}$. Then for every $(x,y) \in \mathscr{F}^{-1}(w) \subseteq S^3_{\sqrt{2h}}$, we have $x_1^2 + y_1^2 > 0$. To see this note that $(x,y) \in \mathscr{F}^{-1}(w)$ if and only if (26) holds. Now $x_1^2 + y_1^2 = w_3 + w_4$. But $(x,y) \in S^3_{\sqrt{2h}}$, so $w_4 = h$. Therefore $x_1^2 + y_1^2 = w_3 + h$. By hypothesis $w_3 \in (-h,h]$ so $x_1^2 + y_1^2 > 0$. Now write the defining equations of w_1 and w_2 as

$$\begin{pmatrix} x_1 & y_1 \\ -y_1 & x_1 \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}. \tag{28}$$

Since the determinant $x_1^2 + y_1^2 > 0$ we may invert $\left(\begin{smallmatrix} x_1 & y_1 \\ -y_1 & x_1 \end{smallmatrix}\right)$ to obtain

$$\begin{pmatrix} x_1 & -y_1 \\ y_1 & x_1 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = (x_1^2 + y_1^2) \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = (h + w_3) \begin{pmatrix} x_2 \\ y_2 \end{pmatrix},$$

which is (27). Thus $\mathscr{F}^{-1}(w) \subseteq \Pi^w$. Hence $\mathscr{F}^{-1}(w) \subseteq \Pi^w \cap S^3_{\sqrt{2h}}$. Now suppose that $w = (0,0,-h)$. Then the defining equations for w_3 and w_4 become

$$\begin{aligned} y_1^2 + x_1^2 - x_2^2 - y_2^2 &= -2h \\ y_1^2 + x_1^2 + x_2^2 + y_2^2 &= 2h, \end{aligned}$$

since $(x,y) \in S^3_{\sqrt{2h}}$. Adding these equations together and dividing by 2 gives $y_1^2 + x_1^2 = 0$, that is, $x_1 = y_1 = 0$. Hence $\mathscr{F}^{-1}(0,0,-h) \subseteq \Pi^{(0,0,-h)} \cap S^3_{\sqrt{2h}}$. Therefore for each $w \in S^2_h$ the fiber $\mathscr{F}^{-1}(w)$ is contained in $\Pi^w \cap S^3_{\sqrt{2h}}$. Because $\mathscr{F}$ is a submersion, $\mathscr{F}^{-1}(w)$ is a smooth compact one dimensional submanifold of $S^3_{\sqrt{2h}}$ without boundary. Hence $\mathscr{F}^{-1}(w)$ is the great circle $\Pi^w \cap S^3_{\sqrt{2h}}$.

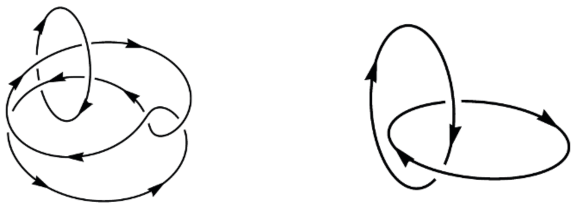

Figure 4.2. Linking number. In the figure on the left the curves have linking number 0, yet can not be pulled apart without being cut. In the figure on the right the curves have linking number $+1$.

Before proving property III we must define the notion of linking number of two smooth oriented disjoint circles γ_1 and γ_2 in an oriented 3-sphere S^3. Intuitively, the circles γ_i are linked if they cannot be pulled apart without being cut, see figure 4.2. A more precise definition goes as follows. For simplicity we will assume that γ_1 bounds a smooth oriented closed 2-disk $\overline{D}_1^2$ in S^3. Since S^3 is simply connected, the circle γ_1 is *null homotopic* and hence is smoothly *contractible* to a point $p \in S^3$. In other words, there is a diffeomorphism $F : \overline{D}_1^2 \subseteq \mathbf{R}^2 \rightarrow S^3 \subseteq \mathbf{R}^4$, called a contraction, such that $F(0) = p$ and such that for every $r \in (0,1]$ the map F restricted to the boundary $\partial \overline{D}_r^2$ of the closed 2-disk $\overline{D}_r^2 = \{x \in \mathbf{R}^2 \mid (x,x) \le r^2\}$ is a diffeomorphism onto γ_1. Orient the 2-disk $\overline{D}_1^2$ so that $F \,|\, \partial \overline{D}_1^2$ is orientation preserving. Furthermore assume that F is *transverse* to γ_2 in S^3, that is, either $\gamma_2 \cap F(D_1^2) = \varnothing$ or for every x in D_1^2 such that $F(x) \in \gamma_2 \cap F(D_1^2)$ we have $T_xF(T_xD_1^2) + T_x\gamma_2 = T_xS^3$. The *linking number* of the circles γ_1 and γ_2 is the *intersection number* of $F(D_1^2)$ with γ_2, that is,

$$\operatorname{Link}(\gamma_1, \gamma_2) = \sum_{x \in \mathscr{T}} \#\big(T_xF(T_xD_1^2), T_x\gamma_2\big) \tag{29}$$

where $\mathscr{T} = \{x \in D_1^2 \,\big|\, F(x) \in \gamma_2 \cap F(D_1^2)\}$ and

$$\#\big(T_xF(T_xD_1^2), T_x\gamma_2\big) = \begin{cases} 1, & \text{if the orientation of } T_xF(T_xD_1^2) \oplus T_x\gamma_2 \\ & \text{is the same as } T_xS^3 \\ -1, & \text{otherwise.} \end{cases}$$

$\triangleright$ Note that the sum in (29) is finite, since F is transverse to γ_2. An argument, which is left as an exercise, shows that the definition of linking number does not depend on the choice of oriented 2-disk $\overline{D}_1^2$ with boundary γ_1.

III. With these preliminaries out of the way we are in a position to prove property III of the Hopf fibration $\mathscr{F}$, namely, that two distinct fibers of $\mathscr{F}$ have linking number 1. Suppose that $w, v \in S_h^2$ and $w \ne v$. Then $\mathscr{F}^{-1}(w) \cap \mathscr{F}^{-1}(v) = \varnothing$. Thus the corresponding 2-planes Π^w and Π^v intersect only at 0, see the proof of property II for the definition of Π^w. Let Π be a 3-plane in $\mathbf{R}^4$ containing Π^w. Then Π^v is not contained in Π. Moreover $\Pi^v \cap \Pi = \ell^v$ is a line in $\mathbf{R}^4$. Let $S^2 = \Pi \cap S^3$ be the *great* 2-*sphere* in S^3 cut out by Π and let $S_w^1 = \Pi^w \cap S^3$ be the great circle on S^3 cut out by Π^w. Furthermore let H^+ be the closed upper *hemisphere* of S^2 with boundary S^1_w. Then H^+ is diffeomorphic to a closed 2-disk $\overline{D}^2$ with boundary S^1_w. To see this just project points of H^+ onto the equatorial plane containing S^1_w. Since Π^v is not contained in Π, the great circle $S^1_v = \Pi^v \cap S^3$ is not contained in S^2. Because the line ℓ^v intersects the great 2-sphere S^2 in two *antipodal points* p_+ and p_-, the circle S^1_v intersects S^2 at p_+ and p_-. Since $\Pi^w \cap \Pi^v = \{0\}$, the points p_+ and p_- do not lie on the *equator* S^1_w of S^2. Hence exactly one of the points $p_\pm$, say p_+, lies in the interior of the hemisphere H^+. Therefore the linking number of the two circles $\mathscr{F}^{-1}(w)$ and $\mathscr{F}^{-1}(v)$ in S^3 is ± 1, since we have not been careful about orientations. If we choose orientations properly we can arrange that the linking number is 1. This completes the proof of the properties of the Hopf fibration. □

We now draw some conclusions about the Hopf fibration from the properties we have just proved. From property I we know that $\mathscr{F}$ is a proper submersion. Therefore the Hopf mapping $\mathscr{F} : S^3_{\sqrt{2h}} \to S^2_h : (x,y) \mapsto w$ defines a locally trivial bundle with fiber S^1, see chapter VIII §2. To find the local trivializations of the bundle $\mathscr{F}$ explicitly we use the identities

$$x_2 = \frac{1}{h+w_3}\,(w_1x_1 - w_2y_1) \quad \text{and} \quad y_2 = \frac{1}{h+w_3}\,(w_2x_1 + w_1y_1) \tag{30}$$

which hold for $(x_1,x_2,y_1,y_2) \in \mathscr{F}^{-1}(w)$ when $w \in (S^2_h \setminus \{(0,0,-h)\}) = U_1$, and the identities

$$x_1 = \frac{1}{h-w_3}\,(w_1x_2 + w_2y_2) \quad \text{and} \quad y_1 = \frac{1}{h-w_3}\,(-w_2x_2 + w_1y_2), \tag{31}$$

which hold when $w \in (S^2_h \setminus \{(0,0,h)\}) = U_2$. Note that $\{U_1, U_2\}$ form an open covering of S^2_h. Let $S^1 = \{(x,y) \in \mathbf{R}^2 \,\big|\, x^2+y^2 = 1\}$. Consider the mappings

$$\begin{aligned} &\tau_1 : U_1 \times S^1 \to \mathscr{F}^{-1}(U_1) : (w_1,w_2,w_3,x,y) \mapsto \\ &\big(x\sqrt{h+w_3}, \tfrac{1}{\sqrt{h+w_3}}(w_1x - w_2y), y\sqrt{h+w_3}, \tfrac{1}{\sqrt{h+w_3}}(w_2x + w_1y)\big) \end{aligned} \tag{32}$$

and

$$\begin{aligned} &\tau_2 : U_2 \times S^1 \to \mathscr{F}^{-1}(U_2) : (w_1,w_2,w_3,x,y) \mapsto \\ &\big(\tfrac{1}{\sqrt{h-w_3}}(w_1x + w_2y), x\sqrt{h-w_3}, \tfrac{1}{\sqrt{h-w_3}}(-w_2x + w_1y), y\sqrt{h-w_3}\big) \end{aligned} \tag{33}$$

Using the definition of the Hopf fibration $\mathscr{F}$, it is easy to check that $\mathscr{F} \circ \tau_1 = \pi_1$ and $\mathscr{F} \circ \tau_2 = \pi_2$, where $\pi_1 : U_1 \times S^1 \to U_1$ and $\pi_2 : U_2 \times S^1 \to U_2$ are the projections onto the first factor. A short calculation shows that

$$\begin{aligned} &\tau_1^{-1} : \mathscr{F}^{-1}(U_1) \to U_1 \times S^1 : (x_1,x_2,y_1,y_2) \mapsto (w_1,w_2,w_3,x,y) = \\ &= \big(x_1x_2 + y_1y_2, x_1y_2 - x_2y_1, \tfrac{1}{2}(y_1^2 + x_1^2 - x_2^2 - y_2^2), \tfrac{x_1}{\sqrt{h+w_3}}, \tfrac{y_1}{\sqrt{h+w_3}}\big), \end{aligned}$$

since $x_1^2 + y_1^2 = h + w_3$ in $\mathscr{F}^{-1}(U_1)$, and also that

$$\begin{aligned} &\tau_2^{-1} : \mathscr{F}^{-1}(U_2) \to U_2 \times S^1 : (x_1,x_2,y_1,y_2) \mapsto (w_1,w_2,w_3,x,y) = \\ &= \big(x_1x_2 + y_1y_2, x_1y_2 - x_2y_1, \tfrac{1}{2}({y_1}^2 + {x_1}^2 - {x_2}^2 - {y_2}^2), \tfrac{x_2}{\sqrt{h-w_3}}, \tfrac{y_2}{\sqrt{h-w_3}}\big), \end{aligned}$$

since ${x_2}^2 + {y_2}^2 = h - w_3$ in $\mathscr{F}^{-1}(U_2)$. Thus τ_1^{-1} and τ_2^{-1} are continuous. Consequently

▷ the mappings τ_1 and τ_2 are *local trivializations* of the bundle $\mathscr{F}$. The Hopf fibration is *not trivial*, that is, the bundle $\mathscr{F} : S^3_{\sqrt{2h}} \to S^2_h$ is not isomorphic to the trivial bundle $S^2_h \times S^1 \to S^2_h$.

(4.3) **Proof**: This follows from the fact that the linking number is a topological invariant and two distinct fibers of the Hopf fibration are linked with linking number one, whereas the fibers of the trivial bundle are unlinked. Another argument uses the observation that the total space $S^3_{\sqrt{2h}}$ of the Hopf fibration is not diffeomorphic to the total space $S^2_h \times S^1$ of the trivial bundle, because the first *homology group* of $S^3_{\sqrt{2h}}$ vanishes, whereas the first homology group of $S^2_h \times S^1$ is $\mathbf{Z}$. For yet another argument see chapter VIII §1. □

Claim: The Hopf bundle $\mathscr{F} : S^3_{\sqrt{2h}} \to S^2_h$ is an S^1 principal bundle.

(4.4) **Proof**: We prove this assertion in several steps.

▷ First we show that the S^1 bundle $\pi_h : H^{-1}(h) \to M_h = H^{-1}(h)/S^1$, where π_h maps each orbit of energy h of the harmonic oscillator vector field X_H to a point, is an S^1 principal bundle.

(4.5) **Proof**: $H^{-1}(h)$ is invariant under the flow φ^H_t of X_H. Since $\varphi^H_{2\pi} = \mathrm{id}_{S^3_{\sqrt{2h}}}$, the flow φ^H_t defines an action of $S^1 = \mathbf{R}/2\pi\mathbf{Z}$ on $H^{-1}(h) = S^3_{\sqrt{2h}}$ given by

$$S^1 \times S^3_{\sqrt{2h}} \to S^3_{\sqrt{2h}} : \big(t,(x,y)\big) \mapsto \varphi^H_t(x,y) = \begin{pmatrix} (\cos t)I_2 & (\sin t)I_2 \\ -(\sin t)I_2 & (\cos t)I_2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Because every orbit of this action has minimal period 2π, the isotropy group of every point on S^3 is the identity element of $S^1 = \mathbf{R}/2\pi\mathbf{Z}$. Therefore the S^1-action is free. It is also proper, since S^1 is compact. Thus the bundle π_h is a principal S^1 bundle, see chapter VII ((2.12)). □

▷ Next we show that the orbit space M_h is diffeomorphic to a 2-sphere .

(4.6) **Proof**: Because the S^1-action defined by the flow of the harmonic oscillator is free and proper, the orbit space M_h is smooth, see chapter VII §2.2. We now use Morse theory to determine the topology of M_h. Consider the smooth function $L|H^{-1}(h) : H^{-1}(h) \to \mathbf{R}$, which is the restriction of the angular momentum L (3) to the energy level set $H^{-1}(h)$. Because $L|H^{-1}(h)$ is invariant under the flow of X_H, it induces a smooth function $\widetilde{L}_h : M_h \to \mathbf{R}$ on M_h. Since the set of critical points of $L|H^{-1}(h)$ consists of two disjoint circles, which are nondegenerate critical submanifolds of $H^{-1}(h)$ of Morse index 0 and 2, see ((2.1)), the function $\widetilde{L}_h$ is a Morse function on M_h with two nondegenerate critical points, one of Morse index 0 and the other of Morse index 2. Thus M_h is homeomorphic to a 2-sphere, see chapter XI ((3.2)). Because M_h is smooth, it is diffeomorphic to a 2-sphere. □

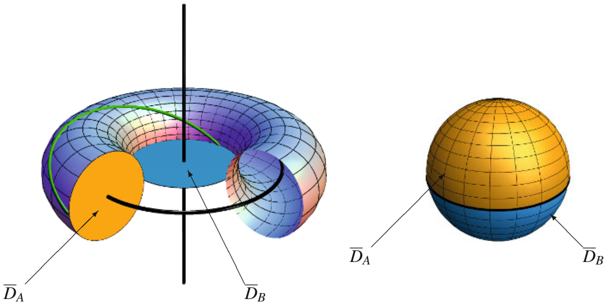

Figure 4.3. Poincaré disks used to construct the orbit space (left). The orbit space $H^{-1}/S^1 = S^2$ (right).

To visualize the orbit space M_h consider figure 4.3. Here the 3-sphere $S^3_{\sqrt{2h}}$ is to be thought of as the *one point compactification* of $\mathbf{R}^3$, the point at infinity having been added. Thus the z-axis is actually a circle. Every S^1 fiber of the Hopf fibration passes transversely through one of the two closed 2-disks $\overline{D}_A$ or $\overline{D}_B$, which have bounding circles A or B, respectively. Corresponding to each point on A there is a unique point $\vartheta(p)$ on B. Gluing the disk $\overline{D}_A$ to the disk $\overline{D}_B$ along their boundary by the diffeomorphism ϑ gives a 2-sphere S^2, which is the orbit space M_h.

▷ The bundle $\pi_h : H^{-1}(h) \rightarrow M_h$ is isomorphic to the Hopf bundle $\mathscr{F} : S^3_{\sqrt{2h}} \rightarrow S^2_h$.

(4.7) **Proof**: This result follows because the map φ making diagram 4.1 commutative is a diffeomorphism. In more detail, the map φ is well defined because each fiber of $\mathscr{F}$ is a single orbit of X_H. Since each fiber of π_h is also a single orbit of X_H, the mapping φ is injective. Clearly, φ is surjective. Its inverse is continuous since S^2_h is a compact Hausdorff space. Hence φ is a homeomorphism. Because the bundle $\mathscr{F}$ is locally trivial, it has a smooth local section. Hence φ is smooth. Its inverse is also smooth because the bundle π_h is locally trivial and hence has a smooth local section. This completes the argument

$$\begin{tikzcd} S^3_{\sqrt{2h}} \arrow[r, "\pi_h"] \arrow[d, "\mathscr{F}"'] & M_h \\ S^2_h \arrow[ur, "\varphi"'] & \end{tikzcd}$$

Diagram 4.1

that the bundle $\pi_h : H^{-1}(h) \rightarrow S^2_h$ is an S^1-principal bundle. □

The preceding result allows us to draw the following conclusions

▷ There is no *global* cross section for the flow of the harmonic oscillator vector field X_H on the energy level set $H^{-1}(h)$.

(4.8) **Proof**: To see this we argue as follows. Suppose that the 2-disk $D \subseteq S^3_{\sqrt{2h}}$ is a global cross section. Since every orbit of X_H on $S^3_{\sqrt{2h}}$ is a circle, it would follow that $S^3_{\sqrt{2h}}$ is homeomorphic to $D^2 \times S^1$. Therefore two distinct orbits of X_H on $H^{-1}(h)$ would be unlinked in $S^3_{\sqrt{2h}}$. But these orbits are two distinct fibers of the Hopf fibration, which are linked in $S^3_{\sqrt{2h}}$. This is a contradiction. Therefore $S^3_{\sqrt{2h}}$ is *not* homeomorphic to $D^2 \times S^1$ as asserted. This proves the result. □

▷ The orbit space $M_h = H^{-1}(h)/S^1$ is *not* a submanifold of $H^{-1}(h)$.

(4.9) **Proof**: See the preceeding argument. □

To determine which principal bundle the Hopf fibration $\mathscr{F}$ is, we calculate its *classifying map*. From the definition of the local trivializations τ_i (32) and (33) we find that the *transition map* between *chart overlaps* is given by

$$\tau_2^{-1} \circ \tau_1 : (U_1 \cap U_2) \times S^1 \to (U_1 \cap U_2) \times S^1 : \big((w, \begin{pmatrix} x \\ y \end{pmatrix})\big) \mapsto \big(w, g_{12}(w) \begin{pmatrix} x \\ y \end{pmatrix}\big)$$

where

$$g_{12} : U_1 \cap U_2 \to \mathrm{SO}(2, \mathbf{R}) = S^1 : (w_1, w_2, w_3) \mapsto \tfrac{1}{\sqrt{h^2 - w_3^2}} \begin{pmatrix} w_1 & -w_2 \\ w_2 & w_1 \end{pmatrix}.$$

Let $S^1_E = \{(w_1, w_2, 0) \in S^2_h \,|\, w_1^2 + w_2^2 = h^2\} \subseteq U_1 \cap U_2$ be the equator of the 2-sphere S^2_h. By definition, the classifying map of the bundle $\mathscr{F}$ is

$$\chi = g_{12}|S^1_E : S^1_E \to \mathrm{SO}(2, \mathbf{R}) : (w_1, w_2) \mapsto \frac{1}{h} \begin{pmatrix} w_1 & -w_2 \\ w_2 & w_1 \end{pmatrix}. \tag{34}$$

Clearly the mapping χ has degree 1.

We now give a way to visualize geometrically the *fibration* of $H^{-1}(h)$ by level sets of the angular momentum L. In figure 4.4 the union of the X_H orbits through $\overline{D}_B$ is the closed *solid torus* $ST_B = \overline{D}_B \times S^1$ with boundary T^2 and the union of the X_H orbits through $\overline{D}_A$ is the closed solid torus $ST_A = \overline{D}_A \times S^1$ with boundary T^2.

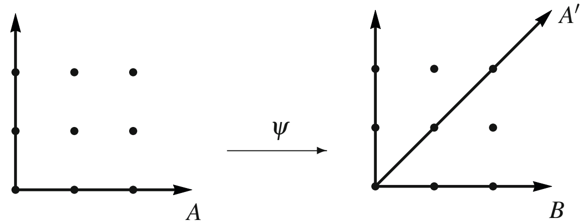

Figure 4.4. The gluing map.

To understand how the 3-sphere $S^3_{\sqrt{2h}}$ is the union the two solid tori ST_A and ST_B we need to know the map $\psi : T^2 \to T^2$ which *glues* the *solid torus* ST_B to the solid torus

ST_A along their common boundary T^2. Using the local trivializations τ_i it follows that $ST_B = \tau_1\big((S^2_h \cap \{w_3 \ge 0\}) \times S^1\big)$, $ST_A = \tau_1\big((S^2_h \cap \{w_3 \le 0\}) \times S^1\big)$, and $T^2 = \tau_1(S^1_E \times S^1) = \tau_2(S^1_E \times S^1)$. Therefore, in the charts provided by the local trivializations, the gluing map ψ is the graph of the transition map $\tau_2^{-1} \circ \tau_1$ restricted to $S^1_E \times S^1$, that is,

$$\psi : S^1_E \times S^1 \to S^1_E \times S^1 : \left(w_1, w_2, \begin{pmatrix} x \\ y \end{pmatrix}\right) \mapsto \left(w_1, w_2, \chi(w_1, w_2) \begin{pmatrix} x \\ y \end{pmatrix}\right).$$

To visualize the gluing map ψ, we identify the 2-torus T^2 with the lattice $\mathbf{Z}^2 \subseteq \mathbf{R}^2$. Taking the S^1 orbits of X_H as vertical and the circles A and B as horizontal, the image of A under the mapping ψ is the line $A' : \{(x, -x) \in \mathbf{R}^2 | x \in \mathbf{R}\}$, see figure 4.4. If we identify the 2-torus T^2 with $\mathbf{R}^2/\mathbf{Z}^2$, we see from figure 4.4 that the gluing map ψ is just the map on T^2 induced by the linear map $\widetilde{\psi}$ of $\mathbf{R}^2$ into itself with matrix $\left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$. The map ψ is well defined because $\widetilde{\psi}(\mathbf{Z}^2) = \mathbf{Z}^2$.

The above treatment of the Hopf fibration gives a description of the fibers of the Hopf mapping $\mathscr{H}$ (24) over the $w_4 = h > 0$ section of its image cone $\mathscr{C}$ (25). To complete the description of the geometry of the Hopf mapping, we look at the fibers of $\mathscr{H}$ over other slices of $\mathscr{C}$. The results are given in table 4.1, which we leave as an exercise to verify.

Section of $\mathscr{C}$	Topology of section	Topology of fiber
1. $w_4 = k > 0$	S^2	S^3
2. $w_3 = k \ne 0$	$\mathbf{R}^2$	$S^1 \times \mathbf{R}^2$
3. $w_2 = k \ne 0$	$\mathbf{R}^2$	$S^1 \times \mathbf{R}^2$
4. $w_1 = k \ne 0$	$\mathbf{R}^2$	$S^1 \times \mathbf{R}^2$
5. $w_3 = 0$	cone on S^1	cone on T^2
6. $w_2 = 0$	cone on S^1	cone on T^2
7. $w_1 = 0$	cone on S^1	cone on T^2

Table 4.1. The fibers of the Hopf map.

5 Invariant theory and reduction

In this section we examine the geometry of the *space of orbits of energy h* of the harmonic oscillator. We will show that this space is a symplectic manifold. This fact can be exploited in several ways. One way is to gain some insight into the geometry of the foliation of the energy surface $H^{-1}(h)$ by integral curves of X_H and to see how the symplectic structure of this foliation depends on the energy. Suppose that we have a Hamiltonian system with an integral, which is the Hamiltonian of the harmonic oscillator. Using this independent first integral, we reduce the original Hamiltonian vector field to a Hamiltonian vector field on the orbit space $H^{-1}(h)/S^1$, which is two dimensions less than the original phase space. We will show how to carry out this reduction process using invariant theory. This procedure has two advantages. First, it allows us to show that any smooth Hamiltonian, which is invariant under the flow of the harmonic oscillator vector field and hence has the harmonic oscillator Hamiltonian as an integral, is a smooth function of four quadratic polynomials. Second, using these polynomials, we can explicitly construct the reduced space $H^{-1}(h)/S^1$ together with an embedding of it in Euclidean space. As a consequence, we obtain the associated Poisson (and symplectic) structure of the reduced space and the reduced vector field, which gives the reduced dynamics. These algebraic techniques will be used repeatedly in succeeding chapters of this book because they give a geometrically faithful model of the reduced space, even when it is *not* a smooth manifold. We may summarize the contents of this section as follows. Let $\mathscr{K}$ be a function which is invariant under the flow of the harmonic oscillator vector field X_H. We reduce the Hamiltonian system $(\mathscr{K}, \mathbf{R}^4, \omega)$ to a Hamiltonian system (K_h, S_h^2, ω_h) on a 2-sphere S_h^2 which is the space formed by collapsing each orbit of X_H of energy h to a point.

We begin by proving

Claim: The Hamiltonian $\mathscr{K}$ is invariant under the flow of the harmonic oscillator vector field X_H if and only if it is an integral X_H.

(5.1) **Proof**: $\mathscr{K}$ is invariant under the flow of the harmonic oscillator vector field if and only if it is constant on the integral curves of X_H if and only if it is an integral of X_H. More formally, let φ_t^H be the flow of X_H. Since $\mathscr{K}$ is invariant under φ_t^H, it follows that $(\varphi_t^H)^*\mathscr{K} = \mathscr{K}$. Differentiating this condition with respect to t and evaluating the result at $t = 0$ gives $L_{X_H}\mathscr{K} = 0$. In other words, $\mathscr{K}$ is an integral of X_H. Conversely, suppose that $\mathscr{K}$ is an integral of X_H. Then

$$\frac{\mathrm{d}}{\mathrm{d}t}(\varphi_t^H)^*\mathscr{K} = \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0}(\varphi_{t+s}^H)^*\mathscr{K} = (\varphi_t^H)^*\Big(\frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0}(\varphi_s^H)^*\mathscr{K}\Big) = (\varphi_t^H)^*(L_{X_H}\mathscr{K}) = 0.$$

Therefore $t \to (\varphi_t^H)^*\mathscr{K}$ is a constant function, that is, $(\varphi_t^H)^*\mathscr{K} = (\varphi_0^H)^*\mathscr{K} = \mathscr{K}$. Thus $\mathscr{K}$ is invariant under the flow of X_H. □

We now describe all the smooth functions which are integrals of X_H. We begin by finding all polynomial integrals of X_H. We show that they are polynomials in the quadratic integrals

$$\begin{aligned} w_1 &= x_1x_2 + y_1y_2 & \qquad w_2 &= x_1y_2 - x_2y_1 \\ w_3 &= \tfrac{1}{2}(y_1^2 + x_1^2 - y_2^2 - x_2^2) & \qquad w_4 &= \tfrac{1}{2}(y_1^2 + x_1^2 + y_2^2 + x_2^2). \end{aligned}$$

Claim: The algebra of *polynomials* which are invariant under the S^1-action given by the flow of the harmonic oscillator vector field X_H is *generated* by the quadratic functions w_i, which satisfy the relation

$$w_4^2 = w_1^2 + w_2^2 + w_3^2, \quad \text{where } w_4 \geq 0. \tag{35}$$

(5.2) **Proof**: We introduce *complex conjugate coordinates*

$$\xi_1 = x_1 + iy_1, \;\; \eta_1 = x_1 - iy_1, \;\; \xi_2 = x_2 + iy_2, \;\; \eta_2 = x_2 - iy_2.$$

Then the algebra $\mathbf{R}[x,y]$ of real polynomials on $\mathbf{R}^4$ becomes the algebra of Hermitian polynomials $\mathbf{HP}[\xi,\eta] = \big\{\sum c_{ij}\xi^i\eta^j \,\big|\, c_{ij} = \overline{c}_{ji}, \text{ where } c_{ij} \in \mathbf{C}\big\}$. Writing the Hamiltonian H in complex conjugate coordinates gives $\widetilde{H}(\xi,\eta) = \frac{1}{2}(\xi_1\eta_1 + \xi_2\eta_2)$. Moreover, we obtain the Hamiltonian vector field on $\mathbf{C}^4$

$$\dot{\xi} = -2i\frac{\partial \widetilde{H}}{\partial \eta} = -i\xi \qquad\qquad \dot{\eta} = 2i\frac{\partial \widetilde{H}}{\partial \xi} = i\eta, \tag{36}$$

where $\xi = (\xi_1, \xi_2)$, $\frac{\partial}{\partial \xi} = \frac{1}{2}(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y})$ and $\eta = (\eta_1, \eta_2)$, $\frac{\partial}{\partial \eta} = \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})$. The flow of $X_{\widetilde{H}}$ is the S^1-action $\cdot$ on $\mathbf{C}^4$ given by

$$\cdot : S^1 \times \mathbf{C}^4 \to \mathbf{C}^4 : \big(s, (\xi, \eta)\big) \mapsto (s\xi, s^{-1}\eta), \tag{37}$$

where $s \in \mathbf{C}$ with $|s| = 1$. A real polynomial is invariant under the flow of X_H if and only if the corresponding Hermitian polynomial is invariant under the flow of $X_{\widetilde{H}}$. A Hermitian polynomial is invariant if and only if for each of its monomials $M = \xi^i\eta^j = \xi_1^{i_1}\xi_2^{i_2}\eta_1^{j_1}\eta_2^{j_2}$ we have

$$s \cdot M = s^{|i|}\xi^i s^{-|j|}\eta^j = s^{|i|-|j|}\xi^i\eta^j = \xi^i\eta^j = M,$$

where $|i| = i_1 + i_2$ and $|j| = j_1 + j_2$. In other words, $|i| = |j|$.

▷ We now show that every S^1 invariant Hermitian monomial M can be written as a product of the invariant quadratic monomials $\sigma_{\ell k} = \xi_\ell \eta_k$, where $\ell = 1, 2$ and $k = 1, 2$.

(5.3) **Proof**: The factors in the monomial $M = \xi_1^{i_1}\xi_2^{i_2}\eta_1^{j_1}\eta_2^{j_2}$ can be displayed as two lists

$$\begin{array}{l} \overbrace{\xi_1 \cdots\cdots \xi_1}^{i_1}\,\overbrace{\xi_2 \cdots \xi_2}^{i_2} \\ \underbrace{\eta_1 \cdots \eta_1}_{j_1}\,\underbrace{\eta_2 \cdots\cdots \eta_2}_{j_2}. \end{array}$$

Because $|i| = |j|$, the above two lists have the same length and hence their entries may be paired off. This pairing expresses M as the product of quadratic monomials $\sigma_{\ell k}$ as claimed. □

Since

$$\begin{aligned}
\sigma_{11} &= \xi_1\eta_1 = x_1^2 + y_1^2 = w_4 + w_3 \\
\sigma_{12} &= \xi_1\eta_2 = (x_1x_2 + y_1y_2) - i(x_1y_2 - x_2y_1) = w_1 - iw_2 \\
\sigma_{21} &= \xi_2\eta_1 = (x_1x_2 + y_1y_2) + i(x_1y_2 - x_2y_1) = w_1 - iw_2 \\
\sigma_{22} &= \xi_2\eta_2 = x_2^2 + y_2^2 = w_4 - w_3,
\end{aligned}$$

every S^1 invariant polynomial is a sum of monomials which are products of w_1, w_2, w_3, w_4 times a real coefficient. From the identity

$$(x_1x_2 + y_1y_2)^2 + (x_1y_2 - x_2y_1)^2 = (x_1^2 + y_1^2)(x_2^2 + y_2^2) \tag{38}$$

it follows that $w_1^2 + w_2^2 + w_3^2 = w_4^2$. Clearly $w_4 \geq 0$. This proves the claim. □

Claim: The only polynomial relation among the generators w_i of the algebra of polynomials invariant under the flow of X_H is $w_1^2 + w_2^2 + w_3^2 - w_4^2 = 0$.

(5.4) **Proof**: Consider the *complexified* Hopf mapping

$$\Phi : \mathbf{C}^4 \to \mathbf{C}^4 : (x_1,x_2,y_1,y_2) \to (w_1,w_2,w_3,w_4) = \\ = \big(x_1x_2+y_1y_2,\, x_1y_2-x_2y_1,\, \tfrac{1}{2}(y_1^2+x_1^2-y_2^2-x_2^2),\, \tfrac{1}{2}(y_1^2+x_1^2+y_2^2+x_2^2)\big).$$

We assert that the image of Φ is equal to the zero set $\mathscr{Z}_F$ of the polynomial $F = w_1^2+w_2^2+w_3^2-w_4^2$. Since $F\circ\Phi = 0$, which follows from (38), the image of Φ is contained in $\mathscr{Z}_F$. To prove the reverse inclusion, let $(w_1,w_2,w_3,w_4) \in \mathscr{Z}_F$. Consider the following cases:

CASE 1. If $w_4+w_3 \neq 0$, let

$$(x_1,x_2,y_1,y_2) = \big(0,-w_2/\sqrt{w_4+w_3},\sqrt{w_4+w_3},w_1/\sqrt{w_4+w_3}\big).$$

CASE 2. If $w_4+w_3 = 0$, let $(x_1,x_2,y_1,y_2) = (0,0,0,\sqrt{2w_4})$.

Which branch of the square root one chooses above is immaterial, as long as it is consistent. In all of the above cases, $\Phi(x_1,x_2,y_1,y_2) = (w_1,w_2,w_3,w_4)$. Thus the image of Φ contains $\mathscr{Z}_F$. Next we show that the polynomial F is irreducible. Suppose not. Then F is the product of two factors. Write $F = (\alpha w_1+\beta)(\gamma w_1+\delta)$, where $\alpha,\beta,\gamma,\delta$ are polynomials in w_2,w_3,w_4. Clearly we can take $\alpha = \gamma = 1$. Since F has no term which is linear in w_1, we must have $\delta = -\beta$. Therefore the expression for F becomes $-\beta^2 = w_2^2+w_3^2-w_4^2$. Consequently the degree of β is at most 1. Since $\beta(0,0,0) = 0$, we may write $\beta = bw_2+cw_3+dw_4$ for some $b,c,d \in \mathbf{C}$. Squaring the preceding formula for β and equating coefficients with the expression for $-\beta^2$ gives $b = \pm i$, $c = \pm i$ and $bc = 0$, which is a contradiction. Let $\mathscr{I}$ be the *ideal* in $\mathbf{C}[w_1,w_2,w_3,w_4]$ generated by F. Since $\mathbf{C}$ is a field, the polynomial ring $\mathbf{C}[w_1,w_2,w_3,w_4]$ is a unique factorization domain and an integral domain. Thus every irreducible element is prime. Since every ideal in an integral domain which is generated by a prime polynomial is a prime ideal, $\mathscr{I}$ is a prime ideal. Suppose that f is a polynomial in $\mathbf{C}[w_1,w_2,w_3,w_4]$ such that $f\circ\Phi = 0$, that is, f is a polynomial relation among the generators w_i. Since the image of Φ is the zero set of $\mathscr{I}$ and f vanishes on the image of Φ, it follows that the zero set of the ideal generated by f contains the image of Φ. By the Hilbert Nullstellensatz there is a positive integer m such that $f^m \in \mathscr{I}$. Since $\mathscr{I}$ is prime, $f \in \mathscr{I}$. □

The flow of X_H defines an algebraic linear action of $\mathrm{SO}(2,\mathbf{R}) = S^1$ on $\mathbf{R}^4$ given by

$$\varphi^H : \mathrm{SO}(2,\mathbf{R}) \times \mathbf{R}^4 \to \mathbf{R}^4 : \left(\begin{pmatrix} a & -b \\ b & a \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} \right) \mapsto \begin{pmatrix} aI_2 & -bI_2 \\ bI_2 & aI_2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$

where $a^2+b^2 = 1$. We may use a theorem of Schwarz to conclude that every *smooth* integral of the harmonic oscillator is a smooth function of the quadratic integrals. Thus we have proved

Claim: For every smooth function $\mathscr{K}$ on $\mathbf{R}^4$, which is invariant under the flow of the harmonic oscillator, there is a smooth function K on $\mathbf{R}^4$ such that $\mathscr{K} = J^*K$, where J is the $\mathrm{U}(2)$-momentum mapping of the harmonic oscillator.

We now turn to constructing a Poisson bracket on $\mathbf{R}^3$ with coordinates (w_1,w_2,w_3). Since the Hamiltonian vector fields X_{w_i} for $i = 1,\ldots,4$ form a Lie algebra which is isomorphic to the Lie algebra $\mathrm{u}(2)$, the quadratic integrals w_i form a Lie algebra under Poisson bracket $\{\, ,\}$ on $C^\infty(T^*\mathbf{R}^2)$ which is isomorphic to $\mathrm{u}(2)$. The bracket relations for this Lie algebra

$\{A,B\}$	w_1	w_2	w_3	w_4	B
w_1	0	$-2w_3$	$2w_2$	0	
w_2	$2w_3$	0	$-2w_1$	0	
w_3	$-2w_2$	$2w_1$	0	0	
w_4	0	0	0	0	
A					

Table 5.1 The structure matrix $\mathscr{W}$ of the Poisson algebra $\mathscr{A}$.

are found by calculating $\{w_1,w_2\} = \omega(X_{w_1},X_{w_2}) = -2w_3$. The rest of the bracket relations are given in table 5.1. Using the quadratic integrals $w = (w_1,w_2,w_3,w_4)$ as coordinates on $\mathbf{R}^4$, the space $C^\infty(\mathbf{R}^4)$ can be made into a *Poisson algebra* $\mathscr{A}$ by defining a *Poisson bracket* by $\{f,g\} = \sum_{i,j=1}^4 \frac{\partial f}{\partial w_j}\frac{\partial g}{\partial w_i}\{w_j,w_i\}$, where $f,g \in C^\infty(\mathbf{R}^4)$, see chapter VI §4. The bracket $\{\, ,\}$ is entirely determined by the bracket relations given in table 5.1, because of the chain rule.

For $K \in C^\infty(\mathbf{R}^4)$ the corresponding *Hamiltonian vector field* X_K is

$$\dot{w}_j = \{w_j,K\} = \sum_i \frac{\partial K}{\partial w_i}\{w_j,w_i\} = -2\sum_{i,k}\frac{\partial K}{\partial w_i}\varepsilon_{jik}w_k \tag{39}$$

for $j = 1,\ldots,4$. Note that for any $f \in C^\infty(\mathbf{R}^4)$ we have $\{f,w_4\} = 0$ and $\{f, w_4^2 - w_1^2 - w_2^2 - w_3^2\} = 0$, because they vanish for $f = w_i$ where $i = 1,\cdots,4$. Thus the functions w_4 and $w_4^2 - w_1^2 - w_2^2 - w_3^2$ are Casimir elements of the Poisson algebra $\mathscr{A}$. Therefore the level set $w_4 = h$ for $h > 0$ defines a smooth submanifold $\mathbf{R}^3 \times \{h\}$ of $\mathbf{R}^4$ diffeomorphic to $\mathbf{R}^3$ which is invariant under the flow of the vector field X_K. Using (39) we see that $X_{K|(\mathbf{R}^3\times\{h\})}$ is given by

$$\dot{w} = -2\,\mathrm{grad}\,K(w) \times w, \tag{40}$$

where $w = (w_1,w_2,w_3)^t \in \mathbf{R}^3$ and all partial derivatives are evaluated with $w_4 = h$.

Now consider the space $\mathscr{B}$ of smooth functions on $\mathbf{R}^3$ which are restrictions of smooth functions on $\mathbf{R}^4$ to $\{w_4 = h\}$. For $K \in \mathscr{A}$ let $\widetilde{K}_h \in \mathscr{B}$ be the restriction of K to $\{w_4 = h\}$. Because w_4 is a Casimir for $\mathscr{A}$, the space $\mathscr{B}$ is a Poisson subalgebra of $\mathscr{A}$ with bracket

$\{A,B\}$	w_1	w_2	w_3	B
w_1	0	$-2w_3$	$2w_2$	
w_2	$2w_3$	0	$-2w_1$	
w_3	$-2w_2$	$2w_1$	0	
A				

Table 5.2 The structure matrix W of the Poisson algebra $\mathscr{B}$. Here the functions w_i are restricted to $\mathbf{R}^3 \times \{h\}$.

relations given in table 5.2. Using the Poisson algebra $\mathscr{B}$, we show that the smooth 2-sphere S_h^2 defined by $w_1^2+w_2^2+w_3^2=h^2$ has a symplectic 2-form

$$\omega_h(w)(u,v) = -\tfrac{1}{2h^2}\,(w,u\times v). \tag{41}$$

Here $w\in S_h^2$, $u,v\in T_wS_h^2$, and $(\,,\,)$ is the Euclidean inner product on $\mathbf{R}^3$.

(5.5) **Proof**: Consider the structure matrix $W(w)$ of the Poisson algebra $\mathscr{B}$ given in table 5.2. Since $\ker W(w) = \mathrm{span}\{w\}$ and $T_wS_h = \mathrm{span}\{w\}^{\perp}$, the matrix $W(w)|T_wS_h$ is invertible. On S_h^2 define the symplectic form $\omega_h(w)(u,v) = u^t\,(W^{-1}(w))^t(v)$ for $w\in S_h^2$ and $u,v\in T_wS_h$. Let $y\in T_wS_h^2$. Then $W(w)y = 2\,w\times y = u$ so that

$$w\times u = w\times(2w\times y) = 2\big(w\,(w,y) - y\,(w,w)\big) = -2y\,(w,w) = -2h^2\,y.$$

Thus $W^{-1}(w)u = -\frac{1}{2h^2}w\times u$, which yields

$$u^t\,(W^{-1}(w))^t v = -\tfrac{1}{2h^2}\,(w\times u)^t v = -\tfrac{1}{2h^2}\,(w\times u, v) = -\tfrac{1}{2h^2}\,(w,u\times v).$$

Therefore $\omega_h(w)(u,v) = -\frac{1}{2h^2}\,(w,u\times v)$. □

On the symplectic manifold (S_h^2,ω_h) the vector field $X_{\widetilde{K}_h}(w) = -2(\mathrm{grad}\,\widetilde{K}_h\times w)$, where $\widetilde{K}_h = K|(\mathbf{R}^3\times\{h\})$, is Hamiltonian because

$$\begin{aligned}\omega_h(w)\big(X_{\widetilde{K}_h}(w),v\big) &= -\tfrac{1}{2h^2}\,(w,-2(\mathrm{grad}\,\widetilde{K}_h\times w)\times v) = \tfrac{1}{h^2}\,\big(w\times(\mathrm{grad}\,\widetilde{K}_h\times w),v\big)\\ &= \tfrac{1}{h^2}\,\big(-w\,(\mathrm{grad}\,\widetilde{K}_h,w)+\mathrm{grad}\,\widetilde{K}_h\,(w,w),v\big) = (\mathrm{grad}\,\widetilde{K}_h,v) = \mathrm{d}\widetilde{K}_h(w)v,\end{aligned}$$

where $w\in S_h^2$ and $v\in T_wS_h^2$.

Up to the factor 2 the integral curves of the vector field X_{K_h}, when the Hamiltonian $\widetilde{K}_h$ is $\frac{1}{2}(I_1^{-1}w_1^2+I_2^{-1}w_2^2+I_3^{-1}w_3^2)$, satisfy *Euler's equations* for the rigid body in momentum coordinates, see chapter III §3.3.

Note that the image under the $\mathrm{U}(2)$-momentum mapping J of the integral curves of $X_{\mathscr{K}}|H^{-1}(h)$ are the integral curves of X_{K_h} on the orbit space $H^{-1}(h)/S^1 = S_h^2$ with symplectic form ω_h. This is precisely what the *regular reduction theorem* says in the case of the harmonic oscillator, see chapter VII §6. Here (S_h^2,ω_h) is the *reduced phase space*, K_h is the *reduced Hamiltonian*, and J is the reduction mapping.

6 Exercises

1. (Complex projective 1-space.) *Complex projective 1-space* $\mathbf{CP}^1$ is defined as the set of equivalence classes of vectors in $\mathbf{C}^2\setminus\{0\}$ under the equivalence relation $\sim$ defined by $(z_1,z_2)\sim(w_1,w_2)$ if there is a $\lambda\in\mathbf{C}^* = \{z\in\mathbf{C}\,|\,|z|=1\}$ such that $(z_1,z_2) = (\lambda w_1,\lambda w_2)$. Denote the equivalence class of (z_1,z_2) by $[z_1:z_2]$. In other words $[z_1:z_2]$ are *homogeneous coordinates* on $\mathbf{CP}^1$. Let

$$U_1 = \{[1:z_2/z_1]\in\mathbf{CP}^1|\,z_1\neq 0\}\quad\text{and}\quad U_2 = \{[z_1/z_2:1]\in\mathbf{CP}^1|\,z_2\neq 0\}$$

with coordinates $w = z_2/z_1$ and $z = z_1/z_2$, respectively. Show that $\{U_1, U_2\}$ form an atlas for $\mathbf{CP}^1$ with transition function

$$\varphi_{12} : U_1 \cap U_2 \to U_1 \cap U_2 : z \to w = 1/z.$$

Show that $\mathbf{CP}^1$ is diffeomorphic to S^2 by verifying that the function

$$f : \mathbf{CP}^1 \to \mathbf{R} : [z_1 : z_2] \to \tfrac{1}{2}(a|z_1|^2 + b|z_2|^2), \qquad a > b > 0$$

is a Morse function with two nondegenerate critical points. Show that

$$\omega = \mathrm{Im}\left(\frac{(z_2\,\mathrm{d}z_1 - z_1\,\mathrm{d}z_2) \wedge (\bar{z}_2\,\mathrm{d}\bar{z}_1 - \bar{z}_1\,\mathrm{d}\bar{z}_2)}{|z_1|^2 + |z_2|^2}\right)$$

is a symplectic form on $\mathbf{CP}^1$.

2. (Linking number.)

a) Let A and B be two smooth circles in S^3. Suppose that W_A and W_B are disjoint tubular neighborhoods of A and B in S^3. Then W_A and W_B are diffeomorphic to $D_A \times S^1$ and $D_B \times S^1$ for some 2-disks D_A and D_B respectively. There are 2-forms η_A and η_B on S^3 which are nonzero on W_A and W_B and zero elsewhere. To show that η_A is closed we argue as follows. Since the normalized volume 3-form vol_{S^3} generates $H^3(S^3)$, it follows that for some λ being a real valued function we have $\mathrm{d}\eta_A = \lambda\,\mathrm{vol}_{S^3}$. By *Stokes' theorem* $\int_{S^3} \mathrm{d}\eta_A = \int_{\partial S^3} \eta_A = 0$, since $\partial S^3 = \varnothing$. Therefore $\lambda = 0$. Similarly η_B is closed. Since W_A and W_B are contractible in S^3, the 2-forms η_A and η_B are exact. So there are 1-forms ξ_A and ξ_B on S^3 such that $\mathrm{d}\xi_A = \eta_A$ and $\mathrm{d}\xi_B = \eta_B$. This also follows because $H^2(S^3) = H^1(S^3) = 0$. Define the *linking number* of A and B as

$$\mathrm{Link}(A,B) = \int_{S^3} \xi_A \wedge \eta_B.$$

Show that $\mathrm{Link}(A,B)$ does not depend on the choice of ξ_A. Also show that $\mathrm{Link}(A,B)$ does not depend on the representative of $\eta_B \in H^2(S^3)$, that is, if $\eta_B' = \eta_B + \mathrm{d}\zeta$ for some 1-form ζ on S^3, then $\mathrm{Link}(A,B) = \int_{S^3} \xi_A \wedge \eta_B'$.

b) Is $\mathrm{Link}(A,B) = \mathrm{Link}(B,A)$?

3. To show that the linking number as defined in the text does not depend on the oriented 2-disk D_1^2 with boundary γ. Let $F : D_1^2 \subseteq \mathbf{R}^2 \to S^3$ be a smooth contraction of D_1^2. Let $G : \widetilde{D}_1^2 \subseteq \mathbf{R}^2 \to S^3$ be another contraction of γ_1. Give the 2-disk $\widetilde{D}_1^2$ the *orientation* opposite to that of the 2-disk D_1^2. Consider the oriented 2-sphere $S^2 \subseteq S^3$ formed from the 2-disks $F(\overline{D}_1^2)$ and $G(\widetilde{D}_1^2)$ by identifying $\partial F(\overline{D}_1^2) = F(\partial \overline{D}_1^2)$ with $\partial G(\widetilde{D}_1^2) = G(\partial \widetilde{D}_1^2)$. The intersection number of S^2 with γ_2 is equal to its intersection number with $F(\overline{D}_1^2)$ minus its intersection number with $G(\widetilde{D}_1^2)$. If we show that the intersection number of S^2 with γ_2 is zero, then we are done. To do this choose a point p in $S^3 \setminus (S^2 \cup \gamma_2)$ and let $\varphi : S^3 \setminus \{p\} \to \mathbf{R}^3$ be *stereographic projection*. Since φ preserves orientation, the intersection number of S^2 and γ_2 is the same as the intersection number of $\varphi(S^2)$ and $\varphi \circ \gamma_2$. Thus we have to show that

the intersection number of an oriented smooth circle $\gamma\colon [0,1] \to \mathbf{R}^3$ which meets an oriented $S^2 \subseteq \mathbf{R}^3$ transversely is zero. If γ does not intersect S^2 then we are done. Suppose that at $\gamma(t_0) \in S^2$ the curve γ has intersection number 1. Then for some sufficiently small $\varepsilon > 0$, $\gamma(t_0 - \varepsilon)$ lies in the bounded component of $\mathbf{R}^3 \setminus S^2$; while $\gamma(t_0 + \varepsilon)$ is in the unbounded component. Reparametrize γ so that γ is defined on $[0,1]$ and begins and ends at $p = \gamma(t_0)$. Since γ is a closed curve, there is a $t_1 \in (0,1)$ such that $\gamma(t_1) \in S^2$. Choose t_1 as small as possible. There are only finitely many since γ intersects S^2 transversely. Then for every $t \in (0,t_1)$, $\gamma(t)$ lies in the unbounded component of $\mathbf{R}^3 \setminus S^2$. Now $q_1 = \gamma(t_1) \neq p$ since γ is a diffeomorphism. Since γ is transverse to S^2 at q_1, $\gamma(t_1 + \varepsilon)$ lies in the bounded component of $\mathbf{R}^3 \setminus S^2$ for sufficiently small $\varepsilon > 0$. Therefore the intersection number of γ at q_1 is -1. A similar argument shows that the next intersection point q_2 of γ with S^2 has intersection number 1. If $q_2 = p$ then we are through; otherwise repeat the argument a finite number of times until $t = 1$ is reached. Since γ has an even number of intersections with S^2, its intersection number is 0. Thus the linking number is well defined.

4. (Degree of a map.) Let (M,σ) and (N,τ) be two connected compact oriented manifolds of dimension r with volume forms σ and τ, respectively. Suppose that $f: M \to N$ is a smooth map. Let $n = f(m)$ be a regular value of f. For each $p \in f^{-1}(n)$ let

$$\operatorname{sign}_p f = \begin{cases} 1, & \text{if } T_pf : (T_pM, \sigma_p) \to (T_nN, \tau_n) \\ & \text{is orientation preserving} \\ -1, & \text{otherwise} \end{cases}$$

Define the *degree* of f by

$$\deg f = \sum_{p \in f^{-1}(n)} \operatorname{sign}_p f.$$

Show that $\int_M f^*\tau = (\deg f) \int_N \tau$. If $f: M \to N$ and $g: M \to N$ are smoothly homotopic, then show that $\deg f = \deg g$.

5. (Hopf invariant.)

a) Let $g: S^3 \to S^2$ be a smooth map and let α be a 2-form on S^2 which generates $H^2(S^2)$. Since α is closed, $g^*\alpha$ is a closed 2-form on S^3. Since $H^2(S^3) = H^1(S^3) = 0$, there is a 1-form β on S^3 such that $g^*\alpha = \mathrm{d}\beta$. Define the *Hopf invariant* of g to be

$$\operatorname{Hopf}(g) = \int_{S^3} \beta \wedge \mathrm{d}\beta.$$

Show that $\operatorname{Hopf}(g)$ does not depend on the choice of β. Moreover, if g and h are homotopic show that $\operatorname{Hopf}(g) = \operatorname{Hopf}(h)$.

b) Let p and q be distinct regular values of the map g. Let D_p and D_q be disjoint closed 2-disks on S^2. Let α_p and α_q be 2-forms on S^2 which are nonzero on D_p and D_q, respectively, and are zero elsewhere. Let $\eta_A = g^*\alpha_p$ and $\eta_B = g^*\alpha_q$. From exercise 2 we know that the linking number of the circles $A = g^{-1}(p)$ and $B = g^{-1}(q)$ in S^3 is given by $\mathrm{Link}(A,B) = \int_{S^3} \xi_A \wedge \eta_B$, where $\eta_B = \mathrm{d}\xi_B$. To show that

$$\mathrm{Link}(A,B) = \mathrm{Hopf}(g) = \int_{S^3} \xi_A \wedge \eta_A$$

we argue as follows. Because $\dim H^2(S^2) = 1$, there is a 1-form β on S^2 such that $\alpha_p - \alpha_q = \mathrm{d}\beta$. Therefore $\eta_A - \eta_B = g^*(\mathrm{d}\beta) = \mathrm{d}(g^*\beta)$. Hence

$$\int_{S^3} \xi_A \wedge (\eta_A - \eta_B) = -\int_{S^3} \mathrm{d}(\xi_A \wedge g^*\beta) + \int_{S^3} g^*(\alpha_p \wedge \beta) = 0.$$

c) Consider the map

$$f : S^3 = \{(z_1, z_2) \in \mathbf{C}^2 \big| |z_1|^2 + |z_2|^2 = 1\} \to \mathbf{CP}^1 : (z_1, z_2) \mapsto [z_1 : z_2].$$

Note that $f(1,0) = [1 : 0]$. Composing f with stereographic projection from the north pole onto the *equatorial* plane gives the Hopf map. To show that for every $[z_1 : z_2] \in \mathbf{CP}^1 = S^2$ the fiber $f^{-1}([z_1 : z_2])$ is a great circle on S^3 we argue as follows. For $c = (c_1, c_2, c_3) \in S^2 \subseteq \mathbf{R}^3$ we see that $(z_1, z_2) \in f^{-1}(c)$ if and only if $|z_1|^2 + |z_2|^2 = 1$, $|z_1|^2 - |z_2|^2 = c_3$, and $2z_1\bar{z}_2 = c_1 + ic_2$. Thus $|z_2|^2 = \frac{1}{2}(1 - c_3) \neq 0$. Choose z_2^0 so that $|z_2^0|^2 = \frac{1}{2}(1 - c_3)$ and z_1^0 satisfies $2z_1^0\bar{z}_2^0 = c_1 + ic_2$. Show that $f^{-1}(c) = \{\zeta(z_1^0, z_2^0) \in S^3 \big| |\zeta| = 1\}$. Thus $f^{-1}(c)$ lies in a complex 1-dimensional subspace of $\mathbf{C}^2$ and hence is a great circle on S^3.

d) Here we show that the Hopf invariant of the Hopf map is 1. Let p be the north pole of S^2 and D_p be the disk on S^2 containing p and bounded by the equator. Let ω be the 2-form on D_p whose pull back under the chart

$$\{u_1^2 + u_2^2 < 1\} \to S^2 \subseteq \mathbf{R}^3 : (u_1, u_2) \mapsto u_3 = \sqrt{1 - u_1^2 - u_2^2}$$

is $\frac{\mathrm{d}u_1 \wedge \mathrm{d}u_2}{2\pi u_3}$. Show that $\int_{D_p} \omega = 1$. Using the relation $\sum_{i=1}^4 x_i \,\mathrm{d}x_i = 0$, which comes from taking the exterior derivative of the defining equation of S^3, show that the pull back of ω by the Hopf map h is $d\beta$ where $\beta = -\frac{1}{\pi}(x_1 dx_2 + x_3 dx_4)$. Finally, using spherical coordinates

$$\begin{cases} x_1 &= \sin\varpi \sin\varphi \cos\theta \\ x_2 &= \sin\varpi \sin\varphi \sin\theta \\ x_3 &= \sin\varpi \cos\varphi \\ x_4 &= \cos\varpi \end{cases}$$

with $0 \leq \varpi \leq \pi$, $0 \leq \varphi \leq \pi$ and $0 \leq \theta \leq 2\pi$, show that

$$\int_{S^3} \beta \wedge \mathrm{d}\beta = \frac{2}{\pi^2} \int_{S^3} x_1 \,\mathrm{d}x_2 \wedge \mathrm{d}x_3 \wedge \mathrm{d}x_4 = 1.$$

6. (Exceptional fibers.) Consider the harmonic oscillator in dimension three. The integrals of motion are the energy and $\mathrm{SO}(3)$ angular momentum.

a) Describe all the fibers of the energy momentum map.

b) Show that the fibers with positive energy and zero angular momentum are smooth Lagrangian submanifolds of $T^*\mathbf{R}^3$ with its standard symplectic form, which are not tori.

7. (n-Dimensional harmonic oscillator.)

a) Let $z = (z_1, \ldots, z_n)$ be coordinates for $\mathbf{C}^n$. Put a Kähler structure on $\mathbf{C}^n$ by defining the Kähler form $\Omega = \frac{i}{2}\sum_{j=1}^n \mathrm{d}z_j \wedge \mathrm{d}\bar{z}_j$. Let $h : \mathbf{C}^n \to \mathbf{R} : z \mapsto h(z, \bar{z})$ be a smooth real valued function. Define the *Kähler Hamiltonian vector field* X_h associated to the *Kähler Hamiltonian* h by $X_h \mathbin{\lrcorner} \Omega = \overline{\mathrm{d}}h$, where $\overline{\mathrm{d}}h = \sum_{j=1}^n \frac{\partial h}{\partial \bar{z}_j}\mathrm{d}\bar{z}_j$. Show that $X_h = -i\sum_{j=1}^n \frac{\partial h}{\partial \bar{z}_j}\frac{\partial}{\partial z_j}$. In particular, if $h : \mathbf{C}^n \to \mathbf{R} : z \mapsto \frac{1}{2}\sum_{j=1}^n z_j \bar{z}_j$ is the harmonic oscillator Hamiltonian, then $X_h = -i\sum_{j=1}^n z_j \frac{\partial}{\partial x_j}$, which is a holomorphic vector field on $\mathbf{C}^n$ whose flow is $\varphi : \mathbf{R} \times \mathbf{C}^n \to \mathbf{C}^n : (t, z) \to e^{-it}z$.

b) Let $\langle z, w\rangle = \sum_{j=1}^n z_j \overline{w}_j$ be the standard *Hermitian inner product* on $\mathbf{C}^n$. An invertible linear map $U : \mathbf{C}^n \to \mathbf{C}^n$ is unitary if $\langle Uz, Uw\rangle = \langle z, w\rangle$ for every $z, w \in \mathbf{C}^n$. The set of all unitary matrices forms the Lie group $\mathrm{U}(n)$ with Lie algebra $\mathrm{u}(n)$ given by the set of all skew Hermitian linear maps $u : \mathbf{C}^n \to \mathbf{C}^n$, that is, $\langle uz, w\rangle + \langle z, uw\rangle = 0$ for every $z, w \in \mathbf{C}^n$. Show that the $n \times n$ skew Hermitian matrices

$$\begin{cases} i(e_j \otimes \bar{e}_k^t + e_k \otimes \bar{e}_j^t), & 1 \le j < k \le n \\ e_j \otimes \bar{e}_k^t - e_k \otimes \bar{e}_j^t, & 1 \le j < k \le n \\ i(e_j \otimes \bar{e}_j^t), & 1 \le j \le n \end{cases} \tag{42}$$

form a basis for $\mathrm{u}(n)$ as a real vector space. Here $\{e_j\}_{j=1}^n$ is the standard basis for $\mathbf{C}^n$. Recall that $i(z \otimes \bar{z}^t) \in \mathbf{C} \otimes (\overline{\mathbf{C}^n})^*$ is the skew Hermitian linear map $w \mapsto i\langle w, z\rangle z$. Write out the matrices in (42) explicitly. Show that

$$k : u(n) \times u(n) \to \mathbf{C} : (u, v) \mapsto \mathrm{tr}\, u\bar{v}^t$$

is a Hermitian inner product on $u(n)$. Since k is Ad_U-invariant for every $U \in \mathrm{U}(n)$, it follows that k is the Killing (Hermitian) metric.

c) The Lie group $\mathrm{U}(n)$ acts on $\mathbf{C}^n$ by $\Phi : U(n) \times \mathbf{C}^n \to \mathbf{C}^n : (U, z) \mapsto Uz$. For every $u = (u_{jk}) \in \mathrm{u}(n)$ the vector field infinitesimally generated by u is

$$X^u(z) = \frac{d}{ds}\Big|_{s=0} \Phi_{\exp su}(z) = \sum_{j,k}^n u_{jk} z_k \frac{\partial}{\partial z_j}.$$

Show that X^u is the Kähler Hamiltonian vector field associated to the function

$$\begin{aligned} j^u : \mathbf{C}^n \to \mathbf{R} : z \mapsto \tfrac{i}{2}\langle uz, z\rangle = \\ = \frac{i}{2}\Big[\sum_{j<k}\Big(\mathrm{Re}\, u_{jk}\,\mathrm{Re}\, z_j\bar{z}_k - \mathrm{Im}\, u_{jk}\,\mathrm{Im}\, z_j\bar{z}_k\Big) + \sum_{j=1}^n u_{jj} z_j \bar{z}_j\Big]. \end{aligned}$$

Show that j^u is an integral of the harmonic oscillator vector field X_h. Define the map $\mathscr{J} : \mathbf{C}^n \mapsto \mathrm{u}(n)^*$ by $\mathscr{J}(z)u = j^u(z)$ for every $z \in \mathbf{C}^n$ and every $u \in \mathrm{u}(n)$. Show that $\mathscr{J}$ intertwines the $\mathrm{U}(n)$-action Φ on $\mathbf{C}^n$ with the coadjoint action of $\mathrm{U}(n)$ on $\mathrm{u}(n)^*$. In other words,

$$\mathscr{J}(Uz)u = \mathrm{Ad}^t_{U^{-1}}(\mathscr{J}(z))u = \mathscr{J}(z)(U^{-1}uU). \tag{43}$$

Thus $\mathscr{J}$ is a *Kähler momentum mapping* for the action Φ. Use the Killing Hermitian metric k to identify $\mathrm{u}(n)^*$ with $\mathrm{u}(n)$. Using the basis (42) of $u(n)$, show that $\mathscr{J}$ becomes

$$J : \mathbf{C}^n \to \mathrm{u}(n) : z \mapsto \tfrac{i}{2}(z \otimes \bar{z}^t) = \tfrac{i}{2}(z_j \bar{z}_k). \tag{44}$$

Verify directly that J intertwines the $\mathrm{U}(n)$ action Φ and the adjoint action of $\mathrm{U}(n)$ on $\mathrm{u}(n)$.

d) Show that $\mathrm{U}(n)$ acts transitively on the set of all 1-dimensional complex subspaces of $\mathbf{C}^n$ with isotropy group at $\mathrm{span}\{e_1\}$ isomorphic to $\mathrm{U}(n-1)$. Deduce that the orbit space $\mathrm{U}(n)/\mathrm{U}(n-1)$ is diffeomorphic to $\mathbf{CP}^{n-1}$, complex projective $(n-1)$-space. For $z \neq 0$ show that $i(z \otimes \bar{z}^t)$ has rank 1 and trace $\langle z, z\rangle$. If $z \neq 0$ deduce that the $\mathrm{U}(n)$ adjoint orbit $\mathscr{O}_\zeta$ through $\zeta = J(z)$ is diffeomorphic to $\mathbf{CP}^{n-1}$. In particular, $\mathbf{CP}^{n-1}$ is defined by

$$\left\{ \begin{array}{l} (z_j\bar{z}_k)(z_\ell \bar{z}_m) = (z_j \bar{z}_m)(z_\ell \bar{z}_k), \quad 1 \le j,k,\ell,m \le n \\[2ex] \qquad \sum_{j=1}^n z_j \bar{z}_j = 1. \end{array} \right. \tag{45}$$

Since (45) is the set of all relations among the quadratic (and hence smooth) integrals of the harmonic oscillator restricted to the level set $h^{-1}(\frac{1}{2})$, the space $h^{-1}(\frac{1}{2})/S^1$ of orbits of the harmonic oscillator of energy $\frac{1}{2}$ is diffeomorphic to $\mathbf{CP}^{n-1}$. Show that

$$J|h^{-1}(\tfrac{1}{2}) : h^{-1}(\tfrac{1}{2}) \to \mathbf{CP}^{n-1} = h^{-1}(\tfrac{1}{2})/S^1$$

is the reduction map for the harmonic oscillator. Use the following argument to find the symplectic form on the reduced space $\mathbf{CP}^{n-1}$. On the orbit $\mathscr{O}_\zeta$ with $\langle z, z\rangle = 1$, there is a symplectic form

$$\omega(\zeta)(v_\zeta, w_\zeta) = k(\zeta, [v_\zeta, w_\zeta]), \tag{46}$$

where $v_\zeta = DJ(z)v$ and $w_\zeta = DJ(z)w$ lie in $T_\zeta \mathscr{O}_\zeta \subseteq u(n)$. Using (46), $\langle z, z\rangle = 1$ and the fact that $v_\zeta = \frac{i}{2}(v_j \bar{z}_k + z_j \bar{v}_k)$, $w_\zeta = \frac{i}{2}(w_j \bar{z}_k + z_j \overline{w}_k)$, show that

$$\omega = \tfrac{1}{4}\mathrm{Im}\big(\langle v, w\rangle \langle z, z\rangle - \langle v, z\rangle \langle w, z\rangle\big). \tag{47}$$

Replacing z by $z/\sqrt{\langle z, z\rangle}$, (47) becomes

$$\omega = \tfrac{1}{4}\mathrm{Im}\left(\frac{\langle v, w\rangle \langle z, z\rangle - \langle v, z\rangle \langle w, z\rangle}{\langle z, z\rangle}\right),$$

which is $\frac{1}{4}$ times the imaginary part of the Fubini-Study Hermitian metric on $\mathbf{CP}^{n-1}$.

Chapter II

Geodesics on S^3

In this chapter we study the geodesic vector field on the tangent bundle of the 3-sphere. We examine its relation to the Kepler vector field, which governs the motion of two bodies in $\mathbf{R}^3$ under gravitational attraction. We give two methods to regularize the flow of the Kepler vector field: one energy surface by energy surface and the other for all negative energies at once.

1 The geodesic vector field

Here we find the geodesic vector field on the 3-sphere and give a formula for its flow.

We begin by discussing the geodesic vector field. Suppose that $\langle\,,\,\rangle$ is the Euclidean inner product on $\mathbf{R}^4$. This induces a *Riemannian metric* g on $\mathbf{R}^4$ defined by $g(x)^\sharp(y)z = \langle y,z\rangle$, where $x \in \mathbf{R}^4$ and $y,z \in T_x\mathbf{R}^4 = \mathbf{R}^4$. Pulling back the canonical symplectic 2-form on $T^*\mathbf{R}^4$ by the map $g^\sharp$, see chapter VI §2, we obtain the symplectic form $\omega_4 = -\,\mathrm{d}\langle y, \mathrm{d}x\rangle$ on $T\mathbf{R}^4$. On $(T\mathbf{R}^4, \omega_4)$ consider the Hamiltonian function

$$\mathscr{H} : T\mathbf{R}^4 \to \mathbf{R} : (x,y) \mapsto \tfrac{1}{2}\langle y,y\rangle. \tag{1}$$

Since an integral curve of the Hamiltonian vector field $X_{\mathscr{H}}$ satisfies $\dot{x} = y$ and $\dot{y} = 0$, it is a straight line on $T\mathbf{R}^4$, except when $y = 0$; then it is a point. Hence $X_{\mathscr{H}}$ describes the *motion* of a particle in $T\mathbf{R}^4$ which is not subject to any *force*. To constrain this free particle so that it moves on the 3-sphere $S^3 = \{x \in \mathbf{R}^4 \,\big|\, \langle x,x\rangle = 1\}$, we add a force $\lambda(x,\dot{x})x$ which is normal to S^3 at the point x. The motion of the particle subject to this constraining force is governed by Newton's equations

$$\ddot{x} = \lambda(x,\dot{x})\,x. \tag{2}$$

Differentiating the defining equation of S^3 twice gives

$$\langle x,\ddot{x}\rangle + \langle \dot{x},\dot{x}\rangle = 0. \tag{3}$$

Substituting (2) into (3) and using the constraint $\langle x,x\rangle = 1$ gives $\lambda(x,\dot{x}) = -\langle \dot{x},\dot{x}\rangle$. Hence the *motion* of the free particle constrained to S^3 is governed by the second order equation

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$$\ddot{x} = -\langle \dot{x}, \dot{x} \rangle x \tag{4}$$

subject to the constraints $\langle x,x\rangle = 1$ and $\langle \dot{x},x\rangle = 0$. Written as a first order equation on the tangent bundle $TS^3 = \{(x,y) \in T\mathbf{R}^4 \,\big|\, \langle x,x\rangle = 1 \;\&\; \langle x,y\rangle = 0\}$ of S^3, the constrained system (4) becomes

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -\langle y,y\rangle x. \end{aligned} \tag{5}$$

This defines the integral curves of the vector field $Y = \langle y, \frac{\partial}{\partial x}\rangle - \langle y,y\rangle \langle x, \frac{\partial}{\partial y}\rangle$ on TS^3. Note that TS^3 is an invariant manifold of (5), thought of as a vector field on $T\mathbf{R}^4$, since the initial conditions $\langle x,x\rangle = 1$ and $\langle x,y\rangle = 0$ are preserved under its flow. The above
$\triangleright$ discussion is not at all Hamiltonian. What we want to do is to show that Y is a Hamiltonian vector field on the phase space (TS^3, Ω_4). Here Ω_4 is a suitable symplectic form.

(1.1) **Proof**: To do this, we use modified Dirac brackets, see chapter VI §4. On the open subset $M = T(\mathbf{R}^4 \setminus \{0\})$ of $T\mathbf{R}^4$ consider the constraint functions

$$c_1 : M \rightarrow \mathbf{R} : (x,y) \mapsto \tfrac{1}{2}(\langle x,x\rangle - 1) \quad \text{and} \quad c_2 : M \rightarrow \mathbf{R} : (x,y) \mapsto \langle x,y\rangle.$$

Let $\{\, , \,\}$ be the *standard Poisson bracket* on $C^\infty(T\mathbf{R}^4)$, the space of smooth functions on the symplectic manifold $(T\mathbf{R}^4, \omega_4)$, see chapter VI §4. Since the matrix $(\{c_i, c_j\})$, which is equal to $\begin{pmatrix} 0 & \langle x,x\rangle \\ -\langle x,x\rangle & 0 \end{pmatrix}$, is invertible on M with inverse $(C_{ij}) = \frac{1}{\langle x,x\rangle}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and 0 is a regular value of the constraint map $\mathscr{C} : M \rightarrow \mathbf{R}^2 : m \mapsto \big(c_1(m), c_2(m)\big)$, the constraint manifold $TS^3 = \mathscr{C}^{-1}(0)$ is a *cosymplectic* submanifold of $(M, \omega_4|M)$. In other words, $\Omega_4 = \omega_4|TS^3$ is a symplectic form on TS^3. For $F \in C^\infty(M)$ let

$$F^* = F - \sum_{i,j=1}^{2} (\{F, c_i\} + F_i) C_{ij} c_j,$$

where the F_i lies in the ideal of $(C^\infty(M), \cdot)$ generated by c_1 and c_2. Define a Poisson bracket $\{\, , \,\}_{TS^3}$ on $C^\infty(TS^3)$ by

$$\{F|TS^3, G|TS^3\}_{TS^3} = \{F^*, G^*\}|TS^3.$$

Note that the Hamiltonian vector field $X_{F|TS^3}$ of the Hamiltonian F constrained to TS^3 is the Hamiltonian vector field X_{F^*} restricted to TS^3. Applying these remarks to the unconstrained Hamiltonian $\mathscr{H}$ (1) on M gives

$$\begin{aligned} \mathscr{H}^* &= \mathscr{H} - \sum_{i,j} (\{\mathscr{H}, c_i\} + \mathscr{H}_i) C_{ij} c_j \\ &= \tfrac{1}{2}\langle y,y\rangle + \langle x,x\rangle^{-1} \big\langle \big(\langle x,y\rangle - \mathscr{H}_1, \langle y,y\rangle - \mathscr{H}_2\big), \big(-\langle x,y\rangle, \tfrac{1}{2}(\langle x,x\rangle - 1)\big)\big\rangle \\ &= \tfrac{1}{2}\big(\langle x,x\rangle\langle y,y\rangle - \langle x,y\rangle^2\big), \end{aligned}$$

where we have chosen $\mathscr{H}_1 = \langle x,y\rangle(1 - \frac{1}{2}\langle x,x\rangle)$ and $\mathscr{H}_2 = -\langle y,y\rangle(\langle x,x\rangle - 1)$.

From Hamilton's equations on $(T\mathbf{R}^4, \omega_4)$ it follows that the integral curves of $X_{\mathscr{H}^*}$ satisfy

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x \\ y \end{pmatrix} = A(x,y)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -\langle x,y\rangle & \langle x,x\rangle \\ -\langle y,y\rangle & \langle x,y\rangle \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}. \tag{6}$$

Using (6) and the definition of TS^3, it is easy to see that the integral curves of $X_{\mathscr{H}^*}|TS^3$ satisfy (5). Because $X_{\mathscr{H}|TS^3} = X_{\mathscr{H}^*}|TS^3$, the *geodesic vector field* on TS^3 is the Hamiltonian vector field X_H on (TS^3, Ω_4) corresponding to the Hamiltonian function

$$H = \mathscr{H}^*|TS^3 : TS^3 \to \mathbf{R} : (x,y) \mapsto \tfrac{1}{2}\langle y,y\rangle. \tag{7}$$

Note that H is the free particle Hamiltonian on $T\mathbf{R}^4$ restricted to TS^3. Thus the integral
$\triangleright$ curves of the geodesic vector field X_H on TS^3 satisfy (5). To find the flow of the geodesic vector field X_H, we first look for *integrals* (= conserved quantities) of the vector field $X_{\mathscr{H}^*}$. From the construction of the Hamiltonian $\mathscr{H}^*$ on $T\mathbf{R}^4$, we know that TS^3 is an invariant manifold of $X_{\mathscr{H}^*}$. Therefore the functions $f_1(x,y) = \frac{1}{2}\langle x,x\rangle$ and $f_2(x,y) = \langle x,y\rangle$ are integrals of $X_{\mathscr{H}^*}$. A calculation shows that $f_3(x,y) = \frac{1}{2}\langle y,y\rangle$ is also an integral of $X_{\mathscr{H}^*}$. The integrals $\{f_1, f_2, f_3\}$ span a Lie subalgebra of $(C^\infty(T\mathbf{R}^4), \{\,,\,\})$, which is isomorphic to $\mathrm{sl}(2,\mathbf{R})$ since $\{f_1, f_2\} = 2f_1$, $\{f_1, f_3\} = f_2$, and $\{f_3, f_2\} = -2f_3$. Because the functions f_i are constant along the integral curves of $X_{\mathscr{H}^*}$, so is the matrix $A(x,y)$ (6). Since $A^2(x,y) = -2\mathscr{H}^*(x,y)I_2$ and $\mathscr{H}^*(x,y) \geq 0$, the flow of $X_{\mathscr{H}^*}$ is

$$\varphi_t^{\mathscr{H}^*}(x,y) = \exp tA(x,y)\begin{pmatrix} x \\ y \end{pmatrix} = \big(\cos(t\sqrt{2\mathscr{H}^*})I_2 + \big(\sin(t\sqrt{2\mathscr{H}^*})/\sqrt{2\mathscr{H}^*}\big)A(x,y)\big)\begin{pmatrix} x \\ y \end{pmatrix}.$$

Restricting $\varphi_t^{\mathscr{H}^*}$ to the invariant manifold TS^3 gives

$$\varphi_t^H(x,y) = \begin{pmatrix} \cos(t\sqrt{2H}) & \sin(t\sqrt{2H})/\sqrt{2H} \\ -\sqrt{2H}\sin(t\sqrt{2H}) & \cos(t\sqrt{2H}) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}, \tag{8}$$

which is the *flow* of the geodesic vector field X_H on TS^3. □

Clearly, all of the integral curves of X_H on the level set $H^{-1}(h)$ with $h > 0$ are periodic
$\triangleright$ of *period* $2\pi/\sqrt{2h}$. In fact, when $y \neq 0$, the image of the integral curve $t \mapsto \varphi_t^H(x,y)$ under the bundle projection map $TS^3 \to S^3 : (x,y) \mapsto x$ is the *geodesic*

$$\gamma_{(x,y)} : \mathbf{R} \to S^3 : t \mapsto x\big(\cos(t\sqrt{2H})\big) + y\big((\sin(t\sqrt{2H})/\sqrt{2H}\big). \tag{9}$$

(1.2) **Proof**: To see that $\gamma_{(x,y)}$ is a geodesic on S^3 it suffices to show that

1. $\gamma_{(x,y)}$ is parametrized up to an affine transformation by arc length.
2. The *acceleration* $\ddot{\gamma}_{(x,y)}$ has no tangential component.

From the equations of motion for geodesics it follows that item 2 holds. Item 1 holds because γ is parametrized. Another argument to prove item 1 goes as follows. Differentiating (9) gives

$$\langle\dot{\gamma}_{(x,y)}, \dot{\gamma}_{(x,y)}\rangle = 2H\sin^2(t\sqrt{2H})\langle x,x\rangle + \cos^2(t\sqrt{2H})\langle y,y\rangle = \langle y,y\rangle = 2H(x,y),$$

which is a constant of motion. This constant is nonzero, since $y \neq 0$. □

The explicit formula (8) for the flow of the geodesic vector field gives *no* qualitative information about how the integral curves are organized into invariant manifolds. To understand the invariant manifolds, it is useful to explain the role of the obvious symmetry of the problem, namely, the group $\mathrm{SO}(4)$ of rigid motions of the 3-sphere. This will be done in the next section.

2 The $\mathrm{SO}(4)$-momentum mapping

In this section we construct the momentum mapping associated to the $\mathrm{SO}(4)$ symmetry of the geodesic vector field on (TS^3, Ω_4) and study its geometric properties.

Recall that $\mathrm{SO}(4)$ is the Lie group of *orthogonal* linear mappings of $(\mathbf{R}^4, \langle\,,\,\rangle)$ into itself with determinant 1. Consider the action of $\mathrm{SO}(4)$ on $\mathbf{R}^4$ given by $\varphi : \mathrm{SO}(4) \times \mathbf{R}^4 \to \mathbf{R}^4 : (A, x) \mapsto Ax$. This action lifts to an action of $\mathrm{SO}(4)$ on $(T\mathbf{R}^4, \omega_4)$ defined by

$$\Phi : \mathrm{SO}(4) \times T\mathbf{R}^4 \to T\mathbf{R}^4 : \big(A, (x,y)\big) \mapsto (Ax, Ay).$$

▷ Φ preserves the 1-form $\theta = \langle y, \mathrm{d}x \rangle$ on $T\mathbf{R}^4$.

(2.1) **Proof**: We compute

$$\Phi_A^* \theta \,=\, \langle Ay, \mathrm{d}Ax \rangle \,=\, \langle Ay, A\,\mathrm{d}x \rangle \,=\, \langle A^t Ay, \mathrm{d}x \rangle \,=\, \langle y, \mathrm{d}x \rangle \,=\, \theta.$$

The second to last equality follows because $A \in \mathrm{SO}(4)$. □

Thus the action Φ is symplectic, for

$$\Phi_A^* \omega_4 = -\Phi_A^*(\mathrm{d}\theta) \,=\, -\,\mathrm{d}(\Phi_A^*\theta) \,=\, -\,\mathrm{d}\theta \,=\, \omega_4.$$

▷ To show that Φ is a Hamiltonian action, we must verify that for every $a \in \mathrm{so}(4)$, the Lie algebra of $\mathrm{SO}(4)$, the vector field

$$X^a(x,y) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \Phi_{\exp ta}(x,y) \,=\, \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \big((\exp ta)x, (\exp ta)y\big) = (ax, ay) \,=\, (X_a(x), ay),$$

which is the infinitesimal generator of Φ in the direction a, is a Hamiltonian vector field on $(T\mathbf{R}^4, \omega_4)$.

(2.2) **Proof**: From the momentum lemma, see chapter VII ((5.7)), it follows that $X^a = X_{J^a}$ where

$$J^a : T\mathbf{R}^4 \to \mathbf{R} : (x,y) \mapsto \theta(x,y)X_a(x) = \langle ax, y \rangle. \tag{10}$$

Thus the action Φ has momentum mapping $J : T\mathbf{R}^4 \to \mathrm{so}(4)^*$ defined by $J(x,y)a = J^a(x,y)$. Choose a basis $\{e_{ij}\}_{1 \le i < j \le 4}$ of $\mathrm{so}(4)$ where the $(k,\ell)^{th}$ entry of the 4×4 matrix e_{ij} is 1 if $(k,\ell) = (i,j)$, -1 if $(k,\ell) = (j,i)$, and 0 otherwise. Then

$$J^{e_{ij}}(x,y) = \langle e_{ij}x, y \rangle \,=\, x_i y_j - x_j y_i = S_{ij}(x,y). \tag{11}$$

▷ The mapping J is coadjoint equivariant.

(2.3) **Proof**: We compute

$$\begin{aligned} J(\Phi_A(x,y))a = J(Ax,Ay)a &= \langle aAx, Ay\rangle = \langle A^{-1}aAx, y\rangle \\ = J(x,y)(Ad_{A^{-1}}a) &= Ad^t_{A^{-1}}(J(x,y))a. \end{aligned}$$ □

Since Φ_A maps TS^3 into itself for every $A \in \mathrm{SO}(4)$, Φ restricts to an action $\widehat{\Phi}$ on TS^3 given by $\widehat{\Phi} : \mathrm{SO}(4) \times TS^3 \to TS^3 : \big(A,(x,y)\big) \mapsto (Ax, Ay)$. For every $a \in \mathrm{so}(4)$ the infinitesimal generator X^a of the SO(4)-action Φ leaves TS^3 invariant because

$$\begin{aligned} \frac{\mathrm{d}\langle x,x\rangle}{\mathrm{d}t} &= 2\langle x,\dot{x}\rangle = 2\langle x, ax\rangle = 0, \\ \frac{\mathrm{d}\langle x,y\rangle}{\mathrm{d}t} &= \langle \dot{x},y\rangle + \langle x,\dot{y}\rangle = \langle x, ay\rangle + \langle ax, y\rangle = 0, \\ \frac{\mathrm{d}\langle y,y\rangle}{\mathrm{d}t} &= 2\langle y,\dot{y}\rangle = 2\langle y, ay\rangle = 0, \end{aligned}$$

since $a^t = -a$. Therefore $X^a|TS^3$ is a vector field on TS^3. The action $\widehat{\Phi}$ preserves the symplectic form Ω_4 on TS^3 because

$$\widehat{\Phi}^*_A \Omega_4 = \widehat{\Phi}^*_A(\omega_4|TS^3) = (\Phi^*_A\omega_4)|TS^3 = \omega_4|TS^3 = \Omega_4.$$

Claim: The action $\widehat{\Phi}$ on (TS^3, Ω_4) is Hamiltonian with *momentum mapping*

$$\mathscr{J} = J|TS^3 : TS^3 \subseteq T\mathbf{R}^4 \to \mathrm{so}(4)^*. \tag{12}$$

(2.4) **Proof**: Because X^a leaves TS^3 invariant and $\Omega_4 = \omega_4|TS^3$, it follows that $X^a|TS^3 = X_{J^a|TS^3}$. Thus $X^a|TS^3$ is the infinitesimal generator of $\widehat{\Phi}$ on TS^3 in the direction a. □

So far the SO(4) symmetry is not related to the geodesic flow on TS^3. But note, the Hamiltonian $\mathscr{H}^*$ is preserved by the action Φ, because for every $A \in \mathrm{SO}(4)$

$$\mathscr{H}^*(\Phi_A(x,y)) = \tfrac{1}{2}\big(\langle Ax,Ax\rangle\langle Ay,Ay\rangle - \langle Ax,Ay\rangle^2\big) = \mathscr{H}^*(x,y).$$

▷ Thus the function J^a (10) is an integral of the vector field $X_{\mathscr{H}^*}$ for every $a \in \mathrm{so}(4)$.

(2.5) **Proof**: For every $a \in \mathrm{so}(4)$ we have $\Phi^*_{\exp ta}\mathscr{H}^* = \mathscr{H}^*$. Therefore

$$0 = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \Phi^*_{\exp ta}\mathscr{H}^* = L_{X^a}\mathscr{H}^* = L_{X_{J^a}}\mathscr{H}^* = -L_{X_{\mathscr{H}^*}}J^a. \tag{13}$$

From the fact that Φ preserves both the Hamiltonian $\mathscr{H}^*$ and the manifold TS^3, it follows that $\widehat{\Phi}$ preserves the geodesic Hamiltonian $H = \mathscr{H}^*|TS^3$. Therefore for every $a \in \mathrm{so}(4)$ the function $J^a|TS^3$ is an integral of the geodesic vector field X_H. □

In order to study the geometry of the momentum mapping $\mathscr{J}$ (12), we transform it into an easier to understand mapping, see (16) below. We begin by recalling that the 4×4 skew symmetric matrices $\{e_{ij}\}_{1 \le i < j \le 4}$ form a basis for the Lie algebra $(\mathrm{so}(4), [\,,\,])$. The covectors $\{e^*_{ij}\}_{1\le i<j\le 4}$, where $e^*_{ij} = e^t_{ij}$, form the standard dual basis for $\mathrm{so}(4)^*$. The Lie bracket $\{\, ,\}_{\mathrm{so}(4)^*}$ on $\mathrm{so}(4)^*$ is defined by $\{e^*_{ij}, e^*_{\ell k}\}_{\mathrm{so}(4)^*} = \sum_{m,n} c^{mn}_{ij,\ell k}\, e^*_{mn}$, where $[e_{ij}, e_{\ell k}] = \sum_{m,n} c^{mn}_{ij,\ell k}\, e_{mn}$.

For $u,v,w \in \mathbf{R}^4$ consider the map $\vartheta : \bigwedge^2 \mathbf{R}^4 \to \mathrm{so}(4) : u\wedge w \mapsto \ell_{u,w}$, where $\ell_{u,w} : \mathbf{R}^4 \to \mathbf{R}^4 : v \mapsto \langle v,w\rangle\, u - \langle v,u\rangle\, w$ is a linear mapping, which is skew symmetric, that is, $\langle \ell_{u,w}x, y\rangle = -\langle x, \ell_{u,w}y\rangle$ for every $x,y\in\mathbf{R}^4$. Using the basis $\{e_i\wedge e_j\}_{1\le i<j\le 4}$ of $\bigwedge^2\mathbf{R}^4$, we see that $\vartheta(e_i\wedge e_j) = e_{ij}$. Thus ϑ is an isomorphism. Consequently the mapping $\vartheta^t : \mathrm{so}(4)^* \to (\bigwedge^2\mathbf{R}^4)^* = \bigwedge^2(\mathbf{R}^4)^* : e^*_{ij} \mapsto e^*_i \wedge e^*_j$. Since

$$\vartheta^t(e^*_{ij})(x,y) = (e^*_i\wedge e^*_j)(x,y) = e^*_i(x)\, e^*_j(y) - e^*_i(y) e^*_j(x) = x_i y_j - x_j y_i = S_{ij}(x,y), \quad (14)$$

for every $x,y\in\mathbf{R}^4$, it follows that $(\bigwedge^2\mathbf{R}^4)^*$ is the space $\mathscr{S}$ of homogeneous quadratic functions on $T\mathbf{R}^4$, which is spanned by $\{S_{ij}\}_{1\le i<j\le 4}$. As a subspace of $C^\infty(T\mathbf{R}^4)$, $\mathscr{S}$ has a Poisson bracket $\{\, ,\}_{\mathscr{S}}$, which is induced from the standard Poisson bracket $\{\, ,\}$ on the space of smooth functions on $(T\mathbf{R}^4, \omega_4)$. In other words, for every $(x,y)\in T\mathbf{R}^4$

$$\{S_{ij}, S_{\ell k}\}_{\mathscr{S}}(x,y) = \omega_4\big(X_{S_{ij}}(x,y), X_{S_{\ell k}}(x,y)\big), \quad (15)$$

where $X_{S_{rs}}$ is the Hamiltonian vector field on $(T\mathbf{R}^4, \omega_4)$ corresponding to the Hamiltonian function S_{rs}. A calculation using (15) gives table 2.1.

$\{A,B\}_{\mathscr{S}}$	S_{12}	S_{13}	S_{14}	S_{23}	S_{24}	S_{34}	B
S_{12}	0	S_{23}	S_{24}	$-S_{13}$	$-S_{14}$	0	
S_{13}	$-S_{23}$	0	S_{34}	S_{12}	0	$-S_{14}$	
S_{14}	$-S_{24}$	$-S_{34}$	0	0	S_{12}	S_{13}	
S_{23}	S_{13}	$-S_{12}$	0	0	S_{34}	$-S_{24}$	
S_{24}	S_{14}	0	$-S_{12}$	$-S_{34}$	0	S_{23}	
S_{34}	0	S_{14}	$-S_{13}$	S_{24}	$-S_{23}$	0	
A							

Table 2.1. The Poisson bracket on $\mathscr{S}$.

Because the functions $f_1 = \frac{1}{2}\langle x,x\rangle$, $f_2 = \langle x,y\rangle$, and $f_3 = \frac{1}{2}\langle y,y\rangle$ are invariant under the $\mathrm{SO}(4)$ action Φ on $T\mathbf{R}^4$, the function J^a (10) is an integral of X_{f_i} for every $a\in\mathrm{so}(4)$. In other words, $\{f_i, J^a\} = 0$ for $i = 1,2,3$ and $a\in\mathrm{so}(4)$. Thus the Lie algebra $(\mathrm{sl}(2,\mathbf{R}), \{\, ,\})$ spanned by $\{f_i\}_{1\le i\le 3}$ and the Lie algebra $(\mathscr{S}, \{\, ,\}_{\mathscr{S}})$ are *dual pairs* in the Lie algebra of homogeneous quadratic functions on $T\mathbf{R}^4$ with Poisson bracket $\{\, ,\}$. In other words, they have the following properties:

1. They centralize $\mathscr{H}^*$, that is, $\{\mathscr{H}^*, f_i\} = 0 = \{\mathscr{H}^*, S_{jk}\}$.
2. They centralize each other, that is, $\{f_i, S_{jk}\} = 0$.

$\triangleright$ We now show that the Lie algebras $(\mathscr{S}, \{\, ,\}_{\mathscr{S}})$ and $(\mathrm{so}(4)^*, \{\, ,\}_{\mathrm{so}(4)^*})$ are isomorphic.

(2.6) **Proof**: From the definition of S_{ij} (11) we obtain $\mathrm{d}S_{ij}(x,y) = -\langle e_{ij}(y), \mathrm{d}x\rangle + \langle e_{ij}(x), \mathrm{d}y\rangle$.

Since $\omega_4^\flat(\mathrm{d}x) = -\frac{\partial}{\partial y}$ and $\omega_4^\flat(\mathrm{d}y) = \frac{\partial}{\partial x}$, we find that

$$X_{S_{ij}}(x,y) = \omega_4^\flat(\mathrm{d}S_{ij})(x,y) = \langle e_{ij}(x), \frac{\partial}{\partial x}\rangle + \langle e_{ij}(y), \frac{\partial}{\partial y}\rangle.$$

Therefore

$$\begin{aligned}\{S_{ij}, S_{\ell k}\}_{\mathscr{S}}(x,y) = (X_{S_{\ell k}} \,\lrcorner\, \mathrm{d}S_{ij})(x,y) \;&=\; -\langle e_{\ell k}(x), e_{ij}(y)\rangle + \langle e_{\ell k}(y), e_{ij}(x)\rangle \\ = \langle (e_{ij}e_{\ell k} - e_{\ell k}e_{ij})x, y\rangle \;&=\; \langle [e_{ij}, e_{\ell k}]x, y\rangle = \vartheta^t([e_{ij}, e_{\ell k}]^*)(x,y) \\ = \vartheta^t(\{e_{ij}^*, e_{\ell k}^*\}_{\mathrm{so}(4)^*})(x,y).&\end{aligned}$$

The last equality above follows by definition of the Poisson bracket $\{\,,\,\}_{\mathrm{so}(4)^*}$. Hence ϑ^t is a Lie algebra isomorphism. $\square$

On $\bigwedge^2\mathbf{R}^4$ define an inner product $B : \bigwedge^2\mathbf{R}^4 \times \bigwedge^2\mathbf{R}^4 \to \mathbf{R} : (u\wedge v, x\wedge y) \mapsto \det\begin{pmatrix}\langle u,x\rangle & \langle u,y\rangle \\ \langle v,x\rangle & \langle v,y\rangle\end{pmatrix}$. Since $\{e_i \wedge e_j\}_{1\le i<j\le 4}$ is an orthonormal basis of $(\bigwedge^2\mathbf{R}^4, B)$, we may identify $\bigwedge^2\mathbf{R}^4$ with

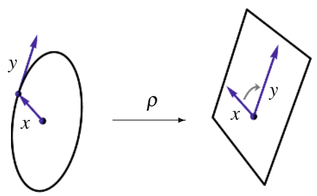

Figure 2.1. The mapping ρ.

$(\bigwedge^2\mathbf{R}^4)^*$. Instead of studying the momentum mapping $\mathscr{J}$ (12) we study the mapping

$$\rho : TS^3 \subseteq T\mathbf{R}^4 \to \bigwedge^2\mathbf{R}^4 : (x,y) \mapsto x\wedge y = \sum_{1\le i<j\le 4} S_{ij}(x,y)\, e_i \wedge e_j, \tag{16}$$

which is nothing but $B^\flat \circ \vartheta^t \circ \mathscr{J}$. The S_{ij} are the *Plücker coordinates* of the oriented 2-plane spanned by $\{x,y\}$ corresponding to the 2-vector $x\wedge y$. In other words, S_{ij} is the 2×2 minor formed from the i^{th} and j^{th} columns of the 2×4 matrix with rows x and y, that is, $S_{ij} = \det\begin{pmatrix} x_i & x_j \\ y_i & y_j\end{pmatrix}$. Because

$$0 = (x\wedge y)\wedge(x\wedge y) \;=\; (S_{12}S_{34} - S_{13}S_{24} + S_{14}S_{23})\, e_1\wedge e_2\wedge e_3\wedge e_4,$$

the Plücker coordinates of $x\wedge y$ satisfy *Plücker's equation*

$$S_{12}S_{34} - S_{13}S_{24} + S_{14}S_{23} = 0. \tag{17}$$

Let C be the set of all nonzero 2-vectors on $\mathbf{R}^4$ whose Plücker coordinates satisfy (17). By
▷ definition $\rho(TS^3) \subseteq C$. Actually, C is the image of ρ.

(2.7) **Proof**: Suppose that $\theta \in C$. Then θ is decomposable, that is, there are vectors $u,v \in \mathbf{R}^4$ such that $\theta = u\wedge v$. To see this, let (S_{ij}) be the Plücker coordinates of θ. Since $\theta \neq 0$

not every S_{ij} is zero. Suppose that S_{12} is nonzero. Let $u = (1, 0, -S_{23}/S_{12}, -S_{24}/S_{12})$ and $v = (0, S_{12}, S_{13}, S_{14})$. Using Plücker's equation (17) it is easy to check that the Plücker coordinates of the 2-vector $u \wedge v$ are (S_{ij}). Therefore $\theta = u \wedge v$. A similar argument, which we omit, works in the other cases. Let $\{x, y\}$ be an orthonormal basis of the 2-plane spanned by $\{u, v\}$. Then $u \wedge v = \lambda\, x \wedge y$ for some nonzero λ. Therefore $\rho(x, \lambda y) = \theta$. □

For $h > 0$ let $H^{-1}(h) = \{(x,y) \in TS^3 \subseteq T\mathbf{R}^4 \,\big|\, \frac{1}{2}\langle y, y\rangle = h\}$ be the h-level set of the geodesic Hamiltonian H (7). Consider the mapping

$$\rho_h : H^{-1}(h) \subseteq TS^3 \to C \subseteq \bigwedge^2 \mathbf{R}^4 : (x,y) \mapsto x \wedge y, \tag{18}$$

which is the restriction of ρ (16) to $H^{-1}(h)$. From the identity

$$\sum_{1 \le i < j \le 4} (x_i y_j - x_j y_i)^2 = \langle x, x\rangle \langle y, y\rangle - \langle x, y\rangle^2 \tag{19}$$

we see that the image of ρ_h is contained in the submanifold C_h of C defined by $\sum_{1 \le i<j\le 4} S_{ij}^2$
▷ $= 2h$. C_h is diffeomorphic to $S^2_{\sqrt{h/2}} \times S^2_{\sqrt{h/2}}$.

(2.8) **Proof**: Adding and subtracting one half times (17) from one quarter times the defining equation of C_h, and using the variables

$$\begin{aligned} \xi_1 &= \tfrac{1}{2}(S_{12} + S_{34}) & \eta_1 &= \tfrac{1}{2}(S_{12} - S_{34}) \\ \xi_2 &= \tfrac{1}{2}(S_{13} - S_{24}) & \eta_2 &= \tfrac{1}{2}(S_{13} + S_{24}) \\ \xi_3 &= \tfrac{1}{2}(S_{14} + S_{23}) & \eta_3 &= \tfrac{1}{2}(S_{14} - S_{23}), \end{aligned} \tag{20}$$

we obtain $\xi_1^2 + \xi_2^2 + \xi_3^2 = h/2$ and $\eta_1^2 + \eta_2^2 + \eta_3^2 = h/2$. □

We now investigate the geometry of the map ρ_h.

Claim: For every $h > 0$, the map $\rho_h : H^{-1}(h) \to C_h$ (18) is a surjective submersion each of whose fibers is a single oriented orbit of the geodesic vector field X_H of energy h.

(2.9) **Proof**: To show that ρ_h is surjective, suppose that $S = (S_{ij}) \in C_h$. Since C_h is contained in $C = \rho(TS^3)$, there is an $(x,y) \in TS^3$ such that $\rho(x,y) = S$. But $2h = \sum_{1\le i<j\le 4} S_{ij}^2$ since $S \in C_h$. From (19), the definition of S_{ij} (14), and the fact that $(x,y) \in TS^3$, we find that $\frac{1}{2}\langle y, y\rangle = h$. Hence $(x,y) \in H^{-1}(h)$.

To show that ρ_h is a submersion, we must verify that the rank of $T_{(x,y)}\rho_h$ is 4 for every $(x,y) \in H^{-1}(h)$, because C_h is 4-dimensional. Towards this goal, let $V_{(x,y)}$ be the space spanned by the Hamiltonian vector fields $X_{S_{ij}}$, $1 \le i < j \le 4$, on $(T\mathbf{R}^4, \omega_4)$ corresponding to the Hamiltonian function $S_{ij} : T\mathbf{R}^4 \to \mathbf{R} : (x,y) \mapsto x_i y_j - x_j y_i$. Since $S_{ij}|TS^3$ is an integral of of the geodesic vector field X_H on TS^3, it follows that $V_{(x,y)} \subseteq \ker \mathrm{d}H(x,y) = T_{(x,y)}H^{-1}(h)$ for every $(x,y) \in H^{-1}(h)$. Now

$$\big(T_{(x,y)}\rho_h\big)|V_{(x,y)} = \big(\mathrm{d}S_{ij}(x,y) X_{S_{\ell k}}(x,y)\big) = (\{S_{ij}, S_{\ell k}\}_{\mathscr{S}}) = \widetilde{P}. \tag{21}$$

Using (20) we see that 6×6 matrix $\widetilde{P}$ is conjugate to the matrix

$$P=\begin{pmatrix} (\{\xi_i,\xi_j\}_{\mathscr{S}}) & 0 \\ 0 & (\{\eta_i,\eta_j\}_{\mathscr{S}}) \end{pmatrix} = \begin{pmatrix} (\sum_k \varepsilon_{ijk}\xi_k) & 0 \\ 0 & (\sum_k \varepsilon_{ijk}\eta_k) \end{pmatrix}.$$

The last equality follows using table 2.1. But $S=(S_{ij})=\rho_h(x,y)\in C_h$. Because ξ and η lie in $S^2_{\sqrt{h/2}}$, each of the 3×3 skew symmetric matrices $(\sum_k \varepsilon_{ijk}\,\xi_k)$ and $(\sum_k \varepsilon_{ijk}\,\eta_k)$ is nonzero. Thus each of these matrices has rank 2. Therefore, the rank of $T_{(x,y)}\rho_h$ is 4 for every $(x,y)\in H^{-1}(h)$. Thus ρ_h is a submersion.

Given $S=(S_{ij})\in C_h$, the fiber $W=\rho_h^{-1}(S)$ is a union of orbits of the geodesic vector field X_H of energy h because $S_{ij}|TS^3$ are integrals of X_H. By definition of ρ_h (18), W is the set of all ordered pairs $\{x,y\}$ of orthogonal vectors in $\mathbf{R}^4$ such that $\langle x,x\rangle=1$, $\langle y,y\rangle=2h$ and the 2-plane Π spanned by $\{x,y\}$ has Plücker coordinates (S_{ij}). Since any two such bases of Π are related by a *counterclockwise* rotation in Π, we find that

$$W=\{(x\cos\theta-y\sin\theta, x\sin\theta+y\cos\theta)\in H^{-1}(h)\,\big|\,\theta\in[0,2\pi]\}.$$

Therefore W is a unique oriented orbit of X_H traced out by an integral curve of X_H. □

Corollary: C_h is the space of orbits of positive energy h of the geodesic vector field X_H on TS^3 with orbit mapping $\rho_h: H^{-1}(h)\to C_h$.

(2.10) **Proof**: The corollary follows immediately from the claim and the definition of orbit space, see chapter VII §2. □

The goal of the following discussion is to construct a symplectic form on C_h. We begin by defining a Poisson bracket $\{\,,\,\}$ on the space $C^\infty(\mathscr{S})$ of smooth functions on the Lie algebra $(\mathscr{S},\{\,,\,\}_{\mathscr{S}})$. For $f,g\in C^\infty(\mathscr{S})$ let

$$\{f,g\}=\sum_{\substack{1\le i<j\le 4\\ 1\le \ell<k\le 4}} \frac{\partial f}{\partial S_{ij}}\frac{\partial g}{\partial S_{\ell k}}\{S_{ij},S_{\ell k}\}_{\mathscr{S}}. \tag{22}$$

As is shown in example 1 of chapter VI §4, $(C^\infty(\mathscr{S}),\{\,,\,\}_{\mathscr{S}})$ is a Lie algebra. On $C^\infty(\mathscr{S})$ define a multiplication $\cdot$ by $(f\cdot g)(s)=f(s)g(s)$ for every $s\in\mathscr{S}$. Then $(C^\infty(\mathscr{S}),\cdot)$ is a commutative ring with unit. Using (22) it is straightforward to check that Leibniz' rule holds, namely $\{f,g\cdot h\}=\{f,g\}\cdot h+\{f,h\}\cdot g$, for every $f,g,h\in C^\infty(\mathscr{S})$. Therefore $\mathscr{A}=(C^\infty(\mathscr{S}),\{\,,\,\}_{\mathscr{S}},\cdot)$ is a Poisson algebra. The functions

$$C_1=\sum_{1\le i<j\le 4} S_{ij}^2-2h \quad\text{and}\quad C_2=S_{12}S_{34}-S_{13}S_{24}+S_{14}S_{23}$$

are *Casimirs* for $\mathscr{A}$. In other words, $\{C_1,f\}=\{C_2,f\}=0$ for every $f\in C^\infty(\mathscr{S})$. From (22) it is enough to show that $\{C_1,S_{ij}\}=\{C_2,S_{ij}\}=0$ for $1\le i<j\le 4$. This is a direct verification using table 2.1. Let $\mathscr{I}$ be the ideal in $(C^\infty(\mathscr{S}),\cdot)$ which is generated by C_1
▷ and C_2. Then $\mathscr{I}$ is a *Poisson ideal* in $\mathscr{A}$, that is, if $g\in\mathscr{I}$, then $\{f,g\}\in\mathscr{I}$ for every $f\in C^\infty(\mathscr{S})$.

(2.11) **Proof**: Since $f\in\mathscr{I}$ there are $f_1,f_2\in C^\infty(\mathscr{S})$ such that $f=f_1C_1+f_2C_2$. Now

$$\begin{aligned}\{f,g\} &= \{f_1,g\}\cdot C_1 + f_1\cdot\{C_1,g\} + \{f_2,g\}\cdot C_2 + f_2\cdot\{C_2,g\}, \text{ by Leibniz' rule}\\ &= \{f_1,g\}\cdot C_1 + \{f_2,g\}\cdot C_2 \in \mathscr{I},\end{aligned}$$

where the equality above follows because C_1 and C_2 are Casimirs. Therefore we can define a Poisson bracket $\{\,,\}_{C_h}$ on $C^\infty(\mathscr{S})/\mathscr{I}$ by $\{f+\mathscr{I}, g+\mathscr{I}\}_{C_h} = \{f,g\}$. In order to be able to identify the space $C^\infty(\mathscr{S})/\mathscr{I}$ with the space $C^\infty(C_h)$ of smooth functions on C_h, we need to know that $\mathscr{I}$ is the set of smooth functions vanishing identically on C_h. This is a consequence of the following general

Fact: Suppose that 0 is a regular value of the smooth map $F : \mathbf{R}^n \to \mathbf{R}^k : z \mapsto \big(F_1(z),\ldots, F_k(z)\big)$. Then $M = F^{-1}(0)$ is a smooth submanifold of $\mathbf{R}^n$ defined by $F_1(z) = \cdots = F_k(z) = 0$. If $G : \mathbf{R}^n \to \mathbf{R}$ is a smooth function, which vanishes identically on M, then there are smooth functions $g_i : \mathbf{R}^n \to \mathbf{R}$, $1 \le i \le k$, such that $G = \sum_{i=1}^k g_i F_i$.

(2.12) **Proof**: Locally the fact follows using Taylor's formula with integral remainder. The global result is obtained by piecing together the local results using a partition of unity. We leave the details to the reader. □

Consequently, we may define the quotient Poisson algebra $\mathscr{B} = \mathscr{A}/\mathscr{I} = (C^\infty(C_h), \{\,,\}_{C_h}, \cdot)$. Because $\{S_{ij}+\mathscr{I}, S_{\ell k}+\mathscr{I}\}_{C_h} = \{S_{ij}, S_{\ell k}\}_{\mathscr{S}}$, the matrix of Poisson brackets $(\{S_{ij}+\mathscr{I}, S_{\ell k}+\mathscr{I}\}_{C_h})$ has rank 4. Therefore C_h is a *cosymplectic manifold*. In other words, the Poisson bracket $\{\,,\}_{C_h}$ is *nondegenerate* and hence defines a symplectic form ω_h on
▷ C_h, see chapter VI §4. Moreover, ω_h satisfies $\rho_h^*\omega_h = \Omega_4|H^{-1}(h)$.

(2.13) **Proof**: For every $(x,y) \in H^{-1}(h)$ we know that $T_{(x,y)}H^{-1}(h)$ is spanned by the vectors $\{X_{S_{ij}}(x,y)\}_{1\le i<j\le 4}$. Since (TS^3, Ω_4) is a cosymplectic submanifold of $(T\mathbf{R}^4, \omega_4)$, we have

$$\begin{aligned}\Omega_4(x,y)\big(X_{S_{ij}}(x,y), X_{S_{\ell k}}(x,y)\big) &= \omega_4\big(X_{S_{ij}}(x,y), X_{S_{\ell k}}(x,y)\big) = \{S_{ij}, S_{\ell k}\}_{\mathscr{S}}(x,y)\\ &= \{S_{ij}, S_{\ell k}\}_{C_h}(\rho_h(x,y)) = \omega_h(\rho_h(x,y))\big(T_{(x,y)}\rho_h X_{S_{ij}}(x,y), T_{(x,y)}\rho_h X_{S_{\ell k}}(x,y)\big)\\ &= (\rho_h^*\omega_h)(x,y)\big(X_{S_{ij}}(x,y), X_{S_{\ell k}}(x,y)\big).\end{aligned}$$ □

We now prove the main result of this section, which describes the geometry of the mapping ρ (16). As a consequence, we know the geometry of the SO(4)-momentum mapping $\mathscr{J}$ (12) of the geodesic vector field X_H on (TS^3, Ω_4).

Claim: The mapping $\rho : TS^3 \subseteq T\mathbf{R}^4 \to C \subseteq \bigwedge^2\mathbf{R}^4 : (x,y) \mapsto x\wedge y$ is a surjective submersion, each of whose fibers is a unique oriented orbit of the geodesic vector field X_H on (TS^3, Ω_4).

(2.14) **Proof**: We have already shown that ρ is surjective ((2.7)). To show that each of its fibers is a unique oriented orbit of X_H we argue as follows. Suppose that $S = (S_{ij}) \in C$. Because S is nonzero, $\sum_{1\le i<j\le 4} S_{ij}^2 = 2h$ for some $h > 0$. Therefore $S \in C_h$. Since the fiber $\rho_h^{-1}(S)$ of ρ_h is a unique oriented orbit of X_H of energy h ((2.9)), so is the fiber $\rho^{-1}(S)$ of ρ because $\rho = \rho_h$ on $H^{-1}(h)$.

To show that ρ is a submersion, first note that by ((2.9)) the map $\rho_h : H^{-1}(h) \to C_h$ is a submersion. Note that $H^{-1}(h)$ and C_h are codimension 1 submanifolds of TS^3 and C, respectively. Since a normal direction to $H^{-1}(h)$ at $(x,y) \in TS^3$ and a normal direction to C_h at $\rho(x,y) \in C$ is spanned by $\operatorname{grad} H(x,y)$ and $\operatorname{grad} F(\rho(x,y))$ respectively, where $F(S_{ij}) = \sum_{1\le i<j\le 4} S_{ij}^2 - 2h = 0$ defines C_h as a submanifold of C, it suffices to show that $\langle T_{(x,y)}\rho \operatorname{grad} H(x,y), \operatorname{grad} F(\rho(x,y))\rangle$ is nonzero. We compute. Clearly $\operatorname{grad} H(x,y) = (0,y)$. Hence $T_{(x,y)}\rho\big(\operatorname{grad} H(x,y)\big) = x \wedge y = \big(S_{ij}(x,y)\big)$. But $\operatorname{grad} F(\rho(x,y)) = 2\big(S_{ij}(x,y)\big)$. Therefore

$$\big\langle T_{(x,y)}\rho\big(\operatorname{grad} H(x,y)\big), \operatorname{grad} F(\rho(x,y))\big\rangle = 2 \sum_{1\le i<j\le 4} S_{ij}^2(x,y) = 4h > 0. \qquad \square$$

This claim has some interesting consequences.

Corollary 1: The *space of orbits* of the geodesic vector field with positive energy is the manifold C. The orbit map is $\rho : TS^3 \to C$, see (16).

(2.15) **Proof**: This follows immediately from the claim and the definition of orbit space, see chapter VII §2. $\square$

Observe that every smooth integral of the geodesic vector field on TS^3 is a smooth function of the integrals S_{ij}. More precisely we prove

Corollary 2: Suppose that $G : TS^3 \subseteq T\mathbf{R}^4 \to \mathbf{R}$ is a smooth integral of the geodesic vector field X_H on (TS^3, Ω_4). Then there is a smooth function $\widehat{G} : C \subseteq \bigwedge^2 \mathbf{R}^4 \to \mathbf{R}$ such that $G = \rho^* \widehat{G}$.

(2.16) **Proof**: Since G is a smooth integral of X_H on TS^3, it is constant on every orbit of X_H on TS^3. Because each fiber of ρ is a unique orbit of X_H on TS^3, G descends to a smooth function $\widehat{G}$ on the orbit space C. But $\rho : TS^3 \to C$ (16) is the orbit map, so $G = \rho^*\widehat{G}$. $\square$

3 The Kepler problem

We investigate the bounded motion of a particle in $\mathbf{R}^3$ which is under the influence of a gravitational field of a second particle fixed at the origin. This is *Kepler's problem*.

3.1 The Kepler vector field

In this subsection we define the Kepler Hamiltonian system $(H, T_0\mathbf{R}^3, \omega_3)$. We then show that the Kepler Hamiltonian vector field X_H conserves energy H, angular momentum $\mathbf{J}$, and the eccentricity vector $\mathbf{e}$. On the set Σ_- of positions and momenta where the values of H are negative, the orbits of X_H are bounded, yet the flow of X_H is incomplete.

On the phase space $T_0\mathbf{R}^3 = (\mathbf{R}^3 \setminus \{0\}) \times \mathbf{R}^3$ with coordinates (q,p) and symplectic form $\omega_3 = \sum_{i=1}^3 \mathrm{d}q_i \wedge \mathrm{d}p_i$, consider the *Kepler Hamiltonian*

$$H : T_0\mathbf{R}^3 \to \mathbf{R} : (q,p) \mapsto \tfrac{1}{2}\langle p,p\rangle - \mu \|q\|^{-1}. \qquad (23)$$

Here $\langle \, , \, \rangle$ is the Euclidean inner product on $\mathbf{R}^3$ and $\|q\|$ is the length of the vector q. The integral curves of the Hamiltonian vector field X_H on $T_0\mathbf{R}^3$ satisfy the equations

$$\begin{aligned} \dot{q} &= p \\ \dot{p} &= -\mu \, \|q\|^{-3} q, \end{aligned} \tag{24}$$

which describe the motion of a particle of mass 1 about the origin under the influence of an inverse $|q|^2$ force — such as Newtonian gravity. We consider the case where the force is attractive, that is, $\mu > 0$. However, much of the following analysis can be carried out without change for $\mu < 0$.

The *Kepler vector field* X_H has some obvious integrals: the *total energy*

$$h = \tfrac{1}{2}\langle p, p\rangle - \mu \|q\|^{-1}, \tag{25}$$

which is nothing but the Hamiltonian H, and the *angular momentum*

$$\mathbf{J} = (J_1, J_2, J_3) = q \times p. \tag{26}$$

Here $\times$ is the *vector product* on $\mathbf{R}^3$.

(3.1) **Proof**: A direct way to see that $\mathbf{J}$ is an integral is to compute

$$\frac{\mathrm{d}\mathbf{J}}{\mathrm{d}t} = \frac{\mathrm{d}q}{\mathrm{d}t} \times p + q \times \frac{\mathrm{d}p}{\mathrm{d}t} = p \times p - \mu \|q\|^{-3} q \times q = 0,$$

where the second to last equality follows using (24).

A more sophisticated way to see this is to note that the SO(3)-action $\mathrm{SO}(3) \times \mathbf{R}^3 \rightarrow \mathbf{R}^3 : (O, q) \mapsto Oq$ lifts to a Hamiltonian action

$$\mathrm{SO}(3) \times T_0\mathbf{R}^3 \rightarrow T_0\mathbf{R}^3 : \big(O, (q, p)\big) \mapsto (Oq, Op).$$

This latter action has the momentum mapping

$$\widetilde{J} : T_0\mathbf{R}^3 \rightarrow \mathrm{so}(3)^* : (q, p) \mapsto \begin{pmatrix} 0 & J_3 & -J_2 \\ -J_3 & 0 & J_1 \\ J_2 & -J_1 & 0 \end{pmatrix},$$

defined by $\widetilde{J}(q, p)X = \langle p, X(q)\rangle$ where $X \in \mathrm{so}(3)$. Now use the map $k^\flat$ associated to the Killing metric $k : \mathrm{so}(3) \times \mathrm{so}(3) \rightarrow \mathbf{R} : (X, Y) \mapsto \frac{1}{2}\mathrm{tr}\, XY^t$ to identify $\mathrm{so}(3)^*$ with $\mathrm{so}(3)$. This identification boils down to taking transposes. Follow this by the map

$$i : \mathrm{so}(3) \rightarrow \mathbf{R}^3 : X = \begin{pmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{pmatrix} \mapsto \mathbf{x} = (x_1, x_2, x_3),$$

which identifies $\mathrm{so}(3)$ with $\mathbf{R}^3$, see chapter III §1. Then $\widetilde{J}$ becomes the usual angular momentum $\mathbf{J} : T_0\mathbf{R}^3 \rightarrow \mathbf{R}^3 : (q, p) \mapsto q \times p$. Since the SO(3) action on $(T_0\mathbf{R}^3, \omega_3)$ leaves the Kepler Hamiltonian H (23) invariant, every component of the angular momentum $\mathbf{J}$ is constant on the integral curves of X_H. □

▷ There is another integral of the Kepler vector field, called the *eccentricity vector*:

$$\mathbf{e} = (e_1, e_2, e_3) = -\|q\|^{-1} q + \mu^{-1}\, p \times (q \times p). \tag{27}$$

(3.2) **Proof**: To see this we calculate

$$\begin{aligned}\frac{\mathrm{d}\mathbf{e}}{\mathrm{d}t} &= -\frac{\mathrm{d}}{\mathrm{d}t}(\|q\|^{-1} q) + \mu^{-1}\,\frac{\mathrm{d}p}{\mathrm{d}t} \times \mathbf{J} = \|q\|^{-3}\,\langle \frac{\mathrm{d}q}{\mathrm{d}t}, q\rangle\, q - \|q\|^{-1}\frac{\mathrm{d}q}{\mathrm{d}t} + \mu^{-1}\,\frac{\mathrm{d}p}{\mathrm{d}t} \times \mathbf{J}\\ &= \|q\|^{-3}\big(\langle q, p\rangle\, q - \langle q, q\rangle\, p\big) - \|q\|^{-3}\, q \times \mathbf{J}, \quad \text{using (24)}\\ &= \|q\|^{-3}\big(q \times (q \times p) - q \times (q \times p)\big) = 0.\end{aligned}$$

□

We now prove some properties of the flow of the Kepler vector field X_H.

Claim: If the energy h is negative, then the image of every integral curve of the Kepler vector field under the bundle projection $\tau : T_0\mathbf{R}^3 \to \mathbf{R}^3 : (q, p) \mapsto q$ is bounded.

(3.3) **Proof**:
CASE 1. $\mathbf{J} = 0$. Since $\mathbf{e}$ is an integral of X_H and $\mathbf{J} = 0$, the direction $\mathbf{e} = -q\|q\|^{-1}$ of the motion is constant. Therefore the motion takes place on the line $q(t) = r(t)\,\mathbf{e}$. From conservation of energy we obtain $h + \mu r^{-1} = \frac{1}{2}\dot{r}^2 \ge 0$. Therefore $\|q(t)\| \le \mu(-h)^{-1}$.

CASE 2. $\mathbf{J} \neq 0$. Since $J^2 = \|q \times p\|^2 = \|q\|^2\|p\|^2 - \langle q, p\rangle^2$, we have

$$h = \tfrac{1}{2}\langle p, p\rangle - \mu\|q\|^{-1} = \tfrac{1}{2}\langle q, p\rangle^2\|q\|^{-2} + \tfrac{1}{2}J^2\|q\|^{-2} - \mu\|q\|^{-1} \ge \tfrac{1}{2}J^2\|q\|^{-2} - \mu\|q\|^{-1}.$$

Now the function $V_J(\|q\|) = \frac{1}{2}J^2\|q\|^{-2} - \mu\|q\|^{-1}$ has a unique nondegenerate minimum at $\|q\| = J^2/\mu$ corresponding to the critical value $-\mu^2/(2J^2)$. Since $\lim_{\|q\|\searrow 0} V_J(\|q\|) \nearrow \infty$ and $\lim_{\|q\|\nearrow\infty} V_J(\|q\|) \nearrow 0$, the function V_J is proper on the set where it has negative values. Therefore $V_J^{-1}([-\mu^2/(2J^2), h])$ is compact. Thus the length of $q(t)$ is bounded, when $h < 0$. □

Claim: The flow of the Kepler vector field X_H is *not complete*.

(3.4) **Proof**: Consider a bounded motion with $\mathbf{J} = 0$ and $h < 0$ which starts at $(r(0), \dot{r}(0)) = (\mu/(-h), 0)$. The time it takes to reach the origin is $T = \int_0^{\mu/(-h)} \frac{\mathrm{d}r}{\sqrt{2\mu r^{-1} + 2h}}$. This is obtained by separating variables in $\frac{1}{2}\dot{r}^2 = h + \mu r^{-1}$ and integrating. Performing the integral gives $T = \frac{\pi}{2}\,\mu(-2h)^{-3/2}$, which is finite. □

3.2 The so(4)-momentum map

Let Σ_- be the open subset of $T_0\mathbf{R}^3$ where the energy H is negative. In this subsection we show that on Σ_- the components of the angular momentum $\mathbf{J}$ and the *modified eccentricity vector* $\widetilde{\mathbf{e}} = -\nu\,\mathbf{e}$, where $\nu = \mu/\sqrt{-2H}$, form a Lie algebra under Poisson bracket which is isomorphic to so(4). This defines a representation of so(4) on the space of Hamiltonian vector fields on $(\Sigma_-, \widetilde{\omega}_3 = \omega_3|\Sigma_-)$ which has a momentum mapping $\widetilde{\mathscr{J}}$. In fact $\widetilde{\mathscr{J}}$ is a surjective submersion from Σ_- to

$$C = \{(\mathbf{J}, \widetilde{\mathbf{e}}) \in \mathbf{R}^6 \,\big|\, \langle \mathbf{J} + \widetilde{\mathbf{e}}, \mathbf{J} + \widetilde{\mathbf{e}}\rangle = \langle \mathbf{J} - \widetilde{\mathbf{e}}, \mathbf{J} - \widetilde{\mathbf{e}}\rangle > 0\} \tag{28}$$

each of whose nonempty fibers is a unique oriented bounded orbit of X_H.

▷ First we show that on $(\Sigma_-, \widetilde{\omega}_3)$ the components of the angular momentum $\mathbf{J}$ and the modified eccentricity vector $\widetilde{\mathbf{e}}$ satisfy the Poisson bracket relations

$$\{J_i, J_j\} = \sum_k \varepsilon_{ijk} J_k, \ \{J_i, \widetilde{e}_j\} = \sum_k \varepsilon_{ijk} \widetilde{e}_k, \ \text{and} \ \{\widetilde{e}_i, \widetilde{e}_j\} = \sum_k \varepsilon_{ijk} J_k. \tag{29}$$

(3.5) **Proof**: We verify only the third equality in (29). Let $\mathbf{A} = \mu\,\mathbf{e}$. Since $\{q_\ell, p_m\} = \delta_{\ell m}$, $\{q_i, q_j\} = 0$, and $\{p_i, p_j\} = 0$, we have

$$\{J_a, \|q\|\} = 0, \ \{q_a, J_b\} = \sum_c \varepsilon_{abc} q_c, \ \text{and} \ \{p_a, J_b\} = \sum_c \varepsilon_{abc} p_c.$$

Using bilinearity and the derivation property of Poisson bracket, expand

$$\{A_i, A_j\} = \Big\{ \sum_{j,k} \varepsilon_{ijk} p_j J_k - \mu \|q\|^{-1} q_i, \sum_{m,n} \varepsilon_{\ell mn} p_m J_n - \mu \|q\|^{-1} q_\ell \Big\}$$

to obtain $\{A_i, A_j\} = -2H \sum_k \varepsilon_{ijk} J_k$. Recall the identity $\sum_i \varepsilon_{ijk} \varepsilon_{i\ell m} = \delta_{mk}\delta_{\ell j} - \delta_{jm}\delta_{\ell k}$. □

▷ The bracket relations (29) define a Lie algebra which is isomorphic to so(4).

(3.6) **Proof**: For $i = 1,2,3$ define $\xi_i = \frac{1}{2}(J_i + \widetilde{e}_i)$ and $\eta_i = \frac{1}{2}(J_i - \widetilde{e}_i)$. In terms of ξ_i and η_i the bracket relations (29) become

$$\{\xi_i, \xi_j\} = \sum_k \varepsilon_{ijk} \xi_k, \ \{\eta_i, \eta_j\} = \sum_k \varepsilon_{ijk} \eta_k, \ \text{and} \ \{\xi_i, \eta_j\} = 0. \tag{30}$$

These relations define the Lie algebra so$(3) \times$ so(3), which is isomorphic to so(4). □

The mappings $J_i \mapsto \mathrm{ad}_{J_i} = -X_{J_i}$ and $\widetilde{e}_i \mapsto \mathrm{ad}_{\widetilde{e}_i} = -X_{\widetilde{e}_i}$ define a representation of so(4) on the space of Hamiltonian vector fields on $(\Sigma_-, \widetilde{\omega}_3)$. In other words, we have a Hamiltonian action of the *Lie algebra* so(4) on $(\Sigma_-, \widetilde{\omega}_3)$. Associated to this Lie algebra action is the mapping

$$\widetilde{\mathscr{J}} : \Sigma_- \to \mathbf{R}^6 : (q,p) \mapsto (\mathbf{J}, \widetilde{\mathbf{e}}) = \big(q \times p, \nu\big(\|q\|^{-1} q - \mu^{-1} p \times (q \times p)\big)\big). \tag{31}$$

Here we have chosen $\{\varepsilon_i\}_{1 \le i \le 6} = \{J_1, J_2, J_3, \widetilde{e}_1, \widetilde{e}_2, \widetilde{e}_2\}$ as a basis for so(4) with Lie bracket $\{\,,\}$. Let $\widetilde{\mathscr{J}}^{\varepsilon_i}$ be the i^{th} component of the mapping $\widetilde{\mathscr{J}}$. Then the bracket relations (29) may be written as $\{\widetilde{\mathscr{J}}^{\varepsilon_i}, \widetilde{\mathscr{J}}^{\varepsilon_j}\} = \widetilde{\mathscr{J}}^{\{\varepsilon_i, \varepsilon_j\}}$. Therefore we say that the map $\widetilde{\mathscr{J}}$ is the *momentum map* of the so(4)-action on $(\Sigma_-, \widetilde{\omega}_3)$.

We now investigate the geometric properties of the mapping $\widetilde{\mathscr{J}}$ (31). We begin by noting that the vectors $\mathbf{J}$ and $\widetilde{\mathbf{e}}$ satisfy

$$\begin{aligned} \langle \mathbf{J}, \widetilde{\mathbf{e}} \rangle &= 0 \\ \langle \mathbf{J}, \mathbf{J} \rangle + \langle \widetilde{\mathbf{e}}, \widetilde{\mathbf{e}} \rangle &= \nu^2 > 0. \end{aligned} \tag{32}$$

The verification of the first equation in (32) is a straightforward. For the second, see (37) below. These relations define a smooth 4-dimensional manifold C_ν, which is diffeomorphic to $S^2_\nu \times S^2_\nu$ because (32) is equivalent to

$$\langle \mathbf{J} + \widetilde{\mathbf{e}}, \mathbf{J} + \widetilde{\mathbf{e}} \rangle = \langle \mathbf{J} - \widetilde{\mathbf{e}}, \mathbf{J} - \widetilde{\mathbf{e}} \rangle = \nu^2 > 0. \tag{33}$$

Write $\nu = \mu/\sqrt{-2h}$ for some $h<0$ and consider the map

$$\widetilde{\mathscr{J}}_h = \widetilde{\mathscr{J}}|H^{-1}(h) : H^{-1}(h) \subseteq \Sigma_- \to C_\nu \subseteq \mathbf{R}^6. \tag{34}$$

Claim: $\widetilde{\mathscr{J}}_h$ is a surjective submersion.

(3.7) **Proof**: Let $(q,p) \in H^{-1}(h)$ and let $V_{(q,p)} = \mathrm{span}\,\{X_{J_j}(q,p), X_{\widetilde{e}_j}(q,p)\}_{1\le j\le 3}$. Since $\mathbf{J}$ and $\widetilde{\mathbf{e}}$ are integrals of X_H, it follows that $V_{(q,p)} \subseteq \ker \mathrm{d}H(q,p)$ which is $T_{(q,p)}H^{-1}(h)$. Therefore

$$D\widetilde{\mathscr{J}}_h(q,p)|V_{(q,p)} = \begin{pmatrix} \mathrm{d}J_j(q,p) \\ \mathrm{d}\widetilde{e}_j(q,p)\end{pmatrix}\Bigg| V_{(q,p)} = \begin{pmatrix} (\{J_i,J_j\}(q,p)) & (\{J_i,\widetilde{e}_j\}(q,p)) \\ (\{\widetilde{e}_i,J_j\}(q,p)) & (\{\widetilde{e}_i,\widetilde{e}_j\}(q,p))\end{pmatrix} = P.$$

On C_ν (32) the rank of P is 4, because P is conjugate to the matrix

$$\begin{pmatrix} (\{\xi_i,\xi_j\}) & 0 \\ 0 & (\{\eta_i,\eta_j\})\end{pmatrix} = \begin{pmatrix} 2\sum_k \varepsilon_{ijk}\,(J_k+\widetilde{e}_k) & 0 \\ 0 & 2\sum_k \varepsilon_{ijk}\,(J_k-\widetilde{e}_k)\end{pmatrix},$$

see (30) and (33). Therefore $\widetilde{\mathscr{J}}_h$ is a submersion.

To show that $\widetilde{\mathscr{J}}_h$ is surjective, let $(\mathbf{J},\widetilde{\mathbf{e}}) \in C_\nu$. Then $e = \|\mathbf{e}\| = \nu^{-1}\,\|\widetilde{\mathbf{e}}\| \in [0,1]$ because $\nu^2 = \langle \mathbf{J},\mathbf{J}\rangle + \langle \widetilde{\mathbf{e}},\widetilde{\mathbf{e}}\rangle \ge \nu^2\langle \mathbf{e},\mathbf{e}\rangle$. Choose

$$(q,p) = \begin{cases} \big(-\nu\mu^{-1}\,(1-e)e^{-1}\widetilde{\mathbf{e}}, -\mu\nu^{-3}\,(e(1-e))^{-1}\,\mathbf{J}\times\widetilde{\mathbf{e}}\big), \text{ when } e\in(0,1) \text{ and } \mathbf{J}\neq 0 \\ \big(-\mu^{-2}\nu^2\,p\times\mathbf{J},p\big), \text{ when } e=0 \text{ and } \mathbf{J}\neq 0. \text{ Here } \langle p,\mathbf{J}\rangle = 0,\ \|p\| = \mu\nu^{-1} \\ \big(-\nu\mu^{-1}\widetilde{\mathbf{e}}, -\nu\mu^{-2}\widetilde{\mathbf{e}}\big), \text{ when } \mathbf{J}=0. \text{ Here } e=1. \end{cases}$$

A straightforward calculation shows that $(q,p) \in H^{-1}(h)$ and $\widetilde{\mathscr{J}}_h(q,p) = (\mathbf{J},\widetilde{\mathbf{e}})$. □

Corollary: For every $c \in C_\nu$ the fiber $\widetilde{\mathscr{J}}_h^{-1}(c)$ is a union of bounded Keplerian orbits.

(3.8) **Proof**: From the fact that $\widetilde{\mathscr{J}}_h$ is a submersion, it follows that

$$\dim\ker D\widetilde{\mathscr{J}}_h(q,p) = \dim T_{(q,p)}H^{-1}(h) - \dim\mathrm{im}\, D\widetilde{\mathscr{J}}_h(q,p) = 5-4 = 1.$$

But $X_H(q,p) \in \ker D\widetilde{\mathscr{J}}_h(q,p)$. Hence for every $c \in C_\nu$

$$T_{(q,p)}\widetilde{\mathscr{J}}_h^{-1}(c) = \ker D\widetilde{\mathscr{J}}_h(q,p) = \mathrm{span}\{X_H(q,p)\}.$$

Therefore $\widetilde{\mathscr{J}}_h^{-1}(c)$ is a union of bounded Keplerian orbits. □

The following claim is a substantial sharpening of the above corollary.

Claim: For every $c \in C_\nu$ the fiber $\widetilde{\mathscr{J}}_h^{-1}(c)$ is

1. an oriented ellipse, when $c \notin C_\nu \cap \{\mathbf{J}=0\}$;
2. a line which is the union of two half open line segments

$$\Big\{\big(\sigma\nu^{-1}\widetilde{\mathbf{e}}, \pm\nu^{-1}\big(\sqrt{2h+2\mu\sigma^{-1}}\big)\widetilde{\mathbf{e}}\big) \in T_0\mathbf{R}^3 \,\Big|\, \sigma \in (0,\mu/(-h)]\Big\},$$

that join smoothly at $(\mu/(-\nu h)\widetilde{\mathbf{e}}, 0)$, when $c \in C_\nu \cap \{\mathbf{J}=0\}$.

(3.9) **Proof**:

CASE 1. $c = (\mathbf{J}, \widetilde{\mathbf{e}}) \in C_\nu \setminus (C_\nu \cap \{\mathbf{J} = 0\})$. Let $h = -\mu^2/2\nu^2$. We have to show that the data $h < 0$, $\mathbf{J} \neq 0$, and $\mathbf{e} = -\nu^{-1}\widetilde{\mathbf{e}}$ determine a unique oriented ellipse which is traced out by the projection $t \mapsto q(t)$ of an integral curve $t \mapsto (q(t), p(t))$ of X_H. Because $\mathbf{J} \neq 0$ and $\langle q(t), \mathbf{J}\rangle = \langle p(t), \mathbf{J}\rangle = 0$, the curves $t \mapsto q(t)$ and $t \mapsto p(t)$ lie in a plane $\Pi \subseteq \mathbf{R}^3$ which is perpendicular to $\mathbf{J}$. Since $\langle \mathbf{J}, \mathbf{e}\rangle = \mathbf{0}$, the eccentricity vector $\mathbf{e}$ also lies in Π. Therefore we may write $\langle q, \mathbf{e}\rangle = \|q\| e \cos f$, where f is the *true anomaly*, namely, the angle $\angle AOP$. From the definition of the eccentricity vector $\mathbf{e}$ (27) it follows that $\langle q, \mathbf{e}\rangle = -\|q\| + \mu^{-1} J^2$. Therefore

$$\|q\| e \cos f = -\|q\| + \mu^{-1} J^2. \tag{35}$$

Suppose that $e = 0$. Then (35) becomes $\|q\| = \mu^{-1} J^2$, which defines a circle $\mathscr{C}$ in Π with center at the origin. Since $0 = \frac{\mathrm{d}\|q(t)\|^2}{\mathrm{d}t} = \langle q, \frac{\mathrm{d}q}{\mathrm{d}t}\rangle = \langle q, p\rangle$, the tangent vector $p(t)$ to $\mathscr{C}$ at

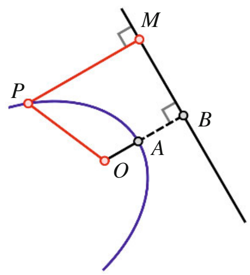

Figure 3.1. Ellipse in the plane Π.

$q(t)$ is perpendicular to $q(t)$. Because $\{q, p, p \times q\}$ is a positively oriented basis of $\mathbf{R}^3$, $\{q, p\}$ is a positively oriented basis for Π. Hence the circle traced out by $t \mapsto q(t)$ is positively oriented. Suppose that $e \neq 0$. Then equation (35) may be written as

$$e\big((\mu e)^{-1} J^2 - \|q\| \cos f\big) = \|q\|. \tag{36}$$

Equation (36) describes the locus of points P in the plane Π for which the ratio of the distance $\overline{OP}$ to the origin to the distance $\overline{PM}$ to the line MB, where $\overline{OB} = (\mu e)^{-1} J^2$, is a constant e, see figure 3.1. Thus the locus is a conic section. To see which conic it is, we calculate the size of e.

$$\begin{aligned} e^2 = \|\mathbf{e}\|^2 &= 1 - 2\mu^{-1}\|q\|^{-1}\,\|q \times p\|^2 + \mu^{-2}\,\|p \times (q \times p)\|^2, \quad \text{using (27)} \\ &= 1 - 2\mu^{-1}\|q\,\|^{-1} J^2 + \mu^{-2}(\|p\|^2 J^2 - \langle p, \mathbf{J}\rangle^2), \quad \text{using } \mathbf{J} = q \times p \\ &= 1 + 2\mu^{-2} J^2 h, \quad \text{since } \langle p, \mathbf{J}\rangle = 0 \text{ and } h = \tfrac{1}{2}\langle p, p\rangle - \mu\|q\|^{-1}. \end{aligned} \tag{37}$$

Since $h < 0$, it follows that $e \in [0, 1)$. Therefore the locus

$$\|q\| = J^2 \mu^{-1}(1 + e \cos f)^{-1} \tag{38}$$

is an *ellipse* in Π with *eccentricity* e and *major semiaxis* lying along $\mathbf{e}$, which is directed from the *center of attraction* O, that is also a *focus*, to the *periapse* A, of length $a =$

▷ $J^2\mu^{-1}(1-e^2)^{-1} = \mu/(-2h)$. When traced out by $t \mapsto q(t)$, this ellipse is oriented in the direction of increasing true anomaly f.

(3.10) **Proof**: From the fact that $\{q,p\}$ is a positively oriented basis of the plane Π, we obtain

$$\begin{aligned} J &= \|q\times p\|, \text{ which is the area of the positively oriented parallelogram spanned by } \{q,p\}. \\ &= \det\begin{pmatrix} \langle q, e^{-1}\mathbf{e}\rangle & \langle q, (Je)^{-1}\mathbf{J}\times\mathbf{e}\rangle \\ \langle p, e^{-1}\mathbf{e}\rangle & \langle p, (Je)^{-1}\mathbf{J}\times\mathbf{e}\rangle \end{pmatrix}, \\ &\qquad \text{since } \{e^{-1}\mathbf{e}, (Je)^{-1}\mathbf{J}\times\mathbf{e}\} \text{ is a positively oriented orthonormal basis of } \Pi \\ &= \|q\|^2\,\frac{\mathrm{d}f}{\mathrm{d}t}. \end{aligned} \tag{39}$$

Equation (39) follows by first differentiating $\langle q, e^{-1}\mathbf{e}\rangle = \|q\|\cos f$ and $\langle q,(Je)^{-1}\mathbf{J}\times\mathbf{e}\rangle = \|q\|\sin f$ along an integral curve of X_H and then using the fact that $p = \frac{\mathrm{d}q}{\mathrm{d}t}$ and $\dot{\mathbf{e}} = \dot{\mathbf{J}} = 0$ to obtain $\langle p, e^{-1}\mathbf{e}\rangle = \frac{\mathrm{d}\|q\|}{\mathrm{d}t}\cos f - \|q\|\sin f\,\frac{\mathrm{d}f}{\mathrm{d}t}$ and $\langle p,(Je)^{-1}\mathbf{J}\times\mathbf{e}\rangle = \frac{\mathrm{d}\|q\|}{\mathrm{d}t}\sin f + \|q\|\cos f\,\frac{\mathrm{d}f}{\mathrm{d}t}$. From (39) we see that $\frac{\mathrm{d}f}{\mathrm{d}t} > 0$. □

CASE 2. $c = (\mathbf{J},\widetilde{\mathbf{e}}) \in C_\nu \cap \{\mathbf{J} = 0\}$. Since $\mathbf{J} = 0$, the modified eccentricity vector $\widetilde{\mathbf{e}} = \nu\,\|q\|^{-1}q$. Because $\widetilde{\mathbf{e}}$ is constant along any integral curve $t \mapsto (q(t),p(t))$ of X_H and $h<0$, the image of $t\mapsto q(t)$ lies along $\widetilde{\mathbf{e}}$ and is the half open line segment $\{\sigma\nu^{-1}\widetilde{\mathbf{e}} \in \Pi \,\big|\, \sigma \in (0,\mu/(-h)]\}$. From $\mathbf{J}=0$ it follows that $p = \lambda\widetilde{\mathbf{e}}$ for some $\lambda\in\mathbf{R}$. In order that $(\sigma\nu^{-1}\widetilde{\mathbf{e}},p) \in H^{-1}(h)$, where $h = -\mu^2/(2\nu)^2$, we must have $\lambda^2\nu^2 = \langle p,p\rangle = 2h + 2\mu\sigma^{-1}$. Therefore $\widetilde{\mathscr{J}}_h^{-1}(c)$ is the line which is the union of the two half open line segments $\{\big(\sigma\nu^{-1}\widetilde{\mathbf{e}}, \pm\nu^{-1}\big(\sqrt{2h+2\mu\sigma^{-1}}\big)\widetilde{\mathbf{e}}\big) \in T_0\mathbf{R}^3 \,\big|\, \sigma\in(0,\mu/(-h)]\}$, which join smoothly at $(\mu/(-h\nu)\widetilde{\mathbf{e}},0)$. □

It is not hard to show that on $\widetilde{\mathscr{J}}_h^{-1}\big(C_\nu \setminus (\{\mathbf{J}=0\}\cap C_\nu)\big)$ the mapping $\widetilde{\mathscr{J}}_h$ is proper, whereas on $\widetilde{\mathscr{J}}_h^{-1}(\{\mathbf{J}=0\}\cap C_\nu)$ it is not.

We now turn to examining the so(4)-momentum mapping $\widetilde{\mathscr{J}}$ (31). Let C be the submanifold of $\mathbf{R}^3\times(\mathbf{R}^3\setminus\{(0,0)\})$ defined by $\langle\mathbf{J},\widetilde{\mathbf{e}}\rangle = 0$.

Claim: The map

$$\widetilde{\mathscr{J}} : \Sigma_- \to C \subseteq \mathbf{R}^6 : (q,p) \mapsto \big(q\times p, \nu(\|q\|^{-1}q - \mu^{-1}\,p\times(q\times p))\big) = (\mathbf{J},\widetilde{\mathbf{e}})$$

is a surjective submersion, each of whose fibers is a unique bounded orbit of the Kepler vector field X_H.

(3.11) **Proof**: First we show that $\widetilde{\mathscr{J}}$ is surjective. Supppose that $c = (\mathbf{J},\widetilde{\mathbf{e}}) \in C$. Then $\|\mathbf{J}+\widetilde{\mathbf{e}}\|^2 = \|\mathbf{J}-\widetilde{\mathbf{e}}\|^2 = \nu^2$ for some $\nu>0$. Hence $(\mathbf{J},\widetilde{\mathbf{e}})\in C_\nu$. Let $h = -\mu^2/(2\nu^2)$. From ((3.7)) it follows that $\widetilde{\mathscr{J}}_h^{-1}(c)$ is nonempty. Hence $\widetilde{\mathscr{J}}^{-1}(c)$ is nonempty. Because $\widetilde{\mathscr{J}}_h^{-1}(c)$ is a unique oriented bounded orbit of the Kepler vector field, $\widetilde{\mathscr{J}}^{-1}(c)$ is as well.

Since C is a 5-dimensional smooth manifold, the map $\widetilde{\mathscr{J}}$ is a submersion if for every

$(q,p) \in \Sigma_-$ the rank of $D\widetilde{\mathscr{J}}(q,p)$ is 5. Actually it suffices to show that for every $(q,p) \in H^{-1}(h)$ the vector $D\widetilde{\mathscr{J}}(q,p)\,\mathrm{grad}\,H(q,p)$ is normal to C_ν at $\widetilde{\mathscr{J}}(q,p)$, because

1. by ((3.7)), $D\widetilde{\mathscr{J}}(q,p)T_{(q,p)}H^{-1}(h) = T_{\widetilde{\mathscr{J}}(q,p)}C_\nu$;
2. a normal space to $H^{-1}(h)$ in Σ_- at (q,p) is spanned by $\mathrm{grad}\,H(q,p)$;
3. as a submanifold of C the manifold C_ν is defined by

$$F(\mathbf{J},\widetilde{\mathbf{e}}) = \langle \mathbf{J},\mathbf{J}\rangle + \langle \widetilde{\mathbf{e}},\widetilde{\mathbf{e}}\rangle - \nu^2 = 0, \qquad (40)$$

where $\nu = \mu(\sqrt{-2H})^{-1/2}$.

Since the normal space to C_ν at $\widetilde{\mathscr{J}}(q,p) = (\mathbf{J},\widetilde{\mathbf{e}}) \in C$ is spanned by $\mathrm{grad}\,F(\mathbf{J},\widetilde{\mathbf{e}}) = 2(\mathbf{J},\widetilde{\mathbf{e}})$, it suffices to check that $\langle D\widetilde{\mathscr{J}}(q,p)\,\mathrm{grad}\,H(q,p),\, \mathrm{grad}\,F(\mathscr{J}(q,p))\rangle$ is nonzero. The following calculation does this.

$$\begin{aligned}
0 \neq \langle \mathrm{grad}\,H(q,p),\, \mathrm{grad}\,H(q,p)\rangle &= DH(q,p)\,\mathrm{grad}\,H(q,p) \\
&= D\big(-\tfrac{1}{2}\mu^2\,(\langle \mathbf{J},\mathbf{J}\rangle + \langle \widetilde{\mathbf{e}},\widetilde{\mathbf{e}}\rangle)^{-1}\big)(q,p)\,\mathrm{grad}\,H(q,p), \\
&\qquad \text{using } H = -\mu^2/(2\nu^2) \text{ and } (40) \\
&= \tfrac{1}{2}\mu^2\,\big(\langle \mathbf{J},\mathbf{J}\rangle + \langle \widetilde{\mathbf{e}},\widetilde{\mathbf{e}}\rangle\big)^{-2}\big(\langle \mathbf{J}, D\mathbf{J}(q,p)\,\mathrm{grad}\,H(q,p)\rangle \\
&\qquad + \langle \widetilde{\mathbf{e}}, D\widetilde{\mathbf{e}}(q,p)\,\mathrm{grad}\,H(q,p)\rangle\big) \\
&= \mu^{-2}H(q,p)^2\,\big\langle D\widetilde{\mathscr{J}}(q,p)\,\mathrm{grad}\,H(q,p), \mathrm{grad}\,F(\widetilde{\mathscr{J}}(q,p))\big\rangle. \qquad \square
\end{aligned}$$

The above result has several useful consequences.

Corollary 1. The smooth manifold C (28) is the *space of orbits* of negative energy of the Kepler vector field X_H and the momentum map $\widetilde{\mathscr{J}} : \Sigma_- \to C$ (31) is the orbit map.

(3.12) **Proof**: The corollary follows from ((3.11)) and the definition of orbit space. $\square$

The next corollary says that every *smooth* integral of the Kepler vector field on Σ_- is a smooth function of the components of angular momentum $\mathbf{J}$ and the modified eccentricity vector $\widetilde{\mathbf{e}}$. More precisely,

Corollary 2. Suppose that $G : \Sigma_- \subseteq T_0\mathbf{R}^3 \to \mathbf{R}$ is a smooth integral of the Kepler vector field X_H. Then there is a smooth function $\widehat{G} : C \subseteq \mathbf{R}^6 \to \mathbf{R}$ such that $G = \widetilde{\mathscr{J}}^*\widehat{G}$.

(3.13) **Proof**: Since G is an integral of X_H on Σ_-, it is constant on each bounded orbit of X_H and hence is constant on the fibers of the momentum map $\widetilde{\mathscr{J}}$. Because C is smooth and is the space of orbits of X_H on Σ_- with orbit mapping $\widetilde{\mathscr{J}}$, G descends to a smooth function $\widehat{G} : C \subseteq \mathbf{R}^6 \to \mathbf{R}$. In other words, $G = \widetilde{\mathscr{J}}^*\widehat{G}$. $\square$

3.3 Kepler's equation

So far we have only used the constants of motion to describe the orbits of the Kepler vector field X_H of negative energy. This means that we cannot tell where on the orbit the particle is at a given time.

In order to give a time parametrization of a bounded Keplerian orbit, we define a new time scale, the *eccentric anomaly* s, by

$$\frac{\mathrm{d}s}{\mathrm{d}t} = \sqrt{-2h}\,\|q\|^{-1}. \tag{41}$$

Before finding a differential equation for $\|q(s)\|$, we use the integrals of energy and angular momentum to find a differential equation for $\|q(t)\|$. Multiplying the energy integral $h = \frac{1}{2}\langle p,p\rangle - \mu\|q\|^{-1}$ by $2\|q\|^2$ gives $\|q\|^2\|p\|^2 = 2\mu\,\|q\| + 2h\,\|q\|^2$. But $\|q\|^2\|p\|^2 = \|q\times p\|^2 + \langle q,p\rangle^2 = J^2 + \langle p,q\rangle^2$. In other words,

$$\|q\|^2\left(\frac{\mathrm{d}\|q\|}{\mathrm{d}t}\right)^2 + J^2 = 2\mu\,\|q\| + 2h\,\|q\|^2. \tag{42}$$

Using (41) to change to the time variable s and dividing by $-2h$ gives

$$\left(\frac{\mathrm{d}\|q\|}{\mathrm{d}s}\right)^2 + a^2(1-e^2) = 2a\,\|q\| - \|q\|^2, \tag{43}$$

since $a = \mu/(-2h) = J^2\mu^{-1}(1-e^2)^{-1}$. Instead of separating variables and immediately integrating (43) we first change variables by $ea\,\rho = a - \|q\|$. Then (43) simplifies to

$$\left(\frac{\mathrm{d}\rho}{\mathrm{d}s}\right)^2 + \rho^2 = 1. \tag{44}$$

Since $\|q(0)\| = a(1-e)$, from the definition of ρ we obtain $\rho(0) = 1$. Therefore

$$\|q(s)\| = a - ae\cos s. \tag{45}$$

To find the relation between the eccentric anomaly time scale s and the physical time scale t, we substitute (45) into (41) and integrate to obtain

$$\sqrt{-2h}(t-\tau) = \sqrt{-2h}\int_\tau^t \mathrm{d}t = \int_0^s (a - ae\cos s)\,\mathrm{d}s = as - ae\sin s. \tag{46}$$

Here τ is a time related to the time of periapse passage. Its precise definition is given below. Dividing (46) by a and using $a = \mu/(-2h) = \nu^2/\mu$ gives *Kepler's equation*

$$s - e\sin s = \mu^2\nu^{-3}(t-\tau) = n\ell, \tag{47}$$

where ℓ is the *mean anomaly* and $n = \mu^2\nu^{-3}$ is the *mean motion*. Note that

$$\begin{aligned}\langle q,p\rangle = \langle q, \frac{\mathrm{d}q}{\mathrm{d}t}\rangle &= \|q\|\frac{\mathrm{d}\|q\|}{\mathrm{d}t} = \|q\|\,\frac{\mathrm{d}\|q\|}{\mathrm{d}s}\frac{\mathrm{d}s}{\mathrm{d}t} \\ &= \sqrt{-2h}\,ae\,\sin s, \quad \text{using (41) and (45)} \\ &= \nu\, e\sin s. \end{aligned} \tag{48}$$

When $t = \tau$ from Kepler's equation it follows that $s = 0$. Let τ' be the physical time corresponding to $s = 2\pi$ in (47). Then $\tau - \tau'$ is the period of elliptical motion, which according to Kepler's equation, is $2\pi n^{-1} = 2\pi \nu^3 \mu^{-2} = 2\pi \mu^{-1/2} a^{3/2}$. This is *Kepler's third law of motion*.

During elliptical motion the particle goes through the periapse periodically. Therefore the time τ in (47) is *not* uniquely determined by the initial condition $(q(0), p(0))$, which defines the integral curve of X_H. We will define τ as follows. In the interval $[-\pi, \pi]$ there are precisely *two* values $\varepsilon \widehat{s}_0$ (with $\varepsilon^2 = 1$) which satisfy $\|q(0)\| = a(1 - e\cos \varepsilon \widehat{s}_0)$. To fix the choice of ε note that from (48) we have $\varepsilon = \langle q(0), p(0)\rangle / (\nu e \sin \widehat{s}_0)$, unless $\widehat{s}_0 = 0$ in which case ε is irrelevant. Set $s_0 = \varepsilon \widehat{s}_0$ and let $\tau = -n^{-1}(s_0 - e\sin s_0)$. In words, we define τ as follows. If at $t = 0$ the particle is in the upper half of the ellipse, then τ is the first time *before* $t = 0$ when the particle passed through the periapse; otherwise it is the first time *on* or *after* $t = 0$ when the particle passes through the periapse.

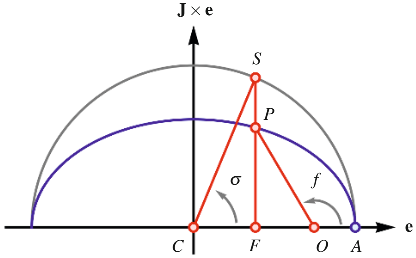

Figure 3.2. The eccentric anomaly.

To describe the classic geometric meaning of the eccentric anomaly s, consider the figure 3.2. Let O be the center of attraction, A the periapse and C the *center of the ellipse* of eccentricity e. The arrow on the ellipse indicates the direction of motion and P is the position of the particle on the ellipse with true anomaly f. Construct a line SP through P which is perpendicular to the line CA. Project P parallel along SP to the point S on the circle $\mathscr{C}$ with center C and radius equal to the distance $\overline{CA}$.

Claim: The *eccentric anomaly* s is the angle $\angle ACS$.

(3.14) **Proof**: Let $\sigma = \angle ACS$. From figure 3.2 we obtain $\overline{CS} = a$ and $\overline{CO} = ae$. Since $\overline{CF} = \overline{CO} + \overline{OF}$, we find that $a\cos\sigma = ae + \|q\|\cos f$. As the orbit is elliptical, we have $\|q\| = a(1-e^2)(1+e\cos f)^{-1}$. This may be rewritten as

$$\|q\| = a - e(ae + \|q\|\cos f) = a - ae\cos\sigma.$$

But $\|q\| = a - ae\cos s$. Hence $\cos s = \cos\sigma$. Since $s = 0$ when $\sigma = 0$, we obtain $\sigma = s$. $\square$

As the point S traces out the circle $\mathscr{C}$ uniformly with speed n, the point P on the ellipse traces out the projection of an integral curve of the Kepler vector field in configuration space.

4 Regularization

In this section we remove the incompleteness of the flow of the Kepler vector field by embedding it into a complete flow. This process is called regularization. We regularize the Kepler problem in two ways: one, called *Moser's regularization*, works on a fixed negative energy level; while the other, called *Ligon-Schaaf regularization*, works on all negative energy level sets at once.

4.1 Moser's regularization

We begin by discussing Moser's regularization. On the phase space $\big(T_0\mathbf{R}^3 = (\mathbf{R}^3 \setminus \{0\}) \times \mathbf{R}^3, \widetilde{\omega}_3 = \big(\sum_{i=1}^3 \mathrm{d}q_i \wedge \mathrm{d}p_i\big)|T_0\mathbf{R}^3\big)$ with coordinates (q,p) consider the Kepler Hamiltonian

$$H(q,p) = \tfrac{1}{2}\|p\|^2 - \|q\|^{-1}. \tag{49}$$

Here $\langle\ ,\ \rangle$ is the Euclidean inner product on $\mathbf{R}^3$ with associated norm $\|\ \|$. We have chosen physical units so that $\mu = 1$. The integral curves of the Hamiltonian vector field X_H associated to H satisfy

$$\begin{aligned} \frac{\mathrm{d}q}{\mathrm{d}t} &= p = \frac{\partial H}{\partial p} \\ \frac{\mathrm{d}p}{\mathrm{d}t} &= -\|q\|^{-3}q = -\frac{\partial H}{\partial q}. \end{aligned} \tag{50}$$

Let $\mathbf{R}_{>0}$ be the multiplicative group of positive real numbers. On $\mathbf{R} \times T_0\mathbf{R}^3$ define an $\mathbf{R}_{>0}$-action by

$$\widetilde{\Psi}_V : \mathbf{R}_{>0} \times (\mathbf{R} \times T_0\mathbf{R}^3) \to \mathbf{R} \times T_0\mathbf{R}^3 : \big(\rho, (t,q,p)\big) \mapsto (\rho^3 t, \rho^2 q, \rho^{-1} p). \tag{51}$$

▷ The equations of motion (50) of the Kepler problem are invariant under the action (51) of the *virial group*.

(4.1) **Proof**: We check this as follows.

$$\frac{\mathrm{d}(\rho^2 q)}{\mathrm{d}(\rho^3 t)} = \rho^{-1} p \quad \text{and} \quad \frac{\mathrm{d}(\rho^{-1} p)}{\mathrm{d}(\rho^3 t)} = -\rho^{-4}\|q\|^{-3} q = -\|\rho^2 q\|^{-3}\rho^2 q. \qquad \square$$

Under the virial group the Kepler Hamiltonian H (49) transforms as $H \mapsto \rho^{-2}H$ and the symplectic form transforms as $\widetilde{\omega}_3 \mapsto \rho\,\widetilde{\omega}_3$.

We now regularize the bounded orbits of the Kepler problem of fixed negative energy. Using the virial group we reduce our considerations to the level set $H^{-1}(-\frac{1}{2})$. First we introduce a new time scale s by $\frac{\mathrm{d}s}{\mathrm{d}t} = \|q\|^{-1}$. Consider the new Hamiltonian

$$\widetilde{F}(q,p) = \|q\|\big(H(q,p) + \tfrac{1}{2}\big) + 1 = \tfrac{1}{2}\|q\|(\|p\|^2 + 1). \tag{52}$$

▷ The integral curves of $X_{\widetilde{F}}$ on $\widetilde{F}^{-1}(1)$ are integral curves of X_H on $H^{-1}(-\frac{1}{2})$, using the time parameter s.

(4.2) **Proof**: Using the time parameter s the integral curves of X_H satisfy

$$\begin{aligned}\frac{\mathrm{d}q}{\mathrm{d}s} &= \frac{\mathrm{d}q}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}s} = p\|q\| = \|q\|\frac{\partial}{\partial p}\big(H(q,p)+\tfrac{1}{2}\big)\\ \frac{\mathrm{d}p}{\mathrm{d}s} &= \frac{\mathrm{d}p}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}s} = -\|q\|^{-2}q = -\|q\|\frac{\partial}{\partial q}\big(H(q,p)+\tfrac{1}{2}\big).\end{aligned} \tag{53}$$

On $H^{-1}(-\frac{1}{2})$ we have $H(q,p)+\frac{1}{2}=0$. So $\|q\|\frac{\partial}{\partial z}\big(H(q,p)+\frac{1}{2}\big) = \frac{\partial}{\partial z}\big(\|q\|\big(H(q,p)+\frac{1}{2}\big)\big)$ for $z=q$ or p. Therefore on $\widetilde{F}^{-1}(1) = H^{-1}(-\frac{1}{2})$ equation (53) is in Hamiltonian form

$$\begin{aligned}\frac{\mathrm{d}q}{\mathrm{d}s} &= \frac{\partial \widetilde{F}}{\partial p}\\ \frac{\mathrm{d}p}{\mathrm{d}s} &= -\frac{\partial \widetilde{F}}{\partial q}.\end{aligned} \tag{54}$$

Hence the integral curves of X_H on $H^{-1}(-\frac{1}{2})$ are the same as the integral curves of $X_{\widetilde{F}}$ on $\widetilde{F}^{-1}(1)$, using the time parameter s. □

Let $\widetilde{K}: T_0\mathbf{R}^3 \to \mathbf{R}$, where

$$\widetilde{K}(q,p) = \tfrac{1}{2}\widetilde{F}(q,p)^2 = \tfrac{1}{8}\|q\|^2(\|p\|^2+1)^2. \tag{55}$$

▷ The integral curves of $X_{\widetilde{K}}$ on $\widetilde{K}^{-1}(\frac{1}{2})$ are the same as the integral curves of $X_{\widetilde{F}}$ on $\widetilde{F}^{-1}(1)$.

(4.3) **Proof**: This follows because on $\widetilde{F}^{-1}(1)$ we have

$$\begin{aligned}\frac{\mathrm{d}q}{\mathrm{d}s} &= \frac{1}{2}\frac{\partial \widetilde{F}^2}{\partial p} = \widetilde{F}\frac{\partial \widetilde{F}}{\partial p} = \frac{\partial \widetilde{K}}{\partial p}\\ \frac{\mathrm{d}p}{\mathrm{d}s} &= -\frac{1}{2}\frac{\partial \widetilde{F}^2}{\partial q} = -\widetilde{F}\frac{\partial \widetilde{F}}{\partial q} = -\frac{\partial \widetilde{K}}{\partial q}.\end{aligned}$$ □

On our way toward defining Moser's regularization map consider stereographic projection $\varphi: S^3_{\mathrm{np}} = S^3 \setminus \{\mathrm{np}\} \to \mathbf{R}^3$ from the north pole $\mathrm{np} = (0,0,0,1)$ of the 3-sphere $S^3 = \{u \in \mathbf{R}^4 \,\big|\, \langle u,u\rangle = \sum_{j=1}^4 u_j^2\}$ to the 3-plane $\mathbf{R}^3 = \mathbf{R}^3 \times \{0\}$ in $\mathbf{R}^4$. For each $u \in S^3_{\mathrm{np}}$ let $q = \varphi(u)$ be the point of intersection of the line joining np to u with $\mathbf{R}^3$. A short calculation shows that $q_i = \frac{u_i}{1-u_4}$ for $i=1,2,3$. Let $T^+S^3_{\mathrm{np}} = \{(u,v) \in T\mathbf{R}^3 \,\big|\, u \in S^3_{\mathrm{np}},\ 0 = \langle u,v\rangle = \sum_{j=1}^4 u_jv_j,\ v \neq 0\}$. Consider the mapping

$$\Phi_M^{-1}: T^+S^3_{\mathrm{np}} \to T_0\mathbf{R}^3 : (u,v) \mapsto (q,p) = \big(-(1-u_4)\widetilde{v} - v_4\widetilde{u}, (1-u_4)^{-1}\widetilde{u}\big), \tag{56}$$

where $u = (\widetilde{u}, u_4)$ and $v = (\widetilde{v}, v_4)$. Then Φ_M^{-1} is the composition of a lift of stereographic projection

$$\widehat{\varphi}: T^+S^3_{\mathrm{np}} \to T_0\mathbf{R}^3 : (u,v) \mapsto (q,p) = \big((1-u_4)^{-1}\widetilde{u}, (1-u_4)\widetilde{v} + v_4\widetilde{u}\big),$$

followed by the momentum reversal $\psi: T_0\mathbf{R}^3 \to T_0\mathbf{R}^3 : (q,p) \mapsto (-p,q)$. When $(u,v) \in$
▷ $T^+S^3_{\mathrm{np}}$ we have the identities

$$\|q\|^2 = \|v\|^2(1-u_4)^2 \tag{57a}$$

$$\|p\|^2 + 1 = 2(1-u_4)^{-1} \tag{57b}$$
$$\langle q, p\rangle = -v_4. \tag{57c}$$

(4.4) **Proof**: Using $\langle u,u\rangle = 1$ and $\langle u,v\rangle = 0$, from (56) we get

$$\begin{aligned}
\|q\|^2 &= \sum_{i=1}^{3} q_i^2 = \sum_{i=1}^{3} \big(v_i(1-u_4)+u_iv_4\big)^2 \\
&= (1-u_4)^2(\|v\|^2 - v_4^2) - 2u_4v_4^2(1-u_4) + v_4^2(1-u_4^2) \\
&= \|v\|^2(1-u_4)^2; \\
\|p\|^2 + 1 &= (1-u_4)^{-2}\sum_{i=1}^{3} u_i^2 + 1 = (1-u_4)^{-2}(1-u_4^2) + 1 = 2(1-u_4)^{-1}; \\
\langle q,p\rangle &= \sum_{i=1}^{3} q_ip_i = -\sum_{i=1}^{3}\big((v_i(1-u_4)+u_iv_4)(1-u_4)^{-1}u_i\big) \\
&= u_4v_4 - v_4(1-u_4)^{-1}(1-u_4^2) = -v_4.
\end{aligned}$$ □

Define *Moser's mapping*

$$\Phi_M : T_0\mathbf{R}^3 \rightarrow T^+S^3_{\mathrm{np}} : (q,p) \mapsto (u,v) = \big((\widetilde{u},u_4),(\widetilde{v},v_4)\big)$$

by

$$\begin{aligned}
\widetilde{u} &= 2p(\|p\|^2+1)^{-1} \quad \text{and} \quad u_4 = (\|p\|^2-1)(\|p\|^2+1)^{-1} \\
\widetilde{v} &= -\tfrac{1}{2}(\|p\|^2+1)q + \langle q,p\rangle p \quad \text{and} \quad v_4 = -\langle q,p\rangle.
\end{aligned} \tag{58}$$

▷ We now show that Φ_M^{-1} (56) is the inverse of Moser's mapping Φ_M (58).

(4.5) **Proof**: Suppose that $(u,v) \in T^+S^3$. Then

$$\begin{aligned}
\Phi_M(\Phi_M^{-1}(u,v)) &= \Phi_M(q,p), \quad \text{where } q = -(1-u_4)\widetilde{v} - v_4\widetilde{u} \text{ and } p = (1-u_4)^{-1}\widetilde{u} \\
&= \big(((1-u_4)(1-u_4)^{-1}\widetilde{u}, (2(1-u_4)^{-1}-2)\tfrac{1}{2}(1-u_4)), \\
&\qquad ((1-u_4)^{-1}[(1-u_4)\widetilde{v}+v_4\widetilde{u}] - v_4\widetilde{u}(1-u_4)^{-1}, v_4)\big) \\
&\qquad \text{using (58) the identities (57a) – (57c)} \\
&= \big((\widetilde{u}, 1-(1-u_4)), (\widetilde{v}+(1-u_4)^{-1}v_4\widetilde{u} - (1-u_4)^{-1}v_4\widetilde{u}, v_4)\big) = (u,v).
\end{aligned}$$

Now suppose that $(q,p) \in T_0\mathbf{R}^3$. Then

$$\begin{aligned}
\Phi_M^{-1}(\Phi_M(q,p)) &= \Phi_M^{-1}(u,v) = \big((-(1-u_4)\widetilde{v} - v_4\widetilde{u}, (1-u_4)^{-1}\widetilde{u}\big), \\
&\qquad \text{where } u,v \text{ are given by (58)} \\
&= \big(-2(\|p\|^2+1)^{-1}[-\tfrac{1}{2}(\|p\|^2+1)q + \langle q,p\rangle p] + 2\langle q,p\rangle(\|p\|^2+1)^{-1}p, \\
&\qquad 2(\|p\|^2+1)^{-1}\tfrac{1}{2}(\|p\|^2+1)p\big) = (q,p).
\end{aligned}$$ □

▷ The restriction $\widetilde{\Phi}$ of Moser's mapping Φ_M (58) to $H^{-1}(-\frac{1}{2})$ is a diffeomorphism of $H^{-1}(-\frac{1}{2})$ onto $T_1S^3_{\mathrm{np}} = \{(u,v) \in T^+S^3 \,\big|\, \|u\|^2 = 1\}$ with inverse $\Phi = \Phi_M^{-1}|T_1S^3_{\mathrm{np}}$.

(4.6) **Proof**: Using $(u,v) \in T_1S^3_{\mathrm{np}}$ and the identities $\|q\|^2 = \|v\|^2(1-u_4)^2$ and $\|p\|^2+1 = 2(1-u_4)^{-1}$ we get $\frac{1}{2}\|p\|^2 - \|q\|^{-1} + \frac{1}{2} = (1-u_4)^{-1} - (\|v\|(1-u_4))^{-1} = 0$, that is, $(q,p) \in H^{-1}(-\frac{1}{2})$. So $\Phi = \Phi_M^{-1}|T_1S^3_{\mathrm{np}}$ maps $T_1S^3_{\mathrm{np}}$ into $H^{-1}(-\frac{1}{2})$. From the fact that $\nu(q,p) = (-2H(q,p))^{-1/2} = 1$ when $(q,p) \in H^{-1}(-\frac{1}{2})$, a straightforward calculation, given in (69a) – (69c), shows that $\widetilde{\Phi} = \Phi_M|H^{-1}(-\frac{1}{2})$ maps $H^{-1}(-\frac{1}{2})$ into $T_1S^3_{\mathrm{np}}$. Because Moser's mapping is a diffeomorphism of $T_0\mathbf{R}^3$ onto $T^+S^3_{\mathrm{np}}$, it follows that $\widetilde{\Phi}$ is a diffeomorphism of $H^{-1}(-\frac{1}{2})$ onto $T_1S^3_{\mathrm{np}}$ with inverse Φ. □

▷ The map Φ_M^{-1} pulls back the 1-form $\big(\sum_{i=1}^3 q_i\,\mathrm{d}p_i\big)|H^{-1}(-\frac{1}{2})$ on $H^{-1}(-\frac{1}{2})$ to the 1-form $-\big(\sum_{j=1}^4 v_j\,\mathrm{d}u_j\big)\,|T_1S^3$ on T_1S^3.

(4.7) **Proof**: On $H^{-1}(-\frac{1}{2})$ we have

$$
\begin{aligned}
(\Phi_M^{-1})^*(\sum_{i=1}^3 q_i\,\mathrm{d}p_i) &= -\sum_{i=1}^3 \big((1-u_4)v_i + v_4u_i\big)\,\mathrm{d}\big((1-u_4)^{-1}u_i\big) = -\sum_{i=1}^3 v_i\,\mathrm{d}u_i \\
&\quad - \sum_{i=1}^3 u_iv_i(1-u_4)^{-1}\,\mathrm{d}u_4 - v_4(1-u_4)^{-1}\sum_{i=1}^3 u_i\,\mathrm{d}u_i - v_4(1-u_4)^{-2}\sum_{i=1}^3 u_i^2\,\mathrm{d}u_4 \\
&\qquad\qquad \text{since } 0 = \mathrm{d}(\textstyle\sum_{j=1}^4 u_j^2) \text{ gives } -u_4\,\mathrm{d}u_4 = \textstyle\sum_{i=1}^3 u_i\,\mathrm{d}u_i \\
&= -\sum_{i=1}^3 v_i\,\mathrm{d}u_i + v_4u_4(1-u_4)^{-1}\,\mathrm{d}u_4 + v_4u_4(1-u_4)^{-1}\,\mathrm{d}u_4 \\
&\quad -(1-u_4^2)(1-u_4)^{-2}\,v_4\,\mathrm{d}u_4, \quad \text{since } \textstyle\sum_{i=1}^3 u_iv_i = -u_4v_4 \text{ and } \sum_{i=1}^3 u_i^2 = 1-u_4^2 \\
&= -\sum_{j=1}^4 v_j\,\mathrm{d}u_j.
\end{aligned}
$$
□

▷ The inverse Φ_M^{-1} (56) of Moser's mapping is symplectic,

(4.8) **Proof**: On $T_1S^3_{\mathrm{np}}$ we get

$$
\begin{aligned}
(\Phi_M^{-1})^*(\sum_{i=1}^3 \mathrm{d}q_i \wedge \mathrm{d}p_i) &= (\Phi_M^{-1})^*(\mathrm{d}\sum_{i=1}^3 q_i\,\mathrm{d}p_i) = \mathrm{d}\big((\Phi_M^{-1})^*(\sum_{i=1}^3 q_i\,\mathrm{d}p_i)\big) \\
&= -\mathrm{d}\big(\sum_{j=1}^4 v_j\,\mathrm{d}u_j\big) = \sum_{j=1}^4 \mathrm{d}u_j \wedge \mathrm{d}v_j.
\end{aligned}
$$
□

On $T^+S^3_{\mathrm{np}}$ let

$$K(u,v) = (\Phi_M^{-1})^*\widetilde{K}(u,v) = \tfrac{1}{2}\|v\|^2, \tag{59}$$

using (57a) and (57b). From ((4.7)) and ((4.8)) it follows that on the energy level $H^{-1}(-\frac{1}{2})$ the Hamiltonian system $(H, T_0\mathbf{R}^3, \omega)$ is equivalent to the Hamiltonian system $\big(K, T^+S^3_{\mathrm{np}}, (\sum_{j=1}^4 \mathrm{d}u_j \wedge \mathrm{d}v_j)\big|T^+S^3_{\mathrm{np}}\big)$ on $K^{-1}(\frac{1}{2})$ via Moser's mapping Φ_M (58). Clearly K extends to a smooth function on $TS^3 \cap K^{-1}(\frac{1}{2})$, whose Hamiltonian vector field X_K defines the geodesic flow on S^3 on $K^{-1}(\frac{1}{2})$. Hence Moser's mapping embeds the incomplete flow of the Kepler vector field X_H on $H^{-1}(-\frac{1}{2})$ into the complete geodesic flow on $K^{-1}(\frac{1}{2})$. Using the virial group we can use Moser's mapping to regularize the Kepler vector field a negative energy level at a time.

▷ We now see how the integrals of the geodesic flow pull back under the of Moser's mapping. We show that under Moser's mapping Φ_M the integral $S_{ij} = u_i v_j - u_j v_i$, $1 \le i < j \le 3$, of X_K on $K^{-1}(\frac{1}{2})$ pulls back to the k^{th} component J_k of the SO(3)-momentum $J = p \times q$ restricted to $H^{-1}(-\frac{1}{2})$. Here $\{i, j, k\} = \{1, 2, 3\}$. The integral $S_{i4} = u_i v_4 - v_i u_4$, $1 \le i \le 3$, of X_K on $K^{-1}(\frac{1}{2})$ pulls back to the the i^{th} component of the eccentricity vector $\mathbf{e} = -\|q\|^{-1} q + p \times (q \times p)$ restricted to $H^{-1}(-\frac{1}{2})$.

(4.9) **Proof**: When $1 \le i < j \le 3$ we get

$$\begin{aligned}\Phi_M^*\big(S_{ij}|K^{-1}(\tfrac{1}{2})\big)(u,v) &= \big(\Phi_M^*(u_i v_j - u_j v_i)\big)|H^{-1}(-\tfrac{1}{2}) \\ &= \big[2p_i(\|p\|^2+1)^{-1}\big(-\tfrac{1}{2}(\|p\|^2+1)q_j + \langle q,p\rangle p_j\big) \\ &\qquad -\big(-\tfrac{1}{2}(\|p\|^2+1)q_i + \langle q,p\rangle p_i\big)2p_j(\|p\|^2+1)^{-1}\big], \text{ using (58)} \\ &= (q_i p_j - p_j q_i)|H^{-1}(-\tfrac{1}{2}) = J_k|H^{-1}(-\tfrac{1}{2}).\end{aligned}$$

Also when $i = 1, 2, 3$ we have

$$\begin{aligned}\Phi_M^*\big(S_{i4}|K^{-1}(\tfrac{1}{2})\big)(u,v) &= \big(\Phi_M^*(u_i v_4 - u_4 v_i)\big)|H^{-1}(-\tfrac{1}{2}) \\ &= \big[-2p_i(\|p\|^2+1)^{-1}\langle q,p\rangle + \tfrac{1}{2}q_i(\|p\|^2-1) \\ &\qquad -\langle q,p\rangle\|p\|^2(\|p\|^2+1)^{-1}p_i + \langle q,p\rangle(\|p\|^2+1)^{-1}p_i\big] \\ &= \big[-p_i\langle q,p\rangle + q_i\|p\|^2 - \tfrac{1}{2}(\|p\|^2+1)q_i\big] \\ &= \big[-\|q\|^{-1}q_i + q_i\langle p,p\rangle - p_i\langle q,p\rangle\big]\,|H^{-1}(-\tfrac{1}{2}) = e_i|H^{-1}(-\tfrac{1}{2}).\end{aligned}$$

The second to last equality follows since $\frac{1}{2}(\|p\|^2+1) = \|q\|^{-1}$ defines $H^{-1}(-\frac{1}{2})$. □

▷ Let

$$\widetilde{J}: T^+S^3 \to \mathbf{R} : (u,v) \mapsto \sum_{1 \le i < j \le 3} (u_i v_j - u_j v_i)^2 = \|\widetilde{u} \times \widetilde{v}\|^2.$$

Then $\widetilde{J}^{-1}(0)$ is the set of all integral curves of the geodesic vector field X_K, which pass through the *collision set* $C = \{(u,v) \in T^+S^3 \,\big|\, u_4 = 1\}$ on T^+S^3.

(4.10) **Proof**: The image of each integral curve of X_K under the bundle projection map is a great circle on S^3. Each great circle intersects the equatorial 2-sphere $\{u_4 = 0\} \cap S^3$ at some point $P = (\widetilde{u}, 0, \widetilde{v}, v_4)$. Suppose that $\widetilde{J}(P) = 0$. Then $\widetilde{u} \times \widetilde{v} = 0$. If $\widetilde{v} \ne 0$, then $\widetilde{u} = \lambda\widetilde{v}$ for some nonzero $\lambda \in \mathbf{R}$. But $(u,v) \in T^+S^3$. So $0 = \langle u, v\rangle = \langle\widetilde{u},\widetilde{v}\rangle + u_4 v_4 = \langle\widetilde{u},\widetilde{v}\rangle$, since $u_4 = 0$. Consequently, $0 = \lambda\langle\widetilde{v},\widetilde{v}\rangle$, which contradicts the fact that $\lambda \ne 0$ and $\widetilde{v} \ne 0$. Therefore $\widetilde{v} = 0$, that is, $P = (\widetilde{u}, 0, 0, v_4)$. For some $\tau > 0$ the integral curve $\gamma : \mathbf{R} \to T^+S^3 : t \mapsto \varphi_t^K(P)$ of X_K passes through the collision set C. To see this we must find τ so that

$$\begin{pmatrix} 0 \\ 1 \\ \widetilde{v}' \\ v_4' \end{pmatrix} = \varphi_\tau^K \begin{pmatrix} \widetilde{u} \\ 0 \\ 0 \\ v_4 \end{pmatrix} = \begin{pmatrix} \cos(\tau\sqrt{2K})\begin{pmatrix} \widetilde{u} \\ 0 \end{pmatrix} + \frac{\sin(\tau\sqrt{2K})}{\sqrt{2K}}\begin{pmatrix} 0 \\ v_4 \end{pmatrix} \\ -\sqrt{2K}\sin(\tau\sqrt{2K})\begin{pmatrix} \widetilde{u} \\ 0 \end{pmatrix} + \cos(\tau\sqrt{2K})\begin{pmatrix} 0 \\ v_4 \end{pmatrix} \end{pmatrix}, \qquad (60)$$

using (8). Noting that $K = K(P) = \frac{1}{2}v_4^2 > 0$, the fourth component of (60) reads $1 = v_4|v_4|^{-1}\sin(\tau\sqrt{2K})$, which gives $\tau = \pi/(2v_4)$, if $v_4 > 0$ or $\tau = 3\pi/(2|v_4|)$, if $v_4 < 0$. Thus the first three components of (60) read $0 = \cos(\tau\sqrt{2K})$, as desired. Consequently, at time $t = \tau$ the integral curve γ, which starts at $P = (\widetilde{u}, 0, 0, v_4)$ passes through the collision set C. Thus $\widetilde{J}^{-1}(0)$ is a subset of C. The collision set is clearly a subset of $\widetilde{J}^{-1}(0)$. □

Under Moser's mapping $\widetilde{J}^{-1}(0)$ corresponds to the set of bounded orbits of the Kepler problem with 0 angular momentum. This is precisely the set of bounded Keplerian orbits which reach the origin of $\mathbf{R}^3$ in finite time.

4.2 Ligon-Schaaf regularization

On the subset Σ_- of phase space $T_0\mathbf{R}^3$, where the Kepler Hamiltonian is *negative*, one can perform regularization in such a way that the embedding is symplectic and the resulting vector field is Hamiltonian with an $\mathrm{SO}(4)$ symmetry, which integrates the $\mathrm{so}(4)$ symmetry of the Kepler Hamiltonian. This symmetry does *not* arise from a lift of a symmetry on configuration space.

We regularize all *negative* energy Keplerian orbits at once using the *Ligon-Schaaf map*

$$\begin{aligned} &\Phi_{LS} : \Sigma_- \subseteq T_0\mathbf{R}^3 \to T^+S^3_{\mathrm{np}} \subseteq T\mathbf{R}^4 : \\ &(q,p) \mapsto \begin{pmatrix} r \\ s \end{pmatrix} = \begin{pmatrix} \cos v_4 & \sin v_4 \\ -\nu(q,p)\sin v_4 & \nu(q,p)\cos v_4 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}, \end{aligned} \tag{61}$$

where

$$\begin{aligned} u &= \big(\nu(q,p)^{-1}\|q\|p, \langle p,p\rangle\|q\| - 1\big) \\ v &= \big(-\|q\|q + \langle q,p\rangle p, -\nu(q,p)^{-1}\langle q,p\rangle\big), \end{aligned} \tag{62}$$

and $\nu(q,p) = (\frac{2}{\|q\|} - \|p\|^2)^{-1/2}$. We start by factoring Φ_{LS}.

Claim: Let

$$S : \Sigma_- \subseteq T_0\mathbf{R}^4 \to T_1S^3_{\mathrm{np}} \times \mathbf{R}_{>0} \subseteq T\mathbf{R}^4 \times \mathbf{R} : (q,p) \mapsto (u,v,\nu), \tag{63a}$$

where $\nu = \nu(q,p)$ and (u,v) is given by (62). Also let

$$L : T_1S^3 \times \mathbf{R}_{>0} \to T^+S^3 \subseteq T\mathbf{R}^4 : (u,v,\nu) \mapsto \begin{pmatrix} \widetilde{r} \\ \nu\widetilde{s} \end{pmatrix}, \tag{63b}$$

where $\begin{pmatrix} \widetilde{r} \\ \widetilde{s} \end{pmatrix} = \begin{pmatrix} \cos v_4 & \sin v_4 \\ -\sin v_4 & \cos v_4 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$. Then $\Phi_{LS} = L \circ S$.

(4.11) **Proof**: Before proving the claim we look at each factor of the Ligon-Schaaf map more carefully starting with the mapping S (63a). On Σ_- we have an $\mathbf{R}_{>0}$-action

$$\Psi^V : \mathbf{R}_{>0} \times \Sigma_- \to \Sigma_- : \big(\rho, (q,p)\big) \mapsto (\rho^2 q, \rho^{-1} p) \tag{64}$$

of the *scaling group*. To see that Ψ^V is well defined suppose that $\rho \in \mathbf{R}_{>0}$ and $(q,p) \in \Sigma_-$. Then at (q,p) the value of the Kepler Hamiltonian H (49) is negative. So

$$H(\rho^2 q, \rho^{-1} p) = \tfrac{1}{2}\|\rho^{-1}p\|^2 - \|\rho^2 q\|^{-1} = \rho^{-2}H(q,p) < 0.$$

▷ Thus for every $\rho \in \mathbf{R}_{>0}$ the map Ψ^V_ρ sends Σ_- into itself. The action Ψ^V is free for if
▷ $(\rho^2 q, \rho^{-1} p) = (q,p)$, then $\rho = 1$. Every orbit of the $\mathbf{R}_{>0}$-action (64) intersects the level set $(H|\Sigma_-)^{-1}(-\frac{1}{2})$ in exactly one point.

(4.12) **Proof**: Suppose that $(q,p) \in \Sigma_-$. Then $m = \Psi^V_{\nu(q,p)^{-1}}(q,p) \in H^{-1}(-\frac{1}{2})$, since

$$H(m) = H(\nu(q,p)^{-2}q, \nu(q,p)p) = \nu(q,p)^2 H(q,p) = -\tfrac{1}{2}$$

for $\nu(q,p)^2 = (-2H(q,p))^{-1}$. Because $H(\Psi^V_\rho(q,p)) = \rho^{-2}H(q,p)$, the Ψ^V-orbit through (q,p) intersects $(H|\Sigma_-)^{-1}(-\frac{1}{2})$ at m only when $\rho = \nu(q,p)^{-1}$. □

▷ The orbit space $\Sigma_-/\mathbf{R}_{>0}$ is diffeomorphic to $(H|\Sigma_-)^{-1}(-\frac{1}{2})$ with orbit map

$$\pi_V : \Sigma_- \subseteq T_0\mathbf{R}^3 \to (H|\Sigma_-)^{-1}(-\tfrac{1}{2}) \subseteq \Sigma_- : (q,p) \mapsto (\widehat{q}, \widehat{p}) = \big(\nu(q,p)^{-2}q, \nu(q,p)p\big). \quad (65)$$

(4.13) **Proof**: We need only show that $(H|\Sigma_-)^{-1}(-\frac{1}{2})$ is a smooth submanifold of $T_0\mathbf{R}^3$, since $\Sigma_- = H^{-1}((-\infty, 0))$ is an open subset of $T_0\mathbf{R}^3$. Suppose that $(q,p) \in T_0\mathbf{R}^3$ is a critical point of H. Then $0 = \mathrm{d}H(q,p) = \langle p, \mathrm{d}p\rangle + \|q\|^{-3}\langle q, \mathrm{d}q\rangle$. This implies $q = 0 = p$, which contradicts the fact that $(q,p) \in T_0\mathbf{R}^3$. Thus every negative real number is a regular value of $H|\Sigma_-$. Consequently, $(H|\Sigma_-)^{-1}(-\frac{1}{2})$ is a smooth manifold. □

Let $T_1S^3_{\mathrm{np}} = \{(u,v) \in T^+S^3_{\mathrm{np}} \,\big|\, \|v\|^2 = 1\}$. Define an $\mathbf{R}_{>0}$-action Ψ^T on $T_1S^3_{\mathrm{np}} \times \mathbf{R}_{>0}$ by

$$\Psi^T : \mathbf{R}_{>0} \times (T_1S^3_{\mathrm{np}} \times \mathbf{R}_{>0}) \to T_1S^3_{\mathrm{np}} \times \mathbf{R}_{>0} : \big(\mu, ((u,v),\lambda)\big) \mapsto ((u,v), \mu\lambda). \quad (66)$$

The action Ψ^T is free for if $\big((u,v),\mu\lambda\big) = \big((u,v),\lambda\big)$, then $\mu = 1$. The space $T_1S^3_{\mathrm{np}} \times \{1\}$
▷ is the orbit space $(T_1S^3_{\mathrm{np}} \times \mathbf{R}_{>0})/\mathbf{R}_{>0}$ of the action Ψ^V. The orbit map is

$$\pi_T : T_1S^3_{\mathrm{np}} \times \mathbf{R}_{>0} \to T_1S^3_{\mathrm{np}} \times \{1\} : \big((u,v),\mu\big) \mapsto \big((u,v),1\big). \quad (67)$$

Claim: Using the restriction of Moser's mapping Φ_M (58) to $(H|\Sigma_-)^{-1}(-\frac{1}{2})$, given by

$$\begin{aligned}\Phi : (H|\Sigma_-)^{-1}(-\tfrac{1}{2}) \subseteq T_0\mathbf{R}^3 \to TS^3_{\mathrm{np}} &: (q,p) \mapsto (u,v) \\ &= \big((\|q\|p, \|p\|^2\|q\| - 1), (-\|q\|^{-1}q + \langle q,p\rangle p, -\langle q,p\rangle)\big),\end{aligned} \quad (68a)$$

and *Moser's fibration*

$$\begin{aligned}&F_M : \Sigma_- \to T_1S^3_{\mathrm{np}} : (q,p) \mapsto \\ &\big((\nu(q,p)^{-1}\|q\|p, \|p\|^2\|q\| - 1), (-\|q\|^{-1}q + \langle q,p\rangle, -\nu(q,p)^{-1}\langle q,p\rangle)\big),\end{aligned} \quad (68b)$$

where $\nu(q,p)^{-2} = \frac{2}{\|q\|} - \|p\|^2$, we obtain the following commutative diagram.

$$
\begin{tikzcd}
\Sigma_- \arrow[r, "S"] \arrow[d, "\pi_V"'] & T_1S^3_{\mathrm{np}} \times \mathbf{R}_{>0} \arrow[d, "\pi_T"] & (q,p) \arrow[r] \arrow[d] & (F_M(q,p), \nu(q,p)) \arrow[d] \\
(H|\Sigma_-)^{-1}(-\frac{1}{2}) \arrow[r, "s"] & T_1S^3_{\mathrm{np}} \times \{1\} & (\widehat{q},\widehat{p}) \arrow[r] & \big(\Phi(\widehat{q},\widehat{p}),1\big)
\end{tikzcd}
$$

Diagram 4.2.1

Morover, the bundle mapping S is an $\mathbf{R}_{>0}$-bundle isomorphism.

(4.14) **Proof**: The next calculation shows that the image of Σ_- under Moser's fibration F_M is contained in $T_1S^3_{\mathrm{np}}$. Using the definition of F_M (68b) and $\nu = \nu(q,p)$ we have

$$
\begin{aligned}
\langle u,u\rangle &= \nu^{-2}\|q\|^2\|p\|^2 + (\|p\|^2\|q\| - 1)^2 \\
&= (-\|p\|^2 + 2\|q\|^{-1})\|q\|^2\|p\|^2 + (\|p\|^2\|q\| - 1)^2 = 1; && (69a) \\
\langle u,v\rangle &= \nu^{-1}\|q\|\langle q,p\rangle\|p\|^2 - \nu^{-1}\langle q,p\rangle + \nu^{-1}\langle q,p\rangle - \nu^{-1}\|q\|\langle q,p\rangle\|p\|^2 = 0; && (69b) \\
\langle v,v\rangle &= \|q\|^{-2}\|q\|^2 - 2\|q\|^{-1}\langle q,p\rangle^2 + \|p\|\langle q,p\rangle^2 + \nu^{-2}\langle q,p\rangle^2 \\
&= 1 - 2\|q\|^{-1}\langle q,p\rangle^2 + \|p\|^2\langle q,p\rangle^2 + 2\|q\|^{-1}\langle q,p\rangle^2 - \|p\|^2\langle q,p\rangle^2 = 1. && (69c)
\end{aligned}
$$

Thus $F_M(\Sigma_-) \subseteq T_1S^3$. Suppose that $u_4 = 1$. Then $\|q\|\|p\|^2 = 2$ using (62). So $-2H(q,p) = \|q\|^{-1}(2 - \|q\|\|p\|^2) = 0$, which contradicts the fact that $H(q,p) < 0$. Consequently, $S(\Sigma_-) \subseteq T_1S^3_{\mathrm{np}} \times \mathbf{R}_{>0}$, since $\nu(q,p) = (-2H(q,p))^{-1/2} > 0$ for $(q,p) \in \Sigma_-$. Diagram 4.2.1 is commutative because $\nu(\widehat{q},\widehat{p}) = 1$ and

$$
\begin{aligned}
\Phi(\widehat{q},\widehat{p}) = \Phi(\nu^{-2}q, \nu p) &= \big((\|\nu^{-2}q\|\nu p, \|\nu p\|^2\|\nu^{-2}q\| - 1), \\
&\qquad (-\|\nu^{-2}q\|^{-1}\nu^{-2}q + \langle \nu^{-2}q, \nu p\rangle \nu p, -\langle \nu^{-2}q, \nu p\rangle)\big) \\
&= \big((\nu^{-1}\|q\|p, \|p\|^2\|q\| - 1), (-\|q\|^{-1}q, +\langle q,p\rangle p, \nu^{-1}\langle q,p\rangle)\big) = F_M(q,p).
\end{aligned}
$$

From ((4.6)) it follows that the mapping s in diagram 4.2.1 is a diffeomorphism.

We have not yet shown that the mapping S is an $\mathbf{R}_{>0}$-bundle isomorphism. To do this, and
$\triangleright$ thus finish proving the ((4.14)), we must show that the mapping S is a fiber preserving $\mathbf{R}_{>0}$-isomorphism and is a diffeomorphism. This assertion follows when we establish:

i) For every $(\widehat{q},\widehat{p}) \in (H|\Sigma_-)^{-1}(-\frac{1}{2})$ and for every $\rho > 0$ we have $S\big(\Psi^V_{\rho^{-1}}(\widehat{q},\widehat{p})\big)$ $= \Psi^T_{\rho^{-1}}\big(s(\widehat{q},\widehat{p})\big)$.

ii) The mapping $S|\pi_V^{-1}(\widehat{q},\widehat{p})$ from the fiber $\pi_V^{-1}(\widehat{q},\widehat{p})$ to the fiber $\pi_T^{-1}\big(s(\widehat{q},\widehat{p})\big)$ is one to one and onto.

(4.15) **Proof**:

i) We compute

$$
\nu(\Psi^V_{\rho^{-1}}(\widehat{q},\widehat{p})) = \nu(\rho^{-2}\widehat{q}, \rho\widehat{p}) = \big(\rho^2(-2H(\widehat{q},\widehat{p}))\big)^{-1/2} = \rho^{-1};
$$

while $F_M(\Psi^V_{\rho^{-1}}(\widehat{q},\widehat{p})) = F_M(\rho^{-2}\widehat{q}, \rho\widehat{p}) = \Phi(\widehat{q},\widehat{p})$. Therefore

$$
S\big(\Psi^V_{\rho^{-1}}(\widehat{q},\widehat{p})\big) = \big(\Phi(\widehat{q},\widehat{p}), \rho^{-1}\big) = \Psi^T_{\rho^{-1}}\big(s(\widehat{q},\widehat{p})\big).
$$

ii) Suppose that $S\big(\Psi^V_{\rho^{-1}}(\widehat{q},\widehat{p})\big) = S\big(\Psi^V_{\sigma^{-1}}(\widehat{q},\widehat{p})\big)$ for some ρ, $\sigma \in \mathbf{R}_{>0}$. Then $\rho = \sigma$, because $\big(\Phi(\widehat{q},\widehat{p}),\rho^{-1}\big) = \big(\Phi(\widehat{q},\widehat{p}),\sigma^{-1}\big)$ by hypothesis. Therefore we get $\Psi^V_{\rho^{-1}}(\widehat{q},\widehat{p}) = \Psi^V_{\sigma^{-1}}(\widehat{q},\widehat{p})$, which shows that $S|\pi_V^{-1}(\widehat{q},\widehat{p})$ is an injective map from the fiber $\pi_V^{-1}(\widehat{q},\widehat{p})$ into the fiber $\pi_T^{-1}(s(\widehat{q},\widehat{p}))$. Suppose that $(\widehat{Q},\widehat{P},\lambda) \in \pi_T^{-1}(s(\widehat{q},\widehat{p}))$. Then

$$(\widehat{Q},\widehat{P},\lambda) = \Psi^V_\lambda\big(\Phi(\widehat{q},\widehat{p}),1\big) = \big(\Phi(\widehat{q},\widehat{p}),\lambda\big) = S(\lambda^2\widehat{q},\lambda^{-1}\widehat{p}).$$

Thus S maps the fiber $\pi_V^{-1}(\widehat{q},\widehat{p})$ onto the fiber $\pi_T^{-1}\big(s(\widehat{q},\widehat{p})\big)$. □

▷ To show that the mapping $S : \Sigma_- \subseteq T_0\mathbf{R}^3 \rightarrow T_1S^3_{\mathrm{np}} \times \mathbf{R}_{>0}$ (63a) is a diffeomorphism we argue as follows.

(4.16) **Proof**: Suppose that $(u,v,\nu) \in T_1S^3_{\mathrm{np}} \times \mathbf{R}_{>0}$. Then $\pi_T(u,v,\nu) = (u,v,1) \in T_1S^3_{\mathrm{np}}$. Since s is a diffeomorphism, there is a unique $(\widehat{q},\widehat{p}) \in (H|\Sigma_-)^{-1}(-\frac{1}{2})$ such that $(\widehat{q},\widehat{p}) = s^{-1}(u,v,1)$. Because S maps the fiber $\pi_V^{-1}(\widehat{q},\widehat{p})$ one to one and onto the fiber $\pi_T^{-1}(u,v,1)$ and the $\mathbf{R}_{>0}$-action Ψ^T is free, there is a unique $\rho \in \mathbf{R}_{>0}$ such that $S(q,p) = S(\Psi^V_\rho(\widehat{q},\widehat{p})) = (u,v,\nu)$. Thus the mapping S is one to one and onto. The next argument shows that for every $(q,p) \in \Sigma_-$ the tangent $T_{(q,p)}S : T_{(q,p)}\Sigma_- \rightarrow T_{S(q,p)}(T_1S^3_{\mathrm{np}} \times \mathbf{R}_{>0})$ of the mapping S is surjective. Suppose that $\gamma : [0,1] \rightarrow T_1S^3_{\mathrm{np}} \times \mathbf{R}_{>0}$ is a smooth curve, which passes through (u,v,ν) at $t = 0$. Then $\pi_T\circ\gamma$ is a smooth curve on $T_1S^3_{\mathrm{np}} \times \{1\}$, which passes through $(u,v,1)$ at $t = 0$. Since s is a diffeomorphism of $(H|\Sigma_-)^{-1}(-\frac{1}{2})$ onto $T_1S^3_{\mathrm{np}} \times \{1\}$, the curve $s^{-1}\circ(\pi_T\circ\gamma)$ on $(H|\Sigma_-)^{-1}(-\frac{1}{2})$ is smooth and passes through $(\widehat{q},\widehat{p})$ at $t = 0$. Therefore $\Psi^V_\rho\circ(s^{-1}\circ(\pi_T\circ\gamma))$ is a smooth curve on Σ_- which passes through (q,p). Consequently, $T_{(q,p)}S$ is surjective. Because $\dim\Sigma_- = \dim(T_1S^3_{\mathrm{np}} \times \mathbf{R}_{>0})$, it follows that $T_{(q,p)}S$ is bijective. In other words, S is a local diffeomorphism. Thus S is a global diffeomorphism since it is injective. □

Thus S is an isomorphism of $\mathbf{R}_{>0}$-bundles and this completes the proof of ((4.14)). □

Claim: Consider the 1-form $\theta = \nu(\langle v,\mathrm{d}u\rangle + \mathrm{d}v_4)|M$ on $M = T_1S^3_{\mathrm{np}} \times \mathbf{R}_{>0} \subseteq T\mathbf{R}^4 \times \mathbf{R}$ with coordinates (u,v,ν). Then

$$S^*\big((\nu\langle v,\mathrm{d}u\rangle)|M\big) = \big(\nu(q,p)\langle q,p\rangle\,\mathrm{d}(\nu(q,p)^{-1}) - \langle q,\mathrm{d}p\rangle\big)|\Sigma_- \tag{70a}$$

$$S^*\big((\nu\,\mathrm{d}v_4)|M\big) = \big(\nu(q,p)\,\mathrm{d}(\nu(q,p)^{-1}\langle q,p\rangle)\big)|\Sigma_-, \tag{70b}$$

that is,

$$S^*\theta = -\big(\langle q,\mathrm{d}p\rangle + \mathrm{d}\langle q,p\rangle\big)|\Sigma_-. \tag{70c}$$

(4.17) **Proof**: Equation (70c) follows from equations (70a) and (70b) because

$$\begin{aligned} S^*(\nu\langle v,\mathrm{d}u\rangle + \nu\,\mathrm{d}v_4) &= \nu(q,p)\langle q,p\rangle\,\mathrm{d}\nu(q,p)^{-1} - \langle q,\mathrm{d}\,p\rangle \\ &\qquad - \nu(q,p)\langle q,p\rangle\,\mathrm{d}\nu(q,p)^{-1} - \mathrm{d}\langle q,p\rangle \\ &= -\langle q,\mathrm{d}p\rangle - \mathrm{d}\langle q,p\rangle. \end{aligned}$$

The following calculation verifies equation (70a). We have

$$S^*(\langle v,\mathrm{d}u\rangle) = \big\langle\big(-\|q\|^{-1}q + \langle q,p\rangle p, -\nu^{-1}\langle q,p\rangle\big),\big(\mathrm{d}(\nu^{-1}\|q\|p),\mathrm{d}(\|q\|\|p\|^2)\big)\big\rangle$$

$$
\begin{aligned}
&= \nu^{-2}\langle q,p\rangle\|q\|(\|q\|^{-1}-\|p\|^2)\,\mathrm{d}\nu - \nu^{-1}\langle q,p\rangle\|q\|(\langle p,\mathrm{d}p\rangle + \|q\|^{-3}\langle q,\mathrm{d}q\rangle)\\
&\qquad - \nu^{-1}\langle q,\mathrm{d}p\rangle\\
&= \nu^{-1}\langle q,p\rangle\|q\|\big(-\|q\|^{-1}\nu^{-1}\,\mathrm{d}\nu + (-2H)(-2H)^{-1}\,\mathrm{d}H\big)\\
&\qquad - \nu^{-1}\langle q,p\rangle\|q\|^{-1}\,\mathrm{d}H - \nu^{-1}\langle q,\mathrm{d}p\rangle\\
&\qquad\quad \text{since } \nu = (-2H)^{-1/2} \text{ implies } \mathrm{d}\nu = (-2H)^{-3/2}\,\mathrm{d}H =\\
&\qquad\quad \nu(-2H)^{-1}\,\mathrm{d}H \text{ and } \mathrm{d}H = \langle p,\mathrm{d}p\rangle + \|q\|^{-3}\langle q,\mathrm{d}q\rangle\\
&= \langle q,p\rangle\,\mathrm{d}\nu^{-1} - \nu^{-1}\langle q,\mathrm{d}p\rangle.
\end{aligned}
$$

On the right hand side of the above equations we have used the abbreviation ν for $\nu(q,p)$. Since $S^*v_4 = -\nu(q,p)^{-1}\langle q,p\rangle$, we obtain equation (70b). □

Corollary: The mapping S is a symplectic diffeomorphism sending Σ_- to M with $S^*(\mathrm{d}\theta) = (\sum_{i=1}^{3}\mathrm{d}q_i\wedge\mathrm{d}p_i)|\Sigma_-$.

(4.18) **Proof**: Take the exterior derivative of both sides of (70c). □

We now look at the factor L (63b) of the Ligon-Schaaf mapping Φ_{LS} (61). On T^+S^3 we have an $\mathbf{R}_{>0}$-action

$$\Psi^D : \mathbf{R}_{>0}\times T^+S^3 \to T^+S^3 : \big(\mu,(r,s)\big) \mapsto (r,\mu s) \tag{71}$$

of a *scaling group*. The action Ψ^D is free, for if $(r,\mu s) = (r,s)$, then $\mu = 1$. Because every orbit of the action Ψ^D intersects T_1S^3 exactly once, the orbit space $T^+S^3/\mathbf{R}_{>0}$ is diffeomorphic to T_1S^3, the unit tangent sphere bundle to S^3, with orbit mapping

$$\pi_D : T^+S^3 \to T_1S^3 : (r,s) \mapsto (r,\|s\|^{-1}s).$$

Claim: Using the mapping

$$L : T_1S^3\times\mathbf{R}_{>0} \to T^+S^3 : (u,v,\nu) \mapsto \big(r(u,v,\nu), s(u,v,\nu)\big),$$

where

$$
\begin{aligned}
r(u,v,\nu) &= \widetilde{r}(u,v) = \cos v_4\,u + \sin v_4\,v\\
s(u,v,\nu) &= \nu\widetilde{s}(u,v) = \nu(-\sin v_4\,u + \cos v_4\,v),
\end{aligned}
\tag{72}
$$

diagram 4.2.2 is commutative

$$
\begin{array}{ccccccc}
T_1S^3\times\mathbf{R}_{>0} & \xrightarrow{\;L\;} & T^+S^3 & \qquad & (u,v,\nu) & \longrightarrow & \big(r(u,v,\nu),s(u,v,\nu)\big)\\
\Big\downarrow{\scriptstyle \pi_T} & & \Big\downarrow{\scriptstyle \pi_D} & & \Big\downarrow & & \Big\downarrow\\
T_1S^3\times\{1\} & \xrightarrow{\;\ell\;} & T_1S^3 & & (u,v,1) & \longrightarrow & \big(\widetilde{r}(u,v),\widetilde{s}(u,v)\big)
\end{array}
$$

Diagram 4.2.2

Moreover, the mapping L is an $\mathbf{R}_{>0}$-bundle isomorphism.

(4.19) **Proof**: The maps in diagram 4.2.2 are properly defined, because if $(u,v) \in T_1S^3$, then for $(\widetilde{r}, \widetilde{s}) = \ell(u,v)$ we have $\|\widetilde{r}\|^2 = 1 = \|\widetilde{s}\|^2$ and $\langle \widetilde{r}, \widetilde{s} \rangle = 0$. So $(r,s) = (\widetilde{r}, \nu\widetilde{s}) = L(u,v,\nu) \in T^+S^3$. From (72) it follows that $\widetilde{r}(u,v) = r(u,v,1)$ and $\widetilde{s}(u,v) = s(u,v,1)$. Consequently, diagram 4.2.2 is commutative.

As a first step toward verifying that the mapping L is an $\mathbf{R}_{>0}$-bundle isomorphism, we
▷ will show that the mapping

$$\ell : T_1S^3 \times \{1\} = T_1S^3 \to T_1S^3 : (u,v) \mapsto \begin{pmatrix} \widetilde{r} \\ \widetilde{s} \end{pmatrix} = T(v_4)\begin{pmatrix} u \\ v \end{pmatrix} \tag{73}$$

with $T(v_4) = \begin{pmatrix} \cos v_4 & \sin v_4 \\ -\sin v_4 & \cos v_4 \end{pmatrix}$ is a diffeomorphism.

(4.20) **Proof**: We start by showing that for every $(u,v) \in T_1S^3$ the tangent mapping $T_{(u,v)}\ell$ of ℓ is a bijective linear mapping of $T_{(u,v)}(T_1S^3)$ into itself.

(4.21) **Proof**: Differentiating (73) gives

$$\begin{pmatrix} \dot{w} \\ \dot{z} \end{pmatrix} = T_{(u,v)}\ell\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = T(v_4)\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} + \dot{v}_4 T(v_4)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = T(v_4)\left[\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} + \dot{v}_4\begin{pmatrix} v \\ -u \end{pmatrix}\right] \tag{74}$$

for $(\dot{u}, \dot{v}) \in T_{(u,v)}(T_1S^3)$ and $\dot{v}_4 \in \mathbf{R}$. Since $T_{(u,v)}(T_1S^3) = \{(\dot{x}, \dot{y}) \in T\mathbf{R}^4 \,\big|\, \langle u, \dot{x} \rangle = 0 = \langle v, \dot{y} \rangle \;\&\; \langle u, \dot{y} \rangle + \langle \dot{x}, v \rangle = 0\}$, the next calculation shows that $(\dot{x}, \dot{y}) = (v, -u) \in T_{(u,v)}(T_1S^3)$.

$$\langle u, \dot{x} \rangle = \langle u, v \rangle = 0, \;\; \langle v, \dot{y} \rangle = -\langle v, u \rangle = 0, \text{ and } \langle u, \dot{y} \rangle + \langle \dot{x}, v \rangle = -\langle u, u \rangle + \langle v, v \rangle = 1 - 1 = 0.$$

Therefore $(\dot{w}, \dot{z})$ given by (74) lies in $T_{(u,v)}(T_1S^3)$. Let $(\dot{x}, \dot{y}) \in T_{(u,v)}(T_1S^3)$. Set $\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = T(v_4)^{-1}\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} - \dot{v}_4\begin{pmatrix} v \\ -u \end{pmatrix}$. Then $(\dot{u}, \dot{v}) \in T_{(u,v)}(T_1S^3)$. Using (74) we get

$$T_{(u,v)}\ell\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = T(v_4)\left(T(v_4)^{-1}\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} - \dot{v}_4\begin{pmatrix} v \\ -u \end{pmatrix}\right) + \dot{v}_4 T(v_4)\begin{pmatrix} v \\ -u \end{pmatrix} = \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix}.$$

Thus $T_{(u,v)}\ell$ is a surjective, and so bijective, linear mapping of $T_{(u,v)}(T_1S^3)$ into itself. □

We now show that the mapping ℓ (73) is a diffeomorphism of T_1S^3 into itself. Since ℓ is smooth and its tangent map is bijective at every point of T_1S^3, from the inverse function theorem it follows that ℓ is a local diffeomorphism. To show that ℓ is a global diffeomorphism it suffices to demonstrate that it is injective. For $s \in [0,1]$ let $\ell^s : T_1S^3 \to T_1S^3 : \begin{pmatrix} u \\ v \end{pmatrix} \mapsto T(sv_4)\begin{pmatrix} u \\ v \end{pmatrix}$. Since $\ell^0 = \mathrm{id}_{T_1S^3}$ and $\ell^1 = \ell$, it follows that ℓ is homotopic to $\mathrm{id}_{T_1S^3}$, whose degree is 1. Hence the degree $\deg \ell$ of ℓ is 1. Induce an orientation on T_1S^3 from the standard orientation of $T\mathbf{R}^4 = \mathbf{R}^8$. For every $(u,v) \in T_1S^3$ the map $T_{(u,v)}\ell$ is bijective and orientation preserving, because $T(v_4) \in \mathrm{SO}(4, \mathbf{R})$. Since T_1S^3 is connected and compact, the mapping ℓ is surjective, being an open mapping. Therefore, every $(x,y) \in T_1S^3$ is a regular value of ℓ. The fiber $F = \ell^{-1}(x,y)$ is a finite set, because it is a discrete closed subset of a compact set. From the definition of degree of smooth mapping we have $1 = \deg \ell = \sum_{p \in F} 1$, see exercise 4 of chapter I. Therefore F has only one element. In other words, the mapping ℓ is injective. This proves ((4.20)). □

To finish the proof that the mapping L (72) is an $\mathbf{R}_{>0}$-bundle isomorphism, we need to show that L is a diffeomorphism and a fiber preserving $\mathbf{R}_{>0}$-isomorphism. We establish the latter assertion by verifying

i) For every $(u,v,1) \in T_1S^3 \times \mathbf{R}_{>0}$ and every $\rho > 0$ we have $L(\Psi^T_\rho(u,v,1)) = \Psi^D_\rho\big(\ell(u,v,1)\big)$.

ii) The mapping $L|\pi_T^{-1}(u,v,1)$ from the fiber $\pi_T^{-1}(u,v,1)$ to the fiber $\pi_D^{-1}\big(\ell(u,v,1)\big)$ is one to one and onto.

(4.22) **Proof**:

i) We compute

$$L\big(\Psi^T_\rho(u,v,1)\big) = L(u,v,\rho) = \big(r(u,v,\rho), s(u,v,\rho)\big) = \big(\widetilde{r}(u,v), \rho\widetilde{s}(u,v)\big)$$
$$= \Psi^D_\rho\big(\widetilde{r}(u,v), \widetilde{s}(u,v)\big) = \Psi^D_\rho\big(\ell(u,v,1)\big).$$

ii) Suppose that $L\big(\Psi^T_\rho(u,v,1)\big) = L\big(\Psi^T_\sigma(u,v,1)\big)$ for some ρ, $\sigma \in \mathbf{R}_{>0}$. Then $\big(\widetilde{r}(u,v), \rho\widetilde{s}(u,v)\big) = \big(\widetilde{r}(u,v), \sigma\widetilde{s}(u,v)\big)$, which implies $\rho = \sigma$. Therefore $\Psi^T_\rho(u,v,1) = \Psi^T_\sigma(u,v,1)$. So $L|\pi_T^{-1}(u,v,1)$ is an injective map from the fiber $\pi_T^{-1}(u,v,1)$ to the fiber $\pi_D^{-1}\big(\ell(u,v,1)\big)$. Suppose that $(\widetilde{R},\widetilde{S},\lambda) \in \pi_D^{-1}\big(\ell(u,v,1)\big)$. Then

$$(\widetilde{R},\widetilde{S},\lambda) = \Psi^D_\lambda\big(\ell(u,v,1)\big) = \big(\widetilde{r}(u,v), \lambda\widetilde{s}(u,v)\big) = \big(r(u,v,\lambda), s(u,v,\lambda)\big) = L(u,v,\lambda).$$

Thus $L|\pi_T^{-1}(u,v,1)$ maps the fiber $\pi_T^{-1}(u,v,1)$ onto the fiber $\pi_D^{-1}\big(\ell(u,v,1)\big)$. □

▷ The mapping L is a diffeomorphism.

(4.23) **Proof**: The argument is similar to the proof of ((4.16)). We include the details. Suppose that $(r,s) \in T^+S^3$. Then $\pi_D(r,s) = (\widetilde{r},\widetilde{s}) \in T_1S^3$. Since the mapping ℓ is a diffeomorphism, there is a unique $(u,v,1) \in T_1S^3 \times \{1\}$ such that $\ell^{-1}(\widetilde{r},\widetilde{s}) = (u,v,1)$. Because L maps the fiber $\pi_T^{-1}(u,v,1)$ one to one and onto the fiber $\pi_D^{-1}(\ell(u,v,1))$, there is a unique $\nu \in \mathbf{R}_{>0}$ such that $L(u,v,\nu) = (r,s)$, since the $\mathbf{R}_{>0}$-action Ψ^T is free. Consequently, the mapping L is one to one and onto. The next argument show that for every $(u,v,\nu) \in T_1S^3 \times \mathbf{R}_{>0}$ the tangent $T_{(u,v,\nu)} : T_{(u,v,\nu)}(T_1S^3 \times \mathbf{R}_{>0}) \to T_{L(u,v,\nu)}(T^+S^3)$ is surjective. Suppose that $\gamma : [0,1] \to T^+S^3 : t \mapsto \gamma(t)$ is a smooth curve in T^+S^3, which passes through $(r,s) = L(u,v,\nu)$. Then $\pi_D \circ \gamma$ is a smooth curve in T_1S^3 which passes through $(\widetilde{r},\widetilde{s})$ at $t = 0$. Since $\ell : T_1S^3 \times \{1\} \to T_1S^3$ is a diffeomorphism, $\ell^{-1} \circ \pi_D \circ \gamma$ is a smooth curve on $T_1S^3 \times \mathbf{R}_{>0}$, which passes through $(u,v,1) = \ell^{-1}(\widetilde{r},\widetilde{s})$ at $t = 0$. Now $L^{-1}(r,s) = (u,v,\nu)$ for some unique $\nu \in \mathbf{R}_{>0}$, since the action Ψ^T is free. So the smooth curve $\Psi^V_\nu \circ \ell^{-1} \circ \pi_D \circ \gamma$ passes through (u,v,ν) at time $t = 0$. Thus $T_{(u,v,\nu)}L$ is surjective. Because $\dim T_{(u,v,\nu)}(T_1S^3 \times \mathbf{R}_{>0}) = \dim T_{L(u,v,\nu)}(T^+S^3)$, the tangent map $T_{(u,v,\nu)}L$ is injective and hence is bijective. Therefore L is a local diffeomorphism. Because L is one to one, it is a global diffeomorphism. □

Thus L is an isomorphism of $\mathbf{R}_{>0}$-bundles and this finishes the proof of ((4.19)). □

To finish the proof of ((4.11)) we show that the image of the Ligon-Schaaf mapping Φ_{LS} (61) is $T^+S^3_{\mathrm{np}}$. To do this we need to show that L maps $T_1S^3_{\mathrm{np}}$ onto $T^+S^3_{\mathrm{np}}$.

(4.24) **Proof**: Suppose that for some $(u,v,\nu) \in T_1S^3_{\mathrm{np}} \times \mathbf{R}_{>0}$ we have $L(u,v,\nu) = (\mathrm{np},s) \in T^+S^3 \subseteq T\mathbf{R}^4$. Here $\mathrm{np} = (0,0,0,1) \in \mathbf{R}^4$. Then $0 = \langle \mathrm{np}, s\rangle = s_4$. So $L(u,v,\nu) = T_\nu(v_4)\begin{pmatrix} u \\ v \end{pmatrix} = \big(\mathrm{np},(\widehat{s},0)\big)^t$, where $T_\nu(v_4) = \begin{pmatrix} \cos v_4 & \sin v_4 \\ -\nu \sin v_4 & \nu\cos v_4 \end{pmatrix}$. Set $\widehat{u} = \mathrm{np}$, $\widehat{v} = (\widetilde{v}, v_4) = (\widehat{s},0)$, and $\widehat{\nu} = 1$. Then $L(\widehat{u},\widehat{v},\widehat{\nu}) = T_1(0)\begin{pmatrix} \widehat{u} \\ \widehat{v} \end{pmatrix} = \big(\mathrm{np},(\widehat{s},0)\big)^t$. But the mapping L is one to one. Thus $(\widehat{u},\widehat{v},1) = (\mathrm{np},(\widehat{s},0),1)$ is the only point of $T_1S^3 \times \mathbf{R}_{>0}$ which maps to $(\mathrm{np},(\widehat{s},0))$ in T^+S^3 under L. This proves the assertion. □

▷ Consider the 1-form $\langle r, \mathrm{d}s\rangle | T^+S^3$ on $T^+S^3 \subseteq T\mathbf{R}^4$ with coordinates (r,s). Then

$$L^*\big(\langle r,\mathrm{d}s\rangle | T^+S^3\big) = -\nu\big(\nu\,\mathrm{d}v_4 + \nu\langle v, \mathrm{d}u\rangle\big)|(T_1S^3 \times \mathbf{R}_{>0}). \tag{75}$$

(4.25) **Proof**: Using the definition of the mapping L (72) we get $\mathrm{d}(L^*s) =$

$$= \nu\,[(-\cos v_4\,\mathrm{d}v_4)u - \sin v_4\,\mathrm{d}u - (\sin v_4\,\mathrm{d}v_4)v + \cos v_4\,\mathrm{d}v] + \big(-\sin v_4\,u + \cos v_4, v\big)\,\mathrm{d}\nu.$$

So

$$\begin{aligned} L^*(\langle r,\mathrm{d}s\rangle) &= \langle L^*r, \mathrm{d}(L^*s)\rangle \\ &= \big(-\sin v_4\cos v_4\langle u,u\rangle + \cos^2 v_4\langle u,v\rangle - \sin^2 v_4\langle v,u\rangle + \sin v_4\cos v_4\langle v,v\rangle\big)\,\mathrm{d}\nu \\ &\qquad + \nu\big(-\cos^2 v_4\langle u,u\rangle - \cos v_4\sin v_4\langle u,v\rangle - \cos v_4\sin v_4\langle v,u\rangle - \sin^2 v_4\langle v,v\rangle\big)\,\mathrm{d}v_4 \\ &\qquad\qquad - \cos v_4\sin v_4\langle u,\mathrm{d}u\rangle + \cos^2 v_4\langle u,\mathrm{d}v\rangle - \sin^2 v_4\langle v,\mathrm{d}u\rangle + \cos v_4\sin v_4\langle v,\mathrm{d}v\rangle \\ &= -\nu\,\mathrm{d}v_4 - \langle v,\mathrm{d}u\rangle. \end{aligned}$$

The last equality above follows because $\langle u,u\rangle = \langle v,v\rangle = 1$ and $\langle u,v\rangle = 0$, which implies $\langle u,\mathrm{d}u\rangle = \langle v,\mathrm{d}v\rangle = 0$ and $\langle u,\mathrm{d}v\rangle + \langle v,\mathrm{d}u\rangle = 0$. □

Corollary: L is a symplectic diffeomorphism sending $M = T_1S^3_{\mathrm{np}} \times \mathbf{R}_{>0}$ to $T^+S^3_{\mathrm{np}}$ with $L^*\big((\sum_{j=1}^4 \mathrm{d}r_j \wedge \mathrm{d}s_j)|T^+S^3_{\mathrm{np}}\big) = \mathrm{d}\theta$, where $\theta = \nu\big(\langle v,\mathrm{d}u\rangle + \mathrm{d}v_4\big)|M$.

(4.26) **Proof**: Take the exterior derivative of both sides of (75). □

Claim: The Ligon-Schaaf map $\Phi_{LS} : \Sigma_- \subseteq T_0\mathbf{R}^3 \to T^+S^3_{\mathrm{np}} \subseteq T\mathbf{R}^4$ (61) has the following properties.

1. It is a symplectic diffeomorphism of $\big(\Sigma_-, (\sum_{i=1}^3 \mathrm{d}q_i \wedge \mathrm{d}p_i)|\Sigma_-\big)$ onto $\big(T^+S^3_{\mathrm{np}}, (\sum_{j=1}^4 \mathrm{d}r_j \wedge \mathrm{d}s_j)|T^+S^3_{\mathrm{np}}\big)$.
2. It pulls back the *Delaunay Hamiltonian* $\mathscr{H} : T^+S^3_{\mathrm{np}} \subseteq T\mathbf{R}^4 \to \mathbf{R} : (r,s) \mapsto -\frac{1}{2}\|s\|^{-2}$ to the Kepler Hamiltonian $H : T_0\mathbf{R}^3 \to \mathbf{R} : (q,p) \mapsto \frac{1}{2}\|p\|^2 - \|q\|^{-1}$.
3. It pulls back the *Delaunay vector field* $X_{\mathscr{H}}$ on $T^+S^3_{\mathrm{np}}$, whose integral curves satisfy

$$\begin{aligned} \frac{\mathrm{d}r}{\mathrm{d}t} &= \|s\|^{-4}s \\ \frac{\mathrm{d}s}{\mathrm{d}t} &= -\|s\|^{-2}r, \end{aligned} \tag{76}$$

and whose flow is

$$\varphi_t^{\mathscr{H}}\begin{pmatrix} r \\ s \end{pmatrix} = \begin{pmatrix} \cos\nu^{-3}t & \nu^{-1}\sin\nu^{-3}t \\ -\nu\sin\nu^{-3}t & \cos\nu^{-3}t \end{pmatrix}\begin{pmatrix} r \\ s \end{pmatrix} \tag{77}$$

with $\nu = \|s\|$, to the Kepler vector field X_H on Σ_-.

4. It intertwines the SO(3)-momentum mapping

$$J : T_0\mathbf{R}^3 \to \mathbf{R}^3 : (q,p) \mapsto p \times q, \tag{78}$$

with the SO(4)-momentum mapping

$$\mathscr{J} : T^+S^3 \subseteq T\mathbf{R}^4 \to \bigwedge^2 \mathbf{R}^4 : (r,s) \mapsto r \wedge s = \sum_{1\le i<j\le 4} (r_is_j - r_js_i)e_i \wedge e_j, \tag{79}$$

that is, $J = \Phi_{LS}^*\mathscr{J}$. Here $\bigwedge^2\mathbf{R}^4$ is identified with so(4) via the mapping which sends $e_i \wedge e_j$ to the 4×4 skew symmetric matrix e_{ij}, whose ij^{th} entry is 1, whose ji^{th} entry is -1, and whose other entries are 0.

(4.27) **Proof**:

1. As Φ_{LS} is the composition of the mappings $S : T_0\mathbf{R}^3 \to T_1S^3_{\mathrm{np}} \times \mathbf{R}_{>0}$ and $L : T_1S^3 \times \mathbf{R}_{>0} \to T^+S^3$, each of which are symplectic diffeomorphisms with $S^*(\mathrm{d}\theta) = (\sum_{i=1}^3 \mathrm{d}q_i \wedge \mathrm{d}p_i)|T_0\mathbf{R}^3$ ((4.18)) and $L^*\big((\sum_{j=1}^4 \mathrm{d}r_j \wedge \mathrm{d}s_j)|T^+S^3_{\mathrm{np}}\big) = \mathrm{d}\,\theta$ ((4.25)), it follows that Φ_{LS} is a symplectic diffeomorphism from $\big(T_0\mathbf{R}^3, (\sum_{i=1}^3 \mathrm{d}q_i \wedge \mathrm{d}p_i)|T_0\mathbf{R}^3\big)$ onto $\big(T^+S^3_{\mathrm{np}}, (\sum_{j=1}^4 \mathrm{d}r_j \wedge \mathrm{d}s_j)|T^+S^3_{\mathrm{np}}\big)$.

2. We compute. From the definition of the mapping S (63a) we have $S^*(-\frac{1}{2}\nu^{-2}) = -\frac{1}{2}\nu(q,p)^{-2} = H(q,p)$; while from the definition of the mapping L (63b) we have $L^*(-\frac{1}{2}\|s\|^{-2}) = -\frac{1}{2}\nu^{-2}$. Therefore $\Phi_{LS}^*\mathscr{H} = H$.

3. Since the Ligon-Schaaf mapping Φ_{LS} exhibits an equivalence between the Kepler Hamiltonian system $\big(H, T_0\mathbf{R}^3, (\sum_{i=1}^3 \mathrm{d}q_i \wedge \mathrm{d}p_i)|T_0\mathbf{R}^3\big)$ and the Delaunay Hamiltonian system $\big(\mathscr{H}, T^+S^3_{\mathrm{np}},\ (\sum_{j=1}^4 \mathrm{d}r_j \wedge \mathrm{d}s_j)|T^+S^3_{\mathrm{np}}\big)$, it pulls back the Delaunay vector field $X_{\mathscr{H}}$ on $T^+S^3_{\mathrm{np}}$ to the Kepler vector field X_H on Σ_-, that is, $\Phi_{LS}^*X_{\mathscr{H}} = X_H$. To find formula (76) for the Delaunay vector field $X_{\mathscr{H}}$ on T^+S^3, we look at the Hamiltonian system $\big(\widetilde{\mathscr{H}}, T\mathbf{R}^4, \sum_{j=1}^4 \mathrm{d}r_j \wedge \mathrm{d}s_j\big)$ with Hamiltonian $\widetilde{\mathscr{H}}(r,s) = -\frac{1}{2}\langle s,s\rangle^{-1}$ constrained to TS^3 with constraint functions $c_1(r,s) = \frac{1}{2}(\langle r,r\rangle - 1)$ and $c_2(r,s) = \langle r,s\rangle$. Since the matrix $(\{c_i,c_j\})$ of Poisson brackets is invertible with inverse given by $(C_{ij}) = \langle r,r\rangle^{-1}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, the manifold TS^3 is cosymplectic with symplectic form $(\sum_{j=1}^4 \mathrm{d}r_j \wedge \mathrm{d}s_j)|TS^3$. We compute the constrained equations of motion using the modified Dirac bracket procedure. Let

$$H^* = \widetilde{\mathscr{H}} - \sum_{i,j=1}^{2} (\{\widetilde{\mathscr{H}}, c_i\} + \widetilde{\mathscr{H}_i})C_{ij}c_j,$$

where $\widetilde{\mathscr{H}_1} = \langle r,s\rangle(\langle s,s\rangle^{-2} - \frac{1}{2}\langle r,r\rangle)$, and $\widetilde{\mathscr{H}_2} = \langle s,s\rangle^{-1}(\langle r,r\rangle - 1)$. Then

$$\begin{aligned} H^* &= -\tfrac{1}{2}\langle s,s\rangle^{-1} - \langle r,r\rangle^{-1}\big\langle\big(-\langle r,r\rangle\langle s,s\rangle^{-1}+\widetilde{\mathscr{H}_1}, \langle s,s\rangle^{-1}+\widetilde{\mathscr{H}_2}\big), \big(-\langle r,s\rangle, \tfrac{1}{2}(\langle r,r\rangle-1)\big)\big\rangle \\ &= -\tfrac{1}{2}\langle s,s\rangle^{-1} - \langle r,s\rangle^2 + \tfrac{1}{2}\langle s,s\rangle^{-1}(\langle r,r\rangle-1). \end{aligned}$$

So

$$\begin{aligned} \frac{\mathrm{d}r}{\mathrm{d}t} &= \frac{\partial H^*}{\partial s} = \langle s,s\rangle^{-2}s - \langle r,s\rangle s + \langle s,s\rangle^{-1}(\langle r,r\rangle-1) \\ \frac{\mathrm{d}s}{\mathrm{d}t} &= -\frac{\partial H^*}{\partial r} = -\langle s,s\rangle^{-1}r + \langle r,s\rangle s. \end{aligned}$$

Therefore the Delaunay vector field $X_{\mathscr{H}} = X_{H^*}|T^+S^3$ has integral curves which satisfy (76). It is straightforward to verify that $\varphi_t^{\mathscr{H}}$ given by (77) is the flow of $X_{\mathscr{H}}$. Comparing (77) with the geodesic flow (8) in §1, one sees that the Delaunay flow is the same as the geodesic flow with time parameter $t = \nu^3 s$, where s is the geodesic time parameter.
4. Since

$$L^*(r\wedge s) = (\cos v_4\, u + \sin v_4\, v)\wedge(-\nu\sin v_4\, u + \nu\cos v_4\, v) = \nu\, u\wedge v,$$

we obtain $L^*\big((r\wedge s)|\mathscr{H}^{-1}(-\frac{1}{2})\big) = (\nu\, u\wedge v)|K^{-1}(\frac{1}{2})$, see (59). Consequently,

$$\Phi^*\big((r\wedge s)|\mathscr{H}^{-1}(-\tfrac{1}{2})\big) = S^*\big((\nu\, u\wedge v)|K^{-1}(\tfrac{1}{2})\big) = \big(\nu(q,p)J\big)|H^{-1}(-\tfrac{1}{2}) = J|H^{-1}(-\tfrac{1}{2}),$$

using $\nu(q,p) = 1$ if $(q,p)\in H^{-1}(-\frac{1}{2})$ and ((4.9)). We now use scaling to obtain the desired result. For every $d>0$ we have $(r,s)\in\mathscr{H}^{-1}(-\frac{1}{2}d^{-2})$ if and only if $\|s\| = d$. Then $(r,d^{-1}s)\in\mathscr{H}^{-1}(-\frac{1}{2})$. So for every $d>0$ we have

$$\begin{aligned} L^*\big((r\wedge s)|\mathscr{H}^{-1}(-\tfrac{1}{2})\big) &= dL^*\big((r\wedge d^{-1}s)|\mathscr{H}^{-1}(-\tfrac{1}{2})\big) \\ &= \big(d\nu(u\wedge d^{-1}v)\big)|K^{-1}(\tfrac{1}{2}) = \big(\nu(u\wedge v)\big)|K^{-1}(\tfrac{1}{2}d^2), \end{aligned}$$

So for every $d>0$,

$$S^*L^*\big((r\wedge s)|\mathscr{H}^{-1}(-\tfrac{1}{2})\big) = \big(d^{-1}\nu(q,p)J\big)|H^{-1}(-\tfrac{1}{2}d^{-2}) = J|H^{-1}(-\tfrac{1}{2}d^{-2}),$$

which implies $\Phi_{LS}^*\mathscr{J} = J$ on Σ_- and thus $\Phi_{LS}^*\mathscr{J} = J$ on $T\mathbf{R}^4$. This follows because $\bigcup_{d>0}\mathscr{H}^{-1}(-\frac{1}{2}d^{-2}) = T^+S^3$, which is an open subset of $T\mathbf{R}^4$ and $\Sigma_- = \bigcup_{d>0}H^{-1}(-\frac{1}{2}d^{-2})$ is an open subset of $T_0\mathbf{R}^3$. Moreover, the components of the momentum mapping $\mathscr{J}$ and J are polynomials.

For every $1\le i<j\le 4$ we have

$$\begin{aligned} L_{X_{\mathscr{H}}}(r_i s_j - r_j s_i) &= \dot r_i s_j + r_i\dot s_j - \dot r_j s_i - r_j\dot s_j \\ &= \|s\|^{-4}s_i s_j - \|s\|^{-2}r_i r_j - \|s\|^{-4}s_j s_i + \|s\|^{-2}r_j r_i, \quad \text{using (76)} \\ &= 0. \end{aligned}$$

Therefore the SO(4)-momentum mapping $\mathscr{J}$ is conserved by the flow of the Delaunay vector field $X_{\mathscr{H}}$. □

5 Exercises

1. (sl$(2,\mathbf{R})$ and the Kepler problem.) For the Kepler problem with rotationally symmetric Hamiltonian $H = \frac{1}{2} p \cdot p - \frac{1}{|q|}$ let $j = q \times p$, $x = q \cdot q$, $y = p \cdot p$, and $z = q \cdot p$.

   a) Show that the functions x, y, z Poisson commute with the components of j. Moreover, show that the Poisson brackets of x,y, and z define a representation of $\mathrm{sl}(2,\mathbf{R})$. Conclude that $\mathrm{so}(3)$ and $\mathrm{sl}(2,\mathbf{R})$ form a dual pair in the Lie algebra of homogeneous quadratic polynomials.

   b) In xyz-space draw that level sets of $j^2 = \text{const.}$ for different values of the constant including zero. These are models for the $\mathrm{SO}(3)$ reduced space.

   c) Draw the intersections of the $h = \text{const.}$ surfaces with a given $j = \text{const.}$ surface to see the integral curves of the reduced dynamics.

   d) Show that the level sets $j^2 = \text{const.}$ are symplectic leaves for the Poisson manifold $\mathrm{sl}(2,\mathbf{R}) = \mathbf{R}^3$ with coordinates (x, y, z)

2. (Geodesics on a hyperboloid.) Consider $H^{3,1} = \{x \in \mathbf{R}^4 \,\big|\, \langle x, x\rangle = -1\}$, which is the set of vectors in $\mathbf{R}^4$ whose Lorentz length squared is -1. Here $\langle x, y\rangle = x_1y_1 + x_2y_2 + x_3y_3 - x_4y_4$ is the Lorentz inner product on $\mathbf{R}^4$. Geometrically, $H^{3,1}$ is a hyperboloid of two sheets. Its tangent bundle $TH^{3,1} = \{(x,y) \in T\mathbf{R}^4 \,\big|\, \langle x,x\rangle = -1 \;\&\; \langle x,y\rangle = 0\}$ is a symplectic manifold with symplectic form $\omega = \omega_4 | TH^{3,1}$. Here $\omega_4 = \sum_{i=1}^4 \mathrm{d}x_i \wedge \mathrm{d}y_i$ is the standard symplectic form on $T\mathbf{R}^4$.

   a) Consider the Hamiltonian system $(H, TH^{3,1}, \omega)$, where
$$H : TH^{3,1} \subseteq T\mathbf{R}^4 \to \mathbf{R} : (x,y) \mapsto \tfrac{1}{2}\langle y, y\rangle \tag{80}$$
   is the Hamiltonian. Show that $TH^{3,1}$ is an invariant manifold of the vector field on $T\mathbf{R}^4$ whose integral curves satisfy
$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= \langle y, y\rangle\, x. \end{aligned} \tag{81}$$
   Show that an integral curve of (81), which starts on $T^+H^{3,1} = \{(x,y) \in TH^{3,1} \big| \langle y,y\rangle > 0\}$, is an integral curve of the Hamiltonian vector field X_H. Verify that the flow of X_H on $T^+H^{3,1}$ is given by
$$\varphi^H : \mathbf{R} \times T^+H^{3,1} \to T^+H^{3,1} : \big(t, (x,y)\big) \mapsto \begin{pmatrix} \cosh t\sqrt{2H} & (\sinh t\sqrt{2H})/\sqrt{2H} \\ \sqrt{2H}\sinh t\sqrt{2H} & \cosh t\sqrt{2H} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$
   Show that the image of an integral curve of X_H, under the bundle projection map $T^+H^{3,1} \subseteq T\mathbf{R}^4 \to H^{3,1} : (x,y) \mapsto x$, is a geodesic on $H^{3,1}$. Verify that every integral curve of X_H on $T^+H^{3,1}$ lies in the 2-plane in $\mathbf{R}^4$ spanned by its initial conditions.

   b) Let $\mathrm{O}(3,1) = \{O \in \mathrm{Gl}(4,\mathbf{R}) \,\big|\, \langle Ox, Oy\rangle = \langle x,y\rangle \text{ for all } x\ y \in \mathbf{R}^4\}$ be the Lorentz group. The Lie algebra of $\mathrm{O}(3,1)$ is $\mathrm{o}(3,1) = \{\xi \in \mathrm{gl}(4,\mathbf{R}) \,\big|\, \langle \xi x, y\rangle + \langle x, \xi y\rangle = 0, \text{ for all } x,\, y \in \mathbf{R}^4\}$. Show that the Lie bracket on $\mathrm{o}(3,1)$ is given by
$$[\xi, \eta] = \xi\,\eta - \eta\,\xi = \begin{pmatrix} i(\sigma \times \tau) + x \otimes y^t - y \otimes x^t & \sigma \times y - \tau \times x \\ (\sigma \times y - \tau \times x)^t & 0 \end{pmatrix},$$

where $\xi = \begin{pmatrix} i(\sigma) & x \\ x^t & 0 \end{pmatrix}$, $\eta = \begin{pmatrix} i(\tau) & y \\ y^t & 0 \end{pmatrix}$, and σ, τ, x, $y \in \mathbf{R}^3$. Here $i : \mathbf{R}^3 \to \mathrm{so}(3,\mathbf{R}) : x \mapsto \left(\begin{smallmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{smallmatrix}\right)$.

Verify that the $\mathrm{O}(3,1)$-action on $H^{3,1}$, given by $\varphi : \mathrm{O}(3,1) \times H^{3,1} \to H^{3,1} : (A,x) \mapsto Ax$, lifts to a Hamiltonian action $\Phi : \mathrm{O}(3,1) \times TH^{3,1} \to TH^{3,1} : \big(A,(x,y)\big) \mapsto (Ax, Ay)$ with coadjoint equivariant momentum mapping $J : TH^{3,1} \subseteq T\mathbf{R}^4 \to \mathrm{o}(3,1)^*$. Here $J(x,y)\xi = J^\xi(x,y)$ with $\xi \in \mathrm{o}(3,1)$ and

$$J^\xi : TH^{3,1} \subseteq T\mathbf{R}^4 \to \mathbf{R} : (x,y) \mapsto \langle \xi x, y \rangle. \tag{82}$$

Observing that the Hamiltonian H (80) is invariant under the action Φ of $\mathrm{O}(3,1)$ on $TH^{3,1}$, deduce that J^ξ is an integral of X_H for every $\xi \in \mathrm{o}(3,1)$.

c) Define the mapping $\vartheta : \Lambda^2\mathbf{R}^4 \to \mathrm{o}(3,1) : v \wedge w \mapsto \ell_{v,w}$, where $\ell_{v,w} : \mathbf{R}^4 \to \mathbf{R}^4 : z \mapsto \langle z,v\rangle w - \langle z,w\rangle v$. Prove the following statements.

1. $\ell_{v,w} \in \mathrm{o}(3,1)$ for every v, $w \in \mathbf{R}^4$.
2. ϑ is a bijective real linear mapping.
3. Consider the action

$$\delta : \mathrm{O}(3,1) \times \Lambda^2\mathbf{R}^4 \to \Lambda^2\mathbf{R}^4 : \big(O, v \wedge w\big) \mapsto Ov \wedge Ow. \tag{83}$$

The mapping ϑ intertwines the action δ with the adjoint action of $\mathrm{O}(3,1)$ on $\mathrm{o}(3,1)$, that is,

$$\vartheta \circ \delta_O = \mathrm{Ad}_O \circ \vartheta = O \circ \vartheta \circ O^{-1} \tag{84}$$

for every $O \in \mathrm{O}(3,1)$.

d) With $\langle x,y\rangle = \sum_{i=1}^3 x_i y_i - x_4 y_4$ for every x, $y \in \mathbf{R}^4$ prove the identity

$$\langle x,x\rangle \, \langle y,y\rangle - \langle x,y\rangle^2 = \sum_{1 \le i < j \le 3} (x_i y_j - x_j y_i)^2 - \sum_{i=1}^{4} (x_i y_4 - x_4 y_i)^2. \tag{85}$$

Let

$$\mathrm{K} : \Lambda^2\mathbf{R}^4 \times \Lambda^2\mathbf{R}^4 \to \mathbf{R} : (v \wedge w, x \wedge y) \mapsto \det \begin{pmatrix} \langle v,x\rangle & \langle w,x\rangle \\ \langle w,x\rangle & \langle w,y\rangle \end{pmatrix}. \tag{86}$$

Show that K is a nondegenerate inner product on $\Lambda^2\mathbf{R}^4$ with $\{e_\ell \wedge e_k\}_{1 \le \ell < k \le 4}$ being an orthonormal basis with respect to which the matrix of K is diagonal. Show that the Morse index of K is 3. Verify that K is invariant under the action δ (83). Let $\mathrm{k} : \mathrm{o}(3,1) \times \mathrm{o}(3,1) \to \mathbf{R} : (\xi,\eta) \mapsto -\frac{1}{2}\mathrm{tr}\, \xi\eta$. Show that k is a nondegenerate inner product on $\mathrm{o}(3,1)$, which is invariant under the adjoint action Ad. With $\xi = \begin{pmatrix} i(\sigma) & x \\ x^t & 0 \end{pmatrix}$ and $\eta = \begin{pmatrix} i(\tau) & y \\ y^t & 0 \end{pmatrix}$, where σ, τ, x, $y \in \mathbf{R}^3$, show that $\mathrm{k}(\xi,\eta) = (\sigma,\tau) - (x,y)$. Here $(\ ,\ )$ is the Euclidean inner product on $\mathbf{R}^3$. Verify that $\vartheta^*\mathrm{k} = \mathrm{K}$.

e) Let $\widetilde{J} : TH^{3,1} \subseteq T\mathbf{R}^4 \rightarrow \Lambda^2\mathbf{R}^4$ be the mapping $K^\flat \circ \vartheta^t \circ J$. Show that for every $(x,y) \in TH^{3,1}$

$$\widetilde{J}(x,y) = x \wedge y = \sum_{1 \le i < j \le 4} K(x \wedge y, e_i \wedge e_j)\, e_i \wedge e_j = \sum_{1 \le i < j \le 4} T_{ij}(x,y)\, e_i \wedge e_j, \quad (87)$$

where $\{e_i\}_{i=1}^4$ is the standard basis of $\mathbf{R}^4$ and $T_{ij}(x,y) = x_i y_j - x_j y_i$. Deduce that $\widetilde{J}$ intertwines the $\mathrm{O}(3,1)$-action Φ on $TH^{3,1}$ with the $\mathrm{O}(3,1)$-action δ on $\Lambda^2\mathbf{R}^4$. Let

$$T_{\sqrt{2h}}H^{3,1} = \{(x,y) \in T^+H^{3,1} \big| \langle x,x\rangle = -1,\ \langle x,y\rangle = 0,\ \text{and}\ \langle y,y\rangle = 2h > 0\}$$

be the bundle of tangent vectors to $H^{3,1}$, whose squared Lorentz length is $2h > 0$. Then $T_{\sqrt{2h}}H^{3,1} = H^{-1}(h)$. Show that $\mathrm{O}(3,1)$ acts transitively on $T_{\sqrt{2h}}H^{3,1}$. Using Plücker coordinates $\{T_{ij}\}_{1 \le i < j \le 4}$ on $\Lambda^2(\mathbf{R}^4)$, show that the image of $T_{\sqrt{2h}}H^{3,1}$ under the mapping $\widetilde{J}$ (87) is the smooth submanifold V_h of $\Lambda^2(\mathbf{R}^4)$ defined by

$$\begin{aligned} T_{12}T_{34} - T_{13}T_{24} + T_{23}T_{14} &= 0 \\ T_{12}^2 + T_{13}^2 + T_{23}^2 - T_{34}^2 - T_{24}^2 - T_{14}^2 &= -2h \end{aligned} \quad (88)$$

Deduce that V_h is diffeomorphic to TS^2. Hint: use the diffeomorphism

$$V_h \subseteq \Lambda^2\mathbf{R}^4 \rightarrow \mathbf{R}^6 : (T_{ij})_{1 \le i < j \le 4} \mapsto \big(T_{12}, T_{13}, T_{23}, T_{34}X^{-1/2}, -T_{24}X^{-1/2}, T_{14}X^{-1/2}\big),$$

where $X = 2h + T_{12}^2 + T_{13}^2 + T_{23}^2 > 0$, since $2h > 0$. Show that $\mathrm{O}(3,1)$ acts transitively on V_h and that V_h is the space of orbits of the geodesic flow on $T^+H^{3,1}$ of energy $h > 0$.

f) Show that V_h is a symplectic manifold. For $u = e_4 \wedge \sqrt{2h}e_1 \in V_h$ let $\mu = \vartheta(u)$. The adjoint orbit $\mathscr{O}_\mu = \{\nu = \mathrm{Ad}_O\mu \in \mathrm{o}(3,1) \big| O \in \mathrm{O}(3,1)\}$ of $\mathrm{O}(3,1)$ through μ is a symplectic manifold with symplectic form $\omega_\nu(\mathrm{ad}_\nu\xi, \mathrm{ad}_\nu\eta) = \mathrm{k}(\nu, [\xi,\eta])$. Since $\vartheta|V_h : V_h \subseteq \Lambda^2(\mathbf{R}^4) \rightarrow \mathscr{O}_\mu \subseteq \mathrm{o}(3,1)$ is a diffeomorphism, $\Omega_h = (\vartheta|V_h)^*\omega_{\mathscr{O}_\mu}$ is a symplectic form on V_h.

We now find an explicit expression for the symplectic form Ω_h. For every $v \in V_h$ show that $\xi_v = T_e\delta_v\xi \in T_vV_h$ for every $\xi \in \mathrm{o}(3,1)$. In fact, $T_vV_h = \mathrm{span}_{\mathbf{R}}\{\xi_v \big| \xi \in \mathrm{o}(3,1)\}$. Infinitesimalizing (84) show that at every $v \in V_h$ we have $\mathrm{ad}_\xi\,\vartheta(v) = T_v\vartheta\,\xi_v$ for every $\xi \in \mathrm{o}(3,1)$. We have $\delta_O^*\Omega_h = \Omega_h$ for every $O \in \mathrm{O}(3,1)$. To see this justify each step of the following calculation.

$$\begin{aligned} \delta_O^*\Omega_h(u)(\xi_u,\eta_u) &= \Omega_h(\delta_O(u))(T_u\delta_O\xi_u, T_u\delta_O\eta_u) \\ &= \omega_\mu(\vartheta\circ\delta_O(u))(T_{Ou}\vartheta\, T_u\delta_O\xi_u, T_{Ou}\vartheta\, T_u\delta_O\eta_u) \\ &= \omega_\mu(\mathrm{Ad}_O\vartheta(u))\big(T_{\vartheta(u)}\mathrm{Ad}_O(\mathrm{ad}_\xi\,\vartheta(u)), T_{\vartheta(u)}\mathrm{Ad}_O(\mathrm{ad}_\xi\,\vartheta(u))\big) \\ &= \omega_\mu(\mathrm{Ad}_O\vartheta(u))\big(T_{\vartheta(u)}\mathrm{ad}_{\mathrm{Ad}_O\xi}\mathrm{Ad}_O\vartheta(u)), T_{\vartheta(u)}\mathrm{ad}_{\mathrm{Ad}_O\eta}\mathrm{Ad}_O\vartheta(u)) \\ &= \mathrm{k}(\mathrm{Ad}_O\vartheta(u), [\mathrm{Ad}_O\xi, \mathrm{Ad}_O\eta]) = \mathrm{k}(\mathrm{Ad}_O\vartheta(u), \mathrm{Ad}_O[\xi,\eta]) \\ &= \mathrm{k}(\vartheta(u), [\xi,\eta]) = \Omega_h(u)(\xi_u,\eta_u). \end{aligned}$$

Write $v = \delta_O u$. Then $T_u\delta_O\xi_u = \xi_v \in T_vV_h$. The above calculation shows that

$$\Omega_h(v)(\xi_v,\eta_v) = \mathrm{k}(\vartheta(u), [\xi,\eta]). \quad (89)$$

To make (89) explicit, show that

$$\xi_u = \xi e_4 \wedge \sqrt{2h} e_1 + e_4 \wedge \sqrt{2h}\,\xi e_1 = (-x_2, -x_3, 0, \sigma_2, -\sigma_3, 0) \in T_u V_h,$$

where $\xi = \begin{pmatrix} i(\sigma) & x \\ x^t & 0 \end{pmatrix}$, with σ, $x \in \mathbf{R}^3$ and we use $\{e_1 \wedge e_2, e_1 \wedge e_3, e_2 \wedge e_3, e_3 \wedge e_4,$ $e_2 \wedge e_4, e_1 \wedge e_4\}$ as a basis for $\Lambda^2(\mathbf{R}^4)$. Let $\eta = \begin{pmatrix} i(\tau) & y \\ y^t & 0 \end{pmatrix}$, where τ, $y \in \mathbf{R}^3$. Note that $\vartheta(u) = -\sqrt{2h}\begin{pmatrix} 0 & e_1 \\ (e_1)^t & 0 \end{pmatrix}$. Justify each step of the following calculation.

$$\begin{aligned} \mathrm{k}(\vartheta(u), [\xi, \eta]) &= -\tfrac{1}{2}\mathrm{tr}\big(\vartheta(u)\,[\xi, \eta]\big) \\ &= \tfrac{1}{2}\sqrt{2h}\,\mathrm{tr}\begin{pmatrix} 0 & e_1 \\ (e_1)^t & 0 \end{pmatrix}\begin{pmatrix} i(\sigma \times \tau) + x \otimes y^t - y \otimes x^t & \sigma \times y - \tau \times x \\ (\sigma \times y - \tau \times x)^t & 0 \end{pmatrix} \\ &= \tfrac{1}{2}\sqrt{2h}\,\mathrm{tr}\begin{pmatrix} e_1 \otimes (\sigma \times y - \tau \times x)^t & * \\ * & (e_1)^t(\sigma \times y - \tau \times x) \end{pmatrix} \\ &= \sqrt{2h}(\sigma \times y - \tau \times x)_1 = \sqrt{2h}(\sigma_2 y_3 - \sigma_3 y_2 - \tau_2 x_3 + \tau_3 x_2). \end{aligned}$$

So

$$\begin{aligned} \Omega_h(Ou)\big(T_u\delta_O(-x_2, -x_3, 0, \sigma_2, -\sigma_3, 0), T_u\delta_O(-y_2, -y_3, 0, \tau_2, -\tau_3, 0)\big) &= \\ = \sqrt{2h}(\sigma_2 y_3 - \sigma_3 y_2 - \tau_2 x_3 + \tau_3 x_2). \end{aligned}$$

3. (Positive energy Keplerian orbits.) This exercise deals with Keplerian orbits of positive energy. Specifically we discuss the changes that need to be made to the treatment of the Kepler problem in §3.2.

   a) First check that the arguments establishing the equation

$$\|q\| = \mu^{-1} J^2 (1 + e\cos f)^{-1} \tag{90}$$

   for the Keplerian orbit with angular momentum $\mathbf{J}$ and eccentricity vector $\mathbf{e}$ as well as the equation

$$e^2 = 1 + 2\mu^{-2} J^2 h \tag{91}$$

   for the magnitude squared of the eccentricity vector continue to hold for positive energy h. When $h > 0$ from (91) it follows that $e > 1$. Deduce that the Keplerian orbit (90) is one branch of a hyperbola. For (90) to hold show that $|f| < f_0 = \pi - \cos^{-1} e^{-1}$. Thus $(\cos f_0, \pm\sin f_0)$ are the directions of the asymptotes of the branch of the hyperbola. From (91) and the facts that $\langle q, \mathbf{J}\rangle = 0$ and $\langle p, \mathbf{J}\rangle = 0$ deduce that a Keplerian orbit of positive energy is a hyperbola, which lies in a 2-plane Π, which is perpendicular to $\mathbf{J}$. Show that $\{\mathbf{e}, \mathbf{J} \times \mathbf{e}\}$ is an orthogonal basis of Π. Let C be the center of the hyperbola, which is the origin of the $\mathbf{e}$-$(\mathbf{J} \times \mathbf{e})$ coordinate system. Let O be the center of attraction, which is a focus of the hyperbola. Show that the periapse A of the hyperbola lies on the $\mathbf{e}$-axis between C and O and that the major semi-axis of the hyperbola OA has length $a = J^2\mu^{-1}(e^2 - 1)^{-1} = \mu(2h)^{-1}$. For $u \in \mathbf{R}$ let $P = (a\cosh u, b\sinh u)$ be a point on the hyperbola. Let $\overline{OP} = \|q\|$ with f the true anomaly of P, that is, f is the angle between $\mathbf{e}$ and the line segment OP. Show that $\overline{CO} = ae = a\cosh u + \|q\|\cos f$. Deduce that the equation of the hyperbolic Keplerian orbit (90) can be written as $\frac{1}{a}\|q\| = e\cosh u - 1$. The minor semi-axis of the hyperbola lies on the $(\mathbf{J}\times\mathbf{e})$-axis. Show that its length is $b = a\sqrt{e^2-1}$ and that $\|q\|\sin f = b\sinh u$.

b) We now determine the analogue of Kepler's equation for a hyperbolic orbit. First we use

$$\frac{\mathrm{d}s}{\mathrm{d}t} = \frac{\sqrt{2h}}{\|q\|} \tag{92}$$

to define the eccentric anomaly s. Following the derivation of equation (43) in §3.3 show that

$$\left(\frac{\mathrm{d}\|q\|}{\mathrm{d}s}\right)^2 + a^2(e^2-1) = 2a\|q\| + \|q\|^2$$

with $q(0) = a(e-1)$. Using the change of variable $ae\rho = \|q\| + a$, show that the above equation becomes

$$-\left(\frac{\mathrm{d}\rho}{\mathrm{d}s}\right)^2 + \rho^2 = 1$$

with $\rho(0) = 1$. Integrating, gives $\rho(s) = \cosh s$. Hence $\|q(s)\| = ae\cosh s - a$, which substituted into (92) and integrating gives the hyperbolic analogue of Kepler's equation

$$n\ell = e\sinh s - s, \tag{93}$$

where $n = \sqrt{2h}\mu^{-1} = \mu^{1/2}a^{-3/2}$ is the mean motion and $\ell = t - \tau$ is the mean anomaly. Here τ is the time at the passage of the periapse.

4. (Hamilton's theorem.) Hamilton's theorem states that the velocity of a particle of mass m subject to an attractive central force with potential $U(|\mathbf{x}|) = -k\frac{1}{|\mathbf{x}|}$, $k > 0$ moves on a circle $\mathscr{C}$, which uniquely determines its Keplerian orbit. Here $|\mathbf{x}|$ is the length of a vector $\mathbf{x} \in \mathbf{R}^3 \setminus \{0\}$ using the Euclidean inner product $\langle\ ,\ \rangle$. Assume that the conserved angular momentum $\mathbf{J} = \mathbf{x} \times m\mathbf{v}$ of the particle is nonzero. The argument outlined in sections a) – c) gives a proof of Hamilton's theorem.

a) Show that the position $\mathbf{x}(t)$ and velocity $\mathbf{v}(t) = \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t}$ of the particle at time t lies in a plane Π, which is perpendicular to $\mathbf{J}$, which we can assume to be the vector $(0,0,j)$, where $j = |\mathbf{J}| > 0$. Using polar coordinates (r,θ) in Π, show that $j = r^2\frac{\mathrm{d}\theta}{\mathrm{d}t}$. Deduce that $\frac{\mathrm{d}\theta}{\mathrm{d}t} > 0$. Consequently, we can reparametrize the curves $t \mapsto \mathbf{x}(t)$ and $t \mapsto \mathbf{v}(t)$ using θ instead of t. Show that this reparametrization preserves the original positive of orientation of these curves given by increasing t.

b) Rewrite Newton's equations of motion

$$m\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = -k\frac{\mathbf{x}}{|\mathbf{x}|^3} \tag{94}$$

using polar coordinates on Π and change the parametrization of the velocity in (94) to θ. Show that we obtain

$$\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}\theta} = (R\cos\theta, R\sin\theta, 0), \tag{95}$$

where $R = k/jm$. Integrating (95) gives

$$\mathbf{v}(\theta) = (-R\sin\theta, R\cos\theta, 0) + \mathbf{c}. \tag{96}$$

Deduce that $\mathbf{v}(\theta) - \mathbf{c}$ moves on a circle $\mathscr{C}$ in Π with center at $\mathbf{c}$ and radius R.

c) Choose coordinates on Π so that $\mathbf{c} = (0, c, 0)$, where $c = |\mathbf{c}| \ge 0$. Let $e = c/R$. Then $\mathbf{v}(\theta) - \mathbf{c} = \big(-R\sin\theta, R(e + \cos\theta), 0\big)$. Using

$$j = \langle J, (0,0,1)\rangle = \langle \mathbf{x}(\theta) \times m\mathbf{v}(\theta), (0,0,1)\rangle,$$

where $\mathbf{x}(\theta) = \big(r(\theta)\cos\theta, r(\theta)\sin\theta, 0\big)$, deduce that

$$r = r(\theta) = \Lambda(1 + e\cos\theta)^{-1}, \tag{97}$$

where $\Lambda = j/mR = j^2/k$. Equation (97) describes a conic section of eccentricity e with a focus at $O = (0,0,0)$.

d) When $0 \le e < 1$, show that $\theta \mapsto \mathbf{v}(\theta) - \mathbf{c}$ traces out the full velocity circle $\mathscr{C}$.

e) When $e > 1$ equation (97) describes a branch of a hyperbola. The following argument shows that $\theta \mapsto \mathbf{v}(\theta) - \mathbf{c}$ traces out a positively oriented arc of $\mathscr{C}$. This arc subtends a positive angle Θ, which is equal to the scattering angle of the hyperbola. Because $e > 1$ for equation (97) to hold $|\theta| < \theta_0 = \pi - \theta_*$, where $\theta_* = \cos^{-1} e^{-1}$. Using conservation of energy show that

$$|\mathbf{v}|^2 = \frac{2h}{m} + \frac{k}{m^2}\frac{1}{|\mathbf{x}|} > \frac{2h}{m}.$$

Hence the velocity of the particle lies outside of the closed 2-disk with center at O and radius $\sqrt{\frac{2h}{m}}$. Show that $\mathbf{v}(\theta) - \mathbf{c}$ lies on the velocity circle $\mathscr{C}$ and the energy circle $\partial\mathscr{E}$, given by $|\mathbf{v}| = \sqrt{\frac{2h}{m}}$, if and only if $0 = \frac{1}{r(\theta)} = \Lambda^{-1}(1 + e\cos\theta)$, that is, if and only if $\theta = \pm\theta_0 = \pm(\pi - \theta_*)$. Show that the velocity vectors $\mathbf{v}(\pm\theta_0) - \mathbf{c}$ are the end points of a closed arc $\mathscr{A}$ on $\mathscr{C}$ and that

$$\mathbf{v}(\pm\theta_0) - \mathbf{c} = \big(-R\sin(\pm\theta_0), R\cos(\pm\theta_0), 0\big) = (\mp R\sin\theta_*, -R\cos\theta_*, 0). \tag{98}$$

From (98) deduce that $\mathbf{v}(\theta_0) - \mathbf{c}$ lies the 3$^{\text{rd}}$ quadrant of Π; while $\mathbf{v}(-\theta_0) - \mathbf{c}$ lies the 4$^{\text{th}}$ quadrant of Π. Deduce that the positive arc $\mathscr{A}$, oriented so that θ increases, has an initial end point at $\mathbf{v}(-\theta_0) - \mathbf{c}$ and a final end point at $\mathbf{v}(\theta_0) - \mathbf{c}$. Show that

$$\mathbf{v}(\theta_0) = \big(R\cos(\tfrac{3}{2}\pi - \theta_*), R\sin(\tfrac{3}{2}\pi - \theta_*), 0\big);$$

while

$$\mathbf{v}(-\theta_0) = \big(R\cos(-(\tfrac{1}{2}\pi - \theta_*)), R\sin(-(\tfrac{1}{2}\pi - \theta_*)), 0\big).$$

Thus the positive angle Θ subtended by the positive arc $\mathscr{A}$ is equal to $2(\pi - \theta_*)$. When $-\theta_0 = -(\pi - \theta_*)$, then $\mathbf{d}_{-\theta_0} = \frac{\mathbf{x}(-\theta_0)}{|\mathbf{x}(-\theta_0)|}$ is the direction of the incoming asymptote of the branch of the hyperbola with center C at $(ae, 0, 0)$ in Π; while when $\theta_0 = \pi - \theta_*$, then $\mathbf{d}_{\theta_0} = \frac{\mathbf{x}(\theta_0)}{|\mathbf{x}(\theta_0)|}$ is the direction of the outgoing asymptote of the branch of the hyperbola. By definition, the scattering angle Ψ of the hyperbolic motion is the counterclockwise rotation about C, which sends $\mathbf{d}_{-\theta_0}$ into $\mathbf{d}_{\theta_0}$. Show that $\Psi = 2(\pi - \theta_*) = \Theta$.

5. (Regularization of positive energy Keplerian orbits.) Let $(H, T_0\mathbf{R}^3, \omega_3)$ be the Kepler Hamiltonian system with $T_0\mathbf{R}^3 = (\mathbf{R}^3 \setminus \{0\}) \times \mathbf{R}^3$ having coordinates (q,p), symplectic form $\omega_3 = \sum_{i=1}^3 \mathrm{d}q_i \wedge \mathrm{d}p_i$, and Hamiltonian

$$H : T_0\mathbf{R}^3 \to \mathbf{R} : (q,p) \mapsto \tfrac{1}{2}(p,p) - \|q\|^{-1}. \tag{99}$$

Here $(\ ,\ )$ is the Euclidean inner product on $\mathbf{R}^3$ with $\|q\|$ being the length of the vector $q \in \mathbf{R}^3$. We look only at a positive energy Keplerian orbit, which in exercise 3 we have shown to be a branch of a hyperbola.

a) To regularize the positive energy Keplerian orbits, we will use an argument analogous to the one given in §4 for the negative energy orbits. Start by using the virial group to show that we may reduce our considerations to the level set $H^{-1}(\frac{1}{2})$. Next introduce a new time scale s by $\frac{\mathrm{d}s}{\mathrm{d}t} = \|q\|^{-1}$. Consider the rescaled Hamiltonian

$$\widetilde{F} : T_0\mathbf{R}^3 \to \mathbf{R} : (q,p) \mapsto \|q\|\big(H(q,p) - \tfrac{1}{2}\big) + 1 = \tfrac{1}{2}\|q\|(\|p\|^2 - 1). \tag{100}$$

Show that the integral curves of $X_{\widetilde{F}}$ on $\widetilde{F}^{-1}(1)$ are the same as the integral curves of X_H on $H^{-1}(\frac{1}{2})$, using the time parameter s. Let

$$\widetilde{K} : T_0\mathbf{R}^3 \to \mathbf{R} : (q,p) \mapsto \tfrac{1}{2}\widetilde{F}^2(q,p) = \tfrac{1}{8}\|q\|^2(\|p\|^2 - 1)^2 \tag{101}$$

be the regularized Hamiltonian. Show that the integral curves of $X_{\widetilde{K}}$ on $\widetilde{K}^{-1}(\frac{1}{2})$ are the same as the integral curves of $X_{\widetilde{F}}$ on $\widetilde{F}^{-1}(1)$, using the time parameter s.

b) Let $\langle\ ,\ \rangle$ be the Lorentz inner product on $\mathbf{R}^4$ given by $\langle u, v\rangle = u_1v_1 + u_2v_2 + u_3v_3 - u_4v_4$. Let $H^{3,1} = \{u \in \mathbf{R}^3 \,\big|\, \langle u,u\rangle = -1\}$. Consider the stereographic projection map

$$\varphi^{-1} : H^{3,1} \subseteq \mathbf{R}^4 \to \mathbf{R}^3 : q \mapsto (1-q_4)^{-1}\widetilde{q} = (1-q_4)^{-1}(q_1,q_2,q_3)$$

from $(\widetilde{0},1)$ with inverse

$$\varphi : \mathbf{R}^3 \to H^{3,1} \subseteq \mathbf{R}^4 : \widetilde{q} \mapsto 2(1-\|\widetilde{q}\|^2)^{-1}\big(\widetilde{q}, -\tfrac{1}{2}(1+\|\widetilde{q}\|^2)\big).$$

The positive energy analogue of Moser's regularization map in §4 is

$$\begin{aligned}\Phi_M^{-1} : TH^{3,1} \subseteq T\mathbf{R}^4 \to T_0\mathbf{R}^3 &:\\ (u,v) \mapsto (q,p) &= \big(-(1-u_4)\widetilde{v} - v_4\widetilde{u}, (1-u_4)^{-1}\widetilde{u}\big),\end{aligned} \tag{102}$$

which is the composition of $T\varphi^{-1}$ followed by momentum reversal $(q,p) \mapsto (-p,q)$. For $(u,v) \in TH^{3,1} = \{(u,v) \in T\mathbf{R}^4 \,\big|\, \langle u,u\rangle = -1 \;\&\; \langle u,v\rangle = 0\}$ show that the following identities hold.

$$\|q\|^2 = \langle v,v\rangle(1-u_4)^2 \tag{103a}$$

$$\|p\|^2 - 1 = -2(1-u_4)^{-1} \tag{103b}$$

$$(q,p) = v_4. \tag{103c}$$

Using the above identities show that the inverse of the regularization mapping Φ_M^{-1} is given by

$$\Phi_M : T_0\mathbf{R}^3 \to TH^{3,1} \subseteq T\mathbf{R}^4 : (q,p) \mapsto \big((\widetilde{u},u_4),(\widetilde{v},v_4)\big),$$

where

$$\begin{cases} \widetilde{u} &= -(\|p\|^2-1)^{-1}(2p) \quad \text{and} \quad u_4 = (\|p\|^2-1)^{-1}(\|p\|^2+1) \\ \widetilde{v} &= \frac{1}{2}(\|p\|^2-1)q-(q,p)p \quad \text{and} \quad v_4 = (q,p). \end{cases} \tag{104}$$

c) Verify that the pull back by the regularization mapping Φ_M^{-1} (102) of the regularized Hamiltonian $\widetilde{K}$ (101) is the geodesic Hamiltonian

$$\mathscr{H} : TH^{3,1} \subseteq \mathbf{R}^4 \to \mathbf{R} : (u,v) \mapsto \tfrac{1}{2}\langle v,v\rangle. \tag{105}$$

Show that $(\Phi_M^{-1})^*\omega_3 = \omega_4|TH^{3,1}$. Deduce that the flow of the regularized Kepler vector field $X_{\widetilde{K}}$ on $\widetilde{K}^{-1}(\frac{1}{2})$ is the flow of the geodesic Hamiltonian vector field $X_{\mathscr{H}}$ on $\mathscr{H}^{-1}(\frac{1}{2})$.

d) Following the proof of ((4.9)) show that

$$\Phi_M^*\big((u_iv_j - v_iu_j)|\mathscr{H}^{-1}(\tfrac{1}{2})\big) = J_k|H^{-1}(\tfrac{1}{2}),$$

where $(i,j,k) = \{1,2,3\}$, and

$$\Phi_M^*\big((u_iv_4 - v_iu_4)|\mathscr{H}^{-1}(\tfrac{1}{2})\big) = e_i|H^{-1}(\tfrac{1}{2}),$$

for $1 \le i \le 3$.

6. (Center of mass and the two body problem.)

   a) For the two body problem in space show that regular reduction by the translation group can be interpreted as passing to a center of mass frame. Do the reduction of the translation and rotational symmetries in one step by using the Euclidean group $E(3)$.

   b) Consider the spherical analogue of the planar two body problem. This is the motion of two particles connected by a spring constrained to move on the surface of a 2-sphere. The rotation group $\mathrm{SO}(3)$ is an obvious symmetry group of the problem, as compared to the Euclidean group $E(2)$ for the planar problem. Construct all the $\mathrm{SO}(3)$ reduced spaces. Show that there is *no notion* of a center of mass *frame*.

   c)* Is the spherical two body problem Liouville integrable?

7. a) Construct an isomorphism between the Lie algebra $\mathrm{so}(4)$ and $\mathrm{so}(3) \times \mathrm{so}(3)$.

   b) Show that the corresponding Lie-Poisson algebras are isomorphic.

   c) Write out Hamilton's equations on the Lie-Poisson algebra corresponding to $\mathrm{so}(3) \times \mathrm{so}(3)$.

8. (Souriau's linearization and regularization.) In the Kepler problem

$$\begin{aligned} \dot{q} &= p \\ \dot{p} &= -\tfrac{1}{r^3}q, \quad r = \|q\|, \end{aligned} \tag{106}$$

let $H = \frac{1}{2}\langle p,p\rangle - \frac{1}{r}$ be the Hamiltonian and define a new time variable s by

$$s = \langle q,p\rangle - 2ht.$$

a) Show that $\frac{ds}{dt} = \frac{1}{r}$. Thus s is the eccentric anomaly.

b) Define a 4-vector ξ by $\xi = \begin{pmatrix} t \\ q \end{pmatrix}$. Let $\Xi = \mathrm{col}(\xi, \xi', \xi'', \xi''')$ be a 4×4 matrix, where $'$ is differentiation with respect to s. Show that Ξ satisfies the *linear* differential equation

$$\Xi' = A\,\Xi \tag{107}$$

where $A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 2h & 0 & 0 \end{pmatrix}$.

c) Solve (107) and thus find $\xi(s)$. Note that because $\xi(s)$ is defined for *all* s and hence for all t by Kepler's equation, it follows that the Kepler problem has been regularized.

9. (Bacry-Györgyi variables and the conformal group.) Using the same notation in the Kepler problem as in exercise 8, set $\alpha = \sqrt{-2h}$, $P = \begin{pmatrix} t''' \\ \alpha^{-1}q''' \end{pmatrix}$ and $Q = \begin{pmatrix} \alpha t'' \\ q'' \end{pmatrix}$. Here we are confining ourselves to the case of bounded motions, namely, $h < 0$.

a) Show that $P^tP = Q^tQ = 1$ and $P^tQ = 0$.

b) Let ζ be the 6×6 matrix

$$\begin{pmatrix} QP^t - PQ^t & P & Q \\ P^t & 0 & 1 \\ Q^t & -1 & 0 \end{pmatrix}$$

c) Show that $\zeta^2 = 0$.

d) Show that the components of ζ satisfy the Poisson bracket relations for the Lie algebra $\mathrm{so}(4,2)$.

e) Show that the map from the regularized phase space of the negative energy orbits $(q,p,h) \to \zeta$ is a symplectic diffeomorphism if we equip the $\mathrm{SO}(4,2)$-coadjoint orbit through ζ with the symplectic structure given in chapter VI §2 example 3. The tricky part of this is deciding which component of the variety $\zeta^2 = 0, \zeta \neq 0$ in $\mathrm{so}(4,2)^*$ you need to map to.

10. (Levi-Civita regularization.)

a) Let $\mathbf{R}^2_0 = \mathbf{R}^2 - \{0\}$. On $T^*\mathbf{R}^2_0 = \mathbf{R}^2_0 \times \mathbf{R}^2$ with coordinates (x,y) and symplectic form $\omega = \sum_{i=1}^3 dx_i \wedge dy_i$ consider the Kepler Hamiltonian

$$H(x,y) = \tfrac{1}{2}(y_1^2 + y_2^2) - (x_1^2 + x_2^2)^{-1/2}. \tag{108}$$

Identify $\mathbf{R}_0^2$ with $\mathbf{C}_0 = \mathbf{C} - \{0\}$ and $T^*\mathbf{R}_0^2$ with $T^*\mathbf{C}_0 = \mathbf{C}_0 \times \mathbf{C}$. Introduce complex coordinates $q = x_1 + ix_2$ and $p = y_1 + iy_2$ on $T^*\mathbf{C}_0$. Show that $\omega = \mathrm{Re}(dq \wedge d\overline{p})$ and that the Kepler Hamiltonian becomes

$$H(q,p) = \tfrac{1}{2}\|p\|^2 - \|q\|^{-1}. \tag{109}$$

b) Using the time rescaling $\frac{\mathrm{d}s}{\mathrm{d}t} = \frac{k}{2|q|}$ show that the integral curves of X_H on the level set $H^{-1}(-k^2/2)$ are a time reparametrization of the integral curves of the vector field $X_{\widetilde{K}}$ on the level set $\widetilde{K}^{-1}(0)$ where

$$\widetilde{K}(q,p) = 2\,\|q\|k^{-1}\left(\tfrac{1}{2}\|p\|^2 - \|q\|^{-1} + \tfrac{1}{2}k^2\right) = k^{-1}\,\|q\|\,\|p\|^2 + k\,\|q\| - 2k^{-1}. \tag{110}$$

c) Define the Levi-Civita map

$$\mathscr{L} : T^*\mathbf{C}_0 \to T^*\mathbf{C}_0 : (u,v) \to (q,p) = \left((2k)^{-1}u^2,\, kv\overline{u}^{-1}\right). \tag{111}$$

Show that $\mathscr{L}$ has the following properties:

1) $\mathscr{L}$ is a smooth two to one surjective submersion with $\mathscr{L}(-u,-v) = \mathscr{L}(u,v)$.

2) $\mathscr{L}^*\left(\mathrm{Re}(q\,\mathrm{d}\overline{p})\right) = \mathrm{Re}(u\,\mathrm{d}\overline{v} - \overline{v}\,\mathrm{d}u)$. Hence $\mathscr{L}$ is symplectic.

3) The Hamiltonian

$$K(u,v) = (\mathscr{L}^*\widetilde{K})(u,v) = \tfrac{1}{2}(|v|^2 + |u|^2) - 2k^{-1} \tag{112}$$

is defined on $K^{-1}(0)$ which is a 3-sphere centered at the origin and having radius $2/\sqrt{k}$. Since $K^{-1}(0)$ is compact, all the integral curves of X_K on $K^{-1}(0)$ are defined for *all* time. Thus K is the Levi-Civita regularization of the Kepler Hamiltonian for negative energy orbits. Note that up to an additive constant, K is the harmonic oscillator Hamiltonian.

d) The Levi-Civita map $\mathscr{L}$ is *not* an equivalence between the Hamiltonian systems $(K, T^*\mathbf{C}_0, \omega)$ and $(\widetilde{K}, T^*\mathbf{C}_0, \omega)$, because it is *not* a diffeomorphism. Show that that vector fields X_K on $K^{-1}(0)$ and $X_{\widetilde{K}}$ on $\widetilde{K}^{-1}(0)$ are $\mathscr{L}$-related, that is, $T\mathscr{L} \circ X_K = X_{\widetilde{K}} \circ \mathscr{L}$. Thus the image of an integral curve of X_K on $K^{-1}(0)$ under the Levi-Civita map $\mathscr{L}$ is an integral curve of $X_{\widetilde{K}}$ on $\widetilde{K}^{-1}(0)$.

e) On $T^*\mathbf{C}_0$ define a $\mathbf{Z}_2$-action generated by $(u,v) \to (-u,-v)$. Show that this action is free, preserves the symplectic form ω, and preserves the Hamiltonian K. Thus there is an induced Hamiltonian $\mathscr{K}$ on $(T^*\mathbf{C}_0/\mathbf{Z}_2, \omega)$. Since the map $\mathscr{L}$ is invariant under the $\mathbf{Z}_2$-action, it induces an equivalence between the Hamiltonian systems $(\mathscr{K}, T^*\mathbf{C}_0/\mathbf{Z}_2, \omega)$ and $(K, T^*\mathbf{C}_0, \omega)$. Thus the regularized energy surface $H^{-1}(-k^2/2)$ of the Kepler Hamiltonian is $\mathscr{K}^{-1}(0) = (S^3_{2/\sqrt{k}})/\mathbf{Z}_2$, which is real projective three space $\mathbf{RP}^3$.

11. (Kustaanheimo-Stiefel regularization.) Let $x = (x_1, x_2, x_3)$ be a vector in $\mathbf{R}_0^3 = \mathbf{R}^3 \setminus \{0\}$ and let $z = (z_1, z_2) \in \mathbf{C}_0^2 = \mathbf{C}^2 \setminus \{0\} = \mathbf{R}^4 \setminus \{0\}$. Define the 2×2 skew Hermitian matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Let $\langle z, w\rangle = z_1\overline{w_1} + z_2\overline{w_2}$ be the standard Hermitian inner product on $\mathbf{C}^2$. Show that the mapping

$$\pi : \mathbf{C}^2_0 \to \mathbf{R}^3_0 : z \mapsto \big(\langle z, \sigma_1(z)\rangle, \langle z, \sigma_2(z)\rangle, \langle z, \sigma_3(z)\rangle\big), \tag{113}$$

is the Hopf map.

a) On $\mathbf{C}^2_0$ define an action

$$\varphi : U(1) \times \mathbf{C}^2_0 \to \mathbf{C}^2_0 : \big(\mathrm{e}^{is}, (z_1, z_2)\big) \mapsto (\mathrm{e}^{is} z_1, \mathrm{e}^{is} z_2).$$

Let $T^*\mathbf{C}^2_0 = (\mathbf{C}^2 - \{0\}) \times \mathbf{C}^2$. Lift φ to a $U(1)$-action

$$\Phi : S^1 \times T^*\mathbf{C}^2_0 \to T^*\mathbf{C}^2_0 : \big(\mathrm{e}^{is}, z, w\big) \mapsto \big(\mathrm{e}^{is} z, \mathrm{e}^{is} w\big).$$

Define a 1-form θ on $T^*\mathbf{C}^2_0$ by $\theta = -2i\,\mathrm{Im}\,\langle w, dz\rangle$. Show that $\Omega = -d\theta$ is a symplectic form on $T^*\mathbf{C}^2_0$ and that Φ is a Hamiltonian action with momentum map

$$\mathscr{I} : T^*\mathbf{C}^2_0 \to \mathbf{R} : (z, w) \mapsto 2\,\mathrm{Re}\,\langle w, z\rangle.$$

Let $\mathscr{I}_0 = \mathscr{I}^{-1}(0) \setminus \{0\}$.

b) The map π (113) lifts to the *Kustaanheimo-Stiefel* map

$$\begin{aligned} \mathscr{K}\mathscr{S} : T^*\mathbf{C}^2_0 \to T^*\mathbf{R}^3_0 : (z, w) \mapsto (x, y) = \\ \big((\langle z, \sigma_j(z)\rangle), \langle z, z\rangle^{-1}(\mathrm{Re}\,\langle w, \sigma_j(z)\rangle)\big), \quad \text{for } j = 1, 2, 3. \end{aligned}$$

The following calculation shows that

$$(\mathscr{K}\mathscr{S})^*(\vartheta|\mathscr{I}_0) = \theta|\mathscr{I}_0, \tag{114}$$

where $\vartheta = \langle y, \mathrm{d}x\rangle$ is the canonical 1-form on $T^*\mathbf{R}^3$. For every $u, w, z \in \mathbf{C}^2$

$$\sum_{j=1}^{3} \langle u, \sigma_j(z)\rangle\, \sigma_j(w) = 2\langle w, z\rangle\, u - \langle u, z\rangle\, w. \tag{115}$$

Interchanging u with z in (115) and subtracting the result from (115) gives

$$i \sum_{j=1}^{3} \mathrm{Im}\,\langle u, \sigma_j(z)\rangle\, \sigma_j(w) = \langle w, z\rangle\, u - \langle w, u\rangle\, z - i\,\mathrm{Im}\,\langle u, z\rangle\, w. \tag{116}$$

Taking the inner product of (116) with z and then adding the result to its complex conjugate gives

$$\sum_{j=1}^{3} \mathrm{Im}\,\langle u, \sigma_j(z)\rangle\, \mathrm{Im}\,\langle z, \sigma_j(w)\rangle = \mathrm{Re}\,\langle z, w\rangle\, \mathrm{Re}\,\langle u, z\rangle - \langle z, z\rangle\, \mathrm{Re}\,\langle u, w\rangle. \tag{117}$$

Replacing w in (117) with $-iw$ gives

$$\sum_{j=1}^{3} \langle u, \sigma_j(z)\rangle\, \langle z, \sigma_j(w)\rangle = \mathrm{Im}\,\langle z, u\rangle\, \mathrm{Re}\,\langle w, z\rangle - \langle z, z\rangle\, \mathrm{Im}\,\langle w, u\rangle. \tag{118}$$

Finally, replacing w by $\mathrm{d}z$ and u by w in (118) gives

$$\sum_{j=1}^{3} \mathrm{Re}\,\langle w, \sigma_j(z)\rangle \,\mathrm{Im}\,\langle z, \sigma_j(\mathrm{d}z)\rangle = \mathrm{Im}\,\langle z, \mathrm{d}z\rangle \,\mathrm{Re}\,\langle z, w\rangle - \langle z, z\rangle \,\mathrm{Im}\,\langle w, \mathrm{d}z\rangle. \qquad (119)$$

Consequently

$$(\mathscr{K}\mathscr{S})^*\vartheta = 2i\left(\frac{\mathrm{Im}\langle z, \mathrm{d}z\rangle \,\mathrm{Re}\,\langle z, w\rangle - \langle z, z\rangle \,\mathrm{Im}\,\langle w, \mathrm{d}z\rangle}{\langle z, z\rangle}\right). \qquad (120)$$

From (120) it follows that $(\mathscr{K}\mathscr{S})^*(\vartheta|\mathscr{I}_0) = \theta|\mathscr{I}_0$.

c) On $T^*\mathbf{R}^3_0$ with coordinates (x,y) and symplectic form $\omega = \sum_i \mathrm{d}x_i \wedge \mathrm{d}y_i$, consider the time rescaled Kepler Hamiltonian

$$\widetilde{K}(x,y) = \tfrac{1}{2}k^{-1}\,\|x\|(\|y\|^2 + k^2)$$

whose μk^{-1}-level set corresponds to the $-k^2/2$-level set of the Kepler Hamiltonian. Setting $u = w$ in (117) show that on $\mathscr{I}_0$ $\|(\mathscr{K}\mathscr{S})^*\|y\|^2 = \langle w, w\rangle\langle z, z\rangle^{-1}$ and $\|(\mathscr{K}\mathscr{S})^*x\|^2 = \langle z, z\rangle$. Therefore on $\mathscr{I}_0$ we obtain the regularized Hamiltonian

$$K = (\mathscr{K}\mathscr{S})^*\widetilde{K} = \tfrac{1}{2}k^{-1}\left(\langle w, w\rangle + k^2\langle z, z\rangle\right). \qquad (121)$$

When $k = 1$ the regularized Hamiltonian is the harmonic oscillator Hamiltonian on $(T^*\mathbf{C}^2, \Omega)$ restricted to the open cone $\mathscr{I}_0$. Show that the regularized Hamiltonian K (121) is invariant under the $U(1)$-action Φ. Since the mapping $\mathscr{K}\mathscr{S}$ is *not* a diffeomorphism, the harmonic oscillator vector field X_K is not equivalent to the Kepler vector field $X_{\widetilde{K}}$. Show that they are $\mathscr{K}\mathscr{S}$-related on $\mathscr{I}_0$, that is, on $\mathscr{I}_0$ we have $T(\mathscr{K}\mathscr{S})\circ X_K = X_{\widetilde{K}}\circ(\mathscr{K}\mathscr{S})$. Moreover, show that after dividing out the S^1-action Φ on $\mathscr{I}_0$ we obtain an equivalence of Hamiltonian systems. Show that the orbit space $\mathscr{I}_0/S^1$ is diffeomorphic to T^+S^3, the tangent bundle to S^3 less its zero section.

12. (Generalized Kepler equation.) Consider the Ligon-Schaaf map

$$LS : \Sigma_- \subseteq T_0\mathbf{R}^3 \to T^+S^3_{np} \subseteq T\mathbf{R}^4 : (q,p) \to (r,s),$$

with $\varphi = \nu^{-1}\langle q, p\rangle$. Show that its inverse is given by

$$\begin{aligned} q &= \mu^{-1}\langle s,s\rangle\left((\sin\varphi - \langle r,r\rangle^{-1/2}s_4)\widetilde{r} + \langle s,s\rangle^{-1/2}(r_4 - \cos\varphi)\widetilde{s}\right) \\ p &= \mu\langle s,s\rangle^{-1/2}\Big(\frac{\widetilde{r}\cos\varphi + \langle s,s\rangle^{-1/2}\widetilde{s}\sin\varphi}{1 - r_4\cos\varphi - \langle s,s\rangle^{-1/2}s_4\sin\varphi}\Big), \end{aligned}$$

where $r = (\widetilde{r}, r_4)$, $s = (\widetilde{s}, s_4)$ and φ is a smooth solution of

$$\varphi - r_4\sin\varphi - s_4\langle s,s\rangle^{-1/2}\cos\varphi = 0.$$

13. a) Show that SO(4)-action on an energy surface of the Delaunay vector field is transitive.

    b) Show that the mapping $\vartheta : \bigwedge^2 \mathbf{R}^4 \rightarrow \mathrm{so}(4)$, defined by $\varphi(u \wedge v)w = \langle v, w \rangle u - \langle v, u \rangle w$ for every $u, v, w \in \mathbf{R}^4$, intertwines the SO(4)-action $\mathrm{SO}(4) \times \bigwedge^2 \mathbf{R}^4 \rightarrow \bigwedge^2 \mathbf{R}^4 : (A, u \wedge v) \rightarrow Au \wedge Av$ with the adjoint action of SO(4) on so(4).

    c) Show that the orbit space (C_h, ω_h) of the flow of the Delaunay vector field on $\widetilde{\mathscr{H}}^{-1}(h)$ is symplectically diffeomorphic to the coadjoint orbit $\mathscr{O}_\mu$ through $\widetilde{\mathscr{J}}(e_1, he_2) = he_{12}^* = \mu \in \mathrm{so}(4)^*$ with its usual symplectic structure $\omega_{\mathscr{O}_\mu}$, see example 3 chapter VI §2.

14. Show that the Hamiltonian vector field X_{e_i} corresponding to the i$^{\text{th}}$ component of the eccentricity vector (27) is incomplete. Give a geometric explanation of this incompleteness. State precisely where the flow of X_{e_i} is defined.

15. Given an initial position and momentum of a Keplerian elliptical orbit, determine the argument of the perihelion, that is, the angle between the line of nodes (= the line of intersection of the plane of the elliptical orbit and the equitorial plane of the celestial sphere) and the line joining the foci of the ellipse.

Chapter III

The Euler Top

Mathematically, the motion of the Euler top is described by geodesics of a left invariant metric on the rotation group $\mathrm{SO}(3)$. Physically, the Euler top is a rigid body moving about its center of mass (which is fixed) without any forces acting on the body.

1 Facts about $\mathrm{SO}(3)$

We begin by reviewing some basic facts about the group of rotations of $\mathbf{R}^3$.

1.1 The standard model

On $\mathbf{R}^3$ with Euclidean inner product $(\,,\,)$, the orthogonal group $\mathrm{O}(3)$ is the group of linear maps which preserve the inner product, that is, $O \in \mathrm{O}(3)$ if and only if for every $x, y \in \mathbf{R}^3$, $(Ox, Oy) = (x, y)$. The *group of rotations* $\mathrm{SO}(3)$ of $(\mathbf{R}^3, (\,,\,))$ is the identity component of $\mathrm{O}(3)$. Equivalently, $O \in \mathrm{SO}(3)$ if and only if $OO^t = I$ and $\det O = 1$. The group $\mathrm{SO}(3)$ is a connected compact Lie group with Lie algebra $T_e\,\mathrm{SO}(3) = \mathrm{so}(3) = \{X \in \mathrm{gl}(3, \mathbf{R}) \,\big|\, X + X^t = 0\}$. $X \in \mathrm{so}(3)$ if and only if it is a 3×3 skew symmetric real matrix, that is, for every $x, y \in \mathbf{R}^3$, $(Xx, y) + (x, Xy) = 0$. The Lie algebra $\mathrm{so}(3)$ has a Lie bracket $[\,,\,]$ defined by the relations

$$[E_1, E_2] = E_1E_2 - E_2E_1 = E_3, \;\; [E_2, E_3] = E_1, \;\; [E_3, E_1] = E_2,$$

where $\{E_1, E_2, E_3\}$ is the *standard basis*

$$E_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Rewritten, the bracket relations read $[E_i, E_j] = \sum_{k=1}^{3} \varepsilon_{ijk} E_k$. Here $\varepsilon_{ijk} = 0$, if i, j and k are not distinct. If i, j and k are distinct, ε_{ijk} is 1 if ijk is an even permutation of 123 and -1 otherwise.

On $\mathrm{so}(3)$ there is an inner product $k : \mathrm{so}(3) \times \mathrm{so}(3) \to \mathbf{R}$, called the *Killing metric*. The

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▷ Killing metric is defined by

$$k(X,Y) = -\tfrac{1}{2}\,\mathrm{tr}\,XY \tag{1}$$

and has the following properties.

1. k is positive definite.
2. For $O \in \mathrm{SO}(3)$ let $\mathrm{Ad}_O X = OXO^{-1} \in \mathrm{so}(3)$. Then for every $X, Y \in \mathrm{so}(3)$ we have $k(\mathrm{Ad}_O X, \mathrm{Ad}_O Y) = k(X,Y)$, that is, k is Ad_O-invariant. In other words, $\mathrm{Ad}^t_{O^{-1}} = k^\sharp \circ \mathrm{Ad}_O \circ k^\flat$.
3. For $X,Y,Z \in \mathrm{so}(3)$ we have $k([Z,X],Y) + k(X,[Z,Y]) = 0$.

(1.1) **Proof**:

1. Let $X = \sum_{i=1}^3 x_i E_i$. Then

$$|X|^2 = k(X,X) \;=\; -\tfrac{1}{2}\,\mathrm{tr}\begin{pmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{pmatrix}^2 = x_1^2 + x_2^2 + x_3^2 \geq 0.$$

Equality holds if and only if $X = 0$.

2. For $O \in \mathrm{SO}(3)$, we have

$$k(\mathrm{Ad}_O X, \mathrm{Ad}_O Y) = -\tfrac{1}{2}\,\mathrm{tr}\,OXYO^{-1} \;=\; -\tfrac{1}{2}\,\mathrm{tr}\,XY = k(X,Y).$$

For every $X, Y \in \mathrm{so}(3)$ we have $k(\mathrm{Ad}_O X, Y) = k(X, \mathrm{Ad}_{O^{-1}} Y)$, because k is $\mathrm{Ad}_{O^{-1}}$-invariant. The preceding equation may be rewritten as

$$(k^\sharp(\mathrm{Ad}_O X))Y = k^\sharp(X)(\mathrm{Ad}_{O^{-1}} Y) \;=\; \big(\mathrm{Ad}^t_{O^{-1}} k^\sharp(X)\big)Y$$

for every $Y \in \mathrm{so}(3)$. Hence $(k^\sharp \circ \mathrm{Ad}_O)X = (\mathrm{Ad}^t_{O^{-1}} \circ k^\sharp)X$ for every $X \in \mathrm{so}(3)$.

3. Since $\frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} \mathrm{Ad}_{\exp tZ} W = \mathrm{ad}_Z W \;=\; [Z,W]$ for every $Z, W \in \mathrm{so}(3)$, differentiating the equation $k(\mathrm{Ad}_{\exp tZ} X, \mathrm{Ad}_{\exp tZ} Y) = k(X,Y)$ with respect to t and setting $t = 0$ gives $k([Z,X],Y) + k(X,[Z,Y]) = 0$. □

Define the linear map

$$i : \mathrm{so}(3) \to \mathbf{R}^3 : X = \begin{pmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{pmatrix} \mapsto x = (x_1, x_2, x_3). \tag{2}$$

▷ The map i allows us to do calculations in $\mathbf{R}^3$ instead of in $\mathrm{so}(3)$. It has the following properties.

1. i is an *isometry* from $(\mathrm{so}(3), k)$ to $(\mathbf{R}^3, (\;,\;))$.
2. i is an isomorphism of the Lie algebra $(\mathrm{so}(3), [\;,\;])$ with the Lie algebra $(\mathbf{R}^3, \times)$, where $\times$ is the vector product on $\mathbf{R}^3$, see exercise 3.
3. i *intertwines* the adjoint action of $\mathrm{SO}(3)$ on $\mathrm{so}(3)$ with the usual action of $\mathrm{SO}(3)$ on $\mathbf{R}^3$, namely, $i(\mathrm{Ad}_O X) = Oi(X)$.

(1.2) **Proof**:

1. For $X = \sum_{i=1}^{3} x_i E_i$ and $Y = \sum_{i=1}^{3} y_i E_i$,

$$k(X,Y) = x_1y_1 + x_2y_2 + x_3y_3 = (x,y) = (i(X), i(Y)).$$

2. For $X = \sum_{i=1}^{3} x_i E_i$ and $Y = \sum_{i=1}^{3} y_i E_i$,

$$i([X,Y]) = i\Big(\sum_{i,j,k=1}^{3} \varepsilon_{ijk} x_i y_j E_k\Big) = \sum_{i,j,k=1}^{3} \varepsilon_{ijk} x_i y_j e_k = x \times y = i(X) \times i(Y) = Xy.$$

3. For $O \in \mathrm{SO}(3)$ define the mapping $A_O : \mathbf{R}^3 \to \mathbf{R}^3 : x \mapsto i(\mathrm{Ad}_O\, i^{-1}(x))$. Then $A_O \in \mathrm{O}(3)$, because

$$(A_O x, A_O y) = k(\mathrm{Ad}_O\, i^{-1}x, \mathrm{Ad}_O\, i^{-1}y) = k(i^{-1}x, i^{-1}y) = (x,y).$$

In fact $A_O \in \mathrm{SO}(3)$, because the map $\sigma : \mathrm{SO}(3) \to \mathrm{O}(3) : O \mapsto A_O$ is continuous and sends the identity element into itself. Since σ is a group homomorphism and

$$T_e\sigma : T_e\,\mathrm{SO}(3) \to T_e O(3) = T_e\,\mathrm{SO}(3) : X \mapsto i \circ \mathrm{ad}_X \circ i^{-1} = X,$$

it follows that σ is the inclusion map. Therefore $O = i \circ \mathrm{Ad}_O \circ i^{-1}$, that is, the map i intertwines the adjoint action of $\mathrm{SO}(3)$ on $\mathrm{so}(3)$ with the usual action of $\mathrm{SO}(3)$ on $\mathbf{R}^3$. □

▷ We now show that every element in $\mathrm{so}(3)$ has a *normal form*. More precisely, we show that for every element $X \in \mathrm{so}(3)$ there is an $O \in \mathrm{SO}(3)$ such that $\mathrm{Ad}_O X = rE_1$, where $r = |X|$.

(1.3) **Proof**: Using the *standard basis* $\{e_1, e_2, e_3\}$ of $\mathbf{R}^3$ the matrix of X is $\sum_{i=1}^{3} x_i E_i$. If $X = 0$ then X is already in normal form. So suppose that $X \neq 0$. Then $r = |X| > 0$ and the unit vector $x = \frac{1}{r}(x_1, x_2, x_3)$ is an eigenvector of X corresponding to the eigenvalue 0. Let Π be the plane orthogonal to the line spanned by the vector x. Π is invariant under X, for if $y \in \Pi$, then $Xy \in \Pi$ because $(Xy, x) = -(y, Xx) = 0$. On $(\Pi, (\ ,\ )|\Pi)$ the mapping $X|\Pi$ is skew symmetric and has characteristic polynomial $\lambda^2 + r^2$. Let $y \in \Pi$ be a vector of unit length. Then $\{y, \frac{1}{r}Xy\}$ is an *orthonormal basis* of Π because $(y, Xy) = -(Xy, y) = -(y, Xy)$ implies that $(y, \frac{1}{r}Xy) = 0$ and $(\frac{1}{r}Xy, \frac{1}{r}Xy) = (y, -\frac{1}{r^2}X^2y) = (y,y) = 1$. Thus the matrix $O^{-1} = \mathrm{col}(x, y, \frac{1}{r}Xy)$ of column vectors is orthogonal. In fact O^{-1} is in $\mathrm{SO}(3)$, because $\det\mathrm{col}(x, y, \frac{1}{r}Xy) = \frac{1}{r}(x \times y, Xy) = \frac{1}{r}(x \times y, x \times y) > 0$. The matrix of X with respect to the ordered orthonormal basis $\{x, y, \frac{1}{r}Xy\}$ is $Y = rE_1$. Clearly $r = |Y| = |X|$. □

▷ As a corollary of the above normal form, we find that for $X \in \mathrm{so}(3)$ the linear map $\mathrm{ad}_X : \mathrm{so}(3) \to \mathrm{so}(3) : Y \mapsto [X,Y]$ has eigenvalues $0, \pm ir$.

(1.4) **Proof**: For $O \in \mathrm{SO}(3)$, it follows that $\mathrm{Ad}_O(\mathrm{ad}_X)(\mathrm{Ad}_O)^{-1} = \mathrm{ad}_{\mathrm{Ad}_O X}$, since

$$\mathrm{Ad}_O(\mathrm{ad}_X)(\mathrm{Ad}_O)^{-1}Y = \mathrm{Ad}_O\,[X, \mathrm{Ad}_{O^{-1}}Y] = [\mathrm{Ad}_O X, Y].$$

Therefore ad_X has the same eigenvalues as $\mathrm{ad}_{\mathrm{Ad}_O X}$. Choosing $O \in \mathrm{SO}(3)$ so that $\mathrm{Ad}_O X = rE_1$, we see that ad_X has the same eigenvalues as $\mathrm{ad}_{rE_1} = r\,\mathrm{ad}_{E_1}$. But $\mathrm{ad}_{E_1} E_1 = 0, \mathrm{ad}_{E_1} E_2 = E_3$, $\mathrm{ad}_{E_1} E_3 = -E_2$. Therefore the matrix of ad_{rE_1} with respect to the basis $\{E_1, E_2, E_3\}$ is rE_1. Consequently ad_X has eigenvalues $0, \pm ir$. □

1.2 The exponential map

We now derive some basic properties of the *exponential mapping*

$$\exp : \mathrm{so}(3) \to \mathrm{gl}(3,\mathbf{R}) : X \mapsto \sum_{n=0}^{\infty} \frac{1}{n!} X^n .$$

▷ We start by showing that the image of exp is contained in SO(3).

(1.5) **Proof**: Since $X \in \mathrm{so}(3)$, it is skew symmetric. So

$$(\exp X)^t = \sum_{n=0}^{\infty} \frac{1}{n!} (X^t)^n = \sum_{n=0}^{\infty} \frac{1}{n!} (-X)^n = \exp(-X) = (\exp X)^{-1}.$$

Therefore the image of exp is contained in O(3). But exp is continuous and $\exp 0 = I$. Hence the image of exp is contained in SO(3). □

▷ Next we show that for a nonzero $X \in \mathrm{so}(3)$

$$\exp X = I + \frac{\sin r}{r} X + \frac{1-\cos r}{r^2} X^2, \tag{3}$$

where $r = |X|$. Since $\lim_{r\to 0} \frac{\sin r}{r} = 1$ and $\lim_{r\to 0} \frac{1-\cos r}{r^2} = \frac{1}{2}$, equation (3) is defined when $X = 0$ and gives $\exp 0 = I$.

(1.6) **Proof**: For $X \in \mathrm{so}(3)$, a calculation shows that its characteristic polynomial is $\lambda^3 + r^2\lambda$. Therefore $X^3 + r^2 X = 0$, from which the formulæ

$$X^{2n+1} = (-1)^n r^{2n} X \quad \text{and} \quad X^{2n+2} = (-1)^n r^{2n} X^2 \quad n \ge 0$$

follow by induction. Substituting these expressions into the power series for exp gives equation (3). □

When $|X| = 1$ we obtain the special case of (3):

$$(\exp sX)y = y + \sin s\,(x \times y) + (1-\cos s)\,(x \times (x \times y)), \tag{4}$$

where $x = i(X)$ and $y \in \mathbf{R}^3$. Because $(\exp sX)x = x$, equation (4) defines a *one parameter group* of rotations about the axis x. If $\{x, y\}$ are orthonormal vectors in $\mathbf{R}^3$, then (4) becomes

$$(\exp sX)y = y\cos s + (x \times y)\sin s, \tag{5}$$

since $x \times (x \times y) = -y$.

Because the function $r^2 : \mathrm{so}(3) \to \mathbf{R} : X \mapsto k(X,X)$ is differentiable, the functions $\sin r/r$ and $(1-\cos r)/r^2$ are differentiable. From (3) it follows that exp is differentiable. Next
▷ we prove the following formula for the derivative of exp:

$$\exp(-X)(D\exp X) = \big(1 - \exp(-\mathrm{ad}_X)\big)/\,\mathrm{ad}_X . \tag{6}$$

The right hand side of (6) is to be thought of as a power series in ad_X.

(1.7) **Proof**: Consider the function

$$Z : \mathbf{R}^2 \to \mathrm{gl}(3,\mathbf{R}) : (s,t) \mapsto Z(s,t) = \exp(-sX - stY)\frac{\partial}{\partial t}\exp(sX + stY).$$

Then $Z(0,0) = 0$, while

$$Z(1,0) = \exp(-X)\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\exp(X+tY) = \exp(-X)(D\exp X)Y.$$

Differentiating $Z(s,t)$ with respect to s gives

$$\begin{aligned}
\tfrac{\partial}{\partial s}Z(s,t) &= \tfrac{\partial}{\partial s}(\exp(-sX-stY))\tfrac{\partial}{\partial t}\exp(sX+stY) + \exp(-sX-stY)\tfrac{\partial^2}{\partial s\partial t}\exp(sX+stY)\\
&= -\exp(-sX-stY)(X+tY)\tfrac{\partial}{\partial t}\exp(sX+stY) + \exp(-sX-stY)\tfrac{\partial^2}{\partial t\partial s}\exp(sX+stY)\\
&= -\exp(-sX-stY)(X+tY)\tfrac{\partial}{\partial t}\exp(sX+stY) + \exp(-sX-stY)\tfrac{\partial}{\partial t}\big((X+tY)\exp(sX+stY)\big)\\
&= \exp(-sX-stY)Y\exp(sX+stY).
\end{aligned}$$

Consequently,

$$Z(1,0) = \int_0^1 \frac{\mathrm{d}}{\mathrm{d}s}Z(s,0)\,\mathrm{d}s = \int_0^1 (\exp -sX)\,Y\,(\exp sX)\,\mathrm{d}s = \int_0^1 \exp(-s\,\mathrm{ad}_X)Y\,\mathrm{d}s.$$

The last equality follows because

1. For every $s \in \mathbf{R}$ the linear map $\gamma(s) : \mathrm{so}(3) \to \mathrm{so}(3) : Y \mapsto (\exp -sX)Y(\exp sX)$ is invertible.
2. The map $\gamma : \mathbf{R} \to \mathrm{Gl}(\mathrm{so}(3),\mathbf{R}) : s \mapsto \gamma(s)$ is a one parameter subgroup.
3. $\gamma'(0) = -\,\mathrm{ad}_X$.

Therefore γ is the one parameter subgroup $s \mapsto \exp(-s\,\mathrm{ad}_X)$. Expanding $\exp(-s\,\mathrm{ad}_X)$ in a power series in s and integrating with respect to s gives

$$Z(1,0) = \big(1 - \frac{1}{2!}\mathrm{ad}_X + \frac{1}{3!}(\mathrm{ad}_X)^2 - \frac{1}{4!}(\mathrm{ad}_X)^3 + \cdots\big)Y = \left(\frac{1-\exp(-\mathrm{ad}_X)}{\mathrm{ad}_X}\right)Y. \quad \square$$

1.3 The solid ball model

In this subsection we will show that the rotation group $\mathrm{SO}(3)$ is homeomorphic to a closed *solid ball* in $\mathbf{R}^3$ of radius π with antipodal points on its bounding 2-sphere identified. Such a homeomorphism can be constructed by sending the rotation O to the vector in the solid ball, which is the axis of rotation of O, normalized so that its length is the amount of rotation O makes in a right handed sense about its rotation axis. Our proof shows that this homeomorphism is defined by the exponential map.

Let $D^3_\pi = \{X \in \mathrm{so}(3) \,|\, k(X,X) < \pi^2\}$ be an open 3-ball in $(\mathrm{so}(3),k)$ of radius π and let $S^2_\pi = \{X \in \mathrm{so}(3) \,|\, k(X,X) = \pi^2\}$ be a 2-sphere of radius π, which is the boundary ∂D^3_π of the 3-ball D^3_π. The proof of the solid ball model of $\mathrm{SO}(3)$ takes three steps.

1. The exponential map $\exp : \overline{D}^3_\pi \subseteq \mathrm{so}(3) \to \mathrm{SO}(3)$ is a diffeomorphism of D^3_π onto its image and is continuous on its closure $\overline{D}^3_\pi$.
2. The image of $\overline{D}^3_\pi$ under the exponential mapping is $\mathrm{SO}(3)$.
3. On $\partial \overline{D}^3_\pi = S^2_\pi$ the exponential map is two to one.

(1.8) **Proof**:

Step 1.

From (6) we see that the derivative of the exponential map is invertible if and only if the linear mapping $Z = (1-\exp(-\mathrm{ad}_X))/\,\mathrm{ad}_X$ is invertible. If $X \neq 0$ then $\pm ir, 0$ are eigenvalues of ad_X. Consequently, we find that $(1-e^{-ir})/ir$, $(e^{ir}-1)/ir$ and 1 are eigenvalues of Z. For $X \in D^3_\pi \setminus \{0\}$, $0 < r < \pi$. Hence Z has no zero eigenvalues. If $X = 0$ then $1, 1, 1$ are eigenvalues of Z. Thus $D\exp X$ is invertible for all $X \in D^3_\pi$. It is clearly continuous in D^3_π.

To show that exp is a diffeomorphism of D^3_π onto its image, we need only verify that $\exp|D^3_\pi$ is one to one. Toward this end, suppose that for some $X, Y \in D^3_\pi$ we have $\exp X = \exp Y$. Let $r = |X|$ and $s = |Y|$. Furthermore, suppose that r and s are greater than zero. The proofs of the other cases are omitted. Then $r, s \in (0, \pi)$. Using (3) we obtain

$$\frac{\sin r}{r} X + \frac{(1-\cos r)}{r^2} X^2 = \frac{\sin s}{s} Y + \frac{(1-\cos s)}{s^2} Y^2. \tag{7}$$

Subtracting (7) from its transpose and then dividing by 2 gives $\frac{\sin r}{r} X = \frac{\sin s}{s} Y$. Therefore

$$\sin^2 r = \frac{\sin^2 r}{r^2} k(X,X) = \frac{\sin^2 s}{s^2} k(Y,Y) = \sin^2 s.$$

Because $r, s \in (0, \pi)$, it follows that $\sin r = \sin s$ and therefore $\frac{1}{r} X = \frac{1}{s} Y$. Adding (7) to its transpose and using the preceding equation gives $\cos r = \cos s$, once we have noted that $X^2 \neq 0$. Therefore $r = s$, since $r, s \in (0, \pi)$. Consequently, $X = Y$.

From (3) we see that the exponential mapping is continuous on $\overline{D}^3_\pi$.

Step 2.

To show that the image of $\overline{D}^3_\pi$ under the exponential mapping is $\mathrm{SO}(3)$, we need the following normal form for rotations. Given $A \in \mathrm{SO}(3)$, there is an $O \in \mathrm{SO}(3)$ such that $\mathrm{Ad}_O A = \exp \theta E_1$ for some $\theta \in [0, \pi]$.

(1.9) **Proof**: Since $A \in \mathrm{SO}(3)$, A has an eigenvalue $+1$. This follows from

$$\det(A - I) = \det\big((I-A)^t A\big) = \det(I - A) = -\det(A - I),$$

which implies $\det(A - I) = 0$. In other words, A leaves the line spanned by the eigenvector x corresponding to the eigenvalue 1 pointwise fixed. This line is called the *axis of rotation* of A. Normalize x so that its length is 1. Let Π be the plane in $\mathbf{R}^3$ orthogonal to x. Π is invariant under A, because if $y \in \Pi$ then $(Ay, x) = (Ay, Ax) = (y, x) = 0$, that is, $Ay \in \Pi$. Therefore $A|\Pi$ is an orthogonal linear map on $(\Pi, (\,,\,)|\Pi)$. Let y be a unit vector in Π. Then $\{y, x \times y\}$ is an orthonormal basis of Π, because $(y, x \times y) = \det \mathrm{col}(y, x, y) = 0$ and $(x \times y, x \times y) = (x,x)(y,y) - (x,y)^2 = 1$. With respect to the orthonormal basis $\{x, y, x \times y\}$

of $\mathbf{R}^3$ the matrix of A is $\left(\begin{smallmatrix} 1 & 0 \\ 0 & \widetilde{A} \end{smallmatrix}\right)$. Since $1 = \det A = \det \widetilde{A} = \det A|\Pi$, the map $A|\Pi$ lies in $\mathrm{SO}(2)$. Hence the matrix of $A|\Pi$ with respect to the orthonormal basis $\{y, x \times y\}$ is $\widetilde{A} = \left(\begin{smallmatrix} \cos\theta' & -\sin\theta' \\ \sin\theta' & \cos\theta' \end{smallmatrix}\right)$ for some $\theta' \in [0, 2\pi]$. If $\theta' \in [0, \pi]$, we are done. Otherwise, $\sin\theta' < 0$. Now use the orthonormal basis $\{-x, y, y \times x\}$. With respect to the orthonormal basis $\{y, y \times x\}$ the matrix of $A|\Pi$ is $\left(\begin{smallmatrix} \cos\theta'' & -\sin\theta'' \\ \sin\theta'' & \cos\theta'' \end{smallmatrix}\right)$ where $\theta'' = 2\pi - \theta' \in [0, \pi]$. Because either $\det \mathrm{col}(x, y, x \times y) = (x \times y, x \times y) > 0$, or $\det \mathrm{col}(-x, y, y \times x) = (-x \times y, y \times x) > 0$, the matrix O^{-1} formed by taking the vectors $\{x, y, x \times y\}$ or $\{-x, y, y \times x\}$ for its columns lies in $\mathrm{SO}(3)$. Moreover, $\mathrm{Ad}_O A = OAO^{-1} = \exp\theta E_1$, where $\theta = \theta'$ or θ''. $\square$

The second step is proved by noting that $A = \mathrm{Ad}_{O^{-1}}(\exp\theta E_1) = \exp(\theta\, \mathrm{Ad}_{O^{-1}} E_1)$ and

$$k(\theta\, \mathrm{Ad}_{O^{-1}} E_1, \theta\, \mathrm{Ad}_{O^{-1}} E_1) = \theta^2 k(E_1, E_1) = \theta^2 \le \pi^2.$$

Step 3.
Suppose that $X, Y \in S^2_\pi$ and $\exp X = \exp Y$. Then using (3) we obtain

$$1 + \frac{2}{\pi^2} X^2 = \exp X = \exp Y = 1 + \frac{2}{\pi^2} Y^2$$

which implies that $X = \pm Y$. Conversely, if $X = \pm Y$ then $\exp X = \exp Y$. Therefore on S^2_π the exponential mapping is two to one with $\exp X = \exp(-X)$. In other words, exp maps antipodal points on S^2_π to the same element of $\mathrm{SO}(3)$.

This establishes the solid ball model of $\mathrm{SO}(3)$. $\square$

1.4 The sphere bundle model

In this subsection we describe the *sphere bundle model* of the rotation group.

Let

$$T_1S^2 = \big\{(x,y) \in T\mathbf{R}^3 = \mathbf{R}^3 \times \mathbf{R}^3 \,\big|\, (x,x) = 1,\ (x,y) = 0 \ \&\ (y,y) = 1\big\}$$

be the unit tangent sphere bundle to the unit 2-sphere S^2. In other words, T_1S^2 is the set of all *ordered pairs of orthonormal vectors* in $\mathbf{R}^3$. T_1S^2 is diffeomorphic to the rotation group $\mathrm{SO}(3)$ via the smooth map

$$\varphi : T_1S^2 \subseteq T\mathbf{R}^3 \to \mathrm{SO}(3) \subseteq \mathbf{R}^9 : (x,y) \mapsto \mathrm{col}(x, y, x \times y), \tag{8}$$

whose smooth inverse is the restriction to $\mathrm{SO}(3)$ of the projection

$$\widehat{\pi} : \mathbf{R}^9 \to T\mathbf{R}^3 : A = \mathrm{col}(a_1, a_2, a_3) \mapsto (a_1, a_2). \tag{9}$$

T_1S^2 can be made into a Lie group by pushing forward the Lie group structure on $\mathrm{SO}(3)$ via the mapping $\widehat{\pi}$ (9). In more detail, define a multiplication $\cdot$ on T_1S^2 by

$$\begin{aligned} (x,y)\cdot(z,w) &= \widehat{\pi}\big(\,\mathrm{col}(x,y,x\times y)\cdot \mathrm{col}(z,w,z\times w)\big) \\ &= \big(z_1x + z_2y + z_3(x\times y), w_1x + w_2y + w_3(x\times y)\big). \end{aligned} \tag{10}$$

Observe that $e = (e_1, e_2)$ is the identity element of $(T_1S^2, \cdot)$.

Since

$$T_{(x,y)}(T_1S^2) = \{(u,v) \in T\mathbf{R}^3 \,\big|\, (x,u) = 0,\ (u,y) + (x,v) = 0 \ \&\ (y,v) = 0\},$$

we find that

$$T_e(T_1S^2) = \{\mathbf{u} = \big((0,u_3,-u_2),(-u_3,0,u_1)\big) \in T\mathbf{R}^3 \,\big|\, (u_1,u_2,u_3) \in \mathbf{R}^3\}. \tag{11}$$

In other words,

$$T_e(T_1S^2) = \left\{ \mathbf{u} = \widehat{\pi}(U) \in T\mathbf{R}^3 \,\middle|\, U = \begin{pmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{pmatrix} \in \mathrm{so}(3) \right\},$$

which has the standard basis $\{\varepsilon_i = \widehat{\pi}(E_i)\}$. Define a Lie bracket $[\,,\,]$ on $T_e(T_1S^2)$ by

$$[\mathbf{u},\mathbf{v}] = \widehat{\pi}([U,V]), \tag{12}$$

for $\mathbf{u}, \mathbf{v} \in T_e(T_1S^2)$. In terms of the standard basis $\{\varepsilon_i\}$ we obtain the bracket relations
▷ $[\varepsilon_i,\varepsilon_j] = \sum_k \varepsilon_{ijk}\varepsilon_k$. The following argument shows that $(T_e(T_1S^2),[\,,\,])$ is the Lie algebra of the Lie group $(T_1S^2,\cdot)$.

(1.10) **Proof**: For every $(x,y) \in T_1S^2$ define left translation by (x,y) as

$$L_{(x,y)} : T_1S^2 \to T_1S^2 : (z,w) \mapsto (x,y)\cdot(z,w) \tag{13}$$

and right translation by (x,y) by $R_{(x,y)} : T_1S^2 \to T_1S^2 : (z,w) \mapsto (z,w)\cdot(x,y)$. Hence we may define the diffeomorphism

$$\mathrm{Int}_{(x,y)} : T_1S^2 \to T_1S^2 : (z,w) \mapsto L_{(x,y)} \circ R_{(x,y)^{-1}}(z,w),$$

which induces the group homomorphism $\mathrm{Int} : T_1S^2 \to \mathrm{Diff}(T_1S^2) : (x,y) \mapsto \mathrm{Int}_{(x,y)}$. Here $(\mathrm{Diff}(T_1S^2),\circ)$ is the group of diffeomorphisms of T_1S^2 with composition $\circ$ as multiplication. Differentiating $\mathrm{Int}_{(x,y)}$ at e, we obtain the linear map

$$\mathrm{Ad}_{(x,y)} : T_e(T_1S^2) \to T_e(T_1S^2) : \mathbf{v} \mapsto T_e\mathrm{Int}_{(x,y)}\mathbf{v} = \widehat{\pi}(\mathrm{Ad}_{\mathrm{col}(x,y,x\times y)}V),$$

which gives rise to the group homomorphism

$$\mathrm{Ad} : T_1S^2 \to \mathrm{Gl}(T_e(T_1S^2),\mathbf{R}) : (x,y) \mapsto \mathrm{Ad}_{(x,y)}.$$

Differentiating Ad at e gives the linear map $\mathrm{ad} : T_e(T_1S^2) \to \mathrm{gl}(T_e(T_1S^2),\mathbf{R}) : \mathbf{u} \mapsto \mathrm{ad}_{\mathbf{u}}$, where

$$\mathrm{ad}_{\mathbf{u}}\mathbf{v} = T_e(\mathrm{Ad}_{(x,y)}\mathbf{v})\mathbf{u} = \widehat{\pi}\big(T_I(\mathrm{Ad}_{\mathrm{col}(x,y,x\times y)}V)U\big) = \widehat{\pi}(\mathrm{ad}_U V) = \widehat{\pi}([U,V]).$$

The Lie bracket on $T_e(T_1S^2)$ is defined by $[\mathbf{u},\mathbf{v}] = \mathrm{ad}_{\mathbf{u}}\mathbf{v}$, which agrees with (12). □

If we identify the $T_e(T_1S^2)$ with $\mathbf{R}^3$ using the mapping

$$i : T_e(T_1S^2) \to \mathbf{R}^3 : \mathbf{u} = \left(\begin{pmatrix} 0 \\ u_3 \\ -u_2 \end{pmatrix}, \begin{pmatrix} -u_3 \\ 0 \\ u_1 \end{pmatrix}\right) \mapsto u = (u_1,u_2,u_3),$$

it follows that i is an isomorphism of the Lie algebra $(T_e(T_1S^2), [\, ,\,])$ with the Lie algebra $(\mathbf{R}^3, \times)$, since $i(\varepsilon_i) = e_i$. In other words, $i([\mathbf{u}, \mathbf{v}]) = i(\mathbf{u}) \times i(\mathbf{v})$ for every $\mathbf{u}, \mathbf{v} \in T_e(T_1S^2)$. Define the Killing metric k on $T_e(T_1S^2)$ as follows. For $\mathbf{u}, \mathbf{v} \in T_e(T_1S^2)$ let

$$k(\mathbf{u}, \mathbf{v}) = (i(\mathbf{u}), i(\mathbf{v})), \tag{14}$$

where $(\, ,\,)$ is the Euclidean inner product on $\mathbf{R}^3$. The Killing metric k is infinitesimally Ad-invariant because for $\mathbf{t}, \mathbf{u}, \mathbf{v} \in T_e(T_1S^2)$

$$k(\mathbf{t}, [\mathbf{u}, \mathbf{v}]) = (i(\mathbf{t}), i(\mathbf{u}) \times i(\mathbf{v})) \, = \, (i(\mathbf{t}) \times i(\mathbf{u}), i(\mathbf{v})) \, = \, k([\mathbf{t}, \mathbf{u}], \mathbf{v}).$$

Since T_1S^2 is connected, it follows that k is Ad-invariant, that is, $k(\mathrm{Ad}_{(x,y)}\mathbf{u}, \mathrm{Ad}_{(x,y)}\mathbf{v}) = k(\mathbf{u}, \mathbf{v})$, for every $(x, y) \in T_1S^2$.

▷ We now turn to discussing the geometry of the sphere bundle model. One of the advantages of the sphere bundle model of $\mathrm{SO}(3)$ is that T_1S^2 is the total space of an S^1-principal bundle over S^2 with bundle projection

$$\tau : T_1S^2 \subseteq T\mathbf{R}^3 \to S^2 \subseteq \mathbf{R}^3 : (x, y) \mapsto x. \tag{15}$$

(1.11) **Proof**: A calculation shows that for every $(v, w) \in T_{(x,y)}(T_1S^2)$ we have $T_{(x,y)}\tau(v, w) = v$. Since $v \in T_xS^2$, it follows that τ is a submersion. In addition, τ is a surjective proper map. Therefore by the Ehresmann theorems of chapter VIII §2, τ is a locally trivial bundle with fiber S^1.

To show that τ is a principal bundle, see chapter VII §2.2, we argue as follows. Consider the $S^1 = \mathbf{R}/2\pi\mathbf{Z}$-action

$$\Psi : S^1 \times T_1S^2 \to T_1S^2 : \big(t, (x, y)\big) \mapsto \big(x, \exp(-tX)y\big) = \big(x, y\cos t - (x \times y)\sin t\big), \tag{16}$$

where X is the skew symmetric matrix $i^{-1}(x)$ and i is the map given by (2). The action Ψ has the following properties.

1. Ψ_t preserves the fibers of the bundle τ. In other words, Ψ_t maps the unit circle in T_xS^2 into itself.
2. The action Ψ is proper, since S^1 is compact.
3. The action Ψ is free, because if $(x, y) = \Psi_t(x, y) \, = \, (x, y\cos t - (x \times y)\sin t)$, then $y = y\cos t - (x \times y)\sin t$. Taking the inner product of both sides of this last equation with y and using the fact that $(y, y) = 1$, yields $1 = \cos t$, that is, $t = 2\pi n$ for some $n \in \mathbf{Z}$. Hence $t = e \in S^1$.

Therefore T_1S^2 is the total space of an S^1 principal bundle over the smooth orbit space $V = T_1S^2/S^1$ with bundle projection $\lambda : T_1S^2 \to V$, see chapter VII ((2.12)). Since τ is invariant under Ψ, it induces a smooth map $\sigma : V \to S^2$ which makes the diagram 1.4.1 commute. The map σ is surjective because τ is. Also σ is injective, because the fiber of τ is a unique S^1 orbit of Ψ by property 1. Therefore σ is a homeomorphism, since V is a compact Hausdorff space. To verify that σ is a diffeomorphism, it suffices to show that for every $v \in V$, the tangent map $T_v\sigma : T_vV \to T_{\sigma(v)}S^2$ is injective, because $\dim T_vV =$

$$\begin{tikzcd} T_1S^2 \arrow[r, "id"] \arrow[d, "\lambda"] & T_1S^2 \arrow[d, "\tau"] \\ V \arrow[r, "\sigma"] & S^2 \end{tikzcd}$$

Diagram 1.4.1

$\dim T_{\sigma(v)}S^2$. Towards this goal suppose that $0 = T_v\sigma(w_v)$ for some $w_v \in T_vV$. Since λ is a surjective submersion, there is a $w_{(x,y)} \in T_{\sigma(v)}(T_1S^2)$ such that $(T_{(x,y)}\lambda)w_{(x,y)} = w_v$ and $\lambda(x,y) = v$. Therefore $0 = (T_v\sigma \circ T_{(x,y)}\lambda)w_{(x,y)} = (T_{(x,y)}\tau)w_{(x,y)}$, since $\tau = \sigma \circ \lambda$. In other words

$$\begin{aligned} w_{(x,y)} \in \ker T_{(x,y)}\tau &= T_{(x,y)}\tau^{-1}(x), && \text{since } \tau \text{ is a smooth bundle} \\ &= T_{(x,y)}\lambda^{-1}(v), && \text{since } \tau^{-1}(x) \text{ and } \lambda^{-1}(v) \text{ are the same } S^1 \text{ orbit} \\ &= \ker T_{(x,y)}\lambda, && \text{since } \lambda \text{ is a smooth bundle.} \end{aligned}$$

Therefore $w_v = 0$. Thus τ and λ are isomorphic bundles. In fact they are isomorphic S^1 principal bundles, since the horizontal arrows in diagram 1.4.1 are diffeomorphisms and the map *id* interwines the S^1-action Ψ on T_1S^2 with itself. □

Now consider the diagram

$$\begin{tikzcd} S^3 \arrow[r, "h"] \arrow[d, "\rho"] & S^2 \\ \mathrm{SO}(3) \arrow[d, "\widehat{\pi}"] & \\ T_1S^2 \arrow[uur, "\tau"'] & \end{tikzcd}$$

Diagram 1.4.2

Here h is the Hopf fibration, see chapter I ((4.2)), $\widehat{\pi}$ is the map (9) and $\rho : S^3 \to \mathrm{SO}(3)$ is a two to one covering map, see (17) below.

Claim: The bundle projection $\tau : T_1S^2 \to S^2$ is double covered by the Hopf fibration h.

(1.12) **Proof**: We start by defining the covering map $\rho : S^3 \to \mathrm{SO}(3)$. This involves an extensive excursion into *quaternions*.

As a real vector space the set of quaternions $\mathbf{H}$ has a basis $\{1, i, j, k\}$. Thus every $q \in \mathbf{H}$

can be written uniquely as $q = q_0\,1 + q_1\,i + q_2\,j + q_3\,k$, where $(q_0, q_1, q_2, q_3) \in \mathbf{R}^4$. We make $\mathbf{H}$ into an associative noncommutative algebra by defining a multiplication $\cdot$ on its basis elements by requiring

1. 1 commutes with i, j and k.
2. $i^2 = j^2 = k^2 = -1$, $i \cdot j = -j \cdot i$, $j \cdot k = -k \cdot j$ and $k \cdot i = -i \cdot k$.
3. $i \cdot j = k$, $j \cdot k = i$ and $k \cdot i = j$.

Using properties 1 – 3 and the distributive law, it follows that multiplication is defined for any two quaternions. For $q \in \mathbf{H}$ define $\overline{q}$ by $\overline{q} = q_0\,1 - q_1\,i - q_2\,j - q_3\,k$. It is straightforward to check that $\overline{q \cdot p} = \overline{p} \cdot \overline{q}$. The 3-sphere $S^3 \subseteq \mathbf{H}$ is the set of all quaternions q such that $\overline{q} \cdot q = 1$. Check that S^3 is a Lie group under quaternionic multiplication with identity element $e = 1$. Identify $\mathbf{R}^3$ with the vector subspace of $\mathbf{H}$ spanned by the vectors i, j and k. For every $q \in S^3 \subseteq \mathbf{H}$ consider the linear map $L_q : \mathbf{R}^3 \to \mathbf{R}^3 : x \mapsto q \cdot x \cdot \overline{q}$
▷ The map L_q is orthogonal.

(1.13) **Proof**: First observe that for $x = x_1\,i + x_2\,j + x_3\,k \in \mathbf{R}^3 \subseteq \mathbf{H}$ we have $x \cdot \overline{x} = x_1^2 + x_2^2 + x_3^2 = (x, x)$, where $(\ ,\ )$ is the Euclidean inner product on $\mathbf{R}^3$. Consequently, for $x, y \in \mathbf{R}^3 \subseteq \mathbf{H}$

$$4(x, y) = (x + y) \cdot (\overline{x + y}) - (x - y) \cdot (\overline{x - y}).$$

Since L_q is a linear map, it suffices to show that it is length preserving. We verify this as follows: $L_q x \cdot \overline{L_q x} = q \cdot x \cdot \overline{x} \cdot \overline{q} = q \cdot \overline{q} \cdot x \cdot \overline{x} = x \cdot \overline{x}$, since $\overline{q} \cdot q = 1$. □

Thus we have a map

$$\rho : S^3 \to \mathrm{O}(3) : q \mapsto L_q. \qquad (17)$$

ρ is a group homomorphism, since

$$L_{p \cdot q} x = p \cdot q \cdot x \cdot \overline{p \cdot q} = p \cdot (q \cdot x \cdot \overline{q}) \cdot \overline{p} = (L_p \circ L_q) x.$$

Because $L_1 = id_{\mathbf{R}^3}$, ρ is continuous, and because S^3 is connected, the image of ρ lies
▷ in $\mathrm{SO}(3)$. The map ρ is a submersion and hence its image is *all* of $\mathrm{SO}(3)$.

(1.14) **Proof**: By its very definition, ρ is smooth. Because ρ is a group homomorphism, it suffices to show that $T_e\rho : T_e S^3 \to T_I\,\mathrm{SO}(3) = \mathrm{so}(3)$ is bijective. Since $(1, i) = (1, j) = (1, k) = 0$, we can identify $T_e S^3 = \{y \in \mathbf{H} \,|\, (y, 1) = 0\}$ with $\mathbf{R}^3 \subseteq \mathbf{H}$. Using this identification we assert that

$$T_e\rho : \mathbf{R}^3 \to \mathrm{so}(3) : x = x_1\,i + x_2\,j + x_3\,k \mapsto -2 \begin{pmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{pmatrix}. \qquad (18)$$

To see this, observe that the curves $t \to e^{ti}$, $t \to e^{tj}$, and $t \to e^{tk}$, which lie on S^3 and pass through 1 represent tangent vectors to S^3 at e in the direction i, j and k respectively, since $\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} e^{ti} = i$, $\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} e^{tj} = j$ and $\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} e^{tk} = k$. Therefore $T_e\rho(i)$ is the linear map on $\mathbf{R}^3$ given by

$$x \to \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} L_{e^{ti}} x = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} e^{ti} \cdot x \cdot e^{-ti} = i \cdot x - x \cdot i = -2(x_3 j - x_2 k).$$

Similarly, $T_e\rho(j)$ is the linear map $x \to -2(-x_3 i + x_1 k)$ and $T_e\rho(k)$ the linear map $x \to -2(x_2 i - x_1 j)$. This proves (18). Consequently the map $T_e\rho$ is bijective. Therefore ρ is a submersion. □

▷ Next we show that the kernel of ρ is $\mathbf{Z}_2$.

(1.15) **Proof**: Suppose that $L_q = id_{\mathbf{R}^3}$. Then for every $x \in \mathbf{R}^3 \subseteq \mathbf{H}$, we have $\overline{q} \cdot x \cdot q = x$, that is, $x \cdot q = q \cdot x$. Letting x be succesively i, j and k in the preceding equation, we see that $q = Q1$ where $Q \in \mathbf{R}$. Since $q \cdot \overline{q} = 1$, we have $Q^2 = 1$. Therefore $q = \pm 1$. Hence $\ker \rho = \mathbf{Z}_2$. □

Thus ρ is a two to one covering map. In other words, the twofold covering group of $\mathrm{SO}(3)$ is S^3. A geometric way of saying this is that if we act on S^3 by the fixed point free proper $\mathbf{Z}_2$-action generated by the map which sends the point q into $-q$, then the smooth orbit space, see chapter VII ((2.9)), is $\mathrm{SO}(3)$. Thus $\mathrm{SO}(3)$ is diffeomorphic to the space formed by identifying antipodal points on S^3, that is, real projective three space $\mathbf{RP}^3$.

$$\begin{tikzcd}
q = y_1 + x_1 i + y_2 j + x_2 k \arrow[r, "h"] \arrow[d, "\rho"] & L_q i = (y_1^2 + x_1^2 - y_2^2 - x_2^2, 2(x_1 y_2 - x_2 y_1), 2(x_1 x_2 + y_1 y_2)) \\
L_q = \mathrm{col}(L_q i, L_q j, L_q k) \arrow[d, "\widehat{\pi}"] & \\
(L_q i, L_q j) \arrow[uur, "\tau"] &
\end{tikzcd}$$

Diagram 1.4.3

We now look at diagram 1.4.3. The top horizontal map h is the Hopf fibration, see chapter I ((4.2)). The first vertical map ρ is the two to one submersion (17), while the second vertical map is the diffeomorphism $\widehat{\pi}$ (9). The diagonal map τ is the bundle projection (15). From diagram 1.4.3 it follows that diagram 1.4.2 commutes. This proves ((1.12)). □

▷ Next we determine the isomorphism type of the bundle τ.

(1.16) **Proof**: For $y \in S^2$, we know that the fiber $h^{-1}(y)$ of the *Hopf bundle* is a great circle on S^3, see chapter I ((4.2)). Let $\widetilde{\rho} = \widehat{\pi} \circ \rho$. From diagram 1.2 it follows that $\widetilde{\rho}(h^{-1}(y)) = \tau^{-1}(y)$. Moreover, the fiber of $\widetilde{\rho}$ is two antipodal points on S^3. Therefore, on fibers $\widetilde{\rho}$ is a two to one covering map. Let S^1_E be the equator of S^2 and let $\chi_h : S^1_E \to S^1 \subseteq S^3$ be the classifying map of the Hopf fibration, see chapter I. Then $\widetilde{\rho} \circ \chi_h$ is a classifying map $\chi_\tau : S^1_E \to S^1 \subseteq T_1 S^2$ of the bundle τ. We compute the degree of χ_τ as follows

$$\deg \chi_\tau = \deg \widetilde{\rho} \cdot \deg \chi_h = 2 \cdot 1 = 2.$$

This determines the homotopy class of χ_τ and hence the isomorphism type of τ. □

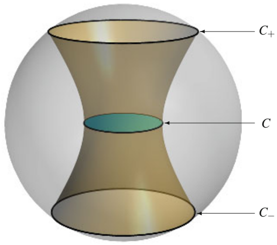

Figure 1.4.1. Solid ball model of SO(3).

We now give a way to visualize the bundle τ, see figure 1.4.1. Think of T_1S^2 as the closed solid ball $\overline{D}^3$ in $\mathbf{R}^3$: $x_1^2+x_2^2+x_3^2 \le 1$ with points on its boundary S^2: $x_1^2+x_2^2+x_3^2 = 1$ identified by the antipodal mapping $\mathbf{R}^3 \to \mathbf{R}^3 : x \mapsto -x$. Look at the piece of the hyperboloid H in $\overline{D}^3$ defined by $x_1^2+x_2^2-x_3^2 = \frac{1}{2}$ and $x_1^2+x_2^2+x_3^2 \le 1$ with waist C being the circle defined by $x_1^2+x_2^2 = \frac{1}{2}$ and $x_3 = 0$. This piece H intersects S^2 in two disjoint circles $C_\pm : x_1^2+x_2^2 = \frac{3}{4}$ with $x_3 = \pm\frac{1}{2}$. Orienting S^2 positively induces a positive orientation on C_+ and C_-. Under the antipodal map, the closed 2-disk $\overline{D}_+$ bounded by the circle C_+ is identified with the closed 2-disk $\overline{D}_-$ bounded by the circle C_-. Because the oriented circles $C_\pm$ agree after identification by the antipodal map, H is a 2-torus T^2 in the solid ball model and *not* a *Klein bottle*. In fact, H bounds a *solid torus* ST_1 formed by identifying the oriented end 2-disks $\overline{D}_+$ and $\overline{D}_-$ of the solid cylinder $x_1^2+x_2^2-x_3^2 \le \frac{1}{2}$ and $x_1^2+x_2^2+x_3^2 \le 1$.

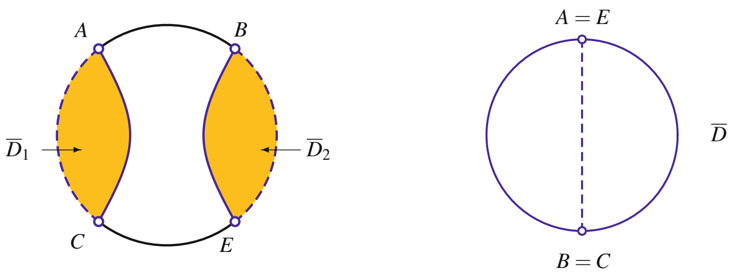

Figure 1.4.2. Solid tori in SO(3).

Claim: $\overline{T_1S^2 \setminus ST_1}$ is a solid torus ST_2.

(1.17) **Proof**: To see this, consider the slice $\{x_1 = 0\} \cap (\overline{T_1S^2 \setminus ST_1})$ given by the shaded region of the left 2-disk in figure 1.4.2. Before identification this slice is two disjoint closed 2-disks $\overline{D}_1$ and $\overline{D}_2$. Under the antipodal map, the boundary piece AC of $\overline{D}_1$ is identified with the boundary piece EB of $\overline{D}_2$, thus forming a closed 2-disk $\overline{D}$, see the right 2-disk in figure 1.4.2. The same argument holds for every slice of the solid ball model through the x_3-axis. Therefore $\overline{T_1S^2 \setminus ST_1}$ is diffeomorphic to $\overline{D} \times S^1$, which is a solid torus ST_2. $\square$

We have decomposed T_1S^2 into the *union of two solid tori* ST_1 and ST_2 which are identified along their common boundary T^2 by a map $\psi : T^2 \to T^2$. We now discuss the geometry of the *gluing map* ψ. Let $\overline{\mathscr{D}}$ be the closed 2-disk defined by $x_1^2 + x_2^2 \le \frac{1}{2}$ & $x_3 = 0$. Let S_1^m and S_2^m be the boundary $\partial\overline{\mathscr{D}}$ of $\overline{\mathscr{D}}$ and ∂D the boundary of $\overline{D}$, respectively, see figure 1.4.3. Clearly $\partial\overline{\mathscr{D}}$ and $\partial\overline{D}$ are closed curves on the 2-tori $\partial(ST_1)$ and $\partial(ST_2)$, respectively. Because $\overline{\mathscr{D}}$ and $\overline{D}$ are contractible in ST_1 and ST_2, respectively, the curves S_1^m and S_2^m are *meridians* on $\partial(ST_1)$ and $\partial(ST_2)$, respectively. Consider the light curve S_1^ℓ on $\partial(ST_1)$ drawn in figure 1.4.3. It is the same as the curve S_2^ℓ on $\partial(ST_2)$. The curve S_1^ℓ is closed, since it joins the points P and P' which are antipodal on S^2 and hence are identified. S_1^ℓ intersects $\overline{D}$ once at P. Similarly, the curve S_2^ℓ intersects $\overline{\mathscr{D}}$ once at Q. Therefore, S_1^ℓ and S_2^ℓ are *not* contractible in ST_1 and ST_2, respectively. Hence S_1^ℓ and S_2^ℓ are *longitudes* on

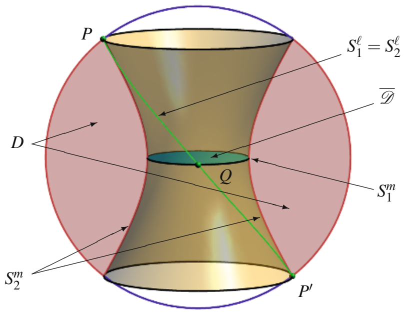

Figure 1.4.3. Solid tori in the solid ball model of SO(3).

$\partial(ST_1)$ and $\partial(ST_2)$, respectively. Thinking of S_2^m as a curve on $\partial(ST_1)$, we see that S_2^m intersects $\overline{\mathscr{D}}$ *twice* with intersection number of the *same* sign. Consequently, S_2^m is homotopic on $\partial(ST_1)$ to *two* times S_1^ℓ. Thus the gluing map $\psi : \partial(ST_2) = T^2 \to \partial(ST_1) = T^2$ takes the longitude S_2^ℓ onto the longitude S_1^ℓ by the identity map and the meridian S_2^m onto a curve on $\partial(ST_1)$ which is homotopic to two times the longitude S_1^ℓ. We now give a map, which up to homotopy, is the gluing map ψ. Identify T^2 with $\mathbf{R}^2/\mathbf{Z}^2$ and consider the linear map

$$\widetilde{\Psi} : \mathbf{R}^2 \to \mathbf{R}^2 : \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \mapsto A\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}.$$

Geometrically we visualize $\widetilde{\Psi}$ as in figure 1.4.4. Since $A \in \mathrm{Sl}(2,\mathbf{Z})$, the map $\widetilde{\Psi}$ preserves $\mathbf{Z}^2$ and hence induces a map $\Psi : T^2 \to T^2$. On T^2 the curves $\{\xi_1 = 0 \bmod 1\}$ and $\{\xi_2 = 0 \bmod 1\}$ are a *longitude* and a meridian, respectively. Under Ψ, the longitude $\{\xi_1 = 0 \bmod 1\}$ is mapped bijectively onto itself; whereas the meridian $\{\xi_2 = 0 \bmod 1\}$ is taken

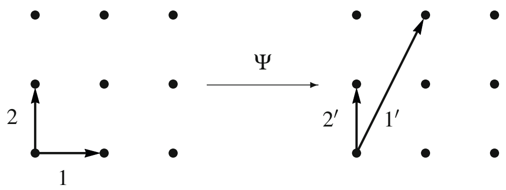

Figure 1.4.4. The gluing map of the solid tori in the solid ball model of SO(3).

onto the curve $\{(\xi_1, 2\xi_1) \in T^2 \,|\, \xi_1 \mod 1\}$, which is homotopic to two times the longitude $\{\xi_1 = 0 \mod 1\}$. Thus homotopically Ψ has the same properties as the gluing map ψ.

2 Left invariant geodesics

In this section we present two models for the Euler top: one based on SO(3) and the other on T_1S^2. Given a left invariant (dual) metric on the cotangent bundle, we consider the Hamiltonian formed by taking a cotangent vector to half its length squared. The pull back by left trivialization of Hamilton's equations on the cotangent bundle gives the Euler-Arnol'd equations, whose solutions describe the motion of the top in space.

2.1 Euler-Arnol'd equations on SO(3)$\times\mathbf{R}^3$

We begin by deriving the *Euler-Arnol'd equations* for the traditional SO(3) model of the Euler top.

Suppose that the initial position of the top is the *frame* $\{e_1, e_2, e_3\}$ in $\mathbf{R}^3$, which we identify with the identity element e of $\mathrm{SO}(3)$. An arbitrary position of the top is given by the frame $\{Be_1, Be_2, Be_3\}$, which is obtained by rotating the initial frame by B. We identify this new frame with the element B in $\mathrm{SO}(3)$. Thus the configuration space of the Euler top is the rotation group $\mathrm{SO}(3)$. The phase space is the cotangent bundle $T^*\mathrm{SO}(3)$ with its canonical symplectic form Ω, see chapter VI §2.

To describe the motion of the Euler top we need a Hamiltonian function on phase space. Towards this end, let ρ be a left invariant *(dual) metric* on $T^*\mathrm{SO}(3)$. In other words, for every $A \in \mathrm{SO}(3)$, $\rho(A)$ is a nondegenerate inner product on $T^*_A\mathrm{SO}(3)$ such that for every $B \in \mathrm{SO}(3)$

$$\rho(BA)(\alpha_{BA}, \beta_{BA}) = \rho(A)(\alpha_A, \beta_A). \tag{19}$$

The cotangent vector α_A is defined by the mapping

$$\mathscr{L} : \mathrm{SO}(3) \times \mathrm{so}(3)^* \to T^*\mathrm{SO}(3) : (A, \alpha) \mapsto (T_A L_{A^{-1}})^t \alpha \,=\, \alpha_A, \tag{20}$$

where $L_A : \mathrm{SO}(3) \to \mathrm{SO}(3) : B \mapsto AB$ is *left* translation. Thus $\mathscr{L}$ is a trivialization of the cotangent bundle $\tau : T^*\mathrm{SO}(3) \to \mathrm{SO}(3) : \alpha_A \mapsto A$. By definition α_A is the momentum of the top at the position A. From left invariance it follows that ρ is completely determined by its value at e. Because ρ is nondegenerate, $\rho(e)$ may be written as

$$\rho(e)(\alpha,\beta) = \kappa((I^{-1})^t\alpha,\beta). \tag{21}$$

Here κ is the (dual) metric on $\mathrm{so}(3)^*$ induced by the Killing metric k (1) on $\mathrm{so}(3)$. In particular, κ is defined by $\kappa(\alpha,\beta) = \beta(k^\flat(\alpha))$. The map $I : \mathrm{so}(3) \to \mathrm{so}(3)$, which is uniquely determined by (21), is k-symmetric and invertible. I is called the *moment of inertia tensor* of the Euler top, see exercise 5. Let I_i be the eigenvalues of I. They are real and nonzero and are called the *principal moments of inertia* of the top. Below we will show that we may suppose that the matrix of $\rho(e)$ with respect to the dual basis $\{E_i^* = k^\sharp(E_i)\}$ of $\mathrm{so}(3)^*$ is $\mathrm{diag}(I_1,I_2,I_3)$.

The motion of the Euler top on $(T^*\mathrm{SO}(3),\Omega)$ is given by the integral curves of the *Hamiltonian vector field* $X_{\mathscr{H}}$ corresponding to the *Hamiltonian*

$$\mathscr{H} : T^*\mathrm{SO}(3) \to \mathbf{R} : \alpha_A \mapsto \tfrac{1}{2}\rho(A)(\alpha_A,\alpha_A). \tag{22}$$

Because $\mathscr{H}$ assigns to a cotangent vector one half its ρ-length squared, the vector field $X_{\mathscr{H}}$ is the geodesic vector field on $T^*\mathrm{SO}(3)$ associated to the *left* invariant metric ρ. The image of an integral curve of $X_{\mathscr{H}}$ under the bundle projection τ is a geodesic on $\mathrm{SO}(3)$ for the left invariant metric ρ, see chapter VI §3.

To write out Hamilton's equations for $X_{\mathscr{H}}$ explicitly, we pull back the Hamiltonian system $(\mathscr{H}, T^*\mathrm{SO}(3),\Omega)$ by left trivialization $\mathscr{L}$ (20). We obtain the *equivalent Hamiltonian system* $(H, \mathrm{SO}(3)\times\mathrm{so}(3)^*,\omega)$, where the Hamiltonian $H = \mathscr{L}^*(\mathscr{H})$ is

$$H : \mathrm{SO}(3)\times\mathrm{so}(3)^* \to \mathbf{R} : (A,\alpha) \mapsto \tfrac{1}{2}\rho(e)(\alpha,\alpha) = \tfrac{1}{2}k\big(k^\flat\big((I^{-1})^t\alpha\big),k^\flat(\alpha)\big) \tag{23}$$

and the symplectic form $\omega = \mathscr{L}^*\Omega$ is

$$\omega(A,\alpha)\big((T_eL_AX,\beta),(T_eL_AY,\gamma)\big) = -\beta(Y)+\gamma(X)+\alpha([X,Y]), \tag{24}$$

for $X,Y \in \mathrm{so}(3)$ and $\beta,\gamma \in \mathrm{so}(3)^*$, see chapter VI §2 example $2'$. Because

$$T_{(A,\alpha)}\big(\mathrm{SO}(3)\times\mathrm{so}(3)^*\big) = \big\{(T_eL_AX = AX,\alpha)\big| X \in \mathrm{so}(3) \text{ and } \alpha \in \mathrm{so}(3)^*\big\},$$

we may write $X_H(A,\alpha) = (T_eL_AX_1,\alpha_1)$ for some $X_1 = X_1(A,\alpha) \in \mathrm{so}(3)$ and some $\alpha_1 = \alpha_1(A,\alpha) \in \mathrm{so}(3)^*$. By definition of Hamiltonian vector field,

$$\mathrm{d}H(A,\alpha)(T_eL_AX_2,\alpha_2) = \omega(A,\alpha)\big(X_H(A,\alpha),(T_eL_AX_2,\alpha_2)\big) \tag{25}$$

for every $(T_eL_AX_2,\alpha_2) \in T_{(A,X)}\big(\mathrm{SO}(3)\times\mathrm{so}(3)^*\big)$. Differentiating H (23) and using the definition of ω (24), equation (25) becomes

$$\rho(e)(\alpha,\alpha_2) = -\alpha_1(X_2)+\alpha_2(X_1)+\alpha([X_1,X_2]) \tag{26}$$

for every $X_2 \in \mathrm{so}(3)$ and every $\alpha_2 \in \mathrm{so}(3)^*$. Setting $X_2 = 0$ in (26) gives $\alpha_2(X_1) = \alpha_2(\rho(e)^\sharp\alpha)$ for every $\alpha_2 \in \mathrm{so}(3)^*$. Hence $X_1 = \rho(e)^\sharp(\alpha)$. Similarly, setting $\alpha_2 = 0$ in (26) and using $X_1 = \rho(e)^\sharp(\alpha)$ gives

$$\alpha_1(X_2) = \alpha([\rho(e)^\sharp(\alpha), X_2]) = (\mathrm{ad}^t_{\rho(e)^\sharp(\alpha)}\alpha)(X_2)$$

for every $X_2 \in \mathrm{so}(3)$. Hence $\alpha_1 = \mathrm{ad}^t_{\rho(e)^\sharp(\alpha)}\alpha$. Consequently, the Hamiltonian vector field of the Euler top on $\mathrm{SO}(3)\times\mathrm{so}(3)^*$ is

$$X_H(A,\alpha) = \big(A\rho(e)^\sharp(\alpha), \mathrm{ad}^t_{\rho(e)^\sharp(\alpha)}\alpha\big). \tag{27}$$

If $\gamma : \mathbf{R} \to \mathrm{SO}(3)\times\mathrm{so}(3)^* : t \mapsto \big(A(t),\alpha(t)\big)$ is an integral curve of X_H and if A_0 is a fixed
▷ rotation, then $\widetilde{\gamma} : \mathbf{R} \to \mathrm{SO}(3)\times\mathrm{so}(3)^* : t \mapsto \big(\widetilde{A}(t),\alpha(t)\big) = \big(A_0A(t),\alpha(t)\big)$ is an integral curve of X_H, because

$$\begin{pmatrix} \frac{\mathrm{d}\widetilde{A}(t)}{\mathrm{d}t} \\ \frac{\mathrm{d}\alpha(t)}{\mathrm{d}t} \end{pmatrix} = \begin{pmatrix} A_0\frac{\mathrm{d}A(t)}{\mathrm{d}t} \\ \frac{\mathrm{d}\alpha(t)}{\mathrm{d}t} \end{pmatrix} = \begin{pmatrix} \big(A_0A(t)\big)\rho(e)^\sharp(\alpha) \\ \mathrm{ad}^t_{\rho(e)^\sharp(\alpha)}\alpha \end{pmatrix} = X_H\big(\widetilde{A}(t),\alpha(t)\big). \qquad \square$$

▷ We now show how to bring the matrix of $\rho(e)$ into diagonal form.

(2.1) **Proof**: Under the mapping

$$j = k^\sharp \circ i^{-1} : \mathbf{R}^3 \to \mathrm{so}(3)^* : e_i \mapsto E_i^*, \tag{28}$$

the inner product $\rho(e)$ on $\mathrm{so}(3)^*$ pulls back to an inner product $\overline{\rho}$ on $\mathbf{R}^3$. For $v,w \in \mathbf{R}^3$ we have

$$\begin{aligned} \overline{\rho}(v,w) &= \rho(e)(j(v),j(w)) = k\big((k^\flat\circ(I^{-1})^t\circ k^\sharp)(i^{-1}v), i^{-1}w\big) \\ &= k\big(i^{-1}((I')^{-1}v), i^{-1}(w)\big), \quad \text{using the } k\text{-symmetry of } I. \\ &\qquad\qquad\qquad\qquad\qquad\quad\ \text{Here } I' = i\circ I\circ i^{-1} \\ &= ((I')^{-1}v, w). \end{aligned}$$

With respect to the standard basis $\{e_i\}$ of $(\mathbf{R}^3, (\,,\,))$ the matrix of $\overline{\rho}$ is invertible and symmetric. Hence there is a rotation O of $\mathbf{R}^3$ such that the matrix of $\overline{\rho}$ with respect to the basis $\{Oe_i\}_{i=1}^3$ is $\mathrm{diag}(I_1^{-1}, I_2^{-1}, I_3^{-1})$. Under the map j, the basis $\{Oe_i\}_{i=1}^3$ becomes the becomes the basis $\{\mathrm{Ad}^t_{O^{-1}}E_i^*\}_{i=1}^3$ of $\mathrm{so}(3)^*$. The matrix of $\rho(e)$ with respect to this basis is $\mathrm{diag}(I_1^{-1}, I_2^{-1}, I_3^{-1})$. Classically, this diagonalization procedure is called transforming the moment of inertia tensor to its *principal axes*. See exercise 5. □

▷ This is not quite what we want because the principal axis transformation does *not* give an equivalent Hamiltonian system for the motion of the Euler top.

(2.2) **Proof**: Consider the diffeomorphism $R_O : \mathrm{SO}(3) \to \mathrm{SO}(3) : A \mapsto AO$ of configuration space given by *right* translation by O, where O is the rotation constructed in ((2.1)). Physically, the initial position of the top is the new frame $\{Oe_1, Oe_2, Oe_3\}$ instead of the frame $\{e_1, e_2, e_3\}$ and its general position is $\{OBe_1, OBe_2, OBe_3\}$ instead of $\{Be_1, Be_2, Be_3\}$.

The diffeomorphism R_O lifts to a symplectic diffeomorphism T^*R_O of $(T^*\,\mathrm{SO}(3),\Omega)$ given by

$$T^*R_O : T^*\,\mathrm{SO}(3) \to T^*\,\mathrm{SO}(3) : \alpha_A \mapsto (T_A R_{O^{-1}})^t \alpha_A.$$

Pushing the left invariant metric ρ forward by T^*R_O gives a new *left* invariant metric ρ'

$$\rho'(A)(\alpha_A,\beta_A) = \rho(AO)\big((T_A R_{O^{-1}})^t\alpha_A,(T_A R_{O^{-1}})^t\alpha_A\big).$$

Pulling ρ' back by the left translation $L_{(AO)^{-1}}$ we obtain $\rho'(e)(\alpha,\beta) = \rho(e)(\mathrm{Ad}^t_{O^{-1}}\alpha,$ $\mathrm{Ad}^t_{O^{-1}}\beta)$. By construction of the rotation O in ((2.1)), the matrix of $\rho'(e)$ with respect to the basis $\{E_i^*\}$ of $\mathrm{so}(3)^*$ is $\mathrm{diag}(I_1^{-1},I_2^{-1},I_3^{-1})$.

Let $(\mathscr{H}',T^*\,\mathrm{SO}(3),\Omega)$ be the Hamiltonian system obtained by pushing forward the original Hamiltonian system $(\mathscr{H},T^*\,\mathrm{SO}(3),\Omega)$ of the Euler top by the symplectic diffeomorphism T^*R_O. Pulling back $(\mathscr{H}',T^*\,\mathrm{SO}(3),\Omega)$ by left trivialization $\mathscr{L}$ (20) we obtain a new Hamiltonian system $(H',\mathrm{SO}(3)\times\mathrm{so}(3)^*,\omega)$ with Hamiltonian

$$H' : \mathrm{SO}(3)\times\mathrm{so}(3)^* \to \mathbf{R} : (A,\alpha) \mapsto \tfrac{1}{2}\rho'(e)(\alpha,\alpha) \tag{29}$$

and symplectic form ω (24). With respect to the basis $\{E_i^*\}$ of $\mathrm{so}(3)^*$, the matrix of ρ' is $\mathrm{diag}(I_1^{-1},I_2^{-1},I_3^{-1})$. □

From now on we will assume that $\rho(e)$ is $\mathrm{diag}(I_1^{-1},I_2^{-1},I_3^{-1})$. Writing $\alpha = \sum_i p_i E_i^*$ and using the relations $(I^{-1})^t E_i^* = I_i^{-1}E_i^*$, $\mathrm{ad}^t_{E_i}E_j^* = -\sum_{k=1}^3 \varepsilon_{ijk}E_k^*$, and $E_i^* = E_i^t$, it follows that the integral curves of X_H (27) on $(\mathrm{SO}(3)\times\mathrm{so}(3)^*,\omega)$ satisfy the *Euler-Arnol'd equations*

$$\begin{aligned}
\frac{\mathrm{d}A}{\mathrm{d}t} &= A\begin{pmatrix} 0 & -I_3^{-1}p_3 & I_2^{-1}p_2 \\ I_3^{-1}p_3 & 0 & -I_1^{-1}p_1 \\ -I_2^{-1}p_2 & I_1^{-1}p_1 & 0 \end{pmatrix}\\
\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} 0 & p_3 & -p_2 \\ -p_3 & 0 & p_1 \\ p_2 & -p_1 & 0 \end{pmatrix} &= \begin{pmatrix} 0 & -(I_2^{-1}-I_1^{-1})p_1p_2 & (I_1^{-1}-I_3^{-1})p_1p_3 \\ (I_2^{-1}-I_1^{-1})p_1p_2 & 0 & -(I_3^{-1}-I_2^{-1})p_2p_3 \\ -(I_1^{-1}-I_3^{-1})p_1p_3 & (I_3^{-1}-I_2^{-1})p_2p_3 & 0 \end{pmatrix}.
\end{aligned}$$

Pulling the Euler-Arnol'd equations back by the diffeomorphism $id\times j$ (28) gives

$$\frac{\mathrm{d}A}{\mathrm{d}t} = A\begin{pmatrix} 0 & -I_3^{-1}p_3 & I_2^{-1}p_2 \\ I_3^{-1}p_3 & 0 & -I_1^{-1}p_1 \\ -I_2^{-1}p_2 & I_1^{-1}p_1 & 0 \end{pmatrix} \tag{30a}$$

$$\frac{\mathrm{d}p}{\mathrm{d}t} = p\times(I')^{-1}p \tag{30b}$$

on $(\mathrm{SO}(3)\times\mathbf{R}^3,\omega' = j^*\omega)$. Here $(I')^{-1}p = (I_1^{-1}p_1,I_2^{-1}p_2,I_3^{-1}p_3)$.

2.2 Euler-Arnol'd equations on $T_1S^2\times\mathbf{R}^3$

In this section we derive the Euler-Arnol'd equations for the Euler top in the nontraditional sphere bundle model.

Before launching into the details, we derive the nontraditional Euler-Arnol'd equations from the traditional ones, (30a) and (30b), as follows. Let $(x,y) \in T_1S^2$ and set $A = \mathrm{col}(x,y,x\times y)$ in (30a). Taking the first two columns of the both sides of (30a) gives

$$\begin{aligned} \dot{x} &= I_3^{-1}p_3 y - I_2^{-1}p_2(x\times y) \\ \dot{y} &= -I_3^{-1}p_3 x + I_1^{-1}p_1(x\times y). \end{aligned} \tag{31a}$$

Clearly (30b) remains unchanged:

$$\dot{p} = p \times (I')^{-1}p. \tag{31b}$$

Of course this derviation does *not* show that the Euler-Arnol'd equations in the sphere ▷ bundle model (31a) – (31b) are in Hamiltonian form. To do this we give an argument which parallels the derivation in the traditional $\mathrm{SO}(3)$ model.

(2.3) **Proof**: First, let ρ be a left invariant *(dual) metric* on the Lie group $(T_1S^2,\cdot)$. Because of left invariance, ρ is determined by its value at the identity element e of T_1S^2. In particular, for every $\alpha,\beta \in T_e^*(T_1S^2)$, the dual of the Lie algebra of T_1S^2,

$$\rho(e)(\alpha,\beta) = \kappa((I^{-1})^t\alpha,\beta). \tag{32}$$

Here κ is the (dual) metric on $T_e^*(T_1S^2)$, induced by the Killing metric k (14) on $T_e(T_1S^2)$, is defined by $\kappa(\alpha,\beta) = \beta(k^\flat(\alpha))$. The linear map $\widehat{I}: T_e(T_1S^2) \to T_e(T_1S^2)$, which is uniquely determined by (32), is invertible and k-symmetric. $\widehat{I}$ is defined by

$$i\big(\widehat{I}(\mathbf{u})\big) = I'\big(i(\mathbf{u})\big) \;=\; (I_1u_1, I_2u_2, I_3u_3),$$

that is, $\widehat{I}(\mathbf{u}) = \big((0, I_3u_3, -I_2u_2), (-I_3u_3, 0, I_1u_1)\big)$.

Let

$$\mathscr{L}: T_1S^2 \times T_e^*(T_1S^2) \to T^*(T_1S^2) : \big((x,y),\alpha\big) \mapsto (T_{(x,y)}L_{(x,y)^{-1}})^t\alpha = \alpha_{(x,y)},$$

where $L_{(x,y)}$ is left translation on $(T_1S^2,\cdot)$ by (x,y). $\mathscr{L}$ is a trivialization of the cotangent bundle

$$\tau: T^*(T_1S^2) \to T_1S^2 : \alpha_{(x,y)} \mapsto (x,y)$$

by *left* translation. On $T^*(T_1S^2)$ with its canonical symplectic form Ω, consider the *Hamiltonian*

$$\mathscr{H}: T^*(T_1S^2) \to \mathbf{R} : \alpha_{(x,y)} \mapsto \tfrac{1}{2}\rho(x,y)\big(\alpha_{(x,y)},\alpha_{(x,y)}\big). \tag{33}$$

Pulling back the *Hamiltonian system* $\big(\mathscr{H}, T^*(T_1S^2), \Omega\big)$ by the left trivialization $\mathscr{L}$ gives the equivalent Hamiltonian system $\big(H, T_1S^2 \times T_e^*(T_1S^2), \omega\big)$, where the Hamiltonian $H = \mathscr{L}^*\mathscr{H}$ is

$$H: T_1S^2 \times T_e^*(T_1S^2) \to \mathbf{R} : \big((x,y),\alpha\big) \mapsto \tfrac{1}{2}\rho(e)(\alpha,\alpha) \tag{34}$$

and symplectic form $\omega = \mathscr{L}^*\Omega$ is

$$\omega(x,y,\alpha)\big((T_eL_{(x,y)}\mathbf{u},\beta),(T_eL_{(x,y)}\mathbf{v},\gamma)\big) = -\beta(\mathbf{v}) + \gamma(\mathbf{u}) + \alpha([\mathbf{u},\mathbf{v}]), \tag{35}$$

for $\mathbf{u}, \mathbf{v} \in T_e(T_1S^2)$ and $\beta, \gamma \in T_e^*(T_1S^2)$.

To compute the Hamiltonian vector field X_H, note that $X_H(x,y,\alpha) = \big(T_eL_{(x,y)}\mathbf{x}_1, \alpha_1\big)$ for some $\mathbf{x}_1 = \mathbf{x}_1(x,y,\alpha) \in T_e(T_1S^2)$ and some $\alpha_1 = \alpha_1(x,y,\alpha) \in T_e^*(T_1S^2)$. From the definition of X_H it follows that

$$\mathrm{d}H(x,y,\alpha)\big(T_eL_{(x,y)}\mathbf{x}_2, \alpha_2\big) = \omega(x,y,\alpha)\big(X_H(x,y,\alpha), (T_eL_{(x,y)}\mathbf{x}_2, \alpha_2)\big), \tag{36}$$

for every $\mathbf{x}_2 \in T_e(T_1S^2)$ and every $\alpha_2 \in T_e^*(T_1S^2)$. Differentiating H and using the definition of ω (35), equation (36) becomes

$$\rho(e)(\alpha, \alpha_2) = -\alpha_1(\mathbf{x}_2) + \alpha_2(\mathbf{x}_2) + \alpha([\mathbf{x}_1, \mathbf{x}_2]), \tag{37}$$

compare with (26). Arguing exactly as in §2.1 we find that

$$X_H(x,y,\alpha) \quad = \quad \big(T_eL_{(x,y)}(\rho(e)^\sharp(\alpha)), \mathrm{ad}^t_{\rho(e)^\sharp(\alpha)}\alpha\big), \tag{38}$$

where $\rho(e)^\sharp(\alpha) = k^\flat\circ(\widehat{I}^{-1})^t$. Hence on $(T_1S^2 \times T_e^*(T_1S^2), \omega)$ the Euler-Arnol'd equations are

$$\frac{\mathrm{d}}{\mathrm{d}t}(x,y) \quad = \quad T_eL_{(x,y)}(\rho(e)^\sharp\alpha) \tag{39a}$$

$$\frac{\mathrm{d}\alpha}{\mathrm{d}t} \quad = \quad \mathrm{ad}^t_{\rho(e)^\sharp(\alpha)}\alpha, \tag{39b}$$

which are Hamilton's equations for X_H. □

To write (39a) and (39b) in a more convenient form, consider the isomorphism

$$j = k^\sharp\circ i^{-1} : \mathbf{R}^3 \to T_e^*(T_1S^2) : e_i \mapsto \varepsilon_i^*.$$

$\{\varepsilon_i^*\}$ is a basis of $T_e^*(T_1S^2)$, which is dual basis to the *standard basis* $\{\varepsilon_i\}$ of $T_e(T_1S^2)$, because

$$\varepsilon_i^*(\varepsilon_\ell) = k^\sharp(i^{-1}(e_i))\varepsilon_\ell = k(i^{-1}(e_i), i^{-1}(e_\ell)) = (e_i, e_\ell) = \delta_i^\ell.$$

Pulling X_H back by the diffeomorphism

$$\psi = \mathrm{id} \times j : T_1S^2 \times \mathbf{R}^3 \to T_1S^2 \times T_e^*(T_1S^2) : (x,y,p) \mapsto (x,y,\sum_i p_i\varepsilon_i^*)$$

gives a Hamiltonian vector field $X_{H'}$ on $(T_1S^2 \times \mathbf{R}^3, \omega' = \psi^*\omega)$ with Hamiltonian

$$H' : T_1S^2 \times \mathbf{R}^3 \to \mathbf{R} : (x,y,p) \mapsto \tfrac{1}{2}\left((I')^{-1}p, p\right) = \tfrac{1}{2}\left(I_1^{-1}p_1^2 + I_2^{-1}p_2^2 + I_3^{-1}p_3^2\right).$$

▷ Here $H' = \psi^*H$. Hamilton's equations for $X_{H'}$ are

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = I_3^{-1}p_3y - I_2^{-1}p_2(x \times y) \\[2ex] \dfrac{\mathrm{d}y}{\mathrm{d}t} = -I_3^{-1}p_3x + I_1^{-1}p_1(x \times y) \end{cases} \tag{40a}$$

$$\frac{\mathrm{d}p}{\mathrm{d}t} = p \times (I')^{-1} p, \tag{40b}$$

where $(I')^{-1} p = (I_1^{-1} p_1, I_2^{-1} p_2, I_3^{-1} p_3)$.

(2.4) **Proof**: With respect to the bases $\{\varepsilon_i^*\}$ of $T_e^*(T_1S^2)$ and $\{\varepsilon_i\}$ of $T_e(T_1S^2)$ the matrix of $\rho(e)^\sharp : T_e^*(T_1S^2) \to T_e(T_1S^2)$ is $\mathrm{diag}(I_1^{-1}, I_2^{-1}, I_3^{-1})$, because

$$\begin{aligned}\varepsilon_\ell^*\big(\rho(e)^\sharp(\varepsilon_i^*)\big) &= \rho(e)\big(\varepsilon_i^*, \varepsilon_\ell^*\big) \,=\, \varepsilon_\ell^*\big(k^\flat((\widehat{I}^{-1})^t \varepsilon_i^*)\big) \\ &= \varepsilon_\ell^*\big((k^\flat \circ (\widehat{I}^{-1})^t \circ k^\sharp)(\varepsilon_i)\big) \,=\, \varepsilon_\ell^*((I')^{-1}\varepsilon_i) \,=\, \varepsilon_\ell^*(I_i^{-1}\varepsilon_i).\end{aligned}$$

Writing $\alpha = \sum_i p_i \varepsilon_\ell^*$, we see that $\rho(e)^\sharp(\alpha) = \sum_i I_i^{-1} p_i \, \varepsilon_i$. Differentiating the definition of left translation $L_{(x,y)}$ (13) gives

$$T_e L_{(x,y)} \mathbf{u} = \widehat{\pi}(\mathrm{col}(x, y, x \times y) \cdot U) \,=\, \big(u_3 y - u_2(x \times y), -u_3 x + u_1(x \times y)\big),$$

for $\mathbf{u} = \sum_i u_i \varepsilon_i \in T_e(T_1S^2)$. Setting $\mathbf{u} = \rho(e)^\sharp(\alpha)$ in the above equation and using (39a) we obtain (40a). To obtain (40b) we first show that $\mathrm{ad}^t_{\varepsilon_i} \varepsilon_j^* = -\sum_k \varepsilon_{ijk}\, \varepsilon_k^*$. We compute

$$(\mathrm{ad}^t_{\varepsilon_i} \varepsilon_j^*)(\varepsilon_\ell) = \varepsilon_j^*([\varepsilon_i, \varepsilon_\ell]) \,=\, \varepsilon_j^*(\sum_k \varepsilon_{i\ell k}\, \varepsilon_k) \,=\, \varepsilon_{i\ell j} \,=\, -\big(\sum_k \varepsilon_{ijk}\, \varepsilon_k^*\big)(\varepsilon_\ell).$$

From (39b) we find that

$$\sum_i \big(\tfrac{\mathrm{d}p}{\mathrm{d}t}\big)_i \varepsilon_i^* = \sum_{ij} p_i (I_j^{-1} p_j)\, \mathrm{ad}^t_{\varepsilon_j} \varepsilon_i^* = \sum_{ijk} p_i (I_j^{-1} p_j)\, \varepsilon_{ijk}\, \varepsilon_k^* \,=\, \sum_i (p \times (I')^{-1} p)_i\, \varepsilon_i^*.$$

Equating components gives (40b). □

Note that the right hand side of (40a) and (40b) defines a vector field on $\mathbf{R}^9$ with coordinates (x, y, p). A calculation shows that $T_1S^2 \times \mathbf{R}^3$ is an invariant manifold of this vector field.

3 Symmetry and reduction

In this section we discuss the SO(3) symmetry of the Euler top. Using the regular reduction theorem, see chapter VII ((6.1)), to remove this symmetry, we obtain a Hamiltonian system on the 2-sphere S^2. We show that the integral curves of the reduced Hamiltonian vector field satisfy *Euler's equations*.

3.1 SO(3) symmetry

In this subsection we discuss the natural SO(3) symmetry of the Euler top.

Because the configuration space of the Euler top is the Lie group SO(3), it has a natural symmetry, namely, the action of SO(3) on itself by left translation

$$L : \mathrm{SO}(3) \times \mathrm{SO}(3) \to \mathrm{SO}(3) : (B, A) \mapsto L_B A = BA. \tag{41}$$

This action lifts to an action

$$\widehat{L}: \mathrm{SO}(3) \times T^*\mathrm{SO}(3) \rightarrow T^*\mathrm{SO}(3) : (B, \alpha_A) \mapsto \alpha_{BA} = (T_A L_{B^{-1}})^t \alpha_A, \tag{42}$$

▷ where $\alpha_A = (T_A L_{A^{-1}})^t \alpha$ for $\alpha \in \mathrm{so}(3)^*$. The action $\widehat{L}$ on $(T^*\mathrm{SO}(3), \Omega)$ is Hamiltonian.

(3.1) **Proof**: Let $\xi \in \mathrm{so}(3)$. The infinitesimal generator $X^\xi(\alpha_A) = \frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} \widehat{L}_{\exp t\xi}(\alpha_A)$ of the action $\widehat{L}$ in the direction ξ is the Hamiltonian vector field $X_{\mathscr{J}^\xi}$ on $(T^*\mathrm{SO}(3), \Omega = -\mathrm{d}\theta)$ where

$$\mathscr{J}^\xi : T^*\mathrm{SO}(3) \rightarrow \mathbf{R} : \alpha_A \mapsto (X^\xi \,\lrcorner\, \theta)(\alpha_A). \tag{43}$$

To see this we observe that the canonical 1-form θ on $T^*\mathrm{SO}(3)$ is invariant under $\widehat{L}$. Therefore

$$0 = L_{X^\xi}\theta = \mathrm{d}(X^\xi \,\lrcorner\, \theta) + X^\xi \,\lrcorner\, \mathrm{d}\theta = \mathrm{d}\mathscr{J}^\xi - X^\xi \,\lrcorner\, \Omega,$$

which implies $X^\xi = X_{\mathscr{J}^\xi}$. □

▷ Next we show that the SO(3)-action $\widehat{L}$ (42) has a coadjoint equivariant momentum mapping

$$\mathscr{J} : T^*\mathrm{SO}(3) \rightarrow \mathrm{so}(3)^* : \alpha_A \mapsto (T_e R_A)^t \alpha, \tag{44}$$

where $R : \mathrm{SO}(3) \times \mathrm{SO}(3) \rightarrow \mathrm{SO}(3) : (B, A) \mapsto R_B A = AB$ is right translation.

(3.2) **Proof**: We begin by finding another expression for the function $\mathscr{J}^\xi$ (43). Let $X_\xi(A) = \frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} L_{\exp t\xi}(A)$ be the infinitesimal generator of L (41) in the direction ξ. Then

$$X_\xi(A) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \exp t\xi\, A = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} R_A \exp t\xi = T_e R_A \xi.$$

From the momentum lemma, see chapter VII ((5.7)), it follows that $\mathscr{J}^\xi(\alpha_A) = \alpha_A(X_\xi(A)) = \big((T_e R_A)^t(\alpha_A)\big)\xi$, for every $\xi \in \mathrm{so}(3)$. Since $\xi \mapsto \mathscr{J}^\xi(\alpha_A)$ is a linear function on so(3), we may define the mapping $\mathscr{J} : T^*\mathrm{SO}(3) \rightarrow \mathrm{so}(3)^*$ by $\mathscr{J}(\alpha_A)\xi = \mathscr{J}^\xi(\alpha_A)$. This
▷ momentum mapping is coadjoint equivariant, because for $B \in \mathrm{SO}(3)$ we have

$$\begin{aligned}\mathscr{J}(\widehat{L}_B \alpha_A) &= (T_e R_{BA})^t \alpha_{BA} = (T_e R_{BA})^t (T_{BA} L_{(BA)^{-1}})^t \alpha \\ &= (T_B L_{B^{-1}} \circ T_e R_B)^t (T_e R_A)^t (T_A L_{A^{-1}})^t \alpha = \mathrm{Ad}^t_{B^{-1}} \mathscr{J}(\alpha_A).\end{aligned}$$

□

Pulling back the SO(3)-action $\widehat{L}$ (42) by the left trivialization mapping $\mathscr{L}$ (20) gives the action

$$\widehat{\ell} : \mathrm{SO}(3) \times \big(\mathrm{SO}(3) \times \mathrm{so}(3)^*\big) \rightarrow \mathrm{SO}(3) \times \mathrm{so}(3)^* : \big(B, (A, \alpha)\big) \mapsto (BA, \alpha). \tag{45}$$

▷ $\widehat{\ell}$ is a Hamiltonian action on $\big(\mathrm{SO}(3) \times \mathrm{so}(3)^*, \omega = \mathscr{L}^*\Omega\big)$ with equivariant momentum mapping

$$J = \mathscr{L}^*\mathscr{J} : \mathrm{SO}(3) \times \mathrm{so}(3)^* \rightarrow \mathrm{so}(3)^* : (A, \alpha) \mapsto \mathrm{Ad}^t_{A^{-1}} \alpha. \tag{46}$$

(3.3) **Proof**: To verify (46) we compute

$$J(A, \alpha) = \mathscr{J}\big(\mathscr{L}(A, \alpha)\big) = \mathscr{J}(\alpha_A) = (T_e R_A)^t \alpha_A = (T_e R_A)^t (T_A L_{A^{-1}})^t \alpha = \mathrm{Ad}^t_{A^{-1}} \alpha.$$

Coadjoint equivariance follows because for $B \in \mathrm{SO}(3)$

$$J\big(\widehat{\ell}_B(A,\alpha)\big) = J(BA,\alpha) = \mathrm{Ad}^t_{(BA)^{-1}}\alpha = \mathrm{Ad}^t_{B^{-1}}(\mathrm{Ad}^t_{A^{-1}}\alpha) = \mathrm{Ad}^t_{B^{-1}}J(A,\alpha). \qquad \square$$

We now show that the natural $\mathrm{SO}(3)$ symmetry gives rise to conserved quantities for the
▷ Euler top. First we observe that the Hamiltonian

$$\mathscr{H} : T^*\,\mathrm{SO}(3) \to \mathbf{R} : \alpha_A \mapsto \tfrac{1}{2}\rho(A)(\alpha_A,\alpha_A)$$

is invariant under the $\mathrm{SO}(3)$-action $\widehat{L}$ (42).

(3.4) **Proof**: We compute. For every $B \in \mathrm{SO}(3)$ we have

$$\begin{aligned}\mathscr{H}(\widehat{L}_B\alpha_A) &= \mathscr{H}(\alpha_{BA}) = \tfrac{1}{2}\rho(BA)(\alpha_{BA},\alpha_{BA}) = \tfrac{1}{2}\rho(L_BA)\big((T_AL_{A^{-1}})^t\alpha_A,(T_AL_{A^{-1}})^t\alpha_A\big)\\ &= \tfrac{1}{2}(\widehat{L}_B^*\rho)(\alpha_A) = \mathscr{H}(\alpha_A),\ \text{since } \rho \text{ is left invariant.} \qquad \square\end{aligned}$$

Thus for every $\xi \in \mathrm{so}(3)$ we see that $\mathscr{H}(\widehat{L}_{\exp t\xi}\alpha_A) = \mathscr{H}(\alpha_A)$. Infinitesimalizing gives

$$0 = \mathrm{d}\mathscr{H}(\alpha_A)X^{\xi}(\alpha_A) = \mathrm{d}\mathscr{H}(\alpha_A)X_{\mathscr{J}^{\xi}}(\alpha_A) = \{\mathscr{J}^{\xi},\mathscr{H}\}(\alpha_A).$$

Thus for every $\xi \in \mathrm{so}(3)$, the ξ-component $\mathscr{J}^{\xi}$ of the momentum mapping $\mathscr{J}$ (44) is
▷ an integral of the Hamiltonian vector field $X_{\mathscr{H}}$ whose integral curves govern the motion of the Euler top. In other words, $\mathscr{J}$ is an $\mathrm{so}(3)^*$-valued integral of $X_{\mathscr{H}}$. A similar argument shows that the mapping J (46) is an $\mathrm{so}(3)^*$-valued integral of the Euler-Arnol'd vector field X_H, whose integral curves are solutions of (30a) and (30b).

3.2 Construction of the reduced phase space

In this subsection we construct the reduced phase space, which is obtained after removing the rotational symmetry of the Euler top.

To start the process of reduction, let μ be a *nonzero* element of $\mathrm{so}(3)^*$. The μ-level set $J^{-1}(\mu)$ of the momentum map J is $\{(A,\mathrm{Ad}^t_A\mu) \in \mathrm{SO}(3)\times\mathrm{so}(3)^* \big| A \in \mathrm{SO}(3)\}$. Thus $J^{-1}(\mu)$ is a smooth submanifold of $\mathrm{SO}(3)\times\mathrm{so}(3)^*$, which is diffeomorphic to $\mathrm{SO}(3)$, because it is the graph of the smooth mapping $\mathrm{SO}(3) \to \mathrm{so}(3)^* : A \mapsto \mathrm{Ad}^t_A\mu$.

We now want to find a subgroup of $\mathrm{SO}(3)$ which acts on $J^{-1}(\mu)$. Because J is coadjoint equivariant, we look at the *isotropy group*

$$\mathrm{SO}(3)_{\mu} = \{B \in \mathrm{SO}(3) \,|\, \mathrm{Ad}^t_{B^{-1}}\mu = \mu\}. \tag{47}$$

Since $\mathrm{Ad}^t_{BA}\mu = \mathrm{Ad}^t_A\mathrm{Ad}^t_B\mu = \mathrm{Ad}^t_A\mu$ for every $B \in \mathrm{SO}(3)_{\mu}$, it follows that $J(BA,\mathrm{Ad}^t_{BA}\mu) = J(BA,\mathrm{Ad}^t_A\mu)$. Therefore restricting the $\mathrm{SO}(3)$-action $\widehat{\ell}$ (45) to $\mathrm{SO}(3)_{\mu}\times J^{-1}(\mu)$ defines the action

$$\Phi : \mathrm{SO}(3)_{\mu}\times J^{-1}(\mu) \to J^{-1}(\mu) : \big(B,(A,\mathrm{Ad}^t_A\mu)\big) \mapsto (BA,\mathrm{Ad}^t_A\mu). \tag{48}$$

To get a better idea of what the isotropy group $\mathrm{SO}(3)_\mu$ means, apply the map $\frac{1}{|\mu|}\, i\circ k^\flat$ to both sides of equation $\mathrm{Ad}^t_{B^{-1}}\mu = \mu$. This gives

$$y = \frac{1}{|\mu|}\, i\circ k^\flat(\mu) \,=\, \frac{1}{|\mu|}\, i(k^\flat\circ \mathrm{Ad}^t_{B^{-1}}\circ k^\sharp)(k^\flat(\mu)) \,=\, (i\circ \mathrm{Ad}_B\circ i^{-1})(y) \,=\, By.$$

Thus $\mathrm{SO}(3)_\mu$ is the set of all rotations which leave the unit vector $y = \frac{1}{|\mu|}\, i\circ k^\flat(\mu)$ fixed. Physically, $|\mu|\, y$ is the angular momentum *vector* of the Euler top with respect to a fixed
$\triangleright$ frame. Let $Y = i^{-1}y$ and note that $|Y| = 1$. Next we show that the image of the one parameter subgroup $\lambda : \mathbf{R} \rightarrow \mathrm{SO}(3)$: $t \mapsto \exp tY$ is the isotropy subgroup $\mathrm{SO}(3)_\mu$.

(3.5) **Proof**: By definition, $\mathrm{SO}(3)_\mu$ is a closed subgroup of $\mathrm{SO}(3)$ and hence is a compact Lie group. Its Lie algebra $\mathrm{so}(3)_\mu = \{X \in \mathrm{so}(3) \,\big|\, \mathrm{ad}^t_X \mu = 0\}$ is one dimensional. To see this apply the map $i\circ k^\flat$ to both sides of the equation defining $\mathrm{so}(3)_\mu$ to obtain

$$0 = i\big(k^\flat\circ \mathrm{ad}^t_X\circ k^\sharp\big)(k^\flat(\mu)) \,=\, -(i\circ \mathrm{ad}_X\circ i^{-1})(|\mu|\, y)) \,=\, -|\mu|\, i(X)\times i(Y),$$

where the second equality follows since ad_X is k-skew symmetric. Therefore X and Y are linearly dependent. Hence $\dim \mathrm{so}(3)_\mu = 1$.

We would be done if we knew that $\mathrm{SO}(3)_\mu$ was connected, because the image of the one parameter group λ is circle. The following argument shows that $\mathrm{SO}(3)_\mu$ is isomorphic to $\mathrm{SO}(2)$ and hence is connected. Write $\mathbf{R}^3 = \mathrm{span}\{y\} \oplus \Pi$, where Π is a plane in $\mathbf{R}^3$ orthogonal to the vector y. Π is invariant under every $B \in \mathrm{SO}(3)_\mu$, because $B \in \mathrm{SO}(3)$ and $By = y$. Moreover, $B|\Pi \in \mathrm{SO}(2)$, since $B \in \mathrm{SO}(3)$ implies that $B|\Pi$ preserves the length of every vector in Π and $1 = \det B = \left(\begin{smallmatrix} 1 & 0 \\ 0 & \det(B|\Pi)\end{smallmatrix}\right) = \det(B|\Pi)$. The smooth map $\sigma : \mathrm{SO}(3)_\mu \rightarrow \mathrm{SO}(2) : B \mapsto B|\Pi$ is a homomorphism of Lie groups. Actually, σ is an isomorphism. To see that σ is surjective, let $\widetilde{B} \in \mathrm{SO}(2)$ and choose $\{w,z\}$ to be a positively oriented orthonormal basis of Π such that $\{y,w,z\}$ is a positively oriented orthonormal basis of $\mathbf{R}^3$. Define the linear map $\mathscr{B} : \mathbf{R}^3 \rightarrow \mathbf{R}^3$ by requiring that $\mathscr{B}$ sends the positively oriented ordered orthonormal basis $\{y,w,z\}$ to the ordered basis $\{y, \widetilde{B}w, \widetilde{B}z\}$. Clearly $\mathscr{B}$ is an extension of $\widetilde{B}$. Since $\widetilde{B} \in \mathrm{SO}(2)$, $\{\widetilde{B}w, \widetilde{B}z\}$ is a positively oriented orthonormal basis of Π. Hence $\{y, \widetilde{B}w, \widetilde{B}z\}$ is a positively oriented orthonormal basis of $\mathbf{R}^3$. Therefore, $\mathscr{B} \in \mathrm{SO}(3)_\mu$. Hence σ is surjective. Because $\mathscr{B}$ is the unique rotation which extends $\widetilde{B}$, it follows that σ is injective. Thus σ is an isomorphism. □

We return to discussing the construction of the *reduced space*. The reduced space $P_\mu = J^{-1}(\mu)/\mathrm{SO}(3)_\mu$ is the space of $\mathrm{SO}(3)_\mu$-orbits on $J^{-1}(\mu)$. Since the action Φ (48) on $J^{-1}(\mu)$ is free and proper, P_μ is a smooth symplectic manifold, see chapter VII ((2.9)).
$\triangleright$ The following argument shows that P_μ is the $\mathrm{SO}(3)$ coadjoint orbit $\mathscr{O}_\mu$ through μ.

(3.6) **Proof**: Consider the *mapping*

$$\pi_\mu : J^{-1}(\mu) \subseteq \mathrm{SO}(3)\times \mathrm{so}(3)^* \rightarrow \mathrm{so}(3)^* : (A, \mathrm{Ad}^t_A\mu) \mapsto \nu = \mathrm{Ad}^t_A\mu. \qquad (49)$$

Because the fiber $\pi_\mu^{-1}(\nu)$ is a single $\mathrm{SO}(3)_\mu$-orbit, the image of π_μ is the orbit space P_μ. By definition, $\pi_\mu(J^{-1}(\mu))$ is the coadjoint orbit $\mathscr{O}_\mu = \{\nu = \mathrm{Ad}^t_A\mu \in \mathrm{so}(3)^* \,\big|\, A \in \mathrm{SO}(3)\}$. From example 3 of chapter VI §2, we know that the symplectic form $\omega_{\mathscr{O}_\mu}$ on $\mathscr{O}_\mu$ is $\omega_{\mathscr{O}_\mu}(\nu)(\mathrm{ad}^t_\xi \nu, \mathrm{ad}^t_\eta \nu) = -\nu([\xi,\eta])$, where $\xi, \eta \in \mathrm{so}(3)$. □

3.3 Geometry of the reduction map

In this subsection we study the geometry of the reduction map π_μ (49).

$\triangleright$ Our main result is that π_μ is double covered by the Hopf fibration.

(3.7) **Proof**: This follows from ((1.12)) once we prove

Claim: The $\mathrm{SO}(3)_\mu$ principal bundle $\pi_\mu : J^{-1}(\mu) \to \mathscr{O}_\mu$ is isomorphic to the S^1 principal bundle $\tau : T_1 S^2 \to S^2$.

(3.8) **Proof**: We find a succession of principal bundle isomorphisms.

1. We begin by looking at the S^1-action $\psi : S^1 \times \mathrm{SO}(3) \to \mathrm{SO}(3) : (s, A) \mapsto (\exp sY)A$, where $Y = \frac{1}{|\mu|} k^\flat(\mu)$. Since ψ is free and proper, $\mathrm{SO}(3)$ is the total space of an S^1-principal bundle over $\mathrm{SO}(3)/S^1 = S^2$ with bundle projection $\sigma^y : \mathrm{SO}(3) \to S^2 : A \mapsto A^{-1}y$, where $y = i(Y)$. We now show that the principal bundles π_μ and σ^y are isomorphic. Consider diagram 3.3.1. Since $\mu = |\mu| k^\sharp \circ i^{-1}(y)$ and $A \in \mathrm{SO}(3)$, it follows that $\mathrm{Ad}^t_A \mu = \beta_1(A^{-1}y)$, where $\beta_1 = k^\sharp \circ i$. Therefore diagram 3.3.1 commutes. Let $\mathscr{B}_1(A) = (A, \mathrm{Ad}^t_A \mu)$. Then

$$\Phi_{\exp sY}(\mathscr{B}_1(A)) = \big((\exp sY)A, \mathrm{Ad}^t_{(\exp sY)A}\mu\big) = \mathscr{B}_1(\psi_s(A)).$$

In other words, $\mathscr{B}_1$ intertwines the actions ψ and Φ. Because $\mathscr{B}_1$ and β_1 are diffeomorphisms, the principal bundles σ^y and π_μ are isomorphic.

$$\begin{tikzcd}
\mathrm{SO}(3) \arrow[r, "\mathscr{B}_1"] \arrow[d, "\sigma^y"'] & J^{-1}(\mu) \arrow[d, "\pi_\mu"] & A \arrow[r] \arrow[d] & (A, \mathrm{Ad}^t_A(\mu)) \arrow[d] \\
S^2_{|\mu|} \arrow[r, "\beta_1 = k^\sharp \circ i"] & \mathscr{O}_\mu & A^{-1}y \arrow[r] & |\mu| \mathrm{Ad}^t_A\big(k^\sharp \circ i^{-1}(y)\big)
\end{tikzcd}$$

Diagram 3.3.1

2. Consider diagram 3.3.2. Because $\mathrm{SO}(3)$ acts transitively on the 2-sphere S^2, there is an $O \in \mathrm{SO}(3)$ such that $Oy = e_1$. Since $(O^{-1}A)^{-1}y = A^{-1}Oy = A^{-1}e_1$, diagram 3.3.2 commutes. The second to last equality holds since $y = O^{-1}e_1$.

$$\begin{tikzcd}
\mathrm{SO}(3) \arrow[r, "\mathscr{B}_2"] \arrow[d, "\sigma^{e_1}"'] & \mathrm{SO}(3) \arrow[d, "\sigma^y"] & A \arrow[r] \arrow[d] & O^{-1}A \arrow[d] \\
S^2 \arrow[r, "id"] & S^2 & A^{-1}e_1 \arrow[r] & A^{-1}e_1
\end{tikzcd}$$

Diagram 3.3.2

Define the S^1 action $\theta : S^1 \times \mathrm{SO}(3) \to \mathrm{SO}(3) : (s,A) \mapsto (\exp sE_1)A$. Since θ is free and proper, $\mathrm{SO}(3)$ is the total space of an S^1 principal bundle over $\mathrm{SO}(3)/S^1 = S^2$ with bundle projection $\sigma^{e_1} : \mathrm{SO}(3) \to S^2 : A \mapsto A^{-1}e_1$. Because

$$\begin{aligned}\mathscr{B}_2(\theta_s(A)) &= (O^{-1}\exp sE_1 O)O^{-1}A = \big(\exp s(\mathrm{Ad}_{O^{-1}}E_1)\big)\mathscr{B}_2(A)\\ &= (\exp sY)(\mathscr{B}_2(A)) = \psi_s(\mathscr{B}_2(A)),\end{aligned}$$

the map $\mathscr{B}_2$ intertwines the actions θ and ψ. Thus the bundles σ^{e_1} and σ^y are isomorphic principal bundles, since the maps $\mathscr{B}_2$ and *id* are diffeomorphisms.

3. To complete the proof of the claim, consider diagram 3.3.3. Clearly this diagram commutes. The S^1-action $\Psi : S^1 \times T_1S^2 \to T_1S^2 : \big(s,(x,y)\big) \mapsto \big(x, y\cos s - (x\times y)\sin s\big)$ defining the principal bundle τ (15) extends to an S^1-action

$$\begin{aligned}&\vartheta : S^1 \times \mathrm{SO}(3) \to \mathrm{SO}(3) :\\ &\big(s, \mathrm{col}\,(x,y,x\times y)^t\big) \mapsto \begin{pmatrix} x^t \\ y^t\cos s - (x\times y)^t \sin s \\ y^t \sin s + (x\times y)^t\cos s\end{pmatrix} = (\exp sE_1)\,\mathrm{col}\,(x,y,x\times y)^t,\end{aligned}$$

ϑ is the *same* as the action θ. Therefore $\mathscr{B}_3(\Psi_s(x,y)) = \theta_s(\mathscr{B}_3(x,y))$, that is, $\mathscr{B}_3$ intertwines the actions θ and Ψ. Since $\mathscr{B}_3$ and *id* are diffeomorphisms, the principal bundles τ and σ^{e_1} are isomorphic.

$$\begin{tikzcd} T_1S^2 \arrow[r, "\mathscr{B}_3"] \arrow[d, "\tau"'] & \mathrm{SO}(3) \arrow[d, "\sigma^{e_1}"] & (x,y) \arrow[r] \arrow[d] & \mathrm{col}\,(x,y,x\times y)^t \arrow[d] \\ S^2 \arrow[r, "id"] & S^2 & x \arrow[r] & x \end{tikzcd}$$

Diagram 3.3.3

Composing the bundle isomorphisms $(\mathscr{B}_3, id)$, $(\mathscr{B}_2, id)$ and $(\mathscr{B}_1, \beta_1)$ shows that the principal bundles τ and π_μ are isomorphic. □

3.4 Euler's equations

In this subsection we study the reduced Hamiltonian vector field of the Euler top. This vector field is obtained by removing the $\mathrm{SO}(3)_\mu$ symmetry of the Hamiltonian vector field X_H (27) restricted to the invariant manifold $J^{-1}(\mu)$. The integral curves of the reduced Hamiltonian vector field satisfy Euler's equations.

▷ We start by constructing the *reduced Hamiltonian*. Observe that the Hamiltonian of the Euler top

$$H : \mathrm{SO}(3) \times \mathrm{so}(3)^* \to \mathbf{R} : (A,\alpha) \mapsto \tfrac{1}{2}\alpha\big((k^\flat(I^{-1})^t\alpha\big), \qquad (50)$$

when restricted to the μ-level set of the momentum J, is invariant under the left $\mathrm{SO}(3)_\mu$-action Φ (48).

(3.9) **Proof**: To see this recall that $J^{-1}(\mu) = \{(A, \nu = \mathrm{Ad}^t_A\mu) \in \mathrm{SO}(3) \times \mathrm{so}(3)^* \big| A \in \mathrm{SO}(3)\}$. Therefore we obtain $(H|J^{-1})(\mu)(A,\nu) = \nu\big(k^\flat(I^{-1})^t\nu\big)$. Since $\Phi_B(A,\nu) = (BA,\nu)$ for every $B \in \mathrm{SO}(3)_\mu$, we see that $H|J^{-1}(\mu)$ is invariant under the action Φ. □

Thus $H|J^{-1}(\mu)$ induces a function H_μ on the $\mathrm{SO}(3)_\mu$-orbit space $J^{-1}(\mu)/\mathrm{SO}(3)_\mu = \mathscr{O}_\mu$ defined by $\pi^*_\mu H_\mu = H|J^{-1}(\mu)$. Here π_μ is the reduction mapping

$$\pi_\mu : J^{-1}(\mu) \subseteq \mathrm{SO}(3) \times \mathrm{so}(3)^* \to \mathscr{O}_\mu \subseteq \mathrm{so}(3)^* : (A,\nu) \mapsto \nu. \tag{51}$$

Hence the reduced Hamiltonian is $H_\mu : \mathscr{O}_\mu \subseteq \mathrm{so}(3)^* \to \mathbf{R} : \nu \mapsto \frac{1}{2}\nu\Big(k^\flat\big((I^{-1})^t(\nu)\big)\Big)$. Since $\mathscr{O}_\mu$ is a *symplectic manifold* with *symplectic form* $\omega_{\mathscr{O}_\mu}$, we obtain the *reduced Hamiltonian system* $(H_\mu, \mathscr{O}_\mu, \omega_{\mathscr{O}_\mu})$.

To see that the integral curves of X_{H_μ} on $\mathscr{O}_\mu$ satisfy Euler's equations, we need another model for the reduced system. Consider the diffeomorphism

$$\widetilde{j} = k^\sharp \circ i^{-1} : S^2_r \subseteq \mathbf{R}^3 \to \mathscr{O}_\mu \subseteq \mathrm{so}(3)^* : p \mapsto \nu \tag{52}$$

where S^2_r is the 2-sphere of radius $r = |\mu|$. Pulling back the reduced Hamiltonian H_μ by $\widetilde{j}$ gives the Hamiltonian

$$H_r : S^2_r \subseteq \mathbf{R}^3 \to \mathbf{R} : p = (p_1,p_2,p_3) \mapsto \tfrac{1}{2}(I_1^{-1}p_1^2 + I_2^{-1}p_2^2 + I_3^{-1}p_3^2). \tag{53}$$

▷ The following calculation shows that $(\widetilde{j})^*\omega_{\mathscr{O}_\mu} = \omega_r$, where ω_r is the symplectic form on S^2_r given by

$$\omega_r(p)(p \times x_1, p \times x_2) = -(p, x_1 \times x_2), \tag{54}$$

for $x_1, x_2 \in \mathbf{R}^3$.

(3.10) **Proof**: For some $A \in \mathrm{SO}(3)$, we may write $p = A(\widetilde{j}^{-1}(\mu))$. Then

$$i^{-1}(p) = (i^{-1} \circ A^{-1} \circ i)\big(k^\flat(\mu)\big) = \mathrm{Ad}_{A^{-1}}(k^\flat(\mu)) = k^\flat(\mathrm{Ad}^t_A\mu) = k^\flat(\nu).$$

Let $\xi = i^{-1}(x_1)$. Then

$$k^\flat(\mathrm{ad}^t_\xi\nu) = -\mathrm{ad}_\xi(k^\flat(\nu)) = \big[i^{-1}(p), i^{-1}(x_1)\big] = i^{-1}(p \times x_1).$$

Similarly, if $\eta = i^{-1}(x_2)$, $k^\flat(\mathrm{ad}^t_\eta\nu) = i^{-1}(p \times x_2)$. We compute $(\widetilde{j})^*\omega_{\mathscr{O}_\mu}$ as follows:

$$\begin{aligned}
((\widetilde{j})^*\omega_{\mathscr{O}_\mu})(p)(p \times x_1, p \times x_2) &= \omega_{\mathscr{O}_\mu}(\widetilde{j}(p))\big(T_p\widetilde{j}(p \times x_1), T_p\widetilde{j}(p \times x_2)\big) \\
&= \omega_{\mathscr{O}_\mu}(\widetilde{j}(p))\big(\widetilde{j}(p \times x_1), \widetilde{j}(p \times x_2)\big), \text{ since } \widetilde{j} \text{ is the restriction of a linear map} \\
&= \omega_{\mathscr{O}_\mu}(\nu)(\mathrm{ad}^t_\xi\nu, \mathrm{ad}^t_\eta\nu) = -\nu([\xi,\eta]) = -\widetilde{j}(p)(i^{-1}(x_1 \times x_2)) \\
&= -k\big(i^{-1}(p), i^{-1}(x_1 \times x_2)\big) = -(p, x_1 \times x_2).
\end{aligned}$$

□

The integral curves of the *reduced Hamiltonian vector field* X_{H_μ} on $(\mathscr{O}_\mu, \omega_{\mathscr{O}_\mu})$ satisfy the

▷ equations $\dot{\nu} = \mathrm{ad}^t_{k^\flat((I^{-1})^t\nu)}\nu$, see chapter VII §6.1 example 2. Pulling X_{H_μ} back by $\widetilde{j}$ gives the vector field X_{H_r} on (S^2_r, ω_r) whose integral curves satisfy *Euler's equations*

$$\dot{p} = p \times (I')^{-1} p \tag{55}$$

in angular momentum coordinates. Here $(I')^{-1}p = (I_1^{-1}p_1, I_2^{-1}p_2, I_3^{-1}p_3)$.

(3.11) **Proof**: Since X_{H_r} is a vector field on S^2_r, we may write $X_{H_r}(p) = p \times X$ for some $X = X(p) \in \mathbf{R}^3$. From the definition of Hamiltonian vector field and the symplectic form ω_r (54) we obtain $\mathrm{d}H_r(p)(p \times x) = -(p, X \times x)$ for every $x \in \mathbf{R}^3$. Differentiating (53) gives $\mathrm{d}H_r(p)(p \times x) = \big((I')^{-1}(p), p \times x\big)$ for every $x \in \mathbf{R}^3$, that is, $((I')^{-1}(p) - X) \times p = 0$. Consequently, for some $\lambda \in \mathbf{R}$ we have $X = \lambda\, p + (I')^{-1}(p)$. In other words, $X_{H_r}(p) = p \times (\lambda\, p + (I')^{-1}(p)) = p \times (I')^{-1}(p)$. Thus the integral curves of X_{H_r} satisfy Euler's equations (55). □

4 Qualitative behavior of the reduced system

In this section we give a global qualitative description of the solutions of Euler's equations (55) on the reduced space S^2_r. For a quantitive treatment see exercise 2. This amounts to finding the topology of the *level sets* of the reduced Hamiltonian H_r (53).

▷ We begin by showing that H_r is a *Morse function* on S^2_r with six critical points.

(4.1) **Proof**: Since S^2_r is compact, H_r has a critical point q. Using Lagrange multipliers, we see that $q = (x, y, z)$ is a solution of the equations

$$\begin{aligned} 0 &= DH_r(q) - \lambda DG(q) = (I_1^{-1}x, I_2^{-1}y, I_3^{-1}z) - \lambda(x, y, z) \\ 0 &= G(q) = \tfrac{1}{2}(x^2 + y^2 + z^2 - r^2). \end{aligned}$$

In other words, q lies on the intersection of an eigenspace of the diagonal matrix $(I')^{-1} = \mathrm{diag}(I_1^{-1}, I_2^{-1}, I_3^{-1})$ with the 2-sphere S^2_r of radius r. From now on we assume that

$$0 < I_3^{-1} < I_2^{-1} < I_1^{-1}. \tag{56}$$

Since the eigenvalues of $(I')^{-1}$ are distinct, the eigenspaces corresponding to the eigenvalues $I_1^{-1}, I_2^{-1}, I_3^{-1}$ are spanned by the vectors re_1, re_2, re_3 respectively, which lie on S^2_r. Hence H_r has six critical points $\pm re_i$ $i = 1, 2, 3$ with Lagrange multiplier I_i^{-1}, $i = 1, 2, 3$, respectively. At the critical point q with Lagrange multiplier λ the Hessian of H_r is

$$\mathrm{Hess}\, H_r(q) = \big(D^2H_r(q) - \lambda D^2G(q)\big)\Big|_{T_qS^2_r} = \mathrm{diag}(I_1^{-1} - \lambda, I_2^{-1} - \lambda, I_3^{-1} - \lambda)\Big|_{T_qS^2_r},$$

see chapter XI §2. Since $T_qS^2_r = \ker DG(q) = \{v \in \mathbf{R}^3 \,\big|\, (v, q) = 0\}$, the tangent space $T_{\pm re_i}S^2_r$ is spanned by $\{e_j, e_k\}$ where $j \neq i$, $k \neq i$ and $k \neq j$. Therefore at $\pm re_1$, $\pm re_2$, and $\pm re_3$, the Hessian of H_r is

$$\mathrm{diag}(I_2^{-1} - I_1^{-1}, I_3^{-1} - I_1^{-1}),\ \mathrm{diag}(I_1^{-1} - I_2^{-1}, I_3^{-1} - I_2^{-1}),\ \text{and}\ \mathrm{diag}(I_1^{-1} - I_3^{-1}, I_2^{-1} - I_3^{-1}),$$

respectively. Because (56) holds, we see that q is a nondegenerate critical point. Thus H_r is a Morse function. In particular, the *Morse index* of H_r at the critical point $\pm re_i$ is 2 if $i = 1$, 1 if $i = 2$, and 0 if $i = 3$. Hence H_r has two maxima, two saddle points and two minima. □

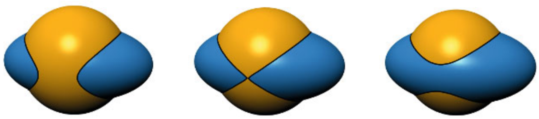

Figure 4.1. The level sets $H_r^{-1}(h)$. In the left figure $\frac{1}{2}I_3^{-1} < h < \frac{1}{2}I_2^{-1}$; in the center figure $h = \frac{1}{2}I_2^{-1}$; and in the right figure $\frac{1}{2}I_2^{-1} < h < \frac{1}{2}I_1^{-1}$.

Geometrically the level set $H_r^{-1}(h)$ of the Hamiltonian is the intersection of a triaxial ellipsoid $\mathscr{E}_h : \frac{1}{2}(I_1^{-1}p_1^2 + I_2^{-1}p_2^2 + I_3^{-1}p_3^2) = h$ with the 2-sphere $p_1^2 + p_2^2 + p_3^2 = r^2$, see figure 4.1.

▷ To show the pictures in figure 4.1 are qualitatively correct we use Morse theory.

(4.2) **Proof**: First we show that the $\frac{1}{2}r^2I_3^{-1}$-level set of H_r on S_r^2 is two points $\{\pm re_3\}$. The point $q = (x, y, z)$ lies on $H_r^{-1}(\frac{1}{2}r^2I_3^{-1})$ if and only if

$$\begin{aligned} \tfrac{1}{2}(I_1^{-1}x^2 + I_2^{-1}y^2 + I_3^{-1}z^2) &= \tfrac{1}{2}r^2I_3^{-1} && \text{(57a)} \\ x^2 + y^2 + z^2 &= r^2. && \text{(57b)} \end{aligned}$$

Multiplying (57b) by I_3^{-1} and subtracting the result from two times (57a) gives

$$(I_1^{-1} - I_3^{-1})x^2 + (I_2^{-1} - I_3^{-1})y^2 = 0. \qquad (58)$$

Since (56) holds, $I_1^{-1} - I_3^{-1} > 0$, and $I_2^{-1} - I_3^{-1} > 0$. Thus (58) yields $x = y = 0$. Consequently, $z = \pm r$.

Because the critical points $\{\pm re_3\}$ are nondegenerate minima, we may apply the Morse lemma, see chapter XI §2, to conclude that for a value of h slightly greater than $\frac{1}{2}r^2I_3^{-1}$, the level set $H_r^{-1}(h)$ is diffeomorphic to two disjoint circles, one in the neighborhood of re_3 and the other in the neighborhood of $-re_3$. Thus $H_r^{-1}(h)$ is *not* connected. A similar argument shows that for h slightly less than $\frac{1}{2}r^2I_1^{-1}$, the level set $H_r^{-1}(h)$ is also the disjoint union of two circles near $\pm re_1$. Since H_r has no critical values in $\mathscr{I} = (\frac{1}{2}r^2I_3^{-1}, \frac{1}{2}r^2I_2^{-1}) \cup (\frac{1}{2}r^2I_2^{-1}, \frac{1}{2}r^2I_1^{-1})$, using the Morse isotopy lemma, see chapter XI §3, we deduce that for $h \in \mathscr{I}$ the h-level set of H_r is diffeomorphic to the disjoint union of two circles.

To describe the remaining level set $H_r^{-1}(\frac{1}{2}r^2I_2^{-1})$ we note that $q = (x, y, z)$ lies in the $\frac{1}{2}r^2I_2^{-1}$-level set of H_r if and only if

$$\begin{aligned} \tfrac{1}{2}(I_1^{-1}x^2 + I_2^{-1}y^2 + I_3^{-1}z^2) &= \tfrac{1}{2}r^2I_2^{-1} && \text{(59a)} \\ x^2 + y^2 + z^2 &= r^2. && \text{(59b)} \end{aligned}$$

Multiplying (59b) by I_2^{-1} and subtracting the result from two times (59a) shows that $0 = (I_2^{-1} - I_3^{-1})z^2 - (I_2^{-1} - I_1^{-1})x^2$. Thus the $\frac{1}{2}r^2 I_2^{-1}$-level set of H_r is the intersection of the two 2-planes $\Pi_\pm = \left\{(x,y,z) \in \mathbf{R}^3 \,\middle|\, (I_2^{-1} - I_3^{-1})^{1/2} z = \pm (I_2^{-1} - I_1^{-1})^{1/2} x\right\}$ with the 2-sphere S_r^2. Since $\Pi_+ \neq \Pi_-$ and $\Pi_+ \bigcap \Pi_-$ is the line spanned by the vector e_2, the level set $H_r^{-1}(\frac{1}{2}r^2 I_2^{-1})$ is an *algebraic variety* $\widetilde{V}$, which is the union of two circles which intersect each other at $\{\pm r e_2\}$. Because $\{\pm r e_2\}$ are nondegenerate saddle points, using the Morse lemma we see that these circles intersect transversely. This completes the verification of figure 4.1. □

The information we have obtained about about the topology of the level sets of H_r is summarized in figure 4.2.

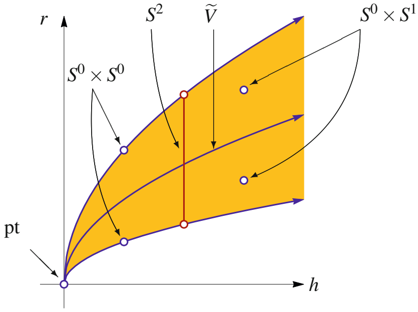

Figure 4.2. Bifurcation diagram for the level sets of the reduced Hamiltonian H_r of the Euler top.

Putting all the pictures in figure 4.1 together gives figure 4.3. Since the connected

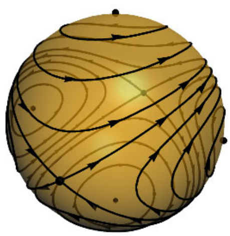

Figure 4.3. Level sets of H_r on S^2.

components of the level sets of H_r on S_r^2 are orbits of the reduced vector field X_{H_r}, figure 4.3 gives a qualitative description of the solutions of Euler's equations of the Euler top. We ▷ now verify that the orientations of the integral curves of the vector field X_{H_r} on S_r^2 in figure 4.3 are correct.

(4.3) **Proof**: By continuity, it suffices to show that the linearization of X_{H_r} at $(0,0,r)$ is an infinitesimal counterclockwise rotation around the positive p_3-axis. Differentiating (55) gives

$$DX_{H_r}(p_1,p_2,p_3) = \begin{pmatrix} 0 & (I_3^{-1}-I_2^{-1})p_3 & (I_3^{-1}-I_2^{-1})p_2 \\ -(I_3^{-1}-I_1^{-1})p_3 & 0 & -(I_3^{-1}-I_1^{-1})p_1 \\ (I_2^{-1}-I_1^{-1})p_2 & (I_2^{-1}-I_1^{-1})p_1 & 0 \end{pmatrix}.$$

Since $T_{(0,0,r)}S_r^2$ is spanned by $\{e_1,e_2\}$, the linearization of X_{H_r} at $(0,0,r)$ is

$$W = DX_{H_r}(0,0,r)|T_{(0,0,r)}S_r^2 = \begin{pmatrix} 0 & -r(I_2^{-1}-I_3^{-1}) \\ r(I_1^{-1}-I_3^{-1}) & 0 \end{pmatrix}.$$

Let $P = \begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix}$ where $u = (I_2^{-1}-I_3^{-1})/(I_1^{-1}-I_3^{-1})$. A calculation shows that $Z = P^{-1}WP = \begin{pmatrix} 0 & -s \\ s & 0 \end{pmatrix}$, where $s = r\big((I_1^{-1}-I_3^{-1})(I_2^{-1}-I_3^{-1})\big)^{1/2}$. Since $s > 0$, Z is an infinitesimal counterclockwise rotation about the p_3-axis. Hence W is also. A similar argument handles each of the remaining cases. □

5 Analysis of the energy momentum mapping

In this section we study the energy momentum mapping $\mathscr{EM}$ of the Euler top:

$$\begin{aligned} \mathscr{EM} : \mathrm{SO}(3) \times \mathrm{so}(3)^* &\to \mathbf{R} \times \mathrm{so}(3)^* : \\ (A,\alpha) &\mapsto \big(H(A,\alpha), J(A,\alpha)\big) = \big(\tfrac{1}{2}\,\alpha(k^\flat(I^{-1})^t\alpha), \mathrm{Ad}^t_{A^{-1}}\alpha\big) \end{aligned}$$

Our main goal is to determine the topology of its fibers $\mathscr{EM}^{-1}(h,\mu)$, because these fibers are invariant under the flow of the Euler-Arnol'd vector field X_H, see table 5.1. We also describe how the fibers $\mathscr{EM}^{-1}(h,\mu)$ foliate a level set $J^{-1}(\mu)$ of constant angular momentum μ, see figure 5.2.

(h,μ)	topology of $H_r^{-1}(h)$	topology of $\mathscr{EM}^{-1}(h,\mu)$
$(0,0)$	point	$\mathrm{SO}(3)$
$h = \frac{1}{2}r^2 I_i^{-1}$, $i = 1,3$	two points	$S^1 \,\dot\cup\, S^1$ whose double cover is once linked.
$h \in \mathscr{I}, r \neq 0$	$S^1 \,\dot\cup\, S^1$	$T^2 \,\dot\cup\, T^2$
$h = \frac{1}{2}r^2 I_2^{-1}$	$\widetilde{V}$, the union of two circles on S^2 which intersect transversely at two points.	$\widetilde{W}$, the union of two T^2 in $\mathrm{SO}(3)$ which intersect along two circles, whose double cover is once linked.

Table 5.1 Topology of the fibers of the energy momentum map $\mathscr{EM}$.

▷ We now reconstruct the fiber $\mathscr{EM}^{-1}(h,\mu)$ from the h-level set of the reduced Hamiltonian H_r. Here $r = |\mu|$.

(5.1) **Proof**: From the definition of the reduced Hamiltonian H_r (53) and the reduction mapping

$$\pi_r : J^{-1}(\mu) \subseteq \mathrm{SO}(3) \times \mathrm{so}(3)^* \to S_r^2 \subseteq \mathbf{R}^3 : (A, \mathrm{Ad}^t_A\mu) \mapsto A^{-1}\big(i \circ k^\flat(\mu)\big), \qquad (60)$$

we find that

$$\pi_r^{-1}(H_r^{-1}(h)) = \big(H|J^{-1}(\mu)\big)^{-1}(h) = H^{-1}(h) \cap J^{-1}(\mu) = \mathscr{E}\mathscr{M}^{-1}(h,\mu).$$

Using table 5.1 we obtain the *bifurcation diagram* figure 5.1. □

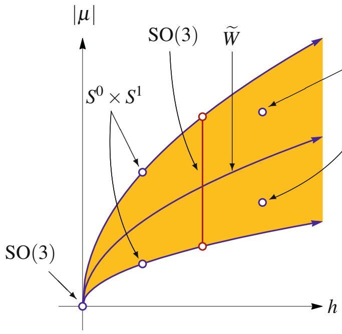

Figure 5.1. Bifurcation diagram for the energy momentum map of the Euler top.

▷ The following argument verifies the entries in the third column of table 5.1.

(5.2) **Proof**: First we find the critical points and critical values of $\mathscr{E}\mathscr{M}$. Suppose that $\mu = 0$. Then $J^{-1}(0) = \{(A,0) \in \mathrm{SO}(3) \times \mathrm{so}(3) \big| A \in \mathrm{SO}(3)\}$, which is diffeomorphic to SO(3). On $J^{-1}(0)$ the Hamiltonian H is the constant function 0. Therefore, every point of $J^{-1}(0)$ is a critical point of $H|J^{-1}(0)$ and hence is a critical point of $\mathscr{E}\mathscr{M}$. The corresponding critical value of $\mathscr{E}\mathscr{M}$ is $(0,0)$.

From now on we suppose that $\mu \neq 0$. Then $J^{-1}(\mu)$ is diffeomorphic to SO(3).

Claim : The function

$$H|J^{-1}(\mu) : J^{-1}(\mu) \rightarrow \mathbf{R} : (A, \mathrm{Ad}_A^t \mu) \rightarrow \tfrac{1}{2}\, k\big(k^\flat(\mathrm{Ad}_A^t \mu), I^{-1}(k^\flat(\mathrm{Ad}_A^t \mu))\big)$$

is an $\mathrm{SO}(3)_\mu$–invariant *Bott–Morse function* with six nondegenerate critical $\mathrm{SO}(3)_\mu$-orbits $\gamma_i^\pm = \pi_r^{-1}(\pm re_i)$ for $i = 1,2,3$, two of Morse index 2, 1 and 0 respectively.

(5.3) **Proof**: By construction $H|J^{-1}(\mu) = (\pi_r)^* H_r$. Therefore p is a critical point of $H|J^{-1}(\mu)$ if and only if $D\big(H|J^{-1}(\mu)\big)(p) = DH_r(\pi_r(p))D\pi_r(p)$ is not surjective. Since π_r is a submersion, p is a critical point if and only if $DH_r(\pi_r(p))$ is not surjective, that is, if and only if $\pi_r(p)$ is a critical point of H_r. By ((4.1)) the set of critical points of H_r is $\{\pm re_i | i = 1,2,3\}$. Therefore the six $\mathrm{SO}(3)_\mu$-orbits $\gamma_i^\pm = \{\pi_r^{-1}(\pm re_i) \big| i = 1,2,3\}$ form the set of critical points of $H|J^{-1}(\mu)$. Since an $\mathrm{SO}(3)_\mu$-orbit is diffeomorphic to a circle, the critical set of $H|J^{-1}(\mu)$ is the union of six circles $\gamma_i^\pm$, $i = 1,2,3$, which correspond to the critical values $\frac{1}{2}r^2 I_i^{-1}$, $i = 1,2,3$, respectively.

We now show that each of the $\gamma_i^\pm$ is a nondegenerate critical manifold, see chapter XI §2. Choose an open neighborhood $\mathscr{U}_i^\pm$ of $\pm re_i$ in S_r^2 such that the bundle $\pi_r : J^{-1}(\mu) \rightarrow S_r^2$, when restricted to $\pi_r^{-1}(\mathscr{U}_i^\pm)$, is a trivial $\mathrm{SO}(3)_\mu$ principal bundle. Then there is a diffeomorphism $\varphi_i^{\pm} : \pi_r^{-1}(\mathscr{U}_i^{\pm}) \subseteq J^{-1}(\mu) \to \mathscr{U}_i^{\pm} \times \mathrm{SO}(3)_\mu$, which intertwines the $\mathrm{SO}(3)_\mu$-action Φ (48) on $\pi_r^{-1}(\mathscr{U}_i^{\pm})$ with the $\mathrm{SO}(3)_\mu$-action on $\mathscr{U}_i \times \mathrm{SO}(3)_\mu$ given by $\big(B,(q,A)\big) \mapsto (q,BA)$. If we parametrize the $\mathrm{SO}(3)_\mu$-orbit $\gamma_i^{\pm}$ by $t \mapsto \Phi_{\exp t\xi_i^{\pm}}(p_i^{\pm})$ where $\xi_i^{\pm} \in \mathrm{so}(3)_\mu$ and $p_i^{\pm} \in \gamma_i^{\pm}$, then $\varphi_i^{\pm}(\gamma_i^{\pm})$ is parametrized by $t \mapsto (\pm re_i, \exp t\xi_i^{\pm})$. Thus for fixed t_0 the set $\mathscr{S}_i^{\pm} = (\varphi_i^{\pm})^{-1}(\mathscr{U}_i^{\pm} \times \{\exp t_0 \xi^{\pm}\})$ is a slice to $\gamma_i^{\pm}$ at $\gamma_i^{\pm}(t_0)$. Since the mapping π_r restricted to $\mathscr{S}_i^{\pm}$ is a diffeomorphism onto $\mathscr{U}_i^{\pm}$, the Morse index of $H|J^{-1}(\mu)$ restricted to $\mathscr{S}_i^{\pm}$ at $\gamma_i^{\pm}(t_0)$ is the same as the Morse index of H_r at $\pm re_i$. Therefore $\gamma_i^{\pm}$ is a nondegenerate critical manifold of $H|J^{-1}(\mu)$. $\square$

We return to verifying the third column of table 5.1. Suppose that h is a regular value of the Hamiltonian H_r. Then for $h \in \mathscr{I} = (\frac{1}{2}r^2 I_3^{-1}, \frac{1}{2}r^2 I_2^{-1}) \dot\cup (\frac{1}{2}r^2 I_2^{-1}, \frac{1}{2}r^2 I_1^{-1})$ the level set $H_r^{-1}(h)$ is diffeomorphic to the disjoint union of two circles S_j^1. Since each S_j^1 is null homotopic in S_r^2, the circle S_j^1 bounds a disk $\overline{D}_j^2$, which is contractible in S_r^2. Therefore, the bundle $\pi_r : J^{-1}(\mu) \to S_r^2$ restricted to $\pi_r^{-1}(\overline{D}_j^2)$ is trivial, that is, $\pi_r^{-1}(\overline{D}_j^2)$ is diffeomorphic to $\overline{D}_j^2 \times S^1$, see chapter VIII §2. Hence, the manifold $\mathscr{EM}^{-1}(h,\mu) = \pi_r^{-1}(H_r^{-1}(h))$ is the disjoint union of two 2-tori $\partial\overline{D}_j^2 \times S^1$. Now suppose that $h = \frac{1}{2}r^2 I_1^{-1}$ or $h = \frac{1}{2}r^2 I_3^{-1}$. Then the h-level set of H_r is the disjoint union of two points. Therefore, the manifold $\mathscr{EM}^{-1}(h,\mu)$ is the disjoint union of two circles, whose double covers are linked once. This follows because the reduction map π_r is double covered by the Hopf map and any two distinct fibers of the Hopf fibration are linked once. Finally, suppose that $h = \frac{1}{2}r^2 I_2^{-1}$. Then $H_r^{-1}(h)$ is the union of two circles C_j which intersect each other transversely at $\pm re_2$. Since each of the circles bounds a contractible disk on S_r^2, each set $\pi_r^{-1}(C_j)$ is a 2-torus T_j^2. Using the Morse lemma, chapter XI §2, we see that these tori intersect each
▷ other transversely in $J^{-1}(\mu)$ along two circles $\gamma_2^{\pm}$. Hence $\gamma_2^{\pm}$ is a hyperbolic periodic orbit of the vector field $X_H|J^{-1}(\mu)$.

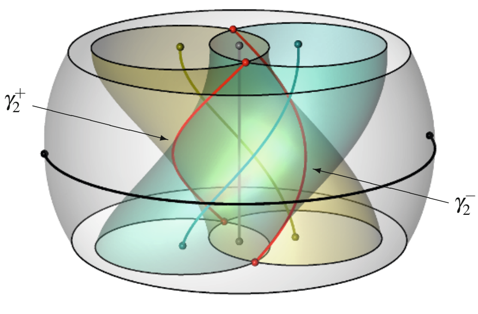

Figure 5.2. The fibration of $J^{-1}(\mu)$ by the level sets of H.

(5.4) **Proof**: Since $(\pi_r)^*\omega_r = \Omega|J^{-1}(\mu)$, the reduction mapping π_r is a symplectic diffeomorphism of the slice $(\mathscr{S}_2^{\pm}, \Omega|(\mathscr{S}_2^{\pm})$ onto $(\mathscr{U}_2^{\pm}, \omega_r|\mathscr{U}_2^{\pm})$. Therefore the vector field $X_H|\mathscr{S}_2^{\pm}$ pushes forward under π_r to the vector field $X_{H_r}|\mathscr{U}_2^{\pm}$, which clearly has a hyperbolic equilibrium point at $\pm re_2$. $\square$

The union of the two 2-tori $\pi_r^{-1}(C_j)$ is the variety $\widetilde{W} = \pi_r^{-1}(\widetilde{V})$, which is the union of the closures of the stable and unstable manifolds of the hyperbolic periodic orbits γ_2^+ and γ_2^-.

▷ The local *unstable manifold* of $\gamma_2^{\pm}$ is *not* twisted.

(5.5) **Proof**: To see this, let $\mathscr{V}_2^{\pm}$ be an open neighborhood of $\pm re_2$ in S_r^2. Then the intersection of $\mathscr{V}_2^{\pm}$ with the closure of the local unstable manifold of $\pm re_2$ is a contractible subset of S_r^2. Consequently, the closure of the local unstable manifold of $\gamma_2^{\pm}$ is a trivial bundle over S^1. Thus it is *not twisted.* □

This completes verification of the third column of table 5.1. □

▷ Figure 5.2 is a qualitatively correct picture of the level sets of H on $J^{-1}(\mu)$.

(5.6) **Proof**: Let $\overline{\mathscr{D}} \subseteq S_r^2$ be a small closed 2-disk about the north pole $(0,0,r)$ of the 2-sphere S_r^2. Using stereographic projection pr from the north pole, we find that the image of $S_r^2 \setminus \mathscr{D}$ is the closed 2-disk E in $\mathbf{R}^2$, see figure 5.3. Since E is contractible, the bundle $\pi_r|\pi_r^{-1}(E)$ is trivial, that is, $\pi_r^{-1}(E)$ is diffeomorphic to the *solid torus* $S^1 \times E$. Thus we obtain a decomposition of $J^{-1}(\mu)$, which is diffeomorphic to SO(3), into the union of two solid tori $ST_1 = \pi_r^{-1}(\overline{\mathscr{D}})$ and $ST_2 = \pi_r^{-1}(E)$.

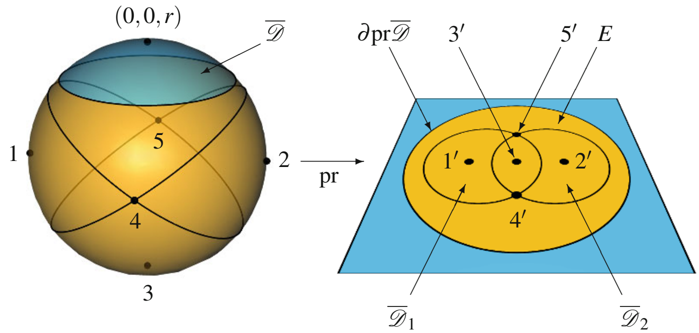

Figure 5.3. Stereographic projection of the level sets of the reduced Hamiltonian H_r.

Let us investigate more carefully how the *solid torus* ST_2 *fits* into SO(3). First remove ST_2 from the solid ball D^3. Clearly, ST_2 is homeomorphic to $S^1 \times (\overline{\mathscr{D}}_1 \cup \overline{\mathscr{D}}_2)$, which is a solid torus on the two overlapping 2-disks $\overline{\mathscr{D}}_1$ and $\overline{\mathscr{D}}_2$. The solid torus ST_2 is formed by

1. Taking the cylinder $\mathscr{C} = [0,1] \times (\overline{\mathscr{D}}_1 \cup \overline{\mathscr{D}}_2)$ and giving it a number of half twists.
2. Placing the result in the 3-disk $D^3 \setminus \mathscr{C}$.
3. Identifying antipodal points on the two end 2-disks of $\mathscr{C}$ and also on $\partial(D^3 \setminus \mathscr{C})$.

In figure 5.4 we illustrate this construction when the cylinder $\mathscr{C}$ has undergone *zero* half twists. The lines AA', BB', and CC' in this figure are the center lines of the cylinders $[0,1] \times \overline{\mathscr{D}}_1$, $[0,1] \times (\overline{\mathscr{D}}_1 \cap \overline{\mathscr{D}}_2)$, and $[0,1] \times \overline{\mathscr{D}}_2$, respectively. After antipodal identification of the end 2-disks, we obtain *two* solid tori with center circles $AA'CC'$ and BB'. In figure 5.4 there are only *three* fibers which are center circles of solid tori. (Do not forget to include the centre circle DD'.) These fibers correspond to critical submanifolds of $H|J^{-1}(\mu)$ of index 0 or 2. According to ((5.3)) there are *four* such critical submanifolds.

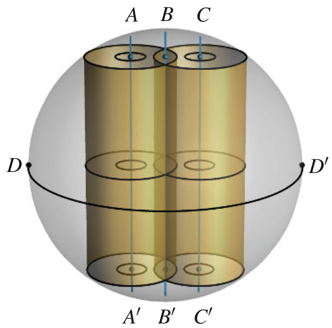

Figure 5.4. Replace the cylinder $\mathscr{C}$ with no half twists.

Hence figure 5.4 does *not* describe the foliation of $J^{-1}(\mu)$ by level sets of H. An obvious generalization of the above argument eliminates the possibility of replacing the cylinder $\mathscr{C}$ with an arbitary *even* number of half twists. Suppose that $\mathscr{C}$ is replaced in $D^3 \setminus \mathscr{C}$ after an *odd* number of half twists *greater* than one. For the sake of argument say three. It is clear that the curves AA', BB', CC' and DD' are center circles of solid tori. However, the double cover of AA' and BB' in S^3 has linking number three, see exercise 10. Thus AA' and BB' do *not* correspond to any critical submanifold of $H|J^{-1}(\mu)$ because the double covers of the critical manifolds of $H|J^{-1}(\mu)$ have linking number one. Thus we can only replace $\mathscr{C}$ with *one* half twist.

Whether this is a clockwise or counterclockwise half twist depends on the *sign* of the linking number. We determine this sign as follows. Orient $\mathrm{SO}(3)$ so that its double cover S^3 is positively oriented. Give the solid tori ST_i in $\mathrm{SO}(3)$ the induced orientation. Orient a 2-disk in ST_i, which is transverse to the center circle, so that its image under the reduction map has the same orientation as a solution to Euler's equations which it contains. This, together with the orientation of ST_i, determines the orientation of the center circles $A'A$ and $B'B$. As integral curves of X_H on $J^{-1}(\mu)$, the curves $A'A$ and $B'B$ are positively oriented. Hence their double covers in S^3, which are integral curves of the harmonic oscillator on an energy surface, are also positively oriented. Thus their linking number in S^3 is $+1$. Therefore the cylinder $\mathscr{C}$ is given a *counterclockwise* half twist when looking in the direction of the positively oriented curve $B'B$.

This completes the verification of figure 5.2. $\square$

6 Integration of the Euler-Arnol'd equations

In this section we *integrate* the Euler-Arnol'd equations

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = I_3^{-1} p_3 y - I_2^{-1} p_2 (x \times y) \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = -I_3^{-1} p_3 x + I_1^{-1} p_1 (x \times y) \end{cases} \tag{61a}$$

$$\frac{\mathrm{d}p}{\mathrm{d}t} = p \times (I')^{-1} p \tag{61b}$$

in the sphere bundle model $(T_1 S^2 \times \mathbf{R}^3, \omega')$, see §2.2. Here $I'p = (I_1 p_1, I_2 p_2, I_3 p_3)$. These solutions describe the motions of the Euler top in space.

We begin by looking at certain invariant manifolds of the Euler-Arnol'd equations. A straightforward calculation shows that the angular momentum

$$J' : T_1 S^2 \times \mathbf{R}^3 \rightarrow \mathbf{R}^3 : (x, y, p) \mapsto Ap = \mathrm{col}(x, y, x \times y) p \tag{62}$$

and energy

$$H' : T_1 S^2 \times \mathbf{R}^3 \rightarrow \mathbf{R} : (x, y, p) \mapsto \tfrac{1}{2} (I_1^{-1} p_1^2 + I_2^{-1} p_2^2 + I_3^{-1} p_3^2). \tag{63}$$

are integrals of these equations.

We now choose a better orthonormal basis $\{\widetilde{e}_1, \widetilde{e}_2, \widetilde{e}_3\}$ of $\mathbf{R}^3$ to study the Euler-Arnol'd equations (61a) and (61b). Let A_0 be a rotation which sends the conserved angular momentum vector $\ell = Ap$ of magnitude $|\ell|$ to the vector $|\ell| e_3$. Let $\widetilde{A} = A_0 A = \mathrm{col}(\widetilde{x}, \widetilde{y}, \widetilde{x} \times \widetilde{y})$. Then $(\widetilde{x}, \widetilde{y}, p)$ is a solution of the Euler-Arnol'd equations, where the angular momentum integral $J' = |\ell| e_3 = \ell$. In what follows we will assume that $\{\widetilde{e}_1, \widetilde{e}_2, \widetilde{e}_3\}$ is the standard basis of $\mathbf{R}^3$ and drop the tildes on the variables x and y.

Suppose that (h, ℓ) is a *regular value* of the energy momentum mapping

$$\mathscr{E}\mathscr{M}' : T_1 S^2 \times \mathbf{R}^3 \rightarrow \mathbf{R} \times \mathbf{R}^3 : (x, y, p) \mapsto \big(H'(x, y, p), J'(x, y, p) \big).$$

Then $(\mathscr{E}\mathscr{M}')^{-1}(h, \ell)$ is diffeomorphic to the disjoint union of two 2-tori. Call one of them $T^2_{h,\ell}$. Then $T^2_{h,\ell}$ is an invariant manifold of the Hamiltonian vector field $X_{H'}$, whose
▷ integral curves satisfy the Euler-Arnol'd equations. We wish to describe $T^2_{h,\ell}$ as a bundle whose projection map is

$$\pi : T^2_{h,\ell} \subseteq T_1 S^2 \times \mathbf{R}^3 \rightarrow S^1_{h,\ell} \subseteq S^2_{|\ell|} \subseteq \mathbf{R}^3 : (x, y, p) \mapsto p. \tag{64}$$

(6.1) **Proof**: Observe that $\pi(T^2_{h,\ell})$ is a connected component of the h-level set of the reduced Hamiltonian $H_{|\ell|} : S^2_{|\ell|} \rightarrow \mathbf{R} : p \mapsto \frac{1}{2} \left((I')^{-1} p, p \right)$ and hence is diffeomorphic to a circle $S^1_{h,\ell}$, since h is a regular value of $H_{|\ell|}$. $S^1_{h,\ell}$ is parametrized by a periodic solution of Euler's equations (61b).

To find an explicit description of $T^2_{h,\ell}$ we begin by solving $J'(x, y, p) = \ell = |\ell| e_3$ for p. Using (62) we obtain $|\ell|^{-1} p = (\mathrm{col}(x, y, x \times y))^t e_3$. In other words,

$$\begin{aligned} x_3 &= |\ell|^{-1} p_1 && \text{(65a)} \\ y_3 &= |\ell|^{-1} p_2 && \text{(65b)} \\ x_1 y_2 - x_2 y_1 &= |\ell|^{-1} p_3. && \text{(65c)} \end{aligned}$$

Substituting (65a) and (65b) into the defining equations of $T_1 S^2$

$$\begin{aligned} x_1^2 + x_2^2 + x_3^2 &= 1 \\ x_1 y_1 + x_2 y_2 + x_3 y_3 &= 0 \\ y_1^2 + y_2^2 + y_3^2 &= 1 \end{aligned} \tag{66}$$

gives

$$\begin{aligned} x_1^2 + x_2^2 &= 1 - |\ell|^{-2} p_1^2 && \text{(67a)} \\ x_1 y_1 + x_2 y_2 &= -|\ell|^{-2} p_1 p_2 && \text{(67b)} \\ y_1^2 + y_2^2 &= 1 - |\ell|^{-2} p_2^2. && \text{(67c)} \end{aligned}$$

Because h is a regular value of the reduced Hamiltonian $H_{|\ell|}$, $p_i \neq \pm |\ell| e_i$ for $i = 1, 2, 3$. Therefore the right hand side of (67a) is nonzero for every $p \in S^1_{h,\ell}$. Write (65c) and (67b) as the linear system

$$\begin{pmatrix} -x_2 & x_1 \\ x_1 & x_2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} |\ell|^{-1} p_3 \\ -|\ell|^{-2} p_1 p_2 \end{pmatrix}. \tag{68}$$

Using (67a), we obtain its solution

$$\begin{aligned} y_1 &= -\tfrac{1}{|\ell|^2 - p_1^2} \left(p_1 p_2 x_1 + |\ell| p_3 x_2 \right) \\ y_2 &= \tfrac{1}{|\ell|^2 - p_1^2} \left(|\ell| p_3 x_1 - p_1 p_2 x_2 \right). \end{aligned} \tag{69}$$

Therefore the 2-torus $T^2_{h,\ell}$ is the set of points $(x_1, x_2, |\ell|^{-1} p_1, y_1, y_2, |\ell|^{-1} p_2, p) \in T_1 S^2 \times \mathbf{R}^3$ where

$$x_1^2 + x_2^2 = 1 - |\ell|^{-2} p_1^2,\ y_1 = -\tfrac{1}{|\ell|^2 - p_1^2} \left(p_1 p_2 x_1 + |\ell| p_3 x_2 \right),\ y_2 = \tfrac{1}{|\ell|^2 - p_1^2} \left(|\ell| p_3 x_1 - p_1 p_2 x_2 \right),$$

and $p = (p_1, p_2, p_3)$ lies in the connected component of

$$I_1^{-1} p_1^2 + I_2^{-1} p_2^2 + I_3^{-1} p_3^2 = 2h \quad \text{and} \quad p_1^2 + p_2^2 + p_3^2 = |\ell|^2,$$

which defines $S^1_{h,\ell}$. This gives the desired description of the bundle π (64). $\square$

$\triangleright$ Because $T^2_{h,\ell}$ is a smooth invariant manifold of $X_{H'}$, the restriction of $X_{H'}$ to $T^2_{h,\ell}$ is a vector field whose integral curves satisfy

$$\begin{aligned}\dot{x}_1 &= \alpha(p)x_1 - \beta(p)x_2 &\text{(70a)}\\ \dot{x}_2 &= \beta(p)x_1 + \alpha(p)x_2 &\text{(70b)}\\ \dot{p} &= p \times (I')^{-1}p. &\text{(70c)}\end{aligned}$$

Here $p \in S^1_{h,\ell}$ and

$$\alpha(p) = -\frac{(I_3^{-1} - I_2^{-1})\,p_1p_2p_3}{|\ell|^2 - p_1^2} \quad \text{and} \quad \beta(p) = \frac{|\ell|(I_2^{-1}p_2^2 + I_3^{-1}p_3^2)}{|\ell|^2 - p_1^2}. \tag{71}$$

(6.2) **Proof**: Consider the Euler-Arnol'd equations, (61a) and (61b), subject to the constraints (66). Restrict these equations to $(J')^{-1}(\ell e_3)$. In other words, impose the additional constraints (65a) – (65c). Using (65a) and (65b) eliminate x_3 and y_3 from (61a). We obtain

$$\begin{aligned}\dot{x}_1 &= I_3^{-1}p_3y_1 - |\ell|^{-1}I_2^{-1}p_2(p_2x_2 - p_1y_2) &\text{(72a)}\\ \dot{x}_2 &= I_3^{-1}p_3y_2 - |\ell|^{-1}I_2^{-1}p_2(p_1y_1 - p_2x_1) &\text{(72b)}\\ \dot{y}_1 &= -I_3^{-1}p_3x_1 + |\ell|^{-1}I^{-1}p_1(p_2x_2 - p_1y_2) &\text{(72c)}\\ \dot{y}_2 &= -I_3^{-1}p_3x_2 + |\ell|^{-1}I_1^{-1}p_1(p_1y_1 - p_2x_1) &\text{(72d)}\end{aligned}$$

together with (70c) and the constraints (65c), (67a – 67b). Restrict (72a – 72d) to $(H')^{-1}(h)$. Since (69) holds on $T^2_{h,\ell}$ (a connected component of $(J')^{-1}(\ell) \cap (H')^{-1}(h)$), equations (72c) and (72d) hold as soon as (72a) and (72b) do. Substituting (69) into (72a) and (72b) to eliminate y_1 and y_2 gives (70a) and (70b) where the functions $\alpha(p)$ and $\beta(p)$ are defined by (71). Clearly, (70c) holds. Since $\pi(T^2_{h,\ell}) = S^1_{h,\ell}$, it follows that $p \in S^1_{h,\ell}$. □

Because of the hierarchical nature of the equations (70a) – (70c), we can solve them as follows. Let $t \mapsto p(t)$ be a solution of Euler's equations (70c), which parametrizes $S^1_{h,\ell}$, see exercise 2. Substituting $t \mapsto p(t)$ into (70a) and (70b) gives the time dependent linear system

$$\begin{aligned}\dot{x}_1 &= \alpha(t)x_1 - \beta(t)x_2\\ \dot{x}_2 &= \beta(t)x_1 + \alpha(t)x_2,\end{aligned} \tag{73}$$

where $\alpha(t) = \alpha(p(t))$ and $\beta(t) = \beta(p(t))$. Introduce polar coordinates $r^2 = x_1^2 + x_2^2$ and $\theta = \tan^{-1}\frac{x_2}{x_1}$. Then (73) becomes

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \beta(t) \quad \text{and} \quad \frac{\mathrm{d}r}{\mathrm{d}t} = \alpha(t). \tag{74}$$

From (67a) we get $r^2(t) = 1 - |\ell|^{-2}p_1^2(t)$. Integrating (74) gives

$$\theta(t) = \int_0^t \beta(s)\,\mathrm{d}s + \theta(0). \tag{75}$$

Therefore,

$$x_1(t) = \sqrt{1-|\ell|^{-2}p_1^2(t)}\,\cos\theta(t) \quad \text{and} \quad x_2(t) = \sqrt{1-|\ell|^{-2}p_1^2(t)}\,\sin\theta(t).$$

Substituting $x_1(t)$ and $x_2(t)$ into (69) and using (65a) and (65b) gives $t \mapsto \big(x(t), y(t), p(t)\big)$, where

$$x(t) = \big(\sqrt{1-|\ell|^{-2}p_1^2(t)}\,\cos\theta(t), \sqrt{1-|\ell|^{-2}p_1^2(t)}\,\sin\theta(t), |\ell|^{-1}p_1(t)\big) \tag{76a}$$

$$\begin{aligned} y(t) = \big(-&\tfrac{|\ell|^{-2}}{\sqrt{1-|\ell|^{-2}p_1^2(t)}}\,[p_1(t)p_2(t)\,\cos\theta(t) + |\ell|p_3(t)\sin\theta(t)], \\ &\tfrac{|\ell|^{-2}}{\sqrt{1-|\ell|^{-2}p_1^2(t)}}\,[|\ell|p_3(t)\cos\theta(t) - p_1(t)p_2(t)\,\sin\theta(t)], |\ell|^{-1}p_2(t)\big). \end{aligned} \tag{76b}$$

Note that $t \mapsto \big(x(t), y(t), p(t)\big)$ is a solution of the Euler-Arnol'd equations which lies in $T^2_{h,\ell} \subseteq T_1S^2 \times \mathbf{R}^3$ and under the bundle projection map π (64) maps onto $t \mapsto p(t)$ which is a solution of Euler's equation of energy h and angular momentum ℓ. □

We now give a geometric description of some of the components of the vector field $X_{H'}$ on the 2-torus $T^2_{h,\ell}$. Let λ be the one parameter subgroup of $(T_1S^2, \cdot)$ defined by

$$\lambda : \mathbf{R} \to T_1S^2 : t \mapsto \big((\cos t, -\sin t, 0), (\sin t, \cos t, 0)\big).$$

Observe that the image of λ is the isotropy group $\mathrm{SO}(3)_\ell$. Considered as a subgroup of $(T_1S^2, \cdot)$, we see that $\mathrm{SO}(3)_\ell$ is diffeomorphic to S^1. The one parameter subgroup λ induces an action

$$\Phi : S^1 \times (T_1S^2 \times \mathbf{R}^3) \to T_1S^2 \times \mathbf{R}^3 : \big(t, (x,y), p\big) \mapsto \big(\lambda(t)\cdot(x,y), p\big) = (\widetilde{R}_t x, \widetilde{R}_t y, p), \tag{77}$$

where $\widetilde{R}_t$ is a counterclockwise rotation about the e_3-axis through an angle t. The momentum mapping of the action Φ is the e_3-component of the momentum mapping J'. The action Φ maps $(J')^{-1}(\ell)$ into itself because for $(x,y,p) \in (J')^{-1}(\ell)$ we have

$$\begin{aligned} J'(\Phi_t(x,y,p)) &= J'(\widetilde{R}_t x, \widetilde{R}_t y, p) = \mathrm{col}(\widetilde{R}_t x, \widetilde{R}_t y, \widetilde{R}_t x \times \widetilde{R}_t y)p \\ &= \widetilde{R}_t(\mathrm{col}(x,y,x\times y)p) = \widetilde{R}_t J'(x,y,p) = \widetilde{R}_t(|\ell|e_3) = |\ell|e_3 = \ell. \end{aligned}$$

Because Φ leaves p fixed, it preserves the Hamiltonian H'. Thus the induced action $\Phi' = \Phi|(S^1 \times T^2_{h,\ell})$ is defined. Also every orbit of the action Φ' is a fiber of the bundle

$$\pi : T^2_{h,\ell} \subseteq (J')^{-1}(\ell) \to S^1_{h,\ell} \subseteq S^2_\ell : (x,y,p) \mapsto p \tag{78}$$

▷ and thus belongs to a *ruling* of $T^2_{h,\ell}$ by circles. The infinitesimal generator of the action Φ' is the vector field $Y = -x_2\frac{\partial}{\partial x_1} + x_1\frac{\partial}{\partial x_2}$. The vector field Y is vertical in the bundle π
▷ because $T\pi\, Y = 0$. For every $(x,y,p) \in T^2_{h,\ell}$ the vertical component of the vector $X_{H'}(x,y,p)$ is $\beta(p)Y(x,y,p)$.

7 The rotation number

In this section we give two ways of finding the rotation number of the flow of the Euler-Arnol'd equations of the Euler top. The first one is an analytic formula based on the integration of the Euler-Arnol'd equations given in §6. The second is follows from a geometric interpretation of the classical Poinsot construction.

7.1 An analytic formula

We begin by deriving an analytic formula for the rotation number. Recall that $S^1_{h,\ell}$ is a connected component of the level set of the reduced Hamiltonian H_r on $S^2_{|\ell|}$ corresponding to the regular value h. Thus $S^1_{h,\ell}$ is parametrized by a periodic solution $t \mapsto p(t)$ of Euler's equations of minimal positive period $T = T(h,\ell)$. Let $t \mapsto \theta(t)$ be the function defined by (75).

Claim: The quantity $\frac{1}{2\pi}\Delta\theta(T) = \frac{1}{2\pi}(\theta(T)-\theta(0))$ is the rotation number of the flow of $X_{H'}$ on $T^2_{h,\ell}$.

(7.1) **Proof**: Because under the bundle map π (78) every integral curve of $X_{H'}|T^2_{h,\ell}$ projects to the periodic integral curve $S^1_{h,\ell}$ of Euler's equations, every fiber of π is a *cross section* for the flow $\varphi^{H'}_t$ of $X_{H'}$ on $T^2_{h,\ell}$. From the fact that along $S^1_{h,\ell}$ the function $t \mapsto \beta(t)$ (71) is strictly positive and bounded away from 0, it follows that the function $t \mapsto \theta(t)$ (75) is strictly increasing and is unbounded as $t \mapsto \pm\infty$. Therefore for every $q \in T^2_{h,\ell}$ the image of the curve $t \mapsto \Phi'_{\theta(t)-\theta(0)}(q)$ is the fiber of the bundle π over $p = \pi(q) \in S^1_{h,\ell}$. Since $t \mapsto p(t) = \pi(\varphi^{H'}_t(q))$ starts at p, parametrizes $S^1_{h,\ell}$, and has minimal positive period T, we see that $\varphi^{H'}_T(q) \in \pi^{-1}(p)$. Therefore for every $q \in T^2_{h,\ell}$ there is a minimal $\tau > 0$ such that

$$\Phi'_{\theta(\tau)-\theta(0)}(q) = \varphi^{H'}_T(q). \tag{79}$$

Below we show that $\tau = T$. Using (79) and the definition of rotation number, it follows that $\frac{1}{2\pi}\Delta\theta(T) = \frac{1}{2\pi}\big(\theta(T)-\theta(0)\big)$ is the rotation number of the flow of $X_{H'}$ on $T^2_{h,\ell}$. □

▷ We now show that $\tau = T$.

(7.2) **Proof**: Let $\big(\xi(t),\eta(t),\rho(t)\big) = \Phi'_{\theta(t)-\theta(0)}(q) = \Phi_{\varphi(t)}(q)$ and let $\big(x(t),y(t),p(t)\big) = \varphi^{H'}_t(q)$. We calculate $\xi(t)$ as follows. From the definition of the action Φ (77), we find that

$$\xi(t) = \big(x_1(0)\cos\varphi(t) - x_2(0)\sin\varphi(t), x_1(0)\sin\varphi(t) + x_2(0)\cos\varphi(t), x_3(t)\big)\ ,$$

where $q = \big(x(0),y(0),p\big)$ and $p(0) = p$. Comparing the above expression for $\xi(t)$ with the expression for $x(t)$ given in (76a) and looking at their third components, we see that $\xi(t) = x(0)$ if and only if $t = nT$, because $t \mapsto p_1(t)$ is periodic with minimal positive period T. Since τ in (79) is positive and minimal, we conclude that $\tau = T$. □

The rotation number has the following physical interpretation. Recall that the action of $\mathrm{SO}(3)$ on itself under left multiplication by A corresponds to changing the space frame $\{e_1,e_2,e_3\}$ to the frame $\{Ae_1,Ae_2,Ae_3\}$. When A is in the isotropy group $\mathrm{SO}(3)_{|\ell|e_3}$, left multiplication by A leaves e_3 fixed. Thus the action of the isotropy group corresponds to the Euler top rotating around the e_3-axis. Hence the rotation number is the amount of revolution around the spatial e_3-axis the top makes in time T. Here the top has energy h and angular momentum $\ell = |\ell| e_3$.

7.2 Poinsot's construction

In this subsection we present a modern version of Poinsot's construction. This construction leads to another geometric description of the rotation number of the Euler top.

Using the sphere bundle model of the preceding section we define the *Poinsot* mapping by

$$P : (J')^{-1}(\ell) \subseteq T_1S^2 \times \mathbf{R}^3 \to \mathbf{R}^3 : (x,y,p) \mapsto \mathrm{col}(x,y,x\times y)(I')^{-1}p, \tag{80}$$

where J' is the angular momentum mapping $J' : T_1S^2 \to \mathbf{R}^3 : (x,y,p) \mapsto \mathrm{col}(x,y,x\times y)$. In the terminology of exercises 4 and 5, the Poinsot mapping assigns to an angular momentum vector of magnitude $|\ell|$ in the *body frame*, which corotates with the body, an *angular velocity* vector in the *space frame* of magnitude $|\ell|$.

Claim: The image of $(\mathscr{E}\mathscr{M})^{-1}(h,\ell)$ under Poinsot's mapping P (80) is contained in the affine subspace $\mathscr{A}_\ell = \{(\Omega_1,\Omega_2,\frac{2h}{|\ell|}) \,\big|\, (\Omega_1,\Omega_2) \in \mathbf{R}^2\}$ of Euclidean 3-space $\big(\mathbf{R}^3, (\ ,\ )\big)$.

(7.3) **Proof**: Suppose that $(x,y,p) \in (\mathscr{E}\mathscr{M})^{-1}(h,\ell)$. Then

$$\begin{aligned}\big(P(x,y,p),\ell\big) &= \big(\mathrm{col}(x,y,x\times y)(I')^{-1}p, |\ell|e_3\big) = \big((I')^{-1}p, \mathrm{col}(x,y,x\times y)^{-1}(|\ell|e_3)\big)\\ &= \big((I')^{-1}p,p\big), \quad \text{since } (x,y,p) \in (J')^{-1}(\ell)\\ &= 2h, \quad \text{since } (x,y,p) \in (H')^{-1}(h). \qquad \square\end{aligned}$$

Let $\mathscr{C}$ be the orbit of the action $\Phi' = \Phi|(S^1 \times T^2_{h,\ell})$ (77) through $(x,y,p) \in T^2_{h,\ell}$.

Claim: The image of the circle $\mathscr{C}$ under the Poinsot mapping $\mathscr{P}$ is a geometric circle in $\mathscr{A}_\ell$ of $\big(\mathbf{R}^3, (\ ,\ )\big)$ with center at $\frac{2h}{|\ell|}e_3$ and radius $\mathrm{rad}(x,y,p) = \|P(x,y,p) - \frac{2h}{|\ell|}e_3\|$. Here $\|\ \|$ is the norm on $\mathbf{R}^3$ associated to the Euclidean inner product $(\ ,\ )$.

(7.4) **Proof**: For every $(x,y,p) \in T^2_{h,\ell}$ we have

$$\begin{aligned}\|P(\Phi'_t(x,y,p)) - \tfrac{2h}{|\ell|}e_3\| &= \|P\big(\mathrm{col}(\widetilde{R}_tx, \widetilde{R}_ty, \widetilde{R}_tx \times \widetilde{R}_ty)\big) - \tfrac{2h}{|\ell|}e_3\| = \|\widetilde{R}_tP(x,y,p) - \tfrac{2h}{|\ell|}e_3\|\\ &= \|\mathrm{col}(x,y,x\times y)(I')^{-1}p - \tfrac{2h}{|\ell|}\widetilde{R}_t^{-1}e_3\| = \|P(x,y,p) - \tfrac{2h}{|\ell|}e_3\|. \quad \square\end{aligned}$$

Corollary: The radius function $\mathrm{rad} : \mathscr{E}\mathscr{M}^{-1}(h,\ell) \to \mathbf{R} : (x,y,p) \mapsto \|P(x,y,p) - \frac{2h}{|\ell|}e_3\|$ is invariant under the S^1-action Φ'.

▷ We now want to show that the image of the 2-torus $T^2_{h,\ell}$ under the Poinsot mapping P is the annulus

$$\mathscr{A} = \Big\{(\Omega_1,\Omega_2,\tfrac{2h}{|\ell|}) \in \mathscr{A}_\ell \,\Big|\, r_{\min} = \min_{T^2_{h,\ell}} \mathrm{rad} \le |\xi| \le \max_{T^2_{h,\ell}} \mathrm{rad} = r_{\max}\Big\}. \tag{81}$$

This entails verifying that $r_{\min} > 0$. As preparation recall that the orbits of the S^1-action Φ are fibers of the reduction mapping

$$\pi_{|\ell|} : (J')^{-1}(\ell) \subseteq T_1S^2 \times \mathbf{R}^3 \to S^2_{|\ell|} : (x,y,p) \mapsto p. \tag{82}$$

Since the radius function rad is invariant under the action Φ, it induces a function rad_ℓ on $\mathscr{S}_{h,\ell} = \pi_{|\ell|}\big(\mathscr{EM}^{-1}(h,\ell)\big)$.

Claim: For every $p \in \mathscr{S}_{h,\ell}$ we have $\mathrm{rad}_\ell(p) = \|Dp - \frac{2h}{|\ell|^2}e_3\|$, where $D = (I')^{-1}$.

(7.5) **Proof**: For every $(x,y,p) \in \pi^{-1}_{|\ell|}(p)$ we have

$$\begin{aligned}
\mathrm{rad}_\ell(p) &= \mathrm{rad}_\ell\big(\pi_{|\ell|}(x,y,p)\big) = \mathrm{rad}(x,y,p)\\
&= \|\,\mathrm{col}(x,y,x\times y)(I')^{-1}p - \tfrac{2h}{|\ell|}e_3\| = \|(I')^{-1}p - \tfrac{2h}{|\ell|^2}\mathrm{col}(x,y,x\times y)^{-1}(\ell)\|\\
&= \|Dp - \tfrac{2h}{|\ell|^2}p\|, \quad \text{since } p \in (J')^{-1}(\ell). \qquad \square
\end{aligned}$$

▷ Next we show that rad_ℓ is strictly positive on $\mathscr{S}_{h,\ell}$.

(7.6) **Proof**: From the preceding claim we see that $\mathrm{rad}_\ell(p) \ge 0$ for every $p \in \mathscr{S}_{h,\ell}$. Suppose that $\mathrm{rad}_\ell(p) = 0$ for some $p \in \mathscr{S}_{h,\ell}$. Then $Dp = \big(\frac{2h}{|\ell|^2}\big)p$. Therefore $\frac{2h}{|\ell|^2}$ is an eigenvalue of D and hence equals I_i^{-1} for some $i \in \{1,2,3\}$, that is, $h = \frac{1}{2}I_i^{-1}|\ell|^2$. Thus (h,ℓ) is a critical value of the energy momentum map $\mathscr{EM}$. This is contrary to our hypothesis. □

To prove the assertion about the image of the Poinsot mapping, from the definition of rad_ℓ it follows that the radius function rad is strictly positive on $\mathscr{EM}^{-1}(h,\ell)$. Since $\mathscr{EM}^{-1}(h,\ell)$ is compact, rad has a positive minimum. Thus $\mathscr{A}$ (81) is an annulus. □

For latter purposes, see ((7.10)), we want to know more about the critical points and critical values of rad_ℓ. Towards this goal, consider the function

$$R : \mathscr{S}_{h,\ell} \subseteq S^2_{|\ell|} \to \mathbf{R} : p \mapsto \big((D - \tfrac{2h}{|\ell|^2})^2 p, p\big).$$

Claim: R and rad_ℓ have the same critical points with the same Morse index.

(7.7) **Proof**: Since R is the square of rad_ℓ, differentiating we obtain

$$DR(p)v = 2\,\mathrm{rad}_\ell(p)\,D\mathrm{rad}_\ell(p)v, \tag{83}$$

for every $p \in S^2_{|\ell|}$ and $v \in T_pS^2_{|\ell|}$. From (83) it follows that R and rad_ℓ have the same critical points, since $\mathrm{rad}_\ell(p)$ is strictly positive on $\mathscr{S}_{h,\ell}$ ((7.6)). Differentiating (83) gives

$$D^2R(p)(v,w) = 2\,D\mathrm{rad}_\ell(p)v\,D\mathrm{rad}_\ell(p)w + 2\,\mathrm{rad}_\ell(p)\,D^2\mathrm{rad}_\ell(p)(v,w), \tag{84}$$

for $p \in S^2_{|\ell|}$ and $v,w \in T_pS^2_{|\ell|}$. If p is a critical point of R on $\mathscr{S}_{h,\ell}$ then (84) becomes

$$D^2R(p)(v,w) = 2\,\mathrm{rad}_\ell(p)\,D^2\mathrm{rad}_\ell(p)(v,w). \tag{85}$$

Since rad_ℓ is strictly positive on $\mathscr{S}_{h,\ell}$, from (85) we see that p is a nondegenerate critical point of R if and only if it is a nondegenerate critical point of rad_ℓ. Moreover, they have the same Morse index. $\square$

▷ We now show that R is a Morse function.

(7.8) **Proof**: Because $\mathscr{S}_{h,\ell}$ is compact and R is continuous, R has a critical point $p=(x_1^0,x_2^0,x_3^0)$. By Lagrange multipliers, p satisfies

$$\big((D-\tfrac{2h}{|\ell|^2})^2+\lambda_1+\lambda_2 D\big)p=0 \tag{86a}$$

$$(p,p)=|\ell|^2 \text{ and } (Dp,p)=2h. \tag{86b}$$

Here $D=\mathrm{diag}(d_1,d_2,d_3)=(I_1^{-1},I_2^{-1},I_3^{-1})$ and $d_1>d_2>d_3>0$ (56).

Suppose that all of the components of p are nonzero. Then (86a) becomes

$$(d_1-\tfrac{2h}{|\ell|^2})^2+\lambda_1+\lambda_2 d_1 \quad = \quad 0 \tag{87a}$$

$$(d_2-\tfrac{2h}{|\ell|^2})^2+\lambda_1+\lambda_2 d_2 \quad = \quad 0 \tag{87b}$$

$$(d_3-\tfrac{2h}{|\ell|^2})^2+\lambda_1+\lambda_2 d_3 \quad = \quad 0. \tag{87c}$$

Subtracting (87a) from (87b) and (87b) from (87c) gives

$$\lambda_2=-(d_1+d_2)+\tfrac{4h}{|\ell|^2} \quad \text{and} \quad \lambda_2=-(d_2+d_3)+\tfrac{4h}{|\ell|^2},$$

since d_1-d_2 and d_2-d_3 are nonzero. Consequently, $d_1=d_3$, which is a contradiction. Therefore at least one of the components of p is zero.

Now suppose that two components of p are zero, say x_1^0 and x_2^0. The other cases are similar and their proof is omitted. Then the equations in (86b) give $h=\frac{1}{2}d_3|\ell|^2$. Hence (h,ℓ) is a critical value of $\mathscr{E}\mathscr{M}$. This is a contradiction. Therefore exactly one component of p is zero.

Suppose that $x_1^0=0$. Again the other cases are handled similarly and the details are left to the reader. Solving (86a) and (86b) we obtain

$$x_1^0=0 \quad x_2^0=\pm\sqrt{\frac{2h-d_3|\ell|^2}{d_2-d_3}}, \quad x_3^0=\pm\sqrt{\frac{d_2|\ell|^2-2h}{d_2-d_3}}, \tag{88}$$

when $h\in\mathscr{I}_1=(\frac{1}{2}d_3|\ell|^2,\frac{1}{2}d_2|\ell|^2)$. The argument when $h\in\mathscr{I}_2=(\frac{1}{2}d_2|\ell|^2,\frac{1}{2}d_1|\ell|^2)$ is left to the reader. Since x_2^0 and x_3^0 are nonzero, (86a) becomes (87b) and (87c). Solving them for the Lagrange multipliers λ_1 and λ_2 we obtain

$$\lambda_1=d_2d_3-\tfrac{4h^2}{|\ell|^4} \quad \text{and} \quad \lambda_2=-(d_2+d_3)+\tfrac{4h}{|\ell|^2}. \tag{89}$$

Therefore (88) is a solution to (86a) and (86b). The set $\mathscr{S}_{h,\ell}=\pi_{|\ell|}(\mathscr{E}\mathscr{M}^{-1}(h,\ell))$ has two connected components $\mathscr{S}^\varepsilon$ $(\varepsilon=\pm)$ one of which is the circle $S^1_{h,\ell}=\pi_{|\ell|}(T^2_{h,\ell})$. A glance at figure 4.3 shows that for fixed ε the *two* critical points $p^{\varepsilon}_{\pm}$

$$x^0_1 = 0 \quad x^0_2 = \pm\sqrt{\frac{2h - d_3|\ell|^2}{d_2 - d_3}}, \quad x^0_3 = \varepsilon\sqrt{\frac{d_2|\ell|^2 - 2h}{d_2 - d_3}},$$

lie on $\mathscr{S}^{\varepsilon}$. The value of R at $p^{\varepsilon}_{\pm}$ is $\frac{4}{|\ell|^2}(\frac{1}{2}d_2|\ell|^2 - h)(h - \frac{1}{2}d_3|\ell|^2)$. At $p^{\varepsilon}_{\pm}$ the Hessian of R is

$$\mathrm{diag}\big((d_1 - \tfrac{2h}{|\ell|^2})^2 + \lambda_1 + \lambda_2 d_1, (d_2 - \tfrac{2h}{|\ell|^2})^2 + \lambda_1 + \lambda_2 d_2, (d_3 - \tfrac{2h}{|\ell|^2})^2 + \lambda_1 + \lambda_2 d_3\big)\Big|_{T_{p^{\varepsilon}_{\pm}}\mathscr{S}^{\varepsilon}}.$$

Since $T_{p^{\varepsilon}_{\pm}}\mathscr{S}^{\varepsilon} = \ker\begin{pmatrix} x_1 & x_2 & x_3 \\ d_1x_1 & d_2x_2 & d_3x_3 \end{pmatrix}\Big|_{(0,x^0_2,x^0_3)}$ is spanned by the vector e_1, using (89) it follows that

$$\mathrm{Hess}_{p^{\varepsilon}_{\pm}} R = \big(d_1 - \tfrac{1}{2}(d_2 + d_2)\big)^2 - \big(\tfrac{1}{2}(d_2 - d_3)\big)^2.$$

But $0 < d_3 < d_2 < d_1$ which implies that $d_1 - \frac{1}{2}(d_2 + d_3) > d_2 - \frac{1}{2}(d_2 + d_3) = \frac{1}{2}(d_2 - d_3) > 0$. Hence $\mathrm{Hess}_{p^{\varepsilon}_{\pm}} R > 0$, that is, $p^{\varepsilon}_{\pm}$ is a nondegenerate minimum of R.

In table 7.1 we summarize information about the critical points of the function R. It follows that R is a Morse function. $\square$

critical points & conditions	critical value	index
1. $\big(0, \pm\sqrt{\frac{2h-d_3\lvert\ell\rvert^2}{d_2-d_3}}, \varepsilon\sqrt{\frac{d_2\lvert\ell\rvert^2-2h}{d_2-d_3}}\big)$, if $h \in \mathscr{I}_1$	$\frac{4}{\lvert\ell\rvert^2}(\frac{1}{2}d_2\lvert\ell\rvert^2 - h)(h - \frac{1}{2}d_3\lvert\ell\rvert^2)$	0
2. $\big(\varepsilon\sqrt{\frac{2h-d_3\lvert\ell\rvert^2}{d_1-d_3}}, 0, \pm\sqrt{\frac{d_1\lvert\ell\rvert^2-2h}{d_1-d_3}}\big)$, if $h \in \mathscr{I}_1$	$\frac{4}{\lvert\ell\rvert^2}(\frac{1}{2}d_1\ell^2 - h)(h - \frac{1}{2}d_3\lvert\ell\rvert^2)$	1
3. $\big(\pm\sqrt{\frac{2h-d_3\lvert\ell\rvert^2}{d_1-d_3}}, 0, \varepsilon\sqrt{\frac{d_1\lvert\ell\rvert^2-2h}{d_1-d_3}}\big)$, if $h \in \mathscr{I}_2$	$\frac{4}{\lvert\ell\rvert^2}(\frac{1}{2}d_1\lvert\ell\rvert^2 - h)(h - \frac{1}{2}d_3\lvert\ell\rvert^2)$	1
4. $\big(\varepsilon\sqrt{\frac{2h-d_2\lvert\ell\rvert^2}{d_1-d_2}}, \pm\sqrt{\frac{d_1\lvert\ell\rvert^2-2h}{d_1-d_2}}, 0\big)$, if $h \in \mathscr{I}_2$	$\frac{4}{\lvert\ell\rvert^2}(\frac{1}{2}d_1\lvert\ell\rvert^2 - h)(h - \frac{1}{2}d_2\lvert\ell\rvert^2)$	0

Table 7.2.1 Critical points of R and their Morse index.

Using ((7.7)) we deduce that $\mathrm{rad}_{\ell} : \mathscr{S}_{h,\ell} \subseteq S^2_{|\ell|} \to \mathbf{R}$ is a Morse function. Fixing ε and looking at table 7.1 we see that rad_{ℓ} has *two* nondegenerate minima and *two* nondegenerate maxima on $\mathscr{S}^{\varepsilon}$ with corresponding critical values $r_{\min}$ and $r_{\max}$. From the above discussion we see that the image of the 2-torus $T^2_{h,\ell}$ under the Poinsot mapping P (80) is a closed annulus $\mathscr{A}$ in the affine plane $\mathscr{A}_{\ell}$, which is bounded by two circles $C_{\min}$ and $C_{\max}$

with center at $\frac{2h}{|\ell|^2}e_3$ and radius $r_{\min}$ and $r_{\max}$. Moreover, both $P^{-1}(C_{\min})$ and $P^{-1}(C_{\max})$ consist of *two* orbits of the S^1-action Φ', because each orbit is the inverse image under the reduction map $\pi_{|\ell|}$ of a critical point of rad_ℓ on $S^1_{h,\ell}$. Since the Hessian of rad_ℓ is definite at these critical points, the Poinsot mapping P has a fold singularity along each orbit in $P^{-1}(C_{\min})$ and $P^{-1}(C_{\max})$. We leave it as an exercise to deduce from these geometric facts that $T^2_{h,\ell}$ cannot be embedded in $\mathbf{R}^3$.

Let $\mathscr{C}_1$ be an orbit of the S^1-action Φ' on $T^2_{h,\ell}$ whose image under the Poinsot mapping P is the oriented circle $C_{\max} \subseteq \mathscr{A}_\ell$. Then $\mathscr{C}_1$ is a cross section for the flow of $X_{H'}|T^2_{h,\ell}$. Hence for $p \in \mathscr{C}_1$ there is a smallest positive time $T = T(h,\ell)$ such that $q = \varphi^{H'}_T(p) \in \mathscr{C}_1$. Here $\varphi^{H'}_t$ is the flow of $X_{H'}$. Let $\frac{1}{2\pi}\Delta\theta(T)$ be the rotation number of the flow of $X_{H'}|T^2_{h,\ell e_3}$. Under the Poinsot mapping P the image of the integral curve $\widetilde{\gamma}: \mathbf{R} \to T^2_{h,\ell} : t \mapsto \varphi^{H'}_t(p)$ of the Hamiltonian vector field $X_{H'}|T^2_{h,\ell}$ is the curve $\Gamma : \mathbf{R} \to \mathscr{A} \subseteq \mathbf{R}^3 : t \mapsto (P\circ\widetilde{\gamma})(t)$, which is called the *herpolhode* corresponding to integral curve $\widetilde{\gamma}$. The angle $\Delta\vartheta(T)$, subtended by the oriented arc on $C_{\max}$ between $P(p)$ and $P(q)$, we call
▷ the *herpolhode angle* . Note that $P(q)$ is *not* the first point after $P(p)$ where the curve Γ meets $C_{\max}$, but is the *second*.

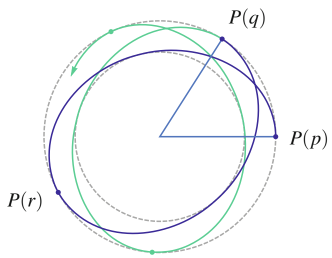

Figure 7.2.1. The herpolhode angle of a solution of the Euler-Arnol'd equations on a 2-torus in $T^2_{h,l}$.

(7.9) **Proof**: To see this, recall that $P^{-1}(C_{\max})$ is the disjoint union of *two* Φ' orbits $\mathscr{C}_1$ and $\mathscr{C}_2$ on $T^2_{h,\ell}$. Before crossing $\mathscr{C}_1$, the curve $\widetilde{\gamma}: t \mapsto \varphi^{H'}_t(p)$, which starts at $p \in \mathscr{C}_1$, must cross $\mathscr{C}_2$ transversely, say at the point r. This follows from the fact that every orbit of the S^1-action Φ' on $T^2_{h,\ell}$ is a cross section for the flow of $X_{H'}|T^2_{h,\ell}$. Thus $P(r) \in C_{\max}$ is the *first* point where $\widetilde{\gamma}$ crosses $\mathscr{C}_2$. A similar argument shows that starting at r the curve $\widetilde{\gamma}$ crosses $\mathscr{C}_1$ for the first time at the point q. Thus $P(q)$ is the second point on $C_{\max}$. □

To find the relationship between the rotation number and the herpolhode angle we look more closely at the geometry of the Poinsot mapping P (80). Recall that the orbits of the S^1-action Φ' give a ruling of the 2-torus $T^2_{h,\ell}$ by circles and that the image of a fiber of the bundle $\pi : (J')^{-1}(\ell) \subseteq T_1S^2 \times \mathbf{R}^3 \to S^2_{|\ell|} : (x,y,p) \mapsto p$ under the Poinsot mapping P is a geometric circle in the annulus $\mathscr{A} = \{(\Omega_1, \Omega_2, \frac{2h}{|\ell|}) \in \mathbf{R}^3 \,\big|\, 0 < r^2_{\min} \le \Omega_1^2 + \Omega_2^2 \le r^2_{\max}\}$. We now see how P maps a circle $\mathscr{S}$ in $T^2_{h,\ell}$, which is transverse to the fibers of the bundle π (78), into the annulus $\mathscr{A}$. More formally, for $p \in S^2_{|\ell|}$ let $\mathscr{C}$ be the circle $\pi^{-1}(p)$ and let $\mathscr{S}$ be a circle in $T^2_{h,\ell}$, which is chosen so that $\{\mathscr{C},\mathscr{S}\}$ is a basis for $\mathrm{H}_1(T^2_{h,\ell},\mathbf{Z})$. Let $\vartheta = \tan^{-1}\frac{\Omega_2}{\Omega_1}$ be the angular variable in the affine plane $\mathscr{A}_\ell$, which contains the annulus $\mathscr{A}$, that measures the amount of rotation about the centre of $\mathscr{A}$. We want to compute $\int_{\mathscr{S}} P^*\,\mathrm{d}\vartheta$.

First we give a precise description of $\mathscr{S}$. Let

$$\sigma : S^2_{|\ell|}\setminus\{\pm\ell,0,0)\} \subseteq \mathbf{R}^3 \to J^{-1}(\ell) \subseteq T_1S^2\times\mathbf{R}^3 :$$
$$p = \begin{pmatrix} p_1\\ p_2\\ p_3\end{pmatrix} \mapsto \big(x,y,p\big) = \left(\begin{pmatrix} \frac{1}{|\ell|}\sqrt{|\ell|^2-p_1^2} & \frac{-p_1p_2}{|\ell|\sqrt{|\ell|^2-p_1^2}}\\ 0 & \frac{p_3}{\sqrt{|\ell|^2-p_1^2}}\\ \frac{1}{|\ell|}p_1 & \frac{1}{|\ell|}p_2 \end{pmatrix}, p\right) \tag{90}$$

$\rhd$ Then σ is a section of the bundle $\pi|\pi^{-1}(S^2_{|\ell|}\setminus\{(\pm|\ell|,0,0)\})$. The curve $\mathscr{S}$ is parametrized by applying the section σ to the closed integral curve $\gamma:\mathbf{R}\to S^2_\ell : t\mapsto p(t)$ of Euler's equations of positive period T and energy $h\neq a|\ell|^2$, $b|\ell|^2$, $c|\ell|^2$. Here $a=I_1^{-1}$, $b=I_2^{-1}$ and $c=I_3^{-1}$. Now

$$P(\sigma(p)) = P(x,y,p) = \mathrm{col}(x,y,x\times y)(I')^{-1}p = \begin{pmatrix} \frac{|\ell|p_1}{\sqrt{|\ell|^2-p_1^2}}\big(a-\frac{2h}{|\ell|^2}\big)\\ \frac{p_2p_3}{\sqrt{|\ell|^2-p_1^2}}(b-c)\\ \frac{2h}{|\ell|}\end{pmatrix}. \tag{91}$$

Since $P(\sigma(p)) = \big((P\circ\sigma)^*(\Omega_1),(P\circ\sigma)^*(\Omega_2),\frac{2h}{|\ell|}\big)$, the point $P(\sigma(p))$ lies the annulus $\mathscr{A}$. Therefore

$$(P\circ\sigma)^*\,\mathrm{d}\vartheta = (P\circ\sigma)^*\,\mathrm{d}\Big(\tan^{-1}\frac{\Omega_2}{\Omega_1}\Big) = \mathrm{d}\Big(\tan^{-1}\frac{(P\circ\sigma)^*(\Omega_2)}{(P\circ\sigma)^*(\Omega_1)}\Big) = \mathrm{d}\Big(D\,\frac{p_2p_3}{p_1}\Big),$$

where $D = \frac{|\ell|(b-c)}{a|\ell|^2-2h}$. We now show

$$\int_{\mathscr{S}} P^*\,\mathrm{d}\vartheta = \begin{cases} 2\pi, & \text{if } c<\frac{2h}{|\ell|^2}<b\\ 0, & \text{if } b<\frac{2h}{|\ell|^2}<a.\end{cases} \tag{92}$$

(7.10) **Proof**: First we observe that

$$\int_{\mathscr{S}} P^*\,\mathrm{d}\vartheta = \int_{\gamma=\sigma^*\mathscr{S}} (P\circ\sigma)^*\,\mathrm{d}\vartheta = \int_\gamma \mathrm{d}\tan^{-1}\Big(D\,\frac{p_2p_3}{p_1}\Big).$$

The last integral above is the variation of the function $t\mapsto\Psi(t)=\tan^{-1}\big(D\,\frac{p_2(t)p_3(t)}{p_1(t)}\big)$ over the closed integral curve $\gamma(t)=\big(p_1(t),p_2(t),p_3(t)\big)$ of Euler's equations on $S^2_{|\ell|}$ of period T and energy h. There are two cases.

CASE 1. $c<\frac{2h}{|\ell|^2}<b$. From exercise 2 we get

$$p_1(t) = A\,\mathrm{cn}(nt;k),\;\; p_2(t)=B\,\mathrm{sn}(nt;k),\;\; \text{and}\;\; p_3(t)=C\,\mathrm{dn}(nt;k),$$

where $A^2 = \frac{2h-c|\ell|^2}{a-c}$, $B^2 = \frac{2h-c|\ell|^2}{b-c}$ and $C^2 = \frac{a|\ell|^2-2h}{a-c}$. Also

$$n = \sqrt{(a|\ell|^2-2h)(b-c)} \text{ and } k = \sqrt{\tfrac{(a-b)(2h-c|\ell|^2)}{(b-c)(a|\ell|^2-2h)}}.$$

Because the integral curves of Euler's equations on $S^2_{|\ell|}$ are homotopic when $h \in (\frac{1}{2}c|\ell|^2, \frac{1}{2}b|\ell|^2)$, the variation of the function Ψ over γ does not depend on h. Therefore we may let $h \searrow \frac{1}{2}c|\ell|^2$. This implies $k \to 0$. So $\mathrm{sn}(nt;k) \to \sin nt$, $\mathrm{cn}(nt;k) \to \cos nt$, and $\mathrm{dn}(nt;k) \to 1$. Moreover, $D\frac{BC}{A} \to \widetilde{D} = \sqrt{\frac{b-c}{a-c}}$ and $n \to \widetilde{n} = \sqrt{(a-c)(b-c)}$. Therefore $\Psi(t) \to \widetilde{\Psi}(t) = \tan^{-1}\big(\widetilde{D}\tan\widetilde{n}t\big)$. The variation of the function $\widetilde{\Psi}$ over $[-\frac{\pi}{2\widetilde{n}}, \frac{3\pi}{2\widetilde{n}}]$ is 2π because the function $\widetilde{D}\tan\widetilde{n}t$ is periodic of period $\frac{2\pi}{\widetilde{n}}$ and is strictly monotonic increasing from $-\infty$ to ∞ on $[-\frac{\pi}{2\widetilde{n}}, \frac{\pi}{2\widetilde{n}}]$ and $[\frac{\pi}{2\widetilde{n}}, \frac{3\pi}{2\widetilde{n}}]$.

CASE 2. $b|\ell|^2 < 2h < a|\ell|^2$. Again from exercise 2 we get

$$p_1(t) = A\,\mathrm{dn}(nt;k),\ \ p_2(t) = B\,\mathrm{sn}(nt;k),\ \text{ and } p_3(t) = C\,\mathrm{cn}(nt;k),$$

where $A^2 = \frac{2h-c|\ell|^2}{a-c}$, $B^2 = \frac{a|\ell|^2-2h}{a-b}$, and $C^2 = \frac{a|\ell|^2-2h}{a-c}$. Also

$$n = \sqrt{(2h-c|\ell|^2)(a-b)} \text{ and } k = \sqrt{\tfrac{(b-c)(a|\ell|^2-2h)}{(a-b)(2h-c|\ell|^2)}}.$$

Because the integral curves of Euler's equations on $S^2_{|\ell|}$ are homotopic when $h \in (\frac{1}{2}b|\ell|^2, \frac{1}{2}a|\ell|^2)$, the variation of the function Ψ over γ does not depend on h. Therefore we may let $h \nearrow \frac{1}{2}a|\ell|^2$. This implies $k \to 0$. So $\mathrm{sn}(nt;k) \to \sin nt$, $\mathrm{cn}(nt;k) \to \cos nt$, and $\mathrm{dn}(nt;k) \to 1$. Moreover, $D\frac{BC}{A} \to \widetilde{D} = \frac{b-c}{\sqrt{(a-b)(a-c)}}$ and $n \to \widetilde{n} = \sqrt{(a-c)(a-b)}$. Therefore $\Psi(t) \to \widetilde{\Psi}(t) = \tan^{-1}\big(\frac{1}{2}\widetilde{D}\sin 2\widetilde{n}t\big)$. The variation of the function $\widetilde{\Psi}$ over $[0, \frac{2\pi}{\widetilde{n}}]$ is 0 because the function $\frac{1}{2}\widetilde{D}\sin 2\widetilde{n}t$ is periodic of period $\frac{2\pi}{\widetilde{n}}$ and is continuous. □

We now prove the relation between the rotation angle of the $\Delta\theta(T)$ of the solution $\widetilde{\gamma}$ of the Euler-Arnol'd equations on $T^2_{h,\ell}$ of period T and the herpolhode angle $\Delta\vartheta(T)$ in the annulus $\mathscr{A}$ of the herpolhode $P\circ\widetilde{\gamma}$.

Claim: We have

$$\Delta\theta(T) = \Delta\vartheta(T) - \begin{cases} 0, & \text{if } b|\ell|^2 < 2h < a|\ell|^2 \\ 2\pi, & \text{if } c|\ell|^2 < 2h < b|\ell|^2. \end{cases} \tag{93}$$

(7.11) **Proof**: We begin by constructing an affine frame on $T^2_{h,\ell}$. Applying the section σ (90) to the periodic integral curve γ of the reduced Hamiltonian vector field $X_{H_{|\ell|}}$ of energy h and period T, gives a closed curve $[0,T] \to T^2_{h,\ell} : t \mapsto \sigma(\gamma(t))$ on $T^2_{h,\ell}$. This curve is transverse to every S^1 orbit of the action Φ', which rule $T^2_{h,\ell}$, and its tangent vector at $\sigma(\gamma(t))$ is $\mathscr{T}_t = T_{\gamma(t)}\sigma X_{H_{|\ell|}}(\gamma(t))$, which is nonzero. Set $\sigma_t = \sigma(\gamma(t))$ and let $Z(\Phi'_s(\sigma_t)) =$

$T_{\sigma_t}\Phi'_s\mathscr{T}_t$ be the push out of the vector $\mathscr{T}_t$ along the ruling $s \mapsto \Phi'_s(\sigma_t)$ of $T^2_{h,\ell}$ through σ_t.

▷ Z is an S^1-invariant vector field on $T^2_{h,\ell}$ because

$$\begin{aligned} T_{\Phi'_s(\sigma_t)}\Phi'_r Z(\Phi'_s(\sigma_t)) &= T_{\Phi'_s(\sigma_t)}\Phi'_r T_{\sigma_t}\Phi'_s\mathscr{T}_t = T_{\sigma_t}(\Phi'_r\circ\Phi'_s)\mathscr{T}_t \\ &= T_{\sigma_t}\Phi'_{r+s}\mathscr{T}_t = Z(\Phi'_{r+s}(\sigma_t)) = Z\big(\Phi'_r(\Phi'_s(\sigma_t))\big). \qquad \square \end{aligned}$$

Since

$$\begin{aligned} T_{\sigma_t}\pi\big(X_{H'}(\Phi'_s(\sigma_t)) - Z(\Phi'_s(\sigma_t))\big) &= X_{H_{|\ell|}}(\Phi'_s(\sigma_t)) - X_{H_{|\ell|}}(\Phi'_s(\sigma_t)) \\ &= X_{H_{|\ell|}}(\sigma_t) - X_{H_{|\ell|}}(\sigma_t) = 0, \end{aligned}$$

the vector field Z is the horizontal component of the vector field $X_{H'}$ on $T^2_{h,\ell}$ with respect to the bundle mapping π (78). Therefore $X_{H'} - Z$ is the vertical component of $X_{H'}|T^2_{h,\ell}$, which is equal to $\beta(\gamma(t))Y(\Phi'_s(\sigma_t))$ at $\Phi'_s(\sigma_t)$. Thus $X_{H'} = Z + \beta Y$ on $T^2_{h,\ell}$.

Now we can describe the relation between the rotation number and the herpolhode angle. Since the Poinsot map P is equivariant, we get $P_*Y = TPY = \frac{\partial}{\partial\vartheta}$. To see this, we compute

$$\begin{aligned} P(\Phi'_t(x,y,p)) &= P(\widetilde{R}_t x, \widetilde{R}_t y, p) = \mathrm{col}(\widetilde{R}_t x, \widetilde{R}_t y, \widetilde{R}_t x \times \widetilde{R}_t y)(I')^{-1}p \\ &= \widetilde{R}_t\,\mathrm{col}(x,y,x\times y)(I')^{-1}p = \widetilde{R}_t P(x,y,p). \qquad \square \end{aligned}$$

The evolution of the herpolhode angle ϑ along the integral curve $\gamma : \mathbf{R} \to S^2_\ell : t \mapsto \gamma(t)$ of $X_{H_{|\ell|}}$ is satisfies

$$\frac{\mathrm{d}\vartheta}{\mathrm{d}t} = \langle \mathrm{d}\vartheta, P_*X_{H'}\rangle = \langle \mathrm{d}\vartheta, P_*Z + \beta\,\frac{\partial}{\partial\vartheta}\rangle.$$

Hence after a period T of γ we get

$$\Delta\vartheta(T) = \vartheta(T) - \vartheta(0) = \int_0^T \langle \mathrm{d}\vartheta, P_*Z\rangle(\gamma(t))\,\mathrm{d}t + \int_0^T \beta(p(t))\,\mathrm{d}t.$$

But by construction of the vector field Z we have $P_*Z(\gamma(t)) = P_*\sigma_*(X_{H_{|\ell|}}(\gamma(t))) = P_*\sigma_*\frac{\mathrm{d}\gamma}{\mathrm{d}t}$. So

$$\int_0^T \langle \mathrm{d}\vartheta, P_*Z\rangle(\gamma(t))\,\mathrm{d}t = \int_0^T \langle(\sigma^*P^*\,\mathrm{d}\vartheta(\gamma(t)), \frac{\mathrm{d}\gamma}{\mathrm{d}t}\rangle\,\mathrm{d}t = \int_{\gamma=\sigma^*\mathscr{S}} \sigma^*P^*\,\mathrm{d}\vartheta = \int_{\mathscr{S}} P^*\,\mathrm{d}\vartheta.$$

Therefore

$$\Delta\vartheta(T) = \int_{\mathscr{S}} P^*\,\mathrm{d}\vartheta + \int_0^T \beta(p(t))\,\mathrm{d}t = \left\{\begin{array}{rl} 0, & \text{if } b|\ell|^2 < 2h < a|\ell|^2 \\ 2\pi, & \text{if } c|\ell|^2 < 2h < b|\ell|^2 \end{array}\right. + \Delta\theta(T). \qquad \square$$

8 A twisting phenomenon

In this section we describe and explain a twisting phenomenon which a tennis racket-like rigid body performs.

Consider a triaxial rigid body with principal axes $\{\widehat{e}_1, \widehat{e}_2, \widehat{e}_3\}$ fixed to the body at its center of mass. Suppose that the corresponding principal moments of inertia I_i satisfy $0 < I_1 < I_2 < I_3$. We say that the rigid body is *tennis racket-like* if it is nearly planar that is,

$$I_3 \approx I_1 + I_2, \tag{94}$$

I_3 is approximately equal to $I_1 + I_2$, and

$$I_1 \ll I_2, \tag{95}$$

I_1 is much less than I_2. Since an old fashioned wooden tennis racket fulfills all of these conditions nicely, we will talk of a tennis racket.

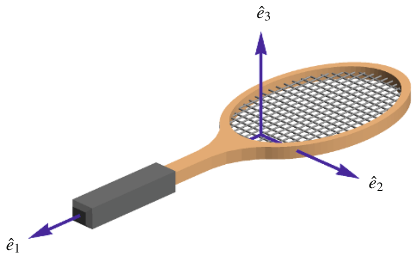

Figure 8.1. The principal axes of a tennis racket.

The following experiment demonstrates the *twisting phenomenon* quite dramatically. Take a tennis racket and mark its faces so that they can be distinguished. Call one rough and the other smooth. Hold the racket horizontally so that the *smooth* face is up. Toss the racket attempting to make it rotate about the intermediate principal axis $\widehat{e}_2$. After one rotation catch the racket by its handle. The *rough* face will almost always be up! Thus the racket has made a near half twist around its handle.

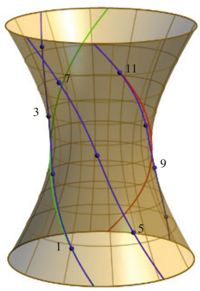

Figure 8.2. A special heteroclinic orbit, connecting γ_+ with γ_-, in the solid ball model of phase space. The numbers correspond to those in figure 8.3. The moments of inertia are: $I_1 = 18$, $I_2 = 16$ and $I_3 = 1$.

The problem, of course, is to explain this twisting phenomenon. In what follows we give a qualitative explanation. Since we are interested in the rotational motion of the racket, we can forget about the motion of its center of mass. In other words we suppose that the center of mass of the racket is fixed. The racket rotating around its intermediate principal axis $\widehat{e}_2$ corresponds to one of the hyperbolic period orbits $\gamma_2^{\pm}$ of the Euler-Arnol'd equations of the Euler top with energy $h = \frac{1}{2}|\ell|^2 I_2^{-1}$ and angular momentum $\ell = |\ell| e_3$, see figure 8.2. In §5 we showed that γ_2^+ with γ_2^- are heteroclinic, that is, the closure of the unstable manifold of γ_2^+ contains γ_2^-. Therefore every motion of the tennis racket which starts “near" γ_2^+ eventually comes “near" γ_2^-. Warning: the racket spinning exactly about its $\widehat{e}_2$-axis does *not* twist at all. Thus we should quantify what we mean by “near". For an estimate of the size of the region about γ_2^+ where the racket does *not* twist, see exercise

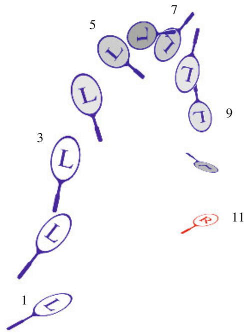

Figure 8.3. A special heteroclinic orbit in configuration space at evenly spaced time intervals.

12. Since γ_1^+ and γ_2^+ are heteroclinic for *every* triaxial rigid body, this qualitative feature of the motion cannot be the whole story why the racket twists. What it does *not* explain is why the racket makes a *twist* when going from γ_2^+ to γ_2^-. To say what we mean by a

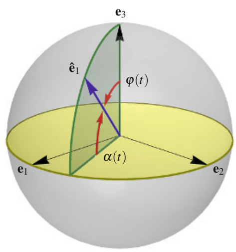

Figure 8.4. The angles $\alpha(t)$ and $\varphi(t)$.

twist, we must have some reference plane *fixed* in space. The following discussion shows that the handle of the racket moves nearly in the plane. Suppose that the racket has energy h and angular momentum $\ell = |\ell| e_3$. Moreover, assume that the principal axis frame $\{\widehat{e}_1, \widehat{e}_2, \widehat{e}_3\}$ and the frame $\{e_1, e_2, e_3\}$ fixed in space are the same at time $t = 0$.

Claim. Let the position of the racket be as given in figure 8.1 and let $\alpha(t)$ be the angle the handle $\widehat{e}_1$ makes with the e_1–e_2 plane at time t. Then

$$0 \le \tan\alpha(t) \le \sqrt{\big(I_1(2I_3h - |\ell|^2)\big)/\big(I_3(|\ell|^2 - 2I_1h)\big)}. \tag{96}$$

(8.1) **Proof**: Since the angular momentum of the rigid body is $\ell = |\ell|e_3$, its component along the $\widehat{e}_1$-axis at time t is $p_1(t)$, where $t \to p(t) = \big(p_1(t), p_2(t), p_3(t)\big)\big)$ is a solution of Euler's equations (55) on $S^2_{|\ell|}$ of energy h. Thus

$$|\ell|^{-1}p_1(t) = |\cos\varphi(t)| = \sin\alpha(t). \tag{97}$$

Using the explicit solution of Euler's equations in terms of Jacobi elliptic functions, see exercise 2, we find that when $h \in [\frac{1}{2}|\ell|^2 I_3^{-1}, \frac{1}{2}|\ell|^2 I_2^{-1}]$,

$$p_1(t) = \sqrt{\big(2h - |\ell|^2 I_3^{-1}\big)/\big(I_1^{-1} - I_3^{-1}\big)}\,\mathrm{cn}(nt;k).$$

Here

$$n = \sqrt{(I_1^{-1} - I_2^{-1})(2h - |\ell|^2 I_3^{-1})} \quad \text{and} \quad k = \sqrt{\frac{(I_2^{-1} - I_3^{-1})(|\ell|^2 I_1^{-1} - 2h)}{(I_1^{-1} - I_3^{-1})(2h - |\ell|^2 I_3^{-1})}}.$$

Since $|\mathrm{cn}(nt;k)| \le 1$, see exercise 1, we obtain

$$0 \le \sin\alpha(t) \le \frac{1}{|\ell|}\sqrt{\big(2h - |\ell|^2 I_3^{-1}\big)/\big(I_1^{-1} - I_3^{-1}\big)}, \tag{98}$$

which yields (96). Similarly when $h \in [\frac{1}{2}|\ell|^2 I_2^{-1}, \frac{1}{2}|\ell|^2 I_1^{-1}]$ we find that

$$p_1(t) = \sqrt{\big(2h - |\ell|^2 I_3^{-1}\big)/\big(I_1^{-1} - I_3^{-1}\big)}\,\mathrm{dn}(\widetilde{n}t;\widetilde{k}).$$

Here

$$\widetilde{n} = \sqrt{(I_2^{-1} - I_3^{-1})(|\ell|^2 I_1^{-1} - 2h)} \quad \text{and} \quad \widetilde{k} = \sqrt{\frac{(I_1^{-1} - I_2^{-1})(2h - |\ell|^2 I_1^{-1})}{(I_2^{-1} - I_3^{-1})(|\ell|^2 I_1^{-1} - 2h)}}.$$

Since $|\mathrm{dn}(\widetilde{n}t;\widetilde{k})| \le 1$, we again obtain (98). □

Corollary: For a tennis racket rotating almost about its intermediate axis the angle $\alpha(t)$ is small for all t.

(8.2) **Proof**: When the racket rotates nearly around its intermediate axis, $h \approx \frac{1}{2}|\ell|^2 I_2^{-1}$. Using (94) the right hand side of (96) is $\approx \frac{I_1}{I_2}$. This is small because of (95). □

Thus the plane swept out by the handle of the racket is fixed in space. With respect to this

▷ plane we can see if the racket twists. To show that the racket does indeed twist, consider its motion along the unstable manifold of γ_2^+ as it goes from a neighborhood of γ_2^+ to a neighborhood of γ_2^-.

(8.3) **Proof**: Let the position of the racket be as given in figure 8.1. At time t the $\widehat{e}_2$-component $p_2(t)$ of the angular momentum $\ell = |\ell| e_3$ of the racket is

$$p_2(t) = |\ell| \cos\varphi(t). \tag{99}$$

Using exercise 2, the solution $t \mapsto p(t) = (p_1(t), p_2(t), p_3(t))$ of Euler's equation of energy $h = \frac{1}{2}|\ell|^2 I_2^{-1}$ which goes from the hyperbolic equilibrium point $|\ell|e_2$ along its unstable manifold to the hyperbolic equilibrium point $-|\ell|e_2$ is given by

$$p(t) = |\mu| \Big(\sqrt{\tfrac{I_2^{-1}-I_3^{-1}}{I_1^{-1}-I_3^{-1}}}\, \mathrm{sech}\, nt, -\tanh nt, \sqrt{\tfrac{I_1^{-1}-I_2^{-1}}{I_1^{-1}-I_3^{-1}}}\, \mathrm{sech}\, nt \Big). \tag{100}$$

Here $n = |\ell| \sqrt{(I_1^{-1}-I_2^{-1})(I_2^{-1}-I_3^{-1})}$. Therefore $\cos\varphi(t) = -\tanh nt$. From (100) we see that at $t = -\infty$, $p(-\infty) = (0, |\mu|, 0)$, while at $t = \infty$, $p(\infty) = (0, -|\ell|, 0)$. Therefore $\varphi(-\infty) = 0$, while $\varphi(\infty) = \pi$. Refering to figure 8.3, this says that the $\widehat{e}_2$-axis of the racket starts with e_3 *vertically above* the invariant e_1–e_2 plane and finishes with $-e_3$ *vertically below*. Thus the racket has made a near *half* twist in going from near γ_2^+ to near γ_2^- along the unstable manifold. □

To explain the experiment described at the beginning of this section in finer detail we should show that

1. The racket has enough time perform a twist, that is, the time it takes the handle to make one revolution is longer than the time it takes to perform a half twist.
2. The racket is likely to be caught after making a half twist.
3. The experiment is repeatable because the handle revolves nearly uniformly.

We refer the reader to the exercise 12 for a treatment of these points.

9 Exercises

1. (Jacobi elliptic functions.) Consider the system of differential equations

$$\begin{cases} \dot{x} &= yz \\ \dot{y} &= -xz \\ \dot{z} &= -k^2 xy, \end{cases} \tag{101}$$

where $0 < k^2 < 1$. Define the *Jacobi elliptic functions* as the integral curve

$$t \mapsto \big(x(t), y(t), z(t)\big) = \big(\mathrm{sn}(t;k), \mathrm{cn}(t;k), \mathrm{dn}(t;k)\big)$$

of (101) which passes through $(0,1,1)$ at $t = 0$.

a) Show that the functions $x^2 + y^2$ and $k^2x^2 + z^2$ are integrals of (101) and deduce that

$$\mathrm{sn}^2(t;k) + \mathrm{cn}^2(t;k) = 1 \text{ and } k^2\mathrm{sn}^2(t;k) + \mathrm{dn}^2(t;k) = 1.$$

Consequently, for all t

$$|\mathrm{sn}(t;k)| \le 1,\ |\mathrm{cn}(t;k)| \le 1 \text{ and } k' = \sqrt{1-k^2} \le \mathrm{dn}(t;k) \le 1.$$

b) Let $x(t) = \mathrm{sn}(t;k)$. From (101) deduce that $\frac{\mathrm{d}t}{\mathrm{d}x} = \frac{1}{\sqrt{(1-x^2)(1-k^2x^2)}} > 0$ for $x \in (-1,1)$. Hence $t(x) = \int_0^x \frac{1}{\sqrt{(1-x^2)(1-k^2x^2)}}\,\mathrm{d}x$ is a smooth inverse of $\mathrm{sn}(t;k)$ on $(-1,1)$. Since $t(\pm 1) = \pm K(k) = \pm K$ is finite, $t(x)$ is continuous on $[-1,1]$. Therefore, $\mathrm{sn}(K;k) = 1$. Hence $\mathrm{cn}(K;k) = 0$ and $\mathrm{dn}(K;k) = k'$.

c) Show that $\mathrm{sn}(t;0) = \sin t$, $\mathrm{cn}(t;0) = \cos t$, $\mathrm{dn}(t;0) = 1$ and $\mathrm{sn}(t;1) = \tanh t$, $\mathrm{cn}(t;1) = \mathrm{sech}\, t$, $\mathrm{dn}(t;1) = \mathrm{sech}\, t$.

d) Define

$$\xi(t) = \frac{\mathrm{cn}(t;k)}{\mathrm{dn}(t;k)},\ \eta(t) = -k'\frac{\mathrm{sn}(t;k)}{\mathrm{dn}(t;k)}, \text{ and } \zeta(t) = k'\frac{1}{\mathrm{dn}(t;k)}.$$

Show that $t \to (\xi(t), \eta(t), \zeta(t))$ is an integral curve of (101) passing through $(1,0,k')$ at $t=0$. Since $t \to \big(\mathrm{sn}(t+K;k), \mathrm{cn}(t+K;k), \mathrm{dn}(t+K;k)\big)$ is also an integral curve of (101) passing through $(1,0,k')$ at $t=0$, deduce that

$$\mathrm{sn}(t+K;k) = \frac{\mathrm{cn}(t;k)}{\mathrm{dn}(t;k)},\ \mathrm{cn}(t+K;k) = -k'\frac{\mathrm{sn}(t;k)}{\mathrm{dn}(t;k)}, \text{ and } \mathrm{dn}(t+K;k) = k'\frac{1}{\mathrm{dn}(t;k)}.$$

Conclude that $\mathrm{sn}(t;k)$ and $\mathrm{cn}(t;k)$ are periodic of period $4K$ while $\mathrm{dn}(t;k)$ is periodic of period $2K$.

e) Using the substitution $x = iy/\sqrt{1-y^2}$ the integral $u = \int_0^x \frac{1}{\sqrt{(1-x^2)(1-k^2x^2)}}\,\mathrm{d}x$ becomes $u = i\int_0^{iy/\sqrt{1-y^2}} \frac{1}{\sqrt{(1-y^2)(1-(k')^2y^2)}}\,\mathrm{d}y$. Let $w = iu$. Then $y = \mathrm{sn}(w;k')$. Therefore

$$\mathrm{sn}(u;k) = i\,\frac{\mathrm{sn}(w;k')}{\mathrm{cn}(w;k')},\ \mathrm{cn}(u;k) = \frac{1}{\mathrm{cn}(w;k')}, \text{ and } \mathrm{dn}(u;k) = \frac{\mathrm{dn}(w;k')}{\mathrm{cn}(w;k')}.$$

Show that $\mathrm{sn}(w;k'), \mathrm{cn}(w;k')$ and $\mathrm{dn}(w;k')$ are periodic of period $4K'$, $4K'$ and $2K'$ respectively where $K' = \int_0^1 \frac{1}{\sqrt{(1-y^2)(1-(k')^2y^2)}}\,\mathrm{d}y$. Thus sn, cn and dn have a second purely imaginary period of $4iK', 4iK'$ and $2iK'$ respectively.

2. (Euler's equations on S^2_ℓ.) Let $\mathbf{x} \in \mathbf{R}^3$ and let $\langle\, ,\, \rangle$ be the Euclidean inner product on $\mathbf{R}^3$. On the 2-sphere $S^2_\ell \subseteq \mathbf{R}^3$ given by $\langle \mathbf{x}, \mathbf{x}\rangle = \ell^2$ show that every vector in $T_{(x,y,z)}S^2_\ell = \{\xi \in \mathbf{R}^3 \,|\, \langle \mathbf{x}, \xi\rangle = 0\}$ can be written as $\mathbf{x} \times \mathbf{p}$ for some $\mathbf{p} \in \mathbf{R}^3$.

   a) Define a 2-form ω_ℓ on S^2_ℓ by $\omega_\ell(\mathbf{x})\big(\mathbf{x}\times\mathbf{p}, \mathbf{x}\times\mathbf{s}\big) = \langle \mathbf{x}, \mathbf{p}\times\mathbf{s}\rangle$, where $\mathbf{p}, \mathbf{s} \in \mathbf{R}^3$. Show that ω_ℓ is the element of surface area of S^2_ℓ and that it is a symplectic form.

   b) On the symplectic manifold (S^2_ℓ, ω_ℓ) consider the Hamiltonian function

$$H : S^2_\ell \subseteq \mathbf{R}^3 \to \mathbf{R} : (x,y,z) \to \tfrac{1}{2}(ax^2 + by^2 + cz^2),$$

where $a > b > c > 0$. Show that the Hamiltonian vector field X_H has integral curves which satisfy Euler's equations

$$\begin{aligned} \dot{x} &= -(b-c)yz \\ \dot{y} &= (a-c)xz \\ \dot{z} &= -(a-b)xy. \end{aligned} \tag{102}$$

Note that (102) defines a vector field on $\mathbf{R}^3$ which has S^2_ℓ as an invariant manifold.

c) Integrate Euler's equations using Jacobi elliptic functions, see exercise 1. In particular show that

i) When $\ell^2 b \ge 2h \ge \ell^2 c$, $\big(x(t), y(t), z(t)\big) = \big(A\,\mathrm{cn}\,(nt;k),\ B\,\mathrm{sn}\,(nt;k),\ C\,\mathrm{dn}\,(nt;k)\big)$, where

$$\begin{aligned} A^2 &= \frac{2h - c\ell^2}{a-c}, \quad B^2 = \frac{2h - c\ell^2}{b-c}, \quad C^2 = \frac{a\ell^2 - 2h}{a-c} \\ n &= \sqrt{(a\ell^2 - 2h)(b-c)} \quad \text{and} \quad k = \sqrt{\frac{(a-b)(2h-c\ell^2)}{(b-c)(a\ell^2-2h)}}. \end{aligned}$$

ii) When $\ell^2 a \ge 2h \ge \ell^2 b$, $\big(x(t), y(t), z(t)\big) = \big(A\,\mathrm{dn}\,(nt;k),\ B\,\mathrm{sn}\,(nt;k),\ C\,\mathrm{cn}\,(nt;k)\big)$, where

$$\begin{aligned} A^2 &= \frac{2h - c\ell^2}{a-c}, \quad B^2 = \frac{a\ell^2 - 2h}{a-b}, \quad C^2 = \frac{a\ell^2 - 2h}{a-c} \\ n &= \sqrt{(a-b)(2h - c\ell^2)} \quad \text{and} \quad k = \sqrt{\frac{(b-c)(a\ell^2-2h)}{(a-b)(2h-c\ell^2)}}. \end{aligned}$$

The signs of A, B and C above are chosen so that $x(t), y(t)$ and $z(t)$ lie in one of the connected components of

$$\begin{cases} x^2 + y^2 + z^2 &= \ell^2 \\ ax^2 + by^2 + cz^2 &= 2h, \end{cases}$$

when $2h \ne \ell^2 b$. When $2h = \ell^2 b$ all choices of sign are possible.

3. Let $\times$ be the vector product and $\langle\,,\,\rangle$ the Euclidean inner product on $\mathbf{R}^3$. For $x, y, z \in \mathbf{R}^3$ show that

a) $x \times (y \times z) = \langle x, z\rangle\, y - \langle x, y\rangle\, z$.

b) $x \times (y \times z) = (x \times y) \times z + y \times (x \times z)$.

c) $\langle x, y \times z\rangle = \det(\mathrm{col}(x, y, z)) = \langle x \times y, z\rangle$.

d) $\langle x \times y, x \times y\rangle + \langle x, y\rangle^2 = \langle x, x\rangle \langle y, y\rangle$.

e) Let $A : \mathbf{R}^3 \to \mathbf{R}^3$ be an invertible linear mapping. Show that $A(x \times y) = \frac{1}{\det A}(Ax \times Ay)$, for any $x, y \in \mathbf{R}^3$.

From b) deduce that $(\mathbf{R}^3, \times)$ is a Lie algebra, which is isomorphic to $(\mathrm{so}(3), [\,,\,])$.

4. (Coriolis' theorem.)

a) We establish some terminology which allows us to state Coriolis' theorem precisely. Let $(V, (\ ,\ ))$ be 3-dimensional Euclidean space with its standard inner product. A *frame of reference* $\mathscr{F}$ is a positively oriented orthonormal basis $\{f_1, f_2, f_3\}$ of V. We say that the vector $v \in V$ *looks like* the vector $x \in \mathbf{R}^3$ *in the frame* $\mathscr{F}$ if and only if $v = \sum_{i=1}^{3} x_i f_i$. Corresponding to the frame $\mathscr{F}$ is its *coframe* $\mathscr{F}^* = \{f_1^*, f_2^*, f_3^*\}$, where $f_i^*(f_j) = \delta_{ij}$. Suppose that $\mathscr{A} = \{a_1, a_2, a_3\}$ is a reference frame such that the vector v looks like the vector $X \in \mathbf{R}^3$, that is, $v = \sum_{i=1}^{3} X_i a_i$. Let $A = (A_{ij})$ be the 3×3 matrix whose ij^{th} entry $A_{ij} = f_i^*(a_j)$, that is, a_j looks like the j^{th} column of A in the frame $\mathscr{F}$. Show that $x = AX$.

b) Let $A : \mathbf{R} \to \mathrm{SO}(3) : t \mapsto A(t) = \mathrm{col}(a_1(t), a_2(t), a_3(t))$. Then $\mathscr{A} = \{a_1(t), a_2(t), a_3(t)\}$ is a reference frame for V whose j^{th} member $a_j(t)$ looks like the j^{th} column of the matrix $A(t)$ with respect to the fixed frame $\mathscr{F}$. We say that $\mathscr{A}$ is a reference frame which *rotates* with respect to the fixed frame $\mathscr{F}$. Let $x : \mathbf{R} \to \mathbf{R}^3 : t \mapsto x(t)$ be a differentiable function. Suppose that $\Xi : \mathbf{R} \to V : t \mapsto \Xi(t)$ is a motion in V so that its position $\Xi(t)$ at time t in the fixed frame $\mathscr{F}$ looks like $x(t)$; while is position in the rotating frame $\mathscr{A}$ looks like $X(t)$. Show that $x(t) = A(t)X(t)$.

c) Differentiating the preceding equation gives

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \dot{A}(t)X + A(t)\frac{\mathrm{d}X}{\mathrm{d}t} = \dot{A}(t)A^{-1}x + A(t)\frac{\mathrm{d}X}{\mathrm{d}t}. \tag{103}$$

The velocity of $t \mapsto \Xi(t)$ at time t with respect to the fixed frame $\mathscr{F}$ is a vector in V which looks like $\frac{\mathrm{d}x}{\mathrm{d}t}$; while with respect to the rotating frame it is a vector in V which looks like $\frac{\mathrm{d}X}{\mathrm{d}t}$. The skew symmetric matrix $A'(t)A^{-1}(t) \in \mathrm{so}(3)$ is an *infinitesimal motion in the fixed frame* at time t. The corresponding vector $\omega = i(\dot{A}(t)A^{-1}(t))$ in $\mathbf{R}^3$ is the *angular velocity at time t with respect to the fixed frame*. Show that

$$\frac{\mathrm{d}x}{\mathrm{d}t} - \omega(t) \times x(t) = A(t)\frac{\mathrm{d}X}{\mathrm{d}t}. \tag{104}$$

This is Coriolis' theorem in the fixed frame.

d) Write $\dot{A}(t)A^{-1}(t) = A(t)\big(A^{-1}\dot{A}(t)\big)A^{-1}(t) = \mathrm{Ad}_{A(t)}\big(A^{-1}(t)\dot{A}(t)\big)$. Deduce that $\omega(t) = A(t)\Omega(t)$, where $\Omega(t) = i(A^{-1}(t)\dot{A}(t))$ is the *angular velocity at time t with respect to the rotating frame*. Show that (103) can be written as

$$\frac{\mathrm{d}x}{\mathrm{d}t} = A(t)\left[\Omega(t) \times X + \frac{\mathrm{d}X}{\mathrm{d}t}\right]. \tag{105}$$

This is Coriolis' theorem in the rotating frame.

5. (Moment of inertia tensor.)

a) A body is a set of points $\mathfrak{B} \subseteq \mathbf{R}^3$ with a mass distribution given by a positive measure $\mathrm{d}m$, whose support is $\mathfrak{B}$ and is not contained in any line through the origin. Assume that the center of mass of the body is at the origin in $\mathbf{R}^3$, that is, the first moments $\int x_i \,\mathrm{d}m$ of $\mathrm{d}m$ are zero for $i = 1, 2, 3$. A body is *rigid* if the Euclidean distance between any two of its points does not change when the body is moving. Let $\xi(t,X)$ be the position of a point X in the body at time t whose position at time 0 is X. Show that for a rigid body there is a unique $A(t) \in \mathrm{O}(3)$ such that $\xi(t,X) = A(t)X$. Suppose that the curve $\mathbf{R} \to \mathrm{O}(3) : t \to A(t)$ is smooth and that the initial position $A(0)$ of the body is id. Then $A(t) \in \mathrm{SO}(3)$.

b) Let $\langle \, , \, \rangle$ be the Euclidean inner product on $\mathbf{R}^3$. Let $\mathscr{A} = \{a_1(t), a_2(t), a_3(t)\}$, where $A(t) = \mathrm{col}\big(a_1(t), a_2(t), a_3(t)\big) \in \mathrm{SO}(3)$, be a frame, which is corotating with the body $\mathfrak{B}$. The kinetic energy (of rotation) of a rigid body $\mathfrak{B}$ in the frame $\mathscr{A}$ is

$$K = \tfrac{1}{2}\int_{\mathfrak{B}} \langle \dot{\xi}(t,X), \dot{\xi}(t,X)\rangle \, \mathrm{d}m = \tfrac{1}{2}\int_{\mathfrak{B}} \langle \dot{A}X, \dot{A}X\rangle \, \mathrm{d}m = \tfrac{1}{2}\int_{\mathfrak{B}} \langle A^{-1}\dot{A}X, A^{-1}\dot{A}X\rangle \, \mathrm{d}m.$$

Now $\Omega(t) = i\big(A^{-1}\dot{A}(t)\big) \in \mathbf{R}^3$ is the angular velocity of $\mathfrak{B}$ at time t with respect to the frame $\mathscr{A}$. Then $(A^{-1}\dot{A})X = i(A^{-1}\dot{A}) \times X = \Omega \times X$. So

$$K = \tfrac{1}{2}\int_{\mathfrak{B}} \langle \Omega \times X, \Omega \times X\rangle \, \mathrm{d}m = \tfrac{1}{2}\int_{\mathfrak{B}} \langle \Omega, \Omega\rangle\langle X, X\rangle - \langle \Omega, X\rangle^2 \, \mathrm{d}m.$$

Let $M = (M_{ij}) = (\int X_i X_j \, \mathrm{d}m)$ be the matrix of second moments of the mass distribution $\mathrm{d}m$ of $\mathfrak{B}$. Then M is a symmetric 3×3 matrix with $M_{ii} \ge 0$ for $i = 1,2,3$. Show that $M = \int_{\mathfrak{B}} X \otimes X^* \, \mathrm{d}m$, where $X \otimes X^* \in \mathbf{R}^3 \otimes (\mathbf{R}^3)^* = \mathrm{gl}(3,\mathbf{R})$ is defined by $(X \otimes X^*)(Y) = X^*(Y)X = \langle X, Y\rangle X$ for every $Y \in \mathbf{R}^3$. Show that

$$K = \tfrac{1}{2}\big(\langle \Omega, \Omega\rangle \, \mathrm{tr}\, M - \langle M\Omega, \Omega\rangle\big) = \tfrac{1}{2}\langle I\Omega, \Omega\rangle,$$

where $I : \mathbf{R}^3 \to \mathbf{R}^3 : \omega \mapsto (\mathrm{tr}\, M)\omega - M\omega$ is the *moment of inertia tensor* of $\mathfrak{B}$.

c) Let $\xi(t,X) = OAX$ be the position of the body at time t, where $O \in \mathrm{SO}(3)$. Show that $OX \otimes (OX)^* = O(X \otimes X^*)O^{-1}$ and deduce that

$$K = \tfrac{1}{2}\big(\langle \Omega, \Omega\rangle \, \mathrm{tr}(OMO^{-1}) - \langle (OMO^{-1})\Omega, \Omega\rangle\big).$$

Choose $O \in \mathrm{SO}(3)$ so that $OMO^t = \mathrm{diag}(M_1, M_2, M_3)$. Such a rotation O is called a *principal axis transformation*. Show that $K = \frac{1}{2}\langle I\Omega, \Omega\rangle$, where $I = \mathrm{diag}(I_1, I_2, I_2)$ with $I_1 = M_2 + M_3$, $I_2 = M_1 + M_3$ and $I_3 = M_1 + M_2$, which are called the *principal moments of inertia* of $\mathfrak{B}$. Show that

$$0 \le I_1 \le I_2 + I_3, \; 0 \le I_2 \le I_1 + I_3, \text{ and } 0 \le I_3 \le I_1 + I_2. \tag{106}$$

d) Let $L = \int_{\mathfrak{B}} X \times (\Omega \times X) \, \mathrm{d}m$. Show that $L = I\Omega$. So L is the *angular momentum of $\mathfrak{B}$ in the corotating frame $\mathscr{A}$*. Using the notation L instead of p, show that we can write Euler's equations $\dot{p} = p \times I^{-1}p$ as $\dot{L} = L \times \Omega$. Let $\ell = \int_{\mathfrak{B}} x \times (\omega \times x) \, \mathrm{d}m$, where $x = AX$ and $\omega = A\Omega$. Show that $\ell = AL$. Hence ℓ is the *angular momentum of $\mathfrak{B}$ in the fixed frame $\mathscr{F} = \{e_1, e_2, e_3\}$*. Using Coriolis' theorem and Euler's equations show that $\dot{\ell} = 0$, that is, ℓ is constant during the motion of $\mathfrak{B}$. Conversely, if we know that $\dot{\ell} = 0$, then using Coriolis' theorem deduce Euler's equations.

6. Show that every $q \in \mathbf{H}$ can be written uniquely as $\alpha + \beta \cdot j$ for some $\alpha, \beta \in \mathbf{C} = \mathbf{R} + \mathbf{R} \cdot i \subseteq \mathbf{H}$. For every $\alpha \in \mathbf{C}$ show that $j \cdot \alpha \cdot j = -\overline{\alpha}$. Verify that the map $\theta : S^3 \subseteq \mathbf{H} \to \mathrm{SU}(2) : \alpha + \beta \cdot j \to \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix}$ is an isomorphism of Lie groups.

7. (Equation of the herpolhode.) Let $\widetilde{\gamma}: \mathbf{R} \rightarrow \mathrm{SO}(3) \times \mathbf{R}^3 : t \mapsto \big(\widetilde{A}(t), p(t)\big)$ be a solution of the Euler-Arnol'd equations of motion of the Euler top of energy h and angular momentum $\ell = |\ell| e_3$. Let P be the Poinsot mapping (80). Then $\Gamma : \mathbf{R} \rightarrow \mathbf{R}^3 : t \mapsto P(\widetilde{\gamma}(t)) = \widetilde{A}(t)(I')^{-1}p(t)$ is the *herpolhode* corresponding to $\widetilde{\gamma}$.

   a) Let $\Omega(t) = (I')^{-1}p(t)$ and set $\omega(t) = \widetilde{A}(t)\Omega(t)$. Then $\omega(t)$ is the angular velocity of the top in the space frame at time t. Show that $\Gamma(t) = \omega(t)$.

   b) Let $M_i = |\ell|^{-1}p_i$. Using the solution of the Euler-Arnol'd equations (61a) and (61b), show that $\widetilde{A}(t) = \mathrm{col}\big(x(t), y(t), (x \times y)(t)\big)$, where

$$\begin{aligned} x(t) &= \big(\sqrt{1-M_1^2}\cos\theta, \sqrt{1-M_1^2}\sin\theta, M_1\big) \\ y(t) &= \big(\tfrac{-1}{\sqrt{1-M_1^2}}\big[M_1M_2\cos\theta + M_3\sin\theta\big], \tfrac{1}{\sqrt{1-M_1^2}}\big[M_3\cos\theta - M_1M_2\sin\theta\big], M_2\big) \\ (x\times y)(t) &= \big(\tfrac{-1}{\sqrt{1-M_1^2}}\big[M_1M_3\cos\theta - M_2\sin\theta\big], \tfrac{-1}{\sqrt{1-M_1^2}}\big[M_2\cos\theta + M_1M_3\sin\theta\big], M_3\big). \end{aligned}$$

   c) Using $\omega(t) = \widetilde{A}(t)\Omega(t)$ and the fact that $\sum_{i=1}^3 M_i\Omega_i = 2|\ell|^{-1}h$, show that

$$\begin{aligned} \begin{pmatrix} \omega_1(t) \\ \omega_2(t) \\ \omega_3(t) \end{pmatrix} &= \begin{pmatrix} \Omega_1 - 2|\ell|^{-1}hM_1 & -(\Omega_2M_3 - \Omega_3M_2) & 0 \\ \Omega_2M_3 - \Omega_3M_2 & \Omega_1 - 2|\ell|^{-1}hM_1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{\cos\theta}{\sqrt{1-M_1^2}} \\ \frac{\sin\theta}{\sqrt{1-M_1^2}} \\ 2|\ell|^{-1}h \end{pmatrix} \\ &= \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} R\cos\theta \\ R\sin\theta \\ 2|\ell|^{-1}h \end{pmatrix} = \begin{pmatrix} R(t)\cos\big(\theta(t)+\varphi(t)\big) \\ R(t)\sin\big(\theta(t)+\varphi(t)\big) \\ 2|\ell|^{1}h \end{pmatrix}, \end{aligned}$$

   where

$$R(t)^2 = \frac{(\Omega_1 - 2|\ell|^{-1}hM_1)^2 + (\Omega_2M_3 - \Omega_3M_2)^2}{1-M_1^2} = \frac{1}{|\ell|^2}\frac{(|\ell|^2I_1^{-1})^2p_1^2 + |\ell|^2(I_2^{-1} - I_3^{-1})^2p_2p_3}{|\ell|^2 - p_1^2}$$

   and

$$\tan\varphi(t) = \frac{\Omega_2M_3 - \Omega_3M_2}{\Omega_1 - 2|\ell|^{-1}hM_1} = \frac{|\ell|(I_2^{-1} - I_3^{-1})p_2(t)p_3(t)}{(|\ell|^2I_1^{-1} - 2h)p_1(t)}.$$

   d) Show that the herpolhode angle is $\Delta\vartheta(T) = \Delta\theta(T) + \Delta\varphi(T)$, where T is the period of a solution of Euler's equations of energy h and angular momentum magnitude $|\ell|$. Using the results of exercise 1 show that

$$\Delta\varphi(T) = \begin{cases} 0, & \text{if } I_2^{-1}|\ell|^2 < 2h < I_1^{-1}|\ell|^2 \\ 2\pi, & \text{if } I_3^{-1}|\ell|^2 < 2h < I_2^{-1}|\ell|^2. \end{cases}$$

8. Let $T^2 \subseteq \mathbf{R}^4$ be a 2-dimensional torus. Suppose that $\pi : T^2 \rightarrow [0,1] \times S^1$ is a smooth surjective mapping such that $\pi^{-1}(\{0\} \times S^1)$ and $\pi^{-1}(\{1\} \times S^1)$ are each the union of two disjoint circles which are the only singularities of π and are fold points. Show that T^2 cannot be embedded in $\mathbf{R}^3$.

9. Consider $T_1S^2 \subseteq \mathbf{R}^6$ as a Lie group. Give a symplectic form on $T\mathbf{R}^6$ whose restriction to $T_1S^2 \times \mathbf{R}^3$ is the canonical symplectic form on $T(T_1S^2)$ pulled back by left trivialization.

10. Consider a vector field on X on a flat 2-torus $T^2 = \mathbf{R}^2/\mathbf{Z}^2$ in $\mathbf{R}^4$ whose flow is a projection of the straight line flow on $\mathbf{R}^4$ with slope p/q, where p,q are relatively prime positive integers. Let $j : T^2 \to S^3$ be an embedding. Give a vector field Y on S^3 such that $j^*(Y|j(T^2)) = X$.

    a) Let $\pi : S^3 \setminus \{(0,0,0,1)\} \to \mathbf{R}^3$ be stereographic projection from the north pole. Show that $\pi(T^2)$ is a 2-torus in $\mathbf{R}^3$. An integral curve γ of Y is called a *p-q torus knot*. Draw a picture of $\pi(\gamma)$ when $(p,q) = (2,3)$. This knot is called the *trefoil*. Draw a picture of the projection of $\pi(\gamma)$ on $\mathbf{R}^2$ using a convention which distinguishes over crossings from under crossings. Here we assume that $\pi(\gamma)$ is oriented. Construct an orientable surface bounded by $\pi(\gamma)$ from its planar projection by replacing every crossover with a rectangle which has been given a counterclockwise half twist and whose edges lie on $\pi(\gamma)$. Fill in the remaining pieces of $\pi(\gamma)$ with rectangles. Do this for the trefoil knot. Find a different filling procedure which produces a nonorientable surface bounded by $\pi(\gamma)$. Show that the 2-1 torus knot bounds a Möbius band.

    b) Suppose that γ and $\gamma\prime$ are two distinct integral curves of the vector field X on T^2. Then γ and γ' are called *parallel p-q torus knots*. Find a formula for the linking number of $\pi(\gamma)$ and $\pi(\gamma')$.

11. (A geometric formula for the rotation number.) Let $\Gamma : [0,T] \to S^2_{|\ell|} : t \mapsto p(t)$ be a periodic solution of period T of Euler's equations (102) on $(S^2_{|\ell|}, \omega = \mathrm{vol}_{S^2_{|\ell|}})$, which has energy $h = \frac{1}{2}(ap_1^2 + bp_2^2 + p_3^2)$ with $a > b > c > 0$ and angular momentum of magnitude ℓ.

    a) Recall that $\beta(p) = \frac{|\ell|(bp_2^2+cp_3^2)}{|\ell|^2-p_1^2}$. Show that $\beta(p) = \frac{2h}{|\ell|} + \frac{2h-a|\ell|^2}{|\ell|}\frac{p_1^2}{|\ell|^2-p_1^2}$. Deduce

$$\Delta\theta(T) = \int_0^T \beta(p(t))\,\mathrm{d}t = \tfrac{2h}{|\ell|}T - \tfrac{a|\ell|^2-2h}{|\ell|}\int_0^T \frac{p_1^2(t)}{|\ell|^2 - p_1^2(t)}\,\mathrm{d}t.$$

    The above formulae expresses the rotation number of the flow of X_H on $T^2_{h,\ell e_3}$ as the sum of a dynamic and a geometric phase.

    b) Write the eastern hemisphere of $S^2_{|\ell|}$ as $\frac{p_1}{|\ell|} = \sqrt{1 - \frac{p_2^2}{|\ell|^2} - \frac{p_3^2}{|\ell|^2}}$ and recall that its volume form σ is $\frac{1}{|\ell|}\frac{\mathrm{d}p_2\wedge\mathrm{d}p_3}{p_1}$. Show that

$$\sigma = -\tfrac{1}{|\ell|}\,\mathrm{d}\Big(p_1\left[\frac{p_2\,\mathrm{d}p_3 - p_3\,\mathrm{d}p_3}{p_2^2+p_3^2}\right]\Big).$$

    Using Stokes' theorem show that the unoriented area $|A|$ of the domain D in S^2_ℓ bounded by the curve Γ is given by

$$\tfrac{1}{|\ell|}\int_0^T p_1(t)\left(\frac{p_2(t)\dot{p}_3 - p_3(t)\dot{p}_2}{p_2^2(t)+p_3^2(t)}\right)\mathrm{d}t.$$

    Using Euler's equations (102) show that $|A|$ is the absolute value of the integral giving the geometric phase of the rotation number $\Delta\theta(T)$.

c) Determine the oriented area A as follows. Note that $\sigma = -|\sigma|$. If $a|\ell|^2 > 2h > b|\ell|^2$, then Γ encloses the positive p_1-axis in $\mathbf{R}^3$ and is traced out in a clockwise fashion. Hence Γ has negative orientation. If $b|\ell|^2 > 2h > c|\ell|^2$ then Γ encloses the positive p_3-axis and it traced out counterclockwise. Thus Γ has a positive orientation. Use a rotation which takes the positive p_3-axis to the positive p_1-axis so that the oriented areas can be compared. Deduce the formula

$$\Delta\theta(T) = \tfrac{2h}{|\ell|}T - \begin{cases} A, & \text{if } a|\ell|^2 > 2h > b|\ell|^2 \\ -A, & \text{if } b|\ell|^2 > 2h > c|\ell|^2. \end{cases}$$

12. (The twisting phenomenon.) Let $\{\widehat{e}_1, \widehat{e}_2, \widehat{e}_3\}$ be the principal axes of a triaxial rigid body with principal moments of inertia $0 < I_1 < I_2 < I_3$. Let $\{e_1, e_2, e_3\}$ be a frame fixed in space which is the principal axis frame at time $t = 0$. Assume that the body is a tennis racket, that is, $I_3 \approx I_1 + I_2$ and $I_1 \ll I_2$.

a) (Almost uniform rotation of the handle.) Because the handle $\widehat{e}_1$ moves nearly in the e_1–e_2 plane, see ((8.1)), the amount that its projection on the e_1–e_2 plane rotates around the e_3-axis is nearly the same as the amount that $\widehat{e}_1$ rotates around e_3. This latter rotation at time t is $\theta(t) = \int_0^t \beta(s)\,ds$, where

$$\beta(t) = |\ell|\left(I_2^{-1}p_2^2(t) + I_3^{-1}p_3^2(t)\right)\left(|\ell|^2 - p_1^2(t)\right)^{-1},$$

and $t \mapsto p(t)$ is a periodic solution of Euler's equations on $S^2_{h,\ell}$ of energy h and period $T = T(h,\ell)$. Prove the following. There is an $M > 0$ such that for every $t \in \mathbf{R}$ we have $|\theta(t) - \overline{\beta}t| \le M$, where $\overline{\beta} = \frac{1}{T}\int_0^T \beta(s)\,ds$. First show that

$$\beta(t) = \frac{2h}{|\ell|} - \frac{1}{|\ell|}\left(|\ell|^2 I_1^{-1} - 2h\right)\frac{p_1^2(t)}{|\ell|^2 - p_1^2(t)}. \tag{107}$$

From the fact the $\beta(t) - \overline{\beta}$ is a periodic function of period T and average value 0 deduce that

$$\left|\int_0^t (\beta(s) - \overline{\beta})\,ds\right| \le \int_0^T |\beta(s) - \overline{\beta}|\,ds = M$$

for every $t \in \mathbf{R}$. For a tennis racket-like body M is small, when h is close to $\frac{1}{2}|\ell|^2 I_2^{-1}$. To see this use the triangle inequality to show that

$$M \le \frac{2}{|\ell|}\left(|\ell|^2 I_1^{-1} - 2h\right)\int_0^T \frac{p_1^2(s)}{|\ell|^2 - p_1^2(s)}\,ds. \tag{108}$$

Using (108) and the fact that $|p_1(t)| \le \sqrt{\left(2h - |\ell|^2 I_3^{-1}\right)/\left(I_1^{-1} - I_3^{-1}\right)}$, see the proof of ((8.1)), deduce the estimate: $M \le 2T\left(2h - |\ell|^2 I_3^{-1}\right)/|\ell|$. When $h \approx |\ell|^2/2I_2$ show that $|\ell|T \approx 2\pi/\sqrt{(I_1^{-1} - I_3^{-1})(I_2^{-1} - I_3^{-1})}$. Note that the right hand side of the preceding formula is the period of rotation of the body around its intermediate axis. Using the fact that the rigid body is tennis racket-like, it follows that $M \approx 4\pi I_1/I_2$ which is small.

Use the same kind argument as in estimating the right hand side of (108) to prove $\overline{\beta} \geq \frac{2h}{|\ell|} - \frac{1}{|\ell|}\left(2h - |\ell|^2 I_3^{-1}\right) = |\ell|/I_3$. Thus the time the projection of $\widehat{e}_1$ on the e_1–e_2 plane needs to make one revolution about e_3 is at *most* $2\pi I_3/|\ell|$.

b) (Enough time to twist.) From (107) it follows immediately that $\overline{\beta} \leq 2h/|\ell|$. When $h = \frac{1}{2}|\ell|^2/I_2$, we find that $\overline{\beta} \leq |\ell|/I_2$. Therefore the time needed for the projection of $\widehat{e}_1$ on the e_1–e_2 plane to make one revolution about e_3 is at *least* $t_* = 2\pi I_2/|\ell|$. For a tennis racket-like body show that t_* is larger than the characteristic twisting time $2\pi/\sqrt{(I_1^{-1} - I_3^{-1})(I_2^{-1} - I_3^{-1})}$.

c) (A long time near γ_2^+.) Show that the racket moving near the unstable (stable) manifold of γ_2^+ (γ_2^-) spends most of its time near γ_2^- (γ_2^+). The hyperbolic character of the periodic orbits $\gamma_2^\pm$ is essential here.

d) (No twist region.) Consider the function

$$F(p_1, p_2, p_3) = (I_1^{-1} - I_2^{-1})p_1^2 + (I_2^{-1} - I_3^{-1})p_3^2$$

on the energy surface $h = \frac{1}{2}\left(I_1^{-1}p_1^2 + I_2^{-1}p_2^2 + I_3^{-1}p_3^2\right)$. The following argument shows that a tennis racket will *not* make a half twist if

$$F(0) < 2I_3 h(I_2^{-1} - I_3^{-1})\,\mathrm{e}^{2nt_0}, \tag{109}$$

where $t_0 = \frac{2\pi I_3}{|\ell|}$ and $n = |\ell|\sqrt{(I_2^{-1} - I_3^{-1})(I_1^{-1} - I_3^{-1})}$. First show that

$$\dot{F} = L_X F \;=\; -4(I_1^{-1} - I_3^{-1})(I_2^{-1} - I_3^{-1})\,p_1 p_2 p_3,$$

where X is the Euler vector field $X(p) = p \times (I')^{-1}p$ on $S^2_{|\ell|}$. Using the inequalities $|p_2| \leq |\ell|$ and $|p_1 p_3| \leq \frac{1}{2}(\sigma^{-1}p_1^2 + \sigma p_3^2)$ with $\sigma = \sqrt{(I_2^{-1} - I_3^{-1})/(I_1^{-1} - I_3^{-1})}$ deduce that $|\dot{F}| \leq 2|\ell|nF$. This inequality integrates to $F(t) \leq F(0)\,\mathrm{e}^{2nt}$. By a) the largest time required for the projection of the handle on the e_1–e_2 plane to make one revolution is $t_0 = 2\pi I_3/|\ell|$. Therefore if $F(0)$ is sufficiently small then $F(t)$ is not very large for $t \in [0, t_0]$. Thus we need a bound on $F(t)$ which excludes the occurence of a half twist. From its definition $\lambda = F(p(t)) = F(t)$ determines a family of ellipses

$$\mathscr{E}_\lambda : (I_1^{-1} - I_2^{-1})p_1^2 + (I_2^{-1} - I_3^{-1})p_3^2 = \lambda.$$

Show that if for every $t \in [0, t_0]$ the curve $\mathscr{E}_\lambda$ lies in the interior of the ellipse

$$\mathscr{E} : 2h = I_1^{-1}p_1^2 + I_3^{-1}p_3^2$$

then the integral curve $t \mapsto p(t)$ of the vector field X on the energy surface does *not* cross the $\{p_2 = 0\}$ plane. From $I_3 - I_2 \leq I_2 - I_1$ deduce the estimate

$$F(t) \geq I_3(I_2^{-1} - I_3^{-1})(I_1^{-1}p_1^2 + I_3^{-1}p_3^2).$$

Show that $\mathscr{E}_\lambda$ lies in the interior of $\mathscr{E}$ if $F(t) < 2hI_3(I_2^{-1} - I_3^{-1})$. Show that if (109) holds on the energy surface then *no* twist occurs.

Chapter IV

The spherical pendulum

In this chapter we treat the spherical pendulum as a constrained Hamiltonian system. We derive Hamilton's equations and show that there is an axial symmetry which gives rise to a conserved angular momentum. Thus the spherical pendulum is a Liouville integrable Hamiltonian system. Using the technique of singular reduction, see chapter VII §7, we remove the axial symmetry to obtain a Hamiltonian system with one degree of freedom which we analyze. From the qualitative description of the reduced system we obtain a complete qualitative picture of the motion of the spherical pendulum. Because of monodromy, the Liouville tori fit together in a nontrivial way. This precludes the existence of global action coordinates, see chapter XI §1.

1 Liouville integrability

First we recall some standard facts about Hamiltonian systems on $T\mathbf{R}^3$, see chapter VI §3. Let $\langle\, ,\rangle$ be the Euclidean inner product and $\times$ the usual vector product on $\mathbf{R}^3$. Let $\zeta = (x, y)$ be canonical coordinates on $T\mathbf{R}^3$, that is, the canonical symplectic 2-form is $\omega = \sum_{i=1}^{3} \mathrm{d}x_i \wedge \mathrm{d}y_i$. Corresponding to a smooth Hamiltonian function $H : T\mathbf{R}^3 \rightarrow \mathbf{R}$ is its Hamiltonian vector field X_H, whose integral curves satisfy Hamilton's equations

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= \frac{\partial H}{\partial y} \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= -\frac{\partial H}{\partial x}. \end{aligned} \tag{1}$$

Using the symplectic form ω, we define a Poisson bracket $\{\, ,\}_{T\mathbf{R}^3}$ on $C^\infty(T\mathbf{R}^3)$ by

$$\{f, g\}_{T\mathbf{R}^3} = \omega(X_f, X_g) = \sum_{i,j} \frac{\partial f}{\partial \zeta_i} \frac{\partial g}{\partial \zeta_j} \{\zeta_i, \zeta_j\}_{T\mathbf{R}^3},$$

whose structure matrix is $(\{\zeta_i, \zeta_j\}_{T\mathbf{R}^3}) = \begin{pmatrix} 0 & I_3 \\ -I_3 & 0 \end{pmatrix}$. In terms of Poisson brackets Hamilton's equations for X_H read

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$$\begin{aligned} \dot{x} &= \{x, H\}_{T\mathbf{R}^3} \\ \dot{y} &= \{y, H\}_{T\mathbf{R}^3}. \end{aligned} \tag{2}$$

We now describe the spherical pendulum as a *constrained Hamiltonian system.* First consider the *unconstrained Hamiltonian system* $(H, T\mathbf{R}^3, \omega)$ with *Hamiltonian*

$$H : T\mathbf{R}^3 \to \mathbf{R} : (x,y) \to \tfrac{1}{2}\langle y, y\rangle + \gamma \langle x, e_3\rangle. \tag{3}$$

The integral curves of X_H give the motion of a particle of unit mass in $\mathbf{R}^3$ under a constant vertical gravitational force. Choosing an appropriate unit of length, we may assume that $\gamma = 1$. Now *constrain* the particle to move on the tangent bundle TS^2 of the 2-sphere $S^2 \subseteq \mathbf{R}^3$, which is defined by $\{(x,y) \in T\mathbf{R}^3 \big| \langle x,x\rangle = 1 \ \& \ \langle x,y\rangle = 0\}$. Then *Newton's equations* of motion are

$$\ddot{x} + e_3 = \lambda x, \tag{4}$$

so that the resultant force on the particle is normal to S^2. We determine the multiplier λ in (4) by differentiating the constraint equation $0 = \langle x,x\rangle - 1$ twice and then using (4). This gives

$$0 = \langle \dot{x}, \dot{x}\rangle + \langle x, \ddot{x}\rangle = \langle \dot{x}, \dot{x}\rangle - \langle x, e_3\rangle + \lambda\langle x,x\rangle = \langle \dot{x}, \dot{x}\rangle - \langle x, e_3\rangle + \lambda.$$

Thus the constrained equations of motion are

$$\ddot{x} = -e_3 + \big(\langle x, e_3\rangle - \langle \dot{x}, \dot{x}\rangle\big)x,$$

which written as a first order system are

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -e_3 + \big(\langle x, e_3\rangle - \langle y, y\rangle\big)x. \end{aligned} \tag{5}$$

Suppose that $(x,y) \in TS^2$ and that $t \mapsto \big(x(t), y(t)\big)$ is a solution of (5) starting at (x,y). Then

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\big(\langle x,x\rangle - 1\big) &= 2\langle x, \dot{x}\rangle = 2\langle x, y\rangle = 0 \\ \frac{\mathrm{d}}{\mathrm{d}t}\langle x, y\rangle &= \langle \dot{x}, y\rangle + \langle x, \dot{y}\rangle = \langle y,y\rangle - \langle e_3, x\rangle + \big(\langle x, e_3\rangle - \langle y,y\rangle\big)\langle x,x\rangle = 0. \end{aligned}$$

$\triangleright$ So $\big(x(t), y(t)\big) \in TS^2$. In other words, TS^2 is an invariant manifold of (5).

Next we show that (5) are Hamilton's equations for the *constrained Hamiltonian system* $(H|TS^2, TS^2, \omega|TS^2)$, which is the *spherical pendulum.*

The following argument shows that the 2-form $\omega|TS^2$ is symplectic.

(1.1) **Proof**: Let $T_0\mathbf{R}^3 = (\mathbf{R}^3 \setminus \{0\}) \times \mathbf{R}^3$. The constraint functions

$$c_1 : T_0\mathbf{R}^3 \to \mathbf{R} : (x,y) \mapsto \langle x,x\rangle - 1 \quad \text{and} \quad c_2 : T_0\mathbf{R}^3 \to \mathbf{R} : (x,y) \mapsto \langle x,y\rangle$$

define the function $\mathscr{C} : T_0\mathbf{R}^3 \to \mathbf{R}^2 : m \mapsto \big(c_1(m), c_2(m)\big)$, whose 0-level set $\mathscr{C}^{-1}(0)$ is the constraint manifold TS^2. Since 0 is a regular value of $\mathscr{C}$, the constraint manifold TS^2

is smooth. For every $(x,y) \in T_0\mathbf{R}^3$ the matrix $\big(\{c_i,c_j\}_{T\mathbf{R}^3}(x,y)\big) = 2\langle x,x\rangle \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ is invertible with inverse $C = (C_{ij}) = \frac{1}{2\langle x,x\rangle}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Consequently, the constraint manifold TS^2 is cosymplectic. Therefore $\omega|TS^2$ is a symplectic form, see chapter VI ((4.7)). □

Using *modified Dirac brackets*, see chapter VI §4, we compute Hamilton's equations for the *constrained Hamiltonian* $H|TS^2$ (3) as follows. Let

$$\begin{aligned} H^* &= H - \sum_{i,j=1}^{2} \big(\{H,c_i\}_{T\mathbf{R}^3} + H_i\big) C_{ij}\, c_j \\ &= \tfrac{1}{2}\langle y,y\rangle + \langle x,e_3\rangle + \tfrac{1}{2\langle x,x\rangle}\big(-2\langle x,y\rangle + H_1\big)\langle x,y\rangle \\ &\qquad - \tfrac{1}{2\langle x,x\rangle}\big(-\langle y,y\rangle + \langle x,e_3\rangle + H_2\big)\big(\langle x,x\rangle - 1\big), \end{aligned} \tag{6}$$

where H_i is in the *ideal* of the commutative algebra $\big(C^\infty(TS^2),\cdot\big)$ of smooth functions on TS^2 generated by the constraints c_j with $j=1,2$. Choose $H_1 = -\langle x,y\rangle\big(\langle x,x\rangle - 2\big)$ and $H_2 = -\big(\langle x,x\rangle - 1\big)\big(\langle y,y\rangle - \langle x,e_3\rangle\big)$. Then

$$H^*(x,y) = \tfrac{1}{2}\langle y,y\rangle + \langle x,e_3\rangle + \tfrac{1}{2}\big(\langle y,y\rangle - \langle x,e_3\rangle\big)\big(\langle x,x\rangle - 1\big) - \tfrac{1}{2}\langle x,y\rangle^2.$$

Using (2), we see that on $T_0\mathbf{R}^3$ the Hamiltonian vector field X_{H^*} has integral curves which satisfy

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= y + y\big(\langle x,x\rangle - 1\big) - \langle x,y\rangle x \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= -e_3 + \tfrac{1}{2}e_3\big(\langle x,x\rangle - 1\big) - \big(\langle y,y\rangle - \langle x,e_3\rangle\big)x + \langle x,y\rangle y. \end{aligned}$$

We have $X_{H^*|TS^2} = X_{H|TS^2}$. Therefore *Hamilton's equations* for the spherical pendulum are given by (5).

We now look at the symmetries of the spherical pendulum. As a physical system in $\mathbf{R}^3$, the spherical pendulum is invariant under a counterclockwise rotation $\widetilde{R}_t = \begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix}$ through an angle t about the positive x_3-axis. Lifting this to $T\mathbf{R}^3$ gives the S^1-action

$$\Phi : S^1 \times T\mathbf{R}^3 \to T\mathbf{R}^3 : (x,y) \mapsto (\widetilde{R}_t x, \widetilde{R}_t y). \tag{7}$$

The infinitesimal generator of the S^1-action Φ is the vector field $Y(x,y) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \Phi_t(x,y)$, whose integral curves satisfy

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= x \times e_3 \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= y \times e_3. \end{aligned} \tag{8}$$

Using (1) we see that Y is the Hamiltonian vector field X_J corresponding to the Hamiltonian function

$$J : T\mathbf{R}^3 \to \mathbf{R} : (x,y) \to \langle x \times y, e_3\rangle = x_1y_2 - x_2y_1. \tag{9}$$

Thus J is the *momentum mapping* of the S^1-action Φ. Since TS^2 is invariant under Φ, the spherical pendulum has an S^1-symmetry, namely, the S^1-action $\Phi|(S^1 \times TS^2)$. This symmetry gives rise to the integral $J|TS^2$, because the Hamiltonian $H|TS^2$ is invariant under Φ, which implies

$$\{J|TS^2, H|TS^2\}_{TS^2} = -L_{X_{J|TS^2}}(H|TS^2) = 0. \tag{10}$$

Hence the spherical pendulum is *Liouville integrable*.

2 Reduction of the S^1 symmetry

We remove the S^1 symmetry $\Phi|(S^1 \times TS^2)$ of the spherical pendulum using singular reduction, see chapter VII §7. The regular reduction theorem does *not* apply, because the S^1-symmetry $\Phi|(S^1 \times TS^2)$ leaves the points $(0,0,\pm 1,0,0,0)$ fixed. After reduction we get a one degree of freedom Hamiltonian system on a *singular* reduced phase space.

2.1 The orbit space $T\mathbf{R}^3/S^1$

As a first step in the reduction process we find the *invariants* of the S^1-action Φ (7).

Claim: The algebra $\mathbf{R}[x,y]^{S^1}$ of polynomials on $T\mathbf{R}^3$, which are invariant under the S^1-action Φ, is *generated* by

$$\begin{aligned} \xi_1 &= x_1y_1 + x_2y_2 & \xi_3 &= \tfrac{1}{2}(y_1^2 + y_2^2 - x_1^2 - x_2^2) & \eta_1 &= x_3 \\ \xi_2 &= x_1y_2 - x_2y_1 & \xi_4 &= \tfrac{1}{2}(y_1^2 + y_2^2 + x_1^2 + x_2^2) & \eta_2 &= y_3. \end{aligned} \tag{11}$$

(2.1) **Proof**: The action Φ fixes every point on the 2-plane

$$\Pi = \big\{(0,0,\eta_1,0,0,\eta_2) \in T\mathbf{R}^3 \big| (\eta_1,\eta_2) = (x_3,y_3) \in \mathbf{R}^2\big\}. \tag{12}$$

Therefore the algebra of polynomials invariant under $\Phi|(S^1 \times \Pi)$ is $\mathbf{R}[x_3,y_3]$, the algebra of polynomials in the variables x_3 and y_3. The action $\Phi|(S^1 \times T(\mathbf{R}^2 \times \{0\}))$ is

$$\widetilde{\Phi} : S^1 \times \mathbf{R}^2 \to \mathbf{R}^2 : (t,\widetilde{x},\widetilde{y}) \mapsto (R_t\widetilde{x}, R_t\widetilde{y}),$$

where $\widetilde{x} = (x_1,x_2)$, $\widetilde{y} = (y_1,y_2)$, and $R_t = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$. In other words, $\widetilde{\Phi}$ is the diagonal action of $\mathrm{SO}(2,\mathbf{R})$ on $\mathbf{R}^2 \times \mathbf{R}^2$. Therefore $\mathbf{R}[x,y]^{S^1} = \mathbf{R}[x_3,y_3] \otimes \mathbf{R}[\widetilde{x},\widetilde{y}]^{S^1}$. It follows that $\mathbf{R}[\widetilde{x},\widetilde{y}]^{S^1}$ is generated by ξ_i (11) for $1 \le i \le 4$, compare with ((5.2)) of chapter I. $\square$

$\triangleright$ The generators ξ_i, $1 \le i \le 4$ and η_j, $1 \le j \le 2$ of $\mathbf{R}[x,y]^{S^1}$ satisfy only one *relation*

$$\xi_1^2 + \xi_2^2 + \xi_3^2 = \xi_4^2, \quad \xi_4 \ge 0, \tag{13}$$

that defines the semialgebraic variety $W \times \mathbf{R}^2$ of $\mathbf{R}^6 = \mathbf{R}^4 \times \mathbf{R}^2$ with coordinates (ξ,η).

(2.2) **Proof**: Use an argument analogous to the one demonstrating ((5.4)) in chapter I. $\square$

Let

$$\varsigma : T\mathbf{R}^3 \to \mathbf{R}^6 : (x,y) \mapsto \big(\xi(x,y), \eta(x,y)\big) = \big(\xi_1(x,y), \ldots, \xi_4(x,y), \eta_1(x,y), \eta_2(x,y)\big) \qquad (14)$$

▷ be the *Hilbert map* of the S^1-action Φ (7). As a set the space of $T\mathbf{R}^3/S^1$ of S^1 orbits is the *semialgebraic variety* $W \times \mathbf{R}^2 = \varsigma(T\mathbf{R}^3)$ defined by (13).

(2.3) **Proof**: We need a better description of the fibers of the Hilbert map ς. Let $\xi \in W$ and suppose that $\varsigma(x,y) = (\xi, 0)$ with $\xi_4 + \xi_3 = y_1^2 + y_2^2 > 0$. Then solving $\left\{ \begin{array}{l} \xi_1 = x_1y_1 + x_2y_2 \\ \xi_2 = x_2y_1 - x_1y_2 \end{array} \right.$ for (x_1, x_2) gives $\left\{ \begin{array}{l} x_1 = (y_1\xi_1 - y_2\xi_2)/(\xi_4 + \xi_3) \\ x_2 = (y_2\xi_1 + y_1\xi_2)/(\xi_3 + \xi_4) \end{array} \right.$. Thus $\varsigma^{-1}(p) \subseteq V_1 \times \{(0,0)\}$ where

$$V_1 = \Big\{(x_1,x_2,y_1,y_2) \in T\mathbf{R}^3 \Big| \, \xi_3 + \xi_4 > 0 \;\&\; \left\{ \begin{array}{l} x_1 = (y_1\xi_1 - y_2\xi_2)/(\xi_4 + \xi_3) \\ x_2 = (y_2\xi_1 + y_1\xi_2)/(\xi_4 + \xi_3) \end{array} \right. \Big\}.$$

Using the relation $\xi_1^2 + \xi_2^2 = (\xi_4 + \xi_3)(\xi_4 - \xi_3)$, a calculation shows that for every $q \in V_1$ we have $\varsigma(q,0) = (\xi, 0)$. Consequently, $V_1 \times \{(0,0)\} = \varsigma^{-1}(\xi, 0)$. Clearly $V_1 \times \{(0,0)\}$ is a single orbit of the S^1-action Φ. Under the hypothesis that $\xi_4 - \xi_3 = x_1^2 + x_2^2 > 0$, a similar argument shows that $V_2 \times \{(0,0)\} = \varsigma^{-1}(\xi, 0)$, where

$$V_2 = \Big\{(x_1,x_2,y_1,y_2) \in T\mathbf{R}^3 \Big| \, \xi_4 - \xi_3 > 0 \;\&\; \left\{ \begin{array}{l} y_1 = (x_1\xi_1 + x_2\xi_2)/(\xi_4 - \xi_3) \\ y_2 = (x_2\xi_1 - x_1\xi_2)/(\xi_4 - \xi_3) \end{array} \right. \Big\}.$$

Note that $\varsigma^{-1}(0,\eta) = \{(0,0,x_3,0,0,y_3) \in T\mathbf{R}^3 \big| (x_3,y_3) = (\eta_1,\eta_2)\}$ is an orbit of Φ, being a fixed point. Thus the image of the Hilbert map ς is the semialgebraic variety $W \times \mathbf{R}^2$ (14). Since each fiber of ς is a single S^1-orbit of Φ, it follows that $W \times \mathbf{R}^2$ is the orbit space $T\mathbf{R}^3/S^1$. □

Claim: The orbit space $T\mathbf{R}^3/S^1$ is homeomorphic to $W \times \mathbf{R}^2$ via the mapping $\overline{\varsigma}$ induced by the Hilbert map ς.

(2.4) **Proof**: Let $\rho : T\mathbf{R}^3 \to T\mathbf{R}^3/S^1$ be the orbit map, which assigns to each $(x,y) \in T\mathbf{R}^3$ the S^1 orbit $\overline{(x,y)} = \{\Phi_t(x,y) \in T\mathbf{R}^3 \big| t \in S^1\}$. Since by definition the Hilbert map ς is invariant under the S^1-action Φ, it induces a map

$$\overline{\varsigma} : T\mathbf{R}^3/S^1 \to W \times \mathbf{R}^2 : \overline{(x,y)} \mapsto \varsigma(x,y).$$

The map $\overline{\varsigma}$ is continuous because the Hilbert map ς is and the orbit map ρ is continuous and open by definition of the topology of the orbit space $T\mathbf{R}^3/S^1$, see chapter VII §2. Since every fiber of the Hilbert map ς is an S^1-orbit and ς is surjective, it follows that the map $\overline{\varsigma}$ is bijective.

To verify that the inverse of $\overline{\varsigma}$ is continuous, it is enough to show that the map ς has a continuous local cross section. Towards this goal consider the mappings

$$\begin{array}{l} \psi_1 : U_1 = (W \times \mathbf{R}^2) \setminus \{\xi_3 = \xi_4\} \to T\mathbf{R}^3 : \\ \qquad (\xi, \eta) \mapsto \Big(0, \sqrt{\xi_4 - \xi_3}, \, \eta_1, \, -\dfrac{\xi_2}{\sqrt{\xi_4 - \xi_3}}, \, \dfrac{\xi_1}{\sqrt{\xi_4 - \xi_3}}, \eta_2\Big) \end{array}$$

and

$$\psi_2 : U_2 = (W \times \mathbf{R}^2) \setminus \{-\xi_3 = \xi_4\} \to T\mathbf{R}^3 :$$
$$(\xi,\eta) \mapsto \big(-\frac{\xi_2}{\sqrt{\xi_4+\xi_3}}, \frac{\xi_1}{\sqrt{\xi_4+\xi_3}}, \eta_1, 0, \sqrt{\xi_4+\xi_3}, \eta_2\big).$$

Clearly ψ_1 and ψ_2 are continuous. Moreover, $\varsigma \circ \psi_1 = \mathrm{id}_{U_1}$ and $\varsigma \circ \psi_2 = \mathrm{id}_{U_2}$. Thus ψ_1 and ψ_2 are continuous local sections of ς. However, $U_1 \cup U_2 = (W \times \mathbf{R}^2) \setminus \Pi$, see (12). To finish the proof we need to show that ψ_1 has a continuous extension $\overline{\psi}_1$ to $U_1 \cup \Pi$, namely, $\overline{\psi}_1|\Pi = \mathrm{id}_\Pi$, while $\overline{\psi}_1|U_1 = \psi_1$. To see that $\overline{\psi}_1$ is continuous we argue as follows. From the relation $\xi_1^2+\xi_2^2+\xi_3^2 = \xi_4^2$ with $\xi_4 \ge 0$ we get $|\xi_i| \le \xi_4$ for $1 \le i \le 3$ and $|\xi_j| \le \sqrt{\xi_4^2 - \xi_3^2}$ for $j = 1,2$. From the first of preceding inequalities we obtain

$$\sqrt{\xi_4 - \xi_3} \le \sqrt{2\xi_4}; \tag{15a}$$

while from the second we get

$$\frac{|\xi_j|}{\sqrt{\xi_4-\xi_3}} \le \sqrt{\xi_4+\xi_3} \le \sqrt{2\xi_4}, \quad \text{for } j = 1,2. \tag{15b}$$

From (15a), (15b), and the definition of the mapping ψ_1 it follows that

$$\overline{\psi}_1(0,0,\eta_1,0,0,\eta_2) = \lim_{\substack{\xi_4 \searrow 0 \\ (\xi,\eta)\in U_1}} \psi_1(\xi,\eta) = (0,0,\eta_1,0,0,\eta_2).$$

Thus $\overline{\psi}_1$ is continuous as desired. □

2.2 The singular reduced space

In this subsection we construct the singular reduced space.

It is convenient to employ another set of *generators* for $\mathbf{R}[x,y]^{S^1}$, namely

$$\begin{array}{lll} \sigma_1 = x_3 & \sigma_3 = y_1^2+y_2^2+y_3^2 & \sigma_5 = x_1^2+x_2^2 \\ \sigma_2 = y_3 & \sigma_4 = x_1y_1+x_2y_2 & \sigma_6 = x_1y_2 - x_2y_1. \end{array} \tag{16}$$

Since $TS^2 \subseteq T\mathbf{R}^3$ is invariant under the S^1-action Φ (7), the defining equations

$$x_1^2+x_2^2+x_3^2 = 1$$
$$x_1y_1+x_2y_2+x_3y_3 = 0$$

of TS^2 may be expressed in terms of invariants as

$$\begin{aligned} \sigma_5 + \sigma_1^2 &= 1 \\ \sigma_4 + \sigma_1\sigma_2 &= 0. \end{aligned} \tag{17a}$$

Therefore the *orbit space* TS^2/S^1 of the S^1-action $\Phi|(S^1 \times TS^2)$ is the semialgebraic variety V defined by (17a) and

$$\sigma_4^2 + \sigma_6^2 = \sigma_5(\sigma_3 - \sigma_2^2), \quad \sigma_3 - \sigma_2^2 \ge 0 \;\&\; \sigma_5 \ge 0, \tag{17b}$$

which comes from the identity $(x_1y_1+x_2y_2)^2+(x_1y_2-x_2y_1)^2=(x_1^2+x_2^2)(y_1^2+y_2^2)$. Using the invariants $(\sigma_1,\ldots\sigma_6)$ as coordinates on $\mathbf{R}^6$, (17b) defines a semialgebraic variety $\Sigma\subseteq\mathbf{R}^6$, which is the image of the *Hilbert mapping*

$$\sigma : T\mathbf{R}^3 \rightarrow \Sigma : (x,y) \mapsto \big(\sigma_1(x,y),\ldots,\sigma_6(x,y)\big) \tag{18}$$

of the S^1-action Φ (7).

Because $J|TS^2$ (9) is the momentum map of the S^1-action $\Phi|(S^1\times TS^2)$, for every $j\in\mathbf{R}$ the reduced space $V_j=(J|TS^2)^{-1}(j)/S^1$ is the semialgebraic subvariety of V defined by (17a), (17b), and

$$\sigma_6 = j. \tag{17c}$$

Eliminating σ_4 and σ_5 from (17b) using (17a) and then using (17c) gives the semialgebraic variety P_j in $\mathbf{R}^3$ with coordinates $(\sigma_1,\sigma_2,\sigma_3)$ defined by

$$\sigma_3(1-\sigma_1^2)-\sigma_2^2-j^2=0, \quad |\sigma_1|\le 1 \;\&\; \sigma_3\ge 0. \tag{19}$$

In other words, the image of V_j under the projection mapping

$$\mu : \mathbf{R}^6 \rightarrow \mathbf{R}^3 : \big(\sigma_1,\ldots,\sigma_6\big) \mapsto (\sigma_1,\sigma_2,\sigma_3) \tag{20a}$$

is the semialgebraic variety P_j. A straightforward calculation shows that the map

$$\nu_j : P_j \subseteq \mathbf{R}^3 \rightarrow \mathbf{R}^6 : (\sigma_1,\sigma_2,\sigma_3) \mapsto \big(\lambda_1,\ldots,\lambda_6\big) = \big(\sigma_1,\sigma_2,\sigma_3,-\sigma_1\sigma_2,1-\sigma_1^2,j\big) \tag{20b}$$

is the inverse of $\mu|V_j$. Since the maps μ and ν_j are continuous and polynomial, the semi-algebraic variety V_j is isomorphic to the singular *reduced phase space* P_j. When $j\neq 0$,

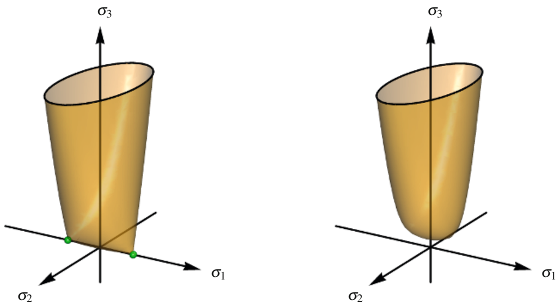


Figure 2.2.1. The singular reduced phase space P_j.
In the left figure $j=0$, while in the right figure $j\neq 0$.

P_j is diffeomorphic to $\mathbf{R}^2$, the diffeomorphism being the graph of the function

$$\sigma_3 = \frac{j^2+\sigma_2^2}{1-\sigma_1^2}, \quad |\sigma_1|<1.$$

When $j = 0$, P_0 is *not* the graph of a function, because it contains the vertical lines $\{(\pm 1, 0, \sigma_3) \in \mathbf{R}^3 | \sigma_3 \geq 0\}$, see figure 2.2.1. P_0 is *not* smooth because $(\pm 1, 0, 0)$ are singular points, which correspond to the fixed points $(0, 0, \pm 1, 0, 0, 0) \in TS^2$ of the action $\Phi|(S^1 \times TS^2)$. However, P_0 is homeomorphic to $\mathbf{R}^2$, because near each of its singular points it is a cone on S^1 with vertex the singular point.

In fact, P_0 is the orbit space of the $\mathbf{Z}_2$-action $\mathbf{Z}_2 \times TS^1 \to TS^1 : (x, y) \mapsto (-x, -y)$ on TS^1, the tangent bundle of S^1. Geometrically, the orbit space $TS^1/\mathbf{Z}_2$ is obtained by taking the piece of the cylinder TS^1 on or above its equatorial zero section and then identifying points on the resulting bounding circle which have the same x-coordinate. Physically, P_0 is the phase space TS^1 of the mathematical pendulum with points identified by the $\mathbf{Z}_2$-action. This identification is necessary because we can not distinguish positive and negative velocities in the spherical pendulum. See exercise 6 of the introduction.

From the above discussion we see that singular reduction of the spherical pendulum not only produces an accurate model of the reduced phase spaces, but also a geometrically faithful representation of the $j \to 0$ limit.

2.3 Differential structure on P_j

In this subsection we define the space $C^\infty(P_j)$ of *smooth functions* on the singular reduced space P_j. We show that the pair $(P_j, C^\infty(P_j))$ is a locally compact subcartesian differential space, see chapter VII §3.

First we look at the Hilbert map $\sigma : T\mathbf{R}^3 \to \Sigma \subseteq \mathbf{R}^6$ (18). We say that $f : \Sigma \subseteq \mathbf{R}^6 \to \mathbf{R}$ is *smooth* if and only if for every $p \in \Sigma$ there is an open subset U_p of p in $\mathbf{R}^6$ and a smooth function $f_p : U_p \subseteq \mathbf{R}^6 \to \mathbf{R}$ such that $f|(\Sigma \cap U_p) = f_p(\Sigma \cap U_p)$. Another way of saying this is: f is a smooth function on Σ if and only if there is a smooth function F on $\mathbf{R}^6$ such that $f = F|\Sigma$. Let $C_i^\infty(\Sigma)$ be the collection of smooth functions on Σ. Using a partition of unity on $\mathbf{R}^6$, one can show that every open subset of $\mathbf{R}^6$ is the inverse image of an open interval under a smooth function. Thus every open subset of Σ in the topology induced from $\mathbf{R}^6$ is the inverse image of an open interval under a smooth function on Σ. Because composing any n-tuple of smooth functions on Σ with a smooth function on $\mathbf{R}^n$ results in a smooth function on Σ, it follows that $C_i^\infty(\Sigma)$ is a differential structure. So the pair $(\Sigma, C_i^\infty(\Sigma))$ is a differential space, which is a locally compact subcartesian because Σ is a closed subset of $\mathbf{R}^6$. Also σ is a continuous map from the differential space $(T\mathbf{R}^3, C^\infty(T\mathbf{R}^3))$ to the differential space $(\Sigma, C_i^\infty(\Sigma))$.

▷ We now show that σ is a smooth mapping from $(T\mathbf{R}^3, C^\infty(T\mathbf{R}^3))$ into $(\Sigma, C_i^\infty(\Sigma))$.

(2.5) **Proof**: Towards this goal define the linear map $\sigma^* : C_i^\infty(\Sigma) \to C^\infty(T\mathbf{R}^3) : f \mapsto f \circ \sigma$. Since the map σ is surjective, it follows that σ^* is injective. To see this, suppose that $\sigma^* f = 0$ for some $f \in C_i^\infty(\Sigma)$. Then for every $(x, y) \in T\mathbf{R}^3$ we have $0 = f(\sigma(x, y))$. Because σ is surjective, it follows that $f = 0$. Therefore $\sigma^*(C_i^\infty(\Sigma))$, which is isomorphic to $C_i^\infty(\Sigma)$, is a subset of $C^\infty(T\mathbf{R}^3)$. Because $\sigma : (T\mathbf{R}^3, C^\infty(T\mathbf{R}^3)) \to (\Sigma, C_i^\infty(\Sigma))$ is a continuous map of differential spaces, it follows that the mapping σ is smooth, see chapter VII §3. □

▷ The above result can be refined somewhat to the statement that $\sigma^* : C_i^\infty(\Sigma) \to C^\infty(T\mathbf{R}^3)^{S^1}$ is an isomorphism. Here $C^\infty(T\mathbf{R}^3)^{S^1}$ is the space of smooth functions on $T\mathbf{R}^3$, which are invariant under the S^1-action Φ (7).

(2.6) **Proof**: For every $f \in C_i^\infty(\Sigma)$, the smooth function $\sigma^* f$ on $T\mathbf{R}^3$ is invariant under the S^1-action Φ. Consequently, $\sigma^*(C_i^\infty(\Sigma)) \subseteq C^\infty(T\mathbf{R}^3)^{S^1}$. Since Φ is a linear action of the compact Lie group S^1 on $T\mathbf{R}^3$, a theorem of Schwarz states that for every smooth S^1-invariant function $\widetilde{f}$ on $T\mathbf{R}^3$ there is a smooth function $\widetilde{F}$ on $\mathbf{R}^6$ with coordinates the invariants σ_i, $1 \le i \le 6$ such that $\widetilde{f} = \sigma^*(\widetilde{F}|\Sigma)$. In other words, the linear mapping σ^* : $C_i^\infty(\Sigma) \to C^\infty(T\mathbf{R}^3)^{S^1}$ is surjective and hence is an isomorphism. □

Since the action Φ is proper, the orbit space $T\mathbf{R}^3/S^1$ is a differential space $\big(T\mathbf{R}^3/S^1, C^\infty(T\mathbf{R}^3/S^1)\big)$, see chapter VII §3.2. Here a function $\overline{f} : T\mathbf{R}^3/S^1 \to \mathbf{R}$ is *smooth* if and only if $\rho^*\overline{f}$ is a smooth S^1-invariant function on $T\mathbf{R}^3$. Recall that $\rho : T\mathbf{R}^3 \to$
▷ $T\mathbf{R}^3/S^1$ is the orbit map of the S^1-action Φ. Thus the linear mapping $\rho^* : C^\infty(T\mathbf{R}^3/S^1) \to C^\infty(T\mathbf{R}^3)^{S^1}$ is an isomorphism.

Since the Hilbert map σ (18) is invariant under the S^1-action Φ, there is an induced map $\overline{\sigma} : T\mathbf{R}^3/S^1 \to \Sigma$ such that $\overline{\sigma}\circ\rho = \sigma$. Also $\overline{\sigma}^* : C_i^\infty(\Sigma) \to C^\infty(T\mathbf{R}^3/S^1)$ is an isomorphism because $\overline{\sigma}^* = (\rho^*)^{-1}\circ\sigma^*$ and both σ^* and ρ^* are isomorphisms. The map

$$\mathbf{R}^6 \to \mathbf{R}^6 : (\xi,\eta) \mapsto (\sigma_1,\ldots,\sigma_6) = \big(\eta_1,\eta_2,\xi_4-\xi_3+\eta_2^2,\xi_1,\xi_4+\xi_3,\xi_2\big)$$

with inverse

$$\mathbf{R}^6 \to \mathbf{R}^6 : (\sigma_1,\ldots,\sigma_6) \mapsto (\xi,\eta) = \big(\sigma_4,\sigma_6,\tfrac{1}{2}(\sigma_5-\sigma_3+\sigma_2^2),\tfrac{1}{2}(\sigma_5+\sigma_3+\sigma_2^2),\sigma_1,\sigma_1\big)$$

is a homeomorphism of $W \times \mathbf{R}^2$ onto Σ. Precomposing with the homeomorphism $\overline{\varsigma}$ ((2.4)) shows that $\overline{\sigma}$ is a homeomorphism. Thus we have proved

Claim: The differential spaces $(\Sigma, C_i^\infty(\Sigma))$ and $(T\mathbf{R}^3/S^1, C^\infty(T\mathbf{R}^3/S^1))$ are diffeomorphic via the diffeomorphism $\overline{\sigma}$.

For every $j \in \mathbf{R}$ the semialgebraic variety V_j of Σ is defined by (17a) and (17c), where $\Sigma \subseteq \mathbf{R}^6$ is defined by (17b). A function f on V_j is *smooth* if and only if there is a smooth function F on $\mathbf{R}^6$ such that $f = F|V_j$. Hence $(V_j, C_i^\infty(V_j))$ is a differential space, which is subcartesian and locally compact because V_j is a closed subset of $\mathbf{R}^6$. □

The mapping $\nu_j : P_j \subseteq \mathbf{R}^3 \to V_j \subseteq \mathbf{R}^6$ (20b) with inverse $\mu|V_j : V_j \subseteq \mathbf{R}^6 \to P_j \subseteq \mathbf{R}^3$ (20a) is a homeomorphism. Define the space $C^\infty(P_j)$ of smooth functions on P_j to be $\nu_j^*(C^\infty(V_j))$. Then the linear map $\nu_j^* : C_i^\infty(V_j) \to C^\infty(P_j)$ is surjective. It is also injective, because the mapping ν_j is surjective. Therefore ν_j^* is an isomorphism. We have proved

Claim: For every $j \in \mathbf{R}$ the differential spaces $(V_j, C_i^\infty(V_j))$ and $(P_j, C^\infty(P_j))$ are diffeomorphic via the diffeomorphism ν_j.

Corollary: The differential space $(P_j, C^\infty(P_j))$ is locally compact and subcartesian.

▷ Another way of defining the space of smooth functions on P_j is to say that $f \in C_\dagger^\infty(P_j)$ if and only if there is a smooth function F on $\mathbf{R}^3$ with coordinates σ_i, $1 \le i \le 3$ such that $f = F|P_j$.

(2.7) **Proof**: Since the mapping $\mu|V_j$ (20a) is injective, it follows that the linear mapping $(\mu|V_j)^* : C^\infty_\dagger(P_j) \rightarrow C^\infty_i(V_j)$ is surjective. To see this let $g \in C^\infty_\dagger(P_j)$. Then $(\mu|V_j)^* g$ is a smooth function on V_j, because the mapping $\mu|V_j$ is the inverse of the mapping ν_j and $\nu_j^*\big((\mu|V_j)^* g\big) = \big((\mu|V_j)\circ\nu_j\big)^* g = g$. The linear mapping $(\mu|V_j)^*$ is injective, because the mapping $\mu|V_j$ is surjective. Therefore $(\mu|V_j)^*$ is an isomorphism, which is the inverse of ν_j. Consequently,

$$C^\infty_\dagger(P_j) = \nu_j^*\big((\mu|V_j)^* C^\infty_\dagger(P_j)\big) = \nu_j^*(C^\infty_i(V_j)) = C^\infty(P_j). \qquad \square$$

2.4 Poisson brackets on $C^\infty(P_j)$

In this subsection we construct a *Poisson bracket* $\{\, ,\, \}_{P_j}$ *on the space of smooth functions* $C^\infty(P_j)$ of the reduced space $(P_j, C^\infty(P_j))$.

We start by noting that the space $C^\infty(T\mathbf{R}^3/S^1)$ of smooth S^1-invariant functions on the
▷ symplectic manifold $(T\mathbf{R}^3, \omega)$ is a Poisson subalgebra of the Poisson algebra $\big(C^\infty(T\mathbf{R}^3), \{\, ,\, \}_{T\mathbf{R}^3}, \cdot\big)$ of smooth functions on $T\mathbf{R}^3$.

(2.8) **Proof**: If $f, g \in C^\infty(T\mathbf{R}^3)^{S^1}$, then applying the S^1-action Φ (7) for every $t \in S^1$ we get

$$\Phi_t^*\{f,g\}_{T\mathbf{R}^3} = \Phi_t^*\big(L_{X_g} f\big) = L_{\Phi_t^* X_g}\Phi_t^* f = L_{X_{\Phi_t^* g}}\Phi_t^* f = \{\Phi_t^* f, \Phi_t^* g\}_{T\mathbf{R}^3} = \{f,g\}_{T\mathbf{R}^3}. \quad \square$$

Using the invariants $\{\sigma_1, \ldots, \sigma_6\}$ (16) the structure matrix $\mathscr{W}_{C^\infty(T\mathbf{R}^3)^{S^1}}$ for the Poisson bracket $\{\, ,\, \}_{T\mathbf{R}^3}$ on $C^\infty(T\mathbf{R}^3)^{S^1}$ is

$\{A,B\}_{T\mathbf{R}^3}$	σ_1	σ_2	σ_3	σ_4	σ_5	σ_6	B
σ_1	0	1	$2\sigma_2$	0	0	0	
σ_2	-1	0	0	0	0	0	
σ_3	$-2\sigma_2$	0	0	$2(\sigma_2^2 - \sigma_3)$	$-4\sigma_4$	0	
σ_4	0	0	$-2(\sigma_2^2 - \sigma_3)$	0	$-2\sigma_5$	0	
σ_5	0	0	$4\sigma_4$	$2\sigma_5$	0	0	
σ_6	0	0	0	0	0	0	
A							

Table 2.4.1. The structure matrix $\mathscr{W}_{C^\infty(T\mathbf{R}^3)^{S^1}}$ for $\{\, ,\, \}_{T\mathbf{R}^3}$ on $C^\infty(T\mathbf{R}^3)^{S^1}$.

Consider $(\sigma_1, \ldots, \sigma_6)$ to be coordinates on $\mathbf{R}^6$. Then the space $C^\infty(\mathbf{R}^6)$ of smooth functions on $\mathbf{R}^6$ has a Poisson bracket $\{\, ,\, \}_{\mathbf{R}^6}$ whose structure matrix $\mathscr{W}_{C^\infty(\mathbf{R}^6)}$ is given by table 2.4.1. From table 2.4.1 we see that the function $C_1 = \sigma_4^2 + \sigma_6^2 - \sigma_5(\sigma_3 - \sigma_2^2)$ is a Casimir element of the Poisson algebra $(C^\infty(\mathbf{R}^6), \{\, ,\, \}_{\mathbf{R}^6}, \cdot)$, that is, $\{C_1, f\}_{\mathbf{R}^6} = 0$ for every $f \in C^\infty(\mathbf{R}^6)$. Since the semialgebraic variety Σ is defined by $C_1 = 0$ and the inequalities $\sigma_3 - \sigma_2^2 \geq 0$ & $\sigma_3 \geq 0$, the structure matrix $\mathscr{W}_{C^\infty_i(\Sigma)}$ is equal to the structure matrix $\mathscr{W}_{C^\infty(\mathbf{R}^6)}$.

Because $TS^2/S^1 \subseteq T\mathbf{R}^3/S^1$, the semialgebraic variety V defined by (17a) and (17b) is a subvariety of Σ. The Poisson bracket $\{\, ,\, \}_V$ on $C^\infty(V)$ may be computed using the

Dirac prescription, since $\big(\{\sigma_5+\sigma_1^2, \sigma_4+\sigma_1\sigma_2\}_{\mathbf{R}^6}\big)|V = 2$. In particular, for every $f,g \in C^\infty(\mathbf{R}^6)$ by definition $\{f|V, g|V\}_V = \big(\{f^*, g^*\}_{\mathbf{R}^6}\big)|V$, where for $h = f$ or g we have

$$h^* = h - \tfrac{1}{2}(\sigma_5+\sigma_1^2-1)\{h, \sigma_4+\sigma_1\sigma_2\}_{\mathbf{R}^6} + \tfrac{1}{2}(\sigma_4+\sigma_1\sigma_2)\{h, \sigma_5+\sigma_1^2-1\}_{\mathbf{R}^6}.$$

Therefore the structure matrix $\mathscr{W}_{C^\infty(V)}$ for the Poisson bracket $\{\,,\,\}_V$ on $C^\infty(V)$ is given in table 2.4.2.

$\{A,B\}_V$	σ_1	σ_2	σ_3	σ_6	B
σ_1	0	$1-\sigma_1^2$	$2\sigma_2$	0	
σ_2	$-(1-\sigma_1^2)$	0	$-2\sigma_1\sigma_3$	0	
σ_3	$-2\sigma_2$	$2\sigma_1\sigma_2$	0	0	
σ_6	0	0	0	0	
A					

Table 2.4.2. The structure matrix $\mathscr{W}_{C^\infty(V)}$ for $\{\,,\,\}_V$ on $C^\infty(V)$.

The reduced space V_j is the semialgebraic subvariety of V defined by $\sigma_6 = j$. As $C_2 = \sigma_6$ is a Casimir element of the Poisson algebra $(C^\infty(V), \{\,,\,\}_V, \cdot)$, the Poisson bracket $\{\,,\,\}_{V_j}$ on $C^\infty(V_j)$ is the restriction of the Poisson bracket $\{\,,\,\}_V$ to V_j. Consequently, the structure matrix $\mathscr{W}_{C^\infty(V_j)}$ of the Poisson bracket $\{\,,\,\}_{V_j}$ on $C^\infty(V_j)$ is equal to the structure matrix $\mathscr{W}_{C^\infty(V)}$ with its last row and column deleted. As V_j is diffeomorphic to the singular reduced space P_j by the diffeomorphism $\nu_j|V_j$ (20b), the structure matrix $\mathscr{W}_{C^\infty(P_j)}$ of the Poisson bracket $\{\,,\,\}_{P_j}$ on $C^\infty(P_j)$ is the same as the structure matrix $\mathscr{W}_{C^\infty(V_j)}$.

It is easier to calculate in the ambient space $\mathbf{R}^3$ with coordinates $(\sigma_1, \sigma_2, \sigma_3)$ than on the reduced space P_j. On $C^\infty(\mathbf{R}^3)$ define a Poisson bracket $\{\,,\,\}_{\mathbf{R}^3}$ whose structure matrix is given in table 2.4.3.

$\{A,B\}_{\mathbf{R}^3}$	σ_1	σ_2	σ_3	B
σ_1	0	$1-\sigma_1^2$	$2\sigma_2$	
σ_2	$-(1-\sigma_1^2)$	0	$-2\sigma_1\sigma_3$	
σ_3	$-2\sigma_2$	$2\sigma_1\sigma_3$	0	
A				

Table 2.4.3. The structure matrix $\mathscr{W}_{C^\infty(\mathbf{R}^3)}$ for $\{\,,\}_{\mathbf{R}^3}$ on $C^\infty(\mathbf{R}^3)$.

An inspection of table 2.4.3 shows that

$$\{\sigma_i, \sigma_j\}_{\mathbf{R}^3} = \sum_{k=1}^{3} \varepsilon_{ijk} \frac{\partial C}{\partial \sigma_k},$$

where $C = \sigma_3(1-\sigma_1^2) - \sigma_2^2 - j^2$. Note that $C = 0$ is the defining equation of the reduced space P_j. Since C is a Casimir element of the Poisson algebra $(C^\infty(\mathbf{R}^3), \{\,,\,\}_{\mathbf{R}^3}, \cdot)$, the Poisson bracket $\{\,,\,\}_{P_j}$ on $C^\infty(P_j)$ is obtained by restricting $\{\,,\,\}_{\mathbf{R}^3}$ to P_j. Thus the structure matrix $\mathscr{W}_{C^\infty(P_j)}$ of the Poisson bracket $\{\,,\,\}_{P_j}$ on $C^\infty(P_j)$ is equal to the structure matrix

$\mathscr{W}_{C^\infty(\mathbf{R}^3)}$. From the definition of the Poisson bracket $\{\, ,\, \}_{\mathbf{R}^3}$ it follows that

$$\{f,g\}_{\mathbf{R}^3} = \sum_{i,j=1}^3 \frac{\partial f}{\partial \sigma_i}\frac{\partial g}{\partial \sigma_j}\{\sigma_i,\sigma_j\}_{\mathbf{R}^3} = \sum_{k=1}^3 \Big(\sum_{i,j=1}^3 \varepsilon_{ijk}\frac{\partial f}{\partial \sigma_i}\frac{\partial g}{\partial \sigma_j}\Big)\frac{\partial C}{\partial \sigma_k} = \langle \nabla f \times \nabla g, \nabla C\rangle,$$

for every $f,g \in C^\infty(\mathbf{R}^3)$. Here $\langle\, ,\, \rangle$ is the Euclidean inner product on $\mathbf{R}^3$ and ∇h is the gradient of $h \in C^\infty(\mathbf{R}^3)$.

2.5 Dynamics on the reduced space P_j

In this subsection we show that after removing the S^1-symmetry, the spherical pendulum gives rise to a *reduced Hamiltonian system* on the reduced phase space $\big(P_j, C^\infty(P_j)\big)$.

Because the Hamiltonian $H|TS^2$ (3) of the spherical pendulum is invariant under the S^1-action $\Phi|(S^1 \times TS^2)$, it induces the function

$$H_j : \mathbf{R}^3 \to \mathbf{R} : (\sigma_1,\sigma_2,\sigma_3) \mapsto \tfrac{1}{2}\sigma_3 + \sigma_1, \tag{21}$$

whose restriction to the reduced space P_j is the *reduced Hamiltonian*. Note that $H_j|P_j \in C^\infty(P_j)$. For every $f \in C^\infty(\mathbf{R}^3)$ the *Hamiltonian derivation*

$$-\mathrm{ad}_{H_j|P_j} : C^\infty(P_j) \to C^\infty(P_j) : f|P_j \mapsto \{f|P_j, H_j|P_j\}_{P_j} = \big(\{f,H_j\}_{\mathbf{R}^3}\big)|P_j,$$

which governs the reduced dynamics, has integral curves which satisfy

$$\frac{\mathrm{d}(f|P_j)}{\mathrm{d}t} = \{f|P_j, H_j|P_j\}_{P_j}. \tag{22}$$

Because the Hamiltonian vector field $X_{H|TS^2}$ of the spherical pendulum has a local flow and the differential space $(P_j, C^\infty(P_j))$ is locally compact and subcartesian, it follows that the derivation $-\mathrm{ad}_{H_j|P_j}$ of $C^\infty(P_j)$ has a local flow, which is a local one parameter group of local diffeomorphisms of P_j, see chapter VII §4. A calculation shows that

$$-\mathrm{ad}_{H_j|P_j} = -(\mathrm{ad}_{H_j})|P_j = \big(\sigma_2\frac{\partial}{\partial \sigma_1} - ((1-\sigma_1^2)+\sigma_1\sigma_3)\frac{\partial}{\partial \sigma_2} - 2\sigma_2\frac{\partial}{\partial \sigma_3}\big)|P_j.$$

The integral curves of the vector field ad_{H_j} on $\mathbf{R}^3$ satisfy

$$\begin{aligned}\frac{\mathrm{d}\sigma_1}{\mathrm{d}t} &= \sigma_2\\ \frac{\mathrm{d}\sigma_2}{\mathrm{d}t} &= -(1-\sigma_1^2) - \sigma_1\sigma_3\\ \frac{\mathrm{d}\sigma_3}{\mathrm{d}t} &= -2\sigma_2,\end{aligned} \tag{23}$$

which leaves the reduced space P_j invariant.

3 The energy momentum mapping

In this section we study the qualitative properties of the *energy momentum mapping*

$$\mathscr{EM} : TS^2 \subseteq T\mathbf{R}^3 \to \mathbf{R}^2 : (x,y) \mapsto (H,J) = \big(\tfrac{1}{2}(y_1^2+y_2^2+y_3^2)+x_3,\, x_1y_2 - x_2y_1\big) \tag{24}$$

of the spherical pendulum. In particular, we will determine its set of critical values, its range, the topology of its fibers, and how these fibers fit together.

3.1 Critical points of $\mathscr{E}\mathscr{M}$

We begin by finding the set *CP* of *critical points* of the energy momentum map $\mathscr{E}\mathscr{M}$, that is, the set of points where the derivative of $\mathscr{E}\mathscr{M}$ has rank < 2.

▷ First we show that the phase space $TS^2 \subseteq T\mathbf{R}^3$ of the spherical pendulum, defined by

$$F_1(x,y) = x_1^2 + x_2^2 + x_3^2 - 1 = 0 \qquad (25a)$$
$$F_2(x,y) = x_1y_1 + x_2y_2 + x_3y_3 = 0 \qquad (25b)$$

is a submanifold of $T\mathbf{R}^3$.

(3.1) **Proof**: The derivative

$$\begin{pmatrix} \mathrm{d}F_1(x,y) \\ \mathrm{d}F_2(x,y) \end{pmatrix} = \begin{pmatrix} 2x_1 & 2x_2 & 2x_3 & 0 & 0 & 0 \\ y_1 & y_2 & y_3 & x_1 & x_2 & x_2 \end{pmatrix}$$

has rank 2 on TS^2 because the minors $[1;4] = 2x^2$, $[2;5] = 2x_2^2$, and $[3;6] = 2x_3^2$ do not all vanish since $x_1^2 + x_2^2 + x_3^2 = 1$. Thus TS^2 is a submanifold of $T\mathbf{R}^3$ with tangent space $T_{(x,y)}(TS^2) = \ker\begin{pmatrix} \mathrm{d}F_1(x,y) \\ \mathrm{d}F_2(x,y) \end{pmatrix}$ for every $(x,y) \in TS^2$. □

Since

$$D(\mathscr{E}\mathscr{M}|TS^2)(x,y) = \begin{pmatrix} \mathrm{d}H(x,y) \\ \mathrm{d}J(x,y) \end{pmatrix} |T_{(x,y)}(TS^2),$$

the rank of $D(\mathscr{E}\mathscr{M}|TS^2) = 0$ if and only if for every $(x,y) \in TS^2$ we have

$$\mathrm{d}H(x,y)|T_{(x,y)}(TS^2) = 0 \quad \text{and} \quad \mathrm{d}J(x,y)|T_{(x,y)}(TS^2) = 0. \qquad (26)$$

Using the Euclidean inner product $\langle\, ,\,\rangle$ on $T\mathbf{R}^3 = \mathbf{R}^6$, we see that the first equation in (26) is equivalent to $\mathrm{d}H(x,y) \perp \ker\begin{pmatrix} \mathrm{d}F_1(x,y) \\ \mathrm{d}F_2(x,y) \end{pmatrix}$. But $\ker\begin{pmatrix} \mathrm{d}F_1(x,y) \\ \mathrm{d}F_2(x,y) \end{pmatrix}$ is equal to

$$\mathrm{span}\{\mathrm{d}F_1(x,y)\}^{\perp} \cap \mathrm{span}\{\mathrm{d}F_2(x,y)\}^{\perp} = \big(\mathrm{span}\{\mathrm{d}F_1(x,y)\} + \mathrm{span}\{\mathrm{d}F_2(x,y)\}\big)^{\perp}.$$

So $\mathrm{d}H(x,y) \in \mathrm{span}\{\mathrm{d}F_1(x,y), \mathrm{d}F_2(x,y)\}$, that is,

$$(0,0,1,y_1,y_2,y_3) \in \mathrm{span}\{2(x_1,x_2,x_3,0,0,0),\, (y_1,y_2,y_3,x_1,x_2,x_3)\}. \qquad (27a)$$

Similarly, the second equation in (26) is equivalent to

$$(y_2,-y_1,0,-x_2,x_1,0) \in \mathrm{span}\{2(x_1,x_2,x_3,0,0,0),\, (y_1,y_2,y_3,x_1,x_2,x_3)\}. \qquad (27b)$$

Equation (27b) holds if and only if there are real numbers λ and μ such that

$$(y_2,-y_1,0,-x_2,x_1,0) = 2\lambda(x_1,x_2,x_3,0,0,0) + \mu(y_1,y_2,y_3,x_1,x_2,x_3), \qquad (28)$$

that is,

$$\begin{array}{lllllll} 0 & = 2\lambda x_1 + \mu y_1 & (30a) & \quad & y_1 & = \mu x_1 & (30d) \\ 0 & = 2\lambda x_2 + \mu y_2 & (30b) & & y_2 & = \mu x_2 & (30e) \\ 1 & = 2\lambda x_3 + \mu y_3 & (30c) & & y_3 & = \mu x_3 & (30f)\,. \end{array}$$

Then (30d) – (30f), (25a), and (25b) give

$$0 = x_1 y_1 + x_2 y_2 + x_2 y_3 = \mu(x_1^2 + x_2^2 + x_3^2) = \mu.$$

Therefore equations (30d) – (30f) become $y_1 = y_2 = y_3 = 0$. If $\lambda = 0$, then the right hand side of equation (28) is the zero vector, whereas the right hand side is a nonzero vector. This is a contradiction. Hence $\lambda \neq 0$. Therefore equations (30a) and (30b) give $x_1 = x_2 = 0$. So $x_3 = \pm 1$, since (25a) holds. Thus $p_\pm = (0,0,\pm 1,0,0,0) \in TS^2$ are the only critical points of $J|TS^2$. Since (27a) holds at $p_\pm$, we have $\mathrm{d}H(p_\pm)|T_{p_\pm}(TS^2) = 0$. Thus $p_\pm$ are the only points on TS^2 where $D(\mathscr{E}\mathscr{M}|TS^2)$ has rank 0.

The derivative $D(\mathscr{E}\mathscr{M}|TS^2)(x,y)$ has rank 1 at $(x,y) \in TS^2$ if and only if $(x,y) \neq p_\pm$ and for some real number λ we have $0 = \big(\mathrm{d}J(x,y) + \lambda\, \mathrm{d}H(x,y)\big)|T_{(x,y)}(TS^2)$, that is, $\mathrm{d}J(x,y) + \lambda\, \mathrm{d}H(x,y) \in \mathrm{span}\{\mathrm{d}F_1(x,y), \mathrm{d}F_2(x,y)\}$. So for some $\alpha, \beta \in \mathbf{R}$ we have

$$(y_2, -y_1, 0, -x_2, x_1, 0) + \lambda(0,0,1,y_1,y_2,y_3) = 2\alpha(x_1,x_2,x_3,0,0,0) + \beta(y_1,y_2,y_3,x_1,x_2,x_3),$$

that is,

$$\begin{array}{rlll@{\qquad}rll}
y_2 & = 2\alpha x_1 + \beta y_1 & (31a) & & -x_2 + \lambda y_1 & = \beta x_1 & (31d) \\
-y_1 & = 2\alpha x_2 + \beta y_2 & (31b) & & x_1 + \lambda y_2 & = \beta x_2 & (31e) \\
\lambda & = 2\alpha x_3 + \beta y_3 & (31c) & & \lambda y_3 & = \beta x_3 & (31f)\,.
\end{array}$$

Then

$$\beta = \beta(x_1^2 + x_2^2 + x_3^2) = x_1(-x_2 + \lambda y_1) + x_2(x_1 + \lambda y_2) + x_3(\lambda y_3) = \lambda(x_1 y_1 + x_2 y_2 + x_3 y_3) = 0.$$

So (31a) – (31f) become

$$\begin{array}{rlll@{\qquad}rll}
y_2 & = 2\alpha x_1 & (32a) & & -x_2 + \lambda y_1 & = 0 & (32d) \\
-y_1 & = 2\alpha x_2 & (32b) & & x_1 + \lambda y_2 & = 0 & (32e) \\
\lambda & = 2\alpha x_3 & (32c) & & \lambda y_3 & = 0 & (32f)\,.
\end{array}$$

Suppose that $\lambda = 0$. Then equations (32d) and (32e) give $x_1 = x_2 = 0$. Thus $x_3 = \pm 1$, which using (32c) implies $\alpha = 0$. Thus equations (32a) and (32b) give $y_1 = y_2 = 0$. So $0 = x_1 y_1 + x_2 y_2 + x_3 y_3 = \pm y_3$. Thus $p_\pm = (0,0,\pm 1,0,0,0)$ solves (31a) – (31f). But this is excluded by hypothesis. Therefore $\lambda \neq 0$. Hence $y_3 = 0$ and

$$\begin{cases} y_1 & = \lambda^{-1} x_2 \\ y_2 & = \lambda^{-1} x_1. \end{cases} \tag{33a}$$

If $\alpha = 0$, then equation (32c) gives $\lambda = 0$, which is a contradiction. Therefore $\alpha \neq 0$. So $x_3 = \frac{\lambda}{2\alpha}$ and

$$\begin{cases} y_1 & = -2\alpha x_2 \\ y_2 & = 2\alpha x_1. \end{cases} \tag{33b}$$

Note that $x_3 \neq \pm 1$, for if $x_3 = \pm 1$, then a short argument shows that $(x,y) = p_\pm$. But this is excluded by hypothesis. From equations (33a) and (33b) we get

$$(2\alpha + \lambda^{-1})x_1 = 0 = (2\alpha + \lambda^{-1})x_2. \tag{34}$$

If $2\alpha \neq -\lambda^{-1}$ then (34) gives $x_1 = x_2 = 0$. But this implies that $(x,y) = p_\pm$, which is excluded. Therefore $2\alpha = -\lambda^{-1}$. So $x_3 = -\lambda^2 < 0$ and

$$\mathrm{RE} = \left\{ \left(x_1, x_2, -\lambda^2, \lambda^{-1}x_2, -\lambda^{-1}x_1, 0\right) \in TS^2 \,\middle|\, x_1^2 + x_2^2 = 1 - \lambda^4 \ \&\ 0 < \lambda^2 < 1 \right\} \quad (35)$$

is the subset of $TS^2 \setminus \{p_\pm\}$ where $D(\mathscr{E}\mathscr{M}|TS^2)$ has rank 1. In other words, $\mathrm{RE} = \{(x,y) \in TS^2 \setminus \{p_\pm\} \big| X_{H|TS^2}(x,y) = \lambda X_{J|TS^2}(x,y)$ for some $\lambda \in \mathbf{R} \setminus \{0\}\}$, that is, RE is the collection of orbits of the vector field $X_{H|TS^2}$ each of which is an orbit of the vector field $X_{J|TS^2}$ that generates the S^1 symmetry of the spherical pendulum. In other words, RE is the set of *relative equilibria*. RE has two connected components: $\mathrm{RE}_+ = \{(x,y) \in \mathrm{RE} \big| \lambda > 0\}$ and $\mathrm{RE}_- = \{(x,y) \in \mathrm{RE} \big| \lambda < 0\}$.

Lemma: On RE_+ we have $J|TS^2 < 0$; while on RE_- we have $J|TS^2 > 0$.

(3.2) **Proof**: Suppose that $(x,y) \in \mathrm{RE}_+$. Using (35) we get $J|TS^2(x,y) = -\lambda^{-1}(x_1^2 + x_2^2) = -\lambda^{-1}(1 - \lambda^4) < 0$, since $0 < \lambda^2 < 1$ and $\lambda > 0$. Similarly, $(J|TS^2)|\mathrm{RE}_- > 0$. □

Consequently, $D(\mathscr{E}\mathscr{M}|TS^2)$ has rank 2 on $TS^2 \setminus (\mathrm{RE} \cup \{p_\pm\})$. Thus we have proved

Claim: The set of critical points CP of the energy momentum map $\mathscr{E}\mathscr{M}|TS^2$ of the spherical pendulum is $\mathrm{RE} \cup \{p_\pm\}$.

3.2 Critical values of $\mathscr{E}\mathscr{M}$

In this subsection we parametrize the set CV of *critical values* of the energy momentum mapping $\mathscr{E}\mathscr{M}$ of the spherical pendulum. By definition CV is the image of the set of critical points CP under $\mathscr{E}\mathscr{M}$.

We give another description of CV. We show that

Claim: The set of critical values of $\mathscr{E}\mathscr{M}$ is the set $\{\Delta = 0\}$, where Δ is the *discriminant* of the polynomial P (39), that is, $\{(h,j) \in \mathbf{R}^2 \big| P$ has a multiple root in $[-1,1]\}$.

(3.3) **Proof**: Because the reduction map

$$\rho_j : (J|TS^2)^{-1}(j) \subseteq T\mathbf{R}^3 \to P_j \subseteq \mathbf{R}^3 : (x,y) \mapsto \left(\sigma_1, \sigma_2, \sigma_2\right) \quad (36)$$

is a smooth map onto the differential space $(P_j, C^\infty(P_j))$, it follows that $(h,j) \in \mathbf{R}^2$ is a critical value of the energy momentum mapping $\mathscr{E}\mathscr{M}$ if and only if h is a critical value of the reduced Hamiltonian $H_j|P_j$. Since P_j is a singular semialgebraic variety P_j, we must define what we mean by a critical value of H_j. In geometric terms, $(h,j) \in \mathrm{CV}$ if and only if the 2-plane

$$\Pi_h : \tfrac{1}{2}\sigma_3 + \sigma_1 = h \quad (37a)$$

intersects the semialgebraic variety P_j, defined by

$$\sigma_2^2 + j^2 = \sigma_3(1 - \sigma_1^2), \quad |\sigma_1| \le 1 \ \&\ \sigma_3 \ge 0, \quad (37b)$$

at a point $\sigma^0 = \left(\sigma_1^0, \sigma_2^0, \sigma_3^0\right)$ with multiplicity greater than one, see figure 3.2.1. To explain this last phrase, consider the equation

$$Q(\sigma_1, \sigma_2) = \sigma_2^2 + j^2 - 2(h - \sigma_1)(1 - \sigma_1^2) = 0, \quad (38)$$

which is obtained by solving for σ_3 in (37a) and then substituting the result into (37b). The point $\sigma^0 \in P_j$ has multiplicity greater than one if and only if the Taylor polynomial of Q at $(\sigma_1^0, \sigma_2^0) \in [-1,1] \times \mathbf{R}$ has no constant or linear terms. This condition is satisfied if and only if $\sigma_2 = \sigma_2^0 = 0$ and the polynomial

$$P(\sigma_1) = 2(h - \sigma_1)(1 - \sigma_1^2) - j^2 = 2\sigma_1^3 - 2h\sigma_1^2 - 2\sigma_1 + 2h - j^2 \tag{39}$$

has a multiple root $\sigma_1^0 \in [-1,1]$. □

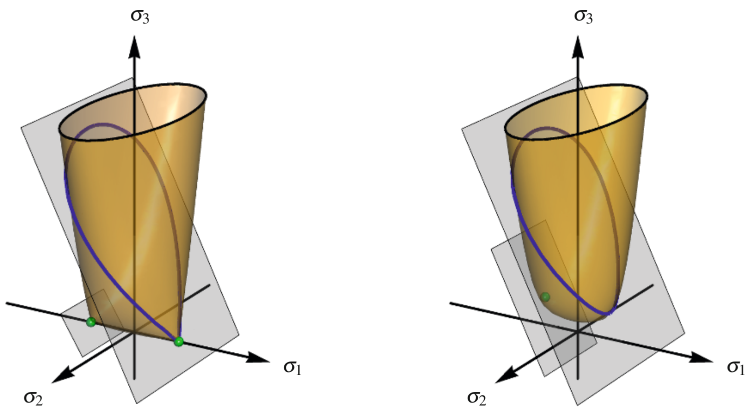


Figure 3.2.1. The critical level sets of the reduced Hamiltonian H_j on the reduced space P_j. In the figure on the left $j = 0$; while in the figure on the right $j \neq 0$.

Another way to formulate the above discussion goes as follows. The point (h, j) is a critical value of $\mathscr{E}\mathscr{M}$ if and only if the line ℓ_h

$$\tfrac{1}{2}\sigma_3 + \sigma_1 = h \tag{40}$$

intersects the curve $\mathscr{F}$

$$\sigma_3(1 - \sigma_1^2) = j^2, \quad |\sigma_1| \leq 1 \;\&\; \sigma_3 \geq 0 \tag{41}$$

at a point $(\sigma_1^0, 0, \sigma_3^0)$ with multiplicity greater than one, see figure 3.2.2. Note that $\mathscr{F}$ is the image of the *fold curve* of the projection map

$$\pi : \mathbf{R}^3 \to \mathbf{R}^2 : (\sigma_1, \sigma_2, \sigma_3) \to (\sigma_1, 0, \sigma_3) \tag{42}$$

restricted to P_j. Geometrically this means that over every point p in the interior of $\pi(P_j)$, the fiber $\pi^{-1}(p)$ consists of two distinct points; while over every point p on $\mathscr{F}$, the fiber $\pi^{-1}(p)$ consists of a point. For the reconstructed level sets of H_j on P_j see figure 3.2.2.

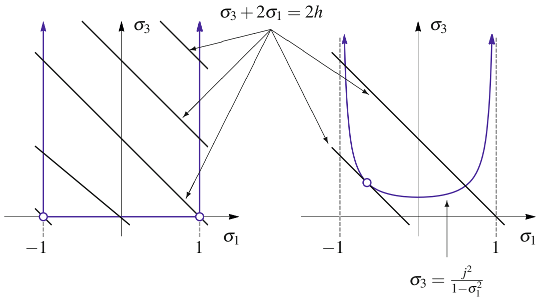


Figure 3.2.1. The critical points of the reduced Hamiltonian H_j on $P_j \cap \{\sigma_2 = 0\}$. In the left figure $j = 0$ while in the right figure $j \neq 0$.

Claim: Let $\Delta = \{(h, j) \in \mathbf{R}^2 \,\big|\, P \text{ has a multiple root in } [-1, 1]\}$ be the discriminant of the polynomial P (39). The *discriminant locus* $\{\Delta = 0\}$ is *parametrized* by

$$\begin{cases} h &= \frac{3}{2}s - \dfrac{1}{2s} \\ j &= \pm\dfrac{1}{\sqrt{-s}}(1-s^2), \end{cases} \qquad \text{for } s \in [-1, 0) \cup \{1\}. \tag{43}$$

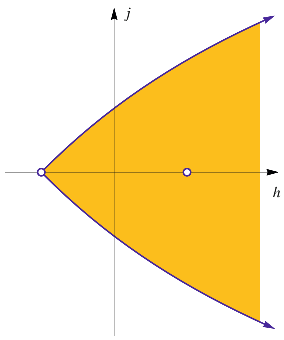


Figure 3.2.3. The discriminant locus $\{\Delta = 0\}$ in the h–j plane. The set of critical values of $\mathscr{E}\mathscr{M}$ is the union of the dark curves, which is $\mathscr{E}\mathscr{M}(RE)$, and points $(-1, 0)$ and $(1, 0)$, which are $\mathscr{E}\mathscr{M}(p_{\mp})$. The shaded region is the set of regular values of $\mathscr{E}\mathscr{M}$.

(3.4) **Proof**: For every $(h, j) \in \Delta$, the polynomial P (39) factors as

$$2(\sigma_1 - s)^2(\sigma_1 - t) = 2\sigma_1^3 - 2(2s+t)\sigma_1^2 + 2(2st + s^2)\sigma_1 - 2ts^2 \tag{44}$$

for some $s \in [-1,1]$ and $t \in \mathbf{R}$. Equating coefficients of (39) and (44) gives

$$2s+t = h \tag{45a}$$

$$2st+s^2 = -1 \tag{45b}$$

$$2ts^2 = j^2-2h. \tag{45c}$$

If $s=0$, then equation (45b) becomes $0=-1$, which is a contradiction. Therefore $s \neq 0$. Eliminating t from (45a) and (45b) gives the expression for h in (43). Eliminating t from equations (45a) and (45c) and then using the first equation in (43) to eliminate h gives the second equation in (43). For j to be real we must have $s \in [-1,0) \cup \{1\}$. □

A *branch* $\mathscr{B}_\pm$ of the discriminant locus $\{\Delta = 0\}$ is parametrized by (43) with the $\pm$ sign fixed and parameter $s \in [-1,0)$.

Claim: $\mathscr{B}_\pm$ is the curve

$$j = \pm \tfrac{2}{9}\big(3-h^2+h\sqrt{h^2+3}\big)\sqrt{h+\sqrt{h^2+3}} = \pm B(h), \quad \text{where } h \geq -1. \tag{46}$$

(3.5) **Proof**: Solving the first equation in (43) for s gives $s = \frac{1}{3}(h-\sqrt{h^2+3})$. Substituting this result into the second equation in (43) gives

$$j = \pm \tfrac{2\sqrt{3}}{9} \frac{1}{\sqrt{\sqrt{h^2+3}-h}}\big(3-h^2+h\sqrt{h^2+3}\big), \tag{47}$$

which simplifies to the expression for $\pm B(h)$ in (46). □

We now verify that figure 3.2.3 is correct. Because $\pm\frac{\mathrm{d}B}{\mathrm{d}h}$ is positive for every $h \geq -1$, the branches intersect at most once. Since $j = \pm B(-1) = 0$, the branches $\mathscr{B}_\pm$ of $\{\Delta = 0\}$ join continuously at $(-1,0)$. Because $\frac{\mathrm{d}j}{\mathrm{d}h}\Big|_{h=-1} = \pm 1$, the branches do *not* join smoothly at $(-1,0)$, but make an angle of $\pi/2$ with each other. The point $(1,0) \in \Delta$, which corresponds to the parameter value $s=1$, does not lie on either of the branches $\mathscr{B}_\pm$. Hence $(1,0)$ is an *isolated* point of Δ and is therefore an *isolated critical value* of the energy momentum mapping $\mathscr{E}\mathscr{M}$.

The critical values $(\pm 1,0)$ of $\mathscr{E}\mathscr{M}$ are special, because they correspond to the critical points $(0,0,\pm 1,0,0,0)$ of $\mathscr{E}\mathscr{M}$ on TS^2, which are fixed points of the S^1-symmetry $\Phi|(S^1 \times TS^2)$ of the spherical pendulum. Under the reduction map ρ_j (36) these points correspond to the singular points $(\pm 1,0,0)$ of the reduced space P_0 and hence are critical points of the reduced Hamiltonian H_0. They do *not* depend on the Hamiltonian, but are a consequence of symmetry *alone*.

3.3 Level sets of the reduced Hamiltonian $H_j|P_j$

Here we describe the qualitative features of the reduced system $(H_j|P_j, P_j, \{\,,\,\}_{P_j})$.

From figure 3.2.2 we can read off the topology of the h-level set of the reduced Hamiltonian $H_j|P_j$ on the reduced space P_j. The results are given in table 3.3.1.

Conditions on (h,j)	Topology of $(H_j\|P_j)^{-1}(h)$
1. $j = \pm B(h), h \ge -1$	a point
2. $\|j\| < B(h), h > -1$ and $(h,j) \ne (1,0)$	a smooth S^1
3. $(1,0)$	a topological S^1 with a conical singular point

Table 3.3.1 The topology of the h-level set of the reduced Hamiltonian $H_j|P_j$ on the reduced phase space P_j.

▷ We now verify the entries in the second column of table 3.3.1.

(3.6) **Proof**:
1. When $|j| = B(h)$ and $h > -1$, the line ℓ_h (40) intersects the curve $\mathscr{F}$ (41) at a nonsingular point $p_{h,j}$. Thus the image of the level set $(H_j|P_j)^{-1}(h))$ under the projection mapping π (42) is $p_{h,j}$ Since $p_{h,j} \in \mathscr{F}$, we infer that $(H_j|P_j)^{-1}(h)$ is a point. So h is the minimum value of $H_j|P_j$ on P_j. Thus the set of (h,j) bounded by the branches $\mathscr{B}_\pm$ and containing the point $(1,0)$ is the range of the energy momentum mapping $\mathscr{EM}$. When $h = -1$, we have $j = \pm B(-1) = 0$. The line ℓ_{-1} meets $\mathscr{F}$ at the point $p_{-1,0} = (-1,0)$. Thus $(H_0|P_0)^{-1}(1)$ is the singular point $(-1,0,0)$ of the reduced space P_0.

2. When $|j| < B(h)$, $-1 < h$ and $(h,j) \ne (1,0)$, the line ℓ_h intersects $\pi(P_j)$ in a closed line segment $L_{h,j}$ whose end points lie on $\mathscr{F}$. Over every point in the interior of $L_{h,j}$ the fiber of the projection map π consists of two distinct points; while over the end points the fiber of π is just a single point. Thus $(H_j|P_j)^{-1}(h) = \pi^{-1}(L_{h,j})$ is a topological circle. Since (h,j) is a regular value of $\mathscr{EM}$, the value h is a regular value for the reduced Hamiltonian $H_j|P_j$. Hence $(H_j|P_j)^{-1}(h)$ is a smooth circle.

3. When $h = 1$, the line ℓ_1 intersects $\pi(P_0)$ in a closed line segment $L_{1,0}$. Thus $(H_0|P_0)^{-1}(1)$ $= \pi^{-1}(L_{1,0})$ is a topological circle. As a semialgebraic variety in $\mathbf{R}^3$, the 1-level set of $H_0|P_0$ is defined by

$$\begin{cases} \sigma_2^2 = \sigma_3(1-\sigma_1^2), & |\sigma_1| \le 1 \;\&\; \sigma_3 \ge 0 \\ 1 = \frac{1}{2}\sigma_3 + \sigma_1. \end{cases} \tag{48}$$

Eliminating σ_3 from (48) yields

$$0 = \sigma_2^2 - 4(1-\sigma_1)^2 + 2(1-\sigma_1)^3, \quad |\sigma_1| \le 1.$$

Hence $(H_0|P_0)^{-1}(1)$ has a nondegenerate tangent cone at the singular point $(1,0,0)$ of P_0. At other points $(H_0|P_0)^{-1}(1)$ is smooth. This completes the verification of the second column of table 3.3.1. □

3.4 Level sets of the energy momentum mapping $\mathscr{EM}$

We are now in position to describe the topology of the *fibers of the energy momentum mapping* $\mathscr{EM}$ of the spherical pendulum. The results are given in table 3.4.1.

Conditions on (h,j)	Topology of $\mathcal{EM}^{-1}(h,j)$
1. $\lvert j\rvert < B(h)$, $-1 < h$ and $(h,j) \neq (1,0)$	T^2, a smooth 2-torus
2. $j = \pm B(h)$, $h > -1$	S^1, a smooth circle
3. $(-1,0)$	a point
4. $(1,0)$	T^*, a 2-torus with a longitudinal circle pinched to a point.

Table 3.4.1 The topology of $\mathcal{EM}^{-1}(h)$.

Before verifying of the entries in the second column of table 3.4.1 we need the

Facts:

1. The reduction mapping

$$\rho_j : J^{-1}(j) \to P_j \subseteq \mathbf{R}^3 : (x,y) \to \big(\sigma_1(x,y), \sigma_2(x,y), \sigma_3(x,y)\big) \qquad (49)$$

is smooth and has fibers $\rho_j^{-1}(p) = \begin{cases} S^1, & \text{if } P_j \text{ is smooth at } p \\ \text{point}, & \text{otherwise.} \end{cases}$

2. For every (h,j) in the image of $\mathcal{EM}$ we have $\mathcal{EM}^{-1}(h,j) = \rho_j^{-1}\big(H_j|P_j)^{-1}(h)\big)$.

(3.7) **Proof** :

1. Let f be a smooth function on the reduced space P_j. Let σ be the Hilbert map of the S^1-action Φ (7) and let $\mu : \mathbf{R}^6 \to \mathbf{R}^3 : (\sigma_1, \ldots, \sigma_6) \mapsto (\sigma_1, \sigma_2, \sigma_3)$ be the projection mapping. Then $(\mu \circ \sigma)^* f$ is a smooth Φ-invariant function on $(J|TS^2)^{-1}(j)$. Hence the linear mapping

$$(\mu \circ \sigma)^* : C^\infty(P_j) \to C^\infty(J^{-1}(j))^{S^1} : f \to \big((\mu \circ \sigma)^* f\big)|J^{-1}(j)$$

is well defined. By construction, the reduction map $\rho_j : J^{-1}(j) \to P_j$ (49) is equal to $(\mu \circ \sigma)|(J|TS^2)^{-1}(j)$. Therefore $(\mu \circ \sigma)^* = \rho_j^*$. By definition ρ_j is a surjective mapping. Therefore ρ_j^* is injective. In other words, $\rho_j^*\big(C^\infty(P_j)\big) \subseteq C^\infty(J^{-1}(j))^{S^1}$. Thus the reduction map ρ_j is a smooth mapping between the differential spaces $\big(P_j, C^\infty(P_j)\big)$ and $\big((J|TS^2)^{-1}(j), C^\infty\big((J|TS^2)^{-1}(j)\big)^{S^1}\big)$, see chapter VII §3.
2. Because the reduction mapping ρ_j is surjective and $\rho_j{}^*(H_j|P_j) = H|J^{-1}(j)$, we have

$$\mathcal{EM}^{-1}(h,j) = H^{-1}(h) \cap J^{-1}(j) = \rho_j^{-1}\big((H_j|P_j)^{-1}(h)\big). \qquad \square$$

If p is a point where the reduced space P_j is nonsingular, then $\mu^{-1}(p)$ is a nonsingular point of V_j and hence is a nonsingular point of the orbit space $W \times \mathbf{R}^2$ (13). From ((2.3)) it follows that $\pi^{-1}(\mu^{-1}(p)) = \pi_j^{-1}(p)$ is a smooth S^1. If p is a singular point of P_j, then $\pi^{-1}(\mu^{-1}(p))$ is a point of $J^{-1}(j)$ where the isotropy group of the action $\Phi_t|J^{-1}(j)$ is *nontrivial*. This can only happen when $j = 0$ and $\pi_0^{-1}(p)$ is a *fixed point* of $\Phi_t|J^{-1}(0)$.

▷ The verification of the second column of table 3.4.1 proceeds as follows.

(3.8) **Proof**:

1. The conditions on (h, j) in the first entry of the first column of table 3.4.1 are equivalent to requiring that (h, j) is a *regular value of the energy momentum mapping*. Hence h is a regular value of the reduced Hamiltonian H_j on P_j. By table 3.3.1 the level set $H_j^{-1}(h)$ is diffeomorphic to S^1. From ((3.5)) it follows that $\mathscr{E}\mathscr{M}^{-1}(h, j) \to H_j^{-1}(h)$ is a smooth bundle with projection map $\rho_j | \mathscr{E}\mathscr{M}^{-1}(h, j)$ and fiber S^1. Since the reduced space P_j is homeomorphic to $\mathbf{R}^2$, the level set $H_j^{-1}(h)$ bounds a 2-disk which is contractible in P_j to a point. Thus the bundle ρ_j is trivial, see chapter VIII §2. In other words $\mathscr{E}\mathscr{M}^{-1}(h, j)$ is a 2-torus $T^2_{h,j}$. By construction $T^2_{h,j}$ is invariant under the flow of the vector fields $X_{H|TS^2}$ and $X_{J|TS^2}$. This completes the verification of the entries in the first row of table 3.4.1.

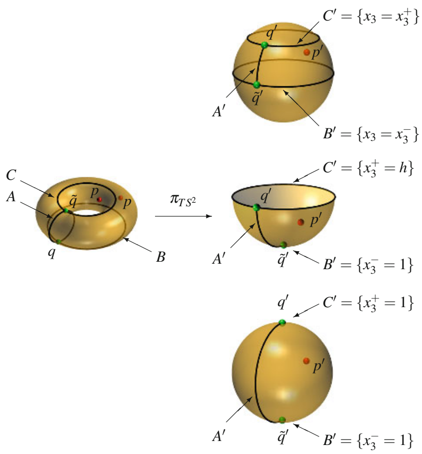


Figure 3.4.1. The image R of the 2-torus $T^2_{h,j}$, where (h, j) is a regular value of $\mathscr{E}\mathscr{M}$, under the bundle projection map π_{TS^2}. In the top figure $j \neq 0$ and $-1 < x_3^- \le x_3 \le x_3^+ < 1$; in the middle figure $j = 0$, $x_3^- = -1$, and $x_3^+ = h < 1$; in the bottom figure $j = 0$, $x_3^- = -1$, $x_3^+ = 1$ and $h > 1$.

Figure 3.4.1 gives a picture of the 2-torus $T^2_{h,j}$ as some sort of fibration over its image under the projection map $\pi_{TS^2} : TS^2 \to S^2 : (x, y) \mapsto x$. Recall that the 2-torus $T^2_{h,j} \subseteq T\mathbf{R}^3$ is defined by

$$x_1^2 + x_2^2 + x_3^2 = 1 \tag{50a}$$
$$x_1 y_1 + x_2 y_2 + x_3 y_3 = 0 \tag{50b}$$
$$\tfrac{1}{2}(y_1^2 + y_2^2 + y_3^2) + x_3 = h \tag{50c}$$
$$x_2 y_1 - x_1 y_2 = j. \tag{50d}$$

▷ When (h, j) is a regular value of the energy momentum map $\mathscr{E}\mathscr{M}$, the image R of the 2-torus $T^2_{h,j} = \mathscr{E}\mathscr{M}^{-1}(h, j)$ under the projection map $\pi_{TS^2} : TS^2 \to S^2$ is given in table 3.4.2.

Conditions on (h,j)	R	Toplogy of R
1. $j \neq 0$	$\{x \in S^2 \big\vert -1 < x_3^- \le x_3 \le x_3^+ < 1\}$	a closed annulus
2. $j = 0$ & $-1 < h < 1$	$\{x \in S^2 \big\vert -1 = x_3^- \le x_3 \le x_3^+ \le h\}$	$\overline{D}^2$, a closed 2-disk
3. $j = 0$ & $h > 1$	$\{x \in S^2 \big\vert -1 = x_3^- \le x_3 \le x_3^+ = 1\}$	S^2

Table 3.4.2 The set R. Here $x_3^{\pm}$ are roots of (50e) in $[-1,1]$.

(3.9) **Proof**:

Substituting (50a) – (50d) into the identity

$$(x_1y_2 - x_2y_1)^2 + (x_1y_1 + x_2y_2)^2 = (x_1^2 + x_2^2)(y_1^2 + y_2^2)$$

and simplifying gives

$$0 \le y_3^2 = P(x_3) = 2(h - x_3)(1 - x_3^2) - j^2 \tag{50e}$$

where $|x_3| \le 1$. Consequently, $x_3 \in [x_3^-, x_3^+]$ where $P(x_3^{\pm}) = 0$, $x_3^- < x_3^+$, and $P(x_3) > 0$ when $x_3 \in (x_3^-, x_3^+)$. Thus the image of the 2-torus $T^2_{h,j}$ under the bundle projection map π_{TS^2} is contained in $\mathrm{R} = \{x \in S^2 \subseteq \mathbf{R}^3 \big\vert x_3^- \le x_3 \le x_3^+\}$.

Suppose that x lies in the interior $\mathrm{int}\,\mathrm{R}$ of R. Then $|x_3^{\pm}| < 1$. Using (50e), which gives $y_3 = \varepsilon\sqrt{P(x_3)}$ with $\varepsilon^2 = 1$, we can solve equations (50b) and (50c) to get

$$\begin{aligned} y_1 &= -(1 - x_3^2)^{-1}\big(jx_2 + \varepsilon x_1x_3\sqrt{P(x_3)}\big) \\ y_2 &= (1 - x_3^2)^{-1}\big(jx_1 - \varepsilon x_2x_3\sqrt{P(x_3)}\big). \end{aligned} \tag{51}$$

▷ This shows that when $x \in \mathrm{int}\,\mathrm{R}$ the fiber $(\pi_{TS^2}|T^2_{h,j})^{-1}(x)$ is the two points $\big(x_1, x_2, x_3.y_1, y_2, \varepsilon\sqrt{P(x_3)}\big)$ in $T^2_{h,j}$ where y_1 and y_2 are given by (51).

We now verify the entries in the third column of table 3.4.2.

1. $j \neq 0$. Then $|x_3| < 1$. So R is a closed annulus in S^2 with boundary $\partial\mathrm{R} = \{(x_1, x_2, x_3^{\pm}) \in S^2\}$, which is the disjoint union of two circles: C', when $x_3 = x_3^+$ and B', when $x_3 = x_3^-$. Because $|x_3^{\pm}| < 1$ and $y_3 = \varepsilon(P(x_3^{\pm}))^{1/2} = 0$, using equations (51) and (50e) we see that

$$(\pi_{TS^2}|T^2_{h,j})^{-1}(\partial R) = \big\{\big(x_1, x_2, x_3^{\pm}, -j\big(1 - (x_3^{\pm})^2\big)^{-1}x_2, j\big(1 - (x_3^{\pm})^2\big)^{-1}x_1, 0\big) \in T^2_{h,j}\big\},$$

which is the disjoint union of two homologous circles: C, when $x_3 = x_3^+$, and B, when $x_3 = x_3^-$, on $T^2_{h,j}$. Each is an orbit of the vector field $X_{J|TS^2}$ on $T^2_{h,j}$. Thus $\pi_{TS^2}(T^2_{h,j}) = \mathrm{R}$.

2. $j = 0$ and $-1 < h < 1$. We have $x_3^- = -1$ and $x_3^+ = h$. Thus R is a closed 2-disk with boundary $C' = \{(x_1, x_2, h) \in S^2\}$. Because $-1 < x_3^+ = h < 1$ and $y_3 = \varepsilon(P(x_3^+))^{1/2} = 0$ we see that

$$C = (\pi_{TS^2}|T^2_{h,0})^{-1}(C') = \{(x_1, x_2, h, 0, 0, 0) \in T^2_{h,0} \big\vert x_1^2 + x_2^2 = 1 - h^2 > 0\},$$

which is an orbit of the vector field $X_{J|TS^2}$ on $T^2_{h,0}$. Thus $\pi_{TS^2}(T^2_{h,j}) = \mathrm{R}$.

3. $j=0$ and $1<h$. We have $x_3^{\pm}=\pm 1$. Thus R is the 2-sphere S^2. Since $(0,0,\pm 1,0,0,0) \in T^2_{h,0}$ and $\pi_{TS^2}(0,0,\pm 1,0,0,0) = (0,0,\pm 1)$, we have shown that $\pi_{TS^2}(T^2_{h,j}) = \mathrm{R}$.

This completes the verification of the entries in the third column of table 3.4.2. □

▷ We now reconstruct the 2-torus $T^2_{h,j}$ from its image R under the mapping $\widetilde{\pi} = \pi_{TS^2}|T^2_{h,j}$.

(3.10) **Proof**:

CASE 1. $j \neq 0$. Then R is a closed annulus with boundary $\partial\mathrm{R}$ two disjoint circles B' and C' on S^2. Suppose that x is a point in the interior $\mathrm{int}\,\mathrm{R}$ of R. Then $\widetilde{\pi}^{-1}(x)$ is two points. If x lies on $\partial\mathrm{R}$, then $\widetilde{\pi}^{-1}(x)$ is a point. In other words, $T^2_{h,j}$ has a fold singularity over $\partial\mathrm{R}$ with fold curve $\widetilde{\pi}^{-1}(\partial\mathrm{R})$. Thus $T^2_{h,j}$ is an S^0-bundle over $\mathrm{int}\,\mathrm{R}$ with S^0 pinched to a point over each point of $\partial\mathrm{R}$. Next we look more closely at the geometry of the mapping $\widetilde{\pi}$. Let A' be the *open* arc $\{(0,(1-x_3^2)^{1/2},x_3) \in S^2\}\big| x_3^- < x_3 < x_3^+\}$ on S^2. Then the closure $\overline{\widetilde{\pi}^{-1}(A')}$ of $\widetilde{\pi}^{-1}(A')$ is a circle A on $T^2_{h,j}$, because for every $x \in A' \subseteq \mathrm{int}\,\mathrm{R}$, the fiber $\widetilde{\pi}^{-1}(x)$ is two points; whereas for $(0,0,x_3^{\pm}) \in (\overline{A'} \setminus A') \subseteq \partial\mathrm{R}$, the fiber $\widetilde{\pi}^{-1}(0,0,x_3^+)$ is the point $q=(0,0,x_3^+,0,0,0)$; while the fiber $\widetilde{\pi}^{-1}(0,0,x_3^-)$ is the point $\widetilde{q}=(0,0,x_3^-,0,0,0)$. Let $B=\widetilde{\pi}^{-1}(B')$ and $C=\widetilde{\pi}^{-1}(C')$. Then B and C are circles on $T^2_{h,j}$, which are homologous. Moreover, either $\{A,B\}$ or $\{A,C\}$ is a basis of $\mathrm{H}_1(T^2_{h,j},\mathbf{Z})$.

CASE 2. $j=0$. This case is more difficult because the geometry of the mapping $\widetilde{\pi}$ is more complicated. Suppose that $-1<h<1$. Let A' be the open arc $\{(0,(1-x_3^2)^{1/2},x_3) \in S^2\big| -1 < x_3 < h\}$ on S^2. For every $x \in A' \subseteq \mathrm{int}\,\mathrm{R}$ the fiber $\widetilde{\pi}^{-1}(x)$ is the two points

$$\{(0,\sqrt{1-x_3^2},x_3,0,-\varepsilon x_3\sqrt{2(h-x_3)},\varepsilon\sqrt{2(h-x_3)(1-x_3^2)})\},$$

where $\varepsilon^2=1$. When $x=(0,\sqrt{1-h^2},h) \in \overline{A'}\setminus A'$, the fiber $\widetilde{\pi}^{-1}(x) \in \overline{\widetilde{\pi}^{-1}(A')} \setminus \widetilde{\pi}^{-1}(A')$ is the point $q=(0,\sqrt{1-h^2},h,0,0,0)$. To find the rest of $\overline{\widetilde{\pi}^{-1}(A')}$ first note that the fiber $\widetilde{\pi}^{-1}(0,0,-1)$ is the circle $B=\{(0,0,-1,y_1,y_2,0) \in T^2_{h,0}\big| y_1^2+y_2^2 = 2(h+1) > 0\}$. Geometrically a point on B is a positive tangent *ray* to S^2 at $B'=(0,0,-1)$ of length $\sqrt{2(h+1)}$. We say that the mapping $\widetilde{\pi}$ *blows up* the point B' to the circle B, because $B=\widetilde{\pi}^{-1}(B')$. Observe that B is homologous to the circle $C=\widetilde{\pi}^{-1}(C')=\{(x_1,x_2,h,0,0,0) \in T^2_{h,0}\big| x_1^2+x_2^2 = 1-h^2 > 0\}$. Now the tangent to the curve $(-1,h) \to S^2 : x_3 \mapsto (0,(1-x_3^2)^{1/2},x_3)$, which parametrizes the arc A', is $(0,-x_3(1-x_3^2)^{-1/2},1)$. The corresponding positive tangent ray of length $\sqrt{2(h+1)}$ to A' is $\big(0,-\sqrt{2(h+1)}x_3,\sqrt{2(h+1)(1-x_3^2)}\big)$. Thus the tangent ray to A' at the point B' is $(0,\sqrt{2(h+1)},0)$, which corresponds to the point $\widetilde{q}=(0,0,-1,0,\sqrt{2(h+1)},0)$ on the circle B. Consequently, $A=\{q,\widetilde{q}\} \cup \widetilde{\pi}^{-1}(A')$ is a circle on $T^2_{h,0}$ which intersects the circle B only at $\widetilde{q}$. Note that $\overline{\widetilde{\pi}^{-1}(A')} = A \cup B$ and that $\{A,B\}$ is a basis for $\mathrm{H}_1(T^2_{h,0},\mathbf{Z})$ as is $\{A,C\}$ since $A \cap C = \{q\}$.

Suppose that $h>1$. If $x \in S^2 \setminus \{(0,0,\pm 1)\} \subseteq \mathrm{int}\,\mathrm{R}$, then the fiber $\widetilde{\pi}^{-1}(x)$ is two points. The fiber $\widetilde{\pi}^{-1}(0,0,-1)$ is the circle $B=\{(0,0,-1,y_1,y_2,0) \in T^2_{h,0}\big| y_1^2+y_2^2 = 2(h-1) > 0\}$; whereas the fiber $\widetilde{\pi}^{-1}(0,0,1)$ is the circle $C=\{(0,0,1,y_1,y_2,0) \in T^2_{h,0}\big| y_1^2+y_2^2 =$

$2(h-1) > 0\}$. Thus the mapping $\widetilde{\pi}$ blows up the points $(0,0,-1)$ and $(0,0,1)$ to the circles B and C, respectively. Consider the open arc A', which is parametrized by

$$(-1,1) \to S^2 \setminus \{(0,0,\pm 1)\} : x_3 \mapsto (0, \sqrt{1-x_3^2}, x_3).$$

Then the tangent ray to A' at $B' = (0,0,-1)$ is $(0, \sqrt{2(h-1)}, 0)$, which corresponds to the point $\widetilde{q} = (0,0,-1,0,\sqrt{2(h-1)},0)$ on B; while tangent ray to A' at $C' = (0,0,1)$ is $(0,-\sqrt{2(h-1)},0)$, which corresponds to the point $q = (0,0,-1,0,-\sqrt{2(h-1)},0)$ on C. Consequently, $A = \{q,\widetilde{q}\} \cup \widetilde{\pi}^{-1}(A')$ is a circle on $T^2_{h,0}$, which intersects B only at $\widetilde{q}$ and C only at q. Note that $\overline{\widetilde{\pi}^{-1}(A')} = A \cup B \cup C$ and that $\{A,B\}$ is a basis for $\mathrm{H}_1(T^2_{h,0}, \mathbf{Z})$ as is $\{A,C\}$ since $A \cap C = \{q\}$.

This completes the reconstruction of the torus $T^2_{h,j}$ from its image R under the bundle projection π_{TS^2}. $\square$

We return to verifying the entries in the second column of table 3.4.1.

2. To verify the second entry we note that the conditions on (h,j) in the first column are precisely those for which the h-level set of the reduced Hamiltonian H_j is a nonsingular point of the reduce space P_j. From ((3.5)) it follows that $\mathscr{EM}^{-1}(h,j)$ is a smooth S^1.

3. When $(h,j) = (-1,0)$ the line ℓ_{-1} (40) intersects the curve $\mathscr{F}$ (41) at the point $(\sigma_1^0, \sigma_3^0) = (-1,0)$. In other words, the 2-plane Π_{-1} (37a) intersects the reduced space P_0 at the singular point $(\sigma_1^0, \sigma_2^0, \sigma_3^0) = (-1,0,0)$. From ((3.5)) it follows that $\mathscr{EM}^{-1}(1,0)$ is the point $(0,0,-1,0,0,0) \in T\mathbf{R}^3$.

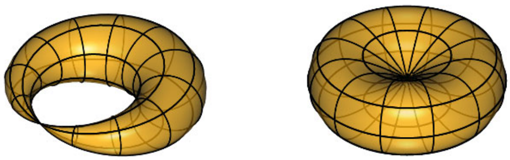


Figure 3.4.2. The pinched 2-torus $T^* = \mathscr{EM}^{-1}(1,0)$. In the left figure the 2-torus is pinched along a longitude; in the right figure the 2-torus is pinched along a merdian.

4. When $(h,j) = (1,0)$ the line ℓ_1 intersects $\rho(P_0)$ in a closed line segment, whose end points lie on the fold curve $\mathscr{F}$. One end point is $\sigma^0 = (\sigma_1^0, \sigma_3^0) = (1,0)$. Thus the 2-plane Π_1 intersects P_0 in a topological circle $H_0^{-1}(1)$ with singular point $p^0 = \rho^{-1}(\sigma^0) = (1,0,0)$. From ((3.5)) it follows that $\rho_0^{-1}(H_0^{-1}(1) \setminus \{p^0\}) \to H_0^{-1}(1) \setminus \{p^0\}$ with projection map $\rho_0 | (H_0^{-1}(1) \setminus \{p^0\})$ is a bundle with fiber S^1. Since $H_0^{-1}(1) \setminus \{p^0\}$ is contractible in P_0, the bundle $\rho_0^{-1}(H_0^{-1}(1) \setminus \{p^0\}) \to H_0^{-1}(1) \setminus \{p^0\}$ is trivial. Thus $\rho_0^{-1}(H_0^{-1}(1) \setminus \{p^0\})$ is topologically a cylinder $S^1 \times \mathbf{R}$. Because p^0 is a singular point of P_0, the fiber $\rho_0^{-1}(p^0)$ is a point. Therefore $\mathscr{EM}^{-1}(1,0)$ is a one point compactification of a cylinder, that is, a cylinder with its ends identified to a point. In other words, it is a 2-torus T^* with a *longitudinal* circle pinched to a point. In ((3.11)) below we show that

$\rho_0^{-1}(p^0) = (0,0,1,0,0,0)$ is a critical point of $H|TS^2$ of Morse index 2. Thus $\rho_0^{-1}(p^0)$ is a conical singular point of $\mathcal{EM}^{-1}(0,1)$.

Another way of describing $\mathcal{EM}^{-1}(1,0)$ is a 2-torus with a *meridial* circle pinched to a point. To see this project $\mathcal{EM}^{-1}(h,0)$ onto the 2-sphere S^2 using the bundle projection π_{TS^2} This gives rise to a 2-disk with boundary $\{x_3 = x_3^+ = h\}$. As $h \nearrow 1$, we get $x_3^+ \nearrow 1$. Hence $\pi_{TS^2}(\mathcal{EM}^{-1}(1,0))$ is S^2. Moreover, for $p \in S^2 \setminus \{(0,0,1)\}$ the fiber $\pi_{TS^2}^{-1}(p)$ consists of two distinct points; while at $(0,0,1)$ the fiber $\pi_{TS^2}^{-1}(0,0,1)$ is a single point. From this information we can reconstruct $\mathcal{EM}^{-1}(1,0)$, see figure 3.4.2. We find that it is a surface of revolution formed by rotating a figure eight about an axis through the crossing point. Topologically this surface is a 2-torus with a meridial circle pinched to a point. The topological equivalence between a torus with a meridial circle pinched to a point and a torus with a longitudinal circle pinched to a point cannot be realized by a homeomorphism of $\mathbf{R}^3$ but can be by a homeomorphism of S^3.

This completes the verification of table 3.4.1 $\square$

▷ Next we determine the topology of the energy surfaces $H^{-1}(h)$ of the spherical pendulum. The results are summarized in table 3.4.3. To verify the entries in the second column of

Conditions	Topology of $H^{-1}(h)$
1. $h = -1$	point
2. $-1 < h < 1$	a smooth 3-sphere, S^3
3. $h = 1$	U, a topological 3-sphere
4. $h > 1$	$\mathbf{RP}^3$, real projective 3-space.

Table 3.4.3 Topology of the level sets $H^{-1}(h)$.

table 3.4.3, we use Morse theory. As a submanifold of $T\mathbf{R}^3$ with coordinates (x,y), the tangent bundle TS^2 of S^2 is defined by

$$F_1(x,y) = x_1^2 + x_2^2 + x_3^2 - 1 = 0 \tag{52a}$$

$$F_2(x,y) = x_1y_1 + x_2y_2 + x_3y_3 = 0. \tag{52b}$$

On $T\mathbf{R}^3$ consider the function $H : T\mathbf{R}^3 \to \mathbf{R} : (x,y) \mapsto \frac{1}{2}(y_1^2 + y_2^2 + y_3^2) + x_3$.
▷ We now show that $H|TS^2$ is a Morse function.

(3.12) **Proof**: Because $H|TS^2$ is proper and is bounded below, it has a critical point $p = (x,y)$. By Lagrange multipliers the critical point p satisfies

$$DH(x,y) + \lambda_1 DF_1(x,y) + \lambda_2 DF_2(x,y) = 0, \text{ and } F_1(x,y) = 0,\ F_2(x,y) = 0. \tag{53}$$

Writing out the first equation in (53) gives

$$\begin{aligned} 2\lambda_1 x_1 + \lambda_2 y_1 &= 0 \quad (54a) & \qquad y_1 + \lambda_2 x_1 &= 0 \quad (54d)\\ 2\lambda_1 x_2 + \lambda_2 y_2 &= 0 \quad (54b) & y_2 + \lambda_2 x_2 &= 0 \quad (54e)\\ 1 + 2\lambda_1 x_3 + \lambda_3 y_3 &= 0 \quad (54c) & y_3 + \lambda_2 x_3 &= 0 \quad (54f) \end{aligned}$$

in addition to (52a) and (52b). Therefore

$$\lambda_2 = \lambda_2(x_1^2+x_2^2+x_3^2) = -(x_1y_1+x_2y_2+x_3y_3) = 0.$$

Hence $y_1 = y_2 = y_3 = 0$. Consequently (54a) – (54c) read

$$2\lambda_1 x_1 = 0 \tag{55a}$$

$$2\lambda_1 x_2 = 0 \tag{55b}$$

$$2\lambda_1 x_3 = -1. \tag{55c}$$

Suppose that $\lambda_1 = 0$. Then (55c) gives $0 = -1$, which is a contradiction. Therefore $\lambda_1 \neq 0$. Hence (55a) and (55b) give $x_1 = x_2 = 0$. From (52a) we obtain $x_3 = \varepsilon$ where $\varepsilon^2 = 1$; while from (55c) we obtain $\lambda_1 = -\frac{1}{2}\varepsilon$. Thus $H|TS^2$ has two critical points $p_\varepsilon = (\varepsilon e_3, 0)$ with Lagrange multipliers $\lambda_1 = \frac{1}{2}\varepsilon$ and $\lambda_2 = 0$.

To show that p_ε is nondegenerate critical point of $H|TS^2$, first note that the tangent space $T_{p_\varepsilon}(TS^2)$ to TS^2 at p_ε is $\ker\begin{pmatrix} DF_1(p_\varepsilon) \\ DF_2(p_\varepsilon)\end{pmatrix} = \ker\begin{pmatrix} 0 & 0 & 2\varepsilon & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \varepsilon \end{pmatrix}$, which is spanned by the vectors $\{e_1, e_2, e_4, e_5\}$. Therefore

$$Q = D^2(H|TS^2)(p_\varepsilon) = \left(D^2H + \lambda_1 D^2F_1 + \lambda_2 D^2F_2\right)\Big|_{T_{p_\varepsilon}(TS^2)}(p_\varepsilon) = \mathrm{diag}(-\varepsilon, -\varepsilon, 1, 1),$$

whereupon the critical points p_ε are nondegenerate. Thus the Morse index of Q is 2 if $\varepsilon = 1$ or 0 if $\varepsilon = -1$. Hence, $p_{-1} = (-e_3, 0)$ is a nondegenerate minimum of H with corresponding minimum value -1, and $p_{+1} = (e_3, 0)$ is a nondegenerate saddle point of index 2 corresponding to the critical value 1. □

▷ We now verify the entries in the second column of table 3.4.3.

(3.11) **Proof**:

1. At the critical value $h = -1$, the level set $(H|TS^2)^{-1}(h)$ is a point.

2. By the Morse lemma, see chapter XI §2, for values of h slightly greater than the minimum value -1, the h-level set of $H|TS^2$ is diffeomorphic to a 3-sphere S^3. Using the Morse isotopy lemma, see chapter XI §3, it follows that the level set $(H|TS^2)^{-1}(h)$ is diffeomorphic to S^3 for every $h \in (-1, 1)$.

4. For $h > 1$ we claim that $(H|TS^2)^{-1}(h)$ is diffeomorphic to the unit tangent S^1 bundle $T_1S^2 \subseteq \mathbf{R}^3 \times \mathbf{R}^3$ of the 2-sphere S^2. Consider the smooth mapping

$$\varphi : (H|TS^2)^{-1}(h) \to T_1S^2 : (x,y) \mapsto (\xi, \eta) = \left(x, \frac{y}{\sqrt{2(h-x_3)}}\right).$$

The level set $(H|TS^2)^{-1}(h) \subseteq \mathbf{R}^3 \times \mathbf{R}^3$ is defined by $\langle x, x\rangle = 1$, $\langle x, y\rangle = 0$, and $h = \frac{1}{2}\langle y, y\rangle + \langle x, e_3\rangle$. A computation shows that $(\xi, \eta) \in \varphi\big((H|TS^2)^{-1}(h)\big)$ satisfies $\langle \xi, \xi\rangle = 1$, $\langle \xi, \eta\rangle = 0$, and $\langle \eta, \eta\rangle = (2h - 2x_3)^{-1}\langle y, y\rangle = 1$. In other words, the image of φ is contained in T_1S^2. Since $\tau : TS^2 \to (H|TS^2)^{-1}(h) : (\xi, \eta) \mapsto \left(\xi, \eta\sqrt{2h - 2\xi_3}\right)$ is a smooth inverse of φ, the mapping φ is a diffeomorphism. Now T_1S^2 is the set of ordered pairs of orthonormal vectors $\xi, \eta \in \mathbf{R}^3$. Extend the ordered pair $\{\xi, \eta\}$ to the positively oriented ordered orthonormal basis $\{\xi, \eta, \xi \times \eta\}$ of $\mathbf{R}^3$. Every such basis may be identified with a rotation of $\mathbf{R}^3$, whose matrix is $\mathrm{col}(\xi, \eta, \xi \times \eta)$. Thus

$$\psi : T_1S^2 \to \mathrm{SO}(3) : (\xi, \eta) \mapsto \mathrm{col}(\xi, \eta, \xi \times \eta)$$

is a smooth map with smooth inverse

$$\sigma : \mathrm{SO}(3) \to T_1S^2 : \mathrm{col}(\xi, \eta, \zeta) \mapsto (\xi, \eta).$$

Hence T_1S^2 is diffeomorphic to $\mathrm{SO}(3)$, which in turn is diffeomorphic to real *projective 3-space* $\mathbf{RP}^3$ by ((1.15)) of chapter III.

3. At the critical value $h = 1$, the level set $(H|TS^2)^{-1}(1)$ is an algebraic subvariety U of $T\mathbf{R}^3$ defined by

$$x_1^2 + x_2^2 + x_3^2 = 1 \tag{56a}$$

$$x_1y_1 + x_2y_2 + x_3y_3 = 0 \tag{56b}$$

$$\tfrac{1}{2}(y_1^2 + y_2^2 + y_3^2) + x_3 = 1 \tag{56c}$$

The variety U is singular only at $p_1 = (0,0,1,0,0,0)$ because the rank of

$$\begin{pmatrix} 2x_1 & 2x_2 & 2x_3 & 0 & 0 & 0 \\ y_1 & y_2 & y_3 & x_1 & x_2 & x_3 \\ 0 & 0 & 1 & y_1 & y_2 & y_3 \end{pmatrix}$$

is < 3 on U only at p_1. Since p_1 is a nondegenerate critical point of $H|TS^2$ with Morse index 2 ((3.11)), from the Morse lemma it follows that there is a neighborhood of p_1 in U which is diffeomorphic to a neighborhood of the vertex 0 of the cone $C = \{(\xi_1, \xi_2, \eta_1, \eta_2) \in \mathbf{R}^4 \big| \xi_1^2 + \xi_2^2 = \eta_1^2 + \eta_2^2\}$. Note that C is a cone on a 2-torus and contains two 2-planes $\{(\xi_1, \xi_2, \varepsilon\xi_1, -\varepsilon\xi_2 \big| (\xi_1, \xi_2) \in \mathbf{R}^2\}$ which intersect transversely at 0. Thus U is a topological manifold, which is smooth except at one point where it has a conical singularity.

We now give a global description of the variety U: first, as a fibration over S^2 with projection mapping $\widehat{\pi} = \pi_{TS^2}|U$. Using (56c) it is straightforward to see that the fiber of $\widehat{\pi}$ over a point in $S^2 \setminus \{(0,0,1)\}$ is an S^1; while over $(0,0,1)$ it is the point p_1. Thus the mapping $\widehat{\pi}$ is proper. Consequently, the variety U is compact. U is also connected, because $\widehat{\pi}$ is a continuous open mapping and its image is S^2, which is connected. Second, we can view U as the disjoint union of two singular closed solid tori whose boundaries are identified. Consider the smooth function $J|U : U \subseteq TS^2 \to \mathbf{R} : (x,y) \mapsto x_1y_2 - x_2y_1$. In ((3.5)) we showed that $(J|U)^{-1}(0) = (J|TS^2)^{-1}(0)$ is a pinched 2-torus T^*. This singular 2-torus T^* is the boundary of the closed singular solid 2-torus $ST_+ = \{(x,y) \in U \big| J(x,y) \ge 0\}$ and is also the boundary of the closed singular solid 2-torus $ST_- = \{(x,y) \in U \big| J(x,y) \le 0\}$. The boundary T_+^* of ST_+ is a 2-torus with a longitudinal circle pinched to a point; while the boundary T_-^* of ST_+ is a 2-torus with a meridial circle pinched to a point, see figure 3.4.3. We now find the map which glues T_+^* to T_-^* so that the closed singular solid tori $ST_\pm$ form the variety U. We think of the singular 2-torus T_+^* as a one point compactification

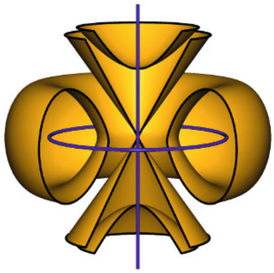


Figure 3.4.3. The variety U in S^3, which we think of as $\mathbf{R}^3$ with a point added at infinity. The upper and lower parts of the solid cone form the singular closed solid torus ST_+, whose boundary T^*_+ is the singular 2-torus with a longitudinal circle pinched to a point, which forms the vertex of the cone. The exterior of the solid cone is again a solid cone which forms the singular closed solid torus ST_-, whose boundary T^*_- is the singular solid torus with meridial circle pinched to a point, which is the vertex of the cone.

of the cylinder $S^1 \times \mathbf{R}$. The 1-dimensional lattice $\mathrm{H}_1(T^*_+, \mathbf{Z})$, which is isomorphic to $\mathbf{Z}$, is formed from the 2-dimensional lattice $\mathbf{Z}^2$ defining the 2-torus $T^2 = \mathbf{R}^2/\mathbf{Z}^2$ by taking its first component $\mathbf{Z} \times \{0\}$. In other words, T^*_+ is the one point compactification of $\mathbf{R}^2/(\mathbf{Z} \times \{0\})$. Similarly, the singular 2-torus T^*_- is the one point compactification of the cylinder $\mathbf{R}^2/(\{0\} \times \mathbf{Z})$. Consider the invertible linear mapping

$$\varphi : \mathbf{Z}^2 \to \mathbf{Z}^2 : \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \end{pmatrix},$$

which sends the lattice $\mathbf{Z} \times \{0\}$ bijectively onto the lattice $\{0\} \times \mathbf{Z}$. Thus the map φ induces a diffeomorphism $\widetilde{\varphi}$ of the cylinder $S^1 \times \mathbf{R}$ onto the cylinder $\mathbf{R} \times S^1$. The map $\widetilde{\varphi}$ extends to a homeomorphism $\overline{\varphi}$ of the one point compactification of $S^1 \times \mathbf{R}$ onto the one point compactification of $\mathbf{R} \times S^1$. In other words, $\overline{\varphi}$ is a homeomorphism of the singular

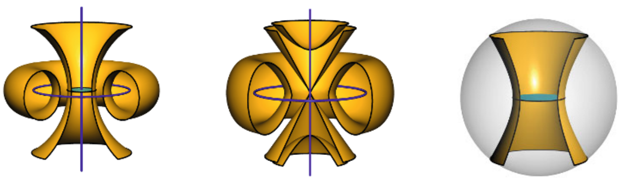


Figure 3.4.4. The bifurcation of the energy surface $(H|TS^2)^{-1}(h)$ of the spherical pendulum as h increases through 1. In the left figure $1 < h < 1$ and $(H|TS^2)^{-1}(h)$ is a smooth 3-sphere; in the middle figure $h = 1$ and $(H|TS^2)^{-1}(h)$ is a topological 3-sphere; in the right figure $h > 1$ and $(H|TS^2)^{-1}(h)$ is a smooth real projective 3-space.

2-torus T^*_+ onto the singular 2-torus T^*_-. So $\overline{\varphi}$ is the desired gluing map. The variety U is homeomorphic to S^3, because every continuous loop in U is contractible to a point, that is, U is simply connected.

In figure 3.4.4 we give a picture of the bifurcation of the energy surfaces of the spherical pendulum as the energy h increases through 1. Each energy surface in figure 3.4.4 is depicted as the union of two closed solid tori. Geometrically, what happens as $h \nearrow 1$ is the circle, bounding the shaded disk in the left figure contracts to the vertex of the cone in the middle figure where $h = 1$. When h increases past 1 the vertex of this cone reappears as a shaded disk in the right figure.

This completes the verification of table 3.4.3. $\square$

We have verified the bifurcation diagram for the energy momentum mapping $\mathscr{EM}$ of the spherical pendulum. From figure 3.4.5 it is clear that for regular values of h the energy level $H^{-1}(h)$ is foliated by 2-tori with two singular S^1 fibers.

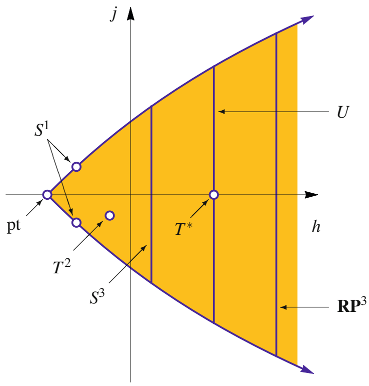


Figure 3.4.5. The bifurcation diagram of the energy momentum mapping $\mathscr{EM}$ of the spherical pendulum.

4 Rotation number and first return time

Suppose that (h, j) is a regular value of $\mathscr{EM}$. In this section we derive formulæ for the rotation number and time of first return to a cross section for the flow of $X_{H|TS^2}$ on the 2-torus $T^2_{h,j}$. We show that the rotation number is a *multivalued* real analytic function on the set of regular values of the energy momentum map, while the first return time is a *single valued* real analytic function.

4.1 Definition of first return time and rotation number

The *first return time* is defined as follows. When (h, j) is a regular value of $\mathscr{EM}$, the differentials $\mathrm{d}(H|TS^2)$, $\mathrm{d}(J|TS^2)$ are linearly independent at every point of $T^2_{h,j} = \mathscr{EM}^{-1}(h, j)$ and therefore so are the vector fields $X_{H|TS^2}$ and $X_{J|TS^2}$. Consider the curve $\mathscr{C}^- = \pi^{-1}_{TS^2}(\{x_3 = x_3^-\})$ on $T^2_{h,j}$, which is the image of the integral curve $t \to \varphi_t^{J|TS^2}(p)$ of $X_{J|TS^2}$ through p. The curve $\mathscr{C}^-$ is a cross section for the flow $\varphi_t^{H|TS^2}$ of $X_{H|TS^2}$ on $T^2_{h,j}$. Note that at every $r \in \mathscr{C}^-$ the vector field $X_{H|TS^2}(r)$ is transverse to $\mathscr{C}^-$. Then observe that the image of the integral curve $\Gamma : t \to \varphi_t^{H|TS^2}(p)$ of $X_{H|TS^2}$ through p under the reduction mapping π_j (49) is a periodic integral curve $t \to \varphi_t^{H_j}(q)$ through $q = \pi_j(p)$ of the reduced vector field X_{H_j} of period $T > 0$. Thus $\pi_j(\varphi_T^H(p)) = \varphi_T^{H_j}(\pi_j(p)) = q$. So $\varphi_T^H(p) \in \mathscr{C}^- = \pi_j^{-1}(q)$. Thus Γ intersects $\mathscr{C}^-$ for the first time at $t = T$. The time $T = T(h, j)$ is called the *first return time*.

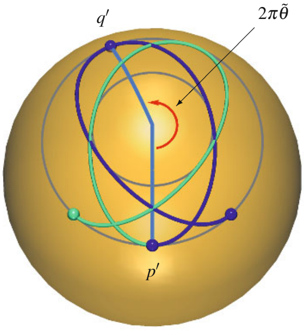


Figure 4.1.1. The rotation number of the image of the integral curve of vector field $X_{H|TS^2}|T^2_{h,l}$ under the bundle projection map π_{TS^2}, which starts at the point $p' = \pi_{TS^2}(p)$ and ends at the point $q' = \pi_{TS^2}(q)$. Both p' and q' lie on the image under π_{TS^2} of an integral curve of $X_{H|TS^2}|T^2_{h,l}$ on $T^2_{h,l}$.

Let $\widetilde{\theta}$ be the smallest positive number such that $\varphi_{2\pi\widetilde{\theta}}^{J|TS^2}(p) = \varphi_T^{H|TS^2}(p)$, see figure 4.1.1. By definition $\widetilde{\theta}$ is the *rotation number* of the flow of $X_{H|TS^2}$ on $T^2_{h,j}$. Because $X_{H|TS^2}$ is invariant under the S^1-action generated by the flow of $X_{J|TS^2}$, the rotation number and the time of first return does not depend on the choice of the point p on $\mathscr{C}^-$. Since $\mathscr{C}^-$ can be an arbitrary integral curve of $X_{J|TS^2}$ on $T^2_{h,j}$, the rotation number and the first return time depend only on (h, j).

We now derive a formula for the rotation number $\widetilde{\theta} = \theta_{h,j}$. Suppose that $j \neq 0$. Then use the bundle map π_{TS^2} to project an integral curve γ of $X_{H|TS^2}$ on $T^2_{h,j}$ onto a curve Γ in the annular region $\pi_{TS^2}(\mathscr{E}\mathscr{M}^{-1}(h, j)) = \mathscr{A}$ of S^2. Let $x_i|\mathscr{A}$ be coordinates on $\mathscr{A}$ with (x_1, x_2, x_3) being coordinates on $\mathbf{R}^3$. Furthermore, let $\theta = \tan^{-1}\frac{x_2}{x_1}$ and x_3 be coordinates
▷ on the *universal covering space* $\widetilde{\mathscr{A}}$ of $\mathscr{A}$. The following argument shows that a lift $\widetilde{\Gamma}$ of Γ to $\widetilde{\mathscr{A}}$ satisfies

[1]
$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{j}{1 - x_3^2} \tag{57a}$$
$$\frac{\mathrm{d}x_3}{\mathrm{d}t} = \varepsilon\sqrt{P(x_3)}, \tag{57b}$$

with $\varepsilon^2 = 1$. How the sign of ε is determined is discussed below.

(4.1) **Proof**: By definition of Lie derivative, we find that

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = L_{X_{H|TS^2}}(\theta|TS^2) = \frac{1}{x_1^2+x_2^2}\big(x_1\frac{\mathrm{d}x_2}{\mathrm{d}t} - x_2\frac{\mathrm{d}x_1}{\mathrm{d}t}\big) = \frac{x_1y_2-x_2y_1}{x_1^2+x_2^2} = \frac{j}{1-x_3^2}.$$

Also,

$$\begin{aligned}\frac{\mathrm{d}x_3}{\mathrm{d}t} &= L_{X_{H|TS^2}}(x_3|TS^2) = \rho_j^*(L_{X_{H_j|P_j}}\sigma_1) = \rho_j^*(\sigma_2), \quad \text{using (23)}\\ &= \pm\rho_j^*\big(\sqrt{\sigma_3(1-\sigma_1^2)-j^2}\big), \quad \text{since } \sigma_2^2-\sigma_3(1-\sigma_1^2)+j^2 \text{ is an integral of } X_{H_j} \text{ with constant value } 0\\ &= \pm\rho_j^*\big(\sqrt{2(h-\sigma_1)(1-\sigma_1^2)-j^2}\big), \text{ since } \gamma(t)\in(H|TS^2)^{-1}(h), \text{ which gives } \sigma_3 = 2(h-\sigma_1)\\ &= \varepsilon\sqrt{P(x_3)},\end{aligned}$$

where $\varepsilon^2=1$ and $P(x_3)=2(h-x_3)(1-x_3^2)-j^2$. The sign ambiguity is handled by the following *sign convention*. Suppose that $\gamma(t_0)=x_3^{\pm}$ and that at $t_0-\delta>0$ for $\delta>0$ and small the value of ε is known. Then at time $t_0+\delta$ the value of ε is the negative of ε at $t_0-\delta$. Since $\dot\theta\neq 0$ and $\dot x_3(t_0)=0$ when $\widetilde\Gamma(t_0)\in\partial\widetilde{\mathscr{A}}$, the curve $\widetilde\Gamma$ has at least first order contact with $\partial\widetilde{\mathscr{A}}$ at $\widetilde\Gamma(t_0)$. Because Γ crosses $\mathscr{C}^{\pm}=\pi_{TS^2}^{-1}(\{x_3=x_3^{\pm}\})$ transversely at $\Gamma(t_0)$ and the mapping $\pi_{TS^2}|T^2_{h,j}$ has a fold singularity at $\gamma(t_0)$, the curve $\widetilde\Gamma$ has second order contact with $\partial\widetilde{\mathscr{A}}$ at $\widetilde\Gamma(t_0)$. The sign convention ensures that the solutions of (57a) and (57b) in $\widetilde{\mathscr{A}}$ are real analytic. □

Now consider the curve $[0,2\pi\theta_{h,j})\to T^2_{h,j}: s\mapsto\varphi_s^{J|TS^2}(p)$, where p lies in a fold curve of the projection $\pi_{TS^2}|T^2_{h,j}$. Suppose that the projected curve $[0,2\pi\theta_{h,j})\to S^2: s\mapsto \pi_{TS^2}(\varphi_s^{J|TS^2}(p))$ is an arc of the small circle $\mathscr{C}^-=\{x_3=x_3^-\}$ on S^2, which joins two successive points of intersection $\Gamma(0)$ and $\Gamma(T)$ with $\mathscr{C}^-$. Let $2\pi\vartheta_{h,j}$ be the angle between
▷ $\Gamma(0)$ and $\Gamma(T)$ as measured from the center of the small circle $\mathscr{C}^-$. Then $\vartheta_{h,j}$ is equal to the rotation number $\theta_{h,j}$.

(4.2) **Proof**: This follows because

$$\begin{aligned}2\pi\vartheta_{h,j} &= \int_0^{2\pi\vartheta_{h,j}}\mathrm{d}\theta = \int_0^{2\pi\theta_{h,j}} L_{X_{J|TS^2}}\theta\,\mathrm{d}s, \quad \text{by definition of rotation number}\\ &= \int_0^{2\pi\theta_{h,j}}\mathrm{d}s = 2\pi\theta_{h,j},\end{aligned}$$

where the second to last equality follows using equation (8) and the definition of θ. □

▷ We now find a formula for $\theta_{h,j}$.

(4.3) **Proof**: Since $P(x_3)>0$ for $x=(x_1,x_2,x_3)$ in the interior of $\mathscr{A}$, from (57b) we see that $\frac{\mathrm{d}x_3}{\mathrm{d}t}\neq 0$. Hence for $t\in(0,T/2)\cup(T/2,T)$ we may parametrize the curve γ by x_3. Chasing down the minus signs, we find

$$2\pi\theta_{h,j} = \int_0^T L_{X_{H|TS^2}}\theta\,\mathrm{d}t = -\int_{x_3^+}^{x_3^-}\frac{\mathrm{d}\theta}{\mathrm{d}x_3}\,\mathrm{d}x_3 + \int_{x_3^-}^{x_3^+}\frac{\mathrm{d}\theta}{\mathrm{d}x_3}\,\mathrm{d}x_3 = 2\int_{x_3^-}^{x_3^+}\frac{\frac{\mathrm{d}\theta}{\mathrm{d}t}}{\frac{\mathrm{d}x_3}{\mathrm{d}t}}\,\mathrm{d}x_3.$$

Using (57a) and (57b) we get

$$2\pi\theta(h,j) = 2\pi\theta_{h,j} = 2j\int_{x_3^-}^{x_3^+} \frac{1}{(1-x_3^2)\sqrt{P(x_3)}}\,\mathrm{d}x_3, \tag{58}$$

which is the desired formula for the rotation number. Some care needs to be taken in interpreting (58) when $j = 0$, for then the integral is infinite. □

We now derive a formula for the first return time T. From the definition of the small circle $\mathscr{C}^- = \{x_3 = x_3^-\}$ of the annulus $\mathscr{A} = \pi_{TS^2}(\mathscr{EM}^{-1}(h,j))$ and the sign convention for ε in (57b), it follows that the time of first return is

$$T(h,j) = \int_{x_3^-}^{x_3^+} \mathrm{d}t - \int_{x_3^+}^{x_3^-} \mathrm{d}t = 2\int_{x_3^-}^{x_3^+} \frac{1}{\frac{\mathrm{d}x_3}{\mathrm{d}t}}\,\mathrm{d}x_3 = 2\int_{x_3^-}^{x_3^+} \frac{1}{\sqrt{P(x_3)}}\,\mathrm{d}x_3, \tag{59}$$

where $-1 < x_3^- < x_3^+ < 1$ are consecutive roots of $P(x_3) = 2(h-x_3)(1-x_3^2) - j^2$.

We now suppose that (h,j) is a regular value of $\mathscr{EM}$ and $j = 0$. Then $0 = x_2y_1 - x_1y_2 = \det\begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix}$, that is, the vectors $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ are linearly dependent. So there is a nonzero real number λ such that $\begin{cases} y_1 = \lambda x_1 \\ y_2 = \lambda x_2 \end{cases}$. From the equations of motion (5) on $T\mathbf{R}^3$ it follows that $\begin{cases} \dot{x}_1 = y_1 = \lambda x_1 \\ \dot{x}_2 = y_2 = \lambda x_2 \end{cases}$. Thus the vertical 2-plane $\Pi \subseteq \mathbf{R}^3$ spanned by the vectors $\begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ is invariant under the image of every integral curve of (5) under the projection map $\pi_{T\mathbf{R}^3} : T\mathbf{R}^3 \to \mathbf{R}^3 : (x,y) \mapsto x$. Since TS^2 is an invariant manifold of (5), it follows that the image of the integral curve $t \mapsto \varphi^{H|TS^2}(p)$ on $T^2_{h,0} = \mathscr{EM}^{-1}(h,0)$
▷ starting at p under the projection map π_{TS^2} is an arc of the great circle $\Pi \cap S^2$ through the north and south poles $(0,0,\pm 1)$ and the point $\frac{1}{\sqrt{x_1^2+x_2^2}}(x_1,x_2,0)$. We need to look at two cases.

CASE 1. $-1 < h < 1$. The image of $\mathscr{EM}^{-1}(h,0)$ under the bundle projection π_{TS^2} is the closed 2-disk D on S^2 with center at $(0,0,-1)$ and boundary the small circle $\mathscr{C}^+ = \{(x_1,x_2,h) \in S^2 \,\big|\, x_1^2+x_2^2 = 1-h^2 > 0\}$. Suppose that $p = (x_1,x_2,h,0,0,0) \in \mathscr{EM}^{-1}(h,0)$, where $\pi_{TS^2}(p) = (x_1,x_2,h) \in \mathscr{C}^+$. Then the time $T(h,0)$ of first return of the integral curve $\gamma : t \mapsto \varphi_t^{H|TS^2}(p)$ of $X_{H|TS^2}$ on $\mathscr{EM}^{-1}(h,0)$ to the closed orbit $C = \pi_{TS^2}^{-1}(\mathscr{C}^+)$ of $X_{J|TS^2}$ starting at p is

$$T(h,0) = 2\int_{-1}^{h} \frac{1}{\sqrt{2(h-x_3)(1-x_3^2)}}\,\mathrm{d}x_3.$$

This follows because the projected curve $\Gamma : t \mapsto \pi_{TS^2}(\varphi_t^{H|TS^2}(p))$ to D satisfies the differential equation $\dot{x}_3 = (2(h-x_3)(1-x_3^2))^{1/2}$ with $x_3^- = -1$ and $x_3^+ = h$, and reaches $\mathscr{C}^+$ at the point $\pi_{TS^2}(p^*) = (-x_1,-x_2,h) \in \Pi \cap \mathscr{C}^+$, where $p^* = (-x_1,-x_2,h,0,\,0,0) \in C$, for the first time at $T(h,0)$. Because $p^* \neq p$, the first return time $T(h,0)$ is *not* the

period of the integral curve γ, which is indeed $2T(h,0)$. We now determine the rotation number $\theta(h,0)$ of the integral curve γ. Because $\Gamma(T(h,0)) = p^*$, which lies in $\Pi \cap \mathscr{C}^+$ and is a half of a full rotation about the positive x_3-axis from p, it follows
▷ that $\pi_{TS^2}(\varphi_{\pm\pi}^{J|TS^2}(p)) = p^*$. Thus the rotation number $\theta(h,0)$ of γ is $\pm\frac{1}{2}$. The next argument shows how to determine the sign.

(4.4) **Proof**: Suppose that (h,j) is a regular value of $\mathscr{EM}$ as $j \searrow 0$. Let C_j be a closed orbit of $X_{J|TS^2}$ on $\mathscr{EM}^{-1}(h,j)$ starting at p_j with $p_j \to p$. Then for all $j>0$ and sufficiently small the curve $\mathscr{C}_j^+ = \pi_{TS^2}(C_j)$ is homotopic in $S^2 \setminus \{(0,0,\pm 1)\}$ to the image under π_{TS^2} of a relative equlibrium in RE_+ corresponding to the energy momentum value $(\mathscr{B}_+^{-1}(j), j)$. Here $\mathscr{B}_+$ is given by (46). The projected relative equilibrium is traversed in an *clockwise* direction about the positive x_3-axis and thus is negatively oriented. Giving $\mathscr{C}^+ = \lim_{j\searrow 0} \mathscr{C}_j^+$ the same orientation as $\mathscr{C}_j^+$, it follows that the plus sign holds, that is, the rotation number of γ is $\theta(h,0) = \frac{1}{2}$. A similar argument shows that if $j \nearrow 0$ then the rotation number of γ is $-\frac{1}{2}$. □

CASE 2. $h>1$. The image of $\mathscr{EM}^{-1}(h,0)$ is S^2. Suppose that $p = (0,0,1,y_1,y_2,0)$, where $y_1^2 + y_2^2 = 2(h-1)$. Then $p \in \mathscr{EM}^{-1}(h,0)$. The time $T(h,0)$ of first return of the integral curve $\gamma : t \mapsto \varphi^{H|TS^2}(p)$ of $X_{H|TS^2}$ on $\mathscr{EM}^{-1}(h,0)$ to the closed orbit C of $X_{J|TS^2}$ on $\mathscr{EM}^{-1}(h,0)$ starting at p is

$$T(h,0) = 2\int_{-1}^{1} \frac{1}{\sqrt{2(h-x_3)(1-x_3^2)}}\,\mathrm{d}x_3.$$

This follows because the projected integral curve $\Gamma : t \mapsto \pi_{TS^2}(\varphi_t^{H|TS^2}(p))$, which starts at $\pi_{TS^2}(p) = (0,0,1) = p^*$, satisfies the differential equation $\dot{x}_3 = (2(h-x_3)(1-x_3^2))^{1/2}$ with $x_3^- = -1$ and $x_3^+ = 1$, and Γ reaches p^* for the first time at $T(h,0)$. Thus γ reaches C at the point p for the first time at $T(h,0)$. So $T(h,0)$ is the *period* of γ. The rotation number $\theta(h,0)$ of γ is ± 1 and the sign is determined as in ((4.4)).

4.2 Analytic properties of the rotation number

Next we investigate the analytic properties of the rotation number.

Let $\mathscr{R}$ be the set of regular values of the energy momentum mapping, see figure 3.2.3. On
▷ $\mathscr{R}^\vee = \mathscr{R} \setminus \{j=0\}$ the rotation number $\theta : \mathscr{R}^\vee \to \mathbf{R} : (h,j) \mapsto \theta_{h,j}$ (58) is *locally* a single valued real analytic function. Because $\mathscr{R}$ is not simply connected, θ need *not* be single valued on all of $\mathscr{R}$.

(4.5) **Proof**: Let $P(z) = 2(h-z)(1-z^2) - j^2$, where $(h,\ell) \in \mathscr{R}^\vee$. Consider the 1-form $\varpi = \frac{1}{(1-z^2)\sqrt{P(z)}}\,\mathrm{d}z$ on $\mathbf{C}^\vee$, the extended complex plane, which is cut along the real axis between x_- and x_+ and again between x_0 and ∞. Here $x_{\pm,0}$ are distinct roots of P with

$$\begin{cases} -1 < x_- < x_+ < h < 1 < x_0, & \text{if } -1 < h < 1 \\ -1 < x_- < x_+ < 1 < h < x_0, & \text{if } h > 1. \end{cases}$$

Write $\sqrt{P(z)} = \sqrt{r_- r_+ r_0}\,\mathrm{e}^{i(\theta_- + \theta_+ + \theta_0)/2}$, where $z - x_{0,\pm} = r_{0,\pm}\mathrm{e}^{i\theta_{0,\pm}}$ and $0 \le \theta_{0,\pm} < 2\pi$. With this choice of complex square root on $\mathbf{C}^\vee$ we see that ϖ is single-valued. It is meromorphic with a first order pole at $z = \pm 1$ whose residue $\operatorname*{Res}_{z=\pm 1} \varpi$ at $z = \pm 1$ is

$$\lim_{z \to \pm 1} (z \mp 1) \frac{1}{(1-z^2)\sqrt{P(z)}} = -\lim_{z \to \pm 1} \frac{1}{z \pm 1} \frac{1}{\sqrt{P(z)}} = \mp \frac{1}{2\sqrt{P(\pm 1)}} = i\,\frac{1}{2|j|}. \tag{60}$$

When $(h_0, j_0) \in \mathscr{R}$ the polynomial P has three distinct real roots: two in $(-1,1)$ and one in $(1,\infty)$. Thus there is an open neighborhood U_0 of (h_0, j_0) in $\mathscr{R}^\vee$ where P has zeroes with the preceding property. For every $(h,j) \in U_0$ there is a positively oriented smooth curve $\mathscr{C}$ in $\mathbf{C}^\vee$ which encircles the cut $[x_-, x_+]$ and avoids the points h and ± 1. We can rewrite (58) as

$$\theta(h,j) = \frac{j}{2\pi} \int_{\mathscr{C}} \frac{1}{(1-z^2)\sqrt{P(z)}}\,\mathrm{d}z. \tag{61}$$

Now complexify h, j, and U_0. With $(h, j_0) \in U_0^{\mathbf{C}}$ we have

$$\frac{\partial \theta}{\partial \overline{h}}(h, j_0) = \frac{j_0}{2\pi} \int_{\mathscr{C}} \frac{1}{1-z^2} \frac{\partial}{\partial \overline{h}} \Big(\frac{1}{\sqrt{2(h-z)(1-z^2) - j_0^2}} \Big)\,\mathrm{d}z = 0.$$

Similarly for fixed h_0 with $(h_0, j) \in U_0^{\mathbf{C}}$ we get $\frac{\partial \theta}{\partial \overline{j}}(h_0, j) = 0$. Using Hartog's theorem we deduce that θ is a complex analytic function on $U_0^{\mathbf{C}}$. This implies that θ is a real analytic function on $U_0 = U_0^{\mathbf{C}} \cap \mathbf{R}^2$. So θ is *locally a real analytic function* on $\mathscr{R}^\vee$. □

To show that θ can be extended to a *locally* single valued real analytic function on *all* of $\mathscr{R}$, it suffices to show that it remains bounded as $j \to 0$. Thus we need to show that
▷ for $(h,j) \in \mathscr{R}^\vee$ with $-1 < h < 1$ we have

$$\lim_{j \to 0^\pm} \theta(h,j) = \pm\tfrac{1}{2}; \tag{62a}$$

while for $(h,j) \in \mathscr{R}^\vee$ with $h > 1$ we have

$$\lim_{j \to 0^\pm} \theta(h,j) = \pm 1. \tag{62b}$$

(4.6) **Proof**: If we have proved

$$\lim_{j \searrow 0} \theta(h,j) = \begin{cases} \frac{1}{2}, & \text{if } -1 < h < 1 \\ 1, & \text{if } h > 1, \end{cases} \tag{63a}$$

then we obtain

$$\lim_{j \nearrow 0} \theta(h,j) = \begin{cases} -\frac{1}{2}, & \text{if } -1 < h < 1 \\ -1, & \text{if } h > 1, \end{cases} \tag{63b}$$

because $\theta(h,-j) = -\theta(h,j)$. Suppose that $(h,j) \in \mathscr{R} \cap \{j > 0\}$ and that $-1 < h < 1$. Let $\mathscr{C}_{-1}$ be a positively oriented smooth curve in $\mathbf{C}^\vee$ which encircles -1. Let $\mathscr{C}_2$ be a positively oriented smooth curve in $\mathbf{C}^\vee$ which encircles the cut $[x_-, x_+]$ and h but avoids $\mathscr{C}_{-1}$. Finally let $\mathscr{C}_3 = \mathscr{C}_{-1} + \mathscr{C}_2$. Then

$$\tfrac{j}{2\pi} \int_{\mathscr{C}_3} \varpi = \tfrac{j}{2\pi} \int_{\mathscr{C}_{-1}} \varpi + \tfrac{j}{2\pi} \int_{\mathscr{C}_2} \varpi = \tfrac{j}{2\pi} \big(2\pi i \operatorname*{Res}_{z=-1} \varpi \big) + \theta(h,j) = -\tfrac{1}{2} + \theta(h,j).$$

To show that $\lim_{j\to 0}\frac{j}{2\pi}\int_{\mathscr{C}_3}\varpi = 0$ we argue as follows. As $j\to 0$ we have $x_3^-\to -1$ and $x_3^+\to h$. Thus the contour $\mathscr{C}_3$ encircles the cut along the real axis between -1 and h and avoids 1 by intersecting the real axis between h and 1. So

$$|\int_{\mathscr{C}_3}\varpi| = 2\int_{-1}^{h}\frac{1}{\sqrt{2(h-x)(1-x^2)}}\,\mathrm{d}x \le \frac{\sqrt{2}}{\sqrt{1-h^2}}\int_{-1}^{h}\frac{1}{\sqrt{h-x}}\,\mathrm{d}x,$$

since $-1\le x\le h$ implies $(h-x)(1-x^2)\ge (h-x)(1-h^2)$. Changing variables by $u^2 = h-x$, the last integral above is $2\sqrt{h+1}$. Thus $\lim_{j\to 0}\frac{j}{2\pi}\int_{\mathscr{C}_3}\varpi = 0$. So we have proved (63a), when $-1<h<1$. To prove (63a) when $h>1$, let $\mathscr{C}_{-1}$ be a positively oriented smooth curve in $\mathbf{C}^\vee$ which encircles -1 and let $\mathscr{C}_1$ be a positively oriented smooth curve in $\mathbf{C}^\vee$ which encircles 1 and avoids h. Let $\mathscr{C}_2$ be a positively oriented smooth curve in $\mathbf{C}^\vee$ which encircles the cut $[x_-,x_+]$ and h but avoids $\mathscr{C}_{\pm 1}$. Finally let $\mathscr{C}_3 = \mathscr{C}_{-1}+\mathscr{C}_1+\mathscr{C}_2$. Then

$$\begin{aligned}\tfrac{j}{2\pi}\int_{\mathscr{C}_3}\varpi &= \tfrac{j}{2\pi}\int_{\mathscr{C}_{-1}}\varpi + \tfrac{j}{2\pi}\int_{\mathscr{C}_1}\varpi + \tfrac{j}{2\pi}\int_{\mathscr{C}_2}\varpi\\ &= \tfrac{j}{2\pi}2\pi i\,[\operatorname*{Res}_{z=-1}\varpi + \operatorname*{Res}_{z=1}\varpi] + \theta(h,j) = -1+\theta(h,j).\end{aligned}$$

To show that $\lim_{j\to 0}\frac{j}{2\pi}\int_{\mathscr{C}_3}\varpi = 0$ we argue as follows. As $j\to 0$ we have $x_3^-\to -1$ and $x_3^+\to 1$. Thus the contour $\mathscr{C}_3$ encircles the cut along the real axis between -1 and 1 and avoids h by intersecting the real axis between 1 and h. So

$$|\int_{\mathscr{C}_3}\varpi| = 2\int_{-1}^{1}\frac{1}{\sqrt{2(h-x)(1-x^2)}}\,\mathrm{d}x \le \frac{\sqrt{2}}{\sqrt{h-1}}\int_{-1}^{1}\frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x = \frac{\sqrt{2}\pi}{\sqrt{h-1}},$$

where the inequality follows since $-1\le x\le 1<h$ gives $(h-x)(1-x^2)\ge(h-1)(1-x^2)$. Thus $\lim_{j\to 0}\frac{j}{2\pi}\int_{\mathscr{C}_3}\varpi = 0$. □

We now turn to describing the *global* real analytic function θ on $\mathscr{R}$ which is defined by *analytically continuing* the locally defined real analytic functions $\theta|\mathscr{U}_i$. Here $\{\mathscr{U}_i\}$ is a suitable open covering of $\mathscr{R}$. Let $\widetilde{\mathscr{R}}$ be the universal covering space of $\mathscr{R}$ with covering mapping $\pi:\widetilde{\mathscr{R}}\to\mathscr{R}$. Then $\widetilde{\mathscr{R}}$ is a real analytic manifold, which is diffeomorphic to a 2-disk D in $\mathbf{R}^2$ because $\mathscr{R}$ is diffeomorphic to an annulus. The covering projection π is real analytic. Each function element $\theta|\mathscr{U}_i$ lifts to a locally defined real analytic function element $\widetilde{\theta}|\widetilde{\mathscr{U}_{i,j}}$ where $\pi(\widetilde{\mathscr{U}_{i,j}}) = \mathscr{U}_i$ and $\bigcup\widetilde{\mathscr{U}_{i,j}} = \pi^{-1}(\mathscr{U}_i)$. Here $\widetilde{\theta}|\widetilde{\mathscr{U}_{i,j}} = (\theta\circ\pi)|\widetilde{\mathscr{U}_{i,j}}$.
▷ Since $\widetilde{\mathscr{R}}$ is simply connected, there is a single valued real analytic function $\widetilde{\theta}$ whose local function elements are $\widetilde{\theta}|\widetilde{\mathscr{U}_{i,j}}$.

(4.7) **Proof**: Let Γ be a simple closed curve in $\widetilde{\mathscr{R}}$. Then Γ bounds a 2-disk D. Since $\widetilde{\mathscr{R}}$ is simply connected, there is a homotopy Γ_s such that $\Gamma_1=\Gamma$ and Γ_0 is a point $p\in D$. Let $S=\big\{s\in[0,1]\,\big|\,\widetilde{\theta}$ is a single valued real analytic function in the disk D_s bounded by $\Gamma_s\big\}$. Then S is nonempty, because for some (i,j) the point $p\in\widetilde{\mathscr{U}_{i,j}}$. Then there is an $s_0>0$ such that $\Gamma_{s_0}\subseteq\widetilde{\mathscr{U}_{i,j}}$. Let $\sigma=\sup\{s\in S\}$. Then $\sigma\ge s_0>0$. Now suppose that $\sigma<1$. Cover Γ_σ with open disks $\Delta_k\subseteq\widetilde{\mathscr{U}_{i(k),j(k)}}$ of radius $\frac{3}{4}\delta$ with center at the point

$p_k \in \Gamma_\sigma$ where for $k \ge 1$ the distance between p_{k-1} and p_k is less than δ. Then the disks Δ_k pairwise overlap and a finite number of them cover Γ_σ. There is an $s_1 < \sigma$ such that $\Gamma_{s_1} \subseteq D_\sigma \cup \bigcup_k \Delta_k$. By definition of σ, the function $\widetilde{\theta}$ is single valued and real analytic in D_{s_1}. By analytic continuation $\widetilde{\theta}$ is single valued and real analytic on $\bigcup_k \Delta_k$. Therefore, $\widetilde{\theta}$ is single valued and real analytic on $D_{s_1} \cup \bigcup_k \Delta_k$, which contains D_σ as a proper subset. There is an $s_2 > \sigma$ such that $\Gamma_{s_2} \subseteq D_\sigma \cup \bigcup_k \Delta_k$. On D_{s_2} the function $\widetilde{\theta}$ is single valued and real analytic. But this contradicts the definition of σ. Therefore $\sigma = 1$. In other words, the function $\widetilde{\theta}$ is real analytic and single valued on Γ and hence on all of $\widetilde{\mathscr{R}}$. □

The above result just states that θ *could* be a *multivalued real analytic function* on $\mathscr{R}$. The following discussion shows that θ *is* multivalued. We begin by proving

Fact: The variation of θ along an oriented closed curve Γ in $\mathscr{R}$ depends only on the homotopy class of Γ.

(4.8) **Proof**: Suppose that Γ_1 and Γ_2 are homotopic positively oriented closed curves in $\mathscr{R}$. Then the curve $\gamma = \Gamma_1 - \Gamma_2$ is null homotopic in $\mathscr{R}$. Since the fundamental group of $\mathscr{R}$ is isomorphic to $\mathbf{Z}$, it is abelian and hence is isomorphic to the first homology group of $\mathscr{R}$. Therefore γ is the boundary of some domain $\mathscr{E}$ in $\mathscr{R}$. Hence

$$\int_{\Gamma_1} \mathrm{d}\theta - \int_{\Gamma_2} \mathrm{d}\theta = \int_{\gamma = \partial\mathscr{E}} \mathrm{d}\theta \, = \, \int_{\mathscr{E}} \mathrm{d}^2\theta \, = 0,$$

where the second to last equality follows by Stokes' theorem. □

Next we prove

Claim: Let Γ be a positively oriented curve which generates the fundamental group of $\mathscr{R}$. As (h, j) makes a circuit around Γ the value of θ *decreases* by 1.

(4.9) **Proof**: Using the above fact we may choose Γ to be the positively oriented non-null homotopic rectangular curve in $\mathscr{R} \cup \{(1,0)\}$ made up of four line segments joining the points (h_1, j_0), (h_0, j_0), $(h_0, -j_0)$, and $(h_1, -j_0)$ with $h_0 < 1 < h_1$ and $j_0 > 0$. Let Γ_1^j be the oriented line segment joining the points (h_1, j) and (h_0, j) and Γ_2^j the oriented line segment joining $(h_0, -j)$ and $(h_1, -j)$. Here $0 < j < j_0$. The curve Γ is homotopic to the curve $\Gamma^j = \Gamma_1^j \cup S_0^j \cup \Gamma_2^j \cup S_1^j$, where S_1^j is the line segment joining $(h_1, -j)$ to (h_1, j) and S_0^j is the line segment joining (h_0, j) to $(h_0, -j)$. Therefore

$$\begin{aligned} \int_\Gamma \mathrm{d}\theta \quad &= \quad \lim_{j \searrow 0} \int_{\Gamma^j} \mathrm{d}\theta \, = \, \lim_{j \searrow 0} \left[2 \int_{\Gamma_1^j} \mathrm{d}\theta + \int_{S_1^j} \mathrm{d}\theta + \int_{S_0^j} \mathrm{d}\theta \right], \\ & \qquad \qquad \text{because } \theta(h, -j) = -\theta(h, j) \text{ when } j \neq 0 \\ &= \quad \lim_{j \searrow 0} \big[2\big(\theta(h_0, j) - \theta(h_1, j)\big) + \big(\theta(h_1, j) - \theta(h_1, -j)\big) \\ & \qquad \qquad + \, \big(\theta(h_0, j) - \theta(h_0, -j)\big)\big] = -1, \end{aligned}$$

where the last equality follows from (62a) and the fact that θ is locally real analytic and hence locally continuous. Hence the variation of θ along Γ in $\mathscr{R}$ is -1. □

4.3 Analytic properties of the first return time

In this subsection we prove some analytic properties of the time T of first return.

▷ On $\mathscr{R}^{\vee}$ the time of first return $T : \mathscr{R}^{\vee} \rightarrow \mathbf{R} : (h, j) \mapsto T(h, j)$ is a local real analytic function.

(4.10) **Proof**: We use the same choice of complex square root on $\mathbf{C}^{\vee}$ and definition of the open neighborhood U_0 of (h_0, j_0) in $\mathscr{R}^{\vee}$ as in ((4.5)). Let $\mathscr{C}$ be a positively oriented closed curve in $\mathbf{C}^{\vee}$ which encircles the cut $[x_3^-, x_3^+]$ along the real axis and avoids the points ± 1 by intersecting the real axis between -1 and x_3^- and again between x_3^+ and 1. Then the 1-form $\frac{1}{\sqrt{P(z)}}\,\mathrm{d}z$ is holomorphic along $\mathscr{C}$ and $T(h, j) = \int_{\mathscr{C}} \frac{1}{\sqrt{P(z)}}\,\mathrm{d}z$. Now complexify h, j and U_0. For fixed complex j_0 with $(h, j_0) \in U_0^{\mathbf{C}}$ we have

$$\frac{\partial T}{\partial \overline{h}}(h, j_0) = \int_{\mathscr{C}} \frac{\partial}{\partial \overline{h}}\Big(\frac{1}{\sqrt{2(h-z)(1-z^2) - j_0^2}}\Big)\,\mathrm{d}z = 0.$$

Similarly for fixed complex h_0 with $(h_0, j) \in U_0^{\mathbf{C}}$ we get $\frac{\partial T}{\partial \overline{j}}(h_0, j) = 0$. Using Hartog's theorem, this shows that T is a complex analytic function on $U_0^{\mathbf{C}}$. Consequently, T is a real analytic function on $U_0 = U_0^{\mathbf{C}} \cap \mathbf{R}^2$. So T is locally a real analytic function on $\mathscr{R}^{\vee}$. □

▷ To show that T can be extended to a real analytic function on all of $\mathscr{R}$, it suffices to show that the function

$$T(h, 0) : (-1, 1) \cup (1, \infty) \rightarrow \mathbf{R} : h \mapsto 2\int_{-1}^{h^* = \min(h,1)} \frac{1}{\sqrt{2(h-x)(1-x^2)}}\,\mathrm{d}x \qquad (64)$$

is real analytic.

(4.11) **Proof**: Making the successive changes of variables $x = \cos 2\theta$ and $u = \sqrt{\frac{2}{h+1}}\cos\theta$ in (64) we get

$$T(h, 0) = -2\sqrt{2}\int_{\pi/2}^{\theta^*} \frac{1}{\sqrt{\frac{h+1}{2} - \cos^2\theta}}\,\mathrm{d}\theta = 2\sqrt{2}\int_0^{u^*} \frac{1}{\sqrt{(1-u^2)(1-\frac{h+1}{2}u^2)}}\,\mathrm{d}u,$$

where $\theta^* = \cos^{-1}\sqrt{\frac{h^*+1}{2}}$ and $u^* = \sqrt{\frac{h^*+1}{h+1}}$. Therefore

$$T(h, 0) = \begin{cases} 2\sqrt{2}K\big(\sqrt{\frac{h+1}{2}}\big), & \text{if } -1 < h < 1 \\ \frac{4}{\sqrt{h+1}}K\big(\sqrt{\frac{2}{h+1}}\big), & \text{if } h > 1. \end{cases}$$

For $0 < k < 1$ the function $K(k) = \int_0^1 \frac{1}{\sqrt{(1-u^2)(1-k^2u^2)}}\,\mathrm{d}u$ is the complete elliptic integral, which is a real analytic function of k. □

We now prove

Claim: The function $T : \mathscr{R} \rightarrow \mathbf{R} : (h, j) \mapsto T(h, j)$ is a *single valued real analytic function*.

(4.12) **Proof**: The same argument used in ((4.9)) shows that the variation of T along an oriented closed curve Γ in $\mathscr{R}$ depends only on the homotopy class of Γ. Choose the curves Γ and Γ^j as in the argument proving ((4.10)). Then

$$\begin{aligned}
\int_{\Gamma} \mathrm{d}T &= \lim_{j\searrow 0} \int_{\Gamma^j} \mathrm{d}T = \lim_{j\searrow 0} \Big[\int_{\Gamma_1^j} \mathrm{d}T + \int_{S_0^j} dT + \int_{\Gamma_2^j} \mathrm{d}T + \int_{S_1^j} \mathrm{d}T \Big] \\
&= \lim_{j\searrow 0} \big[\big((T(h_0, j) - T(h_1, j) \big) + \big(T(h_1, -j) - T(h_0, -j) \big) \\
&\qquad + \big((T(h_0, -j) - T(h_0, j) \big) + \big((T(h_1, j) - T(h_1, -j) \big) \big] \\
&= 0,
\end{aligned}$$

where the last equality follows because T is locally real analytic and hence is locally continuous. Thus T is a single valued real analytic function on $\mathscr{R}$. □

5 Monodromy

In this section we show that over the set $\mathscr{R}$ of regular values the fibers of the energy momentum map $\mathscr{EM}$ of the spherical pendulum fit together in a *nontrivial way*.

5.1 Definition of monodromy

More precisely, let Γ be a closed non-null homotopic curve in $\mathscr{R} \subseteq \mathbf{R}^2$, which bounds a 2-disk in $\mathbf{R}^2$ containing the point $(1,0)$ in its interior. We will show that the 2-torus bundle $\mathscr{EM}^{-1}(\Gamma) \rightarrow \Gamma$ with bundle projection map $\mathscr{EM}$ is *nontrivial*, that is, it is *not* isomorphic to the trivial bundle $T^2 \times S^1 \rightarrow S^1$ with bundle projection map being the projection on the second factor. In other words, the *classifying map* χ of the bundle $\mathscr{EM}^{-1}(\Gamma) \rightarrow \Gamma$, which *glues* together the end 2-tori of the 2-torus bundle $\mathscr{EM}^{-1}(\Gamma \setminus \{pt\}) \rightarrow \Gamma \setminus \{pt\}$, is not homotopic to the identity map. Note that the bundle $\mathscr{EM}^{-1}(\Gamma \setminus \{pt\})$ is trivial because $\Gamma \setminus \{pt\}$ is contractible. In fact, the map χ_* induced on the first *homology group* $\mathrm{H}_1(\mathscr{EM}^{-1}(pt), \mathbf{Z})$ of the end 2-torus $\mathscr{EM}^{-1}(pt)$ by the classifying map χ is $\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$
▷ with respect to a suitably chosen basis. The map χ_* is called the *monodromy map* of the bundle $\mathscr{EM}^{-1}(\Gamma) \rightarrow \Gamma$ and depends only on the homotopy class of the curve Γ in $\mathscr{R}$.

(5.1) **Proof**: To see this suppose that $\widetilde{\Gamma}$ is a closed curve in $\mathscr{R}$, which is homotopic to Γ, then the bundles $\mathscr{EM}^{-1}(\Gamma) \rightarrow \Gamma$ and $\mathscr{EM}^{-1}(\widetilde{\Gamma}) \rightarrow \widetilde{\Gamma}$ are isomorphic. Therefore their classifying maps are homotopic, which implies that their monodromy maps are equal. □

Claim: The bundle $\mathscr{EM}^{-1}(\Gamma) \rightarrow \Gamma$ is *not* trivial.

(5.2) **Proof**: From table 3.4.3 we see that for each $h_1 > 1$ the energy surface $(H|TS^2)^{-1}(h_1)$ is diffeomorphic to $\mathbf{RP}^3$; while for each $-1 < h_0 < 1$ the energy surface $(H|TS^2)^{-1}(h_0)$ is diffeomorphic to S^3. Since $\mathrm{H}_1(\mathbf{RP}^3, \mathbf{Z}) = \mathbf{Z}_2$ and $\mathrm{H}_1(S^3, \mathbf{Z}) = 0$, it follows that $\mathbf{RP}^3$ is not even homeomorphic, let alone diffeomorphic, to S^3. Suppose that the 2-torus bundle $\mathscr{EM}^{-1}(\Gamma) \rightarrow \Gamma$ is trivial. Let Γ_1 be a curve in $\mathscr{R} \cap \{h > 1\}$, which separates $\mathscr{R}$ into two connected components, and let Γ_0 be a curve in $\mathscr{R} \cap \{-1 < h < 1\}$, which does the same thing, see figure 5.1.1. Give Γ_0 and Γ_1 opposite orientations. From the hypothesis that the bundle $\mathscr{EM}^{-1}(\Gamma) \to \Gamma$ is trivial it follows that the manifolds $\mathscr{EM}^{-1}(\Gamma_0)$ and $\mathscr{EM}^{-1}(\Gamma_1)$ are diffeomorphic. But $\mathscr{EM}^{-1}(\Gamma_0)$ is isotopic to $(H|TS^2)^{-1}(h_0)$; while $\mathscr{EM}^{-1}(\Gamma_1)$ is isotopic to $(H|TS^2)^{-1}(h_1)$. Therefore the h_0 and h_1 level sets of $H|TS^2$ are diffeomorphic, which is false. Thus the bundle $\mathscr{EM}^{-1}(\Gamma) \to \Gamma$ is nontrivial. $\square$

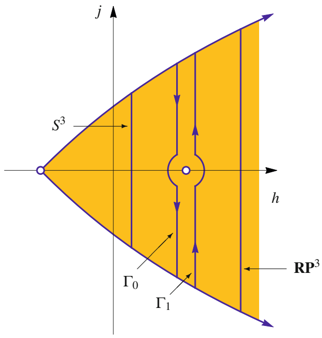


Figure 5.1.1. The geometric situation.

5.2 Monodromy of the bundle of period lattices

In this subsection we construct the bundle $\mathscr{P} \to \Gamma$ of period lattices over the closed curve Γ in the set $\mathscr{R}$ of regular values of $\mathscr{EM}$ such that its transition maps with respect to a collection of suitably chosen trivializations are fixed elements of $\mathrm{Sl}(2,\mathbf{Z})$. Computing the variation of these period lattices as the loop Γ is traversed once determines the monodromy map of the 2-torus bundle $\mathscr{EM}^{-1}(\Gamma) \to \Gamma$.

For each $(h,j) \in \Gamma$ the fiber $\mathscr{P}_{h,j}$ of the bundle $\mathscr{P} \to \Gamma$ of period lattices is the period lattice of the 2-torus $\mathscr{EM}^{-1}(h,j)$. The period lattice $\mathscr{P}_{h,j}$ is obtained in the following way. Let $\varphi_t^{H|TS^2}$ and $\varphi_s^{J|TS^2}$ be the flows of the Hamiltonian vector fields $X_{H|TS^2}$ and $X_{J|TS^2}$, respectively, of the Liouville integrable system $\big(H|TS^2, J|TS^2, TS^2, \omega|TS^2\big)$ determined by the spherical pendulum. Since $\mathscr{EM}^{-1}(h,j)$ is smooth, connected, compact submanifold of TS^2, which is invariant under the commuting flows $\varphi_t^{H|TS^2}$ and $\varphi_s^{J|TS^2}$, there is an $\mathbf{R}^2$-action given by

$$\Phi : \mathbf{R}^2 \times \mathscr{EM}^{-1}(h,j) \to \mathscr{EM}^{-1}(h,j) : \big((s,t),p\big) \mapsto \big(\varphi_s^{J|TS^2} \circ \varphi_t^{H|TS^2}\big)(p). \tag{65}$$

Fix $p_0 \in \mathscr{EM}^{-1}(h,j)$ and let $L_{p_0} = \{(T_1,T_2) \in \mathbf{R}^2 \big| \Phi_{(T_1,T_2)}(p_0) = p_0\}$ be the isotropy group of the action Φ at p_0. Because (h,j) is a regular value of $\mathscr{EM}$, the vector fields $X_{H|TS^2}$ and $X_{J|TS^2}$ give a basis for each tangent space to $\mathscr{EM}^{-1}(h,j)$ and so yield a framing of $\mathscr{EM}^{-1}(h,j)$. Consequently, Φ is a locally transitive action. Since $\mathscr{EM}^{-1}(h,j)$ is connected, the action Φ is transitive. Thus the isotropy group $L = L_{p_0}$ does *not* depend on the point p_0. Because $\mathscr{EM}^{-1}(h,j)$ is compact, L is a discrete subgroup of $(\mathbf{R}^2,+)$, which is a rank 2 is a lattice. The lattice L depends only on (h,j) and is called the *period lattice* $\mathscr{P}_{h,j}$ of $\mathscr{EM}^{-1}(h,j)$. From transitivity it follows that $\mathscr{EM}^{-1}(h,j)$ is diffeomorphic to the orbit space $\mathbf{R}^2/\mathbf{Z}^2$ of the linear action of L on $\mathbf{R}^2$. In other words, $\mathscr{EM}^{-1}(h,j)$ is diffeomorphic to the 2-torus $T^2 = \mathbf{R}^2/\mathbf{Z}^2$.

We now give an explicit description of the period lattice $\mathscr{P}_{h,j}$. For fixed $(h,j) \in \Gamma$ consider the Hamiltonian functions $F_1 = 2\pi J$ and $F_2 = -2\pi\theta(h,j)J + T(h,j)H$ on TS^2. Here we require that the rotation number $\theta(h,j)$ has its principal value, that is, $\theta(h,j) \in [\,0,1)$. From the definition of the rotation number θ and the time T of first return we see that the flows $\varphi_s^{F_1}$ and $\varphi_t^{F_2}$ of the Hamiltonian vector fields X_{F_1} and X_{F_2} on TS^2 have the following properties.

1. They commute.
2. They leave $\mathscr{EM}^{-1}(h,j)$ invariant.
3. They are periodic of period 1 on $\mathscr{EM}^{-1}(h,j)$.

Therefore with respect to the framing

$$\mathscr{EM}^{-1}(h,j) \to T\mathscr{EM}^{-1}(h,j) : p \mapsto \mathrm{span}\{X_{J|TS^2}(p), X_{H|TS^2}(p)\},$$

the period lattice $\mathscr{P}_{h,j}$ is generated by the vectors $\begin{pmatrix} 2\pi \\ 0 \end{pmatrix}$ and $\begin{pmatrix} -2\pi\theta(h,j) \\ T(h,j) \end{pmatrix}$, which do not depend on the point p in $\mathscr{EM}^{-1}(h,j)$.

We now construct the bundle $\mathscr{P} \to \Gamma$ of period lattices over Γ. Because the rotation number θ is *locally* a smooth on $\mathscr{R}$, while the time T of first return is a smooth on all of $\mathscr{R}$, and $\mathscr{R}$ is an open subset of $\mathbf{R}^2$, which retracts onto Γ, there is a *good* open covering $\mathscr{U}$ of $\mathscr{R}$ which restricts to a good open covering of Γ. This means that there is a covering $\{U^\alpha\}_{\alpha\in I}$ of $\mathscr{R}$ by open sets U^α, $\alpha \in I$ such that

1. U^α, $U^\alpha \cap U^\beta$, $U^\alpha \cap \Gamma$, and $U^\alpha \cap U^\beta \cap \Gamma$ are connected and contractible.
2. $U^\alpha \cap U^\beta \cap U^\gamma = \varnothing$.
3. The functions $\theta^\alpha = \theta|U^\alpha$ and $T^\alpha = T|U^\alpha$ are smooth and single valued.

Over $U^\alpha \cap \Gamma$ a parametrization of the bundle $\mathscr{P} \to \Gamma$ is given by the smooth family of lattices

$$\sigma_\alpha : \big(U^\alpha \cap \Gamma\big) \times \mathbf{Z}^2 \to \bigcup_{(h,j)\in U^\alpha\cap\Gamma} \mathscr{P}_{h,j} : \Big((h,j), \begin{pmatrix} n \\ m \end{pmatrix}\Big) \mapsto n\begin{pmatrix} 2\pi \\ 0 \end{pmatrix} + m\begin{pmatrix} -2\pi\theta^\alpha(h,j) \\ T^\alpha(h,j) \end{pmatrix}. \qquad (66)$$

Claim: On the overlap $U^\alpha \cap U^\beta \cap \Gamma$ the transition function for the period lattice bundle $\mathscr{P} \to \Gamma$ is

$$\begin{aligned} \sigma_{\alpha\beta} = \sigma_\beta \circ \sigma_\alpha^{-1} : (U^\alpha \cap U^\beta \cap \Gamma) \times \mathbf{Z}^2 &\to (U^\alpha \cap U^\beta \cap \Gamma) \times \mathbf{Z}^2 : \\ \Big((h,j), \begin{pmatrix} n \\ m \end{pmatrix}\Big) &\mapsto \Big((h,j), g_{\alpha\beta}\begin{pmatrix} n \\ m \end{pmatrix}\Big), \end{aligned} \qquad (67)$$

where $g_{\alpha\beta} \in \mathrm{Sl}(2,\mathbf{Z})$, which is specified in (71) below.

(5.3) **Proof**: For $i = 1,2$ consider the functions $F_i^\alpha : \mathscr{EM}^{-1}(U^\alpha) \to \mathbf{R} : p \mapsto F_i^\alpha(p)$ where

$$F_1^\alpha(p) = 2\pi J(p) \quad \text{and} \quad F_2^\alpha(p) = -2\pi\theta^\alpha(\mathscr{EM}(p))J(p) + T^\alpha(\mathscr{EM}(p))H(p).$$

The flows $\varphi_s^{F_1^\alpha}$ and $\varphi_t^{F_2^\alpha}$ of the Hamiltonian vector fields $X_{F_1^\alpha}$ and $X_{F_2^\alpha}$ on the open subset $\mathscr{E}\mathscr{M}^{-1}(U^\alpha)$ of TS^2 commute, leave $\mathscr{E}\mathscr{M}^{-1}(U^\alpha)$ invariant, and are periodic of period 1. Thus F_1^α and F_2^α are action variables on $\mathscr{E}\mathscr{M}^{-1}(U^\alpha)$, see chapter IX §2. So we have a $T^2 = \mathbf{R}^2/\mathbf{Z}^2$-action

$$\Phi^\alpha : T^2 \times \mathscr{E}\mathscr{M}^{-1}(U^\alpha) \to \mathscr{E}\mathscr{M}^{-1}(U^\alpha) : \big((s,t),p\big) \mapsto (\varphi_s^{F_1^\alpha} \circ \varphi_t^{F_2^\alpha})(p), \tag{68}$$

which is proper and free. Therefore $\mathscr{E}\mathscr{M}^{-1}(U^\alpha)$ is the total space of a *principal* T^2-bundle, see chapter VII ((2.12)), with a trivialization $\tau^\alpha : \mathscr{E}\mathscr{M}^{-1}(U^\alpha) \to U^\alpha \times T^2$ such that $\pi_1 \circ \tau^\alpha = \mathscr{E}\mathscr{M}|\mathscr{E}\mathscr{M}^{-1}(U^\alpha)$, where $\pi_1 : U^\alpha \times T^2 \to U^\alpha$ is projection on the first factor. The trivialization τ^α intertwines the T^2-action (68) with the T^2-action

$$\varphi^\alpha : T^2 \times (U^\alpha \times T^2) \to U^\alpha \times T^2 : \big((s,t),\big((h,j),(s',t')\big)\big) \mapsto \big((h,j),(s+s',t+t')\big),$$

that is,

$$\tau^\alpha\big(\Phi^\alpha_{(s,t)}(p)\big) = \varphi^\alpha_{(s,t)}(\mathscr{E}\mathscr{M}(p)), \tag{69}$$

for every $p \in \mathscr{E}\mathscr{M}^{-1}(h,j)$ and every $(s,t) \in T^2$. From the intertwining property (69) it follows that for every $(h,j) \in U^\alpha$ the tangent $T_{(s,t)}(\tau^\alpha)^{-1}_{(h,j)}$ of the mapping

$$(\tau^\alpha)^{-1}_{(h,j)} : T^2 \to \mathscr{E}\mathscr{M}^{-1}(h,j) : (s,t) \mapsto (\tau^\alpha)^{-1}\big((h,j),(s,t)\big) = r$$

at (s,t) sends the lattice $\mathbf{Z}^2 \subseteq T_{(s,t)}T^2$ onto the lattice in $T_r\mathscr{E}\mathscr{M}^{-1}(h,j)$ with basis $\{X_{F_1^\alpha}(r), X_{F_2^\alpha}(r)\}$. But this latter lattice is just the period lattice $\mathscr{P}_{h,j}$ of $\mathscr{E}\mathscr{M}^{-1}(h,j)$ at r. Since $\mathscr{P}_{h,j}$ does *not* depend on the point $r \in \mathscr{E}\mathscr{M}^{-1}(h,j)$, the mapping $T_{(s,t)}(\tau^\alpha)^{-1}_{h,j}$ does not depend on $(s,t) \in T^2$. Consequently, the inverse of the parametrization σ_α (66)

$$\sigma_\alpha^{-1} : \bigcup_{(h,j)\in U^\alpha\cap\Gamma} \mathscr{P}_{h,j} \to (U^\alpha \cap \Gamma) \times \mathbf{Z}^2 : \mathscr{P}_{h,j} \mapsto \big((h,j), T_p\tau^\alpha(\mathscr{P}_{h,j})\big)$$

does not depend on the point p in $\mathscr{E}\mathscr{M}^{-1}(h,j)$. Therefore for every $(h,j) \in U^\alpha \cap U^\beta \cap \Gamma$ we have the partial transition map

$$\tau^{\alpha\beta}_{(h,j)} = (\pi_2 \circ \tau^\beta) \circ (\tau^\alpha)^{-1}_{(h,j)} : T^2 \mapsto T^2, \tag{70}$$

where $\pi_2 : U^\alpha \times T^2 \to T^2$ is projection on the second factor. This partial transition map has a tangent $T_{(s,t)}\tau^{\alpha\beta}_{(h,j)}$, which does not depend on the point $(s,t) \in T^2$. Moreover, the map $T\tau^{\alpha\beta}_{(h,j)}$ is an invertible linear isomorphism of $\mathbf{R}^2$ onto itself, which preserves the lattice $\mathbf{Z}^2$. The set of all such linear isomorphisms forms the group $\mathrm{Sl}(2,\mathbf{Z})$, which is a discrete subgroup of the Lie group $\mathrm{Sl}(2,\mathbf{R})$. Since $U^\alpha \cap U^\beta \cap \Gamma$ is connected, it follows that the continuous map

$$U^\alpha \cap U^\beta \cap \Gamma \to \mathrm{Sl}(2,\mathbf{Z}) : (h,j) \mapsto T\tau^{\alpha\beta}_{(h,j)} \tag{71}$$

is constant, namely, $g_{\alpha\beta}$. Thus the mapping $\sigma_{\alpha\beta}$ (67) is the *transition map* for the bundle $\mathscr{P} \to \Gamma$ of period lattices. $\square$

▷ The period lattice bundle $\mathscr{P} \rightarrow \Gamma$ is isomorphic to the bundle of first homology groups $\mathrm{H}_1(\mathscr{EM}^{-1}(\Gamma), \mathbf{Z}) \rightarrow \Gamma$ of the fibration $\mathscr{EM}^{-1}(\Gamma) \rightarrow \Gamma$. Here we have $\mathrm{H}_1(\mathscr{EM}^{-1}(\Gamma), \mathbf{Z}) = \coprod_{(h,j)\in\Gamma} H_1(\mathscr{EM}^{-1}(h,j), \mathbf{Z})$.

(5.4) **Proof**: For every $(n,m) \in \mathbf{Z}^2$ and every $p \in \mathscr{EM}^{-1}(h,j)$ with $(h,j) \in U^{\alpha} \cap \Gamma$ consider the mapping

$$\psi^{\alpha} : \mathscr{P}_{h,j} \rightarrow \mathrm{H}_1(\mathscr{EM}^{-1}(h,j), \mathbf{Z}) : (nX_{F_1^{\alpha}} + mX_{F_2^{\alpha}})(p) \mapsto [\gamma_p^{n,m}], \tag{72}$$

where $[\gamma_p^{n,m}]$ is the homology class of the closed curve $\gamma_p^{n,m} : t \mapsto (\varphi_{nt}^{F_1^{\alpha}} \circ \varphi_{mt}^{F_2^{\alpha}})(p)$ on $\mathscr{EM}^{-1}(h,j)$. This class does not depend on the point p, because of the transitivity of the $\mathbf{R}^2$-action Φ (65). In addition, the period lattice $\mathscr{P}_{h,j}$ does not depend on the choice of the point p. Therefore the mapping ψ^{α} is well defined. The closed curve $\gamma_p^{n,m}$ is homotopic to n times the closed curve $\gamma^1 : [0,1] \rightarrow \mathscr{EM}^{-1}(h,j) : t \mapsto \varphi_t^{F_1^{\alpha}}(p)$ followed by m times the curve $\gamma^2 : [0,1] \rightarrow \mathscr{EM}^{-1}(h,j) : t \mapsto \varphi_t^{F_2^{\alpha}}(p)$. Thus $[\gamma_p^{n,m}] = n[\gamma^1] + m[\gamma^2]$. So the mapping ψ^{α} (72) is linear. Since $[\gamma^1]$ and $[\gamma^2]$ generate the lattice $\mathrm{H}_1(\mathscr{EM}^{-1}(h,j), \mathbf{Z})$, the map ψ^{α} is an isomorphism of lattices. Therefore as a bundle map covering the identity map on $U^{\alpha} \cap \Gamma$, the mapping ψ^{α} is an isomorphism of the bundle $\coprod_{(h,j)\in U^{\alpha}\cap\Gamma} \mathscr{P}_{h,j} \rightarrow U^{\alpha} \cap \Gamma$ onto the bundle $\coprod_{(h,j)\in U^{\alpha}\cap\Gamma} \mathrm{H}_1\big(\mathscr{EM}^{-1}(h,j), \mathbf{Z}\big) \rightarrow U^{\alpha} \cap \Gamma$. So the bundle of period lattices over Γ is isomorphic to the bundle of first homology groups assoicated to $\mathscr{EM}^{-1}(\Gamma) \rightarrow \Gamma$. □

Claim: The monodromy map of the bundle $\mathscr{EM}^{-1}(\Gamma) \rightarrow \Gamma$ with bundle projection map $\mathscr{EM}$ is $\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$.

(5.5) **Proof**: We find the monodromy map by computing the *variation* of the period lattice along the positively oriented curve Γ in $\mathscr{R}$. First we choose a good open covering $\{U^{\alpha}\}_{\alpha\in I}$ of the set $\mathscr{R}$ of regular values of $\mathscr{EM}$ so that $\{U^{\alpha}\}_{\alpha=1}^{\ell}$ is a finite good covering of Γ such that $U^{\alpha} \cap U^{\alpha+1} \neq \varnothing$ for every $1 \leq \alpha \leq \ell - 1$. For $(h_0, j_0) \in U^1 \cap \Gamma$ set $\theta^1(h_0, j_0) = 0$. This fixes the global multivalued function θ on $\mathscr{R}$. Then $\theta^{\ell}(h_0, j_0) = -1$ by ((4.9)). For each $1 \leq \alpha \leq \ell - 1$ on the *overlap* $U^{\alpha} \cap U^{\alpha+1}$ we have $\theta^{\alpha} = \theta^{\alpha+1}$ and $T^{\alpha} = T^{\alpha+1}$. Therefore on $U^{\alpha} \cap U^{\alpha+1}$ the transition map $g_{\alpha,\alpha+1}$ for the period lattice bundle is the identity matrix in $\mathrm{Sl}(2, \mathbf{Z})$. In other words, there is *no* variation in the period lattice along Γ when going from $U^{\alpha} \cap \Gamma$ to $U^{\alpha+1} \cap \Gamma$. Of course the period lattice varies on each U^{α}. On the remaining overlap $U^{\ell} \cap U^1$ the period lattice generated by $\{ \begin{pmatrix} 2\pi \\ 0 \end{pmatrix}, \begin{pmatrix} -2\pi\theta^{\ell}(h,j) \\ T^{\ell}(h,j) \end{pmatrix} \}$ is transformed into the period lattice generated by

$$\left\{ \begin{pmatrix} 2\pi \\ 0 \end{pmatrix}, \begin{pmatrix} -2\pi\theta^1(h,j) \\ T^1(h,j) \end{pmatrix} \right\} = \left\{ \begin{pmatrix} 2\pi \\ 0 \end{pmatrix}, \begin{pmatrix} -2\pi\big(\theta^{\ell}(h,j)+1\big) \\ T^{\ell}(h,j) \end{pmatrix} \right\}, \text{ since } T \text{ is single valued on } \mathscr{R}$$

$$= \left\{ \begin{pmatrix} 2\pi \\ 0 \end{pmatrix}, -\begin{pmatrix} 2\pi \\ 0 \end{pmatrix} + \begin{pmatrix} -2\pi\theta^{\ell}(h,j) \\ T^{\ell}(h,j) \end{pmatrix} \right\}.$$

So when going from $U^{\ell} \cap \Gamma$ to $U^1 \cap \Gamma$ the transition map $g_{\ell,1}$ for the period lattice bundle is the matrix $\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$. Hence the *monodromy map* $\chi_* : \mathrm{H}_1(\mathscr{EM}^{-1}(h_0, j_0), \mathbf{Z}) \rightarrow$

$\mathrm{H}_1(\mathscr{EM}^{-1}(h_0, j_0), \mathbf{Z})$ of the bundle $\mathscr{EM}^{-1}(\Gamma) \rightarrow \Gamma$ is the variation of the period lattice around Γ. This variation is just the product $g_{\ell,1} \prod_{\alpha=1}^{\ell-1} g_{\alpha,\alpha+1}$ of the variation on the overlaps, which is $\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$. $\square$

We now give a more *physical argument* to determine the monodromy map of the spherical pendulum. Let Γ be a positively oriented circle in $\mathscr{R}$ with center $(1,0)$ and of radius ε. Let $p_{\pm} = (\pm 1, 0) \in \Gamma$. Let $\Gamma^{\pm} = (\pm\Gamma) \cap \{\pm j \geq 0\}$ be closed oriented semicircular arcs of Γ, which join p_+ to p_-. Let $p^{\pm} \in \mathscr{EM}^{-1}(p_{\pm})$. The integral curve γ of the Hamiltonian vector field $X_{H|TS^2}$, which starts at p^+, is periodic of period $T(1+\varepsilon,0)$, has time of first return $T(1+\varepsilon,0)$, and has rotation number 0. Also the integral curve δ of the Hamiltonian vector field $X_{J|TS^2}$ starting at p_+ is periodic of period 2π with rotation number 0. Therefore the period lattice $\mathscr{P}_{p_+}$ is generated by the vectors $\begin{pmatrix} 2\pi \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ T(1+\varepsilon,0) \end{pmatrix}$. Transported along the arc $(\Gamma^-)^{-1}$ joining p_+ to p_- the period lattice $\mathscr{P}_{p_+}$ becomes the period lattice $\mathscr{P}_{p_-}$, which is generated by the vectors $\begin{pmatrix} 2\pi \\ 0 \end{pmatrix}$ and $\begin{pmatrix} \pi \\ T(1-\varepsilon,0) \end{pmatrix}$. This follows because transporting the first homology class $[\gamma]$ of $\mathscr{EM}^{-1}(p_+)$, represented by the closed curve γ, along $(\Gamma^-)^{-1}$ results in the homology class $[\gamma^-]$, represented by γ^-. Moreover, γ^- is an integral curve of $X_{H|TS^2}$ starting at $p^- \in \mathscr{EM}^{-1}(p_-)$ that is periodic of period $2T(1-\varepsilon,0)$, has first return time $T(1-\varepsilon,0)$, and rotation number $-\frac{1}{2}$, because $\Gamma^- \setminus \{p_\pm\}$ lies in $\mathscr{R} \cap \{j < 0\}$. Also transporting $[\delta]$ along $(\Gamma^-)^{-1}$ gives $[\delta^-]$, which is represented by the integral curve δ^{-1} of $X_{J|TS^2}$ starting at p^- and is periodic of period 2π with rotation number 0. Therefore we have constructed an invertible linear map

$$M^- : \mathscr{P}_{p_+} \rightarrow \mathscr{P}_{p_-} : \left\{ \begin{pmatrix} 2\pi \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ T(1+\varepsilon,0) \end{pmatrix} \right\} \mapsto \left\{ \begin{pmatrix} 2\pi \\ 0 \end{pmatrix}, \begin{pmatrix} \pi \\ T(1-\varepsilon,0) \end{pmatrix} \right\}.$$

Transported along the arc Γ^+ joining p_+ to p_- the period lattice $\mathscr{P}_{p_+}$ becomes the period lattice $\mathscr{P}_{p_-}$, which is generated by the vectors $\{\begin{pmatrix} 2\pi \\ 0 \end{pmatrix}, \begin{pmatrix} -\pi \\ T(1+\varepsilon,0) \end{pmatrix}\}$. This follows because transporting the homology class $[\gamma]$ gives the homology class $[\gamma^+]$, represented by the curve γ^+, which is an integral curve of $X_{H|TS^2}$ starting at p^- that is periodic of period $2T(1-\varepsilon,0)$, has first return time $T(1-\varepsilon,0)$, and rotation number $\frac{1}{2}$, because $\Gamma^+ \setminus \{p_\pm\}$ lies in $\mathscr{R} \cap \{j > 0\}$. Also transporting the homology class $[\delta]$ along Γ^+ gives the homology class $[\delta^-]$, where δ^- is an integral curve of $X_{J|TS^2}$ starting at p^- that is periodic of period 2π with rotation number 0. Therefore we get an invertible linear map

$$M^+ : \mathscr{P}_{p_+} \rightarrow \mathscr{P}_{p_-} : \left\{ \begin{pmatrix} 2\pi \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ T(1+\varepsilon,0) \end{pmatrix} \right\} \mapsto \left\{ \begin{pmatrix} 2\pi \\ 0 \end{pmatrix}, \begin{pmatrix} -\pi \\ T(1-\varepsilon,0) \end{pmatrix} \right\}.$$

So the invertible linear map

$$\begin{aligned} M = (M^+)^{-1} \circ M^- &: \mathscr{P}_{p_+} \rightarrow \mathscr{P}_{p_+} : \\ &\left\{ \begin{pmatrix} 2\pi \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ T(1+\varepsilon,0) \end{pmatrix} \right\} \mapsto \left\{ \begin{pmatrix} 2\pi \\ 0 \end{pmatrix}, \begin{pmatrix} -2\pi \\ T(1-\varepsilon,0) \end{pmatrix} = -\begin{pmatrix} 2\pi \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ T(1+\varepsilon,0) \end{pmatrix} \right\} \end{aligned}$$

has a matrix $\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$. Since δ and γ are closed curves on $\mathscr{EM}^{-1}(p_+)$, they represent the homology classes $[\delta]$ and $[\gamma]$ which form a basis of $\mathrm{H}_1(\mathscr{EM}^{-1}(p_+), \mathbf{Z})$. Thus the monodromy map is

$$M_* : \mathrm{H}_1(\mathscr{EM}^{-1}(p_+),\mathbf{Z}) \to \mathrm{H}_1(\mathscr{EM}^{-1}(p_+),\mathbf{Z}) : \begin{pmatrix} [\delta] \\ [\gamma] \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} [\delta] \\ [\gamma] \end{pmatrix}. \tag{73}$$

Now consider the integral symplectic intersection form $\langle \, , \rangle$ on the first homology group $\mathrm{H}_1(\mathscr{EM}^{-1}(p_+),\mathbf{Z})$, which comes from the natural orientation of $\mathscr{EM}^{-1}(p_+) = T^2 = \mathbf{R}^2/\mathbf{Z}^2$. This intersection form is defined by

$$\langle [\delta],[\delta] \rangle = 0, \ \langle [\gamma],[\gamma] \rangle = 0, \ \text{and} \ \langle [\gamma],[\delta] \rangle = 1.$$

▷ Then the monodromy map M_* (73) associated to a small positively oriented loop Γ around $(1,0)$ in $\mathscr{R}$ satisfies

$$M_*([\lambda]) = [\lambda] - \langle [\lambda],[\delta] \rangle \, [\delta] \tag{74}$$

for every $[\lambda] \in \mathrm{H}_1(\mathscr{EM}^{-1}(p_+),\mathbf{Z})$, where $p_+ \in \Gamma$. Equation (74) is called the *Picard-Lefschetz* formula with δ being the *vanishing cycle*.

6 Exercises

1. (Weierstrass elliptic functions.) Consider the smooth affine elliptic curve E defined by $\{(x,y) \in \mathbf{C}^2 \,\big|\, y^2 = 4(x-e_1)(x-e_2)(x-e_3)\}$, where $e_3 < e_2 < e_1$ and $e_1 + e_2 + e_3 = 0$. The compact Riemann surface corresponding to $\overline{E} \subseteq \mathbf{CP}^1$ is a 2-torus $\mathbf{C}/\Lambda$, where Λ is a lattice.

   a) (The $\wp$ function.) The Weierstrass $\wp$ function corresponding to Λ is

$$\wp(\mu) = \frac{1}{\mu^2} + \sum_{\nu \in \Lambda^* = \Lambda \setminus \{0\}} \left[\frac{1}{(\mu-\nu)^2} - \frac{1}{\nu^2} \right]. \tag{1}$$

   Prove the following properties.

   i. $\wp$ is meromorphic with poles only on Λ, which are second order. In particular, $\wp(\mu) = \frac{1}{\mu^2} + \mathrm{O}(1)$.

   ii. $\wp$ is an even function, that is, $\wp(-\mu) = \wp(\mu)$.

   iii. $\wp$ is doubly periodic, that is, $\wp(\mu+\nu) = \wp(\mu)$ for every $\nu \in \Lambda$.

   iv. For every $a,b \in \mathbf{C}$ we have the addition formula

$$\wp(a+b) = \frac{1}{4} \left(\frac{\wp'(a) - \wp'(b)}{\wp(a) - \wp(b)} \right)^2 - \wp(a) - \wp(b). \tag{2}$$

   Hint: The complex analytic functions $W(\mu) = \wp(\mu+b) + \wp(\mu)$ and $Z(\mu) = \frac{1}{4}\left(\frac{\wp'(\mu)-\wp'(b)}{\wp(\mu)-\wp(b)}\right)^2 - \wp(\mu)$ have poles only at $-b+\Lambda$ and $0+\Lambda$ and satisfy $\frac{1}{(\mu+c+\Lambda)^2} + \mathrm{O}(1)$, where $c = b$ or 0. Deduce that $-Z(\mu) + W(\mu)$ is constant on $\overline{E} \subseteq \mathbf{CP}^1$. From $W(a) = \frac{1}{(a+b)^2} + \wp(a) + \mathrm{O}(1)$ deduce (2).

v.* Show that $\pi : \mathbf{C} \to E \subseteq \mathbf{C} : \mu \mapsto \big(x(\mu), y(\mu)\big) = \big(\wp(\mu), \wp'(\mu)\big)$ parametrizes the curve E.

vi. Let $\mu = -\int_z^\infty \frac{\mathrm{d}x}{y}$. With $x \in \mathbf{C}^\vee = (\mathbf{C} \cup \{\infty\}) \setminus ([e_3, e_2] \cup [e_1, \infty])$ we have chosen the square root in the definition of μ so that for $j = 1, 2, 3$ we have $x - e_j = r_j \mathrm{e}^{i\theta_i}$, where $r_j > 0$ and $0 \le \theta_j \le 2\pi$. Thus $y(x) = 2\sqrt{r_1 r_2 r_3}\,\mathrm{e}^{i(\theta_1+\theta_2+\theta_3)/2}$. Show that the conformal map

$$\wp^{-1} : \{z \in \mathbf{C} \,\big|\, \{\mathrm{Im}\, z > 0\} \to \mathbf{C} : z \mapsto -\int_z^\infty \frac{\mathrm{d}x}{y} \tag{3}$$

transforms the positively oriented upper half plane onto the interior of the positively oriented rectangle R with vertices at 0, $\frac{1}{2}\lambda'$, $-\frac{1}{2}\lambda + \frac{1}{2}\lambda'$, $-\frac{1}{2}\lambda$ being the image of $-\infty$, e_3, e_2, and e_1, respectively. Here

$$\tfrac{1}{2}\lambda = \int_{e_1}^\infty \frac{\mathrm{d}x}{y} > 0 \text{ and } \tfrac{1}{2i}\lambda' = \int_{-\infty}^{e_3} \frac{\mathrm{d}x}{y} > 0. \tag{4}$$

Show that (3) is the inverse of $\wp$.

b) (The ζ function.) The Weierstrass ζ function is defined by

$$\zeta(\mu) = \frac{1}{\mu} + \sum_{\nu \in \Lambda^*} \left(\frac{1}{\mu - \nu} + \frac{1}{\nu} + \frac{\mu}{\nu^2} \right). \tag{5}$$

Show that

i. $\zeta'(\mu) = -\wp(\mu)$.

ii. Show that ζ is an odd function, that is, $\zeta(-\mu) = -\zeta(\mu)$.

iii. The ζ function is quasi-periodic, that is, for every $\nu \in \Lambda$ there is a unique $\eta(\nu) \in \mathbf{C}$ such that $\zeta(\mu + \nu) = \zeta(\mu) + \eta(\nu)$.

iv. Show that for every $\nu \in \Lambda$ we have $\zeta(\frac{1}{2}\nu) = \frac{1}{2}\eta(\nu)$.

v. Show that

$$\zeta(\mu - a) - \zeta(\mu) + \zeta(a) = \tfrac{1}{2} \left(\frac{\wp'(\mu) + \wp'(a)}{\wp(\mu) - \wp(a)} \right). \tag{6}$$

Hint: The functions $W(\mu) = \frac{1}{2}\left(\frac{\wp'(\mu)+\wp'(a)}{\wp(\mu)-\wp(a)}\right)$ and $Z(\mu) = \zeta(\mu - a) - \zeta(\mu)$ both have poles at $0 + \Lambda$ and $a + \Lambda$, which are first order with residue -1 and 1, respectively. Then compete the proof as in exercise a) iv.

Show that the function η has the following properties.

vi. $\nu \mapsto \eta(\nu)$ is a $\mathbf{Z}$-linear function on Λ.

vii. Let $\{\lambda, \lambda'\}$ is a $\mathbf{Z}$-basis for the lattice Λ. Then

$$\lambda\eta(\lambda') - \lambda'\eta(\lambda) = 2\pi i, \tag{7}$$

which is Legendre's relation. Hint: Integrate ζ around a suitable rectangular contour.

c) (The σ function.) Define the Weierstrass σ function by

$$\sigma(\mu) = \mu \prod_{\nu \in \Lambda^*} \Big(1 - \frac{\mu}{\nu}\Big)\, \mathrm{e}^{\frac{\mu}{\nu} + \frac{1}{2}(\frac{\mu}{\nu})^2}. \tag{8}$$

i. Show that $\frac{\mathrm{d}}{\mathrm{d}\mu} \log \sigma = \zeta$.

ii. Show that for every $\nu \in \Lambda$ we have

$$\log\Big(\frac{\sigma(\mu+\nu)}{\sigma(\mu)}\Big) = \tfrac{1}{2}\nu + \eta(\nu)\mu. \tag{9}$$

2. (Formula for rotation number.) Consider the smooth affine elliptic curve $\widetilde{E}_c$ defined by

$$v^2 = \widetilde{p}_c(u) = 2(u^2-1)(c_1-u) - \tfrac{1}{2}c_2^2,$$

where $c = (c_1, c_2) \in (\mathbf{R} \times \mathbf{R}_{\ge 0}) \setminus \{\operatorname{discr} \widetilde{p}_c = 0\}$. The polynomial $\widetilde{p}_c$ has three distinct real roots u_j such that $-1 < u_3 < 0 < u_2 < 1 < u_1$ and is positive on (u_3, u_2). The change of variables

$$u = 2x - \tfrac{1}{3}c_1 \quad \text{and} \quad v = 2y$$

transforms the curve $\widetilde{E}_c$ into the elliptic curve E_c defined by $y^2 = 4x^3 - g_2 x + g_3$, where $g_2 = \frac{1}{3}c_1^2 - 1$ and $g_3 = \frac{1}{3}c_1 - \frac{1}{27}c_1^3 - \frac{1}{4}c_2^2$. E_c is in Weierstrass normal form $y^2 = 4(x-e_1)(x-e_2)(x-e_3)$ where for $j = 1,2,3$ we have $e_j = \frac{1}{2}(u_j + \frac{1}{3}c_1)$ with $\frac{1}{2}(-1+\frac{1}{3}c_1) < e_3 < e_2 < \frac{1}{2}(1+\frac{1}{3}c_1) < e_1$ and $e_1 + e_2 + e_3 = 0$. Let Λ_c be the lattice corresponding to E_c with $\mathbf{Z}$-basis $\{\lambda, \lambda'\}$ with $\lambda, \frac{1}{i}\lambda' \in \mathbf{R}_{>0}$. Let $\wp$ be the Weierstrass elliptic function corresponding to Λ_c, where $\wp(-\frac{1}{2}\lambda) = e_1$, $\wp(-\frac{1}{2}\lambda + \frac{1}{2}\lambda') = e_2$, and $\wp(\frac{1}{2}\lambda') = e_3$. Note that the mapping $\pi : \mathbf{C} \to E_c \subseteq \mathbf{C}^2 : \mu \mapsto \big(x(\mu), y(\mu)\big) = \big(\wp(\mu), \wp'(\mu)\big)$, which is a parametrization of E_c, is also its universal covering map.

a) The angular period $\Theta(c) = \int_{u_3}^{u_2} \frac{2c_2}{1-u^2}\,\frac{\mathrm{d}u}{v}$ is the rotation number of the spherical pendulum when $(c_1, c_2) = (h, j)$. Let $\widetilde{\Phi} = \frac{uv + ic_2}{u^2-1}\,\frac{\mathrm{d}u}{v}$. Show that $\widetilde{\Phi}$ is a meromorphic 1-form on $\overline{E}_c = E_c \cup \{\infty\} \subseteq \mathbf{CP}^1$ with poles only at $\widetilde{P}_1 = (1, ic_2)$, $\widetilde{P}_2 = (-1, -ic_2)$, and ∞ having residues 1, 1, and -2, respectively. Let $\widetilde{\Gamma}$ be a positively oriented closed loop in $\mathbf{C}$, which encircles the cut $[u_3, u_2]$ so that ± 1 lie in its exterior. Show that $\Theta(c) = -i \int_{\widetilde{\Gamma}} \widetilde{\Phi}$. Changing to x, y variables show that $\widetilde{\Phi}$ pulls back to a 1-form Φ on E_c, which can be written as

$$\Phi = \Phi_{P_1} + \Phi_{P_2} = \tfrac{1}{2}\frac{y + y_{P_1}}{x - x_{P_1}}\,\frac{\mathrm{d}x}{y} + \tfrac{1}{2}\frac{y + y_{P_2}}{x - x_{P_2}}\,\frac{\mathrm{d}x}{y}.$$

Here $P_1 = (x_{P_1}, y_{P_1}) = \big(\frac{1}{2}(1+\frac{1}{3}c_1), \frac{1}{2}ic_2\big)$ and $P_2 = (x_{P_2}, y_{P_2}) = \big(\frac{1}{2}(-1+\frac{1}{3}c_1), -\frac{1}{2}ic_2\big)$. Show that the 1-forms Φ_{P_j} are meromorphic on $\overline{E}_c$ with poles only at P_j and ∞, which are first order and have residues 1 and -1, respectively.

b) Let $Q=(x_Q,y_Q)=\big(\wp(a),\wp'(a)\big)$ be a point on E_c. Consider the 1-form Φ_Q. Let Γ' be a closed loop on E_c, whose universal cover is the closed line segment $\overline{\mu_0-\lambda+\lambda',\mu_0-\lambda}$. The following argument shows that

$$I=\int_{\Gamma'}\Phi_Q=\eta(\lambda')a-\zeta(a)\lambda'=\int_{e_2}^{x_Q}[\eta(\lambda')+\lambda' x]\,\frac{\mathrm{d}x}{y}+\pi i. \tag{10}$$

In what follows we use the Weierstrass elliptic functions $\wp$, η, ζ, and σ associated to the lattice Λ_c. Show that the pull back of Φ_Q by the universal covering map is the 1-form

$$\tfrac{1}{2}\Big(\frac{\wp'(\mu)+\wp'(a)}{\wp(\mu)-\wp(a)}\Big)\,\mathrm{d}\mu=[\zeta(\mu-a)-\zeta(\mu)+\zeta(a)]\,\mathrm{d}\mu. \tag{11}$$

Using the definition and properties of the σ function, justify each step of the next calculation.

$$\begin{aligned}
I&=-\log\frac{\sigma(\mu_0-a-\lambda+\lambda')}{\sigma(\mu_0-a-\lambda)}+\log\frac{\sigma(\mu_0-\lambda+\lambda')}{\sigma(\mu_0-\lambda)}-\zeta(a)\lambda'\\
&=-\log\frac{\sigma(\mu_0-a-\lambda+\lambda')}{\sigma(\mu_0-a)}+\log\frac{\sigma(\mu_0-a-\lambda)}{\sigma(\mu_0-a)}\\
&\qquad+\log\frac{\sigma(\mu_0-\lambda+\lambda')}{\sigma(\mu_0)}-\log\frac{\sigma(\mu_0-\lambda)}{\sigma(\mu_0)}-\zeta(a)\lambda'\\
&=-\eta(-\lambda+\lambda')(\mu_0-a)-\tfrac{1}{2}(-\lambda+\lambda')+\eta(-\lambda)(\mu_0-a)-\tfrac{1}{2}\lambda\\
&\qquad+\eta(-\lambda+\lambda')\mu_0+\tfrac{1}{2}(-\lambda+\lambda')-\eta(-\lambda)\mu_0+\tfrac{1}{2}\lambda-\zeta(a)\lambda'\\
&=\eta(\lambda')a-\zeta(a)\lambda'.
\end{aligned}$$

Integrating $\frac{\mathrm{d}I}{\mathrm{d}a}=\eta(\lambda')+\lambda'\wp(a)$ from a to $e=-\frac{1}{2}\lambda$ gives

$$I(a)-I(e)=\int_e^a\frac{\mathrm{d}I}{\mathrm{d}a}\,\mathrm{d}a=\int_e^a[\eta(\lambda')+\lambda'\wp(\mu)]\,\mathrm{d}\mu=\int_{e_1}^{x_Q}[\eta(\lambda')+\lambda' x]\,\frac{\mathrm{d}x}{y}.$$

But

$$\begin{aligned}
I(e)&=\int_{\Gamma'}\Phi_{(e_1,0)}=\eta(\lambda')e-\zeta(e)\lambda'=\eta(\lambda')(-\tfrac{1}{2}\lambda)-\zeta(-\tfrac{1}{2}\lambda)\lambda'\\
&=\tfrac{1}{2}[\eta(\lambda)\lambda'-\eta(\lambda')\lambda]=\pi i,
\end{aligned}$$

using Legendre's relation.

c) Let $Q=(x_Q,y_Q)=\big(\wp(a),\wp'(a)\big)$ be a point on E_c and let Γ be a closed loop on E_c, whose universal covering is the closed line segment $\overline{\mu_0+\lambda',\mu_0-\lambda+\lambda'}$. Using an argument similar to the one in b) show that

$$\int_\Gamma\Phi_Q=\eta(\lambda)a-\zeta(a)=\int_{e_1}^{x_Q}[\eta(\lambda)+\lambda x]\frac{\mathrm{d}x}{y}. \tag{12}$$

d) Let $\tau=\lambda/\lambda'$. Show that

$$a\eta(\lambda)-\zeta(a)\lambda=\tau[a\eta(\lambda')-\zeta(a)\lambda']+2\pi i\,\frac{a}{\lambda'}$$

and that

$$2\pi i\,\frac{a}{\lambda'} = \frac{2\pi i}{\lambda'}\int_{-\frac{1}{2}\lambda+\frac{1}{2}\lambda'}^{a} \mathrm{d}\mu - \tau\pi i + \pi i = \frac{2\pi i}{\lambda'}\int_{e_2}^{x_Q}\frac{\mathrm{d}x}{y} - \tau\pi i + \pi i.$$

Using (10) and the above formulæ deduce

$$a\eta(\lambda) - \zeta(a)\lambda = \tau\int_{e_1}^{x_Q}[\eta(\lambda')+\lambda' x]\,\frac{\mathrm{d}x}{y} + \frac{2\pi i}{\lambda'}\int_{e_2}^{x_Q}\frac{\mathrm{d}x}{y} + \pi i. \tag{13}$$

When $a = -\frac{1}{2}\lambda + \frac{1}{2}\lambda'$ we have

$$\begin{aligned} a\eta(\lambda') - \zeta(a)\lambda' &= (-\tfrac{1}{2}\lambda + \tfrac{1}{2}\lambda')\eta(\lambda') - \zeta(-\tfrac{1}{2}\lambda + \tfrac{1}{2}\lambda')\lambda' \\ &= (-\tfrac{1}{2}\lambda + \tfrac{1}{2}\lambda')\eta(\lambda') - [\tfrac{1}{2}\eta(\lambda') - \tfrac{1}{2}\eta(\lambda)]\lambda' \\ &= \tfrac{1}{2}[\eta(\lambda)\lambda' - \eta(\lambda')\lambda] = \pi i. \end{aligned}$$

Since $x_Q = \wp(a) = e_2$, equation (13) reads $\int_{e_1}^{e_2}[\eta(\lambda')+\lambda' x]\,\frac{\mathrm{d}x}{y} = 0$. From (12) and (13) we obtain

$$\int_\Gamma \Phi_Q = \tau\int_{e_2}^{x_Q}[\eta(\lambda')+\lambda' x]\,\frac{\mathrm{d}x}{y} + \frac{2\pi i}{\lambda'}\int_{e_2}^{x_Q}\frac{\mathrm{d}x}{y} + \pi i. \tag{14}$$

e) Because $x_{P_1} \in (-\infty, e_3)$ and $x_{P_2} \in (e_2, e_1)$ we may choose $a \in \overline{0, \frac{1}{2}\lambda'}$ and $b \in \overline{-\frac{1}{2}\lambda + \frac{1}{2}\lambda', -\frac{1}{2}\lambda}$ so that $P_1 = \big(\wp(a), \wp'(a)\big)$ and $P_2 = \big(\wp(b), \wp'(b)\big)$. Let Γ be the loop on E_c obtained by expressing the loop $\widetilde{\Gamma}$ on $\widetilde{E}_c$ in terms of x, y variables. The universal cover of Γ is the closed line segment $\overline{\lambda', -\lambda + \lambda'}$. Justify each step of the following calculation.

$$\begin{aligned} \Theta(c) &= -i\int_\Gamma \Phi = -i\int_\Gamma \Phi_{P_1} - i\int_\Gamma \Phi_{P_2} \\ &= -i[(a+b)\eta(\lambda) - (\zeta(a)+\zeta(b)\lambda] \qquad (15) \\ &= -i[(a+b)\eta(\lambda) - \zeta(a+b)\lambda + \tfrac{1}{2}ic_2\lambda], \\ &\qquad \text{since } \zeta(a+b) - \zeta(a) - \zeta(b) = \tfrac{1}{2}\big(\tfrac{\wp'(a)-\wp'(b)}{\wp(a)-\wp(b)}\big) = \tfrac{1}{2}ic_2 \\ &= -i\tau\int_{e_2}^{x_{Q'}}[\eta(\lambda')+\lambda' x]\,\frac{\mathrm{d}x}{y} + \frac{2\pi}{\lambda'}\int_{e_2}^{x_{Q'}}\frac{\mathrm{d}x}{y} + \tfrac{1}{2}c_2\lambda + \pi. \qquad (16) \end{aligned}$$

Here

$$x_{Q'} = \wp(a+b) = \frac{1}{4}\Big(\frac{\wp'(a)-\wp'(b)}{\wp(a)-\wp(b)}\Big)^2 - \wp(a) - \wp(b) = -\tfrac{1}{4}c_2^2 + \tfrac{1}{3}c_1. \tag{17}$$

Since $a+b \in \overline{-\frac{1}{2}\lambda, -\frac{1}{2}\lambda + \frac{1}{2}\lambda'}$, we get $\lambda' - a - b \in \overline{-\frac{1}{2}\lambda + \frac{1}{2}\lambda', -\frac{1}{2}\lambda}$. Therefore $\wp(a+b) = \wp(a+b-\lambda') = \wp(\lambda' - a - b) \in [e_2, e_1]$.

3. (Estimate for the rotation number.) If (h, ℓ) is a regular values of the energy momentum map $\mathscr{E}\mathscr{M}$ of the spherical pendulum, then 2π times the rotation number of

the flow on the 2-torus $\mathscr{E}\mathscr{M}^{-1}(h,\ell)$ is

$$\theta(h,\ell) = 2\ell \int_{x_3^-}^{x_3^+} \frac{\mathrm{d}x_3}{(1-x_3^2)\sqrt{2(h-x_3)(1-x_3^2)-\ell^2}}. \tag{18}$$

Note that $\theta(h,-\ell) = -\theta(h,\ell)$, when $\ell \neq 0$. The goal of this exercise is to show that

$$\pi < \theta(h,\ell) < 2\pi, \tag{19}$$

when $\ell > 0$. We will use complex analysis.

a) Let

$$\omega = \frac{\ell\,\mathrm{d}z}{(1-z^2)\sqrt{2(h-z)(1-z^2)-\ell^2}}.$$

Since $-1 < x_3^- < x_3^+ < 1 < x_3^0$ are real roots of $2(h-z)(1-z^2)-\ell^2$, we cut the extended complex plane $\widetilde{\mathbf{C}}$ along the real axis between x_3^- and x_3^+ and again between x_3^0 and ∞. Choosing the square root as in ((4.5)), we see that ω is a meromorphic 1-form on $\mathbf{C}^\vee = \widetilde{\mathbf{C}} \setminus ([x_3^-,x_3^+] \cup [x_3^0,\infty])$ with first order poles at ± 1. Show that $\operatorname*{Res}_{z=\pm 1} \omega = \frac{1}{2}\,i$. Let $\mathscr{C}_1$, $\mathscr{C}_3$, and $\mathscr{C}_4$ be positively oriented closed curves in $\mathbf{C}^\vee$ which enclose $[x_3^-,x_3^+]$ but not ± 1, $[x_3^-,x_3^+]$ and -1 but not $+1$ and $[x_3^-,x_3^+]$ and ± 1, respectively. Show that

$$\begin{aligned}
\theta(h,\ell) &= \int_{\mathscr{C}_1} \omega = \int_{\mathscr{C}_3} \omega - 2\pi i \operatorname*{Res}_{z=-1} \omega \\
&= \int_{\mathscr{C}_4} \omega - 2\pi i \big(\operatorname*{Res}_{z=-1} \omega + \operatorname*{Res}_{z=1} \omega\big) = \int_{\mathscr{C}_4} \omega + 2\pi.
\end{aligned} \tag{20}$$

Let $\mathscr{C}_2$ be the *positively* oriented curve in $\mathbf{C}^\vee$ which encloses the cut $[x_3^0,\infty]$ but not 1. Show that $\mathscr{C}_2$ goes from near ∞ to x_3^- along the underside of the cut and then back to the original starting point. Show that the curves $\mathscr{C}_4$ and $\mathscr{C}_2$ are homotopic in $\mathbf{C}^\vee \setminus \{\pm 1\}$. Deduce that

$$\int_{\mathscr{C}_4} \omega = \int_{\mathscr{C}_2} \omega = -2\int_{x_3^0}^{\infty} \frac{1}{(x_3^2-1)\sqrt{2(h-x_3)(1-x_3^2)-\ell^2}}\,\mathrm{d}x_3. \tag{21}$$

Since $x_3^0 > 1$, the integrand in (21) is positive. Therefore from (20) we obtain $\theta(h,\ell) < 2\pi$.

b) Consider the positively oriented vertical line $L : z = \xi + i\,\eta$ where $\eta \in [-\infty,\infty]$, $\xi \in (x_3^+,1)$ and L is closer to x_3^- than x_3^+, that is, $x_3^- + \xi < x_3^+ - \xi$. Show that L is homotopic to $\mathscr{C}_3$ in $\mathbf{C}^\vee \setminus \{\pm 1\}$. Deduce that

$$\theta(h,\ell) = \int_{\mathscr{C}_1} \omega = \int_L \omega + \pi. \tag{22}$$

Let $z = \xi + i\,\eta$ with $\eta \geq 0$. Define $\theta^{0,\pm}$ and $r^{0,\pm}$ by $z - x_3^{0,\pm} = r^{0,\pm}\,\mathrm{e}^{i\,\theta^{0,\pm}}$, where $0 \leq \theta^{0,\pm} < 2\pi$. Show that $\theta^- + \theta^+ > \pi$. Hence $\pi > \alpha(\eta) = \frac{1}{2}(\theta^0 + \theta^- + \theta^+) >$

$\pi/2$. If $z = \xi + i\eta$ with $\eta < 0$, show that $\alpha(\eta) = \alpha(-\eta)$. Justify each step of the following calculation.

$$\begin{aligned}
\int_L \omega &= \int_{-\infty}^{\infty} \frac{i\ell \, \mathrm{d}\eta}{\left(1-(\xi+i\eta)^2\right) \mathrm{e}^{i\alpha(\eta)} \sqrt{2}\sqrt{r^0 r^- r^+}} \\
&= 2\ell \int_0^{\infty} \left[\frac{(1-\xi^2+\eta^2)\sin\alpha(\eta) - 2\xi\eta\cos\alpha(\eta)}{(1-\xi^2+\eta^2)^2+4\xi^2\eta^2}\right] \frac{\mathrm{d}\eta}{\sqrt{2}\sqrt{r^0 r^- r^+}} \\
&> 0.
\end{aligned}$$

Therefore $\theta(h,\ell) > \pi$.

4. Show that there is no homeomorphism of $\mathbf{R}^3$ which maps a 2-torus with a meridial circle pinched to a point to a 2-torus in $\mathbf{R}^3$ with a longitudinal circle pinched to a point.

5. (Horozov's theorem.) We know that the regular values of the energy momentum map $\mathscr{E\!M}$ of the spherical pendulum is the set $\mathscr{R}$ of $(h,\ell) \in \mathbf{R}^2$ such that

$$h > -1,\ \ell^2 < \frac{4(3+h^2)^{3/2}+4h(9-h^2)}{27},\ \text{and } (h,\ell) \neq (1,0). \tag{23}$$

For $(h,\ell) \in \mathscr{R}$, 2π times the rotation number of the flow of the spherical pendulum on the 2-torus $\mathscr{E\!M}^{-1}(h,\ell)$ is

$$\theta(h,\ell) = 2\ell \int_{x_3^-}^{x_3^+} \frac{\mathrm{d}x_3}{(1-x_3^2)\sqrt{2(h-x_3)(1-x_3^2)-\ell^2}},$$

where $-1 < x_3^- < x_3^+ < 1 < x_3^0$ where $x^{0,\pm}$ are roots of $2(h-x_3)(1-x_3^2)-\ell^2$. The time $T(h,\ell)$ of first return of the flow of the spherical pendulum to a cross section on $\mathscr{E\!M}^{-1}(h,\ell)$ (given by an orbit of the angular momentum vector field) is

$$T(h,\ell) = 2\int_{x_3^-}^{x_3^+} \frac{\mathrm{d}x_3}{\sqrt{2(h-x_3)(1-x_3^2)-\ell^2}}.$$

We know that locally T and θ are real analytic functions in $\mathscr{R}$. The goal of this exercise is to show that they are *coordinates* on $\mathscr{R}$, that is, $\mathrm{d}T \wedge \mathrm{d}\theta \neq 0$. In other words, for every $(h,\ell) \in \mathscr{R}$

$$D = \det \begin{pmatrix} \dfrac{\partial T}{\partial h} & \dfrac{\partial T}{\partial \ell} \\[2ex] \dfrac{\partial \theta}{\partial h} & \dfrac{\partial \theta}{\partial \ell} \end{pmatrix} \neq 0. \tag{24}$$

a) Use a computer to draw the level curves of θ and T. Notice that these curves are like polar coordinates centered at $(1,0)$ with T being the radial coordinate and θ the angular coordinate.

b) Consider the elliptic curve

$$\Gamma_{h,\ell} : y^2 = 2(h-z)(1-z^2) - \ell^2$$

with $(h,\ell) \in \mathscr{R}$. Let $\gamma_{h,\ell}$ be a positively oriented curve in the cut extended complex plane $\mathbf{C}^\vee = \mathbf{C} \setminus ([x_3^-, x_3^+] \cup [x_3^0, \infty])$, which encloses the cut $[x_3^-, x_3^+]$ but not ± 1. Note that $T(h,\ell) = \int_{\gamma_{h,\ell}} \frac{1}{y}\, \mathrm{d}z$ and $\theta(h,\ell) = \int_{\gamma_{h,\ell}} \frac{\ell\, \mathrm{d}z}{(1-z^2)y}$. Since we can homotope the curve $\gamma_{h,\ell}$ in $\mathbf{C}^\vee \setminus \{\pm 1\}$ to another without changing T or θ, it follows that we can compute the partial derivative of T and θ by differentiating under the integral sign. Let $w_0 = \int_{\gamma_{h,\ell}} \frac{1}{y^3}\, \mathrm{d}z$ and $w_1 = \int_{\gamma_{h,\ell}} \frac{z}{y^3}\, \mathrm{d}z$. Show that

$$D = \tfrac{4}{3}(hw_0 - w_1)(hw_1 - w_0) - \ell^2 w_0^2 \tag{25}$$

by checking the following

i). $\frac{\partial T}{\partial h} = -\int_{\gamma_{h,\ell}} \frac{1-z^2}{y^3}\, \mathrm{d}z = -w_0 + \int_{\gamma_{h,\ell}} \frac{z^2}{y^3}\, \mathrm{d}z$. But

$$\int_{\gamma_{h,\ell}} \frac{z^2}{y^3}\, dz = \frac{1}{6} \int_{\gamma_{h,\ell}} \frac{\mathrm{d}(y^2 + 2hz^2 + 2z + \ell^2 - 2h)}{y^3}.$$

Therefore $\frac{\partial T}{\partial h} = \frac{2}{3}(hw_1 - w_0)$.

ii) $\frac{\partial \theta}{\partial \ell} = \int_{\gamma_{h,\ell}} \frac{\ell^2 + y^2}{(1-z^2)y^3}\, \mathrm{d}z = \int_{\gamma_{h,\ell}} \frac{2(h-z)(1-z^2)}{(1-z^2)y^3}\, \mathrm{d}z$. Therefore $\frac{\partial \theta}{\partial \ell} = 2(hw_0 - w_1)$.

iii) $\frac{\partial \theta}{\partial h} = \frac{\partial T}{\partial \ell} = \ell w_0$. From (25), we see that we may assume that $\ell \geq 0$.

c) Using the translation $z = x + \frac{1}{3}h$ and the scaling $y = \alpha v$ and $x = \beta u$, where $3\beta = (3+h^2)^{1/2}$ and $\alpha = \beta^{3/2}$, show that the elliptic curve $\Gamma_{h,\ell}$ becomes the elliptic curve $\Gamma_p : \frac{1}{2}v^2 = u^3 - 3u + p$, where

$$p = p(h,\ell) = \frac{2h(9-h^2)}{(3+h^2)^{3/2}} - \frac{27\ell^2}{2(3+h^2)^{3/2}}.$$

When $(h,\ell) \in \mathscr{R}$ show that $-2 < p(h,\ell) < 2$. Also check that

$$w_0 = \frac{\beta}{\alpha^3} \int_{\gamma_p} \frac{\mathrm{d}u}{v^3} \quad \text{and} \quad w_1 = \frac{\beta}{\alpha^3} \int_{\gamma_p} \frac{\beta u + h/3}{v^3}\, \mathrm{d}u.$$

We now explain how the curve γ_p is chosen. Let $\mathbf{C}^\vee$ be the extended complex plane $\widetilde{\mathbf{C}}$ cut between u_- and u_+ and again between u_0 and ∞. Here $u_{0,\pm}$ are real roots of $u^3 - 3u + p$, $-2 < p < 2$, and $u_- < u_+ < u_0$. Let γ_p be a closed positively oriented curve in $\mathbf{C}^\vee$ which encloses the cut $[u_-, u_+]$ but not ± 1.

d) Let $\theta_0(p) = \int_{\gamma_p} \frac{\mathrm{d}u}{v^3}$ and $\theta_1(p) = \int_{\gamma_p} \frac{u\, \mathrm{d}u}{v^3}$. Show that (θ_0, θ_1) satisfy the Picard Fuchs equation

$$6(4-p^2)\frac{\mathrm{d}}{\mathrm{d}p}\begin{pmatrix}\theta_0\\ \theta_1\end{pmatrix} = \begin{pmatrix}7p & 10\\ 14 & 5p\end{pmatrix}\begin{pmatrix}\theta_0\\ \theta_1\end{pmatrix}.$$

Let $r(p) = \theta_0(p)/\theta_1(p)$. For $p \in (-2,2)$ show that r satisfies the Ricatti equation

$$3(4-p^2)\frac{\mathrm{d}r}{\mathrm{d}p} = 7 - pr - 5r^2 \tag{26}$$

and has the following properties.

i). $r(-2) = 7/5$ and $r(2) = 1$.

ii). $r'(p) < 0$ for every $p \in (-2,2)$.

To see that ii) holds we argue as follows. First we show that $r(p) > 0$ for every $p \in (-2,2)$. Suppose not. Let $p_0 \in (-2,2)$ be the smallest zero of r. Then $r'(p_0) \le 0$. Using (26), we obtain $r'(p_0) = \frac{7}{3(4-p_0{}^2)} > 0$, which is a contradiction. Suppose that $r'(p_0) = 0$, for some $p_0 \in (-2,2)$. Differentiating (26) gives

$$3(4-p_0^2)\,\frac{\mathrm{d}^2 r}{\mathrm{d}p^2}(p_0) = -r(p_0) < 0.$$

Thus every critical point of r is a nondegenerate local maximum. Suppose that for some $p' \in (-2,2)$, $r(p') < 1$. Since $r(-2) = 7/5 > 1 = r(2)$, it follows that r has a minimum in $(-2,2)$. This is a contradiction. Hence $r \ge 1$ on $(-2,2)$. From (26) it follows that $r'(p) = 0$ for $p \in (-2,2)$ if and only if $(p, r(p))$ lie on

$$0 = 7 - xy - 5y^2 \qquad -2 \le x \le 2,\ \&\ y \ge 1. \tag{27}$$

Equation (27) defines a smooth function $x \to y(x)$, which is strictly decreasing on $[-2,2]$ because $y' = \frac{-y}{10y+x} < 0$. Note that $y(-2) = 7/5$ and $y(2) = 1$. Suppose that $r'(p_0) = 0$ for some $p_0 \in (-2,2)$. Then $(p_0, r(p_0))$ satisfies (27). Hence $r(p_0) < 7/5$. But $r(-2) = 7/5$. Because $r(p_0)$ is a nondegenerate local maximum, we see that r has a local minimum in $(-2, p_0)$. This is a contradiction. Hence r has *no* critical points in $(-2,2)$. Since $r(-2) = 7/5$ and $r(2) = 1$, it follows that $r'(p) < 0$ for every $p \in (-2,2)$.

e) Show that $w_0(p)w_1(p) > 0$ for every $p \in (-2,2)$. Argue as follows. Since

$$w_0(p) = 2\int_{u_-}^{u_+} \frac{\mathrm{d}u}{(u^3 - 3u + p)^{3/2}},$$

we see that $w_0(p) \ne 0$ for every $p \in (-2,2)$. Suppose that for some $(h^0, \ell^0) \in \mathscr{R}$ with $p^0 = p(h^0, \ell^0)$ we have $w_1(p^0) = 0$. From the definition of w_1 we obtain

$$r(p^0) = -\frac{h^0}{3\beta} = -h^0\big(3 + (h^0)^2\big)^{-1/2}.$$

Therefore $r(p^0) \le 0$ when $h^0 > 0$ and $0 \le r(p^0) \le \frac{1}{2}$ when $-1 < h \le 0$. This contradicts the fact that $r(p) \in (1, 7/5)$ when $p \in (-2,2)$. Therefore $w_1(p(h,\ell)) \ne 0$ for every $(h,\ell) \in \mathscr{R}$. Setting $h = 0$ in the definition of w_1 gives

$$\frac{w_1(\widetilde{p})}{w_0(\widetilde{p})} = \frac{1}{\beta}\,\frac{\theta_1(\widetilde{p})}{\theta_0(\widetilde{p})} = \frac{1}{\beta}\,\frac{1}{r(\widetilde{p})} > 0,$$

where $\widetilde{p} = p(0,\ell)$. Therefore $w_0(p)w_1(p) > 0$ for every $p \in (-2,2)$.

f) We show that D (25) is nonzero when $(h,\ell) \in \mathscr{R}$ by considering three cases.

CASE 1. $h \le 0$, $\ell > 0$. From (e) and (25) it follows that $D < 0$.

CASE 2. $\ell = 0$. When $\ell = 0$ the spherical pendulum moves in a plane as if it were a mathematical pendulum. From the geometric interpretation of $\theta(h,0)$ ((4.2)) we see that

$$\theta(h,0) = \begin{cases} \pi, \text{ if } -1 < h < 1 \\ 2\pi, \text{ if } h > 1. \end{cases} \tag{28}$$

Therefore $\frac{\partial \theta}{\partial h}(h,0) = 0$. Now

$$T(h,0) = 2\int_{x_3^-}^{x_3^+} \frac{\mathrm{d}x_3}{\sqrt{2(h-x_3)(1-x_3^2)}}$$

is the period of the mathematical pendulum as a function of energy. In exercise 3 of the introduction we show that $\frac{\partial T}{\partial h}(h,0) > 0$. Because θ is a real analytic function on $\mathscr{R}$, we have

$$\theta(h,\ell) = \theta(h,0) + \ell\,\frac{\partial \theta}{\partial \ell}(h,0) + \mathrm{O}(\ell^2).$$

From (28) and exercise 5 it follows that

$$\pi < \pi + \ell\,\frac{\partial \theta}{\partial \ell}(h,0) + \mathrm{O}(\ell^2)$$

for $(h,\ell) \in \mathscr{R}$ and $-1 < h < 1$, $\ell \neq 0$. Therefore $\frac{\partial \theta}{\partial \ell}(h,0) \neq 0$. Similarly, when $h > 1$ we find that $\frac{\partial \theta}{\partial \ell}(h,0) \neq 0$. Therefore $D \neq 0$ when $\ell = 0$. Setting $h = \ell = 0$ in (25) we see that $D < 0$. Hence $D < 0$ when $\ell = 0$.

CASE 3. $h > 0$ and $\ell > 0$. Using (25) show that

$$D = \tfrac{4}{3}\beta h w_0^2 F(p,\nu), \tag{29}$$

where $\nu = \nu(h) = \frac{3\beta}{h}$ and $F(p,\nu) = r(p)\big(r(p) - 2\nu\big) + \nu p - 1$. Consider the mapping

$$\Psi : \mathscr{R} \cap (\mathbf{R}_{>})^2 \to \mathscr{S} = \big\{(p,\nu) \in \mathbf{R}^2 \big| \nu \in (\tfrac{1}{2},\infty)\ \&\ p \in \big(-2,(3\nu^2-1)\nu^{-3}\big)\big\} :$$
$$(h,\ell) \mapsto \big(p(h,\ell),\nu(h)\big).$$

Show that Ψ is a diffeomorphism which maps the half-line $\Psi(\{(h,0)|\, h > 0\})$ bijectively onto the curve $p = \big(3\nu^2-1\big)/\nu^3$, $\nu \in (\frac{1}{2},\infty)$. To show that $F(p,\nu) < 0$ we argue as follows. For every fixed $p_0 \in (-2,2)$, the function $F_{p_0}(\nu) = F(p_0,\nu)$, where $\nu \in \mathscr{S} \cap \{p = p_0\}$ is strictly decreasing. To see this differentiate the definition of F and obtain $\frac{\partial F_p}{\partial \nu} = -2r(p) + p$. At $p = 2$, we know that $r(2) = 1$. Hence $\frac{\partial F_2}{\partial \nu} = 0$. However,

$$\frac{\partial}{\partial p}\Big(\frac{\partial F_p}{\partial \nu}\Big) = -2r'(p) + 1 > 0,$$

since $r'(p) < 0$ for every $p \in (-2,2)$. Hence $\frac{\partial F_{p_0}}{\partial \nu} < 0$ for every $p_0 \in (-2,2)$. For any $(p_0,\nu_0) \in \mathscr{S}$ there is a $\nu_1 \in (\frac{1}{2},\nu_0)$ such that $p_0 = \big(3\nu_1^2 - 1\big)/\nu_1^3$. Since F_{p_0} is strictly decreasing, $F(p_0,\nu_0) < F(p_0,\nu_1)$. Using (29) and the result of case 2 that $D < 0$ when $\ell = 0$, we see that $F(p,\nu_1) < 0$. Therefore $F(p,\nu) < 0$.

Chapter V

The Lagrange top

1 The basic model

Physically, the Lagrange top is a symmetric rigid body spinning about its figure axis whose base point is fixed. A constant vertical gravitational force acts on the center of mass of the top, which lies on its symmetry axis.

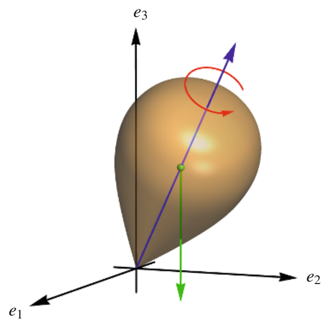


Figure 1.1. The Lagrange top.

Mathematically, the top is a *Hamiltonian system* on the *phase space* $(T\,\mathrm{SO}(3), \Omega_\rho)$. The symplectic form Ω_ρ on $T\,\mathrm{SO}(3)$ is the pull back of the canonical symplectic form Ω on $T^*\,\mathrm{SO}(3)$ by the map $\rho^\sharp$ associated to a *left invariant metric* ρ on $\mathrm{SO}(3)$. The metric ρ, which is uniquely determined by its value at the identity element e, is given by

$$\begin{aligned} \rho(e) : T_e\,\mathrm{SO}(3) \times T_e\,\mathrm{SO}(3) = \mathrm{so}(3) \times \mathrm{so}(3) \rightarrow \mathbf{R} : \\ \big(X,Y\big) \mapsto k(I(X),Y) = I_1 k(X,Y) + (I_3 - I_1)k(X,E_3)\,k(Y,E_3). \end{aligned} \tag{1}$$

Here k is the Killing metric on $\mathrm{so}(3)$, see chapter III §1, and the $I : \mathrm{so}(3) \rightarrow \mathrm{so}(3)$ is a k-symmetric linear map. The matrix of I with respect to the standard k-orthonormal basis $\{E_1, E_2, E_3\}$ is $\mathrm{diag}(I_1, I_1, I_3)$, where I_i are the *principal moments of inertia* of the top.

R.H. Cushman and L.M. Bates, *Global Aspects of Classical Integrable Systems*, DOI 10.1007/978-3-0348-0918-4_5

Because ρ is a metric, I is invertible. We will assume that

$$0 < I_1 < I_3. \tag{2}$$

To make sure that I is the moment of inertia tensor of a physically realizable top, see exercise 5 in chapter III, we require that

$$I_3 \leq 2I_1. \tag{3}$$

The *Hamiltonian*

$$\mathscr{H} : T\,\mathrm{SO}(3) \to \mathbf{R} : X_A \mapsto \mathscr{K}(X_A) + (\tau^*\mathscr{V})(X_A), \tag{4}$$

is the sum of kinetic and potential energy. The kinetic energy $\mathscr{K} : T\,\mathrm{SO}(3) \to \mathbf{R} : X_A \to \frac{1}{2}\rho(A)(X_A, X_A)$ is one half the ρ-length squared of a tangent vector to $\mathrm{SO}(3)$. The potential energy $\mathscr{V} : \mathrm{SO}(3) \to \mathbf{R} : A \mapsto \chi k(\mathrm{Ad}_A E_3, E_3)$ measures the height of the center of mass of the top. To be able to define the Hamiltonian, we must pull back $\mathscr{V}$ by the bundle projection $\tau : T\,\mathrm{SO}(3) \to \mathrm{SO}(3) : X_A \mapsto A$ so that it is a function on $T\,\mathrm{SO}(3)$.

2 Liouville integrability

In this section we show that the Lagrange top is Liouville integrable, see chapter IX §1. To do this we need two additional integrals of motion other than the Hamiltonian. These two extra integrals arise from two rotational symmetries of the top, namely, one about the vertical axis e_3, which is fixed in space, and the other about the *figure axis* (= symmetry axis) fixed in the top.

We now investigate these symmetries more carefully. Let $S^1 = \{B \in \mathrm{SO}(3) \,|\, \mathrm{Ad}_B E_3 = E_3\}$. Then S^1 acts on the *left* on $T\,\mathrm{SO}(3)$ by

$$\Phi^\ell : S^1 \times T\,\mathrm{SO}(3) \to T\,\mathrm{SO}(3) : (B, X_A) \mapsto T_A L_B X_A = X_{BA}. \tag{5}$$

▷ Physically this action corresponds to rotating the top about the vertical axis.

(2.1) **Proof**: To see this let $A(t) \in \mathrm{SO}(3)$ be the configuration of the top at time t with respect to the *fixed frame* $\{e_1, e_2, e_3\}$ in $\mathbf{R}^3$. The figure axis of the top at time t is $A(t)e_3$. Acting on the top on the left by $B \in S^1$, we obtain the new configuration $BA(t)$ Applying the mapping $i : \mathrm{so}(3) \to \mathbf{R}^3$, see chapter III ((1.2)), to the condition $E_3 = \mathrm{Ad}_B E_3$ defining S^1 gives $e_3 = Be_3$. Since $(BA(t)e_3, e_3) = (A(t)e_3, B^{-1}e_3) = (A(t)e_3, e_3)$, the angle between the figure axis and e_3 remains invariant under the left S^1-action. Thus the left S^1-action corresponds to rotating the top about the vertical axis e_3. □

▷ The action Φ^ℓ is Hamiltonian with *momentum mapping*

$$\mathscr{J}_\ell : T\,\mathrm{SO}(3) \to \mathbf{R} : X_A \mapsto \rho(A)(T_e R_A E_3, X_A). \tag{6}$$

Physically, $\mathscr{J}_\ell$ is the angular momentum of the top about the e_3-axis.

(2.2) **Proof**: The formula for the momentum mapping follows from the momentum lemma, see chapter VII ((5.7)). We provide some details. Let θ be the canonical 1-form on $T^*\,\mathrm{SO}(3)$. First we show that the 1-form $\theta_\rho = (\rho^\sharp)^*\theta$ is invariant under the action Φ^ℓ. For $W_{X_A} \in T_{X_A}\big(T\,\mathrm{SO}(3)\big)$ and $B \in S^1$ we have

$$\begin{aligned}\big((\Phi^\ell_B)^*\theta_\rho\big)(X_A)W_{X_A} &= \rho(BA)\big(T\tau(T_{X_A}\Phi^\ell_B)W_{X_A}, X_{AB}\big)\\ &= \rho(BA)\big(T_AL_B(T\tau W_{X_A}), T_AL_BX_A\big), \quad \text{since } \tau\circ\Phi^\ell_B = L_B\circ\tau\\ &= \rho(A)(T\tau W_{X_A}, X_A), \quad \text{by left invariance of } \rho\\ &= \theta_\rho(X_A)W_{X_A}.\end{aligned}$$

Second, note that the Lie algebra $T_eS^1 \subseteq \mathrm{so}(3)$ of S^1 is spanned by E_3. Therefore the infinitesimal generator of the action $S^1\times\mathrm{SO}(3)\to\mathrm{SO}(3) : (B,A)\mapsto BA$ is the vector field

$$X_{\mathrm{SO}(3)}(A) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}(\exp tE_3)A = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} R_A\exp tE_3 = T_eR_AE_3$$

and the infinitesimal generator of Φ^ℓ is $X_{T\,\mathrm{SO}(3)}(X_A) = \frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0}\Phi^\ell_{\exp tE_3}(X_A)$. Since $L_{\exp tE_3}\circ\tau = \tau\circ\Phi^\ell_{\exp tE_3}$, it follows that $T\tau X_{T\,\mathrm{SO}(3)}(X_A) = X_{\mathrm{SO}(3)}(A)$. Therefore

$$\begin{aligned}\mathscr{J}_\ell(X_A) &= \theta_\rho(X_A)(X_{T\,\mathrm{SO}(3)}(X_A))\\ &= \rho(A)(T\tau X_{T\,\mathrm{SO}(3)}(X_A), X_A) = \rho(A)(X_{\mathrm{SO}(3)}(A), X_A).\end{aligned}$$ □

▷ To show that S^1 is a (left) symmetry of the Lagrange top we need only verify that the action Φ^ℓ preserves the Hamiltonian $\mathscr{H}$.

(2.3) **Proof**: For $B \in S^1$, we have

$$\begin{aligned}(\Phi^\ell_B)^*\mathscr{H}(X_A) = \mathscr{H}(X_{BA}) &= \tfrac12\rho(BA)(X_{BA}, X_{BA}) + \chi k(\mathrm{Ad}_{BA}E_3, E_3)\\ &= \tfrac12\rho(A)(X_A, X_A) + \chi k(\mathrm{Ad}_A E_3, \mathrm{Ad}_{B^{-1}}E_3) = \mathscr{H}(X_A).\end{aligned}$$ □

Therefore $\mathscr{J}_\ell$ is constant on the integral curves of the Hamiltonian vector field $X_{\mathscr{H}}$, that is, $\mathscr{J}_\ell$ is an integral of the Lagrange top.

The group $S^1 = \{B \in \mathrm{SO}(3)\,|\,\mathrm{Ad}_BE_3 = E_3\}$ also acts on the *right* on $T\,\mathrm{SO}(3)$ by

$$\Phi^r : T\,\mathrm{SO}(3)\times S^1 \to T\,\mathrm{SO}(3) : (X_A, B)\mapsto X_{AB} = T_AR_BX_A. \tag{7}$$

▷ Physically, this action corresponds to a rotation about the figure axis of the top.

(2.4) **Proof**: To see this, let $A(t)\in\mathrm{SO}(3)$ be the configuration of the top at time t with respect to the fixed frame $\{e_1, e_2, e_3\}$ in $\mathbf{R}^3$. Acting on the right by $B\in S^1$ gives the new configuration $A(t)B$. Since $Be_3 = e_3$, we find that $A(t)Be_3 = A(t)e_3$. Thus the figure axis of the top in the new configuration is the same as in the original configuration. Hence the right S^1-action corresponds to rotating the top about its figure axis. □

▷ The right S^1-action Φ^r is Hamiltonian with *momentum mapping*

$$\mathscr{J}_r : T\,\mathrm{SO}(3)\to\mathbf{R} : X_A \mapsto \rho(A)(T_eL_AE_3, X_A). \tag{8}$$

Physically, $\mathscr{J}_r$ is the angular momentum of the top about its figure axis.

(2.5) **Proof**: The argument follows along the same lines as the proof of ((2.2)). We only show that the 1-form θ_ρ is invariant under Φ^r. First we verify that the metric ρ (1) is *invariant* under the right S^1-action Φ^r, that is,

$$I \circ \mathrm{Ad}_B = \mathrm{Ad}_B \circ I, \tag{9}$$

for every $B \in S^1$. From the definition of $\rho(e)$ and the fact that $B \in S^1$, it follows that

$$\begin{aligned}
&\rho(e)(\mathrm{Ad}_{B^{-1}} X, \mathrm{Ad}_{B^{-1}} Y) = \\
&\qquad = I_1 k(\mathrm{Ad}_{B^{-1}} X, \mathrm{Ad}_{B^{-1}} Y) + (I_3 - I_1) k(\mathrm{Ad}_{B^{-1}} X, E_3)(\mathrm{Ad}_{B^{-1}} Y, E_3) \\
&\qquad = \rho(e)(X,Y), \quad \text{since } k \text{ is Ad-invariant.}
\end{aligned}$$

Equation (9) follows because $\rho(e)(X,Y) = k\big(I(X),Y\big)$. The metric ρ is Φ^r-invariant, because

$$\begin{aligned}
\rho(B)\big(T_e R_B X, T_e R_B Y\big) &= \rho(e)\big(T_B L_{B^{-1}} T_e R_B X, T_B L_{B^{-1}} T_e R_B\big), \quad \text{since } \rho \text{ is } \Phi^\ell\text{-invariant} \\
&= \rho(e)\big(\mathrm{Ad}_{B^{-1}} X, \mathrm{Ad}_{B^{-1}} Y\big) = \rho(e)(X,Y).
\end{aligned}$$

The 1-form θ_ρ is Φ^r-invariant, because

$$\begin{aligned}
(\Phi^r)^* \theta_\rho(X_A) W_{X_A} &= \rho(AB)\big(T\tau(T\Phi^r_B W_{X_A}), X_{AB}\big) = \rho(AB)\big(T_A R_B (T\tau W_{X_A}), T_A R_B X_A\big) \\
&= \theta_\rho(X_A)(W_{X_A}).
\end{aligned}$$

□

▷ To show that S^1 is a (right) symmetry of the Lagrange top it suffices to verify that the Hamiltonian $\mathscr{H}$ is invariant under Φ^r.

(2.6) **Proof**: For $B \in S^1$ we have

$$\begin{aligned}
(\Phi^r_B)^* \mathscr{H}(X_A) = \mathscr{H}(X_{AB}) &= \tfrac{1}{2}\rho(AB)(X_{AB}, X_{AB}) + \chi k(\mathrm{Ad}_{AB} E_3, E_3) \\
&= \tfrac{1}{2}\rho(A)(X_A, X_A) + \chi k(\mathrm{Ad}_A \mathrm{Ad}_B E_3, E_3), \\
&\qquad \text{since } \rho \text{ is } \Phi^r\text{-invariant and } B \in S^1 \\
&= \mathscr{H}(X_A), \quad \text{since } B \in S^1.
\end{aligned}$$

□

Therefore $\mathscr{J}_r$ is an integral of the vector field $X_{\mathscr{H}}$.

To complete the argument that the Lagrange top is *Liouville integrable*, we need only show that the Poisson bracket of any two of the integrals $\{\mathscr{H}, \mathscr{J}_\ell, \mathscr{J}_r\}$ vanishes
▷ identically. Since $\mathscr{J}_\ell$ and $\mathscr{J}_r$ are integrals of $X_{\mathscr{H}}$, their Poisson bracket with $\mathscr{H}$ vanishes identically. We now show that the Poisson bracket of $\mathscr{J}_\ell$ and $\mathscr{J}_r$ vanishes identically.

(2.7) **Proof**: To see this it suffices to show that $\mathscr{J}_r$ is constant on the orbits of Φ^ℓ. For $B \in S^1$ we have

$$\begin{aligned}
((\Phi^\ell_B)^* \mathscr{J}_r)(X_A) = \mathscr{J}_r(X_{BA}) &= \rho(BA)(T_e L_{BA} E_3, X_{BA}) \\
&= \rho(BA)(T_B L_B\, T_e L_A E_3, T_B L_B X_A) \\
&= \rho(A)(T_e L_A E_3, X_A), \quad \text{since } \rho \text{ is } \Phi^\ell\text{-invariant} \\
&= \mathscr{J}_r(X_A).
\end{aligned}$$

□

3 Reduction of right S^1-action

In this section we remove the symmetry about the figure axis of the Lagrange top using the technique of regular reduction, see chapter VII ((6.1)). After reduction we obtain a Hamiltonian system with two degrees of freedom which is equivalent to the magnetic spherical pendulum.

3.1 Reduction to the Euler-Poisson equations

In this subsection we reduce the symmetry given by the *right* S^1-action and obtain a Hamiltonian system for which Hamilton's equations are the Euler-Poisson equations.

To find an initial model for the reduced phase space, we follow the proof of the regular reduction theorem. This constructs the orbit space of the right S^1-action Φ^r (7) on the ▷ a-level set of the right angular momentum mapping $\mathscr{J}_r$. We start by checking that $\mathscr{J}_r^{-1}(a)$ is a smooth manifold for every $a \in \mathbf{R}$.

(3.1) **Proof**: Pulling back the right S^1 Hamiltonian action Φ^r by the left trivialization

$$\mathscr{L} : \mathrm{SO}(3) \times \mathrm{so}(3) \to T\,\mathrm{SO}(3) : (A,X) \to T_e L_A X = X_A \tag{10}$$

gives the right S^1-action

$$\varphi^r : \big(\mathrm{SO}(3) \times \mathrm{so}(3)\big) \times S^1 \to \mathrm{SO}(3) \times \mathrm{so}(3) : \big((A,X),B\big) \mapsto (AB, \mathrm{Ad}_{B^{-1}}X), \tag{11}$$

because

$$(\mathrm{Ad}_{B^{-1}} X)_{AB} = T_e L_{AB}(T_B L_{B^{-1}}\, T_e R_B X) \;=\; T_B L_A\, T_e R_B X = T_A R_B X_A = X_{AB}.$$

Pulling back the right momentum map $\mathscr{J}_r$ (8) by the left trivialization $\mathscr{L}$ (10) shows that the action φ^r has a momentum mapping

$$J_r : \mathrm{SO}(3) \times \mathrm{so}(3) \to \mathbf{R} : (A,X) \mapsto \mathscr{J}_r(X_A)E_3 = \rho(e)(X,E_3) = k(I(X),E_3). \tag{12}$$

Because the derivative of J_r at (A,X)

$$DJ_r(A,X)(V_A,Y) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} J_r(A\exp tV, X+tY) \;=\; \rho(e)(E_3,Y)$$

is a surjective linear mapping from $T_{(A,X)}\big(\mathrm{SO}(3)\times \mathrm{so}(3)\big)$ to $\mathbf{R}$, the level set $J_r^{-1}(a)$ is a smooth manifold. □

The level set $J_r^{-1}(a)$ is invariant under the action φ^r, because for every $B \in S^1$

$$\begin{aligned} J_r(AB, \mathrm{Ad}_{B^{-1}} X) &= k\big(I(\mathrm{Ad}_{B^{-1}} X), E_3\big) \;=\; k\big(\mathrm{Ad}_{B^{-1}}(I(X)), E_3\big), \quad \text{using (9)} \\ &= k\big(I(X), \mathrm{Ad}_B E_3\big) = J_r(A,X). \end{aligned}$$

Therefore φ^r restricts to an S^1-action $\varphi^r|\big(S^1 \times J_r^{-1}(a)\big)$ on $J_r^{-1}(a)$. This induced action is a free, because if $(A,X) = (AB, \mathrm{Ad}_{B^{-1}} X)$, then $A = AB$, that is, B is the identity element e.

Hence the space $J_r^{-1}(a)/S^1$ of S^1 orbits of $\varphi^r|(S^1 \times J_r^{-1}(a))$ is a smooth manifold called the *reduced phase space*, see chapter VII ((6.1)).

This description of the reduced phase space is somewhat abstract. We now provide a more concrete model, namely, $\mathscr{P}^a = \{(Z,W) \in \mathrm{so}(3) \times \mathrm{so}(3) \,\big|\, k(Z,Z) = 1 \ \&\ k(Z,W) = a\}$.

Claim: $\mathscr{P}^a$ is diffeomorphic to $J_r^{-1}(a)/S^1$.

(3.2) **Proof**: Consider the mapping

$$\pi^a : J_r^{-1}(a) \subseteq \mathrm{SO}(3) \times \mathrm{so}(3) \to \mathscr{P}^a : (A,X) \mapsto \big(\mathrm{Ad}_A E_3, \mathrm{Ad}_A I(X)\big) = (Z,W). \qquad (13)$$

The map π^a is surjective. To see this suppose that $(Z,W) \in \mathscr{P}^a$. Then there is an $A \in \mathrm{SO}(3)$ such that $\mathrm{Ad}_A E_3 = Z$. Let $X = I^{-1}(\mathrm{Ad}_{A^{-1}} W)$. Using (13) we see that $\pi^a(A,X) = (Z,W)$. We are done once we can show that $(A,X) \in J_r^{-1}(a)$. This follows because

$$J_r(A,X) = k(I(X),E_3) = k(\mathrm{Ad}_{A^{-1}} W, \mathrm{Ad}_{A^{-1}} Z) = k(Z,W) = a,$$

since $(Z,W) \in \mathscr{P}^a$. Now π^a maps an orbit of $\varphi^r|\big(S^1 \times J_r^{-1}(a)\big)$ onto a point of $\mathscr{P}^a$ because for every $B \in S^1$,

$$\begin{aligned}
\pi^a(AB, \mathrm{Ad}_{B^{-1}} X) &= \big(\mathrm{Ad}_{AB} E_3, \mathrm{Ad}_{AB} I(\mathrm{Ad}_{B^{-1}} X)\big) \\
&= \big(\mathrm{Ad}_A(\mathrm{Ad}_B E_3), \mathrm{Ad}_A \mathrm{Ad}_B I(\mathrm{Ad}_{B^{-1}} X)\big) \\
&= \big(\mathrm{Ad}_A E_3, \mathrm{Ad}_A I(X)\big), \quad \text{since } B \in S^1 \text{ and (9)} \\
&= (Z,W).
\end{aligned}$$

Hence π^a induces a smooth mapping $\sigma^a : J_r^{-1}(a)/S^1 \to \mathscr{P}^a$ such that the diagram 3.1.1 commutes, see chapter VII §2. In diagram 3.1.1 the mapping $\rho^a : J_r^{-1}(a) \to J_r^{-1}(a)/S^1$ is called the orbit map, because it assigns to each point in $J_r^{-1}(a)$ the orbit of the action $\varphi^r|\big(S^1 \times J_r^{-1}(a)\big)$ through the given point.

$$\begin{tikzcd}
J_r^{-1}(a) \arrow[r, "\pi^a"] \arrow[d, "\rho^a"] & \mathscr{P}^a \\
J_r^{-1}(a)/S^1 \arrow[ur, "\sigma^a"'] &
\end{tikzcd}$$

Diagram 3.1.1

▷ We now show that σ^a is a diffeomorphism.

(3.3) **Proof**: Because π^a is surjective, it follows that σ^a is surjective. To show that σ^a is injective, it suffices to verify that the fiber $(\pi^a)^{-1}(Z,W)$ is a single orbit of φ^r. To see this suppose that (A,X) and (C,Y) lie in the fiber $(\pi^a)^{-1}(Z,W)$. Then $\mathrm{Ad}_A E_3 = \mathrm{Ad}_C E_3$, that is, $B = A^{-1}C \in S^1$. Since $\mathrm{Ad}_A I(X) = \mathrm{Ad}_C I(Y)$, it follows that $\mathrm{Ad}_A I(X) =$

$\mathrm{Ad}_A \mathrm{Ad}_B I(Y) = \mathrm{Ad}_A I(\mathrm{Ad}_B Y)$. Consequently, $Y = \mathrm{Ad}_{B^{-1}} X$ because Ad_A and I are invertible. Therefore (C,Y) lies in the same $\varphi^r|\big(S^1 \times J_r^{-1}(a)\big)$ orbit as (A,X). Next we show that the inverse of σ^a is smooth. Because π^a is a proper submersion, the bundle $\pi^a : J_r^{-1}(a) \to \mathscr{P}^a$ is locally trivial, see chapter VIII ((2.1)). Thus for a suitable open subset $\mathscr{U} \subseteq \mathscr{P}^a$ about (Z,W), the bundle π^a restricted to $(\pi^a)^{-1}(\mathscr{U})$ has a smooth cross section $\tau : \mathscr{U} \to (\pi^a)^{-1}(\mathscr{U})$. Therefore on $\mathscr{U}$, we have $\sigma^a \circ (\rho^a \circ \tau) = \pi^a \circ \tau = id_{\mathscr{U}}$. To verify that $(\rho^a \circ \tau) \circ \sigma^a = id_{(\sigma^a)^{-1}(\mathscr{U})}$, let $r \in (\sigma^a)^{-1}(\mathscr{U})$ and set $s = (\rho^a \circ \tau) \circ \sigma^a(r)$. Then $\sigma^a(s) = (\sigma^a \circ \rho^a \circ \tau)\big(\sigma^a(r)\big) = \sigma^a(r)$. But σ^a is injective, so $s = r$ and we are done. Therefore $(\sigma^a)^{-1} = \rho^a \circ \tau$ is smooth. Consequently σ^a is a diffeomorphism. □

By the regular reduction theorem, see chapter VII ((6.1)), the orbit space $J_r^{-1}(a)/S^1$ has a symplectic form $\widetilde{\Omega}^a$ defined by $(\rho^a)^*\widetilde{\Omega}^a = \Omega_\rho | J_r^{-1}(a)$. Define a 2-form Ω^a on $\mathscr{P}^a$ by $(\pi^a)^*\Omega^a = \Omega_\rho | J_r^{-1}(a)$.

Claim: The mapping $\sigma^a : \big(J_r^{-1}(a)/S^1, \widetilde{\Omega}^a\big) \to \big(\mathscr{P}^a, \Omega^a\big)$ in diagram 3.1.1 is a symplectic diffeomorphism.

(3.4) **Proof**: We have already shown that σ^a is a diffeomorphism. To show that it is symplectic, let $\tau : \mathscr{U} \to (\pi^a)^{-1}(\mathscr{U})$ be a cross section for the trivial bundle $\pi^a|\big((\pi^a)^{-1}(\mathscr{U})\big)$. Then by definition $(\rho^a)^*\widetilde{\Omega}^a = \Omega_\rho | J_r^{-1}(a) = (\pi^a)^*\Omega^a$. So on $\mathscr{U}$ we have

$$\Omega^a = (\pi^a \circ \tau)^*\Omega^a = (\rho^a \circ \tau)^*\widetilde{\Omega}^a = \big((\sigma^a)^{-1}\big)^*\widetilde{\Omega}^a.$$

Because σ^a is a diffeomorphism, the 2-form Ω^a is nondegenerate. Moreover, Ω^a is closed because

$$\mathrm{d}\Omega^a = \mathrm{d}\Big(\big((\sigma^a)^{-1}\big)^*\widetilde{\Omega}^a\Big) = \big((\sigma^a)^{-1}\big)^* \mathrm{d}\widetilde{\Omega}^a = 0,$$

since $\widetilde{\Omega}^a$ is symplectic and hence is closed. Thus σ^a is a symplectic diffeomorphism. □

Another way of stating the result of the claim is that $(\mathscr{P}^a, \Omega^a)$ is a model for the reduced
▷ space $(J_r^{-1}(a)/S^1, \widetilde{\Omega}^a)$. For latter use we find an explicit expression for the 2-form Ω^a.

(3.5) **Proof**: We begin by calculating the tangent of the mapping π^a (13). For $(U_A, T) \in T_{(A,X)}\big(\mathrm{SO}(3) \times \mathrm{so}(3)\big)$ we have

$$\begin{aligned} T_{(A,X)}\pi^a(U_A,T) &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \pi^a(A\exp tU, X + tT) \\ &= \Big(\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \mathrm{Ad}_{A\exp tU} E_3, \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \mathrm{Ad}_{A\exp tU}\big(I(X) + tI(T)\big)\Big) \\ &= \big(\mathrm{Ad}_A(\mathrm{ad}_U E_3), \mathrm{Ad}_A(\mathrm{ad}_U I(X) + I(T))\big) \\ &= \big([\mathrm{Ad}_A U, \mathrm{Ad}_A E_3], [\mathrm{Ad}_A U, \mathrm{Ad}_A I(X)] + \mathrm{Ad}_A I(T)\big) = \xi(A,X,U,T). \end{aligned}$$

We compute the 2-form Ω^a as follows.

$$\begin{aligned} &\Omega^a(\mathrm{Ad}_A E_3, \mathrm{Ad}_A I(X))\big(\xi(A,X,U,T), \xi(A,X,R,S)\big) = \\ &\qquad = \Omega^a(\pi^a(A,X))\big(T_{(A,X)}\pi^a(U_A,T), T_{(A,X)}\pi^a(R_A,S)\big) \\ &\qquad = \Omega_\rho(A,X)\big((U_A,T),(R_A,S)\big), \quad \text{by definition of } \Omega^a \end{aligned}$$

$$= -k(I(T),R) + k(I(S),U) + k(I(X),[U,R]),$$
using (8) in §2 of chapter VI and
$\rho(e)^\flat$ to identify $\mathrm{so}(3)^*$ with $\mathrm{so}(3)$.
$$= -k(\mathrm{Ad}_A I(T), \mathrm{Ad}_A R) + k(\mathrm{Ad}_A I(S), \mathrm{Ad}_A I(U)) + k(\mathrm{Ad}_A I(X), [\mathrm{Ad}_A U, \mathrm{Ad}_A R]).$$
$\square$

We are now ready to compute the reduced Hamiltonian $\mathscr{H}^a$ on the reduced space $(\mathscr{P}^a, \Omega^a)$. First we treat the kinetic energy. From the definition of the left trivialization $\mathscr{L}$ (10) it follows that the pull back of the kinetic energy $\mathscr{K}$ by $\mathscr{L}$ is

$$K : \mathrm{SO}(3) \times \mathrm{so}(3) \to \mathbf{R} : (A,X) \mapsto \tfrac{1}{2}k(I(X),X) = \tfrac{1}{2}I_1 k(X,X) + \tfrac{1}{2}(I_3 - I_1)(k(X,E_3))^2.$$

From the definition of K above we find that

$$\tfrac{1}{2}k(I^{-1}(X),X) = \tfrac{1}{2}I_1^{-1}k(X,X) - \tfrac{1}{2}(I_1^{-1} - I_3^{-1})(k(X,E_3))^2. \tag{14}$$

Replacing X with $\mathrm{Ad}_{A^{-1}} W$ in (14) gives

$$k\big(I^{-1}(\mathrm{Ad}_{A^{-1}} W), \mathrm{Ad}_{A^{-1}} W\big) = \tfrac{1}{2}I_1^{-1}k(W,W) - \tfrac{1}{2}(I_1^{-1} - I_3^{-1})\big(k(W, \mathrm{Ad}_A E_3)\big)^2,$$

since k is Ad-invariant. By definition of π^a (13), the reduced kinetic energy is

$$\mathscr{K}^a : \mathscr{P}^a \to \mathbf{R} : (Z,W) \mapsto \tfrac{1}{2}I_1^{-1}k(W,W) - \tfrac{1}{2}(I_1^{-1} - I_3^{-1})a^2 \tag{15}$$

Pulling back the potential energy $\mathscr{V}$ by the left trivialization $\mathscr{L}$, gives the reduced potential energy

$$\mathscr{V}^a : \mathscr{P}^a \to \mathbf{R} : (Z,W) \mapsto \chi\, k(Z,E_3). \tag{16}$$

Thus the *reduced Hamiltonian* $\mathscr{H}^a$ is

$$\mathscr{H}^a : \mathscr{P}^a \subseteq \mathrm{so}(3) \times \mathrm{so}(3) \to \mathbf{R} : (Z,W) \to \tfrac{1}{2}I_1^{-1}k(W,W) + \chi\, k(Z,E_3). \tag{17}$$

In (17) we have omitted the additive constant $-\frac{1}{2}(I_1^{-1} - I_3^{-1})a^2$. In other words, if h is the value of the Hamiltonian $\mathscr{H}$, then the value of the reduced Hamiltonian $\mathscr{H}^a$ is $h^a = h + \frac{1}{2}(I_1^{-1} - I_3^{-1})a^2$.

In order to compute the Hamiltonian vector field of the reduced Hamiltonian $\mathscr{H}^a$ on $(\mathscr{P}^a, \Omega^a)$, it is necessary to use another model for the *reduced Hamiltonian system*
▷ $(\mathscr{H}^a, \mathscr{P}^a, \Omega^a)$. The new model (H^a, P^a, ω^a) is obtained from the old model by pulling back by the map $i : \mathrm{so}(3) \to \mathbf{R}^3$, see chapter III ((1.2)).

(3.6) **Proof**: The new reduced phase space P^a is $\big\{(z,w) \in \mathbf{R}^3 \times \mathbf{R}^3 \big| (z,z) = 1 \;\&\; (z,w) = a\big\}$, where $(\,,\,)$ is the Euclidean inner product on $\mathbf{R}^3$. Here $z = i(Z) = i(\mathrm{Ad}_A E_3)$ and $w = i(W) = i(\mathrm{Ad}_A I(X))$. The new symplectic form $\omega^a = i^*\Omega^a$ is

$$\omega^a(z,w)\big((u \times z, u \times w + t), (r \times z, r \times w + s)\big) = -(t,r) + (u,s) + (w, u \times r), \tag{18}$$

where $u = i(\mathrm{Ad}_A U) = Ai(U)$, $t = i(\mathrm{Ad}_A I(T)) = Ai(I(T))$, $r = i(\mathrm{Ad}_A R) = Ai(R)$ and $s = i(\mathrm{Ad}_A I(S)) = Ai(I(S))$. Here we have used the fact that $i([\mathrm{Ad}_A U,\ \mathrm{Ad}_A E_3]) = i(\mathrm{Ad}_A U) \times$

$i(\mathrm{Ad}_A E_3) = u \times z$ and similarly $i([\mathrm{Ad}_A U, \mathrm{Ad}_A I(X)]) = u \times w$. Note that $(z,t) = 0$ and $(z,s) = 0$, since $(u \times z, u \times w + t)$ and $(r \times z, r \times w + s)$ lie in $T_{(z,w)}P^a$. From (15) we see that the new reduced kinetic energy is

$$K^a = i^* \mathscr{K}^a : P^a \to \mathbf{R} : (z,w) \mapsto \tfrac{1}{2} I_1^{-1}(w,w) - \tfrac{1}{2}(I_1^{-1} - I_3^{-1})a^2; \tag{19}$$

while from (16) it follows that the new reduced potential energy is

$$V^a = i^* \mathscr{V}^a : P^a \to \mathbf{R} : (z,w) \to \chi\,(z, e_3) = \chi\, z_3. \tag{20}$$

Therefore, up to an additive constant the *reduced Hamiltonian* is

$$H^a = i^* \mathscr{H}^a : P^a \subseteq \mathbf{R}^3 \times \mathbf{R}^3 \to \mathbf{R} : (z,w) \mapsto \tfrac{1}{2} {I_1}^{-1}(w,w) + \chi\, z_3. \tag{21}$$

▷ Now we are in position to show that for any smooth Hamiltonian $H : P^a \to \mathbf{R}$ the integral curves of the *Hamiltonian vector field* X_H on (P^a, ω^a) satisfy

$$\begin{aligned} \dot{z} &= \frac{\partial H}{\partial w} \times z \\ \dot{w} &= \frac{\partial H}{\partial w} \times w + \frac{\partial H}{\partial z} \times z. \end{aligned} \tag{22}$$

(3.7) **Proof**: Write $X_H(z,w) = (X_1 \times z, X_1 \times w + X_2)$, where $(z, X_2) = 0$ because $X_H(z,w) \in T_{(z,w)}P^a$. From the definition of Hamiltonian vector field we find that

$$\begin{aligned} \mathrm{d}H(z,w)\big((u \times z, u \times w + v)\big) \quad &= \\ = \omega^a(z,w)\big((X_1 \times z, X_1 \times w + X_2), &(u \times z, u \times w + v)\big), \end{aligned} \tag{23}$$

for every $(u \times z, u \times w + v) \in T_{(z,w)}P^a$. Using $\mathrm{d}H = (\frac{\partial H}{\partial z}, \frac{\partial H}{\partial w})$ and the definition (18) of the symplectic form ω^a, we see that (23) is the same as

$$(\frac{\partial H}{\partial z}, u \times z) + (\frac{\partial H}{\partial w}, u \times w + v) = -(X_2, u) + (X_1, v) + (w, X_1 \times u). \tag{24}$$

Set $u = 0$. Then (24) becomes $(\frac{\partial H}{\partial w}, v) = (X_1, v)$ for every $v \in \mathbf{R}^3$ such that $(z,v) = 0$. This last condition must hold in order that $(u \times z, u \times w + v) \in T_{(z,w)}P^a$. Therefore $X_1 = \frac{\partial H}{\partial w} + \lambda_0 z$ for some $\lambda_0 \in \mathbf{R}$. Set $v = 0$. Then (24) becomes

$$-(\frac{\partial H}{\partial z} \times z, u) - (\frac{\partial H}{\partial w} \times w, u) = -(X_2, u) + (w \times (\frac{\partial H}{\partial w} + \lambda_0 z), u)$$

for every $u \in \mathbf{R}^3$. Hence $X_2 = \frac{\partial H}{\partial z} \times z + \lambda_0 (w \times z)$. Note that $(X_2, z) = 0$ holds automatically. Consequently, the integral curves X_H satisfy

$$\begin{aligned} \dot{z} &= (\frac{\partial H}{\partial w} + \lambda_0 z) \times z = \frac{\partial H}{\partial w} \times z \\ \dot{w} &= \frac{\partial H}{\partial w} \times w + \lambda_0 (z \times w) + \frac{\partial H}{\partial z} \times z + \lambda_0 (w \times z) = \frac{\partial H}{\partial w} \times w + \frac{\partial H}{\partial z} \times z. \end{aligned} \qquad \square$$

Actually (22) defines a system of differential equations on $\mathbf{R}^3 \times \mathbf{R}^3$, called the *Euler-Poisson equations* of H. Note that the reduced space P^a is an invariant manifold of the Euler-Poisson equations.

We now show that if we put a nonstandard Poisson structure on $C^\infty(\mathbf{R}^3 \times \mathbf{R}^3)$, then the Euler-Poisson equations are in Hamiltonian form. Explicitly, define a Poisson bracket $\{\,,\,\}$ on $C^\infty(\mathbf{R}^3 \times \mathbf{R}^3)$ whose structure matrix $W_{C^\infty(\mathbf{R}^3\times\mathbf{R}^3)}$ is given in table 3.1.1.

$\{A,B\}_{\mathbf{R}^3\times\mathbf{R}^3}$	z_1	z_2	z_3	w_1	w_2	w_3	B
z_1	0	0	0	0	z_3	$-z_2$	
z_2	0	0	0	$-z_3$	0	z_1	
z_3	0	0	0	z_2	$-z_1$	0	
w_1	0	z_3	$-z_2$	0	w_3	$-w_2$	
w_2	$-z_3$	0	z_1	$-w_3$	0	w_1	
w_3	z_2	$-z_1$	0	w_2	$-w_1$	0	
A							

Table 3.1.1. Structure matrix $W_{C^\infty(\mathbf{R}^3\times\mathbf{R}^3)}$ for $\{\;,\;\}_{\mathbf{R}^3\times\mathbf{R}^3}$ on $C^\infty(\mathbf{R}^3 \times \mathbf{R}^3)$.

Setting $\zeta = (\zeta_1, \cdots, \zeta_6) = (z, w)$ define the Poisson bracket of $f, g \in C^\infty(\mathbf{R}^3 \times \mathbf{R}^3)$ by

$$\{f, g\}_{\mathbf{R}^3\times\mathbf{R}^3} = \sum_{i,j} \frac{\partial f}{\partial \zeta_i} \frac{\partial g}{\partial \zeta_j} \{\zeta_i, \zeta_j\}_{\mathbf{R}^3\times\mathbf{R}^3}.$$

So $\big(C^\infty(\mathbf{R}^3 \times \mathbf{R}^3), \{\,,\,\}, \cdot\big)$ is a Poisson algebra, see chapter VI §4. For $H \in C^\infty(\mathbf{R}^3 \times \mathbf{R}^3)$ the integral curves of the Hamiltonian vector field $-\mathrm{ad}_H$ satisfy

$$\begin{aligned} \dot z &= -\{H, z\}_{\mathbf{R}^3\times\mathbf{R}^3} = \frac{\partial H}{\partial w} \times z \\ \dot w &= -\{H, w\}_{\mathbf{R}^3\times\mathbf{R}^3} = \frac{\partial H}{\partial w} \times w + \frac{\partial H}{\partial z} \times z. \end{aligned} \tag{25}$$

These are the Euler-Poisson equations (22).

Specializing (25) to the case where the Hamiltonian is the reduced Hamiltonian H^a (21) of the Lagrange top the *reduced Hamiltonian vector field* has integral curves which satisfy

$$\begin{aligned} \dot z &= I_1^{-1} w \times z \\ \dot w &= \chi e_3 \times z. \end{aligned} \tag{26}$$

The solutions of (26) on P^a describe the motion of the Lagrange top after rotation about its figure axis has been removed. This is a model for the motion of the tip of the *figure axis* =(symmetry axis) of the top with a given body angular momentum a.

3.2 The magnetic spherical pendulum

In this subsection we show that after reduction of the right S^1 symmetry, the Lagrange top is equivalent to the *magnetic spherical pendulum* up to a time rescaling.

Physically, the magnetic spherical pendulum is a massive electrically charged particle which moves on a 2-sphere S^2 under the combined influence of a constant vertical gravitational force and a radial magnetic field of strength a due to a *monopole* placed at the center of S^2. The appearance of the magnetic term is due to our choice of zero section of the affine bundle $\mathscr{P}^a$.

Mathematically, the magnetic spherical pendulum is a Hamiltonian system on the phase space (TS^2, Ω_a), where $TS^2 = \big\{(x,y) \in T\mathbf{R}^3 \,\big|\, (x,x) = 1 \;\&\; (x,y) = 0\big\}$ is the tangent bundle of S^2 and Ω_a is the symplectic form

$$\Omega_a(x,y)\big((u,r),(v,s)\big) = -(r,v) + (u,s) + a(x, u \times v), \tag{27}$$

with $(u,r),(v,s) \in T_{(x,y)}(TS^2)$. Note that Ω_a is the sum of the standard symplectic form on TS^2 plus a *magnetic term* $a(x, u \times v)$. Integrating this magnetic term over a domain on S^2 gives a magnetic flux which is proportional to the surface area of the domain. The Hamiltonian of the magnetic spherical pendulum is

$$\widetilde{F} : TS^2 \rightarrow \mathbf{R} : (x,y) \mapsto \tfrac{1}{2} I_1^{-1}(y,y) + \chi\,(x, e_3). \tag{28}$$

Claim: The mapping

$$\varphi : TS^2 \rightarrow P^a : (x,y) \rightarrow (x, x \times y + ax) = (z,w) \tag{29}$$

is an equivalence between the time rescaled magnetic spherical pendulum Hamiltonian system $(\widetilde{F}, TS^2, \Omega_a)$ and the Hamiltonian system (H^a, P^a, ω^a) of the Lagrange top after reduction of the rotational symmetry about its figure axis.

(3.8) **Proof**: It is straightforward to check that the inverse of φ is the smooth mapping $P^a \rightarrow TS^2 : (z,w) \mapsto (z, w \times z)$. Hence φ is a diffeomorphism. We now compute the tangent of φ. From the definition of TS^2 it follows that $(u,r) \in T_{(x,y)}(T\mathbf{R}^3)$ is in $T_{(x,y)}(TS^2)$ if and only if in addition to $(x,x) = 1$ and $(x,y) = 0$, the conditions $(x,u) = 0$ and $(u,y) + (x,r) = 0$ hold. Differentiating (29) gives

$$T_{(x,y)}\varphi : T_{(x,y)}(TS^2) \rightarrow T_{\varphi(x,y)}P^a : (u,r) \mapsto (u, u \times y + x \times r + au)$$

This expression is not useful because the tangent vector $(u, u \times y + x \times r + au)$ at $T_{\varphi(x,y)}P^a$
▷ is not in the form $(\widetilde{u} \times x, \widetilde{u} \times (x \times y + ax) + \widetilde{r})$, where $(x, \widetilde{r}) = 0$. The following argument remedies this by showing that

$$(u, u \times y + x \times r + au) = (\widetilde{u} \times x, \widetilde{u} \times (x \times y + ax) + x \times r) \tag{30}$$

where $\widetilde{u} = x \times u$. Note that $(x, x \times r) = 0$ in (30).

(3.9) **Proof**: Since $(x,x) = 1$ and $(x,u) = 0$, we find that $u = (x \times u) \times x = \widetilde{u} \times x$. Therefore

$$\begin{aligned} u \times y + x \times r + au &= (\widetilde{u} \times x) \times y + x \times r + a\widetilde{u} \times x \\ &= -\widetilde{u}(x,y) + x(\widetilde{u},y) + a\widetilde{u} \times x + x \times r = x(\widetilde{u},y) - y(x,\widetilde{u}) + a\widetilde{u} \times x + x \times r, \\ &\qquad \text{since } (x,y) = 0 \text{ and } (x,\widetilde{u}) = 0 \\ &= \widetilde{u} \times (x \times y + ax) + x \times r. \end{aligned}$$

□

▷ Next we show that $\varphi^*\omega^a = \Omega_a$.

(3.10) **Proof**: We compute

$$\begin{aligned}(\varphi^*\omega^a)(x,y)\big((u,r),(v,s)\big) &= \omega^a(\varphi(x,y))\big(T_{(x,y)}\varphi(u,r), T_{(x,y)}\varphi(v,s)\big)\\ &= \omega^a(x, x\times y + ax)\big((u, u\times y + x\times r + au),(v, v\times y + x\times s + av)\big)\\ &= \omega^a(z,w)\big((\widetilde{u}\times z, \widetilde{u}\times w + \widetilde{r}),(\widetilde{v}\times z, \widetilde{v}\times w + \widetilde{s})\big),\end{aligned}$$

using (30). Here $z = x$, $w = x\times y + ax$, $\widetilde{u} = x\times u$, $\widetilde{v} = x\times v$, $\widetilde{r} = x\times r$ and $\widetilde{s} = x\times s$. Thus

$$\begin{aligned}&(\varphi^*\omega^a)(x,y)\big((u,r),(v,s)\big) =\\ &\quad = -(\widetilde{r},\widetilde{v}) + (\widetilde{u},\widetilde{s}) + (w, \widetilde{u}\times\widetilde{v}), \quad \text{by definition of } \omega^a \text{ (18)}\\ &\quad = -(x\times r, x\times v) + (x\times u, x\times s) + (x\times y + ax, (x\times u)\times(x\times v))\\ &\quad = -(r,v) + (u,s) + a(x, u\times v),\end{aligned}$$

since $(x,u) = (x,v) = 0$ and $(x,x) = 1$. Therefore, $\varphi^*\omega^a = \Omega_a$. □

To finish proving the equivalence we compute φ^*H^a as follows

$$\begin{aligned}(\varphi^*H^a)(x,y) &= \tfrac{1}{2}I_1^{-1}(x\times y + ax, x\times y + ax) + \chi\,(x,e_3)\\ &= \tfrac{1}{2}I_1^{-1}(y,y) + \chi\,(x,e_3) + \tfrac{1}{2}I_1^{-1}a^2 = \widetilde{F}.\end{aligned}$$ □

Introduce a new time scale s by setting $s = I_1^{-1}t$. Then the Hamiltonian system $(\widetilde{F}, TS^2, \Omega_a)$ becomes the *magnetic spherical pendulum* (F, TS^2, Ω_a) with Hamiltonian

$$F(x,y) = I_1(\widetilde{F}(x,y) - 2I_1^{-1}a^2) = \tfrac{1}{2}(y,y) + \lambda\,x_3, \tag{31}$$

where $\lambda = I_1\chi$. □

To find Hamilton's equations for the integral curves of X_F, we consider the magnetic spherical pendulum to be a *constrained* Hamiltonian system. Give the manifold $M = T_0\mathbf{R}^3 = (\mathbf{R}^3\setminus\{0\})\times\mathbf{R}^3$ the *nonstandard* symplectic structure defined by the 2-form

$$\widehat{\Omega}_a(x,y)\big((u,r),(v,s)\big) = -(v,r) + (s,u) + a(|x|^{-3}x, u\times v). \tag{32}$$

For any smooth function $H : T_0\mathbf{R}^3 \to \mathbf{R}$ it is straightforward to check that the Hamiltonian vector field X_H on $(T_0\mathbf{R}^3, \widehat{\Omega}_a)$ has integral curves which satisfy

$$\begin{aligned}\dot{x} &= \frac{\partial H}{\partial y}\\ \dot{y} &= -\frac{\partial H}{\partial x} + a\,|x|^{-3}x\times\frac{\partial H}{\partial y}.\end{aligned}\tag{33}$$

On $(M, \widehat{\Omega}^a)$ define the constraint functions $c_1 : M \to \mathbf{R} : (x,y) \mapsto (x,x) - 1$ and $c_2 : M \to \mathbf{R} : (x,y) \mapsto (x,y)$. Since 0 is a regular value of the constraint map $\mathscr{C} : M \to \mathbf{R}^2 : m \mapsto \big(c_1(m), c_2(m)\big)$, the constraint set $\mathscr{C}^{-1}(0)$ is the smooth manifold TS^2. Because the matrix $(\{c_i, c_j\})$ of Poisson brackets is invertible on M with inverse $(C_{ij}) = \frac{1}{(x,x)}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$, the symplectic form $\widehat{\Omega}_a$ restricted to TS^2 is the symplectic form Ω_a (32) on TS^2. To compute the Hamiltonian vector field of the Hamiltonian

$$\widehat{F}: M \subseteq T\mathbf{R}^3 \to \mathbf{R}: (x,y) \mapsto \tfrac{1}{2}(y,y) + \lambda\,(|x|^{-1}x, e_3) \tag{34}$$

constrained to TS^2, we use the modified Dirac procedure, see chapter VI §4. Note that $\widehat{F}|TS^2$ is the Hamiltonian F of the magnetic spherical pendulum. To start the Dirac procedure let $\widehat{F}^* = \widehat{F} - \sum_{i,j=1}^{3}(\{\widehat{F}, c_i\} + \widehat{F}_i)C_{ij}\,c_j$, where $\widehat{F}_i$ are smooth functions in the ideal of $(C^\infty(M), \cdot)$ generated by the constraint functions c_i. The Poisson bracket $\{\ ,\ \}$ on $C^\infty(M)$ is computed with respect to the symplectic form $\widehat{\Omega}_a$. Because $\{\widehat{F}, c_1\} = -(x,y)$ and $\{\widehat{F}, c_2\} = -(y,y)$, we may choose $\widehat{F}_1(x,y) = -(x,y)\big((x,x)-1\big)$ and $\widehat{F}_2(x,y) = -(y,y)\big((x,x)-1\big)$. Then

$$\widehat{F}^*(x,y) = \tfrac{1}{2}(y,y) + \lambda\,(|x|^{-1}x, e_3) + (y,y)\big((x,x)-1\big) - \tfrac{1}{2}(x,y)^2. \tag{35}$$

Using (33), it follows that the integral curves of the Hamiltonian vector field $X_{\widehat{F}^*}$ on $(M, \widehat{\Omega}_a)$ satisfy

$$\begin{aligned} \dot{x} &= y + \big((x,x)-1\big)y - (x,y)x \\ \dot{y} &= \lambda\,|x|^{-3}x \times (x \times e_3) + (x,y)y - (y,y)x \\ &\qquad + +a\,|x|^{-3}x \times \big(y + ((x,x)-1)y - (x,y)x\big). \end{aligned}$$

Since $\{\widehat{F}^*, F_1\}|TS^2 = \{\widehat{F}^*, F_2\}|TS^2 = 0$, we see that TS^2 is an invariant manifold of $X_{\widehat{F}^*}$. Thus the integral curves of the constrained Hamiltonian vector field $X_F = X_{\widehat{F}^*|TS^2} = X_{\widehat{F}^*}|TS^2$ satisfy the equations

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -\lambda\,e_3 + \big(\lambda\,(x,e_3) - (y,y)\big)x + a\,x \times y, \end{aligned} \tag{36}$$

which are Hamilton's equations for the magnetic spherical pendulum. Note that when $a = 0$ and $\lambda = 1$ (36) reduce to Hamilton's equations for the spherical pendulum, see chapter IV equation (5).

4 Reduction of the left S^1 action

We complete the reduction of the Lagrange top to a one degree of freedom Hamiltonian system by removing the left S^1-action on the reduced level set $J_r^{-1}(a)$ of the momentum of the right S^1-action. Because this left S^1-action has fixed points, the regular reduction theorem does *not* hold. We use invariant theory to carry out singular reduction, see chapter VII §7.

4.1 Induced action on P^a

We show that the left S^1-action Φ^ℓ (5) on $T\,\mathrm{SO}(3)$ induces a diagonal linear S^1-action Δ (40) on $\mathbf{R}^3 \times \mathbf{R}^3$ and then one on the reduced space P^a.

We begin by using the left trivialization map $\mathscr{L}$ (10) to pull back the left S^1-action Φ^ℓ (5) to the S^1-action

$$\varphi^\ell : S^1 \times \big(\mathrm{SO}(3) \times \mathrm{so}(3)\big) \to \mathrm{SO}(3) \times \mathrm{so}(3) : \big(B, (A,X)\big) \mapsto (BA, X). \tag{37}$$

Because the level set $J_r^{-1}(a) = \{(A,X) \in \mathrm{SO}(3) \times \mathrm{so}(3) \,\big|\, k(E_3, I(X)) = a\}$ is invariant under φ^ℓ, the *induced action* $\varphi^\ell|\big(S^1 \times J_r^{-1}(a)\big)$ is defined. Consider the S^1-action

$$\delta^\ell : S^1 \times \big(\mathrm{so}(3) \times \mathrm{so}(3)\big) \to \mathrm{so}(3) \times \mathrm{so}(3) : \big(B, (Z,W)\big) \mapsto (\mathrm{Ad}_B Z, \mathrm{Ad}_B W). \tag{38}$$

The action δ^ℓ leaves the reduced space $\mathscr{P}^a$ invariant; for if $(Z,W) \in \mathscr{P}^a$, then for every $B \in S^1$ we have $k(\mathrm{Ad}_B Z, \mathrm{Ad}_B Z) = k(Z,Z) = 1$ and $k(\mathrm{Ad}_B Z, \mathrm{Ad}_B W) = k(Z,W) = a$. Hence $(\mathrm{Ad}_B Z, \mathrm{Ad}_B Z) \in \mathscr{P}^a$. Therefore the induced action $\delta^\ell|(S^1 \times \mathscr{P}^a)$ is defined.

Claim: The map

$$\pi^a : J_r^{-1}(a) \subseteq \mathrm{SO}(3) \times \mathrm{so}(3) \to \mathscr{P}^a : (A,X) \mapsto \big(\mathrm{Ad}_A E_3, \mathrm{Ad}_A I(X)\big), \tag{39}$$

intertwines the S^1-actions $\varphi^\ell|(S^1 \times J_r^{-1}(a))$ and $\delta^\ell|(S^1 \times \mathscr{P}^a)$. In other words, for every $B \in S^1$ we have $\pi^a \circ \big(\varphi^\ell_B | J_r^{-1}(a)\big) = \big(\delta^\ell_B | \mathscr{P}^a\big) \circ \pi^a$.

(4.1) **Proof**: Let $(A,X) \in J_r^{-1}(a)$. Then

$$\pi^a\big(\varphi^\ell_B(A,X)\big) = \big(\mathrm{Ad}_B(\mathrm{Ad}_A E_3), \mathrm{Ad}_B(\mathrm{Ad}_A I(X))\big) = \delta^\ell_B\big(\pi^a(A,X)\big). \qquad \square$$

Using the identification mapping $i : \mathrm{so}(3) \to \mathbf{R}^3$, see chapter III ((1.2)), the S^1-action δ^ℓ becomes

$$\Delta : S^1 \times (\mathbf{R}^3 \times \mathbf{R}^3) \to \mathbf{R}^3 \times \mathbf{R}^3 : \big(t, (z,w)\big) \mapsto (\widetilde{R}_t z, \widetilde{R}_t w), \tag{40}$$

where $S^1 = \big\{R_t \in \mathrm{SO}(3) \,\big|\, t \in \mathbf{R}\big\}$ with $\widetilde{R}_t = \begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Since the reduced space $P^a = \big\{(z,w) \in \mathbf{R}^3 \times \mathbf{R}^3 \,\big|\, (z,z) = 1 \;\&\; (z,w) = a\big\}$ is invariant under the S^1-action Δ, the
▷ induced action $\Delta|(S^1 \times P^a)$ is defined. It is Hamiltonian with *momentum mapping*

$$J^a_\ell : P^a \subseteq \mathbf{R}^3 \times \mathbf{R}^3 \to \mathbf{R} : (z,w) \to (e_3, w) = w_3. \tag{41}$$

(4.2) **Proof**: The mapping i intertwines the actions $\delta^\ell|\big(S^1 \times \mathscr{P}^a\big)$ and $\Delta|(S^1 \times P^a)$. To find the momentum mapping of the action $\delta^\ell|(S^1 \times \mathscr{P}^a)$ we pull back the momentum mapping $\mathscr{J}_\ell$ (6) of the left S^1-action Φ^ℓ (5) by the left trivialization $\mathscr{L}$. We obtain the momentum map

$$J_\ell : \mathrm{SO}(3) \times \mathrm{so}(3) \to \mathbf{R} : (A,X) \mapsto k(E_3, \mathrm{Ad}_A I(X)) \tag{42}$$

of the left S^1-action φ^ℓ (37). Because $J_r^{-1}(a)$ and the function J_ℓ are invariant under the right S^1-action φ^r (11), the function $J_\ell|J_r^{-1}(a)$ induces a smooth function on the orbit space $J_r^{-1}(a)/S^1 = \mathscr{P}^a$ given by

$$\widetilde{J}^a_\ell : \mathscr{P}^a \subseteq \mathrm{so}(3) \times \mathrm{so}(3) \to \mathbf{R} : (Z,W) \mapsto k(E_3, W), \tag{43}$$

▷ such that $(\pi^a)^* \widetilde{J}^a_\ell = J_\ell | J_r^{-1}(a)$. We now show that $\widetilde{J}^a_\ell$ is the momentum mapping of the S^1-action δ^ℓ (38).

(4.3) **Proof**: Let $(Z,W) \in \mathscr{P}^a$. Then the vector field $Y = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \delta_t^\ell(Z,W)$ is the infinitesimal generator of the S^1-action δ^ℓ. On $J_r^{-1}(a)$ we have

$$(\pi^a)^*(Y \lrcorner \Omega^a) = (\pi^a)^* Y \lrcorner \big((\pi^a)^*\Omega^a\big) = X_{J_\ell} \lrcorner \mathscr{L}^*(\Omega_\rho),$$

since the mapping π^a intertwines the actions φ^ℓ on $J_r^{-1}(a)$ and δ^ℓ on $\mathscr{P}^a$, $(\pi^a)^*\Omega^a = \Omega_\rho | J_r^{-1}(a)$, and X_{J_ℓ} is the infinitesimal generator of the action φ^ℓ, which is Hamiltonian with momentum mapping J_ℓ (42). Therefore $(\pi^a)^*(Y \lrcorner \Omega^a) = \mathrm{d}J_\ell = (\pi^a)^*(\mathrm{d}\widetilde{J}_\ell^a)$, since $(\pi^a)^*\widetilde{J}_\ell^a = J_\ell$ on $J_r^{-1}(a)$. So $Y \lrcorner \Omega^a = \mathrm{d}\widetilde{J}_\ell^a$ on $\mathscr{P}^a$, because the mapping π^a (39) is surjective. Thus the S^1-action δ^ℓ is Hamiltonian with momentum mapping $\widetilde{J}_\ell^a$ (43). □

To finish the proof of ((4.2)) we pull back the function $\widetilde{J}_\ell^a$ by the mapping $i \times i$ to obtain

$$J_\ell^a : P^a \subseteq \mathbf{R}^3 \times \mathbf{R}^3 \rightarrow \mathbf{R} : (z,w) \mapsto (e_3, w) = w_3. \tag{44}$$

J_ℓ^a is the momentum mapping of the S^1-action Δ (40), because $i \times i$ intertwines the actions $\delta^\ell | (S^1 \times \mathscr{P}^a)$ and $\Delta | (S^1 \times P^a)$ and $i \times i$ is a symplectic diffeomorphism of $(\mathscr{P}^a, \Omega^a)$ onto (P^a, ω^a). □

It is interesting to see what all this means in the magnetic spherical pendulum model. The unconstrained Hamiltonian $\widehat{F}$ (34) is invariant under the S^1-action

$$S^1 \times (M = T_0\mathbf{R}^3) \rightarrow M : \big(t, (x,y)\big) \mapsto (\widetilde{R}_t x, \widetilde{R}_t y), \tag{45}$$

where $t \rightarrow \widetilde{R}_t$ is a one parameter group of rotations about the e_3-axis. It is straightforward to check that the infinitesimal generator of this S^1-action is the vector field $X(x,y) = (e_3 \times x, \frac{\partial}{\partial x}) + (e_3 \times y, \frac{\partial}{\partial y})$. Using (33) and $\frac{\partial}{\partial x}(|x|^{-1}x, e_3) = |x|^{-3}(x \times (e_3 \times x))$, we see that X is a Hamiltonian vector field on $(M, \widehat{\Omega}_a)$ corresponding to the Hamiltonian function

$$\widehat{J}_a : M \rightarrow \mathbf{R} : (x,y) \mapsto (x \times y, e_3) + a(|x|^{-1}x, e_3).$$

In other words, the S^1-action (45) is Hamiltonian with momentum mapping $\widehat{J}_a$. Since TS^2 is an invariant manifold of $X_{\widehat{J}_a}$, it follows that the S^1-action (45) restricted to $(TS^2, \widehat{\Omega}_a | TS^2)$ is Hamiltonian with momentum mapping $J_a = \widehat{J}_a | TS^2$. Because the unconstrained Hamiltonian $\widehat{F}$ (34) and the modified Hamiltonian $\widehat{F}^*$ (35) are invariant under the S^1-action (45), it follows that $\{\widehat{F}, J_a\}_{TS^2} = \{\widehat{F}^*, \widehat{J}_a\} | TS^2 = L_X \widehat{F}^* | TS^2 = 0$. Thus

$$J_a : TS^2 \subseteq T\mathbf{R}^3 \rightarrow \mathbf{R} : (x,y) \mapsto (x \times y, e_3) + a(x, e_3)$$

is an integral of the constrained Hamiltonian vector field X_F on $(TS^2, \widehat{\Omega}_a | TS^2)$, where $F = \widehat{F} | TS^2 : (x,y) \mapsto \frac{1}{2}(y,y) + \lambda(x, e_3)$ is the Hamiltonian of the magnetic spherical pendulum.

4.2 The orbit space $(J_\ell^a)^{-1}(b)/S^1$

We now determine the reduced space of the left S^1-action $\Delta | (S^1 \times P^a)$ on P^a (40). This action has fixed points $p_\varepsilon = \varepsilon(0,0,1,0,0,a)$, where $\varepsilon^2 = 1$, because the action Δ (40) fixes every point of the 2-plane $\mathrm{span}\{e_3, e_6\}$ in $\mathbf{R}^3 \times \mathbf{R}^3 = \mathbf{R}^6$ and this 2-plane intersects

P^a at p_ε. Since $p_\varepsilon \in (J^a_\ell)^{-1}(\varepsilon a)$, we can *not* apply the regular reduction theorem to find the reduced phase space of the left S^1-action $\Delta|(S^1 \times P^a)$ on $(J^a_\ell)^{-1}(\varepsilon a)$, because the S^1-action $\Delta|(S^1 \times P^a)$ is *not* free. To construct a model for the *orbit space* $(J^a_\ell)^{-1}(b)/S^1$ for *every* value b of the momentum map J^a_ℓ (44) we use invariant theory.

Claim: The algebra $\mathbf{R}[z,w]^{S^1}$ of polynomials on $\mathbf{R}^3 \times \mathbf{R}^3$, which are invariant under the S^1-action Δ (40), is generated by the monomials

$$\begin{aligned} \pi_1 &= z_3, \quad & \pi_3 &= z_1w_1 + z_2w_2, \quad & \pi_5 &= z_1^2 + z_2^2, \\ \pi_2 &= w_3, \quad & \pi_4 &= z_2w_1 - z_1w_2, \quad & \pi_6 &= w_1^2 + w_2^2. \end{aligned} \tag{46}$$

(4.4) **Proof**: See chapter IV ((2.1)). □

To find a model for the orbit space $(\mathbf{R}^3 \times \mathbf{R}^3)/S^1$ of the action Δ let

$$\pi : \mathbf{R}^3 \times \mathbf{R}^3 \to \mathbf{R}^6 : (z,w) \mapsto \big(\pi_1(z,w), \ldots, \pi_6(z,w)\big). \tag{47}$$

▷ π is the *Hilbert map* of the S^1-action Δ. The image of π is the real semialgebraic variety $V = \mathbf{R}^2 \times W$ in $\mathbf{R}^6$ with coordinates $\big((\pi_1,\pi_2),(\pi_3,\ldots,\pi_6)\big)$ defined by

$$\pi_3^2 + \pi_4^2 = \pi_5\pi_6, \quad \text{where } \pi_5 \geq 0 \;\&\; \pi_6 \geq 0. \tag{48}$$

(4.5) **Proof**: See chapter IV ((2.2)). □

▷ V is homeomorphic to the orbit space $(\mathbf{R}^3 \times \mathbf{R}^3)/S^1$ via the induced mapping

$$\overline{\pi} : (\mathbf{R}^3 \times \mathbf{R}^3)/S^1 \to V. \tag{49}$$

(4.6) **Proof**: See chapter IV ((2.3)). □

The next step is to find the orbit space $(J^a_\ell)^{-1}(b)/S^1$. From (44) we see that the level set $(J^a_\ell)^{-1}(b)$ is defined by

$$\begin{aligned} z_1^2 + z_2^2 + z_3^2 &= 1 \\ z_1w_1 + z_2w_2 + z_3w_3 &= a \\ w_3 &= b. \end{aligned} \tag{50}$$

Let $\Sigma_{a,b}$ be the semialgebraic variety in $\mathbf{R}^6$ defined by

$$\begin{aligned} \pi_3^2 + \pi_4^2 &= \pi_5\pi_6, \quad \pi_5 \geq 0 \;\&\; \pi_6 \geq 0 \\ \pi_5 + \pi_1^2 &= 1 \\ \pi_3 + \pi_1\pi_2 &= a \\ \pi_2 &= b. \end{aligned} \tag{51}$$

The left hand side of last three equations in (51) comes from expressing the polynomials on the left hand side of (50) in terms of the invariants (46). From ((4.3)) it follows that the set $\Sigma_{a,b} = \pi\big((J^a_\ell)^{-1}(b)\big)$ is the space of S^1 orbits of the S^1-action $\Delta|\big(S^1 \times (J^a_\ell)^{-1}(b)\big)$.

Claim: The semialgebraic variety P^a_b of $\mathbf{R}^3$ with coordinates σ_i, $1 \leq i \leq 3$, defined by

$$0 = G(\sigma_1,\sigma_2,\sigma_3) = (1-\sigma_1^2)\sigma_3 - \sigma_2^2 - (a - b\sigma_1)^2 \quad |\sigma_1| \leq 1 \;\&\; \sigma_3 \geq 0 \tag{52}$$

is homeomorphic to the variety $\Sigma_{a,b}$ (51), where $\sigma_1 = \pi_1$, $\sigma_2 = \pi_4$, and $\sigma_3 = \pi_6$.

(4.7) **Proof**: Consider the mapping

$$\begin{aligned} &\lambda : \mathbf{R}^3 \rightarrow \mathbf{R}^6 : \\ &\qquad \sigma = (\sigma_1, \sigma_2, \sigma_3) \mapsto (\pi_1, \pi_2, \pi_3, \pi_4, \pi_5, \pi_6) = (\sigma_1, b, a - b\sigma_1, \sigma_2, 1 - \sigma_1^2, \sigma_3). \end{aligned} \tag{53}$$

A quick check shows that the mapping $\pi : \mathbf{R}^6 \rightarrow \mathbf{R}^3 : \pi \mapsto (\pi_1, \pi_4, \pi_6)$, when restricted to $\Sigma_{a,b}$, is the inverse of $\lambda | P_b^a$. Consequently, the varieties P_b^a and $\Sigma_{a,b}$ are homeomorphic, using the topology induced from $\mathbf{R}^3$ and $\mathbf{R}^6$, respectively. □

▷ The variety P_b^a is homeomorphic to the orbit space $(J_\ell^a)^{-1}(b)/S^1$.

(4.8) **Proof**: Consider diagram 4.2.1, where $\widetilde{\pi} = (\lambda^{-1} \circ \pi) | (J_\ell^a)^{-1}(b)$ and $\widetilde{\rho}$ is the orbit map, which assigns to each S^1 orbit of $\Delta | \big(S^1 \times (J_\ell^a)^{-1}(b)\big)$ a point in the orbit space $(J_\ell^a)^{-1}(b)/S^1$.

$$\begin{tikzcd} (J_\ell^a)^{-1}(b) \arrow[r, "\widetilde{\pi}"] \arrow[d, "\widetilde{\rho}"] & P_b^a \\ (J_\ell^a)^{-1}(b)/S^1 \arrow[ur, "\widetilde{\sigma}"'] & \end{tikzcd}$$

Diagram 4.2.1.

Since the continuous mapping $\widetilde{\pi}$ is invariant under the S^1-action Δ on $(J_\ell^a)^{-1}(b)$, it induces the continuous map $\widetilde{\sigma} : (J_\ell^a)^{-1}(b)/S^1 \rightarrow P_b^a$, which makes diagram 4.2.1 commute. Because each fiber of the bundle map $\widetilde{\pi}$ is a single S^1 orbit of Δ on $(J_\ell^a)^{-1}(b)$ and $\widetilde{\pi}$ has a continuous local cross section, it follows that $\widetilde{\sigma}$ is invertible and its inverse is continuous. Hence $\widetilde{\sigma}$ is a homeomorphism. □

Therefore P_b^a is a model for the singular reduced space $(J_\ell^a)^{-1}(b)/S^1$. Figures 4.2.1 and
▷ 4.2.2 depict the reduced space P_b^a. We check that these figures are qualitatively correct.

(4.9) **Proof**: When $b \neq \varepsilon a$, the reduced space P_b^a is a smooth manifold diffeomorphic to $\mathbf{R}^2$.

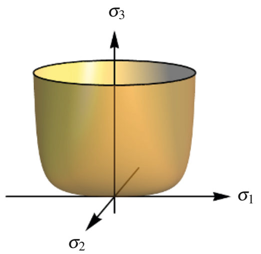


Figure 4.2.1. The reduced space P_b^a when $b \neq \varepsilon a$.

To see this suppose that $\widetilde{\sigma} = (\widetilde{\sigma}_1, \widetilde{\sigma}_2, \widetilde{\sigma}_3)$ is a singular point of P^a_b. Then

$$(0,0,0) = DG(\widetilde{\sigma}) = \big(2b(a - b\widetilde{\sigma}_1) - 2\widetilde{\sigma}_1\widetilde{\sigma}_3, -2\widetilde{\sigma}_2, 1 - \widetilde{\sigma}_1^2\big). \tag{54}$$

We obtain $\widetilde{\sigma_1} = \varepsilon$, $\widetilde{\sigma_2} = 0$, $\widetilde{\sigma_3} = \varepsilon b(a - \varepsilon b)$, where $\varepsilon^2 = 1$. But $G(\widetilde{\sigma}) = 0$, which gives $b = \varepsilon a$. This contradicts our hypothesis. Hence the reduced space P^a_b is a smooth manifold, when $b \neq \varepsilon a$. In fact P^a_b is diffeomorphic to $\mathbf{R}^2$, because it is the graph of the smooth function

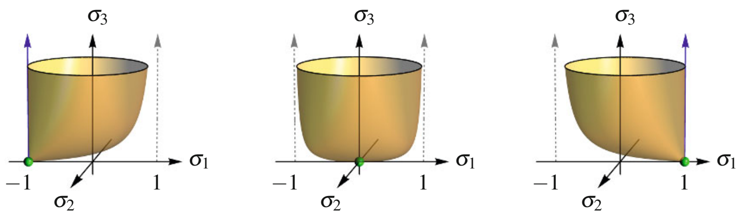
$$\mathscr{G} : (-1,1) \times \mathbf{R} \to \mathbf{R} : (\sigma_1, \sigma_2) \mapsto \big(\sigma_2^2 + (a - b\sigma_1)^2\big)(1 - \sigma_1^2)^{-1}, \quad |\sigma_1| < 1.$$

To see this solve the defining equation (52) of P^a_b for σ_3. We check that $|\sigma_1| < 1$ as follows. Suppose that $\sigma_1 = \varepsilon$ with $\varepsilon^2 = 1$. Then equation (52) becomes $0 = \sigma_2^2 + (a - \varepsilon b)^2$, which implies $b = \varepsilon a$. But this is a contradiction. Therefore $|\sigma_1| < 1$, because $|\sigma_1| \leq 1$ in (52).

Now suppose that $b = \varepsilon a$. Then $(\varepsilon, 0, 0)$ is the only singular point of $P^a_{\varepsilon a}$, if $a \neq 0$; while $(\pm 1, 0, 0)$ are the only singular points of P^0_0 when $a = 0$. At each singular point $(\varepsilon, 0, 0)$ the variety $P^a_{\varepsilon a}$ has a nondegenerate *tangent cone* with Morse index 1 given by

$$0 = a^2(1 - \varepsilon\sigma_1)^2 + \sigma_2^2 - 2(1 - \varepsilon\sigma_1)\sigma_3.$$

Since $\sigma_3 \geq 0$ on $P^a_{\varepsilon a}$, each conical singularity of $P^a_{\varepsilon a}$ is topologically a cone on a circle. Thus $P^a_{\varepsilon a}$ is homeomorphic, but *not* diffeomorphic, to $\mathbf{R}^2$. □



$b = -a,\ a \neq 0$ $\qquad$ $a = 0$ $\qquad$ $b = a,\ a \neq 0$

Figure 4.2.2. The reduced phase space $P^a_{\varepsilon a}$.

Since the Hamiltonian $H^a : P^a \to \mathbf{R} : (z, w) \mapsto \frac{1}{2}I_1^{-1}(w, w) + \chi z_3$ is invariant under the left S^1-action $\Delta|\big(S^1 \times (J^a_\ell)^{-1}(b)\big)$ (40), there is an *induced Hamiltonian* on P^a_b given by

$$H^a_b : P^a_b \subseteq \mathbf{R}^3 \to \mathbf{R} : (\sigma_1, \sigma_2, \sigma_3) \mapsto \tfrac{1}{2}I_1^{-1}\sigma_3 + \chi\sigma_1. \tag{55}$$

More precisely, $(\pi^a_b)^* H^a_b = H^a|(J^a_\ell)^{-1}(b)$, where

$$\begin{aligned} &\pi^a_b : (J^a_\ell)^{-1}(b) \subseteq P^a \to P^a_b : \\ &\qquad (z, w) \mapsto \big(\sigma_1(z, w), \sigma_2(z, w), \sigma_3(z, w)\big) = (z_3, z_2w_1 - z_1w_2, w_1^2 + w_2^2) \end{aligned} \tag{56}$$

is the $\Delta|\big(S^1 \times (J^a_\ell)^{-1}(b)\big)$ *orbit mapping* $(\lambda^{-1} \circ \pi)|(J^a_\ell)^{-1}(b)$. Note that π^a_b maps the $h^a + \frac{1}{2}I_1^{-1}b^2$ level set of H^a onto the h^a_b level set of H^a_b.

4.3 Some differential spaces

We start by looking at the Hilbert map $\pi : \mathbf{R}^3 \times \mathbf{R}^3 \to \mathbf{R}^6$ (47) of the S^1-action Δ (40).

Claim: The linear mapping $\pi^* : C^\infty(\mathbf{R}^6) \to C^\infty(\mathbf{R}^6 \times \mathbf{R}^6)^{S^1} : f \mapsto f \circ \pi$ is surjective.

(4.10) **Proof**: Treat the invariant polynomials π_i, $1 \le i \le 6$, (46) as coordinates on $\mathbf{R}^6$. Since the S^1-action Δ is a linear action of a compact Lie group S^1 on $\mathbf{R}^3 \times \mathbf{R}^3$, by Schwarz's theorem for each smooth S^1-invariant function f on $\mathbf{R}^3 \times \mathbf{R}^3$ there is a smooth function F on $\mathbf{R}^6$ such that $\pi^* F = f$. Hence π^* is surjective. $\square$

The semialgebraic variety V (48) is the image of the Hilbert map π (47). Let $\mathscr{F}$ be the family of functions $f : V \subseteq \mathbf{R}^6 \to \mathbf{R}$ such that the function $\pi^* f$ is smooth on $\mathbf{R}^3 \times \mathbf{R}^3$. Let $C^\infty(V)$ be the space of smooth functions on V generated by $\mathscr{F}$. $C^\infty(V)$ is a differential structure on V, see ((3.2)) in chapter VII, which contains $\mathscr{F}$, see ((3.1)) chapter VII. In fact, $\mathscr{F} = C^\infty(V)$, see ((3.11)) in chapter VII.

Claim: The linear mapping $\pi^* : C^\infty(V) \to C^\infty(\mathbf{R}^3 \times \mathbf{R}^3)^{S^1}$ is an isomorphism of vector spaces.

(4.11) **Proof**: Because $\pi(\mathbf{R}^3 \times \mathbf{R}^3) = V$, the mapping π^* is injective. To see this we argue as follows. If $\pi^* f = 0$ for some $f \in C^\infty(V)$, then $f(\pi(p)) = 0$ for every $p \in \mathbf{R}^3 \times \mathbf{R}^3$. Hence $f = 0$ on V. By Schwarz's theorem, for each smooth S^1-invariant function f on $\mathbf{R}^3 \times \mathbf{R}^3$ there is a smooth function F on $\mathbf{R}^6$ such that $f = \pi^*(F|V)$. Thus the mapping π^* is surjective and hence is an isomorphism. $\square$

We give another differential structure on V. We say that a function $f : V \subseteq \mathbf{R}^6 \to \mathbf{R}$ is a member of the family $\widetilde{\mathscr{F}}$ if and only if there is a smooth function $F : \mathbf{R}^6 \to \mathbf{R}$ such that $f = F|V$. Let $C_i^\infty(V)$ be the space of smooth functions on V generated by $\widetilde{\mathscr{F}}$, see chapter VII §3. Then $C_i^\infty(V)$ is a differential structure on V, see ((3.2)) in chapter VII, which contains $\widetilde{\mathscr{F}}$, see ((3.1)) in chapter VII. In fact, $\widetilde{\mathscr{F}} = C_i^\infty(V)$, see ((3.1.3)) in chapter VII. The identity map on V is a homeomorphism, using ((3.14)) in chapter VII, and the ▷ differential space topologies on the differential spaces $(V, C^\infty(V))$ and $(V, C_i^\infty(V))$. The mapping $\mathrm{id}_V^* : C_i^\infty(V) \to C^\infty(V)$ is an isomorphism of vector spaces.

(4.12) **Proof**: The mapping id_V^* is well defined, for suppose that $f \in C_i^\infty(V)$. Then there is an $F \in C^\infty(\mathbf{R}^6)$ such that $F|V = f$. So $\pi^* f = \pi^*(F|V) \in C^\infty(\mathbf{R}^3 \times \mathbf{R}^3)^{S^1}$, that is, $f \in C^\infty(V)$. Hence $f = \mathrm{id}_V^* f$, which shows that the map id_V^* is well defined and is surjective. It is injective, because the map id_V is surjective. Thus id_V^* is an isomorphism. $\square$

Claim: The mapping id_V is a diffeomorphism of the differential space $(V, C^\infty(V))$ onto the differential space $(V, C_i^\infty(V))$.

(4.13) **Proof**: This follows immediately from ((4.12)) and the fact that id_V is a homeomorphism of V onto itself, using the differential space topologies on $(V, C^\infty(V))$ and $(V, C_i^\infty(V))$. $\square$

Let $(\mathbf{R}^3 \times \mathbf{R}^3)/S^1$ be the space of orbits of the action Δ with orbit map $\rho : \mathbf{R}^3 \times \mathbf{R}^3 \to (\mathbf{R}^3 \times \mathbf{R}^3)/S^1$, which assigns to each $p \in \mathbf{R}^3 \times \mathbf{R}^3$ the S^1-orbit of Δ that passes through p. A function $\overline{f} : (\mathbf{R}^3 \times \mathbf{R}^3)/S^1 \to \mathbf{R}$ is smooth if the S^1-invariant function $\rho^* \overline{f}$ on $\mathbf{R}^3 \times \mathbf{R}^3$ is

smooth. Let $C^\infty\big((\mathbf{R}^3\times\mathbf{R}^3)/S^1\big)$ be the space of smooth functions on $(\mathbf{R}^3\times\mathbf{R}^3)/S^1$. The
▷ linear mapping $\rho^*: C^\infty\big((\mathbf{R}^3\times\mathbf{R}^3)/S^1\big)\to C^\infty(\mathbf{R}^3\times\mathbf{R}^3)^{S^1}$ is an isomorphism.

(4.14) **Proof**: Because the orbit mapping ρ is surjective, the linear mapping ρ^* is injective. If $f\in C^\infty(\mathbf{R}^3\times\mathbf{R}^3)^{S^1}$, then it induces a smooth function $\overline{f}:(\mathbf{R}^3\times\mathbf{R}^3)/S^1\to\mathbf{R}$ such that $f=\overline{f}\circ\rho=\rho^*\overline{f}$. Hence the mapping ρ^* is surjective and thus is an isomorphism. □

By definition the Hilbert mapping π (47) is invariant under the S^1-action Δ. Therefore
▷ it induces a mapping $\overline{\pi}:(\mathbf{R}^3\times\mathbf{R}^3)/S^1\to V$ such that $\pi=\overline{\pi}\circ\rho$. The linear mapping $\overline{\pi}^*: C^\infty(V)\to C^\infty\big((\mathbf{R}^3\times\mathbf{R}^3)/S^1\big)$ is an isomorphism.

(4.15) **Proof**: This follows immediately because the linear maps ρ^* and π^* are isomorphisms and $\overline{\pi}^*=(\rho^*)^{-1}\circ\pi^*$. □

Since the S^1-action Δ (40) is proper, the orbit space and its collection of smooth functions $\big((\mathbf{R}^3\times\mathbf{R}^3)/S^1, C^\infty\big((\mathbf{R}^3\times\mathbf{R}^3)/S^1\big)\big)$ is a differential space, see ((3.6)) in chapter VII.

Claim: The mapping $\overline{\pi}$ is a diffeomorphism of the differential space $\big((\mathbf{R}^3\times\mathbf{R}^3)/S^1$, $C^\infty\big((\mathbf{R}^3\times\mathbf{R}^3)/S^1\big)$ onto the differential space $(V, C^\infty(V))$.

(4.16) **Proof**: Because the mapping $\overline{\pi}^*: C^\infty(V)\to C^\infty\big((\mathbf{R}^3\times\mathbf{R}^3)/S^1\big)$ is an isomorphism, it suffices to show that the mapping $\overline{\pi}$ is a homeomorphism from the orbit space $(\mathbf{R}^3\times\mathbf{R}^3)/S^1$ onto the semialgebraic variety V. On $C^\infty\big((\mathbf{R}^3\times\mathbf{R}^3)/S^1\big)$ and $C^\infty(V)$ we use the differential space topology, see chapter VII §3 and §4. We argue as follows. Let U be an open subset of V. For every $p\in U$ there are open intervals I_i, $1\le i\le n$, such that $\cap_{i=1}^n f_i^{-1}(I_i)$ is an open subset of V containing p and contained in U. Now

$$\overline{\pi}^{-1}\big(\bigcap_{i=1}^n f_i^{-1}(I_i)\big)=\bigcap_{i=1}^n \overline{\pi}^{-1}\big(f_i^{-1}(I_i)\big)=\bigcap_{i=1}^n (\overline{\pi}^* f_i)^{-1}(I_i),$$

where the first equality holds because $\overline{\pi}$ is injective, since each of its fibers is a single orbit of the action Δ. But $\overline{\pi}^* f\in C^\infty\big((\mathbf{R}^3\times\mathbf{R}^3)/S^1\big)$. So $\overline{\pi}^{-1}\big(\bigcap_{i=1}^n f_i^{-1}(I_i)\big)$ is an open subset of $\overline{\pi}^{-1}(U)\subseteq V$ containing $\overline{\pi}(p)$. Thus $\overline{\pi}^{-1}(U)$ is an open subset of $(\mathbf{R}^3\times\mathbf{R}^3)/S^1$ in the differential space topology. Hence the mapping $\overline{\pi}$ is continuous.
Let $\overline{p}\in\overline{U}$, where $\overline{U}$ is an open subset of $(\mathbf{R}^3\times\mathbf{R}^3)/S^1$ in the differential space topology. Then there are open intervals I_i, $1\le i\le n$, and functions $\overline{F}_i\in C^\infty\big((\mathbf{R}^3\times\mathbf{R}^3)/S^1\big)$, $1\le i\le n$, such that $\cap_{i=1}^n\overline{F}_i^{-1}(\overline{U})$ is an open subset of $\overline{U}$ containing $\overline{p}$. Because $\overline{\pi}$ is injective we find that

$$\overline{\pi}\big(\bigcap_{i=1}^n \overline{F}_i^{-1}(\overline{U})\big)=\bigcap_{i=1}^n \overline{\pi}\big(\overline{F}_i^{-1}(I_i)\big)=\bigcap_{i=1}^n ((\overline{\pi}^{-1})^*\overline{F}_i)^{-1}(I_i).$$

So $\overline{\pi}\big(\bigcap_{i=1}^n\overline{F}_i^{-1}(I_i)\big)$ is an open subset of $\overline{\pi}(\overline{U})\subseteq V$ containing $\overline{\pi}(\overline{p})$. Consequently, $\overline{\pi}(\overline{U})$ is an open set. So the map $\overline{\pi}^{-1}$ is continuous. Thus $\overline{\pi}$ is a homeomorphism. □

In §3 we have shown that for every $a\in\mathbf{R}$ the first reduced phase space $P^a=\{(z,w)\in\mathbf{R}^3\times\mathbf{R}^3\,\big|\,(z,z)=1\ \&\ (z,w)=a\}$ is a smooth S^1-invariant symplectic submanifold with symplectic form ω^a. We say that f is a smooth S^1-invariant function on P^a if there is a smooth S^1-invariant function F on $\mathbf{R}^3\times\mathbf{R}^3$ such that $f=F|P^a$. Let $C^\infty(P^a)^{S^1}$ be the set of smooth S^1-invariant functions on P^a.

The S^1-action $\Delta|(S^1 \times P^a)$ is Hamiltonian with momentum mapping $J^a_\ell : P^a \to \mathbf{R} : (z,w) \mapsto (w,e_3) = w_3$. For every $b \in \mathbf{R}$ the level set $(J^a_\ell)^{-1}(b)$ is a smooth S^1-invariant submanifold of P^a. Hence the space $C^\infty\big((J^a_\ell)^{-1}(b)\big)^{S^1}$ of smooth S^1-invariant functions on $(J^a_\ell)^{-1}(b)$ is isomorphic to $C^\infty(P^a)^{S^1}/\mathscr{I}$, where $\mathscr{I} = \mathscr{I}\big((J^a_\ell)^{-1}(b)\big)$ is the ideal of smooth S^1-invariant functions on P^a, which vanish identically on $(J^a_\ell)^{-1}(b)$. Let $\widetilde{\rho} = \rho|(J^a_\ell)^{-1}(b) : (J^a_\ell)^{-1}(b) \to (J^a_\ell)^{-1}(b)/S^1$ be the restriction of the S^1-orbit map $\rho : \mathbf{R}^3 \times \mathbf{R}^3 \to (\mathbf{R}^3 \times \mathbf{R}^3)/S^1$ to $(J^a_\ell)^{-1}(b)$. We say that $\overline{f}$ is a smooth function on $(J^a_\ell)^{-1}(b)/S^1$ if there is an $F \in C^\infty(\mathbf{R}^3 \times \mathbf{R}^3)^{S^1}$ such that $\overline{f} = F|(J^a_\ell)^{-1}(b)$. Note that $F|(J^a_\ell)^{-1}(b) \in$
$\triangleright$ $C^\infty((J^a_\ell)^{-1}(b))^{S^1}$. The induced linear map $\widetilde{\rho}^* : C^\infty((J^a_\ell)^{-1}(b)/S^1) \to C^\infty((J^a_\ell)^{-1}(b))^{S^1}$ is an isomorphism.

(4.17) **Proof**: Because the mapping $\widetilde{\rho}$ is surjective, it follows that the linear mapping $\widetilde{\rho}^*$ is injective. To verify that $\widetilde{\rho}^*$ is surjective, let $f \in C^\infty((J^a_\ell)^{-1}(b))^{S^1}$. Then there is an $F \in C^\infty(\mathbf{R}^3 \times \mathbf{R}^3)^{S^1}$ such that $f = F|(J^a_\ell)^{-1}(b)$. Since $F|(J^a_\ell)^{-1}(b)$ is an S^1-invariant smooth function on $(J^a_\ell)^{-1}(b)$, it induces a function $\overline{F}$ on $(J^a_\ell)^{-1}(b)/S^1$ such that $\widetilde{\rho}^*(\overline{F}) = F|(J^a_\ell)^{-1}(b) = f$. By definition $\overline{F} \in C^\infty((J^a_\ell)^{-1}(b)/S^1)$. Hence $\widetilde{\rho}^*$ is surjective. □

Since the S^1-action $\Delta|\big(S^1 \times (J^a_\ell)^{-1}(b)\big)$ is proper, $\big((J^a_\ell)^{-1}(b)/S^1, C^\infty\big((J^a_\ell)^{-1}(b)/S^1\big)\big)$ is a differential space, see ((3.8)) chapter VII. The topology on the orbit space $(J^a_\ell)^{-1}(b)/S^1$ is the differential space topology.

Recall that $\Sigma_{a,b} = \pi\big((J^a_\ell)^{-1}(b)\big)$, where π is the Hilbert mapping (47). We say that the function f is a member of the family $\mathscr{F}$ if there is a smooth S^1-invariant function F on $\mathbf{R}^3 \times \mathbf{R}^3$ such that $\tau^* f = F|J^a_\ell)^{-1}(b)$. Here $\tau = \pi|J^a_\ell)^{-1}(b) : J^a_\ell)^{-1}(b) \subseteq \mathbf{R}^3 \times \mathbf{R}^3 \to \Sigma_{a,b} \subseteq \mathbf{R}^6$. Let $C^\infty(\Sigma_{a,b})$ be the space of smooth functions on $\Sigma_{a,b}$ generated by $\mathscr{F}$. Then $C^\infty(\Sigma_{a,b})$ is a differential structure on $\Sigma_{a,b}$, see chapter VII ((3.2)); the topology generated by $\mathscr{F}$ is the same as the topology induced from $\mathbf{R}^6$, see ((3.13)); and $\mathscr{F} = C^\infty(\Sigma_{a,b})$, see ((3.3)). Because the mapping τ is S^1-invariant and surjective with each fiber being a single S^1-orbit of the action $\Delta|\big(S^1 \times (J^a_\ell)^{-1}(b)\big)$, it follows that the induced mapping $\overline{\tau} : (J^a_\ell)^{-1}(b)/S^1 \to \Sigma_{a,b}$, where $\overline{\tau}\circ\widetilde{\rho} = \tau$, is a homeomorphism, being the restriction of the homeomorphism $\overline{\pi}$ to $(J^a_\ell)^{-1}(b)/S^1$. Here we use the differential space topology on $(J^a_\ell)^{-1}(b)/S^1$ and $\Sigma_{a,b}$, which by ((3.10)) and ((3.14)) of chapter VII are the same as the quotient topology and the topology induced from $\mathbf{R}^6$, respectively. The induced mapping
$\triangleright$ $\overline{\tau}^* : C^\infty(\Sigma_{a,b}) \to C^\infty\big((J^a_\ell)^{-1}(b)/S^1\big)$ is an isomorphism.

(4.18) **Proof**: Suppose that $f \in C^\infty(\Sigma_{a,b})$. Then there is an $F \in C^\infty(P^a)^{S^1}$ such that $g = F|(J^a_\ell)^{-1}(b)$ $= f\circ\tau$. But $g \in C^\infty((J^a_\ell)^{-1}(b))^{S^1}$. So it induces $\overline{g} \in C^\infty((J^a_\ell)^{-1}(b)/S^1)$. From $\overline{g}\circ\widetilde{\rho} = g = f\circ\tau = f\circ\overline{\tau}\circ\widetilde{\rho}$, we obtain $\overline{g} = f\circ\overline{\tau} = \overline{\tau}^*(f)$, because $\widetilde{\rho}$ is surjective. Thus the linear mapping $\overline{\tau}^*$ is well defined and is surjective. It is injective, since the map $\overline{\tau}$ is surjective, and hence is an isomorphism. □

Thus we have proved

Claim: The mapping $\overline{\tau}$ is a diffeomorphism of the differential space $\big((J^a_\ell)^{-1}(b)/S^1,$ $C^\infty\big((J^a_\ell)^{-1}(b)/S^1\big)\big)$ onto the differential space $\big(\Sigma_{a,b}, C^\infty(\Sigma_{a,b})\big)$.

Consider the mappings

$$\lambda : \mathbf{R}^3 \to \mathbf{R}^6 : (\sigma_1, \sigma_2, \sigma_3) \mapsto (\sigma_1, b, a - b\sigma_1, \sigma_2, 1 - \sigma_1^2, \sigma_3) \tag{57}$$

and

$$\mu : \mathbf{R}^6 \to \mathbf{R}^3 : (\pi_1, \ldots, \pi_6) \mapsto (\pi_1, \pi_4, \pi_6). \tag{58}$$

Then $P_b^a = \mu(\Sigma_{a,b})$. Since $(\lambda \circ \mu)|\Sigma_{a,b} = \mathrm{id}_{\Sigma_{a,b}}$ and $(\mu \circ \lambda)|P_b^a = \mathrm{id}_{P_b^a}$, it follows that $\mu|\Sigma_{a,b}$ is a homeomorphism of $\Sigma_{a,b}$ onto P_b^a. Here we use the topology on $\Sigma_{a,b}$ and P_b^a induced from $\mathbf{R}^6$ and $\mathbf{R}^3$, respectively. Observe that $\lambda|P_b^a$ is the inverse of $\mu|\Sigma_{a,b}$. Let $C^\infty(P_b^a) = (\lambda|P_b^a)^* C^\infty(\Sigma_{a,b})$ be the space of *smooth functions* on P_b^a. The result ((3.16)) in chapter VII shows that the differential space topology on $\Sigma_{a,b}$ and P_b^a is the same as the topology induced from $\mathbf{R}^6$ and $\mathbf{R}^3$, respectively. Thus the mapping $\mu|\Sigma_{a,b}$ is a homeomorphism using the induced topologies. By definition the linear mapping $(\lambda|P_b^a)^* : C^\infty(\Sigma_{a,b}) \to C^\infty(P_b^a)$ is surjective. It is injective because the map $\lambda|P_b^a : P_b^a \to \Sigma_{a,b}$ is surjective. Hence $(\lambda|P_b^a)^*$ is an isomorphism. This proves

Claim: The mapping $\lambda|P_b^a$ from the differential space $\big(P_b^a, C^\infty(P_b^a)\big)$ onto the differential space $\big(\Sigma_{a,b}, C^\infty(\Sigma_{a,b})\big)$ is a diffeomorphism.

Corollary: The differential spaces $\big(V, C^\infty(V)\big)$, $\big(\Sigma_{a,b}, C^\infty(\Sigma_{a,b})\big)$, and $\big(P_b^a, C^\infty(P_b^a)\big)$ are subcartesian and locally compact.

(4.19) **Proof**: Because V, $\Sigma_{a,b}$, and P_b^a are semialgebraic varieties, they are locally closed in the topology induced from their ambient real vector space, namely, $\mathbf{R}^6$, $\mathbf{R}^6$, and $\mathbf{R}^3$, respectively. Hence the differential spaces $\big(V, C^\infty(V)\big)$, $\big(\Sigma_{a,b}, C^\infty(\Sigma_{a,b})\big)$, and $\big(P_b^a, C^\infty(P_b^a)\big)$ are subcartesian and locally compact, see §3.2 in chapter VII. □

We give another differential structure on $\Sigma_{a,b}$, which is a semialgebraic variety in $\mathbf{R}^6$. We say that a function $f : \Sigma_{a,b} \subseteq \mathbf{R}^6 \to \mathbf{R}$ is a member of the family $\widetilde{\mathscr{F}}$ if there is a smooth function $F : \mathbf{R}^6 \to \mathbf{R}$ such that $f = F|\Sigma_{a,b}$. Let $C_i^\infty(\Sigma_{a,b})$ be the space of smooth functions on $\Sigma_{a,b}$ generated by the family $\widetilde{\mathscr{F}}$. By ((3.2)) in chapter VII, $C_i^\infty(\Sigma_{a,b})$ is a differential structure on $\Sigma_{a,b}$. By ((3.16)) in chapter VII we know that the topology on $\Sigma_{a,b}$ generated by $\widetilde{F}$ is the same as that induced from $\mathbf{R}^6$ and from ((3.3)) in chapter VII it follows that $\widetilde{\mathscr{F}} = C_i^\infty(\Sigma_{a,b})$. Using the differential space topology of $(\Sigma_{a,b}, C^\infty(\Sigma_{a,b}))$ and $(\Sigma_{a,b}, C_i^\infty(\Sigma_{a,b}))$ from ((3.18)) of chapter VII it follows that the identity map $\mathrm{id}_{\Sigma_{a,b}}$ on $\Sigma_{a,b}$ is a homeomorphism from the differential space $(\Sigma_{a,b}, C^\infty(\Sigma_{a,b}))$ onto the differential space $(\Sigma_{a,b}, C_i^\infty(\Sigma_{a,b}))$. The mapping $\mathrm{id}_{\Sigma_{a,b}}^* : C_i^\infty(\Sigma_{a,b}) \to C^\infty(\Sigma_{a,b})$ is an isomorphism of vector spaces.

▷ The mapping $\mathrm{id}_{\Sigma_{a,b}}^* : C_i^\infty(\Sigma_{a,b}) \to C^\infty(\Sigma_{a,b})$ is an isomorphism.

(4.20) **Proof**: The mapping $\mathrm{id}_{\Sigma_{a,b}}^*$ is well defined; for suppose that $f \in C_i^\infty(\Sigma_{a,b})$. Then there is an $F \in C^\infty(\mathbf{R}^6)$ such that $F|\Sigma_{a,b} = f$. So $\pi^* f = \pi^*(F)|(J_\ell^a)^{-1}(b)$, where $\pi^*(F) \in C^\infty(\mathbf{R}^3 \times \mathbf{R}^3)^{S^1}$. So $f \in C^\infty(\Sigma_{a,b})$. Hence $f = \mathrm{id}_{\Sigma_{a,b}}^* f$, which shows that the linear mapping $\mathrm{id}_{\Sigma_{a,b}}^*$ is well defined and is surjective. It is injective because the map $\mathrm{id}_{\Sigma_{a,b}}$ is surjective. □

We have proved

Claim: The identity map $\mathrm{id}_{\Sigma_{a,b}}$ is a diffeomorphism of the differential space $\big(\Sigma_{a,b}, C^{\infty}(\Sigma_{a,b})\big)$ onto the *differential space* $\big(\Sigma_{a,b}, C^{\infty}_i(\Sigma_{a,b})\big)$.

We give another definition of differential structure on P^a_b, which is a semialgebraic variety in $\mathbf{R}^3$. We say that the function $f : P^a_b \subseteq \mathbf{R}^3 \to \mathbf{R}$ is a member of the family $\widetilde{\mathscr{F}}$ if there is a smooth function $F : \mathbf{R}^3 \to \mathbf{R}$ such that $f = F|\mathbf{R}^3$. Let $C^{\infty}_i(P^a_b)$ be the space of smooth functions on P^a_b generated by $\widetilde{\mathscr{F}}$. By ((3.2)) in chapter VII, $C^{\infty}_i(P^a_b)$ is a differential structure on P^a_b. From ((3.17)) in chapter VII it follows that $\widetilde{\mathscr{F}} = C^{\infty}_i(P^a_b)$. Using the differential space topology on $(P^a_b, C^{\infty}(P^a_b))$ and $(P^a_b, C^{\infty}_i(P^a_b))$ from ((3.18)) it follows that the identity map $\mathrm{id}_{P^a_b}$ on P^a_b is a homeomorphism from $(P^a_b, C^{\infty}(P^a_b))$ onto $(P^a_b, C^{\infty}_i(P^a_b))$. The induced mapping $\mathrm{id}^*_{P^a_b} : C^{\infty}_i(P^a_b) \to C^{\infty}(P^a_b)$ is an isomorphism of vector spaces.

(4.21) **Proof**: The proof follows the pattern of ((4.18)) and is left to the reader. □

Because $\mathrm{id}_{P^a_b}$ is a homeomorphism of P^a_b into itself using the differential space topology on the differential spaces $(P^a_b, C^{\infty}(P^a_b)$ and $(P^a_b, C^{\infty}_i(P^a_b))$, we have proved

Claim: The identity map $\mathrm{id}_{P^a_b}$ is a diffeomorphism of the differential space $\big(P^a_b, C^{\infty}(P^a_b)\big)$ onto the differential space $\big(P^a_b, C^{\infty}_i(P^a_b)\big)$.

4.4 Poisson structure on $\widetilde{C}^{\infty}(P^a_b)$

In this subsection we find a Poisson structure $\{\,,\,\}_{P^a_b}$ on the space $C^{\infty}_i(P^a_b)$ of smooth functions on P^a_b, which is equivalent to the Poisson structure $\{\,,\,\}$ on the smooth functions $C^{\infty}(\Sigma_{a,b})$ of the singular reduced space Σ_{a_b} given by the singular reduction theorem, see chapter VII §7.

We begin by constructing the Poisson bracket $\{\,,\,\}_{P^a_b}$ on $C^{\infty}_i(P^a_b)$. On $C^{\infty}(\mathbf{R}^3)$, where $\mathbf{R}^3$ has coordinates σ_i, $1 \le i \le 3$, (53) define a Poisson bracket $\{\,,\,\}_{\mathbf{R}^3}$ by the structure matrix $W_{C^{\infty}(\mathbf{R}^3)}$ given in table 4.4.1.

$\{A,B\}_{\mathbf{R}^3}$	σ_1	σ_2	σ_3	B
σ_1	0	$(1-\sigma_1^2)$	$2\sigma_2$	
σ_2	$-(1-\sigma_1^2)$	0	$2b(a-b\sigma_1)-2\sigma_1\sigma_3$	
σ_3	$-2\sigma_2$	$-2b(a-b\sigma_1)+2\sigma_1\sigma_3$	0	
A				

Table 4.4.1 The structure matrix $W_{C^{\infty}(\mathbf{R}^3)}$ for $\{\,,\,\}_{\mathbf{R}^3}$.

From table 4.4.1 we get $\{\sigma_i, \sigma_j\}_{\mathbf{R}^3} = \sum_{k=1}^{3} \varepsilon_{ijk} \frac{\partial G}{\partial \sigma_k}$, where

$$G(\sigma_1,\sigma_2,\sigma_3) = (1-\sigma_1^2)\sigma_3 - \sigma_2^2 - (a-b\sigma_1)^2 \tag{59}$$

is the defining equation of P^a_b (52). For $f,g \in C^\infty(\mathbf{R}^3)$ we have

$$\{f,g\}_{\mathbf{R}^3} = \sum_{i,j=1}^{3} \frac{\partial f}{\partial \sigma_i}\frac{\partial g}{\partial \sigma_j}\{\sigma_i,\sigma_j\}_{\mathbf{R}^3} = (\operatorname{grad} f \times \operatorname{grad} g, \operatorname{grad} G).$$

The proof that $\{\ ,\ \}_{\mathbf{R}^3}$ satisfies the Jacobi identity is left as an exercise.

Next we define a Poisson bracket $\{\ ,\ \}_{P^a_b}$ on $C^\infty_i(P^a_b)$. Let $\mathscr{I} = \mathscr{I}(P^a_b)$ be the subset of smooth functions on $\mathbf{R}^3$, which vanish identically on P^a_b. Then $\mathscr{I}$ is an ideal in the associative commutative algebra $\big(C^\infty(\mathbf{R}^3),\cdot\big)$, where $\cdot$ is pointwise multiplication of smooth
▷ functions. $\mathscr{I}$ is a *Poisson ideal* in the Poisson algebra $\mathscr{E} = \big(C^\infty(\mathbf{R}^3), \{\ ,\ \}_{\mathbf{R}^3},\cdot\big)$, that is, if $f \in \mathscr{I}$ and $g \in C^\infty(\mathbf{R}^3)$ then $\{f,g\}_{\mathbf{R}^3} \in \mathscr{I}$.

(4.22) **Proof**: For $g \in C^\infty(\mathbf{R}^3)$ let X_g be the derivation $-\mathrm{ad}_g$ of the Poisson algebra $\mathscr{E}$. For $p \in G^{-1}(0)$ with $G \in C^\infty(\mathbf{R}^3)$ given by (59) there is an $\alpha_p > 0$ such that $\gamma_p : I = [0,\alpha_p) \to \mathbf{R}^3 : t \mapsto \varphi^g_t(p)$ is the maximal integral curve of the vector field X_g starting at p. Look at the function $\mathscr{G} : I \to \mathbf{R} : t \mapsto G(\varphi^g_t(p))$. Then

$$\frac{\mathrm{d}\mathscr{G}}{\mathrm{d}t} = \frac{\mathrm{d}G}{\mathrm{d}t}(\varphi^g_t(p)) = (L_{X_g}G)(\varphi^g_t(p)) = \{G,g\}_{\mathbf{R}^3}(\varphi^g_t(p)) = 0,$$

since G is a Casimir in $\mathscr{E}$. Therefore $\mathscr{G}$ is the constant function on I. But $\mathscr{G}(0) = G(p) = 0$, since $p \in G^{-1}(0)$. Thus $G(\varphi^g_t(p)) = 0$ for every $t \in I$, that is, $\gamma_p(I) \subseteq G^{-1}(0)$.

Let $p \in P^a_b$. The set $J = \{t \in [0,\alpha_p) \,\big|\, \varphi^g_t(p) \in P^a_b\}$ contains 0, since $\varphi^g_0(p) = p \in P^a_b$ by hypothesis. Set $t' = \sup_{t\in[0,\alpha_p)}\{\varphi^g_t(p) \in P^a_b\}$. Suppose that $t' < \alpha_p$. By definition of t' we have $\varphi^g_t(p) \in P^a_b$ for every $t \in [0,t')$. Since P^a_b is a closed subset of $\mathbf{R}^3$, we get $p_{t'} = \varphi^g_{t'}(p) = \lim_{t\nearrow t'}\varphi^g_t(p) \in P^a_b$. Therefore $t' \in J \subseteq I$. Let $\sigma = \mu\circ\tau : (J^a_\ell)^{-1}(b) \subseteq \mathbf{R}^3\times\mathbf{R}^3 \to P^a_b \subseteq \mathbf{R}^3$, where $\tau = \pi|(J^a_\ell)^{-1}(b)$, π is the Hilbert map (47), and μ is the projection mapping (58). Then σ is surjective with image P^a_b. Let $q_{t'} \in (J^a_\ell)^{-1}(b)$ such that $\sigma(q_{t'}) = p_{t'}$. Since P^a_b is a semialgebraic variety, it is locally closed in
▷ the induced topology, chapter VII §3. But σ is an open mapping, using the topology on $(J^a_\ell)^{-1}(b)$ induced from $\mathbf{R}^3\times\mathbf{R}^3$ and the topology on P^a_b induced from $\mathbf{R}^3$.

(4.23) **Proof**: The orbit mapping $\widetilde{\rho} : (J^a_\ell)^{-1}(b) \subseteq \mathbf{R}^3\times\mathbf{R}^3 \to (J^a_\ell)^{-1}(b)/S^1 \subseteq (\mathbf{R}^3\times\mathbf{R}^3)/S^1$ is an open mapping, because $\widetilde{\rho} = \rho|(J^a_\ell)^{-1}(b)$ and ρ is. The mapping $\overline{\tau} : (J^a_\ell)^{-1}(b)/S^1 \to \Sigma_{a,b} \subseteq \mathbf{R}^6$, induced by $\tau = \pi|(J^a_\ell)^{-1}(b)$, is a homeomorphism. Thus $\tau = \overline{\tau}\circ\widetilde{\rho}$ is an open mapping. But $\mu|\Sigma_{a,b} : \Sigma_{a,b} \subseteq \mathbf{R}^6 \to P^a_b \subseteq \mathbf{R}^3$ is a homeomorphism. Consequently, the map $\sigma = \mu\circ\tau$ is an open mapping. □

So for every open neighborhood U of $q_{t'}$ in $(J^a_\ell)^{-1}(b)$ there is an open neighborhood W of $p_{t'}$ in P^a_b such that $\sigma(U) = W$.

Consider the derivation $X_{\sigma^*g} = -\mathrm{ad}_{\sigma^*g}$ of the Poisson algebra $\mathscr{D} = (C^\infty((J^a_\ell)^{-1}(b)) = C^\infty(\mathbf{R}^3\times\mathbf{R}^3)^{S^1}|(J^a_\ell)^{-1}(b), \{\ ,\ \}_{\mathbf{R}^3\times\mathbf{R}^3},\cdot)$. Let $\varphi^{\sigma^*g}_t$ be the flow of the vector field X_{σ^*g} on $\mathbf{R}^3\times\mathbf{R}^3$. Because π^* and μ^* are Poisson epimorphisms, see ((4.31)) and ((4.37)), the mapping σ^* is a Poisson epimorphism of $\mathscr{E}$ onto $\mathscr{D}$, being equal to $(\pi^*\circ\mu^*)|(J^a_\ell)^{-1}(b)$. Thus σ intertwines the flows $\varphi^{\sigma^*g}_t$ and φ^g_t, that is, $\sigma\circ\varphi^{\sigma^*g}_t = \varphi^g_t\circ\sigma$. Let $q = \varphi^{\sigma^*g}_{-t'}(q_{t'})$.

Then $\sigma(q) = \varphi^g_{-t'}(\sigma(q_{t'})) = \varphi^g_{-t'}(p_{t'}) = p$. There is an open set U in $(J^a_\ell)^{-1}(b)$ containing $q_{t'}$. So there is an open interval $I' \subseteq I$, which contains t', such that $\varphi^{\sigma^* g}_t(q) \in U$ for every $t \in I'$. Thus there is a $t'' \in I'$ with $t'' > t'$ for which $\varphi^{\sigma^* g}_{t''}(q) \in U$. So

$$\varphi^g_{t''}(p) = \varphi^g_{t''}(\sigma(q)) = \sigma(\varphi^{\sigma^* g}_{t''}(q)) \in \sigma(U) = W.$$

But this contradicts the definition of t'. Hence $t' = \alpha_p$, that is, $\varphi^g_t(p) \in P^a_b$ for every $t \in [0, \alpha_p)$.

For every $p \in P^a_b$ we have $\{f,g\}_{\mathbf{R}^3}(p) = \frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} f(\varphi^g_t(p))$. But $f(\varphi^g_t(p)) = 0$ for every $t \in [0, \alpha_p)$, since $f \in \mathscr{I}$. So $\{f,g\}_{\mathbf{R}^3}(p) = 0$ for every $p \in P^a_b$, that is, $\{f,g\}_{\mathbf{R}^3} \in \mathscr{I}$. Thus $\mathscr{I}$ is a Poisson ideal of the Poisson algebra $\mathscr{E}$. □

▷ We now show that $\mathscr{D}$ is a Poisson algebra.

(4.24) **Proof**: Let $\mathscr{I} = \mathscr{I}((J^a_\ell)^{-1}(b))$ be the ideal in $(C^\infty(\mathbf{R}^3 \times \mathbf{R}^3)^{S^1}, \cdot)$ of functions, which
▷ vanish identically on $(J^a_\ell)^{-1}(b)$. Then $\mathscr{I}$ is a Poisson ideal of the Poisson algebra $\mathscr{A}' = (C^\infty(\mathbf{R}^3 \times \mathbf{R}^3)^{S^1}, \{\ ,\ \}_{\mathbf{R}^3 \times \mathbf{R}^3}, \cdot)$.

(4.25) **Proof**: To see this let $f \in \mathscr{I}$ and $g \in C^\infty(\mathbf{R}^3 \times \mathbf{R}^3)^{S^1}$. Suppose that $X_g = -\mathrm{ad}_g$ is a derivation in the Poisson algebra $\mathscr{A}'$. For $p \in (J^a_\ell)^{-1}(b)$ let $\gamma_p : [0, \alpha_p) \to (J^a_\ell)^{-1}(b)$ be the maximal integral curve of X_g starting at p. For $i =, 1, 2, 3$ let $\mathscr{C}_i$ be the Casimirs of $\mathscr{A}'$, the intersection of whose 0-level sets define $(J^a_\ell)^{-1}(b)$, see (50). Then for each $i = 1, 2, 3$ we have

$$\frac{\mathrm{d}\mathscr{C}_i}{\mathrm{d}t}(\gamma_p(t)) = (L_{X_g}\mathscr{C}_i)(\gamma_p(t)) = \{\mathscr{C}_i, g\}_{\mathbf{R}^3 \times \mathbf{R}^3}(\gamma_p(t)) = 0,$$

since $\mathscr{C}_i$ is a Casimir of $\mathscr{A}'$. Thus for each $i = 1, 2, 3$, $\mathscr{C}_i$ is constant on γ_p. But $\mathscr{C}_i(p) = 0$ for each $i = 1, 2, 3$. So $\gamma_p(I) \subseteq (J^a_\ell)^{-1}(b)$. In other words, $(J^a_\ell)^{-1}(b)$ is an invariant manifold for the vector field X_g for every $g \in C^\infty(\mathbf{R}^3 \times \mathbf{R}^3)^{S^1}$.
Now for every $p \in (J^a_\ell)^{-1}(b)$ we have

$$\{f,g\}_{\mathbf{R}^3 \times \mathbf{R}^3}(p) = L_{X_g} f(p) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} f(\gamma_p(t)) = 0,$$

since f vanishes identically on $(J^a_\ell)^{-1}(b)$. Therefore $\{f,g\}_{\mathbf{R}^3 \times \mathbf{R}^3} \in \mathscr{I}$, that is, $\mathscr{I}$ is a Poisson ideal of $\mathscr{A}'$. □

We now complete the proof of ((4.24)). Since $\mathscr{I}$ is a Poisson ideal of the Poisson algebra $\mathscr{A}'$, we deduce that $\mathscr{A}'/\mathscr{I} = (C^\infty(\mathbf{R}^3 \times \mathbf{R}^3)^{S^1}/\mathscr{I}, \{\ ,\ \}_{\mathbf{R}^3 \times \mathbf{R}^3}, \cdot)$ is a Poisson algebra, which is equal to $\mathscr{D}$, because

$$C^\infty(\mathbf{R}^3 \times \mathbf{R}^3)^{S^1}/\mathscr{I} = C^\infty((J^a_\ell)^{-1}(b))^{S^1} = C^\infty(\mathbf{R}^3 \times \mathbf{R}^3)^{S^1}|(J^a_\ell)^{-1}(b). \qquad \square$$

Let $\iota : P^a_b \to \mathbf{R}^3$ be the inclusion mapping. Then the induced map $\iota^* : C^\infty(\mathbf{R}^3) \to C^\infty_i(P^a_b)$ is surjective with kernel $\mathscr{I}$; for if $0 = \iota^* F$ with $F \in C^\infty(\mathbf{R}^3)$, then for every $p \in P^a_b$ we have $0 = F(\iota(p)) = F(p)$, that is, $F \in \mathscr{I}$. Thus $\ker \iota^* \subseteq \mathscr{I}$. Suppose that $F \in \mathscr{I}$. Then $F \in C^\infty(\mathbf{R}^6)$ and for every $p \in P^a_b$ we have $0 = F(p) = F(\iota(p)) = (\iota^* F)(p)$. Thus $F \in$

$\ker \iota^*$. So $\mathscr{I} \subseteq \ker \iota^*$ verifying that $\mathscr{I} = \ker \iota^*$. Therefore ι^* induces the isomorphism $\widehat{\iota} : C^\infty(\mathbf{R}^3)/\mathscr{I} \to C_i^\infty(P_b^a) : F + \mathscr{I} \mapsto \iota^* F = F|P_b^a$. On $C_i^\infty(P_b^a)$ define a *Poisson bracket* $\{\, ,\, \}_{P_b^a}$ by

$$\{F|P_b^a, H|P_b^a\}_{P_b^a} = \{F,H\}_{\mathbf{R}^3}|P_b^a, \tag{60}$$

▷ for every $F,H \in C^\infty(\mathbf{R}^3)$. Note that $F|P_b^a$, $H|P_b^a \in C_i^\infty(P_b^a)$ by definition. The bracket $\{\, ,\, \}_{P_b^a}$ is well defined.

(4.26) **Proof**: We argue as follows. Using the isomorphism $\widehat{\iota}$ we can write $F|P_b^a$ as $F+\mathscr{I}$ and $H|P_b^a$ as $H+\mathscr{I}$. Then

$$\{F+\mathscr{I}, H+\mathscr{I}\}_{\mathbf{R}^3} = \{F,H\}_{\mathbf{R}^3} + \big(\{\mathscr{I},H\}_{\mathbf{R}^3} + \{F,\mathscr{I}\}_{\mathbf{R}^3} + \{\mathscr{I},\mathscr{I}\}_{\mathbf{R}^3}\big) = \{F,H\}_{\mathbf{R}^3} + \mathscr{I}$$

since $\mathscr{I}$ is a Poisson ideal in $\mathscr{H}$. Therefore

$$\{F+\mathscr{I}, H+\mathscr{I}\}_{\mathbf{R}^3}|P_b^a = \{F,H\}_{\mathbf{R}^3}|P_b^a = \{F|P_b^a, H|P_b^a\}_{P_b^a}.$$

where the first equality above holds because $\mathscr{I}|P_b^a = 0$. In other words, the Poisson bracket $\{\, ,\, \}_{P_b^a}$ does not depend on the choice of function in $C^\infty(\mathbf{R}^3)/\mathscr{I}$ which is used to represent a function in $C_i^\infty(P_b^a)$. Thus $\{\, ,\, \}_{P_b^a}$ is well defined. □

We now construct the Poisson bracket $\{\, ,\, \}$ on $C^\infty(P_b^a)$ given by the singular reduction theorem, see chapter VII §7. We start by noting that the space $C^\infty(\mathbf{R}^3 \times \mathbf{R}^3)^{S^1}$ of smooth functions on $\mathbf{R}^3 \times \mathbf{R}^3$, which are invariant under the S^1-action Δ (40), is a Lie subalgebra of the Lie algebra $\big(C^\infty(\mathbf{R}^3 \times \mathbf{R}^3), \{\, ,\, \}_{\mathbf{R}^3\times\mathbf{R}^3}\big)$, see table 3.1.1. The first reduced phase space P^a is the submanifold of $\mathbf{R}^3 \times \mathbf{R}^3$ defined by

$$\begin{aligned} C_1(z,w) &= z_1^2 + z_2^2 + z_3^2 - 1 = 0 \\ C_2(z,w) &= z_1 w_1 + z_2 w_2 + z_3 w_3 - a = 0. \end{aligned} \tag{61}$$

A function on P^a is smooth if it is the restriction to P^a of a smooth function on $\mathbf{R}^3 \times \mathbf{R}^3$. Since the functions C_1 and C_2 in (61) are invariant under the S^1-action Δ, so is the submanifold P^a. Thus a smooth function on P^a is invariant under the action $\Delta|(S^1 \times P^a)$ if it is the restriction of an Δ-invariant function on $\mathbf{R}^3 \times \mathbf{R}^3$ to P^a. Let $C^\infty(P^a)^{S^1}$ be the set of smooth S^1-invariant functions on P^a. Then $\big(C^\infty(P^a)^{S^1}, \{\, ,\, \}_{P^a}\big)$ is a Lie subalgebra of $\big(C^\infty(P^a), \{\, ,\, \}_{P^a}\big)$. Recall that P^a has a symplectic form ω^a (18). Moreover, the functions C_1 and C_2, which define P^a, are Casimirs in the Poisson algebra $\mathscr{A} = (C^\infty(\mathbf{R}^3 \times \mathbf{R}^3), \{\, ,\, \}_{\mathbf{R}^3\times\mathbf{R}^3}, \cdot)$. On $C^\infty(P^a)$ define a Poisson bracket $\{\, ,\, \}_{P^a}$ by

$$\{F|P^a, H|P^a\}_{P^a} = \{F,H\}_{\mathbf{R}^3\times\mathbf{R}^3}|P^a,$$

▷ for every $F,H \in C^\infty(\mathbf{R}^3 \times \mathbf{R}^3)$. The Poisson bracket $\{\, ,\, \}_{P^a}$ on $C^\infty(P^a)$ is the same as the standard one $\{\, ,\, \}$ on $C^\infty(P^a)$ using the symplectic form ω^a on P^a.

(4.27) **Proof**: For $1 \le i,j \le 3$ consider the functions $z_i = (z, e_i)$ and $w_j = (w, e_j)$ on $\mathbf{R}^3 \times \mathbf{R}^3$ restricted to P^a. Using (22) we find that the corresponding Hamiltonian vector fields X_{z_i}

and X_{w_j} on (P^a, ω^a) are $(e_i \times z, \frac{\partial}{\partial w})$ and $(e_j \times z, \frac{\partial}{\partial z}) + (e_j \times w, \frac{\partial}{\partial w})$, respectively. Therefore on P^a we get

$$\{z_i, z_j\}_{P^a} = L_{X_{z_j}} z_i = 0$$

$$\{z_i, w_j\}_{P^a} = L_{X_{w_j}} z_i = (e_j \times z, e_i) = \sum_{k=1}^{3} \varepsilon_{ijk} z_k$$

$$\{w_i, w_j\}_{P^a} = L_{X_{w_j}} w_i = (e_j \times w, e_i) = \sum_{k=1}^{3} \varepsilon_{ijk} w_k,$$

which agree with the entries in table 3.1.1. $\square$

Consequently, $\mathscr{B} = \big(C^\infty(P^a), \{\ ,\ \}_{P^a}, \cdot\big)$ is a Poisson algebra with the Poisson algebra $\mathscr{B}' = \big(C^\infty(P^a)^{S^1}, \{\ ,\ \}_{P^a}, \cdot\big)$ being a subalgebra.

Let $\mathscr{I} = \mathscr{I}((J^a_\ell)^{-1}(b))$ the set of $\Delta|(S^1 \times P^a)$-invariant smooth functions on P^a, which ▷ vanish identically on $(J^a_\ell)^{-1}(b) = \{(z,w) \in P^a \,\big|\, w_3 = b\}$. Then $\mathscr{I}$ is a Poisson ideal in the Poisson algebra $\mathscr{B}'$.

(4.28) **Proof**: Let Y be the infinitesimal generator of the action $\Delta|(S^1 \times P^a)$, which has momentum mapping J^a_ℓ (44). Then $Y = X_{J^a_\ell}$ on (P^a, ω^a). Let $f \in C^\infty(P^a)^{S^1}$ and let φ^f_t be the flow of the Hamiltonian vector field X_f on (P^a, ω^a). For $p \in (J^a_\ell)^{-1}(b)$ we have

$$-\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} J^a_\ell(\varphi^f_t(p)) = -(L_{X_f} J^a_\ell)(p) = -\{J^a_\ell, f\}_{P^a}(p) = \{f, J^a_\ell\}_{P^a}(p) = (L_{X_{J^a_\ell}} f)(p)$$

$$= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} f(\varphi^Y_t(p)), \quad \text{where } \varphi^Y_t \text{ is the flow of } Y$$

$$= 0, \quad \text{since } f \text{ is invariant under } \Delta|(S^1 \times P^a).$$

Therefore J^a_ℓ is constant on the integral curve $t \mapsto \varphi^f_t(p)$ of X_f, which starts at p. So $J^a_\ell(\varphi^f_t(p)) = J^a_\ell(p) = b$, that is, $\varphi^f_t(p) \in (J^a_\ell)^{-1}(b)$. Consequently, $(J^a_\ell)^{-1}(b)$ is an invariant manifold of X_f. For every $h \in \mathscr{I}$ we get $\{h, f\}_{P^a}(p) = \frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} h(\varphi^f_t(p)) = 0$, since $\varphi^f_t(p) \in (J^a_\ell)^{-1}(b)$. Thus $\{h, f\}_{P^a} \in \mathscr{I}$. $\square$

For $f, g \in C^\infty(\Sigma_{a,b})$ define a Poisson bracket $\{\ ,\ \}$ on $C^\infty(\Sigma_{a,b})$ by

$$\tau^*\{f, g\} = \{F, G\}_{P^a} | (J^a_\ell)^{-1}(b),$$

where $\tau^* f = F|(J^a_\ell)^{-1}(b)$, $\tau^* g = G|(J^a_\ell)^{-1}(b)$, and $F, G \in C^\infty(P^a)^{S^1}$. Here $\tau = \pi|(J^a_\ell)^{-1}(b)$, where π is the Hilbert map (48). Because $\mathscr{I} = \mathscr{I}((J^a_\ell)^{-1}(b))$ is a Poisson ideal in the Poisson algebra $\mathscr{B}' = \big(C^\infty(P^a)^{S^1}, \{\ ,\ \}_{P^a}, \cdot\big)$, the bracket $\{\ ,\ \}$ is well defined. In more detail, suppose that $\tau^* g = G'|(J^a_\ell)^{-1}(b)$ for some $G' \in C^\infty(P^a)^{S^1}$. Then $G - G' \in \mathscr{I}$. So $\{F, G - G'\}_{P^a}|(J^a_\ell)^{-1}(b) = 0$, which implies that $\{\ ,\ \}$ does not depend on the choice of G representing $\tau^* g$. By skew symmetry of $\{\ ,\ \}_{P^a}$, which implies the skew symmetry of $\{\ ,\ \}$, it follows that $\{\ ,\ \}$ does not depend on the choice of F representing $\tau^* f$. The Jacobi identity holds for $\{\ ,\ \}$ because $(C^\infty(P^a, \{\ ,\ \}_{P^a}))$ is a Lie algebra.

The goal of the next few paragraphs is to construct a Poisson isomorphism between the Poisson algebras $\widetilde{\mathscr{C}} = \big(C_i^\infty(\Sigma_{a,b}), \{\ ,\ \}_{\Sigma_{a,b}}, \cdot\big)$ and $\mathscr{C} = \big(C^\infty(\Sigma_{a,b}), \{\ ,\ \}, \cdot\big)$. Here $\{\ ,\ \}$ is the Poisson bracket coming from the singular reduction theorem ((7.9)) in chapter VII, when one reduces the Hamiltonian S^1-action $\Delta|(S^1 \times P^a)$ on (P^a, ω^a) with momentum mapping J_ℓ^a and reduction mapping $\tau : (J_\ell^a)^{-1}(b) \to \Sigma_{a,b}$.

Look at the Hilbert map $\pi : \mathbf{R}^3 \times \mathbf{R}^3 \to \mathbf{R}^6$ (47) of the S^1-action Δ (40). Here we consider the S^1-invariant polynomials π_i, $1 \le i \le 6$ (46) to be coordinates on $\mathbf{R}^6$. Define a Poisson bracket $\{\ ,\ \}_{\mathbf{R}^6}$ on $C^\infty(\mathbf{R}^6)$ by the structure matrix $W_{C^\infty(\mathbf{R}^6)}$ given in table 4.4.2.

$\{A,B\}_{\mathbf{R}^6}$	π_1	π_2	π_3	π_4	π_5	π_6	B
π_1	0	0	0	π_5	0	$2\pi_4$	
π_2	0	0	0	0	0	0	
π_3	0	0	0	$-\pi_2\pi_5$	0	$-2\pi_2\pi_4$	
π_4	$-\pi_5$	0	$\pi_2\pi_5$	0	$2\pi_1\pi_5$	$2(\pi_2\pi_3 - \pi_1\pi_6)$	
π_5	0	0	0	$-2\pi_1\pi_5$	0	$-4\pi_1\pi_4$	
π_6	$-2\pi_4$	0	$2\pi_2\pi_4$	$-2(\pi_2\pi_3 - \pi_1\pi_6)$	$4\pi_1\pi_4$	0	
A							

Table 4.4.2 The structure matrix $W_{C^\infty(\mathbf{R}^6)}$ for $\{\,,\,\}_{\mathbf{R}^6}$ on $C^\infty(\mathbf{R}^6)$.

Claim: The map $\pi^* : \mathscr{F} = \big(C^\infty(\mathbf{R}^6), \{\ ,\ \}_{\mathbf{R}^6}, \cdot\big) \to \mathscr{A}' = \big(C^\infty(\mathbf{R}^3 \times \mathbf{R}^3)^{S^1}, \{\ ,\ \}_{\mathbf{R}^3 \times \mathbf{R}^3}, \cdot\big)$ is an epimorphism of Poisson algebras.

(4.29) **Proof**: Using table 4.4.2 a straightforward calculation shows that for every $1 \le i, j \le 6$ we have $\pi^*(\{\pi_i, \pi_j\}_{\mathbf{R}^6}) = \{\pi^*\pi_i, \pi^*\pi_j\}_{\mathbf{R}^3 \times \mathbf{R}^3}$. This implies that for every $f, g \in C^\infty(\mathbf{R}^6)$ we have

$$\begin{aligned}
\pi^*(\{f,g\}_{\mathbf{R}^6}) &= \pi^*\Big(\sum_{i,j=1}^{6} \frac{\partial f}{\partial \pi_i}\frac{\partial g}{\partial \pi_j}\{\pi_i \pi_j\}_{\mathbf{R}^6}\Big) = \sum_{i,j=1}^{6} \frac{\partial(\pi^* f)}{\partial(\pi^*\pi_i)}\frac{\partial(\pi^* g)}{\partial(\pi^*\pi_j)}\pi^*(\{\pi_i \pi_j\}_{\mathbf{R}^6}) \\
&= \sum_{i,j=1}^{6} \frac{\partial(\pi^* f)}{\partial(\pi^*\pi_i)}\frac{\partial(\pi^* g)}{\partial(\pi^*\pi_j)}\{\pi^*\pi_i, \pi^*\pi_j\}_{\mathbf{R}^3\times\mathbf{R}^3} = \{\pi^* f, \pi^* g\}_{\mathbf{R}^3\times\mathbf{R}^3}. \qquad \square
\end{aligned}$$

The image of the submanifold $(J_\ell^a)^{-1}(b)$ of $\mathbf{R}^3 \times \mathbf{R}^3$ under the Hilbert map π is the semi-algebraic variety $\Sigma_{a,b}$ of $\mathbf{R}^6$ defined by

$$\begin{aligned}
&\mathscr{C}_1 = \pi_3^2 + \pi_4^2 - \pi_5\pi_6 = 0, \quad \pi_5 \ge 0 \ \&\ \pi_6 \ge 0 \\
&\mathscr{C}_2 = \pi_5 + \pi_1^2 - 1 = 0 \\
&\mathscr{C}_3 = \pi_3 + \pi_1\pi_2 - a = 0 \\
&\mathscr{C}_4 = \pi_2 - b = 0.
\end{aligned}$$

The functions $\mathscr{C}_i$, $1 \le i \le 4$, are Casimirs in the Poisson algebra $\mathscr{F}$. Let $\mathscr{I} = \mathscr{I}(\Sigma_{a,b})$ be the ideal of smooth functions in the associative algebra $\big(C^\infty(\mathbf{R}^6), \cdot\big)$, which vanish
▷ identically on $\Sigma_{a,b}$. Then $\mathscr{I}$ is a Poisson ideal in the Poisson algebra $\mathscr{F}$.

(4.30) **Proof**: The argument is similar to the proof of ((4.23)) and is included for completeness.

For $g \in C^\infty(\mathbf{R}^6)$ let X_g be the derivation $-\mathrm{ad}_g$ of $(C^\infty(\mathbf{R}^6,\cdot)$. Let $\mathscr{C} : \mathbf{R}^6 \to \mathbf{R}^4 : p \mapsto \big(\mathscr{C}_1(p),\ldots,\mathscr{C}_4(p)\big)$. Then $\mathscr{C}^{-1}(0) = \Sigma_{a,b}$, which is connected being the image of the connected set $(J^a_\ell)^{-1}(b)$ under the continuous mapping π. Let $p \in \mathscr{C}^{-1}(0)$. For some $\alpha_p > 0$ let $\gamma_p : I = [0,\alpha_p) \to \mathbf{R}^6 : t \mapsto \varphi^g_t(p)$ be the maximal integral curve of the vector field X_g starting at p. Look at the mapping $\mathscr{G}_i : [0,\alpha_p) \to \mathbf{R} : t \mapsto \mathscr{C}_i(\varphi^g_t(p))$ for each $1 \le i \le 4$. Then

$$\frac{\mathrm{d}\mathscr{G}_i}{\mathrm{d}t}(t) = \frac{\mathrm{d}\mathscr{C}_i}{\mathrm{d}t}(\varphi^g_t(p)) = \{\mathscr{C}_i, g\}_{\mathbf{R}^6}(\varphi^g_t(p)) = 0,$$

since $\mathscr{C}_i$ is a Casimir in the Poisson algebra $\mathscr{F}$. Therefore $\mathscr{G}_i$ is a constant function. But $p \in \mathscr{C}^{-1}(0)$. So $\mathscr{G}_i(p) = \mathscr{C}_i(p) = 0$. Consequently for every $1 \le i \le 4$ we have $\mathscr{C}_i(\varphi^g_t(p)) = 0$, that is, $\mathscr{C}(\varphi^g_t(p)) = 0$. So $\gamma_p(I) \in \mathscr{C}^{-1}(0)$.

Let $p \in \Sigma_{a,b}$. The set $J = \{t \in [0,\alpha_p) \,\big|\, \varphi^g_t(p) \in \Sigma_{a,b}\}$ contains 0, since $\varphi^g_0(p) = p \in \Sigma_{a,b}$ by hypothesis. Set $t' = \sup_{t\in[0,\alpha_p)}\{\varphi^g_t(p) \in \Sigma_{a,b}\}$. Suppose that $t' < \alpha_p$. By definition of t' we have $\varphi^g_t(p) \in \Sigma_{a,b}$ for every $t \in [0,t')$. Since $\Sigma_{a,b}$ is a closed subset of $\mathbf{R}^6$, we get $p_{t'} = \varphi^g_{t'}(p) = \lim_{t\nearrow t'} \varphi^g_t(p) \in \Sigma_{a,b}$. Therefore $t' \in J$. Let $\tau = \pi|(J^a_\ell)^{-1}(b) : (J^a_\ell)^{-1}(b) \subseteq \mathbf{R}^3 \times \mathbf{R}^3 \to \Sigma_{a,b} \subseteq \mathbf{R}^6$, where π is the Hilbert map (47). Then τ is surjective with image $\Sigma_{a,b}$. Let $q_{t'} \in (J^a_\ell)^{-1}(b)$ such that $\tau(q_{t'}) = p_{t'}$. Since $\Sigma_{a,b}$ is a semialgebraic variety, it is locally closed in the induced topology, see chapter VII §3. But τ is an open mapping, using the induced topology on $\Sigma_{a,b}$, see ((4.23)). So for every open neighborhood U of $q_{t'}$ in $(J^a_\ell)^{-1}(b)$ there is an open neighborhood W of $p_{t'}$ in $\Sigma_{a,b}$ such that $\tau(U) = W$. Now look at the derivation $X_{\tau^* g} = -\mathrm{ad}_{\tau^* g}$ of the Poisson algebra $\mathscr{D} = (C^\infty((J^a_\ell)^{-1}(b))^{S^1}, \{\,,\,\}_{\mathbf{R}^3\times\mathbf{R}^3}, \cdot)$. Let $\varphi^{\tau^* g}_t$ be the flow of the vector field $X_{\tau^* g}$ on $(J^a_\ell)^{-1}(b)$. The mapping τ^* is a Poisson epimorphism of the Poisson algebra $\mathscr{F}$ onto the Poisson algebra $\mathscr{D}$, see ((4.31)). τ intertwines the flows $\varphi^{\tau^* g}_t$ and φ^g_t, that is, $\tau\circ\varphi^{\tau^* g}_t = \varphi^g_t\circ\tau$. Let $q = \varphi^{\tau^* g}_{-t'}(q_{t'})$. Then $\tau(q) = \varphi^g_{-t'}(\tau(q_{t'})) = \varphi^g_{-t'}(p_{t'}) = p$. Because U is an open subset of $(J^a_\ell)^{-1}(b)$ containing $q_{t'}$, there is an open interval $I' \subseteq I$, which contains t', such that $\varphi^{\tau^* g}_t(q) \in U$ for every $t \in I'$. Thus there is a $t'' \in I'$ with $t'' > t'$ for which $\varphi^{\tau^* g}_{t''}(q) \in U$. So

$$\varphi^g_{t''}(p) = \varphi^g_{t''}(\tau(q)) = \tau(\varphi^{\tau^* g}_{t''}(q)) \in \tau(U) = W.$$

But this contradicts the definition of t'. Hence $t' = \alpha_p$, that is, $\varphi^g_t(p) \in \Sigma_{a,b}$ for every $t \in [0,\alpha_p)$.

Let $f \in \mathscr{I} = \mathscr{I}(\Sigma_{a,b})$. For every $p \in \Sigma_{a,b}$ we have $\{f,g\}_{\mathbf{R}^6}(p) = \frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} f(\varphi^g_t(p))$. But $f(\varphi^g_t(p)) = 0$ for every $t \in [0,\alpha_p)$, since $f \in \mathscr{I}$. Therefore $\{f,g\}_{\mathbf{R}^6}(p) = 0$ for every $p \in \Sigma_{a,b}$, that is, $\{f,g\}_{\mathbf{R}^6} \in \mathscr{I}$. □

Let $i : \Sigma_{a,b} \to \mathbf{R}^6$ be the inclusion mapping. Then the induced linear map $i^* : C^\infty(\mathbf{R}^6) \to C^\infty_i(\Sigma_{a,b})$ is surjective with kernel $\mathscr{I}$; because if $\widetilde{f} \in C^\infty_i(\Sigma_{a,b})$, then there is $F \in C^\infty(\mathbf{R}^6)$ such that $\widetilde{f} = F|\Sigma_{a,b}$. For $p \in \Sigma_{a,b}$ we have $(i^*(F))(p) = F(i(p))$. So $i^*F = F|\Sigma_{a,b} = \widetilde{f}$. Hence i^* is surjective. Next we show that $\ker i^* = \mathscr{I}$. Suppose that $i^*F = 0$. Then $F|\Sigma_{a,b} = 0$, that is, $F \in \mathscr{I}$. So $\ker i^* \subseteq \mathscr{I}$. If $F \in \mathscr{I}$, then $0 = F|\Sigma_{a,b} = i^*F$. So $F \in \ker i^*$, that is, $\mathscr{I} \subseteq \ker i^*$. Thus the mapping $\widehat{i}: C^\infty(\mathbf{R}^6)/\mathscr{I} \to C_i^\infty(\Sigma_{a,b}) : F + \mathscr{I} \mapsto i^*F = F|\Sigma_{a,b}$ is an isomorphism.

On $C_i^\infty(\Sigma_{a,b})$ define a Poisson bracket $\{\, , \,\}_{\Sigma_{a,b}}$ by

$$\{F|\Sigma_{a,b}, H|\Sigma_{a,b}\}_{\Sigma_{a,b}} = \{F,H\}_{\mathbf{R}^6}|\Sigma_{a,b}, \tag{62}$$

▷ for every $F,H \in C^\infty(\mathbf{R}^6)$. The bracket $\{\, , \,\}_{\Sigma_{a,b}}$ is well defined.

(4.31) **Proof**: The argument follows the pattern of the proof ((4.26)) and uses the fact that $\mathscr{I}$ is a Poisson ideal of $\mathscr{F}$ ((4.30)). □

▷ The identity mapping id_Σ on $\Sigma_{a,b}$ induces the isomorphism $\mathrm{id}^*_{\Sigma_{a,b}} : C_i^\infty(\Sigma_{a,b}) \to C^\infty(\Sigma_{a,b})$.

(4.32) **Proof**: Let $f \in C_i^\infty(\Sigma_{a,b})$. Then there is $F \in C^\infty(\mathbf{R}^6)$ such that $f = F|\Sigma_{a,b}$. Now $\tau^* f = \pi^*(F)|(J^a_\ell)^{-1}(b)$, where $\pi^*(F) \in C^\infty(\mathbf{R}^3 \times \mathbf{R}^3)^{S^1}$. So $\pi^*(F)|(J^a_\ell)^{-1}(b)$ is the restriction of $\pi^*(F)|P^a$ in $C^\infty(P^a)^{S^1}$ to $(J^a_\ell)^{-1}(b)$. This implies $f \in C^\infty(\Sigma_{a,b})$. Therefore $f = \mathrm{id}^*_{\Sigma_{a,b}} f$. Hence the mapping $\mathrm{id}^*_{\Sigma_{a,b}}$ is well defined and is surjective. It is injective because the mapping $\mathrm{id}_{\Sigma_{a,b}}$ is surjective. □

▷ The mapping $\mathrm{id}^*_{\Sigma_{a,b}}$ is an isomorphism of the Poisson algebra $\widetilde{\mathscr{C}} = \big(C_i^\infty(\Sigma_{a,b}), \{\, , \,\}_{\Sigma_{a,b}}, \cdot\big)$ onto the Poisson algebra $\mathscr{C} = \big(C^\infty(\Sigma_{a,b}), \{\, , \,\}, \cdot\big)$.

(4.33) **Proof**: We need only show that for every $f, g \in C_i^\infty(\Sigma_{a,b})$ we have

$$\mathrm{id}^*_{\Sigma_{a,b}}(\{f,g,\}_{\Sigma_{a,b}}) = \{\mathrm{id}^*_{\Sigma_{a,b}} f, \mathrm{id}^*_{\Sigma_{a,b}} g\}. \tag{63}$$

There are $\widetilde{f}, \widetilde{g} \in C^\infty(\mathbf{R}^6)$ such that $f = \widetilde{f}|\Sigma_{a,b}$ and $g = \widetilde{g}|\Sigma_{a,b}$. So

$$\mathrm{id}^*_{\Sigma_{a,b}}(\{f,g\}_{\Sigma_{a,b}}) = \mathrm{id}^*_{\Sigma_{a,b}}(\{\widetilde{f},\widetilde{g}\}_{\mathbf{R}^6}|\Sigma_{a,b}) = \{\widetilde{f},\widetilde{g}\}_{\mathbf{R}^6}|\Sigma_{a,b}. \tag{64}$$

Now $\tau^* f = \tau^*(\widetilde{f}|\Sigma_{a,b}) = (\pi^*\widetilde{f})|(J^a_\ell)^{-1}(b)$, where $\pi^*\widetilde{f} \in C^\infty(\mathbf{R}^3 \times \mathbf{R}^3)^{S^1}$. Also we have $\tau^* f = \tau^*(\mathrm{id}^*_{\Sigma_{a,b}} f) = \widetilde{F}|(J^a_\ell)^{-1}(b)$ for some $\widetilde{F} \in C^\infty(\mathbf{R}^3 \times \mathbf{R}^3)^{S^1}$. So $\tau^*\widetilde{f} = \widetilde{F} + \mathscr{I}$, where $\mathscr{I} = \mathscr{I}((J^a_\ell)^{-1}(b))$ is the Poisson ideal in $\mathscr{A}' = (C^\infty(\mathbf{R}^3 \times \mathbf{R}^3)^{S^1}, \{\, , \,\}_{\mathbf{R}^3\times\mathbf{R}^3}, \cdot)$ of functions in $\mathscr{A}'$, which vanish identically on $(J^a_\ell)^{-1}(b)$. Therefore

$$\begin{aligned}
\tau^*(\{\widetilde{f},\widetilde{g}\}_{\mathbf{R}^6}|\Sigma_{a,b}) &= \{\tau^*\widetilde{f}, \tau^*\widetilde{g}\}_{\mathbf{R}^3\times\mathbf{R}^3}|(J^a_\ell)^{-1}(b), \quad \text{since } \tau\big((J^a_\ell)^{-1}(b)\big) = \Sigma_{a,b} \\
&= \{\widetilde{F},\widetilde{G}\}_{\mathbf{R}^3\times\mathbf{R}^3}|(J^a_\ell)^{-1}(b), \quad \text{since } \mathscr{I} \text{ is a Poisson ideal} \\
&= \{\widetilde{F}|P^a, \widetilde{G}|P^a\}_{P^a}|(J^a_\ell)^{-1}(b), \quad \text{by definition of } \{\, , \,\}_{P^a} \\
&= \pi^*(\{\mathrm{id}^*_{\Sigma_{a,b}} f, \mathrm{id}^*_{\Sigma_{a,b}} g\}), \quad \text{by the singular reduction theorem} \\
&\qquad \text{((7.9)) in chapter VII.}
\end{aligned}$$

Since τ maps $(J^a_\ell)^{-1}(b)$ onto $\Sigma_{a,b}$ and is the restriction of the mapping π to $(J^a_\ell)^{-1}(b)$, we obtain

$$\{\widetilde{f},\widetilde{g}\}_{\mathbf{R}^6}|\Sigma_{a,b} = \{\mathrm{id}^*_{\Sigma_{a,b}} f, \mathrm{id}^*_{\Sigma_{a,b}} g\}. \tag{65}$$

Combining (64) and (65) gives (63) as desired. $\square$

Claim: The mapping $\lambda : \mathbf{R}^3 \rightarrow \mathbf{R}^6 : (\sigma_1, \sigma_2, \sigma_3) \mapsto (\sigma_1, b, a - b\sigma_1, \sigma_2, 1 - \sigma_1^2, \sigma_3)$ (57) induces the mapping $\lambda^* : C^{\infty}(\mathbf{R}^6) \rightarrow C^{\infty}(\mathbf{R}^3)$, which is a Poisson map, that is, for every $f, g \in C^{\infty}(\mathbf{R}^6)$ we have

$$\{\lambda^* f, \lambda^* g\}_{\mathbf{R}^3} = \lambda^*(\{f, g\}_{\mathbf{R}^6}). \tag{66}$$

(4.34) **Proof**: To verify (66) it suffices to show that for every $1 \le i, j \le 6$ we have

$$\{\lambda^* \pi_i, \lambda^* \pi_j\}_{\mathbf{R}^3} = \lambda^*(\{\pi_i, \pi_j\}_{\mathbf{R}^6}). \tag{67}$$

Using tables 4.4.1 and 4.4.2 and the fact that

$$\lambda^*\pi_1 = \sigma_1,\ \lambda^*\pi_2 = b,\ \lambda^*\pi_3 = a - b\sigma_1,\ \lambda^*\pi_1 = \sigma_2,\ \lambda^*\pi_1 = 1 - \sigma_1^2,\ \lambda^*\pi_1 = \sigma_3,$$

which follows from the definition of the mapping λ, a straightforward calculation shows that (67) holds. Thus (67) holds because λ^* pulls back the structure matrix $W_{C^{\infty}(\mathbf{R}^6)}$ to the structure matrix $W_{C^{\infty}(\mathbf{R}^3)}$. $\square$

Corollary: The mapping $\lambda^* : \mathscr{F} = (C^{\infty}(\mathbf{R}^6), \{\ ,\ \}_{\mathbf{R}^6}, \cdot) \rightarrow \mathscr{E} = (C^{\infty}(\mathbf{R}^3), \{\ ,\ \}_{\mathbf{R}^3}, \cdot)$ is an epimorphism of Poisson algebras with right inverse $\mu^* : C^{\infty}(\mathbf{R}^3) \rightarrow C^{\infty}(\mathbf{R}^6)$, where $\mu : \mathbf{R}^6 \rightarrow \mathbf{R}^3 : (\pi_1, \ldots, \pi_6) \mapsto (\pi_1, \pi_4, \pi_6)$.

(4.35) **Proof**: It suffices to show that the linear mapping λ^* is an surjective. For $g \in C^{\infty}(\mathbf{R}^6)$ set $f = \mu^* g$. Then $f \in C^{\infty}(\mathbf{R}^3)$. So $\lambda^* f = \lambda^*(\mu^* g) = (\mu \circ \lambda)^* g = g$, since $\mu \circ \lambda = \mathrm{id}_{\mathbf{R}^3}$. Consequently, λ^* is surjective. $\square$

▷ The mapping $(\lambda|P^a_b)^* : C^{\infty}_i(\Sigma_{a,b}) \rightarrow C^{\infty}_i(P^a_b)$ is a *linear Poisson map*, that is for every f, $g \in C^{\infty}_i(\Sigma_{a,b})$ we have

$$\{(\lambda|P^a_b)^* f, (\lambda|P^a_b)^* g\}_{P^a_b} = (\lambda|P^a_b)^*(\{f, g\}_{\Sigma_{a,b}}). \tag{68}$$

(4.36) **Proof**: Because $f \in C^{\infty}_i(\Sigma_{a,b})$, there is an $F \in C^{\infty}(\mathbf{R}^6)$ such that $f = F|\Sigma_{a,b}$. So we get $(\lambda|P^a_b)^*(F|\Sigma_{a,b}) = (F \circ \lambda)|\Sigma_{a,b} = \lambda^*(F)|P^a_b \in C^{\infty}_i(P^a_b)$, since $\lambda^*(F) \in C^{\infty}(\mathbf{R}^3)$. Similarly $g = G|\Sigma_{a,b}$ for some $G \in C^{\infty}(\mathbf{R}^6)$ and $(\lambda|P^a_b)^*(G|\Sigma_{a,b}) = \lambda^*(G)|P^a_b \in C^{\infty}_i(P^a_b)$. Now we compute

$$\begin{aligned}
\{(\lambda|P^a_b)^* f, (\lambda|P^a_b)^* g\}_{P^a_b} &= \{\lambda^*(F)|P^a_b, \lambda^*(G)|P^a_b\}_{P^a_b} \\
&= \{\lambda^*(F), \lambda^*(G)\}_{\mathbf{R}^3}|P^a_b = \big(\lambda^*(\{F, G\}_{\mathbf{R}^6})\big)|P^a_b \\
&= \big(\lambda^*(\{F, G\}_{\mathbf{R}^6}|\Sigma_{a,b})\big)|P^a_b = \big(\lambda^*(\{F|\Sigma_{a,b}, G|\Sigma_{a,b}\}_{\Sigma_{a,b}})\big)|P^a_b \\
&= (\lambda|P^a_b)^*(\{f, g\}_{\Sigma_{a,b}}).
\end{aligned}$$

$\square$

Note that the map $(\lambda|P^a_b)^*$ is injective, because $\lambda|P^a_b : P^a_b \rightarrow \Sigma_{a,b}$ is surjective. Recall that $C^{\infty}(P^a_b) = (\lambda|P^a_b)^* C^{\infty}(\Sigma_{a,b})$ by definition. Define a *Poisson bracket* $\{\ ,\ \}$ on $C^{\infty}(P^a_b)$ by

$$(\lambda|P^a_b)^*(\{f, g\}) = \{(\lambda|P^a_b)^* f, (\lambda|P^a_b)^* g\}, \tag{69}$$

where $f, g \in C^{\infty}(\Sigma_{a,b})$. The bracket on the left hand side of (69) is the bracket on $C^{\infty}(\Sigma_{a,b})$; whereas the bracket on the right hand side is on $C^{\infty}(P^a_b)$. In other words, the Poisson bracket $\{\ ,\ \}$ on $C^\infty(\Sigma_{a,b})$ given by the singular reduction theorem is equivalent to the Poisson bracket $\{\ ,\ \}$ on $C^\infty(P^a_b)$, using the *linear Poisson isomorphism* $(\lambda|P^a_b)^*$.

Claim: The map $\mathrm{id}^*_{P^a_b} : \widetilde{\mathscr{G}} = (C^\infty_i(P^a_b), \{\ ,\ \}_{P^a_b}, \cdot) \to \mathscr{G} = (C^\infty(P^a_b), \{\ ,\ \}, \cdot)$ is an isomorphism of Poisson algebras.

(4.37) **Proof**: It suffices to show that $\mathrm{id}^*_{P^a_b}(\{f,g\}_{P^a_b}) = \{\mathrm{id}^*_{P^a_b} f, \mathrm{id}^*_{P^a_b} g\}$ for every $f,g \in C^\infty_i(P^a_b)$. Note that $\mathrm{id}_{P^a_b} = (\lambda|P^a_b)^{-1} \circ \mathrm{id}_{\Sigma_{a,b}} \circ (\lambda|P^a_b)$. We compute

$$\begin{aligned}
\{\mathrm{id}^*_{P^a_b} f, \mathrm{id}^*_{P^a_b} g\} &= \{\big((\lambda|P^a_b)^* \circ \mathrm{id}^*_{\Sigma_{a,b}} \circ ((\lambda|P^a_b)^{-1})^*\big) f, \big((\lambda|P^a_b)^* \circ \mathrm{id}^*_{\Sigma_{a,b}} \circ ((\lambda|P^a_b)^{-1})^*\big) g\} \\
&= (\lambda|P^a_b)^* (\{(\mathrm{id}^*_{\Sigma_{a,b}} \circ ((\lambda|P^a_b)^{-1})^*) f, (\mathrm{id}^*_{\Sigma_{a,b}} \circ ((\lambda|P^a_b)^{-1})^*) g\}) \\
&= \big((\lambda|P^a_b)^* \circ \mathrm{id}^*_{\Sigma_{a,b}}\big)(\{((\lambda|P^a_b)^{-1})^* f, ((\lambda|P^a_b)^{-1})^* g\}_{\Sigma_{a,b}}) \\
&= \big((\lambda|P^a_b)^* \circ \mathrm{id}^*_{\Sigma_{a,b}} \circ ((\lambda|P^a_b)^{-1})^*\big)(\{f,g\}_{P^a_b}) = \mathrm{id}^*_{P^a_b}(\{f,g\}_{P^a_b}). \qquad \square
\end{aligned}$$

5 The Euler-Poisson equations

In this section we describe the invariant manifolds of the Euler-Poisson vector field X_{H^a} (22) on the reduced space P^a by studying the geometry of the energy momentum mapping:

$$EM^a : P^a \to \mathbf{R}^2 : (z,w) \mapsto \big(H^a(z,w), J^a_\ell(z,w)\big) = \big(\tfrac{1}{2} I_1^{-1}(w,w) + \chi z_3, w_3\big) \tag{70}$$

Because the reduced energy H^a and and angular momentum J^a_ℓ are integrals of X_{H^a}, the fiber $(EM^a)^{-1}(h^a,b)$ is invariant under the flow of X_{H^a}. To understand these invariant sets, we need to know the following about the energy momentum mapping EM^a.

1. What its critical points, critical values, and its range are.
2. What the topology of every fiber $(EM^a)^{-1}(h^a,b)$ is.
3. How these fibers foliate an energy level set.

We also study the qualitative properties of the image of the integral curves of X_{H^a} restricted to $(EM^a)^{-1}(h^a,b)$ under the projection $\tau^a : P^a \subseteq \mathbf{R}^3 \times \mathbf{R}^3 \to S^2 \subseteq \mathbf{R}^3 : (z,w) \mapsto z$, when (h^a,b) is a regular value of EM^a. These curves on S^2 describe the motion of the tip of the figure axis of the top.

5.1 The twice reduced system

In this subsection we study the qualitative behavior of the *twice reduced Hamiltonian system* $(H^a_b, P^a_b, \{\ ,\ \}_{P^a_b})$.

Recall that the *twice reduced space* P^a_b is the semialgebraic variety in $\mathbf{R}^3$ defined by

$$G(\sigma) = \sigma_3(1-\sigma_1^2) - \sigma_2^2 - (a - b\sigma_1)^2 = 0, \quad \text{where } |\sigma_1| \le 1 \;\&\; \sigma_3 \ge 0 \tag{71}$$

and the *twice reduced Hamiltonian* is

$$H^a_b : P^a_b \subseteq \mathbf{R}^3 \to \mathbf{R} : (\sigma_1,\sigma_2,\sigma_3) \mapsto \tfrac{1}{2} I_1^{-1} \sigma_3 + \chi \sigma_1. \tag{72}$$

In order to determine the topology of the level set $(H^a_b)^{-1}(h^a_b)$, we first find the critical points and critical values of H^a_b. Because P^a_b is not necessarily smooth, we use an algebraic definition of critical point. We say that $\widetilde{\sigma} = (\widetilde{\sigma}_1, \widetilde{\sigma}_2, \widetilde{\sigma}_3)$ is a critical point of H^a_b corresponding to the critical value h^a_b if and only if the 2-plane Π defined by $\frac{1}{2}I_1{}^{-1}\sigma_3 + \chi\,\sigma_1 = h^a_b$ intersects the reduced space P^a_b at $\widetilde{\sigma}$ with *multiplicity* greater than one. Let $\alpha = I_1 h^a_b$ and $\beta = I_1\chi$ and form the polynomial

$$P(\sigma_1, \sigma_2) = 2(\alpha - \beta\sigma_1)(1-\sigma_1^2) - \sigma_2^2 - (a - b\sigma_1)^2$$

by eliminating σ_3 from (71) using the definition of Π. The point $\widetilde{\sigma}$ has multiplicity greater than one if and only if the Taylor polynomial of P at $(\widetilde{\sigma}_1, \widetilde{\sigma}_2)$ has no constant or linear terms. Therefore, $\widetilde{\sigma}$ is a critical point of H^a_b corresponding to the critical value h^a_b if and only if $\widetilde{\sigma}_2 = 0$ and $\widetilde{\sigma}_1$ is a multiple root of

$$W(\sigma_1) = (2\alpha - \sigma_1)(1-\sigma_1^2) - (a - b\sigma_1)^2 \tag{73}$$

in $[-1, 1]$. Here we have set $\beta = \frac{1}{2}$, which can be arranged by a suitable choice of physical

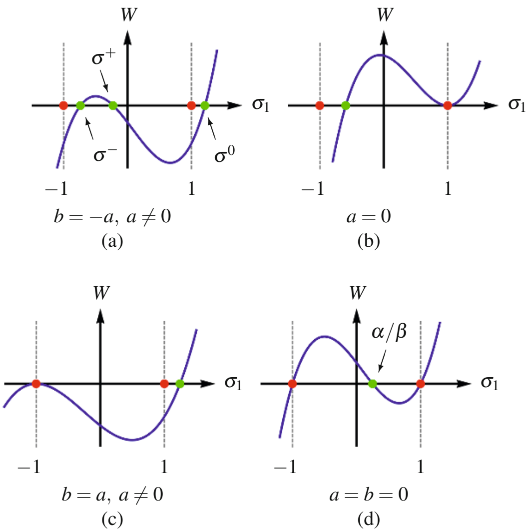


Figure 5.1.1. The graph of W.

$\triangleright$ units. Note that $\widetilde{\sigma}_3 = 2\alpha - \widetilde{\sigma}_1$. Every critical point of the twice reduced H^a_b lies on the curve $\mathscr{C} = \{\sigma_2 = 0\} \cap P^a_b$, which is the fold curve of the projection map $\rho : \mathbf{R}^3 \to \mathbf{R}^2 : \sigma = (\sigma_1, \sigma_2, \sigma_3) \mapsto (\sigma_1, \sigma_3)$ restricted to P^a_b.

(5.1) **Proof**: The image of P^a_b under the mapping ρ is the set of points (σ_1, σ_3) in $\mathbf{R}^2$ which satisfy $\sigma_3(1-\sigma_1^2) - (a - b\sigma_1)^2 \ge 0$, $|\sigma_1| \le 1$ & $\sigma_3 \ge 0$. Over every point in the interior of $\rho(P^a_b)$ the fiber of $\rho|P^a_b$ consists of two distinct nonsingular points of P^a_b; while over every point on the boundary $\rho(\mathscr{C})$ of $\rho(P^a_b)$, the fiber is a single point. As is easily checked, table 5.1.1 gives all the possibilities for $\rho(\mathscr{C})$. $\square$

Conditions	$\rho(\mathscr{C})$
1. $b \neq \pm a,\ a \neq 0$	$\sigma_3 = \frac{(a-b\sigma_1)^2}{1-\sigma_1^2}, \quad \lvert\sigma_1\rvert < 1$
2. $b = a,\ a \neq 0$	$\{\sigma_3 = \frac{a^2(1-\sigma_1)}{1+\sigma_1},\ -1 < \sigma_1 \leq 1\} \cup \{(1,0,\sigma_3) \mid \sigma_3 \geq 0\}$
3. $b = -a,\ a \neq 0$	$\{\sigma_3 = \frac{a^2(1+\sigma_1)}{1-\sigma_1},\ -1 \leq \sigma_1 < 1\} \cup \{(-1,0,\sigma_3) \mid \sigma_3 \geq 0\}$
4. $b = a = 0$	$\{(\pm 1,0,\sigma_3) \mid \sigma_3 \geq 0\} \cup \{(\sigma_1,0,0) \mid \lvert\sigma_1\rvert \leq 1\}$

Table 5.1.1.

From (71) and the definition of the 2-plane Π, the point $\widetilde{\sigma} = (\widetilde{\sigma}_1, 0, \widetilde{\sigma}_3)$ is a critical point

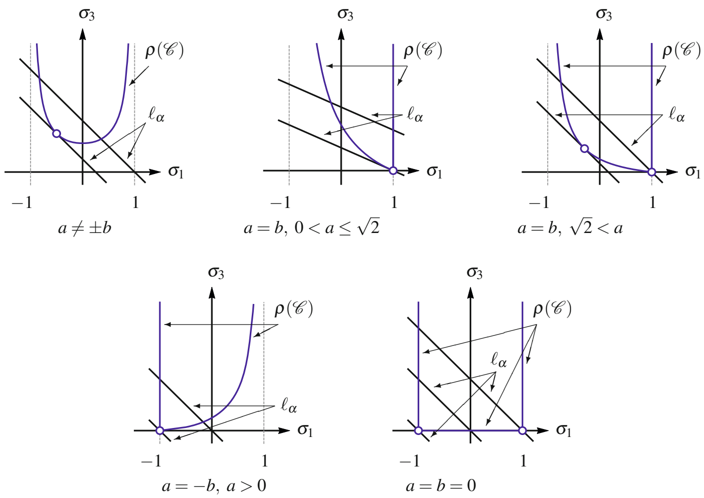


Figure 5.1.2. Intersection of the line ℓ_α with the image of the fold curve $\rho(\mathscr{C})$. The points $(\tilde{\sigma}_1, \tilde{\sigma}_3)$ are large dots. The corresponding critical point of H_b^a is $(\tilde{\sigma}_1, 0, \tilde{\sigma}_3)$ with critical value α.

of $H_b^a|\mathscr{C}$ if and only if the line $\ell_\alpha : \sigma_3 + \sigma_1 = 2\alpha$ in the σ_1–σ_3 plane is tangent to $\rho(\mathscr{C})$ or passes through a singular point of $\rho(\mathscr{C})$. Figure 5.1.2 gives the geometric possibilities.

Next we give a geometric description of the set of critical values of H_b^a, see figure 5.1.4. In the preceding paragraphs we showed that $h_b^a = I_1^{-1}\alpha$ is a critical value of H_b^a if and only if

$$W(\sigma) = \sigma^3 - (2\alpha + b^2)\sigma^2 + (2ab-1)\sigma + 2\alpha - a^2 \tag{74}$$

has a multiple zero in $[-1,1]$. The set of $(a,b,\alpha) \in \mathbf{R}^3$ where W has a multiple root in $[-1,1]$ is the *discriminant* of W. The *discriminant locus* of W is $\{\Delta_W = 0\}$. A good way to present this locus is to give a parametrization

$$\mathscr{P} : U_\pm \subseteq [-1,1] \times \mathbf{R} \to \{\Delta_W = 0\} : (s,a) \mapsto \big(a, b_\pm(s,a), \alpha_\pm(s,a)\big). \tag{75}$$

Claim: The domain $U_\pm$ of the parametrization $\mathscr{P}$ (75) is defined by

$$U_\pm = \{(s,a) \in ([-1,1]\times \mathbf{R} \,\big|\, (1-s^2)^2(a^2-2s) \geq 0\} \setminus \{(0,a) \in \mathbf{R}^2 \,\big|\, \mp a \geq 0\}.$$

In other words

$$U_+ = \Big\{(s,a) \in [-1,1]\times \mathbf{R} \,\Big|\, s \in \mathscr{I}_a = \begin{cases} [-1,0)\cup\{1\}, \text{ if } a = 0, \\ (I_a \setminus \{0\})\cup\{1\}, & \text{if } a > 0 \\ I_a \cup \{1\}, & \text{if } a < 0, \end{cases}$$

and

$$U_- = \Big\{(s,a) \in [-1,1]\times \mathbf{R} \,\Big|\, s \in \widetilde{\mathscr{I}}_a = \begin{cases} [-1,0)\cup\{1\}, \text{ if } a = 0, \\ I_a \cup \{1\}, & \text{if } a > 0, \\ (I_a \setminus \{0\})\cup\{1\}, & \text{if } a < 0, \end{cases}$$

where $I_a = [-1, \min\{\frac{1}{2}a^2, 1\}]$.

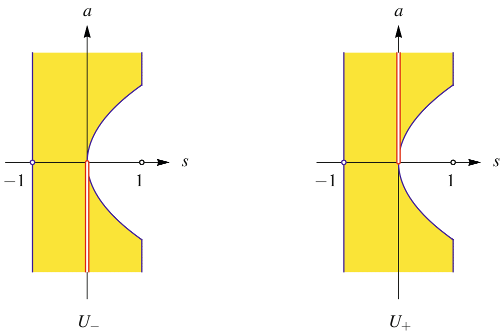


Figure 5.1.3. In the domain U_-, which is the shaded area in the left figure, the nonpositive a-axis is removed; while in the domain U_+, which is the shaded area in the right figure, the nonnegative a-axis is removed.

Then

1. when $(s,a) \in U_+$, we have

$$b_+(s,a) = \begin{cases} \frac{1}{\sqrt{-2s}}(1-s^2), & s \in \mathscr{I}_0 \\ \frac{1}{2s}\big[a(1+s^2)+(1-s^2)\sqrt{a^2-2s}\big], & s \in \mathscr{I}_a,\ a \neq 0; \end{cases}$$

2. when $(s,a) \in U_-$, we have

$$b_-(s,a) = \begin{cases} -\frac{1}{\sqrt{-2s}}(1-s^2), & s \in \widetilde{\mathscr{I}}_0 \\ \frac{1}{2s}\big[a(1+s^2)-(1-s^2)\sqrt{a^2-2s}\big], & s \in \widetilde{\mathscr{I}}_a,\ a \neq 0; \end{cases}$$

3. when $(s,a) \in U_\pm$ we have

$$\alpha_\pm(s,a) = \begin{cases} \frac{1}{4}s(1+s^2),\ s \in \mathscr{I}_0 \\ \frac{1}{4}\big[a^2(1-s^2) \mp a(1-s^2)\sqrt{a^2-2s} + s(1+s^2)\big],\ s \in \mathscr{I}_a,\ a \neq 0. \end{cases}$$

(5.2) **Proof**: Note that $(s,t) \in [-1,1] \times \mathbf{R}$ lies in Δ_W if and only if W (74) can be factored as

$$(\sigma - s)^2(\sigma - t) = \sigma^3 - (2s+t)\sigma^2 + (s^2+2st)\sigma - s^2 t. \tag{76}$$

Equating coefficients in (74) and (76) gives

$$2s + t = 2\alpha + b^2 \tag{77a}$$

$$s^2 + 2st = 2ab - 1 \tag{77b}$$

$$s^2 t = a^2 - 2\alpha. \tag{77c}$$

To find $b(s,a)$ add (77a) and (77c) to obtain $t = \frac{1}{1+s^2}(a^2+b^2-2s)$, which then substituted into (77b) gives

$$Q(b) = 2sb^2 - 2(1+s^2)ab + 2sa^2 + (1-s^2)^2 = 0. \tag{78}$$

Suppose that $s \neq 0$, then Q has real roots if and only if $\operatorname{discr} Q = 4(1-s^2)^2(a^2-2s) \geq 0$, that is, if and only if $s \in (I_a \cup \{1\}) \setminus \{0\}$. Solving (78) for b gives the expressions for $b_\pm$. To determine the precise domain of definition of $b_\pm$ we need to look more closely what happens on the a-axis. When $s = 0$, (78) becomes $2ab = 1$. Since $\lim_{s\to 0} b_\pm(s,a) = (2a)^{-1}$ when $\mp a > 0$, we can extend the domain of definition of $b_\pm$ from $\{(s,a) \in ([-1,1] \setminus \{0\}) \times \mathbf{R} \,\big|\, \operatorname{discr} Q \geq 0\}$ to $U_\pm$.

To find $\alpha(s,a)$ we use $t = \frac{1}{1+s^2}(a^2+b^2-2s)$ to eliminate t from (77a). We obtain

$$\alpha = -\tfrac{s^2}{2(1+s^2)}b^2 + \tfrac{1}{2(1+s^2)}a^2 + \tfrac{s^3}{1+s^2}. \tag{79}$$

Using the expressions for $b_\pm$ in claim (79) gives the expression for $\alpha_\pm$. $\square$

For a fixed, the curves $s \mapsto \gamma_\pm(s,a) = \big(b_\pm(s,a), \alpha_\pm(s,a)\big)$ are pieces of a parametrization

$s \in \partial U_\pm$	$\gamma_\pm(s,a)$
1. -1	$(-a, -\frac{1}{2})$
2. $\frac{1}{2}a^2$, $0 < \lvert a\rvert < \sqrt{2}$	$\big(\frac{1}{4a}(4+a^4), \frac{a^2}{32}(12-a^4)\big)$
3. 1	$(a, \frac{1}{2})$

Table 5.1.2. Points where γ_+ joins γ_-.

of an a-slice of $\{\Delta_W = 0\}$. We now determine how the the curves $\gamma_\pm$ fit together. A straightforward calculation gives table 5.1.2, which lists the points where γ_+ and γ_- join. We make the following observations. When $\mp a \geq 0$ is fixed, γ_- is defined and continuous on $I_a \cup \{1\}$. When $\pm a \geq 0$, $\lim_{s\to 0^\pm} b_\pm = \mp\infty$ and $\lim_{s\to 0^\pm} \alpha_\pm = 0$. When $|a| < \sqrt{2}$, $\{1\}$ is an isolated point of $I_a \cup \{1\}$. Hence $(a, \frac{1}{2})$ is an *isolated point* of $\gamma_\pm$. From these facts and table 5.1.2 we see that each of the curves γ_i defined below is continuous.

1. When $a = 0$, let γ_1 be the curve formed by joining $\gamma_+|[-1,0)$ to $\gamma_-|[-1,0)$ at $s = -1$.
2. When $0 < |a| < \sqrt{2}$, let γ_2 be the curve formed by joining $\gamma_+|[-1,0)$ to $\gamma_-|(-1, \frac{1}{2}a^2]$ at $s = -1$ and joining $\gamma_-|(-1, \frac{1}{2}a^2]$ to $\gamma_+|(0, \frac{1}{2}a^2]$ at $s = \frac{1}{2}a^2$.

3. When $|a| \geq \sqrt{2}$, let γ_3 be the curve formed by joining $\gamma_+|[-1,0)$ to $\gamma_-|[-1,1]$ at $s=-1$ and $\gamma_-|[-1,1]$ to $\gamma_+|(0,1]$ at $s=1$.

Gathering the above information together we have proved

Claim: An a-slice of the discriminant locus $\{\Delta_W = 0\}$ of W (74) is $\gamma_1 \cup \{(0,\frac{1}{2})\}$, if $a=0$; $\gamma_2 \cup \{(a,\frac{1}{2})\}$, if $0<|a|<\sqrt{2}$; and γ_3, if $|a| \geq \sqrt{2}$.

Recall that $\alpha = I_1 h^a_b$. Applying the map $\mathbf{R}^2 \to \mathbf{R}^2 : (b,\alpha) \mapsto (b, I_1{}^{-1}\alpha) = (b, h^a_b)$ to a fixed a-slice of $\{\Delta_W = 0\}$ gives the set of *critical values* of the twice reduced Hamiltonian H^a_b for a fixed value of a, see figure 5.1.4. From the algebraic definition of critical point, it is clear that a point of the a-slice of Δ_W, which lies on the curve $\gamma_\pm$ is a minimum of $I_1 h^a_b$. Thus for a fixed value of a, the range of H^a_b is the set of (b, h^a_b) values which lie on or above the curves given in figure 5.1.4.

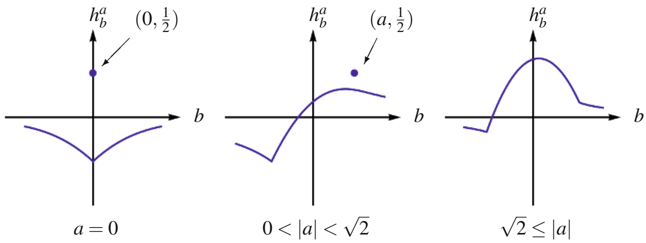


Figure 5.1.4. The critical values of the twice reduced Hamiltonian H^a_b for fixed a.

$\triangleright$ Next we determine the topology of the *level sets* of the (twice) reduced Hamiltonian H^a_b. The results are given in table 5.1.3.

(5.3) **Proof**: To find the topology of the level sets of H^a_b, we return to the geometric situation sketched in figure 5.1.2. Because the image of the fold curve $\mathscr{C}$ under the projection ρ bounds a convex subset C of $\mathbf{R}^2$, the line ℓ_α intersects $\rho(\mathscr{C})$ in at most two points p_1 and p_2, where p_2 lies to the right of p_1. Let $[p_1,p_2]$ be the segment of ℓ_α lying in C. Then $[p_1,p_2] = \rho\big((H^a_b)^{-1}(h^a_b)\big)$.

We now verify the entries in the second column to table 5.1.3.

1 and 2. The first and second entries follow from the fact that the fiber of ρ over a singular point of $\rho(\mathscr{C})$ is a single point of $\mathscr{C}$.

3 and 4. For the third and fourth entry we observe that $\rho^{-1}([p_1,p_2])$ is a *pinched* S^0 bundle, that is, a two point bundle over the interior of $[p_1,p_2]$ with fiber over p_1 and p_2 pinched to a point. Thus the h^a_b-level set of H^a_b is a topological circle. If h^a_b is a regular value of H^a_b, then $(H^a_b)^{-1}(h^a_b)$ is smooth. Otherwise, $p_1 = (-1,0)$ or $p_2 = (1,0)$ is the only singular point of $\rho(\mathscr{C})$ because the line ℓ_1 has negative slope. When $p_1 = (-1,0)$ is a singular point, then $b=-a$ and $h^a_{-a} = -\frac{1}{2}I_1^{-1}$. Hence the h^a_{-a}-level set of H^a_{-a} is a point. When $p_2 = (1,0)$ is a singular point, then $b=a$ with $|a|<\sqrt{2}$ and $h^a_a = \frac{1}{2}I_1^{-1}$. Hence the h^a_a-level set of H^a_a is the semialgebraic variety

$$\begin{aligned} \sigma_1 + \sigma_3 &= 1 \\ \sigma_2^2 + a^2(1-\sigma_1)^2 - \sigma_3(1-\sigma_1^2) &= 0, \qquad |\sigma_1| \leq 1 \;\&\; \sigma_3 \geq 0. \end{aligned}$$

Project $(H_a^a)^{-1}(h_a^a)$ onto the σ_1–σ_2 plane by eliminating σ_3. This gives

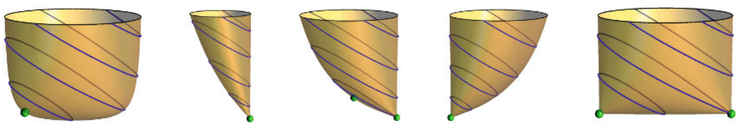
$$0 = \sigma_2^2 - (2-a^2)(1-\sigma_1)^2 + (1-\sigma_1)^3, \qquad |\sigma_1| \le 1,$$

which has a conical singular point $(1,0)$. Hence $\rho^{-1}(p_2)$ is a conical singularity of $(H_a^a)^{-1}(h_a^a)$. $\square$

$[p_1, p_2]$	Topology of $(H_b^a)^{-1}(h_b^a)$	Conditions on roots of W, where $\beta = \frac{1}{2}$
1. $p_1 = p_2$ is nonsingular point of $\rho(\mathscr{C})$	point	A double real root in $(-1,1)$ and no other root in $[-1,1]$
2. $p_1 = p_2$ is a singular point of $\rho(\mathscr{C})$	point	A triple real root at $+1$ or a double root at -1 or $+1$ and no other root in $[-1,1]$
3. $p_1 \ne p_2$ and p_2 is a singular point of $\rho(\mathscr{C})$	A topological S^1 which is smooth except for one conical singular point	A double root at 1 with $\|a\| < \sqrt{2}$, and one other simple real root in $[-1,1]$
4. $p_1 \ne p_2$, p_1 and p_2 are nonsingular points of $\rho(\mathscr{C})$	A smooth S^1	Two simple real roots in $[-1,1]$

Table 5.1.3. Topology of the level sets of $H_b^a | P_b^a$.

A connected component of a level set of H_b^a on P_b^a is an orbit of the *Hamiltonian derivation* $-\mathrm{ad}_{H_b^a|P_b^a}$. This derivation is a vector field $X_{H_b^a}$ on the locally compact subcartesian differential space P_b^a, see chapter VII §4. If p is a critical point of $(H_b^a)^{-1}(h_b^a)$, which is a *nonsingular* point of P_b^a, then p is an equilibrium point of $X_{H_b^a}$. When this is the case, the usual definitions of elliptic and hyperbolic equilibrium point apply. On the



$a = \pm b$ $\quad$ $a = b \ne 0$, $|a| \le \sqrt{2}$ $\quad$ $a = b \ne 0$, $|a| > \sqrt{2}$ $\quad$ $a = -b \ne 0$ $\quad$ $a = b = 0$

Figure 5.1.5. Integral curves of the vector field $-\mathrm{ad}_{H_b^a|P_b^a}$ on P_b^a.

other hand, if p is a critical point of $(H_b^a)^{-1}(h_b^a)$, which is a *singular* point of P_b^a, we use the following definitions. We say that p is an *elliptic equilibrium point* of $-\mathrm{ad}_{H_b^a|P_b^a}$ if for every $\widetilde{h}$ slightly larger than h_b^a, the $\widetilde{h}$-level set of H_b^a is a smooth circle, which shrinks to p as $\widetilde{h} \to (h^a_b)^+$. We say that p is a *hyperbolic equilibrium point* of $-\mathrm{ad}_{H^a_b|P^a_b}$ if the connected component of the h^a_b-level set of H^a_b containing p has points which do not lie in any sufficiently small neighborhood of p. In figure 5.1.4 we sketch the orbits of $-\mathrm{ad}_{H^a_b|P^a_b}$.

5.2 The energy momentum mapping

In this subsection we study the geometry of the energy momentum mapping EM^a (70) of the Euler-Poisson vector field X_{H^a} (22). First we reconstruct the topology of the (h^a,b)-level set of EM^a using table 5.1.3 and some basic facts about the geometry of the reduction mapping π^a_b (56). The results are summarized in table 5.2.1. Second we determine the topology of the level set $(H^a)^{-1}(h^a)$, see table 5.2.2 and show how it is foliated by level sets of J^a_ℓ.

Topology of $(H^a_b)^{-1}(h^a_b)$	Topology of $(EM^a)^{-1}(h^a,b)$	Conditions
1. a nonsingular point of P^a_b	smooth S^1	$(a,b,\alpha) \in \Delta_W$, $b \neq \pm a$, $a \neq 0$
2. a singular point of P^a_b	point	$b = a$, $\lvert a\rvert \geq \sqrt{2}$, $\alpha = \frac{1}{2} + \frac{1}{2}a^2$, or $b = -a$, $\alpha = -\frac{1}{2} + \frac{1}{2}a^2$
3. a topological circle with a conical singular point	a 2-torus with a fiber pinched to a point	$b = a$, $\alpha = \frac{1}{2} + \frac{1}{2}a^2$, $\lvert a\rvert < \sqrt{2}$
4. a smooth S^1	a smooth 2-torus	(h^a,b) regular value of EM^a

Table 5.2.1. Topology of the level sets of $(EM^a)^{-1}(h^a,b)$. Here $h^a = h^a_b + \frac{1}{2}I_1^{-1}b^2$, $\alpha = I_1 h^a_b$, and $\beta = I_1\chi = \frac{1}{2}$.

Knowing the geometry of the reduction mapping $\pi^a_b : (J^a_\ell)^{-1}(b) \to P^a_b$ and the topology of the h^a_b-level set of the twice reduced Hamiltonian H^a_b, we can reconstruct the topology of the (h^a,b)-level set of EM^a because $(EM^a)^{-1}(h^a,b) = (H^a)^{-1}(h^a) \cap (J^a_\ell)^{-1}(b) =$
▷ $(\pi^a_b)^{-1}\big((H^a_b)^{-1}(h^a_b)\big)$. Here $h^a_b = h^a - \frac{1}{2}I_1^{-1}b^2$. We now verify table 5.2.1.

(5.4) **Proof**: The first and third columns of table 5.2.1 are the same as the first and third columns of table 5.1.3, respectively.

We now check the entries in the second column. Suppose that the h^a_b-level set of H^a_b is a point p. If p is a nonsingular point of P^a_b, then the fiber $(\pi^a_b)^{-1}(p)$ is a smooth S^1; otherwise it is a point, because it is a fixed point of the left S^1 action δ^ℓ (38).

4. To verify the fourth entry suppose that $(h^a,b) = (h^a_b + \frac{1}{2}I_1^{-1}b^2, b)$ is a regular value of EM^a which lies in its range. Then h^a is a regular value of H^a, which implies that h^a_b is a regular value of H^a_b. Therefore the h^a_b-level set of H^a_b is a smooth S^1. Let $M = (EM^a)^{-1}(h^a,b)$. Since π^a_b is a proper smooth submersion of M onto $(H^a_b)^{-1}(h^a_b)$, M is a smooth S^1 bundle over S^1, see chapter VIII §2. The total space of this bundle is connected, because the base and fiber are connected. Moreover, M is orientable because it is the preimage of 0 under the mapping $\mathbf{R}^3 \times \mathbf{R}^3 \to \mathbf{R}^4 : (z,w) \mapsto \big(G_1(z,w),\ldots,G_4(z,w)\big)$,

where

$$
\begin{aligned}
G_1(z,w) &= z_1^2+z_2^2+z_3^2-1\\
G_2(z,w) &= z_1w_1+z_2w_2+z_3w_3-a\\
G_3(z,w) &= \tfrac12 I_1^{-1}(w_1^2+w_2^2+w_3^2)+\chi z_3-h^a,\\
G_4(z,w) &= w_3-b.
\end{aligned}
$$

Therefore M is a smooth 2-torus. An alternative way to see this, which mimics part of the proof of the existence of action angle coordinates, see chapter IX §2, goes as follows. Because (h^a,b) is a regular value of EM^a, at each $m\in M$ the differentials $\mathrm{d}H^a(m)$ and $\mathrm{d}J^a_\ell(m)$ are linearly independent. Therefore, the vector fields $X_{H^a}|M$ and $X_{J^a_\ell}|M$ are tangent to M and are linearly independent at each $m\in M$. Since M is compact and $\{H^a,J^a_\ell\}_{P^a}=0$, the flows of $X_{H^a}|M$ and $X_{J^a_\ell}|M$ define an $\mathbf{R}^2$-action on M which is transitive. Again because M is compact, the isotropy group L of the $\mathbf{R}^2$-action is a rank 2 lattice. Therefore M is diffeomorphic to $\mathbf{R}^2/L$ which is a 2-torus.

3. We verify the third entry as follows. Suppose that $(H^a_b)^{-1}(h^a_b)$ is a topological circle with one singular point $p=(1,0,0)$. Then $q=(\pi^a_b)^{-1}(p)$ is the only singular point of the variety M. The map π^a_b is a proper smooth submersion of $M\setminus\{q\}$ onto $(H^a_b)^{-1}(h^a_b)\setminus\{p\}$. Since $(H^a_b)^{-1}(h^a_b)\setminus\{p\}$ is diffeomorphic to $\mathbf{R}$, it follows that $M\setminus\{q\}$ is diffeomorphic to a cylinder $S^1\times\mathbf{R}$, see chapter IX §2. An alternative argument which establishes this goes as follows. Consider the vector fields $X_{H^a}|(M\setminus\{q\})$ and $X_{J^a_\ell}|(M\setminus\{q\})$. Since the integral curves of $X_{J^a_\ell}$ on P^a are periodic and q is an equilibrium point, the flow of $X_{J^a_\ell}|(M\setminus\{q\})$ is periodic and hence is complete. On the other hand, the only way that the flow of the vector field $X_{H^a}|(M\setminus\{q\})$ could be incomplete is for one of its integral curves to reach q in finite time. But this is impossible, because q is an equilibrium point of X_{H^a}. Consequently, the flows of $X_{H^a}|(M\setminus\{q\})$ and $X_{J^a_\ell}|(M\setminus\{q\})$ define a transitive $\mathbf{R}^2$-action on $M\setminus\{q\}$, which has a rank 1 lattice L as isotropy group. Therefore $M\setminus\{q\}$ is diffeomorphic to $\mathbf{R}^2/L$ which is a cylinder $S^1\times\mathbf{R}$. Hence M is the one point compactification of a cylinder, that is, a two dimensional torus with a meridial circle pinched to a point. A more detailed argument shows that q is a hyperbolic equilibrium point for X_{H^a}. Thus q is a conical singularity of M being the transverse intersection of the stable and unstable manifolds of q. This completes the verification of table 5.2.1. □

Topology of $(H^a)^{-1}(h^a)$	Conditions
1. point	$h^a=-\chi+\frac12 I_1^{-1}a^2$
2. a smooth three sphere, S^3	$-\chi+\frac12 I_1^{-1}a^2<h^a<\chi+\frac12 I_1^{-1}a^2$
3. a topological three sphere with one conical singular point	$h^a=\chi+\frac12 I_1^{-1}a^2$
4. unit tangent S^1 bundle over S^2	$h^a>\chi+\frac12 I_1^{-1}a^2$

Table 5.2.2. Topology of the level sets $(H^a)^{-1}(h^a)$.

We now determine the topology of the level sets of H^a, see table 5.2.2. We use Morse theory to verify the entries in table 5.2.2. The space P^a is the preimage of the regular value $(0,0)$ of the mapping $\mathbf{R}^3\times\mathbf{R}^3\to\mathbf{R}^2:(z,w)\mapsto\big(F_1(z,w),F_2(z,w)\big)$, where $F_1(z,w)=z_1^2+$

$z_2^2+z_3^2-1$ and $F_2(z,w)=z_1w_1+z_2w_2+z_3w_3-a$. Consider the function $H:\mathbf{R}^3\times\mathbf{R}^3\to$
▷ $\mathbf{R}:(z,w)\mapsto \frac{1}{2}I_1^{-1}(w_1^2+w_2^2+w_3^2)+\chi z_3$. The following argument shows that $H^a=H|P^a$ is a Morse function.

(5.5) **Proof**: H^a is bounded below and is proper. Therefore, H^a has a critical point $(z,w)\in P^a$, which we find using Lagrange multipliers. At (z,w) we have

$$DH(z,w)+\lambda_1 DF_1(z,w)+\lambda_2 DF_2(z,w)=0 \quad \text{and} \quad F_1(z,w)=F_2(z,w)=0.$$

In other words,

$$\begin{array}{rlrl} 2\lambda_1 z_1+\lambda_2 w_1=0, & & \lambda_2 z_1+I_1^{-1}w_1=0, \\ 2\lambda_1 z_2+\lambda_2 w_2=0, & & \lambda_2 z_2+I_1^{-1}w_2=0, \\ 2\lambda_1 z_3+\lambda_2 w_3+\chi=0, & & \lambda_2 z_3+I_1^{-1}w_3=0, \end{array}$$

$z_1^2+z_2^2+z_3^2=1$, and $z_1w_1+z_2w_2+z_3w_3=a$. Therefore

$$\lambda_2=\lambda_2(z_1^2+z_2^2+z_3^2) = -I_1^{-1}(z_1w_1+z_2w_2+z_3w_3) = -I_1^{-1}a,$$

which gives $w_i=az_i$ for $i=1,2,3$. Hence

$$\begin{aligned} (2\lambda_1-I_1^{-1}a^2)z_1 &= 0\\ (2\lambda_1-I_1^{-1}a^2)z_2 &= 0\\ (2\lambda_1-I_1^{-1}a^2)z_3+\chi &= 0. \end{aligned}$$

If $2\lambda_1-I_1^{-1}a^2=0$, then $\chi=0$, which is a contradiction. Therefore $2\lambda_1-I_1^{-1}a^2\neq 0$, which gives $z_1=z_2=0$. Consequently, $z_3=\varepsilon$ with $\varepsilon^2=1$ and $w_1=w_2=0$ and $w_3=\varepsilon a$. Thus we have shown that H^a has two critical points $p_\varepsilon=\varepsilon(e_3,ae_3)$ with Lagrange multipliers $\lambda_1=\frac{1}{2}(-\varepsilon\chi+I_1^{-1}a^2)$ and $\lambda_2=-I_1^{-1}a$.

Next we show that the critical points p_ε are nondegenerate. The tangent space $T_{p_\varepsilon}P^a$ is $\ker\begin{pmatrix} DF_1(p_\varepsilon)\\ DF_2(p_\varepsilon)\end{pmatrix}=\ker\begin{pmatrix} 0 & 0 & 2\varepsilon & 0 & 0 & 0\\ 0 & 0 & \varepsilon a & 0 & 0 & \varepsilon\end{pmatrix}$, which is spanned by $\{e_1,e_2,e_4,e_5\}$. Therefore

$$D^2H^a(p_\varepsilon)= \big(D^2H+\lambda_1D^2F_1+\lambda_2D^2F_2\big)\Big|_{T_{p_\varepsilon}P^a}(p_\varepsilon)=\begin{pmatrix} I_1^{-1}a^2-\varepsilon\chi & 0 & -aI_1^{-1} & 0\\ 0 & I_1^{-1}a^2-\varepsilon\chi & 0 & -aI_1^{-1}\\ -aI_1^{-1} & 0 & I_1^{-1} & 0\\ 0 & -aI_1^{-1} & 0 & I_1^{-1}\end{pmatrix}.$$

Since $\det D^2\big(H^a(p_\varepsilon)\big)=\chi^2I_1^{-2}\neq 0$, the critical points p_ε are nondegenerate. Hence H^a is a Morse function. The characteristic polynomial of $D^2H^a(p_\varepsilon)$ is the square of the polynomial $\lambda^2-((a^2+1)I_1^{-1}-\varepsilon\chi)\lambda-\varepsilon\chi I_1^{-1}$. Thus $D^2H^a(p_\varepsilon)$ has two negative and two positive eigenvalues when $\varepsilon=1$ and four positive eigenvalues when $\varepsilon=-1$. Therefore $p_1=(e_3,ae_3)$ is a nondegenerate saddle point of H^a of Morse index 2 corresponding to the critical value $\chi+\frac{1}{2}I_1^{-1}a^2$; whereas $p_{-1}=(-e_3,-ae_3)$ is a nondegenerate minimum corresponding to the critical value $-\chi+\frac{1}{2}I_1^{-1}a^2$. □

▷ We are now ready to determine the topology of the level sets of H^a using Morse theory.

(5.6) **Proof**:

1. When $h^a = -\chi + \frac{1}{2}I_1^{-1}a^2$, the level set $(H^a)^{-1}(h^a)$ is the point p_{-1}.

2. By the Morse lemma, see chapter XI §2, near p_{-1}, the function H^a is equal to its second derivative at p_{-1} up to a smooth change of coordinates. Therefore for h^a values slightly greater than $-\chi + \frac{1}{2}I_1^{-1}a^2$ the h^a-level set of H^a is diffeomorphic to a three dimensional sphere S^3. By the Morse isotopy lemma, see chapter XI §3, for every $h^a \in (-\chi + \frac{1}{2}I_1^{-1}a^2, \chi + \frac{1}{2}I_1^{-1}a^2)$ the level set $(H^a)^{-1}(h^a)$ is diffeomorphic to S^3.

4. For $h^a > \chi + \frac{1}{2}I_1^{-1}a^2$ the h^a-level set of H^a is diffeomorphic to the unit tangent S^1 bundle T_1S^2 over S^2. In chapter III §1 we have seen that the unit tangent circle bundle T_1S^2 of S^2 is diffeomorphic to real projective 3-space $\mathbf{RP}^3$. To check this we view $(H^a)^{-1}(h^a)$ as a bundle over S^2 by applying the mapping $\varphi : P^a \rightarrow TS^2 : (z,w) \mapsto (x,y) = (z, w - az)$. The total space $\varphi\big((H^a)^{-1}(h^a)\big)$ of this new bundle over S^2 is defined by

$$x_1^2 + x_2^2 + x_3^2 = 1$$

$$x_1y_1 + x_2y_2 + x_3y_3 = 0 \qquad (80a)$$

$$\tfrac{1}{2}I_1^{-1}(y_1^2 + y_2^2 + y_3^2) + \chi x_3 = h^a - \tfrac{1}{2}I_1^{-1}a^2. \qquad (80b)$$

Fix a point x on S^2. Then (80b) defines a 2-sphere $y_1^2 + y_2^2 + y_3^2 = r^2$, which has positive radius $r = \big(2I_1\big(h^a - (\chi x_3 + \frac{1}{2}I_1^{-1}a^2)\big)\big)^{1/2}$, since $h^a > \chi + \frac{1}{2}I_1^{-1}a^2 \ge \chi x_3 + \frac{1}{2}I_1^{-1}a^2$. Intersecting this 2-sphere with the 2-plane (80a) defines a circle which is tangent to S^2 at x. This circle is the fiber over x of the projection map $TS^2 \rightarrow S^2 : (x,y) \mapsto x$. Hence $\varphi\big((H^a)^{-1}(h^a)\big)$ is diffeomorphic to T_1S^2. Thus $(H^a)^{-1}(h^a)$ is diffeomorphic to T_1S^2.

3. At the critical value $h^a = \chi + \frac{1}{2}I_1^{-1}a^2$, the h^a-level set of H^a is the algebraic variety U, which is defined by

$$z_1^2 + z_2^2 + z_3^2 = 1$$

$$z_1w_1 + z_2w_2 + z_3w_3 = a$$

$$\tfrac{1}{2}I_1^{-1}(w_1^2 + w_2^2 + w_3^2) + \chi z_3 = \chi + \tfrac{1}{2}I_1^{-1}a^2,$$

which is singular only at $p_1 = (e_3, ae_3)$. Because p_1 is a nondegenerate critical point of Morse index 2, the level set $(H^a)^{-1}(h^a)$ is locally diffeomorphic to the zero level set of $D^2H^a(p_1)$, that is, to a neighborhood of the vertex of the cone defined by $z_1^2 - z_2^2 + w_1^2 - w_2^2 = 0$. Thus p_1 is a conical singularity of U. The variety U is homeomorphic to the

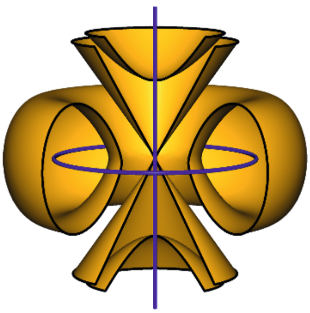


Figure 5.2.1. The variety U in S^3.

3-sphere S^3, see figure 5.2.1. To see this we think of S^3 as $\mathbf{R}^3$ with a point added at infinity. The upper and lower parts of the solid cone form the singular closed solid torus ST_+, whose boundary T^*_+ is the singular 2-torus with a longitudinal circle pinched to a point, which forms the vertex of the cone. The exterior of the solid cone is again a solid cone which forms the singular closed solid torus ST_-, whose boundary T^*_- is the singular solid torus with meridial circle pinched to a point, which is the vertex of the cone. S^3 is the union of the solid tori $ST_\pm$, see the proof of point 3 of ((3.11)) in chapter IV.

This completes the verification of table 5.2.2. □

From table 5.2.2 we see that as the value of h^a increases through the critical value $\chi + \frac{1}{2}I_1^{-1}a^2$, the topology of the h^a-level set of H^a changes, see figure 3.4.4 in chapter IV. This bifurcation is due to monodromy about the isolated critical value $(\chi + \frac{1}{2}I_1^{-1}a^2, a)$ of the energy momentum mapping EM^a when $|a| < 2\sqrt{\chi I_1}$, see figure 5.1.4. The same argument used to compute the monodromy in the spherical pendulum, see chapter IV ((5.1)), shows that for a small circle S^1 in the set of regular values of EM^a with center at the isolated critical value the 2-torus bundle $(EM^a)^{-1}(S^1)$ is *nontrivial*.

We now discuss how the level sets of angular momentum foliate an energy level set. For $h^a \in (-\chi + \frac{1}{2}I_1^{-1}a^2, \chi + \frac{1}{2}I_1^{-1}a^2)$ the energy surface $(H^a)^{-1}(h^a) = S^3$ is foliated by 2-tori $(J^a_\ell)^{-1}(b) \cap S^3$ as in the harmonic oscillator. When $h^a > \chi + \frac{1}{2}I_1^{-1}a^2$ the energy surface $(H^a)^{-1}(h^a) = \mathbf{RP}^3$ is foliated by 2-tori so that its twofold covering is the same as the foliation of S^3 by 2-tori of the harmonic oscillator, see figure 5.2 in chapter III.

5.3 Motion of the tip of the figure axis

In this subsection we discuss the qualitative behavior of the integral curves of the Euler-Poisson vector field X_{H^a} (26) on the smooth 2-torus $T^2_{h^a,b} = (EM^a)^{-1}(h^a, b)$. Here (h^a, b) is a regular value of the energy momentum mapping EM^a (70). Thinking of the reduced space P^a, defined by $z_1^2 + z_2^2 + z_3^2 = 1$ and $z_1w_1 + z_2w_2 + z_3w_3 = a$, as a bundle over S^2 with bundle projection map

$$\tau : P^a \subseteq \mathbf{R}^3 \times \mathbf{R}^3 \to S^2 \subseteq \mathbf{R}^3 : (z, w) \mapsto z, \tag{81}$$

we can view the 2-torus $T^2_{h^a,b} \subseteq P^a$ as a bundle over $\mathrm{R} = \tau(T^2_{h^a,b}) \subseteq S^2$. Physically, the image of an integral curve of $X_{H^a}|T^2_{h^a,b}$ under the projection τ is the curve in space traced out by the tip of the figure axis of the Lagrange top. We classify the possible motions of the tip.

We will begin by describing the various subsets R of S^2 which are the image of $\tau|T^2_{h^a,b}$.

Claim: Let $z_3^\pm$ be roots of the polynomial V, see (83) below, which lie in $[-1, 1]$. Then R is one of the following sets.

1. When $b \neq \pm a$ & $a \neq 0$, R is the closed annulus $\mathscr{B} = \{z \in S^2 \subseteq \mathbf{R}^3 \,\big|\, -1 < z_3^- \le z_3 \le z_3^+ < 1\}$ with boundary $\partial\mathscr{B} = \{z_3 = z_3^\pm\}$.

2. When $b=-a$ & $a \neq 0$, R is the closed 2-disk $\overline{D}^2_- = \{z \in S^2 \subseteq \mathbf{R}^3 \,\big|\, -1 \le z_3 \le z_3^+ < 1\}$, which contains the south pole sp $=(0,0,-1)$ of S^2, but not the north pole np $=(0,0,1)$, and has boundary $\partial D_- = \{z_3 = z_3^+\}$.

3. When $b=a$ & $a \neq 0$, R is the closed 2-disk $\overline{D}^2_+ = \{z \in S^2 \subseteq \mathbf{R}^3 \,\big|\, -1 < z_3^- \le z_3 \le 1\}$, which contains the north pole np $=(0,0,1)$ of S^2, but not the south pole sp $=(0,0,-1)$, and has boundary $\partial D_+ = \{z_3 = z_3^-\}$.

4. When $b=a=0$, R is all of S^2.

(5.7) **Proof**: By construction, the 2-torus $T^2_{h^a,b} \subseteq \mathbf{R}^3 \times \mathbf{R}^3$ is given by

$$z_1^2+z_2^2+z_3^2 = 1 \tag{82a}$$

$$z_1w_1+z_2w_2+z_3w_3 = a \tag{82b}$$

$$\tfrac{1}{2}I_1^{-1}(w_1^2+w_2^2+w_3^2)+\chi z_3 = h^a \tag{82c}$$

$$w_3 = b. \tag{82d}$$

Substituting (81a) – (81d) into $(z_1w_2-z_2w_1)^2 = (z_1^2+z_2^2)(w_1^2+w_2^2)-(z_1w_1+z_2w_2)^2$ gives

$$0 \le (z_1w_2-z_2w_1)^2 = 2(\alpha-\beta z_3)(1-z_3^2)-(a-bz_3)^2 = V(z_3) \tag{83}$$

with $|z_3| \le 1$. Here $\beta = I_1\chi$ and $\alpha = I_1(h^a - \frac{1}{2}b^2) = I_1 h^a_b$. Since (h^a,b) is a regular value in the image of EM^a, the polynomial V has two simple roots $z_3^{\pm}$ in $[-1,1]$, see table 5.1.3. Therefore, $\tau(T^2_{h^a,b}) \subseteq \mathrm{R} = \{z \in S^2 \,\big|\, -1 \le z_3^- \le z_3 \le z_3^+ \le 1\}$.

To finish the argument it suffices to show that $\mathrm{R} \subseteq \tau(T^2_{h^a,b})$. A nice way to do this, which also explains the geometry of the projection mapping $\tau|T^2_{h^a,b}$, is to apply the diffeomorphism

$$\begin{aligned}&\varphi: T^2_{h^a,b} \subseteq \mathrm{R} \times \mathbf{R}^3 \to \mathrm{R} \times \mathbf{R}^3 :\\ &\quad (z,w) \mapsto (\xi,\eta) = \big(z_1,z_2,z_3,-w_2+\mu(z_3)z_2, w_1-\mu(z_3)z_1, w_3\big)\end{aligned} \tag{84}$$

where

$$\mu(z_3) = \begin{cases} (a-bz_3)(1-z_3^2)^{-1}, \text{ if } b \neq \pm a \text{ and } a \neq 0 \\ a(1+z_3)^{-1}, \text{ if } b=a \text{ and } a \neq 0 \\ a(1-z_3)^{-1}, \text{ if } b=-a \text{ and } a \neq 0 \\ 0, \text{ if } b=a=0. \end{cases}$$

A calculation shows that $\varphi(T^2_{h^a,b})$ is the 2-torus T^2 in $\mathbf{R}^3 \times \mathbf{R}^3$ with coordinates (ξ,η), which is defined by

$$\xi_1^2+\xi_2^2 = 1-\xi_3^2, \tag{85a}$$

$$\xi_1\eta_2-\xi_2\eta_1 = 0 \tag{85b}$$

$$\eta_1^2+\eta_2^2 = \nu(\xi_3) \tag{85c}$$

$$\eta_3 = b, \tag{85d}$$

where

$$\nu(\xi_3) = \begin{cases} V(\xi_3)(1-\xi_3^2)^{-1}, \text{ if } b \neq \pm a \text{ and } a \neq 0 \\ \left[2(\alpha - \beta\xi_3)(1+\xi_3) - a^2(1-\xi_3)\right](1+\xi_3)^{-1}, \text{ if } b = a \text{ and } a \neq 0 \\ \left[2(\alpha - \beta\xi_3)(1-\xi_3) - a^2(1+\xi_3)\right](1-\xi_3)^{-1}, \text{ if } b = -a \text{ and } a \neq 0 \\ 2(\alpha - \beta\xi_3), \text{ if } b = a = 0. \end{cases}$$

Let π be the projection $T^2 \subseteq \mathrm{R} \times \mathbf{R}^3 \to \mathrm{R} : (\xi, \eta) \mapsto \xi$. For $\xi \in \mathrm{R}$, the fiber $\pi^{-1}(\xi)$ is diffeomorphic to the fiber $\tau^{-1}(\varphi^{-1}(\xi))$.

Since $\nu(\xi_3) \geq 0$, it follows that $\xi_3 \in [\xi_3^-, \xi_3^+]$, where $V(\xi_3^\pm) = 0$, $\xi_3^- < \xi_3^+$, and $V(\xi_3) > 0$ when $\xi_3 \in (\xi_3^-, \xi_3^+)$, Therefore $\pi(T^2) \subseteq \mathrm{R} = \{\xi \in S^2 \subseteq \mathbf{R}^3 \,\big|\, \xi \in [\xi_3^-, \xi_3^+]\}$. Suppose that $\xi \in \operatorname{int}\mathrm{R}$, the interior of R. Solving (85b) and (85c) gives

$$\eta_1 = \varepsilon\Big(\frac{\nu(\xi_3)}{1-\xi_3^2}\Big)^{1/2}\xi_1 \quad \text{and} \quad \eta_2 = \varepsilon\Big(\frac{\nu(\xi_3)}{1-\xi_3^2}\Big)^{1/2}\xi_2, \tag{86}$$

where $\varepsilon^2 = 1$. Thus when $\xi \in \operatorname{int}\mathrm{R}$, the fiber $\pi^{-1}(\xi) = (\xi, \eta_1, \eta_2, b)$, where η_1 and η_2 are given by (86).

1. When $b \neq \pm a$ & $a \neq 0$, we have $|\xi_3| < 1$. Therefore R is a closed annulus $\mathscr{B}$ in S^2 with boundary $\partial\mathrm{R} = \{(\xi_1, \xi_2, \xi_3^\pm) \in S^2\}$, which is the disjoint union of two circles: C', where $\xi_3 = \xi_3^+$ and B', where $\xi_3 = \xi_3^-$. Because $\nu(\xi_3^\pm) = 0$, equation (85c) implies that $\eta_1 = \eta_2 = 0$. Equation (85d) gives $\eta_3 = b$. Therefore $\partial(\pi^{-1}(R)) = \{(\xi_1, \xi_2, \xi_3^\pm, 0, 0, b) \in T^2\}$ is the disjoint union of two circles: $C = \pi^{-1}(C')$, when $\xi_3 = \xi_3^+$ and $B = \pi^{-1}(B')$, when $\xi_3 = \xi_3^-$. Each circle is an orbit of the vector field $\varphi_* X_{J_\ell^a}$ on T^2.

2. When $b = -a$ & $a \neq 0$, we have $\mathrm{sp} = (0,0,-1) \in \mathrm{R}$ but $\mathrm{np} = (0,0,1) \notin \mathrm{R}$, since $\xi_3^- = 1$ but $-1 < \xi_3^+ < 1$. Thus R is a closed 2-disk $\overline{D}^2_-$ with boundary $C' = \{(\xi_1, \xi_2, \xi_3^+) \in S^2\}$. Because $\nu(\xi_3^+) = 0$, it follows that $\pi^{-1}(C')$ is the circle $C = \{(\xi_1, \xi_2, \xi_3^+, 0, 0, b) \in T^2 \,\big|\, \xi_1^2 + \xi_2^2 = 1 - (\xi_3^+)^2 > 0\}$.

3. When $b = a$ & $a \neq 0$, we have $\mathrm{np} = (0,0,1) \in \mathrm{R}$ but $\mathrm{sp} = (0,0,-1) \notin \mathrm{R}$, because $\xi_3^+ = 1$ but $-1 < \xi_3^- < 1$. Thus R is a closed 2-disk $\overline{D}^2_+$ with boundary $B' = \{(\xi_1, \xi_2, \xi_3^-) \in S^2\}$. Because $\nu(\xi_3^-) = 0$, it follows that $\pi^{-1}(B')$ is the circle $B = \{(\xi_1, \xi_2, \xi_3^-, 0, 0, b) \in T^2 \,\big|\, \xi_1^2 + \xi_2^2 = 1 - (\xi_3^-)^2 > 0\}$.

4. When $b = a = 0$, both np and sp lie in R. So $\mathrm{R} = S^2$. □

▷ We now reconstruct the 2-torus T^2 from its image R under the projection map π.

(5.8) **Proof**:

CASE 1. $b \neq \pm a$ & $a \neq 0$. Then R is a closed annulus $\mathscr{B}$ with boundary $\partial\mathrm{R}$ which is two disjoint small circles $B' = \{\xi_3 = \xi_3^-\} \cap S^2$ and $C' = \{\xi_3 = \xi_3^+\} \cap S^2$ on S^2. If $\xi \in \operatorname{int}\mathrm{R}$, then $\pi^{-1}(\xi)$ is two points; while if $\xi \in \partial\mathrm{R}$, then $\pi^{-1}(\xi)$ is a point. In other words, T^2 has a fold singularity over $\partial\mathrm{R}$ with fold curve $\pi^{-1}(\partial\mathrm{R})$, which is the disjoint union of two circles: $C = \{(\xi_1, \xi_2, \xi_3^+, \eta_1, \eta_2, b) \in T^2 \,\big|\, \xi_1^2 + \xi_3^2 = 1 - (\xi_3^+)^2 \text{ \& } \eta_1^2 + \eta_2^2 = \nu(\xi_3^+)\}$ and $B = \{(\xi_1, \xi_2, \xi_3^-, \eta_1, \eta_2, b) \in T^2 \,\big|\, \xi_1^2 + \xi_3^2 = 1 - (\xi_3^-)^2 \text{ \& } \eta_1^2 + \eta_2^2 = \nu(\xi_3^-)\}$.

Consequently, T^2 is an S^0 bundle over $\mathrm{int}\,\mathrm{R}$ with S^0 pinched to a point over each point of $\partial\mathrm{R}$. Let A' be the *open* arc $\{(0,(1-\xi_3^2)^{1/2},\xi_3) \in S^2 \,\big|\, \xi_3^- < \xi_3 < \xi_3^+\}$. Then the closure $\overline{\pi^{-1}(A')}$ of $\pi^{-1}(A')$ in T^2 is a circle A, because for every $\xi \in A' \subseteq \mathrm{int}\,\mathrm{R}$ the fiber $\pi^{-1}(\xi)$ is two points; whereas for $(0,0,\xi_3^{\pm}) \in (\overline{A'} \setminus A') \subseteq \partial R$ the fiber $\pi^{-1}(0,0,\xi_3^+)$ is the point $q = (0,0,\xi_3^+,0,0,b)$; while the fiber $\pi^{-1}(0,0,\xi_3^-)$ is the point $\widetilde{q} = (0,0,\xi_3^-,0,0,b)$. Since $\pi(q) = q'$ and $\pi(\widetilde{q}) = \widetilde{q}'$, we get $\widetilde{q} \in B$ and $q \in C$. So B and C are circles on T^2, which are homologous. Moreover, either $\{A,B\}$ or $\{A,C\}$ is a basis of $\mathrm{H}_1(T^2,\mathbf{Z})$.

CASE 2. When $b = -a \;\&\; a \neq 0$. Then R is a closed 2-disk $\overline{D}_+^2$ with boundary the circle $C' = \{(\xi_1,\xi_2,\xi_3^+) \in S^2 \,\big|\, \xi_1^2 + \xi_2^2 = 1 - (\xi_3^-)^2 > 0\}$, which contains the south pole $\widetilde{q}' = (0,0,-1)$. Let A' be the open arc $\{(0,(1-\xi_3^2)^{1/2},\xi_3) \in S^2 \,\big|\, -1 < \xi_3 < \xi_3^+ < 1\}$. For each $\xi \in A' \subseteq \mathrm{int}\,\mathrm{R}$ the fiber $\pi^{-1}(\xi)$ is two points, namely, $(0,(1-\xi_3^2)^{1/2},\xi_3,\eta_1,\eta_2,b)$, where for $i = 1,2$ we have $\eta_i = \varepsilon\big(\nu(\xi_3)(1-\xi_3^2)^{-1}\big)^{1/2}\xi_i$ and $\varepsilon^2 = 1$. When $\xi = (0,(1-(\xi_3^+)^2)^{1/2},\xi_3^+)$, the fiber $\pi^{-1}(\xi)$ is the point $q = (0,(1-(\xi_3^+)^2)^{1/2},\xi_3^+,0,0,b)$ on C. We now find the rest of $\overline{\pi^{-1}(A')}$. The tangent to the curve $(-1,\xi_3^+) \to S^2 : \xi_3 \mapsto (0,(1-\xi_3^2)^{1/2},\xi_3)$, which parametrizes the open arc A', is $(0,-\xi_3(1-\xi_3^2)^{-1/2},1)$. The corresponding positive tangent ray of length $\sqrt{2(\alpha+\beta)} = \nu(-1)$ to A' at $\widetilde{q}'$ is $(0,\sqrt{2(\alpha+\beta)},0)$. Thus the corresponding affine ray at $\widetilde{q}'$ is $(0,\sqrt{2(\alpha+\beta)},b)$, which is associated to the point $\widetilde{q} = (0,0,-1,0,\sqrt{2(\alpha+\beta)},b)$ on the circle $B = \{(0,0,-1,\eta_1,\eta_2,b) \in T^2 \,\big|\, \eta_1^2 + \eta_2^2 = \nu(-1) = 2(\alpha+\beta) > 0\}$. Consequently, $A = \{q,\widetilde{q}\} \cup \pi^{-1}(A')$ is a circle on T^2, which intersects the circle B only at the point $\widetilde{q}$. At $\widetilde{q}' = (0,0,-1)$ the fiber $\pi^{-1}(\widetilde{q}')$ is the circle B. Thus the map π blows up the point $\widetilde{q}'$ to a circle B. Note that $\overline{\pi^{-1}(A')} = A \cup B$ and that $\{A,B\}$ is a basis for $\mathrm{H}_1(T^2,\mathbf{Z})$ as is $\{A,C\}$, since $A \cap C = \{q\}$.

CASE 3. $b = a \;\&\; a \neq 0$. Then R is a closed 2-disk $\overline{D}_+^2$ with boundary the circle $B' = \{(\xi_1,\xi_2,\xi_3^-) \in S^2 \,\big|\, \xi_1^2 + \xi_2^2 = 1 - (\xi_3^-)^2 > 0\}$, which contains the north pole $q' = (0,0,1)$. Let A' be the open arc $\{(0,(1-\xi_3^2)^{1/2},\xi_3) \in S^2 \,\big|\, -1 < \xi_3^- < \xi_3 < 1\}$. For each $\xi \in A' \subseteq \mathrm{int}\,\mathrm{R}$ the fiber $\pi^{-1}(\xi)$ is the two points $(0,(1-\xi_3^2)^{1/2},\xi_3,\eta_1,\eta_2,b)$, where for $i = 1,2$ $\eta_i = \varepsilon\big(\nu(\xi_3)(1-\xi_3^2)^{-1}\big)^{1/2}\xi_i$ and $\varepsilon^2 = 1$. When $\xi = (0,(1-(\xi_3^-)^2)^{1/2},\xi_3^-)$, the fiber $\pi^{-1}(\xi)$ is the point $\widetilde{q} = (0,(1-(\xi_3^-)^2)^{1/2},\xi_3^-,0,0,b)$ on the circle B. We now find the rest of $\overline{\pi^{-1}(A')}$. The tangent to the curve $(-1,\xi_3^+) \to S^2 : \xi_3 \mapsto (0,(1-\xi_3^2)^{1/2},\xi_3)$, which parametrizes the open arc A', is $(0,-\xi_3(1-\xi_3^2)^{-1/2},1)$. The corresponding positive tangent ray of length $\sqrt{2(\alpha-\beta)} = \nu(1)$ to A' at q' is $(0,\sqrt{2(\alpha-\beta)},0)$. Thus the corresponding affine ray at q' is $(0,\sqrt{2(\alpha-\beta)},b)$, which is associated to the point $q = (0,0,-1,0,\sqrt{2(\alpha-\beta)},b)$ on the circle $C = \{(0,0,-1,\eta_1,\eta_2,b) \in T^2 \,\big|\, \eta_1^2 + \eta_2^2 = \nu(1) = 2(\alpha-\beta) > 0\}$. Consequently, $A = \{q,\widetilde{q}\} \cup \pi^{-1}(A')$ is a circle on T^2, which intersects the circle B only at the point $\widetilde{q}$. At $q' = (0,0,1)$ the fiber $\pi^{-1}(q')$ is the circle C. Thus the map π blows up the point q' to a circle C. Note that $\overline{\pi^{-1}(A')} = A \cup B$ and that $\{A,B\}$ is a basis for $\mathrm{H}_1(T^2,\mathbf{Z})$ as is $\{A,C\}$, since $A \cap C = \{q\}$.

CASE 4. $b = a = 0$. Then R is the 2-sphere S^2 with north pole $q' = (0,0,1)$ and south pole $\widetilde{q}' = (0,0,-1)$. Let A' be the open arc $\{(0,(1-\xi_3^2)^{1/2},\xi_3) \in S^2 \,\big|\, -1 < \xi_3 < 1\}$. For $\xi \in A' \subseteq \mathrm{int}\,\mathrm{R}$, the fiber $\pi^{-1}(\xi)$ is two distinct points. At q' the fiber $\pi^{-1}(q')$ is the circle $C = \{(0,0,-1,\eta_1,\eta_2,b) \in T^2 \,\big|\, \eta_1^2 + \eta_2^2 = \nu(-1) = 2(\alpha+\beta) > 0\}$; while at

$\widetilde{q}' = (0,0,-1)$ the fiber $\pi^{-1}(\widetilde{q}')$ is the circle $B = \{(0,0,-1,\eta_1,\eta_2,b) \in T^2 \,\big|\, \eta_1^2+\eta_2^2 = \nu(1) = 2(\alpha-\beta) > 0\}$. The map π blows up the point q' to the circle C and the point $\widetilde{q}'$ to the circle B. Thus $A = \{q,\widetilde{q}\} \cup \pi^{-1}(A')$ is a circle on T^2, which intersects the circle C only at q and intersects the circle B only at $\widetilde{q}$. Thus $\overline{\pi^{-1}(A'} = A \cup B \cup C$. Moreover, $\{A,C\}$ and $\{A,B\}$ form a basis of $\mathrm{H}_1(T^2,\mathbf{Z})$. $\square$

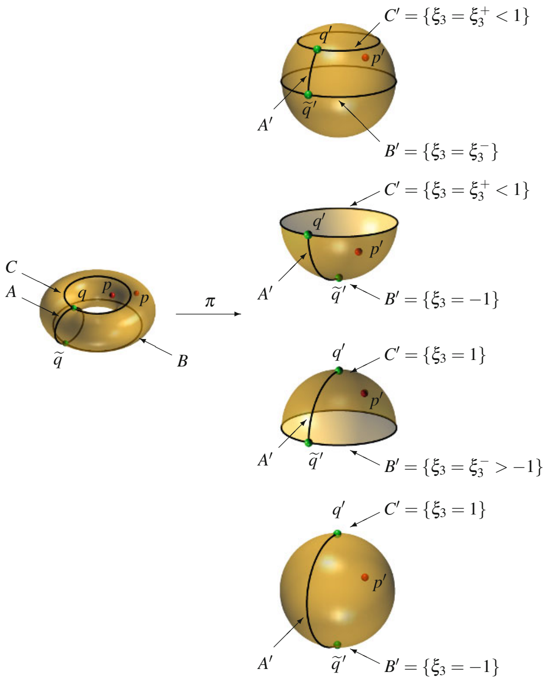


Figure 5.3.1. The image R of the 2-torus $T^2 = \varphi(T^2_{h^a,b})$ under the projection map π. Here (h^a,b) is a regular value of EM^a. In the top right figure $b \neq \pm a$ & $a \neq 0$; in the next to top right figure $b = a \pm a$ & $a \neq 0$; in the next to bottom right figure $b = -a \pm a$ & $a \neq 0$; in the bottom right figure $b \neq a = 0$.

Next we look at the projection of the vector field $X_{H^a}|T^2_{h^a,b}$ on R under the bundle projection τ (81). Consider the strip $\widetilde{\mathscr{A}} = [\sigma^-,\sigma^+] \times \mathbf{R}$ with coordinates (σ,φ) where $z = (z_1,z_2,z_3) \in \mathrm{R} \subseteq S^2$, $\sigma = z_3$, $\sigma^\pm = z_3^\pm$, and $\varphi = \tan^{-1}\frac{z_1}{z_2}$. Then $\widetilde{\mathscr{A}}$ is the universal covering space of $\mathscr{A}$ with covering map $\widetilde{\pi} : \widetilde{\mathscr{A}} \subseteq \mathbf{R}^2 \to \mathscr{A} \subseteq S^2 : (\sigma,\varphi) \mapsto z = \big(\cos\varphi, \sin\varphi, \sigma\big)^t$. Here $\mathscr{A}$ is either the closed annulus $\mathscr{B}$, the blow up of the closed 2-disk $\overline{D}^2_\pm$, or S^2 blown up at the north or south pole. Let $p \in T^2_{h^a,b}$ and let $\Gamma : t \mapsto \varphi_t^{H^a}(p)$ be the integral curve of $X_{H^a}|T^2_{h^a,b}$ through p. Set $\gamma : \mathbf{R} \to \mathscr{A} : t \mapsto \tau(\Gamma(t))$. Let $\widetilde{\gamma} : \mathbf{R} \to \widetilde{\mathscr{A}} : t \mapsto \widetilde{\tau}(\Gamma(t))$ be the lift of γ to $\widetilde{\mathscr{A}}$. Here $\widetilde{\tau} : P^a \to \widetilde{\mathscr{A}}$ is the map defined by $\widetilde{\pi} \circ \widetilde{\tau} = \tau$. So $\widetilde{\pi} \circ \widetilde{\gamma} = \gamma$.

Claim: The curve $\widetilde{\gamma}: \mathbf{R} \to \widetilde{\mathscr{A}}: t \mapsto \big(\sigma(t), \varphi(t)\big)$ satisfies

$$\dot{\sigma} = \eta I_1^{-1}\sqrt{V(\sigma)}, \quad \text{with } \eta^2 = 1 \tag{87a}$$

$$\dot{\varphi} = I_1^{-1}(b - a\sigma)(1-\sigma^2)^{-1} \tag{87b}$$

where V is given by (83).

(5.9) **Proof**: We compute. Equation (87a) is obtained as follows:

$$\begin{aligned} \dot{\sigma} &= \tau\big(L_{X_{H^a}|T^2_{h^a,b}} z_3\big) = \tau\big((L_{X_{H^a}} z_3)|T^2_{h^a,b}\big) = \tau\big(\big(I_1^{-1}(z_2 w_1 - z_1 w_2)\big)|T^2_{h^a,b}\big), \\ &\qquad \text{using the Euler-Poisson equations } \dot{z} = I_1^{-1} w \times z,\ \dot{w} = \chi e_3 \times z \text{ for } H^a \\ &= \eta I_1^{-1}\sqrt{V(\tau(z_3))}, \quad \text{using (83)} \\ &= \eta I_1^{-1}\sqrt{V(\sigma)}. \end{aligned}$$

Equation (87b) is obtained as follows.

$$\begin{aligned} \dot{\varphi} &= \tau\big(L_{X_{H^a}|T^2_{h^a,b}} \varphi\big) = \tau\big((z_1\dot{z}_2 - z_2\dot{z}_1)(z_1^2+z_2^2)^{-1}\big), \quad \text{by definition of } \varphi \\ &= \tau\Big(I_1^{-1}\big(w_3(z_1^2+z_2^2) - z_3(z_1w_1+z_2w_2)\big)(z_1^2+z_2^2)^{-1}\Big), \\ &\qquad \text{using the Euler-Poisson equations for } H^a \\ &= I_1^{-1}(b-a\sigma)(1-\sigma^2)^{-1}, \ \text{since } \Gamma \text{ lies on } T^2_{h^a,b} \text{ (82a) – (82d).} \end{aligned}$$

$\square$

The sign ambiguity in (87a) is handled in the following way. The integral curve Γ crosses each of the curves $\mathscr{C}^{\pm} = \tau^{-1}(\{z_3 = \sigma^{\pm}\})$ transversely, because $\mathscr{C}^{\pm}$ is an integral curve of $X_{J^a_\ell}|T^2_{h^a,b}$ and the vector fields X_{H^a} and $X_{J^a_\ell}$ are linearly independent on $T^2_{h^a,b}$. Thus the curve $t \mapsto \widetilde{\gamma}(t) = \big(\sigma(t), \varphi(t)\big)$ does not stop when it reaches the boundary of $\widetilde{\mathscr{A}}$, say at time t_0. In other words, $\frac{\mathrm{d}\widetilde{\gamma}}{\mathrm{d}t}(t_0) \neq 0$.

Claim: Suppose that $\frac{\mathrm{d}\widetilde{\gamma}}{\mathrm{d}t}(t_0) \neq 0$ and $q_0 = \widetilde{\gamma}(t_0) \in \partial\widetilde{\mathscr{A}}$, while $\widetilde{\gamma}(t)$ lies in the interior $\operatorname{int}\widetilde{\mathscr{A}}$ of $\widetilde{\mathscr{A}}$ for all t slightly smaller than t_0. Then one of the following holds.

1. $\dot{\varphi}(t_0) = 0$. Then $\frac{\mathrm{d}\widetilde{\gamma}}{\mathrm{d}t}(t_0)$ is an outward normal to $\partial\widetilde{\mathscr{A}}$ at $\widetilde{\gamma}(t_0)$.
2. $\dot{\varphi}(t_0) \neq 0$. Then $\frac{\mathrm{d}\widetilde{\gamma}}{\mathrm{d}t}(t_0)$ is tangent to $\partial\widetilde{\mathscr{A}}$ at $\widetilde{\gamma}(t_0)$.

(5.10) **Proof**:

1. Since $\frac{\mathrm{d}\widetilde{\gamma}}{\mathrm{d}t} = (\dot{\sigma}, \dot{\varphi})$ is nonzero at t_0 and $\dot{\varphi}(t_0) = 0$, it follows that $\dot{\sigma}(t_0) \neq 0$.

2. Observe that continuity implies that $\dot{\varphi} \neq 0$ for points in $\widetilde{\mathscr{A}}$ near q_0. Hence near q_0 we may parametrize $\widetilde{\gamma}$ by φ instead of t. Using (87a) and (87b) we obtain

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\varphi} = \frac{\dot{\sigma}}{\dot{\varphi}} = \eta\,\frac{\sqrt{V(\sigma)}}{I_1\dot{\varphi}}.$$

Therefore $\frac{\mathrm{d}\sigma}{\mathrm{d}\varphi} = 0$ at q_0. $\square$

Suppose that $\widetilde{\gamma}(t_0) = \sigma^-$ or σ^+ and for some sufficiently small $\varepsilon > 0$ the value of η at time $t_0 - \varepsilon$ is known. Then at time $t_0 + \varepsilon$ the value of η is defined to be the negative of its value at $t_0 - \varepsilon$. From ((5.9)) it follows that this *sign convention* ensures that the image of $\widetilde{\gamma}$ lies in $\widetilde{\mathscr{A}}$ for all time and is continuous. With this convention in force, equations (87a) and (87b) do *not* define a vector field on $\widetilde{\mathscr{A}}$, because $\widetilde{\gamma}$ may have self intersections. Even worse, $\widetilde{\gamma}$ may *not* be smooth, see figure 5.3.6. When $\widetilde{\mathscr{A}}$ is the universal covering space of the closed annulus $\mathscr{B}$, the fiber $\tau^{-1}(\partial\mathscr{B})$ is the union of two fold curves $\mathscr{C}^{\pm}$ of $\tau|T^2_{h^a,b}$. In this case, $\widetilde{\gamma}$ has second order contact with $\partial\widetilde{\mathscr{A}}$ at $\widetilde{\gamma}(t_0)$ and is real analytic, although it still may have self intersections, see figure 5.3.3.

We now determine the rotation number Θ of the flow of X_{H^a} restricted to the 2-torus $T^2_{h^a,b}$. The closed curve $\mathscr{C} = \mathscr{C}^- = \tau^{-1}(\{\sigma = \sigma^-\})$ is the image of a periodic integral curve of $X_{J^a_\ell}$ on $T^2_{h^a,b} \subseteq (J^a_\ell)^{-1}(b)$, since $X_{J^a_\ell}$ is the infinitesimal generator of the S^1-action $\Delta|(S^1 \times (J^a_\ell)^{-1}(b))$, which leaves the function $z_3|T^2_{h^a,b}$ invariant. Because the vector fields X_{H^a} and $X_{J^a_\ell}$ are linearly independent at each point of $T^2_{h^a,b}$, it follows that $\mathscr{C}$ is a cross section for the flow $\varphi^{H^a}_t$ of $X_{H^a}|T^2_{h^a,b}$. Thus for any $p \in \mathscr{C}$ there is a smallest $T = T(h^a,b) > 0$ such that $\varphi^{H^a}_T(p) \in \mathscr{C}$. Let $\varphi^{J^a_\ell}_s$ be the flow of $X_{J^a_\ell}$. By definition of the rotation number Θ we have $\varphi^{J^a_\ell}_{2\pi\Theta}(p) = \varphi^{H^a}_T(p)$. Because the flows $\varphi^{H^a}|T^2_{h,\ell_t}$ and $\varphi^{J^a_\ell}|T^2_{h,\ell_s}$ commute with the S^1-action $\Delta|(S^1 \times T^2_{h,\ell})$, the rotation number Θ does not depend on the choice of the point p on the cross section $\mathscr{C}$ nor does it depend on the choice of integral curve of $X_{J^a_\ell}|T^2_{h,\ell}$ giving the cross section. Therefore Θ is a function of h and ℓ alone.

From the choice of the cross section $\mathscr{C}$ it follows that $\tau(\mathscr{C})$ is the boundary component of the closed annulus $\mathscr{A}$ corresponding to the lower boundary $\{\sigma = \sigma^-\}$ of the closed strip $\widetilde{\mathscr{A}}$. Let $\widehat{\Gamma}: [0,2\pi] \to T^2_{h^a_b,b} : s \mapsto \varphi^{J^a_\ell}_s(p)$. The lift of $\tau\circ\widehat{\Gamma}$ to $\widetilde{\mathscr{A}}$ parametrizes the line segment on $\{\sigma = \sigma^-\}$ which joins $\widetilde{\gamma}(0)$ to $\widetilde{\gamma}(T)$. Let ϑ be the difference in the φ-coordinates of $\widetilde{\gamma}(T)$ and $\widetilde{\gamma}(0)$.

Claim: $\vartheta/2\pi$ is equal to the rotation number Θ.

(5.11) **Proof**: We compute.

$$\vartheta/2\pi = \int_0^{\vartheta/2\pi} \mathrm{d}\varphi = \int_0^{\Theta} (L_{X_{J^a_\ell}}\varphi)\,\mathrm{d}s = \int_0^{\Theta} \mathrm{d}s = \Theta.$$

The second to last equality above follows from the Euler-Poisson equations $\dot{z} = e_3 \times z$ and $\dot{w} = e_3 \times w$ for $X_{J^a_\ell}$ and the definition of the coordinate φ. □

▷ Here is an explicit formula for the rotation number Θ.

$$2\pi\Theta = 2\int_{\sigma^-}^{\sigma^+} \frac{b - a\sigma}{(1-\sigma^2)\sqrt{V(\sigma)}}\,\mathrm{d}\sigma. \tag{88}$$

(5.12) **Proof**: If σ lies in the interior $\mathrm{int}\,\widetilde{\mathscr{A}}$ of $\widetilde{\mathscr{A}}$, then $V(\sigma) > 0$. Therefore $\dot{\sigma} \neq 0$ for every $\sigma \in \mathrm{int}\,\widetilde{\mathscr{A}}$. Suppose that $\sigma(0) = \sigma^-$. Then there is a $t' > 0$ such that $\sigma(t') = \sigma^+$. Moreover, we may parametrize $\widetilde{\gamma}$ by σ in (σ^-,σ^+) instead of $t \in (0,t') \cup (t',T)$. In (87a) choose

$\eta = -1$ for $t = \varepsilon$. Then

$$2\pi\Theta = \int_0^T \left(L_{X_{H^a}|T^2_{h^a,b}}\varphi\right) \mathrm{d}t = -\int_{\sigma^+}^{\sigma^-} \frac{\mathrm{d}\varphi}{\mathrm{d}\sigma}\,\mathrm{d}\sigma + \int_{\sigma^-}^{\sigma^+} \frac{\mathrm{d}\varphi}{\mathrm{d}\sigma}\,\mathrm{d}\sigma = 2\int_{\sigma^-}^{\sigma^+} \frac{\dot\varphi}{\dot\sigma}\,\mathrm{d}\sigma,$$

which using (87a) and (87b) gives (88). □

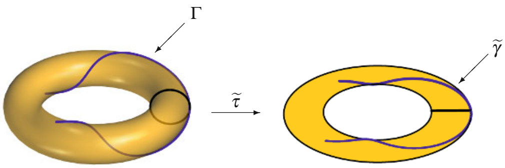


Figure 5.3.2. Tangential singularity.

We now describe the qualitative behavior of a solution $\widetilde{\gamma}$ of (87a) and (87a). We assume that the sign convention is in force. Then $\widetilde{\gamma}$ is defined for all time and lies in $\widetilde{\mathscr{A}}$. If at $q_0 \in \partial\widetilde{\mathscr{A}}$ we have $\dot\varphi \neq 0$, then we say that $\widetilde{\gamma}$ has a *tangential singularity* at q_0. Until further notice we shall assume that $\widetilde{\gamma}$ has only tangential singularities.

To classify the possible qualitative behaviors of $\widetilde{\gamma}$, it suffices to look at a time interval when $\widetilde{\gamma}$ has three successive tangential singularities, say $q_0 = (\sigma^-, \varphi^0) = \widetilde{\gamma}(0)$, $q_1 = (\sigma^+, \varphi^1) = \widetilde{\gamma}(t')$, and $q_2 = (\sigma^-, \varphi^2) = \widetilde{\gamma}(T)$, see figure 5.3.3. The reason why this suffices is that the image of $\widetilde{\gamma}$ in $\widetilde{\mathscr{A}}$ is invariant under the translation mapping

$$\mathrm{trans} : \widetilde{\mathscr{A}} \to \widetilde{\mathscr{A}} : (\sigma, \varphi) \mapsto (\sigma, \varphi + 2\pi\Theta).$$

In more detail, during an elapsed time T the φ-coordinate of $\widetilde{\gamma}$ has increased by $2\pi\Theta$. Hence the image of $\widetilde{\gamma}$ is invariant under the translation map trans.

Claim: The possible qualitative behaviors of $\widetilde{\gamma}$ in $\widetilde{\mathscr{A}}$ with only tangential singularities are given in figure 5.3.3.

(5.13) **Proof**: We assume that $\dot\varphi > 0$ at q_0. The argument when $\dot\varphi < 0$ is similar and is omitted. Since $\dot\sigma \neq 0$ in int$\widetilde{\mathscr{A}}$, the curves $\widetilde{\gamma}_1 = \widetilde{\gamma}|(0,t')$ and $\widetilde{\gamma}_2 = \widetilde{\gamma}|(t',T)$ may be parametrized by σ instead of t. As curves parametrized by σ, $\widetilde{\gamma}_1$ and $\widetilde{\gamma}_2$ are each a graph of a smooth function F_1 and F_2, respectively, which are defined on (σ^-, σ^+).

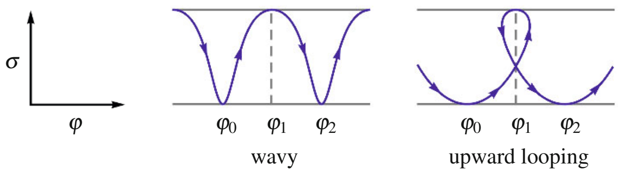


Figure 5.3.3. Possible behaviors of $\tilde\gamma$ having only tangential singularities.

CASE I. Suppose that $\dot\varphi > 0$ at q_1. Then equation (87b) confirms that $\dot\varphi > 0$ throughout $\widetilde{\mathscr{A}}$. Because the sign of $\dot\varphi$ does not change as $\widetilde{\gamma}$ passes through q_1, whereas the sign of $\dot\sigma$ does by our sign convention, the function F_1 is strictly increasing; whereas the function

F_2 is strictly decreasing. Thus the closures of the graphs of F_1 and F_2 intersect only at q_1. Since $\widetilde{\gamma}$ has a tangential singularity at q_1, $\widetilde{\gamma}$ is convex downward at q_1. A similar argument shows that $\widetilde{\gamma}$ is convex upward at q_0 and q_2. Therefore $\gamma|[0,T]$ is a *wave*. Since $\varphi^2 > \varphi^0$, the rotation number $\Theta = (\varphi^2 - \varphi^0)/2\pi$ of $\widetilde{\gamma}$ is positive.

CASE II. Suppose that $\dot{\varphi} < 0$ at q_1. From (87a) and (87b) it follows that $\frac{\mathrm{d}\varphi}{\mathrm{d}\sigma}$ vanishes exactly once in (σ^-, σ^+). Suppose that $a = 0$. From (87b) we see that $\dot{\varphi}$ has the same sign as b throughout $\widetilde{\mathscr{A}}$. But this is contrary to the hypothesis that $\dot{\varphi}$ changes sign in $\widetilde{\mathscr{A}}$. Therefore, $a \neq 0$. Suppose that $a < 0$. From the fact that $\dot{\varphi} > 0$ at q_0 it follows that $b - a\sigma^- > 0$. Therefore $b/a < \sigma^-$. From the fact that $\dot{\varphi} < 0$ at q_1, it follows that $b - a\sigma^+ < 0$. Therefore $b/a > \sigma^+$. This is a contradiction because $\sigma^+ > \sigma^-$. Consequently $a > 0$. Since $\frac{\mathrm{d}\varphi}{\mathrm{d}\sigma}$ has only one zero at $\sigma^* = b/a$, each of the functions F_1 and F_2 has only one critical point at σ^* in (σ^-, σ^+). Now $\frac{\mathrm{d}F_1}{\mathrm{d}\sigma}$ and $\frac{\mathrm{d}F_2}{\mathrm{d}\sigma}$ have opposite signs near q_1. To be specific suppose that $\frac{\mathrm{d}F_1}{\mathrm{d}\sigma} < 0$ near q_1. The argument for the other case is similar to the one given below and is omitted. Then $\frac{\mathrm{d}F_2}{\mathrm{d}\sigma} > 0$ near q_1. It follows that F_1 has a strict maximum at σ^*, whereas F_2 has a strict minimum there. Because $\frac{\mathrm{d}F_1}{\mathrm{d}\sigma} < 0$ and $\frac{\mathrm{d}F_2}{\mathrm{d}\sigma} > 0$ in $(\sigma^*, \sigma^+]$, we find that

$$\begin{aligned}\varphi^1 - F_1(\sigma^*) = F_1(\sigma^+) - F_1(\sigma^*) &= \int_{\sigma^*}^{\sigma^+} \frac{\mathrm{d}F_1}{\mathrm{d}\sigma}\,\mathrm{d}\sigma < \int_{\sigma^*}^{\sigma^+} \frac{\mathrm{d}F_2}{\mathrm{d}\sigma}\,\mathrm{d}\sigma \\ &= F_2(\sigma^+) - F_2(\sigma^*) = \varphi^1 - F_2(\sigma^*),\end{aligned}$$

that is, $F_1(\sigma^*) > F_2(\sigma^*)$. Therefore the point $q^* = (\sigma^*, F_2(\sigma^*))$ does *not* lie on the graph of F_1. There are three possible locations for q_2 relative to the graph of F_1, see figure 5.3.4. Each of the three locations gives rise to a different possible qualitative behavior.

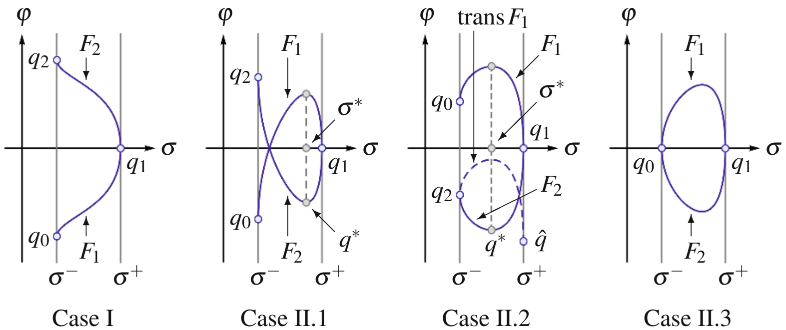


Figure 5.3.4. The geometric situation.

1. Suppose that q_2 lies above the graph of F_1. Then q_2 and q^* lie on opposite sides of the graph of F_1. Thus the graph of $F_2|(\sigma^-, \sigma^*]$ crosses the graph of F_1 at least once. It crosses exactly once because $F_2|(\sigma^-, \sigma^*]$ is strictly monotonic. Therefore in $\widetilde{\mathscr{A}}$ the curve $\widetilde{\gamma}|[0,T]$ makes an *upward pointing loop*. Note that the rotation number $\Theta = (\varphi^2 - \varphi^0)/2\pi$ of $\widetilde{\gamma}$ is positive.

2. Suppose that q_2 lies below the graph of F_1. Then q_2 and q^* lie on the same side of the

graph of F_1. Note that $F_1(\sigma^*) > F_1(\sigma^-) = \varphi^0$, while $F_2(\sigma^*) < F_2(\sigma^-) = \varphi^2$. Applying the translation mapping trans to the graph of F_1 gives $\mathrm{trans}\,(q_0) = q_2$ and $\mathrm{trans}\,(q_1) = \widehat{q}$. Since $F_1|(\sigma^-, \sigma^*\,]$ is strictly increasing, so is $\mathrm{trans}\,(F_1|(\sigma^-, \sigma^*\,])$. Because $F_2|(\sigma^-, \sigma^*\,]$ is strictly decreasing and

$$\mathrm{trans}\,(F_1(\sigma^*)) = F_1(\sigma^*) + \big(F_2(\sigma^-) - F_1(\sigma^-)\big) = \big(F_1(\sigma^*) - F_1(\sigma^-)\big) + F_2(\sigma^-) > F_2(\sigma^-),$$

it follows that $\mathrm{trans}\,(F_1(\sigma^*))$ lies above the graph of F_2. Since $F_2(\sigma^+) = q_1 > \mathrm{trans}\,(q_1) = \widehat{q} = \mathrm{trans}\,(F_1(\sigma^+))$, the point $\widehat{q}$ lies below the graph of F_2. Thus we find that the graph of $\mathrm{trans}\,(F_1|[\sigma^*, \sigma^+))$ crosses the graph of F_2 at least once. It crosses exactly once because $F_1|[\sigma^*, \sigma^+)$ is strictly monotonic and hence $\mathrm{trans}\,(F_1|[\sigma^*, \sigma^+))$ is strictly monotonic. Therefore in $\widetilde{\mathscr{A}}$ the curve $\widetilde{\gamma}|[0,T]$ makes a *downward pointing loop*. Note that the rotation number Θ of $\widetilde{\gamma}$ is *negative*.

3. Suppose that q_2 lies on the graph of F_1. Then $q_2 = q_0$. Thus the closures of the graphs of F_1 and F_2 form a smooth closed curve, which does not have any self intersections. Thus $\widetilde{\gamma}|[0,T]$ forms a *wheel*. The rotation number Θ of $\widetilde{\gamma}$ is zero.

▷ We now show that downward looping and wheeling motion do *not* occur, because the rotation number Θ is positive when $b/a \in (-1,1)$ and $a > 0$.

(5.14) **Proof**: Cut the extended complex plane along the real axis from σ^- to σ^+ and then again between σ^0 to ∞ thus forming the cut extended complex plane $\mathbf{C}^\vee$. Let $z - \sigma^{\pm,0} = r_{\pm,0}\mathrm{e}^{i\theta_{\pm,0}}$ and choose a complex square root so that

$$\sqrt{(z-\sigma^-)(z-\sigma^+)(z-\sigma^0)} = \sqrt{r_- r_+ r_0}\,\mathrm{e}^{\frac{1}{2}i(\theta_- + \theta_+ + \theta_0)}.$$

For $i = 1,2,3$ let $\mathscr{C}_i$ be a closed positively oriented curves in $\mathbf{C}^\vee$, where $\mathscr{C}_1$ encircles the cut $[\sigma^-, \sigma^+]$ but avoids the points ± 1 and the cut $[\sigma^0, \infty)$; $\mathscr{C}_2$ encircles the cut $[\sigma^-, \sigma^+]$ and the points ± 1 but avoids the curve $\mathscr{C}_1$; and $\mathscr{C}_3$ encircles the cut $[\sigma^0, \infty)$ but avoids the curve $\mathscr{C}_2$. Note that the real root σ^0 of V, which does not lie in $[-1,1]$, is strictly greater than 1. Let $\omega = \frac{b-az}{(1-z^2)\sqrt{V(z)}}$. Then ω is a meromorphic 1-form on $\mathbf{C}^\vee$ with first order poles at $\varepsilon = \pm 1$, which have residue $\operatorname*{Res}_{z=\varepsilon} \omega = \frac{1}{2}i$. This follows from the choice of square root and the fact that $b/a \in [-1,1]$. Hence by the residue theorem we get

$$\int_{\mathscr{C}_2} \omega = 2\pi i \operatorname*{Res}_{z=-1} \omega + 2\pi i \operatorname*{Res}_{z=1} \omega + \int_{\mathscr{C}_1} \omega = -2\pi + \int_{\mathscr{C}_1} \omega.$$

By Cauchy's theorem

$$\int_{\mathscr{C}_2} \omega = \int_{\mathscr{C}_3} \omega = 2a \int_{\sigma_0}^{\infty} \frac{x - b/a}{(x^2-1)\sqrt{V(x)}}\,\mathrm{d}x > 0,$$

since $b/a \in [-1,1]$, $a > 0$ and $\sigma^0 > 1$. This proves the assertion because $2\pi\Theta = \int_{\mathscr{C}_1} \omega$. □

This completes the verification of figure 5.3.4. □

We now translate the motions of $\widetilde{\gamma}$ on $\widetilde{\mathscr{A}}$ into motions γ of the tip of the figure axis of the top on $\mathscr{A}$. There are two cases: when $\widetilde{\mathscr{A}}$ is the universal covering space of a closed annulus $\mathscr{B}$ or the universal covering space of the closed 2 disk $\overline{D}_{\pm}$ blown up at its center. In the case of the closed annulus, the translation is straightforward and the results are given in figure 5.3.3. In the case of a disk $\overline{D}^2_+$ consider the curve γ given in figure 5.3.5 (a), which

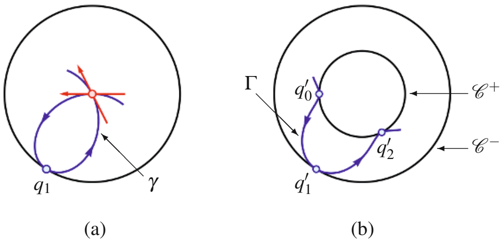


Figure 5.3.5. (a) The motion of the tip of the figure axis in the closed 2-disk $\overline{D}^2_+$. (b) The blow up of the motion of (a) in the annulus.

represents a motion of the tip of the figure axis of the top which passes through the north pole np of S^2. Recall that under the blow up map each positive tangent ray to γ at np of a suitable length after a translation corresponds to a point on the circle $\mathscr{C}^+$, which is the blow up of np. The point q_1 on $\partial\overline{D}^2_+$, after applying the blow up map, corresponds to the point q_1' on $\mathscr{C}^-$. The blow up of the curve γ in $\overline{D}^2_+$ is the curve Γ in the annulus bounded by $\mathscr{C}^{\pm}$. The curve Γ is tangent to $\mathscr{C}^+$ at q_0' and q_2', which correspond to the two distinct positive tangent rays to γ at np. Thus Γ is a *wavy* motion. From this discussion we see that the motion of the tip of the figure axis of the top through the north pole is *not* the limit of cuspy motion in an annulus whose upper boundary shrinks to the north pole. For more details see the discussion after ((5.15)) below.

We now turn to discussing singularities of $\widetilde{\gamma}$ which are *not* tangential. If $\dot{\varphi} = 0$ at $q_0 \in \partial\widetilde{\mathscr{A}}$, then we say that $\widetilde{\gamma}$ has a *nontangential singularity* at q_0. From ((5.9)) we see that $\frac{\mathrm{d}\widetilde{\gamma}}{\mathrm{d}t}(t_0)$ is an outward pointing normal to $\partial\widetilde{\mathscr{A}}$ at $q_0 = \widetilde{\gamma}(t_0)$. If in addition $\dot{\varphi}$ is not identically zero on $\mathscr{A}$, we say that q is *nondegenerate*; otherwise we say that it is *degenerate*.

Claim: A nondegenerate nontangential singularity of $\widetilde{\gamma}$ can only occur on the upper boundary $\{\sigma = \sigma^+\}$ of $\widetilde{\mathscr{A}}$.

Before giving a formal proof, here is a physical argument using the magnetic spherical pendulum model. When $\dot{\varphi} = 0$, the figure axis of the top is not moving. Hence the *Lorentz force* on the electrically charged particle from the magnetic monopole field is zero. Thus only downward gravity is acting. Since the particle is moving in an annulus, this can only happen at the upper boundary. Here is the formal proof.

(5.15) **Proof**: Suppose that $\widetilde{\gamma}$ has a nondegenerate nontangential singularity at $q_0 = (\sigma^-, \varphi_0)$. From the definition of this kind of singularity and using (87b) it follows that $a \neq 0$, $\sigma^- = b/a$ and $|b/a| < 1$. Because σ^- is a root of V, from (83) we obtain $\sigma_2 = 0$. But $\tau^{-1}(q_0)$ lies on an integral curve of $X_{H^a}|T^2_{h^a,b}$. Hence its image under the reduction map

π^a_b (56) lies on an integral curve of $-\mathrm{ad}_{H^a_b}$, which intersects $P^a_b \cap \{\sigma_2 = 0\}$. Since (h^a, b) is a regular value of EM^a, from figure 5.1.2 we see that the line ℓ_α in the σ_1–σ_3 plane defined by $\sigma_3 + \sigma_1 = 2\alpha$ intersects the image of the fold curve $\rho(\mathscr{C})$ defined by $\sigma_3 = (b - a\sigma_1)(1-\sigma_1^2)^{-1}$ with $|\sigma_1| < 1$ in two distinct points (σ^+, σ_3^+) and (σ^-, σ_3^-) with $\sigma^- < \sigma^+$, because $\sigma^\pm$ are simple roots of V in $(-1,1)$. Since $\sigma^- = b/a$, it follows that $\sigma_3^- = 0$. However, the slope of the line ℓ_α is negative and $\rho(\mathscr{C})$ lies in the half plane $\sigma_3 \geq 0$. Therefore $(\sigma^-, \sigma_3^-) = (b/a, 0)$ cannot be the left most point of intersection of ℓ_α with $\rho(\mathscr{C})$. This contradicts the definition of σ^-. □

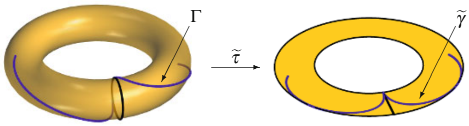


Figure 5.3.6. Appearance of a cusp.

Claim: Suppose that $q_1 = (\sigma^+, \varphi^1) \in \partial\widetilde{\mathscr{A}}$ is a nondegenerate nontangential singularity of $\widetilde{\gamma}$. Then $\widetilde{\gamma}$ has a cusp at q_1.

(5.16) **Proof**: From the definition of nondegenerate nontangential singularity and equations (87a) and (87b) we find that $a \neq 0$, $\sigma^+ = b/a$ and $|b/a| < 1$. Parametrizing $\widetilde{\gamma}$ near q_1 by σ, we obtain

$$\frac{\mathrm{d}\varphi}{\mathrm{d}\sigma} = \frac{a(\sigma^+ - \sigma)}{(1-\sigma^2)\sqrt{V(\sigma)}}. \tag{89}$$

Since $T^2_{h^a,b}$ is a smooth 2-torus, V has two simple real roots $\sigma^\pm$ in $(-1,1)$ and one real root $\sigma_0 > 1$. Therefore $V'(\sigma^+) < 0$. Expanding $(1-\sigma^2)^{-1}$ and $V(\sigma)^{-1/2}$ in a Taylor series about σ^+, equation (89) becomes

$$\frac{\mathrm{d}\varphi}{\mathrm{d}\sigma} = \frac{a}{(1-(\sigma^+)^2)\sqrt{-V'(\sigma^+)}}(\sigma^+ - \sigma)^{1/2} + \mathrm{O}\big((\sigma^+ - \sigma)\big),$$

which integrated gives

$$\varphi(\sigma) = -\frac{2a}{3(1-(\sigma^+)^2)\sqrt{-V'(\sigma^+)}}(\sigma^+ - \sigma)^{3/2} + \mathrm{O}\big((\sigma^+ - \sigma)^2\big).$$

Therefore γ has a cusp at q_1. □

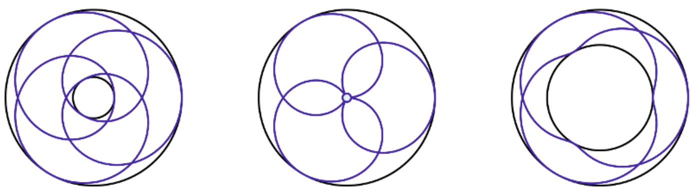


Figure 5.3.7. Transition between wavy and looping motion of the tip of the top without becoming cuspy.

From the proof of ((5.15)) we see that $\widetilde{\gamma}$ has a *cusp singularity* if and only if V has two simple roots σ^- and $\sigma^+ = b/a$ in $(-1,1)$ with $0 < |b| < |a|$. Consequently the tip of the figure axis of the top makes a *cuspy motion* only when $\widetilde{\mathscr{A}}$ is the universal covering space of the closed annulus $\mathscr{B}$ and only at those energy momentum values (h^a, a, b) where $\alpha = \frac{b}{a}\beta$, since $(b/a, 0)$ lies on ℓ_α. Recall that $\alpha = I_1 h^a - \frac{1}{2}b^2$ and $\beta = I_1\chi$. Thus the north pole of S^2 is *never* a cusp point of the motion of the tip of the figure axis.

In figure 5.3.7 we have sketched how the tip of the figure axis can pass from a wavy to an upward looping motion without passing through a cuspy motion by a continuous variation in the parameter (h^a, a, b) and the initial condition. The motion of the tip in the middle drawing in figure 5.3.7 is *not* smooth when parametrized by arc length, even though the motion disregarding the time parametrization *is*. This lack of smoothness is due to the geometric behavior of the projection mapping τ.

Type of motion	Conditions	The roots $\sigma^\pm$
1. wavy	$b \neq \pm a,\ a \neq 0,\ b/a > 0$	$\sigma^\pm$ are simple roots of V in $(-1,1)$. $\frac{d\varphi}{d\sigma}$ has same sign at $\sigma^\pm$.
2. upward looping	$b \neq \pm a,\ a \neq 0, b/a < 0$	$\sigma^\pm$ are simple roots of V in $(-1,1)$. $\sigma^- < b/a$ & $\sigma^+ > 0$. $\frac{d\varphi}{d\sigma}$ has opposite signs at $\sigma^\pm$.
3. cuspy	$0 < \lvert b\rvert < \lvert a\rvert,\ \alpha = \beta\dfrac{b}{a}$	$\sigma^\pm$ are simple roots of V in $(-1,1)$. $\sigma^+ = a/b$. $\frac{d\varphi}{d\sigma} = 0$ at σ^+.
4. through north pole	$b = a,\ a \neq 0,\ \alpha \neq \beta$	σ^- and $\sigma^+ = 1$ are simple roots of V in $(-1,1]$. $\frac{d\varphi}{d\sigma} \neq 0$ at $\sigma^\pm$.
5. through south pole	$b = -a,\ a \neq 0,\ \alpha \neq -\beta$	$\sigma^- = -1$ and σ^+ are simple roots of V in $[-1,1)$. $\frac{d\varphi}{d\sigma} \neq 0$ at $\sigma^\pm$.
6. arc of great circle	$b = a = 0,\ \lvert\alpha\rvert < \beta$	$\sigma^- = -1$ and $\sigma^+ = \alpha/\beta$ are simple roots of V in $[-1,1)$.
7. great circle	$b = a = 0,\ \alpha > \beta$	$\sigma^\pm = \pm 1$ are simple roots of V.

Table 5.3.1. Types of motion of the tip of the figure axis. Here (h^a, b) is a regular value of the energy momentum mapping EM^a.

Suppose that $\widetilde{\gamma}$ has a *degenerate* nontangential singularity in $\widetilde{\mathscr{A}}$. Then $\widetilde{\gamma}$ moves periodically with period T along a vertical line joining the two boundary components of

$\widetilde{\mathscr{A}}$. Since $\dot{\varphi} = 0$ on all of $\widetilde{\mathscr{A}}$, from (87b) it follows that $a = b = 0$. Hence $V(\sigma) = 2(\alpha - \beta\sigma)(1-\sigma^2)$. Suppose that $\alpha/\beta < 1$. Then $V(\sigma) \geq 0$ for $[-1, \alpha/\beta]$. Hence $\sigma^- = -1$ and $\sigma^+ = \alpha/\beta$. Thus $\widetilde{\mathscr{A}}$ is the universal covering space of the closed 2-disk $\overline{D}^2_-$ blown up at the south pole. So the tip of the figure axis periodically traces out an arc of a great circle on S^2 that passes through the south pole of S^2. Its rotation number is $\pm\frac{1}{2}$. If $\alpha/\beta > 1$, then $\sigma^{\pm} = \pm 1$. In this case, $\widetilde{\mathscr{A}}$ is the universal covering space of S^2 blown up at the north and south poles. The tip of the figure axis traverses a great circle through the north and south poles of S^2. Its rotation number is ± 1.

In table 5.3.1 we summarize the classification of motions of the tip of the figure axis of the Lagrange top.

6 The energy momentum mapping

In this section we investigate the geometry of the *energy momentum mapping*

$$\widetilde{\mathscr{E}\mathscr{M}} : T\,\mathrm{SO}(3) \to \mathbf{R}^3 : X_A \mapsto \big(\mathscr{H}(X_A), \mathscr{J}_r(X_A)\,\mathscr{J}_\ell(X_A)\big)$$

of the Lagrange top. Here $\mathscr{H}$ is the Hamiltonian (4), $\mathscr{J}_r$ the angular momentum (8) of the right S^1-action, and $\mathscr{J}_\ell$ the angular momentum (6) of the left S^1-action. Because the left trivialization mapping $\mathscr{L}$ (10) of $T\,\mathrm{SO}(3)$ is a diffeomorphism, it suffices to look at the geometry of

$$\mathscr{E}\mathscr{M} : \mathrm{SO}(3) \times \mathrm{so}(3) \to \mathbf{R}^3 : (A,X) \mapsto \big(H(A,X), J_r(A,X), J_\ell(A,X)\big). \tag{90}$$

Here $H = \mathscr{L}^*\mathscr{H}$ is given by (98), $J_r = \mathscr{L}^*\mathscr{J}_r$ by (12), and $J_\ell = \mathscr{L}^*\mathscr{J}_\ell$ by (42). If we understand the topology of the fibers $\mathscr{E}\mathscr{M}^{-1}(h,a,b)$ and how they fit together to form $H^{-1}(h)$, then we have a complete qualitative picture of the invariant manifolds of the Hamiltonian vector field X_H, whose integral curves give the motions of the Lagrange top.

6.1 Topology of $\mathscr{E}\mathscr{M}^{-1}(h,a,b)$ and $H^{-1}(h)$

In this subsection we reconstruct the topology of $\mathscr{E}\mathscr{M}^{-1}(h,a,b)$ from the *topology* of the level set $(H^a_b)^{-1}(h^a_b)$ of the second reduced Hamiltonian (72) and the geometry of the reduction mapping $\pi : J_r^{-1}(a) \to P^a_b$. Here the map π is the composition of the reduction map $\pi^a : J_r^{-1}(a) \to P^a$ (13) of the right S^1-action (11) and the reduction map $\pi^a_b : (J^a_\ell)^{-1}(b) \subseteq P^a \to P^a_b$ (56) of the induced left S^1 action (40) on P^a. Reconstruction is possible because $\mathscr{E}\mathscr{M}^{-1}(h,a,b) = \pi^{-1}\big((H^a_b)^{-1}(h^a_b)\big)$, where $h^a_b = h - \frac{1}{2}I_1^{-1}(b^2-a^2) -$
▷ $\frac{1}{2}I_3^{-1}a^2$. The results are given in table 6.1.1. We now verify the entries in the third column of table 6.1.1.

(6.1) **Proof**:
1and 5. If h^a_b is a regular value of H^a_b or $a = -b$ and $h^a_b = -\chi + \frac{1}{2}I_1^{-1}a$, then the level set $(H^a_b)^{-1}(h^a_b)$ bounds a contractible 2-disk in P^a_b. Therefore $\pi^{-1}\big((H^a_b)^{-1}(h^a_b)\big)$ is a trivial 2-torus bundle over S^1, that is, $\mathscr{E}\mathscr{M}^{-1}(h,a,b)$ is a smooth 3-torus.

2. Over every nonsingular point of P^a_b the fiber of the reduction mapping π is a smooth 2-torus.
4. Over a singular point $\widetilde{\sigma}$ of $P^a_{\varepsilon a}$ with $\varepsilon^2 = 1$ the fiber of π is a circle. This follows because the fiber $(\pi^a_b)^{-1}(\widetilde{\sigma})$ is a point p, being a fixed point of the left S^1-action on P^a; while the fiber of $(\pi^a)^{-1}(p)$ is the smooth circle $\pi^{-1}(\widetilde{\sigma})$.

Conditions	Topology of $(H^a_b)^{-1}(h^a_b)$	Topology of $\mathscr{EM}^{-1}(h,a,b)$
1. h^a_b is a regular value of H^a_b	a smooth S^1	a smooth 3-torus T^3
2. V has a multiple root in $(-1,1)$	a nonsingular point of P^a_b	a smooth 2-torus T^2
3. $a=b$, $h^a_b = \chi + \frac{1}{2}I_1^{-1}a^2$, $\lvert a\rvert < 2\sqrt{\beta}$	a topological S^1 with a conical singular point	a 3-torus with a normal crossing along an S^1
4. $a=b$, $h^a_b = \chi + \frac{1}{2}I_1^{-1}a^2$, $\lvert a\rvert \ge 2\sqrt{\beta}$	a singular point of P^a_a	a smooth S^1
5. $a=-b$, $h^a_b = -\chi + \frac{1}{2}I_1^{-1}a^2$	a smooth S^1	a smooth 3-torus T^3

Table 6.1.1. Topology of $\mathscr{EM}^{-1}(h,a,b)$

3. To verify the third entry suppose that $(H^a_a)^{-1}(h^a_a)$ is a topological circle $\mathscr{C}$ with a conical singular point $\widetilde{\sigma} = (1,0,0)$, which is the singular point of P^a_a. Since $\mathscr{C}\setminus\{\widetilde{\sigma}\}$ is contractible, $\pi^{-1}(\mathscr{C}\setminus\{\widetilde{\sigma}\})$ is diffeomorphic to $T^2\times\mathbf{R}$. The fiber $\pi^{-1}(\widetilde{\sigma})$ is a nondegenerate critical circle of $H|\big(J_r^{-1}(a)\cap J_\ell^{-1}(b)\big)$ of Morse index 2 because p is a nondegenerate critical point of $H^a|(J^a_\ell)^{-1}(b)$ of Morse index 2, see ((6.4)). By the Morse lemma the local stable and unstable manifolds of $H^a|(J^a_\ell)^{-1}(b)$ at p are diffeomorphic to $x_1^2 - x_2^2 - x_3^2 = 0$, which is a double cone $C = C(S^0\times S^1)$ on S^1. The local stable and unstable manifolds of $\pi^{-1}(\widetilde{\sigma})$ are untwisted, because the local stable and unstable manifolds of p are contractible and a fiber of π^a is an S^1. Therefore a neighborhood of $\pi^{-1}(\widetilde{\sigma})$ in the fiber $\pi^{-1}(\mathscr{C})$ is diffeomorphic to $S^1\times C$, that is, is a double cone on T^2. Consequently, $\pi^{-1}(\mathscr{C})$ is a 3-torus with a normal crossing along the circle $\pi^{-1}(\widetilde{\sigma})$. In other words, $\pi^{-1}(\mathscr{C})$ is a product of an S^1 with a 2-torus with a meridial circle pinched to a point. □

For later purposes, see figure 5.1.5, we note that the singular point $\sigma_\varepsilon = \varepsilon(1,0,0)\in P^a_{\varepsilon a}$ is a local minimum of $H^a_{\varepsilon a}$, when $\varepsilon = 1$, $b=a$ and $|a| > 2\sqrt{\beta}$, or when $\varepsilon = -1$ and $b=-a$, because σ_ε is an elliptic equilibrium point of $-\mathrm{ad}_{H^a_b}$. Under these restrictions it follows that $\pi^{-1}(\sigma_\varepsilon)$ is a local minimum for H restricted to $J_r^{-1}(a)\cap J_\ell^{-1}(\varepsilon a)$.

We want to determine the topology of the level sets of the Hamiltonian

$$H : \mathrm{SO}(3)\times\mathrm{so}(3)\to\mathbf{R} : (A,X)\mapsto \tfrac{1}{2}k(I(X),X) + \chi k(\mathrm{Ad}_{A^{-1}}E_3, E_3).$$

Let $S^1 = \{B\in\mathrm{SO}(3)\,\big|\,\mathrm{Ad}_{B^{-1}}E_3 = E_3\}$ and consider the S^1-action

$$\varphi^\ell : S^1\times\big(\mathrm{SO}(3)\times\mathrm{so}(3)\big)\to\mathrm{SO}(3)\times\mathrm{so}(3) : (B,(A,X))\mapsto(BA,X) \tag{91}$$

with orbit mapping

$$\widetilde{\rho} : \mathrm{SO}(3) \times \mathrm{so}(3) \to S^2 \times \mathbf{R}^3 : (A,X) \mapsto \big(Ae_3, i(X)\big) = (z,w). \tag{92}$$

Since $H(BA,X) = H(A,X)$, it follows that H is invariant under the S^1-action φ^ℓ. Therefore H induces the smooth function

$$\widehat{H} : S^2 \times \mathbf{R}^3 \to \mathbf{R} : (z,w) \mapsto \tfrac{1}{2} I_1^{-1}(w_1^2 + w_2^2) + I_3^{-1} w_3^2 + \chi z_3,$$

▷ where $\widetilde{\rho}^* \widehat{H} = H$. We now show that $\widehat{H}$ is a Morse function.

(6.2) **Proof**: Consider the function $\widetilde{H} : \mathbf{R}^3 \times \mathbf{R}^3 \to \mathbf{R} : (z,w) \mapsto \frac{1}{2}(I^{-1}w, w) + \chi(z, e_3)$ constrained to $F^{-1}(0)$, where $F : \mathbf{R}^3 \times \mathbf{R}^3 \to \mathbf{R} : (z,w) \mapsto (z,z) - 1$. Here $I^{-1} = \mathrm{diag}\,(I_1^{-1}, I_1^{-1}, I_3^{-1})$. Because the fibers of $\widetilde{H}|F^{-1}(0)$ are compact, the function $\widetilde{H}|F^{-1}(0)$ is proper and hence has a critical point (z,w). By Lagrange multipliers, at the critical point (z,w) we have $D\widetilde{H}(z,w) + \lambda DF(z,w) = 0$ and $(z,z) = 1$, that is,

$$\begin{cases} 2\lambda z_1 &= 0 \\ 2\lambda z_2 &= 0 \\ \chi + 2\lambda z_3 &= 0 \\ I^{-1} w &= 0 \end{cases} \quad \& \quad z_1^2 + z_2^2 + z_3^2 = 1.$$

If $\lambda = 0$, then $\chi = 0$, which is a contradiction. Therefore $\lambda \neq 0$, which implies $z_1 = z_2 = 0$. So $z_3 = \varepsilon$, where $\varepsilon^2 = 1$ and $\lambda = -\frac{1}{2}\varepsilon\chi = \lambda_{p_\varepsilon}$. Since I is invertible, we get $w = 0$. Therefore $p_\varepsilon = (\varepsilon e_3, 0)$ is a critical point of $\widetilde{H}|F^{-1}(0)$. Now

$$T_{p_\varepsilon}(F^{-1}(0)) = \ker DF(p_\varepsilon) = \ker(\varepsilon e_3, 0) = \mathrm{span}\{e_1, e_2, e_4, e_5, e_6\}.$$

So

$$\begin{aligned} \mathrm{Hess}_{p_\varepsilon}(\widetilde{H}|F^{-1}(0)) &= \big(D^2\widetilde{H}(p_\varepsilon) + \lambda_{p_\varepsilon} D^2F(p_\varepsilon)\big)\Big|_{T_{p_\varepsilon}(F^{-1}(0))} \\ &= \big(\mathrm{diag}\,(0,0,0,I_1^{-1},I_1^{-1},I_3^{-1}) - \tfrac{1}{2}\varepsilon\chi\,\mathrm{diag}\,(2,2,2,0,0,0)\Big|_{\mathrm{span}\{e_1,e_2,e_4,e_5,e_6\}} \\ &= \mathrm{diag}\,(-\varepsilon\chi, -\varepsilon\chi, I_1^{-1}, I_1^{-1}, I_3^{-1}), \end{aligned}$$

which is invertible and has Morse index 0 at p_{-1} and Morse index 2 at p_1. Thus $\widehat{H} = \widetilde{H}|F^{-1}(0)$ is a Morse function on $F^{-1}(0) = S^2 \times \mathbf{R}^3$. □

h	Topology of $\widehat{H}^{-1}(h)$
1. $-\chi$	point
2. $-\chi < h < \chi$	a smooth S^4
3. χ	$U = (S^2 \times \overline{D}^2) \dot\cup_{S^2 \times S^1} C(S^2 \times S^1)$
4. $h > \chi$	a smooth $S^2 \times S^2$

Table 6.1.2. Topology of $\widehat{H}^{-1}(h)$. The level set $U = \widehat{H}^{-1}(\chi)$ in the third entry is the disjoint union of the product of a 2-sphere and a closed 2-disk with a closed cone on a product of a 2-sphere S^2 and a circle S^1, which are glued together along their common boundary, which is a product of a 2-sphere S^2 and S^1.

(6.3) **Proof**: We now verify the entries in the second column of table 6.1.2.

1. Because p_1 is a nondegenerate minimum of $\widehat{H}$ on $S^2 \times \mathbf{R}^3$ with corresponding critical value $-\chi$, it follows that $\widehat{H}^{-1}(-\chi)$ is a point.

2. Using the Morse lemma near p_1, there is a smooth local change of coordinates so that $\widehat{H}+\chi$ is equal to its positive definite Hessian at p_{-1}. Therefore for every h slightly greater than $-\chi$, the level set $\widehat{H}^{-1}(h)$ is diffeomorphic to a 4-sphere S^4. By the Morse isotopy lemma for all $-\chi < h < \chi$ the level set $\widehat{H}^{-1}(h)$ is diffeomorphic to S^4.

4. For each $h > \chi$ let $\mu : \widetilde{H}^{-1}(h) \subseteq S^2 \times \mathbf{R}^3 \to S^2 : (z,w) \mapsto z$. Then μ is a smooth submersion. For each $z \in S^2$ the fiber of the map μ is defined by $(I^{-1}w,w) = 2(h-\chi z_3)$ where $w \in \mathbf{R}^3$. Because $h > \chi$ and $|z_3| \le 1$, we have $2(h-\chi z_3) > 0$. Therefore for each $z \in S^2$ the fiber $\mu^{-1}(z)$ is diffeomorphic to S^2, which is compact. Hence μ is a proper map and thus exhibits $\widehat{H}^{-1}(h)$ as a local trivial bundle over S^2 with fiber S^2. Recall that smooth bundles over S^2 are classified by homotopy classes of maps of an equatorial S^1 of the base S^2 into the fiber S^2. Since S^2 is simply connected, its fundamental group $\pi_1(S^2)$ is the identity element. Therefore the bundle defined by the mapping μ is trivial. In other words, $\widehat{H}^{-1}(h)$ is diffeomorphic to $S^2 \times S^2$.

3. Consider the function

$$\mathscr{F} : \mathbf{R}^3 \times \mathbf{R}^3 \to \mathbf{R}^2 : (z,w) \mapsto \big(\widetilde{H}(z,w), F(z,w)\big) = \big(\tfrac{1}{2}(I^{-1}w,w) + \chi z_3, (z,z)\big).$$

We look at the level set $U = \widehat{H}^{-1}(\chi) = \mathscr{F}^{-1}(\chi,1)$. First we show that $\widehat{H}^{-1}(\chi) \setminus \{p_1\}$ is a smooth 4-dimensional manifold. This follows because

$$D\mathscr{F}(z,w) = \begin{pmatrix} D\widetilde{H}(z,w) \\ DF(z,w) \end{pmatrix} = \begin{pmatrix} 0 & 0 & \chi & I_1^{-1}w_1 & I_1^{-1}w_2 & I_3^{-1}w_3 \\ 2z_1 & 2z_2 & 2z_3 & 0 & 0 & 0 \end{pmatrix}$$

has rank 1 only at $(0,0,z_3,0,0,0)$, $z_3 \ne 0$, that is, only at $p_1 \in \widehat{H}^{-1}(\chi)$.

By the Morse lemma there is a smooth local change of coordinates near p_1 such that $\widehat{H}-\chi$ is equal to its Hessian $\operatorname{diag}(-\chi,-\chi,I_1^{-1},I_1^{-1},I_3^{-1})$ near p_1. Thus the level set $\widehat{H}^{-1}(h)$ near p_1 is homeomorphic to the cone C in $\mathbf{R}^5$ with coordinates (x_1,x_2,x_4,x_5,x_6) defined by $-x_1^2 - x_2^2 + x_4^2 + x_5^2 + x_6^2 = 0$. Let $\overline{D}^4$ be the closed 4-disk in $\mathbf{R}^4$ defined by $x_1^2+x_2^2+x_4^2+x_5^2+x_6^2 \le 1$. Writing $\overline{D}^4 = \bigcup_{0\le r\le 1} S_r^4 = \bigcup_{0\le r\le 1}\{x_1^2+x_2^2+x_4^2+x_5^2+x_6^2 \le r^2\}$ and noting that $C \cap S_r^4 = S^1_{r/\sqrt{2}} \times S^2_{r/\sqrt{2}} = \{x_1^2+x_2^2 = \frac{1}{2}r^2\} \cup \{x_4^2+x_5^2+x_6^2 = \frac{1}{2}r^2\}$, it follows that $C \cap \overline{D}^4$ is topologically a cone $C(S^1 \times S^2)$ on $S^1 \times S^2$ with vertex at p_1.

Consider the map

$$\widetilde{\mu} = \mu|(\widehat{H}^{-1}(\chi)\setminus\{p_1\}) : \widehat{H}^{-1}(\chi)\setminus\{p_1\} \subseteq S^2 \times \mathbf{R}^3 \to S^2_{\sqrt{\chi}} \setminus \{\chi e_3\} : (z,w) \mapsto z.$$

$\widetilde{\mu}$ is a smooth submersion, which is proper because each of its fibers is a 2-sphere S^2, which is compact. Thus $\widehat{H}^{-1}(\chi)\setminus\{p_1\}$ is a locally trivial bundle over $S^2_{\sqrt{\chi}} \setminus \{\chi e_3\}$ with fiber S^2. This bundle is trivial because $S^2_{\sqrt{\chi}} \setminus \{\chi e_3\}$ is contractible. Consequently $\widehat{H}^{-1}(\chi)\setminus\{p_1\}$ is topologically $D^2 \times S^2$, the product of an open 2-disk D^2 and a 2-sphere S^2. Let D^2 be an open 2-disk in $S^2_{\sqrt{\chi}}$ which contains the point χe_3. Then $\widetilde{\mu}^{-1}(S^2_{\sqrt{\chi}} \setminus D^2)$

is a smooth submanifold of $\widehat{H}^{-1}(\chi)$, which is diffeomorphic to the product of a closed 2-disk and a 2-sphere and has a boundary $S^1 \times S^2$. To form $U = \widehat{H}^{-1}(\chi)$ take the disjoint union of $\widetilde{\mu}^{-1}(S^2_{\sqrt{\chi}} \setminus D^2)$ and the cone $C(S^1 \times S^2)$ and glue them together along their common boundary $S^1 \times S^2$.

This completes the verification of table 6.1.2. □

We now give another description of the algebraic variety $U = \widehat{H}^{-1}(\chi)$. First we rescale the variables (z,w) by $(\sqrt{\chi}z, \sqrt{I_1/2}w_1, \sqrt{I_1/2}w_2, \sqrt{I_3/2}w_3)$. In the rescaled variables, which we call (z,w), the variety U is diffeomorphic to the real algebraic variety $\widetilde{U}$ in $S^2 \times \mathbf{R}^3$ defined by

$$w_1^2 + w_2^2 + w_3^2 + z_3 = 1 \tag{93a}$$

$$z_1^2 + z_2^2 + z_3^2 = 1 \tag{93b}$$

$\widetilde{U}$ is singular only at $q = (e_3, 0)$. Consider the S^1-action

$$\Phi : S^1 \times (\mathbf{R}^3 \times \mathbf{R}^3) \to \mathbf{R}^3 \times \mathbf{R}^3 : (t,(z,w)) \mapsto (\widetilde{R}_t z, \widetilde{R}_t w), \tag{94}$$

where $\widetilde{R}_t = \begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix}$, whose algebra of invariant polynomials is generated by

$$\begin{array}{llllll} \pi_1 & = z_3 & \pi_3 & = z_1w_1 + z_2w_2 & \pi_5 & = \frac{1}{2}(w_1^2 + w_2^2 - z_1^2 - z_2^2) \\ \pi_2 & = w_3 & \pi_4 & = z_1w_2 - z_2w_1 & \pi_6 & = \frac{1}{2}(w_1^2 + w_2^2 + z_1^2 + z_2^2) \end{array}$$

subject to the relation

$$\pi_3^2 + \pi_4^2 = \pi_6^2 - \pi_5^2, \quad \pi_6 \geq 0. \tag{95a}$$

The variety $\widetilde{U}$ is invariant under the S^1-action Φ, which leaves the point q fixed. The orbit space $\widetilde{U}/S^1$ is the semialgebraic variety in $\mathbf{R}^6$ with coordinates $(\pi_1, \ldots, \pi_6)$ defined by (95a) and

$$\pi_5 + \pi_6 + \pi_2^2 + \pi_1 = 1 \tag{95b}$$

$$\pi_6 - \pi_5 + \pi_1^2 = 1. \tag{95c}$$

Solving (95b) for $\pi_6 + \pi_5$ and (95c) for $\pi_6 - \pi_5$ and substituting the result into (95a) gives the semialgebraic variety $\widetilde{V}$ in $\mathbf{R}^6$ defined by

$$\pi_3^2 + \pi_4^2 = (1 - \pi_1 - \pi_2^2)(1 - \pi_1^2) \tag{96a}$$

together with $1 - \pi_1 - \pi_2^2 \geq 0$ (96b) and $1 - \pi_1^2 \geq 0$ (96c). The variety $\widetilde{V}$ is homeomorphic to the orbit space $\widetilde{U}/S^1$. Moreover, the differential spaces $(\widetilde{U}/S^1, C^\infty(\widetilde{U})^{S^1})$ and $(\widetilde{V}, \widetilde{C}^\infty(\widetilde{V}))$ are diffeomorphic.

We now determine the topology of $\widetilde{V}$. Let $\overline{\mathscr{D}}$ be the subset of $\mathbf{R}^2$ with coordinates (π_1, π_2) defined by (96b) and (96c), see figure 6.1.1.

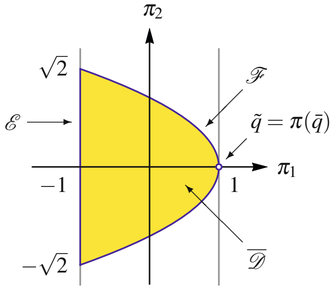


Figure 6.1.1. The darkened region is $\overline{\mathscr{D}}$, whose boundary is $\mathscr{E} \cup \mathscr{F}$, where $\mathscr{E} = \{(-1,\pi_2) : |\pi_2| \le \sqrt{2}\}$ and $\mathscr{F} = \{(-\pi_2^2+1,\pi_2) : |\pi_2| \le \sqrt{2}\}$.

Consider the surjective mapping

$$\pi : \widetilde{V} \subseteq \mathbf{R}^4 \to \overline{\mathscr{D}} \subseteq \mathbf{R}^2 : (\pi_1,\pi_2,\pi_3,\pi_4) \mapsto (\pi_1,\pi_2).$$

First we determine the geometry of the map π. Let $\mathscr{E} = \{(-1,\pi_2) \in \partial\overline{\mathscr{D}} \,\big|\, |\pi_2| \le \sqrt{2}\}$. We find the fiber $\pi^{-1}(\pi_1,\pi_2)$ for $(\pi_1,\pi_2) \in \mathscr{E}$ as follows. Since $\pi_1 = -1$ on $\mathscr{E}$ from (96c) it follows that $\pi_6 - \pi_5 = 0$. Then (96a) reads $\pi_3^2 + \pi_4^2 = 0$, which implies $\pi_3 = \pi_4 = 0$. Therefore for $|\pi_2| \le \sqrt{2}$ we get $\pi^{-1}(-1,\pi_2) = (-1,\pi_2,0,0)$. Let $\mathscr{F} = \{(-\pi_2^2+1,\pi_2) \in \partial\overline{\mathscr{D}} \,\big|\, |\pi_2| \le 2\}$. When $\pi_1 = -\pi_1^2+1$ then (95a) reads $\pi_3^2+\pi_4^2 = 0$, which gives $\pi_3 = \pi_4 = 0$. Therefore for $|\pi_2| \le 2$ we get $\pi^{-1}(-\pi_2^2+1,\pi_2) = (-\pi_2^2+1,\pi_2,0,0)$. Note that $\partial\overline{\mathscr{D}} = \mathscr{E} \cup \mathscr{F}$, which is topologically an S^1. The above argument shows that $\pi^{-1}(\partial\overline{\mathscr{D}})$ is topologically an S^1 in $\widetilde{V}$. The map $\widetilde{\pi} = \pi|(\widetilde{V} \setminus \pi^{-1}(\partial\overline{\mathscr{D}}))$ is a surjective submersion of $\widetilde{V} \setminus \pi^{-1}(\partial\overline{\mathscr{D}})$ onto $\operatorname{int}\overline{\mathscr{D}}$. For every $(\pi_1,\pi_2) \in \operatorname{int}\overline{\mathscr{D}}$ we have $1-\pi_1-\pi_2^2 > 0$ and $1-\pi_1^2 > 0$. Therefore the fiber $\widetilde{\pi}^{-1}(\pi_1,\pi_2)$, which is defined by $\pi_3^2+\pi_4^2 = (1-\pi_1-\pi_2^2)(1-\pi_1^2)$, is diffeomorphic to a circle S^1. Since S^1 is compact, the mapping $\widetilde{\pi}$ is proper. Therefore for every contractible subset $\mathscr{C}$ of $\overline{\mathscr{D}}$ it follows that $\widetilde{\pi}^{-1}(\mathscr{C})$ is topologically $\mathscr{C} \times S^1$. Let $\pi_2^0 \in [0,\sqrt{2})$. Consider the closed horizontal line segment $\ell_{\pi_2^0} = \{(\pi_1,\pi_2^0) \in \overline{\mathscr{D}} \,\big|\, -1 \le \pi_1 \le -(\pi_2^-)^2+1\}$ in $\overline{\mathscr{D}}$ with endpoints $Q_1 = (-1,\pi_2^0) \in \mathscr{E}$ and $Q_2 = (-(\pi_2^0)^2+1,\pi_2^0) \in \mathscr{F}$. From what we have already shown, we see that $\widetilde{\pi}^{-1}(\ell_{\pi_2^0} \setminus \{Q_1,Q_2\})$ is topologically a cylinder $(0,1) \times S^1$ with each of its endpoints pinched to a point. In other words, $\widetilde{\pi}^{-1}(\ell_{\pi_2^0})$ is topologically a 2-sphere S^2. When $\pi_2^0 = \sqrt{2}$, the fiber $\widetilde{\pi}^{-1}(-1,\sqrt{2})$ is a point. Therefore the preimage under the map $\widetilde{\pi}$ of the closed half 2-disk $\overline{\mathscr{D}} \cap \{\pi_2 \ge 0\}$ is a closed cone $\overline{C(S^2)}$ on S^2 with boundary $\widetilde{\pi}^{-1}(\ell_0)$, which is an S^2. Thus $\overline{\mathscr{D}} \cap \{\pi_2 \ge 0\}$ is topologically a closed 3-disk $\overline{D}_1^3$ with boundary $\widetilde{\pi}^{-1}(\ell_0)$. A similar argument shows that $\widetilde{\pi}^{-1}(\overline{\mathscr{D}} \cap \{\pi_2 \le 0\})$ is also a closed 3-disk $\overline{D}_2^3$ with boundary $\widetilde{\pi}^{-1}(\ell_0)$. Thus $\widetilde{V} = \widetilde{\pi}^{-1}(\overline{\mathscr{D}}) = \overline{D}_1^3 \coprod_{S^2} \overline{D}_2^3$, that is, $\widetilde{V}$ is the disjoint union of two closed 3-disks $\overline{D}_1^2$ and $\overline{D}_2^2$ glued together along their common boundary $\widetilde{\pi}^{-1}(\ell_0)$, which is an S^2. So $\widetilde{V}$ is a topological 3-sphere S^3.

Because the map $\pi|(\widetilde{V} \setminus \{\overline{q}\})$ is smooth except at $\overline{q} = (1,0,0,0)$, it is a proper submersion of $\widetilde{V} \setminus \{\overline{q}\}$ onto $\overline{\mathscr{D}} \setminus \{\widetilde{q}\}$, where $\widetilde{q} = (1,0)$. The following argument shows that in a neighborhood of $\overline{q}$, the variety $\widetilde{V}$ has a conical singularity, which is a cone $C(S^2)$

on S^2 with vertex at $\overline{q}$. For each $\pi_1^0 \in [0,1]$ consider the closed vertical line segment $L_{\pi_1^0} = \{(\pi_1^0, \pi_2^0) \in \overline{\mathscr{D}} \,\big|\, |\pi_2| \le \sqrt{1-\pi_1}\}$. Then $\pi^{-1}(\mathrm{int}\, L_{\pi_1^0})$ is topologically a cylinder $(0,1)\times S^1$; while the fiber over each of its endpoints is a point. Hence $\pi^{-1}(L_{\pi_1^0})$ is a topological 2-sphere S^2. Therefore the preimage under the map π of the right half disk
▷ $\overline{\mathscr{D}} \cap \{\pi_1 \ge 0\}$ is a closed cone $\overline{C(S^2)}$ on S^2 with vertex at $\overline{q}$. This completes the proof that the semialgebraic variety $\widetilde{V}$ is a topological 3-sphere, which is smooth except at one point where it has a singularity, which is a cone on S^2.

We now look at the variety $\widetilde{U}$. Let

$$\rho : \widetilde{U} \subseteq \mathbf{R}^3 \times \mathbf{R}^3 \to \widetilde{U}/S^1 = \widetilde{V} : (z,w) \mapsto \big(\pi_1(z,w), \ldots, \pi_4(z,w)\big). \tag{97}$$

Then ρ is the orbit map of the S^1-action Φ (94). Except at the singular point p_1 of $\widetilde{U}$, the map ρ is a smooth surjection of $\widetilde{U}\setminus\{p_1\}$ onto $\widetilde{V}\setminus\{\overline{q}\}$ with fiber S^1. Therefore ρ is a proper map, which defines a locally trivial S^1-fibration of $\widetilde{U}\setminus\{p_1\}$ over $\widetilde{V}\setminus\{\overline{q}\}$. Since $\widetilde{V}$ is topologically a 3-sphere, $\widetilde{V}\setminus\{\overline{q}\}$ is contractible. Consequently, $\widetilde{U}\setminus\{p_1\}$ is topologically the product of $\widetilde{V}\setminus\{\overline{q}\}$ and S^1. Because an open neighborhood of $\overline{q}$ in $\widetilde{V}$ is topologically a cone $C(S^2)$ on S^2, which is contractible, $\rho^{-1}(C(S^2))$ is topologically a cone $C(S^2\times S^1)$
▷ on $S^2\times S^1$. Thus we have shown that $\widetilde{U} = (\overline{D}^3 \times S^1) \bigcup_{S^2\times S^1} C(S^2\times S^1)$, that is, $\widetilde{U}$ is the disjoint union of the product of a closed 3-disk and S^1 with a cone on $S^2\times S^1$ glued together along their common boundary $S^2\times S^1$. This completes our alternative description of the algebraic variety $U = \widehat{H}^{-1}(\chi)$.

Next we turn to reconstructing the *topology of the level sets of the Hamiltonian*

$$H : \mathrm{SO}(3)\times \mathrm{so}(3) \to \mathbf{R} : (A,X) \mapsto \tfrac{1}{2}k(I(X),X) + \chi k(\mathrm{Ad}_{A^{-1}} E_3, E_3). \tag{98}$$

The results are given in table 6.1.3.

h	Topology of $H^{-1}(h)$
1. $-\chi$	smooth S^1
2. $-\chi < h < \chi$	a smooth $S^1\times S^4$
3. χ	$W = U\times S^1$, see table 6.1.2 for U
4. $h > \chi$	a smooth $\mathrm{SO}(3)\times S^2$

Table 6.1.3. Topology of $H^{-1}(h)$.

▷ We now verify the entries in the second column of table 6.1.3.

(6.4) **Proof**: From the fact that $\widehat{H}$ is a Morse function with two nondegenerate critical points p_{-1} of index 0 and p_1 of index 2, it follows that H is a Bott-Morse function on $\mathrm{SO}(3)\times\mathrm{so}(3)$ with two nondegenerate critical circles: one $\widetilde{\rho}^{-1}(p_{-1})$ of index 0 and the other $\widetilde{\rho}^{-1}(p_1)$ of index 2. Here the mapping $\widetilde{\rho}$ is given by (97).

1. Because $-\chi$ is the minimum value of $\widehat{H}$, corresponding to the critical point p_1, we find that $H^{-1}(\widetilde{\rho}^{-1}(p_{-1})) = \widehat{H}^{-1}(p_{-1}) = -\chi$. In other words, $H^{-1}(-\chi)$ is a smooth circle $\widetilde{\rho}^{-1}(p_{-1})$, which is an orbit of the S^1-action φ^{ℓ}.

2. By an equivariant version of the Morse lemma, $\widetilde{\rho}^{-1}(p_{-1})$ has a tubular neighborhood of the form $S^1 \times N$ where N is a neighborhood of 0 in $\mathbf{R}^5$ in a normal slice to the S^1 orbit $\widetilde{\rho}^{-1}(p_{-1})$ of the action φ^{ℓ}. Restricting $H+\chi$ to the normal slice gives $z_1^2+z_2^2+w_1^2+w_2^2+w_3^2 = \mathrm{Hess}_{p_{-1}}\widehat{H}$. Therefore for h values slightly greater than $-\chi$, the h-level set of H is diffeomorphic to $S^1 \times S^4$. From an equivariant version of the Morse isotopy lemma, it follows that for every $-\chi < h < \chi$ the h-level set of H is diffeomorphic to $S^1 \times S^4$.
4. We now show that for $h > \chi$ the h-level set $H^{-1}(h)$ is the total space of an S^2-bundle over SO(3). Consider the projection mapping $\tau : \mathrm{SO}(3) \times \mathrm{so}(3) \rightarrow \mathrm{SO}(3) : (A,X) \mapsto A$. Fix $A \in \mathrm{SO}(3)$ and suppose that $(A,X) \in H^{-1}(h)$. Then

$$\tfrac{1}{2}k(I(X),X) = h - \chi k(\mathrm{Ad}_{A^{-1}}E_3, E_3) \geq h - \chi > 0,$$

where the first inequality follows because

$$(k(\mathrm{Ad}_{A^{-1}}E_3, E_3))^2 \leq k(\mathrm{Ad}_{A^{-1}}E_3, \mathrm{Ad}_{A^{-1}}E_3)\, k(E_3,E_3) = 1,$$

using the Cauchy-Schwarz inequality. Thus the fiber $(\tau|H^{-1}(h))^{-1}(A)$ is a 2-sphere contained in $\{A\} \times \mathrm{so}(3)$. Since SO(3) is a Lie group, the level set $H^{-1}(h)$ is a trivial S^2 bundle over SO(3).
3. From $H = \widehat{H}\circ\widetilde{\rho}$ it follows that $H^{-1}(\chi) = \widetilde{\rho}^{-1}(\widehat{H}^{-1}(h))$. But $\widehat{H}^{-1}(h) = U$. So we get $H^{-1}(\chi) = \widetilde{\rho}^{-1}(U) = W$. Applying equivariant Morse theory at $\widetilde{\rho}^{-1}(p_1)$, we find that near $\widetilde{\rho}^{-1}(p_1)$ the level set W is a locally trivial bundle ν over S^1 with fiber $C(S^1 \times S^2)$. A closer look at the geometry of W shows that topologically $W = S^1 \times U$. In particular, the bundle ν over S^1 is trivial. We start our analysis by observing that the orbit map $\widetilde{\rho} : \mathrm{SO}(3) \times \mathrm{so}(3) \rightarrow S^2 \times \mathbf{R}^3$ of the free S^1-action φ^{ℓ} (91) is smooth, surjective and proper. Moreover, $\widetilde{\rho}$ maps the smooth manifold $W \setminus \{\widetilde{\rho}^{-1}(p_1)\}$ onto the smooth manifold $U \setminus \{p_1\}$. Because a neighborhood $\mathscr{U}$ of the singular point p_1 of U is a cone $C(S^2 \times S^1)$ on $S^2 \times S^1$ with vertex at p_1, the set $\mathscr{U}$ is contractible. Thus topologically we have $\widetilde{\rho}^{-1}(\mathscr{U}) = C(S^2 \times S^1) \times S^1$. Here $\{p_1\} \times S^1$ is the S^1-orbit $\widetilde{\rho}^{-1}(p_1)$. Since $U = (\overline{D}^2 \times S^1) \cup_{S^2 \times S^1} C(S^2 \times S^1)$ and $\overline{\mathscr{U}}$ have a common boundary, which is topologically $S^2 \times T^2$, the variety W is the disjoint union of the total space $\mathscr{V}$ of an S^1-bundle μ over $\overline{D}^3 \times S^1$ and $C(S^2 \times S^1) \times S^1$ glued together along their common boundary $S^2 \times T^2$. Below we show that the bundle μ is trivial. This implies that $W = U \times S^1$, as desired.

We now show that $\mathscr{V}$ is topologically $\overline{D}^3 \times T^2$. Consider the mapping $\sigma = \rho\circ\widetilde{\rho}$, see (92) and (97). σ sends W into $\widetilde{V}$, and is a smooth, surjective, and proper mapping of $W \setminus \{\widetilde{\rho}^{-1}(p_1)\}$ onto $\widetilde{V} \setminus \{q\}$ with fiber T^2. Now $\mathscr{V} = \sigma^{-1}(\pi^{-1}(\overline{D} \cap \{\pi_2 \geq 0\}))$. But $V \setminus \{q\}$ is contractible and $\pi^{-1}(\overline{\mathscr{D}} \cap \{\pi_2 \geq 0\})$ is a closed 3-disk $\overline{\mathscr{D}}^3$ in $\widetilde{V}$. Thus $\mathscr{V} = \sigma^{-1}(\overline{D}^3)$ is topologically $\overline{D}^3 \times T^2$.

This completes the verification of table 6.1.3. □

6.2 The discriminant locus

In this section we examine the set of critical values of the energy momentum mapping $\mathscr{E}\mathscr{M}$ (90) of the Lagrange top.

From the reduction to a one degree of freedom Hamiltonian system, see section 4.4, it follows that the set of critical values of $\mathcal{EM}$ is very closely related to the set of critical values of the twice reduced Hamiltonian H^a_b (72). More precisely, (h,a,b) is a critical value of $\mathcal{EM}$ if and only if $h^a_b = h - \frac{1}{2}I_1^{-1}(b^2-a^2) - \frac{1}{2}I_3^{-1}a^2$ is a critical value of H^a_b. In other words, the polynomial $\widetilde{W}(\sigma) = 2(\alpha - \beta\sigma)(1-\sigma^2) - (a-b\sigma)^2$ with $\alpha = I_1 h^a_b$ and $\beta = I_1\chi$ has a multiple root in $[-1,1]$, see section 6.1. Thus (a,b,α) lies in the *discriminant* $\Delta_{\widetilde{W}}$ of $\widetilde{W}$. In the following we describe the singularities of the discriminant locus ▷ $\{\Delta_{\widetilde{W}}\} = 0\}$ of $\widetilde{W}$. We start by finding a parametrization of the discriminant locus $\{\Delta_{\widetilde{W}} = 0\}$ of the polynomial $\widetilde{W}$.

(6.5) **Proof**: To simplify the discussion we choose physical units so that $\beta = \frac{1}{2}$. Then $\widetilde{W}$ becomes

$$W(\sigma) = \sigma^3 - (2\alpha + b^2)\sigma^2 + (2ab-1)\sigma + 2\alpha - a^2. \tag{99}$$

Note that (a,b,α) lies in the discriminant Δ_W of W if and only if for $(s,t) \in [-1,1]\times\mathbf{R}$ the polynomial W can be factored as

$$(\sigma - s)^2(\sigma - t) = \sigma^3 - (2s+t)\sigma^2 + (s^2+2st)\sigma - s^2t.$$

Therefore

$$2\alpha + b^2 = 2s + t \tag{100a}$$

$$2ab - 1 = s^2 + 2st \tag{100b}$$

$$a^2 - 2\alpha = s^2 t. \tag{100c}$$

Thinking of (s,t) as parameters, we solve (100a, b, c) for a, b, α as follows. Adding (100a) and (100c) to eliminate α together with (100b) gives

$$a^2 + b^2 = 2s + t + s^2 t \tag{101a}$$

$$2ab = s^2 + 2st + 1. \tag{101b}$$

Adding and subtracting (101a) and (101b) gives

$$0 \le (a+b)^2 = (s+1)^2(t+1) \quad \text{and} \quad 0 \le (a-b)^2 = (1-s)^2(t-1). \tag{102}$$

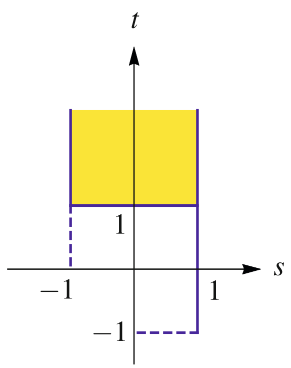


Figure 6.2.1. The set $\mathcal{S}$ of allowable values of (s,t).

For the inequalities in (102) to hold, (s,t) must lie in the set $\mathscr{S}$, see figure 6.2.1. More precisely, one of the following holds: $(s,t) \in (-1,1) \times (1,\infty)$, or $(1,t) \in \{1\} \times [-1,\infty)$, or $(-1,t) \in \{-1\} \times [1,\infty)$, or $(s,1) \in [-1,1] \times \{1\}$. Extracting the square root of both sides of the equations in (102) gives

$$a+b = \varepsilon_1(s+1)\sqrt{t+1} \text{ and } a-b = \varepsilon_2(1-s)\sqrt{t-1},$$

where $\varepsilon_1^2 = \varepsilon_2^2 = 1$. Therefore

$$a(s,t) = \tfrac{1}{2}\varepsilon_1(s+1)\sqrt{t+1} + \tfrac{1}{2}\varepsilon_2(1-s)\sqrt{t-1} \tag{103a}$$

$$b(s,t) = \tfrac{1}{2}\varepsilon_1(s+1)\sqrt{t+1} - \tfrac{1}{2}\varepsilon_2(1-s)\sqrt{t-1}. \tag{103b}$$

Substituting (103a) into (100c) we obtain

$$\alpha(s,t) = \tfrac{1}{4}\Big(2s + (1-s^2)t + \varepsilon_1\varepsilon_2(1-s^2)\sqrt{t^2-1}\Big). \tag{103c}$$

Therefore for each choice of ε_1 and ε_2 the map

$$\mathscr{P} : \mathscr{S} \subseteq \mathbf{R}^2 \to \mathbf{R}^3 : (s,t) \mapsto \big(a(s,t), b(s,t), \alpha(s,t)\big) \tag{104}$$

parametrizes a piece of the discriminant locus $\{\Delta_W = 0\}$. $\square$

Claim: The parametrization (104) is smooth except possibly when $(s,t) \in \partial\mathscr{S}$.

(6.6) **Proof**: Define a smooth function $F : \mathbf{R}^5 \to \mathbf{R}^3$ by

$$(s,t,a,b,\alpha) \mapsto \big(2\alpha + b^2 - 2s - t, 2ab - 1 - s^2 - 2st, a^2 - 2\alpha - s^2 t\big).$$

Then

$$DF(s,t,a,b,\alpha) = \begin{pmatrix} -2 & -1 & 0 & 2b & 2 \\ -2s-2t & -2s & 2b & 2a & 0 \\ -2st & -s^2 & 2a & 0 & -2 \end{pmatrix}.$$

Since the [345]-minor of DF is $8(b^2 - a^2)$, DF has rank 3 when $a^2 \neq b^2$. By the implicit function theorem, the level set $F^{-1}(0)$ is the graph of the mapping (104), except possibly for those values of $(s,t) \in \mathscr{S}$ where $a(s,t)^2 = b(s,t)^2$. Using (102) it is straightforward to check that $a(s,t) = b(s,t)$ if and only if $s = 1 \,\&\, t \in [-1,\infty)$ or $s \in [-1,1) \,\&\, t = 1$. Similarly, we see that $a(s,t) = -b(s,t)$ if and only if $s = -1 \,\&\, t \in [1,\infty)$ or $s = 1 \,\&\, t = -1$. In other words, $a(s,t)^2 = b(s,t)^2$ if and only if $(s,t) \in \partial\mathscr{S}$. $\square$

We now investigate the $\{a = \pm b\}$ slices of the discriminant locus $\{\Delta_W = 0\}$. We begin with the $\{a = b\}$ slice. Let $s = 1$. Then (103a) – (103c) become

$$a = \varepsilon\sqrt{1+t},\ b = \varepsilon\sqrt{1+t},\ \alpha = \tfrac{1}{2} \tag{105a}$$

for $\varepsilon^2 = 1$ and $t \geq -1$. Let $t = 1$. Then (103a) and (103c) become

$$a = \tfrac{1}{2}\varepsilon\sqrt{2}(1+s),\ b = \tfrac{1}{2}\varepsilon\sqrt{2}(1+s),\ \alpha = \tfrac{1}{4} + \tfrac{1}{2}s - \tfrac{1}{4}s^2 \tag{105b}$$

for $\varepsilon^2 = 1$ and $s \in [-1,1]$. Thus $\{b = a\} \cap \{\Delta_W = 0\}$ is parametrized by (105a) and (105b), see figure 6.2.2.

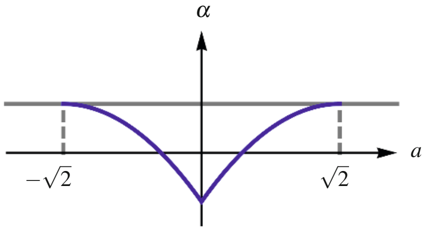


Figure 6.2.2. The $b = a$ slice of $\Delta_w = 0$.

For the $\{a = -b\}$ slice let $s = -1$. Then (103a) – (103c) become

$$a = \varepsilon\sqrt{t-1},\ b = -\varepsilon\sqrt{t-1},\ \alpha = -\tfrac{1}{2} \tag{106a}$$

with $\varepsilon^2 = 1$ and $t \geq 1$. For $s = 1$ and $t = 1$ we obtain the point

$$a = 0,\ b = 0,\ \alpha = \tfrac{1}{2}. \tag{106b}$$

Thus $\{b = -a\} \cap \{\Delta_W = 0\}$ is parametrized by (106a) and (106b), see figure 6.2.3.

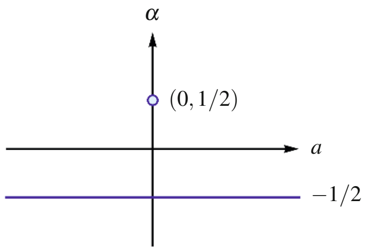


Figure 6.2.3. The $b = -a$ slice of $\Delta_w = 0$.

▷ Next we analyze the geometry of the discriminant locus $\{\Delta_W = 0\}$ near $\{b = \pm a\}$.

(6.7) **Proof**: First suppose that (a,b,α) lies on the line ℓ_ε parametrized by $a \mapsto (a, \varepsilon a, \frac{1}{2}\varepsilon)$ with $\varepsilon^2 = 1$. Set $\sigma = v + \varepsilon$ and introduce new variables $x = \varepsilon - 2\alpha$, $y = a - \varepsilon b$, $z = b$, which turns the line ℓ_ε into the z-axis. Then $W(\sigma)$ (99) becomes

$$U(v) = v^3 + (2\varepsilon + x - z^2)v^2 + 2(\varepsilon x + yz)v - y^2.$$

Recall that a cubic polynomial $p(u) = au^3 + bu^2 + cu + d$ with $a \neq 0$ has a multiple zero if and only if its *discriminant* D_p vanishes, that is, if and only if

$$D_p = b^2c^2 - 4ac^3 - 4b^3d - 27a^2d^2 + 18abcd = 0. \tag{107}$$

Therefore U has a multiple zero if and only if

$$\begin{aligned} D_U &= 4(2\varepsilon + x - z^2)^2(\varepsilon x + yz)^2 - 32(\varepsilon x + yz)^3 - 27y^4 \\ &\quad + 4y^2(2\varepsilon + x - z^2)^3 - 36y^2(2\varepsilon + x - z^2)(\varepsilon x + yz) = 0. \end{aligned} \tag{108}$$

Since the constant and linear terms of the Taylor polynomial of D_U at $(0,0,z)$ vanish, the z-axis is in the locus of singular points of $\{D_U = 0\}$. To find the tangent cone to $\{D_U = 0\}$ at $(0,0,z)$ in a plane parallel to the x–y plane, which is normal to the z-axis at $(0,0,z)$, we need to know the terms of degree two in x and y in the Taylor polynomial of D_U. A calculation gives

$$(2\varepsilon - z^2)^2(x^2 + 2\varepsilon zxy + 2\varepsilon y^2). \tag{109}$$

Let $\varepsilon = 1$. At each point of the segment $I_< = \{|z| < \sqrt{2}\}$ of the z-axis the tangent cone to $\{D_U = 0\}$ in a normal slice consists of a point. Since $(a,b,\alpha) = (0,0,\frac{1}{2})$ is an isolated point of the $\{a = -b\} \cap \{\Delta_W = 0\}$, it follows that $(x,y,z) = (0,0,0)$ is an isolated point of the intersection of $\{D_U = 0\}$ and the normal slice. Therefore the line segment $I_<$

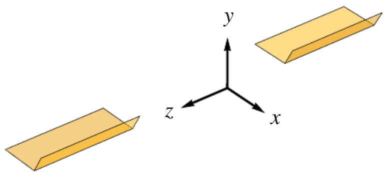


Figure 6.2.4. The crease singularity of $\Delta_w = 0$.

is *isolated* in $\{D_U = 0\}$. In other words, the line segment parametrized by $a \mapsto (a,a,\frac{1}{2})$ for $|a| < \sqrt{2}$ forms an isolated one dimensional piece of $\{\Delta_W = 0\}$ called the *thread*. This thread is perhaps the most remarkable feature of the set of critical values of the energy momentum mapping of the Lagrange top. At each point of the segment $I_> = \{|z| > \sqrt{2}\}$ of the z-axis when $\varepsilon = -1$ or on the z-axis when $\varepsilon = 1$ the tangent cone to $\{D_U = 0\}$ in the normal slice consists of two intersecting lines. Hence the tangent cone to $\{D_U = 0\}$ at each point in $I_>$ is two transversely intersecting 2-planes. Recall that for (a,b,α) on the line ℓ_{-1} or on the line segments ℓ_1 when $|a| > \sqrt{2}$, the Hamiltonian $H|(J_r^{-1}(a) \cap J_\ell^{-1}(\varepsilon a))$ assumes its minimum value. Therefore the tangent cone to $\{\Delta_W = 0\}$ forms a *crease*, see figure 6.2.4.

Second suppose that (a,b,α) lies along the curve $\mathscr{C}_\varepsilon = \{\Delta_W = 0\} \cap \{b = a\}$, which is parametrized by $a \mapsto \big(a, a, \frac{1}{2}(-a^2 + 2\sqrt{2}\varepsilon a - 1)\big)$, where $\varepsilon^2 = 1$ and $a \in I$. Here I is the interval $[0, \sqrt{2})$ if $\varepsilon = 1$ or the interval $(-\sqrt{2}, 0]$ if $\varepsilon = -1$. Set $\sigma = v + 1$ and introduce new variables $x = 2\alpha + (a^2 - 2\sqrt{2}\varepsilon a + 1)$, $y = a - b$, and $z = b$, which turns $\mathscr{C}_\varepsilon$ into the segment I of the z-axis. Then $W(\sigma)$ (99) becomes

$$\begin{aligned} U(v) &= v^3 + (4 - x + y^2 + 2yz - 2\varepsilon\sqrt{2}y - 2\varepsilon\sqrt{2}z)v^2 \\ &\quad + 2(2 - x + y^2 + 3yz + z^2 - 2\varepsilon\sqrt{2}y - 2\varepsilon\sqrt{2}z)v - y^2. \end{aligned} \tag{110}$$

A calculation shows that the constant term of the Taylor polynomial of D_U vanishes at every z in the segment I of the z-axis, whereas the linear terms in the normal slice do not. Thus $\{D_U = 0\}$ is smooth along I. So the discriminant locus $\{\Delta_W = 0\}$ is smooth on $\mathscr{C}_\varepsilon$.

Finally, suppose that $(a,b,\alpha) = (\varepsilon\sqrt{2}, \varepsilon\sqrt{2}, \frac{1}{2})$. Set $\sigma = v+1$ and introduce new variables $x = 1-2\alpha$, $y = a-b$ and $z = b-\varepsilon\sqrt{2}$. Then $(\varepsilon\sqrt{2}, \varepsilon\sqrt{2}, \frac{1}{2})$ becomes the point $(x,y,z) = (0,0,0)$ and $W(\sigma)$ (99) becomes

$$U(v) = v^3 + (x - 2\varepsilon\sqrt{2}z - z^2)v^2 + 2(x + \sqrt{2}\varepsilon y + yz)v - y^2. \tag{111}$$

We want to describe the discriminant locus $\{\Delta_U = 0\}$, that is, the set of (x,y,z) where U has a multiple root in $[0,2]$.

Claim: Let D_F be the discriminant of the cubic polynomial

$$F(u) = au^3 - bu^2 + cu - d^2, \tag{112}$$

where $a > 0$ and the constant term is a square. Let D_G be the discriminant of the special quartic polynomial

$$G(u) = u^4 - \frac{b}{2a}u^2 + \frac{d}{\sqrt{a}}u + (\frac{b^2}{16a^2} - \frac{c}{4a}). \tag{113}$$

Then the discriminant loci of F and G are equal, that is, $\{D_F = 0\} = \{D_G = 0\}$.

(6.8) **Proof**: Using (107) we see that the discriminant of F is $a^2 D_F = 4s^3 - 27t^2$, where $s = \frac{1}{3}b^2 - ac$ and $t = \frac{2}{27}b^3 - \frac{1}{3}abc + a^2d^2$. For a special quartic polynomial $Q(u) = u^4 + Au^2 + Bu + C$ its *discriminant* is $D_Q = 4S^3 - 27T^2$, where $S = A^2 + 12C$ and $T = 2A^3 + 27B^2 - 72AC$. For the polynomial G we find that $S = 3a^{-2}(\frac{1}{3}b^2 - ac)$ and $T = 27a^{-3}(\frac{2}{27}b^3 - \frac{1}{3}abc + a^2d^2)$. Hence $\frac{a^6}{27}D_G = a^2 D_F$. □

From ((6.8)) it follows that the locus $\{D_U = 0\}$ of multiple zeroes of the cubic polynomial U (111) is the same as the locus of multiple zeroes of the special quartic polynomial

$$Y(v) = v^4 + \tfrac{1}{2}(x - 2\varepsilon\sqrt{2}z - z^2)v^2 + yv + \tfrac{1}{16}(x - 2\varepsilon\sqrt{2}z - z^2)^2 - \tfrac{1}{2}(x + \varepsilon\sqrt{2}y + yz),$$

which is the well known *swallowtail surface*, see figure 6.2.5. The *double line* of the

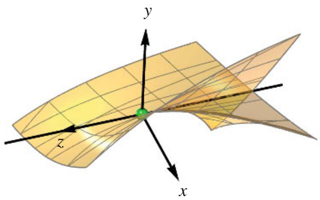


Figure 6.2.5. The swallowtail surface $\{D_Y = 0\}$ near $(0,0,0)$, when $\varepsilon = 1$. When $\varepsilon = -1$, the surface must be reflected in the x-y plane.

the special quartic Q is the union of the *line of self intersection* given by $B = 0$ & $A^2 - 4C = 0$ & $A \le 0$ and the *whisker* given by $B = 0$ & $A^2 - 4C = 0$ & $A \ge 0$. The double line of Y is the z–axis, on which $Y(v) = \big(v^2 - \frac{1}{4}(2\varepsilon\sqrt{2}z + z^2)\big)^2$. On the z-axis Y has a zero of multiplicity 4 at $z = 0$ and at $z = -2\varepsilon\sqrt{2}$, which are *swallowtail points*. Y has a double real root on the z-axis when $z \in I_1$, where I_1 is either $(-\infty, -2\sqrt{2}) \cup (0,\infty)$ when $\varepsilon = 1$ or $(-\infty, 0) \cup (0, 2\sqrt{2})$ when $\varepsilon = -1$. Y has a double purely imaginary root on the z-axis when $z \in I_2$, where I_2 is either $(-2\sqrt{2}, 0)$, when $\varepsilon = 1$, or $(0, 2\sqrt{2})$, when $\varepsilon = -1$. Therefore the line of self intersection of $\{D_U = 0\}$ is the union of the segments of the z-axis where $z \in I_1$; whereas the whisker of $\{D_U = 0\}$ is the segment of the z-axis where

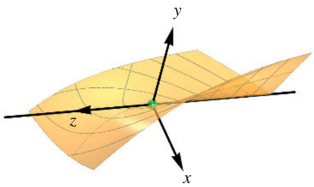


Figure 6.2.6. The discriminant locus $\Delta_W = 0$ near the swallowtail point $(\sqrt{2}, \sqrt{2}, 1/2)$, when $\varepsilon = 1$. When $\varepsilon = -1$, the surface must be reflected in the x-y plane.

$z \in I_2$. For the *discriminant locus* $\{\Delta_W = 0\}$ the above discussion translates into the following. The points $(a, b, \alpha) = (\varepsilon\sqrt{2}, \varepsilon\sqrt{2}, \frac{1}{2})$ are swallowtail points. The whisker is the thread, which is parametrized by $a \mapsto (a, a, \frac{1}{2})$ where $|a| < \sqrt{2}$. The line of self intersection is parametrized by $a \mapsto (a, a, \frac{1}{2})$ where $|a| > \sqrt{2}$. Since the line of self intersection of $\{D_Y = 0\}$ is a crease singularity, the discriminant locus $\{\Delta_W = 0\}$ is missing the *tail* of the swallowtail surface, see figure 6.2.6. Another way of saying this is that the tail of the swallowtail surface of $\{D_Y = 0\}$ corresponds to multiple roots of Y which do not lie in $[0, 2]$. In figure 6.2.6 we have sketched the discriminant locus $\{\Delta_W = 0\}$ near a swallowtail point. $\square$

Under the mapping $(a, b, \alpha) \mapsto (a, b, h) = \big(a, b, I_1^{-1}\alpha + \frac{1}{2}I_1^{-1}(b^2 - a^2) + \frac{1}{2}I_3^{-1}a^2\big)$ the image of the discriminant locus $\{\Delta_W = 0\}$ is the set of critical values of $\mathscr{E}\mathscr{M}$, see figure 6.2.7.

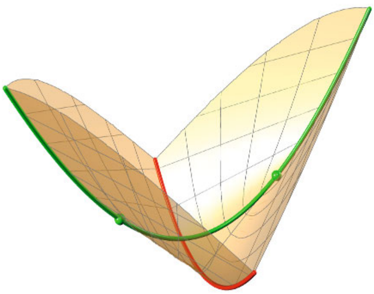


Figure 6.2.7. The set of critical values of the energy momentum mapping $\mathscr{E}\mathscr{M}$ of the Lagrange top.

We end this subsection by describing the motion of the top which corresponds to a fixed critical value of the energy momentum mapping $\mathscr{E}\mathscr{M}$. Suppose that the critical value (h,a,b) lies on the smooth two dimensional piece of the set of critical values. In phase space, the motion takes place on a two dimensional torus, see table 6.1.1. In physical space the top is spinning at a constant speed about its figure axis, which makes a fixed angle with the vertical axis, while turning uniformly about the vertical axis. In other words, the top is undergoing *regular precession*. If the critical value lies on a crease in the $\{b=-a\}$ plane, then the top is spinning at a constant speed about its figure axis which is pointing vertically downward. If the critical value lies on the crease in the $\{b=a\}$ plane, then the top is spinning at a constant speed about a figure axis, which is pointing vertically upward. Here the top is said to be *sleeping*. This motion is stable because the total energy of the top on the intersection of the appropriate level sets of the angular momenta is at a minimum. If the critical value lies on the thread, then the top is either spinning at a constant speed about its figure axis which is vertical or moving so that it is asymptotic to this motion as time goes to $\pm\infty$. Here the top is said to be *waking*. This motion is unstable except at the points where the thread attaches itself to the two dimensional piece of the set of critical values. At the points of attachment the motion is stable because it is surrounded by *bounded* motions which either lie on tori or on the stable or unstable manifolds of hyperbolic periodic orbits.

6.3 The period lattice

Let $\mathscr{R}$ be the set of regular values which lie in the image of the energy momentum mapping $\mathscr{E}\mathscr{M}$. We have shown that if $(h,a,b)\in\mathscr{R}$, then $\mathscr{E}\mathscr{M}^{-1}(h,a,b)$ is a smooth 3-torus $T^3_{h,a,b}$. On $T^3_{h,a,b}$ the motion of the top is the superposition of three circular motions, namely, a constant *spin* about its figure axis; a constant *precession* of its figure axis about the vertical axis; and a variable up and down *nutating* motion of its figure axis. In this subsection we look more closely at how these smooth 3-tori are defined by discussing the concept of period lattice $\mathscr{P}(h,a,b)$ associated to the smooth 3-torus $T^3_{h,a,b}$ where $(h,a,b)\in\mathscr{R}$.

The *period lattice* $\mathscr{P}(h,a,b)$ is a lattice over $\mathbf{Z}$ generated by certain Hamiltonian vector fields X_F. Specifically, for some open neighborhood $\mathscr{U}$ of (h,a,b) in $\mathscr{R}$, the function F lies in the free $C^\infty(\mathscr{U})$-module generated by the angular momenta J_r, J_ℓ, and the energy H. Moreover, the flow of the vector field $X_F|T^3_{h',a',b'}$ is periodic of period 1 for every $(h',a',b')\in\mathscr{U}$.

$\triangleright$ The following argument constructs a basis for the period lattice $\mathscr{P}(h,a,b)$.

(6.9) **Proof**: Clearly, the vector fields $2\pi X_{J_r}$ and $2\pi X_{J_\ell}$ on $T^3_{h,a,b}$ are elements of a basis for $\mathscr{P}(h,a,b)$. To construct the third basis element, note that $T^3_{h,a,b}=(\pi^a_b\circ\pi^a)^{-1}\big((H^a_b)^{-1}(h^a_b)\big)$, where $\pi^a: J_r^{-1}(a)\rightarrow P^a$ is the reduction map of the S^1-action $\varphi^r|\big(S^1\times J_r^{-1}(a)\big)$ (11) and $\pi^a_b:(J^a_\ell)^{-1}(b)\subseteq P^a\rightarrow P^a_b$ is the reduction map of the induced S^1-action δ^ℓ (38) on P^a. On the smooth 2-torus $T^2_{h^a,b}=(\pi^a_b)^{-1}\big((H^a_b)^{-1}(h^a_b)\big)$ the vector field $2\pi X_{J^a_\ell}$ has a periodic flow of period 1. Note that $\pi^a_b(T^3_{h,a,b})=T^2_{h^a,b}$. Here $h^a=h^a_b+\frac{1}{2}I_1^{-1}b^2$ and $\pi^a_b((H^a)^{-1}(h^a))=(H^a_b)^{-1}(h^a_b)$. Let Θ_ℓ be the rotation number of the flow of $X_{H^a}|T^2_{h^a,b}$

and let T be the period of the flow of $-\mathrm{ad}_{H^a_b}$ on the circle $(H^a_b)^{-1}(h^a_b)$. From the definition of rotation number it follows that the vector field $TX_{H^a} - 2\pi\Theta_\ell X_{J^a_\ell}$ on $T^2_{h^a,b}$ has a periodic flow of period 1. Choose a point p on $T^3_{h,a,b}$ and let $\mathscr{C}$ be the image of an integral curve of X_{J_r} through p. Since X_{J_r} has a periodic flow on $T^3_{h,a,b}$ of period 1, $\mathscr{C}$ is a closed curve. Consider an integral curve $t \mapsto \gamma(t)$ of the vector field $Y = X_H - \frac{2\pi\Theta_\ell}{T}X_{J_\ell}$ which starts at p. Since $t \mapsto \pi^a_b \circ \pi^a(\gamma(t))$ is a periodic integral curve of $-\mathrm{ad}_{H^a_b}$ of period T, it follows that T is the least positive time such that $\gamma(T) \in \mathscr{C}$. Thus $\mathscr{C}$ is a cross section for the flow of Y on $T^3_{h,a,b}$. During the time T the integral curve of X_{J_r}, which starts at p, has travelled through an angle $2\pi\Theta_r$. By construction, the vector field $TX_H - 2\pi\Theta_\ell X_{J_\ell} - 2\pi\Theta_r X_{J_r}$ has periodic flow on $\mathscr{E}\mathscr{M}^{-1}(h,a,b)$ of period 1. Thus the period lattice $\mathscr{P}(h,a,b)$ of the smooth 3-torus $\mathscr{E}\mathscr{M}^{-1}(h,a,b)$ has a basis $\{X_{2\pi J_r},\, X_{2\pi J_\ell},\, X_{TH-2\pi\Theta_\ell J_\ell - 2\pi\Theta_r J_r}\}$. $\square$

To give a geometric interpretation of the rotation number Θ_r, we reduce the Lagrange top to the Hamiltonian system $(H^a_b, P^a_b, \{\,,\,\}_{P^a_b})$ with one degree of freedom. This time we first reduce by the *left* S^1-action and then by the induced *right* S^1-action. Because the argument follows along the same lines as that given in §3.1 and §4, we give only the high points, leaving the details to the reader. Recall that $S^1 = \{B \in \mathrm{SO}(3) \,|\, \mathrm{Ad}_B E_3 = E_3\}$. The left S^1-action on $T\,\mathrm{SO}(3)$, given by $\Phi_\ell : S^1 \times T\,\mathrm{SO}(3) \to T\,\mathrm{SO}(3) : \big(B, X_A\big) \mapsto X_{BA}$, becomes the action

$$\varphi_\ell : S^1 \times \big(\mathrm{SO}(3) \times \mathrm{so}(3)\big) \to \mathrm{SO}(3) \times \mathrm{so}(3) : \big(B, (A,X)\big) \mapsto (BA, X),$$

after pulling back by the left trivialization $\mathscr{L}$ (10). The action Φ_ℓ is Hamiltonian with momentum mapping $\mathscr{J}_\ell : T\mathrm{SO}(3) \to \mathbf{R} : X_A \mapsto \rho(e)(\mathrm{Ad}_{A^{-1}} E_3, X)$. Under $\mathscr{L}$, the map $\mathscr{J}_\ell$ pulls back to the momentum mapping

$$J_\ell : \mathrm{SO}(3) \times \mathrm{so}(3) \to \mathbf{R} : (A,X) \mapsto k\big(\mathrm{Ad}_{A^{-1}} E_3, I(X)\big)$$

of the action φ_ℓ. Since every value b is a regular value of J_ℓ, the level set $J_\ell^{-1}(b)$ is a smooth manifold. Because φ_ℓ acts freely and properly on $J_\ell^{-1}(b)$, the orbit space $J_\ell^{-1}(b)/S^1$ is a smooth manifold. Let $\mathscr{P}_b = \{(Z,W) \in \mathrm{SO}(3) \times \mathrm{so}(3) \,\big|\, k(Z,Z) = 1 \;\&\; k(Z,W) = b\}$. From the fact that the orbit map

$$\pi_b : J_\ell^{-1}(b) \to \mathscr{P}_b : (A,X) \mapsto \big(\mathrm{Ad}_{A^{-1}} E_3, I(X)\big) = (Z,W)$$

is a submersion and every fiber $\pi_b^{-1}(Z,W)$ is a single φ_ℓ orbit, it follows that the orbit space $J_\ell^{-1}(b)/S^1$ is diffeomorphic to $\mathscr{P}_b$. By the regular reduction theorem, the *reduced space* $\mathscr{P}_b$ has a symplectic form

$$\begin{aligned}\Omega_b(Z,W)\big(T\pi_b(-[U, \mathrm{Ad}_{A^{-1}} E_3], I(R)), T\pi_b(-[V, \mathrm{Ad}_{A^{-1}} E_3], I(S))\big) &= \\ = -k(I(R),V) + k(I(S),U) + k(I(Z),[U,V]),&\end{aligned}$$

where the arguments of $\Omega_b(Z,W)$ lie in $T_{(z,w)}P^a$. Under $\mathscr{L}$ the Hamiltonian of the Lagrange top

$$\mathscr{H} : T\,\mathrm{SO}(3) \to \mathbf{R} : X_A \mapsto \tfrac{1}{2}\rho(A)(X_A, X_A) + \chi k(\mathrm{Ad}_{A^{-1}} E_3, E_3)$$

pulls back to $H : \mathrm{SO}(3) \times \mathrm{so}(3) \to \mathbf{R} : (X,A) \mapsto \frac{1}{2}k(I(X),X) + \chi k(\mathrm{Ad}_{A^{-1}} E_3, E_3)$. Since H is invariant under the S^1-action φ_ℓ, it induces a function

$$\widetilde{H}_b : \mathscr{P}_b \to \mathbf{R} : (Z,W) \mapsto \tfrac{1}{2}k(I^{-1}(W),W) + \chi k(Z,E_3).$$

If h is the value of H, then $h_b = h$ is the value of the *reduced Hamiltonian* $\widetilde{H}_b$.

Using the identification map $i : \mathrm{so}(3) \to \mathbf{R}^3$, see chapter III ((1.2)), we obtain a second model (H_b, P_b, ω_b) of the *reduced Hamiltonian system* $(\widetilde{H}_b, \mathscr{P}_b, \Omega_b)$. Here the reduced phase space is $P_b = \{(z,w) \in \mathbf{R}^3 \times \mathbf{R}^3 \,\big|\, (z,z) = 1 \;\&\; (z,w) = b\}$ with reduction map

$$\pi_b : J_\ell^{-1}(b) \to P_b : (A,X) \mapsto (z,w) = \big(i(Z), i(W)\big). \tag{114}$$

The symplectic form is

$$\omega_b(z,w)\big((-u \times z, \xi), (-v \times z, \eta)\big) = -(\xi, v) + (\eta, u) + (w, u \times v).$$

The arguments of $\omega_b(z,w)$ lie in $T_{(z,w)}P_b$. The *reduced Hamiltonian* is

$$H_b : P_b \subseteq \mathbf{R}^3 \times \mathbf{R}^3 \to \mathbf{R} : (z,w) \mapsto \tfrac{1}{2}I_1^{-1}(w_1^2 + w_2^2) + \tfrac{1}{2}I_3^{-1}w_3^2 + \chi z_3.$$

On $\mathbf{R}^3 \times \mathbf{R}^3$ Hamilton's equations are

$$\dot{z} = z \times I^{-1}(w) \tag{115a}$$

$$\dot{w} = z \times \chi e_3 + w \times I^{-1}(w), \tag{115b}$$

where $I^{-1}(w) = (I_1^{-1}w_1, I_2^{-1}w_2, I_3^{-1}w_3)$, are the *left Euler-Poisson equations* for H_b. They are the negative of the right Euler-Poisson equations (22) of §3.1. Restricting the left Euler-Poisson equations to P_b gives the Hamiltonian vector field X_{H_b} of the reduced Hamiltonian H_b.

The right Hamiltonian action $\Phi^r : T\,\mathrm{SO}(3) \times S^1 \to T\,\mathrm{SO}(3) : (X_A, B) \mapsto X_{AB}$ with *momentum mapping* $\mathscr{J}_r : T\,\mathrm{SO}(3) \to \mathbf{R} : X_A \mapsto \rho(A)(X_A, T_eL_AE_3)$, when pulled back by the left trivialization $\mathscr{L}$ gives the right Hamiltonian action

$$\varphi^r : \big(\mathrm{SO}(3) \times \mathrm{so}(3)\big) \times S^1 \to \mathrm{SO}(3) \times \mathrm{so}(3) : \big((A,X),B\big) \mapsto (AB, \mathrm{Ad}_{B^{-1}} X)$$

with momentum mapping $J_r : \mathrm{SO}(3) \times \mathrm{so}(3) \to \mathbf{R} : (A,X) \mapsto k(I(X),E_3)$. Note that the induced right S^1-action on P_a is the restriction of the diagonal action

$$\Delta : S^1 \times (\mathbf{R}^3 \times \mathbf{R}^3) \to \mathbf{R}^3 \times \mathbf{R}^3 : \big(t,(z,w)\big) \mapsto (\widetilde{R}_t z, \widetilde{R}_t w), \tag{116}$$

where $\widetilde{R}_t = \begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Since $\mathscr{J}_r$ is invariant under the left S^1 action Φ_ℓ, it *induces the function* $J_r^b : P_b \to \mathbf{R} : (z,w) \mapsto w_3$, which is an integral of the reduced vector field X_{H_b}. Let $EM_b : P_b \to \mathbf{R}^2 : (z,w) \mapsto \big(H_b(z,w), J_r(z,w)\big)$ be the *energy momentum mapping* of X_{H_b}.

After this preparation we are ready to give a geometric interpretation of the angle $2\pi\Theta_r$.

Claim: Let (h_b, a) be a regular value of EM_b. On the smooth 2-torus $T^2_{h_b,a}$, which is the (h_b, a)-level set of EM_b, the rotation number of the flow of X_{H_b} is Θ_r.

(6.10) **Proof**: Note that the image of the 3-torus $T^3_{h,a,b}$ under the reduction map π_b is a smooth 2-torus $T^2_{h_b,a}$. Since $T\pi_b X_H = X_{H_b} \circ \pi_b$, $T\pi_b X_{J_r} = X_{J_r^b} \circ \pi_b$, and $T\pi_b X_{J_\ell} = 0$, we see that the image of the vector field $Y = TX_H - 2\pi\Theta_\ell X_{J_\ell} - 2\pi\Theta_r X_{J_r}$ on $T^3_{h,a,b}$ under $T\pi_b$ is the vector field $Z = (TX_{H_b} - 2\pi\Theta_r X_{J_r^b}) \circ \pi_b$ on $T^2_{h_b,a}$. The vector field Z has a periodic flow of period 1 on $T^2_{h_b,a}$ because the vector field Y belongs to the period lattice $\mathscr{P}(h,a,b)$ associated to the 3-torus $T^3_{h,a,b}$. Therefore Θ_r is the rotation number of X_{H_b} on $T^2_{h_b,a}$. □

▷ In order to compute the monodromy in the next subsection, we need an explicit formula for the rotation number Θ_r, see (119).

(6.11) **Proof**: We find a formula for Θ_r using the same technique as we used to find the rotation number Θ_ℓ, see §5.3. Look at the reduced space P_b as a bundle over S^2 with bundle projection $\tau : P_b \to S^2 : (z, w) \mapsto z$. On $\tau(T^2_{h_b,a})$ introduce coordinates $\varphi = \tan^{-1}\frac{z_2}{z_1}$ and z_3. The equations on $\tau(T^2_{h_b,a})$, which are satisfied by the image of an integral curve of $X_{H_b}|T^2_{h_b,a}$ under τ, are obtained as follows. From the third component of the first Euler-Poisson equation (115a) we find that

$$\begin{aligned}
\dot{z}_3 &= I_1^{-1}(z_1 w_2 - z_2 w_1) \\
&= \eta I_1^{-1}\big((z_1^2+z_2^2)(w_1^2+w_2^2) - (z_1 w_1 + z_2 w_2)^2\big)^{1/2}, \quad \text{where } \eta^2 = 1 \\
&= \eta I_1^{-1}\big((1-z_3^2)\big(2I_1(h - \tfrac{1}{2}I_3^{-1}a^2 - \chi z_3)\big) - (b - a z_3)^2\big)^{1/2}
\end{aligned} \tag{117}$$

since

$$\begin{aligned}
z_1^2 + z_2^2 + z_3^2 &= 1, \\
z_1 w_1 + z_2 w_2 + z_3 w_3 &= b, \\
\tfrac{1}{2}I_1^{-1}(w_1^2 + w_2^2) + \tfrac{1}{2}I_3^{-1}w_3^2 + \chi z_3 &= h_b = h \\
w_3 &= a
\end{aligned}$$

are the defining equations for $T^2_{h_b,a}$. From the definition of the angle φ we get $\dot{\varphi} = L_{X_{H_b}}\varphi = (z_1\dot{z}_2 - z_2\dot{z}_1)(z_1^2+z_2^2)^{-1}$. Using (115a) we obtain

$$\begin{aligned}
\dot{\varphi} &= \left[I_1^{-1}z_3(z_1w_1 + z_2w_2) - I_3^{-1}w_3(z_1^2+z_2^2)\right](z_1^2+z_2^2)^{-1} \\
&= \left[I_1^{-1}z_3(b - az_3) - I_3^{-1}a(1-z_3^2)\right](1-z_3^2)^{-1}.
\end{aligned} \tag{118}$$

Choosing $\eta = -1$ and using (117) and (118) shows that the rotation number Θ_r is given by

$$\begin{aligned}
2\pi\Theta_r = 2\int_{z_3^-}^{z_3^+} \frac{\dot{\varphi}}{\dot{z}_3}\,\mathrm{d}z_3 &= 2\int_{z_3^-}^{z_3^+} \frac{z_3(b - az_3)\,\mathrm{d}z_3}{(1-z_3^2)\sqrt{2(\widetilde{\alpha} - \beta z_3)(1-z_3^2) - (b-az_3)^2}} \\
&\quad - 2a\frac{I_1}{I_3}\int_{z_3^-}^{z_3^+} \frac{\mathrm{d}z_3}{\sqrt{2(\widetilde{\alpha} - \beta z_3)(1-z_3^2) - (b-az_3)^2}},
\end{aligned} \tag{119}$$

where $\widetilde{\alpha} = I_1(h - \frac{1}{2}I_3{}^{-1}a^2)$ and $\beta = I_1\chi$. In addition, $z_3^{\pm} \in [-1,1]$ are roots of the polynomial $2(\widetilde{\alpha} - \beta z_3)(1 - z_3^2) - (b - az_3)^2$. $\square$

6.4 Monodromy

In this subsection we examine how the 3-torus fibers of $\mathscr{EM}$ fit together. Let $\widetilde{\Gamma}$ be a loop in the set of regular values $\mathscr{R}$ of $\mathscr{EM}$, which bounds a disk in $\mathbf{R}^3$ that intersects the *thread* at one point. The curve $\widetilde{\Gamma}$ is not null homotopic in $\mathscr{R}$. As $\widetilde{\Gamma}$ is traced out once, we will show that the *variation* of the period lattice associated to the 3-torus $T^3_{h,a,b}$, $(h,a,b) \in \widetilde{\Gamma}$, is nonzero and gives rise to monodromy. Another way to say this is that the level sets of $\mathscr{EM}$ over points in $\widetilde{\Gamma}$ fit together to form a *nontrivial* smooth 3-torus bundle over $\widetilde{\Gamma}$.

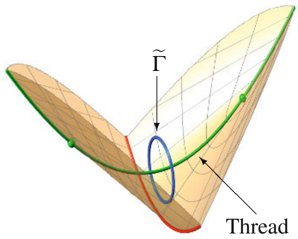


Figure 6.4.1. The loop Γ.

Next we turn to discussing monodromy, which we now define. Let $\widetilde{\Gamma}$ be a loop as given above. Since $\mathscr{EM}$ restricted to $\mathscr{EM}^{-1}(\widetilde{\Gamma})$ is a proper submersion, it follows that $\mathscr{EM}^{-1}(\widetilde{\Gamma})$ is a smooth 3-torus bundle over $\widetilde{\Gamma}$. Because $I = \widetilde{\Gamma} \setminus \{\mathrm{pt}\}$ is contractible, the bundle $\mathscr{EM}^{-1}(I)$ is trivial. Therefore the bundle $\mathscr{EM}^{-1}(\widetilde{\Gamma})$ is determined by an orientation preserving diffeomorphism χ of the fiber $T^3 = \mathscr{EM}^{-1}(\mathrm{pt})$ into itself, called the *monodromy map*, see chapter X §1 or chapter IV §5. The map χ induces the map $\chi_* : H^1(T^3,\mathbf{Z}) \to H^1(T^3,\mathbf{Z})$, which is the *monodromy* of the bundle $\mathscr{EM}^{-1}(\widetilde{\Gamma})$. The goal of the rest of this subsection is to show that $\chi_* = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$. The map χ is not homotopic to the identity map on $\mathscr{EM}^{-1}(\mathrm{pt})$, because χ_* is *not* conjugate in $\mathrm{Sl}(3,\mathbf{Z})$ to the identity map. Consequently, the bundle is *nontrivial*. This fact was unknown classically and represents a new qualitative feature of the Lagrange top.

To compute the monodromy, we need to find the variation in the period lattice as (h,a,b) runs over the closed curve $\widetilde{\Gamma}$. The loop $\widetilde{\Gamma}$ may be smoothly homotoped within $\mathscr{R}$ to a loop Γ in the $\{b = 0\}$ plane which bounds a disk that intersects the thread at $(0,0,\frac{1}{2})$. This homotopy does not change the diffeomorphism type of the three torus bundle. In other words, the monodromy maps of the bundles $\mathscr{EM}^{-1}(\widetilde{\Gamma})$ and $\mathscr{EM}^{-1}(\Gamma)$ are homotopic and
▷ hence the bundles have the same monodromy. We now find the variation in the period lattice $\mathscr{P}(h,a,0)$ associated to the 3-torus fiber $T^3_{h,a,0}$ as $(h,a,0)$ runs once around the curve Γ, see figure 6.4.2.

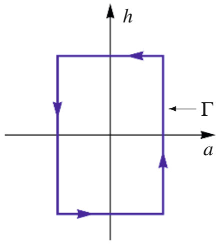


Figure 6.4.2. The curve Γ.

(6.12) **Proof**: The period lattice $\mathscr{P}(h,a,0)$ has a basis

$$\{2\pi X_{J_r},\ 2\pi X_{J_\ell},\ TX_H - 2\pi\Theta_r^0 X_{J_r} - 2\pi\Theta_\ell^0 X_{J_\ell}\}, \tag{120}$$

where $\Theta_\ell^0 = \Theta_\ell\Big|_{b=0}$ and $\Theta_r^0 = \Theta_r\Big|_{b=0}$. From (88) we find that

$$2\pi\Theta_\ell^0 = -2a\int_{\sigma^-}^{\sigma^+} \frac{\sigma\,\mathrm{d}\sigma}{(1-\sigma^2)\sqrt{2(\alpha-\beta\sigma)(1-\sigma^2)-a^2}},$$

where $\alpha = I_1 h^a = I_1\big(h+\frac{1}{2}(I_1^{-1}-I_3^{-1})a^2\big)$ and $\beta = I_1\chi$. From (119) we get

$$\begin{aligned}2\pi\Theta_r^0 = -2a\int_{z_3^-}^{z_3^+} &\frac{z_3^2\,\mathrm{d}z_3}{(1-z_3^2)\sqrt{2(\widetilde{\alpha}-\beta z_3)(1-z_3^2)-a^2z_3^2}}\\ &-2aI_1I_3^{-1}\int_{z_3^-}^{z_3^+}\frac{\mathrm{d}z_3}{\sqrt{2(\widetilde{\alpha}-\beta z_3)(1-z_3^2)-a^2z_3^2}},\end{aligned}$$

where $\widetilde{\alpha} = I_1(h-\frac{1}{2}I_3^{-1}a^2)$. We calculate the variation in the period lattice along Γ by taking the limit of Γ as $a \searrow 0$, see figure 6.4.2. Using an argument similar to the one used in the spherical pendulum, see chapter IV §4, we find that $\widetilde{\Theta_\ell} = \lim_{a\searrow 0}\Theta_\ell^0$ and $\widetilde{\Theta_r} = \lim_{a\searrow 0}\Theta_r^0$ are equal to $-\frac{1}{2}$ if $h \in (-\chi,\chi)$ or -1 if $h\in(\chi,\infty)$. Thus the variation in Θ_ℓ^0 and Θ_r^0 after one loop around Γ is -1. Hence the initial basis (120) of the period lattice $\mathscr{P}(h,a,0)$ becomes the basis

$$\{2\pi X_{J_r},\ 2\pi X_{J_\ell}, TX_H - 2\pi(\Theta_r^0-1)X_{J_r} - 2\pi(\Theta_\ell^0-1)X_{J_\ell}\}, \tag{121}$$

after running around Γ once. Thus the *variation* in the bases of $\mathscr{P}(h,a,0)$ is given by the matrix $\begin{pmatrix}0&0&0\\0&0&0\\1&1&0\end{pmatrix}$. Therefore the monodromy of $\mathscr{EM}^{-1}(\Gamma)$, which is the matrix taking the initial basis of $\mathscr{P}(h,a,0)$ into the final basis, is $\chi_* = \begin{pmatrix}1&0&0\\0&1&0\\1&1&1\end{pmatrix}$. Conjugating χ_* by $P = \begin{pmatrix}1&0&0\\1&1&0\\0&0&1\end{pmatrix} \in \mathrm{Sl}(3,\mathbf{Z})$ gives $P\chi_*P^{-1} = \begin{pmatrix}1&0&0\\0&1&0\\0&1&1\end{pmatrix}$, which is a standard form for the monodromy mapping of the 3-torus bundle $\mathscr{EM}^{-1}(\Gamma)$.

7 Hamiltonian Hopf bifurcation

Consider the family of motions of the Lagrange top where the figure axis of the top is vertical and the spin $|a|$ about the figure axis is increased through $|a| = 2\sqrt{\beta}$. These motions are interesting because the top goes from unstable to stable motion. In physical terms, the top has become *gyroscopically stabilized*. Mathematically, this family of motions corresponds to the set of critical values of the energy momentum mapping where the thread attaches itself to the two dimensional piece of the discriminant locus and then becomes a crease. The goal of this section is to show that the top has undergone a *Hamiltonian Hopf bifurcation*, which is the mathematical explanation of gyroscopic stabilization.

7.1 The linear Hamiltonian Hopf bifurcation

In this subsection we carry out a linear analysis of the motion of the top when its figure axis is vertical. We show that the left Euler-Poisson equations undergo a *linear Hamiltonian Hopf bifurcation*.

After reducing the left S^1-action on $\mathrm{SO}(3) \times \mathrm{so}(3)$, the motion of the Lagrange top is governed by the *left* Euler-Poisson equations on $\mathbf{R}^3 \times \mathbf{R}^3$

$$\begin{aligned} \dot{z} &= z \times I^{-1}(w) \\ \dot{w} &= z \times \chi e_3 + w \times I^{-1}(w). \end{aligned} \tag{122}$$

Here $I = \mathrm{diag}(I_1, I_1, I_3)$ with $0 < I_3 \le 2I_1$, $0 < I_1 < I_3$ and $\chi > 0$. The Euler-Poisson equations are in Hamiltonian form on $(\mathbf{R}^3 \times \mathbf{R}^3, \{\,,\,\}_{\mathbf{R}^3 \times \mathbf{R}^3})$ with Hamiltonian

$$F_b : \mathbf{R}^3 \times \mathbf{R}^3 \to \mathbf{R} : (z, w) \to \tfrac{1}{2} I_1^{-1}(w_1^2 + w_2^2) + \tfrac{1}{2} I_3^{-1} w_3^2 + \chi z_3.$$

The structure matrix $W_{C^\infty(\mathbf{R}^3 \times \mathbf{R}^3)}$ of the Poisson bracket $\{\,,\,\}_{\mathbf{R}^3 \times \mathbf{R}^3}$ is given in table 7.1.1.

$\{A,B\}$	z_1	z_2	z_3	w_1	w_2	w_3	B
z_1	0	0	0	0	z_3	$-z_2$	
z_2	0	0	0	$-z_3$	0	z_1	
z_3	0	0	0	z_2	$-z_1$	0	
w_1	0	z_3	$-z_2$	0	w_3	$-w_2$	
w_2	$-z_3$	0	z_1	$-w_3$	0	w_1	
w_3	z_2	$-z_1$	0	w_2	$-w_1$	0	
A							

Table 7.1.1. The structure matrix $W_{C^\infty(\mathbf{R}^3 \times \mathbf{R}^3)}$ of the Poisson bracket $\{\,,\,\}_{\mathbf{R}^3 \times \mathbf{R}^3}$.

To simplify matters we rescale the time by setting $s = I_1 t$. Then (122) becomes

$$\begin{aligned} \dot{z} &= z \times \widetilde{I}^{-1}(w) \\ \dot{w} &= z \times \beta e_3 + w \times \widetilde{I}^{-1}(w), \end{aligned} \tag{123}$$

where $\widetilde{I} = \mathrm{diag}(1, 1, \gamma)$, $1 > \gamma = I_1/I_3 \ge \frac{1}{2}$ and $\beta = I_1 \chi > 0$. The rescaled Euler-Poisson equations (123) are in Hamiltonian form on $(\mathbf{R}^3 \times \mathbf{R}^3, \{\,,\,\}_{\mathbf{R}^3 \times \mathbf{R}^3})$ with Hamiltonian

$$\widetilde{F}_b : \mathbf{R}^3 \times \mathbf{R}^3 \to \mathbf{R} : (z, w) \to \tfrac{1}{2}(w_1^2 + w_2^2) + \tfrac{1}{2}\gamma w_3^2 + \beta z_3. \tag{124}$$

After reduction of the left S^1-action, see §6.3, the motion of the top takes place on the reduced phase space $P_b = \{(z,w) \in \mathbf{R}^3 \times \mathbf{R}^3 \big| (z,z) = 1 \;\&\; (z,w) = b\}$, which is defined by the 0-level sets of $C_1 = z_1^2 + z_2^2 + z_3^2 - 1$ and $C_2 = z_1 w_1 + z_2 w_2 + z_3 w_3 - b$. Sleeping or waking motion of the top occurs when the left and right S^1 angular momenta are equal, that is, $b = a$. After reduction, these motions correspond to the point $p_a = (e_3, ae_3)$ on P_a, which is an equilibrium point of $X_{\widetilde{F}_a}|P_a$. Linearizing $X_{\widetilde{F}_a}$ about p_a and then restricting to

$$T_{p_a}P_a = \ker \begin{pmatrix} 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & a & 0 & 0 & 1 \end{pmatrix} = \mathrm{span}\{e_1, e_2, e_4, e_5\}$$

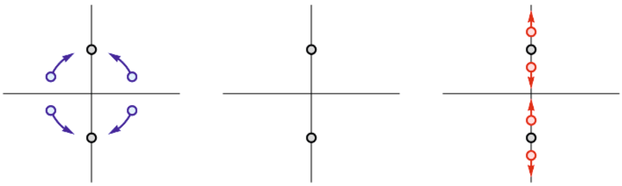
gives the smooth family of 4×4 matrices

$$a \to R_a = \begin{pmatrix} 0 & a\gamma & 0 & -1 \\ -a\gamma & 0 & 1 & 0 \\ 0 & \beta & 0 & a(\gamma-1) \\ -\beta & 0 & -a(\gamma-1) & 0 \end{pmatrix}. \tag{125}$$

Since the structure matrix $\widetilde{W}_a = W(p_a)|T_{p_a}P_a$ is invertible,

$$\Omega_a = (\widetilde{W}_a^{-1})^t = \begin{pmatrix} 0 & -a & 0 & 1 \\ a & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

is a symplectic form on $T_{p_a}P_a$. The matrix R_a is infinitesimally symplectic with respect to Ω_a, that is, $R_a^t \Omega_a + \Omega_a R_a = 0$, or equivalently $R_a \in \mathrm{sp}(\Omega_a, \mathbf{R})$.



$|a| < 2\sqrt{\beta}$ $\qquad$ $|a| = 2\sqrt{\beta}$ $\qquad$ $|a| > 2\sqrt{\beta}$

Figure 7.1.1. Movement of the eigenvalues of R_a for a near $a_0 = 2\sqrt{\beta}$.

We now look more closely at the smooth family $a \mapsto R_a$. A calculation shows that the characteristic polynomial of R_a is $\lambda^4 + 2\alpha\lambda^2 + \delta^2$, where

$$\alpha = \tfrac{1}{2}a^2(\gamma^2 + (1-\gamma)^2) - \beta \quad \text{and} \quad \delta = -a^2\gamma(1-\gamma) + \beta.$$

So $\alpha + \delta = \frac{1}{2}a^2(2\gamma - 1)^2 \geq 0$ and $\alpha - \delta = \frac{1}{2}(a^2 - 4\beta)$. Thus the eigenvalues of R_a are

$$\begin{cases} \pm i\sqrt{\alpha \pm \sqrt{\alpha^2 - \delta^2}}, \text{ when } |a| > 2\sqrt{\beta} \\ \pm \frac{1}{2}\sqrt{2}(\sqrt{\delta - \alpha} \pm i\sqrt{\delta + \alpha}), \text{ when } |a| < 2\sqrt{\beta} \\ \pm i(2\gamma - 1)\sqrt{\beta}, \text{ when } |a| = 2\sqrt{\beta}. \end{cases}$$

The roots $\pm i(2\gamma-1)\sqrt{\beta}$ are of multiplicity two. As a varies near $a_0 = 2\sqrt{\beta}$, the movement of the eigenvalues of R_a is given in figure 7.1.1.

To understand the fusing of the eigenvalues at $a = a_0$, we examine

$$R = R_{a_0} = \begin{pmatrix} 0 & 2\gamma\sqrt{\beta} & 0 & -1 \\ -2\gamma\sqrt{\beta} & 0 & 1 & 0 \\ 0 & \beta & 0 & 2(\gamma-1)\sqrt{\beta} \\ -\beta & 0 & -2(\gamma-1)\sqrt{\beta} & 0 \end{pmatrix}.$$

The matrix R can be written uniquely as a commuting sum of a semisimple matrix $\mathscr{S} = \begin{pmatrix} 0 & r & 0 & 0 \\ -r & 0 & 0 & 0 \\ 0 & 0 & 0 & r \\ 0 & 0 & -r & 0 \end{pmatrix}$ and a nilpotent matrix $\mathscr{N} = \begin{pmatrix} 0 & \sqrt{\beta} & 0 & -1 \\ -\sqrt{\beta} & 0 & 1 & 0 \\ 0 & \beta & 0 & -\sqrt{\beta} \\ -\beta & 0 & \sqrt{\beta} & 0 \end{pmatrix}$. Here $r = (2\gamma-1)\sqrt{\beta}$. Because R is infinitesimally symplectic with respect to the symplectic form

$$\Omega = \Omega_{a_0} = \begin{pmatrix} 0 & -2\sqrt{\beta} & 0 & 1 \\ 2\sqrt{\beta} & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix},$$

it follows that $\mathscr{S}$ and $\mathscr{N}$ are also infinitesimally symplectic.

To find the infinitesimally symplectic normal form of R, let $e = (0,0,1,0)$ and set $f = e - \frac{1}{2r^2}\Omega(e,\mathscr{S}e)\mathscr{N}\mathscr{S}e$. From $\mathscr{S}^2 + r^2 = 0$, $\mathscr{N}^2 = 0$ and $\Omega(e,\mathscr{N}e) = -1$, it follows that $\Omega(f,\mathscr{N}f) = -1$ and $\Omega(f,\mathscr{S}f) = \Omega(\mathscr{N}f,\mathscr{S}f) = 0$. Consequently $\{f, \frac{1}{r}\mathscr{S}f, -\mathscr{N}f, -\frac{1}{r}\mathscr{S}\mathscr{N}f\}$ is a symplectic basis of $\mathbf{R}^4$ with respect to the standard symplectic form $\omega = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$. The infinitesimally symplectic normal form of R is

$$\mathscr{R} = P_0^{-1} R P_0 = \begin{pmatrix} 0 & -r & 0 & 0 \\ r & 0 & 0 & 0 \\ -1 & 0 & 0 & -r \\ 0 & -1 & r & 0 \end{pmatrix},$$

where $P_0 = \mathrm{col}(f, \frac{1}{r}\mathscr{S}f, -\mathscr{N}f, -\frac{1}{r}\mathscr{S}\mathscr{N}f)$.

To see that the smooth family $a \mapsto R_a$ undergoes a linear Hamiltonian Hopf bifurcation at $a = a_0$, we first transform it using the smooth coordinate change

$$a \to Q_a = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{1}{2}a & 0 & 0 & 1 \\ 0 & \frac{1}{2}a & -1 & 0 \end{pmatrix}. \tag{126}$$

The transformed smooth family is $a \mapsto U_a = Q_a^{-1} R_a Q_a$, where

$$U_a = \begin{pmatrix} 0 & \frac{1}{2}a(2\gamma-1) & 1 & 0 \\ -\frac{1}{2}a(2\gamma-1) & 0 & 0 & 1 \\ \frac{1}{4}(4\beta-a^2) & 0 & 0 & \frac{1}{2}a(2\gamma-1) \\ 0 & \frac{1}{4}(4\beta-a^2) & -\frac{1}{2}a(2\gamma-1) & 0 \end{pmatrix} \tag{127}$$

and the transformed symplectic form $Q_a^t \Omega_a Q_a$ is ω. At $a = a_0$ the real symplectic linear map $Q = Q_{a_0}^{-1} P_0 \in \mathrm{Sp}(\omega, \mathbf{R})$ conjugates U_{a_0} into the normal form $\mathscr{R}$, that is, $Q^{-1} U_{a_0} Q = \mathscr{R}$. Using Q we transform $a \mapsto U_a$ into the smooth family $a \mapsto V_a = Q^{-1} U_a Q$, where $V_{a_0} = \mathscr{R}$.

The following result gives a *smooth normal form* for the family $a \mapsto V_a$ of infinitesimally symplectic linear maps on $(\mathbf{R}^4, \omega)$.

Claim: Suppose that $\gamma : \mathbf{R} \rightarrow \mathrm{sp}(\omega, \mathbf{R}) : \mu \mapsto B_\mu$ is a smooth family of real infinitesimally symplectic matrices on $(\mathbf{R}^4, \omega)$ with

$$B_0 = \begin{pmatrix} 0 & -b & 0 & 0 \\ b & 0 & 0 & 0 \\ -1 & 0 & 0 & -b \\ 0 & -1 & b & 0 \end{pmatrix}.$$

For every μ in some open interval I containing 0 there is a smooth family $I \rightarrow \mathrm{Sp}(\omega, \mathbf{R}) : \mu \mapsto P_\mu^{-1}$ of real linear symplectic mappings which transforms the smooth family γ into the smooth normal form $\Gamma : I \rightarrow \mathrm{sp}(\omega, \mathbf{R})$:

$$\mu \mapsto P_\mu B_\mu P_\mu^{-1} = \begin{pmatrix} 0 & -(b+\nu_1(\mu)) & \nu_2(\mu) & 0 \\ b+\nu_1(\mu) & 0 & 0 & \nu_2(\mu) \\ -1 & 0 & 0 & -(b+\nu_1(\mu)) \\ 0 & -1 & b+\nu_1(\mu) & 0 \end{pmatrix}.$$

Here $\nu_i : I \rightarrow \mathbf{R} : \mu \mapsto \nu_i(\mu)$ is a smooth function with $\nu_i(0) = 0$.

(7.13) **Proof**: Consider the mapping

$$\varphi : \mathbf{R}^2 \times \mathrm{Sp}(\omega, \mathbf{R}) \rightarrow \mathrm{sp}(\omega, \mathbf{R}) : \big((\nu_1, \nu_2), P \big) \rightarrow P^{-1}(B_0 + \nu_1 X_S + \nu_2 X_M) P.$$

Here

$$X_S = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad X_M = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

which are both infinitesimally symplectic. Partially differentiating φ with respect to (ν_1, ν_2) we obtain $D_1\varphi(0, I)(s, t)^t = sX_S + tX_M$. Moreover, the partial derivative of φ at $(0, I)$ with respect to P is $D_2\varphi(0, I)X = \mathrm{ad}_{B_0} X$ where $X \in T_I \mathrm{Sp}(\omega, \mathbf{R}) = \mathrm{sp}(\omega, \mathbf{R})$. Using the fact that

$$\mathrm{sp}(\omega, \mathbf{R}) = \mathrm{im}\, \mathrm{ad}_{B_0} \oplus \mathrm{span}\{X_S, X_M\}, \tag{128}$$

which is proved in ((7.14)) below, we see that $D\varphi(0, I)$ is surjective. From the implicit function theorem, it follows that the image of φ contains an open neighborhood $\mathscr{U}$ of B_0. Choose an open interval I containing 0 such that for every $\mu \in I$, the matrix $B(\mu) \in \mathscr{U}$. Again from the implicit function theorem, there are smooth functions $\nu_i : I \rightarrow \mathbf{R} : \mu \mapsto \nu_i(\mu)$ with $\nu_i(0) = 0$ and $P : I \rightarrow \mathrm{Sp}(\omega, \mathbf{R}) : \mu \mapsto P(\mu)$ such that

$$B_\mu = \varphi\big(\nu_1(\mu), \nu_2(\mu), P_\mu\big) = P_\mu^{-1}\big(B_0 + \nu_1(\mu) X_S + \nu_2(\mu) X_M\big) P_\mu$$

for every $\mu \in I$. This proves the claim except for (128). $\square$

(7.14) **Proof**: To prove (128) we show that $\{X_S, X_M\}$ spans $\ker \mathrm{ad}_{X_S} \cap \ker \mathrm{ad}_{X_M}$ and that a complement to $\mathrm{im}\,\mathrm{ad}_{B_0}$ is given by $\ker \mathrm{ad}_{X_S} \cap \ker \mathrm{ad}_{X_M}$. First observe that $\ker \mathrm{ad}_{X_S}$ is a Lie subalgebra of $\mathrm{sp}(\omega, \mathbf{R})$ with basis $\{X_M, X_N, X_T, X_S\}$ where

$$X_N = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad X_T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

This follows because $[X_T, X_M] = X_T X_M - X_M X_T = 2X_M$, $[X_T, X_N] = -2X_N$, $[X_M, X_N] = X_T$, and $[X_S, X_M] = [X_S, X_N] = [X_S, X_T] = 0$. From these bracket relations we see that the matrix of $\mathrm{ad}_{X_M} \,|\, \ker \mathrm{ad}_{X_S}$ with respect to the basis $\{X_M, X_N, X_T, X_S\}$ is

$$\begin{pmatrix} 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Therefore the intersection of $\ker \mathrm{ad}_{X_M}$ and $\ker \mathrm{ad}_{X_S}$ is spanned by $\{X_M, X_S\}$. Similarly,

$$\mathrm{ad}_{X_N} \,|\, \ker \mathrm{ad}_{X_S} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

So $\ker \mathrm{ad}_S = \big(\ker \mathrm{ad}_{X_M} \cap \ker \mathrm{ad}_{X_S}\big) \oplus \big(\mathrm{im}\,\mathrm{ad}_{X_N} \cap \ker \mathrm{ad}_{X_S}\big)$. Since X_S is semisimple, we have $\mathrm{sp}(\omega, \mathbf{R}) = \ker \mathrm{ad}_{X_S} \oplus \mathrm{im}\,\mathrm{ad}_{X_S}$. Note that in ((7.13)) $B_0 = bX_S - X_N$. From $X_S X_N = X_N X_S$ and the fact that X_N is nilpotent we obtain $\mathrm{im}\,\mathrm{ad}_{B_0} = \mathrm{im}\,\mathrm{ad}_{X_S} \oplus \big(\mathrm{im}\,\mathrm{ad}_{X_N} \cap \ker \mathrm{ad}_{X_S}\big)$. Consequently, we get $\mathrm{sp}(\omega, \mathbf{R}) = \mathrm{im}\,\mathrm{ad}_{B_0} \oplus \big(\ker \mathrm{ad}_{X_M} \cap \ker \mathrm{ad}_{X_S}\big)$. This proves (128). □

In the course of the above proof we have shown that the tangent space at B_0 to the $\mathrm{Sp}(\omega, \mathbf{R})$-orbit $\mathscr{O} = \{P^{-1} B_0 P \,|\, P \in \mathrm{Sp}(\omega, \mathbf{R})\}$ through B_0 is $\mathrm{im}\,\mathrm{ad}_{B_0}$. Thus the plane spanned by X_S and X_M is *transverse* to $\mathscr{O}$ at B_0.

Since $V_{a_0} = \mathscr{R}$, the *smooth normal form* of the family $a \mapsto V_a$ near a_0 is the family $a \mapsto Y_a$ where

$$Y_a = \begin{pmatrix} 0 & -(r+\nu_1(a)) & \nu_2(a) & 0 \\ r+\nu_1(a) & 0 & 0 & \nu_2(a) \\ -1 & 0 & 0 & -(r+\nu_1(a)) \\ 0 & -1 & r+\nu_1(a) & 0 \end{pmatrix}. \tag{129}$$

To compute the smooth functions ν_1 and ν_2 in (129) observe that the families $a \mapsto Y_a$ and $a \mapsto R_a$ are smoothly conjugate. Therefore the characteristic polynomial of Y_a

$$\lambda^4 + 2(r+\nu_1)^2 \lambda^2 + ((r+\nu_1)^2 + \nu_2)^2$$

is equal to the characteristic polynomial of R_a

$$\lambda^4 + 2\big(\tfrac{1}{2}a^2(\gamma^2 + (\gamma-1)^2) - \beta\big)\lambda^2 + \big(\beta - a^2\gamma(1-\gamma)\big)^2.$$

Equating coefficients and solving gives

$$\nu_1(a) = \sqrt{\tfrac{1}{2}a^2(\gamma^2 + (\gamma-1)^2) - \beta} - (2\gamma - 1)\sqrt{\beta}$$

$$\nu_2(a) = -\tfrac{1}{2}(a^2 - 4\beta).$$

Note that $\nu_1(a_0) = \nu_2(a_0) = 0$. Hence $Y_{a_0} = \mathscr{R}$. Since

$$\frac{dY_a}{da}\Big|_{a=a_0} = \frac{d\nu_1}{da}\Big|_{a=a_0} X_S + \frac{d\nu_2}{da}\Big|_{a=a_0} X_M \quad \text{and} \quad \frac{d\nu_2}{da}\Big|_{a=a_0} = -a_0 \neq 0,$$

the curve $a \mapsto Y_a$ crosses the orbit $\mathscr{O} = \{P^{-1}Y_{a_0}P \,|\, P \in \mathrm{Sp}(\omega, \mathbf{R})\}$ transversely at Y_{a_0}. Thus the curve $a \mapsto V_a$ crosses $\mathscr{O}$ transversely at V_{a_0}. Hence the eigenvalues of a small smooth perturbation of the curve $a \mapsto R_a$, such that the new curve lies in $\mathrm{sp}(\omega, \mathbf{R})$ and passes through R_{a_0}, has the same behavior as those of $a \mapsto R_a$, see figure 7.1.1. We say that the curve $a \mapsto R_a$ undergoes a linear Hamiltonian Hopf bifurcation at $a = a_0$.

7.2 The nonlinear Hamiltonian Hopf bifurcation

In this subsection we show that, after reduction of the left S^1-action, the Lagrange top undergoes a *nonlinear Hamiltonian Hopf bifurcation* when the left and right angular momenta are equal and the spin $|a|$ about the figure axis increases through $2\sqrt{\beta}$. This entails finding a smooth family $a \mapsto \psi_a$ of local symplectic diffeomorphisms, which for every a near $2\sqrt{\beta}$ transforms the reduced Hamiltonian $\widetilde{F}_a$ (124) into a Hamiltonian H_a whose 4-jet at the origin is in normal form to second order. Specifically, we show that $H_a = H_a^2 + \varepsilon^2 H_a^4 + \mathrm{O}(\varepsilon^4)$, where

$$H_a^2 = -\tfrac{1}{2}a(2\gamma - 1)(\xi_1\eta_2 - \xi_2\eta_1) + \tfrac{1}{2}(\eta_1^2 + \eta_2^2) + \tfrac{1}{8}(a^2 - 4\beta)(\xi_1^2 + \xi_2^2)$$

and

$$\begin{aligned} H_a^4 \;=\; & \tfrac{1}{12}(a^2 + 2\beta)(\xi_1^2 + \xi_2^2)^2 + \tfrac{1}{8}a(14\gamma - 9)(\xi_1\eta_2 - \xi_2\eta_1)(\xi_1^2 + \xi_2^2) \\ & + \tfrac{1}{6}(3\gamma - 2)(\xi_1\eta_2 - \xi_2\eta_1)^2. \end{aligned}$$

Since $\frac{1}{12}(a^2 + 2\beta) > 0$, the Hamiltonian H_a undergoes a *nonlinear* Hamiltonian Hopf bifurcation as a increases through $2\sqrt{\beta}$. The construction of the symplectic diffeomorphism ψ_a uses techniques from normal form theory. The argument is not straightforward as we need a constructive version of Darboux's theorem.

To start our analysis we choose a special parametrization of the reduced space P_a near $p_a = (e_3, ae_3)$ given by

$$\begin{aligned} \varphi_a : U = D_1^2 \times \mathbf{R}^2 \subseteq \mathbf{R}^4 \to P_a : (x_1, x_2, y_1, y_2) \mapsto (z_1, z_2, z_3, w_1, w_2, w_3) \\ = \big(x_1, x_2, (1 - x_1^2 - x_2^2)^{1/2}, y_1, y_2, (a - x_1y_1 - x_2y_2)(1 - x_1^2 - x_2^2)^{-1/2}\big). \end{aligned} \tag{130}$$

Here $\varphi_a(0) = p_a$ and D_1^2 is the 2-disc $\{x_1^2 + x_2^2 < 1\}$. The reason why φ_a is special is that it intertwines the right S^1-action on P_a, see (116), with the S^1-action

$$S^1 \times \mathbf{R}^4 \to \mathbf{R}^4 : \big(s, (x, y)\big) \mapsto (R_s x, R_s y), \tag{131}$$

where $R_s = \begin{pmatrix} \cos s & -\sin s \\ \sin s & \cos s \end{pmatrix}$. Because the reduced Hamiltonian $\widetilde{F}_a$ (124) is invariant under the right S^1-action on P_a, the Hamiltonian

$$\widehat{H}_a = (\varphi_a)^* \widetilde{F}_a : U \subseteq \mathbf{R}^4 \to \mathbf{R} : (x,y) \mapsto \tfrac{1}{2}(y_1^2 + y_2^2) + \beta(1 - x_1^2 - x_2^2)^{1/2}$$
$$+ \tfrac{1}{2}\gamma\big(a - x_1y_1 - x_2y_2\big)^2(1 - x_1^2 - x_2^2)^{-1} \qquad (132)$$

is invariant under the S^1-action (131) on $\mathbf{R}^4$.

We now find the induced symplectic structure on an open neighborhood of 0 in U. Recall that the reduced space P_a is defined by the 0-level sets of the Casimirs $z_1^2 + z_2^2 + z_3^2 - 1$ and $z_1w_1 + z_2w_2 + z_3w_3 - a$ of the Poisson algebra $(C^\infty(\mathbf{R}^3 \times \mathbf{R}^3), \{\,,\,\}_{\mathbf{R}^3 \times \mathbf{R}^3}, \cdot)$. Thus the Poisson bracket $\{\,,\,\}_{P_a}$ on $C^\infty(P_a)$ is the restriction of $\{\,,\,\}_{\mathbf{R}^3 \times \mathbf{R}^3}$ on $C^\infty(\mathbf{R}^3 \times \mathbf{R}^3)$ to $C^\infty(P_a)$. On $C^\infty(U)$ define a Poisson bracket $\{\,,\,\}_U$ by pulling back $\{\,,\,\}_{P_a}$ by φ_a, that is, for $f, g \in C^\infty(U)$

$$\{f,g\}_U = \varphi_a^*\big(\{(\varphi_a^{-1})^* f, (\varphi_a^{-1})^* g\}_{P_a}\big).$$

The structure matrix $\widehat{W}^a_{C^\infty(U)}$ of $\{\,,\,\}_U$ on $C^\infty(U)$ is given in table 7.2.1.

$\{A,B\}$	x_1	x_2	y_1	y_2	B
x_1	0	0	0	$\sqrt{1-x_1^2-x_2^2}$	
x_2	0	0	$-\sqrt{1-x_1^2-x_2^2}$	0	
y_1	0	$\sqrt{1-x_1^2-x_2^2}$	0	$(a-x_1y_1-x_2y_2)(1-x_1^2-x_2^2)^{-1/2}$	
y_2	$-\sqrt{1-x_1^2-x_2^2}$	0	$-(a-x_1y_1-x_2y_2)(1-x_1^2-x_2^2)^{-1/2}$	0	
A					

Table 7.2.1. The structure matrix $\widehat{W}^a_{C^\infty(U)}$ of $\{\,,\,\}_U$.

Since $\widehat{W}^a_{C^\infty(U)}(0)$ is invertible, the Poisson bracket $\{\,,\,\}_U$ defines a symplectic structure

$$\widehat{\Omega}_a(x,y) \;=\; ((\widehat{W}^a_{C^\infty(U)}(0))^{-1})^t = (a - x_1y_1 - x_2y_2)(1 - x_1^2 - x_2^2)^{-3/2}\,\mathrm{d}x_1 \wedge \mathrm{d}x_2$$
$$-(1 - x_1^2 - x_2^2)^{-1/2}(\mathrm{d}x_1 \wedge \mathrm{d}y_2 - \mathrm{d}x_2 \wedge \mathrm{d}y_1) \qquad (133)$$

on an open neighborhood of 0 in $\mathbf{R}^4$. Note that $\widehat{\Omega}_a$ is invariant under the S^1-action (131) on $\mathbf{R}^4$. Introduce new variables (ξ, η) by $(x,y)^t = Q_a(\xi,\eta)^t$, where Q_a is given by (126). The pulled back symplectic form $Q_a^*\widehat{\Omega}_a = \widetilde{\Omega}_a$ is

$$\widetilde{\Omega}_a(\xi,\eta) \;=\; \big(a - \tfrac{1}{2}a(\xi_1^2 + \xi_2^2) - (\xi_1\eta_2 - \xi_2\eta_1)\big)(1 - \xi_1^2 - \xi_2^2)^{-3/2}\,\mathrm{d}\xi_1 \wedge \mathrm{d}\xi_2$$
$$+(1 - \xi_1^2 - \xi_2^2)^{-1/2}(\mathrm{d}\xi_1 \wedge \mathrm{d}\eta_1 + \mathrm{d}\xi_2 \wedge \mathrm{d}\eta_2 - a\,\mathrm{d}\xi_1 \wedge \mathrm{d}\xi_2). \quad (134)$$

Thus $\widetilde{\Omega}_a(0) = Q_a^t\widehat{\Omega}_a(0)Q_a$ is the standard symplectic form $\omega = \mathrm{d}\xi_1 \wedge \mathrm{d}\eta_1 + \mathrm{d}\xi_2 \wedge \mathrm{d}\eta_2$ on $\mathbf{R}^4$. Since Q_a commutes with $\mathrm{diag}(R_s, R_s)$, the 2-form $\widetilde{\Omega}_a$ is invariant under the S^1-action (131) on $\mathbf{R}^4$. Pulling back the Hamiltonian $\widehat{H}_a$ by Q_a gives the S^1-invariant Hamiltonian

$$\begin{aligned}\widetilde{H}_a(\xi,\eta) &= \tfrac{1}{8}a^2(\xi_1^2+\xi_2^2)+\tfrac{1}{2}a(\xi_1\eta_2-\xi_2\eta_1)+\tfrac{1}{2}(\eta_1^2+\eta_2^2)\\ &\quad +\tfrac{1}{2}\gamma\big(a-\tfrac{1}{2}a(\xi_1^2+\xi_2^2)-(\xi_1\eta_2-\xi_2\eta_1)\big)^2(1-\xi_1^2-\xi_2^2)^{-1}\\ &\quad +\beta\,(1-\xi_1^2-\xi_2^2)^{1/2}. \end{aligned} \tag{135}$$

Introduce a formal small parameter ε and replace the variables ξ and η by $\varepsilon\xi$ and $\varepsilon\eta$. Then the blown up Hamiltonian $\widetilde{H}'_a(\xi,\eta)=\frac{1}{\varepsilon^2}\widetilde{H}_a(\varepsilon\xi,\varepsilon\eta)$ and the blown up symplectic form $\widetilde{\Omega}'_a(\xi,\eta)=\frac{1}{\varepsilon^2}\widetilde{\Omega}_a(\varepsilon\xi,\varepsilon\eta)$ have 4-jet at the origin given by $\widetilde{H}'_a=\widetilde{H}^2_a+\varepsilon^2\widetilde{H}^4_a+O(\varepsilon^4)$, where

$$\begin{aligned}\widetilde{H}^2_a &= \tfrac{1}{2}(\eta_1^2+\eta_2^2)+\tfrac{1}{8}(a^2-4\beta)(\xi_1^2+\xi_2^2)\\ &\quad -\tfrac{1}{2}a(2\gamma-1)(\xi_1\eta_2-\xi_2\eta_1) \end{aligned}\tag{136}$$

$$\begin{aligned}\widetilde{H}^4_a &= \tfrac{1}{8}(a\gamma^2-\beta)(\xi_1^2+\xi_2^2)^2-\tfrac{1}{2}a\gamma(\xi_1\eta_2-\xi_2\eta_1)(\xi_1^2+\xi_2^2)\\ &\quad +\tfrac{1}{2}\gamma(\xi_1\eta_2-\xi_2\eta_1)^2 \end{aligned}\tag{137}$$

and $\widetilde{\Omega}'_a=\widetilde{\Omega}^0_a+\varepsilon^2\widetilde{\Omega}^2_a+O(\varepsilon^4)$, where

$$\widetilde{\Omega}^0_a = \mathrm{d}\xi_1\wedge\mathrm{d}\eta_1+\mathrm{d}\xi_2\wedge\mathrm{d}\eta_2 = \omega \tag{138}$$

$$\begin{aligned}\widetilde{\Omega}^2_a &= \big(\tfrac{1}{2}a(\xi_1^2+\xi_2^2)-(\xi_1\eta_2-\xi_2\eta_1)\big)\,\mathrm{d}\xi_1\wedge\mathrm{d}\xi_2\\ &\quad +\tfrac{1}{2}(\xi_1^2+\xi_2^2)(\mathrm{d}\xi_1\wedge\mathrm{d}\eta_1+\mathrm{d}\xi_2\wedge\mathrm{d}\eta_2). \end{aligned}\tag{139}$$

We now drop the prime.

From (138) we see that to zeroth order the symplectic form $\widetilde{\Omega}_a$ is the standard constant symplectic form ω. In other words, $\widetilde{\Omega}_a$ is flat to zeroth order. Darboux's theorem, see chapter VI ((4.8)), states that there is a coordinate change which makes $\widetilde{\Omega}_a$ flat to all orders. However, the usual proof, see chapter VII exercise 13, does not give a constructive way to find this coordinate change. The goal of the following discussion is to show how to construct a coordinate change which removes the second order terms in $\widetilde{\Omega}_a$. We begin by giving a constructive proof of the Poincaré lemma.

Claim: Let A be a diagonalizable linear vector field on $\mathbf{R}^n$ with all of its eigenvalues strictly negative. Let φ_t be the flow of A. Suppose that β is a closed p-form with $p\ge 1$ in a closed ball $\overline{B}_r$ of radius r about 0 in $\mathbf{R}^n$. Then the $(p-1)$-form $\alpha=-\int_0^\infty\varphi_t^*(A\,\lrcorner\,\beta)\,\mathrm{d}t$ satisfies $\beta=\mathrm{d}\alpha$ in $\overline{B}_r$.

(7.15) **Proof**: Note that $\varphi_\infty=\lim_{t\to\infty}\varphi_t=0$. The integral defining α exists because every coefficient of $A\,\lrcorner\,\beta$ is bounded on $\overline{B}_r$, while the pull back of the differentials by φ_t decay exponentially. To finish the argument we compute

$$\begin{aligned}\beta = -(\varphi_\infty^*\beta-\varphi_0^*\beta) &= -\int_0^\infty\frac{\mathrm{d}}{\mathrm{d}t}(\varphi_t^*\beta)\,\mathrm{d}t = -\int_0^\infty\frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0}(\varphi_{s+t}^*\beta)\,\mathrm{d}t\\ &= -\int_0^\infty\varphi_t^*(L_A\beta)\,\mathrm{d}t = -\int_0^\infty\varphi_t^*\big(A\,\lrcorner\,\mathrm{d}\beta+\mathrm{d}(A\,\lrcorner\,\beta)\big)\,\mathrm{d}t\end{aligned}$$

$$= -\,\mathrm{d}\Big(\int_0^\infty \varphi_t^*(A \lrcorner \beta)\,\mathrm{d}t\Big), \quad \text{since } \mathrm{d}\beta = 0 \text{ and } \varphi_t^*\,\mathrm{d} = \mathrm{d}\varphi_t^*$$
$$= \mathrm{d}\alpha. \qquad \square$$

Suppose that β is a p-form with coefficients which are homogeneous polynomials of degree ℓ. Choose A to be the vector field $-x_1\frac{\partial}{\partial x_1} - \cdots - x_n\frac{\partial}{\partial x_n}$. The flow of A is $\varphi_t(x) = e^{-t}x$. The above proof of the Poincaré lemma shows that $\alpha = -\frac{1}{\ell+p}(A \lrcorner \beta)$.

Before proceeding to state and prove the formal power series version of the Darboux theorem, we prove the following claim which gives the basic computational tools of normal form theory.

Claim:
1. Let X be a vector field on $\mathbf{R}^n$ on which we have coordinates $x = (x_1, \ldots, x_n)$. Then the formal power series in ε given by $(\exp \varepsilon L_X)x = \sum_{n\ge 0}\frac{\varepsilon^n}{n!}L_X^n x$ is the *formal flow* of X. Here $L_X x$ is the vector whose i^{th} component is the Lie derivative $L_X x_i$ of the function x_i with respect to X.
2. Suppose that Q is a smooth geometric quantity on $\mathbf{R}^n$ such as a function, a vector field, or a differential form. Then the formal pull back of Q by the time ε map of the flow of X is given by the formal power series $(\exp \varepsilon L_X)^*Q = \sum_{n\ge 0}\frac{\varepsilon^n}{n!}L_X^n Q$.

(7.16) **Proof**:
1. From the power series for exp it is easy to see that $\varepsilon \mapsto \exp \varepsilon L_X$ is a one parameter group. Therefore $\varepsilon \mapsto (\exp \varepsilon L_X)x$ is a one parameter group of invertible formal power series mappings on $\mathbf{R}^n$. Since $\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0}(\exp \varepsilon L_X)x = L_X x = X(x)$, the one parameter group $\varepsilon \mapsto \exp \varepsilon L_X$ is the formal flow of X.
2. We compute

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}(\exp \varepsilon L_X)^*Q = \frac{\mathrm{d}}{\mathrm{d}\eta}\Big|_{\eta=0}\big(\exp(\varepsilon+\eta)L_X\big)^*Q = (\exp \varepsilon L_X)^*\big(\frac{\mathrm{d}}{\mathrm{d}\eta}\Big|_{\eta=0}(\exp \eta L_X)^*Q\big)$$
$$= (\exp \varepsilon L_X)^*(L_X Q).$$

Therefore by induction $\frac{\mathrm{d}^n}{\mathrm{d}\varepsilon^n}(\exp \varepsilon L_X)^*Q = (\exp \varepsilon L_X)^*(L_X^n Q)$. Hence

$$(\exp \varepsilon L_X)^*Q = \sum_{n\ge 0}\frac{\varepsilon^n}{n!}\frac{\mathrm{d}^n}{\mathrm{d}\varepsilon^n}\Big|_{\varepsilon=0}(\exp \varepsilon L_X)^*Q = \sum_{n\ge 0}\frac{\varepsilon^n}{n!}L_X^n Q. \qquad \square$$

The following claim shows how to flatten a formal power series closed 2-form to second order.

Claim: Let $\Omega = \Omega_0 + \varepsilon\Omega_1 + \cdots$ be a formal power series closed 2-form on $\mathbf{R}^{2n}$ with Ω_0 a constant symplectic form. By the Poincaré lemma, there is a formal power series 1-form $\alpha = \alpha_0 + \varepsilon\alpha_1 + \cdots$ such that $\Omega = \mathrm{d}\alpha$. Define a vector field X by $X \lrcorner \Omega_0 = -\alpha_1$. Then changing coordinates by the time ε map of the flow of X flattens Ω to second order.

(7.17) **Proof**: Applying the coordinate change $\exp \varepsilon L_X$ to the 1-form α gives

$$\widehat{\alpha} = (\exp \varepsilon L_X)^*\alpha = \alpha + \varepsilon L_X\alpha + \mathrm{O}(\varepsilon^2) = \alpha_0 + \varepsilon(\alpha_1 + L_X\alpha_0) + \mathrm{O}(\varepsilon^2)$$

$$= \alpha_0 + \varepsilon\, \mathrm{d}(X \lrcorner\, \alpha_0) + \varepsilon(\alpha_1 + X \lrcorner\, \mathrm{d}\alpha_0) + \mathrm{O}(\varepsilon^2) = \alpha_0 + \varepsilon\, \mathrm{d}(X \lrcorner\, \alpha_0) + \mathrm{O}(\varepsilon^2),$$

since $\Omega_0 = \mathrm{d}\alpha_0$ and $X \lrcorner\, \Omega_0 = -\alpha_1$. Hence

$$\begin{aligned}(\exp \varepsilon L_X)^*\Omega = (\exp \varepsilon L_X)^*(\mathrm{d}\alpha) &= \mathrm{d}\big((\exp \varepsilon L_X)^*\alpha\big) \\ = \mathrm{d}\widehat{\alpha} &= \mathrm{d}\alpha_0 + \mathrm{O}(\varepsilon^2) = \Omega_0 + \mathrm{O}(\varepsilon^2).\end{aligned} \qquad \square$$

With these tools at hand we flatten $\widetilde{\Omega}_a$ to second order as follows. Clearly $\widetilde{\Omega}_a^0 = \mathrm{d}\alpha^0$ where $\alpha^0 = \xi_1\, \mathrm{d}\eta_1 + \xi_2\, \mathrm{d}\eta_2$. Using the Poincaré lemma and the homogeneity of $\widetilde{\Omega}_a^2$, we find that $\widetilde{\Omega}_a^2 = \mathrm{d}\alpha_a^2$, where $\alpha_a^2 = -\frac{1}{4}(A \lrcorner\, \widetilde{\Omega}_a^2)$ and $A = -\xi_1 \frac{\partial}{\partial \xi_1} - \xi_2 \frac{\partial}{\partial \xi_2} - \eta_1 \frac{\partial}{\partial \eta_1} - \eta_2 \frac{\partial}{\partial \eta_2}$. A calculation gives

$$\begin{aligned}\alpha_a^2 \;=\; & \big[-\tfrac{1}{8}\eta_1(\xi_1^2+\xi_2^2) - \tfrac{1}{8}a\xi_2(\xi_1^2+\xi_2^2) + \tfrac{1}{4}\xi_2(\xi_1\eta_2 - \xi_2\eta_1)\big]\, \mathrm{d}\xi_1 \\ & + \big[-\tfrac{1}{8}\eta_2(\xi_1^2+\xi_2^2) + \tfrac{1}{8}a\xi_1(\xi_1^2+\xi_2^2) - \tfrac{1}{4}\xi_1(\xi_1\eta_2 - \xi_2\eta_1)\big]\, \mathrm{d}\xi_2 \\ & -\tfrac{1}{8}\xi_1(\xi_1^2+\xi_2^2)\, \mathrm{d}\eta_1 - \tfrac{1}{8}\xi_2(\xi_1^2+\xi_2^2)\, \mathrm{d}\eta_2.\end{aligned}$$

Clearly α^0 is S^1 invariant; while α_a^2 is S^1 invariant since A and $\widetilde{\Omega}_a^2$ are. Put $X_a^2 = -(\widetilde{\Omega}_a^0)^\flat(\alpha_a^2)$. Because $-\mathrm{d}\xi_i = (\widetilde{\Omega}_a^0)^\sharp(\frac{\partial}{\partial \eta_i})$ and $\mathrm{d}\eta_i = (\widetilde{\Omega}_a^0)^\sharp(\frac{\partial}{\partial \xi_i})$, we find that

$$\begin{aligned}X_a^2 \;=\; & \big[-\tfrac{1}{8}\eta_1(\xi_1^2+\xi_2^2) - \tfrac{1}{8}a\xi_2(\xi_1^2+\xi_2^2) + \tfrac{1}{4}\xi_2(\xi_1\eta_2 - \xi_2\eta_1)\big]\, \frac{\partial}{\partial \eta_1} \\ & + \big[-\tfrac{1}{8}\eta_2(\xi_1^2+\xi_2^2) + \tfrac{1}{8}a\xi_1(\xi_1^2+\xi_2^2) - \tfrac{1}{4}\xi_1(\xi_1\eta_2 - \xi_2\eta_1)\big]\, \frac{\partial}{\partial \eta_2} \\ & -\tfrac{1}{8}\xi_1(\xi_1^2+\xi_2^2)\frac{\partial}{\partial \xi_1} - \tfrac{1}{8}\xi_2(\xi_1^2+\xi_2^2)\frac{\partial}{\partial \xi_2}.\end{aligned}$$

Note that X_a^2 is S^1 invariant. Pulling $\widetilde{\Omega}_a$ back by the time ε^2 map of the flow of X_a^2 flattens $\widetilde{\Omega}_a$ to fourth order, that is, $\overline{\Omega}_a = (\exp \varepsilon^2 L_{X_a^2})^*\widetilde{\Omega}_a = \widetilde{\Omega}_a^0 + O(\varepsilon^4) = \omega + O(\varepsilon^4)$. Since X_a^2 is S^1 invariant, the time ε^2 map of its flow commutes with the S^1-action on $\mathbf{R}^4$. Therefore in the flattened coordinates the new Hamiltonian is S^1 invariant and is given by

$$\overline{H}_a = (\exp \varepsilon^2 L_{X_a^2})^*\widetilde{H}_a = \widetilde{H}_a^2 + \varepsilon^2(\widetilde{H}_a^4 + L_{X_a^2}\widetilde{H}_a^2) + O(\varepsilon^4) = \overline{H}_a^2 + \varepsilon^2 \overline{H}_a^4 + O(\varepsilon^4),$$

where

$$\begin{aligned}\overline{H}_a^2 &= \tfrac{1}{2}(\eta_1^2+\eta_2^2) + \tfrac{1}{8}(a^2 - 4\beta)(\xi_1^2+\xi_2^2) - \tfrac{1}{2}a(2\gamma-1)(\xi_1\eta_2 - \xi_2\eta_1) \\ \overline{H}_a^4 &= A\,(\xi_1^2+\xi_2^2)^2 + B\,(\xi_1^2+\xi_2^2)(\xi_1\eta_2 - \xi_2\eta_1) + \tfrac{1}{4}(2\gamma-1)(\xi_1\eta_2 - \xi_2\eta_1)^2 \\ &\quad - \tfrac{1}{8}(\xi_1^2+\xi_2^2)(\eta_1^2+\eta_2^2),\end{aligned}$$

$A = \frac{1}{32}a^2$, and $B = \frac{1}{16}a(14\gamma - 9)$. Flattening $\overline{\Omega}_a$ at order four or higher does not change the terms through second order in $\overline{H}_a$. Since the occurrence of the Hamiltonian Hopf bifurcation depends only on the terms in $\overline{H}_a$ up through second order in $\overline{H}_a$, we will treat $\overline{H}_a$ as a Hamiltonian on $(\mathbf{R}^4, \omega)$.

We now find the normal form of $\overline{H}_a$ to second order using representation theory. The algebra of S^1 invariant polynomials on $(\mathbf{R}^4, \omega)$ is generated by $M = \frac{1}{2}(\xi_1^2 + \xi_2^2)$, $N =$

$\frac{1}{2}(\eta_1^2+\eta_2^2)$, $T=\xi_1\eta_1+\xi_2\eta_2$, and $S=\xi_1\eta_2-\xi_2\eta_1$. Note that $T^2+S^2=4MN$. Using the Poisson bracket $\{\,,\,\}$ associated to the symplectic form ω, we find that $\{M,N,T,S\}$ form a Lie algebra $\mathscr{L}$ with bracket relations $\{T,M\}=-2M$, $\{T,N\}=2N$, $\{M,N\}=T$, and $\{S,M\}=\{S,N\}=\{S,T\}=0$. In other words, $\mathscr{L}$ is isomorphic to $\mathrm{sl}(2,\mathbf{R})\times\mathbf{R}$. Let $\mathscr{Q}$ be the vector space of homogeneous quadratic polynomials on $\mathscr{L}$, which is spanned by $\{M^2,MN,MT,MS,N^2,NT,NS,T^2,TS,S^2\}$. On $\mathscr{Q}$ we have a representation of $\mathrm{sl}(2,\mathbf{R})$ given by $\mathrm{ad}:\mathrm{sl}(2,\mathbf{R})\to\mathrm{gl}(\mathscr{Q},\mathbf{R}):X\mapsto\mathrm{ad}_X$. From the theory of representations of $\mathrm{sl}(2,\mathbf{R})$ we know that $\mathscr{Q}=\ker\mathrm{ad}_M\oplus\mathrm{im}\,\mathrm{ad}_N$. Explicitly, $\ker\mathrm{ad}_M$ is spanned by $\{M^2,MS,S^2,4MN-T^2\}$ and $\mathrm{im}\,\mathrm{ad}_N$ is spanned by $\{\mathrm{ad}_N M^2,\mathrm{ad}_N^2 M^2,\,\mathrm{ad}_N^3 M^2,\mathrm{ad}_N^4 M^2,\mathrm{ad}_N(MS),\mathrm{ad}_N^2(MS)\}=\{-2MT,2T^2+4MN,12NT,24N^2,2ST,2NS\}$. We have

$$6MN=(T^2+2MN)+(4MN-T^2)\;=\;-\mathrm{ad}_N(MT)+S^2.$$

Writing $\overline{H}_a$ in terms of $\{M,N,S,T\}$ gives

$$\begin{aligned}\overline{H}_a^2&=N+\tfrac{1}{4}(a^2-4\beta)M-\tfrac{1}{2}a(2\gamma-1)S\\ \overline{H}_a^4&=4A\,M^2+2BMS+\tfrac{1}{4}(2\gamma-1)S^2-\tfrac{1}{2}MN\end{aligned}$$

To remove the term $-\frac{1}{2}MN$ from $\overline{H}_a$ and thus bring $\overline{H}_a$ into normal form to second order, we apply the change of coordinates $\exp\varepsilon^2\mathrm{ad}_{\frac{1}{12}MT}$ and obtain

$$\begin{aligned}H_a&=(\exp\varepsilon^2\mathrm{ad}_{\frac{1}{12}MT})^*\overline{H}_a=\overline{H}_a^2+\varepsilon^2(\overline{H}_a^4+\mathrm{ad}_{\frac{1}{12}MT}\overline{H}_a^2)+\mathrm{O}(\varepsilon^4)\\ &=\overline{H}_a^2+\varepsilon^2(\overline{H}_a^4-\tfrac{1}{12}\mathrm{ad}_{\overline{H}_a^2}MT)+\mathrm{O}(\varepsilon^4)\\ &=\overline{H}_a^2+\varepsilon^2[\overline{H}_4^a-\tfrac{1}{12}\mathrm{ad}_N(MT)-\tfrac{1}{24}(a^2-4\beta)M^2]\\ &=H_a^2+\varepsilon^2H_a^4+O(\varepsilon^4).\end{aligned}$$

Here $H_a^2=\overline{H}_a^2$ and

$$\begin{aligned}H_a^4&=\big(4A-\tfrac{1}{24}(a^2-4\beta)\big)M^2+2BMS+\tfrac{1}{6}(3\gamma-2)S^2\\ &=\tfrac{1}{12}(a^2+2\beta)M^2+\tfrac{1}{8}a(14\gamma-9)MS+\tfrac{1}{6}(3\gamma-2)S^2.\end{aligned}$$

Thus the 4-jet of H_a at the origin is in the proper form for a *nonlinear Hamiltonian Hopf bifurcation*.

8 Exercises

1. Let (h,j) be a regular value for the energy momentum mapping $\mathscr{EM}$ of the magnetic spherical pendulum. Suppose that $\Gamma:\mathbf{R}\to\mathscr{EM}^{-1}(h,j)\subseteq TS^2$ is an integral curve of the Hamiltonian vector field X_H of the magnetic spherical pendulum. Let $\pi:TS^2\to S^2$ be the bundle projection.

   a) Show that the curvature of $\gamma=\pi\circ\Gamma$ is a decreasing function of its height on S^2. Deduce that γ has no downward pointing cusps.

b)* Does the rotation number $\Theta(h,j)$ of the flow of X_H on $\mathscr{E}\mathscr{M}^{-1}(h,j)$ satisfy an estimate of the form $C_1 < \Theta(h,j) < C_2$ where C_i are positive constants which do not depend on h or j?

2. Give an argument (not a calculation) why the coefficient of $(\xi_1^2+\xi_2^2)^2$ in the normal form for the nonlinear Hamiltonian Hopf bifurcation of the Lagrange top does *not* depend on the parameter $\gamma = I_1/I_3$.

3. Give a geometric explanation how a cusp catastrophe (= the discriminant locus) of the generic cubic polynomial $F(u) = au^3 - bu^2 + cu - d^2$, where $a > 0$ and becomes a swallowtail surface (= the discriminant locus) of the special quartic

$$G(u) = u^4 - \frac{b}{2a}u^3 + \frac{d}{\sqrt{a}}u + \left(\frac{b}{16a^2} - \frac{c}{4a}\right).$$

In particular, explain why it is essential that the constant term in F is a square.

4. Lift the S^1-action

$$S^1 \times \mathbf{R}^2 \to \mathbf{R}^2 : \big(t, x = (x_1, x_2)\big) \mapsto R_t\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

to a Hamiltonian action $S^1 \times T^*\mathbf{R}^2 \to T^*\mathbf{R}^2 : \big(t,(x,y)\big) \mapsto (R_t x, R_t y)$ on $(T^*\mathbf{R}^2, \omega_4 = \mathrm{d}x_1 \wedge \mathrm{d}y_1 + \mathrm{d}x_2 \wedge \mathrm{d}y_2)$ with momentum mapping $J : T^*\mathbf{R}^2 \to \mathbf{R} : (x,y) \mapsto x_1y_2 - x_2y_1$. Also consider the symplectic action of $\mathbf{Z}_2$ on $(T^*\mathbf{R}, \omega_2 = \mathrm{d}x_1 \wedge \mathrm{d}y_1)$ which is generated by $(x_1,y_1) \to (-x_1,-y_1)$. The goal of this exercise is to show that the singular reduced spaces $J^{-1}(0)/S^1$ and $T^*\mathbf{R}/\mathbf{Z}_2$ are isomorphic.

a) Show that the map $\psi : T^*\mathbf{R} \to J^{-1}(0) \subseteq T^*\mathbf{R}^2 : (x_1,y_1) \mapsto (x_1,0,y_1,0)$ induces a homeomorphism between $T^*\mathbf{R}/\mathbf{Z}_2$ and $J^{-1}(0)/S^1$.

b) Because $\psi^*\omega_4 = \omega_2$, show that $\psi^* : C^\infty(T^*\mathbf{R}^2)^{S^1} \to C^\infty(T^*\mathbf{R})^{\mathbf{Z}_2} : f \mapsto f \circ \psi$ is an injective Poisson map.

c) Using Schwarz's theorem show that the polynomials x_1^2, y_1^2, and x_1y_1 generate $C^\infty(T^*\mathbf{R})^{\mathbf{Z}_2}$. Because

$$\psi^*(x_1^2+x_2^2) = x_1^2,\ \psi^*(y_1^2+y_2^2) = y_1^2, \text{ and } \psi^*(x_1y_1+x_2y_2) = x_1y_1,$$

deduce that ψ^* is surjective and hence is an isomorphism of singular reduced spaces.

5. Consider the Lagrange top after reduction of the left S^1-action, see §6.3. In other words we look at the reduced Hamiltonian system (H_a, P_a, ω_a), which is invariant under the Hamiltonian right S^1-action given by (116). The goal of this exercise is to carry out singular reduction of this right S^1 symmetry near the fixed point $p_a = (e_3, ae_3)$.

Let φ_a be the coordinate change (130), which maps $D_1 \times \mathbf{R}^2 = \{x_1^2 + x_2^2 = 1\} \times \mathbf{R}^2$ into a neighborhood of p_a in P_a. Recall that φ_a intertwines the right S^1-action (116) on P_a with the S^1-action (131) on $\mathbf{R}^4$. In addition, φ_a pulls back the Hamiltonian system (H_a, P_a, ω_a) to the Hamiltonian system $(\widehat{H}_a, \mathbf{R}^4, \widehat{\Omega}_a)$, where $\widehat{H}_a$ is given by (132) and $\widehat{\Omega}_a$ by (133).

a) Near p_a show that the pull back by φ_a of the momentum map $J_r^a : P_a \subseteq \mathbf{R}^3 \times \mathbf{R}^3 \to \mathbf{R} : (z, w) \mapsto w_3 - a$ of the right S^1-action on P_a is the momentum map

$$\widehat{J}_r^a : D_1 \subseteq \mathbf{R}^4 \to \mathbf{R} : (x, y) \mapsto a - (x_1 y_1 + x_2 y_2)\big(1 - (x_1^2 + x_2^2)\big)^{-1/2}$$

of the S^1-action on $(\mathbf{R}^4, \widehat{\Omega}_a)$.

b) Let $\pi : (\widehat{J}^a)^{-1}(0) \to \widetilde{P}_a = (\widehat{J}^a)^{-1}(0)/S^1$ be the reduction map. Consider the symplectic map

$$\psi : T^*\mathbf{R} = \mathbf{R}^2 \to (\widehat{J}^a)^{-1}(0) \subseteq T^*\mathbf{R}^2 = \mathbf{R}^4 : (q, p) \mapsto \big(q, 0, a(1 - (1 - q^2)^{1/2})q^{-1}, p\big).$$

Consider the linear $\mathbf{Z}_2$-action on $\mathbf{R}^2$ generated by $\zeta : \mathbf{R}^2 \to \mathbf{R}^2 : (q, p) \mapsto (-q, -p)$. Show that $\vartheta = \pi \circ \psi$ is an orbifold chart for $\widetilde{P}_a$, that is,

1. The mapping ϑ is $\mathbf{Z}_2$-invariant.
2. The induced map $\overline{\vartheta} : \mathbf{R}^2/\mathbf{Z}_2 \to \widetilde{P}_a$ is a diffeomorphism of differential spaces.

In this chart show that $\widetilde{\omega} = \psi^*\widehat{\Omega}_a = -(1 - q^2)^{-1/2}\, \mathrm{d}q \wedge \mathrm{d}p$ is a symplectic form on $\widetilde{P}_a$.

c) Since $\widehat{H}_a$ is invariant under the S^1-action, there is an induced Hamiltonian $\widetilde{H}_a$ on the singular reduced space $\widetilde{P}_a$. In the orbifold chart $(\mathbf{R}^2, \vartheta)$ show that

$$\widetilde{H}_a(q, p) = \psi^*\widehat{H}_a(q, p) = \tfrac{1}{2}p^2 + \tfrac{1}{2}a^2\big(1 - (1 - q^2)^{1/2}\big)^2 q^{-2}.$$

d) Analyze the behavior of the one parameter family of one degree of freedom Hamiltonian systems $(\widetilde{H}_a, \mathbf{R}^2, \widetilde{\omega})$ near $(0, 0)$. Can one see if this family has monodromy?

Part II. Theory

Chapter VI

Fundamental concepts

In this chapter we describe the basic mathematical structures needed to do Hamiltonian mechanics. We begin with a section on symplectic linear algebra. The motion of a Hamiltonian system takes place on a symplectic manifold, that is, a manifold with a closed nondegenerate 2-form, called a symplectic form. The symplectic form allows one to turn the differential of a function, called a Hamiltonian, into a vector field whose integral curves satisfy Hamilton's equations. An algebraic way of treating Hamiltonian mechanics is via Poisson brackets. When the vector space of smooth functions on a symplectic manifold, which is a Lie algebra under Poisson bracket, is made into an algebra using pointwise multiplication of smooth functions, we obtain a Poisson algebra. The symplectic formulation of mechanics can be recovered from this Poisson algebra.

1 Symplectic linear algebra

In this section we treat the fundamentals of symplectic linear algebra.

Let V be a finite dimensional real vector space. A skew symmetric bilinear form $\sigma : V \times V \to \mathbf{R}$ is said to be *nondegenerate* if the linear mapping $\sigma^\sharp : V \to V^* : v \mapsto \sigma^\sharp(v)$ is bijective. Here $\sigma^\sharp(v)$ is the linear map $\sigma^\sharp(v) : V \to \mathbf{R} : w \mapsto \sigma(v,w)$. A vector space V on which is defined a nondegenerate skew symmetric bilinear form σ is called a *symplectic vector space*. σ is called the *symplectic form* on V.

Example 1. Let W be a finite dimensional real vector space and let $V = W \times W^*$. Define a bilinear form σ on V by $\sigma\big((w,\alpha),(w',\alpha')\big) = \alpha(w') - \alpha'(w)$, for every $(w,\alpha),(w',\alpha') \in W \times W^*$. (V,σ) is a symplectic vector space. Clearly σ is skew symmetric. We need only show that σ is nondegenerate. Suppose that for some $(w,\alpha) \in W \times W^*$ we have $0 = \sigma^\sharp\big((w,\alpha)\big)(w',\alpha')$ for every $(w',\alpha') \in W \times W^*$. In particular, for every $w' \in W$ we have $0 = \sigma^\sharp\big((w,\alpha)\big)(w',0) = \sigma((w,\alpha),(w',0)) = \alpha(w')$, that is, $\alpha = 0$. Similarly, for every $\alpha' \in W^*$ we have $0 = \sigma^\sharp\big((w,\alpha)\big)(0,\alpha') = -\alpha'(w)$, that is, $w = 0$. Hence the linear map $\sigma^\sharp$ is injective. Since $\dim V^* = \dim V$, $\sigma^\sharp$ is surjective and hence bijective. $\square$

Let (V,σ) be a symplectic vector space and let W be a subspace of V. The *symplectic*

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perpendicular W^σ of W in V is the set of all vectors $v \in V$ such that $\sigma(v,w) = 0$ for every $w \in W$. Below we assemble the basic properties of symplectic perpendicularity.

Fact: Let U, W be subspaces of (V, σ).

a) If $U \subseteq W$, then $W^\sigma \subseteq U^\sigma$.

b) $U^\sigma \cap W^\sigma = (U+W)^\sigma$

c) $\dim V = \dim U + \dim U^\sigma$.

d) $U = (U^\sigma)^\sigma$.

e) $(U \cap W)^\sigma = U^\sigma + W^\sigma$.

(1.1) **Proof**:

a) Suppose that $v \in W^\sigma$. Then $\sigma(v,w) = 0$ for every $w \in W$ and hence for every $w \in U$, because $U \subseteq W$. Therefore $v \in U^\sigma$.

b) Suppose that $v \in U^\sigma \cap W^\sigma$. Then $\sigma(v,u) = 0$ for every $u \in U$ and $\sigma(v,w) = 0$ for every $w \in W$. So $0 = \sigma(v, u+w) = \sigma(v,z)$ for every $z \in U+W$. In other words, $v \in (U+W)^\sigma$, that is, $U^\sigma \cap W^\sigma \subseteq (U+W)^\sigma$. Because $U \subseteq U+W$ and $W \subseteq U+W$ from a) it follows that $(U+W)^\sigma \subseteq U^\sigma$ and $(U+W)^\sigma \subseteq W^\sigma$. Consequently $(U+W)^\sigma \subseteq U^\sigma \cap W^\sigma$. Thus b) holds.

c) Consider the linear mapping $\widetilde{\sigma} : U \rightarrow \big(V/U^\sigma\big)^* : u \mapsto \sigma^\sharp(u)$ and let $v \in V$. Since $\sigma^\sharp(u)(v+U^\sigma) = \sigma(u, v+U^\sigma) = \sigma(u,v)$, the mapping $\widetilde{\sigma}$ is well defined. Suppose that $\widetilde{\sigma}(u) = 0$ for some $u \in U$. Then for every $v + U^\sigma \in V/U^\sigma$ we have $0 = \widetilde{\sigma}(u)(v+U^\sigma) = \sigma(u,v)$ for every $v \in V$, which implies that $u = 0$, since σ is nondegenerate. Thus the linear mapping $\widetilde{\sigma}$ is injective. So $\dim U \le \dim \big(V/U^\sigma\big)^* = \dim \big(V/U^\sigma\big) = \dim V - \dim U^\sigma$. Next consider the mapping $\widehat{\sigma} = \iota \circ \sigma^\sharp : V \rightarrow U^*$, where $\iota : V^* \rightarrow U^*$ is the inclusion mapping. Now $0 = \widehat{\sigma}(v)$ for some $v \in V$ if and only if for each $u \in U$ we have $0 = \sigma(v,u)$, that is, if and only if $v \in U^\sigma$. Thus $U^\sigma = \ker \widehat{\sigma}$. Consequently, the induced map $\sigma^\vee : V/U^\sigma \rightarrow U^*$ is injective. So $\dim U^* \ge \dim V/U^\sigma$, which gives $\dim U = \dim U^* \ge \dim V - \dim U^\sigma$. The above inequalities show that $\dim V = \dim U + \dim U^\sigma$, that is, c) holds.

d) Suppose that $u \in U$ and $v \in U^\sigma$. Then $0 = \sigma(u,v)$, which implies that $u \in (U^\sigma)^\sigma$. Thus $U \subseteq (U^\sigma)^\sigma$. By c) we have $\dim U + \dim U^\sigma = \dim V = \dim U^\sigma + \dim (U^\sigma)^\sigma$, which gives $\dim U = \dim (U^\sigma)^\sigma$. Therefore $U = (U^\sigma)^\sigma$.

e) We compute

$$\begin{aligned}
(U \cap W)^\sigma &= \big((U^\sigma)^\sigma \cap (W^\sigma)^\sigma\big)^\sigma, \quad \text{using d)} \\
&= \big((U^\sigma + W^\sigma)^\sigma\big)^\sigma, \quad \text{using b)} \\
&= U^\sigma + W^\sigma, \quad \text{using d).}
\end{aligned}$$ □

A subspace W of a symplectic vector space (V, σ) is *isotropic* if and only if $W \subseteq W^\sigma$. In other words, W is isotropic if and only if $\sigma | (W \times W)$ vanishes identically.

Example 2. Let u be a nonzero vector in V and let U be the subspace of V spanned by u. Then U is isotropic, because $\sigma(u,u) = -\sigma(u,u)$ by skew symmetry of σ. Hence $\sigma(u,u) = 0$. □

Let W be an isotropic subspace of the symplectic vector space (V,σ). W is *Lagrangian* if and only if $W = W^{\sigma}$.

Fact: Every isotropic subspace of a symplectic vector space (V,σ) is contained in a Lagrangian subspace.

(1.2) **Proof**: Let W be an isotropic subspace of (V,σ). If $W = W^{\sigma}$ we are done. Otherwise, because W is properly contained in W^{σ}, there is a nonzero vector v in W^{σ} which is not in W. Let U be the space spanned by v. Then U is isotropic, that is, $U \subseteq U^{\sigma}$. As $U \subseteq W^{\sigma}$, we deduce that $U \subseteq U^{\sigma} \cap W^{\sigma}$. From ((1.1a)) and ((1.1d)) it follows that $W \subseteq U^{\sigma}$. Because W is isotropic, $W \subseteq W^{\sigma}$ and hence $W \subseteq U^{\sigma} \cap W^{\sigma}$. Therefore $U + W \subseteq U^{\sigma} \cap W^{\sigma} = (U+W)^{\sigma}$, that is, $U + W$ is isotropic. Because W is properly contained in $U + W$, after a finite number of repetitions of the above argument we have constructed a Lagrangian subspace of (V,σ) which contains W. □

Example 3. If (V,σ) is the symplectic vector space constructed in example 1, it follows that $W \times \{0\}$ and $\{0\} \times W^*$ are Lagrangian subspaces. □

A *subspace* W of a symplectic vector space (V,σ) is *symplectic* if and only if $\sigma|(W \times W)$ is nondegenerate. Below we collect together some properties of symplectic subspaces.

Fact:

a) W is symplectic if and only if $W \cap W^{\sigma} = \{0\}$.

b) If W is a symplectic subspace, then so is W^{σ}.

c) Every symplectic vector space (V,σ) is the direct sum of σ-perpendicular 2-dimensional symplectic subspaces $(V_i, \sigma|(V_i \times V_i))$, which have a basis $\{e_i, f_i\}$ with respect to which the matrix of $\sigma|(V_i \times V_i)$ is $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

d) If Y is a subspace of W which is complementary to $W \cap W^{\sigma}$ in W, then Y is symplectic. In fact, Y is a maximal symplectic subspace of (V,σ) contained in W.

(1.3) **Proof**:

a) Suppose that $w' \in W \cap W^{\sigma}$. Then $\sigma(w', w) = 0$ for every $w \in W$. In other words, $\sigma^{\sharp}(w') = 0 \in W^*$. As $\sigma|(W \times W)$ is nondegnerate, it follows that $w' = 0$. Hence $W \cap W^{\sigma} = \{0\}$. Conversely, suppose that $\sigma^{\sharp}(w') = 0 \in W^*$ for some $w' \in W$. Then for every $w \in W$, we have $0 = \sigma(w', w)$, that is, $w' \in W^{\sigma}$. Hence $w' \in W \cap W^{\sigma} = \{0\}$. Therefore the map $\sigma^{\sharp} : W \to W^*$ is injective and hence surjective because $\dim W = \dim W^*$. Thus $\sigma|(W \times W)$ is nondegenerate.

b) Since W is symplectic, $\{0\} = W \cap W^{\sigma} = (W^{\sigma})^{\sigma} \cap W^{\sigma}$, using ((1.1d)). From a) it follows that W^{σ} is symplectic.

c) Let v be a nonzero vector in V. Suppose that for every $v' \in V$, $\sigma(v, v') = 0$. Then $v \in V^{\sigma}$. Since σ is nondegenerate, it follows that $V \cap V^{\sigma} = \{0\}$ and hence $v = 0$, which is a contradiction. Thus there is a $v' \in V$ such that $\sigma(v, v') = r \neq 0$. Let V_1 be the space spanned by $\{v, \frac{1}{r}v'\}$. The matrix of $\sigma|(V_1 \times V_1)$ is $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, which is invertible. In other words, $\big(V_1, \sigma|(V_1 \times V_1)\big)$ is a symplectic subspace of (V,σ). Set $W = V_1^{\sigma}$. Repeat the above argument on the symplectic vector space $\big(W, \sigma|(W\times W)\big)$. After a finite number of repetions we find that $V = \sum_{i=1}^{p} \oplus V_i$ where $\big(V_i, \sigma|(V_i\times V_i)\big)$ are 2 dimensional symplectic subspaces, which are mutually σ-perpendicular and hence the dimension of V is even.

d) Let $U = \ker\sigma|(W\times W) = \{w\in W \,\big|\, \sigma(w,w') = 0 \text{ for every } w'\in W\}$. Then by definition $U = W\cap W^{\sigma}$. The skew symmetric bilinear form $\sigma|(W\times W)$ on W induces a skew symmetric bilinear form $\widetilde{\sigma}$ on W/U defined by $\widetilde{\sigma}(w+U, w'+U) = \sigma(w,w')$. This definition is all right because $\sigma(w,U) = 0$ for every $w\in W$ and U is a $\sigma|(W\times W)$-isotropic subspace of W. In addition, $\widetilde{\sigma}$ is nondegenerate; for if $0 = \widetilde{\sigma}(w+U, w'+U)$ for every $w'+U\in W/U$, then $0 = \sigma(w,w')$ for every $w'\in W$, that is, $w\in\ker\sigma|(W\times W) = U$. So $w+U = 0+U$. Let Y be a complementary subspace to U in W. Consider the linear map $\varphi : Y\to W/U : y\mapsto y+U$. Then φ is an isomorphism such that $\varphi^*\widetilde{\sigma} = \sigma|(Y\times Y)$. Consequently, the skew symmetric bilinear form $\sigma|(Y\times Y)$ on Y is nondegenerate, that is, Y is a symplectic subspace of (V,σ).

Suppose that Y' is a symplectic subspace of (V,σ) which is contained in W and properly contains Y. Then $\dim Y' > \dim Y$ and hence $Y'\cap W\cap W^{\sigma} \neq \{0\}$. This assertion follows because

$$\dim Y + \dim(W\cap W^{\sigma}) = \dim W = \dim Y' + \dim W\cap W^{\sigma} - \dim(Y'\cap W\cap W^{\sigma}),$$

and hence $\dim Y'\cap W\cap W^{\sigma} = \dim Y' - \dim Y > 0$. Let $z\in Y'\cap W\cap W^{\sigma}$. Then $0 = \sigma(z,y')$ for every $y'\in Y'\subseteq W$. Since Y' is symplectic, we deduce that $z = 0$, that is, $Y'\cap W\cap W^{\sigma} = \{0\}$. This is a contradiction. Therefore Y is a maximal symplectic subspace of (V,σ) contained in W. □

The following decomposition is called the *Witt decomposition*,.

Fact: Let (V,σ) be a symplectic vector space and let W be a subspace of V. Then V may be written as $V = X\oplus Y\oplus Z$ where X, Y and Z are σ-perpendicular symplectic subspaces such that $W = X\oplus(W\cap W^{\sigma})$, $W^{\sigma} = Y\oplus(W\cap W^{\sigma})$, and $W\cap W^{\sigma}$ is a Lagrangian subspace of Z.

(1.4) **Proof**: Choose $X\subseteq W$ complementary to $W\cap W^{\sigma}$, and $Y\subseteq W^{\sigma}$ complementary to $W\cap W^{\sigma}$. Then ((1.3d)) shows that both X and Y are symplectic. Let $Z = (X\oplus Y)^{\sigma}$. Then Z is symplectic as well. Hence $V = X\oplus Y\oplus Z$ is a decomposition of V into σ-perpendicular symplectic subspaces. $W\cap W^{\sigma}\subseteq Z$ by construction. Let T be an isotropic subspace of Z with the properties that $T\cap W\cap W^{\sigma} = \{0\}$ and $T\oplus(W\cap W^{\sigma}) = R$ is symplectic. Then $R\oplus X\oplus Y$ is symplectic. To see that $R\oplus X\oplus Y = V$ observe that

$$W = X\oplus(W\cap W^{\sigma})\subseteq X\oplus R\subseteq X\oplus Y\oplus R$$

and

$$W^{\sigma} = Y\oplus(W\cap W^{\sigma})\subseteq Y\oplus R\subseteq X\oplus Y\oplus R.$$

Therefore $(R\oplus X\oplus Y)^{\sigma}\subseteq W\cap W^{\sigma}\subseteq R$. As $R\subseteq R\oplus X\oplus Y$, we find that $(R\oplus X\oplus Y)^{\sigma}\subseteq R^{\sigma}$. So $(R\oplus X\oplus Y)^{\sigma}\subseteq R\cap R^{\sigma} = \{0\}$, because R is symplectic. Therefore $V = R\oplus X\oplus Y$. Hence $\dim R = \dim Z$. As $R\subseteq Z$, we get $R = Z$.

We now show that $W\cap W^{\sigma}$ is a Lagrangian subspace of Z. Since $Z = T\oplus(W\cap W^{\sigma})$, it follows that

$$\dim Z = \dim(W\cap W^{\sigma}) + \dim T \le \dim(W\cap W^{\sigma}) + \tfrac{1}{2}\dim Z,$$

because T is an isotropic subspace of the symplectic subspace Z. Therefore $\frac{1}{2}\dim Z \le \dim W \cap W^{\sigma}$. But $W \cap W^{\sigma}$ is an isotropic subspace of Z. Hence $\dim(W \cap W^{\sigma}) \le \frac{1}{2}\dim Z$. Thus $\frac{1}{2}\dim Z = \dim(W \cap W^{\sigma})$, that is, $W \cap W^{\sigma}$ is a Lagrangian subspace of Z. □

2 Symplectic manifolds

In this section we define the concept of a symplectic manifold.

A *symplectic manifold* (M, ω) is a pair consisting of a smooth manifold M with a 2-form ω which is

1. closed, that is, $\mathrm{d}\omega = 0$;
2. nondegenerate, that is, $\omega(p)$ is an nondegenerate skew symmetric bilinear form on T_pM at each $p \in M$. In other words, $\big(T_pM, \omega(p)\big)$ is a symplectic vector space.

Example 1. Let V be a real vector space and set $M = V \times V^*$. On M define the constant 2-form ω by $\omega\big((v,\alpha),(v',\alpha')\big) = \alpha(v') - \alpha'(v)$, see example 1 §1. Then (M, ω) is a symplectic manifold.

Example 2. Consider the cotangent bundle T^*N of a smooth manifold N. This is the *phase space* of classical mechanics, which is typically the space of all positions and momenta of a physical system. To show that T^*N is a symplectic manifold we first define the *canonical* 1*-form* θ by

$$\theta_\alpha(v_\alpha) = \alpha(T\tau v_\alpha). \tag{1}$$

Here $\tau : T^*N \to N$ is the map which assigns to every covector α in T^*N its point of attachment $\tau(\alpha)$ in N and v_α is a vector in the tangent space $T_\alpha(T^*N)$ to T^*N at α. The *canonical 2-form* or *symplectic structure* Ω on T^*N is the 2-form $\Omega = -\mathrm{d}\theta$. Therefore Ω is closed. We have not yet shown that Ω is nondegenerate.

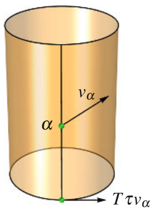

Figure 2.1. The canonical 1-form on T^*N.

First we prove some general facts. Let $\varphi : N \to N$ be a diffeomorphism of N and let
▷ $\widehat{\varphi} : T^*N \to T^*N : \alpha \mapsto (T_{\varphi(\tau(\alpha))}\varphi^{-1})^t\alpha$. Then by definition $\tau \circ \widehat{\varphi} = \varphi \circ \tau$. Also $\widehat{\varphi}^*\theta = \theta$.

(2.1) **Proof**: For every $\alpha \in T^*N$ and every $v_\alpha \in T_\alpha(T^*N)$ we have

$$\begin{aligned}\big((\widehat{\varphi}^*\theta)(\alpha)\big)v_\alpha &= \theta(\widehat{\varphi}(\alpha))(T_\alpha\widehat{\varphi}(v_\alpha)), \quad \text{by definition of pull back}\\ &= \widehat{\varphi}(\alpha)\big(T_{\widehat{\varphi}(\alpha)}\tau(T_\alpha\widehat{\varphi}(v_\alpha))\big), \quad \text{by definition of } \theta\end{aligned}$$

$$
\begin{aligned}
&= (T_{\varphi(\tau(\alpha))}\varphi^{-1})^t \alpha\big(T_{\widehat{\varphi}(\alpha)}\tau(T_\alpha\widehat{\varphi}(v_\alpha))\big), \quad \text{by definition of } \widehat{\varphi} \\
&= \alpha\big((T_{\varphi(\tau(\alpha))}\varphi^{-1})(T_\alpha\varphi(T_\alpha\tau(v_\alpha)))\big), \quad \text{since } \tau\circ\widehat{\varphi} = \varphi\circ\tau \text{ gives } T\tau\circ T\widehat{\varphi} = T\varphi\circ T\tau. \\
&= \alpha(T_\alpha\tau(v_\alpha)) = \theta(\alpha)v_\alpha. \qquad \square
\end{aligned}
$$

On our way towards showing that Ω is nondegenerate, we first define a connection on T^*N. Let X be a vector field on N with local flow φ_t^X and let $\widehat{X}$ be the vector field on T^*N, whose infinitesimal generator is $\varphi_t^{\widehat{X}} = \widehat{\varphi_t^X}$. Then infinitesimalizing $\tau(\varphi_t^{\widehat{X}}(\alpha)) = \varphi_t^X(\tau(\alpha))$ gives $T\tau(\widehat{X}(\alpha)) = X(\tau(\alpha))$ for every $\alpha \in T^*N$. Thus $\widehat{X}$ is a horizontal vector field on T^*N for an Ehresmann connection on the bundle $\tau : T^*N \rightarrow N$, whose vertical distribution is defined by $\alpha \mapsto \ker T_\alpha\tau$, see (2) below and chapter VIII. Given a 1-form β on N define the vector field X^β by

$$X^\beta(\alpha) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\big(\alpha + t\beta(\tau(\alpha))\big).$$

Here $\varphi_t^{X^\beta}(\alpha) = \alpha + t\beta(\tau(\alpha))$ is the flow of X^β on T^*N. Because $T_\alpha\tau(X^\beta(\alpha)) = 0$ for every $\alpha \in T^*N$, we say that X^β is a vertical vector field on T^*N. At every point $\alpha \in T^*N$ we have

$$T_\alpha(T^*N) = \mathrm{span}_{\mathbf{R}}\{X^\beta(\alpha)\big|\beta \in \Omega^1(N)\} \oplus \mathrm{span}_{\mathbf{R}}\{\widehat{X}(\alpha)\big|X \in \mathscr{X}(N)\}, \tag{2}$$

where $\mathscr{X}(N)$ is the set of vector fields on N and $\Omega^1(N)$ is the set of 1-forms on N.

We are now in a position to prove

Claim: The canonical 2-form Ω on T^*N is nondegenerate.

(2.2) **Proof**: Since $T\tau(X^\beta(\alpha)) = 0$, it follows that for every $\alpha \in T^*N$ we have

$$\theta(\alpha)(X^\beta(\alpha)) = \alpha\big(T\tau(X^\beta(\alpha))\big) = 0. \tag{3}$$

Recall that X is a vector field on N. Let $f^X(\alpha) = \alpha(X(\tau(\alpha))) = \theta(\alpha)(\widehat{X}(\alpha))$. Then f^X is a smooth function on T^*N. Now

$$
\begin{aligned}
(L_{X^\beta}f^X)(\alpha) &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} f^X\big(\alpha + t\beta(\tau(\alpha))\big) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} (\alpha + t\beta(\tau(\alpha)))X(\tau(\alpha)), \quad \text{since } \tau(\alpha + t\beta(\tau(\alpha))) = \tau(\alpha) \\
&= \beta(\tau(\alpha))X(\tau(\alpha)) = \langle\beta|X\rangle(\tau(\alpha)).
\end{aligned}
$$

So $L_{X^\beta}f^X = \langle\beta|X\rangle\circ\tau$. For every 1-form β_1 and β_2 on N we have

$$
\begin{aligned}
\mathrm{d}\theta(X^{\beta_1}, X^{\beta_2}) &= X^{\beta_1} \mathbin{\lrcorner} \mathrm{d}\big(\theta(X^{\beta_2})\big) - X^{\beta_2} \mathbin{\lrcorner} \mathrm{d}\big(\theta(X^{\beta_1})\big) - [X^{\beta_1}, X^{\beta_2}] \mathbin{\lrcorner} \theta \\
&= -\langle\theta|[X^{\beta_1}, X^{\beta_2}]\rangle, \quad \text{since } \theta(X^{\beta_i}) = 0, \text{ using (3)} \\
&= 0,
\end{aligned} \tag{4}
$$

since $[X^{\beta_1}, X^{\beta_2}] = 0$. This follows because the flows $\varphi_t^{X^{\beta_1}}$ and $\varphi_t^{X^{\beta_2}}$ commute. Also

$$\mathrm{d}\theta(X^\beta, \widehat{X}) = X^\beta \mathbin{\lrcorner} \mathrm{d}\big(\theta(\widehat{X})\big) - \widehat{X} \mathbin{\lrcorner} \mathrm{d}\big(\theta(X^\beta)\big) - [X^\beta, \widehat{X}] \mathbin{\lrcorner} \theta$$

$$= L_{X^\beta} f^X - \langle \theta | [X^\beta, \widehat{X}] \rangle, \quad \text{using (3) and the definition of } f^X.$$

Since $\theta(X^\beta) = 0$ we get

$$0 = L_{\widehat{X}}(\theta(X^\beta)) = (L_{\widehat{X}}\theta)X^\beta + \theta([\widehat{X}, X^\beta]) = \theta([X^\beta, \widehat{X}]),$$

because $(\varphi_t^{\widehat{X}})^*\theta = \theta$ implies $L_{\widehat{X}}\theta = 0$. Therefore

$$\mathrm{d}\theta(X^\beta, \widehat{X}) = L_{X^\beta} f^X = \langle \beta | X \rangle \circ \tau. \tag{5}$$

Let $(x_1, \ldots, x_m)$ be local coordinates on N near n. Then $\{\mathrm{d}x_1(n), \ldots, \mathrm{d}x_m(n)\}$ are coodinates for T_n^*N and $\big\{ \frac{\partial}{\partial x_1}\big|_n, \ldots, \frac{\partial}{\partial x_m}\big|_n \big\}$ are coordinates for T_nN. With respect to the basis $\big\{ X^{\mathrm{d}x_i}(\alpha_n), \widehat{\frac{\partial}{\partial x_i}}\big|_{\alpha_n} \big\}_{i=1}^m$ of $T_{\alpha_n}(T^*N)$ the matrix $\big(\mathrm{d}\theta(\alpha_n)(X^{\mathrm{d}x_i}(\alpha_n), \widehat{\frac{\partial}{\partial x_j}}\big|_{\alpha_n}) \big)$ is the matrix $\big(\langle \mathrm{d}x_i(n) | \frac{\partial}{\partial x_j}\big|_n \rangle \big) = \big(\delta_{ij} \big)$. Thus the matrix of the 2-form Ω at α_n is $\left(\begin{smallmatrix} 0 & I \\ -I & D_{\alpha_n} \end{smallmatrix}\right)$, where $D_{\alpha_n} = \big(\mathrm{d}\theta(\alpha_n)(\widehat{\frac{\partial}{\partial x_i}}\big|_{\alpha_n}, \widehat{\frac{\partial}{\partial x_j}}\big|_{\alpha_n}) \big)$. Since the matrix of Ω is invertible for every $\alpha_n \in T^*N$, the 2-form Ω is nondegenerate. $\square$

Example 2′: Suppose that T^*N is the cotangent bundle T^*G of a Lie group G. Let $\mathfrak{g}^*$ be the dual to its Lie algebra $\mathfrak{g}$. Then $\tau : T^*G \to G : \alpha_g \to g$ is a *trivial* bundle with bundle projection τ. A trivialization of T^*G is given, for example, by left translation

$$\mathscr{L} : G \times \mathfrak{g}^* \to T^*G : (g, \alpha) \mapsto (T_g L_{g^{-1}})^t \alpha = \alpha_g, \tag{6}$$

where $L_g : G \to G : h \mapsto gh$ is *left translation* by G. The map $\mathscr{L}$ identifies a covector at a point with a left invariant 1-form, which may be thought of as an element of $\mathfrak{g}^*$. Pulling back the canonical 1-form θ on T^*G by the mapping $\mathscr{L}$ gives the 1-form $\vartheta = \mathscr{L}^*\theta$ on $G \times \mathfrak{g}^*$. Similarly, pulling back the canonical 2-form Ω on T^*G gives the 2-form $\omega = \mathscr{L}^*\Omega$ on $G \times \mathfrak{g}^*$.

We calculate the 1-form ϑ as follows. First we compute the tangent of the mapping $\tau \circ \mathscr{L}$. For $\xi \in \mathfrak{g}$ and $\beta \in \mathfrak{g}^*$, the mapping

$$\varphi_t^{(\xi,\beta)} : G \times \mathfrak{g}^* \to G \times \mathfrak{g}^* : (g, \alpha) \mapsto (g \exp t\xi, \alpha + t\beta)$$

is the flow of the vector field $X^{(\xi,\beta)}(g, \alpha) = (T_e L_g \xi, \beta)$ on $G \times \mathfrak{g}^*$. Using the left trivialization $\mathscr{L}$, we pull back the action L of left translation to obtain a G-action ℓ on $G \times \mathfrak{g}^*$ defined by $\ell_g(h, \alpha) = (L_g h, \alpha)$. Note that $X^{(\xi,\beta)}$ is invariant under the action ℓ because

$$\varphi_t^{(\xi,\beta)}\big(\ell_g(h, \alpha)\big) = \varphi_t^{(\xi,\beta)}(gh, \alpha) = (gh \exp t\xi, \alpha + t\beta) = \ell_g\big(\varphi_t^{(\xi,\beta)}(h, \alpha)\big).$$

Therefore the tangent of $\tau \circ \mathscr{L}$ is

$$T_{(g,\alpha)}(\tau \circ \mathscr{L})(T_e L_g \xi, \beta) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \tau \circ \mathscr{L}\big(\varphi_t^{(\xi,\beta)}(g, \alpha)\big) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} L_g \exp t\xi = T_e L_g \xi.$$

Consequently,

$$\begin{aligned}\vartheta(g,\alpha)(T_eL_g\xi,\beta) &= \theta(\mathscr{L}(g,\alpha))\big(T_{(g,\alpha)}\mathscr{L}(T_eL_g\xi,\beta)\big)\\ &= \alpha_g\big(T_{(g,\alpha)}(\tau\circ\mathscr{L})(T_eL_g\xi,\beta)\big) = \alpha_g(T_eL_g\xi) = \alpha(\xi).\end{aligned}\tag{7}$$

To compute the 2-form ω, we use the formula

$$-\,\mathrm{d}\vartheta(X^{(\xi,\beta)},X^{(\eta,\gamma)}) = -X^{(\xi,\beta)}\,\lrcorner\,\mathrm{d}(X^{(\eta,\gamma)}\lrcorner\,\vartheta) + X^{(\eta,\gamma)}\,\lrcorner\,\mathrm{d}(X^{(\xi,\beta)}\lrcorner\,\vartheta) + [X^{(\xi,\beta)},X^{(\eta,\gamma)}]\lrcorner\,\vartheta$$

for the exterior derivative of ϑ. We now calculate each term on the right hand side of the above equation. For the first term, we find that

$$\begin{aligned}&\big(X^{(\xi,\beta)}\lrcorner\,\mathrm{d}(X^{(\eta,\gamma)}\lrcorner\,\vartheta)\big)(g,\alpha) = L_{X^{(\xi,\beta)}}(X^{(\eta,\gamma)}\lrcorner\,\vartheta)(g,\alpha)\\ &\quad= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}(\varphi_t^{(\xi,\beta)})^*(X^{(\eta,\gamma)}\lrcorner\,\vartheta)(g,\alpha)\\ &\quad= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\vartheta(g\exp t\xi,\alpha+t\beta)\big(X^{(\eta,\gamma)}(g\exp t\xi,\alpha+t\beta)\big)\\ &\quad= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\vartheta(g\exp t\xi,\alpha+t\beta)(T_eL_{g\exp t\xi}\eta,\gamma) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}(\alpha+t\beta)(\eta) = \beta(\eta).\end{aligned}$$

The second term is calculated similarly. The third term follows once we notice that

$$\begin{aligned}[X^{(\xi,\beta)},X^{(\eta,\gamma)}](g,\alpha) &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\big(\varphi^{(\eta,\gamma)}_{-\sqrt{t}}\circ\varphi^{(\xi,\beta)}_{-\sqrt{t}}\circ\varphi^{(\eta,\gamma)}_{\sqrt{t}}\circ\varphi^{(\xi,\beta)}_{\sqrt{t}}\big)(g,\alpha)\\ &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\big(g\exp\sqrt{t}\xi\,\exp\sqrt{t}\eta\,\exp-\sqrt{t}\xi\,\exp-\sqrt{t}\eta,\alpha\big)\\ &= (T_eL_g\,[\xi,\eta]\,,0),\end{aligned}$$

as then

$$\vartheta([X^{(\xi,\beta)},X^{(\eta,\gamma)}])(g,\alpha) = \vartheta(g,\alpha)(T_eL_g\,[\xi,\eta]\,,0) = \alpha([\xi,\eta]).$$

Therefore the 2-form ω on $G\times\mathfrak{g}^*$ is given by

$$\omega(g,\alpha)\big((T_eL_g\xi,\beta),(T_eL_g\eta,\gamma)\big) = -\beta(\eta)+\gamma(\xi)+\alpha([\xi,\eta]).\tag{8}$$

From (8) it is immediate that ω is a nondegenerate 2-form on $G\times\mathfrak{g}^*$. Since $\omega = \mathscr{L}^*\Omega$ and Ω is closed, it follows that ω is closed. Hence ω is symplectic. Because $\mathscr{L}$ is a diffeomorphism, we deduce that Ω is symplectic. □

Example 3. Another important example of a symplectic manifold is an orbit $\mathscr{O}_\mu$ of the coadjoint action of a Lie group G on the dual $\mathfrak{g}^*$ of its Lie algebra $\mathfrak{g}$. In more detail, the *coadjoint action* of G on $\mathfrak{g}^*$ is defined by

$$\mathrm{Ad}^*: G\times\mathfrak{g}^*\to\mathfrak{g}^*:(g,\nu)\to\mathrm{Ad}^t_{g^{-1}}\,\nu\tag{9}$$

where the *adjoint action* Ad of G on its Lie algebra $\mathfrak{g}$ is

$$\mathrm{Ad}: G\times\mathfrak{g}\to\mathfrak{g}:(g,\xi)\mapsto\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}g\exp t\xi\,g^{-1}.$$

▷ The *coadjoint orbit* $\mathscr{O}_\mu$ through $\mu \in \mathfrak{g}^*$ is $\{\nu = \mathrm{Ad}^*_g \mu \in \mathfrak{g}^* \,|\, g \in G\}$. The following calculation shows that the tangent space $T_\nu \mathscr{O}_\mu$ to $\mathscr{O}_\mu$ at ν is $\{\mathrm{ad}^t_\xi \nu \,|\, \xi \in \mathfrak{g}\}$. It is reasonable that this should be the case, because this equality is obtained by differentiating the defining relation of $\mathscr{O}_\mu$. Here ad_ξ is the tangent to the mapping Ad in the direction ξ, that is,

$$\mathrm{ad}_\xi : \mathfrak{g} \to \mathfrak{g} : \eta \mapsto \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \mathrm{Ad}_{\exp t\xi}\, \eta = [\xi, \eta],$$

where $[\,,\,]$ is the Lie bracket on $\mathfrak{g}$.

(2.3) **Proof**: For $\xi \in \mathfrak{g}$ the curve $\gamma^\xi : \mathbf{R} \to \mathscr{O}_\mu : s \mapsto \mathrm{Ad}^t_{\exp -s\xi}\, \nu$ lies in $\mathscr{O}_\mu$ and passes through ν at $s = 0$. γ^ξ represents the tangent vector $-\mathrm{ad}^t_\xi \nu$ to the coadjoint orbit $\mathscr{O}_\mu$ at the point ν because $\frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0} \gamma^\xi(s) = \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0} \mathrm{Ad}^t_{\exp -s\xi}\, \nu = -\mathrm{ad}^t_\xi\, \nu$. Therefore $T_\nu \mathscr{O}_\mu \subseteq \{\mathrm{ad}^t_\xi \nu \,|\, \xi \in \mathfrak{g}\}$. To prove the reverse inclusion, for some $\xi \in \mathfrak{g}$ consider the curve $\Gamma^\xi : \mathbf{R} \to \mathscr{O}_\mu : s \mapsto \exp(s\,\mathrm{ad}^t_\xi)\mu$. The image of Γ^ξ is contained in $\mathscr{O}_\mu$ because $\exp(s\,\mathrm{ad}^t_\xi)\mu = \mathrm{Ad}^t_{\exp s\xi}\mu$. Since $\Gamma^\xi(0) = \mu$ and $\frac{\mathrm{d}}{\mathrm{d}s}\Gamma^\xi(0) = \mathrm{ad}^t_\xi$, we are done. □

For $\xi \in \mathfrak{g}$ define the vector field X^ξ on $\mathscr{O}_\mu$ by $X^\xi(\nu) = -\mathrm{ad}^t_\xi\, \nu$. As is easily checked the flow φ^ξ_s of X^ξ on $\mathscr{O}_\mu$ is

$$\varphi^\xi_s(\nu) = \mathrm{Ad}^t_{\exp -s\xi}\, \nu. \tag{10}$$

On $\mathscr{O}_\mu$ define a 2-form Ω by

$$\Omega(\nu)(X^\xi(\nu), X^\eta(\nu)) = -\nu([\xi, \eta]). \tag{11}$$

Claim: Ω is a symplectic form on the coadjoint orbit $\mathscr{O}_\mu$.

(2.4) **Proof**: First we show that Ω is closed. Recall that the exterior derivative of an n-form Θ is

$$\begin{aligned} \mathrm{d}\Theta(X_0, X_1, \ldots, X_n) &= \textstyle\sum_{i=0}^n (-1)^i\, \mathrm{d}(X_i \lrcorner\, \Theta)(X_0, \ldots, \widehat{X_i}, \ldots, X_n) \\ &\quad + \textstyle\sum_{0 \le i < j \le n} (-1)^{i+j} ([X_i, X_j] \lrcorner\, \Theta)(X_0, \ldots, \widehat{X_i}, \ldots, \widehat{X_j}, \ldots, X_n). \end{aligned} \tag{12}$$

Applying (12) to the 2-form Ω gives

$$\begin{aligned} \mathrm{d}\Omega(X^\xi, X^\eta, X^\zeta) &= \mathrm{d}(X^\xi \lrcorner\, \Omega)(X^\eta, X^\zeta) + \mathrm{d}(X^\eta \lrcorner\, \Omega)(X^\zeta, X^\xi) \\ &\quad + \mathrm{d}(X^\zeta \lrcorner\, \Omega)(X^\xi, X^\eta) - \Omega([X^\xi, X^\eta], X^\zeta) - \Omega([X^\zeta, X^\xi], X^\eta) - \Omega([X^\eta, X^\zeta], X^\xi). \end{aligned} \tag{13}$$

Again using (12), this time to compute the exterior derivative of the 1-form $X^\xi \lrcorner\, \Omega$, we obtain

$$\mathrm{d}(X^\xi \lrcorner\, \Omega)(X^\eta, X^\zeta) = \mathrm{d}(X^\eta \lrcorner\, (X^\xi \lrcorner\, \Omega))X^\zeta - \mathrm{d}(X^\zeta \lrcorner\, (X^\xi \lrcorner\, \Omega))X^\eta - [X^\eta, X^\zeta] \lrcorner\, (X^\xi \lrcorner\, \Omega).$$

Similarly by cyclically permuting the variables in the above equation gives

$$\mathrm{d}(X^\eta \lrcorner\, \Omega)(X^\zeta, X^\xi) = \mathrm{d}(X^\zeta \lrcorner\, (X^\eta \lrcorner\, \Omega))X^\xi - \mathrm{d}(X^\xi \lrcorner\, (X^\eta \lrcorner\, \Omega))X^\zeta - [X^\zeta, X^\xi] \lrcorner\, (X^\eta \lrcorner\, \Omega)$$

and

$$\mathrm{d}(X^\zeta \lrcorner\, \Omega)(X^\xi, X^\eta) = \mathrm{d}(X^\xi \lrcorner\, (X^\zeta \lrcorner\, \Omega))X^\eta - \mathrm{d}(X^\eta \lrcorner\, (X^\zeta \lrcorner\, \Omega))X^\xi - [X^\xi, X^\eta] \lrcorner\, (X^\zeta \lrcorner\, \Omega).$$

Substituting the above three equations into (13) gives

$$\begin{aligned} \mathrm{d}\Omega(X^\xi, X^\eta, X^\zeta) &= \mathrm{d}(X^\eta \lrcorner (X^\xi \lrcorner \Omega))X^\zeta - \mathrm{d}(X^\zeta \lrcorner (X^\xi \lrcorner \Omega))X^\eta \\ &\quad - \mathrm{d}(X^\xi \lrcorner (X^\eta \lrcorner \Omega))X^\zeta + \mathrm{d}(X^\zeta \lrcorner (X^\eta \lrcorner \Omega))X^\xi \\ &\quad + \mathrm{d}(X^\xi \lrcorner (X^\zeta \lrcorner \Omega))X^\eta - \mathrm{d}(X^\eta \lrcorner (X^\zeta \lrcorner \Omega))X^\xi. \end{aligned} \tag{14}$$

We calculate the first term in (14) as follows

$$\begin{aligned} \mathrm{d}(X^\eta \lrcorner X^\xi \lrcorner \Omega)(\nu)X^\zeta(\nu) &= \big(L_{X^\zeta}\Omega(X^\xi, X^\eta)\big)(\nu) \\ &= \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0} \Omega(\varphi_s^\zeta(\nu))\big(X^\xi(\varphi_s^\zeta(\nu)), X^\eta(\varphi_s^\zeta(\nu))\big) \\ &= -\frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0} \varphi_s^\zeta(\nu([\xi,\eta])), \quad \text{using (11)} \\ &= \mathrm{ad}^t_\zeta \, \nu([\xi,\eta]), \quad \text{using (10)} \\ &= \nu([\zeta,[\xi,\eta]]). \end{aligned}$$

The other terms in (14) follow similarly. Therefore

$$\begin{aligned} \mathrm{d}\Omega(X^\xi, X^\eta, X^\zeta) &= \nu([\zeta,[\xi,\eta]]) - \nu([\eta,[\xi,\zeta]]) - \nu([\zeta,[\eta,\xi]]) \\ &\qquad + \nu([\xi,[\eta,\zeta]]) + \nu([\eta,[\zeta,\xi]]) - \nu([\xi,[\zeta,\eta]]) \\ &= 2\nu\big([\zeta,[\xi,\eta]] + [\xi,[\eta,\zeta]] + [\eta,[\zeta,\xi]]\big) \\ &= 0 \end{aligned}$$

by the Jacobi identity. Thus $\mathrm{d}\Omega = 0$, which is what we wanted to show.

To see that Ω is nondegenerate, suppose that $0 = \Omega(\nu)(X^\xi(\nu), X^\eta(\nu))$ for every $X^\eta(\nu) = -\mathrm{ad}^t_\eta \, \nu \in T_\nu \mathscr{O}_\mu$. Then from the definition of Ω it follows that for every $\eta \in \mathfrak{g}$, we have

$$0 = \nu([\xi,\eta]) = \nu(\mathrm{ad}_\xi \, \eta) = (\mathrm{ad}^t_\xi \, \nu)(\eta) = -X^\xi(\nu)(\eta),$$

that is, $X^\xi(\nu) = 0$, since $X^\xi(\nu)$ annihilates all of $\mathfrak{g}^*$. Consequently Ω is nondegenerate. Note that Ω is invariant under the coadjoint action of G on $\mathscr{O}_\mu$. □

3 Hamilton's equations

In this section we define the concept of Hamiltonian vector field.

On a symplectic manifold (M,ω), the nondegeneracy of the symplectic form ω implies that the mapping $\omega^\sharp(p) : T_pM \to T_p^*M$ defined by $\omega^\sharp(p)(v)w = \omega(p)(v,w)$ for $v, w \in T_pM$ is an isomorphism for every $p \in M$. Denote the inverse of $\omega^\sharp(p)$ by $\omega^\flat(p)$. For each smooth function f on M we may define the *Hamiltonian vector field* X_f of the *Hamiltonian function* f by $X_f(p) = \omega^\flat(p)\big(\mathrm{d}f(p)\big)$, or equivalently $X_f \lrcorner \, \omega = \mathrm{d}f$.

Example 1. Consider the cotangent bundle T^*N with its canonical symplectic form Ω. Suppose that $f : T^*N \to \mathbf{R}$ is a smooth function. We calculate a local expression for the

Hamiltonian vector field X_f as follows. Let U be an open subset of $\mathbf{R}^n$. Then locally T^*N is $T^*U = U \times (\mathbf{R}^n)^*$ with bundle projection $\tau : T^*U \to U : (u,\alpha) \to u$. From the definition of the canonical 2-form Ω it follows that for every $(u,\alpha) \in T^*U$

$$\Omega^\sharp(u,\alpha) : T_{(u,\alpha)}(T^*U) = \mathbf{R}^n \times (\mathbf{R}^n)^* \to T^*_{(u,\alpha)}(T^*U) = (\mathbf{R}^n)^* \times \mathbf{R}^n : (v,\beta) \mapsto \big(\Omega^\sharp(u,\alpha)\big)(v,\beta),$$

where $\Omega^\sharp(u,\alpha)\begin{pmatrix} v \\ \beta \end{pmatrix}$ is the linear function on $T_{(u,\alpha)}(T^*U)$ whose value at (w,γ) is given by $= -\beta(w) + \gamma(v)$, that is, $\Omega^\sharp(u,\alpha)\begin{pmatrix} v \\ \beta \end{pmatrix} = (-\beta, v)$. Consequently, the local expression for $\Omega^\flat$ is $\Omega^\flat(u,\alpha)(\gamma,w) = \begin{pmatrix} w \\ -\gamma \end{pmatrix}$. Locally, the 1-form $\mathrm{d}f$ is

$$\mathrm{d}f : T^*U \subseteq \mathbf{R}^n \times (\mathbf{R}^n)^* \to (\mathbf{R}^n)^* \times \mathbf{R}^n : (u,\alpha) \mapsto \big(D_1 f(u,\alpha), D_2 f(u,\alpha)\big).$$

Thus on T^*N the local expression for the Hamiltonian vector field X_f corresponding to f is given by

$$\begin{aligned} &X_f : T^*U \to \mathbf{R}^n \times (\mathbf{R}^n)^* : \\ &\quad (u,\alpha) \mapsto \Omega^\flat(u,\alpha)\big(D_1 f(u,\alpha), D_2 f(u,\alpha)\big) = \big(D_2 f(u,\alpha), -D_1 f(u,\alpha)\big). \end{aligned}$$

An integral curve $\gamma : I \to T^*U$ of X_f satisfies *Hamilton's equations*

$$\frac{\mathrm{d}\gamma(t)}{\mathrm{d}t} = X_f(\gamma(t)) = \begin{pmatrix} D_2 f(\gamma(t)) \\ -D_1 f(\gamma(t)) \end{pmatrix}. \qquad \square$$

Example 2. We derive an expression for the Hamiltonian vector field X_K on TN associated to the Hamiltonian function $K : TN \to \mathbf{R} : v \mapsto \frac{1}{2} g(x)(v,v)$, where g is a smooth nondegenerate *metric* on N and $v \in T_x N$. On TN we use the symplectic form $\omega = (g^\sharp)^*\Omega$, where Ω is the canonical 2-form on T^*N. The mapping $g^\sharp(x) : T_x N \to T^*_x N$ is defined by $g^\sharp(x)(v) : T_x N \to \mathbf{R} : w \mapsto g(x)(v,w)$. From the local expression of X_K given in (15) below we see that its integral curves are geodesics for the metric g. Thus X_K is the geodesic vector field associated to the metric g.

We begin by deriving a local expression for the 1-form ϑ on TN defined by $\vartheta(v)(w_v) = g(x)(T_v\tau\, w_v, v)$. Here $v \in T_x N$, $w_v \in T_v N$ and $\tau : TN \to N$ is the bundle projection. Then $\vartheta = (g^\sharp)^*\theta$, where θ is the canonical 1-form on T^*N. Let U be an open subset of $\mathbf{R}^n$. Then locally TN is $TU = U \times \mathbf{R}^n$ with bundle projection $\tau : TU \to U : (x,v) \mapsto x$. Moreover, the tangent of τ is

$$T\tau : T(TU) = (U \times \mathbf{R}^n) \times (\mathbf{R}^n \times \mathbf{R}^n) \to TU = U \times \mathbf{R}^n : \big((x,v),(w,z)\big) \mapsto (x,w).$$

Therefore, $\vartheta(x,v)(w,z) = g^\sharp(x)(v)w$, where $\vartheta(x,v)$ is a linear mapping from $\mathbf{R}^n \times \mathbf{R}^n$ into $\mathbf{R}$. To find the symplectic form ω on TN we note that $\omega = (g^\sharp)^*\Omega = -(g^\sharp)^*\,\mathrm{d}\theta = -\mathrm{d}(g^\sharp)^*\theta = -\,\mathrm{d}\vartheta$. Thus it suffices to compute the exterior derivative of ϑ. This we do as follows. By definition of exterior derivative

$$\begin{aligned} -\big(\mathrm{d}\vartheta(x,v)(w_1,z_1)\big)(w_2,z_2) &= -D_1\vartheta(x,v)w_1(w_2,z_2) + D_2\vartheta(x,v)z_1(w_2,z_2) \\ &= -Dg^\sharp(x)w_1(v,w_2) - g^\sharp(x)(z_1)w_2 + Dg^\sharp(x)w_2(v,w_1) + g^\sharp(x)(z_2)w_1 \end{aligned}$$

$$= \omega(x,v)\big((w_1,z_1),(w_2,z_2)\big),$$

which is the desired local expression for the symplectic form ω on TN. Now we compute the local expression for the Hamiltonian vector field X_K. On TU the function K is given by $K : U \times \mathbf{R}^n \subseteq \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R} : (x,v) \mapsto \frac{1}{2} g(x)(v,v) = \frac{1}{2} g^\sharp(x)(v)v$. Thus the 1-form $\mathrm{d}K$ is

$$\mathrm{d}K(x,v) : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R} : (w_1,z_1) \to D_1K(x,v)w_1 + D_2K(x,v)z_1 \;=\; \tfrac{1}{2} Dg^\sharp(x)w_1(v,v) + g^\sharp(x)(v)z_1.$$

Let X_K be the vector field on TU given by

$$X_K : U \times \mathbf{R}^n \to \mathbf{R}^n \times \mathbf{R}^n : (x,v) \mapsto \big(X^1(x,v), X^2(x,v)\big).$$

Then for any vector field Y on TU equal to (Y^1,Y^2) we have $\omega(X_K,Y) = \mathrm{d}K(Y)$, by definition of X_K, that is,

$$\begin{aligned} -Dg^\sharp(x)X^1(v,Y^1) - g^\sharp(x)(X^2)Y^1 + Dg^\sharp(x)Y^1(v,X^1) + g^\sharp(x)(Y^2)X^1 &= \\ = \tfrac{1}{2} Dg^\sharp(x)Y^1(v,v) + g^\sharp(x)(v)Y^2&. \end{aligned}$$

Setting $Y^1 = 0$ in the above equation gives $g^\sharp(x)(Y^2)X^1 = g^\sharp(x)(v)Y^2$. In other words, $g(x)(Y^2,X^1) = g(x)(v,Y^2)$ for every Y^2. Thus by symmetry and nondegeneracy of $g(x)$ we obtain $X^1(x,v) = v$. Setting $Y^2 = 0$ and using $X^1 = v$ gives

$$-Dg^\sharp(x)v(v,Y^1) - g^\sharp(x)(X^2)Y^1(v,v) + Dg^\sharp(x)Y^1(v,v) \;=\; \tfrac{1}{2} Dg^\sharp(x)Y^1(v,v),$$

which rewritten, using the symmetry property $(Dg^\sharp(x)X)(Y,Z) = (Dg^\sharp(x)X)(Z,Y)$, is

$$g(x)(X^2,Y^1) = \tfrac{1}{2}\big(Dg^\sharp(x)v(Y^1,v) - Dg^\sharp(x)v(v,Y^1) - Dg^\sharp(x)Y^1(v,v)\big) = -\widetilde{\Gamma}(x,v)Y^1.$$

Therefore we obtain $X^2(x,v) = -g(x)^\flat\big(\widetilde{\Gamma}(x,v)\big) = -\Gamma(x,v)$. Hence the local expression of the vector field X_K is

$$X_K(x,v) = (v, -\Gamma(x,v)). \tag{15}$$

Since the components of Γ are the usual *Christoffel symbols* of the metric g, the image of the integral curves of the *geodesic vector field* X_K under the bundle projection τ are *geodesics* on N for the metric g. □

Let $\varphi : M \to M$ be a diffeomorphism of the symplectic manifold (M,ω). If φ preserves ω, that is, $\varphi^*\omega = \omega$, then φ is a *symplectic diffeomorphism.*

Claim: Let f be a smooth function on (M,ω). The *flow* φ_t^f of a Hamiltonian vector field X_f is a local one parameter group of symplectic diffeomorphisms of (M,ω).

(3.1) **Proof**: From the definition of the *flow* of a vector field it follows that $t \mapsto \varphi_t^f$ is a local one parameter group. To show that φ_t^f is symplectic, we calculate

$$\begin{aligned} L_{X_f}\omega &= X_f \mathbin{\lrcorner} \mathrm{d}\omega + \mathrm{d}(X_f \mathbin{\lrcorner} \omega) = \mathrm{d}^2 f, \quad \text{using the definition of } X_f \text{ and the fact that } \omega \text{ is closed} \\ &= 0. \end{aligned}$$

By definition of *Lie derivative* $\frac{\mathrm{d}}{\mathrm{d}t}(\varphi_t^f)^*\omega = (\varphi_t^f)^*(L_{X_f}\omega) = 0$. We get $(\varphi_t^f)^*\omega = \omega$. $\square$

A basic property of a symplectic diffeomorphism is that it maps a Hamiltonian vector field into another Hamiltonian vector field.

Claim: For every smooth function on the symplectic manifold (M,ω) and for every symplectic diffeomorphism φ of (M,ω) into itself we have

$$\varphi^* X_f = X_{\varphi^* f}. \tag{16}$$

(3.2) **Proof**: We calculate

$$X_{\varphi^* f} \mathbin{\lrcorner} \omega = \mathrm{d}(\varphi^* f) = \varphi^* \mathrm{d}f = \varphi^*(X_f \mathbin{\lrcorner} \omega) = \varphi^* X_f \mathbin{\lrcorner} \varphi^*\omega = \varphi^* X_f \mathbin{\lrcorner} \omega,$$

since φ is symplectic. Equation (16) follows because ω is nondegenerate. $\square$

Example 3. We derive Hamilton's equations on the cotangent bundle of a Lie group, which has been trivialized using left translation. Let G be a Lie group and (T^*G,Ω) its cotangent bundle with canonical 2-form Ω. Suppose that $\mathscr{H}: T^*G \to \mathbf{R}$ is a smooth function. The Hamiltonian vector field $X_{\mathscr{H}}$ on (T^*G,Ω) corresponding to $\mathscr{H}$ is defined by

$$\mathrm{d}\mathscr{H}(\alpha)v_\alpha = \Omega(\alpha)(X_{\mathscr{H}}(\alpha), v_\alpha) \tag{17}$$

for every $v_\alpha \in T_\alpha(T^*G)$. Trivialize T^*G using the mapping $\mathscr{L}: G\times\mathfrak{g}^* \to T^*G$ (6) coming from left translation. Pulling the Hamiltonian $\mathscr{H}$ back by $\mathscr{L}$ gives the Hamiltonian $H : G\times\mathfrak{g}^* \to \mathbf{R} : (g,\alpha) \mapsto \mathscr{H}(\alpha_g)$. Note that $\mathscr{L}^*\Omega = \omega$ (8). Using ((3.2)) we find that the Hamiltonian vector field X_H on $(G\times\mathfrak{g}^*,\omega)$ is given by $X_H = \mathscr{L}^*(X_{\mathscr{H}})$.

We now derive Hamilton's equations, which are satisfied by the integral curves of the vector field X_H on $G\times\mathfrak{g}^*$. In this context Hamilton's equations are called the *Euler-Arnol'd equations*. Partially differentiating H in the directions $T_eL_g\eta \in T_gG$ and $\beta \in \mathfrak{g}^*$ gives

$$\mathrm{d}H(g,\alpha)(T_eL_g\eta,\beta) = D_1H(g,\alpha)T_eL_g\eta + \beta\big(D_2H(g,\alpha)\big),$$

since $D_2H(g,\alpha)\beta = \beta\big(D_2H(g,\alpha)\big)$ for $D_2H(g,\alpha) \in \mathfrak{g}^{**} = \mathfrak{g}$. Because $X_H(g,\alpha) \in T_gG \times \mathfrak{g}^*$, we may write

$$X_H(g,\alpha) = \big(T_eL_gX(g,\alpha), \Lambda(g,\alpha)\big). \tag{18}$$

By definition of X_H, we have

$$\mathrm{d}H(g,\alpha)(T_eL_g\eta,\beta) = \omega(g,\alpha)\big((T_eL_gX(g,\alpha),\Lambda(g,\alpha)),(T_eL_g\eta,\beta)\big)$$

for any $\eta \in \mathfrak{g}$ and any $\beta \in \mathfrak{g}^*$. Now using (8) the preceding equality reads

$$D_1H(g,\alpha)T_eL_g\eta + \beta(D_2H(g,\alpha)) = -\Lambda(g,\alpha)\eta + \beta(X(g,\alpha)) + \alpha([X(g,\alpha),\eta]). \tag{19}$$

Setting $\eta = 0$ in (19) gives $\beta(D_2H(g,\alpha)) = \beta(X(g,\alpha))$ for every $\beta \in \mathfrak{g}^*$. Therefore

$$X(g,\alpha) = D_2H(g,\alpha). \tag{20}$$

Setting $\beta = 0$ in (19) gives $D_1 H(g,\alpha) T_e L_g \eta = -\Lambda(g,\alpha)\eta + (\mathrm{ad}^t_X \alpha)\eta$, for any $\eta \in \mathfrak{g}$. Using (20), we obtain

$$\Lambda(g,\alpha) = -(T_e L_g)^t (D_1 H(g,\alpha)) + \mathrm{ad}^t_{D_2 H(g,\alpha)} \alpha. \tag{21}$$

Substituting (20) and (21) into (18) gives the *Euler-Arnol'd vector field X_H* on $G \times \mathfrak{g}^*$. The integral curves of X_H satisfy the Euler-Arnol'd equations

$$\begin{aligned} \dot{g} &= T_e L_g (D_2 H(g,\alpha)) \\ \dot{\alpha} &= -(T_e L_g)^t (D_1 H(g,\alpha)) + \mathrm{ad}^t_{D_2 H(g,\alpha)} \alpha. \end{aligned} \tag{22}$$

If the Hamiltonian $\mathscr{H} : T^*G \to \mathbf{R}$ is left invariant, then so is H. Therefore, $D_1 H = 0$, so the Euler-Arnol'd equations for a left invariant Hamiltonian are

$$\begin{aligned} \dot{g} &= T_e L_g (D_2 H(g,\alpha)) \\ \dot{\alpha} &= \mathrm{ad}^t_{D_2 H(g,\alpha)} \alpha. \end{aligned}$$

□

4 Poisson algebras and manifolds

In this section we define the notions of a Poisson algebra and a Poisson manifold. This leads to an algebraic formulation of Hamiltonian mechanics.

A *Poisson algebra* $(\mathscr{A}, \{\,,\,\}, \cdot)$ is a real Lie algebra under the *bracket* $\{\,,\,\}$ which is also a commutative ring with unit under the multiplication $\cdot$. In addition, for every $f,g,h \in \mathscr{A}$ *Leibniz' rule*: $\{f, g\cdot h\} = h\cdot\{f,g\} + g\cdot\{f,h\}$ holds. Leibniz' rule simply states that for every $f \in \mathscr{A}$ the linear mapping $\mathrm{ad}_f : \mathscr{A} \to \mathscr{A} : g \mapsto \{f,g\}$ is a derivation. We call ad_f the *Hamiltonian derivation* associated to f. Thinking of ad_f as a formal vector field, its *formal flow* is given by $\varphi^f_t = \exp(t\,\mathrm{ad}_f) = \sum_{n=0}^{\infty} \frac{1}{n!} t^n \mathrm{ad}^n_f$. A map $\varphi : \mathscr{A} \to \mathscr{B}$ is a *homomorphism* of Poisson algebras if it is a homomorphism of the Lie algebra and commutative ring structures. A bijective homomorphism of the Poisson algebra $\mathscr{A}$ is an automorphism of $\mathscr{A}$. It is easy to check that the formal flow φ^f_t is a one parameter group of automorphisms of $\mathscr{A}$, that is, $(\varphi^f_t)^*(g\cdot h) = (\varphi^f_t)^* g \cdot (\varphi^f_t)^* h$ and $(\varphi^f_t)^*\{g,h\} = \{(\varphi^f_t)^* g, (\varphi^f_t)^* h\}$. An element $f \in \mathscr{A}$ is a *Casimir* if and only if $\{f,g\} = 0$ for every $g \in \mathscr{A}$. Clearly the unit element of $\mathscr{A}$ is a Casimir. If the only Casimir elements are real multiples of the unit element of $\mathscr{A}$, then $\mathscr{A}$ is a *nondegenerate* Poisson algebra.

We now give some examples of Poisson algebras.

Example 1. Let $\mathfrak{g}^*$ be the dual of the finite dimensional Lie algebra $\mathfrak{g}$ with Lie bracket $[\,,\,]$. For every $f,g \in C^\infty(\mathfrak{g}^*)$ and every $\mu \in \mathfrak{g}^*$ define a Poisson bracket $\{\,,\,\}_{\mathfrak{g}^*}$ by

$$\{f,g\}_{\mathfrak{g}^*}(\mu) = \mu([\mathrm{d}f(\mu), \mathrm{d}g(\mu)]). \tag{23}$$

The right hand side of (23) is well defined since the differential $\mathrm{d}f(\mu)$ of f at μ is a linear form on $\mathfrak{g}^*$, which we identify with an element of $\mathfrak{g}$. To check that $(C^\infty(\mathfrak{g}^*), \{\,,\,\}_{\mathfrak{g}^*})$ is a Poisson algebra, where $\cdot$ is the usual product of smooth functions, we first verify Leibniz' rule:

$$\{f, g\cdot h\}_{\mathfrak{g}^*}(\mu) \quad = \quad \mu\big([\mathrm{d}f(\mu), \mathrm{d}(g\cdot h)(\mu)]\big)$$

$$\begin{aligned} &= \mu\big([\mathrm{d}f(\mu), g(\mu)\,\mathrm{d}h(\mu) + h(\mu)\,\mathrm{d}g(\mu)]\big) \\ &= \big(g\cdot\{f,h\}_{\mathfrak{g}^*} + h\cdot\{f,g\}_{\mathfrak{g}^*}\big)(\mu). \end{aligned}$$

We now need only show that $\{\,,\,\}_{\mathfrak{g}^*}$ satisfies the Jacobi identity. This is not entirely straightforward.

Let $\{x_i\}$, $i = 1,\ldots n$ be a basis for $\mathfrak{g}$. Identifying $(\mathfrak{g}^*)^*$ with $\mathfrak{g}$ we can think of $\{x_i\}$ as coordinates on $\mathfrak{g}^*$. Since $\mathrm{ad}_f : \mathfrak{g}^* \to \mathfrak{g}^* : h \mapsto \{f,h\}_{\mathfrak{g}^*}$ is a derivation on $\mathfrak{g}^*$, it follows that $\mathrm{ad}_f = \sum_{k=1}^n f_k\,\partial_k$ for some $f_k \in C^\infty(\mathfrak{g}^*)$. Here $\partial_k = \frac{\partial}{\partial x_k}$. Evaluating the preceding expression for ad_f on x_j and using the fact that $\partial_k x_j = \delta_k^j$, we obtain $f_k = \{f, x_k\}_{\mathfrak{g}^*}$, that is,

$$\mathrm{ad}_f = \sum_{k=1}^n \{f, x_k\}_{\mathfrak{g}^*}\,\partial_k. \tag{24}$$

Replacing f in (24) by x_j gives

$$\mathrm{ad}_{x_j} = \sum_{k=1}^n \{x_j, x_k\}_{\mathfrak{g}^*}\,\partial_k. \tag{25}$$

Hence

$$\begin{aligned} \{f,g\}_{\mathfrak{g}^*} &= \mathrm{ad}_f g = \sum_k \{f,x_k\}_{\mathfrak{g}^*}\,\partial_k g \\ &= -\sum_k \mathrm{ad}_{x_k} f\,\partial_k g = \sum_{j,k}\{x_j,x_k\}_{\mathfrak{g}^*}\,\partial_j f\,\partial_k g. \end{aligned} \tag{26}$$

In other words, $(\{x_j,x_k\}_{\mathfrak{g}^*})$ is the *structure matrix* of the Poisson bracket $\{\,,\,\}_{\mathfrak{g}^*}$.

Claim: $(C^\infty(\mathfrak{g}^*), \{\,,\,\}_{\mathfrak{g}^*})$ is a Lie algebra if and only if the coordinate functions x_i satisfy the *Jacobi identity*, that is, if and only if

$$\{\{x_i,x_j\}_{\mathfrak{g}^*}, x_k\}_{\mathfrak{g}^*} = \{\{x_i,x_k\}_{\mathfrak{g}^*}, x_j\}_{\mathfrak{g}^*} + \{x_i, \{x_j,x_k\}_{\mathfrak{g}^*}\}_{\mathfrak{g}^*} \tag{27}$$

for every $1 \le i, j, k \le n$.

(4.1) **Proof**: Before launching into the proof we first verify the identity

$$\sum_k \{x_k,x_i\}_{\mathfrak{g}^*}\,\{\partial_k f, x_j\}_{\mathfrak{g}^*} = \sum_k \{x_k,x_j\}_{\mathfrak{g}^*}\,\{\partial_k f, x_i\}_{\mathfrak{g}^*}. \tag{28}$$

We calculate.

$$\begin{aligned} \sum_k \{x_k,x_i\}_{\mathfrak{g}^*}\,\{\partial_k f, x_j\}_{\mathfrak{g}^*} &= -\sum_k \{x_k,x_i\}_{\mathfrak{g}^*}\,\mathrm{ad}_{x_j}\partial_k f \\ &= -\sum_{k,\ell}\{x_k,x_i\}_{\mathfrak{g}^*}\,\{x_j,x_\ell\}_{\mathfrak{g}^*}\,\partial_\ell\partial_k f = \sum_k \{x_k,x_j\}_{\mathfrak{g}^*}\,\{\partial_k f, x_i\}_{\mathfrak{g}^*}. \end{aligned}$$

The last equality follows by interchanging the order of summation and partial differentiation, and using (25) and the skew symmetry of $\{\,,\,\}_{\mathfrak{g}^*}$.

We now prove the claim. Let $f, g, h \in C^\infty(\mathfrak{g}^*)$. Then

$$\begin{aligned}
\{\{f,g\},h\}_{\mathfrak{g}^*} &= \{\sum_{j,k}\{x_j,x_k\}_{\mathfrak{g}^*}\,\partial_j f\,\partial_k g, h\}_{\mathfrak{g}^*}, \quad \text{using (26)} \\
&= \sum_{j,k,\ell}\Big[\{\{x_j,x_k\}_{\mathfrak{g}^*},h\}_{\mathfrak{g}^*}\,\partial_j f\,\partial_k g + \{x_j,x_k\}_{\mathfrak{g}^*}\,\{\partial_j f,h\}_{\mathfrak{g}^*}\,\partial_k g \\
&\qquad + \{x_j,x_k\}_{\mathfrak{g}^*}\,\{\partial_k g,h\}_{\mathfrak{g}^*}\,\partial_j f\Big], \quad \text{using Leibniz' rule} \\
&= \sum_{j,k,\ell}\{\{x_j,x_k\}_{\mathfrak{g}^*},x_\ell\}_{\mathfrak{g}^*}\,\partial_\ell h\,\partial_j f\,\partial_k g + \overbrace{\sum_{j,k,\ell}\{x_j,x_k\}_{\mathfrak{g}^*}\,\{\partial_j f,x_\ell\}_{\mathfrak{g}^*}\,\partial_\ell h\,\partial_k g}^{I} \\
&\qquad + \overbrace{\sum_{j,k,\ell}\{x_j,x_k\}_{\mathfrak{g}^*}\,\{\partial_k g,x_\ell\}_{\mathfrak{g}^*}\,\partial_j f\,\partial_\ell h}^{II}, \quad \text{using (24).}
\end{aligned}$$

Interchanging h and g in the above formula gives

$$\begin{aligned}
\{\{f,g\},h\}_{\mathfrak{g}^*} &= \sum_{j,k,\ell}\{\{x_j,x_k\}_{\mathfrak{g}^*},x_\ell\}_{\mathfrak{g}^*}\,\partial_\ell g\,\partial_j f\,\partial_k h \\
&+ \overbrace{\sum_{j,k,\ell}\{x_j,x_k\}_{\mathfrak{g}^*}\{\partial_j f,x_\ell\}_{\mathfrak{g}^*}\,\partial_\ell g\,\partial_k h}^{I'} + \overbrace{\sum_{j,k,\ell}\{x_j,x_k\}_{\mathfrak{g}^*}\,\{\partial_k h,x_\ell\}_{\mathfrak{g}^*}\,\partial_j f\,\partial_\ell g}^{III}.
\end{aligned}$$

Since $\{f,\{g,h\}\}_{\mathfrak{g}^*} = -\{f,\{h,g\}\}_{\mathfrak{g}^*} = \{\{h,g\},f\}_{\mathfrak{g}^*}$, interchanging f and h gives

$$\begin{aligned}
\{f,\{g,h\}\}_{\mathfrak{g}^*} &= \sum_{j,k,\ell}\{\{x_j,x_k\}_{\mathfrak{g}^*},x_\ell\}_{\mathfrak{g}^*}\,\partial_\ell f\,\partial_j h\,\partial_k g \\
&+ \overbrace{\sum_{j,k,\ell}\{x_j,x_k\}_{\mathfrak{g}^*}\,\{\partial_j h,x_\ell\}_{\mathfrak{g}^*}\,\partial_\ell f\,\partial_k g}^{III'} + \overbrace{\sum_{j,k,\ell}\{x_j,x_k\}_{\mathfrak{g}^*}\,\{\partial_k g,x_\ell\}_{\mathfrak{g}^*}\,\partial_j h\,\partial_\ell f}^{II'}.
\end{aligned}$$

Using (28) one can show that the terms $I = I'$, $II = II'$, and $III = -III'$. We will prove only the last equality.

$$\begin{aligned}
III &= -\sum_{j,\ell}\sum_{k}\{x_k,x_j\}_{\mathfrak{g}^*}\,\{\partial_k h,x_\ell\}_{\mathfrak{g}^*}\,\partial_j f\,\partial_\ell g \\
&= -\sum_{j,\ell}\sum_{k}\{x_k,x_\ell\}_{\mathfrak{g}^*}\,\{\partial_k h,x_j\}_{\mathfrak{g}^*}\,\partial_j f\,\partial_\ell g, \quad \text{using (28)} \\
&= -III',
\end{aligned}$$

replacing k by j, ℓ by k and j by ℓ. Therefore

$$\begin{aligned}
&\{\{f,g\},h\}_{\mathfrak{g}^*} - \{\{f,h\},g\}_{\mathfrak{g}^*} - \{f,\{g,h\}\}_{\mathfrak{g}^*} = \\
&\quad = \sum_{j,k,\ell}\Big[\{\{x_j,x_k\}_{\mathfrak{g}^*},x_\ell\}_{\mathfrak{g}^*} - \{\{x_j,x_\ell\}_{\mathfrak{g}^*},x_k\}_{\mathfrak{g}^*} - \{x_j,\{x_\ell,x_k\}_{\mathfrak{g}^*}\}_{\mathfrak{g}^*}\Big]\,\partial_j f\,\partial_k g\,\partial_\ell h,
\end{aligned}$$

which proves the claim. □

▷ We now verify that (27) holds.

(4.2) **Proof**: By definition

$$\{x_i, x_j\}_{\mathfrak{g}^*}(\mu) = \mu\big([\mathrm{d}x_i, \mathrm{d}x_j]\big) = \mu\big([x_i, x_j]\big), \tag{29}$$

because x_ℓ are linear functions on $\mathfrak{g}^*$. Since $(\mathfrak{g}, [\,,\,])$ is a Lie algebra the Jacobi identity $[[x_i,x_j],x_k] = [[x_i,x_k],x_j] + [x_i,[x_j,x_k]]$ holds. Using (29) it follows that the Jacobi identity

$$\{\{x_j,x_k\}_{\mathfrak{g}^*}, x_\ell\}_{\mathfrak{g}^*} = \{\{x_j,x_\ell\}_{\mathfrak{g}^*}, x_k\}_{\mathfrak{g}^*} + \{x_j, \{x_\ell, x_k\}_{\mathfrak{g}^*}\}_{\mathfrak{g}^*}$$

holds. □

Hence by ((4.1)) $(C^\infty(\mathfrak{g}^*), \{\,,\,\}_{\mathfrak{g}^*})$ is a Lie algebra.

▷ When $\mathfrak{g}$ is a semisimple Lie algebra, the Poisson algebra $(C^\infty(\mathfrak{g}^*), \{\,,\,\}_{\mathfrak{g}^*})$ is *not* nondegenerate.

(4.3) **Proof**: We need only construct a Casimir element of $(C^\infty(\mathfrak{g}^*), \{\,,\,\}_{\mathfrak{g}^*}, \cdot)$ to prove the assertion. Recall that $\mathfrak{g}$ is *semisimple* if and only if the *Killing form* $k(X,Y) = \mathrm{tr}(\mathrm{ad}_X \,\mathrm{ad}_Y)$ is a nondegenerate symmetric bilinear form on $\mathfrak{g}$. The Killing form satisfies $k(\mathrm{ad}_Z X, Y) = -k(X, \mathrm{ad}_Z Y)$ for every $X, Y, Z \in \mathfrak{g}$ because

$$\begin{aligned} k(\mathrm{ad}_Z X, Y) &= \mathrm{tr}(\mathrm{ad}_{[Z,X]}\,\mathrm{ad}_Y) = \mathrm{tr}\big(\mathrm{ad}_Z\,\mathrm{ad}_X - \mathrm{ad}_X\,\mathrm{ad}_Z)\,\mathrm{ad}_Y\big) \\ &= \mathrm{tr}\big(\mathrm{ad}_X(\mathrm{ad}_Y\,\mathrm{ad}_Z - \mathrm{ad}_Z\,\mathrm{ad}_Y)\big), \\ &\qquad \text{since } \mathrm{tr}(ABC) = \mathrm{tr}(BCA) \text{ for linear maps } A,B,C. \\ &= \mathrm{tr}(\mathrm{ad}_X\,\mathrm{ad}_{[Y,Z]}) = -k(X, \mathrm{ad}_Z Y). \end{aligned}$$

Using the isomorphism $k^\flat = (k^\sharp)^{-1} : \mathfrak{g}^* \to \mathfrak{g}$ to identify $\mathfrak{g}^*$ with $\mathfrak{g}$, the Poisson bracket $\{\,,\,\}_{\mathfrak{g}^*}$ (23) on $C^\infty(\mathfrak{g}^*)$ becomes the Poisson bracket $\{\,,\,\}_{\mathfrak{g}}$ on $C^\infty(\mathfrak{g})$ defined by $\{f,g\}_{\mathfrak{g}}(X) = k\big(X, [k^\flat(\mathrm{d}f(X)), k^\flat(\mathrm{d}g(X))]\big)$. The following calculation shows that the smooth function $g : \mathfrak{g} \to \mathbf{R} : X \mapsto \frac{1}{2}k(X,X)$ is a Casimir element of $(C^\infty(\mathfrak{g}), \{\,,\,\}_{\mathfrak{g}}, \cdot)$.

$$\begin{aligned} \{f,g\}_{\mathfrak{g}}(X) &= k\big(X, [k^\flat(\mathrm{d}f(X)), X)]\big), \quad \text{by definition of } g \text{ and } \{\,,\,\}_{\mathfrak{g}} \\ &= -k\big([X,X], k^\flat(\mathrm{d}f(X))\big) = 0. \end{aligned}$$

□

Example 2. Let (M, ω) be a smooth symplectic manifold. On the space of smooth functions $C^\infty(M)$ define a *Poisson bracket* $\{\,,\,\}$ by $\{f,g\} = \omega(X_f, X_g)$ for every $f, g \in C^\infty(M)$. Here X_f, X_g are the Hamiltonian vector fields corresponding to the Hamiltonians f, g, respectively. It follows that $\{f,g\} = L_{X_g} f = -L_{X_f} g$. Thus ad_f is the derivation $-L_{X_f}$ which may be identified with the *Hamiltonian vector field* $-X_f$, since M is a smooth manifold.

Claim: $(C^\infty(M), \{\,,\,\}, \cdot)$ is a Poisson algebra.

(4.4) **Proof**: Clearly the bracket $\{\,,\,\}$ is linear in each argument and is skew symmetric. It remains to show that the derivation property and the Jacobi identity hold. To see that the bracket is a derivation in each slot, note that

$$\{f \cdot g, h\} = L_{X_h}(f \cdot g) = (L_{X_h} f) \cdot g + f \cdot (L_{X_h} g) = \{f,h\} \cdot g + f \cdot \{g,h\}.$$

Thus the derivation property holds because Leibniz' rule holds for the Lie derivative. To show that the Jacobi identity $\{f,\{g,h\}\} = \{\{f,g\},h\} + \{g,\{f,h\}\}$ holds we use the fact that the symplectic form ω is closed. Recall, see (12), that for smooth vector fields X, Y, Z on M

$$\begin{aligned} \mathrm{d}\omega(X,Y,Z) = \quad & -\mathrm{d}(X \lrcorner\, \omega)(Y,Z) + \mathrm{d}(Y \lrcorner\, \omega)(Z,X) + \mathrm{d}(Z \lrcorner\, \omega)(X,Y) \\ & -\omega([X,Y],Z) - \omega([Y,Z],X) - \omega([Z,X],Y). \end{aligned} \tag{30}$$

Letting $X = X_f$, $Y = X_g$ and $Z = X_h$, we see that the first three terms in (30) vanish because $\mathrm{d}(X_f \lrcorner\, \omega) = \mathrm{d}(\mathrm{d}f) = 0$. To deal with the next three terms note that $X_{\{f,g\}} = -[X_f, X_g]$, because

$$\begin{aligned} -[X_f, X_g] = L_{X_g} X_f &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} (\varphi_t^g)^* X_f = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} X_{(\varphi_t^g)^* f}, \quad \text{by ((3.2))} \\ &= X_{L_{X_g} f} = X_{\{f,g\}}. \end{aligned}$$

Thus $\omega\big([X_f, X_g], X_h\big) = -\omega(X_{\{f,g\}}, X_h) = -\{\{f,g\},h\}$. So

$$\mathrm{d}\omega(X_f, X_g, X_h) = \{\{f,g\},h\} - \{\{f,h\},g\} + \{\{g,h\},f\}. \tag{31}$$

Consequently, the closedness of ω implies that Jacobi's identity holds. □

Corollary: If $\varphi : M \to M$ is a symplectic diffeomorphism of (M, ω), then

$$\varphi^*\{f,g\} = \{\varphi^* f, \varphi^* g\}$$

for every $f, g \in C^\infty(M)$.

(4.5) **Proof**: For every $m \in M$ we compute

$$\begin{aligned} \{\varphi^* f, \varphi^* g\}(m) &= \omega(m)\big(X_{\varphi^* f}(m), X_{\varphi^* g}(m)\big) = \omega(m)\big(\varphi^* X_f(m), \varphi^* X_g(m)\big) \quad \text{by ((3.2))} \\ &= \omega(\varphi^{-1}(\varphi(m)))\big(T\varphi^{-1} X_f(\varphi(m)), T\varphi^{-1} X_g(\varphi(m))\big) \\ &= \big((\varphi^{-1})^*\omega\big)(\varphi(m))\big(X_f(\varphi(m)), X_g(\varphi(m))\big) = \{f,g\}(\varphi(m)) \\ &= \varphi^*(\{f,g\})(m). \end{aligned}$$ □

▷ M is a *Poisson manifold* if there is a Poisson bracket $\{\,,\}$ on $C^\infty(M)$ such that $(C^\infty(M), \{\,,\}, \cdot)$ is a Poisson algebra. We now show that the Poisson bracket $\{\,,\}$ is determined by the brackets of local coordinate functions on M.

(4.6) **Proof**: For a fixed f the map $g \to \{f,g\}$ is a derivation on $C^\infty(M)$, and hence may be represented by a vector field Y_f. Thus in local coordinates we have an expression of the form $\{f,g\} = \sum_i Y_f^i \partial_i g$, where ∂_i is the partial derivative with respect to the i^{th} coordinate function x_i and $Y_f^i = L_{Y_f} x_i$. By the skew symmetry of the Poisson bracket we have a similar expression with f and g interchanged. Therefore we have a local expression $\{f,g\} = \sum_{i,j} W_{ij} \partial_i f \partial_j g$. This tells us that the W_{ij} are components of a skew symmetric contravariant tensor W of order 2. Because the Poisson bracket $\{\,,\}$ satisfies the Jacobi identity, the components of W satisfy

$$\sum_\ell \left(\frac{\partial W_{jk}}{\partial x_\ell} W_{i\ell} + \frac{\partial W_{ij}}{\partial x_\ell} W_{k\ell} + \frac{\partial W_{ki}}{\partial x_\ell} W_{j\ell} \right) = 0. \tag{32}$$

If we compute the bracket of our local coordinate functions $(x_1,\ldots,x_n)$ using the local expression for the Poisson bracket , we find that $\{x_i,x_j\} = W_{ij}$. $\square$

The *rank* of the Poisson structure at a point is the rank of the *structure matrix* W at the point.

Claim: A Poisson manifold M is symplectic if and only if the matrix of the Poisson structure tensor W is everywhere invertible.

(4.7) **Proof**: Suppose that W is everywhere invertible. Let $X_f = W(\cdot,\mathrm{d}f)$, and define ω by

$$\omega(X_f,X_g) = W(\mathrm{d}f,\mathrm{d}g) = \mathrm{d}f(X_g) = \{f,g\}. \tag{33}$$

We only need check that ω is closed and nondegenerate. Since at any point $p \in M$ we have $\operatorname{span}\{\mathrm{d}f(p)\big| f \in C^\infty(M)\} = T_p^*M$ and the matrix (W_{ij}) is invertible, we see that ω is nondegenerate. As a corollary, the vector fields X_f span T_pM as well. They really are Hamiltonian vector fields since (33) implies that $\mathrm{d}f = W^{-1}(X_f,\cdot) = X_f \lrcorner\, \omega$. To see that ω is closed, we refer to ((4.4)), where we showed that a symplectic manifold (M,ω) gave a Lie algebra structure to $C^\infty(M)$ such that

$$\mathrm{d}\omega(X_f,X_g,X_h) = \{f,\{g,h\}\} + \{\{g,h\},f\} + \{\{h,f\},g\}. \tag{34}$$

Because W is a Poisson structure tensor, (32) holds and thus the right hand side of (34) vanishes. Hence $\mathrm{d}\omega = 0$.

Now suppose that (M,ω) is a symplectic manifold. Let $\{x_1,\ldots,x_n\}$ be local coordinates on (M,ω). Then the structure matrix W has entries $W_{ij} = \omega(X_{x_i},X_{x_j})$. For $f,g \in C^\infty(M)$ define

$$\{f,g\} = \sum_{ij} W_{ij}\,\partial_i f\,\partial_i g = W(\mathrm{d}f,\mathrm{d}g).$$

Thus $\{f,g\} = \omega(X_f,X_g)$. Since ω is closed, the Poisson bracket $\{\,,\}$ satisfies the Jacobi identity ((4.4)) and thus (32). Suppose that $W(\mathrm{d}f,\mathrm{d}g) = 0$ for every $g \in C^\infty(M)$. Then $0 = \omega(m)(X_f(m),X_g(m))$ for every $m \in M$. Since $\operatorname{span}\{\mathrm{d}g(m)\big| g \in C^\infty(M)\} = T_m^*M$ and $\omega(m)$ is nondegenerate, the vector fields $X_g(m)$ span T_mM. From the nondegeneracy of $\omega(m)$ we deduce that $X_f(m) = 0$ for every $m \in M$. Therefore $\mathrm{d}f = 0$, that is, W is nondegenerate. This implies that the matrix $(W_{ij}(m))$ is invertible for every $m \in M$. $\square$

We now prove a result which gives the local structure of Poisson manifolds

Claim: Suppose that $(M,\{\,,\})$ is a Poisson manifold of dimension n, whose structure tensor W in a neighborhood of p has constant rank k. Then k is even, say $k = 2\ell$, and there are local coordinates about p of the form $x = (\xi_1,\ldots,\xi_\ell,\eta_1,\ldots,\eta_\ell,\,\zeta_{2\ell+1},\ldots,\zeta_{n-2\ell})$ such that

$$W(x) = \left(\begin{array}{cc|c} 0 & I_\ell & \\ -I_\ell & 0 & \\ \hline & & 0_{n-2\ell} \end{array}\right).$$

(4.8) **Proof**: Pick a chart so that p is mapped to the origin. If the rank of $W(0)$ is zero, then put $W(x) = 0$ and we are done. If not, then there are smooth functions $\widetilde{\xi}_1$ and η_1 in a neighborhood of 0 such that $\{\widetilde{\xi}_1, \eta_1\}(0) \neq 0$. This implies $L_{X_{\eta_1}} \widetilde{\xi}_1(0) \neq 0$, and in particular, that the vector field $X_{\eta_1}(0) \neq 0$. We may find a system of local coordinates $(\xi_1, \ldots, \xi_n)$ about 0 so that $X_{\eta_1} = \frac{\partial}{\partial \xi_1}$ by rectifying the vector field X_{η_1}. Then in this neighborhood $\{\xi_1, \eta_1\} = 1$. If the rank of W at 0 is 2 we are done. Otherwise, let $N = \{x | \, \xi_1 = \eta_1 = 0\}$. In a possibly smaller neighborhood, N is a smooth submanifold containing the origin as $\mathrm{d}\xi_1 \wedge \mathrm{d}\eta_1 \neq 0$ at 0. Let $\varphi_t^{\xi_1}$ and $\varphi_s^{\eta_1}$ denote the flow of X_{ξ_1} and X_{η_1}, respectively. We find that they commute as $\left[X_{\xi_1}, X_{\eta_1}\right] = -X_{\{\xi_1, \eta_1\}} = 0$. For small enough t and s we see that $\mathscr{U} = \{u \in M \big| \, u = \varphi_t^{\xi_1} \circ \varphi_s^{\eta_1}(v), \, v \in N\}$ is an open neighborhood of N about 0. Choosing coordinates $(\overline{x}_1, \ldots, \overline{x}_{n-2})$ on N, we extend them to functions on $\mathscr{U}$ by setting $x_i(u) = \overline{x}_i(v)$. Then $(\xi_1, \eta_1, x_1, \ldots, x_{n-2})$ are coordinates on $\mathscr{U}$. Furthermore, the functions x_i are invariant under the flows $\varphi_t^{\xi_1}$ and $\varphi_s^{\eta_1}$. Therefore $\{\xi_1, x_i\} = 0 = \{\eta_1, x_i\}$. As

$$\begin{aligned}
\frac{\partial}{\partial \xi_1}(\{x_i, x_j\}) = L_{X_{\eta_1}}(\{x_i, x_j\}) &= \{L_{X_{\eta_1}} x_i, x_j\} + \{x_i, L_{X_{\eta_1}} x_j\} \\
= \{\{x_i, \eta_1\}, x_j\} + \{x_i, \{x_j, \eta_1\}\} &= 0,
\end{aligned}$$

$\{x_i, x_j\}$ does not depend on ξ_1. Similarly, $\{x_i, x_j\}$ does not depend on η_1. Therefore $\{x_i, x_j\}$ depends only on $(x_1, \ldots, x_{n-2})$. To finish the argument we just have to check that we can repeat this argument on N. Identify $C^\infty(N)$ with the smooth functions on $\mathscr{U}$ which are invariant under the flows of X_{ξ_1} and X_{η_1}. For these functions define a Poisson structure $\{\, , \,\}_N$ on N by $\{f|N, g|N\}_N = \{f, g\}|N$. Thus we may continue the argument inductively on $(C^\infty(N), \{\, , \,\}_N)$. □

When W is invertible, that is, $k = n$, the above result gives a local structure theorem for symplectic manifolds.

An important consequence of ((4.8)) is that in (ξ, η) coordinates about 0 the Hamiltonian derivation $-\mathrm{ad}_f$ for $f \in C^\infty(\mathbf{R}^n)$ has a local flow which satisfies Hamilton's equations

$$\begin{aligned}
\dot{\xi}_j &= -\{f, \xi_j\} = \frac{\partial f}{\partial \eta_j} \\
\dot{\eta}_j &= -\{f, \eta_j\} = -\frac{\partial f}{\partial \xi_j}
\end{aligned}$$

for $j = 1, \ldots, \ell$. Moreover the flow of $-\mathrm{ad}_f$ is a local one parameter group of symplectic diffeomorphisms of $(\mathbf{R}^n, \omega)$.

Suppose that $(M, \{\, , \,\}_1)$ and $(N, \{\, , \,\}_2)$ are Poisson manifolds. Then $\varphi : M \to N$ is a *Poisson map* if $\varphi^*\{f, h\}_2 = \{\varphi^* f, \varphi^* h\}_1$ for every $f, h \in C^\infty(M)$. In other words, the map $\varphi^* : C^\infty(N) \to C^\infty(M)$ is a homomorphism of Poisson algebras.

Claim: Let $f \in C^\infty(N)$ and $\varphi : M \to N$ be a Poisson map. If γ is an integral curve of the Hamiltonian vector field $X_{\varphi^* f}$ on M, then $\varphi \circ \gamma$ is an integral curve of X_f.

(4.9) **Proof**: For every $h \in C^\infty(N)$ we have

$$\frac{\mathrm{d}(\varphi^* h)}{\mathrm{d}t}(\gamma(t)) = \{\varphi^* h, \varphi^* f\}_1(\gamma(t)), \tag{35}$$

because γ is an integral curve of $X_{\varphi^* f}$. Since φ is a Poisson map, (35) is equivalent to

$$\frac{\mathrm{d}}{\mathrm{d}t}(h(\varphi(\gamma(t))) = \varphi^*\{h,f\}_2(\gamma(t)) = \{h,f\}_1(\varphi(\gamma(t))).$$

Thus $\varphi \circ \gamma$ is an integral curve of X_f. □

Let $(M, \{\,,\,\})$ be a Poisson manifold. A submanifold N of M is a *Poisson submanifold* if we can define a Poisson bracket $\{\,,\,\}_N$ on N by $\{\overline{f}, \overline{g}\}_N = \{f,g\}|N$, where f and g are smooth extensions to M of $\overline{f}, \overline{g} \in C^\infty(N)$. Equivalently, N is a Poisson submanifold of M if the inclusion map $i : N \to M$ is a Poisson map.

An important consequence of the local classification of Poisson manifolds ((4.8)) is the following. Define an equivalence relation $\sim$ on M by declaring $p \sim q$ if q can be reached from p by a piecewise smooth curve each segment of which is an integral curve of a Hamiltonian vector field on M. Then $\sim$ is an equivalence relation and an equivalence class is called a *leaf* of the Poisson structure on M.

Claim: Every leaf on M is a smooth Poisson submanifold of M with a nondegenerate Poisson bracket.

(4.10) **Proof**: Because the rank of a tensor is lower *semicontinuous*, we need only show that the rank of a Poisson structure is invariant under the flow of a Hamiltonian vector field. Then we can use the local classification of Poisson manifolds at each point in the leaf, because the constant rank hypothesis holds on a leaf. The invariance of the rank follows from the calculation

$$\begin{aligned}
\{\{g,f\},h\} + \{g,\{h,f\}\} &= \{\{g,h\},f\} = L_{X_f}\{g,h\} = L_{X_f}(W(\mathrm{d}g,\mathrm{d}h)) \\
&= (L_{X_f}W)(\mathrm{d}g,\mathrm{d}h) + W(L_{X_f}\mathrm{d}g,\mathrm{d}h) + W(\mathrm{d}g, L_{X_f}\mathrm{d}h) \\
&= (L_{X_f}W)(\mathrm{d}g,\mathrm{d}h) + W(\mathrm{d}\{g,f\},\mathrm{d}h) + W(\mathrm{d}g,\mathrm{d}\{h,f\}) \\
&= (L_{X_f}W)(\mathrm{d}g,\mathrm{d}h) + \{\{g,f\},h\} + \{g,\{h,f\}\}.
\end{aligned}$$

Since this holds for any f,g,h, it follows that $L_{X_f}W = 0$. Thus the flow of X_f acts by automorphisms of the Poisson structure. Hence the rank of W is invariant under the flow of X_f. □

We now discuss how to deal with Poisson brackets and constraints. Let $\{c_i\}_{i=1}^k$ be smooth functions on a symplectic manifold (M,ω) and suppose that $c \in \mathbf{R}^k$ is a regular value of the map $\mathscr{C} : M \to \mathbf{R}^k : p \mapsto \big(c_1(p), \ldots, c_k(p)\big)$. Then $N = \mathscr{C}^{-1}(c)$ is a smooth submanifold of M, called the *constraint manifold* defined by the *constraint functions* $\{c_i\}_{i=1}^k$. If the matrix $\big(\{c_i, c_j\}(p)\big)$ of Poisson brackets of the constraints is invertible for every $p \in N$, then we say that N is a *cosymplectic* submanifold of M.

Claim: If N is a cosymplectic submanifold of a symplectic manifold (M,ω), then $\omega|N$ is a symplectic form on N.

(4.11) **Proof**: For each $p \in N$ let V_p be the space spanned by the vectors $\{X_{c_1}(p), \dots, X_{c_k}(p)\}$. For every $v \in T_pN$ we have $0 = \mathrm{d}c_i(p)v = \omega(p)(X_{c_i}(p), v)$, which implies $X_{c_i}(p) \in (T_pN)^\omega$ and hence $T_pN \subseteq V_p^\omega$. Since the matrix $\big(\{c_i, c_j\}(p)\big) = \big(\omega(p)(X_{c_i}(p), X_{c_j}(p))\big)$ is invertible, V_p is a symplectic subspace of $(T_pM, \omega(p))$ of dimension k, which must be even. But

$$\begin{aligned}\dim T_pN &= \dim T_pM - \dim \operatorname{span}\{\mathrm{d}c_1(p), \dots, \mathrm{d}c_k(p)\} = \dim T_pM - k \\ &= \dim T_pM - \dim V_p = \dim V_p^\omega.\end{aligned}$$

Therefore $T_pN = V_p^\omega$. Hence T_pN is a symplectic subspace of $(T_pM, \omega(p))$, that is, $\omega|N$ is nondegenerate. Since $\mathrm{d}(\omega|N) = (\mathrm{d}\omega)|N = 0$, $\omega|N$ is a symplectic form on N. $\square$

Using the symplectic form $\omega|N$ on N, we may define a Poisson bracket $\{\,,\,\}_N$ on $C^\infty(N)$ in the standard way. We now discuss a way to compute the bracket $\{\,,\,\}_N$ using the bracket $\{\,,\,\}$ on M. We make use of *Dirac brackets*, which are defined as follows. Let $C = (C_{ij})$ be the *inverse* of the matrix $(\{c_i, c_j\})$ of Poisson brackets of the constraint functions. Since N is cosymplectic, C is defined in an open neighborhood $\mathscr{U}$ of N. Let $F, G \in C^\infty(M)$. For every $u \in \mathscr{U}$ define the Dirac bracket $\{\,,\,\}^*$ by

$$\{F, G\}^*(u) = \{F, G\}(u) - \sum_{i,j=1}^{k} \{F, c_i\}(u) C_{ij}(u) \{c_j, G\}(u). \tag{36}$$

To keep the notation simple, in what follows we will suppress the variable u. Before we can relate the Dirac bracket to the bracket $\{\,,\,\}_N$, we make some preliminary observations. First note that $C^\infty(N) = C^\infty(M)/\mathscr{I}$, where $\mathscr{I}$ is the ideal of smooth functions which vanish identically on N. In other words, $f \in C^\infty(N)$ if there is a function $F \in C^\infty(M)$ such that $F|N = f$. Since the ideal $\mathscr{I}$ is generated by the functions $\{c_i\}_{i=1}^k$, the function f is represented by $F + \sum_{i=1}^k \lambda_i c_i = F + \mathscr{I}$. Second, observe that the constraint function c_i is a Casimir for the Dirac bracket. This follows because for every $F \in C^\infty(M)$

$$\{F, c_k\}^* = \{F, c_k\} - \sum_{ij} \{F, c_i\} C_{ij} \{c_j, c_k\} = \{F, c_k\} - \sum_i \{F, c_i\} \delta_{ik} = 0.$$

From the above discussion we see that $\{\,,\,\}^*|N$ is well defined on $C^\infty(N)$ because on N we have

$$\begin{aligned}\{F + \mathscr{I}, G\}^* &= \{F + \sum_{i=1}^{k} \lambda_i \cdot c_i, G\}^* = \{F, G\}^* + \sum_i \{\lambda_i, G\}^* c_i, \\ &\qquad \text{since } \{\,,\,\}^* \text{ is a derivation and } \{c_i, G\}^* = 0. \\ &= \{F, G\}^* + \mathscr{I}.\end{aligned}$$

▷ The following argument shows that

$$\{F|N, G|N\}_N = \{F, G\}^*|N \tag{37}$$

for $F, G \in C^\infty(M)$. This is the basic property of the Dirac bracket.

(4.12) **Proof**: Let $F^* = F - \sum_{i,j}\{F,c_i\}C_{ij}c_j$. Then

$$\{F^*,c_k\} = \{F,c_k\} - \sum_{i,j}\{F,c_i\}C_{ij}\{c_j,c_k\} + \mathscr{I} = \{F,c_k\} - \sum_{i}\{F,c_i\}\delta_{ik} + \mathscr{I} = \mathscr{I},$$

that is, $\{F^*,c_k\}|N = 0$. Consequently, N is an invariant manifold of X_{F^*}. For every $p \in N$ and every $Y_p \in T_pN$ we have

$$\omega(p)(X_{F^*}(p),Y_p) = \mathrm{d}F^*(p)Y_p = \mathrm{d}F(p)Y_p = \omega(p)(X_{F^*|N}(p),Y_p).$$

From the nondegeneracy of $\omega|N$ it follows that

$$X_{F^*}|N = X_{F^*|N}. \tag{38}$$

The next calculation shows that $\{F^*,G^*\} = \{F,G\}^* + \mathscr{I}$. Using the definition of F^* and G^* we have

$$\begin{aligned}
\{F^*,G^*\} \quad &= \quad \{F - \sum_{i,j}\{F,c_i\}C_{ij}c_j, G - \sum_{k,\ell}\{G,c_k\}C_{k\ell}c_\ell\} \\
&= \{F,G\} - \sum_{k,\ell}\{G,c_k\}C_{k\ell}\{F,c_\ell\} - \sum_{i,j}\{F,c_i\}C_{ij}\{c_j,G\} \\
&\qquad + \sum_{ijk\ell}\{F,c_i\}\{G,c_k\}C_{ij}C_{k\ell}\{c_j,c_\ell\} + \mathscr{I}, \quad \text{by bilinearity and Leibniz' rule} \\
&= \{F,G\} - \sum_{i,j}\{F,c_i\}C_{ij}\{c_j,G\} + \mathscr{I}, \quad \text{using } \textstyle\sum C_{ij}\{c_j,c_k\} = \delta_{ik}. \\
&= \{F,G\}^* + \mathscr{I}.
\end{aligned}$$

After these preliminaries we verify (37) as follows.

$$\begin{aligned}
\{F,G\}^*|N &= \{F^*,G^*\}|N \\
&= (\omega|N)(X_{F^*}|N, X_{G^*}|N), \quad \text{by definition of } \{\,,\,\}. \\
&= (\omega|N)(X_{F^*|N}, X_{G^*|N}) = (\omega|N)(X_{F|N}, X_{G|N}), \\
&\qquad \text{using (38) and the definition of } F^* \text{ and } G^* \\
&= \{F|N, G|N\}_N, \quad \text{by definition of } \{\,,\,\}_N. \qquad \square
\end{aligned}$$

Note that in the preceding argument we have *not* used the Jacobi identity for Dirac brackets. Since the Poisson bracket $\{\,,\,\}_N$ satisfies the Jacobi identity, the Dirac bracket $\{\,,\,\}^*|N$ also satisfies the Jacobi identity.

The construction of Dirac brackets in ((4.12)) can be modified by letting

$$F^* = F - \sum_{i,j}(\{F,c_i\} + F_i)C_{ij}c_j, \tag{39}$$

where F_i lies in the ideal generated by $\{c_1,\ldots,c_k\}$. It is straightforward to check that (38) holds for the modified F^*. This construction we will call *modified* Dirac brackets. In concrete examples it is often necessary to use the freedom in the choice of F_i to simplify F^*.

5 Exercises

1. (Geodesic vector field.) Let g be a Riemannian metric on a smooth manifold M. In local coordinates $x = (x^1, \ldots x^n)$ the metric may be written as $g = \sum g_{ij}\,\mathrm{d}x^i \otimes \mathrm{d}x^j$. Let $v = (x^1, \ldots, x^n, v^1, \ldots v^n)$ be natural coordinates on TM. Show that the pullback θ_g by the map $g^\sharp$ to TM of the canonical 1-form θ on T^*M may be written as $\theta_g(v) = \sum g_{ij} v^i\,\mathrm{d}x^j$. Moreover, the pullback Ω_g to TM by $g^\sharp$ of the canonical 2-form Ω on T^*M may be written as
$$\Omega_g(v) = -\,\mathrm{d}\theta_g = -\sum \frac{\partial g_{kj}}{\partial x^i} v^k\,\mathrm{d}x^i \wedge \mathrm{d}x^j - \sum g_{ij}\,\mathrm{d}v^i \wedge \mathrm{d}x^j.$$
Show that the Hamiltonian vector field Z_g corresponding to the Hamiltonian function $E : TM \to \mathbf{R} : v \to \frac{1}{2} g(v,v)$ is the geodesic vector field.

2. Let $W = \sum_{i,j} W_{ij} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}$ be the structure tensor for a nondegenerate Poisson structure $\{\,,\}$ on $\mathbf{R}^{2n}$ with coordinates $(x_1, \ldots, x_{2n})$. In other words $W(\mathrm{d}x_i, \mathrm{d}x_j) = \{x_i, x_j\}$. Show that $X_{x_j} = \sum_i W_{ij} \frac{\partial}{\partial x_i}$. Define a 2-form by $\Omega = \sum_{k,\ell} W^{k\ell}\,\mathrm{d}x_\ell \wedge \mathrm{d}x_k$, where (W^{ij}) is the inverse of the matrix (W_{ij}). Show that $\Omega(X_{x_i}, X_{x_j}) = \{x_i, x_j\}$. Furthermore show that
$$\sum_\ell \frac{\partial W_{jk}}{\partial x_\ell} W_{i\ell} = \sum_\ell \frac{\partial W_{ij}}{\partial x_\ell} W_{\ell k} + \sum_\ell \frac{\partial W_{ik}}{\partial x_\ell} W_{j\ell}. \tag{40}$$
is equivalent to the Jacobi identity for the Poisson structure $\{\,,\}$. Let
$$\Gamma_j^{k\ell} = \sum_{mn} \frac{\partial W_{mn}}{\partial x_j} W^{mk} W^{n\ell}.$$
Show that (40) can be written as $\Gamma_j^{k\ell} + \Gamma_k^{\ell j} + \Gamma_\ell^{jk} = 0$. Deduce that Ω is closed.

3. Let G be a Lie group with a nondegenerate metric which is invariant under left and *right* translation. Show that the image of every geodesic on T^*G under the bundle projection is a one parameter subgroup of G, if it passes through $\alpha_e \in T_e^*G$.

4. Find the Euler-Arnol'd equations for geodesics of a left invariant metric on the Lie groups $\mathrm{Sl}_2(\mathbf{R})$ and $S^3 = \mathrm{SU}(2)$.

Chapter VII

Systems with symmetry

In this chapter we discuss Hamiltonian systems with symmetry. By a symmetry of a Hamiltonian system (H, M, ω) we mean a proper action of a Lie group G on a symplectic manifold (M, ω), which has a momentum mapping $J : M \to \mathfrak{g}^*$ and preserves the Hamiltonian H. We will show that symmetries of a Hamiltonian system give rise to conserved quantities which are constant along integral curves of the Hamiltonian vector field X_H of H. Using the technique of singular reduction we remove the G symmetry of (H, M, ω) by constructing a lower dimensional Hamiltonian system $(H_\mu, M_\mu, \{\,,\,\}_\mu)$ for *each* μ in the image of the momentum mapping J.

1 Smooth group actions

In this section we treat some basic properties of smooth group actions on manifolds. We discuss in detail the properties of proper actions.

Let G be a *Lie group*, that is, G is a smooth manifold which is a group such that multiplication is a smooth map. A (left) *action* Φ of G on a smooth manifold M is a smooth mapping

$$\Phi : G \times M \to M : (g, m) \mapsto \Phi(g, m) = \Phi_g(m) = \Phi_m(g) = g \cdot m \tag{1}$$

such that for every $g, h \in G$ and every $m \in M$ we have $\Phi_{gh}(m) = \Phi_g(\Phi_h(m))$, while for the identity element e in G we have $\Phi_e(m) = m$. A succinct way of expressing these conditions is to say that the mapping $G \to \mathrm{Diff}(M) : g \mapsto \Phi_g$ is a homomorphism of G into the group of diffeomorphisms of M.

For $m \in M$ let $\mathscr{O}_m$ be the *orbit* of m under the action of G, that is, $\mathscr{O}_m = \{\Phi_g(m) \big| \, g \in G\}$. Sometimes, when it is convenient, and not at all confusing, we will write $G \cdot m = \mathscr{O}_m$. Let $\mathfrak{g}$ be the Lie algebra of G. Then $T_e\Phi_m(\mathfrak{g})$ is the tangent space to the orbit $G \cdot m$ at m.

Denote the *isotropy group* of m by $G_m = \{g \in G \big| \, \Phi(g, m) = m\}$. Note that G_m is a Lie group because it is a closed subgroup of G. The Lie algebra of G_m is $\mathfrak{g}_m$.

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Claim: Let L be a submanifold of G through e such that $\mathfrak{g} = \mathfrak{g}_m \oplus T_eL$ and let S be a submanifold of M through m such that $T_mM = T_m\mathscr{O}_m \oplus T_mS$. Then there is an open neighborhood $L_0 \times S_0$ of (e,m) in $L \times S$ such that $\Phi|(L_0 \times S_0)$, the restriction of the action Φ to $L_0 \times S_0$, is a diffeomorphism onto an open neighborhood of m in M.

(1.1) **Proof**: For $(\xi, v) \in T_eL \times T_mS$ we have

$$\begin{aligned} T_{(e,m)}(\Phi|(L\times S))(\xi,v) &= T_e\Phi_m\xi + T_m\Phi_e v, \text{ by the formula for partial derivatives} \\ &= T_e\Phi_m\xi + v, \text{ since } \Phi_e \text{ is the identity map.} \end{aligned}$$

Now $T_{(e,m)}\Phi|(L\times S)$ is injective; for if $0 = T_e\Phi_m\xi + v$, then $v \in T_m\mathscr{O}_m \cap T_mS = \{0\}$, which follows from our hypothesis, and thus $T_e\Phi_m\xi = 0$. But by hypothesis we have $\ker T_e\Phi_m \cap T_eL = \{0\}$. Hence $\xi = 0$. Moreover $T_{(e,m)}\Phi|(L\times S)$ is surjective because $T_mM = T_m\mathscr{O}_m + T_mS = T_e\Phi_m T_eL + T_mS$. Applying the inverse function theorem concludes the argument. □

An immediate consequence of the above result is the following.

Corollary: If $\Phi_\ell(S_0) \cap S_0 \neq \varnothing$ for some $\ell \in L_0$, then $\ell = e$.

(1.2) **Proof**: By hypothesis, there is an $s \in S_0$ such that $\Phi_\ell(s) = s' \in S_0$. But then $\Phi_e(s') = \Phi_\ell(s)$. In other words, $\Phi(\ell, s) = \Phi(e, s')$. Since Φ is a local diffeomorphism on $L_0 \times S_0$, we obtain $\ell = e$. □

In order to ensure that orbit spaces, treated in the following section, are reasonably well behaved, some condition must be imposed on the action Φ. A condition which works well is that the action Φ be *proper*, that is, the map $G \times M \to M \times M : (g,m) \mapsto (m, g\cdot m)$ is a *proper map*. As G and M are both manifolds, this means that the inverse image of a compact set under this map is compact. One might think that it would be more natural to require that the map $G \times M \to M : (g,m) \mapsto g\cdot m = \Phi(g,m)$ be proper. But what we need is a condition which ensures that nearby points in M may only be mapped into each other by elements of G which are close to the identity. Actually, properness is somewhat stronger than this.

Claim: If the G-action is proper, then the isotropy group G_m is compact.

(1.3) **Proof**: This follows immediately by looking at the inverse image of (m,m). □

A nice feature of a proper action of a Lie group near a fixed point is that it is locally linearizable. In particular, we have

Claim: For any point $m \in M$ there is a G_m-invariant neighborhood U of $m \in M$ and a diffeomorphism $\psi : U \subseteq M \to T_mM$ such that

1. ψ maps the point m to the origin and its tangent at m is the identity.
2. ψ is G_m-equivariant, that is, for every $g \in G_m$ and $p \in U$ we have

$$\psi(\Phi_g(p)) = T_m\Phi_g(\psi(p)).$$

In other words, the diffeomorphism ψ locally identifies the G_m-action on M near m with a linear action of G_m on the vector space T_mM.

(1.4) **Proof**: For every chart $(\varphi, \widetilde{U})$ on M with $\varphi(m) = 0$ there is a G_m-invariant open set U containing m with $U \subseteq \widetilde{U}$, see the exercises. Working in this chart we may identify the image of the chart map with the tangent space to M at m. We are reduced to the following situation. There is a compact group G (= isotropy group of 0) acting, not necessarily linearly, on $\mathbf{R}^n$ and the origin is a fixed point. For $p \in \mathbf{R}^n$ define a map $\psi : U \subseteq \mathbf{R}^n \to U' \subseteq \mathbf{R}^n$ by $\psi(p) = \int_G D_2\Phi(g^{-1},0) \cdot \Phi(g,p)\, \mathrm{d}g$ with $\mathrm{d}g$ being Haar measure on G with $\mathrm{vol}(G) = 1$. In other words, $\mathrm{d}g$ is a bi-invariant volume form on G. Now for any $h \in G$ and $p \in U$

$$
\begin{aligned}
D_2\Phi(h,0) \cdot \psi(p) &= \int_G D_2\Phi(h,0) \cdot D_2\Phi(g^{-1},0) \cdot \Phi(g,p)\, \mathrm{d}g \\
&= \int_G D_2\Phi(hg^{-1},0) \cdot \Phi(g,p)\, \mathrm{d}g \\
&= \int_G D_2\Phi(k^{-1},0) \cdot \Phi(kh,p)\, \mathrm{d}(kh), \quad \text{changing integration variable by } g = kh \\
&= \int_G D_2\Phi(k^{-1},0) \cdot \Phi(kh,p)\, \mathrm{d}k \\
&= \int_G D_2\Phi(k^{-1},0) \cdot \Phi(k,\Phi(h,p))\, \mathrm{d}k = \psi(\Phi(h,p)).
\end{aligned}
$$

Thus we have produced a map ψ which intertwines the action of G with an action by linear maps. Since $T_0\psi = \int_G D_2\Phi(g^{-1},0)\, D_2\Phi(g,0)\, \mathrm{d}g = id$, the map ψ is a local diffeomorphism. □

Corollary. Let H be a compact subgroup of G and let $M_{[H]} = \{m \in M \big| H \subseteq G_m\}$. Then every connected component of $M_{[H]}$ is a smooth submanifold of M. Moreover, the tangent space $T_pM_{[H]}$ at $p \in M_{[H]}$ is the set $(T_pM)^H$ of H-fixed vectors of T_pM, that is, $(T_pM)^H = \{v_p \in T_pM \big| T_p\Phi_h v_p = v_p, \text{ for every } h \in H\}$.

(1.5) **Proof**: Let V be an open neighborhood of 0 in T_pM which is mapped diffeomorphically onto the open neighborhood U of p in M by ψ^{-1}. The map ψ^{-1} intertwines the linear H-action $\widehat{\Phi} : H \times T_pM \to T_pM : (h, v_p) \mapsto T_p\Phi_h v_p$ with the H-action $\widetilde{\Phi} = \Phi|(H \times M)$. Since $(T_pM)^H$ is clearly a vector subspace of T_pM, to show that $M_{[H]}$ is a submanifold of M it suffices to verify that $\psi^{-1}\big((T_pM)^H \cap V\big) = U \cap M_{[H]}$. If $v \in (T_pM)^H \cap V$, then $m = \psi^{-1}(v) \in U$ and $\widehat{\Phi}_h(v) = v$ for every $h \in H$. Since ψ^{-1} intertwines the H-actions $\widehat{\Phi}$ and $\widetilde{\Phi}$, it follows that $\Phi_h(m) = m$ for every $h \in H$, that is, $H \subseteq G_m$. So $m \in M_{[H]}$. Therefore $\psi^{-1}\big((T_pM)^H \cap V\big) \subseteq U \cap M_{[H]}$. Now suppose that $m \in U \cap M_{[H]}$. Then $\psi(m) = v \in V$ and $\Phi_h(m) = m$ for every $h \in H$. Since ψ intertwines the H-actions $\widetilde{\Phi}$ and $\widehat{\Phi}$, we obtain $\widehat{\Phi}_h(v) = v$ for every $h \in H$, that is, $v \in (T_pM)^H$. Therefore $\psi(U \cap M_{[H]}) \subseteq (T_pM)^H \cap V$. This proves the desired equality and thus the corollary. □

An important property of proper actions is the existence of a slice. A *slice* at m for the action Φ is a smooth submanifold S of M through m such that

1. S is transverse and complementary to the orbit $\mathscr{O}_m$ of m at the point m, that is, $T_mM = T_m\mathscr{O}_m \oplus T_mS$.
2. For every $p \in S$, the submanifold S is transverse to $\mathscr{O}_p$, that is, $T_pM = T_p\mathscr{O}_p + T_pS$.

3. S is G_m-invariant.
4. For $p \in S$ and $g \in G$, if $\Phi_g(p) \in S$ then $g \in G_m$.

▷ We now show that slices exist for proper group actions.

(1.6) **Proof**: We begin by constructing a candidate slice S_ε and then show that properties 1 through 4 hold.

3. For every $k \in G_m$ we have $\Phi_k \circ \Phi_g(m) = \Phi_k \circ \Phi_g \circ \Phi_{k^{-1}}(m) = \Phi_{kgk^{-1}}(m)$. Therefore for every $\xi \in \mathfrak{g}$

$$T_m\Phi_k \circ T_e\Phi_m\xi = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \Phi_k\big(\Phi_m(\exp t\xi)\big) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \Phi_{k(\exp t\xi)k^{-1}}(m).$$

Since the one parameter subgroups $t \mapsto k(\exp t\xi)k^{-1}$ and $t \mapsto \exp t\,\mathrm{Ad}_k\,\xi$ have the same tangent vector $\mathrm{Ad}_k\,\xi$ at $t=0$, we obtain $T_m\Phi_k \circ T_e\Phi_m\xi = T_e\Phi_m(\mathrm{Ad}_k\,\xi)$. This implies that $T_m\Phi_k$ leaves $T_m\mathscr{O}_m$ invariant. With respect to a G_m-invariant positive definite inner product on T_mM, the orthogonal complement $(T_m\mathscr{O}_m)^\perp$ is a G_m-invariant subspace. Using the local linearizing diffeomorphism ψ of ((1.4)) the submanifold $S_\varepsilon = \psi^{-1}\big((T_m\mathscr{O}_m)^\perp \cap B_\varepsilon\big)$, where B_ε is a ball of radius ε in T_mM, is G_m-invariant. Thus S_ε satisfies property 3.

1. From the argument proving property 3, it follows that $T_mS_\varepsilon = (T_m\mathscr{O}_m)^\perp$. Hence S_ε has property 1.

2. Property 2 is an open condition in S_ε as we see from the following argument. Consider the map $\widetilde{\Phi} : G \times S \to M : (g,s) \mapsto \Phi_g(s)$. Then $T_{(e,s)}\widetilde{\Phi}(\mathfrak{g} \times T_sS) = T_s\mathscr{O}_s + T_sS$. By property 1 we see that $T_{(e,m)}\widetilde{\Phi}$ is surjective. This is an open condition on S, that is, for every $s \in S$ near m, the map $T_{(e,s)}\widetilde{\Phi}$ is surjective. So $T_sM = T_s\mathscr{O}_s + T_sS$. Thus property 2 holds on S_ε.

4. Now suppose that property 4 does not hold for any $\varepsilon > 0$. Then there is a sequence $\{m_j\}$ with $m_j \in S_{1/j}$, $m_j \to m$, and a sequence $g_j \notin G_m$ with $g_j \cdot m_j \in S_{1/j}$, which implies $g_j \cdot m \to m$. By properness, there is a subsequence $g_{j_k} \to g$, which we may assume is just g_j itself. Replacing g_j by $g^{-1}g_j$ we may suppose that $\{g_j\}$ converges to e, but $g_j \notin G_m$. Choosing H to be a submanifold of G which is transverse to G_m, there is an open neighborhood $V \times W$ of (e,e) in $H \times G_m$ such that the multiplication $(h,k) \to hk$ is a diffeomorphism onto an open neighborhood of e in G. Thus we may assume that $g_j = h_jk_j$. As $g_j \notin G_m$ and $k_j \in G_m$, we conclude $h_j \neq e$ for all j. Arguing as above once more with $S_{1/j}$ in place of H, we obtain $g_j \cdot m_j = h_j \cdot k_j \cdot m_j \in S_{1/j}$. Because $S_{1/j}$ is G_m-invariant, $k_j \cdot m_j \in S_{1/j}$. But ((1.2)) implies that $h_j = e$. This contradiction establishes there is an $\varepsilon > 0$ such that property 4 holds. Hence S_ε is a slice. □

Claim. Give $M_{[H]}$ the topology induced from that of M. Then $M_H = \{m \in M \,\big|\, G_m = H\}$ is an open subset of $M_{[H]}$. So every connected component of M_H is a submanifold of M.

(1.7) **Proof**: Suppose that $p \in M_H$. Let S_p be a slice to the G-action at p. The map $\varphi : S_p \times L \to M : (s,\eta) \mapsto \Phi_{\exp\eta}(s)$ with $\varphi(p,0) = p$ and L a complement to $\mathfrak{h}$ in $\mathfrak{g}$ is a diffeomorphism of an open neighborhood W of $(p,0)$ in $S_p \times L$ onto an open neighborhood U of p in M. Let $q \in M_{[H]} \cap U$. Then there is a $g \in \exp L$ and an $s \in S_p$ such that $\Phi_g(s) = q$. Therefore $g^{-1}G_q\,g = G_s \subseteq H$, where the inclusion follows because S_p is a slice. But $q \in M_{[H]}$. Therefore $H \subseteq G_q \subseteq gHg^{-1}$. Since H and gHg^{-1} have the same Lie algebras, it follows that $H = gHg^{-1}$ if H is connected. Because the map $H \mapsto gHg^{-1}$ is a diffeomorphism,

H and gHg^{-1} have the same number of connected components, which is finite since H is compact. From $g^{-1} G_q g = G_s \subseteq H$ it follows that every connected component of H is a connected component of gHg^{-1}. Therefore H and gHg^{-1} have the same connected components. Hence $H = gHg^{-1}$, which implies that $G_q = H$ using $g^{-1} G_q g = G_s \subseteq H$. Therefore, $q \in M_H \cap U$, that is, $M_{[H]} \cap U \subseteq M_H \cap U$. But $M_H \subseteq M_{[H]}$. So $M_{[H]} \cap U = M_H \cap U$. $\square$

Let $M_{(H)} = G \cdot M_H$. Then $M_{(H)}$ is the set of points of M of *orbit type* H. From the definition of M_H it follows that $M_{(H)} = \{m \in M \big| \text{there is a } g \in G \text{ such that } gG_m g^{-1} = H\}$.
$\triangleright$ The following argument shows that $M_{(H)}$ is a submanifold of M.

(1.8) **Proof**: Let $m \in M_{(H)}$. Since $M_{(H)} = G \cdot M_H$, there is a $p \in M_H$ and a $g \in G$ such that $\Phi_g(p) = m$. Let L be a complement to $\mathfrak{h}$ in $\mathfrak{g}$. Let S_p be a slice to the G-action Φ at p. Then the map $\varphi : S_p \times L \to M : (s, \eta) \mapsto \Phi_{\exp \eta}(s)$ with $\varphi(0, p) = p$ is a local diffeomorphism at p. Since $\vartheta = \psi \times \mathrm{id}_L : T_pS_p \times L \mapsto S_p \times L$ with $\vartheta(0,0) = (0, p)$ is a local diffeomorphism, where ψ^{-1} is a local diffeomorphism which intertwines the H-action $\widehat{\Phi}$ on T_pM with the H-action $\widetilde{\Phi} = \Phi|(H \times M)$ on M, see ((1.4))), the map

$$\theta : T_pS_p \times L \to M : (v_p, \eta) \mapsto \Phi_{\exp \eta}(\psi^{-1}(v_p))$$

with $\theta(0,0) = p$ is a local diffeomorphism of an open neighborhood $W \times V \subseteq T_pS_p \times L$ of $(0,0)$ onto an open neighborhood $U = \theta(W \times V)$ of p in M. Now $\psi(M_H \cap S_p \cap U) = \psi(M_{[H]} \cap S_p \cap U) = Y$ is the intersection of an open neighborhood of 0 in T_pS_p with $T_p(M_{[H]} \cap S_p)$. Since $G \cdot M_H = M_{(H)}$, we obtain $\theta(W \times Y) = \Phi_{\exp W}(M_H \cap S_p \cap U) \cap U = M_{(H)} \cap U$. Therefore $M_{(H)}$ is a submanifold of M. $\square$

On the set of all closed subgroups of G define a relation $\asymp$ by saying that $K \asymp H$ if and only if K is conjugate to H in G. Clearly $\asymp$ is an equivalence relation on the set of all closed subgroups of G. Thus as H runs over the set of conjugacy classes of closed
$\triangleright$ subgroups of G, the orbit types $M_{(H)}$ partition M. The *orbit type decomposition* of M is locally finite if the action of G is proper.

(1.9) **Proof**: We prove the local finiteness of the orbit type decomposition by induction on the dimension of the group G and the dimension of the manifold M. If G is a finite group acting on a manifold, the number of conjugacy classes of subgroups is finite. Hence the orbit type decomposition is finite. Suppose that M is a finite number of points, then the orbit type decomposition is finite. With these special cases out of the way, we can begin the induction argument. Let $G \cdot m$ be the G orbit through m of the proper group action Φ on M. Since Φ is proper there is a G_m-invariant slice S through the point m. Note that the dimension of the isotropy group G_m is at most equal to the dimension of G. Because G_m is compact and leaves m fixed, the G_m-action on S is locally equivalent to the linear isotropy action $\widehat{\Phi} : G_m \times T_mS \to T_mS$. Suppose that the action $\widehat{\Phi}$ is trivial, that is, $\widehat{\Phi}_h(v_m) = v_m$ for every $h \in G_m$ and every $v_m \in T_mS$. In other words, if $p \in S$ then the isotropy group of a point p on the G-orbit is equal to G_m. Since every G-orbit near $G \cdot m$ intersects S, the isotropy group is conjugate in G to G_m, that is, it lies in the equivalence class of the orbit type decomposition corresponding to (G_m). Hence the orbit type decomposition near $G \cdot m$ is locally finite. Now suppose that the action $\widehat{\Phi}$ is not trivial. Since G_m is compact, there is a G_m-invariant Euclidean inner product $\langle \, , \, \rangle$ on T_mS. Let $\mathscr{S}_r$ be the sphere of radius

r in $(T_mS, \langle\,,\,\rangle)$. Note that the dimension of $\mathscr{S}_r$ is strictly less than the dimension of M. Since the action $\widehat{\Phi}$ is linear, it induces a proper action $\Psi : G_m \times \mathscr{S}_r \rightarrow \mathscr{S}_r$. Repeating the above argument on the G_m-action Ψ either reduces the dimension of the group or the dimension of the manifold on which it acts. This completes the induction step and hence the proof. $\square$

2 Orbit spaces

In this section we describe the space of orbits M/G of a proper group action of G on a smooth manifold M. In general, M/G is only a topological space. It need not be a topological manifold. If in addition we assume that the action is free, that is, the isotropy group at each point is the identity element, then the orbit space is a smooth manifold.

2.1 Orbit space of a proper action

In this subsection we define the orbit space of a Lie group acting properly on a smooth manifold and give some of its properties.

Let $\Phi : G \times M \rightarrow M$ be an action of a Lie group G on a smooth manifold M. We begin by constructing the orbit space M/G set theoretically. Let $\overline{m} = G \cdot m$ be the orbit of the action Φ through the point $m \in M$. Define the relation $\sim$ on M by saying that $m \sim m'$ if and only if m and m' lie on the same G-orbit of Φ. It is easy to check that $\sim$ is an equivalence relation on M and thus partitions M into G-orbits. The *orbit space* M/G is the set of all G-orbits of Φ on M. Thus the *orbit map* $\pi : M \rightarrow M/G : m \mapsto \overline{m}$ is surjective.

We put a topology on M/G, called the *quotient topology*, by saying that $\mathscr{U} \subseteq M/G$ is *open* if and only if $\pi^{-1}(\mathscr{U})$ is an open subset of M. Let S be a slice at m to the G-action Φ. Then S is transverse to the G-orbit $G \cdot m$ at m. For a suitable open neighborhood $S_0 \subseteq S$ of m, the set $\pi(S_0)$ is an open subset of M/G.

▷

(2.1) **Proof**: It suffices to show that the union of all G-orbits through S_0, namely, $\pi^{-1}(\pi(S_0)) = G \cdot S_0$, is an open subset of M. From ((1.1)) it follows that there is an open neighborhood G_0 of e in G and a G_m-invariant open neighborhood S_0 in the slice S at m such that $\Phi_{G_0}(S_0)$ is an open neighborhood of m in M. Since G is the union of translates $\{g \cdot G_0 \,|\, g \in G\}$, the set $G \cdot S_0$ is the union of open sets $\Phi_{g \cdot G_0}(S_0)$ and hence is open. $\square$

▷ Let S be a slice for a proper G-action at the point m in M. Then the orbit spaces $(G \cdot S)/G$ and S/G_m are homeomorphic.

(2.2) **Proof**: Consider the map $\varphi : (G \cdot S)/G \rightarrow S/G_m : G \cdot s \mapsto (G \cdot s) \cap S$. By property 4 of a slice, $(G \cdot s) \cap S = G_m \cdot s$. Hence φ is well defined. Because $G_m \subseteq G$, it follows that the G_m orbit $G_m \cdot s$ is contained in the G-orbit $G \cdot s$. Hence the map $\vartheta : S/G_m \rightarrow (G \cdot S)/G : G_m \cdot s \mapsto G \cdot s$ is well defined. It is easy to see that $\vartheta \circ \varphi = id_{(G \cdot S)/G}$ and $\varphi \circ \vartheta = id_{S/G_m}$. Therefore φ is bijective.

We now show that φ is a homeomorphism. Let $\rho : S \to S/G_m$ be the orbit map for the G_m-action on S and let $\pi : G\cdot S \to (G\cdot S)/G$ be the orbit map for the G-action on $G\cdot S$. Let $\mathscr{U}$ be an open set in S/G_m containing the orbit $G_m\cdot s$. Then $\rho^{-1}(\mathscr{U}) = U$ is an open set in S containing $G_m\cdot s$. Since $G\cdot U$ is an open set in $G\cdot S$, $\pi(G\cdot U)$ is an open set in $(G\cdot S)/G$ containing $G\cdot s$. Thus φ is continuous. In fact φ is an open map. For suppose that $\mathscr{V}$ is an open set in $(G\cdot S)/G$ containing $G\cdot s$. Then $\pi^{-1}(\mathscr{V}) = V$ is an open set in $G\cdot S$. Hence $U = V\cap S$ is an open set in S containing $G_m\cdot s$. Therefore $\rho(U)$ is an open set in S/G_m. Hence φ is a homeomorphism. □

Claim: If $\{(m,\Phi_g(m)) \in M\times M \,\big|\, g\in G \text{ and } m\in M\}$ is closed, then the orbit space M/G is Hausdorff.

(2.3) **Proof**: Suppose that M/G is not Hausdorff, that is, $\overline{m} \ne \overline{m}'$ cannot be separated. Let U_k and V_k be nested neighborhood bases of m and m' respectively. As $\overline{m}$ and $\overline{m}'$ cannot be separated, there is a $p_k \in (G\cdot U_k)\cap(G\cdot V_k)$ for each k. Set $p_k = g_k\cdot m_k = h_k\cdot m'_k$, where $m_k\in U_k$ and $m'_k \in V_k$. Then $h_k^{-1}g_km_k \to m'$ and consequently the sequence $\{(m_k, h_k^{-1}g_km_k)\}$ converges to (m,m'). By hypothesis $m' = \Phi_g(m)$ for some $g\in G$. This implies $\overline{m} = \overline{m}'$, which is a contradiction. □

Corollary: If the G-action Φ is proper, then M/G is Hausdorff.

(2.4) **Proof**: Let $\{(m_k, \Phi_{g_k}(m_k))\}$ be a sequence of points in $M\times M$ which converges to (m,n). By properness of the action there is a subsequence $\{(g_{k_\ell}, m_{k_\ell})\}$ which converges to (g,m) in $G\times M$. By continuity of Φ, $g_{k_\ell}\cdot m_{k_\ell} \to g\cdot m$. This implies that $\{(m, g\cdot m)\}\subseteq M\times M$ is a closed set. Hence by the claim, M/G is Hausdorff. □

Example 1. Let H be a closed subgroup of G. Suppose that $(g_k, h_kg_k) \to (g,j)$ in $G\times G$. Then $h_k = (h_kg_k)g_k^{-1} \to jg^{-1}$ since G is a Lie group. Thus the action of the closed subgroup H on G is proper. Consequently, the orbit space G/H is Hausdorff.

Claim: Suppose that G acts properly on M, and that $\{U_\alpha\}$ is a G-invariant open covering of M. Then there is a smooth G-invariant partition of unity subordinate to $\{U_\alpha\}$.

(2.5) **Proof**: As M is locally compact and σ-compact, and M/G is Hausdorff, it follows that M/G is locally compact and paracompact, and hence normal. We can choose a set of points m_k with slices S_k through m_k so that $\{G\cdot S_k\}$ is a locally finite open covering of M subordinate to $\{U_\alpha\}$ and $\pi(S_k)$ is a locally finite open covering of M/G. The local compactness of M/G implies that the slices S_k may be chosen in such a way that there are relatively compact sets $V_k \subseteq S_k$ so that $\{\pi(V_k)\}$ also cover M/G. On each S_k we may construct a smooth function $\widetilde{f}_k$ which is positive on V_k and has compact support in S_k. Define $\overline{f}_k$ to be the average of $\widetilde{f}_k$ over the compact isotropy group G_{m_k}. Let f_k be $f_k(g\cdot m) = \overline{f}_k(m)$ if $m\in S_k$, otherwise $=0$. Then f_k is a smooth nonnegative function on M. The proof is completed by setting $h_k = f_k/\sum_k f_k$. □

Corollary: Suppose that N is a G-invariant subset of M and f is a smooth function on M such that $f|N$ is constant on G-orbits. Then there is a smooth G-invariant function F on M such that $F|N = f|N$.

(2.6) **Proof**: First we define a function F_k on each $G\cdot S_k$ by the following rule. If $N\cap S_k = \varnothing$, then set $F_k = 0$. Otherwise, set $F_k|S_k =$ average of $f|S_k$ over G_{m_k}. Note that $F_k|(S_k \cap N) = f|(S_k \cap N)$. Extend $F_k|S_k$ to $G \cdot S_k$ by the G-action. Letting $F = \sum_k h_k F_k$ where $\{h_k\}$ is a G-invariant partition of unity subordinate to $\{G \cdot S_k\}$ does the job. □

Corollary: Smooth G-invariant functions on M separate orbits, that is, if m and m' are on distinct G-orbits of M, then there is a smooth G-invariant function F such that $F|\mathscr{O}_m = 1$ and $F|\mathscr{O}_{m'} = 0$.

(2.7) **Proof**: Let S_m be a slice to the G-action at m. Suppose that the G-orbit $G \cdot m'$ does not intersect S_m. Then let $N = G \cdot S_m$ and apply the proof of the above corollary to the function f which is identically 1 on $\mathscr{O}_m = G \cdot m$. This gives a smooth function F on M which is 1 on $\mathscr{O}_m$ and 0 on $\mathscr{O}_{m'}$. Now suppose that $q \in \mathscr{I} = (G \cdot m') \cap S_m$ at q. Then $q \neq m$ since the G-orbits $G \cdot m$ and $G \cdot m'$ are distinct. By property 4) of a slice, $\mathscr{I}$ is just $G_m \cdot q$, which is compact and does not contain m. Hence there is an open set U in S_m which contains m but does not intersect $\mathscr{I}$. Again apply the proof of the above corollary taking $N = G \cdot U$ and letting f be identically equal to 1 on $G \cdot m$. This gives a smooth function F on M which is 1 on $\mathscr{O}_m$ and 0 on $\mathscr{O}_{m'}$. □

2.2 Orbit space of a free action

In this subsection we show that the orbit space of a proper free group action is a smooth manifold. In fact, it is the base of a principal bundle whose total space is the original manifold.

Before proving the result, we consider a germane example, namely, the quotient of a Lie group by a closed subgroup.

Example 1. Let H be a closed subgroup of a Lie group G. Then the orbit space $G/H = \{Hg | g \in G\}$ is a smooth manifold.

(2.8) **Proof**: Choose $L \subseteq \mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{h} \oplus L$. Then there are open neighborhoods $U^{\mathfrak{h}}$ and U^L of 0 in $\mathfrak{h}$ and L such that $\varphi : U^{\mathfrak{h}} \times U^L \to G : (h, \ell) \mapsto \exp h \exp \ell$ with $\varphi(0,0) = e$ is a local diffeomorphism. Since H is a closed subgroup of G, there is an open neighborhood V of e in G such that $V \cap H = \exp U^{\mathfrak{h}}$. Let U be an open neighborhood of 0 in U^L with compact closure $\overline{U}$ satisfying $\exp(\overline{U})\exp(-\overline{U}) \subseteq V$. To show that the H-orbit mapping $\pi : \exp(\overline{U}) \subseteq G \to \pi(\exp(\overline{U})) \subseteq G/H$ is a homeomorphism, it suffices to show that π is one to one, since $\pi(\exp(\overline{U}))$ is compact and Hausdorff. If $\pi(\exp u) = \pi(\exp u')$ for some $u, u' \in \overline{U}$, then there is an $h \in U^{\mathfrak{h}}$ such that $\exp h \exp u = \exp u' = \exp 0 \exp u'$. Because φ is a local diffeomorphism, it follows that $u = u'$ and $h = 0$. Hence $\pi | \exp(\overline{U})$ is a homeomorphism. From the definition of π it follows that the set $\pi(\exp U^{\mathfrak{h}} \exp U) = \pi(\exp U)$ is open and contains $\overline{e} = \pi(e)$ in G/H. Exactly the same argument applies to the map $\varphi_g = R_g \circ \varphi$, where $R_g : G \to G : h \mapsto hg$. This allows us to conclude that every $\overline{g} = \pi(g)$ in G/H has an open neighborhood homeomorphic to an open set in L. Consequently G/H is a topological manifold.

To show that G/H is a smooth manifold, we construct charts as follows. Let $\mathscr{U}_g \subseteq G/H$ be a set $\mathscr{U}_g = \pi(\exp U\, g) = \overline{e} \exp U\, g$. Note that G acts on the *right* on G/H by $(Hg, k) \mapsto Hgk$. Define the mapping ψ_g by $\psi_g : \mathscr{U}_g \subseteq G/H \to U \subseteq L : \overline{e} \exp u\, g \mapsto u$. The proof is complete once we have shown that $\{(\mathscr{U}_g, \psi_g) \,|\, g \in G\}$ form an atlas for G/H with smooth transition functions. To do this form the local section maps $\sigma_g : \mathscr{U}_g \to G : \overline{e}\exp u g \mapsto \exp u g$. Clearly, $\pi \circ \sigma_g = \mathrm{id}_{\mathscr{U}_g}$. So $\sigma_g(\mathscr{U}_g) \subseteq \pi^{-1}(\mathscr{U}_g)$. Furthermore, $\sigma_g \circ \psi_g^{-1} : U \to G : u \mapsto \exp u g$ is a diffeomorphism onto its image $\sigma_g(\mathscr{U}_g)$ in G. Now $\psi_g = (\psi_g \circ \sigma_g^{-1}) \circ \sigma_g$. Since $\psi_g \circ \sigma_g^{-1}$ is a diffeomorphism and σ_g is a homeomorphism, we conclude that ψ_g is a homeomorphism.

To prove the smoothness of the transition function $\psi_{g'} \circ \psi_g^{-1} : \psi_g(\mathscr{U}_g \cap \mathscr{U}_{g'}) \to \psi_g'(\mathscr{U}_g \cap \mathscr{U}_{g'})$ we argue as follows. Let $\overline{k} \in \mathscr{U}_g \cap \mathscr{U}_{g'}$. Then there is a unique element $h \in H$ such that $hk = k'$, where $k = \sigma_g(\overline{k})$ and $k' = \sigma_{g'}(\overline{k})$. Since multiplication by h is a smooth map, $h^{-1} \cdot \sigma_{g'}(\mathscr{U}_{g'})$ is a smooth submanifold of G containing k and is transverse to Hk. Using the local diffeomorphism $\widetilde{\varphi}_k : \exp U^{\mathfrak{h}} \times U \to V : (h, \ell) \mapsto (h \exp \ell)k$, we see that the image of the submanifold $h^{-1} \cdot \sigma_{g'}(\mathscr{U}_{g'})$ under $\widetilde{\varphi}_k^{-1}$ is the graph of a smooth mapping $\rho : U \subseteq U^L \to H$. Therefore the transition mapping is $\psi_{g'} \circ \psi_g(u) = \pi_2 \circ \widetilde{\varphi}_{k'}^{-1}\big(h\widetilde{\varphi}_k(\rho(u), u)\big)$, which is smooth. $\square$

It may help to think of this proof in the following way. In order to show that the transition map $\psi_{g'} \circ \psi_g^{-1}$ is a diffeomorphism we have by design made it a homeomorphism. The multiplication by h is a technical device to get things in a single chart so that we can deduce that the transition map is a local diffeomorphism. A homeomorphism which is everywhere a local diffeomorphism is a diffeomorphism.

In general, to ensure that the orbit space M/G of a G-action Φ is a smooth manifold, we need to put very strong conditions on the action. For example, one might hope that if the dimension of each orbit $G \cdot m$ is equal to the dimension of the group G, in particular there are no fixed points, and G acts properly, then M/G is a smooth manifold. Unfortunately this need *not* be true, see the example in section 7.1. Recall that a G-action on M is *free* if and only if for every $m \in M$ the isotropy group G_m is the identity element of G.

Claim: If the action $\Phi : G \times M \to M$ is free and proper, then the orbit space M/G is a smooth manifold. Moreover, for every $\overline{m} \in M/G$ there is an open neighborhood $\mathscr{U}$ of $\overline{m}$ in M/G and a diffeomorphism

$$\psi_U : U = \pi^{-1}(\mathscr{U}) \subseteq M \to G \times S : m \mapsto \big(\vartheta_U(m), s_U(m)\big) \tag{2}$$

such that for every $g \in G$ and $m \in U$

$$\psi_U(g \cdot m) = \big(g\vartheta_U(m), s_U(m)\big). \tag{3}$$

(2.9) **Proof**:
Step 1.
Our first task is construct the diffeomorphisms ψ_U, because these maps will allow us to put a smooth structure on M/G. To do this, let $\widetilde{S}$ be a smooth submanifold through $m \in M$ such that $T_m\widetilde{S} \oplus T_e\Phi_m\mathfrak{g} = T_mM$. As this is an open condition in $\widetilde{S}$, there is an open neighborhood $S \subseteq \widetilde{S}$ about m such that $T_sS \oplus T_e\Phi_s\mathfrak{g} = T_sM$ for each $s \in S$.

Claim: For each $(g,s) \in G \times S$ the action $\Phi|(G \times S)$ has a bijective tangent map at (g,s).

(2.10) **Proof**: To see that $T_{(e,s)}\Phi|(G \times S)$ is injective, suppose that $0 = T_{(e,s)}\Phi|(G \times S)(\xi, v) = T_e\Phi_s\xi + v$ for some $(\xi, v) \in \mathfrak{g} \times T_sS$. Then $v \in T_e\Phi_s\mathfrak{g} \cap T_sS = \{0\}$, so $v = 0$ and $T_e\Phi_s\xi = 0$.

Therefore $T_e\Phi_s(t\xi) = 0$ for all $t \in \mathbf{R}$. This implies that s is a fixed point for Φ restricted to the one parameter subgroup $t \mapsto \exp t\xi$. Because Φ is free, $\exp t\xi = e$ for all $t \in \mathbf{R}$. Therefore $\xi = 0$. To see that $T_{(e,s)}\Phi|(G \times S)$ is surjective, note that

$$\operatorname{rank} T_{(e,s)}\Phi|(G \times S) = \dim T_sS + \dim T_e\Phi_s\,\mathfrak{g} - \dim(T_e\Phi_s\mathfrak{g} \cap T_sS) = \dim T_sM.$$

Since $\Phi(g,s) = \Phi_g \circ \Phi(e,s)$, we find that $T_{(g,s)}\Phi|(G \times S) = T_s\Phi_g \circ T_{(e,s)}\Phi|(G \times S)$. However, Φ_g is a diffeomorphism. Thus $T_{(g,s)}\Phi$ is bijective. $\square$

From the inverse function theorem it follows that the mapping $\Phi|(G \times S)$ is a local diffeomorphism of $G \times S$ onto an open subset U of M.

Claim: Shrinking S if necessary, we can arrange that $\Phi|(G \times S)$ is injective and hence is a diffeomorphism.

(2.11) **Proof**: Suppose that for no $S_0 \subseteq S$ the map $\Phi|(G \times S_0)$ is one to one. Let $\{U_k\}$ be a nested neighborhood base of m in S. Since $\Phi|(G \times S_0)$ is not one to one, there is an $s_k \in U_k$ and a $g_k \in G$ bounded away from e such that $g_k \cdot s_k \in U_k$. Therefore $g_k \cdot s_k \to m$. By properness of Φ there is a subsequence $g_{k_\ell} \to g$ and $g \cdot m = m$. This contradicts the freeness of Φ. $\square$

The diffeomorphism

$$\psi_U = (\Phi|(G \times S))^{-1} : U \subseteq M \to G \times S : m \to \big(\vartheta_U(m), s_U(m)\big) \tag{4}$$

intertwines the G-action Φ on U with the G action $\bullet$ on $G \times S$ given by $g\bullet(h,s) = (gh,s)$, that is,

$$\psi_U(g \cdot m) = g\bullet\big(\vartheta_U(m), s_U(m)\big). \tag{5}$$

Step 2.
Now we want to show that M/G is a *topological manifold*. We already know that M/G is paracompact. So it remains to show that each point has a neighborhood homeomorphic to Euclidean space and that the transition maps are continuous. To do this, let $\psi_{\mathscr{U}} : \mathscr{U} \subseteq M/G \to S$ be the mapping with makes diagram 2.1 commute, that is, $\psi_{\mathscr{U}} \circ \pi = \pi_2 \circ \psi_U$. Here π_2 is projection on the second factor. The map $\psi_{\mathscr{U}}$ is well defined because of the way ψ_U intertwines the G-actions. Now π and π_2 are not only continuous but also open

$$\begin{tikzcd}
U \subseteq M \arrow[r, "\psi_U"] \arrow[d, "\pi"'] & G \times S \arrow[d, "\pi_2"] \\
\mathscr{U} \subseteq M/G \arrow[r, "\psi_{\mathscr{U}}"] & S
\end{tikzcd}$$

Diagram 2.1

mappings and ψ_U is a diffeomorphism. If V is an open subset of S, then $\psi_{\mathscr{U}}^{-1}(V) = \pi \circ \psi_U^{-1} \circ \pi_2^{-1}(V)$ is open. So $\psi_{\mathscr{U}}$ is continuous. The map $\psi_{\mathscr{U}}$ is one to one because

$$\psi_{\mathscr{U}}^{-1}(\mathrm{pt}) = \pi\big(\psi_U^{-1}(G \times \{\mathrm{pt}\})\big) = \pi(\text{single } G\text{-orbit}) = \mathrm{pt}.$$

Chasing the diagram in the opposite direction, the same reasoning as above shows that $\psi_{\mathscr{U}}^{-1}$ is continuous. Hence $\psi_{\mathscr{U}}$ is a homeomorphism. By our construction of the charts $(\psi_{\mathscr{U}}, \mathscr{U})$, on overlaps the transition map is a homeomorphism.

Step 3.
To complete the proof that M/G is a smooth manifold, we only have to check that the transition maps are diffeomorphisms. Let T be a submanifold of M passing through $m' = h \cdot m$ such that the map $\psi_V : V \subseteq M \to G \times T$ is a diffeomorphism satisfying (4) and $\psi_V^{-1}(e,t) \in T \subseteq M$ for every $t \in T$. Since $U = \Phi(G \times S)$ and $V = \Phi(G \times T)$, there are open subsets $\widehat{S} \subseteq S$, $\widehat{T} \subseteq T$ such that $U \cap V = \Phi(G \times \widehat{S}) = \Phi(G \times \widehat{T})$. We wish to show that the map $\theta : \widehat{S} \to \widehat{T} : s \mapsto t = \widehat{T} \cap (G \cdot s)$ is a diffeomorphism. To see this choose s and t so that $s = g \cdot t$. Then $g \cdot \widehat{T}$ intersects $\widehat{S}$ at s and both $\widehat{S}$ and $g \cdot \widehat{T}$ intersect $G \cdot s$ transversely. In a local product chart (U, ψ_U) adapted to (e,s) so that $s \in \widehat{S}$ has coordinates $(e,0)$, $g \cdot \widehat{T}$ is a smooth manifold which intersects the origin and is transverse to $G \times \{0\}$. This means that $\widehat{S}$ is diffeomorphic by θ to $g \cdot \widehat{T}$ about s. Hence $\widehat{S}$ is diffeomorphic to $\widehat{T}$. Just like in the example 1, we have a homeomorphism that is everywhere a local diffeomorphism. Hence the mapping θ is a diffeomorphism. This tells us that the transition map between the charts $(\psi_{\mathscr{U}}, \mathscr{U} \cap \mathscr{V})$ and $(\psi_{\mathscr{V}}, \mathscr{U} \cap \mathscr{V})$ of M/G is the smooth map $\psi_{\mathscr{V}} \circ \psi_{\mathscr{U}}^{-1} : s \mapsto t = \theta(s)$ because the charts $(\psi_{\mathscr{U}}, \mathscr{U} \cap \mathscr{V})$ and $(\psi_{\mathscr{V}}, \mathscr{U} \cap \mathscr{V})$ are naturally induced from the charts $(\psi_U, U \cap V)$ and $(\psi_V, U \cap V)$ by restricting the image of ψ_U to $\{e\} \times \widehat{S}$ and the image of ψ_V to $\{e\} \times \widehat{T}$. This concludes the proof that M/G is a smooth manifold. □

Let (P, G, N) be a *left principal G-bundle over N*. This means that

1. G acts smoothly on P and there is a surjective submersion $\pi : P \to N$, called the *bundle projection* such that $\pi(p) = \pi(g \cdot p)$.
2. For each $p \in P$ there is an open neighborhood $\mathscr{U} \subseteq N$ of $\overline{p} = \pi(p)$ and a G-equivariant diffeomorphism of the form
$$\psi_{\mathscr{U}} : \pi^{-1}(\mathscr{U}) \to G \times \mathscr{U} : p \to \big(\vartheta_{\mathscr{U}}(p), \widehat{\psi}_{\mathscr{U}}(\overline{p})\big),$$
such that $\pi \circ \psi_{\mathscr{U}}^{-1} = id_{\mathscr{U}}$. The group action on $G \times \mathscr{U}$ is multiplication by G on the first factor. The pair $(\psi_{\mathscr{U}}, \pi^{-1}(\mathscr{U}))$ is called a *local trivialization* of the bundle $\pi : P \to N$ at p.
3. For every pair of local trivializations given by $(\psi_{\mathscr{U}}, \pi^{-1}(\mathscr{U}))$ and $(\psi_{\mathscr{V}}, \pi^{-1}(\mathscr{V}))$ the transition mapping $\psi_{\mathscr{V}} \circ \psi_{\mathscr{U}}^{-1} : G \times (\mathscr{U} \cap \mathscr{V}) \to G \times (\mathscr{U} \cap \mathscr{V})$ is a diffeomorphism of the form
$$\psi_{\mathscr{V}} \circ \psi_{\mathscr{U}}^{-1}(g,n) = \big(g\vartheta_{\mathscr{U}\mathscr{V}}(n), n\big)$$
where $\vartheta_{\mathscr{U}\mathscr{V}} : \mathscr{U} \cap \mathscr{V} \to G$ is smooth.

Note this is a *left* principal G-bundle. The modifications for a *right* principal G-bundle are straightforward.

Example 2. If H is a closed subgroup of the Lie group G and π is the orbit mapping $\pi : G \to G/H : g \mapsto Hg$ then (G, H, π) is a right principal H-bundle. To see this we use the same notation as in example 1. Define the map $\psi_{\mathscr{U}_G}$ by $\psi_{\mathscr{U}_g} : \pi^{-1}(\mathscr{U}_g) \to H \times \mathscr{U}_g :$

$k \mapsto \big(k(\sigma_g \circ \pi(k))^{-1}, \pi(k)\big)$, where $\sigma_g : \mathscr{U}_g \to G : \overline{e}\exp u\, g \mapsto \exp u\, g$ and $u \in U \subseteq \mathscr{L}$. The map $\psi_{\mathscr{U}_g}$ is well defined because Hk intersects $\sigma_g(\mathscr{U}_g)$ exactly once, and consequently is a diffeomorphism. Hence $\pi : G \to G/H$ is a smooth bundle with local trivializations $\{\big(\psi_{\mathscr{U}_g}, \pi^{-1}(\mathscr{U}_g)\big)\}$. From the definition of the local trivialization it follows that the transition maps are

$$\psi_{\mathscr{U}_{g'}} \circ \psi_{\mathscr{U}_g}^{-1} : H \times (\mathscr{U}_g \cap \mathscr{U}_{g'}) \to H \times (\mathscr{U}_g \cap \mathscr{U}_{g'}) : (h,u) \mapsto (h',u) = \big(h(\sigma_g(u))(\sigma_{g'}(u))^{-1}, u)\big).$$

This completes the example. □

Example 2 should motivate the proof of our next

Claim: Let G be a Lie group which acts freely and properly on a smooth manifold M. Then M is a principal G-bundle over the orbit space M/G with orbit mapping $\pi : M \to M/G$ being the bundle projection.

(2.12) **Proof**: We use the same notation as in ((2.9)). Observe that the map

$$\varphi_U : U = \pi^{-1}(\mathscr{U}) \to G \times M/G : m \mapsto \big(\vartheta_U(m), \psi_{\mathscr{U}}^{-1}(s_U(m))\big)$$

is a diffeomorphism. Here ψ_U is the map defined in step 2 of ((2.11)). So $\varphi_U = (id \times \psi_{\mathscr{U}}^{-1}) \circ \psi_U$. Because it intertwines the G-action, φ_U is a local trivialization. To finish the argument, we need only show that if (φ_U, U) and (φ_V, V) are local trivializations of the bundle $\pi : M \to M/G$, then the transition map $\varphi_V \circ \varphi_U^{-1}$ is of the form given in point 3 of the definition of principal bundle. The reason for this is that by design there is a canonical diffeomorphism of $S \subseteq M$, with $\mathscr{U} \subseteq M/G$. The computation is

$$\varphi_V \circ \varphi_U^{-1}(u)(g,u) = \varphi_V\big(\Phi_g(\varphi_U^{-1}(e,u))\big) = g\bullet\varphi_V \circ \varphi_U^{-1}(e,s) = \big(g \cdot \vartheta_{UV}(u), u\big).$$

Here the value of the *transition map* $\vartheta_{UV}(u)$ is defined to be $\pi_1 \circ \varphi_V \circ \varphi_U^{-1}(e,u)$ where π_1 is the projection onto the first factor of $G \times M/G$. □

Example 3. The normalizer $N(H)$ of H in G is $\{g \in H \,\big|\, gHg^{-1} = H\}$. Clearly, $H \subseteq N(H)$. Moreover, $N(H)$ is a closed subgroup of G and hence is a Lie group. Because H is a
▷ normal subgroup of $N(H)$, the coset space $N(H)/H$ is a Lie group. Suppose that $M_H = \{m \in M \,\big|\, G_m = H\}$ is not empty. Then $g \in N(H)$ if and only if $g \cdot M_H = M_H$.

(2.13) **Proof**: Suppose that $g \in N(H)$ and $m \in M_H$. Then $g \cdot m \in M_H$, because $G_{g\cdot m} = gG_mg^{-1} = G_m$, since $g \in N(H)$ and $H = G_m$. Thus $g \cdot m \in M_H$. So $g \cdot M_H \subseteq M_H$. Now $g^{-1} \cdot m \in M_H$, since $G_{g^{-1}\cdot m} = g^{-1}G_m(g^{-1})^{-1} = G_m$, because $g^{-1} \in N(H)$. Thus $g^{-1} \cdot M_H \subseteq M_H$. So $M_H = g \cdot (g^{-1} \cdot M_H) \subseteq g \cdot M_H \subseteq M_H$, which shows that $g \cdot M_H = M_H$. Now suppose that $g \cdot M_H = M_H$. Then for every $m \in M_H$ we have $g \cdot m \in M_H$, that is, $G_{g\cdot m} = G_m$. So $gG_mg^{-1} = G_{g\cdot m} = G_m$, that is, $g \in N(H)$. □

Claim: The orbit space of the free and proper action of $N(H)/H$ on the smooth manifold M_H is the smooth manifold $\overline{M}_{(H)}$. The $N(H)/H$-orbit mapping is $\pi|M_H : M_H \to \overline{M}_{(H)}$.

(2.14) **Proof**: The proof is a consequence of the following results.

▷ $N(H)/H$ acts freely and properly on M_H.

(2.15) **Proof**: Because $N(H)$ is a closed subgroup of G, M_H is a closed subset M, and the G-action on M is proper, it follows that the action of $N(H)/H$ on M_H is proper. To prove freeness let $m \in M_H$ and suppose that $g \in N(H)_m$. Then $g \cdot m = m$. So $g \in G_m = H$. Thus $N(H)_m \subseteq H$. Now suppose that $g \in H = G_m$. Then $g \cdot m = m$ and $g \in N(H)$. So $g \in N(H)_m$. Thus $N(H)_m = H$, that is, $\big(N(H)/H\big)_m = \{e\}$. □

We now look at the mapping $\pi|M_H : M_H \to \overline{M} = M/G$, where $\pi : M \to \overline{M}$ is the G-orbit
▷ mapping. First we show that the image of $\pi|M_H$ is the orbit type $\overline{M}_{(H)} = \pi(M_{(H)})$ in the G-orbit space $\overline{M}$.

(2.16) **Proof**: Suppose that $\overline{m} \in \overline{M}_{(H)}$. Then there is an $m \in M_{(H)}$ such that $\pi(m) = \overline{m}$. But $M_{(H)} = G \cdot M_H$. Thus there is $g \in G$ such that $g \cdot m \in M_H$. So $(\pi|M_H)(g \cdot m) = \pi(g \cdot m) = \pi(m) = \overline{m}$, that is $\overline{m} \in \operatorname{im} \pi|M_H$. Consequently, $\overline{M}_{(H)} \subseteq \operatorname{im} \pi|M_H$. Now suppose that $\overline{m} \in \operatorname{im} \pi|M_H$. Then there is $m \in M_H$ such that $\overline{m} = \pi(m)$. But $M_{(H)} = G \cdot M_H \supseteq e \cdot M_H = M_H$. So $m \in M_{(H)}$, which implies $\overline{m} \in \overline{M}_{(H)}$. Thus $\operatorname{im} \pi|M_H \subseteq \overline{M}_{(H)}$. □

▷ Next we show that each fiber of the mapping $\pi|M_H$ is a unique $N(H)/H$-orbit on M_H.

(2.17) **Proof**: Let $\overline{m} \in \overline{M}_{(H)}$. Then $\big(\pi|M_H\big)^{-1}(\overline{m}) \subseteq M_H \cap M_{(H)}$. Let m', $m'' \in \big(\pi|M_H\big)^{-1}(\overline{m})$. Then $m', m'' \in M_H \cap M_{(H)}$ and $\pi(m') = \pi(m'') = \overline{m}$. So there is $g \in G$ such that $m'' = g \cdot m'$. Since m' and m'' lie in M_H, it follows that $g \in N(H)$. Thus $\big(\pi|M_H\big)^{-1}(\overline{m}) = N(H) \cdot m'$ with $\pi(m') = \overline{m}$. □

The claim ((2.14)) follows by applying ((2.9)). □

Example 4. Let G be a Lie group with a closed subgroup H. Suppose that $\rho : H \to \mathrm{Gl}(V)$ is a homomorphism. Define an action of H on $G \times V$ by

$$\Psi : H \times (G \times V) \to G \times V : (h, (g, v)) \to (gh^{-1}, \rho(h)v).$$

Then Ψ is free and proper. Hence the orbit space $G \times_H V$ is a smooth manifold. The map $\pi_1 : G \times V \to G$ intertwines the H-action Ψ with the action of H on G. Hence it induces a map $\pi : G \times_H V \to G/H$ on the orbit spaces. π is a locally trivial bundle over G/H with fiber V and total space $G \times_H V$. It is an *associated bundle* of the principal bundle $G \to G/H$. □

Recall that $M_H = \{m \in M \,\big|\, G_m = H\}$. Let $m \in M$. For the proper action $\Phi : G \times M \to M$ we have $T_e\Phi_m\mathfrak{g} = T_m(G \cdot m)$. For every $g \in G_m$ the linear transformation $T_m\Phi_g : T_mM \to T_mM$ leaves the subspace $T_m(G \cdot m)$ invariant. Hence we obtain an induced linear action $*$ of the compact group $H = G_m$ on the vector space $E = T_mM/T_m(G \cdot m)$. Let B be an H-invariant open subset of E. On $G \times B$ we have an action of H defined by

$$\mu : H \times (G \times B) \to G \times B : (h, (g, b)) \mapsto (gh^{-1}, h * b). \tag{6}$$

This action is free and proper. Therefore the orbit space $G \times_H B$ of the H-action μ is a smooth manifold. Because the G-action on $G \times B$ defined by

$$\nu : G \times (G \times B) \to G \times B : (g', (g, b)) \mapsto (g'g, b) \tag{7}$$

commutes with the H-action (6), it induces a G-action on $G \times_H B$.

Claim (**Tube theorem**.) Let $m \in M$ and set $H = G_m$. There is a G-invariant open neighborhood U of m in M, an open H-invariant neighborhood B of the origin in E, and a diffeomorphism $\varphi : G \times_H B \to U$, which intertwines the G-action on $G \times_H B$ with the G-action on U.

(2.18) **Proof**: B is identified with a submanifold S of M containing m, called a slice, by a diffeomorphism ψ, see ((1.4)). φ is induced by the mapping $G \times B \to M : (g,b) \mapsto g \cdot \psi(b)$, which is a diffeomorphism. □

3 Differential spaces

In this subsection we discuss the differential geometry of the space of orbits of a proper action using the concept of a differential space.

3.1 Differential spaces

A *differential space* is a pair $(P, C^\infty(P))$, where P is a topological space and $C^\infty(P)$ is a set of continuous real valued functions having the following properties.

1. The sets $f^{-1}(I)$, with $f \in C^\infty(P)$ and I an open interval in $\mathbf{R}$, form a subbasis for the topology of P.
2. For every positive integer n, every $F \in C^\infty(\mathbf{R}^n)$, and every $f_1, \ldots, f_n \in C^\infty(P)$, we have $F \circ \mathfrak{f} \in C^\infty(P)$, where $\mathfrak{f}(p) = \big(f_1(p), \ldots, f_n(p)\big) \in \mathbf{R}^n$ for every $p \in P$.
3. If $f : P \mapsto \mathbf{R}$ has the property that for every $p \in P$ there is an open neighborhood U_p of p in P and a function $f_p \in C^\infty(P)$ such that $f|U_p = f_p|U_p$, then $f \in C^\infty(P)$.

The set $C^\infty(P)$ is called the *differential structure* of the differential space $(P, C^\infty(P))$.

Example. Let P be a smooth manifold with $C^\infty(P)$ its collection of smooth functions. Then $(P, C^\infty(P))$ is a differential space. □

A way of constructing a differential structure on a *set* P goes as follows. Let $\mathscr{F}$ be a family of functions on P. Endow P with a topology $\mathscr{T}$ generated by a subbasis $\{f^{-1}(I) \,\big|\, f \in \mathscr{F}$ and I an open interval in $\mathbf{R}\}$. We say that $h \in C^\infty_{\mathscr{F}}(P)$ if and only if for each $p \in P$ there is set $U \in \mathscr{T}$ containing p, functions $f_1, \ldots, f_n \in \mathscr{F}$, and a function $F \in C^\infty(\mathbf{R}^n)$ such that $h(u) = \big(F \circ (f_1, \ldots, f_n)\big)(u) = F\big(f_1(u), \ldots, f_n(u)\big)$ for every $u \in U$. We call $C^\infty_{\mathscr{F}}(P)$ the space of *smooth functions on P generated by* $\mathscr{F}$. We call the topology $\mathscr{T}$ the *differential space topology on P generated by* $\mathscr{F}$.

Fact: $\mathscr{F} \subseteq C^\infty_{\mathscr{F}}(P)$.

(3.1) **Proof**: Let $f \in \mathscr{F}$ and $p \in P$. Take I to be an open interval in $\mathbf{R}$ containing $f(p)$. Then $U = f^{-1}(I)$, which contains p by construction, is an open subset of P, that is, $U \in \mathscr{T}$. Let $F = \mathrm{id}_{\mathbf{R}}$. Then $F \in C^\infty(\mathbf{R})$. Since $f(u) = (F \circ f)(u)$ for every $u \in U$, it follows that $f \in C^\infty_{\mathscr{F}}(P)$. □

Claim: $C^\infty_{\mathscr{F}}(P)$ is a differential structure on P, which we call the *differential space structure generated by* $\mathscr{F}$.

(3.2) **Proof**: We verify that $C^\infty_{\mathscr{F}}(P)$ satisfies the definition of differential structure.

Point 1 is satisfied by the choice of topology $\mathscr{T}$ on P.

To prove point 2 let $h_1,\ldots,h_n \in C^\infty_{\mathscr{F}}(P)$ and let $F \in C^\infty(\mathbf{R}^n)$. By definition for each $p \in P$ there is a $U \in \mathscr{T}$ containing p, functions $f_{ij_i} \in \mathscr{F}$ for $1 \le j_i \le m_i$ such that $h_i|U = \big(F_i\circ(f_{i1},\ldots,f_{im_i})\big)|U$ for every $1 \le i \le n$ for some functions $F_i \in C^\infty(\mathbf{R}^{m_i})$ for $1 \le i \le n$. Then

$$\begin{aligned}\big(F\circ(h_1,\ldots,h_n)\big)|U &= F\circ(h_1|U,\ldots,h_n|U)\\ &= F\circ\big(F_1\circ(f_{11},\ldots,f_{1m_1}),\ldots,F_n\circ(f_{n1},\ldots,f_{1m_n})\big)|U\\ &= F\circ(F_1,\ldots,F_n)\circ\big((f_{11},\ldots,f_{1m_1}),\ldots,(f_{n1},\ldots,f_{nm_n})\big)|U.\end{aligned}$$

Since $F\circ(F_1,\ldots,F_n) \in C^\infty(\mathbf{R}^m)$, where $m = m_1+\cdots+m_n$, and $f_{ij_i} \in \mathscr{F}$ for every $1 \le j_i \le m_i$ and $1 \le i \le n$, it follows that $F\circ(F_1,\ldots,F_n) \in C^\infty_{\mathscr{F}}(P)$.

To verify point 3 suppose that the function $h : P \to \mathbf{R}$ has the property that for every $p \in P$ there is a $U \in \mathscr{T}$ and a function $h_p \in C^\infty_{\mathscr{F}}(P)$ such that $h|U = h_p|U$. By construction of $C^\infty_{\mathscr{F}}(P)$ there is a $U_p \in \mathscr{T}$ containing p, functions $f^1_p,\ldots,f^n_p \in \mathscr{F}$, and a function $F_p \in C^\infty(\mathbf{R}^n)$ such that $h_p|U_p = \big(F_p\circ(f^1_p,\ldots,f^n_p)\big)|U_p$. Thus $h|(U\cap U_p) = h_p|(U\cap U_p) = \big(F_p\circ(f^1_p,\ldots,f^n_p)\big)|(U\cap U_p)$. So $h \in C^\infty_{\mathscr{F}}(P)$. Hence $C^\infty_{\mathscr{F}}(P)$ is a differential structure on P. □

When $\mathscr{F} = C^\infty(P)$ we say that $\mathscr{T}$ is the *differential space topology on P*.

Fact: If $C^\infty(P)$ is a differential structure on P and $\mathscr{F} = C^\infty(P)$, then $C^\infty_{\mathscr{F}}(P) = C^\infty(P)$.

(3.3) **Proof**: By ((3.1)) we have $C^\infty(P) = \mathscr{F} \subseteq C^\infty_{\mathscr{F}}(P)$. Let $p \in P$ and let U_p be a subset of P containing p in the topology generated by $\mathscr{F}$. Then U_p is an open set in the differential space topology on the differential space $(P, C^\infty(P))$. Suppose that $f \in C^\infty_{\mathscr{F}}(P)$. Then there are functions $f_1,\ldots,f_n \in \mathscr{F} = C^\infty(P)$ and a smooth function F on $\mathbf{R}^n$ such that for every $u \in U_p$ we have $f(u) = \big(F\circ(f_1,\ldots,f_n)\big)|U_p$. But $F\circ(f_1,\ldots,f_n) \in C^\infty(P)$, because $C^\infty(P)$ is a differential structure. Thus $f|U_p = F\circ(f_1,\ldots,f_n)|U_p$, which implies that $f \in C^\infty(P)$, because $C^\infty(P)$ is a differential structure. So $C^\infty_{\mathscr{F}}(P) \subseteq C^\infty(P)$. □

If $(P, C^\infty(P))$ and $(Q, C^\infty(Q))$ are differential spaces, then a *smooth mapping* φ from $(P, C^\infty(P))$ to $(Q, C^\infty(Q))$ is a continuous mapping $\varphi : P \to Q$ such that $\varphi^*(C^\infty(Q)) \subseteq C^\infty(P)$. The map φ is a *diffeomorphism* from $(P, C^\infty(P))$ to $(Q, C^\infty(Q))$ if φ is a homeomorphism from P onto Q and both φ and φ^{-1} are smooth. This is equivalent to the condition that φ is a homeomorphism from P onto Q such that $\varphi^*(C^\infty(Q)) = C^\infty(P)$, because

$$C^\infty(Q) \supseteq (\varphi^{-1})^*(C^\infty(P)) \supseteq (\varphi^{-1})^*(\varphi^*(C^\infty(Q))) = (\varphi\circ\varphi^{-1})^*(C^\infty(Q)) = C^\infty(Q).$$

Differential spaces and smooth mappings form a category.

Let $(P, C^\infty(P))$ be a differential space and let N be a subset of P. Define $C^\infty_i(N)$ to be the set of all functions $f : N \mapsto \mathbf{R}$ with the property that for every $n \in N$ there is an open

neighborhood U_n of n *in* P and an $f_n \in C^\infty(P)$ such that $f|(U_n \cap N) = f_n|(U_n \cap N)$. If we provide N with the topology induced from that of P, then $(N, C_i^\infty(N))$ is a differential
$\triangleright$ space, called a *differential subspace* of the differential space $(P, C^\infty(P))$. The inclusion map $i: \big(N, C_i^\infty(N)\big) \to \big(P, C^\infty(P)\big)$ is a smooth map.

(3.4) **Proof**: Let $f \in C^\infty(P)$. We need only show that $i^*f \in C_i^\infty(N)$ for then $i^*(C^\infty(P)) \subseteq C_i^\infty(N)$. Suppose that $n \in N$. Let U_n be an open neighborhood of n in P. Then $(i^*f)|(U_n \cap N) = f|(U_n \cap N)$. So $i^*f \in C_i^\infty(N)$. □

$\triangleright$ If N is a *closed* subset of P, then $C_i^\infty(N) = C^\infty(P)|N$.

(3.5) **Proof**: Let $f \in C_i^\infty(N)$. Since N is a closed subset of P, by the Whitney extension theorem there is an $F \in C^\infty(P)$ such that $f = F|N$. Thus $C_i^\infty(N) \subseteq C^\infty(P)|N$. Conversely, suppose that $f = F|N$ for some smooth function F on P. Then for every $p \in P$ and every open neighborhood U_p of p in P we have $f|(U_p \cap N) = F|(U_p \cap N)$. Therefore $f \in C_i^\infty(N)$. So $C^\infty(P)|N \subseteq C_i^\infty(N)$. □

If U is an open subset of P, then $C_i^\infty(U) = \{f|U \,\big|\, f \in C^\infty(P)\} = C^\infty(U)$. A differential space $(P, C^\infty(P))$ is a *smooth manifold* if for every $p \in P$ there is a nonnegative integer n, an open subset U of P containing p, and $f_1, \ldots, f_n \in C^\infty(P)$ such that the map $\mathfrak{f}: U \subseteq P \to V = \mathfrak{f}(U) \subseteq \mathbf{R}^n : p \mapsto \mathfrak{f}(p) = \big(f_1(p), \ldots, f_n(p)\big)$ is a diffeomorphism from the differential space $(U, C^\infty(U))$ onto the differential space $(V, C_i^\infty(V))$, seen as a differential subspace of the differential space $(\mathbf{R}^n, C^\infty(\mathbf{R}^n))$. In other words, $(P, C^\infty(P))$ is locally diffeomorphic to an open subset of $\mathbf{R}^n$ with its differential structure being given by restricting smooth
$\triangleright$ functions to this open subset. If the topological space P of the differential space $(P, C^\infty(P))$ is Hausdorff, locally compact and paracompact, then for every open covering $\mathscr{U}$ of P there is a partition of unity in $C^\infty(P)$, which is subordinate to $\mathscr{U}$.

(3.6) **Proof**: If $p \in P$ and U is an open neighborhood of p in P, then it follows from property 1 of a differential structure that there is a positive integer n, an open subset W of $\mathbf{R}^n$, and $f_1, \ldots, f_n \in C^\infty(P)$ such that $\mathfrak{f}^{-1}(W) \subseteq U$. There is a cutoff function $F \in C^\infty(\mathbf{R}^n)$ such that $F = 1$ on an open neighborhood of $\mathfrak{f}(p)$ in W and the support of F is a compact subset K of W. From property 2 of a differential structure it follows that $F \circ \mathfrak{f} \in C^\infty(P)$. Also the support of $F \circ \mathfrak{f}$ is $\mathfrak{f}^{-1}(K)$, which is a closed subset of U. The set $\mathfrak{f}^{-1}(K)$ is compact if the closure of U in P is compact. Therefore, if the topological space P is locally compact, then there are cutoff functions in $C^\infty(P)$.

If X is a topological space with $\mathscr{F}(X)$ a space of real valued functions on X and $\mathscr{U} = \{U_j\}_{j \in J}$ is an open covering of X, then a *partition of unity in* $\mathscr{F}(X)$ *subordinate to* $\mathscr{U}$ is a collection $\{\chi_j\}_{j \in J}$ of functions in $\mathscr{F}(X)$ having the following properties.

1. For each $j \in J$, the support $\operatorname{supp}\chi$ of the function χ_j is a compact subset of some $U_j \in \mathscr{U}$.
2. The supports $\{\operatorname{supp}\chi_j\}_{j \in J}$ form a locally finite family of compact subsets of X, whose union is X.
3. $\sum_{j \in J} \chi_j = 1$ on X.

A Hausdorff topological space X is called *paracompact* if every open covering of X has a locally finite refinement. If X is Hausdorff and *locally compact*, then it is paracompact if and only if every connected component of X is equal to the union of a countable collection of compact subsets.

Cutoff functions in $\mathscr{F}(X)$ can be used to obtain a partition of unity on X. $\square$

For $C^\infty(P)$ to be the space of smooth functions on P, we should have a *sheaf of locally defined "smooth functions"*. Towards this goal, let U be an open subset of P. Then we have already defined $C^\infty(U)$. Moreover, the mapping $U \to C^\infty(U)$, where U ranges over all open subsets U of P, defines a sheaf of functions on P. The following claim shows that this sheaf behaves like the sheaf of smooth functions on a manifold.

Claim: Let N be a subset of a smooth paracompact manifold M. N is an embedded submanifold of M if and only if the identity map from the differential space $(N, C_i^\infty(N))$ into the differential space $(N, C^\infty(N))$ is a diffeomorphism of differential spaces, that is, if and only if $C_i^\infty(N) = C^\infty(N)$.

3.2 An orbit space as a differential space

Let $\overline{M} = M/G$ be the orbit space of a proper G-action on a smooth manifold M with G-orbit map $\pi : M \to \overline{M}$ and quotient topology, namely, $\overline{U}$ is an open subset of $\overline{M}$ if and only if $\pi^{-1}(\overline{U})$ is an open subset of M.

For every open subset $\overline{U}$ of $\overline{M}$ a function $\overline{f} : \overline{U} \subseteq \overline{M} \to \mathbf{R}$ is *smooth* if $\pi^*\overline{f}(U) :$
▷ $\pi^{-1}(\overline{U}) \subseteq M \to \mathbf{R}$ is smooth. Let $C^\infty(\overline{U})$ be the space of smooth functions on $\overline{U}$. For every $\overline{m} \in \overline{M}$ and every open neighborhood $\overline{U}$ of $\overline{m}$ in $\overline{M}$ there is a *cutoff function* $\overline{\chi} \in C^\infty(\overline{M})$ such that $\overline{\chi} = 1$ on an open neighborhood of $\overline{m}$ and is 0 on the complement of a compact neighborhood of $\overline{m}$ in $\overline{U}$.

(3.7) **Proof**: We use the notation of the tube theorem ((2.18)). From the tube theorem with $U = \pi^{-1}(\overline{U})$ and $H = G_m$ with $m \in M$ such that $\overline{m} = \pi(m)$ it follows that the mapping $\rho : B/H \to U/G : b*H \mapsto \varphi(b) \cdot G$ is a homeomorphism. Because H is a compact Lie group which acts linearly on $E = T_mM/T_m(G \cdot m)$, we may average an arbitrary inner product on E over H to obtain an H -invariant inner product β on E with norm $\| \; \|$. Since B is an H-invariant open subset of E containing the origin, there is an $\varepsilon > 0$ such that $\{\|x\| \le \varepsilon\} \subseteq B$. Moreover, there is a smooth function $\psi : \mathbf{R} \to \mathbf{R} : r \mapsto \psi(r)$, which is 1 in an open neighborhood of 0 in $\mathbf{R}$ and is 0 when $r \ge \varepsilon$, Then $f : H \times B \to \mathbf{R} : (h,b) \mapsto \psi(\|b\|)$ is a smooth H invariant function. Therefore f induces a smooth function $\widetilde{f} : B/H \to \mathbf{R}$, which corresponds to a function $\overline{g} = \widetilde{f} \circ \rho^{-1} \in C^\infty(\overline{U})$ with support in $\overline{U}$. Extending $\overline{g}$ by 0 outside $\overline{U}$ gives the cutoff function $\overline{\chi} \in C^\infty(\overline{M})$. $\square$

Corollary: Paracompactness of M implies that the Hausdorff space $\overline{M}$ is paracompact.

(3.8) **Proof**: To see this let $\overline{C}$ be a connected component of $\overline{M}$. Then $\overline{C} = \pi(C)$ for some connected component C of M. Because M is paracompact, C is equal to the union of a countable number of compact sets K_i. Since the G-orbit map π is continuous, $\pi(K_i)$ is compact and their union is $\overline{C}$. $\square$

Cutoff functions can now be used to obtain a partition of unity in $C^\infty(\overline{M})$. $\square$

Claim: $(\overline{M} = M/G, C^\infty(\overline{M}) = C^\infty(M/G))$ is a differential space and the G-orbit map π is a smooth mapping from $(M, C^\infty(M))$ to the differential space $(\overline{M}, C^\infty(\overline{M}))$.

(3.9) **Proof**: We verify that properties 1–3 defining a differential structure hold for $C^\infty(\overline{M})$.

1. If $\overline{\chi}$ is a cutoff function on $\overline{M}$ as constructed in ((3.7)), then $\overline{\chi}^{-1}(1/2,3/2) \subseteq \overline{U}$. This proves property 1.

2. Let $\overline{f}_1,\ldots,\overline{f}_n \in C^\infty(\overline{M})$ and let $F \in C^\infty(\mathbf{R}^n)$. Then $\overline{f}_j \circ \pi \in C^\infty(M)^G$, the space of smooth G-invariant functions on M. Hence $F \circ \overline{\mathfrak{f}} \circ \pi \in C^\infty(M)^G$, where $\overline{\mathfrak{f}}(\overline{m}) = (\overline{f}_1(\overline{m}),\ldots,\overline{f}_n(\overline{m}))$ for every $\overline{m} = \pi(m) \in \overline{M}$. Therefore $F \circ \overline{\mathfrak{f}} \in C^\infty(\overline{M})$. This proves property 2.

3. Let $\overline{f} : \overline{M} \mapsto \mathbf{R}$. Suppose that for every $\overline{m} \in \overline{M}$ there is an open neighborhood $\overline{U}_{\overline{m}}$ of $\overline{m}$ in $\overline{M}$ and an $\overline{f}_{\overline{m}} \in C^\infty(\overline{M})$ such that $\overline{f}|\overline{U}_{\overline{m}} = \overline{f}_{\overline{m}}|\overline{U}_{\overline{m}}$. Then $\overline{f} \circ \pi$ is G-invariant and is equal to the smooth function $\overline{f}_{\overline{m}} \circ \pi$ on $U_m = \pi^{-1}(\overline{U}_m)$. Here $m \in \overline{m}$. Because the $\{U_m\}_{m \in M}$ form an open covering of M, it follows that $\overline{f} \circ \pi \in C^\infty(M)$. Therefore $\overline{f} \circ \pi \in C^\infty(M)^G$, which implies that $\overline{f} \in C^\infty(\overline{M})$. This proves property 3.

Thus $C^\infty(\overline{M})$ is a differential structure. The G-orbit map $\pi : M \to \overline{M}$ is a smooth map from $(M, C^\infty(M))$ to $(\overline{M}, C^\infty(\overline{M}))$ because $\pi^*(C^\infty(\overline{M})) = C^\infty(M)^G \subseteq C^\infty(M)$. □

Corollary: $C_i^\infty(\overline{U}) = C^\infty(\overline{U})$, for any open subset $\overline{U}$ of $\overline{M}$.

(3.10) **Proof**: Let $\overline{f} : \overline{U} \mapsto \mathbf{R}$ be an element of $C_i^\infty(\overline{U})$. Then locally $\overline{f}$ agrees with an element of $C^\infty(\overline{M})$. Using the argument which proved property 3 in ((3.9)), it follows that $\overline{f} \circ \pi \in C^\infty(U)^G$, where $U = \pi^{-1}(\overline{U})$. Hence $\overline{f} \in C^\infty(\overline{U})$. Conversely, suppose that $\overline{f} : \overline{U} \to \mathbf{R}$ such that $\overline{f} \circ \pi \in C^\infty(U)^G$. Using ((3.7)) we know that for every $\overline{m} \in \overline{U}$ there is a cutoff function $\overline{\chi}_{\overline{m}} \in C^\infty(\overline{M})$ whose support is a compact subset $\overline{K}$ of $\overline{U}$ and which is 1 in an open neighborhood of $\overline{m}$ in $\overline{M}$. Define $\overline{f}_{\overline{m}} : \overline{M} \to \mathbf{R}$ by $\overline{f}_{\overline{m}} = \overline{\chi}_{\overline{m}} \cdot \overline{f}$ on $\overline{U}$ and 0 on $\overline{M} \setminus \overline{U}$. Then $\overline{f}_{\overline{m}} = 0$ on $\overline{M} \setminus \overline{K} = \overline{W}$. Therefore $U = \pi^{-1}(\overline{U})$ and $W = \pi^{-1}(\overline{W})$ are open subsets of $M = U \cup W$. On U we have $\overline{f}_{\overline{m}} \circ \pi = (\overline{\chi}_{\overline{m}} \circ \pi) \cdot (\overline{f} \circ \pi)$, which is smooth and G-invariant; while on W we have $\overline{f}_{\overline{m}} \circ \pi = 0$, which is also smooth and G-invariant. Therefore $\overline{f}_{\overline{m}} \in C^\infty(\overline{M})$. But $\overline{f} = \overline{f}_{\overline{m}}$ in an open neighborhood of $\overline{m} \in \overline{M}$. Therefore $\overline{f} \in C_i^\infty(\overline{U})$. Consequently, $C^\infty(\overline{U}) = C_i^\infty(\overline{U})$. □

Corollary: The differential space topology on $\overline{M}$ coincides with its orbit space topology.

(3.11) **Proof**: Let $\overline{V}$ be an open neighborhood of $\overline{m}$ in the orbit space topology on $\overline{M}$. Using ((3.7)) there is a cutoff function $\overline{\chi} \in C^\infty(\overline{M})$ such that $\overline{\chi} = 1$ in an open neighborhood of $\overline{m}$ and is equal to 0 on the complement of a compact neighborhood of $\overline{m}$ in $\overline{V}$. Then $\overline{\chi}^{-1}(\frac{1}{2},\frac{3}{2}) \subseteq \overline{V}$, that is $\overline{V}$ contains an open set in the differential space topology on $\overline{M}$. Now let I be a nonempty open interval and let $\overline{U} = f^{-1}(I)$ for some smooth function f on $\overline{M}$. Then $\overline{U}$ is an open set in the differential space topology on $\overline{M}$. Since $\pi^* f$ is a smooth G-invariant function on M, it follows that $\pi^{-1}(\overline{U}) = (\pi^* f)^{-1}(I)$ is an open subset of M. Because π is an open mapping $\pi(\pi^{-1}(\overline{U}))$ is an open subset of $\overline{M}$ in the orbit space topology contained in $\overline{U}$. Consequently, the orbit space topology on $\overline{M}$ and the differential space topology on $\overline{M}$ coincide. □

Fact: Let $G \times M \to M : (g,m) \mapsto g \cdot m$ be a proper action of a Lie group G on a smooth manifold M and let N be a closed G-invariant subset of M endowed with the differential structure $C_i^\infty(N)$. For each G-invariant function $f \in C_i^\infty(N)$ there is a G-invariant extension $h \in C^\infty(M)$.

(3.12) **Proof**: By definition a function $f : N \to \mathbf{R}$ lies in $C_i^\infty(N)$ if for each $n \in N$ there is an open neighborhood U_n of n of n in N and a function $h_1 \in C^\infty(M)$ such that $f|(U_n \cap N) = h_1|(U_n \cap N)$. Since the action of G on M is proper, there is a slice S_n at n for this action. Without loss of generality we may assume that $S_n \subseteq U_n$. The intersection $U_n \cap S_n$ is a closed subset of S_n. Since the isotropy group G_n is compact and preserves S_n we may average the function $h_2 = h_1|(U_n \cap S_n)$ over G_n and obtain a G_n-invariant extension h_3 of $f|(U_n \cap S_n)$ to S_n. The product $S_n \times \mathscr{O}_n$, where $\mathscr{O}_n = G \cdot n$ is the G-orbit through n, is a G-invariant neighborhood of n in M. Using the projection map $S_n \times \mathscr{O}_n \rightarrow S_n$ we pull back h_3 to a smooth G-invariant function h_n on $S_n \times \mathscr{O}_n$. Since N is a closed subset of M, its complement $M \setminus N$ is open. The open subsets $M \setminus N$ and $\{S_n \cap N\}_{n \in N}$ form a covering of M by G-invariant open sets. Using a locally finite subcovering and a subordinate G-invariant partition of unity, we extend h_n to a smooth G-invariant function h on M. □

Using the notation of the tube theorem ((2.18)) we have

Claim: $(B/H, C^\infty(B/H))$ and $(U/G, C^\infty(U/G))$ are diffeomorphic differential spaces.

(3.13) **Proof**: On $G \times B$ we have the G-action ν (7) with G-orbit map π_ν. Since every G-orbit on $G \times B$ intersects $\{e\} \times B$ exactly once, the G-orbit space $(G \times B)/G$ is diffeomorphic to B. In particular, the map $i : B \rightarrow (G \times B)/G : b \mapsto \pi_\nu(e, b)$ is a diffeomorphism, whose inverse is given by the smooth map $\widetilde{\pi} : (G \times B)/G \rightarrow B$, induced from the G-invariant map $\pi : G \times B \rightarrow B : (g, b) \mapsto b$. On $G \times B$ we have an H-action μ (6) whose orbit space is $G \times_H B$. In addition, we have an action of $G \times H$ given by

$$\lambda : (G \times H) \times (G \times B) \rightarrow G \times B : ((g', h), (g, b)) \mapsto \big((g')^{-1} g, h * b\big).$$

Because the actions μ and ν commute, there is an induced H-action on the orbit space $(G \times B)/G$ and an induced G-action on $G \times_H B$, whose orbit spaces $((G \times B)/G)/H$ and $(G \times_H B)/G$ are equal, respectively, to the orbit space $(G \times B)/(G \times H)$ of the action λ. Let $\varphi : G \times_H B \rightarrow U$ be the G-equivariant diffeomorphism given by the tube theorem ((2.18)). Then φ induces the homeomorphism $\widetilde{\varphi} : (G \times_H B)/G = (G \times B)/(G \times H) \rightarrow U/G$ and the isomorphism $\varphi^* : C^\infty(U) \rightarrow C^\infty(G \times B)^H$, which restricts to the isomorphism $\widetilde{\varphi}^* : C^\infty(U)^G \rightarrow C^\infty(G \times B)^{G \times H}$. The diffeomorphism i, defined above, induces the homeomorphism $\widetilde{i} : B/H \rightarrow ((G \times B)/G)/H = (G \times B)/(G \times H)$ and the isomorphism $i^* : C^\infty(G \times B)^G \rightarrow C^\infty(B)$, which restricts to the isomorphism $\widetilde{i^*} : C^\infty(G \times B)^{G \times H} \rightarrow C^\infty(B)^H = C^\infty(B/H)$. Therefore $\widetilde{\varphi} \circ \widetilde{i} : B/H \rightarrow U/G$ is a homeomorphism and $\widetilde{i^*} \circ \widetilde{\varphi}^* : C^\infty(U/G) \rightarrow C^\infty(B/H)$ is an isomorphism. In other words, $(B/H, C^\infty(B/H))$ and $(U/G, C^\infty(U/G))$ are diffeomorphic differential spaces. □

3.3 Subcartesian spaces

In this section we will show that the orbit space of a proper group action is a subcartesian differential space.

A differential space $(P, C^\infty(P))$ is *subcartesian* if P is a Hausdorff topological space and $(P, C^\infty(P))$ is locally diffeomorphic to $(N, C_i^\infty(N))$, where N is a subset of $\mathbf{R}^n$. In other words, $(P, C^\infty(P))$ is subcartesian if for each $p \in P$ there is an open neighborhood U of p in P, a nonnegative integer n, a subset N of $\mathbf{R}^n$, and a diffeomorphism ψ from $(U, C^\infty(U))$ onto $(N, C_i^\infty(N))$.

A subcartesian differential space $(P, C^\infty(P))$ is *locally compact* if and only if for every $p \in P$ there is an open neighborhood U of p in P and a diffeomorphism $\varphi : U \subseteq P \to V \subseteq \mathbf{R}^n$, which maps U onto a locally closed subset V.

Using the notation of the tube theorem ((2.18)), we now investigate the orbit space of the linear action of the compact Lie group $H = G_m$ on the vector space $E = T_mM/T_m(G \cdot m)$. From classical invariant theory we know that the algebra $P(E)^H$ of *H-invariant polynomial functions* on E is finitely generated, that is, there is a positive integer n and polynomials $p_1, \ldots, p_n \in P(E)$ such that every $p \in P(E)^H$ can be written as $p = F(p_1, \ldots, p_n)$, where F is a polynomial on $\mathbf{R}^n$. Because the action of H on E is linear, we can choose p_i to be homogeneous of degree $d_i > 0$. We may also suppose that n is minimal. We then say that $\{p_i\}_{i=1}^n$ is a *Hilbert basis* of $P(E)^H$ and that

$$\sigma : E \to \mathbf{R}^n : x \mapsto \big(p_1(x), \ldots, p_n(x)\big) \tag{8}$$

is the *Hilbert map* corresponding to the given Hilbert basis. Because there can be nontrivial relations among the generators p_i, neither the Hilbert basis of E nor the Hilbert map is unique. Since elements of $P(E)^H$ separate H-orbits on E and σ is H-invariant, the Hilbert map σ induces a continuous bijective map

$$\widetilde{\sigma} : E/H \to \Sigma = \sigma(E) \subseteq \mathbf{R}^n. \tag{9}$$

The *Tarski-Seidenberg theorem* states that the image of a semialgebraic set under a polynomial mapping is *semialgebraic*. Therefore Σ is a semialgebraic subset of $\mathbf{R}^n$. Because $\sigma(t \cdot x) = \sigma(t^{d_1}x_1, \ldots, t^{d_n}x_n) = (t^{d_1}p_1(x), \ldots, t^{d_n}p_n(x))$ the set Σ is *quasi homogeneous* in the sense that if $y \in \Sigma$ then $t \cdot y \in \Sigma$. In particular, Σ is contactible to the origin in $\mathbf{R}^n$. Averaging an arbitrary inner product on E over the orbits of the H-action, we obtain an H-invariant inner product β on E. Since the quadratic function $E \to \mathbf{R} : x \mapsto \beta(x,x)$ is H-invariant, there is a polynomial $P : \mathbf{R}^n \to \mathbf{R}$ such that $\beta(x,x) = P(\sigma(x))$ for every $x \in E$. Consequently, the Hilbert mapping σ (8) is proper. This implies that Σ is a closed subset of $\mathbf{R}^n$. Because the induced mapping $\widetilde{\sigma}$ (9) is continuous, bijective, and proper, it is a homeomorphism from the locally compact Hausdorff space E/H onto Σ. So Σ is locally compact. Thus we have proved

Claim: The orbit space E/H of the H-action on E is homeomorphic to the image Σ of the Hilbert map σ.

We give the *set* Σ a differential structure as follows. Let $\mathscr{F}$ be the family of functions $f : \Sigma \subseteq \mathbf{R}^n \to \mathbf{R}$ such that $\sigma^* f : E \to \mathbf{R}$ is a smooth function on the finite dimensional real vector space E. Note that $\sigma^* f$ is H-invariant. Let $C^\infty_{\mathscr{F}}(\Sigma)$ be the space of smooth functions on Σ generated by $\mathscr{F}$. From ((3.2)) we see that $C^\infty_{\mathscr{F}}(\Sigma)$ is a differential structure on Σ.

▷ Let $\mathscr{T}_1$ be the topology on Σ generated by the family $\mathscr{F}$. Then the topology $\mathscr{T}_1$ on Σ is the same as the topology induced on Σ from $\mathbf{R}^n$.

(3.14) **Proof**: Let $U \in \mathscr{T}_1$. Then for every $p \in U$ there is a collection of open intervals I_i, $1 \le i \le n$, and smooth functions f_i, $1 \le i \le n$, in $\mathscr{F}$ such that $\cap_{i=1}^n f_i^{-1}(I_i) \in \mathscr{T}_1$ and contains p. Since $f_i \in \mathscr{F}$, there is an $F_i \in C^\infty(E)^H$ such that $F_i = \sigma^* f_i$. By Schwarz's theorem there is a smooth function g_i on $\mathbf{R}^n$ such that $\sigma^*(g_i|\Sigma) = F_i$. So $\sigma^* f_i = \sigma^*(g_i|\Sigma)$, which implies $g_i|\Sigma = f_i$, since the Hilbert mapping σ is surjective. Therefore

$$\bigcap_{i=1}^n f_i^{-1}(I_i) = \bigcap_{i=1}^n (g|\Sigma)_i^{-1}(I_i) = \big(\bigcap_{i=1}^n g_i^{-1}(I_i)\big) \cap \Sigma.$$

But $\cap_{i=1}^n g_i^{-1}(I_i)$ is an open subset of $\mathbf{R}^n$. So U is an open subset of Σ in the topology induced from $\mathbf{R}^n$.

Conversely, let $\widetilde{U}$ be an open subset of Σ induced from the topology of $\mathbf{R}^n$. Then there is an open subset U of $\mathbf{R}^n$ such that $\widetilde{U} = U \cap \Sigma$. Using a partition of unity on $\mathbf{R}^n$, there is a smooth function $f : \mathbf{R}^n \to \mathbf{R}$ and an open interval I such that $U = f^{-1}(I)$. So $\sigma^*(f|\Sigma) = F \in C^\infty(E)^H$, that is, $f|\Sigma \in \mathscr{F} \subseteq C^\infty(\Sigma)$. Consequently, $(f|\Sigma)^{-1}(I) = f^{-1}(I) \cap \Sigma = U \cap \Sigma = \widetilde{U}$. Thus $\widetilde{U} \in \mathscr{T}_1$. □

Fact: $C^\infty_{\mathscr{F}}(\Sigma) \subseteq \mathscr{F}$.

(3.15) **Proof**: Let $f \in C^\infty_{\mathscr{F}}(\Sigma)$. Then for every $p \in \Sigma$ there is a $\widetilde{U}_p \in \mathscr{T}_1$, which contains p, functions $f_1, \ldots, f_{m_p} \in \mathscr{F}$, and a function $h \in C^\infty(\mathbf{R}^{m_p})$ such that

$$f|\widetilde{U}_p = \big(h \circ (f_1, \ldots, f_{m_p})\big)|\widetilde{U}_p. \tag{10}$$

Because $f_i \in \mathscr{F}$, the functions $\sigma^* f_i \in C^\infty(E)^H$ for every $1 \le i \le m_p$. Pulling both sides of (10) back by the Hilbert mapping σ gives

$$\begin{aligned}\sigma^* f|\sigma^{-1}(\widetilde{U}_p) &= \sigma^*\big(h\circ(f_1, \ldots, f_{m_p})\big)|\sigma^{-1}(\widetilde{U}_p) = h\circ(\sigma^* f_1|\sigma^{-1}(\widetilde{U}_p), \ldots, \sigma^* f_{m_p}|\sigma^{-1}(\widetilde{U}_p)) \\ &= \big(h\circ(\sigma^* f_1, \ldots, \sigma^* f_{m_p})\big)|\sigma^{-1}(\widetilde{U}_p).\end{aligned}$$

Thus on $\sigma^{-1}(\widetilde{U}_p)$ the function $\sigma^* f$ is smooth being equal to the smooth H-invariant function $h\circ(\sigma^* f_1, \ldots, \sigma^* f_{m_p})$ on E. By ((3.14)) there is an open subset U_p of p in $\mathbf{R}^n$ such that $\widetilde{U}_p = U_p \cap \Sigma$. Since the Hilbert map $\sigma : E \to \Sigma$ is continuous and surjective, the set $\sigma^{-1}(U_p \cap \Sigma)$ is an H-invariant open subset of E. Consequently, $\mathscr{V} = \{V_p = \sigma^{-1}(U_p \cap \Sigma)\}_{p \in \Sigma}$ is an open covering of E by H-invariant open sets. By ((2.5)) there is a locally finite covering $\{V_{p'}\}_{p' \in \Sigma' \subseteq \Sigma}$ of E subordinate to $\mathscr{V}$ and H-invariant smooth functions χ_p with compact support $\operatorname{supp} \chi_{p'}$ in $V_{p'}$ such that $\cup_{p' \in \Sigma'} \operatorname{supp} \chi_{p'} = E$ and $\sum_{p' \in \Sigma'} \chi_{p'} = 1$ on E. Let $G = \sum_{p' \in \Sigma'} \chi_{p'} \cdot \big(h\circ(f_1, \ldots, f_{m_{p'}})\big)$. Then G is a smooth H-invariant function on E. Since

$$G|\operatorname{supp} \chi_{p'} = \big(h\circ(f_1, \ldots, f_{m_{p'}})\big)|\operatorname{supp} \chi_{p'} = \sigma^* f|\operatorname{supp} \chi_{p'}$$

for every $p' \in \Sigma'$, it follows that $G = \sigma^* f$ on E. In other words, $f \in \mathscr{F}$. □

We define another differential structure on the set Σ. Let $\widetilde{\mathscr{F}}$ be the family of functions $f : \Sigma \subseteq \mathbf{R}^n \to \mathbf{R}$ such that there is a smooth function $F : \mathbf{R}^n \to \mathbf{R}$ with $F|\Sigma = f$. Let $\widetilde{C^\infty}(\Sigma)$ be the space of smooth functions on Σ generated by $\widetilde{\mathscr{F}}$. Then from ((3.2)) we see that $\widetilde{C^\infty}(\Sigma)$ is a differential structure on Σ. By definition $\widetilde{C^\infty}(\Sigma) = C^\infty(\mathbf{R}^n)|\Sigma = C^\infty_i(\Sigma)$, where the second equality follows from ((3.4)).

▷ Let $\mathscr{T}_2$ be the topology on Σ generated by the family $\widetilde{\mathscr{F}}$. Then the topology $\mathscr{T}_2$ is the same as the topology on Σ induced from $\mathbf{R}^n$.

(3.16) **Proof**: Let $\widetilde{U}$ be an open subset of Σ in the topology induced from $\mathbf{R}^n$. Then $\widetilde{U} = U \cap \Sigma$ for some open subset U of $\mathbf{R}^n$. Using a partition of unity on $\mathbf{R}^n$, one can show that there is an open interval I and a smooth function $F : \mathbf{R}^n \to \mathbf{R}$ such that $F^{-1}(I) = U$. So $(F|\Sigma)^{-1}(I) = F^{-1}(I) \cap \Sigma = U \cap \Sigma = \widetilde{U}$. Thus $\widetilde{U} \in \mathscr{T}_2$.

Conversely, suppose that $U \in \mathscr{T}_2$. Then for every $p \in U$ there are open intervals I_i, $1 \le i \le n$, and smooth functions $f_i \in \widetilde{C^\infty}(\Sigma)$, $1 \le i \le n$, such that $\cap_{i=1}^n f^{-1}(I_i) \in \mathscr{T}_2$, contains p, and is contained in U. But $f_i = F_i|\Sigma$ for some smooth function F_i on $\mathbf{R}^n$. We get

$$\bigcap_{i=1}^n f^{-1}(I_i) = \big(\bigcap_{i=1}^n f^{-1}(I_i)\big) \cap \Sigma = \big(\bigcap_{i=1}^n (F_i|\Sigma)^{-1}(I_i)\big) \cap \Sigma = \big(\bigcap_{i=1}^n F_i^{-1}(I_i)\big) \cap \Sigma,$$

where $\cap_{i=1}^n F_i^{-1}(I_i)$ is an open subset of $\mathbf{R}^n$. Therefore U is an open subset of Σ in the topology induced from $\mathbf{R}^n$. □

Fact: $\widetilde{C^\infty}(\Sigma) \subseteq \widetilde{\mathscr{F}}$.

(3.17) **Proof**: Let $f \in \widetilde{C^\infty}(\Sigma)$. Then for every $p \in \Sigma$ there is a $\widetilde{U}_p \in \mathscr{T}_2$ containing p, functions $f_1, \ldots, f_{m_p} \in \widetilde{\mathscr{F}}$, and a function $F \in C^\infty(\mathbf{R}^{m_p})$ such that $f|\widetilde{U}_p = \big(F\circ(f_1, \ldots, f_{m_p})\big)|\widetilde{U}_p$. By ((3.14)) there is an open subset U_p of $\mathbf{R}^n$ containing p such that $\widetilde{U}_p = U_p \cap \Sigma$. Since $f_i \in \widetilde{\mathscr{F}}$ for $1 \le i \le m_p$, there is a smooth function g_i on $\mathbf{R}^n$ such that $f_i = g_i|\Sigma$. Therefore

$$\begin{aligned} f|\widetilde{U}_p &= \big(F\circ(f_1, \ldots, f_{m_p})\big)|\widetilde{U}_p = F\circ(f_1|\widetilde{U}_p, \ldots f_{m_p}|\widetilde{U}_p) \\ &= F\circ\big((g_1|\Sigma)|U_p, \ldots (g_{m_p}|\Sigma)|U_p\big) = \big(F\circ(g_1, \ldots, g_{m_p})|\Sigma\big)|\widetilde{U}_p, \end{aligned}$$

where $F\circ(g_1, \ldots, g_{m_p}) \in C^\infty(\mathbf{R}^n)$. So on $\widetilde{U}_p$ the function f is the restriction of a smooth function on $\mathbf{R}^n$ to Σ. We now use a partition of unity on $\mathbf{R}^n$ to piece these local results together. Since Σ is a closed subset of $\mathbf{R}^n$, the collection $\mathscr{U}_{\Sigma\cup\{\infty\}} = \{U_p\}_{p\in\Sigma} \cup U_\infty$, where $U_\infty = \mathbf{R}^n \setminus \Sigma$, is an open cover of $\mathbf{R}^n$. Because $\mathbf{R}^n$ is paracompact, there is a subordinate open covering $\{U_{p'}\}_{p'\in\Sigma'\subseteq(\Sigma\cup\{\infty\})}$ and smooth functions $\chi_{p'}$ with compact support $\operatorname{supp}\chi_{p'}$ in $U_{p'}$ such that $\{\operatorname{supp}\chi_{p'}\}_{p'\in\Sigma'}$ is a locally finite covering of $\mathbf{R}^n$ and $\sum_{p'\in\Sigma'} \chi_{p'} = 1$ on $\mathbf{R}^n$. Let $G = \sum_{p'\in\Sigma'} \chi_{p'} \cdot \big(F\circ(g_1, \ldots, g_{m_{p'}})\big)$. Then G is a smooth function on $\mathbf{R}^n$ such that $G|(\Sigma \cap \operatorname{supp}\chi_{p'}) = f|(\Sigma \cap \operatorname{supp}\chi_{p'})$ for every $p' \in \Sigma'$. So $f = G|\Sigma$, that is, $f \in \widetilde{\mathscr{F}}$. □

Claim: The identity map on Σ is a homeomorphism from Σ, using the differential space topology $\mathscr{T}_1$, onto Σ, using the differential space topology $\mathscr{T}_2$.

(3.18) **Proof**: The claim follows because the topologies $\mathscr{T}_1$ and $\mathscr{T}_2$ on Σ are the same: each being the same as the topology on Σ induced from $\mathbf{R}^n$. □

We now show that $(\Sigma, C_i^\infty(\Sigma))$ is a locally compact subcartesian differential space. We use the notation of the tube theorem ((2.18)). Shrinking B, and thereby also U, if necessary, we may assume that $B = \{x \in E \,\big|\, \beta(x,x) < c\}$ for some $c > 0$. Then the map $\widetilde{\sigma}$ (9) is a homeomorphism from B/H onto $\sigma(B) = \{y \in \Sigma = \sigma(E) \big| P(y) < c\}$. Here P is an H-invariant polynomial which expresses β in terms of the invariant polynomials $\{p_i\}_{i=1}^n$ on E. Note that $\sigma(B)$ is an open semialgebraic subset of the closed semialgebraic subset Σ of $\mathbf{R}^n$.

Claim: Let $\phi : B/H \to U/G$ be the diffeomorphism given in ((3.13)). The map $\widetilde{\sigma} \circ \phi^{-1} : (U/G, C^\infty(U/G)) \to (\sigma(B), C_i^\infty(\sigma(B)))$ is a *diffeomorphism* of differential spaces.

(3.19) **Proof**: Let $f \in C_i^\infty(\sigma(B))$ and $b \in B$. Then there is an open neighborhood U of $\sigma(b)$ in $\mathbf{R}^n$ and a $g \in C^\infty(\mathbf{R}^n)$ such that $f = g$ on $\sigma(B) \cap U$. Therefore $f \circ \sigma = g \circ \sigma$ is smooth on the open neighborhood $\sigma^{-1}(U)$ of b in B. Because this holds for every $b \in B$, we get $f \circ \sigma \in C_i^\infty(B)$. Since $f \circ \sigma$ is H-invariant, it follows that $\sigma^*(C_i^\infty(\sigma(B))) \subseteq C^\infty(B)^H$. Conversely, the theorem of Schwarz states that $C^\infty(B)^H \subseteq \sigma^*(C^\infty(\mathbf{R}^n))$. Let $i : \sigma(B) \to \mathbf{R}^n$ be the inclusion mapping. Then $i^* : C^\infty(\mathbf{R}^n) \to C_i^\infty(\sigma(B))$ is surjective, that is, $i^*(C^\infty(\mathbf{R}^n)) = C_i^\infty(\sigma(B))$. Since $\sigma = i \circ \sigma$, we obtain

$$C^\infty(B)^H \subseteq \sigma^*(C^\infty(\mathbf{R}^n)) = \sigma^* \circ i^*(C^\infty(\mathbf{R}^n)) = \sigma^*(C_i^\infty(\sigma(B))).$$

So $C^\infty(B)^H = \sigma^*(C_i^\infty(\sigma(B)))$. Since the mapping $\widetilde{\sigma} : B/H \to \sigma(B)$ is a homeomorphism, we deduce that it is a diffeomorphism from $(B/H, C^\infty(B/H))$ onto $(\sigma(B), C_i^\infty(\sigma(B)))$. The claim follows using ((3.13)). □

Because the open sets $\{U/G\}$ form an open covering of the orbit space M/G we have proved

Corollary: For a proper action of a Lie group G on a smooth manifold M, the differential space $(M/G, C^\infty(M/G) = C^\infty(M)^G)$ is a locally compact subcartesian space. More precisely, M/G has a covering by open subsets each of which is diffeomorphic as a differential space to an open subset of a closed semialgebraic set.

3.4 Stratification of an orbit space by orbit types

In this subsection we investigate the stratification of the orbit space $\overline{M} = M/G$ of a proper action of G on a manifold M by orbit types $\overline{M}_{(H)} = M_{(H)}/G$.

Using the notation of the tube theorem ((2.18)), consider the G-action

$$G \times (G \times_H E) \to G \times_H E : \big(g', \rho(g,x)\big) \mapsto \rho(g'g, x),$$

where $\rho : G \times E \to G \times_H E$ is the orbit map of the H-action

$$H \times (G \times E) \to G \times E : (h, (g,x)) \mapsto (gh^{-1}, h * x),$$

▷ see (6) for the definition of the action $*$. Then $G_{\rho(g,x)} = gH_x g^{-1}$.

(3.20) **Proof**: To see this suppose that $g' \in G_{\rho(g,x)}$. Then $g' \cdot \rho(g,x) = \rho(g,x)$ or $\rho(g'g,x) = \rho(g,x)$. Therefore there is an $h \in H_x$ such that $(g'gh^{-1}, h * x) = (g,x)$, that is, $g'gh^{-1} = g$ and $h * x = x$. In other words, $g' \in gH_x g^{-1}$. So $G_{\rho(g,x)} \subseteq gH_x g^{-1}$. Conversely, suppose that $g' \in gH_x g^{-1}$. Then for some $h^{-1} \in H_x$, we have $g'g = gh^{-1}$. Therefore $g'gh = g$ and $h * x = x$. Thus $g' \cdot \rho(g,x) = \rho(g,x)$, that is, $g' \in G_{\rho(g,x)}$. So $gH_x g^{-1} \subseteq G_{\rho(g,x)}$. □

Let $E^H = \{x \in E \,\big|\, h*x = x \text{ for every } h \in H\} = \{x \in E \,\big|\, H_x = H\}$.

Fact: The orbit type $(G \times_H E)_{(H)}$ equal to E^H.

(3.21) **Proof**: Suppose that $x \in E^H$. Then $H = H_x$. Therefore for each $g \in G$, we have $G_{\rho(g,x)} = gH_x g^{-1} = gHg^{-1}$. In other words, $\rho(g,x) \in (G \times_H E)_{(H)}$. Now if $\rho(g,x) \in (G \times_H E)_{(H)}$, then $G_{\rho(g,x)} \in (H)$, the G conjugacy class of H. But $G_{\rho(g,x)} = gH_x g^{-1}$, so $H_x \in (H)$. However, $H_x \subseteq H$ and H_x has the same dimension and number of connected components as H, since H_x is conjugate to H in G. Therefore $H_x = H$, that is, $x \in E^H$. □

Claim: Each connected component of each orbit type $\overline{M}_{(H)}$ in the G-orbit space $\overline{M}$ is a smooth manifold, when regarded as a differential subspace of the differential space $(\overline{M}, C^\infty(\overline{M}))$. In other words, $(\overline{M}_{(H)}, C^\infty_i(\overline{M}_{(H)}))$ is a smooth manifold.

(3.22) **Proof**: The fact that $\pi(M_H) = \overline{M}_{(H)}$ in combination with ((3.13)) gives $\overline{U}_{(H)} = U_{(H)}/G = \phi((B \cap E^H)/H)$. Here ϕ is the diffeomorphism given by ((3.13)) and $(B \cap E^H)/H$ is the image of $B \cap E^H$ under the H orbit map on B.

Because H acts linearly on E, it follows that E^H is a linear subspace of E. Let F be the orthogonal complement of E with respect to the H-invariant inner product β. Then F is H-invariant and $F \cap E^H = \{0\}$. The mapping $E^H \times F \to E : (x,y) \mapsto x+y$ is a linear isomorphism, which is H-equivariant if we let H act on $E^H \times F$ by

$$H \times (E^H \times F) \to E^H \times F : (h,(x,y)) \mapsto (x, h*y).$$

If $x = (x_1, \ldots, x_\ell)$ is a coordinate system on E^H and $y = (y_1, \ldots, y_r)$ is one on F, then every polynomial $p \in P(E^H \times F)$ can be written uniquely as $p(x,y) = \sum_\alpha x^\alpha q_\alpha(y)$, where $\alpha = (\alpha_i, \ldots, \alpha_\ell)$, $x^\alpha = x_1^{\alpha_1} \cdots x_\ell^{\alpha_\ell}$ and $q_\alpha(y)$ is a polynomial in $y_1, \ldots, y_r$. Since p lies in $P(E^H \times F)^H$ implies 1) that $q_\alpha \in P(F)^H$ for every α and 2) that $P(E)^H$ is isomorphic to $P(E^H \times F)^H$, it follows that $\{x_1, \ldots, x_\ell, q_1, \ldots, q_m\}$ is a Hilbert basis of $P(E)^H$, where $\{q_j\}_{j=1}^m$ is a Hilbert basis of $P(F)^H$.

The image of $B \cap E^H$ under the Hilbert map

$$\sigma : E = E^H \times F \to \mathbf{R}^n = \mathbf{R}^\ell \times \mathbf{R}^m : (x,y) \mapsto (x, q_1(y), \ldots, q_m(y)) \tag{11}$$

is an open subset of $\mathbf{R}^\ell \times \{0\}$, which is a smooth ℓ-dimensional submanifold. Because the differential spaces $\big((B \cap E^H)/H, C^\infty(B \cap E^H)^H\big)$ and $(\sigma(B \cap E^H), C^\infty(\sigma(B \cap E^H)))$ are diffeomorphic by ((3.19)) and $(\sigma(B \cap E^H), C^\infty(\sigma(B \cap E^H)))$ is diffeomorphic to $(U_{(H)}/G, C^\infty(U_{(H)}/G))$ by ((3.13)), we have proved the claim. □

The orbit types for the action of H on $E = E^H \times F$ are of the form $E^H \times R$, where R is an orbit type for the linear H-action on F. Furthermore, the $\mathbf{R}_{>0}$-action on E of multiplication by $t > 0$ commutes with the linear action of H on F. Therefore $t \cdot R = R$. Let S^{r-1} be the unit sphere in F with respect to the H-invariant inner product β. Then $R \to R \cap S^{r-1}$ is a bijective map from all orbit types $R \neq \{0\}$ to orbit types of the induced H-action on S^{r-1}. Using induction on the dimension of S^{r-1}, it follows that there are only finitely many H orbit types on S^{r-1}. Consequently, there are only finitely many H-orbit types for the action of H on E.

From the quasihomogeneity of the Hilbert map σ (11), it follows that $\sigma(E)$ is invariant under the transformation $(x_1,\ldots,x_\ell,q_1,\ldots,q_m) \mapsto (x_1,\ldots,x_\ell,t^{d_1}q_1,\ldots,t^{d_m}q_m)$, where $d_j = \deg q_j$. Note that when $(q_,\ldots,q_m) \neq (0,\ldots,0)$ then

$$(x_1,\ldots,x_\ell,t^{d_1}q_1,\ldots,t^{d_m}q_m) \neq (x_1,\ldots,x_\ell,s^{d_1}q_1,\ldots,s^{d_m}q_m),$$

$\triangleright$ when $s > 0$ and $s \neq t$. This shows that each orbit type in $\sigma(E)$, which is different from $\mathbf{R}^\ell \times \{0\}$ is equal to a product of $\mathbf{R}^\ell$ with a submanifold of $\mathbf{R}^m$ of dimension greater than or equal to 1. Therefore each connected component of an orbit type in the orbit space near a given connected component of a given orbit type has dimension greater than the dimension of the given connected component of the given orbit type. This proves

Claim: The connected components of orbit types in the orbit space $\overline{M}$ define a stratification $\mathscr{S}$ of $\overline{M}$, called the *orbit type stratification of the orbit space*.

Given a semialgebraic variety, it has a *primary stratification* given by iteratively forming the semialgebraic set of singular points of the preceding semialgebraic variety.

Claim: In the local model of $\sigma(B) \subseteq \mathbf{R}^n$ of the orbit space U/G given in ((3.19)), the orbit type stratification of B/H coincides with the primary stratification of the semialgebraic set $\sigma(B)$.

3.5 Minimality of $\mathscr{S}$

In this subsection we show that the orbit type stratification $\mathscr{S}$ of the orbit space is *minimal* when stratifications are partially ordered by inclusion of the strata.

We begin with some observations. Let $q \in P(F)^H$ be a homogeneous polynomial of degree 1. Then $q \in F^*$, which shows that $\beta^\sharp(q) \in F$. The inner product β on F induces a bijective linear map $\beta^\sharp : F \to F^*$ given by $\beta^\sharp(y)y' = \beta(y,y')$. The inverse of $\beta^\sharp$ is $\beta^\flat$. Since β is H-invariant, it follows that $\beta^\flat(q) \in F^H = \{0\}$. Therefore $q = 0$. Consequently, $d_j = \deg q_j$, where $\{q_j\}_{j=1}^m$ form a Hilbert basis of $P(F)^H$, is greater than or equal to 2. Because the polynomial $F \to \mathbf{R} : y \mapsto \beta(y,y)$ is H-invariant and of degree 2, we may choose $q_1(y) = \beta(y,y)$ for every $y \in F$.

Let $S_F = \{y \in F \big| q_1(y) = 1\}$ be the unit sphere in F with respect to the inner product β. For each j with $2 \leq j \leq m$ let C_j be the maximum of $|q_j(y)|$ with $y \in S_F$. When $y \in F \setminus \{0\}$, let $t = q_1(y)^{1/2}$. Then $t^{-1}y \in S_F$. Since $q_j(y) = q_j(t(t^{-1}y)) = t^{d_j}q_j(t^{-1}y)$ and $|q_j(t^{-1}y)| \leq C_j$, we obtain $|q_j(y)| \leq q_1(y)^{d_j/2}C_j$, which also holds when $y = 0$. Therefore $\sigma(E)$ is contained in

$$\{(x,q) \in \mathbf{R}^\ell \times \mathbf{R}^m \big| q_1 \geq 0 \ \& \ |q_j| \leq C_j q_1^{d_j/2} \text{ for every } 2 \leq j \leq m\}. \tag{12}$$

Here σ is the Hilbert map given in (11).

Claim: Let I be an open interval in $\mathbf{R}$ containing 0 and let $\gamma : I \to \sigma(E) \subseteq \mathbf{R}^\ell \times \mathbf{R}^m : t \mapsto \gamma(t) = (x(t),q(t))$. Suppose that $q_1(0) = 0$ and that $q : I \to \mathbf{R}^m : t \mapsto q(t)$ is differentiable at $t = 0$. Then $q'(0) = 0$.

(3.23) **Proof**: From the fact that $\gamma(I) \subseteq \sigma(E)$ and the inequality $|q_j| \le C_j q_1^{d_j/2}$ (12), it follows that $q_1(t) \ge 0$ for every $t \in I$. Because $q_1(0) = 0$, we see that the function $t \mapsto q_1(t)$ attains its minimum value at 0. Therefore $q_1'(0) = 0$, that is, $\frac{q_1(t)}{t} \to 0$ as $t \to 0$. For each j where $2 \le j \le m$, the fact that $\gamma(I) \subseteq \sigma(E)$ and inequality (12) imply

$$\frac{|q_j(t)|}{|t|} \le C_j \frac{q_1(t)^{d_j/2}}{|t|} = C_j \left| \frac{q_1(t)}{t} \right|^{d_j/2} |t|^{d_j/2-1}. \tag{13}$$

The right hand side of (13) converges to 0 as $t \to 0$, because $\frac{q_1(t)}{t} \to 0$ as $t \to 0$ and $\frac{1}{2} d_j - 1 \ge 0$. Therefore $q_j'(0) = 0$ for every $2 \le j \le m$. □

From ((3.23)) we see that if N is a C^1 submanifold of $\mathbf{R}^n = \mathbf{R}^\ell \times \mathbf{R}^m$ such that $N \subseteq \sigma(E)$ and $0 \in N$, then $T_0 N \subseteq \mathbf{R}^\ell \times \{0\}$. In particular, $\dim N \le \ell$. From an earlier argument we know that all the orbit type strata in $\sigma(E)$ different from $\mathbf{R}^\ell \times \{0\}$ have dimension strictly greater than ℓ. Therefore no union of $\mathbf{R}^\ell \times \{0\}$ with different strata in $\sigma(E)$ can be a C^1 manifold through the origin. This proves

Claim: The orbit type stratification of the orbit space M/G of a proper G-action on M is *minimal*, that is, no union of different strata can be a connected smooth manifold in the differential space $(M/G, C^\infty(M/G))$.

Corollary: If M/G is connected, then the differential space $(M/G,\, C^\infty(M/G))$ is a smooth manifold if and only if there is exactly one orbit type.

4 Vector fields on a differential space

In this section we define what it means to be a vector field on a differential space and then look at vector fields on the orbit space of a proper action.

4.1 Definition of vector field

Let $\mathscr{A}$ be an algebra over $\mathbf{R}$ with multiplication $\cdot$. A *derivation* of $\mathscr{A}$ is a linear mapping δ of $\mathscr{A}$ into itself such that Leibniz' rule holds, namely, $\delta(f \cdot g) = (\delta f) \cdot g + f \cdot (\delta g)$ for every $f, g \in \mathscr{A}$. We denote the set of all derivations of $\mathscr{A}$ by $\mathrm{Der}(\mathscr{A})$. Note that $\mathrm{Der}(\mathscr{A})$ is a Lie algebra with bracket $[\delta, \delta'] = \delta \circ \delta' - \delta' \circ \delta$ for every $\delta, \delta' \in \mathrm{Der}\, \mathscr{A}$.

Example: Let M be a smooth manifold with $C^\infty(M)$ its space of smooth functions. If X is a smooth vector field on M, then for every $f \in C^\infty(M)$ the *Lie derivative* $L_X f$ of f with respect to X, namely, $L_X f : M \to \mathbf{R} : m \mapsto \langle \mathrm{d} f(m) | X(m) \rangle$, is a smooth function on M. Moreover, the linear mapping $L_X : C^\infty(M) \to C^\infty(M)$ is a derivation. □

Let $(M, C^\infty(M))$ be a differential space, which is not necessarily a smooth manifold, and let $\delta \in \mathrm{Der}(C^\infty(M))$. An *integral curve* of δ is a smooth mapping $\gamma : I \subseteq \mathbf{R} \to M$, where I is an interval, such that

$$\frac{\mathrm{d} f(\gamma(t))}{\mathrm{d} t} = \delta(f)(\gamma(t)), \quad \text{for every } f \in C^\infty(M) \text{ and every } t \in I.$$

Claim: Let $(M, C^\infty(M))$ be a locally compact subcartesian differential space and let $\delta \in \mathrm{Der}(C^\infty(M))$. Suppose that for every $m \in M$, there is an *open* interval $I \subseteq \mathbf{R}$ containing 0 and an integral curve $\gamma_m : I \subseteq \mathbf{R} \to M$ of δ such that $\gamma_m(0) = m$. Then:

1. For each $m \in M$ there is a unique integral curve $\gamma : I_m \to M$ of δ, which is defined on a maximal open interval I_m in $\mathbf{R}$ containing 0 such that $\gamma(0) = m$.
2. The set $D = \{(t,m) \in R \times M \,\big|\, t \in I_m\}$ is an open subset of $\mathbf{R} \times M$ and the map $\varphi : D \subseteq \mathbf{R} \times M \to M : (t,m) \mapsto \gamma_m(t)$ is smooth.
3. For each $t \in \mathbf{R}$, the set $D_t = \{m \in M \,\big|\, t \in I_m\}$ is an open subset of M and the mapping $\varphi_t : D_t \to M : m \mapsto \gamma_m(t)$ is smooth. φ_t is called the *flow of δ at time t*.
4. If $s,t \in R$, $m \in D_s$, $\varphi_s(q) \in D_t$, then $m \in D_{t+s}$ and $\varphi_t(\varphi_s(m)) = \varphi_{t+s}(m)$. Therefore $\varphi_t : D_t \to M$ is a diffeomorphism with inverse φ_{-t}.

(4.1) **Proof**:

1. Using the fact that M is locally compact, we may identify a suitable open neighborhood of m in M with a locally closed subset V of $\mathbf{R}^n$. Let $\{x_i\}_{i=1}^n$ be coordinate functions on $\mathbf{R}^n$. There is an open neighborhood U_i of m in $\mathbf{R}^n$ and $\delta_i \in \mathrm{Der}(C^\infty(U_i))$ such that $\delta(x_i) = \delta_i | U_i$. Let $U = \cap_{i=1}^n U_i$ and let $X_\delta : U \to \mathbf{R}^n$ be a smooth vector field on U such that $L_{X_\delta} x_i = \delta_i | U$. Shrinking U if necessary, we may assume that $V \cap U$ is a closed subset of U. Let $\gamma : I \to V \cap U$ be an integral curve of δ. Then

$$\frac{\mathrm{d}\gamma_i(t)}{\mathrm{d}t} = \frac{\mathrm{d}x_i(\gamma(t))}{\mathrm{d}t} = \delta(x_i)(\gamma(t)) = \delta_i(\gamma(t))$$

shows that γ is an integral curve of the derivation L_{X_δ}, thought of as a smooth vector field on $\mathbf{R}^n$. Therefore the local existence and uniqueness of integral curves of the smooth vector field implies the existence and uniqueness of smooth integral curves of X_δ with prescribed initial value. Consequently, for each $m \in V \cap U$ there is a unique integral curve $\gamma : I_m \to V \cap U$ of the derivation δ on a maximal open interval I_m in $\mathbf{R}$ containing 0 with $\gamma(0) = m$.

Let $\widetilde{\varphi}_t$ be the flow at time t of the vector field X_δ on U. Let $\widetilde{I}$ be the maximal domain of definition of the integral curve $t \mapsto \widetilde{\gamma}(t) = \widetilde{\varphi}_t(m)$ of X_δ with $\widetilde{\gamma}(0) = m$. Next we show that $I_m = \widetilde{I}$. Now $\widetilde{\gamma}|I_m = \gamma$, so $I_m \subseteq \widetilde{I}$. Suppose that $s = \sup I_m \in \widetilde{I}$ and let $r = \widetilde{\gamma}(s)$. Then $r = \lim_{t \nearrow s} \widetilde{\gamma}(t) = \lim_{t \nearrow s} \gamma(t) \in V \cap U$, because $\gamma(t) \in V \cap U$ for every $t \in I_m$ and $V \cap U$ is closed in U. The hypothesis that every integral curve of δ is defined on an open interval, implies that there is an open interval J in $\mathbf{R}$ containing 0 and an integral curve $\widehat{\gamma} : J \to V \cap U$ of δ such that $\widehat{\gamma}(0) = r$. Therefore

$$\widehat{\gamma}(t-s) = \varphi_{t-s}(r) = \varphi_{t-s}(\varphi_s(q)) = \widetilde{\gamma}(t)$$

for all $t \in I_m \cap (s+J)$. From uniqueness it follows that $\widetilde{\gamma}$ and $t \mapsto \widehat{\gamma}(t-s)$ piece together to form an integral curve of δ, which is defined on $I_m \cap (s+J)$. This contradicts the maximality of I_m. A similar argument shows that if $\inf I_m \in \widetilde{I}$, then we obtain a contradiction. Therefore $I_m = \widetilde{I}$. This proves assertion 1.

2. Let $\widetilde{D} \subseteq \mathbf{R} \times U$ be the domain of definition of the flow $\widetilde{\varphi} : \widetilde{D} \to U$ of the smooth vector field X_δ on U. Let $\varphi : D \to V \cap U$ be as defined in assertion 2 of the claim with M replaced

by $V \cap U$. The argument of the preceding paragraph shows that $D = \widetilde{D} \cap (\mathbf{R} \times (V \cap U))$ and that $\varphi = \widetilde{\varphi}|D$. Because $\widetilde{D}$ is an open subset of $\mathbf{R} \times U$ and the flow $\widetilde{\varphi}$ is smooth, it follows that D is an open subset of $\mathbf{R} \times (V \cap U)$ and that the mapping $\varphi : D \rightarrow V \cap U$ is smooth as a map of differential spaces. Because the preceding two properties are local, we have shown that the set D, given in statement 2 of the proposition, is an open subset of $\mathbf{R} \times M$ and that the mapping $\varphi : D \rightarrow M$ is smooth. This proves assertion 2.

Assertions 3 and 4 follow using the same arguments as for a smooth vector field on a smooth manifold. $\square$

Example: Consider the set $S = \{(x_1, x_2) \in \mathbf{R}^2 \mid x_1^2 + (x_2 - 1)^2 < 1 \text{ or } x_2 = 0\}$. The vector field $X = \frac{\partial}{\partial x_1}$ on $\mathbf{R}^2$ restricts to a derivation L_X of $C^\infty(S)$. For every $x = (x_1, x_2) \in S$, $\varphi_t^X(x) = (x_1 + t, x_2)$ for all $t \in \mathbf{R}$. Its restriction to S induces $\widetilde{\varphi}_t$ given by $\widetilde{\varphi}_t(x_1, x_2) = (x_1 + t, x_2)$, where

$$\begin{cases} t \in (-x_1 - \sqrt{1-(x_2-1)^2}, -x_1 + \sqrt{1-(x_2-1)^2}), & \text{if } x_2 > 0 \\ t \in \mathbf{R}, & \text{if } x_2 = 0. \end{cases}$$

Hence, every integral curve of X on S has an open domain. Nevertheless, $\widetilde{\varphi}_t$ fails to be a local one-parameter group of local diffeomorphisms of S, because S is not a locally closed subset of $\mathbf{R}^2$. $\square$

Claim ((4.1)) motivates the following definition. Let $(M, C^\infty(M))$ be a locally compact subcartesian space and let $\delta \in \mathrm{Der}(C^\infty(M))$. We call δ a *vector field* on $(M, C^\infty(M))$ if and only if for every $m \in M$ there is an *open* interval I in $\mathbf{R}$ containing 0 and an integral curve $\gamma : I \subseteq \mathbf{R} \rightarrow M$ of δ such that $\gamma(0) = m$. Let $\mathscr{X}(M, C^\infty(M))$ be the set of all smooth *vector fields* on the differential space $(M, C^\infty(M))$.

4.2 Vector fields on a stratified differential space

Let $(M, C^\infty(M))$ be a differential space. A *stratification* $\mathscr{S}$ of $(M, C^\infty(M))$ is a collection of differential subspaces $\{S\}$, of which each stratum S is a smooth submanifold of $(M, C^\infty(M))$, such that

1. $\mathscr{S}$ is a locally finite partition of M.
2. For each $S \in \mathscr{S}$ the closure of S in M is the union of S and $\{S' \in \mathscr{S} \,\big|\, \dim S' < \dim S\}$.

Denote a differential space $(M, C^\infty(M))$ with the stratification $\mathscr{S}$ by $(M, \mathscr{S}, C^\infty(M))$.

A *stratified vector field* W on $(M, \mathscr{S}, C^\infty(M))$ is a map which assigns to each stratum S of the stratification $\mathscr{S}$ a smooth vector field W_S on S. If $f \in C^\infty(M)$ and $S \in \mathscr{S}$, then $f|S \in C^\infty(S)$. Therefore $L_{W_S}(f|S) \in C^\infty(S)$.

Fact: For every $f \in C^\infty(M)$ the function $\partial_W f$, which assigns to each $S \in \mathscr{S}$ the smooth function $L_{W_S}(f|S)$ in $C^\infty(S)$, is a smooth function on M. Moreover, ∂_W lies in $\mathrm{Der}(C^\infty(M))$.

(4.2) **Proof**: Let $m \in M$. Because $f \in C^\infty(M)$, for every open neighborhood U of m in M, we have $f|U \in C^\infty(U)$. Since the stratification $\mathscr{S}$ is a locally finite partition of M, there are a finite number of strata $\{S_j\}_{j \in J}$ such that $S_j \cap U \neq \varnothing$. Let $F = \sum_{j \in J} L_{W_{S_j}}(f|(U \cap S_j))$. Then

$F \in C^\infty(U)$ because $L_{W_{S_j}}(f|(U \cap S_j)) \in C^\infty(U \cap S_j) \subseteq C^\infty(U)$, since S_j is an embedded submanifold of $(M, C^\infty(M))$. From $(\partial_W f)|(U \cap S_j) = \sum_{j \in J} L_{W_{S_j}}(f|(U \cap S_j)) = F|(U \cap S_j)$ it follows that $L_W f \in C_i^\infty(U) = C^\infty(U)$. Therefore $\partial_W f \in C^\infty(M)$. Clearly ∂_W is a derivation on $C^\infty(M)$. □

A stratified vector field W on $(M, \mathscr{S}, C^\infty(M))$ is *smooth* if and only if $\partial_W f \in C^\infty(M)$ for every $f \in C^\infty(M)$. Let $\mathscr{X}^\infty(M, \mathscr{S})$ be the *set of all smooth stratified vector fields* on $(M, \mathscr{S}, C^\infty(M))$.

Claim: The map $\partial : \mathscr{X}^\infty(M, \mathscr{S}) \rightarrow \mathrm{Der}(C^\infty(M)) : W \mapsto \partial_W$ is an injective homomorphism of Lie algebras. Moreover,

$$\partial(\mathscr{X}^\infty(M, \mathscr{S})) \subseteq \mathscr{X}(M, C^\infty(M)), \tag{14}$$

if the differential space $(M, C^\infty(M))$ is locally compact and subcartesian.

(4.3) **Proof**: First we show that the mapping ∂ is injective. Let $W \in \mathscr{X}^\infty(M, \mathscr{S})$ and suppose that $\partial_W = 0$. Let $S \in \mathscr{S}$, $f \in C^\infty(S)$, and $m \in S$. Then there is an open neighborhood U of m in M and $g \in C^\infty(M)$ such that $f|(S \cap U) = g|(S \cap U)$. Because $\partial_W g = 0$, we obtain $L_{W_S}(f|(S \cap U)) = 0$. Since this holds for every $m \in S$, we see that $L_{W_S} f = 0$. Because $L_{W_S} : \mathscr{X}^\infty(S) \rightarrow \mathrm{Der}(C^\infty(S))$ is an isomorphism for smooth manifolds, it follows that $W_S = 0$. Since this holds for every $S \in \mathscr{S}$, we deduce that $W = 0$.

We now show that (14) holds. Let $W \in \mathscr{X}^\infty(M, \mathscr{S})$ and $m \in M$. Then there is an $S \in \mathscr{S}$ with $m \in S$. The smooth vector field W_S on S has a smooth integral curve $\gamma : I \rightarrow S$, where I is an open interval in $\mathbf{R}$ containing 0 and $\gamma(0) = m$. For any $f \in C^\infty(M)$, we know that $f|S$ is a smooth function on the smooth manifold S. Therefore

$$\begin{aligned} \frac{\mathrm{d} f(\gamma(t))}{\mathrm{d} t} &= \mathrm{d}(f|S)(\gamma(t))\, \gamma'(t) = \mathrm{d}(f|S)(\gamma(t))\, W_S(\gamma(t)) \\ &= L_{W_S}(f|S)(\gamma(t)) = \partial_W f(\gamma(t)). \end{aligned}$$

So γ is an integral curve of the derivation ∂_W with $\gamma(0) = m$. Because this holds for every $m \in M$, the derivation ∂_W is a smooth vector field on $(M, C^\infty(M))$, that is, $\partial_W \in \mathscr{X}(M, C^\infty(M))$. □

4.3 Vector fields on an orbit space

Let G be a Lie group which acts smoothly and properly on a smooth manifold M with orbit map $\pi : M \rightarrow \overline{M} = M/G$. Then $(M/G, C^\infty(M/G))$ is a differential space, which has a stratification $\mathscr{S}$ by orbit types.

Claim: We have $\pi_*(\mathscr{X}(M)^G) \subseteq \mathscr{X}^\infty(\overline{M}, \mathscr{S})$.

(4.4) **Proof**: Let $X \in \mathscr{X}(M)^G$, that is, X is a smooth G-invariant vector field on M. Then $X_{(H)} = X|M_{(H)}$ is a smooth vector field on the orbit type $M_{(H)}$. Let $\overline{X}_{(H)} = \pi_* X_{(H)}$. Then $\overline{X}_{(H)}$ is a smooth vector field on the orbit type $\overline{M}_{(H)} = \pi(M_{(H)})$ in the orbit space $\overline{M}$. The map which assigns to the orbit type $\overline{M}_{(H)}$ the smooth vector field $\overline{X}_{(H)}$ defines a smooth stratified vector field $\overline{X}$ on $(\overline{M}, \mathscr{S}, C^\infty(\overline{M}))$ such that $\pi_* X = \overline{X}$. □

Claim: Using the map $\partial : \mathscr{X}^\infty(\overline{M},\mathscr{S}) \to \mathrm{Der}(C^\infty(\overline{M}))$ defined in §4.2, we have

$$\partial \mathscr{X}^\infty(\overline{M},\mathscr{S}) = \mathscr{X}(\overline{M},C^\infty(\overline{M})). \tag{15}$$

(4.5) **Proof**: We begin by observing that the orbit space $\overline{M}$ is locally compact, since M is and the orbit map π is continuous, open and surjective. In addition, the differential space $(\overline{M},C^\infty(\overline{M}))$ is subcartesian.

Let $\overline{X}$ be a smooth vector field on $\overline{M}$. For any point $\overline{m} \in \overline{M}$, we use the diffeomorphism $\varphi = \widetilde{\sigma} \circ \phi^{-1}$ of ((3.19)) to identify an open neighborhood $\overline{U}$ of $\overline{m}$ in $\overline{M}$ with the image $\sigma(B) \subseteq \mathbf{R}^n$ of the Hilbert map. We also use the decomposition $\mathbf{R}^n = \mathbf{R}^\ell \times \mathbf{R}^m$ so that the orbit types in $\sigma(E)$ are of the form $\mathbf{R}^\ell \times R$, where R is an orbit type for the action of H on F. We may assume that $\widetilde{\sigma}(\overline{m}) = (0,0) \in \mathbf{R}^\ell \times \mathbf{R}^m$ and that its orbit type corresponds locally to $\mathbf{R}^\ell \times \{0\}$.

Let $X(x,q) = (\dot{x},\dot{q})$ be a smooth vector field in an open neighborhood of 0 in $\mathbf{R}^n$ induced by the smooth vector field $\overline{X}$. Let $\gamma(t) = \big(x(t),q(t)\big)$ be an integral curve of $\overline{X}$ defined in an open interval I of $\mathbf{R}$ containing 0 such that $q(0) = 0$. Because $\gamma(t) \in \sigma(E)$ for all $t \in I$, from ((3.23)) it follows that $\frac{\mathrm{d}q(\gamma(t))}{\mathrm{d}t}(0) = 0$. Because integral curves of $\overline{X}$ are integral curves of X, it follows that $\dot{q} = 0$ when $q = 0$. In other words, the vector field X is tangent to $S = \mathbf{R}^\ell \times \{0\}$. Thus $X|S$ near 0 is a smooth vector field W on S. Therefore the integral curves of $\overline{X}$, which are integral curves of X, remain on S when they start on S.

Let f be a smooth function on an open neighborhood of the 0 in $\mathbf{R}^n$. For any integral curve γ of $\overline{X}$, which lies on S, we have

$$\overline{X}(f)(\gamma(t)) = \frac{\mathrm{d}f(\gamma(t))}{\mathrm{d}t} = \partial_W f(\gamma(t)),$$

see ((4.3)). Consequently, $\overline{X}(f)|S = \partial_W(f|S)$. Thus, for every $S \in \mathscr{S}$ and every $\overline{m} \in S$, there is an open neighborhood $\overline{U}$ of $\overline{m}$ in $\overline{M}$ and a smooth vector field W_T in $T = S \cap U$ such that for every $f \in C^\infty(\overline{U})$ we have $\overline{X}(f)|T = \partial_{W_T}(f|T)$. Because the preceding equality determines W_T uniquely in terms of $\overline{X}$, the vector fields W_T patch together to form a smooth vector field W_S on S. Moreover, every $f \in C^\infty(\overline{M})$ we have $\overline{X}(f)|S = \partial_{W_S}(f|S)$. Because the preceding equality holds for every $S \in \mathscr{S}$, we obtain a mapping $\widetilde{W} : \mathscr{S} \to \mathscr{X}^\infty(\overline{M},\mathscr{S}) : S \mapsto W_S$, which defines a smooth stratified vector field $\widetilde{W}$ on $\overline{M}$ such that $\overline{X} = \partial_{\widetilde{W}}$. Consequently, $\mathscr{X}(\overline{M},C^\infty(\overline{M})) \subseteq \partial \mathscr{X}^\infty(\overline{M},\mathscr{S})$. The inclusion $\partial \mathscr{X}^\infty(\overline{M},\mathscr{S}) \subseteq \mathscr{X}(\overline{M},C^\infty(\overline{M}))$ follows from ((4.3)) with $(M,C^\infty(M))$ replaced by $(\overline{M},C^\infty(\overline{M}))$. □

In view of ((4.4)) and ((4.5)), we call the set of smooth vector fields on $\overline{M}$ the space $\mathscr{X}(\overline{M},C^\infty(\overline{M}))$ of *smooth vector fields* on the G orbit space $\overline{M}$. These vector fields depend *only* on the differential structure $C^\infty(\overline{M})$ of $\overline{M}$; whereas the space of induced vector fields $\pi_*(\mathscr{X}^\infty(M)^G)$ depends on the manifold structure of M, and the space $\mathscr{X}^\infty(\overline{M},\mathscr{S})$ of stratified vector fields on $\overline{M}$ depends on the stratification $\mathscr{S}$ of $\overline{M}$ by orbit types.

Examples:

1. Let $M = \mathbf{R}$ and $G = \mathbf{Z}_2 = \{\pm 1\}$. Then $p : \mathbf{R} \to \mathbf{R}_{\geq 0} \subseteq \mathbf{R} : x \mapsto x^2$ is a $\mathbf{Z}_2$-invariant polynomial, which freely generates $C^\infty(\mathbf{R})^{\mathbf{Z}_2}$. In other words, every smooth even function is a smooth function of p. Therefore $p^* : C^\infty(\mathbf{R}_{\geq 0}) \to C^\infty(\mathbf{R})^{\mathbf{Z}_2}$ is an isomorphism,

which gives rise to a diffeomorphism of the differential space $(\mathbf{R}/\mathbf{Z}_2, C^\infty(\mathbf{R}/\mathbf{Z}_2))$ onto the differential space $(\mathbf{R}_{\ge 0}, C^\infty(\mathbf{R}_{\ge 0}))$. The derivation L_W on $C^\infty(\mathbf{R}_{\ge 0})$ is a smooth vector field on $C^\infty(\mathbf{R}_{\ge 0})$ if and only if the flow φ_t of W maps 0 into $\mathbf{R}_{\ge 0}$ for every t in an open neighborhood of 0 in $\mathbf{R}$. This holds if and only if $W(0) = 0$. Thus *not* every derivation of $C^\infty(\mathbf{R}/\mathbf{Z}_2)$ is a smooth vector field on $\mathbf{R}/\mathbf{Z}_2$.

2. Let $M = \mathbf{C}$ and $G = S^1 = \{z \in C \,\big|\, |z| = 1\}$. Suppose that S^1 acts on $\mathbf{C}$ by multiplication. Then $C^\infty(\mathbf{C})^{S^1}$ is generated freely by the real valued function $p : \mathbf{C} \to \mathbf{R}_{\ge 0} \subseteq \mathbf{R} : z \mapsto z\bar{z}$. Thus $p^* : C^\infty(\mathbf{R}_{\ge 0}) \to C^\infty(\mathbf{C}/S^1)$ is an isomorphism. So *not* every derivation of $C^\infty(\mathbf{C}/S^1)$ is a vector field on $\mathbf{C}/S^1$, because the inclusion map $i : \mathbf{R} \to \mathbf{C}$ induces a diffeomorphism of the differential space $(\mathbf{C}/S^1, C^\infty(\mathbf{C}/S^1))$ onto the differential space $(\mathbf{R}/\mathbf{Z}_2, C^\infty(\mathbf{R}/\mathbf{Z}_2))$.
□

5 Momentum mappings

In this section we define the concept of a momentum mapping of a Hamiltonian group action on a smooth symplectic manifold (M, ω).

5.1 General properties

Let Φ be an action of a Lie group G on a smooth manifold M. For every ξ in the Lie algebra $\mathfrak{g}$ of G define a vector field X^ξ on M by $X^\xi(m) = \frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} \Phi_m(\exp t\xi) = T_e\Phi_m \xi$ for every $m \in M$. The vector field X^ξ is the *infinitesimal generator* of Φ in the direction ξ.

Claim: The linear map $\mathfrak{g} \to \mathscr{X}(M) : \xi \mapsto X^\xi$ is an antihomomorphism of the Lie algebra $\mathfrak{g}$ to the Lie algebra $\mathscr{X}(M)$ of vector fields on M.

(5.1) **Proof**: Linearity is obvious. To verify that the mapping is an antihomomorphism of Lie algebras we compute

$$
\begin{aligned}
[X^\xi, X^\eta](m) &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \Phi^*_{\exp t\xi} X^\eta(m) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} T\Phi_{\exp -t\xi} X^\eta(\Phi_{\exp t\xi}(m)) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0} \Phi_{\exp -t\xi} \circ \Phi_{\exp s\eta} \circ \Phi_{\exp t\xi}(m) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0} \Phi_m(\exp -t\xi \, \exp s\eta \, \exp t\xi) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0} \Phi_m\big(\exp s(\mathrm{Ad}_{\exp -t\xi}\eta)\big) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} X^{\mathrm{Ad}_{\exp -t\xi}\eta}(m) = X^{-[\xi,\eta]}(m) = -X^{[\xi,\eta]}(m). \qquad \square
\end{aligned}
$$

Let (M, ω) be a symplectic manifold. The G-action Φ on (M, ω) is a *Hamiltonian G-action* if for every $\xi \in \mathfrak{g}$ the infinitesimal generator X^ξ is a Hamiltonian vector field on (M, ω). In other words, for every $\xi \in \mathfrak{g}$, there is a smooth function $J^\xi : M \to \mathbf{R}$ such that $X^\xi = X_{J^\xi}$.

▷ If G is connected, then Φ_g is a symplectic diffeomorphism for every $g \in G$.

(5.2) **Proof**: Let $\xi \in \mathfrak{g}$. Then $L_{X^\xi}\omega = X^\xi \mathbin{\lrcorner} \mathrm{d}\omega + \mathrm{d}(X^\xi \mathbin{\lrcorner} \omega) = \mathrm{d}(\mathrm{d}J^\xi) = 0$. Therefore

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi^*_{\exp t\xi}\omega = \Phi^*_{\exp t\xi}\big(\frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0}\Phi^*_{\exp s\xi}\omega\big) = \Phi^*_{\exp t\xi}(L_{X^\xi}\omega) = 0,$$

which gives $\Phi^*_{\exp t\xi}\omega = \omega$ for every $\xi \in \mathfrak{g}$. Since G is connected, it is generated by an open neighborhood of e. Therefore $\Phi^*_g\omega = \omega$ for every $g \in G$. □

To avoid limiting ourselves to actions of connected Lie groups, we supplement the definition of Hamiltonian G-action Φ by requiring that Φ_g be a symplectic diffeomorphism of (M,ω) for every $g \in G$. We define a mapping $J : M \to \mathfrak{g}^*$ by $J(m)\xi = J^\xi(m)$, where $\xi \in \mathfrak{g}$ and $m \in M$. This makes sense because $\xi \mapsto J^\xi(m)$ is a linear form on $\mathfrak{g}$ for every fixed $m \in M$. The mapping J is a *momentum map* of the Hamiltonian action Φ.

Claim: If (M,ω) is connected, then the momentum mapping is determined up to an additive constant $\mu_0 \in \mathfrak{g}^*$.

(5.3) **Proof**: Suppose that $\widetilde{J}$ is another momentum map for the Hamiltonian action Φ. Let $\mathscr{J} = J - \widetilde{J}$ and fix $m \in M$. For every $\xi \in \mathfrak{g}$ and every $v_m \in T_mM$ we have

$$\begin{aligned}(T_m\mathscr{J})(v_m)\xi &= \mathrm{d}\mathscr{J}^\xi(m)v_m = \mathrm{d}J^\xi(m)v_m - \mathrm{d}\widetilde{J}^\xi(m)v_m \\ &= \omega(m)\big(X_{J^\xi}(m) - X_{\widetilde{J}^\xi}(m), v_m\big) \\ &= \omega(m)\big(X^\xi(m) - X^\xi(m), v_m\big) = 0.\end{aligned}$$

Hence $T_m\mathscr{J}$ vanishes for every $m \in M$. Since M is connected, it follows that $\mathscr{J} = \mu_0$ for some fixed $\mu_0 \in \mathfrak{g}^*$. □

The momentum map $J : M \to \mathfrak{g}^*$ of the Hamiltonian G-action Φ on (M,ω) is *coadjoint equivariant* provided that for every $g \in G$ and every $m \in M$

$$J(\Phi_g(m)) = \mathrm{Ad}^t_{g^{-1}}J(m). \tag{16}$$

▷ If $J : M \to \mathfrak{g}^*$ is a coadjoint equivariant momentum mapping, then

$$J^{[\xi,\eta]}(m) = \{J^\xi, J^\eta\}(m) \tag{17}$$

for every ξ, $\eta \in \mathfrak{g}$ and every $m \in M$.

(5.4) **Proof**: For every $\zeta \in \mathfrak{g}$ infinitesimalizing (16) gives $\frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0} J(\Phi_m(\exp s\zeta)) = \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0}\mathrm{Ad}^t_{\exp -s\zeta}J(m)$, that is, $T_mJX^\zeta(m) = -\mathrm{ad}^t_\zeta J(m)$. Consequently for every ξ, $\eta \in \mathfrak{g}$ we get

$$\begin{aligned}-J^{[\xi,\eta]}(m) &= -J(m)(\mathrm{ad}_\xi\eta) = \big(-\mathrm{ad}^t_\xi J(m)\big)\eta \\ &= \big(T_mJX^\xi(m)\big)\eta = \mathrm{d}J^\eta(m)X^\xi(m), \quad \text{since } J(m)\zeta = J^\zeta(m) \\ &= \mathrm{d}J^\eta(m)X_{J^\xi}(m) = \{J^\eta, J^\xi\}(m).\end{aligned}$$

□

Suppose that the Hamiltonian G-action Φ preserves a Hamiltonian function H on M, that is, for every $g \in G$, we have $\Phi_g^* H = H$. Then Φ is a *symmetry* of the Hamiltonian system (M, ω, H). Symmetries of Hamiltonian systems give rise to conserved quantities.
▷ More precisely, for every $\xi \in \mathfrak{g}$ the function J^ξ is an *integral* of X_H, that is, $L_{X_H} J^\xi = 0$.

(5.5) **Proof**: The map Φ preserves the Hamiltonian H, that is, $\Phi^*_{\exp t\xi} H = H$ for every $\xi \in \mathfrak{g}$. Therefore

$$\begin{aligned} 0 &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \Phi^*_{\exp t\xi} H = L_{X^\xi} H, \quad \text{by definition of } X^\xi \\ &= L_{X_{J^\xi}} H, \quad \text{because } \Phi \text{ is a Hamiltonian action} \\ &= \{H, J^\xi\}, \quad \text{by definition of Poisson bracket } \{\,,\,\} \\ &= -\{J^\xi, H\} = -L_{X_H} J^\xi. \qquad \square \end{aligned}$$

We now give some examples of momentum mappings.

Example 1: Suppose that ψ is an action of a Lie group G on a configuration space M, which is a smooth manifold. Then ψ *lifts* to an action Ψ of G on the phase space T^*M. Ψ is defined by $\Psi_g(\alpha_m) = (T_m \psi_{g^{-1}})^t \alpha_m$ for every $\alpha_m \in T^*_m M$. The action Ψ covers the action ψ, that is, $\tau(\Psi_g(\alpha_m)) = \psi_g(\tau(\alpha_m))$, where $\tau : T^*M \rightarrow M : \alpha_m \rightarrow m$ is the bundle projection. To show that Ψ is a Hamiltonian action we need the following

Fact: For every $g \in G$, the mapping Ψ_g preserves the canonical 1-form θ on T^*M, see chapter VI §2.

(5.6) **Proof**: For $v_{\alpha_m} \in T_{\alpha_m}(T^*M)$ and $\alpha_m \in T^*_m M$,

$$\begin{aligned} \big((\Psi_g)^*\theta\big)(\alpha_m) v_{\alpha_m} &= \theta(\Psi_g \alpha_m) T\Psi_g \, v_{\alpha_m} \\ &= (\Psi_g \alpha_m) T\tau \circ T\Psi_g \, v_{\alpha_m}, \quad \text{by definition of } \theta \\ &= \alpha_m(T\psi_{g^{-1}} \circ T\psi_g \circ T\tau \, v_{\alpha_m}), \quad \text{by definition of the action } \Psi \\ &= \theta(\alpha_m) v_{\alpha_m}. \qquad \square \end{aligned}$$

From this follows the *momentum lemma*

Claim: For every $\xi \in \mathfrak{g}$, the infinitesimal generator X^ξ of the G-action Ψ in the direction ξ is the Hamiltonian vector field on (T^*M, Ω) corresponding to the Hamiltonian function $J^\xi : T^*M \rightarrow \mathbf{R} : \alpha_m \mapsto \alpha_m(X_\xi(m))$. Here $X_\xi(m) = \frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} \psi_{\exp t\xi}(m)$ is the infinitesimal generator of the G-action ψ on M and Ω is the canonical 2-form on T^*M.

(5.7) **Proof**: Since the action Ψ preserves the 1-form θ, $L_{X^\xi}\theta = 0$. But $L_{X^\xi}\theta = X^\xi \mathbin{\lrcorner} \mathrm{d}\theta + \mathrm{d}(X^\xi \mathbin{\lrcorner} \theta)$. Therefore $X^\xi \mathbin{\lrcorner} \Omega = -X^\xi \mathbin{\lrcorner} \mathrm{d}\theta = \mathrm{d}(X^\xi \mathbin{\lrcorner} \theta)$. Thus X^ξ is the Hamiltonian vector field on (T^*M, Ω) corresponding to the Hamiltonian function

$$J^\xi : T^*M \rightarrow \mathbf{R} : \alpha_m \mapsto (X^\xi \mathbin{\lrcorner} \theta)(\alpha_m) = \alpha_m(T_{\alpha_m}\tau X^\xi(\alpha_m)) = \alpha_m(X_\xi(m)).$$

The last equality above follows by differentiating $\tau(\Psi_{\exp t\xi}(\alpha_m)) = \psi_{\exp t\xi}(\tau(\alpha_m)) = \psi_{\exp t\xi}(m)$ and evaluating the resulting expression at $t = 0$. $\square$

Claim: The Hamiltonian G-action Ψ has a coadjoint equivariant momentum mapping

$$J : T^*M \to \mathfrak{g}^* : \alpha_m \mapsto (T\tau \circ T_e \Psi_{\alpha_m})^t \alpha_m. \tag{18}$$

(5.8) **Proof**: Since

$$J(\alpha_m)\xi = \alpha_m\big(T_{\alpha_m}\tau X^{\xi}(\alpha_m)\big) = \alpha_m(X_{\xi}(m)) = J^{\xi}(\alpha_m)$$

and $X^{\xi}(m) = X_{J^{\xi}}(m)$, the mapping J (18) is a momentum map for the G-action Ψ on T^*M. The map J is coadjoint equivariant, because

$$\begin{aligned} J(\Psi_g(\alpha_m))\xi &= \theta(\Psi_g(\alpha_m))X^{\xi}(\Psi_g(\alpha_m)) \\ &= \Psi_g(\alpha_m)\big(T\tau \circ T\Psi_g X^{\mathrm{Ad}_{g^{-1}}\xi}(\alpha_m)\big), \quad \text{see below} \\ &= \alpha_m\big(T\psi_{g^{-1}} \circ T\psi_g \circ T\tau X^{\mathrm{Ad}_{g^{-1}}\xi}(\alpha_m)\big) = J(\alpha_m)\,\mathrm{Ad}_{g^{-1}}\xi \\ &= \mathrm{Ad}^t_{g^{-1}}(J(\alpha_m))\xi. \end{aligned}$$

The second equality above follows from the definition of θ and the ensuing calculation

$$\begin{aligned} X^{\xi}(\Psi_g(\alpha_m)) &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \Psi_{\exp t\xi} \circ \Psi_g(\alpha_m) \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \Psi_g \circ \Psi_{g^{-1}(\exp t\xi)g}(\alpha_m) = T\Psi_g X^{\mathrm{Ad}_{g^{-1}}\xi}(\alpha_m). \end{aligned} \qquad \square$$

Example 2: Let (V,σ) be a real symplectic vector space. A linear mapping $A : V \to V$ is *symplectic* if it preserves σ, that is, if $A^*\sigma = \sigma$. Note that a linear symplectic map is invertible; for if $Av = 0$, then $0 = \sigma(Av, Aw) = \sigma(v,w)$ for every $w \in V$. Hence $v = 0$, since σ is nondegenerate. The set $\mathrm{Sp}(V,\sigma)$ of all real symplectic linear mappings on (V,σ), is a Lie group because it is a closed subgroup of the Lie group of all invertible linear mappings of V to itself. $\mathrm{Sp}(V,\sigma)$ is called the *real symplectic group*. The Lie algebra $\mathrm{sp}(V,\sigma)$ of $\mathrm{Sp}(V,\sigma)$ is the set of *infinitesimally symplectic* linear mappings $a : V \to V$, that is, $\sigma^{\sharp} \circ a + a^t \circ \sigma^{\sharp} = 0$.

Define a linear action Φ of $\mathrm{Sp}(V,\sigma)$ on V by $\Phi : \mathrm{Sp}(V,\sigma) \times V \to V : (A,v) \mapsto Av$. For $\xi \in \mathrm{sp}(V,\sigma)$ the infinitesimal generator X^{ξ} of Φ in the direction ξ is ξ, because for every $v \in V$ we have $X^{\xi}(v) = \frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} \Phi_{\exp t\xi}(v) = \xi(v)$. Define the mapping $J : V \to \mathrm{sp}(V,\sigma)^*$ by $J(v)\xi = J^{\xi}(v) = \frac{1}{2}\sigma(\xi(v),v)$ for every $\xi \in \mathrm{sp}(V,\sigma)$ and every $v \in V$.

Claim: J is a coadjoint equivariant momentum mapping for the linear action Φ.

(5.9) **Proof**: By definition J^{ξ} is a homogeneous quadratic function on V. Differentiating the definition of J gives $\mathrm{d}J^{\xi}(v)w = \sigma(\xi(v),w) = \sigma(X^{\xi}(v),w)$ for every $v,w \in V$. In other words, X^{ξ} is a Hamiltonian vector field on (V,σ) corresponding to the Hamiltonian J^{ξ}. Suppose that $A \in \mathrm{Sp}(V,\sigma)$. Then for every $\xi \in \mathrm{sp}(V,\sigma)$ and every $v \in V$ we have

$$\begin{aligned} J(Av)\xi = J^{\xi}(Av) &= \tfrac{1}{2}\sigma\big(\xi(Av),Av\big) = \tfrac{1}{2}\sigma\big((A^{-1}\xi A)(v),v\big) \\ &= J^{\mathrm{Ad}_{A^{-1}}\xi}(v) = J(v)\mathrm{Ad}_{A^{-1}}\xi = \big(\mathrm{Ad}^t_{A^{-1}}(J(v))\big)\xi. \end{aligned}$$

Hence J is coadjoint equivariant. $\square$

Example 3: Let G be a Lie group with Lie algebra $\mathfrak{g}$. The group G acts naturally on itself by left translation, namely, $L : G \times G \rightarrow G : (h,g) \mapsto L_h g = hg$. The action L induces an action Φ on the cotangent bundle T^*G given by $\Phi : G \times T^*G \rightarrow T^*G : (h, \alpha_g) \mapsto \alpha_{hg} = (T_g L_{h^{-1}})^t \alpha_g$. Φ preserves the canonical 1-form θ on T^*G and therefore the canonical
▷ symplectic form $\Omega = -\,\mathrm{d}\theta$. For $\xi \in \mathfrak{g}$, the vector field $X^\xi(\alpha_g) = \frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} \Phi_{\exp t\xi}(\alpha_g)$ on (T^*G, Ω) is Hamiltonian with Hamiltonian function $\mathscr{J}^\xi : T^*G \rightarrow \mathbf{R} : \alpha_g \mapsto (T_e R_g)^t(\alpha_g)\xi$.

(5.10) **Proof**: To see this, note that for $\xi \in \mathfrak{g}$ the infinitesimal generator of the action L in the direction ξ is $X_\xi(g) = \frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} L_{\exp t\xi}\, g = \frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} R_g \exp t\xi = T_e R_g \xi$. Here $R_g : G \rightarrow G : h \mapsto hg$ is right translation by g. Therefore, by the momentum lemma ((5.7)),

$$\mathscr{J}^\xi(\alpha_g) = \alpha_g(T_e R_g \xi) = (T_e R_g)^t(\alpha_g)\xi.$$

Thus the function $\mathscr{J} : T^*G \rightarrow \mathfrak{g}^* : \alpha_g \mapsto (T_e R_g)^t(\alpha_g)$ is the momentum mapping of the action Φ. The mapping $\mathscr{J}$ intertwines the action Φ on T^*G with the coadjoint action of G on $\mathfrak{g}^*$. In other words, $\mathscr{J}(\Phi_h(\alpha_g)) = \mathrm{Ad}^t_{h^{-1}}\, \mathscr{J}(\alpha_g)$. $\square$

Using the left trivialization $\mathscr{L} : G \times \mathfrak{g}^* \rightarrow T^*G : (g, \alpha) \mapsto (T_g L_{g^{-1}})^t \alpha = \alpha_g$, the G-action Φ on T^*G pulls back to the G-action $\varphi = \mathscr{L}^*\Phi$ on $G \times \mathfrak{g}^*$ defined by $\varphi_h(g, \alpha) = (hg, \alpha)$ for $h \in G$. Because φ_h preserves the 1-form $\vartheta = \mathscr{L}^*\theta$, the vector field $X^\xi(g, \alpha) = \frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} \varphi_{\exp t\xi}(g, \alpha) = (T_e R_g \xi, 0)$ is Hamiltonian with corresponding Hamiltonian function

$$\begin{aligned}\big(X^\xi \mathbin{\lrcorner} \vartheta\big)(g, \alpha) &= \vartheta(g, \alpha) X^\xi(g, \alpha) = \alpha_g(T_e R_g \xi) \\ &= \alpha(T_g L_{g^{-1}} T_e R_g \xi) = \alpha(\mathrm{Ad}_{g^{-1}}\, \xi) = (\mathrm{Ad}^t_{g^{-1}}\, \alpha)\xi.\end{aligned}$$

Therefore, the pull back of the momentum mapping $\mathscr{J}$ by left trivialization $\mathscr{L}$ is the momentum mapping $J : G \times \mathfrak{g}^* \rightarrow \mathfrak{g}^* : (g, \alpha) \mapsto \mathrm{Ad}^t_{g^{-1}}\, \alpha$ of the action φ. Clearly the mapping J intertwines the G-action φ on $G \times \mathfrak{g}^*$ with the coadjoint action of G on $\mathfrak{g}^*$. $\square$

Example 3$'$: Let H be a closed subgroup of a Lie group G. Then H is a Lie group. Let H act on G by left translation. The lift of this action to T^*G has a momentum mapping

$$J_H : T^*G \rightarrow \mathfrak{h}^* : \alpha_g \mapsto (T_e R_g)^t \alpha_g,$$

which is coadjoint equivariant. The lift of left translation by G on itself to T^*G has a coadjoint equivariant momentum mapping

$$J_G : T^*G \rightarrow \mathfrak{g}^* : \alpha_g \mapsto (T_e R_g)^t \alpha_g.$$

Because H is a Lie subgroup of G, we have an inclusion map $i : \mathfrak{h} \rightarrow \mathfrak{g}$ of Lie algebras. By duality this becomes the projection $\pi = i^t : \mathfrak{g}^* \rightarrow \mathfrak{h}^*$. Clearly, $J_H = \pi \circ J_G$. $\square$

Example 4: Let $\mathscr{O}_\mu$ be the orbit through $\mu \in \mathfrak{g}^*$ of the coadjoint action of G on the dual $\mathfrak{g}^*$ of its Lie algebra $\mathfrak{g}$, see chapter VI §2. Clearly $\Phi : G \times \mathscr{O}_\mu \rightarrow \mathscr{O}_\mu : (g, \nu) \mapsto \mathrm{Ad}^t_{g^{-1}} \nu$ is a G-action. For every $\xi \in \mathfrak{g}$, the vector field $X^\xi(\nu) = \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0} \mathrm{Ad}^t_{\exp -s\xi}\, \nu = -\mathrm{ad}^t_\xi\, \nu$ is the infinitesimal generator of the action Φ in the direction ξ. Recall that in chapter VI ((2.4)) we showed that $\mathscr{O}_\mu$ is a symplectic manifold with symplectic form $\omega_{\mathscr{O}_\mu}(\nu)\big(X^\xi(\nu),X^\eta(\nu)\big) = -\nu([\xi,\eta])$. We now show that Φ is a Hamiltonian action on $(\mathscr{O}_\mu,\omega_{\mathscr{O}_\mu})$. First we verify that Φ_g is a symplectic diffeomorphism. We calculate.

$$\begin{aligned}(\Phi_g^*\omega_{\mathscr{O}_\mu})(\nu)\big(X^\xi(\nu),X^\eta(\nu)\big) &= \omega_{\mathscr{O}_\mu}(\mathrm{Ad}^t_{g^{-1}}\nu)\big(\mathrm{Ad}^t_{g^{-1}}(\mathrm{ad}^t_\xi(\nu)),\mathrm{Ad}^t_{g^{-1}}(\mathrm{ad}^t_\eta(\nu))\big)\\ &= \omega_{\mathscr{O}_\mu}(\mathrm{Ad}^t_{g^{-1}}\nu)\big(\mathrm{ad}^t_{\mathrm{Ad}_g\xi}(\mathrm{Ad}^t_{g^{-1}}\nu),\mathrm{ad}^t_{\mathrm{Ad}_g\eta}(\mathrm{Ad}^t_{g^{-1}}\nu)\big)\\ &= -(\mathrm{Ad}^t_{g^{-1}}\nu)([\mathrm{Ad}_g\xi,\mathrm{Ad}_g\eta]) = -\nu([\xi,\eta])\\ &= \omega_{\mathscr{O}_\mu}(\nu)\big(X^\xi(\nu),X^\eta(\nu)\big).\end{aligned}$$

Next we show that X^ξ is a Hamiltonian vector field on $(\mathscr{O}_\mu,\omega_{\mathscr{O}_\mu})$. Observe that the definition of $\omega_{\mathscr{O}_\mu}$ may be rewritten as

$$\omega_{\mathscr{O}_\mu}(\nu)\big(X^\xi(\nu),X^\eta(\nu)\big) = (\mathrm{ad}^t_\eta\nu)\xi = -X^\eta(\nu)(\xi) = -\xi(X^\eta(\nu)), \tag{19}$$

where we have identified $\mathfrak{g}^{**}$ with $\mathfrak{g}$. Now consider the linear function $J^\xi : \mathfrak{g}^* \to \mathbf{R} : \alpha \mapsto -\alpha(\xi)$. Then $\mathrm{d}J^\xi(\alpha)\beta = -\beta(\xi) = -\xi(\beta)$, again identifying $\mathfrak{g}^{**}$ with $\mathfrak{g}$. Thus (19) may be written as $\omega_{\mathscr{O}_\mu}(\nu)\big(X^\xi(\nu),X^\eta(\nu)\big) = \mathrm{d}J^\xi(\nu)X^\eta(\nu)$. Since $T_\nu\mathscr{O}_\mu$ is spanned by the vectors $X^\eta(\nu)$ as η ranges over $\mathfrak{g}$, it follows that X^ξ is the Hamiltonian vector field X_{J^ξ}. Thus $J : \mathscr{O}_\mu \subseteq \mathfrak{g}^* \to \mathfrak{g}^* : \nu \mapsto -\nu$ is a momentum mapping for the coadjoint action Φ of G on $\mathfrak{g}^*$. The momentum mapping J is obviously coadjoint equivariant. □

We need the next result in the following section. Suppose that $\mu \in \mathfrak{g}^*$ lies in the image of a coadjoint equivariant momentum mapping $J : M \to \mathfrak{g}^*$ of a Hamiltonian action of a Lie group G on a symplectic manifold (M,ω).

Fact: For every $m \in M$ we have

$$T_m(G\cdot m) = (\ker T_mJ)^{\omega(m)} \tag{20a}$$

and for every $m \in J^{-1}(\mu)$

$$T_m(G_\mu\cdot m) = (\ker T_mJ)\cap(\ker T_mJ)^{\omega(m)}. \tag{20b}$$

(5.11) **Proof**: We prove (20a) as follows. From the definition of the momentum mapping it follows that

$$\omega(m)(X^\xi(m),v) = \mathrm{d}J^\xi(m)v = (T_mJ(v))\xi, \tag{21}$$

for every $\xi \in \mathfrak{g}$. If $v \in \ker T_mJ$, then from (21) we find that $\omega(m)(X^\xi(m),v) = 0$ for every $\xi \in \mathfrak{g}$. In other words, $v \in T_m(G\cdot m)^{\omega(m)}$. Conversely, if $v \in T_m(G\cdot m)^{\omega(m)}$, then $\omega(m)(X^\xi(m),v) = 0$ for every $\xi \in \mathfrak{g}$. Hence from (21) it follows that $(T_mJ(v))\xi = 0$ for every $\xi \in \mathfrak{g}$, that is, $v \in \ker T_mJ$. This establishes (20a).

To prove (20b) we begin by showing that

$$T_m(G_\mu\cdot m) = T_m(G\cdot m)\cap\ker T_mJ. \tag{22}$$

To verify the inclusion $\subseteq$ we argue as follows. Let $v_m \in T_m(G_\mu\cdot m)$. Then for some ξ in the Lie algebra $\mathfrak{g}_\mu$ of G_μ, $v_m = X^\xi(m)$. Differentiating the relation $J(\Phi_{\exp s\xi}(m)) =$

$\mathrm{Ad}^t_{\exp -s\xi}(J(m))$ with respect to s and then setting $s=0$ gives

$$T_mJ X^\xi(m) = -X^\xi_{\mathfrak{g}^*}(J(m)) = -X^\xi_{\mathfrak{g}^*}(\mu) = 0, \tag{23}$$

where $X^\xi_{\mathfrak{g}^*}(\mu) = -\frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0} \mathrm{Ad}^t_{\exp -s\xi}\mu$. The last equality in (23) follows because $\xi \in \mathfrak{g}_\mu$. Therefore $X^\xi(m) \in \ker T_mJ$, which proves $T_m(G_\mu \cdot m) \subseteq \ker T_mJ$. Since $G_\mu \subseteq G$, we have $T_m(G_\mu \cdot m) \subseteq T_m(G \cdot m)$. Therefore the inclusion $\subseteq$ in (22) holds. To prove the reverse inclusion $\supseteq$ suppose that $v_m \in T_m(G\cdot m) \cap \ker T_mJ$. Then there is a ξ in the Lie algebra of $\mathfrak{g}$ such that $v_m = X^\xi(m) = T_e\Phi_m\xi$. Because $v_m \in \ker T_mJ$, it follows from $T_mJX^\xi(m) = X^\xi_{\mathfrak{g}^*}(\mu)$ that $X^\xi_{\mathfrak{g}^*}(\mu) = 0$, that is, $\xi \in \mathfrak{g}_\mu$, the Lie algebra of G_μ. Consequently, $v_m \in T_m(G_\mu \cdot m)$. This proves (22). Substituting (20a) into (22) gives (20b). □

5.2 Normal form

In this subsection we find a local normal form for a coadjoint equivariant momentum mapping of a proper Hamiltonian action near a given point in its zero level set. This normal form is used to show that the zero level set of the momentum mapping is locally arcwise connected.

Let $\Phi : G \times M \to M$ be a proper Hamiltonian action of a Lie group G on a symplectic manifold (M,ω) with coadjoint equivariant momentum map $J : M \to \mathfrak{g}^*$. For $\mu \in \mathfrak{g}^*$ in the image of J let m be a point in the level set $J^{-1}(\mu)$. Let G_m be the isotropy group of m under the G-action Φ with Lie algebra $\mathfrak{g}_m$ and let G_μ be the isotropy group of μ under the coadjoint action of G on $\mathfrak{g}^*$ with Lie algebra $\mathfrak{g}_\mu$. Note that the linear $\omega(m)$-symplectic action

$$\widehat{\Phi} : G_m \times T_mM \to T_mM : (h, v_m) \to T_m\Phi_h v_m \tag{24}$$

has a G_m-coadjoint equivariant momentum mapping $\widehat{J} : T_mM \to \mathfrak{g}^*_m : v_m \mapsto \widehat{J}(v_m)$, where $\widehat{J}(v_m)\xi = \frac{1}{2}\omega(m)(X^\xi(v_m), v_m)$ for $\xi \in \mathfrak{g}_m$, see example 2 §5.1. Here X^ξ is the infinitesimal generator of the G_m-action $\widehat{\Phi}$ in the direction ξ.

Before we can state the local normal form of the momentum mapping J, we need the following decomposition of T_mM.

Fact:

$$T_mM = \mathfrak{h} \oplus \big(X \oplus T^*_m(G_\mu \cdot m)\big), \tag{25}$$

where X is a complement to $W = (\ker T_mJ) \cap (\ker T_mJ)^{\omega(m)}$ in $\ker T_mJ$ and $\mathfrak{h}$ is a subspace of $\mathfrak{g}$ which is isomorphic to $\mathfrak{g}/\mathfrak{g}_m$, that is, $\mathfrak{h}$ is isomorphic to $T_m(G\cdot m)$.

(5.12) **Proof**: Using the Witt decomposition ((1.4)) of chapter VI starting with the subspace $\ker T_mJ$ of the symplectic vector space $(T_mM, \omega(m))$, we obtain the decomposition $T_mM = X \oplus Y \oplus Z$, where X, Y and Z are the $\omega(m)$-symplectic subspaces of T_mM defined by $\ker T_mJ = X \oplus W$, $(\ker T_mJ)^{\omega(m)} = Y \oplus W$, and $Z = (X \oplus Y)^{\omega(m)}$. From the Witt decomposition it follows that W is a Lagrangian subspace of Z. Therefore Z is isomorphic to $W \oplus W^*$. Thus we may rewrite $T_mM = X \oplus Y \oplus Z$ as

$$T_mM = (Y \oplus W) \oplus (X \oplus W^*) = (\ker T_mJ)^{\omega(m)} \oplus (X \oplus W^*). \tag{26}$$

To finish the proof of (25) we only need note that $T_m(G\cdot m) = (\ker T_mJ)^{\omega(m)}$ (20a) and that $T_m(G_\mu \cdot m) = (\ker T_mJ) \cap (\ker T_mJ)^{\omega(m)} = W$ (20b). $\square$

Corollary: When $\mu = 0$ the decomposition (25) reads

$$T_mM = \mathfrak{h} \oplus X \oplus W^*, \tag{27}$$

where $\mathfrak{h} = W = T_m(G\cdot m)$.

(5.13) **Proof**: When $\mu = 0$ the isotropy group G_μ equals G. Therefore from (20a) and (20b) we obtain $(\ker T_mJ)^{\omega(m)} = T_m(G\cdot m) = T_m(G_\mu \cdot m) = (\ker T_mJ) \cap (\ker T_mJ)^{\omega(m)}$. $\square$

Corollary: The spaces X, Y, W and W^* in the Witt decomposition

$$T_mM = X \oplus Y \oplus (W \oplus W^*) \tag{28}$$

can be choosen to be G_m-invariant.

(5.14) **Proof**: Since $G\cdot m$ is invariant under the Hamiltonian G_m-action $\Phi|(G_m \times M)$ on M, it follows that $\ker T_mJ = T_m(G\cdot m)^{\omega(m)}$ is invariant under the linear $\omega(m)$-symplectic action $\widehat{\Phi}$. Thus $(\ker T_mJ)^{\omega(m)}$ and $W = \ker T_mJ \cap (\ker T_mJ)^{\omega(m)}$ are also invariant under $\widehat{\Phi}$. Let γ be an inner product on T_mM. Then averaging over G_m, which is compact because the G-action is proper, we may assume that γ is $\widehat{\Phi}$-invariant. Let X, Y be the orthogonal complement of W in $\ker T_mJ$ and $(\ker T_mJ)^{\omega(m)}$, respectively. Then X and Y are $\widehat{\Phi}$-invariant $\omega(m)$-symplectic subspaces of $(T_mM, \omega(m))$. Therefore Z, which is the orthogonal complement of $X \oplus Y$ in T_mM, is a $\widehat{\Phi}$-invariant $\omega(m)$-symplectic subspace. Hence we have obtained the $\widehat{\Phi}$-invariant Witt decomposition $T_mM = X \oplus Y \oplus Z$. Since W is a $\widehat{\Phi}$-invariant Lagrangian subspace of Z, its orthogonal complement $W^\perp$ in Z is $\widehat{\Phi}$-invariant. Since $W^\perp$ is isotropic and hence Lagrangian, it is isomorphic to W^*. This proves (28). $\square$

We now are in position to state the local normal form theorem.

Claim: Let $m \in J^{-1}(0)$. Using the decomposition (27) we write the tangent space T_mM to M at m as the sum $\mathfrak{h} \oplus X \oplus W^*$. Let (η, x, α) be coordinates on T_mM with respect to this decomposition. Then there is a local diffeomorphism $\vartheta : T_mM \to M$ with $\vartheta(0) = m$ and $T_0\vartheta = \mathrm{id}_{T_mM}$ such that for every (η, x, α) sufficiently close to 0, we have

$$\vartheta^* J(\eta, x, \alpha) = \mathrm{Ad}^t_{\exp -\eta}(\widehat{J}(x) + \alpha). \tag{29}$$

Here $\widehat{J}$ is the momentum mapping for the linear G_m-action $\widehat{\Phi}$ (24) on $(T_mM, \omega(m))$.

(5.15) **Proof**:

Step 1.

Since the G-action Φ is proper, the isotropy group G_m is compact. Let γ be a G_m-invariant Riemannian metric on M. Then the *exponential map* $\mathrm{Exp} : T_mM \to M$ associated to the metric γ is a local diffeomorphism with $\mathrm{Exp}(0) = m$ such that $T_0\,\mathrm{Exp} = \mathrm{id}_{T_mM}$. Because γ is G_m-invariant, the map Exp intertwines the linear G_m-action $\widehat{\Phi}$ on T_mM with the G_m-action $\Phi|(G_m \times M)$ on M. Pulling back the symplectic form ω on M by the exponential map gives a symplectic form $\widehat{\sigma}$ on a neighborhood of 0 in T_mM with $\widehat{\sigma}(0) = \omega(m)$. Since the map $\Phi_h : M \rightarrow M$ is a symplectic diffeomorphism for every $h \in G_m$, it follows that $\widehat{\Phi}_h : T_mM \rightarrow T_mM$ is a linear symplectic mapping of $(T_mM, \widehat{\sigma})$. The equivariant Darboux theorem, see exercise 13, shows that there is a local diffeomorphism $\psi : T_mM \rightarrow T_mM$ with $\psi(0) = 0$, which pulls back the symplectic form $\widehat{\sigma}$ to a *constant* symplectic form $\omega(m)$, commutes with the linear G_m-action $\widehat{\Phi}$, and has $T_0\,\psi = \mathrm{id}_{T_mM}$. Thus the local diffeomorphism $\varphi = \mathrm{Exp}\circ\psi : T_mM \rightarrow M$ symplectically identifies a neighborhood of 0 in T_mM with a neighborhood of m in M so that the ω-symplectic G_m-action $\Phi|(G_m \times M)$ becomes the linear $\omega(m)$-symplectic G_m-action $\widehat{\Phi}$ on T_mM.

Step 2.
Using the local diffeomorphism $\varphi : T_mM \rightarrow M$ constructed in step 1, pull back the G-coadjoint equivariant momentum map $J : M \rightarrow \mathfrak{g}^*$ with $J(m) = 0$ to obtain a locally defined G_m-equivariant map $\mathscr{J} = \varphi^*J : T_mM \rightarrow \mathfrak{g}^*$ with $\mathscr{J}(0) = 0$. Split $\mathscr{J}$ into the sum of two maps as follows. Since the isotropy group G_m is closed, it is a Lie subgroup of G. Thus the Lie algebra $\mathfrak{g}_m$ of G_m is a subalgebra of $\mathfrak{g}$. Hence we have an inclusion map $i' : \mathfrak{g}_m \rightarrow \mathfrak{g}$. By duality we obtain a projection map $\pi' = (i')^t : \mathfrak{g}^* \rightarrow \mathfrak{g}_m^*$. Since G_m is compact, there is a G_m-coadjoint invariant inner product on $\mathfrak{g}^*$. Let $\mathfrak{h}^* = (\mathfrak{g}_m^*)^{\perp}$. Then $\mathfrak{g}^* = \mathfrak{g}_m^* \oplus \mathfrak{h}^*$ and there is a projection $\pi'' : \mathfrak{g}^* \rightarrow \mathfrak{h}^*$ such that $\pi' + \pi'' = \mathrm{id}_{\mathfrak{g}^*}$. Write $\mathscr{J} = \pi'\circ\mathscr{J} + \pi''\circ\mathscr{J} = \mathscr{J}' + \mathscr{J}''$. Then $\mathscr{J}' : T_mM \rightarrow \mathfrak{g}_m^*$ and $\mathscr{J}'' : T_mM \rightarrow \mathfrak{h}^*$ are locally defined G_m-coadjoint equivariant maps with $\mathscr{J}'(0) = \pi'(0) = 0$ and $\mathscr{J}''(0) = \pi''(0) = 0$.

Step 3.
We now analyze the mapping $\mathscr{J}'$. Since $\mathscr{J}'$ is G_m-coadjoint equivariant and $\mathscr{J}'(0) = 0$, it follows that $\mathscr{J}'$ is the canonical momentum map $\widehat{J}$ of the linear symplectic action $\widehat{\Phi}$ on $(T_mM, \omega(m))$. Note that the mapping $\rho : \mathfrak{g}_m \rightarrow \mathrm{sp}(T_mM, \omega(m)) : \xi \rightarrow X^{\xi}$ is an antihomomorphism of Lie algebras. We now study ρ more closely. Because the Witt decomposition $T_mM = X \oplus (W \oplus W^*)$ ((5.13)) has summands which are invariant under the linear G_m-action $\widehat{\Phi}$ and because the linear Hamiltonian vector field X^{ξ} is the infinitesimalization of this action in the direction ξ, it follows that each summand of the Witt decomposition is *invariant* under X^{ξ} for every $\xi \in \mathfrak{g}_m$. Since W and W^* are $\widehat{\Phi}$-invariant Lagrangian subspaces of Z, W and W^* are invariant Lagrangian subspaces of X^{ξ} for every $\xi \in \mathfrak{g}_m$.

▷ For every $\alpha \in W^*$, we have $\widehat{J}(\alpha) = 0$.

(5.16) **Proof**: Since W^* is an X^{ξ}-invariant subspace for every $\xi \in \mathfrak{g}_m$, it follows that $X^{\xi}(\alpha) \in W^*$ for every $\xi \in \mathfrak{g}_m$. Now $\omega(m)(X^{\xi}(\alpha), \alpha) = 0$, because W^* is Lagrangian. Hence $\widehat{J}(\alpha)\xi = 0$ for every $\xi \in \mathfrak{g}_m$. □

▷ Let $\alpha \in W^*$ and $x \in X$, then $\widehat{J}(\alpha + x) = \widehat{J}(x)$.

(5.17) **Proof**: Let $\xi \in \mathfrak{g}_m$. We compute:

$$\begin{aligned} 2\widehat{J}(\alpha + x)\xi &= \omega(m)(X^{\xi}(\alpha), \alpha) + \omega(m)(X^{\xi}(\alpha), x) + \omega(m)(X^{\xi}(x), \alpha) + \omega(m)(X^{\xi}(x), x) \\ &= \omega(m)(X^{\xi}(x), x), \quad \text{since } X^{\xi}(\alpha) \in W^*,\ X^{\xi}(x) \in X,\ W^* \subseteq X^{\omega(m)}, \text{ and} \\ &\qquad\qquad\qquad\qquad\quad W \text{ is Lagrangian} \\ &= 2\widehat{J}(x)\xi. \end{aligned}$$

□

Step 4.

▷ The mapping $\mathscr{J}'' : T_mM \to \mathfrak{h}^*$ with $\mathscr{J}''(0) = 0$ is a local submersion at 0.

(5.18) **Proof**: It suffices to show that $T_0(\mathscr{J}''|W^*)$ is injective, because $\mathfrak{h}^* = W^*$. Suppose that there is an $\alpha \in W^*$ such that $0 = T_0\mathscr{J}''(\alpha) = \pi''(T_0\mathscr{J}(\alpha)) = \pi''(T_mJ(\alpha))$. In other words, $T_mJ(\alpha) \in \mathfrak{g}_m^*$. Suppose that $T_mJ(\alpha)$ is nonzero, then there is a $\zeta \in \mathfrak{g}_m$ such that $1 = T_mJ(\alpha)\zeta = \omega(m)(X^\zeta(\alpha), \alpha)$. But $\omega(m)(X^\zeta(\alpha), \alpha) = 0$, since W^* is a $\mathfrak{g}_m$-invariant Lagrangian subspace of $(T_mM, \omega(m))$. This is a contradiction. Therefore $T_mJ(\alpha) = 0$, that is, $\alpha \in \ker T_mJ = X \oplus W$. From the decomposition (27) it follows that $(X \oplus W) \cap W^* = \{0\}$. Therefore $\alpha = 0$. Hence $T_0(\mathscr{J}''|W^*)$ is injective. □

Let (x, w, α) be coordinates on $T_mM = X \oplus W \oplus W^*$. From the above assertion and the implicit function theorem it follows that there is a local diffeomorphism $\theta : T_mM \to T_mM$ with $\theta(0) = 0$ such that $\mathscr{J}''\circ\theta$ is the projection $T_mM \to \mathfrak{h}^* = W^* : (x, w, \alpha) \to \alpha$ and $\theta|(X \oplus W) = id_{X\oplus W}$. In other words, for every $(x, w, \alpha) \in X \oplus W \oplus W^*$ sufficiently close to 0 we have $\mathscr{J}''\big(\theta(x, w, \alpha)\big) = \alpha$ and $\theta(x, w, \alpha) = \big(x, w, \theta_3(x, w, \alpha)\big)$. Therefore

$$\begin{aligned}\theta^*\mathscr{J}(x, 0, \alpha) &= \mathscr{J}'(\theta(x, 0, \alpha)) + \mathscr{J}''(\theta(x, 0, \alpha)) = \mathscr{J}'(x, 0, \theta_3(x, 0, \alpha)) + \alpha \\ &= \widehat{J}(x) + \alpha, \quad \text{using ((5.16)).}\end{aligned}$$

Since $X \oplus W^*$ is transverse to the G-orbit $G \cdot m$ at m and $\mathfrak{h}$ is $T_m(G \cdot m)$, we can use the map φ constructed in step 1 to define

$$\vartheta : (X \oplus W^*) \times \mathfrak{h} \to M : \big((x, \alpha), \eta\big) \mapsto \Phi_{\exp \eta}\big(\varphi(\theta(x, 0, \alpha))\big).$$

Note that $\vartheta(0, 0, 0) = m$. Moreover, ϑ is a local diffeomorphism because $T_{(0,0,0)}\vartheta = \mathrm{id}_{T_mM}$. Thus for every u in M near m there is a unique (x, α, η) in $(X \oplus W) \times \mathfrak{h}$ such that $u = \vartheta(x, \alpha, \eta)$. Hence

$$J(u) = J\big(\exp \eta \cdot \varphi(\theta(x, \alpha))\big) = \mathrm{Ad}^t_{\exp -\eta}\big(\theta^*\mathscr{J}(x, 0, \alpha)\big) = \mathrm{Ad}^t_{\exp -\eta}(\widehat{J}(x) + \alpha),$$

which is the desired normal form. □

Corollary: For every $\mu \in \mathfrak{g}^*$ the level set $J^{-1}(\mu)$ is locally arcwise connected.

Using the following device, called the *shifting trick*, we reduce the proof of the corollary to the case of the 0-level of a coadjoint equivariant momentum mapping of a proper G-action on $M \times \mathscr{O}_\mu$. Consider the manifold $M \times \mathscr{O}_\mu$ with symplectic form $\Omega = \pi_1^*\omega - \pi_2^*\omega_{\mathscr{O}_\mu}$. Here ω is the symplectic form on M and $\omega_{\mathscr{O}_\mu}$ is the symplectic form on the G coadjoint orbit $\mathscr{O}_\mu$ given in example 3 of chapter VI §2. The map π_i is the projection on the i^{th} factor of $M \times \mathscr{O}_\mu$. Define a G-action on $M \times \mathscr{O}_\mu$ by $\big(g, (m, \nu)\big) \mapsto (\Phi_g(m), \mathrm{Ad}^t_{g^{-1}}\nu)$. The G-action is proper and has a coadjoint equivariant momentum mapping $J_{M\times\mathscr{O}_\mu}$: $M \times \mathscr{O}_\mu \to \mathfrak{g}^*$, where $J_{M\times\mathscr{O}_\mu}(m, \nu)\xi = J^\xi(m) - \nu(\xi)$ for every $\xi \in \mathfrak{g}$.

Fact: Since $C^\infty(J^{-1}_{M\times\mathscr{O}_\mu}(0)/G) = C_i^\infty(J^{-1}_{M\times\mathscr{O}_\mu}(0))^G$ and $C^\infty(J^{-1}(\mathscr{O}_\mu))/G) = C_i^\infty(J^{-1}(\mathscr{O}_\mu))^G$, we find that $\big(J^{-1}_{M\times\mathscr{O}_\mu}(0)/G, C^\infty(J^{-1}_{M\times\mathscr{O}_\mu}(0)/G)\big)$ and $\big(J^{-1}(\mathscr{O}_\mu)/G,\, C^\infty(J^{-1}(\mathscr{O}_\mu)/G)\big)$ are diffeomorphic differential spaces.

(5.19) **Proof**: We determine the diffeomorphism between the differential spaces as follows. Restricting the domain of the smooth mapping $F_1 : M \times \mathscr{O}_\mu \to M : (m,\nu) \mapsto m$ to the closed subset $J^{-1}_{M\times\mathscr{O}_\mu}$ of $M \times \mathscr{O}_\mu(0)$ gives a smooth mapping $F_2 : J^{-1}_{M\times\mathscr{O}_\mu}(0) \to M$. Since $J_{M\times\mathscr{O}_\mu}(m,\nu) = 0$ is equivalent to $J(m) = \nu$, it follows that $\mathrm{im}\, F_2 = J^{-1}(\mathscr{O}_\mu)$. Cutting the map F_2 down to its image, we obtain a map $F : J^{-1}_{M\times\mathscr{O}_\mu}(0) \to J^{-1}(\mathscr{O}_\mu)$. We now show that F is smooth. Suppose that $f \in C^\infty_i(J^{-1}(\mathscr{O}_\mu))$. For each $p \in J^{-1}(\mathscr{O}_\mu)$ there is an open neighborhood U of p in $M \times \mathscr{O}_\mu$ and an $h \in C^\infty(M \times \mathscr{O}_\mu)$ such that $f|\big(J^{-1}(\mathscr{O}_\mu \cap U)\big) = h|(\mathscr{O}_\mu \cap U)$. Moreover, $V = F^{-1}(U)$ is an open subset of $M \times \mathscr{O}_\mu$. For every $q \in V$ we have $F(q) \in J^{-1}(\mathscr{O}_\mu) \cap U$ since the image of F is $J^{-1}(\mathscr{O}_\mu)$. Also $f \circ F(q) = h(F(q)) = (h \circ F)(q)$. So $(f \circ F)|V = (h \circ F)|V$. Since $h \circ F \in C^\infty(M \times \mathscr{O}_\mu)$, it follows that $f \circ F \in C^\infty_i(J^{-1}_{M\times\mathscr{O}_\mu}(0))$. Thus F is smooth. We now construct a smooth inverse to the mapping F. Consider the smooth map $H_1 : M \to M \times \mathfrak{g}^* : m \mapsto (m, J(m))$. Restricting the domain of H_1 to $J^{-1}(\mathscr{O}_\mu)$ gives a smooth map $H_2 : \big(J^{-1}(\mathscr{O}_\mu), C^\infty_i(J^{-1}(\mathscr{O}_\mu))\big) \to M \times \mathfrak{g}^*$, whose image is $\big(J^{-1}_{M\times\mathscr{O}_\mu}(0), C^\infty_i(J^{-1}_{M\times\mathscr{O}_\mu}(0))\big)$. Cutting H_2 down to its image gives a smooth map $H : \big(J^{-1}(\mathscr{O}_\mu), C^\infty_i(J^{-1}(\mathscr{O}_\mu))\big) \to \big(J^{-1}_{M\times\mathscr{O}_\mu}(0),\, C^\infty_i(J^{-1}_{M\times\mathscr{O}_\mu}(0))\big)$. For each $m \in J^{-1}(\mathscr{O}_\mu)$ we have $F(H(m)) = F(m, J(m)) = m$; while for each $(m, J(m)) \in J^{-1}_{M\times\mathscr{O}_\mu}(0)$ we have $H(F(m,J(m))) = H(m) = (m, J(m))$. Therefore $H = F^{-1}$, which implies that the differential spaces $\big(J^{-1}(\mathscr{O}_\mu), C^\infty_i(J^{-1}(\mathscr{O}_\mu))\big)$ and $\big(J^{-1}_{M\times\mathscr{O}_\mu}(0),\, C^\infty_i(J^{-1}_{M\times\mathscr{O}_\mu}(0))\big)$ are diffeomorphic. The maps F_1, F_2, F and H_1, H_2, H intertwine the actions of G. Hence they pass to smooth maps on the corresponding G-orbit spaces, namely, $\widetilde{F}_1$, $\widetilde{F}_2$, $\widetilde{F}$ and $\widetilde{H}_1$, $\widetilde{H}_2$, $\widetilde{H}$. Since $\widetilde{H} = \widetilde{F}^{-1}$, it follows that

$$\widetilde{F} : \big(J^{-1}_{M\times\mathscr{O}_\mu}/G, C^\infty_i(J^{-1}_{M\times\mathscr{O}_\mu}(0))^G\big) \to \big(J^{-1}(\mathscr{O}_\mu)/G, C^\infty_i(J^{-1}(\mathscr{O}_\mu))^G\big)$$

is a diffeomorphism of differential spaces. □

(5.20) **Proof of corollary**: First we show that $J^{-1}_{M\times\mathscr{O}_\mu}(0)$ and $J^{-1}(\mu)$ are locally diffeomorphic. Let $\mathscr{U}$ be a neighborhood of $\mu \in \mathscr{O}_\mu$. Suppose that $\sigma : \mathscr{U} \to G$ is a section of the bundle $G \to G/G_\mu$ such that $\sigma(\mu) = e$ and $\mathrm{Ad}^t_{\sigma(\nu)^{-1}}\mu = \nu$, see §2 example 1. The map $\varphi : M \times \mathscr{U} \to M \times \mathscr{U} : (m,\nu) \mapsto \big(\sigma(\nu)\cdot m, \nu\big)$ is a local diffeomorphism. For every $\xi \in \mathfrak{g}^*$ we have

$$\begin{aligned} J_{M\times\mathscr{O}_\mu}(\varphi(m,\nu))\xi &= J_{M\times\mathscr{O}_\mu}(\sigma(\nu)\cdot m, \nu)\xi \\ &= (J(\sigma(\nu)\cdot m) - \nu)(\xi) = \big(\mathrm{Ad}^t_{\sigma(\nu)^{-1}}J(m) - \nu\big)\xi. \end{aligned}$$

Thus $J_{M\times\mathscr{O}_\mu}(\varphi(m,\nu)) = 0$ if and only if $J(m) = \mathrm{Ad}^t_{\sigma(\nu)^{-1}}\nu = \mu$ and $\nu \in \mathscr{U}$. In other words $(J_{M\times\mathscr{O}_\mu})^{-1}(0)$ and $J^{-1}(\mu) \times \mathscr{O}_\mu$ are locally diffeomorphic. Because $\mathscr{O}_\mu$ is locally arcwise connected, it follows that $(J_{M\times\mathscr{O}_\mu})^{-1}(0)$ is locally arcwise connected if and only if $J^{-1}(\mu)$ is. Thus it suffices to prove the corollary when $\mu = 0$. Applying the normal form ((5.15)) to the value 0 of the coadjoint equivariant momentum mapping J, we see that $u = (x, \alpha, \eta) \in J^{-1}(0)$ if and only if $0 = \mathrm{Ad}^t_{\exp -\eta}\big((\widehat{J}|X)(x) + \alpha\big)$, that is, if and only if $0 = (\widehat{J}|X)(x)$ and $\alpha = 0$. But $\widehat{J}$ is the canonical quadratic momentum map of a linear symplectic G_m-action. Hence $(\widehat{J}|X)^{-1}(0)$ is a cone in X, which is locally arcwise connected. Therefore $J^{-1}(0)$ is locally arcwise connected. □

6 Regular reduction

Reduction is the basic technique in symplectic geometry for removing symmetry from a Hamiltonian system. In the regular case the *reduction theorem* reads

Theorem (regular reduction): Let Φ be a *free* proper action of a Lie group G on the symplectic manifold (M,ω), which has a coadjoint equivariant momentum mapping $J : M \to \mathfrak{g}^*$. Suppose that $\mu \in \mathfrak{g}^*$ lies in the image of J. Then the *reduced space* $M_\mu = J^{-1}(\mu)/G_\mu$ is a smooth symplectic manifold with symplectic form ω_μ defined by $\pi_\mu^*\omega_\mu = i^*\omega$. Here $\pi_\mu : J^{-1}(\mu) \to M_\mu$ is the orbit map of the G_μ-action $\Phi|(G_\mu \times J^{-1}(\mu))$ on M, which is called the *reduction map*, and $i : J^{-1}(\mu) \to M$ is the inclusion map.

6.1 The standard approach

Here we give the standard approach to proving the regular reduction theorem.

(6.1) **Proof**: First we show that μ is a regular value of J. Since G acts freely and properly on M and the isotropy group G_μ is a closed Lie subgroup of of G, which leaves the closed level set $J^{-1}(\mu)$ invariant, it follows that G_μ acts freely and properly on $J^{-1}(\mu)$. Thus $\dim \mathfrak{g}_\mu = 0$. Let $m \in J^{-1}(\mu)$. Since $T_m(G_\mu \cdot m) = \ker T_mJ \cap \big(\ker T_mJ\big)^{\omega(m)} = W_m$ by ((5.11)), we obtain $0 \le \dim W_m = \dim \mathfrak{g}_\mu - \dim \mathfrak{g}_m$. Thus $\dim \mathfrak{g}_m = 0$, since $\dim \mathfrak{g}_\mu = 0$. Hence $\dim W_m = 0$, that is, $W_m = \{0\}$. So $\dim T_mM = \dim\ker T_mJ + \dim\big(\ker T_mJ\big)^{\omega(m)}$. But using ((5.11)) we get $\big(\ker T_mJ\big)^{\omega(m)} = T_m(G\cdot m) = \dim \mathfrak{g} - \dim \mathfrak{g}_m = \dim \mathfrak{g}$. Consequently, $\dim\ker T_mJ + \dim \mathfrak{g} = \dim T_mM = \dim\ker T_mJ + \dim\operatorname{im} T_mJ$, which implies $\dim\operatorname{im} T_mJ = \dim \mathfrak{g} = \dim \mathfrak{g}^*$. In other words, μ is a regular value of J. Hence $J^{-1}(\mu)$ is a smooth submanifold of M, which may not be connected.

The reduced space M_μ is a smooth manifold, because G_μ acts freely and properly on the smooth manifold $J^{-1}(\mu)$. To finish the proof we need only construct the symplectic form ω_μ on M_μ. Since μ is a regular value of J, we have $T_mJ^{-1}(\mu) = \ker T_mJ$. Therefore

$$W_m = T_mJ^{-1}(\mu) \cap (T_mJ^{-1}(\mu))^{\omega(m)} = \ker\big(\omega(m)|T_mJ^{-1}(\mu)\big)$$

is an $\omega(m)$-isotropic subspace of the symplectic vector space $(T_mM, \omega(m))$. Let V_m be a complement to W_m in $T_mJ^{-1}(\mu)$. From ((1.3d)) of chapter VI, it follows that V_m is an $\omega(m)$-symplectic subspace of $(T_mM, \omega(m))$. From ((5.11)) we get $W_m = T_m(G_\mu \cdot m)$. Thus V_m is isomorphic to $T_mJ^{-1}(\mu)/W_m = T_{\pi_\mu(m)}M_\mu$. Since $\ker T_m\pi_\mu = W_m$, the symplectic form $\omega(m)|V_m$ pushes down under $T_m\pi_\mu$ to a nondegenerate 2-form $\omega_\mu(\pi_\mu(m))$ on $T_{\pi_\mu(m)}M_\mu$. In other words, for every $m \in J^{-1}(\mu)$ we have $(\pi_\mu^*\omega_\mu)(m) = i^*\omega(m)$. Now

$$\pi_\mu^*(\mathrm{d}\omega_\mu) = \mathrm{d}(\pi_\mu^*\omega_\mu) = \mathrm{d}(i^*\omega) = i^*(\mathrm{d}\omega) = 0,$$

because ω is closed. Hence $\mathrm{d}\omega_\mu = 0$, since π_μ is a submersion. Therefore, ω_μ is a symplectic form on M_μ. □

Example 1: Consider the Hamiltonian G-action

$$\Phi : G \times T^*G \to T^*G : (h, \alpha_g) \mapsto \alpha_{hg} = (T_eL_{h^{-1}})^t\alpha_g$$

on the cotangent bundle T^*G of G with its canonical symplectic form Ω. The action Φ is the lift to T^*G of the action $L : G \times G \to G : (h,g) \mapsto L_h g = hg$ of left multiplication on G. From example 3 of §5.1 we see that Φ has a coadjoint equivariant momentum mapping $\mathscr{J} : T^*G \to \mathfrak{g}^* : \alpha_g \mapsto (T_eR_g)^t(\alpha_g)$. Pulling back by the left trivialization

$$\mathscr{L} : G \times \mathfrak{g}^* \to T^*G : (g,\alpha) \mapsto (T_eL_{g^{-1}})^t\alpha = \alpha_g, \tag{30}$$

which is a symplectic diffeomorphism of $(G \times \mathfrak{g}^*, \omega = \mathscr{L}^*\Omega)$ with (T^*G, Ω), the action Φ becomes the G-action

$$\varphi : G \times (G \times \mathfrak{g}^*) \to G \times \mathfrak{g}^* : \big(h,(g,\alpha)\big) \mapsto (hg,\alpha). \tag{31}$$

The action φ is Hamiltonian with momentum mapping $J = \mathscr{L}^* \mathscr{J} : G \times \mathfrak{g}^* \to \mathfrak{g}^* : (g,\alpha) \mapsto \mathrm{Ad}^t_{g^{-1}}\alpha$. We now apply the regular reduction theorem to the action φ. First observe that the action φ (31) is free and proper. Next note that for every $\mu \in \mathfrak{g}^*$ the μ-level set of the momentum mapping J is the smooth manifold $\{(g, \mathrm{Ad}^t_g \mu) \in G \times \mathfrak{g}^* \,\big|\, g \in G\}$. Indeed, $J^{-1}(\mu)$ is diffeomorphic to G, because it is the graph of the smooth mapping $g \mapsto \mathrm{Ad}^t_g \mu$. Now $G_\mu = \{h \in G \,\big|\, \mathrm{Ad}^t_{h^{-1}}\mu = \mu\}$ is the isotropy group of the coadjoint action of G on $\mathfrak{g}^*$ at μ. Note that $J^{-1}(\mu)$ is G_μ-invariant by the coadjoint equivariance of J. Since the G_μ-action $\varphi|(G_\mu \times J^{-1}(\mu))$ on $J^{-1}(\mu)$ is free and proper, the orbit space $J^{-1}(\mu)/G_\mu$ is a smooth manifold with G_μ-orbit map π_μ. G_μ is a closed subgroup of G, which implies that it is a smooth submanifold of G. Moreover, the orbit space G/G_μ is a smooth manifold. Because the diffeomorphism $\chi : G \to J^{-1}(\mu) : g \mapsto (g, \mathrm{Ad}^t_g \mu)$ intertwines the G_μ-action on G with the G_μ-action on $J^{-1}(\mu)$, the orbit spaces $J^{-1}(\mu)/G_\mu$ and G/G_μ
▷ are diffeomorphic. Next we show that $J^{-1}(\mu)/G_\mu$ is *symplectically* diffeomorphic to the coadjoint orbit $\mathscr{O}_\mu$.

(6.2) **Proof**: Consider the diagram 6.1.1. Here σ is the map induced by π_μ and ρ is the

$$\begin{tikzcd}
J^{-1}(\mu) \arrow[r, "\pi_\mu"] \arrow[d, "\rho"'] & J^{-1}(\mu)/G_\mu \\
\mathscr{O}_\mu \arrow[ur, "\sigma"'] &
\end{tikzcd}$$

Diagram 6.1.1

restriction of the projection map $(g,\alpha) \mapsto \alpha$ to $J^{-1}(\mu)$. Explicitly, $\rho(g, \mathrm{Ad}^t_g \mu) = \mathrm{Ad}^t_g \mu$. The image of $J^{-1}(\mu)$ under ρ is the coadjoint orbit $\mathscr{O}_\mu$ through μ. The map σ is injective because every fiber of the mapping ρ is exactly one G_μ-orbit. To see this, suppose that $\rho(g, \mathrm{Ad}^t_g \mu) = \rho(h, \mathrm{Ad}^t_h \mu)$. Then $\mathrm{Ad}^t_g \mu = \mathrm{Ad}^t_h \mu$ or $\mu = \mathrm{Ad}^t_{g^{-1}} \mathrm{Ad}^t_h \mu = \mathrm{Ad}^t_{hg^{-1}} \mu$. Whereupon $hg^{-1} \in G_\mu$. Consequently, $\varphi_{hg^{-1}}(g, \mathrm{Ad}^t_g \mu) = (h, \mathrm{Ad}^t_h \mu)$, by definition of the φ action. Thus $(g, \mathrm{Ad}^t_g \mu)$ and $(h, \mathrm{Ad}^t_h \mu)$ lie on the same G_μ-orbit. Since π_μ is surjective, σ is surjective and hence is bijective. Because π_μ and ρ are smooth and because the bundle ρ has a smooth local cross section, it follows that σ is smooth. Since $J^{-1}(\mu)$ is a principal G_μ-bundle, it has a smooth local cross section. Therefore σ^{-1} is smooth. Hence σ is a diffeomorphism.

Now we show that the symplectic form $\sigma^*\omega_\mu$ on the coadjoint orbit $\mathscr{O}_\mu$ is $\omega_{\mathscr{O}_\mu}$. First we find the tangent of the mapping ρ. Differentiating the curve $s \mapsto \rho(g\exp s\xi, \mathrm{Ad}^t_{g\exp s\xi}\mu) = \mathrm{Ad}^t_{g\exp s\xi}\mu$ and evaluating the derivative at $s=0$ gives $T_{(g,\nu)}\rho(T_eL_g\xi, \mathrm{ad}^t_\xi\nu) = \mathrm{ad}^t_\xi\nu$ with $\nu = \mathrm{Ad}^t_g\mu$, since $\mathrm{Ad}^t_{hg} = \mathrm{Ad}^t_g\mathrm{Ad}^t_h$ and $\frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0}\mathrm{Ad}^t_{\exp t\xi}\eta = \mathrm{ad}^t_\xi\eta$. We compute $\sigma^*\omega_\mu$ as follows.

$$
\begin{aligned}
&(\sigma^*\omega_\mu)(\nu)\big(\mathrm{ad}^t_\xi\nu, \mathrm{ad}^t_\eta\nu\big) = \\
&\quad = (\sigma^*\omega_\mu)(\rho(g,\nu))\big(T_{(g,\nu)}\rho(T_eL_g\xi, \mathrm{ad}^t_\xi\nu), T_{(g,\nu)}\rho(T_eL_g\eta, \mathrm{ad}^t_\eta\nu)\big) \\
&\quad = \big((\sigma\circ\rho)^*\omega_\mu\big)(g,\nu)\big((T_eL_g\xi, \mathrm{ad}^t_\xi\nu), (T_eL_g\eta, \mathrm{ad}^t_\eta\nu)\big) \\
&\quad = \omega(g,\nu)\big((T_eL_g\xi, \mathrm{ad}^t_\xi\nu), (T_eL_g\eta, \mathrm{ad}^t_\eta\nu)\big), \\
&\qquad\qquad \text{because } \sigma\circ\rho = \pi_\mu \text{ and } \pi^*_\mu\omega_\mu = \omega|J^{-1}(\mu) \\
&\quad = -(\mathrm{ad}^t_\xi\nu)\eta + (\mathrm{ad}^t_\eta\nu)\xi + \nu([\xi,\eta]), \\
&\qquad\qquad \text{by construction of } \omega\text{, see chapter VI §2 equation (8)} \\
&\quad = -\nu([\xi,\eta]) + \nu([\eta,\xi]) + \nu([\xi,\eta]) = -\nu([\xi,\eta]).
\end{aligned}
$$

Hence the reduced phase space $(J^{-1}(\mu)/G_\mu, \omega_\mu)$ is symplectically diffeomorphic to the coadjoint orbit $(\mathscr{O}_\mu, \omega_{\mathscr{O}_\mu})$. □

One of the main motivations for the reduction theorem is to remove symmetries from a Hamiltonian system (H,M,ω). We say that G is a symmetry of (H,M,ω) if H is invariant under a proper free Hamiltonian action of G on the symplectic manifold (M,ω) with coadjoint equivariant momentum map $J : M \to \mathfrak{g}^*$. The Hamiltonian $H|J^{-1}(\mu)$ induces a smooth function H_μ on the reduced space M_μ, called the *reduced Hamiltonian*, which satisfies $\pi^*_\mu H_\mu = i^*H$. Thus we have constructed a *reduced Hamiltonian system* $(H_\mu, M_\mu, \omega_\mu)$. Its importance lies in the fact that the vector field X_H on $J^{-1}(\mu)$ is π_μ-related to the *reduced vector field* X_{H_μ}, that is, $T\pi_\mu \circ X_H = X_{H_\mu}\circ\pi_\mu$. This is a precise statement of what it means to use the symmetry to reduce the number of variables in a symmetric Hamiltonian system.

(6.3) **Proof**: To see that $\pi^*_\mu H_\mu = i^*H$ holds and that the reduced equations are Hamiltonian, first look at the given equation of motion on $J^{-1}(\mu)$, namely, $X_H \mathbin{\lrcorner} \omega = \mathrm{d}H$. As everything is G_μ-invariant, we can push X_H and H down to M_μ. To push ω down to M_μ, we need to know that ω is a pull back on $J^{-1}(\mu)$. Hence ω must vanish on vectors tangent to G_μ-orbits. From ((5.11)) we know that $\ker\omega|J^{-1}(\mu) = T_m(G_\mu\cdot m)$. Thus we can push ω down to M_μ. Consequently, the equations of motion pass to the quotient to give $X_{H_\mu}\mathbin{\lrcorner}\omega_\mu = \mathrm{d}H_\mu$, because both sides of $X_H \mathbin{\lrcorner}\omega = \mathrm{d}H$ vanish on vectors tangent to G_μ-orbits. □

Example 2: An interesting special case of the regular reduction theorem occurs when the Hamiltonian $\mathscr{H}$ on T^*G is a quadratic function associated to a left invariant metric ρ on G. In this case the image of the integral curves of the Hamiltonian system under the bundle projection are geodesics on G, see chapter VI §3 example 3. Explicitly, if ρ^* is the metric dual to ρ, it is left invariant. The Hamiltonian

$$\mathscr{H} : T^*G \to \mathbf{R} : \alpha_g = T_eL^t_{g^{-1}}\alpha \mapsto \tfrac{1}{2}\rho^*(g)(\alpha_g,\alpha_g) = \tfrac{1}{2}\rho^*(e)(\alpha,\alpha)$$

is left invariant. Pulling $\mathscr{H}$ back by the left trivialization $\mathscr{L}$ (30) gives the Hamiltonian $H : G \times \mathfrak{g}^* \to \mathbf{R} : (g, \alpha) \mapsto \frac{1}{2}\rho^*(e)(\alpha, \alpha)$. The integral curves of X_H are the pull back of integral curves of $X_{\mathscr{H}}$ by the left trivialization $\mathscr{L}$. On $G \times \mathfrak{g}^*$ the integral curves of X_H satisfy the Euler-Arnol'd equations

$$\begin{aligned} \dot{g} &= T_eL_g(\rho^*(e))^\sharp(\alpha) \\ \dot{\alpha} &= \mathrm{ad}^t_{(\rho^*(e))^\sharp(\alpha)}\alpha, \end{aligned}$$

since $D_2H(g, \alpha)\beta = \rho^*(e)(\alpha, \beta) = (\rho^*(e))^\sharp(\alpha)\beta$, see chapter VI §3 example 3. Because H is invariant under the G-action φ (31) on $G \times \mathfrak{g}$, we may use regular reduction, see example 1. We obtain a reduced phase space $(\mathscr{O}_\mu, \omega_{\mathscr{O}_\mu})$ with a reduced Hamiltonian $H_\mu : \mathscr{O}_\mu \subseteq \mathfrak{g}^* \to \mathbf{R} : \nu = \mathrm{Ad}^t_g\mu \mapsto H_\mu(\nu)$. Here $H_\mu(\nu) = H(e, \nu) = \frac{1}{2}\rho^*(e)(\nu, \nu)$. Since $\mathrm{d}H_\mu(\nu)\mathrm{ad}^t_\xi\nu = (\rho^*(e))^\sharp(\nu)\mathrm{ad}^t_\xi\nu$, the integral curves of the reduced Hamiltonian vector field X_{H_μ} on the reduced space $(\mathscr{O}_\mu, \omega_{\mathscr{O}_\mu})$ satisfy *Euler's equations*

$$\dot{\nu} = \mathrm{ad}^t_{(\rho^*(e))^\sharp(\nu)}\nu. \qquad \square$$

6.2 An alternative approach

In this subsection we give an alternative approach to regular reduction. We first construct another model $\mathscr{M}_\mu$ for the reduced space M_μ and then construct a Poisson bracket on its space of smooth functions.

Recall that μ is a regular value of the momentum map J and that $\pi : M \to M/G$ is the G-orbit map. Consider the set $\mathscr{M}_\mu = \pi\big(J^{-1}(\mu)\big) = J^{-1}(\mu)/G$. Let $\mathscr{O}_\mu = \{\mathrm{Ad}^t_{g^{-1}}\mu \,\big|\, g \in G\}$ be the G-coadjoint orbit through $\mu \in \mathfrak{g}^*$. Then

$$\pi^{-1}(\mathscr{M}_\mu) = G \cdot J^{-1}(\mu) = \bigcup_{g \in G} g \cdot J^{-1}(\mu) = \bigcup_{g \in G} J^{-1}(\mathrm{Ad}^t_{g^{-1}}\mu) = J^{-1}(\mathscr{O}_\mu).$$

So $\mathscr{M}_\mu = J^{-1}(\mathscr{O}_\mu)/G$. Note that $\mathscr{M}_\mu$ is a closed subset of the reduced space M_μ if the
▷ coadjoint orbit $\mathscr{O}_\mu$ is a closed subset of $\mathfrak{g}^*$. We now show that $\mathscr{M}_\mu$ is a smooth manifold.

(6.4) **Proof**: Because G acts freely and properly on M, the orbit map π of the G-action Φ is a submersion onto the orbit space M/G. Since $J^{-1}(\mu)$ is a smooth submanifold of M, it follows that $\widehat{\pi} = \pi | J^{-1}(\mu)$ is a surjective mapping onto $\mathscr{M}_\mu \subseteq M/G$ with surjective derivative. Therefore $\mathscr{M}_\mu$ is a smooth manifold. $\square$

Claim: $\mathscr{M}_\mu$ is diffeomorphic to the reduced space $M_\mu = J^{-1}(\mu)/G_\mu$.

(6.5) **Proof**: Consider diagram 6.2.1. The maps π_μ and π are the G_μ and G-orbit maps,

$$\begin{tikzcd} J^{-1}(\mu) \arrow[d, "\pi_\mu"'] \arrow[dr, "\widehat{\pi}"] & \\ M_\mu \arrow[r, "\sigma"] & \mathscr{M}_\mu \end{tikzcd}$$

Diagram 6.2.1

respectively, and $\widehat{\pi} = \pi | J^{-1}(\mu)$. For every $q \in J^{-1}(\mu)$ define the map $\sigma : M_\mu \rightarrow \mathscr{M}_\mu$ by $G_\mu \cdot q \mapsto G \cdot q$. Diagram 6.2.1 is commutative, because $(\sigma \circ \pi_\mu)(q) = \sigma(G_\mu \cdot q) = G \cdot q = \pi(q) = \widehat{\pi}(q)$ for every $q \in J^{-1}(\mu)$.

$\triangleright$ The mapping σ is bijective.

(6.6) **Proof**: Let $g \in G_\mu$ and $q \in J^{-1}(\mu)$. The map σ is injective for if $\sigma(G_\mu \cdot q) = \sigma(G_\mu \cdot q')$ for some $q, q' \in J^{-1}(\mu)$, then $G \cdot q = G \cdot q'$. So there is a $g \in G$ such that $q' = g \cdot q$. Consequently, $\mu = J(q') = \mathrm{Ad}^t_{g^{-1}} J(q) = \mathrm{Ad}^t_{g^{-1}} \mu$, which shows that $g \in G_\mu$. Therefore $G_\mu \cdot q = G_\mu \cdot q'$, that is, σ is injective. To see that σ is surjective, suppose that $G \cdot q \in \mathscr{M}_\mu$ for some $q \in J^{-1}(\mu)$. Then $G_\mu \cdot q \in M_\mu$ and $\sigma(G_\mu \cdot q) = G \cdot q$. $\square$

$\triangleright$ The map σ is a diffeomorphism.

(6.7) **Proof**: Let $g \in G_\mu$ and $q \in J^{-1}(\mu)$, then $g \cdot q \in J^{-1}(\mu)$, because $J(g \cdot q) = \mathrm{Ad}^t_{g^{-1}} J(q) = \mathrm{Ad}^t_{g^{-1}} \mu = \mu$. Therefore $\widehat{\pi}(g \cdot q) = \pi(g \cdot q) = \pi(q) = \widehat{\pi}(q)$, that is, the map $\widehat{\pi}$ is G_μ-invariant. Since $\widehat{\pi}$ is smooth, the induced map σ is smooth. Because σ is bijective by ((6.6)), to check that it is a diffeomorphism it suffices to verify that for every $\overline{q} = \pi_\mu(q)$ the map $T_{\overline{q}}\sigma$ is injective and that $\dim T_{\overline{q}} M_\mu = \dim T_{\widehat{q}} \mathscr{M}_\mu$, where $\widehat{q} = \widehat{\pi}(q)$. Suppose that for some $\overline{v} \in T_q M_\mu$ we have $(T_{\overline{q}}\sigma)\overline{v} = 0$. Since π_μ is a submersion, there is a $v \in T_q J^{-1}(\mu)$ such that $T_q \pi_\mu v = \overline{v}$. Thus $0 = T_q(\sigma \circ \pi_\mu) v = T_q \widehat{\pi} v$, that is, $v \in \ker T_q \widehat{\pi}$. Because diagram 6.2.1 commutes, the map π_μ is a surjective submersion with fiber G_μ, and the map σ is bijective, it follows that the map $\widehat{\pi}$ is a surjective submersion with fiber G_μ. More precisely, $\widehat{\pi}^{-1}(\widehat{q}) = \pi_\mu^{-1}(\overline{q})$. Consequently, $\ker T_q \widehat{\pi} = T_q \widehat{\pi}^{-1}(\widehat{q}) = \ker T_q \pi_\mu$. Hence $T_q \pi_\mu v = 0$, which implies $\overline{v} = 0$. Thus $T_{\overline{q}}\sigma$ is injective. From

$$\dim T_q J^{-1}(\mu) = \dim \mathrm{im}\, T_q \widehat{\pi} + \dim \ker T_q \widehat{\pi} = \dim T_{\widehat{q}} \mathscr{M}_\mu + \dim \mathfrak{g}_\mu$$

and

$$\dim T_q J^{-1}(\mu) = \dim \mathrm{im}\, T_q \pi_\mu + \dim \ker T_q \pi_\mu = \dim T_{\overline{q}} M_\mu + \dim \mathfrak{g}_\mu$$

we deduce that $\dim T_{\widehat{q}} \mathscr{M}_\mu = \dim T_{\overline{q}} M_\mu$, since $\widehat{q} = \sigma(\overline{q})$. Thus σ is a diffeomorphism. $\square$

With an eye towards singular reduction, we want to construct a Poisson bracket $\{\,,\}_{\mathscr{M}_\mu}$ on $C^\infty(\mathscr{M}_\mu)$. We begin by constructing a Poisson bracket $\{\,,\}_{M_\mu}$ on $C^\infty(M_\mu)$. Because (M_μ, ω_μ) is a smooth symplectic manifold, the Poisson bracket $\{\,,\}_{M_\mu}$ on $C^\infty(M_\mu)$ is defined in the standard way, namely, for every $f, h \in C^\infty(M_\mu)$ we set $\{f, h\}_{M_\mu}(\overline{q}) = \omega_\mu(\overline{q})(X_f(\overline{q}), X_h(\overline{q}))$ for every $\overline{q} \in M_\mu$. We now define a symplectic form $\omega_{\mathscr{M}_\mu}$ on $\mathscr{M}_\mu$. Because the map $\widehat{\pi} = \sigma \circ \pi_\mu$ is a surjective submersion, we define $\omega_{\mathscr{M}_\mu}$ by setting $\widehat{\pi}^* \omega_{\mathscr{M}_\mu} = \omega | J^{-1}(\mu) = i^* \omega$, where $i : J^{-1}(\mu) \rightarrow M$ is the inclusion mapping. In other words, for every $u_1, u_2 \in T_q J^{-1}(\mu) \subseteq T_q M$ we have

$$\omega_{\mathscr{M}_\mu}\big(\widehat{\pi}(q)\big)\big(T_q \widehat{\pi} u_1, T_q \widehat{\pi} u_2\big) = \omega(q)\big(u_1, u_2\big). \tag{32}$$

To see that $\omega_{\mathscr{M}_\mu}$ is well defined suppose that there is $u_1' \in T_q J^{-1}(\mu)$ such that $T_q \widehat{\pi} u_1' = T_q \widehat{\pi} u_1$. Then $u_1' - u_1 \in \ker T_q(\sigma \circ \pi_\mu) = \ker T_q \pi_\mu$, since σ is a diffeomorphism. But $\ker T_q \pi_\mu = T_q(G_\mu \cdot q) = T_q J^{-1}(\mu) \cap T_q J^{-1}(\mu)^{\omega(q)}$. So $\omega(q)\big(u_1' - u_1, u_2\big) = 0$. Thus $\omega_{\mathscr{M}_\mu}$ is a well defined 2-form on $\mathscr{M}_\mu$. It is closed, since $\widehat{\pi}^* \mathrm{d}\omega_{\mathscr{M}_\mu} = \mathrm{d}\widehat{\pi}^* \omega_{\mathscr{M}_\mu} = \mathrm{d}(i^* \omega) =$

$i^*(\mathrm{d}\omega) = 0$. The last equality follows because ω is a symplectic form on M. To see that $\omega_{\mathscr{M}_\mu}(p)$ is nondegenerate at every $p \in \mathscr{M}_\mu$ we argue as follows. Suppose that there is a $v_1 \in T_p\mathscr{M}_\mu$ such that $\omega_{\mathscr{M}_\mu}(p)\big(v_1, v_2\big) = 0$ for every $v_2 \in T_p\mathscr{M}_\mu$. Since $\widehat{\pi}$ is a surjective submersion, there is a $q \in J^{-1}(\mu)$ such that $\widehat{\pi}(q) = p$ and there are u_1, $u_2 \in T_qJ^{-1}(\mu)$ such that $v_1 = T_q\widehat{\pi}\,u_1$ and $v_2 = T_q\widehat{\pi}\,u_2$. Therefore for every $u_2 \in T_qJ^{-1}(\mu)$ we get

$$0 = \omega_{\mathscr{M}_\mu}(\widehat{\pi}(q))\big(T_q\widehat{\pi}(q)\,u_1, T_q\widehat{\pi}(q)\,u_2\big) = \omega(q)(u_1, u_2).$$

In other words, $u_1 \in T_qJ^{-1}(\mu)^{\omega(q)}$. Thus $u_1 \in T_qJ^{-1}(\mu) \cap T_qJ^{-1}(\mu)^{\omega(q)} = T_q(G_\mu \cdot q) = \ker T_q\pi_\mu$, which implies $v_1 = T_q\widehat{\pi}(q)\,u_1 = T_q(\sigma \circ \pi_\mu)u_1 = 0$. Hence $\omega_{\mathscr{M}_\mu}(p)$ is nondegenerate. Thus $\omega_{\mathscr{M}_\mu}$ is a symplectic form on $\mathscr{M}_\mu$. □

Corollary: The map $\sigma : (M_\mu, \omega_{M_\mu}) \to (\mathscr{M}_\mu, \omega_{\mathscr{M}_\mu})$ is a symplectic diffeomorphism.

(6.8) **Proof**: We need only show that σ is a symplectic mapping. By definition $\widehat{\pi}^*\omega = (\sigma \circ \pi_\mu)^*\omega_{\mathscr{M}_\mu} = i^*\omega$. By regular reduction we have $\pi_\mu^*\omega_{M_\mu} = i^*\omega$. So $\pi_\mu^*\big(\sigma^*(\omega_{\mathscr{M}_\mu})\big) = \pi_\mu^*\omega_{M_\mu}$, which implies $\sigma^*(\omega_{\mathscr{M}_\mu}) = \omega_{M_\mu}$, since π_μ is a surjective submersion. □

Let $\{\,,\,\}_{\mathscr{M}_\mu}$ be the standard Poisson bracket on $C^\infty(\mathscr{M}_\mu)$ coming from the symplectic form $\omega_{\mathscr{M}_\mu}$ on $\mathscr{M}_\mu$. From ((6.8)) it follows that $\sigma^*\{f, h\}_{\mathscr{M}_\mu} = \{\sigma^* f, \sigma^* h\}_{M_\mu}$ for every f, $h \in C^\infty(\mathscr{M}_\mu)$. Consequently,

Corollary: The map $\sigma^* : \big(C^\infty(\mathscr{M}_\mu), \{\,,\,\}_{\mathscr{M}_\mu}, \cdot\big) \to \big(C^\infty(M_\mu), \{\,,\,\}_{M_\mu}, \cdot\big)$ is an isomorphism of Poisson algebras.

7 Singular reduction

In this section we examine the reduction of symmetry of a Hamiltonian systems when we relax the hypotheses of the regular reduction theorem. In particular, we assume that the action $\Phi : G \times M \to M : (g, m) \mapsto \Phi_g(m) = g \cdot m$ of the Lie group G on the symplectic manifold (M, ω) is *proper*, Hamiltonian, and has a coadjoint equivariant momentum mapping $J : M \to \mathfrak{g}^*$.

7.1 Singular reduced space and reduced dynamics

Singular reduction gives rise to a reduced space which is a locally compact subcartesian differential space that may not be a smooth manifold.

As a set the singular reduced space $\mathscr{M}_\mu$ at the value $\mu \in \mathfrak{g}^*$ in the image of the momentum mapping J is the orbit space $\pi(J^{-1}(\mu))$, where $\pi : M \to \overline{M} = M/G$ is the G-orbit mapping. Using the topology on $J^{-1}(\mu)$ induced from the topology of M and the quotient topology on $\mathscr{M}_\mu$, we say that a continuous function f on $\mathscr{M}_\mu$ is *smooth* if and only if there is an smooth G-invariant function F on M such that $\pi^* f = F|J^{-1}(\mu)$. Let $C^\infty(\mathscr{M}_\mu)$ be the space of smooth functions on $\mathscr{M}_\mu$. By definition $C^\infty(\mathscr{M}_\mu) = C^\infty(M)^G|J^{-1}(\mu)$. From ((3.9)) we see that $C^\infty(M)^G$ is a differential structure on $\overline{M}$. Therefore $C^\infty(M)^G|J^{-1}(\mu)$ is a differential structure on $\mathscr{M}_\mu$. So $\big(\mathscr{M}_\mu, C^\infty(\mathscr{M}_\mu)\big)$ is a differential space.

Because the set $J^{-1}(\mu)$ is G_μ-invariant, the action $\Phi|(G_\mu \times J^{-1}(\mu))$ of G_μ on $J^{-1}(\mu)$ is defined. Since $J^{-1}(\mu)$ is a closed subset of M, this action is proper with orbit space M_μ, which, using the quotient topology, is a Hausdorff topological manifold with G_μ-orbit map $\pi_\mu : M \to M_\mu$. We say that a continuous function f on M_μ is *smooth* if and only if there is a smooth G_μ-invariant function $\widetilde{F}$ on $J^{-1}(\mu)$ such that $\pi^* f = \widetilde{F}$. Since $J^{-1}(\mu)$ is a closed G_μ-invariant subset of M, using ((2.6)) $\widetilde{F}$ may be extended to a smooth G_μ-invariant function F on M. Consequently, the set $C^\infty(M_\mu) = C_i^\infty(J^{-1}(\mu))^{G_\mu}$ of smooth functions on M_μ is equal to $C^\infty(M)^{G_\mu}|J^{-1}(\mu)$, which by ((3.9)) is a differential structure on M_μ. To procede we need

Claim: The map $\sigma : M_\mu \to \mathscr{M}_\mu : G_\mu \cdot q \mapsto G \cdot q$ is a diffeomorphism of the differential space $\big(M_\mu, C^\infty(M_\mu)\big)$ onto the differential space $\big(\mathscr{M}_\mu, C^\infty(\mathscr{M}_\mu)\big)$.

(7.1) **Proof**: In ((6.6)) we have shown that the map σ is bijective. Now we show that σ is a homeomorphism. Observe that the G_μ-invariant map $\widehat{\pi} = \pi|J^{-1}(\mu)$ is a continuous open map. Therefore, the induced map σ is continuous. Let U be an open subset of M_μ. Then $\pi_\mu^{-1}(U)$ is an open subset of $J^{-1}(\mu)$. Since $\pi|J^{-1}(\mu) : J^{-1}(\mu) \to \mathscr{M}_\mu$ is an open mapping, it follows that $(\sigma^{-1})^{-1}(U) = \widehat{\pi}(\pi^{-1}(U))$ is an open subset of $\mathscr{M}_\mu$. Consequently, σ^{-1} is a continuous mapping. Hence σ is a homeomorphism. From ((3.11)) it follows that the quotient topology on M_μ and $\mathscr{M}_\mu$ is equivalent to the differential space topology on $C^\infty(M_\mu)$ and $C^\infty(\mathscr{M}_\mu)$, respectively. Thus the mapping σ is a homeomorphism of the differential space $\big(M_\mu, C^\infty(M_\mu)\big)$ onto the differential space $\big(\mathscr{M}_\mu, C^\infty(\mathscr{M}_\mu)\big)$. To show that σ is a diffeomorphism we must verify that $\sigma^*(C^\infty(\mathscr{M}_\mu)) = C^\infty(M_\mu)$. First we show that the mapping σ is smooth, that is, $\sigma^*(C^\infty(\mathscr{M}_\mu)) \subseteq C^\infty(M_\mu)$. For $h \in C^\infty(\overline{M})$ consider the function $h|\mathscr{M}_\mu$. For $q \in J^{-1}(\mu)$ with $\overline{q} = \pi(q) \in \overline{M}$, we have

$$\begin{aligned}\sigma^*(h|\mathscr{M}_\mu)\big(\pi_\mu(q)\big) &= (h|\mathscr{M}_\mu)\big(\sigma(\pi_\mu(q))\big) = h(\overline{q}), \quad \text{since } \widehat{\pi} = \sigma \circ \pi_\mu = \pi|J^{-1}(\mu) \\ &= h(\pi(q)) = \big((\pi^* h)|J^{-1}(\mu)\big)(q).\end{aligned}$$

Since the function $\pi^* h \in C^\infty(M)$ is G-invariant, the function $(\pi^* h)|J^{-1}(\mu)$ is G_μ-invariant and hence pushes forward to a smooth function on M_μ. Consequently, $\sigma^*(h|\mathscr{M}_\mu) \in C^\infty(M_\mu)$. Thus $\sigma^*(C^\infty(\mathscr{M}_\mu)) \subseteq C^\infty(M_\mu)$. Before showing that the mapping σ^{-1} is smooth we prove

Lemma: For every $f \in C^\infty(J^{-1}(\mu))^{G_\mu}$ and every $q \in J^{-1}(\mu)$ there is a G-invariant open neighborhood W of q in M and a function $F_W \in C^\infty(M)^G$ such that $f|(W \cap J^{-1}(\mu)) = F_W|(W \cap J^{-1}(\mu))$.

(7.2) **Proof**: Let $f \in C^\infty(J^{-1}(\mu))^{G_\mu}$. For each $q \in J^{-1}(\mu)$ there is an open neighborhood U_q of q in M and a smooth function F on M such that $f|(U_q \cap J^{-1}(\mu)) = F|(U_q \cap J^{-1}(\mu))$. Let $S_q \subseteq U_q$ be a slice for the G-action Φ on M at q. Then S_q is invariant under the action $\Phi|(G_q \times M)$. Here G_q is the isotropy group at q. Now $G_q \cap G_\mu$ acts on $S_q \cap J^{-1}(\mu)$ and leaves $f|(S_q \cap J^{-1}(\mu))$ invariant. Let W_1 and W_2 be G_q-invariant open neighborhoods of q in S_q such that $\overline{W}_1 \subseteq W_2$. There is a nonnegative function $\chi \in C^\infty(S_q)$ such that $\chi|\overline{W}_1 = 1$ and whose support $\operatorname{supp}\chi$ is contained in W_2. Since G_q is compact we may average $\chi \cdot (F|S_q)$ over G_q to obtain a G_q invariant function $\widetilde{F}$ on S_q whose support $\operatorname{supp}\widetilde{F}$ is contained in W_2. Now $G \cdot S_q$ is a G-invariant open neighborhood of q in M. Since $\widetilde{F}$ is a G_q-invariant smooth function on S_q, it extends to a unique smooth G-invariant function $\widehat{F}$ on $G \cdot S_q$. Let $W = G \cdot W_1$ and let F_W be a smooth function on M, which is equal to $\widehat{F}$ on $G \cdot S_q$. Thus $\widehat{F}$ vanishes off of $G \cdot W_2$. The function F_W on M is G-invariant and agrees with f on $W \cap J^{-1}(\mu)$. □

We return to the proof that σ^{-1} is smooth. Let $f \in C^\infty(M_\mu)$. Then $\pi_\mu^* f$ is a smooth G_μ-invariant function on $J^{-1}(\mu)$. By lemma ((7.2)) for every $q \in J^{-1}(\mu)$ there is a G-invariant open neighborhood W of q in M and a smooth G-invariant function F_W on M such that $\pi_\mu^* f$ agrees with F_W on $W \cap J^{-1}(\mu)$. Since F_W is G-invariant and smooth, it induces a smooth function $\overline{F}_W$ on $\overline{M}$ such that $\pi^* \overline{F}_W = F_W$. For each $q \in W \cap J^{-1}(\mu)$ we get

$$\begin{aligned} f(\pi_\mu(q)) &= (\pi_\mu^* f)(q) = F_W(q) = (\overline{F}_W | \mathscr{M}_\mu)(\pi(q)) \\ &= (\overline{F}_W | \mathscr{M}_\mu)\big((\sigma \circ \pi_\mu)(q)\big) = \sigma^*(\overline{F}_W | \mathscr{M}_\mu)(\pi_\mu(q)), \end{aligned}$$

where $\overline{F}_W | \mathscr{M}_\mu \in C^\infty(\mathscr{M}_\mu)$. So

$$f | \big(M_\mu \cap \pi_\mu(W)\big) = (\sigma^*(\overline{F}_W | \mathscr{M}_\mu)) | \big(M_\mu \cap \pi_\mu(W)\big). \tag{33}$$

Cover M by G-invariant open sets W such that (33) holds. Using a G-invariant partition of unity subordinate to this covering, we conclude that $f = \sigma^*(\overline{F}_W | \mathscr{M}_\mu)$. So $f \in \sigma^*(C^\infty(\mathscr{M}_\mu))$. Thus $\sigma^*(C^\infty(\mathscr{M}_\mu)) \supseteq C^\infty(M_\mu)$. Hence the mapping σ^{-1} is smooth. So σ is a diffeomorphism. □

Corollary: The differential space $\big(\mathscr{M}_\mu, C^\infty(\mathscr{M}_\mu)\big)$ is locally compact and subcartesian.

(7.3) **Proof**: This follows once we show that the differential space $\big(M_\mu, C^\infty(M_\mu)\big)$ is locally compact and subcartesian, because by ((7.1)) the differential spaces $(M_\mu, C^\infty(M_\mu))$ and $(\mathscr{M}_\mu, C^\infty(\mathscr{M}_\mu))$ are diffeomorphic. We start by noting that G_μ is a closed subgroup of G, which acts properly on M_μ because G does. In what follows we use the following notation. Let $m \in J^{-1}(\mu) \subseteq M$. Set $G = G_\mu$ and $H = G_m = (G_\mu)_m$. Let U be a G-invariant open neighborhood of m in M and let B be an H-invariant open neighborhood of 0 in $E = T_m M / T_m(G_m \cdot m)$. From ((3.13)) we see that the differential spaces $(U/G, C^\infty(U/G))$ are $(\sigma(B), C^\infty(\sigma(B)))$ are diffeomorphic. Here B is an open ball about the origin in E defined by an H-invariant inner product and $\sigma : E \to \mathbf{R}^n$ is the Hilbert map of the H-action on E. Let $\psi : B \to U$ with $\psi(0) = m$ be the diffeomorphism given by ((1.4)), which intertwines the H-action on B with the G-action on U. Since the momentum map $J|U : U \subseteq M \to \mathfrak{g}^*$ is G-invariant, the map $\mathscr{J}_U = \psi^*(J|U) : B \to \mathfrak{g}^*$ is H-invariant. Choose a basis $\{\alpha_i\}_{i=1}^m$ for $\mathfrak{g}^*$. Then the components of the mapping $\mathscr{J}_U : B \to \mathbf{R}^m$ are H-invariant smooth functions. Because $C^\infty(B)^H = \sigma^*(C^\infty(\sigma(B)))$, there is a smooth function $j_U : \sigma(B) \subseteq \mathbf{R}^n \to \mathbf{R}^m$ such that $\sigma^* j_U = \mathscr{J}_U$. Consequently, the differential space $\big((U \cap J^{-1}(\mu))/G, C^\infty((U \cap J^{-1}(\mu))/G)\big)$ is diffeomorphic to the differential space $\big(\sigma(B) \cap j_U^{-1}(\mu), C^\infty(\sigma(B) \cap j_U^{-1}(\mu))\big)$. Note that $\sigma(B) \cap j_U^{-1}(\mu)$ is a closed subset of $\mathbf{R}^n$. Since $\{(U \cap J^{-1}(\mu))/G\}$ is an open covering of $M_\mu = J^{-1}(\mu)/G_\mu \subseteq M/G_\mu$, it follows that the differential space $(M_\mu, C^\infty(M_\mu))$ is subcartesian and locally compact. □

We now define a Poisson bracket $\{\,,\}_{M_\mu}$ on $C^\infty(M_\mu)$. Let f_μ, $h_\mu \in C^\infty(M_\mu)$. Then $\pi_\mu^* f_\mu$, $\pi_\mu^* h_\mu \in C^\infty(J^{-1}(\mu))$. By ((7.2)) for every $q \in J^{-1}(\mu)$ there exists a G-invariant open neighborhood W of q in M and F_W, $H_W \in C^\infty(M)^G$ such that $\pi_\mu^* f|(W \cap J^{-1}(\mu)) = F_W|(W \cap J^{-1}(\mu))$ and $\pi_\mu^* h|(W \cap J^{-1}(\mu)) = H_W|(W \cap J^{-1}(\mu))$. For every $q \in W \cap J^{-1}(\mu)$ let

$$\{f_\mu, h_\mu\}_{M_\mu}(\pi_\mu(q)) = \{F_W, H_W\}_M(q).$$

Here $\{\,,\}_M$ is the standard Poisson bracket on $C^\infty(M)^G$ coming from the symplectic 2-form ω. To see that $\{\,,\}_{M_\mu}$ is well defined, consider the set $\mathscr{I}_W = \mathscr{I}(W \cap J^{-1}(\mu))$ in the associative commutative algebra $\big(C^\infty(M)^G, \cdot\big)$ of G-invariant smooth functions on M which vanish identically on $W \cap J^{-1}(\mu)$. Then $\mathscr{I}_W$ is a Poisson ideal of the Poisson algebra $\mathscr{A} = \big(C^\infty(M)^G, \{\,,\}_M, \cdot\big)$.

(7.4) **Proof**: Suppose that $\xi \in \mathfrak{g}$ and $f \in C^\infty(M)^G$. For every $q \in J^{-1}(\mu)$ we have

$$\{f, J^\xi\}_M(q) = (L_{X_{J^\xi}} f)(q) = (L_{X^\xi} f)(q) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} f(\exp t\xi \cdot q) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} f(q) = 0,$$

where the second to last equality above follows because $f \in C^\infty(M)^G$. Thus for every $\xi \in \mathfrak{g}$, the smooth function J^ξ is constant along the integral curves $t \mapsto \varphi_t^f(q)$ of the vector field X_f on M starting at $q \in J^{-1}(\mu)$. So $t \mapsto \varphi_t^f(q)$ lies in $J^{-1}(\mu)$ because for every $\xi \in \mathfrak{g}$ we have

$$J\big(\varphi_t^f(q)\big)\xi = J^\xi\big(\varphi_t^f(q)\big) = J^\xi(q) = J(q)\xi = \mu(\xi).$$

Let $h \in \mathscr{I}_W$ and suppose that $q \in W \cap J^{-1}(\mu)$. Since W is an open subset of M there is a $t_q' > 0$ such that $\varphi_t^f(q) \in W \cap J^{-1}(\mu)$ for every $t \in (-t_q', t_q')$. Consequently,

$$\{h, f\}_M(q) = (L_{X_f} h)(q) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} h(\varphi_t^f(q)) = 0,$$

because $(-t_q', t_q') \to W \cap J^{-1}(\mu) : t \mapsto \varphi_t^f(q)$ and $h \in \mathscr{I}_W$. Thus $\{h, f\}_M \in \mathscr{I}_W$. So $\mathscr{I}_W$ is a Poisson ideal of the Poisson algebra $\mathscr{A}$. □

We now show that $\{\,,\}_{M_\mu}$ is well defined. Shrinking the G-invariant neighborhood W of q if necessary, let $H_W' \in C^\infty(M)^G$ such that $(\pi_\mu^* h_\mu)|(W \cap J^{-1}(\mu)) = H_W'|(W \cap J^{-1}(\mu))$. By construction, the smooth G-invariant function $H_W - H_W'$ on M vanishes identically on the set $W \cap J^{-1}(\mu)$. In other words, $H_W - H_W' \in \mathscr{I}_W$. Therefore $\{F_W, H_W - H_W'\}_M \in \mathscr{I}_W$, since $\mathscr{I}_W$ is a Poisson ideal of $\mathscr{A}$. So on $M_\mu \cap \pi_\mu(W)$ the Poisson bracket $\{f_\mu, h_\mu\}_{M_\mu}$ does not depend on the choice of smooth G-invariant function H_W on M which represents $\pi_\mu^* h_\mu$ on $W \cap J^{-1}(\mu)$. Since $\{\,,\}_{M_\mu}$ is skew symmetric, a similar argument shows that $\{f_\mu, h_\mu\}_{M_\mu}$ does not depend on the choice of smooth G-invariant function F_W on M representing $\pi_\mu^* f_\mu$ on $W \cap J^{-1}(\mu)$. Thus the Poisson bracket $\{\,,\}_{M_\mu}$ on $C^\infty(M_\mu)$ is well defined on $M_\mu \cap \pi_\mu(W)$ and hence on all of M_μ, since we can cover $J^{-1}(\mu)$ by open G-invariant neighborhoods W in M such that ((7.2)) holds.

Claim: The Poisson bracket $\{\,,\}_{M_\mu}$ on $C^\infty(M_\mu)$ is nondegenerate.

In order to prove the claim we need the following result.

Fact: Let $\alpha_m \in T^*_m M$ be a covector such that $\alpha_m(X_{J^\xi(m)}) = 0$ for every $\xi \in \mathfrak{g}$. Then there is a smooth G-invariant function f on M such that $\mathrm{d}f(m) = \alpha_m$.

(7.5) **Proof**: Let S_m be a slice at $m \in M$ to the G-action Φ on M. Let G_m be the isotropy group at m. There is a compactly supported smooth function f_{S_m} on S_m such that $\mathrm{d}f_{S_m}(m) = \alpha_m$. To see this note that by hypothesis $\alpha_m(X^\xi(m)) = 0$ for every $\xi \in \mathfrak{g}$, that is, $\alpha_m | T_m(G\cdot m) = 0$. Since $T_mM = T_mS_m \oplus T_m(G\cdot m)$ and $\mathrm{d}f_{S_m}|T_m(G\cdot m) = 0$, because f_{S_m} is a smooth function on S_m, we only need to find a smooth function f_{S_m} on S_m such that $\mathrm{d}f_{S_m}(m)|T_mS_m = \alpha_m|T_mS_m$. This is straightforward to arrange. We continue with the construction of the function f. As G_m is compact, we may average f_{S_m} over the G_m-action on S_m to obtain a G_m-invariant smooth function $\overline{f}_{S_m} = \int_{G_m} \Phi^*_g f_{S_m}\,\mathrm{d}\mu(g)$ on S_m with compact support. Here $\mathrm{d}\mu(g)$ is Haar measure on G_m which has been normalized so that $\mathrm{vol}\, G_m = 1$. For each $\xi \in \mathfrak{g}_m$ we have $\mathrm{d}(\Phi^*_{\exp t\xi} f_{S_m})(m) = \mathrm{d}(f_{S_m}(\exp t\xi \cdot m)) = \mathrm{d}f_{S_m}(m)$. Therefore $\mathrm{d}\overline{f}_{S_m}(m) = \mathrm{d}f_{S_m}(m)$. Now extend $\overline{f}_{S_m}$ to a smooth G-invariant function f on M with support contained in $G\cdot S_m$ by first setting $f(g\cdot s) = \overline{f}_{S_m}(s)$ for every $s \in S_m$ and every $g \in G$ and then setting $f|(M \setminus G\cdot S_m) = 0$. By construction we have

$$\mathrm{d}f(m)|T_mS_m = \mathrm{d}\overline{f}_{S_m}(m)|T_mS_m = \mathrm{d}f_{S_m}(m)|T_mS_m = \alpha_m|T_mS_m.$$

Since f is G-invariant, it follows that $\mathrm{d}f(m)|T_m(G\cdot m) = 0$. From our hypothesis we have $\alpha_m|T_m(G\cdot m) = 0$. Since $T_mM = T_mS_m \oplus T_m(G\cdot m)$ it follows that $\mathrm{d}f(m) = \alpha_m$. $\square$

(7.6) **Proof of the claim**: We need to show that if for some $h \in C^\infty(M)^G$ we have $\{h,f\}_M|J^{-1}(\mu) = 0$ for every $f \in C^\infty(M)^G$, then $h|J^{-1}(\mu)$ is locally constant. To verify that $h|J^{-1}(\mu)$ is locally constant, it suffices to show that $\frac{\mathrm{d}}{\mathrm{d}t}(h\circ\gamma)(t) = 0$ for every C^1 curve $t \mapsto \gamma(t)$ with $\gamma(0) = q \in J^{-1}(\mu)$ and $\gamma(t) \in J^{-1}(\mu)$, because $J^{-1}(\mu)$ is locally arcwise connected ((5.20)). By hypothesis for every $q \in J^{-1}(\mu)$ we have $0 = \{f,h\}_M(q) = \mathrm{d}f(q)X_h(q)$. From fact ((7.5)) it follows that

$$\mathrm{span}\{\mathrm{d}f(q) \in T^*_qM \,\big|\, f \in C^\infty(M)^G\} = T_q(G\cdot q)^\circ = \{\alpha_q \in T^*_qM \,\big|\, \alpha_q|T(G\cdot q) = 0\}.$$

Therefore $X_h(q) \in \mathrm{span}\{\mathrm{d}f(q) \in T^*_qM \,\big|\, f \in C^\infty(M)^G\}^\circ = \big((T_q(G\cdot q))^\circ\big)^\circ = T_q(G\cdot q)$ for every $q \in J^{-1}(\mu)$. From the definition of the momentum map J we get

$$\omega(\gamma(t))\big(X^\xi(\gamma(t)), \frac{\mathrm{d}\gamma}{\mathrm{d}t}\big) = \mathrm{d}J^\xi(\gamma(t))\frac{\mathrm{d}\gamma}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}(J^\xi\circ\gamma)(t) = 0, \tag{34}$$

for every $\xi \in \mathfrak{g}$. The last equality in (34) follows because $\gamma(t) \in J^{-1}(\mu)$ by hypothesis. Therefore $\frac{\mathrm{d}\gamma}{\mathrm{d}t} \in \big(T_{\gamma(t)}(G\cdot\gamma(t))\big)^{\omega(\gamma(t))}$. So

$$0 = \omega(\gamma(t))\big(X_h(\gamma(t)), \frac{\mathrm{d}\gamma}{\mathrm{d}t}\big) = \mathrm{d}h(\gamma(t))\frac{\mathrm{d}\gamma}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}(h\circ\gamma)(t). \qquad \square$$

In order to have dynamics on the singular reduced space $\mathscr{M}_\mu$ we define a Poisson bracket $\{\,,\,\}_{\mathscr{M}_\mu}$ on $C^\infty(\mathscr{M}_\mu)$ by $\{f,h\}_{\mathscr{M}_\mu} = (\sigma^{-1})^*\{\sigma^*f, \sigma^*h\}_{M_\mu}$, where $f, h \in C^\infty(\mathscr{M}_\mu)$ and σ is the diffeomorphism of differential spaces given by $\big(M_\mu, C^\infty(M_\mu)\big) \rightarrow \big(\mathscr{M}_\mu, C^\infty(\mathscr{M}_\mu)\big)$:
▷ $G_\mu\cdot q \mapsto G\cdot q$ for every $q \in J^{-1}(\mu)$. We now show that $\mathscr{B} = \big(C^\infty(\mathscr{M}_\mu), \{\,,\,\}_{\mathscr{M}_\mu}, \cdot\big)$ is a Poisson algebra.

(7.7) **Proof**: First we show that $\big(C^\infty(\mathscr{M}_\mu), \{\,,\}_{\mathscr{M}_\mu}\big)$ is a Lie algebra. The bracket $\{\,,\}_{\mathscr{M}_\mu}$ is skew symmetric since

$$\{h,f\}_{\mathscr{M}_\mu} = (\sigma^{-1})^*\{\sigma^* f, \sigma^* h\}_{M_\mu} = -(\sigma^{-1})^*\{\sigma^* h, \sigma^* h\}_{M_\mu} = \{f,h\}_{\mathscr{M}_\mu},$$

for every f, $h \in C^\infty(\mathscr{M}_\mu)$. It is bilinear because σ^* is linear and $\{\,,\}_{M_\mu}$ is bilinear. The Jacobi identity holds because for every f, h, $k \in C^\infty(\mathscr{M}_\mu)$ we have

$$\begin{aligned}
&\{f,\{h,k\}_{\mathscr{M}_\mu}\}_{\mathscr{M}_\mu} - \{\{f,h\}_{\mathscr{M}_\mu},k\}_{\mathscr{M}_\mu} - \{h,\{f,k\}_{\mathscr{M}_\mu}\}_{\mathscr{M}_\mu} = \\
&\quad = (\sigma^{-1})^*\{\sigma^* f, \sigma^*\{h,k\}_{\mathscr{M}_\mu}\}_{M_\mu} - (\sigma^{-1})^*\{\sigma^*\{f,h\}_{\mathscr{M}_\mu}, \sigma^* k\}_{M_\mu} - (\sigma^{-1})^*\{\sigma^* h, \sigma^*\{f,k\}_{\mathscr{M}_\mu}\}_{M_\mu} \\
&\quad = (\sigma^{-1})^*\big(\{\sigma^* f, \{\sigma^* h, \sigma^* k\}_{M_\mu}\}_{M_\mu} - \{\{\sigma^* f, \sigma^* h\}_{M_\mu}, \sigma^* k\}_{M_\mu} - \{\sigma^* h, \{\sigma^* f . \sigma^* k\}_{M_\mu}\}_{M_\mu}\big) \\
&\quad = 0,
\end{aligned}$$

since the Jacobi identity holds for $\{\,,\}_{M_\mu}$. Thus $\big(C^\infty(\mathscr{M}_\mu), \{\,,\}_{\mathscr{M}_\mu}\big)$ is a Lie algebra. It is straightforward to check that $\big(C^\infty(\mathscr{M}_\mu), \cdot\big)$ is a commutative, associative algebra. Since

$$\begin{aligned}
\{f, h\cdot k\}_{\mathscr{M}_\mu} &= (\sigma^{-1})^*\{\sigma^* f, \sigma^*(h\cdot k)\}_{M_\mu} \\
&= (\sigma^{-1})^*\big(\{\sigma^* f, \sigma^* h\}_{M_\mu}\cdot \sigma^* k + \sigma^* h \cdot \{\sigma^* f, \sigma^* k\}_{M_\mu}\big) \\
&= \{f,h\}_{\mathscr{M}_\mu}\cdot k + h\cdot\{f,k\}_{\mathscr{M}_\mu},
\end{aligned}$$

it follows that $\mathscr{B}$ is a Poisson algebra. □

By definition $\sigma^*\{f,h\}_{\mathscr{M}_\mu} = \{\sigma^* f, \sigma^* h\}_{M_\mu}$ for every f, $h \in C^\infty(\mathscr{M}_\mu)$. Since the map $\sigma : \big(M_\mu, C^\infty(M_\mu)\big) \to \big(\mathscr{M}_\mu, C^\infty(\mathscr{M}_\mu)\big)$ is a diffeomorphism of differential spaces it fol- ▷ lows that $\sigma^*\{(\sigma^*)^{-1}F, (\sigma^*)^{-1}H\}_{\mathscr{M}_\mu} = \{F,H\}_{M_\mu}$ for every F, $H \in C^\infty(M_\mu)$. Thus $\sigma^* : \mathscr{B} = \big(C^\infty(\mathscr{M}_\mu), \{\,,\}_{\mathscr{M}_\mu}, \cdot\big) \to \mathscr{A} = \big(C^\infty(M_\mu), \{\,,\}_{M_\mu}, \cdot\big)$ is an isomorphism of Poisson algebras. So the mapping $\sigma : \big(M_\mu, C^\infty(M_\mu), \{\,,\}_{M_\mu}\big) \to \big(\mathscr{M}_\mu, C^\infty(\mathscr{M}_\mu), \{\,,\}_{\mathscr{M}_\mu}\big)$ is a Poisson diffeomorphism of *Poisson differential spaces*.

Corollary: The bracket $\{\,,\}_{\mathscr{M}_\mu}$ on $C^\infty(\mathscr{M}_\mu)$ is nondegenerate.

(7.8) **Proof**: Suppose that for some $h \in C^\infty(\mathscr{M}_\mu)$ we have $\{h,f\}_{\mathscr{M}_\mu} = 0$ for every $f \in C^\infty(\mathscr{M}_\mu)$. Then $0 = \{\sigma^* h, \sigma^* f\}_{M_\mu}$ for every $\sigma^* f \in C^\infty(M_\mu)$. But $\sigma^* : C^\infty(\mathscr{M}_\mu) \to C^\infty(M_\mu)$ is a linear isomorphism. Since $\{\,,\}_{M_\mu}$ is nondegenerate, it follows that $\sigma^* h$ is locally constant on M_μ, which implies that h is locally constant on $\mathscr{M}_\mu$. So $\{\,,\}_{\mathscr{M}_\mu}$ is nondegenerate. □

We summarize the preceding discussion in the following

Theorem (singular reduction): Let $\Phi : G \times M \to M$ be an action of a Lie group G on a smooth symplectic manifold (M,ω). Suppose that this action is proper with orbit map $\pi : M \to M/G$ and is Hamiltonian with coadjoint equivariant momentum map $J : M \to \mathfrak{g}^*$. For every $\mu \in \mathfrak{g}^*$ in the image of J the singular reduced space $\mathscr{M}_\mu = \pi(J^{-1}(\mu))$ with its space of smooth functions $C^\infty(\mathscr{M}_\mu)$ is a locally compact subcartesian differential space with a nondegenerate Poisson bracket $\{\,,\}_{\mathscr{M}_\mu}$ on $C^\infty(\mathscr{M}_\mu)$.

Corollary: If in addition to the hypotheses of the singular reduction theorem we have a smooth Hamiltonian function H on M, which is G-invariant, then for every μ in the image of the G-momentum mapping J there is an induced smooth function $H_\mu : \mathscr{M}_\mu \to \mathbf{R}$, called the *reduced Hamiltonian*, which gives rise to a *reduced vector field* X_{H_μ} on $\mathscr{M}_\mu$, which is the derivation $-\mathrm{ad}_{H_\mu}$ of the Poisson algebra $(C^\infty(\mathscr{M}_\mu, \{\ ,\ \}_\mu, \cdot)$.

(7.9) **Proof**: Let $\widehat{\pi} = \pi | J^{-1}(\mu)$. Then $\widehat{\pi} : J^{-1}(\mu) \to \mathscr{M}_\mu$ is a smooth surjective mapping. Because the flow $\varphi_t^{H_\mu}$ of the Hamiltonian vector field X_H is G-invariant it preserves the level set $J^{-1}(\mu)$ of the momentum map J and induces a one parameter group of diffeomorphisms $\varphi_t^{H_\mu}$ of the locally compact subcartesian space $(\mathscr{M}_\mu, C^\infty(\mathscr{M}_\mu))$. Note that $\widehat{\pi} \circ (\varphi_t^H | J^{-1}(\mu)) = \varphi_t^{H_\mu} \circ \widehat{\pi}$. To finish the argument we need only show that integral curves of derivation $-\mathrm{ad}_{H_\mu}$ on the Poisson algebra $(C^\infty(\mathscr{M}_\mu), \{\ ,\ \}_\mu, \cdot)$ $C^\infty(\mathscr{M}_\mu)$ are induced from integral curves of the vector field X_H on $J^{-1}(\mu)$, because then $-\mathrm{ad}_{H_\mu}$ is a vector field. To see this we argue as follows. For every $f_\mu \in C^\infty(\mathscr{M}_\mu)$ we have

$$\begin{aligned}
\widehat{\pi}^*\big((\varphi^{H_\mu})_t^*\{f_\mu, H_\mu\}_\mu\big) &= \big(\varphi_t^H | J^{-1}(\mu)\big)^* \widehat{\pi}^*\{f_\mu, H_\mu\}_\mu \\
&= \big(\varphi_t^H | J^{-1}(\mu)\big)^* \big(\{\widehat{\pi}^* f_\mu, \widehat{\pi}^* H_\mu\} | J^{-1}(\mu)\big) \\
&\qquad \text{by definition of the Poisson bracket on } C^\infty(\mathscr{M}_\mu) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\big((\varphi_t^H | J^{-1}(\mu))^* \widehat{\pi}^* f_\mu\big) \\
&\qquad \text{because } \varphi_t^H | J^{-1}(\mu) \text{ is the flow of } X_H \text{ on } J^{-1}(\mu) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\big(\widehat{\pi}^*(\varphi_t^{H_\mu})^* f_\mu\big) = \widehat{\pi}^*\Big(\frac{\mathrm{d}}{\mathrm{d}t}(\varphi_t^{H_\mu})^* f_\mu\Big), \\
&\qquad \text{because the mapping } \widehat{\pi} \text{ is smooth.}
\end{aligned}$$

Consequently, for every $f_\mu \in C^\infty(\mathscr{M}_\mu)$ we get $(\varphi_t^{H_\mu})^*\{f_\mu, H_\mu\}_\mu = \frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0}(\varphi_t^{H_\mu})^* f_\mu$, since the mapping $\widehat{\pi}$ is surjective. Therefore $t \mapsto \varphi_t^{H_\mu}$ is the flow of the derivation $-\mathrm{ad}_{H_\mu}$. $\square$

Example: The 2:1 resonance. On $(\mathbf{R}^4, \omega = \sum_{i=1}^2 \mathrm{d}x_i \wedge \mathrm{d}y_i)$ consider the Hamiltonian $S^1 = \mathbf{R}/2\pi\mathbf{Z}$-action

$$\Phi : S^1 \times \mathbf{R}^4 \to \mathbf{R}^4 : \big(t, (x,y)\big) \mapsto \begin{pmatrix} \cos t & 0 & \sin t & 0 \\ 0 & \cos 2t & 0 & \sin 2t \\ -\sin t & 0 & \cos t & 0 \\ 0 & -\sin 2t & 0 & \cos 2t \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \tag{35}$$

which has a momentum mapping

$$H : \mathbf{R}^4 \to \mathbf{R} : (x,y) \to \tfrac{1}{2}\left[(y_1^2 + x_1^2) + 2\,(y_2^2 + x_2^2)\right]. \tag{36}$$

Here we have identified the dual of the Lie algebra of S^1 with $\mathbf{R}$. The action Φ restricted to $S^1 \times H^{-1}(h)$ is proper since S^1 is compact. However, it is not free, because for $h > 0$ the isotropy group $S^1_{(0,\sqrt{h},0,0)} = \{t \in \mathbf{R}/2\pi\mathbf{Z} \,\big|\, (0,\sqrt{h},0,0) = (0,\sqrt{h}\cos 2t, 0, \sqrt{h}\sin 2t)\} = \{0 \bmod 2\pi,\ \pi \bmod 2\pi\}$ is isomorphic to $\mathbf{Z}_2$. To exhibit the singular reduced space $M_h = H^{-1}(h)/S^1$ as a semialgebraic variety we use invariant theory. Introducing complex conjugate coordinates $z_j = x_j + i y_j$, $\bar{z}_j = x_j - i y_j$ for $j = 1, 2$, the action Φ becomes the

S^1-action

$$\widetilde{\Phi} : S^1 \times \mathbf{C}^4 \to \mathbf{C}^4 : \big(t,(z,\overline{z})\big) \mapsto (\mathrm{e}^{it} z_1, \mathrm{e}^{2it} z_2, \mathrm{e}^{-it}\overline{z}_1, \mathrm{e}^{-2it}\overline{z}_2). \tag{37}$$

The Hermitian monomial $z^j\overline{z}^k = z_1^{j_1} z_2^{j_2} \overline{z}_1^{k_1} \overline{z}_2^{k_2}$ is invariant under $\widetilde{\Phi}$ if and only if $j_1 + 2j_2 - k_1 - 2k_2 = 0$.

Claim: The algebra of Hermitian polynomials $\sum c_{jk} z^j \overline{z}^k$, where $c_{jk} = \overline{c_{kj}} \in \mathbf{C}$, which are invariant under $\widetilde{\Phi}$, is generated by the Hermitian monomials $z_1\overline{z}_1$, $z_2\overline{z}_2$, $z_1^2\overline{z}_2$, $z_2\overline{z}_1^2$, which satisfy the relation $(z_1^2\overline{z}_2)(z_2\overline{z}_1^2) = (z_1\overline{z}_1)^2(z_2\overline{z}_2)$ and inequalities $z_1\overline{z}_1 \ge 0$ and $z_2\overline{z}_2 \ge 0$.

(7.10) **Proof**: To show that the monomials $z_1\overline{z}_1$, $z_2\overline{z}_2$, $z_1^2\overline{z}_2$, and $z_2\overline{z}_1^2$ generate the algebra of invariant Hermitian polynomials, we need to determine all $j,k \in (\mathbf{Z}_{\ge 0})^2$ such that $j_1 + 2j_2 - k_1 - 2k_2 = 0$. Let $n_1 = j_1 - k_1$ and $n_2 = j_2 - k_2$. Then $n_1 + 2n_2 = 0$. Suppose that $n_2 \ge 0$. Then $n_2 = j_2 - k_2$ and $2n_2 = -n_1 = k_1 - j_1$. Hence the monomial

$$z_1^{j_1} z_2^{k_2+n_2} \overline{z}_1^{j_1+2n_2} \overline{z}_2^{k_2} = (z_1\overline{z}_1)^{j_1} (z_2\overline{z}_2)^{k_2} (z_2\overline{z}_1^2)^{n_2}$$

is invariant. Thus $z_1\overline{z}_1$, $z_2\overline{z}_2$ and $z_2\overline{z}_1^2$ are generators. Now suppose that $n_2 \le 0$. Then $n = -n_2 \ge 0$, $n = -n_2 = k_2 - j_2$ and $2n = -2n_2 = n_1 = j_1 - k_1$. Hence the monomial

$$z_1^{2n+k_1} z_2^{j_2} \overline{z}_1^{k_1} \overline{z}_2^{n+j_2} = (z_1\overline{z}_1)^{k_1} (z_2\overline{z}_2)^{j_2} (z_1^2\overline{z}_2)^{n}$$

is invariant. Thus $z_1\overline{z}_1$, $z_2\overline{z}_2$ and $z_1^2\overline{z}_2$ are generators. Consequently, the algebra of $\widetilde{\Phi}$-invariant Hermitian polynomials is generated by the Hermitian monomials $z_1\overline{z}_1$, $z_2\overline{z}_2$, $z_1^2\overline{z}_2$, and $z_2\overline{z}_1^2$. □

Translating the above result into real coordinates, we have shown that the algebra of real polynomials in x,y which are invariant under the S^1-action Φ (35) is generated by the monomials

$$\sigma_1 = \mathrm{Re}\, z_1^2\overline{z}_2 = x_2(x_1^2 - y_1^2) + y_2(2x_1y_1) \qquad \sigma_2 = \mathrm{Im}\, z_1^2\overline{z}_2 = x_2(2x_1y_1) - y_2(x_1^2 - y_1^2)$$

$$\sigma_3 = y_2^2 + x_2^2 \qquad \sigma_4 = y_1^2 + x_1^2\,,$$

which satisfy the relation

$$\sigma_1^2 + \sigma_2^2 = \sigma_3\sigma_4^2, \quad \sigma_3 \ge 0,\ \sigma_4 \ge 0. \tag{38}$$

Claim: The orbit space $\mathbf{R}^4/S^1$ of the S^1-action Φ is the semialgebraic variety V defined by (38).

(7.11) **Proof**: Consider the Hilbert map

$$\varsigma : \mathbf{R}^4 \to \mathbf{R}^4 : (x,y) \mapsto \big(\sigma_1(x,y), \sigma_2(x,y), \sigma_3(x,y), \sigma_4(x,y)\big).$$

It suffices to show that for every $\xi = (\xi_1,\xi_2,\xi_3,\xi_4) \in V$ the fiber $\varsigma^{-1}(\xi)$ is a single S^1 orbit of Φ. Suppose that $\xi_4 > 0$. Then solving the linear equations

$$\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} x_1^2 - y_1^2 & 2x_1y_1 \\ 2x_1y_1 & -(x_1^2 - y_1^2) \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$$

gives $x_2(x_1,y_1) = \frac{1}{\xi_4^2}[(x_1^2-y_1^2)\xi_1 + (2x_1y_1)\xi_2]$ and $y_2(x_1,y_1) = \frac{1}{\xi_4^2}[(2x_1y_1)\xi_1 - (x_1^2-y_1^2)\xi_2]$. Thus the fiber $\varsigma^{-1}(\xi)$ is $\big\{\big(x_1,y_1,x_2(x_1,y_1),y_2(x_1,y_1)\big) \in \mathbf{R}^4 \,\big|\, x_1^2+y_1^2 = \xi_4\big\}$, since

$$(x_2(x_1,y_1))^2 + (y_2(x_1,y_1))^2 = \tfrac{1}{\xi_4^2}(\xi_1^2+\xi_2^2) \,=\, \xi_3.$$

Suppose that $\xi_4 = 0$, then $x_1 = y_1 = 0$. Hence $\xi_1 = \xi_2 = 0$. So the fiber $\varsigma^{-1}(0,0,\xi_3,0) = \{(0,0,x_2,y_2) \in \mathbf{R}^4 \,|\, x_2^2+y_2^2 = \xi_3\}$. Therefore for every $\xi \in V$, the fiber $\varsigma^{-1}(\xi)$ is a single S^1 orbit of Φ. □

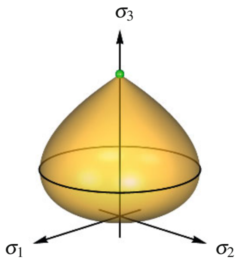

Figure 7.1. The reduced phase space M_h.

Since the momentum mapping H (36) is S^1-invariant, the reduced space $M_h = H^{-1}(h)/S^1$ is defined by (38) and $\sigma_4 + 2\sigma_3 = 2h$. After eliminating σ_4, we find that M_h is the semi-algebraic variety defined by

$$\sigma_1^2 + \sigma_2^2 = 4\sigma_3(h-\sigma_3)^2, \qquad 0 \le \sigma_3 \le h,$$

M_h is a compact surface, which is homeomorphic to S^2 with a conical singularity at $(0,0,h)$, since we can rewrite its defining equation as

$$\sigma_1^2 + \sigma_2^2 - 4h(h-\sigma_3)^2 + 4(h-\sigma_3)^3 = 0,$$

see figure 7.1. □

7.2 Stratification of the singular reduced space

In this subsection we show that the image under the G-orbit map π of connected components of the intersection of the μ-level set of the momentum map J with the manifold $M_H = \{m \in M \,\big|\, G_m = H\}$, where H is a compact subgroup of G, form a stratification of the singular reduced space $\mathscr{M}_\mu = \pi(J^{-1}(\mu))$ by smooth connected symplectic manifolds.

First we construct the strata of the singular reduced space $\mathscr{M}_\mu$. We start by proving

▷ $(M_H, \omega|M_H)$ is a smooth symplectic submanifold of (M,ω).

(7.12) **Proof**: Let $p \in M_H$. We begin by showing that $T_pM_H = (T_pM)^H = \{v_p \in T_pM \,\big|\, T_p\Phi_h v_p = v_p$ for all $h \in H\}$ is a symplectic subspace of $\big(T_pM, \omega(p)\big)$. Because ω is a G-invariant symplectic form on M, the form $\omega(p)$ is an $H = G_p$-invariant symplectic form on T_pM. Since H is compact there is an H-invariant inner product $\gamma(p)$ on T_pM. Define the linear mapping $A_p : T_pM \to T_pM$ by $A_p = \gamma^\flat(p)\circ\omega^\sharp(p)$. Then $j_p = A_p(-A_p^2)^{-1/2}$ is an H-invariant almost complex structure on T_pM such that $j_p^2 = -\mathrm{id}_{T_pM}$ and $\omega(p)(v_p, w_p) = \gamma(p)(j_p(v_p), w_p)$ for every v_p, $w_p \in T_pM$. The following argument shows that $(T_pM)^H$ is j_p-invariant. From the definition $v_p \in (T_pM)^H$ if and only if $T_p\Phi_h v_p = v_p$ for every $h \in H$. Because j_p is H-invariant, we get $T_p\Phi_h j_p(v_p) = j_p(T_p\Phi_h v_p) = j_p(v_p)$. So $j_p(v_p) \in (T_pM)^H$, that is, $(T_pM)^H$ is j_p-invariant. The next argument shows that $(T_pM)^H$ is an $\omega(p)|T_pM_H$ symplectic subspace of $\big(T_pM, \omega(p)\big)$. For some $v_p \in (T_pM)^H$ suppose that $0 = \omega(p)(v_p, w_p)$ for every $w_p \in (T_pM)^H$. Then $0 = \gamma(p)(j_p(v_p), w_p)$ for every $w_p \in (T_pM)^H$. This gives $j_p(v_p) = 0$ since $\gamma(p)$ is nondegenerate on $(T_pM)^H$. But j_p is an invertible linear map of $(T_pM)^H$ into itself, so $v_p = 0$, that is, $\omega(p)|(T_pM)^H$ is nondegenerate. Since ω is a closed 2-form on M it follows that $\omega|M_H$ is a closed 2-form on M_H. Therefore $(M_H, \omega|M_H)$ is a smooth symplectic manifold, which by ((1.7)) is a submanifold of (M, ω). □

Let N be a connected component of M_H. The stability group Stab_N of N is $\{g \in G \,\big|\, g\cdot n \in N$ for every $n \in N\}$. Now Stab_N is a closed subgroup, and hence is a Lie subgroup, of the Lie group $N(H) = \{g \in G \,\big|\, g \cdot M_H = M_H\}$. Note that H acts trivially on N, for if $n \in N$ then $H \cdot n = n$ because $N \subseteq M_H$. Moreover, H is a normal subgroup of Stab_N. Let $\mathfrak{h}$, $\mathfrak{s}_N$, and $\mathfrak{g}_N$ be the Lie algebras of the Lie groups H, Stab_N, and $G_N = \mathrm{Stab}_N/H$, respectively. Because H acts trivially on N, we may consider G_N to be a subgroup of $\mathrm{Stab}_N \subseteq G$. The Lie group G_N acts freely and properly on N. Moreover, it preserves the symplectic form $\omega_N = \omega|N$.

Claim: The G_N-action $\Phi|(G_N \times N)$ on (N, ω_N) has a momentum mapping $J_N : N \to \mathfrak{g}_N^*$.

(7.13) **Proof**: First we define the mapping J_N. For each $\xi \in \mathfrak{g}$ we have $X^\xi \mathbin{\lrcorner} \omega = \mathrm{d}J^\xi$. Because $J : M \to \mathfrak{g}^*$ is a momentum mapping. If $\xi \in \mathfrak{h}$ then the vector field X^ξ vanishes identically on N, since H acts trivially on N. Therefore $\mathrm{d}J^\xi = 0$ on N. Let $\kappa : \mathfrak{h} \to \mathfrak{g}$ be the inclusion mapping. Then its transpose $\kappa^t : \mathfrak{g}^* \to \mathfrak{h}^*$ is surjective. The preceding argument shows that the mapping $\kappa^t\circ(J|N) : N \to \mathfrak{h}^*$ is constant. Let $\lambda : \mathfrak{h} \to \mathfrak{s}_N$ be the inclusion mapping. Then its transpose $\lambda^t : \mathfrak{s}_N^* \to \mathfrak{h}^*$ is surjective. Hence there is $j_N \in \mathfrak{s}_N^*$ such that $\lambda^t(j_N) = \kappa^t\circ(J|N)$. Let $\nu : \mathfrak{s}_N \to \mathfrak{g}$ be the inclusion mapping and let $\eta : \mathfrak{s}_N \to \mathfrak{g}_N = \mathfrak{s}_N/\mathfrak{h}$ be the canonical projection map. Then $\ker\eta$ is $\mathfrak{h}$. Every element of $\ker\eta$ is mapped to 0 by $\nu^t\circ(J|N) - j_N : N \to \mathfrak{s}_N^*$, because $\nu\circ\lambda = \kappa$ implies that $\lambda^t\circ\big(\nu^t\circ(J|N) - j_N\big) = 0^* \in \mathfrak{h}^*$. Hence there is a unique mapping $J_N : N \to \mathfrak{g}_N^*$ such that

$$\eta^t\circ J_N = \nu^t\circ(J|N) - j_N. \tag{39}$$

We now show that J_N is a momentum mapping for the G_N-action on N. For each $\xi \in \mathfrak{s}_N$ we have $\nu(\xi) \in \mathfrak{g}$. The G-action on M restricted to the one parameter group $t \mapsto \exp t\nu(\xi)$ is generated by the vector field $X_M^{\nu(\xi)}$. Similarly, the G_N-action on N restricted to the one parameter group $t \mapsto \exp t\eta(\xi)$ is generated by the vector field $X_N^{\eta(\xi)}$ on N. On N the G-action restricted to $t \mapsto \exp\nu(\xi)$ and the G_N-action restricted to $t \mapsto \exp\eta(\xi)$ coincide. So $X_N^{\eta(\xi)} = X_M^{\nu(\xi)}|N$. Restricting $X_M^{\nu(\xi)} \mathbin{\lrcorner} \omega = \mathrm{d}J^{\nu(\xi)}$ to N gives

$$X_N^{\eta(\xi)} \mathbin{\lrcorner} \omega_N = \mathrm{d}(J|N)^{\nu(\xi)} = \mathrm{d}\big((\nu^t\circ(J|N))(\xi)\big) = \mathrm{d}\big((\eta^t\circ J_N + j_N)(\xi)\big) = \mathrm{d}J_N^{\eta(\xi)}.$$

The last equality above follows because j_N is a fixed element of $\mathfrak{s}_N^*$. Hence $J_N : N \to \mathfrak{g}_N^*$ is a momentum mapping for the G_N-action on N. $\square$

▷ For every $\alpha \in \mathfrak{g}_N^*$, which lies in the image of J_N, every connected component of $J_N^{-1}(\alpha)$ is a smooth submanifold of N.

(7.14) **Proof**: Since the G_N-action on N is free, it follows that $X_N^{\xi}(n) \ne 0$ for every $\xi \in \mathfrak{g}_N$ and every $n \in N$. Therefore $\mathrm{d}(J_N)^{\xi}(n) \ne 0$ for every $\xi \in \mathfrak{g}_N$ and every $n \in N$. Thus for every $n \in N$ the mapping $\mathrm{d}J_N(n) : T_nN \to \mathfrak{g}_N^*$ is onto. So for every $\alpha \in \mathfrak{g}_N^*$, which lies in the image of J_N, the α-level set $J_N^{-1}(\alpha)$ is a submanifold of N, which may not be connected. $\square$

Claim. For every $n \in N$, the connected component of $N \cap J^{-1}(J(n))$ and $J_N^{-1}(J_N(n))$ containing the point n coincide.

(7.15) **Proof**: We have $\eta^t \circ J_N(n) = \nu^t(J(n)) - j_N(n)$ for every $n \in N$. Moreover, we have shown that j_N is a fixed element of $\mathfrak{s}_N^*$. Since $\eta^t : \mathfrak{g}_N^* \to \mathfrak{s}_N^*$ is injective and $\nu^t : \mathfrak{g}^* \to \mathfrak{s}_N^*$ is surjective, it follows that the connected component of $N \cap J^{-1}(J(n))$ containing the point n is a subset of the connected component of $J_N^{-1}(J_N(n))$ containing the point n. Since the connected components of $J_N^{-1}(J_N(n))$ are submanifolds of N, to complete the proof it suffices to show that the function J is constant on each connected component of $J_N^{-1}(J_N(n))$, for then a connected component of $J_N^{-1}(J_N(n))$ containing the point n is a subset of a connected component of $N \cap J^{-1}(J(n))$ containing the point n.

Towards this goal let $\mathfrak{n} = \{\xi \in \mathfrak{g} \,\big|\, \mathrm{Ad}_h\xi = \xi \text{ for every } h \in H\}$ and let $\mathfrak{l} = \{\xi \in \mathfrak{g} \,\big|\, \overline{\xi} = \int_H \mathrm{Ad}_h\xi \, \mathrm{d}h = 0, \text{ where } \int_H \mathrm{d}h = 1\}$. Then $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{l}$, since for every $\xi \in \mathfrak{g}$ we have $\xi = \overline{\xi} + (\xi - \overline{\xi}) \in \mathfrak{n} + \mathfrak{l}$; while if $\xi \in \mathfrak{n} \cap \mathfrak{l}$, then $\xi = \overline{\xi} = 0$. Now decompose the momentum mapping $J|N : N \to \mathfrak{g}^*$ into components: $J^{\mathfrak{n}} : N \to \mathfrak{n}^*$ and $J^{\mathfrak{l}} : N \to \mathfrak{l}^*$. If $\xi = \xi_{\mathfrak{n}} + \xi_{\mathfrak{l}} \in \mathfrak{n} \oplus \mathfrak{l}$, then $(J|N)^{\xi} = (J^{\mathfrak{n}})^{\xi_{\mathfrak{n}}} + (J^{\mathfrak{l}})^{\xi_{\mathfrak{l}}}$.

▷ For every $\zeta \in \mathfrak{l}$ we have $X^{\zeta}(n) \in (T_nN)^{\omega(n)}$ for every $n \in N$.

(7.16) **Proof**: Because $\omega|N$ is H-invariant, for every $v_n \in T_nN$ we get

$$
\begin{aligned}
\omega(n)\big(X^{\zeta}(n), v_n\big) &= \int_H \omega(h\cdot n)\big(T_n\Phi_h X^{\zeta}(n), T_n\Phi_h v_n\big)\, \mathrm{d}h \\
&= \int_H \omega(n)\big(X^{\mathrm{Ad}_h\zeta}(n), v_n\big)\, \mathrm{d}h, \\
&\qquad \text{since } H \text{ acts trivially on } N \text{ and } T_nN = (T_nM_H)^H \\
&= \omega(n)\big(X^{\int_H \mathrm{Ad}_{h^{-1}}\zeta\, \mathrm{d}h}(n), v_n\big) = \omega(n)\big(X^{\overline{\zeta}}(n), v_n\big) \\
&= 0, \quad \text{since } \overline{\zeta} = 0 \text{ because } \zeta \in \mathfrak{l}.
\end{aligned}
$$
$\square$

Fact: $\xi \in \mathfrak{n}$ if and only if $X^{\xi}(n) \in T_nN$ for every $n \in N$.

(7.17) **Proof**: Suppose that $\xi \in \mathfrak{n}$. Then for every $h \in H$ we have $T_n\Phi_h X^{\xi}(n) = X^{\mathrm{Ad}_h\xi}(n)$. Since $\xi \in \mathfrak{n}$, by definition $\mathrm{Ad}_h\xi = \xi$ for every $h \in H$. So $T_n\Phi_h X^{\xi}(n) = X^{\xi}(n)$ for every $h \in H$, that is, $X^{\xi}(n) \in (T_nM_H)^H = T_nN$. Conversely, suppose that $X^{\xi}(n) \in T_nN$. Then for every

$h \in H$ we have $X^{\xi}(n) = T_n\Phi_h X^{\xi}(n) = X^{\mathrm{Ad}_h \xi}(n)$. This implies

$$X^{\xi}(n) = \int_H X^{\mathrm{Ad}_h \xi}(n)\, \mathrm{d}h = X^{\int_H \mathrm{Ad}_h \xi}(n) = X^{\overline{\xi}}(n).$$

Consequently, $X^{\xi - \overline{\xi}}(n) = 0$ for all $n \in N$, that is, $\Phi_{\exp t(\xi - \overline{\xi})}(n) = n$. In other words, $t \mapsto \exp t(\xi - \overline{\xi})$ is a one parameter subgroup of H. So $\xi - \overline{\xi} = \eta \in \mathfrak{h}$. This yields $\overline{\eta} = \overline{\xi - \overline{\xi}} = \overline{\xi} - \overline{\xi} = 0$. Thus $\eta \in \mathfrak{l}$. Suppose that $\eta \neq 0$. Then $0 \neq X^{\eta}(n) \in (T_nN)^{\omega(n)}$. But $\overline{\xi} \in \mathfrak{n}$, which implies that $X^{\overline{\xi}}(n) \in T_nN$. So $X^{\xi}(n) = X^{\overline{\xi}}(n) + X^{\eta}(n) \notin T_nN$, which contradicts our hypothesis. Therefore $\eta = 0$, that is, $\xi = \overline{\xi} \in \mathfrak{n}$. □

We return to proving ((7.15)). For every $\xi \in \mathfrak{n}$ the vector field $X_{J^{\xi}}$ is tangent to N. Hence the G-action Φ restricted to the one parameter group $t \mapsto \exp t\nu(\xi)$ preserves N. So $\xi \in \mathfrak{s}_N$, the Lie algebra of the stability group Stab_N of N. Using (39) we get $(J^{\mathfrak{n}})^{\xi} = J^{\nu(\xi)} = \eta^t \circ J_N^{\eta(\xi)} + j_N(\xi)$, where $\nu : \mathfrak{s}_N \to \mathfrak{g}$ is the inclusion mapping and $\eta : \mathfrak{s}_N \to \mathfrak{g}_N = \mathfrak{s}_N/\mathfrak{h}$ is the canonical projection map. So $J^{\mathfrak{n}} = \eta^t \circ J_N + j_N$. Since j_N is constant on N, it follows that $J^{\mathfrak{n}}$ is constant on level sets of J_N. For every $n \in N$ and every $\xi \in \mathfrak{l}$ we have $X_{J^{\xi}}(n) \in (T_nN)^{\omega(n)}$. Thus we get $\mathrm{d}J^{\xi}(n)w_n = \omega(n)(X_{J^{\xi}}(n), w_n) = 0$ for every $w_n \in T_nN$. Hence for every $n \in N$ we have $\mathrm{d}J^{\mathfrak{l}}(n) = 0$. So $J^{\mathfrak{l}} : N \to \mathfrak{l}^*$ is constant on N. Thus $J|N = J^{\mathfrak{n}} + J^{\mathfrak{l}}$ is constant on level sets of J_N and hence on connected components of $(J_N)^{-1}(J_N(n))$. Hence for every $n \in N$ the connected components of $(J_N)^{-1}(J_N(n))$ and $N \cap J^{-1}(J(n))$, which contain the point n, coincide. □

Claim: For every $n \in N$ the connected component of $N \cap J^{-1}(J(n))$, which contains the point n is a smooth submanifold of N.

(7.18) **Proof**: Since the connected component of $(J_N)^{-1}(J_N(n))$ and $N \cap J^{-1}(J(n))$, which contains the point n, are equal and every connected component of $(J_N)^{-1}(J_N(n))$ is a smooth submanifold of N, it follows that the connected component of $N \cap J^{-1}(J(n))$, which contains the point n is a smooth submanifold of N. □

We now apply the argument in the proof the regular reduction theorem ((6.1)) to the α level set of the G_N-momentum mapping J_N on the smooth symplectic manifold $(N, \omega|N)$ to construct a connected component of a stratum of the singular reduced space.

Because the following arguments are local, we use the shifting trick ((5.19)) to replace the study of the μ-level set J with the α level set $J_N^{-1}(\alpha)$, where $\alpha = J_N(n)$ for some $n \in N$ such that $\widehat{J}(n) = (J|N)(n) = 0$.

To start we must verify that for every $q \in J_N^{-1}(\alpha)$ we have

$$T_q(J_N^{-1}(\alpha)) \cap T_q(J_N^{-1}(\alpha))^{\omega(q)} = T_q\mathscr{O}_q, \tag{40}$$

where $\mathscr{O}_q = G_N \cdot q$. Because the momentum mapping J_N is not necessarily coadjoint equivariant, equation (40) does *not* follow from ((5.11)).

(7.19) **Proof**: From ((7.15)) we see that $T_q(\widehat{J}^{-1}(0)) = T_q(J_N^{-1}(\alpha))$. Therefore (40) is equivalent to

$$T_q(\widehat{J}^{-1}(0)) \cap T_q(\widehat{J}^{-1}(0))^{\omega(q)} = T_q\mathscr{O}_q, \tag{41}$$

for every $q \in \widehat{J}^{-1}(0)$. To establish (41) let S_q be a slice to the G_N-action on $\widehat{J}^{-1}(0)$ at q, which is defined because $\widehat{J}^{-1}(0)$ is invariant under the G_N-action on N. From the definition of slice it follows that

$$T_q(\widehat{J}^{-1}(0) \cap S_q) \oplus T_q\mathscr{O}_q = T_q(\widehat{J}^{-1}(0)). \tag{42}$$

By construction $T_q(S_q \cap N) = (T_qS_q)^{H_N}$, the set of $H_N = (G_N)_q$-fixed vectors in T_qS_q. Then we obtain $T_q(\widehat{J}^{-1}(0) \cap S_q) = \ker T_q\widehat{J} \cap (T_qS_q)^{H_N} \subseteq \ker T_q\widehat{J}$. Using $\ker T_q\widehat{J} = T_q(\widehat{J}^{-1}(0)) = T_q(J_N^{-1}(\alpha)) = \ker T_qJ_N$, which follows from the fact that J_N is a G_N-momentum mapping, we get

$$T_q\mathscr{O}_q = (\ker T_qJ_N)^{\omega q} = (\ker T_q\widehat{J})^{\omega(q)} \subseteq T_q(\widehat{J}^{-1}(0) \cap S_q)^{\omega(q)}. \tag{43}$$

Because $\widehat{J}^{-1}(0)$ is G_N-invariant, the G_N-orbit $\mathscr{O}_q$ through $q \in N$ is contained in $\widehat{J}^{-1}(0)$. Consequently, $T_q\mathscr{O}_q \subseteq T_q(\widehat{J}^{-1}(0)) = \ker T_q\widehat{J}$. So $(\ker T_q\widehat{J})^{\omega(q)} \subseteq (T_q\mathscr{O}_q)^{\omega(q)}$, which implies

$$T_q\mathscr{O}_q \subseteq (T_q\mathscr{O}_q)^{\omega(q)}, \tag{44}$$

for every $q \in \widehat{J}^{-1}(0)$, using (43). The next argument shows that $T_q(\widehat{J}^{-1}(0) \cap S_q)$ is a symplectic subspace of $(T_qN, \omega(q))$. Let j_q be an H_N-invariant almost complex structure on T_qN, which is constructed in the same way as in the proof of ((7.12)). Then $\omega(q)(v_q, w_q) = \gamma(q)(j_q(v_q), w_q)$ for every v_q, $w_q \in T_qN$, where γ is an H_N-invariant Riemannian metric on N. For every subspace W of T_qN we have $(W^{\perp})^{\omega(q)} = j_q(W)$, because $(W^{\perp})^{\omega(q)} = \{u_q \in W \,\big|\, \omega(q)(u_q, v_q) = 0 \text{ for every } v_q \in W^{\perp}\} = \{u_q \in T_qN \,\big|\, j_q(u_q) \in$
$\triangleright$ $(W^{\perp})^{\perp} = W\} = j_q(W)$. Now $\omega(q)$ is nondegenerate on $T_q(\widehat{J}^{-1}(0) \cap S_q)$.

(7.20) **Proof**: First we show that $T_q(\widehat{J}^{-1}(0) \cap S_q)$ is a j_q-invariant subspace of T_qN. By construction of the slice S_q we have $T_qS_q = (T_q\mathscr{O}_q)^{\perp}$. So

$$j_q(\ker T_q\widehat{J}) = \big((\ker T_q\widehat{J})^{\perp}\big)^{\omega(q)} = \big((\ker T_q\widehat{J})^{\omega(q)}\big)^{\perp} = (T_q\mathscr{O}_q)^{\perp} = T_qS_q,$$

which gives $\ker T_q\widehat{J} \cap T_qS_q = j_q(T_qS_q) \cap T_qS_q$. Thus $\ker T_q\widehat{J} \cap T_qS_q$ is j_q-invariant. Now

$$T_q(\widehat{J}^{-1}(0) \cap S_q) = \ker T_q\widehat{J} \cap (T_qS_q)^{H_N} = (\ker T_q\widehat{J} \cap T_qS_q) \cap (T_qN)^{H_N}.$$

But $(T_qN)^{H_N}$ is j_q-invariant. Consequently, $T_q(\widehat{J}^{-1}(0) \cap S_q)$ is j_q-invariant, being the intersection of j_q-invariant subspaces. Now suppose that for some $u_q \in T_q(\widehat{J}^{-1}(0) \cap S_q)$ we have $\omega(q)(u_q, v_q) = 0$ for every $v_q \in T_q(\widehat{J}^{-1}(0) \cap S_q)$. Then $0 = \gamma(q)(j_q(u_q), v_q)$ for every $v_q \in T_q(\widehat{J}^{-1}(0) \cap S_q)$. But $\gamma(q)|T_q(\widehat{J}^{-1}(0) \cap S_q)$ is nondegenerate. So $j_q(u_q) = 0$. However, j_q is an invertible linear mapping of $T_q(\widehat{J}^{-1}(0) \cap S_q)$ into itself. So $u_q = 0$, which implies that $\omega(q)$ is nondegenerate on $T_q(\widehat{J}^{-1}(0) \cap S_q)$. $\square$

We now complete the proof of equation (41). Since $T_q(\widehat{J}^{-1}(0) \cap S_q)$ is an $\omega(q)$-symplectic subspace of $(T_qN, \omega(q))$, so is $T_q(\widehat{J}^{-1}(0) \cap S_q)^{\omega(q)}$. Thus their intersection is the zero vector. The Witt decomposition of the symplectic vector space $\big(T_q(\widehat{J}^{-1}(0) \cap S_q)^{\omega(q)}, \omega(q)\big)$ with respect to the $\omega(q)$-isotropic subspace $T_q\mathscr{O}_q$ is $T_q(\widehat{J}^{-1}(0) \cap S_q)^{\omega(q)} = Z \oplus Z^{\omega(q)}$, where $T_q\mathscr{O}_q{}^{\omega(q)} = Z \oplus \big(T_q\mathscr{O}_q \cap (T_q\mathscr{O}_q)^{\omega(q)}\big) = Z \oplus T_q\mathscr{O}_q$, Z is a symplectic subspace of

$T_q(\widehat{J}^{-1}(0) \cap S_q)$, and $T_q\mathscr{O}_q$ is a Lagrangian subspace of $Z^{\omega(q)}$. Since $Z^{\omega(q)} = T_q\mathscr{O}_q \oplus T_q^*\mathscr{O}_q$, it follows that

$$T_q(\widehat{J}^{-1}(0) \cap S_q)^{\omega(q)} = Z \oplus T_q\mathscr{O}_q \oplus T_q^*\mathscr{O}_q = (T_q\mathscr{O}_q)^{\omega(q)} \oplus T_q^*\mathscr{O}_q. \tag{45}$$

Before finishing the proof of equation (41) we verify

Fact. Let X, Y, and Z be subspaces of a finite dimensional real vector space V such that $Y \subseteq Z$ and $X \cap Z = \{0\}$. Then

$$(X \oplus Y) \cap Z = Y. \tag{46}$$

(7.21) **Proof**: Since $Y \subseteq X \oplus Y$ and $Y \subseteq Z$, we get $Y \subseteq (X \oplus Y) \cap Z$. Now suppose that $w \in X \oplus Y$ and $w \in Z$. Then for some $x \in X$ and $y \in Y$ we can write $w = x + y$. So $x = w - y \in Z$, because $w \in Z$ and $y \in Y \subseteq Z$. Thus $x \in X \cap Z = \{0\}$, which shows that $w = y \in Y$. Consequently, $(X \oplus Y) \cap Z \subseteq Y$, which verfies (46). □

Returning to the proof of (41), we have

$$\begin{aligned} T_q(\widehat{J}^{-1}(0)) \cap T_q(\widehat{J}^{-1}(0))^{\omega(q)} &= (T_q(\widehat{J}^{-1}(0) \cap S_q) \oplus T_q\mathscr{O}_q) \cap (T_q(\widehat{J}^{-1}(0) \cap S_q) \oplus T_q\mathscr{O}_q)^{\omega(q)} \\ &= (T_q(\widehat{J}^{-1}(0) \cap S_q) \oplus T_q\mathscr{O}_q) \cap T_q(\widehat{J}^{-1}(0) \cap S_q)^{\omega(q)} \cap (T_q\mathscr{O}_q)^{\omega(q)} \\ &= T_q\mathscr{O}_q, \end{aligned}$$

using (46), since $T_q(\widehat{J}^{-1}(0) \cap S_q) \cap \Big(T_q(\widehat{J}^{-1}(0) \cap S_q)^{\omega(q)}\Big) \cap (T_q\mathscr{O}_q)^{\omega(q)}\Big) = \{0\}$; while from $T_q\mathscr{O}_q \subseteq T_q(\widehat{J}^{-1}(0) \cap S_q)^{\omega(q)}$ using (45), and $T_q\mathscr{O}_q \subseteq (T_q\mathscr{O}_q)^{\omega(q)}$, using (44), we get $T_q\mathscr{O}_q \subseteq T_q(\widehat{J}^{-1}(0) \cap S_q)^{\omega(q)} \cap (T_q\mathscr{O}_q)^{\omega(q)}$. □

We summarize what we have already shown. We have a free and proper action of a Lie group G_N on a smooth connected symplectic manifold (N, ω_N). This group action on (N, ω_N) is Hamiltonian with a momentum mapping $J_N : N \rightarrow \mathfrak{g}_N^*$ such that for $\alpha \in \mathfrak{g}_N^*$ $J_N^{-1}(\alpha)$ is a smooth submanifold of N such that $T_qJ_N^{-1}(\alpha) \cap \big(T_qJ_N^{-1}(\alpha)\big)^{\omega(q)} = G_N \cdot q$ for
▷ every $q \in J_N^{-1}(\alpha)$. Let $\pi_N : N \rightarrow N/G_N$ be the G_N-orbit mapping. Then $\pi(J_N^{-1}(\alpha))$ is a smooth symplectic submanifold of the smooth manifold N/G_N with symplectic form $\widetilde{\omega}$ where $\pi_N^*(\widetilde{\omega}) = \omega_N | J_N^{-1}(\alpha)$.

(7.22) **Proof**: The G_N-action on N leaves the connected component C of $N \cap J^{-1}(0)$ containing the point n, which is a smooth submanifold of N, invariant. To see this note that the G-momentum mapping J is G-coadjoint equivariant, which implies that J is G_N-coadjoint equivariant. Consequently, $J^{-1}(0)$ is G_N-invariant. By definition G_N leaves N invariant. Because for some $\alpha \in \mathfrak{g}_N^*$, C is the connected component of $J_N^{-1}(\alpha)$ containing the point n. Thus G_N-action on C leaves C invariant. This action is free and proper, preserves the symplectic form $\omega_C = \omega_N | C$, and is Hamiltonian with momentum mapping $J_N | C$. Let $\pi_C = \pi_N | C : C \rightarrow \widetilde{C} = C/G_N$ be the G_N-orbit map on N restricted to C. Because π_C is a surjective submersion, it follows that $\widetilde{C}$ is a connected smooth submanifold of N/G_N.

We now show that $\widetilde{C}$ is a symplectic manifold. First observe that for every $q \in C$ we have $\ker \omega_C(q) = T_qJ_N^{-1}(\alpha) \cap \big(T_qJ_N^{-1}(\alpha)\big)^{\omega(q)} = W_q$, which is an $\omega_C(q)$-isotropic subspace of $(T_qC, \omega_C(q)) = (T_qN, \omega(q))$. Thus there is an $\omega_C(q)$-symplectic subspace X_q of T_qN such that $T_qC = X_q \oplus W_q$. Since $W_q = T_q(G_N \cdot q)$ we obtain $\ker T_q\pi_C = W_q$. Consequently, $T_q\pi_C$

maps X_q bijectively onto $Y_{\pi_C(q)} = T_{\pi_C(q)}\widetilde{C}$. On $Y_{\pi_C(q)}$ define a skew symmetric bilinear form $\widetilde{\omega}_{\widetilde{C}}(\pi_C(q))(\widetilde{x}_{\pi_C(q)}, \widetilde{y}_{\pi_C(q)}) = \omega_C(q)(x_q, y_q)$, where x_q, $y_q \in X_q$, $T_q\pi_C x_q = \widetilde{x}_{\pi_C(q)}$, and $T_q\pi_C y_q = \widetilde{y}_{\pi_C(q)}$. In other words, $\pi_C^*\widetilde{\omega}_{\widetilde{C}} = \omega_C$. Because $T_q\pi_C : X_q \to T_{\pi_C(q)}X_q = T_{\pi_C(q)}\widetilde{C}$ is a bijective linear mapping, we obtain that $\widetilde{\omega}_{\widetilde{C}}$ is a nondegenerate 2-form on $\widetilde{C}$. Since $\pi_C^*(\mathrm{d}\widetilde{\omega}_{\widetilde{C}}) = \mathrm{d}\,\pi_C^*(\widetilde{\omega}_{\widetilde{C}}) = \mathrm{d}\omega_C = 0$ and π_C is a submersion, it follows that $\mathrm{d}\widetilde{\omega}_{\widetilde{C}} = 0$. Thus $\widetilde{\omega}_{\widetilde{C}}$ is a symplectic form on $\widetilde{C}$. □

The above result completes the proof of

Claim: The singular reduced space $(\mathscr{M}_\mu, C^\infty(\mathscr{M}_\mu))$ is stratified by the connected component of the smooth symplectic manifolds $\mathscr{M}_{(H)} = \pi(J^{-1}(\mu)\cap M_H) = \pi(J^{-1}(\mu)\cap M_{(H)})$, where $H = G_m$ for some $m \in M$ and $M_{(H)} = G\cdot M_H$ is the orbit type corresponding to H.

▷ We now show that the flow of a Hamiltonian derivation corresponding to a smooth function on the singular reduced space $\mathscr{M}_\mu$ preserves the stratification of $\mathscr{M}_\mu$.

(7.23) **Proof**: First use the shifting trick to reduce to the case where $\mu = 0$. Let f_0 be a smooth function on the symplectic stratum $\mathscr{M}_{(H)}$ of $\mathscr{M}_0$. Then $\pi^* f_0 = f|J_{(H)}$ for some smooth G-invariant function f on M. Here $J_{(H)} = G\cdot(J^{-1}(0))\cap M_H)$. Let φ_t^f be the flow of the vector field X_f. First we verify that $\varphi_t^f(J_{(H)}) \subseteq J_{(H)}$. It suffices to show that $\varphi_t^f(J^{-1}(0)) \subseteq J^{-1}(0)$ and $\varphi_t^f(M_H) \subseteq M_H$. The proof of the former inclusion follows because $f \in C^\infty(M)^G$. To prove the latter inclusion, suppose that $p \in M_H$. Because $f \in C^\infty(M)^G$, the flow φ_t^f and the G-action Φ_g commute for every $g \in G$. Thus for every $h \in H$

$$\Phi_h(\varphi_t^f(p)) = \varphi_t^f(\Phi_h(p)) \, = \, \varphi_t^f(p). \tag{47}$$

In other words, $\varphi_t^f(p) \in M_{[H]} = \{m \in M \,\big|\, H \subseteq G_m\}$. Because M_H is relatively open in $M_{[H]}$ ((1.7)), there is an open neighborhood U of p in M such that $U\cap M_H = U\cap M_{[H]}$. Let $q \in U\cap M_H$. Then repeating the argument in (47) shows that $\varphi_t^f(q) \in U\cap M_{[H]}$, that is, $\varphi_t^f(U\cap M_H) \subseteq U\cap M_{[H]}$. Because φ_t^f is a homeomorphism of M, it induces a homeomorphism of M_H in the relative topology. Therefore $\varphi_t^f(U\cap M_H)$ is a relative open subset of $M_{[H]}$ containing $\varphi_t^f(p)$. Consequently, $\varphi_t^f(p) \in M_H$. Thus we obtain $\varphi_t^f(J_{(H)}) \subseteq J_{(H)}$. Because the G-action Φ_g intertwines the flow φ_t^f of the vector field X_f with the flow $\varphi_t^{f_0}$ of the Hamiltonian derivation $-\mathrm{ad}_{f_0}$, it follows that the flow $\varphi_t^{f_0}$ maps the stratum $\mathscr{M}_{(H)}$ of $\mathscr{M}_0$ into itself. □

▷ The inclusion map $\widetilde{\imath} : (\mathscr{M}_{(H)}, C^\infty(\mathscr{M}_{(H)})) \to (\mathscr{M}_0, C_i^\infty(\mathscr{M}_0))$ is a Poisson map.

(7.24) **Proof**: Since $J_{(H)}$ is a G-invariant symplectic submanifold of (M,ω), it follows that the inclusion map $i : J_{(H)} \to M$ induces a homomorphism of Poisson algebras

$$i^* : (C^\infty(M)^G, \{\,,\,\}|J_{(H)}, \cdot) \to (C^\infty(M)^G|J_{(H)}, \{\,,\,\}, \cdot). \tag{48}$$

Suppose that $f_0, g_0 \in C^\infty(\mathscr{M}_0)$, then $f_0|\mathscr{M}_0$, $g_0|\mathscr{M}_0 \in C_i^\infty(\mathscr{M}_{(H)})$. Moreover, there are functions $f, g \in C^\infty(M)^G$ such that $\pi^*(f_0|\mathscr{M}_{(H)}) = f|J_{(H)}$ and $\pi^*(g_0|\mathscr{M}_{(H)}) = g|J_{(H)}$.

Therefore for every $q \in J_{(H)}$ we have

$$\begin{aligned}\{f_0|\mathscr{M}_{(H)}, g_0|\mathscr{M}_{(H)}\}_{\mathscr{M}_{(H)}}(\pi(q)) &= \{f|J_{(H)}, g|J_{(H)}\}(q), \quad \text{by definition of } \{\,,\,\}_{\mathscr{M}_{(H)}}.\\ &= \{f,g\}(q), \quad \text{using (48)}\\ &= \{f_0, g_0\}_0(\pi(q)), \quad \text{by definition of } \{\,,\,\}_0.\end{aligned}$$

Thus the induced mapping

$$\widetilde{\imath}^* : (C^\infty(\mathscr{M}_0), \{\,,\,\}_{\mathscr{M}_0}) \to (C_i^\infty(\mathscr{M}_{(H)})), \{\,,\,\}_{\mathscr{M}_{(H)}})$$

is a Poisson map. Here $\{\,,\,\}_{\mathscr{M}_{(H)}}$ is the standard Poisson bracket associated to the symplectic form $\widetilde{\omega}_{\mathscr{M}_{(H)}}$ on the smooth stratum $\mathscr{M}_{(H)}$. □

Claim: The decomposition of the singular reduced space $\mathscr{M}_\mu$ into symplectic strata is encoded in the Poisson algebra $(C^\infty(\mathscr{M}_\mu), \{\,,\,\}_\mu, \cdot)$.

(7.25) **Proof**: To see this recall that on a connected symplectic manifold the group generated by the time one maps of the flows of a Hamiltonian vector fields corresponding to smooth Hamiltonian functions acts transitively. From this and the fact that flows of Hamiltonian derivations corresponding to smooth Hamiltonian functions on $\mathscr{M}_\mu$ preserve the decomposition of $\mathscr{M}_\mu$ into symplectic strata, it follows that the connected components of symplectic strata of $\mathscr{M}_\mu$ are equivalence classes of the relation: m_μ is equivalent to m'_μ if and only if there are smooth functions $f^1_\mu, \ldots, f^n_\mu$ on $\mathscr{M}_\mu$ such that the composition of the time one maps of the flows of $-\mathrm{ad}_{f^i_\mu}$ maps m_μ to m'_μ. □

More precisely we have proved

Corollary: Suppose that $\mathscr{M}_\mu$ and $\mathscr{M}'_\mu$ are singular reduced spaces and that $\varphi : \mathscr{M}_\mu \to \mathscr{M}'_\mu$ is a homeomorphism. If $\varphi^* : C^\infty(\mathscr{M}'_\mu) \to C^\infty(\mathscr{M}_\mu)$ is an isomorphism of the Poisson algebra $(C^\infty(\mathscr{M}'_\mu), \{\,,\,\}_\mu, \cdot)$ onto the Poisson algebra $(C^\infty(\mathscr{M}_\mu), \{\,,\,\}_\mu, \cdot)$, then φ maps a symplectic stratum of the locally compact differential space $(\mathscr{M}_\mu, C^\infty(\mathscr{M}_\mu))$ diffeomorphically onto a symplectic stratum of the locally compact differential space $(\mathscr{M}'_\mu, C^\infty(\mathscr{M}'_\mu))$.

8 Exercises

1. Let G be a compact Lie group which acts linearly on $\mathbf{R}^n$. For $y, z \in \mathbf{R}^n$ suppose that the G-orbit through y is disjoint from the G-orbit through z. Show that there is a G-invariant polynomial P on $\mathbf{R}^n$ such that $P(y) \neq P(z)$.

2. Let G be a compact Lie group acting linearly on $\mathbf{R}^n$. Suppose that $\sigma_1, \ldots, \sigma_r$ generate the algebra of G-invariant polynomials on $\mathbf{R}^n$. Then the fiber of the Hilbert map $\Phi : \mathbf{R}^n \to \mathbf{R}^r : x \mapsto \big(\sigma_1(x), \ldots, \sigma_r(x)\big)$ is a single G-orbit.

3. Give an example of a proper action of a Lie group on a smooth manifold whose fixed point set is a union of submanifolds of *different* dimension.

4. Give a counterexample to the statement: For every symplectic form Ω on $T^*\mathbf{R}^n$, which is invariant under an action of a compact Lie group G, there is a local diffeomorphism φ about 0, which commutes with the G-action, such that $\varphi^*\Omega$ is the *standard* symplectic form on $T^*\mathbf{R}^n$ near 0.

5. (Linear Hamiltonian vector fields.)

a) Let (V,ω) be a real symplectic vector space. Let $\mathrm{Sp}(\omega,\mathbf{R})$ be the group of all linear symplectic isomorphisms of (V,ω). In other words, $P \in \mathrm{Sp}(\omega,\mathbf{R})$ if and only if $P \in \mathrm{Gl}(V,\mathbf{R})$ and $P^*\omega = \omega$. Show that $\mathrm{Sp}(\omega,\mathbf{R})$ is a closed subgroup of $\mathrm{Gl}(V,\mathbf{R})$ and hence is a Lie group. Let $\mathrm{sp}(\omega,\mathbf{R})$ be the set of all $p \in \mathrm{gl}(V,\mathbf{R})$ such that $\omega(pv,w)+\omega(v,pw) = 0$ for every $v,w \in V$. Show that $\mathrm{sp}(\omega,\mathbf{R})$ is the Lie algebra of $\mathrm{Sp}(\omega,\mathbf{R})$ with Lie bracket $[\xi,\eta] = \xi\eta - \eta\xi$. Let $\{e_1,\ldots,e_n,f_1,\ldots,f_n\}$ be a basis of V such that the matrix of ω is $J_{2n} = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$. Show that $R = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $\in \mathrm{Sp}(\omega,\mathbf{R}) = \mathrm{Sp}(2n,\mathbf{R})$ if and only if $R^tJ_{2n}R = J_{2n}$ if and only if $a^tc = c^ta$, $b^td = d^tb$ and $a^td - c^tb = 1$. Similarly show that $r = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{sp}(\omega,\mathbf{R}) = \mathrm{sp}(2n,\mathbf{R})$ if and only if $r^tJ_{2n} + J_{2n}r = 0$ if and only if $d = -a^t$, $b = b^t$ and $c = c^t$.

b) Let $H : V \to \mathbf{R}$ be a homogeneous quadratic function on V, that is, there is a symmetric bilinear form $\widetilde{H}$ on V such that $H(v) = \frac{1}{2}\widetilde{H}(v,v)$. Let X_H be the *linear Hamiltonian vector field* associated to the Hamiltonian H on (V,ω). Show that $X_H \in \mathrm{sp}(\omega,\mathbf{R})$. Conversely, show that $\xi \in \mathrm{sp}(\omega,\mathbf{R})$ is the linear Hamiltonian vector field associated to the homogeneous quadratic function $v \to \frac{1}{2}\omega(\xi v,v)$. Let $\mathscr{Q}$ be the vector space of homogeneous quadratic functions on V. For $F,G \in \mathscr{Q}$ define their *Poisson bracket* by $\{F,G\} = \omega(X_F,X_G)$. Show that $(\mathscr{Q},\{\,,\,\})$ is a Lie algebra which is isomorphic to $(\mathrm{sp}(\omega,\mathbf{R}),[\,,\,])$.

c) The *linear symmetry group* G of the Hamiltonian system (H,V,ω) where $H \in \mathscr{Q}$ is the set of all $Q \in \mathrm{Sp}(\omega,\mathbf{R})$ such that $Q^*H = H$. Show that G is a closed subgroup of $\mathrm{Sp}(\omega,\mathbf{R})$ and hence is a Lie group. Show that the Lie algebra $\mathfrak{g}$ of G is the set of all $\xi \in \mathrm{sp}(\omega,\mathbf{R})$ such that $[\xi,X_H] = 0$. In other words, $\mathfrak{g}$ is the Lie subalgebra of all $F \in \mathscr{Q}$ such that $\{F,H\} = 0$.

d) Show that the linear action $\Phi : G \times V \to V$: $(Q,v) \to Qv$ is Hamiltonian with *momentum mapping* $J : V \to \mathfrak{g}^*$ where $J(v)\xi = \frac{1}{2}\omega(\xi v,v)$ for every $\xi \in \mathfrak{g}$. Show that J is coadjoint equivariant. Verify that for every $\xi \in \mathfrak{g}$ the function $J^\xi : V \to \mathbf{R}$: $v \to J(v)\xi$ is an integral of X_H.

6. (The $1^p : -1^q$ semisimple resonance.) Consider the quadratic Hamiltonian

$$H : \mathbf{R}^{2n} \to \mathbf{R} : (x,y) \to \tfrac{1}{2}\left\langle \begin{pmatrix} \mathscr{I} & 0 \\ 0 & \mathscr{I} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix}\right\rangle,$$

where $\mathscr{I} = \mathrm{diag}(I_p, -I_q)$.

a) Show that the linear symmetry group G of the Hamiltonian system $(H,\mathbf{R}^{2n},\omega)$ is the set of all $R \in \mathrm{Sp}(2n,\mathbf{R})$ such that $R = \begin{pmatrix} a & b \\ -\mathscr{I}b\mathscr{I} & \mathscr{I}a\mathscr{I} \end{pmatrix}$, where $a,b \in \mathrm{gl}(n,\mathbf{R})$ satisfy

$$\begin{aligned}(a\mathscr{I})^t(\mathscr{I}a)+(\mathscr{I}b)^t(b\mathscr{I}) &= I_n\\ (b\mathscr{I})^t(\mathscr{I}a) &= (\mathscr{I}a)^t(b\mathscr{I}).\end{aligned}$$

b) Let $\mathscr{I}$ be the matrix of a Hermitian inner product on $\mathbf{C}^n$ with respect to the standard basis. Let $\mathrm{U}(p,q)$ be the set of all $S\in \mathrm{gl}(n,\mathbf{C})$ such that $\overline{S}^t\mathscr{I}S=\mathscr{I}$. Show that the matrix $S=\begin{pmatrix} a & b\\ c & d\end{pmatrix}$, where $a\in \mathrm{gl}(p,\mathbf{C})$, $d\in \mathrm{gl}(q,\mathbf{C})$, b is a $q\times p$ and c is a $p\times q$ complex matrix, lies in $\mathrm{U}(p,q)$ if and only if

$$\begin{aligned}\overline{a}^t a-\overline{c}^t c &= I_p\\ \overline{d}^t d-\overline{d}^t d &= I_q\\ \overline{a}^t b-\overline{c}^t d &= 0.\end{aligned}$$

Verify that the map

$$\Theta: G\to \mathrm{U}(p,q): R=\begin{pmatrix} a & b\\ -\mathscr{I}b\mathscr{I} & \mathscr{I}a\mathscr{I}\end{pmatrix}\to S=a+ib\mathscr{I}$$

is an isomorphism of Lie groups.

c) Let $\mathrm{u}(p,q)$ be the set of all $s\in \mathrm{gl}(n,\mathbf{C})$ such that $\overline{s}^t\mathscr{I}+\mathscr{I}s=0$. Show that $\mathrm{u}(p,q)$ is the Lie algebra of $\mathrm{U}(p,q)$. Show that the matrix $s=\begin{pmatrix} a & b\\ c & d\end{pmatrix}$, where $a\in \mathrm{gl}(p,\mathbf{C})$, $d\in \mathrm{gl}(q,\mathbf{C})$, b is a $q\times p$ and c is a $p\times q$ complex matrix, lies in $\mathrm{u}(p,q)$ if and only if $a=-\overline{a}^t$, $c=-\overline{b}^t$, and $d=-\overline{d}^t$. Show that the Lie algebra $\mathfrak{g}$ of the symmetry group G is the set of all $r\in \mathrm{sp}(2n,\mathbf{R})$ such that $r=\begin{pmatrix} \mathscr{I}a & b\\ -b & \mathscr{I}a\end{pmatrix}$, where $a,b\in \mathrm{gl}(n,\mathbf{R})$ satisfy $(\mathscr{I}a)^t=-\mathscr{I}a$ and $b=b^t$. Show that the map

$$\theta: \mathfrak{g}\to \mathrm{u}(p,q): r=\begin{pmatrix} \mathscr{I}a & b\\ -b & \mathscr{I}a\end{pmatrix}\to a+ib\mathscr{I}$$

is an isomorphism of Lie algebras.

d) Define a Hermitian inner product $\langle\, ,\rangle$ on $\mathrm{u}(p,q)$ by $\langle u,w\rangle=-\frac{1}{2}\mathrm{tr}(\mathscr{I}\,\overline{w}^t\,\mathscr{I}u)$. Show that $\langle\, ,\rangle$ is invariant under Ad_U for every $U\in \mathrm{U}(p,q)$ and that it is nondegenerate. Show that the map

$$J:\mathbf{C}^n=\mathbf{C}^p\times \mathbf{C}^q\to \mathrm{u}(p,q): (z,w)\to \begin{pmatrix} i\,(z\otimes \overline{z}^t) & z\otimes \overline{w}^t\\ -w\otimes \overline{z}^t & i\,(w\otimes \overline{w}^t)\end{pmatrix}$$

is a Hermitian momentum mapping for the linear $\mathrm{U}(p,q)$-action

$$\Phi: \mathrm{U}(p,q)\times \mathbf{C}^n\to \mathbf{C}^n: (U,\zeta)\to U\zeta.$$

Here we have identified $\overline{\mathrm{u}(p,q)}^{\,*}$ with $\mathrm{u}(p,q)$ using $\langle, \rangle$. Writing

$$J(z,w)=i\,(z,iw)\otimes \overline{(z,iw)}^{\,t}$$

deduce that the rank of $J(z,w)$ is at most 1. Hence

$$(z_j\bar{z}_k)(w_\ell\overline{w}_m) = (z_j\overline{w}_m)(\bar{z}_k w_\ell)$$

are the only relations among the quadratic Hermitian integrals

$$\begin{cases} z_j\bar{z}_k, & 1 \le j,k \le p; \\ z_j\overline{w}_\ell,\ \bar{z}_j w_\ell, & 1 \le j \le p \,\&\, 1 \le \ell \le q; \\ w_\ell\overline{w}_m, & 1 \le \ell, m \le q \end{cases}$$

of the Hermitian vector field X_H. Show that the $\mathrm{U}(p,q)$-adjoint orbit through $J(z,0)$, $z \ne 0$, $J(0,w)$, $w \ne 0$ and $J(z,w)$, $z \ne 0 \,\&\, w \ne 0$ is diffeomorphic to $\mathbf{CP}^{p-1}$, $\mathbf{CP}^{q-1}$, $\mathbf{CP}^{n-1}$, respectively. Find an expression for the symplectic form on each of these adjoint orbits.

7. Let $S^2_\ell = \{x \in \mathbf{R}^3 \,|\, \langle x,x\rangle = \ell^2,\ \ell > 0\}$ be the 2-sphere of radius ℓ in $\mathbf{R}^3$ with Euclidean inner product $\langle\,,\,\rangle$. Consider the action

$$\Phi : S^1 \times (S^2_\ell \times S^2_\ell) \to S^2_\ell \times S^2_\ell : \big(t,(x,y)\big) \to (R_t x, R_t y),$$

where $R_t = \begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Let $\Omega = \pi_1^*\sigma_2 + \pi_2^*\sigma_2$, where π_i for $i = 1,2$ is the projection onto the i^{th} factor and σ_2 is the standard volume form on S^2_ℓ.

a) Show that Ω is a symplectic form on $S^2_\ell \times S^2_\ell$ and the S^1-action Φ is Hamiltonian with momentum map

$$J : S^2_\ell \times S^2_\ell \to \mathbf{R} : (x,y) \to x_3 + y_3.$$

Show that ℓ is a regular value of J. Using Morse theory show that $J^{-1}(\ell)$ is homeomorphic, and thus diffeomorphic, to a 3-sphere.

b) Show that $\pm\ell(e_3,e_3)$ are fixed points of the action Φ restricted to $J^{-1}(\ell)$. Thus the reduced space $\mathscr{M}_\ell = J^{-1}(\ell)/S^1$ has singularities. Use invariant theory to construct a concrete model of $\mathscr{M}_\ell$ as a semialgebraic variety in $\mathbf{R}^3$. (Hint: first find the generators of the algebra of polynomials which are invariant under the action Φ. Next show that they satisfy one relation and two inequalities. Using the fact that Φ is an action on $S^2_\ell \times S^2_\ell$ obtain additional relations.)

c) Find the structure matrix of the Poisson bracket on $\mathscr{M}_\ell$ and show that it is induced from a Poisson bracket on $\mathbf{R}^3$. Write out the Hamiltonian derivation of a smooth function on $\mathscr{M}_\ell$.

8. (The $p:q$ resonance.) Let $p,q \in \mathbf{Z}_\ge$ with $\gcd(p,q) = 1$. Consider the resonant harmonic oscillator on $(T\mathbf{R}^2, \omega)$ given by the Hamiltonian

$$H_2 : T\mathbf{R}^2 \to \mathbf{R} : (x,y) \mapsto \tfrac{1}{2}\big[p(y_2^2 + x_2^2) + q(y_1^2 + x_1^2)\big]. \tag{49}$$

a) Show that all orbits of X_{H_2} are periodic of period $2\pi pq$.

b) Show that the projection of every orbit of energy $1/2$ on configuration space $\mathbf{R}^2$ is a closed curve, contained in the rectangle $\mathscr{R} : |x_1| \le 1/q \ \& \ |x_2| \le 1/p$, and is tangent to the horizontal sides of $\mathscr{R}$ q-times and the vertical sides of $\mathscr{R}$ p times.

c) Show that the algebra of polynomials of $T\mathbf{R}^2$ which are invariant under the flow φ_t^H of X_{H_2} is generated by the polynomials

$$\begin{cases} \sigma_1 &= y_1^2 + x_1^2 \qquad\qquad \sigma_3 = \mathrm{Re}\,[(x_1 + i\,y_1)^p (x_2 - i\,y_2)^q] \\ \sigma_2 &= y_2^2 + x_2^2 \qquad\qquad \sigma_4 = \mathrm{Im}\,[(x_1 + i\,y_1)^p (x_2 - i\,y_2)^q], \end{cases}$$

subject to the relation

$$\sigma_3^2 + \sigma_4^2 = \sigma_1^p \sigma_2^q \qquad \sigma_1 \ge 0, \ \& \ \sigma_2 \ge 0. \tag{50}$$

d) Show that the orbit space $P_h^{p,q}$ of the S^1-action generated by X_{H_2} is defined by

$$\sigma_3^2 + \sigma_4^2 = \left(\frac{2h - \sigma_1}{p}\right)^q \sigma_1^p \qquad 0 \le \sigma_1 \le 2h.$$

Draw a picture of $P_h^{p,q}$ for $(p,q) = (1,3), (2,3)$ and $(1,4)$. Find the structure matrix for the Poisson algebra of smooth functions on $P_h^{p,q}$.

9. a) Let $\Phi : G \times V \to V$ be a linear Hamiltonian action of a compact Lie group on a symplectic vector space (V, ω), see exercise 5. Let $J : V \to \mathfrak{g}^*$ be the quadratic momentum mapping of Φ. Choose a set of generators $\{\sigma_1, \ldots, \sigma_k\}$ of the algebra of invariant polynomials. Embed the reduced space $\mathscr{M}_0 = J^{-1}(0)/G$ into $\mathbf{R}^k$ using the Hilbert map $\pi : V \to \mathbf{R}^k : x \to \big(\sigma_1(x), \ldots, \sigma_k(x)\big)$. Then there exists a Poisson bracket on $\mathbf{R}^k$ that restricts to the Poisson structure on $\mathscr{M}_0$.

   b) Show that the conclusion of a) holds for the linear Hamiltonian action on $(\mathbf{R}^4, \omega)$ given by the $p:q$-resonance, see exercise 8.

   c)* Is this also true for the $p_1 : p_2 : \ldots : p_n$-resonance?

   d)* Consider the linear Hamiltonian action on $(T^*(\mathbf{R}^3)^n, \omega)$ given by lifting the diagonal action of SO(3) on n-copies of $\mathbf{R}^3$. Let $J : T^*(\mathbf{R}^3)^n \to \mathrm{so}(3)^*$ be the momentum mapping. Does the conclusion of a) hold for the Poisson structure on the reduced space $J^{-1}(0)/\,\mathrm{SO}(3)$?

10. Let $\Phi : G \times M \to M : (g, m) \mapsto g \cdot m$ be a Hamiltonian action of a Lie group G on the smooth symplectic manifold (M, ω). Then for every $\xi \in \mathfrak{g}$, the Lie algebra of G, the infinitesimal generator $X^{\xi}(m) = \frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} \Phi_m(\exp t\xi)$ is a Hamiltonian vector field corresponding to the Hamiltonian function $J^{\xi} : M \to \mathbf{R}$, that is, for every $m \in M$ we have $X^{\xi}(m) \mathbin{\lrcorner} \omega(m) = \mathrm{d}J^{\xi}(m)$. The rest of this exercise shows that there is an affine action $A : G \times \mathfrak{g}^* \to \mathfrak{g}^*$ of G such that

$$J(\Phi_g(m)) = A(g, J(m)) \tag{51}$$

for every $g \in G$ and every $m \in M$.

a) Show that the mapping $\sigma : G \rightarrow \mathfrak{g}^* : g \mapsto J(\Phi_g(m)) - \mathrm{Ad}^t_{g^{-1}} J(m)$ does not depend on the choice of $m \in M$. Verify each step of the following calculation. For each $\xi \in \mathfrak{g}$ we have

$$\begin{aligned} \mathrm{d}\big((J \circ \Phi_g)^{\xi}\big)(m) &= T_m\Phi_g(X^{\xi}(m)) \mathbin{\lrcorner} \omega(m) = X^{\mathrm{Ad}_g\xi}(m) \\ &= \mathrm{d}J^{\mathrm{Ad}_g\xi}(m) = \mathrm{d}(\mathrm{Ad}_{g^{-1}} J)^{\xi}(m), \end{aligned}$$

that is, $\mathrm{d}\left(J \circ \Phi_g - \mathrm{Ad}^t_{g^{-1}} J\right)(m) = 0$ for every $m \in M$. Hence on connected components of M, the function $J \circ \Phi_g - \mathrm{Ad}^t_{g^{-1}} J$ is constant.

b) Show that for every $g,\, g' \in G$ we have $\sigma(gg') = \sigma(g) + \mathrm{Ad}^t_{g^{-1}}(\sigma(g'))$. Verify each step of the following calculation.

$$\begin{aligned} \sigma(gg') &= J \circ \Phi_{gg'} - \mathrm{Ad}^t_{(gg')^{-1}} J \\ &= (J \circ \Phi_g - \mathrm{Ad}^t_{g^{-1}} J) \circ \Phi_g + \mathrm{Ad}^t_{g^{-1}} (J \circ \Phi_{g'} - \mathrm{Ad}^t_{(g')^{-1}} J) \\ &= (J \circ \Phi_g - \mathrm{Ad}^t_{g^{-1}} J) + \mathrm{Ad}^t_{g^{-1}} (J \circ \Phi_{g'} - \mathrm{Ad}^t_{(g')^{-1}} J) \\ &= \sigma(g) + \mathrm{Ad}^t_{g^{-1}} \sigma(g'). \end{aligned}$$

c) Define the map $A : G \times \mathfrak{g}^* \rightarrow \mathfrak{g}^* : (g, \alpha) \mapsto \mathrm{Ad}^t_{g^{-1}} \alpha + \sigma(g)$. Show that A is an action of G on $\mathfrak{g}^*$. Verify that $A(e, \alpha) = \alpha$ for every $\alpha \in \mathfrak{g}^*$. Using the definition of A show that $A(gg', \alpha) = A(g, A(g', \alpha))$ as follows

$$\begin{aligned} A(gg', \alpha) &= \mathrm{Ad}^t_{(gg')^{-1}} \alpha + \sigma(gg') = \mathrm{Ad}^t_{g^{-1}} (\mathrm{Ad}^t_{(g')^{-1}} \alpha) + \sigma(g) + \mathrm{Ad}^t_{g^{-1}} \sigma(g') \\ &= \mathrm{Ad}^t_{g^{-1}} \left(\mathrm{Ad}^t_{(g')^{-1}} \alpha + \sigma(g')\right) + \sigma(g) \\ &= \mathrm{Ad}^t_{g^{-1}} A(g', \alpha) + \sigma(g) = A(g, A(g', \alpha)). \end{aligned}$$

d) Show that (51) holds. We have

$$\begin{aligned} A(g, J(m)) &= \mathrm{Ad}^t_{g^{-1}} J(m) + \sigma(g) \\ &= \mathrm{Ad}^t_{g^{-1}} J(m) + J(\Phi_g(m)) - \mathrm{Ad}^t_{g^{-1}} J(m) = J(\Phi_g(m)). \end{aligned} \tag{52}$$

11. (Reduction in stages.)

a) Define the notion of a momentum mapping of a smooth Poisson action of a Lie group on a singular reduced space.

b) Let G_1 and G_2 be compact Lie groups acting in a Hamiltonian fashion on the smooth symplectic manifold (M, ω) with corresponding momentum mappings $J_i : M \rightarrow \mathfrak{g}^*$ for $i = 1, 2$. Suppose that the actions commute, that is, the group $G_1 \times G_2$ acts in a Hamiltonian way on (M, ω) with momentum mapping

$$J : M \rightarrow \mathfrak{g}_1^* \times \mathfrak{g}_2^* : m \rightarrow (J_1(m), J_2(m)).$$

Show that we may assume that J is $G_1 \times G_2$ coadjoint equivariant. Show that $J^{-1}(0)/(G_1 \times G_2) = \widetilde{J}_2^{-1}(0)/G_2$ where $\widetilde{J}_2$ is the restriction of the function J_2 to $J_1^{-1}(0)/G_1$. First try proving the special case when all the actions are free and proper and all the reduced spaces are smooth.

12. Show that an integral for a Hamiltonian vector field on $(T^*\mathbf{R}^n, \omega)$ comes from an action on the configuration space $\mathbf{R}^n$ lifted to an action on the cotangent bundle $T^*\mathbf{R}^n$ if and only if it is linear in the momenta and the action fixes the zero section.

13. (Equivariant Darboux theorem.) Let $G \times V \rightarrow V$ be a linear action of a compact Lie group on a real vector space V. Suppose that this action preserves the symplectic forms ω_0 and ω_1 on V where $\omega_0(0) = \omega_1(0)$. Then there are G-invariant neighborhoods $\mathscr{U}_0$ and $\mathscr{U}_1$ of 0 and a diffeomorphism $\varphi : \mathscr{U}_0 \rightarrow \mathscr{U}_1$, which maps 0 to 0 and commutes with the G-action such that $\varphi^*\omega_1 = \omega_0$. We outline a proof.

a) First we find a G-invariant 1-form ζ on V with the following properties

1. $\zeta(0) = 0$.
2. $\mathrm{d}\zeta = \omega_0 - \omega_1$ on an open G-invariant neighborhood $\mathscr{U}_0$ of 0 in V.

Let $\psi_t(v) = (1-t)v$ be a 1-parameter group of radial contractions on V. Then $\psi_0 = \mathrm{id}_V$, $\psi_1 = 0$, $\psi_t(0) = 0$, and ψ_t commutes with the linear action of G on V. Let $\eta(v) = \frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} \psi_t(v)$ be the vector field on V generated by ψ_t. Justify each step of the following calculation.

$$
\begin{aligned}
-(\omega_1 - \omega_0) = \psi_1^*(\omega_1 - \omega_0) - \psi_0^*(\omega_1 - \omega_0) &= \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t}\big(\psi_t^*(\omega_1 - \omega_0)\,\mathrm{d}t \\
&= \int_0^1 \psi_t^*\big(L_\eta(\omega_1 - \omega_0)\big)\,\mathrm{d}t \\
&= \int_0^1 \psi_t^*\big(\mathrm{d}(\eta \,\lrcorner\, (\omega_1 - \omega_0))\big)\,\mathrm{d}t, \quad \text{since } \mathrm{d}(\omega_1 - \omega_0) = 0 \\
&= \mathrm{d}\Big(\int_0^1 \psi_t^*(\eta \,\lrcorner\, (\omega_1 - \omega_0))\,\mathrm{d}t\Big) = \mathrm{d}\zeta.
\end{aligned}
$$

Note that $\zeta(0) = 0$ and that ζ is G-invariant because the integrand in the last equality is G-invariant.

b) Next we construct a diffeomorphism φ which fixes 0, commutes with the action of G and pulls back ω_1 to ω_0. Define a time dependent vector field ξ on $\mathscr{U}_0$ such that when t is held fixed we have $\xi_t = (t\omega_1 + (1-t)\omega_0)^\flat \zeta$. Let ϑ be the flow of ξ. By shrinking $\mathscr{U}_0$ we can arrange that $\mathscr{U}_0 \times [0,1]$ is in the domain of ϑ. The next calculation shows that on $[0,1]$ the curve $t \mapsto \vartheta_t^*(t\omega_1 + (1-t)\omega_0)$ is constant. Justify each step of the calculation

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\vartheta_t^*(t\omega_1 + (1-t)\omega_0) &= \vartheta_t^*\big(L_\xi(t\omega_1 + (1-t)\omega_0)\big) \\
&= \vartheta_t^*\big(L_{\xi_t + \frac{\partial}{\partial t}}(t\omega_1 + (1-t)\omega_0)\big), \quad \text{since } \xi = \xi_t + \tfrac{\partial}{\partial t} \\
&= \vartheta_t^*\big(L_{\xi_t}(t\omega_1 + (1-t)\omega_0) + \omega_1 - \omega_0\big) \\
&= \vartheta_t^*\big(\mathrm{d}(\xi_t \,\lrcorner\, (t\omega_1 + (1-t)\omega_0)) + \xi_t \,\lrcorner\, \mathrm{d}(t\omega_1 + (1-t)\omega_0) + \omega_1 - \omega_0\big) \\
&= \vartheta_t^*(\mathrm{d}\zeta + \omega_1 - \omega_0) = 0.
\end{aligned}
$$

Therefore on $\mathscr{U}_0$ we have $\omega_0 = \vartheta_1^*\omega_1$. Set $\varphi = \vartheta_1$.

Chapter VIII

Ehresmann connections

1 Basic Properties

In this section we define the notion of an Ehresmann connection associated to a surjective submersion $\pi : M \to N$. A connection permits a curve in N to be *locally lifted* to a horizontal curve in M. An Ehresmann connection is *good* if every smooth curve in N has a global horizontal lift. For good connections we define the notions of parallel translation and holonomy.

Let $\pi : M \to N$ be a submersion. Consider two smooth distributions on M called vertical and horizontal. The *vertical* distribution is defined as $\mathrm{Vert} : M \to TM : m \mapsto \mathrm{Vert}_m = \ker T_m\pi$; while the *horizontal* distribution $\mathrm{Horz} : M \to TM : m \mapsto \mathrm{Horz}_m \subseteq T_mM$, is defined by a subspace Horz_m of T_mM which is complementary to Vert_m for each $m \in M$. These distributions give a smooth splitting $M \to TM : m \mapsto \mathrm{Horz}_m \oplus \mathrm{Vert}_m$ of T_mM, which is called an *Ehresmann connection* $\mathscr{K}$ associated to the submersion π provided that $T_m\pi(\mathrm{Horz}_m) = T_{\pi(m)}N$ for every $m \in M$.

To see how an Ehresmann connection allows smooth curves in N to be locally lifted to smooth curves in M, we define the notion of a local horizontal lift of a curve. Let $J = [0, \varepsilon)$ with $\varepsilon > 0$ and let $\gamma : J \to N$ be a smooth curve in N. Suppose that $m_0 \in \pi^{-1}(\gamma(0))$. A smooth curve $\widetilde{\gamma} : J \to M$ is called the *local horizontal lift* of γ starting at m_0 if and only if
$\triangleright$ $\widetilde{\gamma}(0) = m_0$, $(\pi \circ \widetilde{\gamma})(t) = \gamma(t)$, and $\frac{\mathrm{d}\widetilde{\gamma}}{\mathrm{d}t} \in \mathrm{Horz}_{\widetilde{\gamma}(t)}$ for every $t \in J$. From these conditions it follows that the local horizontal lift $\widetilde{\gamma}$ satisfies a system of ordinary differential equations with a given initial condition and hence is unique.

(1.1) **Proof**: Let $(p, v_p) \in TM$. Choose local coordinates $(x_1, \ldots, x_m, \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_m})$ on TM near (p, v_p) such that $\{\frac{\partial}{\partial x_{n+1}}, \ldots, \frac{\partial}{\partial x_m}\}$ span $\ker T_x\pi$. Let $(y_1, \ldots, y_n, \frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_n})$ be local coordinates on TN near $\big(\pi(p), T_p\pi\, v_p\big)$. In these coordinates $\gamma : J \to N : t \mapsto \big(\gamma_1(t), \ldots, \gamma_n(t)\big)$ and $\widetilde{\gamma} : J \to M : t \mapsto \big(\widetilde{\gamma}_1(t), \ldots, \widetilde{\gamma}_m(t)\big)$. There is an $m \times n$ matrix $(a_{ij}(y))$ such that for every $1 \le i \le m$ we have $T\pi\left(\frac{\partial}{\partial x_i}\right) = \sum_{j=1}^{n} a_{ij}(y)\, \frac{\partial}{\partial y_j}$. Because π is a submersion, the matrix $(a_{ij}(y))$ has rank n. In order that $\pi(\widetilde{\gamma}(t)) = \gamma(t)$ for $t \in J$, we need

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$T\pi\left(\frac{d\widetilde{\gamma}}{dt}(t)\right) = \frac{d\gamma}{dt}(t)$. Equivalently,

$$\sum_{i=1}^{m} \widetilde{\gamma}_i'(t)\, a_{ij}(\gamma(t)) = \gamma_j'(t) \tag{1}$$

for every $1 \le j \le n$. Because the projection mapping $\rho : TM \to \ker T\pi : \left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_m}\right) \mapsto \left(\frac{\partial}{\partial x_{n+1}}, \ldots, \frac{\partial}{\partial x_m}\right)$ is surjective, there is an $m \times (m-n)$ matrix $(b_{ij}(y))$ of rank $m-n$ such that $\rho\left(\frac{\partial}{\partial x_i}\right) = \sum_{k=1}^{m-n} b_{ik}(y)\, \frac{\partial}{\partial x_k}$. If $\widetilde{\gamma}$ is locally horizontal, then $\rho\left(\frac{d\widetilde{\gamma}}{dt}(t)\right) = 0$ for every $t \in J$, that is,

$$\sum_{i=1}^{m} \widetilde{\gamma}_i'(t)\, b_{ik}(\gamma(t)) = 0 \tag{2}$$

for every $1 \le k \le m-n$. Hence (1) and (2) may be written as

$$\sum_{i=1}^{m} \widetilde{\gamma}_i'(t)\, c_{ij}(\gamma(t)) = \begin{pmatrix} \gamma_j'(t) \\ 0_{m-n} \end{pmatrix} \tag{3}$$

where for $1 \le i \le m$ we have $c_{ij}(y) = \begin{cases} a_{ij}(y), & \text{if } 1 \le j \le n \\ 0, & \text{if } n+1 \le j \le m. \end{cases}$ Since the $m \times m$ matrix $(c_{ij}(y))$ has rank m, we may solve (3) and obtain

$$\widetilde{\gamma}_i'(t) = \sum_{j=1}^{n} c^{ij}(\gamma(t))\, \gamma_j'(t) \tag{4}$$

for $1 \le i \le m$ and $t \in J$. Here $(c^{ij}(y))$ is the inverse of $(c_{ij}(y))$. Equation (4) is the desired differential equation for the local horizontal lift. Given the initial condition $\widetilde{\gamma}(0)$, equation (4) has a unique solution. $\square$

A smooth curve $\widetilde{\gamma} : [0,1] \to M$ is a *global horizontal lift* of a curve $\gamma : [0,1] \to N$ if it is a local horizontal lift for every $t \in [0,1]$.

Example 1: Consider the Ehresmann connection associated to the surjective submersion $\pi : \mathbf{R}^2 \to \mathbf{R} : (x,y) \mapsto x$ defined by the vertical distribution $\mathrm{Vert} : \mathbf{R}^2 \to T_{(x,y)}\mathbf{R}^2 : (x,y) \mapsto \mathrm{span}\,\{(0,1)\}$ and the horizontal distribution $\mathrm{Horz} : \mathbf{R}^2 \to T_{(x,y)}\mathbf{R}^2 : (x,y) \mapsto \mathrm{span}\,\{(1,y^2)\}$. Suppose that $\gamma : [0,1] \to \mathbf{R} : t \mapsto t$. Then the local horizontal lift $\widetilde{\gamma} : J \to \mathbf{R}^2 : t \mapsto \big(\widetilde{\gamma}_1(t), \widetilde{\gamma}_2(t)\big)$ of γ starting at $(0,2)$ satisfies

$$\widetilde{\gamma}_1(t) = \pi(\widetilde{\gamma}(t)) = \gamma(t) = t, \tag{5}$$

$\widetilde{\gamma}_1(0) = 0$, $\widetilde{\gamma}_2(0) = 2$, and $\frac{d\widetilde{\gamma}(t)}{dt} \in \mathrm{span}\,\{(1, (\widetilde{\gamma}_2(t))^2\}$, which is equivalent to

$$\frac{d\widetilde{\gamma}_1}{dt} = \lambda(t) \quad \text{and} \quad \frac{d\widetilde{\gamma}_2}{dt} = \lambda(t)\,\big(\widetilde{\gamma}_2(t)\big)^2 \tag{6}$$

for some smooth function $\lambda : J \to \mathbf{R}$. Differentiating (5) gives $1 = \dfrac{d\widetilde{\gamma}_1}{dt} = \lambda(t)$. Hence the second equation in (6) is

$$\frac{d\widetilde{\gamma}_2}{dt} = \big(\widetilde{\gamma}_2(t)\big)^2 \tag{7}$$

with initial condition $\widetilde{\gamma}_2(0) = 2$. Integrating (7) gives $\widetilde{\gamma}_2(t) = (\frac{1}{2} - t)^{-1}$. Thus the local horizontal lift $\widetilde{\gamma}$ of γ exists only on $[0, \frac{1}{2})$. Hence γ does *not* have a global horizontal lift, since $\widetilde{\gamma}$ is not defined on all of $[0,1]$. □.

We say that an Ehresmann connection $\mathscr{K}$ associated to the surjective submersion $\pi : M \to N$ is *good* if every smooth curve $\gamma : [0,1] \to N$ has a horizontal lift $\widetilde{\gamma} : [0,1] \to M$ starting at *any* $m_0 \in \pi^{-1}(\gamma(0))$. The following claim gives a criterion when there is a good Ehresmann connection.

Claim: Let M be a connected smooth manifold. If $\pi : M \to N$ is a proper surjective submersion, then there is a good Ehresmann connection $\mathscr{K}$ associated to π.

(1.2) **Proof**: Since M has a Riemannian metric we may define the horizontal distribution at $m \in M$ as the orthogonal complement of the vertical distribution in T_mM. In other words, $m \mapsto \mathrm{Horz}_m = (\ker T_m\pi)^{\perp}$, where $m \mapsto \mathrm{Vert}_m = \ker T_m\pi$. Clearly this defines an Ehresmann connection associated to π.

To show that this connection is good, let $\gamma : [0,1] \to N$ be a smooth curve. Consider the set $\mathscr{I}$ of $t \in [0,1]$ such that $\gamma|[0,t]$ has a horizontal lift $\widetilde{\gamma}$ defined on $[0,t]$ for every starting point $m \in \pi^{-1}(\gamma(0))$. First we show that $\mathscr{I}$ is nonempty. Let $m_0 \in \pi^{-1}(\gamma(0))$. Because the local horizontal lift $\widetilde{\gamma}$ of γ starting at m_0 satisfies a smooth set of differential equations, there is an open neighborhood $U_{m_0} \subseteq \pi^{-1}(\gamma(0))$ of m_0 whose closure is compact and a positive time T_{m_0} such that for every $m \in U_{m_0}$ the local horizontal lift of γ starting at m is defined on $(-T_{m_0}, T_{m_0})$. Since π is a proper map, the fiber $\pi^{-1}(\gamma(0))$ is compact. Therefore the open covering $\{U_m | m \in \pi^{-1}(\gamma(0))\}$ has a finite subcovering $\{U_{m_i}\}_{i=1}^{r}$. Let $T = \min_i \{T_{m_i}\} > 0$. Then the local horizontal lift of γ starting at any point of the fiber $\pi^{-1}(\gamma(0))$ is defined on $(-T, T)$. In other words, $0 \in \mathscr{I}$. Let τ be the least upper bound for $\mathscr{I}$. Suppose that $\tau < 1$. Repeating the argument above with the fiber $\pi^{-1}(\gamma(0))$ replaced by the fiber $\pi^{-1}(\gamma(\tau))$, we find a $T' > 0$ such that every local horizontal lift of γ starting at any $m_\tau \in \pi^{-1}(\gamma(\tau))$ is defined on $(-T', T')$. For every $m_0 \in \pi^{-1}(\gamma(0))$ the horizontal lift of γ starting at m_0 has left end point at some $m_\tau \in \pi^{-1}(\gamma(\tau))$. This horizontal lift joins smoothly to the local horizontal lift of γ at m_τ, because the domains of the lifts overlap and local horizontal lifts are unique. The new horizontal lift of γ formed from this joining process starts at m_0 and has left end point in $\pi^{-1}(\gamma(T' + \tau))$. But this contradicts the definition of τ. Hence $\mathscr{I} = [0,1]$. □

In what follows we will assume that $\mathscr{K}$ is a good Ehresmann connection associated to the submersion π. Let $\gamma : [0,1] \to N$ be a smooth path and let $\widetilde{\gamma}_m$ be a horizontal lift of γ starting at $m \in \pi^{-1}(\gamma(0))$ with respect to the connection $\mathscr{K}$. The mapping

$$\mathscr{P}_\gamma : \pi^{-1}(\gamma(0)) \to \pi^{-1}(\gamma(1)) : m \mapsto \widetilde{\gamma}_m(1)$$

is called *parallel translation* along γ with respect to $\mathscr{K}$. Note that for the Ehresmann connection defined in example 1 no fiber of π can be even locally parallel transported along any curve. From smooth dependence of solutions of differential equations on initial conditions, we have

Claim: Let $\pi : M \to N$ be a surjective submersion with a good Ehresmann connection $\mathscr{K}$. Then along γ the parallel translation $\mathscr{P}_\gamma$ is a diffeomorphism.

If γ is a closed path, that is, $\gamma(0)=\gamma(1)$, then the map $\mathscr{P}_\gamma$ is the *holonomy* of the Ehresmann connection $\mathscr{K}$ along the curve γ. Fix a point $n\in N$. Then the set of diffeomorphisms $\{\mathscr{P}_\gamma\}$, where γ is a smooth closed curve in N with $\gamma(0)=n$, generates a group under composition called the *holonomy group* of the Ehresmann connection $\mathscr{K}$ at n.

Example 2. Consider the mapping

$$\begin{aligned}&\pi:\mathbf{R}^4\to\mathbf{R}^3:\\ &\qquad (x_1,x_2,x_3,x_4)\to(w_1,w_2,w_3)=\big(x_1^2+x_2^2-x_3^2-x_4^2,2(x_1x_4+x_2x_3),2(x_2x_4-x_1x_3)\big).\end{aligned}$$

Then the image of the unit 3-sphere S^3 under π is the unit 2-sphere S^2. Moreover, the mapping $\mathscr{F}=\pi|S^3:S^3\to S^2$ is the Hopf fibration. Define an Ehresmann connection $\mathscr{K}$ associated to $\mathscr{F}$ as follows. The vertical distribution of $\mathscr{K}$ at $x\in S^3$ is

$$\mathrm{Vert}_x=(\ker T_x\pi)\cap T_xS^3=\ker\begin{pmatrix}x_1 & x_2 & -x_3 & -x_4\\ x_4 & x_3 & x_2 & x_1\\ -x_3 & x_4 & -x_1 & x_2\\ x_1 & x_2 & x_3 & x_4\end{pmatrix}=\mathrm{span}\{(-x_2,x_1,x_4,-x_3)^t\}.$$

The horizontal distribution of $\mathscr{K}$ at $x\in S^3$ is the intersection of $\big(\mathrm{span}(-x_2,x_1,x_4,-x_3)^t\big)^\perp$ with T_xS^3. Here $X^\perp$ is the orthogonal complement of the subspace X of $\mathbf{R}^4$ with respect to the Euclidean inner product on $\mathbf{R}^4$. Therefore

$$\mathrm{Horz}_x=\mathrm{span}\{(x_3,x_4,-x_1,-x_2)^t,(-x_4,x_3,-x_2,x_1)^t\}.$$

It follows from the definition of Vert_x and Horz_x that $\mathrm{Vert}_x\oplus\mathrm{Horz}_x=T_xS^3$ for every $x\in S^3$. Because the rank of $D\pi(x)$ is 2 at every $x\in\mathbf{R}^4\setminus\{0\}$, we have $T_x\pi(\mathrm{Horz}_x)=T_{\mathscr{F}(x)}S^2$ for every $x\in S^3$. Therefore $\mathscr{K}$ is an Ehresmann connection associated to $\mathscr{F}$. Since $\mathscr{F}$ is a proper submersion, this connection is good.

Let $\gamma:[0,T]\to S^2:t\mapsto w(t)$ be a smooth curve with $\gamma(0)=w$. Then γ lifts to a horizontal curve $\widetilde{\gamma}:[0,T]\to S^3:t\mapsto x(t)$ with $\widetilde{\gamma}(0)=x$. In other words, $\mathscr{F}(x(t))=\mathscr{F}(\widetilde{\gamma}(t))=\gamma(t)=w(t)$ and $\frac{\mathrm{d}\widetilde{\gamma}(t)}{\mathrm{d}t}\in\mathrm{Horz}_{\widetilde{\gamma}(t)}$ for every $t\in[0,T]$. The latter condition for a horizontal lift may be rewritten as

$$\left\{\begin{aligned}\dot{x}_1&=\lambda_1x_3-\lambda_2x_4\\ \dot{x}_2&=\lambda_1x_4+\lambda_2x_3\\ \dot{x}_3&=-\lambda_1x_1-\lambda_2x_2\\ \dot{x}_4&=\lambda_2x_1-\lambda_1x_2,\end{aligned}\right. \tag{8}$$

for some smooth functions $\lambda_1=\lambda_1(t)$ and $\lambda_2=\lambda_2(t)$. Assume that x lies in the open set $U_1=\{x\in S^3\,\big|\,x_3^2+x_4^2>0\}=S^3\setminus\mathscr{F}^{-1}(1,0,0)$. Using $x_3^2+x_4^2=\frac{1}{2}(1-w_1)$ and the definition of $\mathscr{F}$, we eliminate λ_1 and λ_2 from the first two equations in (8), and obtain

$$\left\{\begin{aligned}\dot{x}_3&=(1-w_1)^{-1}(w_3\dot{x}_1-w_2\dot{x}_2)\\ \\ \dot{x}_4&=-(1-w_1)^{-1}(w_2\dot{x}_1+w_3\dot{x}_2).\end{aligned}\right. \tag{9}$$

From the definition of $\mathscr{F}$, we have $\begin{pmatrix} x_4 & x_3 \\ -x_3 & x_4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} w_2 \\ w_3 \end{pmatrix}$. Because $x \in U_1$, these equations may be solved to give

$$\begin{cases} x_1 = x_1(x_3,x_4) = (1-w_1)^{-1}(-w_3x_3 + w_2x_4) \\ x_2 = x_2(x_3,x_4) = (1-w_1)^{-1}(w_2x_3 + w_3x_4). \end{cases} \tag{10}$$

Note that

$$\begin{aligned} \tau_1 : \mathscr{F}(U_1) \times S^1 = (S^2 \setminus \{w_1 = 1\}) \times \{x_3^2 + x_4^2 = \tfrac{1}{2}(1-w_1)\} \rightarrow U_1 : \\ \big((w_1,w_2,w_3),(x_3,x_4)\big) \mapsto \big(x_1(x_3,x_4), x_2(x_3,x_4), x_3, x_4\big) \end{aligned}$$

is a local trivialization of the Hopf fibration. Differentiating (10) and substituting the result into (9) gives

$$\begin{cases} 2(1-w_1)\dot{x}_3 = -\left[(w_2\dot{w}_2 + w_3\dot{w}_3)x_3 - (w_3\dot{w}_2 - w_2\dot{w}_3)x_4 + (1+w_1)\,\dot{w}_1x_3\right] \\ 2(1-w_1)\dot{x}_4 = \left[(w_2\dot{w}_3 - w_3\dot{w}_2)x_3 + (w_2\dot{w}_2 + w_3\dot{w}_3)x_4 + (1+w_1)\,\dot{w}_1x_4\right]. \end{cases} \tag{11}$$

These are the equations satisfied by the horizontal lift of γ with respect to the connection $\mathscr{K}$ in the local trivialization τ_1.

Using (11) we compute the holonomy of the connection $\mathscr{K}$ along the equator

$$\gamma : [0,2\pi] \rightarrow S^2 \setminus \{w_1 = 1\} : t \mapsto w(t) = (0, \cos t, \sin t),$$

traversed in a *counterclockwise* direction. Since $\dot{w}_1 = 0$, $w_2\dot{w}_2 + w_3\dot{w}_3 = 0$, and $w_2\dot{w}_3 - w_3\dot{w}_2 = 1$, equation (11) becomes

$$\begin{cases} \dot{x}_3 = \frac{1}{2}x_4 \\ \dot{x}_4 = -\frac{1}{2}x_3. \end{cases} \tag{12}$$

Using the fact that $x_3^2 + x_4^2 = \frac{1}{2}$ on γ, equation (12) integrates to

$$\begin{pmatrix} x_3(t) \\ x_4(t) \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} \cos t/2 & \sin t/2 \\ -\sin t/2 & \cos t/2 \end{pmatrix}\begin{pmatrix} x_3(0) \\ x_4(0) \end{pmatrix}.$$

Therefore parallel transport of the fiber $\mathscr{F}^{-1}(0,0,1)$ along the closed curve γ is

$$\mathscr{P}_t^1 : \mathscr{F}^{-1}(0,0,1) \rightarrow \mathscr{F}^{-1}(\gamma(t)) : \tau_1\big(0,0,1,x_3(0),x_4(0)\big) = \begin{pmatrix} x_4(0) \\ x_3(0) \\ x_3(0) \\ x_4(0) \end{pmatrix} \mapsto$$

$$\tau_1\big(0,\cos t,\sin t,x_3(t),x_4(t)\big) = \frac{1}{\sqrt{2}}\begin{pmatrix} \cos 3t/2 & -\sin 3t/2 & 0 & 0 \\ \sin 3t/2 & \cos 3t/2 & 0 & 0 \\ 0 & 0 & \cos t/2 & \sin t/2 \\ 0 & 0 & -\sin t/2 & \cos t/2 \end{pmatrix}\begin{pmatrix} x_4(0) \\ x_3(0) \\ x_3(0) \\ x_4(0) \end{pmatrix}.$$

In other words, $\mathscr{P}_t^1$ is a clockwise rotation of the circle $\mathscr{F}^{-1}(0,0,1)$ through an angle $t/2$. Consequently, the holonomy $\mathscr{P}_{2\pi}^1$ of the connection $\mathscr{K}$ along the curve γ is a clockwise rotation through an angle π.

Let $U_2 = \{x \in S^3 \, \big| \, x_1^2 + x_2^2 > 0\} = S^3 \setminus \mathscr{F}^{-1}(-1,0,0)$. Then $\mathscr{F}(U_2) = S^2 \setminus \{w_1 = -1\}$. The map

$$\tau_2 : \mathscr{F}(U_2) \times S^1 = (S^3 \setminus \{w_1 = -1\}) \times \{x_1^2 + x_2^2 = \tfrac{1}{2}(1+w_1)\} \to U_2 : \\ \{\big((w_1,w_2,w_3),(x_1,x_2)\big) \mapsto \big(x_1,x_2,x_3(x_1,x_2),x_4(x_1,x_2)\big),$$

where

$$\begin{cases} x_3(x_1,x_2) = (1+w_1)^{-1}\,(-w_3x_1 + w_2x_2) \\ x_4(x_1,x_2) = (1+w_1)^{-1}\,(w_2x_1 + w_3x_2), \end{cases}$$

is a local trivialization of the Hopf fibration. The horizontal lift with respect to $\mathscr{K}$ of the equator $\{w_1 = 0\} \cap S^2$ traversed in the *clockwise* direction, that is, the horizontal lift of the curve $-\gamma : [0,2\pi] \to S^2 : t \mapsto \gamma(2\pi - t) = (0, \cos t, -\sin t)$, satisfies

$$\begin{cases} \dot{x}_1 & = -\tfrac{1}{2}x_2 \\ \dot{x}_2 & = \tfrac{1}{2}x_1. \end{cases}$$

Thus the parallel translation of $\mathscr{F}^{-1}(0,0,1)$ along the curve $-\gamma$ is the map

$$\mathscr{P}_t^2 : \mathscr{F}^{-1}(0,0,1) \to \mathscr{F}^{-1}\big(-\gamma(t)\big) = \mathscr{F}^{-1}\big(\gamma(2\pi - t)\big)$$

which is a clockwise rotation through an angle $2\pi - t/2$. Thus the holonomy $\mathscr{P}_{2\pi}^2$ of the connection $\mathscr{K}$ along the curve γ is a counterclockwise rotation through an angle $-\pi$.

We note that the classifying map of the Hopf fibration is

$$\chi : S^1 = \{w_1 = 0\} \cap S^2 \to S^1 = \mathscr{F}^{-1}(0,0,1) : t \mapsto (\mathscr{P}^2_{-2\pi+t})^{-1} \circ \mathscr{P}^1_{-t},$$

which is a counterclockwise rotation of S^1 through an angle t. Hence the degree of the mapping χ is 1. $\square$

2 The Ehresmann theorems

In this section we prove the Ehresmann fibration and trivialization theorems.

We begin with the *fibration theorem*.

Claim: Let $\pi : M \to N$ be a proper surjective submersion. Then π is a *locally trivial* fibration.

(2.1) **Proof**: Because π is a proper surjective submersion, there is a good Ehresmann connection $\mathscr{K}$ associated to π. Give N a Riemannian metric. For each $n \in N$, let B_n be an open ball in T_nN about 0 where the exponential map $\exp_n : B_n \subseteq T_nN \to N$ is a diffeomorphism. Let $U_n = \exp_n B_n$. Then U_n is a open neighborhood of n in N. For $n' \in U_n$ let

$$\gamma_{n,n'} : [0,1] \to U_n \subseteq N : t \mapsto \exp_n t v_{n'},$$

be the geodesic joining n to n'. In other words, $\gamma_{n,n'}(1) = n'$. Because $\exp_n$ is a diffeomorphism, the vector $v_{n'} \in B_n \subseteq T_nN$ is uniquely determined. Let $\mathscr{P}_{\gamma_{n,n'}} : \pi^{-1}(n) \to \pi^{-1}(n')$ be parallel translation along $\gamma_{n,n'}$ using the Ehresmann connection $\mathscr{K}$.

Consider the mapping

$$\tau : U_n \times \pi^{-1}(n) \to \pi^{-1}(U_n) : (n', m) \mapsto \mathscr{P}_{\gamma_{n,n'}}(m)$$

and the projection mapping $\pi_1 = \pi|\pi^{-1}(U_n) : \pi^{-1}(U_n) \to U_n : m \mapsto \pi(m)$. Then τ is a trivialization of the fibration π_1, because for every $m \in \pi^{-1}(n)$ and every $n' \in U_m$ we have

$$\pi_1 \circ \tau(n', m) = \pi_1(\mathscr{P}_{\gamma_{n,n'}}(m)) = n', \tag{13}$$

by definition of parallel translation. To finish the argument we need only show that τ is a diffeomorphism. Define the smooth mappings

$$\rho : U_n \times \pi^{-1}(U_n) \to \pi^{-1}(U_n) : (n', m) \mapsto \mathscr{P}_{-\gamma_{n,n'}}(m)$$

and

$$\sigma : \pi^{-1}(U_n) \to U_n \times \pi^{-1}(n) : m \mapsto \big(\pi_1(m), \rho(\pi_1(m), m)\big).$$

The following calculation shows that $\sigma \circ \tau = \mathrm{id}_{U_n \times \pi^{-1}(n)}$. For every $n' \in U_n$ and every $m \in \pi^{-1}(n)$ we have

$$\begin{aligned}
\sigma \circ \tau(n', m) &= \sigma\big(\mathscr{P}_{\gamma_{n,n'}}(m)\big) \\
&= \Big(\pi_1(\mathscr{P}_{\gamma_{n,n'}}(m)), \rho\big(\pi_1(\mathscr{P}_{\gamma_{n,n'}}(m)), \rho\big(\pi_1(\mathscr{P}_{\gamma_{n,n'}}(m)), \mathscr{P}_{\gamma_{n,n'}}(m)\big)\big)\Big) \\
&= \big(n', \rho(n', \mathscr{P}_{\gamma_{n,n'}}(m))\big), \quad \text{by (13)} \\
&= \big(n', \mathscr{P}_{-\gamma_{n,n'}} \circ \mathscr{P}_{\gamma_{n,n'}}(m)\big) = (n', m).
\end{aligned}$$

Now $\tau \circ \sigma = \mathrm{id}_{\pi^{-1}(U_n)}$, because for every $m \in \pi^{-1}(U_n)$

$$\begin{aligned}
\tau(\sigma(m)) &= \tau\big(\pi_1(m), \rho(\pi_1(m), m)\big) = \mathscr{P}_{\gamma_{n,\pi_1(m)}}(\rho(\pi_1(m), m)) \\
&= \mathscr{P}_{\gamma_{n,\pi_1(m)}} \circ \mathscr{P}_{-\gamma_{n,\pi_1(m)}}(m) = m.
\end{aligned}$$

Therefore τ is a diffeomorphism. □

We now prove the *trivialization theorem*.

Claim: Suppose that $\pi : M \to N$ is a proper surjective submersion and that N is smoothly contractible, that is, there is a one parameter family of mappings $F : [0,1] \times N \to N$ such that

1. F_t is a diffeomorphism for every $t \in [0,1]$.
2. $F_0 = id_N$.
3. For every $n \in N$, $F_1(n) = n_0$ for some fixed n_0 in N.

Then the fibration π is trivial, that is, M is diffeomorphic to $N \times \pi^{-1}(n_0)$ and this diffeomorphism maps fibers of π onto fibers of $\pi_1 : N \times \pi^{-1}(n_0) \to N : (n, m) \mapsto n$.

(2.2) **Proof**: For $n \in N$ consider the curve $\gamma_n : [0,1] \to N : t \to F_t(n)$, which smoothly joins the point n to n_0. Define the map $\varphi : N \times \pi^{-1}(n_0) \to M : (n, m) \mapsto \mathscr{P}_{\gamma_n}(m)$. Following the same argument used in the proof of the Ehresmann fibration theorem, we see that φ is a diffeomorphism which is fiber preserving. □

3 Exercises

1. (Levi-Civita connection.) Let g be a Riemannian metric on M. Let $\tau_M : TM \rightarrow M$ be the tangent bundle of M with natural coordinates (x^i, v^i). The second tangent bundle $\tau_{TM} : T(TM) \rightarrow TM$ has natural coordinates (x^i, v^i, X^i, V^i). On TM define two distributions

$$\mathrm{Horz}^g : TM \rightarrow T(TM) : (x^i, v^i) \mapsto \mathrm{span}\{(x^i, v^i, X^i, -\sum_{j,k} \Gamma^i_{jk} v^j X^k)\}$$

and

$$\mathrm{Vert}^g : TM \rightarrow T(TM) : (x^i, v^i) \mapsto \mathrm{span}\{(x^i, v^i, 0, V^i + \sum_{j,k} \Gamma^i_{jk} v^j X^k)\}.$$

Show that Horz^g and Vert^g define an Ehresmann connection $\mathscr{K}^g$ on the bundle τ_{TM}. $\mathscr{K}^g$ is called *Levi-Civita connection*. Prove the following properties of $\mathscr{K}^g$.

a) The geodesic vector field $Z_g(x^i, v^i) = (v^i, -\sum_{j,k} \Gamma^i_{j,k} v^j v^k)$ lies in Horz^g, that is, for every $(x, v) \in TM$ we have $Z_g(x, v) \in \mathrm{Horz}^g_{(x,v)}$.

b) $\mathscr{K}^g$ is symmetric, that is, $j \circ \mathrm{Horz}^g = \mathrm{Horz}^g$, where $j(x, v, X, V) = (x, X, v, V)$ is the canonical involution on $T(TM)$.

c) Let $K_g : T(TM) \rightarrow TM : (x^i, v^i, X^i, V^i) \mapsto (x^i, V^i + \sum_{j,k} \Gamma^i_{jk} v^j X^k)$. Show that for every $X_1, X_2 \in T_{(x,v)}(TM)$ the symplectic 2-form Ω on TM is $\Omega_g(x, v)(X_1, X_2) = g(x)\big(K_g(X_2), T\tau_M(X_1)\big) - g(x)\big(K_g(X_1), T\tau_M(X_2)\big)$. Thus $\mathrm{Horz}^g_{(x,v)}$ and $\mathrm{Vert}^g_{(x,v)}$ are *Lagrangian* subspaces of the symplectic vector space $\big(T_{(x,v)}(TM), \Omega(x, v)\big)$.

2. (Connections on a principal bundle.)

a) Let $\Phi : G \times P \rightarrow P : (g, m) \mapsto \Phi_g(m) = g \cdot m$ be a free and proper action of a Lie group G on a manifold P. Then P is a (left) *principal G-bundle* over the G-orbit space M with bundle projection $\pi : P \rightarrow M$ given by the orbit map. The bundle π is locally trivial, that is, for every $m \in M$ there is an open neighborhood U and a diffeomorphism

$$\tau : \pi^{-1}(U) \rightarrow G \times U : p \mapsto \big(\varphi(p), \pi(p)\big)$$

such that $\varphi(g \cdot p) = g\varphi(p)$. Show that $\pi^{-1}(\pi(p)) = \{g \cdot p \in P \big| g \in G\}$. Let $\mathfrak{g}$ be the Lie algebra of G. For a fixed $p \in P$, let $\sigma_p : \mathfrak{g} \rightarrow T_pP : \xi \rightarrow X_\xi(p)$, where $X_\xi(p) = \frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} \exp t\xi \cdot p$. Show that $\mathrm{im}\, \sigma_p = T_p\big(\pi^{-1}(p)\big)$.

b) A smooth $\mathfrak{g}$-valued 1-form ϑ on P is a smooth section of the bundle $L(TP, \mathfrak{g}) = \dot{\bigcup}_{p \in P} L(T_pP, \mathfrak{g})$ with bundle projection $\rho(L(T_pP, \mathfrak{g})) = p$. A *connection* on a (left) principal G-bundle $\pi : P \rightarrow M$ is a $\mathfrak{g}$-valued 1-form ϑ such that

1. $\vartheta(p)(\sigma_p(\xi)) = \xi$ for every $\xi \in \mathfrak{g}$ and every $p \in P$.
2. For every $g \in G$ and every $v_p \in T_pP$, we have $\vartheta\big(\Phi_g(p)\big)(T_e\Phi_g v_p) = \mathrm{Ad}_g\,\big(\vartheta(p)(v_p)\big)$.

Consider the distributions $\mathrm{Vert}^{\vartheta} : P \to TP : p \mapsto \mathrm{span}\{\sigma_p(\xi) \in T_pP \,\big|\, \xi \in \mathfrak{g}\}$ and $\mathrm{Horz}^{\vartheta} : P \to TP : p \mapsto \ker \vartheta(p)$. Show that $\mathrm{Vert}^{\vartheta}$ and $\mathrm{Horz}^{\vartheta}$ define an Ehresmann connection on the bundle π. Show that $\mathrm{Horz}^{\vartheta}$ is G-invariant, that is, for every $p \in P$ and every $g \in G$, we have $T_e\Phi_g(\mathrm{Horz}^{\vartheta}_p) = \mathrm{Horz}^{\vartheta}_{\Phi_g(p)}$. Conversely, show that an Ehresmann connection on a principal G-bundle π whose horizontal distribution is G-invariant determines a $\mathfrak{g}$-valued 1-form ϑ which satisfies 1) and 2) above.

c)* Show that every connection 1-form on a principal bundle defines an Ehresmann connection where every curve has a horizontal lift.

3. (Connections on S^1 principal bundles.) Let $\psi : S^1 \times P \to P$ be a proper free action on a smooth manifold P and let X be the infinitesimal generator of ψ. The orbit space M of this action is a smooth manifold. The orbit map $\pi : P \to M$ gives P the structure of an S^1 principal bundle. Let θ be a 1-form on P which is ψ-invariant and for which $X \mathbin{\lrcorner} \theta = 1$. Then θ is a *connection 1-form* on P.

a) Show that $X \mathbin{\lrcorner} \mathrm{d}\theta = 0$. Since $\ker T\pi = \mathrm{span}\{X\} \subseteq \ker \mathrm{d}\theta$ and $\mathrm{d}\theta$ is ψ-invariant, deduce that there is a 2-form Ω on M such that $\pi^*\Omega = \mathrm{d}\theta$. Ω is called the *curvature form* of θ.

b) (Infinitesimal holonomy formula.) Let $\varphi : V \subseteq M \to U \subseteq \mathbf{R}^n$ be a chart for M such that $\tau : \pi^{-1}(V) \subseteq P \to U \times S^1$ is a trivialization of the principal bundle π. Let $\gamma : I \times [0,1] \to U : (s,t) \to \gamma(s,t) = \gamma_s(t)$ be a family of smooth curves with $\gamma_s(0) = u$ for every $s \in I = (-\varepsilon, \varepsilon)$. For every $s \in I$ let $\Gamma_s : [0,1] \to \pi^{-1}(U)$ be the horizontal lift of γ_0 with respect to the connection 1-form θ with $\Gamma_s(0) = p \in \pi^{-1}(u)$ fixed. In other words, $\pi(\Gamma_s(t)) = \gamma_s(t)$ and $\frac{\partial \Gamma_s}{\partial t}(t) \mathbin{\lrcorner} \theta(\Gamma_s(t)) = 0$ for every $(s,t) \in I \times [0,1]$. Suppose that $\Gamma_s(1) \in \pi^{-1}(u)$ for every $s \in I$. Then the *holonomy of θ along γ_s as measured from the point p* is the element $\mathrm{hol}^{\gamma_s}_p$ of $S^1 = \mathbf{R}/2\pi\mathbf{Z}$ such that $\psi_{\mathrm{hol}^{\gamma_s}_p}(p) = \Gamma_s(1)$. Let $\frac{\partial}{\partial s}\mathrm{hol}^{\gamma_s}_p$ be the *infinitesimal holonomy vector field* on the fiber $\pi^{-1}(u)$. Then

$$\frac{\partial}{\partial s}\mathrm{hol}^{\gamma_s}_p(s) = \int_0^1 \Omega(\gamma_s(t))\left(\frac{\partial \gamma_s}{\partial t}(t), \frac{\partial \gamma_s}{\partial s}(t)\right) \mathrm{d}t. \tag{14}$$

Each step of the following calculation takes place in the image of the trivialization τ. Supply a justification.

$$\begin{aligned}
0 &= \frac{\mathrm{d}}{\mathrm{d}s}\big(\theta(\Gamma_s(t))\frac{\partial \Gamma_s}{\partial t}\big) = \big(D\theta(\Gamma_s(t))\frac{\partial \Gamma_s}{\partial s}\big)\frac{\partial \Gamma_s}{\partial t} + \theta(\Gamma_s(t))\frac{\partial^2 \Gamma_s}{\partial s\,\partial t} \\
&= \big(D\theta(\Gamma_s(t))\frac{\partial \Gamma_s}{\partial s}\big)\frac{\partial \Gamma_s}{\partial t} - \big(D\theta(\Gamma_s(t))\frac{\partial \Gamma_s}{\partial t}\big)\frac{\partial \Gamma_s}{\partial s} + \frac{\mathrm{d}}{\mathrm{d}t}\big(\theta(\Gamma_s(t))\frac{\partial \Gamma_s}{\partial s}\big) \\
&= -\mathrm{d}\theta(\Gamma_s(t))(\frac{\partial \Gamma_s}{\partial t}, \frac{\partial \Gamma_s}{\partial s}) + \frac{\mathrm{d}}{\mathrm{d}t}\big(\theta(\Gamma_s(t))\frac{\partial \Gamma_s}{\partial s}\big) \\
&= -\Omega(\gamma_s(t))(\frac{\partial \gamma_s}{\partial t}, \frac{\partial \gamma_s}{\partial s}) + \frac{\mathrm{d}}{\mathrm{d}t}\big(\theta(\Gamma_s(t))\frac{\partial \Gamma_s}{\partial s}\big).
\end{aligned}$$

Integrating over $t \in [0,1]$ gives

$$\theta(\Gamma_s(1))\frac{\partial \Gamma_s}{\partial s}(1) = \int_0^1 \Omega(\gamma_s(t))(\frac{\partial \gamma_s}{\partial t}, \frac{\partial \gamma_s}{\partial s})\,\mathrm{d}t.$$

But $\frac{\partial \Gamma_s}{\partial s}(1) = \frac{\partial}{\partial s}\mathrm{hol}_p^{\gamma_s} X(\Gamma_s(1))$ and $X(\Gamma_s(1)) \lrcorner\, \theta(\Gamma_s(1)) = 1$. This proves (14).

c) (Hopf bundle.) Let $S^3 = \{(z_1, z_2) = (x_1 + iy_1, x_2 + iy_2) \in \mathbf{C}^2 \,\big|\, |z_1|^2 + |z_2|^2 = 1\}$. Consider the proper free $S^1 = \mathbf{R}/2\pi\mathbf{Z}$ action $\psi : S^1 \times S^3 \to S^3 : \big(t, (z_1, z_2)\big) \mapsto (\mathrm{e}^{-it} z_1, \mathrm{e}^{-it} z_2)$. Its orbit space is $S^2 = \{(w_1, w_2, w_3) \in \mathbf{R}^3 \,\big|\, w_1^2 + w_2^2 + w_3^2 = 1\}$ and its orbit map

$$\begin{aligned} h : S^3 \to S^2 : (z_1, z_2) \mapsto & \big(2\,\mathrm{Im}\,\overline{z}_1 z_2, 2\,\mathrm{Re}\,\overline{z}_1 z_2, |z_1|^2 - |z_2|^2\big) = \\ & = \big(2(x_1y_2 - x_2y_1), 2(x_1x_2 + y_1y_2), x_1^2 + y_1^2 - x_2^2 - y_2^2\big) \end{aligned}$$

is the Hopf fibration. On $\mathbf{C}^2 = \mathbf{R}^4$ consider the 1-form

$$\theta = -\frac{1}{2i}(\overline{z}_1\,\mathrm{d}z_1 + \overline{z}_1\,\mathrm{d}z_2 - z_1\,\mathrm{d}\overline{z}_1 - z_2\,\mathrm{d}\overline{z}_2) = y_1\,\mathrm{d}x_1 - x_1\,\mathrm{d}y_1 + y_2\,\mathrm{d}x_2 - x_2\,\mathrm{d}y_2.$$

Show that $\vartheta = \theta|S^3$ is a connection 1-form on the principal S^1 bundle h. Show that the distributions $\mathrm{Horz} : S^3 \to TS^3 : z \mapsto \ker\vartheta(z)$ and $\mathrm{Vert} : S^3 \to TS^3 : z \mapsto \mathrm{span}\{X(z)\}$, where $\ker\vartheta(z)$ is spanned by $X_1(z) = -y_2\frac{\partial}{\partial x_1} - x_2\frac{\partial}{\partial y_1} + y_1\frac{\partial}{\partial x_2} + x_1\frac{\partial}{\partial y_2}$, $X_2(z) = x_2\frac{\partial}{\partial x_1} - y_2\frac{\partial}{\partial y_1} - x_1\frac{\partial}{\partial x_2} + y_1\frac{\partial}{\partial y_2}$, and $X(z) = y_1\frac{\partial}{\partial x_1} - x_1\frac{\partial}{\partial y_1} + y_2\frac{\partial}{\partial x_2} - x_2\frac{\partial}{\partial y_2}$, define the Ehresmann connection associated to ϑ.

The following argument shows that the curvature Ω of the connection 1-form ϑ is $\frac{1}{2}\mathrm{vol}_{S^2}$. Here vol_{S^2} is the standard volume form on S^2 defined by $\mathrm{vol}_{S^2}(w)(u,v) = (w, u \times v)$, where $w \in S^2$ and $u, v \in T_wS^2$. Note that $\int_{S^2}\mathrm{vol}_{S^2} = 4\pi$. Since $H^3(S^2) = 0$, Ω is a closed 2-form. Because $\dim H^2(S^2) = 1$, there is a $\lambda \in \mathbf{R}$ such that $\Omega = \lambda\,\mathrm{vol}_{S^2}$. To determine λ we must orient S^3. Let $n(z) = x_1\frac{\partial}{\partial x_1} + y_1\frac{\partial}{\partial y_1} + x_2\frac{\partial}{\partial x_2} + y_2\frac{\partial}{\partial y_2}$ be the outward normal to S^3 in $\mathbf{R}^4$ at $z = (x_1, y_1, x_2, y_2)$. Show that $n \lrcorner\, \mathrm{vol}_{\mathbf{R}^4} = -\frac{1}{2}\theta \wedge \mathrm{d}\theta$. From the fact that $-\frac{1}{2}(\theta \wedge \mathrm{d}\theta)(z)\big(X(z), X_1(z), X_2(z)\big) = 1$, it follows that $\mathrm{vol}_{S^3} = -\frac{1}{2}\vartheta \wedge \mathrm{d}\vartheta$. Since $\det\big(n(z), X(z), X_1(z), X_2(z)\big) = 1$, the basis $\{X(z), X_1(z), X_2(z)\}$ of T_zS^3 is positively oriented with respect to vol_{S^3}. Show that T_zh maps $\{X(z), X_1(z), X_2(z)\}$ onto the frame $\{2v_1, 2v_2\}$ on $T_{h(z)}S^2$ where

$$v_1 = (x_1^2 + x_2^2 - y_1^2 - y_2^2)\frac{\partial}{\partial w_1} + 2(x_1y_1 - x_2y_2)\frac{\partial}{\partial w_2} - 2(x_1y_2 + x_2y_1)\frac{\partial}{\partial w_3}$$

and

$$v_2 = 2(x_1y_1 + x_2y_2)\frac{\partial}{\partial w_1} + 2(x_1x_2 + y_1y_2)\frac{\partial}{\partial w_2} + (x_1^2 + y_1^2 - x_2^2 - y_2^2)\frac{\partial}{\partial w_3}.$$

Since $\det(h(z), v_1, v_2) = -1$, the frame $\{2v_1, 2v_2\}$ is *negatively* oriented with respect to vol_{S^2}. Hence $(h^*\mathrm{vol}_{S^2})(z)(X_1(z), X_2(z)) = -4$. From the fact that $\mathrm{d}\vartheta(z)(X_1(z), X_2(z)) = -2$, it follows that $\lambda = \frac{1}{2}$. Therefore $\Omega = \frac{1}{2}\mathrm{vol}_{S^2}$.

The following argument computes the holonomy of the connection 1-form ϑ along the equator $\gamma_{\pi/2}$ of S^2. Let D be the closed upper hemisphere, which is positively oriented with respect to vol_{S^2} and is bounded by the positively oriented equator $\gamma_{\pi/2}$. For each $s \in [0, \pi/2]$ let Π_s be the plane in $\mathbf{R}^3$ passing through $u = (1,0,0) \in S^2$ with normal vector $(\cos s, 0, \sin s)$. For $s \in (0, \pi/2]$ let $\gamma_s : [0,1] \rightarrow D$ be a parametrization of the ellipse $\Pi_s \cap D$. Note that $\gamma_0(t) = u$ for every $t \in [0,1]$. Thus we have a smooth family of curves $\gamma : [0, \pi/2] \times [0,1] \rightarrow D : (s,t) \rightarrow \gamma_s(t)$. Since the horizontal lift Γ_s of γ_s with respect to the connection ϑ starts at $p \in h^{-1}(u)$, the holonomy $\mathrm{hol}_p^{\gamma_0}$ along γ_0 is 0. Justify each step of the following calculation.

$$\begin{aligned}\mathrm{hol}_p^{\gamma_{\pi/2}} &= \int_0^{\pi/2} \frac{d}{ds}\mathrm{hol}_p^{\gamma_s}\, ds = \int_0^{\pi/2}\int_0^1 \Omega(\gamma_s(t))\Big(\frac{\partial \gamma_s(t)}{\partial t}, \frac{\partial \gamma_s(t)}{\partial s}\Big)\, \mathrm{d}t\, \mathrm{d}s \\ &= \int_{[0,\pi/2]\times[0,1]} \gamma^*\Omega = \int_D \Omega = \tfrac{1}{2}\mathrm{vol}_{S^2}(D) = \pi.\end{aligned}$$

Generalize this argument to show that the holonomy along any closed positively oriented curve γ on S^2 is equal to one half the area on S^2 of the 2-disk bounded by γ.

d) Consider the S^1 principal bundle $\pi : T_1S^2 \subseteq T\mathbf{R}^3 \rightarrow S^2 \subseteq \mathbf{R}^3 : (x,y) \mapsto x$ on the unit tangent bundle T_1S^2 to the 2-sphere S^2 with S^1-action

$$\psi : S^1 \times T_1S^2 \rightarrow T_1S^2 : \big(t,(x,y)\big) \mapsto \big(x, y\cos t + (x \times y)\sin t\big).$$

Let $\theta(x,y) = (x \times y, \frac{\partial}{\partial y})$ be a 1-form on $T\mathbf{R}^3$ where $(\ ,\ )$ is the Euclidean inner product on $\mathbf{R}^3$. Show that $\vartheta = \theta | T_1S^2$ is a connection 1-form for the principal bundle π. The vector fields $X_1(x,y) = (x, \frac{\partial}{\partial y}) - (y, \frac{\partial}{\partial x})$ and $X_2(x,y) = (x \times y, \frac{\partial}{\partial x})$, when restricted to T_1S^2 span $\ker \vartheta(x,y)$. Let ω be the symplectic form on TS^2 given by restricting the standard symplectic form on $T\mathbf{R}^3$. Orient TS^2 using the volume form $\mathrm{vol}_{TS^2} = \omega \wedge \omega$. With respect to the outward normal $n(x,y) = (y, \frac{\partial}{\partial y})$ at $(x,y) \in T_1S^2$ in TS^2, we have $n \,\lrcorner\, \mathrm{vol}_{TS^2} = \mathrm{vol}_{T_1S^2}$. Show that $T_{(x,y)}\pi$ maps the positively oriented frame $\{(x \times y, \frac{\partial}{\partial y}), X_1(x,y), X_2(x,y)\}$ of $T_{(x,y)}(T_1S^2)$ onto the positively oriented frame $\{(y, \frac{\partial}{\partial x}), (x \times y, \frac{\partial}{\partial x})\}$ of T_xS^2. Here we are using the standard volume form vol_{S^2} to orient S^2. Show that the curvature 2-form Ω of ϑ is equal to vol_{S^2}. Hence the holonomy around the positively oriented equator, as measured from a point in the fiber over a fixed point on the equator, is 2π. Generalize this argument to show that the holonomy along any closed positively oriented curve on S^2 is equal to the area on S^2 of the 2-disk bounded by γ.

Chapter IX

Action angle coordinates

Here we prove the existence of local action angle coordinates for a Liouville integrable Hamiltonian system near a compact connected fiber of its integral mapping.

1 Liouville integrable systems

A Hamiltonian system (f_1, M, ω) on a smooth symplectic manifold (M, ω) of dimension $2n$ is a *Liouville integrable* system with n-degrees of freedom if there are *Poisson commuting* functions $(f_1, \ldots, f_n)$, that is, $\{f_i, f_j\} = 0$, whose differentials are linearly independent on an open dense subset W of M and whose associated Hamiltonian vector fields X_{f_i} are complete.

An important result for Liouville integrable systems is the following.

Claim: If (f_1, M, ω) is an n-degree of freedom Liouville integrable Hamiltonian system with integrals $(f_1, \ldots, f_n)$, then every connected component of a fiber of the *integral map* $f : M \rightarrow \mathbf{R}^n : p \mapsto \big(f_1(p), \ldots, f_n(p)\big)$, corresponding to a regular value in its image, is diffeomorphic to a product of a k-torus $\mathbf{T}^k$ and $\mathbf{R}^{n-k}$ for some $0 \le k \le n$.

(1.1) **Proof**: The proof of the claim follows along the lines of step 2 of the proof of the existence of local action angle coordinates in §2. The $\mathbf{R}^n$-action Ψ (1) on the connected component F_q of the fiber $f^{-1}(q)$ of the integral map corresponding to a regular value q exists because the flows of X_{f_i} on F_q are complete. Because this action is transitive, the isotropy groups are all the same group and are discrete. Using the lemma ((2.2)) we see that F_q is diffeomorphic to $\mathbf{T}^{n-k} \times \mathbf{R}^k$ for some $1 \le k \le n$. □

2 Local action angle coordinates

Here we prove the existence of local action angle coordinates.

Claim: Let (M, ω) be a symplectic manifold of dimension $2n$. Suppose that $(f_1, \ldots, f_n)$ are Poisson commuting functions. Consider the integral map $f : M \rightarrow \mathbf{R}^n : p \mapsto \big(f_1(p), \ldots,$

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$f_n(p)\big)$. Suppose that q is a regular value in the image of f and that $f^{-1}(q)$ has a compact connected component F_q. Then there is a neighborhood V of F_q, an open set U of $\mathbf{R}^n$, and a diffeomorphism $V \to U \times \mathbf{T}^n : p \mapsto (j_1, \ldots j_n, \phi_1, \ldots \phi_n)$, where $\mathbf{T}^n$ is the n-torus $\mathbf{R}^n/\mathbf{Z}^n$. The coordinates j_i, called the *actions*, are smooth functions of the integrals f_i as are the coordinates ϕ_i, which are called the *angles*. In *action angle coordinates* the symplectic form ω may be written as $\omega = \sum_{i=1}^{n} \mathrm{d}j_i \wedge \mathrm{d}\phi_i$. For fixed $j = (j_1, \ldots, j_n)$ the symplectic form ω vanishes identically on the n-torus $\{j\} \times \mathbf{T}^n$. Because the dimension of this *torus* is $\frac{1}{2}\dim M$, it is a *Lagrangian* manifold.

We may reformulate the existence of action angle coordinates geometrically as follows. The open neighborhood V has the structure of a symplectic principal bundle with structure group $\mathbf{T}^n$, Lagrangian fibers, and a Hamiltonian action of the structure group with momentum map, which is the projection map of the bundle. In addition, a section of this bundle may be chosen to be Lagrangian so that the action angle variables form a symplectic chart.

The proof of the claim takes several steps.

1. Show that F_q has a neighborhood V diffeomorphic to $U \times F_q$.
2. Show that F_q is diffeomorphic to $\mathbf{T}^n$.
3. Define the group action.
4. Define the action variables and show that the action variables are a momentum map for the group action. This shows that the group action is Hamiltonian.
5. Construct a Lagrangian section of V.

(2.1) **Proof**: The steps of the proof are numbered as in the outline above.

Step 1.
Since q is a regular value of f and F_q is a compact component of $f^{-1}(q)$, there is an open ball $U \subseteq \mathbf{R}^n$ containing q which is contained in the set of regular values of f. Because F_q is a smooth compact submanifold of M, there is an open neighborhood V of F_q whose closure is compact such that $f(\overline{V}) \subseteq U$. Therefore $\widetilde{f} = f|V : V \to U$ is a proper submersion. Applying the Ehresmann fibration and trivialization theorems, see chapter VIII ((2.1)) and ((2.2)), and shrinking U and V if necessary, it follows that $\widetilde{f}$ is a trivial smooth fibration, that is, $\tau : V \to U \times \widetilde{f}^{-1}(u_0)$, where $F_q = \widetilde{f}^{-1}(u_0)$ with $u_0 = q \in U$ such that $\tau^{-1}\big(\{u\} \times \widetilde{f}^{-1}(u_0)\big) = \widetilde{f}^{-1}(u)$.

Step 2.
Let ψ_t^i be the flow of the Hamiltonian vector field X_{f_i}. This flow leaves the fibers of the integral map f, and hence those of $\widetilde{f}$, invariant. Because $\widetilde{f}^{-1}(u)$ is compact, the flow ψ_t^i on $\widetilde{f}^{-1}(u)$ is complete. Since $\{f_i, f_j\} = 0$, the flows ψ_t^i and ψ_s^j commute. Thus we have an $\mathbf{R}^n$-action

$$\Psi : \mathbf{R}^n \times \widetilde{f}^{-1}(u) \to \widetilde{f}^{-1}(u) : (t, p) = \big((t_1, \ldots, t_n), p\big) \mapsto \psi_{t_1}^1 \circ \cdots \circ \psi_{t_n}^n(p). \qquad (1)$$

Since the vector fields $X_{f_i}|\widetilde{f}^{-1}(u)$ are linearly independent, $D_1\Psi$ has rank n. Hence the orbits of Ψ are open. Because $\widetilde{f}^{-1}(u)$ is foliated by the orbits and is connected, there is only a single orbit, that is, Ψ is *transitive*. Let $\widetilde{p} \in \widetilde{f}^{-1}(u)$ and let $I_{\widetilde{p}} = \{t \in \mathbf{R}^n \big| \Psi(t,\widetilde{p}) = \widetilde{p}\}$, be the isotropy group of the action Ψ at $\widetilde{p}$. Because $\widetilde{f}^{-1}(u)$ is compact $I_{\widetilde{p}} \neq \{0\}$. By transitivity, the isotropy groups at any two points in $\widetilde{f}^{-1}(u)$ are conjugate and hence are equal since Ψ is an abelian action. Denote this isotropy group by $\mathscr{P}_f(u)$ and call it the *period lattice* of Ψ at u. Because $D_1\Psi$ has rank n, the period lattice $\mathscr{P}_f(u)$ is a zero dimensional Lie subgroup of $\mathbf{R}^n$ and hence is *discrete*. We claim that $\mathscr{P}_f(u)$ is a *lattice*, that is, it is a free $\mathbf{Z}$-module with n generators. To see this we prove

Lemma: Let Λ be a nonzero discrete subgroup of $\mathbf{R}^n$. Then for some $1 \le k \le n$ there are k linearly independent vectors $\lambda_1, \ldots, \lambda_k$, called *generators*, such that

$$\Lambda = \mathbf{Z}\lambda_1 + \cdots + \mathbf{Z}\lambda_k. \tag{2}$$

(2.2) **Proof**: Since $\Lambda \neq 0$, there is a nonzero vector $\overline{\lambda}_1 \in \Lambda$. Let $\mathrm{span}\{\overline{\lambda}_1\}$ be the subspace of $\mathbf{R}^n$ spanned by $\overline{\lambda}_1$. Since $\Lambda \cap \mathrm{span}\{\overline{\lambda}_1\}$ is a discrete group, there is a $\widehat{\lambda}_1 \in \Lambda \cap \mathrm{span}\{\overline{\lambda}_1\}$ such that the open line segment $\{t\widehat{\lambda}_1 | t \in (0,1)\}$ does *not* intersect $\Lambda \cap \overline{\lambda}_1$. Set $\lambda_1 = \widehat{\lambda}_1$. Then $\mathbf{Z}\lambda_1 = \Lambda \cap \mathrm{span}\{\lambda_1\}$. If $\Lambda = \mathbf{Z}\lambda_1$, then $k = 1$ and we are done.

Otherwise, let $\overline{\lambda}_2 \in \Lambda \setminus \mathbf{Z}\lambda_1$. As before let $\widehat{\lambda}_2$ be a generator of $\Lambda \cap \mathrm{span}\{\overline{\lambda}_2\}$ and let $P_{\lambda_1\widehat{\lambda}_2} = \{s_1\lambda_1 + s_2\widehat{\lambda}_2 \big| s_1, s_2 \in [0,1]\}$ be the *parallelogram* spanned by λ_1 and $\widehat{\lambda}_2$. Suppose that $P_{\lambda_1\widehat{\lambda}_2} \cap \Lambda$ contains an infinite number of points $\widetilde{\lambda}_i$. Select one, say $\overline{\lambda}_{i_0}$. A finite number of $\mathbf{Z}^2$-translates of $P_{\lambda_1\widehat{\lambda}_{i_0}}$ cover the parallelogram $P_{\lambda_1\widehat{\lambda}_2}$. Hence one of these translates contains an infinite number of points of Λ. Since Λ is closed under $\mathbf{Z}^2$-translations, we conclude that $P_{\lambda_1\widehat{\lambda}_{i_0}}$ contains an infinite number of points of $P_{\lambda_1\widehat{\lambda}_2} \cap \Lambda$. Repeat the above argument on the parallelogram $P_{\lambda_1\widehat{\lambda}_{i_0}}$. We obtain a sequence of points $\widehat{\lambda}_i \in P_{\lambda_1\widehat{\lambda}_2} \cap \Lambda$ which converges. This contradicts the discreteness of Λ. Hence the parallelogram $P_{\lambda_1\widehat{\lambda}_2}$ contains only *finitely* many points of Λ. Therefore it contains a parallelogram $P_{\lambda_1\lambda_2}$ such that $P_{\lambda_1\lambda_2} \cap \Lambda$ contains only the vertices $\{0, \lambda_1, \lambda_2, \lambda_1 + \lambda_2\}$. Suppose that $\mathbf{Z}\lambda_1 + \mathbf{Z}\lambda_2$ is not equal to $\mathrm{span}\{\lambda_1, \lambda_2\} \cap \Lambda$. Then there is a $\lambda = r_1\lambda_1 + r_2\lambda_2 \in \Lambda$ with $r_i \in \mathbf{R} \setminus \mathbf{Z}$. Hence for some $n_i \in \mathbf{Z}$ with $r_i - n_i \in (0,1)$, the vector $\mu = \lambda - n_1\lambda_1 - n_2\lambda_2$ lies in $P_{\lambda_1\lambda_2}$ and is not a vertex. This contradicts the definition of $P_{\lambda_1\lambda_2}$. Therefore $\mathbf{Z}\lambda_1 + \mathbf{Z}\lambda_2 = \mathrm{span}\{\lambda_1, \lambda_2\} \cap \Lambda$. If $\Lambda = \mathbf{Z}\lambda_1 + \mathbf{Z}\lambda_2$, then $k = 2$ and we are done.

Otherwise repeat a similar argument as above with parallelograms replaced by parallelopipeds. Eventually we obtain a linearly independent set of vectors $\{\lambda_1, \ldots, \lambda_k\}$ such that (2) holds. □

Corollary: If $\mathbf{R}^n/\Lambda$ is compact, then $k = n$.

(2.3) **Proof**: Suppose that $1 \le k < n$. Then there is a nonzero vector $\lambda \in \mathbf{R}^n$ such that $\lambda \notin \mathrm{span}\{\lambda_1, \ldots, \lambda_k\}$. Thus for every $t \in \mathbf{R} \setminus \{0\}$ the element $\mu_t = t\lambda + \mathbf{Z}\lambda_1 + \cdots + \mathbf{Z}\lambda_k$ of $\mathbf{R}^n/\Lambda$ is a nonzero. Since $\mathbf{R}^n/\Lambda$ is compact, there is a subsequence $\{t_i\}$ in $[1,2]$ such that $\{\mu_{t_i}\}$ converges to $\mu \in \mathbf{R}^n/\Lambda$. Taking another subsequence we may assume that t_i converges to $t' \in [1,2]$ and that μ_{t_i} converges to $\mu = \sum_{i=1}^k r_i\lambda_i + \mathbf{Z}\lambda_1 + \cdots + \mathbf{Z}\lambda_k$ for some $r_i \in [0,1)$. Therefore $\lambda = \frac{1}{t'}\big(\mu + \mathbf{Z}\lambda_1 + \cdots + \mathbf{Z}\lambda_k\big) \in \mathrm{span}\{\lambda_1, \ldots, \lambda_k\}$, which contradicts the fact that $\lambda \notin \Lambda$. Conseqently $k = n$. □

From the corollary it follows that the period lattice $\mathscr{P}_f(u)$ is a lattice. Therefore $\widetilde{f}^{-1}(u)$ is diffeomorphic to an n-torus $\mathbf{R}^n/\mathscr{P}_f(u)$.

Step 3.
Because the fibration $\widetilde{f}$ is trivial, for every $p_0 \in \widetilde{f}^{-1}(u_0)$ there is a smooth section $\sigma : U \to \widetilde{f}^{-1}(U) \subseteq V$ with $\sigma(u_0) = p_0$, which is a diffeomorphism onto its image. We now show that the isotropy group varies smoothly with u. Consider the function

$$\Theta : \mathscr{W} \times U \to \mathscr{U} \subseteq \mathbf{R}^n : (t,u) \mapsto \sigma^{-1} \circ \Psi(t, \sigma(u)) - u,$$

where $\mathscr{W}$ is an open subset of $\mathbf{R}^n$ about $t_0 \in I_{p_0} \setminus \{0\}$ and $\mathscr{U}$ is an open subset of $\mathbf{R}^n$. Since $\Psi(t_0, p_0) = p_0$, it follows from the continuity of σ and Ψ that both $\mathscr{W}$ and $\mathscr{U}$ can be chosen so that Θ is well defined. Since σ is a diffeomorphism onto its image and $D_1\Psi$ is surjective, $D_1\Theta$ is surjective. Thus 0 is a regular value of Θ and $(t_0, u_0) \in \Theta^{-1}(0)$. Therefore, near (t_0, u_0) the set $\Theta^{-1}(0)$ is the graph of a smooth function $T_0 : U \to \mathbf{R}^n$. Let t_0 run through a basis $\{t_0^i\}$ of the period lattice $\mathscr{P}_f(u_0)$. Using the preceeding argument for $i = 1, \ldots, n$, we obtain smooth functions $T^i : U \to \mathbf{R}^n : u \mapsto (T_1^i(u), \ldots, T_n^i(u))$ such that $\{T^1(u), \ldots, T^n(u)\}$ form a basis of $\mathscr{P}_f(u)$ which depends smoothly on $u \in U$ and $\{T^i(u_0)\} = \{t_0^i\}$.

Let Y_i be a smooth vector field on V whose flow is

$$\Phi^i : \mathbf{R} \times V \to V : (t,p) \mapsto \Psi\big((tT_1^i(u), \ldots, tT_n^i(u)), p\big).$$

Then $Y_i(p) = \sum_{j=1}^n T_j^i(f(p)) X_{f_i}(p)$. The flow of Y_i is periodic with period one. The vector fields $\{Y_i\}$ are linearly independent and $[Y_i, Y_j] = 0$ since $\{f_i, f_j\} = 0$. Thus the $\{\Phi^i\}$ are n commuting flows and define an action of the n-dimensional torus group $\mathbf{T}^n = \mathbf{R}^n/\mathbf{Z}^n$ on $\widetilde{f}^{-1}(U)$.

By restricting U if necessary, we can arrange that $0 = [\omega] \in \mathrm{H}^2(\widetilde{f}^{-1}(U), \mathbf{R})$. To see this we argue as follows. In a neighborhood V of $\widetilde{f}^{-1}(u_0)$ which retracts to $\widetilde{f}^{-1}(u_0)$, (this exists by virtue of the previous statements about the local fibration) any two-cycle Σ is homotopic to a cycle Σ' in the fiber $\widetilde{f}^{-1}(u_0)$. We have $\int_\Sigma \omega = \int_{\Sigma'} \omega = 0$, where the first equality holds because ω is closed and the second because $\widetilde{f}^{-1}(u_0)$ is isotropic. This shows that the cohomology class of ω vanishes. Thus by *de Rham's theorem* there exists a smooth 1-form ζ on V such that $\omega = -\,\mathrm{d}\zeta$.

Step 4.
Define $j_a(p) = \int_0^1 (\zeta(Y_a)) \circ \Phi_a^t(p)\, \mathrm{d}t$. We assert that the torus action $\Phi = (\Phi_1, \ldots, \Phi_n)$ is symplectic and has a momentum map $j = (j_1, \ldots, j_n)$.

To establish the assertion, it is enough to demonstrate this for each Φ_a individually because the torus group is abelian. This is established if we can show $\mathrm{d}j_a(p)Z(p) = \omega(p)(Y_a(p), Z(p))$ for each $p \in V$ and each $Z(p) \in T_pV$. In other words, the vector field Y_a is the Hamiltonian vector field X_{j_a} corresponding to the action j_a. Extend Z to a vector field, also denoted by Z. We can arrange things so that $\Phi_a^* Z = Z$. In particular, $[Y_a, Z] = 0$. Differentiating the definition of j_a and remembering that Z is Φ-invariant gives $\mathrm{d}j_a(Z) = \int_0^1 \big(\mathrm{d}(\zeta(Y_a))Z\big) \circ \Phi_a^t\, \mathrm{d}t$. In the preceding formula we have dropped the reference to p. However,

$$\mathrm{d}(\zeta(Y_a))Z = L_Z(\zeta(Y_a)) \,=\, (L_Z\zeta)Y_a + \zeta(L_ZY_a). \tag{3}$$

The second term in (3) vanishes since $[Z,Y_a]=0$. Now

$$(L_Z\zeta)(Y_a) = \mathrm{d}(Z \,\lrcorner\, \zeta)Y_a + (Z \,\lrcorner\, \mathrm{d}\zeta)Y_a \,=\, L_{Y_a}(\zeta(Z)) + \omega(Y_a,Z).$$

Since

$$\int_0^1 L_{Y_a}(\zeta(Z))\circ\Phi_a^t\,\mathrm{d}t = \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t}(\zeta(Z))\circ\Phi_a^t\,\mathrm{d}t = 0,$$

we have $\mathrm{d}j_a(Z) = \int_0^1 \omega(Y_a,Z)\circ\Phi_a^t\,\mathrm{d}t$. Next we show that the integrand in the preceding formula is invariant under the flow Φ_a. We calculate

$$L_{Y_a}\big(\omega(Y_a,Z)\big) = L_{Y_a}\big((Y_a \,\lrcorner\, \omega)(Z)\big) \,=\, \big(Y_a \,\lrcorner\, L_{Y_a}\omega\big)(Z)$$

since $[Y_a,Z]=0$. Furthermore

$$\begin{aligned} Y_a \,\lrcorner\, L_{Y_a}\omega &= Y_a \,\lrcorner\, \big(Y_a \,\lrcorner\, \mathrm{d}\omega + \mathrm{d}(Y_a \,\lrcorner\, \omega)\big) \,=\, Y_a \,\lrcorner\, \mathrm{d}(Y_a \,\lrcorner\, \omega) \\ &= Y_a \,\lrcorner\, \mathrm{d}\Big(\sum_{c=1}^n T_a^c(X_{f_c} \,\lrcorner\, \omega)\Big) \,=\, Y_a \,\lrcorner\, \Big(\sum_{c=1}^n \mathrm{d}T_a^c \wedge \mathrm{d}f_c\Big) \\ &= 0, \end{aligned}$$

because both the T_a^c and f_c are constant on the fibers to which Y_a is tangent. Since the integrand is invariant, we have $\mathrm{d}j_a(Z) = \omega(Y_a,Z)$, which is precisely what is required to show that we have a momentum map. Note that the momentum map has rank n because

$$(\mathrm{d}j_1\wedge\cdots\wedge\mathrm{d}j_n)(Z_1,\ldots,Z_n) = (\omega\wedge\cdots\wedge\omega)(Y_1,\ldots,Y_n,Z_1,\ldots,Z_n),$$

and we can certainly choose vector fields $Z_1,\ldots,Z_n$ so that the vectors $Y_a(p)$ and $Z_a(p)$ are a basis for T_pV.

Step 5.
In order to choose angle variables correctly, it is useful to review first the situation so far. The results of the previous sections show that $V\subseteq M$ is the total space of a symplectic principal bundle ρ whose fibers are Lagrangian n-tori $\mathbf{T}^n$. The action of the structure group $\mathbf{T}^n$ is Hamiltonian whose momentum map is the projection map of the bundle, namely, $\rho: V\to U\subseteq\mathbf{R}^n : p\mapsto j(p)=(j_1,\ldots,j_n)$. About a point $j(p)$ in the base, we choose a section $\sigma: U\to U\times\mathbf{T}^n = V : j\mapsto \big(j,\sigma(j)\big)$ of the bundle ρ, whose image determines the origin for the angular coordiantes θ on $\mathbf{T}^n$. On $\mathbf{T}^n$ the group operation is addition.

We now compute the symplectic form ω in the (j,θ) coordinates. First we calculate the structure matrix $\mathscr{W} = \begin{pmatrix} \{j_k,j_\ell\} & \{j_k,\theta_\ell\} \\ \{\theta_k,j_\ell\} & \{\theta_k,\theta_\ell\}\end{pmatrix}$ of the Poisson bracket on V. We have $\{j_k,\theta_\ell\} = -\{\theta_\ell,j_k\} = -L_{X_{j_k}}\theta_\ell = -L_{X^{\theta_k}}\theta_\ell$, since j is the momentum map of the Hamiltonian action of $\mathbf{T}^n$ on V. Therefore $\{j_k,\theta_\ell\} = -\frac{\partial}{\partial\theta_k}\theta_\ell = -\delta_{k\ell}$. Also $\{j_k,j_\ell\} = L_{X_{j_\ell}}j_k =$

$L_{X^{\theta_\ell}} j_k = 0$, since j_k is an integral of X^{θ_ℓ}. So the structure matrix $\mathscr{W}$ is $\begin{pmatrix} 0 & -I \\ I & -F \end{pmatrix}$, where $F = (F_{k,\ell}) = (-\{\theta_k, \theta_\ell\})$. The matrix of ω is $(\mathscr{W}^{-1})^t = \begin{pmatrix} F & -I \\ I & 0 \end{pmatrix}$, that is, $\omega = \sum_{\ell,k} F_{\ell,k}\, \mathrm{d}j_\ell \wedge \mathrm{d}j_k + \sum_\ell \mathrm{d}j_\ell \wedge \mathrm{d}\theta_\ell$. Because

$$\tfrac{\partial}{\partial \theta_m}(\{\theta_k, \theta_\ell\}) = L_{X_{j_m}}\{\theta_k, \theta_\ell\} = \{L_{X_{j_m}}\theta_k, \theta_\ell\} + \{\theta_k, L_{X_{j_m}}\theta_\ell\} = \{\delta_{m,k}, \theta_\ell\} + \{\theta_k, \delta_{m,k}\} = 0,$$

the functions $F_{\ell,k}$ depend only on $j_1, \ldots, j_n$. Thus the 2-form $F = \sum_{\ell,k} F_{\ell,k}\, \mathrm{d}j_\ell \wedge \mathrm{d}j_k$ on U is closed. Shrinking U we may apply the Poincaré lemma to obtain a 1-form $A = \sum_\ell A_\ell\, \mathrm{d}j_\ell$ such that $F = \mathrm{d}A$. Adjusting the section σ, by defining new angle coordinates ϕ_ℓ, using the map $\phi_\ell = \theta_\ell - A_\ell$, moves σ to a Lagrangian section of the bundle ρ. In the coordinates $(j_1, \ldots, j_\ell, \phi_1, \ldots, \phi_\ell)$ we have

$$-\sum_{\ell=1}^{n} \mathrm{d}\phi_\ell \wedge \mathrm{d}j_\ell = -\sum_{\ell=1}^{n} (\mathrm{d}\theta_\ell - \mathrm{d}A_\ell) \wedge \mathrm{d}j_\ell = \sum_{k,\ell=1}^{n} F_{\ell,k}\, \mathrm{d}j_\ell \wedge \mathrm{d}j_k + \sum_{\ell=1}^{n} \mathrm{d}j_\ell \wedge \mathrm{d}\theta_\ell = \omega.$$

This concludes the proof of local action angle coordinates. □

3 Exercise

1. (Period energy relation.) Consider the Hamiltonian system (P, ω, H), where all the integral curves of the Hamiltonian vectorfield X_H on P are periodic of positive period T. In other words, P is a smooth circle bundle, where the fiber is a periodic integral curve of X_H.

   a) Show that the period and the energy H are functionally related, that is, $\mathrm{d}T \wedge \mathrm{d}H = 0$. Colloquially, the period is constant on a level set of the Hamiltonian. Hint: rescale X_H and show that the resulting vector field is still Hamiltonian.

   b) Compute the period energy relation for the Kepler problem. Check its relation with Kepler's third law of motion.

2. (Action angle coordinates.) The following version of the action angle coordinate theorem emphasizes the integral affine structure of the base. Let $F : M \to B$ be a surjective submersion, whose fibers are compact connected Lagrangian submanifolds of the symplectic manifold (M, ω). Show that for each $b \in B$ there is an open neighborhood $V \subseteq B$ of b and coordinates $(I, \varphi) = (I_1, \ldots, I_n, \varphi_1, \ldots, \varphi_n)$ on $F^{-1}(V)$ such that

   a) For $1 \le j \le n$ each I_j factors through F, that is, there are smooth functions x_j on B such $I_j = x_j \circ F$ and $\{\mathrm{d}x_j\}_{j=1}^n$ are linearly independent. Each φ_j takes values in $\mathbf{T} = \mathbf{R}/\mathbf{Z}$.

   b) The coordinates (I, φ) are symplectic, that is, $\omega|(F|F^{-1}(V)) = \sum_{j=1}^n \mathrm{d}I_j \wedge \mathrm{d}\varphi_j$.

   c) The coordinates I are unique up to an integer affine transformation in $\mathbf{R}^n \rtimes \mathrm{Gl}(n, \mathbf{Z})$.

   d) Given I, the coordinates φ are determined by the choice of a Lagrangian submanfold on which all the φ_j are zero.

Chapter X

Monodromy

Since the construction of action angle variables is in general *local*, it is of some interest to see whether it can be extended to a global one. We do not describe all the obstructions to having global action angle variables. In this chapter we will discuss the crudest obstruction to the existence of global action angle coordinates called *monodromy*.

1 The period lattice bundle

Suppose that we are in the following situation. Let M be a symplectic manifold foliated by connected Lagrangian n-tori which are the fibers of a surjective smooth map $F: M \rightarrow B$. Then F is a fiber bundle over B with total space M and fiber an n-torus. We wish to determine if F is a product bundle. This we do by constructing a *bundle of period lattices* associated to the bundle F. The total space of the associated bundle is $\mathscr{P}$, which is the disjoint union $\coprod_{b \in B} \mathscr{P}_F(b)$ of the period lattice $\mathscr{P}_F(b)$ at b of the bundle F. The base of the
$\triangleright$ bundle is B and the projection mapping is $\mathscr{F}: \mathscr{P} \rightarrow B: \mathscr{P}_b = \mathscr{P}_F(b) \mapsto b$. We construct a local trivialization for $\mathscr{F}$ as follows.

(1.1) **Proof**: Let $\{U_\alpha\}$ be an open covering of B such that on $F^{-1}(U_\alpha)$ local action angle coordinates $(j^1_\alpha, \ldots j^n_\alpha, \phi^1_\alpha, \ldots \phi^n_\alpha)$ exist. In addition, assume that $\{U_\alpha\}$ is a *good covering* of B in the sense of Leray, that is,

1. U_α and $U_\alpha \cap U_\beta$ are connected and contractible.
2. $U_\alpha \cap U_\beta \cap U_\gamma = \varnothing$.

The flows $\psi_t^{j^i_\alpha}$ of the Hamiltonian vector fields $X_{j^i_\alpha}$ on $F^{-1}(U_\alpha)$ generate a $\mathbf{T}^n = \mathbf{R}^n/\mathbf{Z}^n$-action

$$\Psi^\alpha : \mathbf{T}^n \times F^{-1}(U_\alpha) \rightarrow F^{-1}(U_\alpha) : \big((t_1, \ldots, t_n), p\big) \mapsto \psi_{t_1}^{j^1_\alpha} \circ \cdots \circ \psi_{t_n}^{j^n_\alpha}(p),$$

which is free and proper. Therefore, $F^{-1}(U_\alpha)$ is a principal $\mathbf{T}^n$ bundle, see chapter VII ((2.12)). Moreover, $F^{-1}(U_\alpha)$ is trivial because U_α is contractible, see chapter VIII §2. The trivialization $\tau_\alpha: F^{-1}(U_\alpha) \rightarrow U_\alpha \times \mathbf{T}^n$ intertwines the $\mathbf{T}^n$-action Ψ^α with the affine action of $\mathbf{T}^n$ on itself given by $\mathbf{T}^n \times \mathbf{T}^n \rightarrow \mathbf{T}^n : (\bar{s}, \bar{t}) \mapsto \overline{s+t}$. Here $\bar{t} = t \bmod 1$. From this

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intertwining property we find that for fixed $u_\alpha \in U_\alpha$ the tangent of $\tau^{-1}_{\alpha,u_\alpha} : \mathbf{T}^n \to F^{-1}(u_\alpha) :$ $t \mapsto \tau^{-1}_\alpha(u_\alpha, t) = p$ maps the lattice $\mathbf{Z}^n \subseteq T_t\mathbf{T}^n$ into the lattice in $T_pF^{-1}(u_\alpha)$ spanned by the vectors $\{X_{j^1_\alpha}(p), \ldots, X_{j^n_\alpha}(p)\}$. This latter lattice is the period lattice $\mathscr{P}_F(u_\alpha)$ of F at $F(p) = u_\alpha$. Since $\mathscr{P}_F(u_\alpha)$ does *not* depend on the choice of $p \in F^{-1}(u_\alpha)$, the tangent of the map $\tau^{-1}_{\alpha,u_\alpha}$ does *not* depend on the point in $\mathbf{T}^n$. Thus a local trivialization of $\mathscr{F}|\mathscr{F}^{-1}(U_\alpha)$ is given by

$$\rho_\alpha : \bigcup_{p\in F^{-1}(U_\alpha)}^{\cdot} \mathscr{P}_F(F(p)) \to U_\alpha \times \mathbf{Z}^n : p \mapsto \big(F(p), T_p\tau_\alpha \mathscr{P}_F(F(p))\big). \qquad \square$$

▷ To complete the construction of the bundle $\mathscr{F}$, we find its transition mappings. Towards this end let

$$\tau_\beta \circ \tau_\alpha^{-1} : (U_\alpha \cap U_\beta) \times \mathbf{T}^n \to (U_\alpha \cap U_\beta) \times \mathbf{T}^n : (u,t) \mapsto \big(\varphi_{\alpha\beta}(u), \tau^{\alpha\beta}(u,t)\big)$$

be the transition map for the bundle $F|F^{-1}(U_\alpha \cap U_\beta)$. For $u \in U_\alpha \cap U_\beta$, the partial transition mapping $\tau^{\alpha\beta}_u : \mathbf{T}^n \to \mathbf{T}^n : t \mapsto \tau^{\alpha\beta}(u,t)$ has a tangent map $T_t\tau^{\alpha\beta}_u$ which does *not* depend on the point $t \in \mathbf{T}^n$. Moreover, $T\tau^{\alpha\beta}_u$ is a linear isomorphism of $\mathbf{R}^n$, which preserves the lattice $\mathbf{Z}^n$. Since $U_\alpha \cap U_\beta$ is connected, we may assume that this isomorphism is orientation preserving. The collection of all such isomorphisms forms the group $\mathrm{Sl}(n,\mathbf{Z})$, which is a *discrete* subgroup of $\mathrm{Sl}(n,\mathbf{R})$. Because $U_\alpha \cap U_\beta$ is connected, it follows from discreteness and continuity that the map $U_\alpha \cap U_\beta \to \mathrm{Sl}(n,\mathbf{Z}) : u \mapsto T\tau^{\alpha\beta}_u$ is constant, say $g^{\alpha\beta}$. Consequently, the transition map for the period lattice bundle $\mathscr{F}|\mathscr{F}^{-1}(U_\alpha \cap U_\beta)$ is

$$\rho_\beta \circ \rho_\alpha^{-1} : (U_\alpha \cap U_\beta) \times \mathbf{Z}^n \to (U_\alpha \cap U_\beta) \times \mathbf{Z}^n : (u,z) \mapsto \big(\varphi_{\alpha\beta}(u), g^{\alpha\beta}(z)\big). \qquad \square$$

This completes the construction of the bundle $\mathscr{F}$ of period lattices associated to F.

Next we describe the monodromy representation associated to the bundle of period lattices. Let $\gamma : [0,1] \to B$ be a smooth closed curve in B with starting point b. Restricting the bundle $\mathscr{F}$ to γ gives

$$\mathscr{F}|\gamma : \mathscr{P}|\gamma = \bigcup_{u\in\gamma([0,1])} \mathscr{P}_F(u) \to \gamma : \mathscr{P}_F(u) \mapsto u,$$

which is a smooth bundle of period lattices over γ. Since γ is diffeomorphic to a circle, the isomorphism class of the bundle $\mathscr{F}|\gamma$ is determined by an automorphism $g(b)$ of the fiber $\mathscr{P}_b$ over b. This isomorphism class depends only on the homotopy class of γ in B. Therefore we have the map $\pi : \pi_1(B) \to \mathrm{Sl}(n,\mathbf{Z}) : \gamma \mapsto g(\gamma(0))$, which is the *monodromy representation* of the fundamental group of the base B of the bundle $\mathscr{F}$. If π is *not* the identity, that is, if for some smooth loop γ in B the bundle $\mathscr{F}|\gamma$ is *not* trivial, then the bundle of period lattices $\mathscr{F}$ is said to have *monodromy*.

The bundle of period lattices is isomorphic to the bundle $\mathrm{H}_1(F^{-1}(B),\mathbf{Z})$ over B whose fiber at $b \in B$ is the first homology group of the fiber $F^{-1}(b)$ of the n-torus bundle F. The proof of this assertion is left as an exercise.

We now prove

Claim: The n-torus bundle $F: M \to B$ is not trivial if the associated period lattice bundle $\mathscr{F}: \mathscr{P} \to B$ has monodromy.

(1.2) **Proof**: Suppose that the bundle F is trivial. Then over every smooth closed loop γ in B, the bundle $F|\gamma$ is trivial. Hence the classifying map $\chi : F^{-1}(\gamma(0)) \to F^{-1}(\gamma(0))$ of $F|\gamma$ is homotopic to the identity map. Thus χ induces the map

$$\chi_* : \mathrm{H}_1(F^{-1}(\gamma(0)), \mathbf{Z}) \to \mathrm{H}_1(F^{-1}(\gamma(0)), \mathbf{Z}),$$

which is the identity. Therefore the monodromy representation of the period lattice bundle $\mathscr{F}: \mathscr{P} \to B$ associated to the n-torus bundle $F: M \to B$ is the identity. In other words, the bundle $\mathscr{F}$ has no monodromy. Taking the contrapositive proves the claim. □

2 The geometric monodromy theorem

In this section we give a geometric conditions so that the integral mapping of a Liouville integrable system with a focus-focus equilibrium point has monodromy.

The origin 0 of $\mathbf{R}^4$ is a *focus-focus* equilibrium point of the Liouville integrable system $(h_1, h_2, \mathbf{R}^4, \omega = \mathrm{d}x \wedge \mathrm{d}p_x + \mathrm{d}y \wedge \mathrm{d}p_y)$ if and only if

1. The vector fields X_{h_1} and X_{h_2} vanish at 0, that is, 0 is an equilibrium point of X_{h_1} and X_{h_2}.
2. The space spanned by the linearized Hamiltonian vector fields $DX_{h_1}(0)$ and $DX_{h_2}(0)$ is conjugate by a real linear symplectic mapping of $(\mathbf{R}^4, \omega)$ into itself to the Cartan subalgebra of $\mathrm{sp}(4, \mathbf{R})$ spanned by X_{q_1} and X_{q_2}, where $q_1 = xp_x + yp_y$ and $q_2 = xp_y - yp_x$.

From point 2 we may assume that $h_i = q_i + r_i$ for $i = 1, 2$, where r_i is a smooth function on $\mathbf{R}^4$, which is flat to 2^{nd} order at 0, that is, $r_i \in \mathscr{O}(2)$.

The remainder of this section is devoted to proving

Theorem (Geometric monodromy). Let $(h_1, h_2, \mathbf{R}^4, \omega)$ be a Liouville integrable system with a focus-focus equilibrium point at $0 \in \mathbf{R}^4$. Consider the integral map

$$F : \mathbf{R}^4 \to \mathbf{R}^2 : z \mapsto \big(h_1(z), h_2(z)\big) = (c_1, c_2). \tag{1}$$

Note that $h_1(0) = h_2(0) = 0$. Suppose that F has the following properties.

1. There is an open neighborhood U of the origin in $\mathbf{R}^2$ such that 0 is the only critical value of the integral map F in U.
2. For every $c \in U \setminus \{0\}$ the fiber $F^{-1}(c)$ is compact and connected, and hence is diffeomorphic to a smooth 2-torus T_c^2.
3. The singular fiber $F^{-1}(0)$ is compact and connected. Moreover, for every $z \in F^{-1}(0) \setminus \{0\}$ the rank of $DF(z)$ is 2.

Then the smooth 2-torus bundle $\rho = F|F^{-1}(\Gamma) : F^{-1}(\Gamma) \to \Gamma$ over the smooth circle Γ in $U \setminus \{0\}$ is *nontrivial*. For $c_0 \in \Gamma$, using a suitable basis of $\mathrm{H}_1(F^{-1}(c_0), \mathbf{Z})$, the classifying map of the bundle ρ is $\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$. Here the 2-torus $F^{-1}(c_0)$ is identified with $\mathbf{R}^2/\mathbf{Z}^2$.

2.1 The singular fiber

In this subsection we show that the singular fiber of the integral mapping F (1) is homeomorphic to a once pinched 2-torus.

Let $\varphi_t^{h_1}$ and $\varphi_s^{h_2}$ be the flows of the vector fields X_{h_1} and X_{h_2}, respectively. The hyperbolicity of X_{h_1} at 0 implies that there is an open ball B in $\mathbf{R}^4$ centered at 0 having radius r such that the local stable $W_s^B(0)$ and unstable $W_u^B(0)$ manifolds of 0 in B are smooth connected manifolds, whose tangent space at 0 is the ∓ 1 eigenspace of the linear mapping X_{q_1}, respectively. The global stable (unstable) manifold $W_{s,u}(0)$ of 0 is $\bigcup_{\mp t \geq 0} \varphi_t^{h_1}(W_{s,u}^B(0))$. When $z \in W_{s,u}(0)$ as $t \to \pm\infty$ we have $F(z) = F(\varphi_t^{h_1}(z)) \to F(0) = 0$.
$\triangleright$ Thus $W_{s,u}(0) \subseteq F^{-1}(0)$. The following reasoning shows that $F^{-1}(0) = W_s(0) = W_u(0)$.

(2.1) **Proof**: Since $F^{-1}(0)$ is a compact subset of $\mathbf{R}^4$, which is locally invariant under the flow $\varphi_t^{h_1}$, it is globally invariant. Thus $\varphi_t^{h_1}|F^{-1}(0)$ is defined for every $t \in \mathbf{R}$. Because of hypothesis 3, the set $F^{-1}(0)^\times = F^{-1}(0) \setminus \{0\}$ is a smooth 2-dimensional submanifold of $\mathbf{R}^4$. So $\varphi_t^{h_1}(W_{s,u}^B(0)) \setminus \{0\})$ is an open subset of $F^{-1}(0)^\times$. Thus $W_{s,u}(0)^\times = W_{s,u}(0) \setminus \{0\} = \bigcup_{\mp t \geq 0} \varphi_t^{h_1}(W_{s,u}^B(0) \setminus \{0\})$. The compact set $F^{-1}(0)$ is invariant under the flow $\varphi_s^{h_2}$. Because $\{h_1, h_2\} = 0$, the flows $\varphi_t^{h_1}$ and $\varphi_s^{h_2}$ commute, that is, $\varphi_t^{h_1} \circ \varphi_s^{h_2} = \varphi_s^{h_2} \circ \varphi_t^{h_1}$. Thus $F^{-1}(0)$ is invariant under the $\mathbf{R}^2$-action

$$\Xi : \mathbf{R}^2 \times \mathbf{R}^4 \to \mathbf{R}^4 : \big((t,s), z\big) \mapsto \varphi_t^{h_1} \circ \varphi_s^{h_2}(z). \tag{2}$$

So the $\mathbf{R}^2$-action $\Xi_{(t,s)}|F^{-1}(0)$ on $F^{-1}(0)$ is defined. Because $0 \in \mathbf{R}^2$ is an isolated critical value of F by hypothesis 1, it follows that $0 \in \mathbf{R}^4$ is an isolated equilibrium point of X_{h_1} and X_{h_2}. Thus 0 is an isolated fixed point of the $\mathbf{R}^2$-action $\Xi_{(t,s)}|F^{-1}(0)$. Moreover, if $z \in W_{s,u}(0)$, then $\varphi_t^{h_1}(\varphi_s^{h_2}(z)) = \varphi_s^{h_2}(\varphi_t^{h_1}(z)) \to \varphi_s^{h_2}(0) = 0$ where $t \to \pm\infty$. So $W_{s,u}(0)$ is invariant under the flow $\varphi_s^{h_2}$. Because 0 is a fixed point of the $\mathbf{R}^2$-action $\Xi_{(t,s)}|F^{-1}(0)$, it follows that $W_{s,u}(0)^\times = W_{s,u}(0) \setminus \{0\}$ is invariant under both flows $\varphi_t^{h_1}$ and $\varphi_s^{h_2}$. By hypothesis 2 the vector fields X_{h_1} and X_{h_2} are linearly independent at each point of $F^{-1}(0)^\times$. Consequently, each orbit $\mathscr{O}$ of the $\mathbf{R}^2$-action $\Xi_{(t,s)}|F^{-1}(0)$ is open. Because the complement of $\mathscr{O}$ in $F^{-1}(0)^\times$ is the union of other $\mathbf{R}^2$ orbits of $\Xi_{(t,s)}|F^{-1}(0)$, it is also open. Thus $\mathscr{O}$ is a connected component of $F^{-1}(0)^\times$. A similar argument shows that $W_{s,u}(0)^\times$ is an $\mathbf{R}^2$-orbit in $F^{-1}(0)^\times$. The orbit $\mathscr{O}$ is open and closed in $F^{-1}(0)^\times$, which implies that it is open in $F^{-1}(0) = F^{-1}(0)^\times \cup \{0\}$. If 0 is not in the closure of $\mathscr{O}$ in $\mathbf{R}^4$, then $\mathscr{O}$ is closed in $F^{-1}(0)$. Hence $\mathscr{O} = F^{-1}(0)$, because $F^{-1}(0)$ is connected. But $0 \in F^{-1}(0)$, which is a contradiction. Thus $\mathscr{O} \cup \{0\}$ is a closed subset of $\mathbf{R}^4$.

So 0 is the unique limit point in $\mathbf{R}^4 \setminus \mathscr{O}$ of the $\mathbf{R}^2$-orbit $\mathscr{O}$. Thus for any $\mathscr{O}$ in $F^{-1}(0)^\times$, we know that the closure of $\mathscr{O}$ in $\mathbf{R}^4$ is $\mathscr{O} \cup \{0\}$. In particular, this holds when $\mathscr{O} = W_{s,u}(0)$. If $\mathscr{O}$ is an orbit of the $\mathbf{R}^2$-action $\Xi_{(t,s)}|F^{-1}(0)$ and $z \in \mathscr{O}$, then the mapping $(s,t) \mapsto \varphi_s^{h_2}(\varphi_t^{h_1}(z))$ induces a diffeomorphism of $\mathbf{R}^2/\Gamma_z$ onto $\mathscr{O}$, where $\Gamma_z = \{(s,t) \in \mathbf{R}^2 \,\big|\, \varphi_s^{h_2}(\varphi_t^{h_1}(z)) = z\}$ is the isotropy group of z. Γ_z is an additive subgroup of $\mathbf{R}^2$, which does not depend on z. Therefore we will write $\Gamma_{\mathscr{O}}$ instead of Γ_z. Suppose that $\mathscr{O}$ is an $\mathbf{R}^2$-orbit of the action $\Xi_{(t,s)}|F^{-1}(0)$ in $F^{-1}(0)^\times$ and that $\Gamma_{\mathscr{O}} \cap (\mathbf{R} \times \{0\}) \neq \varnothing$. Then the flow $\varphi_t^{h_1}$ of X_{h_1} would be periodic with period $t_c > 0$. Because periodic integral curves of X_{h_1}, which lie in $\mathscr{O}$ and start near 0 leave a fixed neighborhood of 0, have an arbitarily large period, we deduce that a periodic solution of X_{h_1}, which starts near 0, must stay close to 0. Because X_{h_1} is hyperbolic at 0 it does not have any periodic solutions which remain close to 0 other than 0. This is a contradiction. So $\Gamma_{\mathscr{O}} \cap (\mathbf{R} \times \{0\}) = \varnothing$.

Combined with the fact that 0 is the only limit point in $\mathbf{R}^4 \setminus \mathscr{O}$ of $\mathscr{O}$ and that $\mathscr{O}$ is contained in the compact subset $F^{-1}(0)$ of $\mathbf{R}^4$, it follows that for every $z \in \mathscr{O}$ we have $\varphi_t^{h_1}(z) \to 0$ as $t \to \pm\infty$. In other words, $z \in W_{s,u}(0)$. Because $W_{s,u}(0)^\times$ are $\mathbf{R}^2$-orbits of the action $\Xi_{(t,s)}|F^{-1}(0)$ in $F^{-1}(0)^\times$, we find that for every $\mathbf{R}^2$-orbit $\mathscr{O}$ we have $\mathscr{O} = W_s(0)^\times = W_u(0)^\times$. Consequently, $F^{-1}(0)^\times = W_s(0)^\times = W_u(0)^\times$, which shows that $W_s(0) = W_u(0) = F^{-1}(0)$. □

We now prove

Claim In a suitable open neighborhood of 0 in $\big(\mathbf{R}^4, \omega = -\,\mathrm{d}\alpha = -\,\mathrm{d}(p_x\,\mathrm{d}x + p_y\,\mathrm{d}y)\big)$ there is a Hamiltonian vector field X_I, associated to the Hamiltonian function $I = h_2 + \mathscr{O}(2)$, which has a periodic flow χ_u. The Hamiltonian I Poisson commutes with the Hamiltonians h_1 and h_2.

The proof of the claim procedes with in several steps. First we construct the function I.

By the focus-focus Morse lemma, see the exercise 7, there is a near identity local diffeomorphism $\Phi : (\mathbf{R}^4, 0) \mapsto (\mathbf{R}^4, 0) : w \mapsto z = \Phi(w)$ with $\Phi = \mathrm{id} + \mathscr{O}(1)^2$ such that $\Phi^* q_i = h_i$ for $i = 1,2$ with $q_1 = xp_x + yp_y$ and $q_2 = xp_y - yp_x$. Let Y_2 be the vector field $\Phi^* X_{q_2}$. Then the flow of Y_2 is $\psi_s = \Phi^{-1} \circ \varphi_s^{q_2} \circ \Phi$. Hence the integral curve Γ_w of Y_2 starting at $w \in B \setminus \{0\}$ is periodic of period 2π. This follows because $\Gamma_w(s) = \psi_s(w) = \Phi^{-1}\big(\varphi_s^{q_2}(\Phi(w))\big) = \Phi^{-1}(\gamma_z(s))$, where γ_z is an integral curve of $X_{q_2} = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} - p_y\frac{\partial}{\partial p_x} + p_x\frac{\partial}{\partial p_y}$ starting at $z \neq 0$, which is periodic of period 2π. Since

$$L_{Y_2} h_i = L_{\Phi^* X_{q_2}} h_i = \Phi^*\big(L_{X_{q_2}} (\Phi^{-1})^* h_i\big) = \Phi^*(L_{X_{q_2}} q_i) = 0,$$

the flow ψ_s of Y_2 preserves the level sets of the integral map F (1).

Let

$$I : (\mathbf{R}^4, 0) \to \mathbf{R} : w \mapsto I(w) = \begin{cases} \frac{1}{2\pi}\int_{\Gamma_w} \alpha, & \text{when } w \neq 0 \\ 0, & \text{when } w = 0. \end{cases} \tag{3}$$

▷ Then $I = \Phi^* K$, where

$$K(z) = \begin{cases} \frac{1}{2\pi}\int_{\gamma_z} (\Phi^{-1})^*\alpha, & \text{when } z \neq 0 \\ 0, & \text{when } z = 0. \end{cases} \tag{4}$$

(2.2) **Proof**: The conclusion is obvious when $w = 0$. Therefore suppose that $w \neq 0$. We compute

$$\begin{aligned} I(w) &= \frac{1}{2\pi}\int_{\Gamma_w} \alpha = \frac{1}{2\pi}\int_0^{2\pi} \langle \alpha \,|\, \frac{\mathrm{d}\psi_t}{\mathrm{d}t} \rangle (\Gamma_w(t))\,\mathrm{d}t = \frac{1}{2\pi}\int_0^{2\pi} \langle \alpha(\psi_t(w)) \,\big|\, Y_2(\psi_t(w)) \rangle\,\mathrm{d}t \\ &= \frac{1}{2\pi}\int_0^{2\pi} \langle \alpha\big(\Phi^{-1}(\varphi_t^{q_2}(z))\big) \,\big|\, T\Phi X_{q_2}(\varphi_t^{q_2}(z)) \rangle\,\mathrm{d}t, \ \text{since } Y_2 = \Phi^* X_{q_2} \\ &= \frac{1}{2\pi}\int_0^{2\pi} \langle (\Phi^{-1})^*\alpha \,\big|\, X_{q_2} \rangle (\varphi_t^{q_2}(z)) \rangle\,\mathrm{d}t = \frac{1}{2\pi}\int_0^{2\pi} \langle (\Phi^{-1})^*\alpha \,\big|\, \frac{\mathrm{d}\varphi_t^{q_2}}{\mathrm{d}t} \rangle\rangle(z)\,\mathrm{d}t \end{aligned}$$

$$= \frac{1}{2\pi}\int_{\gamma_z}(\Phi^{-1})^*\alpha = K(z) = K(\Phi(w)). \qquad \square$$

$\triangleright$ Next we show that $I : (\mathbf{R}^4, 0) \to \mathbf{R}$ is smooth near 0.

(2.3) **Proof**: Let $z_1 = x + iy$ and $z_2 = p_x + ip_y$. Then $q_2 = \mathrm{Im}\,\bar{z}_1 z_2$ and the flow $\varphi_t^{q_2}$ of X_{q_2} is $\big(t, (z_1, z_2)\big) \mapsto (\mathrm{e}^{it}z_1, \mathrm{e}^{it}z_2)$. Let $D = \{\zeta \in \mathbf{C} \,\big|\, |\zeta| \le 1\}$ be the closed unit disk in $\mathbf{C}$ with boundary $\partial D = \{\zeta \in \mathbf{C} \,\big|\, |\zeta| = 1\} = S^1$ the unit circle in $\mathbf{C}$. Define the map $j : D \times \mathbf{C}^2 \to \mathbf{C}^2 : \big(\zeta, (z_1, z_2)\big) \mapsto (\zeta z_1, \zeta z_2)$ with $j_z : D \to \mathbf{C}^2 : \zeta \mapsto (\zeta z_1, \zeta z_2)$. Using Stokes' theorem we have $K(z) = \int_{\partial D} j_z^*(\Phi^{-1})^*\alpha = \int_D j_z^*(\Phi^{-1}(\omega))$. Since D is compact and ω, j_z are smooth, it follows that K is smooth near 0. Thus $I = \Phi^* K$ is smooth near 0. $\square$

The next few results show that the function I (3) is an action for the integrable system $\big(h_1, h_2, \mathbf{R}^4, \omega\big)$.

$\triangleright$ The function I Poisson commutes with h_i for $i = 1, 2$ on $F^{-1}(c)$, where c is a regular value of the integral map F (1). Note that $F^{-1}(c)$ is a smooth 2-torus T_c^2.

(2.4) **Proof**: We compute.

$$\{I, h_i\} = L_{X_{h_i}} I = \int_{\Gamma_w} L_{X_{h_i}}\alpha,$$ because we can move Γ_w by a homotopy in $F^{-1}(c)$ without changing the integral. The new integral does not depend on w. So we can take the Lie derivative under the integral sign

$$= \int_{\Gamma_w} X_{h_i} \mathbin{\lrcorner} \mathrm{d}\alpha + \mathrm{d}(X_{h_i} \mathbin{\lrcorner} \alpha)$$

$$= \int_{\Gamma_w} \mathrm{d}(-h_i + X_{h_i} \mathbin{\lrcorner} \alpha) = 0,$$ since Γ_w is a closed curve. $\square$

$\triangleright$ For all values of c close to but equal to 0 and for all $w \in T_c^2$, the tangent vectors $X_I(w)$ and $X_{h_1}(w)$ to T_c^2 at w are linearly independent.

(2.5) **Proof**: Since $\Phi = \mathrm{id} + \mathscr{O}(1)^2$, we get $\Phi^{-1} = \mathrm{id} + \mathscr{O}(1)^2$. So $(\Phi^{-1})^*\alpha = \alpha + \mathscr{O}(1)$, which gives

$$K(z) = \frac{1}{2\pi}\int_{\gamma_z}(\Phi^{-1})^*\alpha = \Big(\frac{1}{2\pi}\int_{\gamma_z}\alpha\Big) + \mathscr{O}(2) = q_2 + \mathscr{O}(2). \tag{5}$$

Therefore

$$I = \Phi^* K = \Phi^* q_2 + \mathscr{O}(2) = h_2 + \mathscr{O}(2). \tag{6}$$

Since the vector fields X_{h_1} and X_{h_2} are linearly independent near, but not at the origin, so are the vector fields X_I and X_{h_1}. In particular, the latter vector fields are linearly independent on the 2-torus T_c, when c is a regular value of F close to but not equal to 0. $\square$

$\triangleright$ For all c close to but not equal to 0, the flow χ_u of X_I on T_c^2 is periodic of period T_I.

(2.6) **Proof**: Since $\{I, h_i\} = 0$ on T_c^2, we get $[X_{h_1}, X_I] = -X_{\{h_1, I\}} = 0$ on T_c^2. Therefore the flows $\chi_u|T_c^2$ and $\varphi_t^{h_1}|T_c^2$ commute and are well defined for all $(u, t) \in \mathbf{R}^2$, because T_c^2 is compact. Thus we have an $\mathbf{R}^2$-action

$$\Lambda : \mathbf{R}^2 \times T_c^2 \to T_c^2 : \big((u, t), p\big) \mapsto \chi_u \circ \varphi_t^{h_1}(p),$$

which is transitive since $X_I(p)$ and $X_{h_1}(p)$ span $T_p(T^2_c)$ and T^2_c is connected. Thus the isotropy group $\Gamma_p = \{(T_I, T_{h_1}) \in \mathbf{R}^2 \,\big|\, \Lambda_{(T_I,T_{h_1})}(p) = p\}$ is a rank 2 lattice, which does not depend on the point p. Consequently, the flow χ_u of X_I on T^2_c is periodic of period $T_I = T_I(c)$, which is a smooth function of c. $\square$

(2.7) **Proof of claim**: We are now in position to prove the claim. From equation (6) it follows that $T_I = T_{h_2} + \mathscr{O}(1) = 2\pi + \mathscr{O}(1)$. Therefore on a bounded open neighborhood of 0 in $\mathbf{R}^4$ the period T_I function of X_I is bounded. Hence the limit of a periodic orbit of X_I on T^2_c as $c \to 0$ is periodic. This completes the proof of the claim. $\square$.

We return to describing the singular fiber $F^{-1}(0)$ of the integral map F (1). We prove

Claim: $F^{-1}(0)$ is homeomorphic to a pinched 2-torus, that is, a smooth 2-torus $T^2 = S^1 \times S^1$ with one of its generating circles pinched to a point. In other words, $F^{-1}(0)$ is homomorphic to the one point $\{0\}$ compactification a cylinder $S^1 \times \mathbf{R}$. The singular fiber at 0 has two transverse tangent planes.

(2.8) **Proof**: Since the action I Poisson commutes with the integrals h_i for $i = 1,2$, the flow χ_u of X_I leaves the fiber $F^{-1}(0) \cap V$ invariant. Here $V \subseteq B$ is an open neighborhood of 0, which is invariant under the $S^1 = \mathbf{R}/T_I\mathbf{Z}$-action generated by χ_u. Note that $W_s(0) \cap V$ is invariant under the flow χ_u.

We now extend the S^1-action $\chi_u|(W_s(0) \cap V)$ to all of $W_s(0)$. Let $p \in W_s(0)$. Then there is an open neighborhood V_p of 0 in $\mathbf{R}^4$ and a time $t_p > 0$ such that $\varphi^{h_1}_{t_p}(V_p) \subseteq V$. For every $\widetilde{p} \in V_p$ let $\widehat{\chi}_u(\widetilde{p}) = \varphi^{h_1}_{-t_p} \circ \chi_u \circ \varphi^{h_1}_{t_p}(\widetilde{p})$. Then $\widehat{\chi}_u$ defines an S^1-action on an open neighborhood $\mathscr{V} = \bigcup_{p \in F^{-1}(0)} V_p$ of $F^{-1}(0)$. To see that the mapping $\widehat{\chi}_u$ of $\mathscr{V}$ into itself is well defined, suppose that $q \in V_p \cap V_{p'}$, where $p' \in F^{-1}(0) \setminus \{p\}$. Then there is a $t'_{p'} > 0$ such that $\varphi^{h_1}_{t'_{p'}}(V_{p'}) \subseteq V$. Now $\varphi^{h_1}_{u'}(V_{p'} \cap V_p) \subseteq \varphi^{h_1}_{u}(V_{p'} \cap V_p)$, when $u' > u > 0$, since the flow $\varphi^{h_1}_t|W_s(0)$ is contracting when $t > 0$. So we may suppose that $t'_{p'} > t_p$. Thus

$$\varphi^{h_1}_{-t'_{p'}} \circ \chi_u \circ \varphi^{h_1}_{t'_{p'}}(q) = \varphi^{h_1}_{-t_p} \circ \big(\varphi^{h_1}_{t_p - t'_{p'}} \circ \chi_u \circ \varphi^{h_1}_{t'_{p'} - t_p}\big) \circ \varphi^{h_1}_{t_p}(q) = \varphi^{h_1}_{-t_p} \circ \chi_u \circ \varphi^{h_1}_{t_p}(q) = \widehat{\chi}_u(q),$$

where the second to last equality follows because the flows χ_u and $\varphi^{h_1}_t$ commute on V. To simplify our notation we drop the hat on the S^1-action $\widehat{\chi}_u$ on $\mathscr{V}$. Therefore we have an $\mathbf{R}^2$-action on $\mathscr{V}$ defined by

$$\widetilde{\Xi} : \mathbf{R}^2 \times \mathscr{V} \to \mathscr{V} : \big((u,t), z\big) \mapsto \chi_u \circ \varphi^{h_1}_t(z). \tag{7}$$

Note that $F^{-1}(0)$ is invariant under the action $\widetilde{\Xi}$ and that 0 is the only fixed point of the flow χ_u on $F^{-1}(0)$. So we have an $\mathbf{R}^2$-action $\widetilde{\Xi}_{(u,t)}|F^{-1}(0)^{\times}$. Let $\Gamma_{\mathscr{O}}$ be the isotropy group for the $\mathbf{R}^2$-orbit $\mathscr{O} = F^{-1}(0)^{\times}$. Because $F^{-1}(0)^{\times}$, which is diffeomorphic to $\mathbf{R}^2/\Gamma_{\mathscr{O}}$, is not compact, the rank of the lattice $\Gamma_{\mathscr{O}}$ can not be equal to 2. But $\Gamma_{\mathscr{O}} \neq \varnothing$. So $\Gamma_{\mathscr{O}}$ is isomorphic to $\mathbf{Z}$. Since the flow $\chi_u|(V \setminus \{0\})$ is periodic of period T_I, it follows that $\Gamma_{\mathscr{O}} = T_I\mathbf{Z}$. Consequently, $F^{-1}(0)^{\times}$ is diffeomorphic to the cylinder $\mathbf{R}^2/\Gamma_{\mathscr{O}} = (\mathbf{R}/T_I\mathbf{Z}) \times \mathbf{R} = S^1 \times \mathbf{R}$. Because $F^{-1}(0) = F^{-1}(0)^{\times} \cup \{0\}$ and $\{0\}$ is the only limit point of the closure of $F^{-1}(0)^{\times} = \mathscr{O}$ in $\mathbf{R}^4$, it follows that $F^{-1}(0)$ is homeomorphic to the one point compactification of the cylinder $S^1 \times \mathbf{R}$. In other words, $F^{-1}(0)$ is homeomorphic to a smooth 2-torus $S^1 \times S^1$ with a generating circle pinched to a point. Since the stable and unstable manifolds of the linear vector field $DX_{h_1}(0)$ at 0 are the coordinate 2-planes $\{0\} \times \mathbf{R}^2$ and $\mathbf{R}^2 \times \{0\}$, respectively, in $\mathbf{R}^4$, the limits of the tangent planes to $F^{-1}(0)^{\times}$ at 0 exist and are transverse. Thus $F^{-1}(0)$ is a smooth 2-sphere S^2, which is immersed in $\mathbf{R}^4$ with a normal crossing at 0. □

2.2 Nearby regular fibers

In this subsection we study the fibers of the integral map F (1) which are close to the singular fiber $F^{-1}(0)$. The main goal is to prove

Claim: There is an open neighborhood W of $F^{-1}(0)$ which is invariant under the flows $\varphi_t^{h_1}$ and χ_u and an open neighborhood U of 0 in $\mathbf{R}^2$ such that the mapping

$$F|(W \setminus F^{-1}(0)) : W \setminus F^{-1}(0) \to U \setminus \{0\} : p \mapsto \big(h_1(p), h_2(p)\big) \tag{8}$$

is a smooth locally trivial fibration. For $c \in U \setminus \{0\}$ the fiber $F^{-1}(c) \cap W$ is a smooth 2-torus that is an orbit of the $\mathbf{R}^2$-action $\widetilde{\Xi}$ (7).

(2.9) **Proof**: Let B be an open ball in $\mathbf{R}^4$ centered at 0, which is chosen small enough so that its boundary ∂B intersects $F^{-1}(0)$ in two circles $W^B_{s,u}(0) \cap \partial B$. We can also arrange that all the orbits of $\mathbf{R}^2$-action $\widetilde{\Xi}$ (7) are 2-dimensional and all of its orbits near 0 intersect ∂B in two circles which are close to the circles $W^B_{s,u}(0) \cap \partial B$. The set $F^{-1}(0) \setminus B$ is a compact smooth manifold with boundary $W_{s,u}(0) \cap \partial B$. Thus $F^{-1}(0) \setminus B$ is diffeomorphic to a compact cylinder $S^1 \times [0,1]$. Using a suitable 2-dimensional fibration transverse to $F^{-1}(0) \setminus B$ and the facts that the rank of $DF(z)$ is 2 for every $z \in F^{-1}(0) \setminus B$ and that the tangent vectors $X_{h_1}(z)$ and $X_I(z)$ are linearly independent, we find that there is an open neighborhood $\mathscr{N} \subseteq \mathscr{V}$ of $F^{-1}(0) \setminus B$ in $\mathbf{R}^4 \setminus B$, an open neighborhood U of 0 in $\mathbf{R}^2$, and a diffeomorphism $\nu : \mathscr{N} \to \big(F^{-1}(0) \setminus B\big) \times U$ such that

a) $F|\mathscr{N} = \pi_2 \circ \nu$, where $\pi_2 : \big(F^{-1}(0) \setminus B\big) \times U \to U$ is the projection map on the 2nd factor.

b) For every $z \in \mathscr{N}$ the vectors $X_{h_1}(z)$ and $X_I(z)$ are tangent to $(F|\mathscr{N})^{-1}(c)$, where $F(z) = c$ and $c \in U$, and are linearly independent.

c) The fibers of $F|\mathscr{N}$ intersect ∂B in two circles, which are close to the circles $(F^{-1}(0) \setminus B) \cap \partial B$.

Consider the *local* $\mathbf{R}^2$-action $\widetilde{\Xi}_{(u,t)}|\overline{B}$. Its orbits through $\mathscr{N} \cap B$ intersect ∂B in two circles, which coincide with the intersection of the fibers of $F|\mathscr{N}$ and ∂B, because F is constant on $\mathbf{R}^2$-orbits. Let W be the union of $\mathbf{R}^2$-orbits in $\overline{B}$, which intersect $\mathscr{N} \cap \partial B$, $\mathscr{N}$, and $\{0\}$. The union of $\mathbf{R}^2$-orbits through $\mathscr{N} \cap \partial B$ is an open subset of $\overline{B}$, which contains a small open neighborhood of 0 in $\mathbf{R}^4$. This follows because by hyperbolicity the integral curves of X_{h_1} at distance δ from 0 enter and leave B at points on ∂B, which are at a distance $\mathrm{O}(\delta)$ from $\partial B \cap W_{s,u}(0)$. Thus W is an open neighborhood of $F^{-1}(0)$ in $\mathbf{R}^4$ such that $F^{-1}(c) \cap W$ is equal to the union of $F^{-1}(c) \cap \mathscr{N}$ and the $\mathbf{R}^2$-orbit in $\overline{B}$ through $F^{-1}(c) \cap \partial B$ for every $c \in U \setminus \{0\}$. Hence the smooth mapping $F|W : W \to U \setminus \{0\}$ is proper and surjective with connected fibers. The invariance of F under the local $\mathbf{R}^2$-action

$\widetilde{\Xi}|\overline{B}$ together with the fact that for every $z \in \mathscr{N} \cap \partial B$ the rank of $DF(z)$ is 2, implies that for each point z on an $\mathbf{R}^2$-orbit, which intersects $\mathscr{N} \cap \partial B$, the rank of $DF(z)$ is 2. Therefore the map $F|(W \setminus F^{-1}(0))$ is a surjective submersion onto $U \setminus \{0\}$. Consequently, the fibration given by (8) is a locally trivial fibration with compact connected fibers.

We now show that each fiber of the fibration (8) is a smooth 2-torus. Let $c \in U \setminus \{0\}$. Since the flows $\varphi_t^{h_1}$ and χ_u^I leave the fibers $F^{-1}(c) \cap W$ invariant and their flows are complete, they define an $\mathbf{R}^2$-action

$$\widehat{\Xi} : \mathbf{R}^2 \times \big(W \setminus F^{-1}(0)\big) \to W \setminus F^{-1}(0) : \big((u,t),z\big) \to \chi_u \circ \varphi_t^{h_1}(z) \tag{9}$$

on $W \setminus F^{-1}(0)$. Because the vector fields X_{h_1} and Y, whose flow is χ_u, are linearly independent on $W \setminus (F^{-1}(0) \cap W)$, the $\mathbf{R}^2$-orbit $\mathscr{O}$ is an open subset of a fiber $F^{-1}(c) \cap W$ of the fibration (8). Since $F^{-1}(c) \cap W$ is connected, we get $\mathscr{O} = F^{-1}(c) \cap W$. Now $\mathscr{O}$ is diffeomorphic to $\mathbf{R}^2/\Gamma_{\mathscr{O}}$ and $\mathscr{O} = F^{-1}(c) \cap W$ is compact. Therefore $\Gamma_c = \Gamma_{\mathscr{O}}$ is a 2-dimensional lattice in $\mathbf{R}^2$. Consequently, $\mathscr{O} = F^{-1}(c) \cap W$ is a smooth 2-dimensional torus T_c^2. □

Let Σ_ξ be the image of a smooth local section $\sigma : U \setminus \{0\} \to W \setminus F^{-1}(0)$ of the fibration (8) at $\xi \in F^{-1}(0)^\times$, which is invariant under the flow of X_I. Because T_I is a primitive period of the integral curve $u \mapsto \chi_u(\xi)$ of X_I, it is a primitive period for every integral curve $u \mapsto \chi_u(\xi')$ of X_I, where $\xi' \in \Sigma_\xi \cap F^{-1}(U \setminus \{0\})$ and U is a sufficiently small open neighborhood of 0 in $\mathbf{R}^2$. Thus for every $c \in U \setminus \{0\}$, we have $(T_I, 0) \in \Gamma_c$.

Let $\xi_1 \in W_s^B(0)$ and $\xi_2 \in W_u^B(0)$. Let $\Sigma_{\xi_i} \subseteq B$ be the image of a smooth section of the smooth fibration (8) at ξ_i, which is invariant under the action χ_u. For $c \in U \setminus \{0\}$ it follows that $F^{-1}(c) \cap W \cap \Sigma_{\xi_i}$ is diffeomorphic to a circle $\mathscr{C}_i(c)$, which is an orbit of the action χ_u. For each $z(c) \in \mathscr{C}_2(c)$ there is a smallest positive time $t_1(c)$ such that the integral curve $t \mapsto \varphi^{h_1}_{t_1(c)}\big(z(c)\big)$ of X_{h_1} starting at $z(c)$ lies in $\mathscr{C}_1(c)$. In other words,
▷ $y(c) = \varphi^{h_1}_{t_1(c)}\big(z(c)\big) \in \mathscr{C}_1(c)$. As $z(c)$ traces out $\mathscr{C}_2(c)$ once, $y(c)$ traces out $\mathscr{C}_1(c)$ once.

(2.10) **Proof**: Since the circles $\mathscr{C}_i(c)$ bound a subset of $F^{-1}(c)$, which is diffeomorphic to a closed cylinder $S^1 \times [0,1] = \bigcup_{z(c) \in \mathscr{C}_2(c)} \{\varphi_t^{h_1}\big(z(c)\big) \in F^{-1}(c) \,\big|\, t \in [0, t_1(c)]\}$, the result follows by construction. □

2.3 Monodromy

In this subsection we show that the torus bundle of (8) has mondromy.

For $c \in U \setminus \{0\}$ let $z(c) \in \mathscr{C}_2(c) = F^{-1}(c) \cap W \cap \Sigma_{\xi_2}$. Then $\gamma_1(c) : S^1 \to F^{-1}(c) : u \mapsto \chi_u\big(z(c)\big)$ is a circle, which represents the first member $\delta_1(c)$ of an ordered basis of $\mathrm{H}_1 = \mathrm{H}_1\big(F^{-1}(c), \mathbf{Z}\big)$. The mapping $U \setminus \{0\} \to \bigcup_{c \in U \setminus \{0\}} \mathrm{H}_1(F^{-1}(c), \mathbf{Z}) : c \mapsto \delta_1(c)$ is a smooth function. When c encircles $\{0\}$ in $U \setminus \{0\}$ we return to the same element of H_1. Thus $\delta_1(c)$ is an eigenvector of the monodromy mapping $\mathscr{M} : \mathrm{H}_1 \to \mathrm{H}_1$ corresponding to the eigenvalue 1

Look at the $\mathbf{R}^2$-action $\widehat{\Xi}$ (9) on $W \setminus F^{-1}(0)$. Consider the map which assigns to each (u,t) in the isotropy group Γ_c the homology class of the curve $\gamma : [0,1] \to F^{-1}(c) : s \mapsto$

$\chi_{su} \circ \varphi^{h_1}_{st}(z)$, where $z \in F^{-1}(c) \cap W$. This mapping induces an isomorphism taking Γ_c to $\mathrm{H}_1(F^{-1}(c), \mathbf{Z})$. We know that $(T_I, 0) \in \Gamma_c$ corresponds to $\delta_1(c)$. To find the second member $\delta_2(c)$ of the ordered basis of $\mathrm{H}_1(F^{-1}(c), \mathbf{Z})$ observe that the projection map $(u,t) \mapsto t$ sends Γ_c onto a subset of Γ_c of the form $\{0\} \times t_c \mathbf{Z}$, where $t_c > 0$ is a unique real number. An element $(-u, t_c) \in \Gamma_c$ corresponds to the homology class $\delta_2(c)$. In terms of integral curves of $X_I|(F^{-1}(c) \cap B)$ and $X_{h_1}|F^{-1}(c)$ this means the following. Let $\mathscr{C}_1(c) \subset F^{-1}(c) \cap B$ be an S^1 orbit of the flow χ_u of $X_I|(F^{-1}(c) \cap B)$ through $\widetilde{z}(c)$. Since the vector field $X_{h_1}|F^{-1}(c)$ is transverse to $\mathscr{C}_1(c)$, there is a smallest positive time $t(c)$ such that $\widetilde{w}(c) = \varphi^{h_1}_{t(c)}\big(\widetilde{z}(c)\big) \in \mathscr{C}_1(c)$. Moreover, there is a $u(c) \in S^1 = \mathbf{R}/T_I\mathbf{Z}$ such that $\chi_{-u(c)}\big(\varphi^{h_1}_{t(c)}(\widetilde{z}(c))\big) = \widetilde{z}(c)$. So $\big(-u(c), t(c)\big) \in \Gamma_c$, which corresponds to $\delta_2(c)$.

We now investigate how $\delta_2(c)$ varies as a function of $c \in U \setminus \{0\}$. Let $t_2(c)$ be the first positive time such that $\varphi^{h_1}_{t_2(c)}(y(c)) = w(c) \in \mathscr{C}_2(c)$, where $y(c) = \varphi^{h_1}_{t_1(c)}\big(z(c)\big) \in \mathscr{C}_1(c)$ and $z(c) \in \mathscr{C}_2(c)$. Then $t(c) = t_1(c) + t_2(c)$. As $z(c)$ winds around the circle $\mathscr{C}_2(c)$ once, using ((2.10)) it follows that $y(c)$ winds around the circle $\mathscr{C}_1(c)$ once. Because $\varphi^{h_1}_{t_2(c)}$ is a continuous surjective mapping of $\mathscr{C}_1(c)$ to $\mathscr{C}_2(c)$, it follows that the winding number of the mapping

$$\mu : U \setminus \{0\} \to \mathscr{C}_2(c) : c \mapsto w(c) = \varphi^{h_1}_{t_2(c)}(y(c)) \tag{10}$$

is an integer k. Therefore after c encircles 0 once in U, the resulting homology class $\mathscr{M}\delta_2(c) + \mathrm{k}\,\delta_1(c)$ becomes $\delta_2(c)$ Consequently, the monodromy operator $\mathscr{M} : \mathrm{H}_1 \to \mathrm{H}_1$ sends $\delta_1(c)$ to $\delta_1(c)$ and $\delta_2(c)$ to $\delta_2(c) - \mathrm{k}\,\delta_1(c)$. In other words, with respect to the ordered basis $\{\delta_1(c), \delta_2(c)\}$ of $\mathrm{H}_1\big(F^{-1}(c), \mathbf{Z}\big)$ the matrix of $\mathscr{M}$ is $\begin{pmatrix} 1 & -\mathrm{k} \\ 0 & 1 \end{pmatrix}$. We have
▷ shown that the monodromy integer k is determined by the *local* behavior of the integral curves of the vector fields X_{h_1} and X_I in the open ball B centered at 0 in $\mathbf{R}^4$.

To compute the integer k we use the inverse of the local diffeomorphism Φ of $(\mathbf{R}^4, \omega = -\mathrm{d}\alpha)$ into itself, given by the focus-focus Morse lemma to reduce the calculation to the focus-focus case near the origin. Let $B' = \{z = (x, y, p_x, p_y) \in \mathbf{R}^4 \,\big|\, x^2 + y^2 + p_x^2 + p_y^2 < r^2\}$ be an open ball in $\mathbf{R}^4$ centered at the origin such that $\widetilde{V} = \Phi^{-1}(B')$ is an open subset of $\mathbf{R}^4$ containing 0 such that $B \subseteq \widetilde{V}$. Let $\alpha' = (\Phi^{-1})^*\alpha$ and set $\omega' = -\mathrm{d}\alpha' = (\Phi^{-1})^*\omega$. Since $\Phi = \mathrm{id} + \mathscr{O}(1)^2$, it follows that ω' is nondegenerate and hence is a symplectic form on B'. Thus the map $\Phi : (\widetilde{V}, \omega|\widetilde{V}) \to (B', \omega'|B')$ is a symplectic diffeomorphism, that is, $\Phi^*(\omega'|B') = \omega|\widetilde{V}$. Pull back the Hamiltonian vector fields X_{h_1} and X_I on $(\widetilde{V}, \omega|\widetilde{V})$ by Φ^{-1} to Hamiltonian vector fields

$$(\Phi^{-1})^* X_{h_1} = X'_{(\Phi^{-1})^* h_1} = X'_{q_1} \text{ and } (\Phi^{-1})^* X_I = X'_{(\Phi^{-1})^* I} = X'_K,$$

see (4). The local flows $(\varphi')^{q_1}_t$ of X'_{q_1} and $(\chi')^K_u$ of X'_K in B' preserve the level sets $Q^{-1}(c) \cap B'$, where $Q(z) = \big(q_1(z), q_2(z)\big)$, since for $i = 1, 2$ we have

$$L_{X'_{q_1}} q_i = L_{(\Phi^{-1})^* X_{h_1}} (\Phi^{-1})^* h_i = (\Phi^{-1})^* (L_{X_{h_1}} h_i) = 0, \quad \text{because } \{h_1, h_i\} = 0$$

and

$$L_{X'_K} q_i = L_{(\Phi^{-1})^* X_I} (\Phi^{-1})^* h_i = (\Phi^{-1})^* (L_{X_I} h_i) = 0, \quad \text{because } \{I, h_i\} = 0.$$

Note that $Q^{-1}(c)\cap B' = \Phi^{-1}\big(F^{-1}(c)\cap \widetilde{V}\big)$. Moreover, the local flow $(\chi')^K_u$ is periodic of period T_I. Let $\widetilde{\delta}_1(c) = (\Phi^{-1})^*\delta_1(c)$ and observe that the homology class $\mathrm{k}\,\widetilde{\delta}_1(c)$ in $\mathrm{H}_1(Q^{-1}(c),\mathbf{Z})$ is represented by the closed curve

$$\mu' : U\setminus\{0\} \to \Phi^{-1}\big(\mathscr{C}_2(c)\big) : c \mapsto \Phi^{-1}\circ\mu(c), \tag{11}$$

which has winding number k.

Let $\delta\in[0,1]$. Consider the homotopy $\alpha^\delta = \alpha + (1-\delta)\big(\alpha'-\alpha\big)$ of 1-forms on B', where $\alpha' = (\Phi^{-1})^*\alpha$. Then $\alpha^0 = \alpha'$ and $\alpha^1 = \alpha$. Consequently, we obtain the homotopy $\omega^\delta = \omega + (1-\delta)\big((\Phi^{-1})^*\omega - \omega\big)$ of 2-forms such that $\omega^0 = (\Phi^{-1})^*\omega = \omega'$ and $\omega^1 = \omega$. Now $\Phi = \mathrm{id} + \mathscr{O}(1)^2$ implies $\Phi^{-1}(\alpha) = \alpha + \mathscr{O}(1)$. So $\alpha^\delta = \alpha + \mathscr{O}(1)$ and thus $\omega^\delta = \omega + \mathscr{O}(1)$. Thus ω^δ is nondegenerate on B'. For $z\in B'$ let

$$K^\delta(z) = \begin{cases} \frac{1}{2\pi}\int_{\gamma_z^\delta}\alpha^\delta, & \text{if } z\neq 0\\ 0, & \text{if } z = 0.\end{cases}$$

Then $K^0 = K$ (4) and $K^1 = \frac{1}{2\pi}\int_{\gamma_z}\alpha = q_2$. Moreover $K^\delta = \frac{1}{2\pi}\int_{\gamma_z}\big(\alpha+\mathscr{O}(1)\big) = q_2 + \mathscr{O}(2)$ on B'. Let X_f^δ be the Hamiltonian vector field on $(\mathbf{R}^4,\omega^\delta)$ associated to the smooth function f on $\mathbf{R}^4$. For $i=1,2$ we have

$$L_{X^\delta_{q_i}}K^\delta = \tfrac{1}{2\pi}\int_{\gamma_z} L_{X^\delta_{q_i}}\alpha^\delta = \tfrac{1}{2\pi}\int_{\gamma_z}\Big[X^\delta_{q_i}\,\lrcorner\,\mathrm{d}\alpha^\delta + \mathrm{d}(X^\delta_{q_i}\,\lrcorner\,\alpha^\delta)\Big] = \tfrac{1}{2\pi}\int_{\gamma_z}\mathrm{d}\big(-q_i + X^\delta_{q_i}\,\lrcorner\,\alpha^\delta\big) = 0,$$

since γ_z is a closed curve in B'. Thus $\{K^\delta, q_i\}_\delta = 0$. This implies that the integral curves of the vector fields $X^\delta_{K^\delta}$ and $X^\delta_{q_1}$, which start on $Q^{-1}(c)\cap B'$, lie on $Q^{-1}(c)\cap B'$ for all time since $Q^{-1}(c)\cap B'$ is compact, being diffeomorphic to the smooth 2-torus $\Phi^{-1}(F^{-1}(c))$ in $\Phi^{-1}(B')$. From $X^\delta_{q_i} = X_{q_i} + \mathscr{O}(1)$ in B' it follows that $X^\delta_{q_1}(p)$ and $X^\delta_{K^\delta}(p)$ are linearly independent at every $p\in Q^{-1}(c)\cap B'$. Thus the flows $(\varphi^\delta)^{q_1}_t$ and $(\chi^\delta)^{K^\delta}_u$ of the vector fields $X^\delta_{q_1}$ and $X^\delta_{K^\delta}$ on $Q^{-1}(c)\cap B'$, respectively, define an $\mathbf{R}^2$-action

$$\mathbf{R}^2\times Q^{-1}(c)\cap B' \to Q^{-1}(c)\cap B' : \big((t,u),p\big)\mapsto (\varphi^\delta)_t\circ(\chi^\delta)^{K^\delta}_u(p),$$

whose isotropy group does not depend on p and is a rank 2 lattice, because $Q^{-1}(c)\cap B'$ is compact. Thus the flow $(\chi^\delta)^{K^\delta}_u$ of the vector field $X^\delta_{K^\delta}$ on $Q^{-1}(c)\cap B'$ is periodic for all $c\in U\setminus\{0\}$ of period $T^\delta_{K^\delta} = T^\delta_{K^\delta}(c)$, which is a smooth function of c. Note that $T^0_{K^0}(c) = T_K(c) = T_I(c)$.

Observe that the circle $C_2(c) = (\Phi^{-1})^*\mathscr{C}_2(c)$ in $Q^{-1}(c)\cap B'$, which is a periodic orbit of $X^0_{K^0} = X'_K$ on $(B',\omega^0|B' = \omega'|B')$ of period $T_K = T_I$, deforms during the homotopy to a circle $C_2^\delta(c)$, which is a periodic orbit of $X^\delta_{K^\delta}$ on $Q^{-1}(c)\cap B'$. Since the local flow $(\varphi^\delta)^{q_1}_t$ of $X^\delta_{q_1}$ on $(B',\omega^\delta|B')$ is defined, we can follow the definition of the mapping μ' (11) to define the deformed mapping $\mu^\delta : U\setminus\{0\}\to C_2^\delta(c)$. In more detail, let $y^\delta(c)\in C_1(c) = \Phi^{-1}(\mathscr{C}_1(c))$ such that $y^\delta(c) = \Phi^{-1}(y(c))$, where $y(c)\in\mathscr{C}_1(c)$. Define $t_2^\delta(c)$ to be the first positive time so that $(\varphi^\delta)^{q_1}_{t_2^\delta(c)}(y^\delta(c))\in C_2(c)$. Then

$$\mu^\delta : U\setminus\{0\}\to C_2(c) : c\mapsto (\varphi^\delta)^{q_1}_{t_2^\delta(c)}(y^\delta(c)) = \Phi^{-1}\big((\varphi^\delta)^{h_1}_{t_2^\delta(c)}(y(c))\big).$$

Note that $\mu^0 = \mu'$ and that the winding number of μ^δ is k for every $\delta \in [0,1]$, since the mapping μ^δ is obtained from the mapping μ' by a homotopy. At $\delta = 1$ we are looking at the focus-focus model $(q_1, q_2, \mathbf{R}^4, \omega)$ near the origin. In the following paragraph we calculate the winding number of μ^1.

In complex coordinates $z = (z_1, z_2)$ on $\mathbf{C}^2 = \mathbf{R}^4$, where $z_1 = x + iy$ and $z_2 = p_x + ip_y$ we have $q_1(z) = \mathrm{Re}\,\bar{z}_1 z_2$ and $q_2(z) = \mathrm{Im}\,\bar{z}_1 z_2$. Let $Q = q_1 + iq_2 = \bar{z}_1 z_2$ and $c = c_1 + ic_2$. The flow of X_{q_1} is $\varphi_t^{q_1}(z) = (\mathrm{e}^t z_1, \mathrm{e}^{-t} z_2)$ and the flow of X_{q_2} is $\varphi_s^{q_2}(z) = (\mathrm{e}^{is} z_1, \mathrm{e}^{is} z_2)$. Choose a fixed $0 < \varepsilon < r$, where r is the radius of the ball $B' = \{(z_1, z_2) \in \mathbf{C}^2 \,\big|\, |z_1|^2 + |z_2|^2 < r^2\}$ centered at $0 \in \mathbf{R}^4$. Note that the local stable manifold $W_s^{B'}(0)$ at 0 of X_{q_1} is $\{0\} \times \mathbf{C}$; while local unstable manifold $W_u^{B'}(0)$ at 0 of X_{q_1} is $\mathbf{C} \times \{0\}$. Let

$$C_1(c) = \{(z_1, z_2) \in \mathbf{C}^2 \,\big|\, \bar{z}_1 z_2 = c \;\&\; |z_1| = \varepsilon\} = \{(z_1, \varepsilon^{-2} c z_1) \in \mathbf{C}^2 \,\big|\, |z_1| = \varepsilon\}.$$

Then $C_1(c)$ is a circle in $Q^{-1}(c) \cap (\{|z_1| = \varepsilon\} \times \mathbf{C}) \cap B'$, being the image of the integral curve of X_{q_2} which starts at $y_0(c) = (\varepsilon, \varepsilon^{-1} c)$. Similarly, let

$$C_2(c) = \{(z_1, z_2) \in \mathbf{C}^2 \,\big|\, \bar{z}_1 z_2 = c \;\&\; |z_2| = \varepsilon\} = \{(\varepsilon^{-2} \bar{c} z_2, z_2) \in \mathbf{C}^2 \,\big|\, |z_2| = \varepsilon\}.$$

Then $C_2(c)$ is a circle in $Q^{-1}(c) \cap (\{\mathbf{C} \times |z_2| = \varepsilon\}) \cap B'$, being the image of the integral curve of X_{q_2} which starts at $(\varepsilon^{-1}\bar{c}, \varepsilon)$. Let $y(c) = \big(y_1(c), y_2(c)\big) \in C_1(c)$. Consider the integral curve $\gamma : \mathbf{R} \to Q^{-1}(c) \cap B' : t \mapsto \varphi_t^{q_1}\big(y(c)\big) = \big(\mathrm{e}^t y_1(c), \mathrm{e}^{-t} y_2(c)\big)$ of X_{q_1}. Now $\varphi_t^{q_1}\big(y(c)\big) \in C_2(c)$ if and only it $|\mathrm{e}^{-t} y_2(c)| = \varepsilon$. So the first time that the integral curve γ intersects $C_2(c)$ is $t_2(c) = \ln \frac{1}{\varepsilon} |y_2(c)|$. In particular, when $y(c) = y_0(c) = (\varepsilon, \varepsilon^{-1} c) \in C_1(c)$, we obtain $t_2^0(c) = \ln \varepsilon^{-2} |c|$. So $w_0(c) = \varphi^{q_1}_{t_2^0(c)}(y_0(c)) = \big(\mathrm{e}^{t_2^0(c)} \varepsilon, \mathrm{e}^{-t_2^0(c)} \varepsilon^{-1} c\big) = (\varepsilon^{-1}|c|, \varepsilon \frac{c}{|c|})$. Since $C_1(c)$ is an orbit of X_{q_2}, any point $y(c) \in C_1(c)$ may be written as $\mathrm{e}^{is_c} y_0(c)$ for some $s_c \in [0, 2\pi)$. Therefore we find that

$$\begin{aligned} \mu^1 : U \setminus \{0\} \to C_2(c) : c \mapsto w(c) &= \varphi^{q_1}_{t_2(c)}(y(c)) = \varphi^{q_1}_{t_2(c)}(\mathrm{e}^{is_c} y_0(c)) \\ &= \big(\mathrm{e}^{is_c} \mathrm{e}^{t_2(c)} \varepsilon, \mathrm{e}^{is_c} \mathrm{e}^{-t_2(c)} \varepsilon^{-1} c\big) = (\mathrm{e}^{is_c} \varepsilon^{-1} |c|, \mathrm{e}^{is_c} \varepsilon \tfrac{c}{|c|}), \end{aligned}$$

since $t_2(c) = \ln |\varepsilon^{-1} \mathrm{e}^{is_c} \varepsilon^{-1} c| = t_2^0(c)$. Because $s_c \in [0, 2\pi)$, the mapping μ^1 has winding number 1, that is, the integer k is 1.

This completes the proof of the geometric monodromy theorem. □

3 The hyperbolic circular billiard

In this section we discuss the simplest Hamiltonian system with monodromy, namely, the hyperbolic circular billiard.

3.1 The basic model

The hyperbolic circular billiard is the Hamiltonian system $\big(\mathscr{P}, \omega, H\big)$ with phase space $\mathscr{P} = \mathscr{D} \times \mathbf{R}^2 = \{(x, y) \in \mathbf{R}^2 \,\big|\, x^2 + y^2 < r_{\max}^2\} \times \mathbf{R}^2$, having coordinates $z = (x, y, p_x, p_y)$, symplectic form $\omega = \mathrm{d}x \wedge \mathrm{d}p_x + \mathrm{d}y \wedge \mathrm{d}p_y$, and Hamiltonian

$$H : \mathscr{P} \to \mathbf{R} : z \mapsto \tfrac{1}{2}(p_x^2 + p_y^2) - \tfrac{1}{2}V(x,y).$$

Here $V(x,y) = \begin{cases} x^2+y^2, & \text{when } (x,y) \in \mathscr{D} \\ \infty, & \text{when } (x,y) \in \partial\overline{\mathscr{D}}. \end{cases}$ The Hamiltonain vector field X_H governs the motion of a particle of mass 1 in a *repulsive* quadratic potential of height $\frac{1}{2}r_{\max}^2$ with a *wall* at the circle $\partial\overline{\mathscr{D}} = \{(x,y) \in \mathbf{R}^2 \,\big|\, x^2+y^2 = r_{\max}^2\}$.

To fix the motion of the particle, we must specify what happens when the particle reaches the point $P = (x,y)$ on the wall $\partial\overline{\mathscr{D}}$. We assume that at P the momentum $p = (p_x, p_y)$ of the particle undergoes a reflection in the line ℓ_P tangent to $\partial\overline{\mathscr{D}}$ at P. This reflection is uniquely determined by requiring that the radial component p_r of the momentum p at P becomes $-p_r$, the radial component of the momentum $p' = (p'_x, p'_y)$ after reflection. Let

$$\varphi_q = \tan^{-1}\frac{y}{x}, \quad \varphi_p = \tan^{-1}\frac{p_y}{p_x}, \quad \text{and} \quad \varphi'_{p'} = \tan^{-1}\frac{p'_y}{p'_x}.$$

Moreover, let α be the positive angle between the vector p and the line ℓ_P and let β be

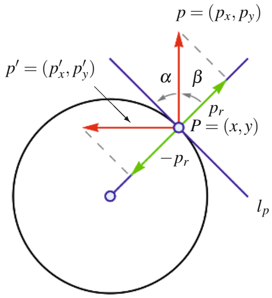


Figure 3.1.1. The geometric situation.

the positive angle between the vectors (x,y) and p, whose tails have been moved to P. The following relations are geometrically obvious, see figure 3.1.1.

$$\begin{pmatrix} p'_x \\ p'_y \end{pmatrix} = \begin{pmatrix} \cos 2\alpha & -\sin 2\alpha \\ \sin 2\alpha & \cos 2\alpha \end{pmatrix} \begin{pmatrix} p_x \\ p_y \end{pmatrix}, \quad \varphi_p = \varphi_q + \beta, \;\; \varphi_{p'} = \varphi_p + 2\alpha, \;\; 2(\alpha+\beta) = \pi.$$

At the point P we have the mapping

$$\begin{aligned} \Phi : \partial\overline{\mathscr{D}} \times \mathbf{R}^2 \to \partial\overline{\mathscr{D}} \times \mathbf{R}^2 : (x,y,p_x,p_y) \mapsto (x,y,p'_x,p'_y) &= \\ = (x,y,p_x\cos 2\alpha - p_y \sin 2\alpha, p_x \sin 2\alpha + p_y \cos 2\alpha)&, \end{aligned} \tag{12}$$

which corresponds to the reflection at P in the wall $\partial\overline{\mathscr{D}}$ in configuration space.

The phase space $\mathscr{P}$ is invariant under the S^1-symmetry given by

$$\Psi : S^1 \times \mathscr{P} \to \mathscr{P} : \big(t,(x,y,p_x,p_y)\big) \mapsto \left(R_t\begin{pmatrix} x \\ y \end{pmatrix}, R_t\begin{pmatrix} p_x \\ p_y \end{pmatrix}\right),$$

where $R_t = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$. This S^1-action is Hamiltonian with momentum mapping

$$J : \mathscr{P} \subseteq \mathbf{R}^4 \rightarrow \mathbf{R} : (x, y, p_x, p_y) \mapsto x p_y - y p_x. \tag{13}$$

Since the Hamiltonian H is invariant under the S^1-action Ψ, the angular momentum J is a conserved quantity.

Fact. The angle of reflection α is a smooth function of $r_{\max}$, the energy e of the particle, and the magnitude j of its angular momentum. Explicitly,

$$\tan \alpha = \tfrac{1}{j}\sqrt{r_{\max}^2(2e + r_{\max}^2) - j^2}. \tag{14}$$

(3.11) **Proof**: The identity $(xp_x + yp_y)^2 + (xp_y - yp_x)^2 = (x^2 + y^2)(p_x^2 + p_y^2)$ for the given motion in $\mathscr{P}$ of the particle at the point P reads $(xp_x + yp_y)^2 + j^2 = r_{\max}^2(p_x^2 + p_y^2)$. Since the energy of the particle at P is $e = \frac{1}{2}(p_x^2 + p_y^2) - \frac{1}{2}r_{\max}^2$, we get $xp_x + yp_y = \sqrt{r_{\max}^2(2e + r_{\max}^2) - j^2}$. Now $\alpha = \pi/2 - (\varphi_p - \varphi_q)$. So $\tan\alpha = \cot(\varphi_p - \varphi_q)$. But

$$\tan(\varphi_p - \varphi_q) = \tan\left(\tan^{-1}\frac{p_y}{p_x} - \tan^{-1}\frac{y}{x}\right) = \frac{xp_y - yp_x}{xp_x + yp_y} = \frac{j}{xp_x + yp_y},$$

from which (14) follows. □

We extend the motion of the particle on $\mathscr{P} = \mathscr{D} \times \mathbf{R}^2$ to $\overline{\mathscr{P}} = \overline{\mathscr{D}} \times \mathbf{R}^2$ by noting that the map Φ (12) preserves energy and angular momentum. The extended motion is only *continuous* at the point of reflection.

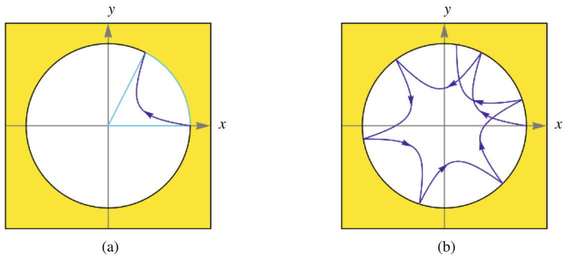


Figure 3.1.2. The motion of the circular billiard in configuration space.

3.2 Reduction of the S^1 symmetry

We extend the S^1-symmetry Ψ on $\mathscr{P}$ real analytically to $\overline{\mathscr{P}} \subseteq \mathbf{R}^4$. To construct the reduced space we use invariant theory. The algebra of S^1-invariant polynomials on $\overline{\mathscr{P}}$ is generated by the polynomials

$$R = \tfrac{1}{2}(x^2 + y^2),\ T = \tfrac{1}{2}(p_x^2 + p_y^2),\ J = xp_y - yp_x,\ \text{and } K = \tfrac{1}{2}(xp_y + yp_x)$$

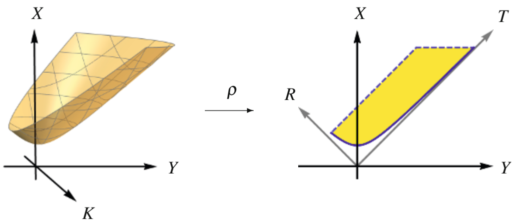
restricted to $\overline{\mathscr{P}}$. These polynomials satisfy the relation

$$TR = K^2 + \tfrac{1}{4}J^2,\ 0 \le R \le R_{\max} = \tfrac{1}{2}r_{\max}^2\ \&\ T \ge 0. \tag{15}$$

The semialgebraic variety in $\mathbf{R}^4$ with coordinates (T,R,J,K) defined by (15) is the orbit space $\overline{P} = \overline{\mathscr{P}}/S^1$. The reduced space $\overline{P}_j = (J^{-1}(j) \cap \overline{\mathscr{P}})/S^1 \subseteq \mathbf{R}^3$ is defined by

$$C(T,R,K) = TR - K^2 - \tfrac{1}{4}j^2 = 0,\ 0 \le R \le R_{\max}\ \&\ T \ge 0, \tag{16}$$

see figure 3.2.1(a). The image of $\overline{P}_j$ under the projection mapping $\rho : \mathbf{R}^3 \to \mathbf{R}^2 : (K,Y,X) \mapsto (X,Y)$ is given in figure 3.2.1(b).



(a) (b)

Figure 3.2.1. The reduced space $\overline{P}_j$ is shown in (a) and its image under the mapping ρ is shown in (b). In both cases $X = \frac{1}{2}(T+R)$ and $Y = \frac{1}{2}(T-R)$.

The S^1-reduced Hamiltonian is

$$H_j : \overline{P}_j \subseteq \mathbf{R}^3 \to \mathbf{R} : (T,R,K) \mapsto T - R. \tag{17}$$

3.3 Partial reconstruction

After partial reconstruction, corresponding to the motion of the particle up to the first reflection, we obtain table 3.3.1, which gives the topology of the e-level set $H_j^{-1}(e)$ of the

Conditions	Topology of $H_j^{-1}(e)$
1. $j \ne 0,\ \ e = \frac{1}{4}R_{\max}^{-1}j^2 - R_{\max}$	point $= (\frac{1}{4}R_{\max}^{-1}j^2, R_{\max}, 0)$
2. $j \ne 0,\ \ e > \frac{1}{4}R_{\max}^{-1}j^2 - R_{\max}$	a closed interval with end points in Γ_j
3. $j = 0,\ \ e = -R_{\max}$	point $= (0, R_{\max}, 0)$
4. $j = 0,\ \ -R_{\max} < e < 0$ or $e > 0$	a closed interval with end points in Γ_0
4. $j = 0,\ \ e = 0$	a closed interval with an angle at $(0,0,0)$ whose end points lie in Γ_0

Table 3.3.1. The topology of $H_j^{-1}(e)$.

reduced Hamiltonian H_j. Let Γ_j be the parabola

$$\overline{P}_j \cap \{R = R_{\max}\} = \{(R_{\max}^{-1}K^2 + \tfrac{1}{4}R_{\max}^{-1}j^2, R_{\max}, K) \in \overline{P} \,\big|\, K \in \mathbf{R}\}.$$

When the level set $H_j^{-1}(e)$ is a point, this point is a critical point $\overline{p}$ of the reduced Hamiltonian H_j on the reduced space $\overline{P}_j$ and e is a critical value of H_j. It reconstructs to a relative equilibrium of the hyperbolic circular billiard. When $\overline{P}_j$ is smooth at $\overline{p}$, the relative equilibrium is a circle; otherwise, it is a point. The locus of critical values of H_j in the energy momentum plane is the union of the parabola $e = \frac{1}{4}R_{\max}^{-1} j^2 - R_{\max}$ and the point $(0,0)$.

3.4 Full reconstruction

The discussion in §3.3 describes the topology of the fibers of the energy momentum mapping $\mathscr{E}\mathscr{M}$ of the hyperbolic circular billiard up to an including the first contact of the particle with the wall. To describe what happens after reflection at the wall, which is given by the mapping Φ (12), we need to know what happens to the functions H, J, and K.

Fact. We have $\Phi^* T = T$, $\Phi^* R = R$, $\Phi^* J = J$, and $\Phi^* K = -K$.

(3.12) **Proof**: Clearly $\Phi^* R = R$. We compute

$$\begin{aligned}(\Phi^* H)(z) &= H(z') = \tfrac{1}{2}\big((p_x')^2 + (p_y')^2 - \tfrac{1}{2}(x^2+y^2)\big)\\ &= \tfrac{1}{2}(p_x^2+p_y^2) - \tfrac{1}{2}(x^2+y^2), \text{ because } p' = R_{2\alpha}p, \text{ where } R_{2\alpha} \in \mathrm{SO}(2)\\ &= H(z).\end{aligned}$$

But $H = T - R$. So $\Phi^* T = T$. Next because the vector p' is obtained by a planar rotation of the vector p, the vectors $J(z') = (p_x', p_y') \times (x,y)$ and $J(z) = (p_x, p_y) \times (x,y)$ have the same direction. We now show that $J(z')$ and $J(z)$ have the same magnitude. By definition

$$|J(z')| = |xp_y' - yp_x'| = |(xp_y - yp_x)\cos 2\alpha - (xp_x + yp_y)\sin 2\alpha|.$$

But $\tan\alpha = (xp_y - yp_x)(xp_x + yp_y)^{-1}$. So $xp_y - yp_x = D\sin\alpha$ and $xp_x + yp_y = D\cos\alpha$, where $D = \sqrt{(xp_y - yp_x)^2 + (xp_x + yp_y)^2} = \sqrt{x^2+y^2}\,\sqrt{p_x^2+p_y^2}$, which gives

$$|J(z')| = D|\sin\alpha\cos 2\alpha - \cos\alpha\sin 2\alpha| = D|\sin\alpha| = |xp_y - yp_x| = |J(z)|.$$

Therefore $\Phi^* J = J$. Lastly,

$$\begin{aligned}-K(z') &= -\langle(-x,-y),(p_x',p_y')\rangle = \sqrt{x^2+y^2}\,\sqrt{(p_x')^2+(p_y')^2}\cos\beta\\ &= \sqrt{x^2+y^2}\,\sqrt{p_x^2+p_y^2}\cos\beta = \langle(x,y),(p_x,p_y)\rangle = K(z).\end{aligned} \qquad \square$$

So the mapping Φ (12) induces the mapping

$$\overline{\Phi} : \overline{P}_j \subseteq \mathbf{R}^3 \to \overline{P}_j \subseteq \mathbf{R}^3 : (R,T,K) \mapsto (R,T,-K). \qquad (18)$$

When (e,j) is a regular value of the reduced Hamiltonian H_j, the map $\overline{\Phi}$ interchanges the ends of $H_j^{-1}(e)$. So we may use $\overline{\Phi}$ to identify the ends of $H_j^{-1}(e)$ to obtain a topological circle. Consequently, after reconstruction, the (e,j)-level set of the energy momentum mapping of the *full motion* of the particle is a topological 2-torus. Similarly, we may identify the ends of $H_0^{-1}(0)$ to obtain a topological circle passing through the origin.

Hence after reconstruction we find that the $(0,0)$-level set of the energy momentum map of the full motion is topologically a once pinched 2-torus, that is, a 2-torus with the circle pinched to a point, which is mapped to the origin under the S^1-reduction mapping. When (e,j) lies on the parabola $e = \frac{1}{4}R_{\max}^{-1}j^2 - R_{\max}$, then e is a critical value of the reduced Hamiltonian H_j (17) and $H_j^{-1}(e)$ is a relative equilibrium, see §3.3. This verifies the entries in table 3.4.1.

Conditions	Topology of (e,j)-level set
1. $j \neq 0$, $e = \frac{1}{4}R_{\max}^{-1}j^2 - R_{\max}$	S^1
2. $j \neq 0$, $e > \frac{1}{4}R_{\max}^{-1}j^2 - R_{\max}$	T^2
3. $j = 0$, $e = -R_{\max}$	point $= (0,0,0,0)$
4. $j = 0$, $-R_{\max} < e < 0$ or $e > 0$	T^2
4. $j = 0$, $e = 0$	T^*, a once pinched 2-torus

Table 3.4.1. The topology of (e,j)-level set of full energy momentum mapping.

Thus the energy momentum mapping of the hyperbolic circular billiard satisfies the hypotheses of the monodromy theorem. So it has *nontrivial* monodromy around a small loop in the set of regular values, which encircles the origin.

3.5 The first return time and rotation angle

In this subsection we compute the first return time and rotation number of a full motion of the hyperbolic billiard on the smooth 2-torus $T^2_{e,j} = \mathscr{EM}^{-1}(e,j)$.

In order to compute the time T_{per} it takes an integral curve $\gamma: [-T_{\mathrm{per}}/2, T_{\mathrm{per}}/2] \to H_j^{-1}(e):$ $t \mapsto \gamma(t)$ of the reduced equations of motion to go from one end of $H_j^{-1}(e)$ to the other, we first find an explicit expression for γ. We begin by determining the reduced equations of motion on the reduced space P_j. Since

$$\{T,R\} = -2K = \frac{\partial C}{\partial K}, \ \{R,K\} = R = \frac{\partial C}{\partial T}, \ \text{and } \{K,T\} = T = \frac{\partial C}{\partial R},$$

where C is given by (16), it follows that the vector field $-\mathrm{ad}_{H_j}$ on $\mathbf{R}^3$ is

$$\begin{aligned} \dot{T} &= \{T,H_j\} = \{T, T-R\} = 2K \\ \dot{R} &= \{R,H_j\} = 2K \\ \dot{K} &= \{K,H_j\} = T+R. \end{aligned} \tag{19}$$

The reduced phase space $\overline{P}_j$ is an invariant subset of the vector field $-\mathrm{ad}_{H_j}$. So the reduced vector field X_{H_j} on $\overline{P}_j$ has integral curves which satisfy (19). Equations (19) are equivalent to

$$\ddot{K} = \dot{T} + \dot{R} = 4K \tag{20a}$$
$$\dot{R} = 2K \tag{20b}$$
$$\dot{H}_j = \dot{T} - \dot{R} = 0. \tag{20c}$$

Equation (20c) integrates to

$$T = R + e. \tag{21}$$

Fact. Let

$$T_{\mathrm{per}} = \cosh^{-1}\left[\frac{2R_{\max}+e}{\sqrt{e^2+j^2}}\right]. \tag{22a}$$

Then the integral curve of the reduced equations of motion (19), which starts at the end $(R_{\max}+e, \frac{1}{2}\sqrt{4R_{\max}^2+4eR_{\max}-j^2}-\frac{e}{2}, \frac{1}{2}\sqrt{4R_{\max}^2+4eR_{\max}-j^2})$ of $H_j^{-1}(e)$ at time $-T_{\mathrm{per}}/2$, is given by

$$\begin{aligned}\gamma : [-T_{\mathrm{per}}/2, T_{\mathrm{per}}/2] &\to H_j^{-1}(e) \subseteq \overline{P}_j : t \mapsto \gamma(t) = \big(T(t), R(t), K(t)\big)\\ &= \big(\tfrac{1}{2}\sqrt{e^2+j^2}\cosh 2t + \tfrac{e}{2}, \tfrac{1}{2}\sqrt{e^2+j^2}\cosh 2t - \tfrac{e}{2}, \tfrac{1}{2}\sqrt{e^2+j^2}\sinh 2t\big) \end{aligned} \tag{22b}$$

(3.13) **Proof**: First, we show that $t \mapsto \gamma(t)$ is a solution of (20a) – (20c). This follows because $\dot{R}(t) = \sqrt{e^2+j^2}\sinh 2t = 2K(t)$, $\ddot{K}(t) = 4K(t)$ and $\dot{H}_j = \dot{T} - \dot{R} = \sqrt{e^2+j^2}\sinh 2t - \sqrt{e^2+j^2}\sinh 2t = 0$. Now $\gamma(t) \in H_j^{-1}(e)$ for every $t \in [-T_{\mathrm{per}}/2, T_{\mathrm{per}}/2]$ because

$$T(t)R(t) - K^2(t) = \tfrac{1}{4}(e^2+j^2)\cosh^2 2t - \tfrac{1}{4}e^2 - \tfrac{1}{4}(e^2+j^2)\sinh^2 2t = \tfrac{1}{4}j^2,$$

$T(t) \geq 0$ for every $t \in [-T_{\mathrm{per}}/2, T_{\mathrm{per}}/2]$ and $0 \leq R(t) \leq R(T_{\mathrm{per}}/2) = R_{\max}$. Moreover, $T(t) - R(t) = e$. Since $R_{\max} = \frac{1}{2}\sqrt{e^2+j^2}\cosh 2(T_{\mathrm{per}}/2) - e/2$, we find that $\gamma(T_{\mathrm{per}}/2)$ is an end point of $H_j^{-1}(e)$ in $\overline{P}_j$. □

In the full motion the ends of $H_j^{-1}(e)$ are joined together and correspond to a circle on the unreduced 2-torus $T_{e,j}^2$, which is an an integral curve $\mathscr{C}$ of the angular momentum vector field $X_J|T_{e,j}^2$. Thus the unreduced motion on $T_{e,j}^2$, which is sent to γ by the reduction mapping, starts at a point on $\mathscr{C}$ and after a time T_{per} returns for the first time to $\mathscr{C}$.

Next we compute the rotation angle on $T_{e,j}^2$ of the full motion of X_H. Now

$$\dot{\varphi}_q = L_{X_H}\varphi_q = \{\varphi_q, H\} = \{\tan^{-1}\frac{y}{x}, \tfrac{1}{2}(p_x^2+p_y^2) - \tfrac{1}{2}(x^2+y^2)\} = \frac{xp_y - yp_x}{x^2+y^2} = \frac{j}{2R}.$$

So the rotation angle $\Theta(e,j)$, when $j \neq 0$, of the flow of X_H on $T_{e,j}^2$ is

$$\begin{aligned}\Theta(e,j) &= \int_0^{T_{\mathrm{per}}} \dot{\varphi}_q \,\mathrm{d}t = 2\int_{R_{\min}}^{R_{\max}} \frac{j}{2R}\frac{\mathrm{d}R}{\dot{R}}\\ &= 2j\int_{R_{\min}}^{R_{\max}} \frac{\mathrm{d}R}{2R(2K)}, \qquad \text{using (20c). From (22a) we get } R_{\min} = -e/2 + \tfrac{1}{2}\sqrt{e^2+j^2}\\ &= \frac{j}{2}\int_{R_{\min}}^{R_{\max}} \frac{\mathrm{d}R}{R\sqrt{(R+e/2)^2 - \frac{1}{4}(e^2+j^2)}}, \qquad \text{since } K^2 = TR - \tfrac{1}{4}j^2 = (R+e)R - \tfrac{1}{4}j^2, \text{ using (21).}\end{aligned}$$

We now calculate the integral giving $\Theta(e,j)$. Successively let $R = u^{-1}$, $v = \frac{2e}{j} - ju$, and $\alpha^2 = 4(1+\frac{e^2}{j^2})$. Then

$$\frac{j\,\mathrm{d}R}{2R\sqrt{(R+e/2)^2 - \frac{1}{4}(e^2+j^2)}} = \frac{-j\,\mathrm{d}u}{\sqrt{4(1+\frac{e^2}{j^2}) - (\frac{2e}{j} - ju)^2}} = \frac{\mathrm{d}v}{\sqrt{\alpha^2 - v^2}} = \mathrm{d}\big(\sin^{-1}\frac{v}{\alpha}\big).$$

Therefore

$$\begin{aligned}
\Theta(e,j) &= \sin^{-1}\left(\frac{2\frac{e}{j} - jR^{-1}}{2\sqrt{1+\frac{e^2}{j^2}}}\right)\Bigg|_{R_{\min}}^{R_{\max}} = \pm\Big(\sin^{-1}\left(\frac{eR_{\max} - \frac{1}{2}j^2}{R_{\max}\sqrt{e^2+j^2}}\right) + \frac{\pi}{2}\Big), \text{ if } \pm j > 0,\\
&\qquad\qquad \text{since } \frac{eR_{\min} - \frac{1}{2}j^2}{R_{\min}\sqrt{e^2+j^2}} = -1\\
&= 2\tan^{-1}\left[\frac{e+\sqrt{e^2+j^2}}{j}\sqrt{\frac{2R_{\max}+e-\sqrt{e^2+j^2}}{2R_{\max}+e+\sqrt{e^2+j^2}}}\right]. \qquad (23)
\end{aligned}$$

Taking limits in (23) we find that $\Theta(e,0^+) = \pi$, when $j \searrow 0$ and $\Theta(e,0^-) = -\pi$, when $j \nearrow 0$. Note that $\Theta(e,-j) = -\Theta(e,j)$.

3.6 The action functions

In this subsection we find the action functions for the hyperbolic billiard.

Because the vector field $\frac{1}{2\pi}X_J$ on $T^2_{e,j}$ has a periodic flow $\varphi^J_{2\pi t}$ of period 1, the rescaled angular momentum $\frac{1}{2\pi}J|T^2_{e,j}$ (13) is a globally defined action.

The second action for the hyperbolic circular billiard is given by $I = \int_\Gamma \langle p, \mathrm{d}q\rangle$, where $\Gamma : \mathbf{R} \rightarrow T^2_{e,j} : s \mapsto \varphi^H_{T_{\mathrm{per}}(e,j)s} \circ \varphi^J_{-\frac{\Theta(e,j)}{2\pi}s}(m)$ for $m \in T^2_{e,j}$ is an integral curve of the vector field $T_{\mathrm{per}}(e,j)X_H - \frac{\Theta(e,j)}{2\pi}X_J$ on $T^2_{e,j}$. Because $\varphi^H_{T_{\mathrm{per}}(e,j)} \circ \varphi^J_{-\frac{\Theta(e,j)}{2\pi}}(m) = m$ for every $m \in T^2_{e,j}$, Γ is a closed curve on $T^2_{e,j}$. Moreover, Γ is homotopic on $T^2_{e,j}$ to the curve $\Gamma_1 : [0,1] \rightarrow T^2_{e,j} : t \mapsto \varphi^H_{T_{\mathrm{per}}(e,j)t}(m)$ followed by $\Gamma_2 : [0,1] \rightarrow T^2_{e,j} : t \mapsto \varphi^J_{-\frac{\Theta(e,j)}{2\pi}t}\big(\varphi^H_{T_{\mathrm{per}(e,j)}}(m)\big)$.

We now compute the action I. By definition

$$\begin{aligned}
I(e,j) &= \int_\Gamma \langle p, \mathrm{d}q\rangle = \int_{\Gamma_1}\langle p, \mathrm{d}q\rangle + \int_{\Gamma_2}\langle p, \mathrm{d}q\rangle\\
&= \textstyle\int_0^1 \langle p, L_{T_{\mathrm{per}}X_H} q\rangle\,\mathrm{d}t + \int_0^1 \langle p, L_{-\frac{\Theta(e,j)}{2\pi}X_J} q\rangle\,\mathrm{d}t = \int_0^{T_{\mathrm{per}}} \langle p, L_{X_H} q\rangle\,\mathrm{d}t + \int_0^{-\frac{\Theta(e,j)}{2\pi}} \langle p, L_{X_J} q\rangle\,\mathrm{d}t\\
&= \textstyle\int_0^{T_{\mathrm{per}}} \langle p, \{q,H\}\rangle\,\mathrm{d}t + \int_0^{-\frac{\Theta(e,j)}{2\pi}} \langle p, \{q,J\}\rangle\,\mathrm{d}t = \int_0^{T_{\mathrm{per}}} \langle p, p\rangle\,\mathrm{d}t + \int_0^{-\frac{\Theta(e,j)}{2\pi}} \langle (p_x,p_y), (-y,x)\rangle\,\mathrm{d}t\\
&= \int_0^{T_{\mathrm{per}}} 2T\,\mathrm{d}t - \frac{j}{2\pi}\,\Theta(e,j)
\end{aligned}$$

But $\dot{K} = \{K,H\} = T + R = 2T - e$, since $H = T - R = e$. So

$$I(e,j) = \int_{-T_{\mathrm{per}}/2}^{T_{\mathrm{per}}/2} (\dot{K} + e)\,\mathrm{d}t - \tfrac{j}{2\pi}\,\Theta(e,j) = 2\big(K(T_{\mathrm{per}}/2) - K(0)\big) + eT_{\mathrm{per}} - \tfrac{j}{2\pi}\,\Theta(e,j)$$

$$= \sqrt{4R_{\max}(R_{\max}+e) - j^2} + eT_{\mathrm{per}}(e,j) - \tfrac{j}{2\pi}\Theta(e,j), \tag{24}$$

where $T_{\mathrm{per}}(e,j)$ is given by (22a) and $\Theta(e,j)$ by (23).

4 Exercises

1. (Mondromy as holonomy.) Let P^{2n} be the phase space of a Liouville integrable system with integral map $F : P \to \mathbf{R}^n$.

   a) Let $f : U \to \mathbf{R}^n$ be a local set of action variables on the open set $U \subseteq \mathbf{R}^n$. Pull back the standard flat connection $\nabla_X Y = DY \cdot X$ on $\mathbf{R}^n$ by the mapping f. Show that the pulled back connection ∇^*_X in U is $\nabla^*_X Y = DY \cdot X + Df^{-1} \cdot D^2 f(X,Y)$. Deduce that $\nabla^*_X Y$ defines a covariant derivative in U.

   b) Let $g : V \to \mathbf{R}^n$ be another set of local action variables. If we define ∇ in V as in part a), show that the two covariant derivatives agree on $U \cap V$. (Hint: how are the two sets of action variables related?)

   c) Show that the construction in a) and b) defines a flat connection on the set $\mathscr{R}$ of regular values of F, where one can define local action variables.

   d) Show that the holonomy of the connection defined in part c) has the same matrix representation, in a suitable basis, as the monodromy of the bundle $F|F^{-1}(\mathscr{R}) : F^{-1}(\mathscr{R}) \to \mathscr{R}$.

2. (Particle in a champagne bottle potential.) On $(T\mathbf{R}^2, \omega = \mathrm{d}x_1 \wedge \mathrm{d}y_1 + \mathrm{d}x_2 \wedge \mathrm{d}y_2)$ consider the Hamiltonian

   $$H : T\mathbf{R}^2 \to \mathbf{R} : (x,y) \mapsto \tfrac{1}{2}(y_1^2 + y_2^2) - \tfrac{1}{2}(x_1^2 + x_2^2) + \tfrac{1}{2}(x_1^2 + x_2^2)^2.$$

   The Hamiltonian vector field X_H describes the motion of a particle in a circularly symmetric double well potential. The Hamiltonian H is invariant under the lift of this S^1-action to $T\mathbf{R}^2$. Show that $(0,0)$ is an isolated critical value of the energy momentum mapping $\mathscr{EM}$. Let Γ be a circle in the range of $\mathscr{EM}$ which encircles $(0,0)$. Suppose that Γ lies in the set of regular values of $\mathscr{EM}$. Show that the 2-torus bundle $\mathscr{EM}^{-1}(\Gamma)$ has monodromy $\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$.

3. (Monodromy as toral accumulation.) On $T^*\mathbf{R}^3$ with coordinates $(q,p) = (q_1, q_2, q_3, p_1, p_2, p_3)$ and symplectic form $\omega = \sum_{i=1}^3 \mathrm{d}q_i \wedge \mathrm{d}p_i$ the three dimensional champagne bottle system is given by the Hamiltonian

   $$H : T^*\mathbf{R}^3 \to \mathbf{R} : (q,p) \mapsto \tfrac{1}{2}(p_1^2 + p_2^2 + p_3^2) + (q_1^2 + q_2^2 + q_3^2)^2 - (q_1^2 + q_2^2 + q_3^2).$$

   Rotational invariance of H implies that the angular momentum $J : T^*\mathbf{R}^3 \to \mathbf{R} : (q,p) \mapsto q \times p$ is conserved.

   a) Find the topology of every fiber of the energy momentum mapping. In particular, show that the fiber (h,j) with $h > 0$ and $j = 0$ is diffeomorphic to a circle bundle over the real projective plane. This bundle is the circle bundle associated to the canonical line bundle over the projective plane.

b) Show that the the exceptional fibers with $h = h_{\min} \neq 0$ and $j = 0$ are smooth Lagrangian manifolds.

c) Show that the three dimensional champagne bottle Hamiltonian system does *not* have monodromy.

d)* Thinking of the three dimensional champagne bottle as a superintegrable Hamiltonian system, explain how the monodromy can be seen as a Chern class of a bundle of isotropic 2-tori over a 2-sphere. Show that these 2-tori accumulate at $j = 0$ and induce the monodromy of the 2-dimensional champagne bottle subsystem.

4. (Fractional monodromy.) On $(T^*\mathbf{R}^2 = \mathbf{R}^4, \omega)$ with coordinates $(x,y) = (x_1,x_2,y_1,y_2)$ and symplectic form $\omega = \mathrm{d}x_1 \wedge \mathrm{d}y_1 + \mathrm{d}x_2 \wedge \mathrm{d}y_2$ consider the Hamiltonian system $(H, \mathbf{R}^4, \omega)$, where for ε positive and small we have

$$H : \mathbf{R}^4 \to \mathbf{R} : (x,y) \mapsto \sqrt{2}\big(2x_1y_1x_2 + (x_1^2 - y_1^2)y_2\big) + \varepsilon(x_1^2+y_1^2)(x_2^2+y_2^2). \qquad (25)$$

H is invariant under the flow of the Hamiltonian vector field X_J where $J : \mathbf{R}^4 \to \mathbf{R} : (x,y) \mapsto \frac{1}{2}(x_1^2+y_1^2) - (x_1^2+y_2^2)$ is the Hamiltonian of the $1 : -2$ resonant oscillator.

a) Show that the system $(H, J, \mathbf{R}^4, \omega)$ is Liouville integrable with integral mapping $F : \mathbf{R}^4 \to \mathbf{R}^2 : (x,y) \mapsto \big(J(x,y), H(x,y)\big) = (j,h)$.

b) Show that the set of critical values of F is the union of the image $\mathscr{C} = \{(j,0) \in \mathbf{R}^2 \,\big|\, 0 < -j < \frac{1}{2}\varepsilon^{-2}\}$ of the curve $\mathscr{C}_1 : [0,\infty) \to \mathbf{R^2} : s \mapsto (-s,0)$ and the image $\mathscr{D}$ of the curve $\mathscr{C}_1 : [\frac{1}{2}\varepsilon^{-2},\infty) \to \mathbf{R^2} : \big(3s + \mathrm{O}(\varepsilon), -24\varepsilon s^2 + \mathrm{O}(\varepsilon)\big)$. The union of the curves $\mathscr{C}_i$ for $i = 1,2$ parametrizes the discriminant locus of the polynomial $\pi_2^2 + \big(h - (\pi_1^2 - j^2)\big) - (\pi_1^2 - j^2)(\pi_1 + j) = 0$, where $\pi_1 \geq |j|$. Here

$$j = \tfrac{1}{2}|z_1|^2 - |z_2|^2, \;\; \pi_1 = \tfrac{1}{2}|z_1|^2 + |z_2|^2, \;\; \pi_2 = \sqrt{2}\,\mathrm{Re}(z_1^2 z_2), \;\text{ and }\; \pi_3 = \sqrt{2}\,\mathrm{Im}(z_1^2 z_2)$$

with $z_k = x_k + iy_k$ for $k = 1,2$ are generators of the algebra of polynomials on $\mathbf{R}^4$, which are invariant under the S^1 generated by the flow of X_J. They are subject to the relation $G(\pi) = -2\pi_2^2 - 2\pi_3^2 + 4(\pi_1 - j)(\pi_1 + j)^2 = 0$ with $\pi_1 \geq |j|$.

c) Show that the image $\mathscr{I}$ of the integral map F is the region of $\mathbf{R}^2$ on or above the union of the graphs of $\mathscr{C}$ and $\mathscr{D}$ and that the set $\mathscr{R}_{\mathrm{reg}}$ of regular values of F is $\mathscr{I} \setminus (\mathscr{C} \cup \mathscr{D})$. Prove each of the following facts. The fiber of F over a point on $\mathscr{C} \setminus \{(0,0)\}$ is a *curled* 2-torus, formed by taking a compact cylinder on a figure eight and then identifying its ends after making a half a revolution. The fiber over $(0,0)$ is a once piched 2-torus. The fiber over every point of $\mathscr{D}$ is a circle. The fiber over every point in $\mathscr{R}_{\mathrm{reg}}$ is a smooth 2-torus. Use this information to draw a picture of the bifurcation diagram of F.

d) Use invariant theory to carry out reduction of the S^1-symmetry generated by X_J. Show that the orbit space $J^{-1}(j)/S^1$ is the semialgebraic variety P_j in $\mathbf{R}^3$ defined by $G(\pi) = 2(\pi_1 + j)^2(\pi_1 - j) - 2\pi_2^2 - 2\pi_3^2 = 0$ and $\pi_1 \geq |j|$. The S^1-invariant Hamiltonian H induces a function $H_j : \mathbf{R}^3 \to \mathbf{R} : \pi = (\pi_1, \pi_2, \pi_3) \mapsto \pi_3 + \varepsilon(\pi_1^2 - j^2)$. The reduced Hamiltonian is $H_j|P_j$. Using $\{\pi_1, \pi_2\} = \frac{\partial G}{\partial \pi_3} = -4\pi_3$, $\{\pi_2, \pi_3\} = \frac{\partial G}{\partial \pi_1} =$

$2(\pi_1+j)(3\pi_1-j)$, and $\{\pi_3,\pi_1\} = \frac{\partial G}{\partial \pi_2} = -4\pi_2$, show that the reduced equations of motion are the restriction of

$$\begin{aligned}\dot{\pi}_1 &= \{\pi_1, H_j\} = 4\pi_2 \\ \dot{\pi}_2 &= \{\pi_2, H_j\} = 2(\pi_1+j)(3\pi_1-j) + 8\varepsilon\pi_1\pi_3 \\ \dot{\pi}_3 &= \{\pi_3, H_j\} = -8\varepsilon\pi_1\pi_2\end{aligned}$$

to P_j.

e) On the smooth 2-torus $F^{-1}(j,h) = J^{-1}(j)\cap H^{-1}(h)$ where $(j,h)\in\mathscr{R}_{\mathrm{reg}}$ show that the time of first return of an integral curve of $X_H|T^2_{(j,h)}$ to the periodic orbit γ of $X_J|T^2_{(j,h)}$ is

$$T(j,h) = \tfrac{1}{2}\int_{\pi_1^-}^{\pi_1^+} \frac{\mathrm{d}\pi_1}{\pi_2} = \tfrac{1}{2}\int_{\pi_1^-}^{\pi_1^+} \frac{\mathrm{d}\pi_1}{\sqrt{S_{j,h}(\pi_1)}},$$

where $S_{j,h}(\pi_1) = \big((\pi_1^2-j^2)(\pi_1+j) - \big(h-\varepsilon(\pi_1^2-j^2)\big)^2\big)^{1/2}$ and $S_{j,h}(\pi_1^{\pm}) = 0$. Show that $T(j,h)\to\infty$ as $(j,h)\to\mathscr{C}$. Verify that the rotation angle of the flow of $X_H|T^2_{(j,h)}$ is

$$\Theta(j,h) = 4h\int_{\pi_1^-}^{\pi_1^+} \frac{1}{j+\pi_1}\,\frac{\mathrm{d}\pi_1}{\sqrt{S_{j,h}(\pi_1)}}$$

and that for $j\in(-\frac{1}{2}\varepsilon^{-2},0)$ we have $\lim_{h\to 0^{\pm}}\Theta(j,h) = \pm\pi/2 + \sin^{-1}(\varepsilon\sqrt{2|j|})$.

f) Let Γ be a circle in the image of F which encloses the origin and avoids hitting $\mathscr{D}$. The bundle $\rho: F^{-1}(\Gamma)\to\Gamma$ of fibers of F is not locally trivial because the fiber over $c_0\in\Gamma\cap\mathscr{C}$ is not a smooth 2-torus. In spite of this we can still compute the monodromy of the induced bundle $\rho_*: \bigcup_{c\in\Gamma}\mathrm{H}_1(F^{-1}(c),\mathbf{Z})\to\Gamma$ by *continuously* transporting a suitable basis of $\mathrm{H}_1(F^{-1}(c),\mathbf{Z})$ as c traverses Γ from c_0 to c_0.

We now construct a continuous family of cycles of $F^{-1}(c)$ as c traces out Γ using integral curves of vector fields. The vector field X_H is not suitable on $F^{-1}(c_0)$ because $T(c)\to\infty$ as $c\to c_0$. Look at the vector field X on $\mathbf{R}^4\setminus\{x=y_1=0\}$ defined by $(x_1^2+y_1^2)^{-1}X_H$. Show that X is incomplete on $\mathbf{R}^4\setminus\{x=y_1=0\}$, commutes with X_J and leaves $F^{-1}(c)\setminus\{x=y_1=0\}$ invariant for every c in the image $\mathscr{I}$ of F. Moreover, the time τ of first return of X on $F^{-1}(c)$ with respect to X_J is smooth for every $c\in\mathscr{I}$ and is *continuous* on $\mathscr{C}$. Show that the rotation angle $\widehat{\Theta}$ of X on $F^{-1}(c)$ is equal to the rotation angle $\Theta(c)$ of X_H on $F^{-1}(c)$ for every $c\in\mathscr{R}_{\mathrm{reg}}$. Let $X_1(\xi) = 2\pi X_J(\xi)$ and $X_2(\xi) = \tau(F(\xi))X(\xi) - \Theta(F(\xi))X(\xi)$ for every $\xi\in\mathbf{R}^4\setminus\{x=y_1=0\}$. Show that on $F^{-1}(c)$ for $c\in\mathscr{R}_{\mathrm{reg}}$ the vector fields X_k, $k=1,2$ are smooth, linearly independent, and have flows which are periodic of period 1. For $k=1,2$ let $\beta_k(c): [0,1]\to T^2_c = F^{-1}(c): t\mapsto\varphi_t^{X_k}(\xi)$ where $c\in\mathscr{R}_{\mathrm{reg}}$. Then the homology classes $[\beta_k(c)]$ for $k=1,2$ form a basis of $\mathrm{H}_1(T^2_c,\mathbf{Z})$. Let $[0,1]\to\mathbf{R}^2: u\mapsto\Gamma(u)$ be a parametrization of the curve Γ with $u^*\in(0,1)$ such that $\Gamma(u^*) = c_0$. Look at the vector field $Z^u(\xi) = -\vartheta(u)X_J(\xi) + \tau(\Gamma(u))X(\xi)$ where $\xi\in\mathbf{R}^4\setminus\{x=y_1=0\}$. Here

$$\vartheta: [0,1]\to\mathbf{R}: u\mapsto\begin{cases}\Theta(\Gamma(u))-\pi, & \text{if } u\in[0,u^*) \\ \lim_{u\to u^*}\Theta(\Gamma(u)), & \text{if } u=u^* \\ \Theta(\Gamma(u)), & \text{if } u\in(u^*,1].\end{cases}$$

Show that the function ϑ is continuous at u^*. Let $\gamma_1(u) = \beta_1(\Gamma(u))$. To define the cycle $\gamma_2(c)$ we need the following construction. Consider the initial points $\xi_{\pm}(u) = \big(\pm\big(\pi_1^+(u)+j(u)\big)^{1/2}, 0, 0, \frac{h(u)-\varepsilon(\pi_1^+(u)^2-j(u))^2}{\sqrt{2}(\pi_1^+(u)+j(u))}\big)$, where $\Gamma(u) = \big(j(u), h(u)\big)$ and $\pi^+(u) = \max_{H^{-1}_{j(u)}(h(u))}\pi_1$. Let $\gamma_2(u)$ be the union of the curves $\gamma_2^{\pm}(u)$, where $\gamma_2^{\pm}(u) : [-1/2, 1/2] \to F^{-1}(\Gamma[0,1]) : t \mapsto \varphi_t^{Z^u}(\xi_{\pm}(u))$. Show that when $u \in [0, u^*)$ the curves $\gamma^{\pm}(u)$ join together smoothly to form a closed curve because $\gamma_2^+(\mp 1/2) = \gamma_2^-(\pm 1/2) = \chi^{\pm}$; when $u = u^*$ we have $\chi^+ = \chi^- = \chi^*$. So $\gamma_2(u^*)$ is smooth closed curve with a normal crossing at χ^*; when $u \in (u^*, 1]$ we have $\gamma_2^{\pm}(-1/2) = \gamma_2^{\pm}(1/2) = \widetilde{\chi}^{\pm}$ and $\widetilde{\chi}^+ \neq \widetilde{\chi}^-$. So $\gamma_2(u)$ is a disjoint union of two smooth closed curves. Show that the homology classes in $\mathrm{H}_1(F^{-1}(\Gamma(u)), \mathbf{Z})$, which vary continuously as u crosses u^* are precisely those which lie in the index 2 sublattice $\mathscr{H}_1(T^2_{\Gamma(u)}, \mathbf{Z})$ of $\mathrm{H}_1(F^{-1}(\Gamma(u)), \mathbf{Z})$ spanned by $[\gamma_1(u)]$ and $[\gamma_2(u)]$, when $u \neq u^*$.

g) We now compute the monodromy map $\mu : \mathscr{H}_1(T^2_{\Gamma(0)}, \mathbf{Z}) \to \mathscr{H}_1(T^2_{\Gamma(0)}, \mathbf{Z})$ along the curve Γ. The cycle basis $\{\beta_1(c), \beta_2(c)\}$ of T_c^2 with $c = \Gamma(u)$ with $u \neq u^*$ is independent of the parameter u. So $\{\beta_1(c_0), \beta_2(c_0)\}$ is a basis of $\mathrm{H}_1(F^{-1}(c_0), \mathbf{Z})$. Show that $[\gamma_2(0)] = [\beta_1(0)] + 2[\beta_2(0)]$ and $[\gamma_2(1)] = 2[\beta_2(1)] = 2[\beta_2(0)]$ by drawing a picture of $\gamma_2(u)$ on $F^{-1}(\Gamma(u)) = T^2_{\Gamma(u)}$ when $u \neq u^*$. Since $[\gamma_1(0)] = [\gamma_1(1)]$ show that the monodromy mapping μ sends the basis $\{[\beta_1(0)], [\beta_1(0)] + 2[\beta_2(0)]\}$ to the basis $\{[\beta_1(0)], 2[\beta_2(0)]\}$. Thus with respect to the basis $\{[\beta_1(0)], 2[\beta_2(0)]\}$ of $\mathscr{H}_1(T^2_{\Gamma(0)}, \mathbf{Z})$ the matrix of the linear fractional monodromy μ is $\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$. Observe that the off diagonal entry in the fractional monodromy matrix can not be removed by a change of basis of $\mathscr{H}_1(T^2_{\Gamma(0)}, \mathbf{Z})$.

h) (Geometric fractional monodromy theorem.) Let $(F_1, F_2, \mathbf{R}^4 = T^*\mathbf{R}^2, \omega)$ be a two degree of freedom Liouville integrable system with a proper integral mapping $F : \mathbf{R}^4 \to \mathbf{R}^2 : (x,y) \mapsto \big(F_1(x,y), F_2(x,y)\big)$ with the following properties

1. 0 is a critical value of F such that $F^{-1}(0)$ is connected and $F_1|F^{-1}(0)$ has no critical points.
2. The critical points of F on $F^{-1}(0)$ are transversely nondegenerate.
3. The critical set $\mathscr{C}$ of F on $F^{-1}(0)$ is exactly one periodic orbit γ_0 of X_{F_1}.
4. If Σ is a two dimensional Poincaré section for γ_0, then $p = \Sigma \cap \gamma_0$ is a hyperbolic critical point of $F|\Sigma$.
5. F is invariant under a globally defined Hamiltonian S^1-action.
6. There is a closed path $\Gamma : [0,1] \to \mathscr{R}_{\mathrm{reg}} : u \mapsto \Gamma(u)$ such that Γ crosses the set $\mathscr{CV}$ of critical values of F exactly once at $u^* \in (0,1)$ and $\Gamma(u) \in \mathscr{R}_{\mathrm{reg}}$ if $u \neq u^*$. Here $\mathscr{R}_{\mathrm{reg}}$ is the set of regular values of F in its image.

Then the set of cycles of $\mathrm{H}_1(F^{-1}(\Gamma(u)), \mathbf{Z})$ with $u \in [0,1]$ that can be continued through $\mathscr{CV}$ form an index 2 subgroup $\mathscr{H}_1(F^{-1}(c), \mathbf{Z})$ of $\mathrm{H}_1(F^{-1}(c), \mathbf{Z})$. With respect to a suitable basis of $\mathscr{H}_1(F^{-1}(\Gamma(0), \mathbf{Z})$ the matrix of the monodromy mapping is $\begin{pmatrix} 1 & 0 \\ k-\frac{1}{2} & 1 \end{pmatrix}$ for some integer k. Check that the integral map in sections a) –

g) above satisfy the hypotheses of the geometric fractional monodromy theorem. Prove this theorem trying to generalize the results of this example. Why does the integer k appear in the matrix of the monodromy mapping?

5. (Quadratic spherical pendulum.) On $TS^2 \subseteq T\mathbf{R}^3$ with symplectic form $\omega|TS^2$ consider the Hamiltonian

$$H : TS^2 \subseteq T\mathbf{R}^3 \to \mathbf{R} : (x,y) \mapsto \tfrac{1}{2}(y_1^2 + y_2^2 + y_3^2) + \tfrac{1}{2}bx_3^2 + cx_3 + d.$$

The Hamiltonian vector field X_H describes the motion of a particle on a 2-sphere S^2 under a force coming from a potential $V(x) = \frac{1}{2}bx_3^2 + cx_3 + d$. The Hamiltonian is invariant under the Hamiltonian S^1-action $S^1 \times TS^2 \to TS^2 : \big(t,(x,y)\big) \mapsto (\widetilde{R}_t x, \widetilde{R}_t y)$, where $\widetilde{R}_t = \begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix}$, which has momentum mapping $J : TS^2 \to \mathbf{R} : (x,y) \mapsto x_1y_2 - x_2y_1$.

a) Verify that the energy momentum mapping

$$\mathscr{E}\mathscr{M} : TS^2 \to \mathbf{R}^2 : (x,y) \mapsto \big(H(x,y), J(x,y)\big)$$

is a proper mapping. Show that discriminant Δ of the energy momentum mapping is $\{(h,j) \in \mathbf{R}^2 \,\big|\, P(z) = 2\big(h - V(z)\big)(1-z^2) - j^2$ has a double root$\}$. Show that the discriminant locus $\{\Delta = 0\}$ is parametrized by the curve $\gamma_\pm^\ell : I_j \to \mathbf{R}^2 : s \mapsto \big(h(s), \pm j(s)\big)$, where

$$h(s) = bs^2 + \tfrac{3}{2}cs - \tfrac{1}{2}cs^{-1} - \tfrac{1}{2}b + d \text{ and } j(s) = \pm(1-s^2)\sqrt{-s^{-1}(bs+c)} \qquad (26)$$

where

CASE I. $0 < |b| \le c$ and $s \in I_1 = [-1,0) \cup \{1\}$.

CASE II. $0 < c < b$ and $s \in I_2 = \{-1\} \cup [-cb^{-1},0) \cup \{1\}$.

CASE III. $0 < c < b$ and $s \in I_3 = \{-1\} \cup [-cb^{-1},0) \cup \{1\}$.

When $c = 0$ the discriminant locus is the union of the curve $2(h-d) = j^2$ and the point $(\frac{1}{2}b, 0)$ when $b > 0$ or the union of the curves $2(h-d) = -|b| \pm 2j\sqrt{|b|}$ with $|j| \le -b$ and $2(h-d) \ge b$ when $b < 0$. Note that when $b = 0$ the quadratic spherical pendulum is the spherical pendulum.

b) In case I show that the bifurcation diagram of the energy momentum map is the same as that of the spherical pendulum. Compute the monodromy of the 2-torus bundle over a circle in the set of regular values enclosing the isolated critical value. In case III show that the energy momentum map has two isolated critical values where the fiber corresponding to $(\frac{1}{2}b \pm |c|, 0)$ is a singly pinched 2-torus. Again compute the monodromy of the bundle over a circle enclosing one of the isolated critical values. What is it when the circle encloses both critical values? When $c = 0$ the fiber corresponding to $(\frac{1}{2}b, 0)$ is a doubly pinched 2-torus. Compute the monodromy of the bundle of 2-tori over a circle which encloses the critical value.

The remainder of this exercise considers the case II. The set of critical values consists of the union of the image of the curves $\gamma_\pm^1$ parametrized by $s \in [-1,0)$ which join together at $P = (\frac{1}{2}b - c + d, 0)$, and the image of the curves $\gamma^2_\pm$ which join the point $B_0 = (0, d - c^2 b^{-1})$ to $D = (0, \frac{1}{2}b + c + d)$. Show that these curves are smooth except at one point $B_\pm = \big(\pm j(s^*), h(s^*)\big)$ with $s^* \in (cb^{-1}, 1)$ where they have a cusp. Thus the union of the image of γ^2_- and γ^2_+ bound a triangle $\mathscr{T}$ with cusps at the vertices at $B_{0,\pm}$ and curved smooth sides $\overline{\mathrm{B_0 B_\pm}} = \gamma^2_\pm([-cb^{-1}, s^*])$ and $\overline{B_- B_+} = \gamma^2_-([s^*, 1)) \cup (-\gamma^2_+)([s^*, 1))$.

c) Using invariant theory and singular reduction show that the range of the energy momentum mapping $\mathscr{R}$ lies in the interior of the region bounded by the union of the image of γ^1_- and γ^1_+. The triangle $\mathscr{T}$ lies in the interior of the region $\mathscr{R}$. Verify the entries in the second column of the following table.

Conditions on p	Fiber of $\mathscr{EM}^{-1}(p)$
$p = P$	$\{\mathrm{pt}\}$
$p \in \gamma^1_\pm([-1, 0))$	S^1
$p \in \overline{B_- B_+}$	bitorus = a compact cylinder on a figure eight with its ends identified
$p \in \overline{B_0 B_\pm}$	$S^1 \dot\cup T^2_\pm$
$p \in B_{0,\pm}$	$\{\mathrm{pt}\} \cup T^2_\pm$
$p \in \mathrm{int}\, \mathscr{T}$	$T^2_- \dot\cup T^2_+$
$p \in \mathscr{R} \setminus \mathscr{T}$	T^2

Because of the geometry of the energy momentum mapping $\mathscr{EM}$ given in the table above, we say that the quadratic spherical pendulum is an integrable system with a *swallowtail*.

d) On each connected component of $\mathscr{EM}^{-1}(\mathrm{int}\, \mathscr{T})$ construct an action function I, whose limit on the side $\mathscr{C} = \overline{B_- B_+}$ of the triangle $\mathscr{T}$ with $\mathscr{EM}$ fiber a bitorus coincides with the limit of the action on $\mathscr{R} \setminus (\partial \mathscr{R} \cup \mathscr{T})$.

e) Consider the *bipath* in $\mathscr{R} \setminus \partial \mathscr{R}$ made up of two continuous paths $\Gamma^\pm$, which start at the point $A \in \mathscr{R} \setminus (\partial \mathscr{R} \cup \mathscr{T})$, cross the locus $\mathscr{C}$ at the point C, and when in $\mathrm{int}\, \mathscr{T}$ have inverse image under $\mathscr{EM}$ which lie in $\dot\bigcup T^2_\pm$. Then $\Gamma^\pm$ exits $\mathscr{T}$ and returns in $\mathscr{R} \setminus (\partial \mathscr{R} \cup \mathscr{T})$ to the point A. At A we begin with a basis $\{b(A), g_0(A)\}$ of an index 2 subgroup $\mathscr{H}_1(T^2_A = \mathscr{EM}^{-1}(A), \mathbf{Z})$ of $\mathrm{H}_1(T^2_A, \mathbf{Z})$. Here $b(A)$ is the cycle on T^2_A generated by a periodic orbit of X_J and $g_0(A)$ is the cycle on T^2_A generated by a periodic orbit of X_I. Parallel transport these cycles along $\Gamma^\pm$. As we approach $C \in \mathscr{C}$ outside of $\mathscr{T}$ show that g_0 approaches a limit $g_0(C)$. Let $g_0(C^\pm)$ be the limit of cycles in $T^2_\pm$ as we approach C along $\Gamma^\pm$ from $\mathrm{int}\, \mathscr{T}$. Show that $g_0(C) = g_0(C^-) + g_0(C^+)$ in $\mathrm{H}_1(\mathscr{EM}^{-1}(C), \mathbf{Z})$. Show that when $g_0(C^\pm)$ is parallel transported along $\Gamma^\pm$ back to the point A it returns to $g_0(A)$. Show that as the cycle $b(A)$ is parallel transported from A along $\Gamma^\pm$ back to A it returns to $b(A)$.

f) Starting at A with the basis $\{2b(A), g_0(A)\}$ of cycles on T^2_A after parallel translation along $\Gamma^\pm$ and returning to A, we obtain *two* copies of the same fiber T^2_A with the same homology basis $\{b(A), g_0(A)\}$. *Merging* these bases by adding to get the basis $\{2b(A), 2g_0(A)\}$ of $\mathscr{H}_1(T^2_A, \mathbf{Z})$. This gives the *bidromy transformation*

$\mu : \mathscr{H}_1(T_A^2, \mathbf{Z}) \to \mathscr{H}_1(T_A^2, \mathbf{Z})$, which sends the basis $\{2b(A), g_0(A)\}$ to the basis $\{2b(A), 2g_0(A)\}$. Thus μ has the *bidromy matrix* $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 2 \end{smallmatrix}\right)$. Let $k \in \mathbf{Z}$. Using the basis $\{2b(A), g_k(A) = g_0(A) + kb(A)\}$ of $\mathscr{H}_1(T_A^2, \mathbf{Z})$ show that the bidromy matrix is $\left(\begin{smallmatrix} 1 & 0 \\ -\frac{k}{2} & 2 \end{smallmatrix}\right)$. Deduce that bidromy is different from fractional monodromy.

6. (Scattering monodromy.) Consider the hyperbolic oscillator (Burkes' egg), which is a Hamiltonian system on $T^*\mathbf{R}^2 = \mathbf{R}^4$ with coordinates $(x, y) = (x_1, x_2, y_1, y_2)$ and *nonstandard* symplectic form $\Omega = \mathrm{d}x_1 \wedge \mathrm{d}x_2 - \mathrm{d}y_1 \wedge \mathrm{d}y_2$ corresponding to the Hamiltonian
$$v : \mathbf{R}^4 \to \mathbf{R} : (x, y) \mapsto x_1 y_1 + x_2 y_2. \tag{27}$$
a) Show that $u : \mathbf{R}^4 \to \mathbf{R} : (x, y) \mapsto \frac{1}{2}(x_1^2 + x_2^2 - y_1^2 - y_2^2)$ is an integral of the Hamiltonian vector field X_v and has a periodic flow φ_r^u of period 2π. The system $(v, u, \mathbf{R}^4, \Omega)$ is Liouville integrable with energy momentum mapping $\mathrm{EM} : \mathbf{R}^4 \to \mathbf{R}^2 : (x, y) \mapsto \big(v(x, y), u(x, y)\big) = (h, \ell)$, which has $(0, 0)$ as an *isolated* critical value. Show that the bundle $\mathrm{EM}|\big(\mathbf{R}^4 \setminus \mathrm{EM}^{-1}(0, 0)\big) : \mathbf{R}^4 \setminus \mathrm{EM}^{-1}(0, 0) \to \mathbf{R}^2 \setminus \{(0, 0)\}$ is *trivial* because the action angle coordinate mapping
$$\begin{aligned} \widetilde{\Psi} : \big((S^1 \times \mathbf{R}) \times (\mathbf{R}^2 \setminus \{(0,0)\}), \widehat{\Omega} = \mathrm{d}t \wedge \mathrm{d}h + \mathrm{d}r \wedge \mathrm{d}\ell\big) &\to \big(\mathbf{R}^4 \setminus \mathrm{EM}^{-1}(0, 0), \Omega\big) : \\ \big((r, t), (h, \ell)\big) &\mapsto \varphi_t^v \circ \varphi_r^u(\ell + \tfrac{1}{2}, h, h, -\ell + \tfrac{1}{2}) \end{aligned} \tag{28}$$
is a symplectic diffeomorphism. Thus the hyperbolic oscillator has *no* Hamiltonian monodromy.

b) Consider the circle $S = \{(h, \ell) \in \mathbf{R}^2 \,\big|\, h^2 + \ell^2 = R^2,\ R > 0\}$. Show that the mapping
$$\sigma : S \to \mathrm{EM}^{-1}(S) : (h, \ell) \mapsto (\ell + \tfrac{1}{2}, h, h, -\ell + \tfrac{1}{2}) \tag{29}$$
is a section of the bundle $\rho = \mathrm{EM}|\big(\mathrm{EM}^{-1}(S)\big) : \mathrm{EM}^{-1}(S) \to S$. Look at the family of curves $s \mapsto \Gamma_{\sigma(s)}$ on $\mathrm{EM}^{-1}(S)$, where $\Gamma_{\sigma(s)}(t) = \varphi_t^v(\sigma(s))$ is an integral curve of X_v which starts at $\sigma(s)$. Show that the image of $\Gamma_{\sigma(s)}$ under the cotangent bundle projection map $\pi : \mathbf{R}^4 \to \mathbf{R}^2 : (x, y) \mapsto x$ is the curve
$$t \mapsto \gamma_s(t) = \big((\ell + \tfrac{1}{2})\cosh t + (-\ell + \tfrac{1}{2})\sinh t, h(\cosh t - \sinh t)\big),$$
whose image is the branch $\mathscr{H}_s$ of a hyperbola given by $x_2(hx_1 - \ell x_2) = \frac{1}{2}h^2$, where $\pm x_2 \geq 0$ if $\pm h \geq 0$. Show that the *outgoing* asymptotic direction of $\mathscr{H}_s$ is $(1, 0)$; while the *incoming* asymptotic direction is $(h^2 + \ell^2)^{-1/2}(\ell, h)$. Define the *scattering angle* ϑ_s to be the angle corresponding to the counter clockwise rotation which sends the outgoing asymptotic direction of $\mathscr{H}_s$ to its incoming asymptotic direction. Show that $\vartheta_s = \tan^{-1}\frac{h}{\ell}$. Let $s^* = (0, -R) \in S$. Show that
$$\vartheta_{s_+^*} = \lim_{s \in S \cap \{h \geq 0\} \to s^*} \vartheta_s = 2\pi \quad \text{and} \quad \vartheta_{s_-^*} = \lim_{s \in S \cap \{h \leq 0\} \to s^*} \vartheta_s = 0. \tag{30}$$
Thus as s traverses the circle S in a clockwise fashion starting and finishing at $s^* = (0, -R)$, the scattering angle ϑ_s increases by 2π. This variation in the scattering angle is called the *scattering monodromy* of the hyperbolic oscillator. In

the remainder of this exercise we give a *phase space* interpretation of scattering monodromy.

c) The map $\widetilde{\Psi}$ (28) restricts to a diffeomorphism $\Psi : (S^1 \times \mathbf{R}) \times S \to \mathrm{EM}^{-1}(S)$, which intertwines the S^1-action

$$\psi : S^1 \times \big((S^1 \times \mathbf{R}) \times S\big) \to (S^1 \times \mathbf{R}) \times S : \big(r', ((r,t),s)\big) \mapsto \big((r'+r,t),s\big)$$

with the S^1-action $\varphi^u_{r'}|\mathrm{EM}^{-1}(S)$. Consequently, we get the S^1-principal bundle

$$\widetilde{\rho} : \mathrm{EM}^{-1}(S) \to \mathrm{EM}^{-1}(S)/S^1 = S \times \mathbf{R} : p = \Psi\big((r,t),s\big) \mapsto (s,t) = \big(\mathrm{EM}(p),t\big),$$

which is trivial, because $\widetilde{\sigma}(s,t) = \varphi^v_t(\sigma(s))$ is a section. Let $S^\times = S \setminus \{s^* = (0,-R)\}$. Consider the trivial S^1-principal bundle $\rho^\times : \mathrm{EM}^{-1}(S^\times) \to S^\times \times \mathbf{R}$ given by $\rho^\times = \rho|\mathrm{EM}^{-1}(S^\times)$. Let $\mathrm{d}\vartheta = (x_1^2+x_2^2)^{-1}\big(x_1\,\mathrm{d}x_2 - x_2\,\mathrm{d}x_1\big)$ be the angle 1-form on $\mathbf{R}^2 \setminus \{(0,0)\}$. Let $\theta = -\pi^*(\mathrm{d}\vartheta)|\mathrm{EM}^{-1}(S^\times)$ be a 1-form on $\mathrm{EM}^{-1}(S^\times)$, where $\pi : \mathbf{R}^4 \to \mathbf{R}^2 : (x,y) \mapsto x$. Show that θ is a closed 1-form with $X_u \,\lrcorner\, \theta = 1$, which is invariant under the S^1-action $\varphi^u_r|\mathrm{EM}^{-1}(S^\times)$. Hence θ is a connection 1-form on the S^1-principal bundle $\rho^\times$. The *scattering phase* $\Theta(s)$ of the curve $\Gamma_{\sigma(s)}$ on $\mathrm{EM}^{-1}(S^\times)$ is $\int_{\Gamma_{\sigma(s)}} \theta$. Verify each step of the following calculation.

$$\begin{aligned}\Theta(s) &= \int_{\Gamma_{\sigma(s)}} \theta = -\int_{\Gamma_{\sigma(s)}} \pi^*\,\mathrm{d}\vartheta = -\int_{\gamma_s = \pi(\Gamma_{\sigma(s)})} \mathrm{d}\vartheta \\ &= -\int_{-\infty}^{\infty} \gamma_s^*\,\mathrm{d}\vartheta = \vartheta(\gamma_s(-\infty)) - \vartheta(\gamma_s(\infty)) = \vartheta_s.\end{aligned}$$

The last equality above follows because the curve γ_s traces out the branch $\mathscr{H}_s$ of the hyperbola, whose incoming asymptotic direction is $(h^2+\ell^2)^{-1/2}(\ell,h)$ has angle $\vartheta(\gamma_s(-\infty)) = \tan^{-1}\frac{h}{\ell}$, and whose outgoing asymptotic direction is $(1,0)$ has angle $\vartheta(\gamma_s(\infty)) = 0$.

d) Let $\widetilde{S}^\times$ be the image of $S^\times$ under the section σ precomposed with the injection map $i_s : S \to S \times \mathbf{R} : s \mapsto (s,0)$. Then $\Gamma_{\sigma(s)}$ is the integral curve of X_v on $\mathrm{EM}^{-1}(\widetilde{S}^\times)$, which starts at $\sigma(s) \in \widetilde{S}^\times$. The *infinitesimal elevation* of the curve $\Gamma_{\sigma(s)}$ on $\mathrm{EM}^{-1}(\widetilde{S}^\times)$ with respect to the 1-form θ is $\frac{\mathrm{d}\theta}{\mathrm{d}t} = (X_v \,\lrcorner\, \theta)\big(\Gamma_{\sigma(s)}(t)\big)$. The *elevation* of $\Gamma_{\sigma(s)}$ is $\int_{-\infty}^{\infty} \dot{\theta}\,\mathrm{d}t$. Verify each step of the following calculation.

$$\begin{aligned}\Theta(s) &= \int_{\Gamma_{\sigma(s)}} \theta = \int_{-\infty}^{\infty} \theta\big(\Gamma_{\sigma(s)}(t)\big)\frac{\mathrm{d}\Gamma_{\sigma(s)}}{\mathrm{d}t}\,\mathrm{d}t = \int_{-\infty}^{\infty} \theta\big(\Gamma_{\sigma(s)}(t)\big)X_v\big(\Gamma_{\sigma(s)}(t)\big)\,\mathrm{d}t \\ &= \int_{-\infty}^{\infty} (X_v \,\lrcorner\, \theta)\big(\Gamma_{\sigma(s)}(t)\big)\,\mathrm{d}t = \int_{-\infty}^{\infty} \dot{\theta}\,\mathrm{d}t = \int_0^{\infty} \dot{\theta}\,\mathrm{d}t + \int_{-\infty}^{0} \dot{\theta}\,\mathrm{d}t \\ &= \int_0^{\infty} \dot{\theta}\,\mathrm{d}t - \int_0^{\infty} \dot{\theta}\,\mathrm{d}(-t) = p_+(s) - p_-(s),\end{aligned}$$

where $p_+(s)$ is the elevation of the curve $[0,\infty) \to \mathrm{EM}^{-1}(S^\times) : t \mapsto \varphi^v_t\big(\sigma(s)\big)$ and $p_-(s)$ is the elevation of the curve $[0,\infty) \to \mathrm{EM}^{-1}(S^\times) : t \mapsto \varphi^v_{-t}\big(\sigma(s)\big)$. Using (30) show that

$$\lim_{s \to s^*_+} \big(p_+(s) - p_-(s)\big) = 2\pi \quad \text{and} \quad \lim_{s \to s^*_-} \big(p_+(s) - p_-(s)\big) = 0. \tag{31}$$

We now give a geometric interpretation of the elevations $p_{\pm}(s)$. We say that two integral curve of X_v on $\mathrm{EM}^{-1}(S)$ are *equivalent* at $\pm\infty$ if they are asymptotic to each other as $t \to \pm\infty$. An equivalence class is called an *end* at $\pm\infty$. Show that each integral curve of X_v on $\mathrm{EM}^{-1}(S)$ has exactly two ends, noting that because the flows of the vector fields X_u and X_v on $\mathrm{EM}^{-1}(S)$ define an affine structure on $\mathrm{EM}^{-1}(s)$ for each $s \in S$ and distinct level sets of EM are disjoint, no distinct integral curves of X_v on $\mathrm{EM}^{-1}(S)$ are asymptotic to each other. Therefore the integral curve $\Gamma_{\sigma(s)}$ has exactly two ends $p_{\pm}(s^*_{\pm})$, which lie on the circle $\mathscr{C}_{\pm}$ at $\pm\infty$ consisting of the ends of the integral curves of X_v, which start on $\widetilde{S}$. Since

$$\Theta(s^*_+) - \Theta(s^*_-) = \big(p_+(s^*_+) - p_-(s^*_+)\big) - \big(p_+(s^*_-) - p_-(s^*_-)\big) = 2\pi,$$

using (31), again we have shown that the scattering monodromy of the hyperbolic oscillator is 2π.

e) The following argument shows that the scattering monodromy of the family $s \mapsto \Gamma_{\sigma(s)}$ does not depend on the choice of the connection 1-form on the S^1-principal bundle $\rho^{\times}$. Let ψ be a connection 1-form on $\rho^{\times}$ whose scattering phase on each integral curve of X_v is uniformly bounded. Because the period of the 1-form $i^*_s(\psi - \theta)$ over every integral curve of X_u on $\mathrm{EM}^{-1}(s)$ with $s \in S$ vanishes, the 1-form $i^*_s(\psi - \theta)$ is exact, that is, there is a smooth function f_s such that $i^*_s\psi = i^*_s\theta + \mathrm{d}f_s$. The function f_s is S^1-invariant and the mapping $S^{\times} \to C^0(S^{\times}) : s \mapsto f_s$ is continuous and uniformly bounded. Show that the infinitesimal elevation of the curve $\Gamma_{\sigma(s)}$ with respect to $i^*_s\psi$ is $\frac{\mathrm{d}i^*_s\psi}{\mathrm{d}t} = \frac{\mathrm{d}\theta}{\mathrm{d}t} + \frac{\mathrm{d}f_s}{\mathrm{d}t}$. This yields the scattering phase

$$\Phi(s) = \int_{-\infty}^{\infty} i^*_s\dot{\psi}\,\mathrm{d}t = \int_{-\infty}^{\infty} \dot{\theta}\,\mathrm{d}t + \int_{-\infty}^{\infty} \dot{f}_s\,\mathrm{d}t = \Theta(s) + \big(f_s(p_+) - f_s(p_-)\big).$$

Taking the limit as $s \to s^*_{\pm}$, using (31) and the fact that $f_{s^*_{\pm}}$ is a continuous 2π periodic function on the circle $\mathscr{C}_-$ show that $\Phi(s^*_+) - \Phi(s^*_-) = \Theta(s^*_+) - \Theta(s^*_-) = 2\pi$, as desired.

f) Letting $r_{\max} \to \infty$ show that the hyperbolic circular billiard of §3 becomes the hyperbolic oscillator. Taking the limit as $R_{\max} \to \infty$ show that the rotation number of the circular billiard becomes the scattering angle of the oscillator. Give a geometric argument which shows why the monodromy in the hyperbolic circular billiard becomes the scattering monodromy in the hyperbolic oscillator.

g) Using Levi-Civita regularization, show that the two degree of freedom Kepler problem for a *repulsive* gravitational potential becomes the hyperbolic harmonic oscillator. Deduce that the 2-degree of freedom scattering Kepler problem has scattering monodromy.

7. (Burke's egg (poached).) The problem is the same as in the hyperbolic oscillator except that we add a magnetic term to the symplectic structure. More explicitly, the phase space of the problem is $T^*\mathbf{R}^2$ with coordinates (x_1, x_2, y_1, y_2) and symplectic form $\Omega = (x_1^2 + x_2^2)\,dx_1 \wedge dx_2 + dx_1 \wedge dy_1 + dx_2 \wedge dy_2$. The Hamiltonian is $H(x,y) = \frac{1}{2}\big(y_1^2 + y_2^2\big) - (x_1^2 + x_2^2)$. Prove the following.

   a) The problem is Liouville integrable.

b) The origin is an isolated critical value of the energy momentum mapping.

c) The topology of the fiber of a regular value of the energy momentum mapping is a 2-torus T^2. Show that every fiber of the energy momentum mapping is *compact*. Show that the projection of every orbit of X_H onto configuration space is *bounded*.

d) Show that the monodromy is *nontrivial* even though the topology of the energy surfaces does *not* change when the value of the energy passes through 0.

8. (Focus-focus Morse lemma.) The focus-focus model is an integrable Hamiltonian system on $T^*\mathbf{R}^2$ with coordinates (x, y, p_x, p_y), symplectic form $\omega_0 = \mathrm{d}x \wedge \mathrm{d}p_x + \mathrm{d}y \wedge \mathrm{d}p_y$, and integrals $q_1 = xp_x + yp_y$ and $q_2 = xp_y - yp_x$. The goal of this exercise is to prove the focus-focus Morse lemma:

Let Ω be an open neighborhood of 0 in $T^*\mathbf{R}^2 = \mathbf{R}^4$. For $i = 1, 2$ suppose that $h_i \in C^\infty(\Omega)$, where $h_i = q_i + r_i$ with $r_i \in \mathscr{O}(2)$, that is, $r_i \in C^\infty(\Omega)$ and is flat to 2nd order at 0. Then there is a local diffeomorphism $\Phi : (\mathbf{R}^4, 0) \to (\mathbf{R}^4, 0)$, which is near the identity, that is, $\Phi = \mathrm{id} + \mathscr{O}(1)^2$, such that $\Phi^* h_i = q_i$ for $i = 1, 2$.

We need two preliminary results

a) For every $u = (u_1, u_2) \in C^\infty(\Omega)^2$ with $u_i \in \mathscr{O}(2)$ for $i = 1, 2$, there is a smooth vector field $Y = Y(u) \in \mathscr{X}(\Omega)$ such that

$$\langle (\mathrm{d}q_1, \mathrm{d}q_2) \,\big|\, Y(u) \rangle = u. \tag{32}$$

Here $\mathscr{X}(\Omega)$ is the set of smooth vector fields on Ω. We outline the proof of (32). Let $Y = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + c\frac{\partial}{\partial p_x} + d\frac{\partial}{\partial p_y}$, where $a, b, c, d \in C^\infty(\Omega)$. A calculation shows that (32) is equivalent to $u = u_1 + i\,u_2 = \overline{\alpha} z_2 + \beta \overline{z}_1$, where $\alpha = a + i\,b$, $\beta = c + i\,d$, $z_1 = x + i\,y$, and $z_2 = p_x + i\,p_y$. Note that $u = u(z_1, z_2, \overline{z}_1, \overline{z}_2)$ is a smooth complex valued function near 0 and $u(0) = 0$. Consequently,

$$\overline{\alpha} z_2 + \beta \overline{z}_1 + \alpha \overline{z}_2 + \overline{\beta} z_1 = u + \overline{u} = z_2 v_2 + \overline{z}_1 \overline{v}_1 + \overline{z}_2 \overline{v}_2 + z_1 v_1, \tag{33}$$

for some smooth complex valued functions v_1, v_2 on Ω. Use Taylor's theorem with integral remainder and the fact that $u(0) = 0$ to prove the second equality in (33). Set $\alpha = \overline{v}_2$ and $\beta = v_1$ solves (33). Then $a = \mathrm{Re}\, v_2$, $b = -\mathrm{Im}\, v_2$, $c = \mathrm{Re}\, v_1$, and $d = \mathrm{Im}\, v_1$ solves (32).

b) Let $A = \Psi \circ \mathrm{d}R$, where $R \in \mathscr{O}(2)$. Here $\Psi(u) = Y(u)$, where Y is the solution of (32). Then for every $t \in [0, 1]$ the linear operator $\mathrm{id} - tA : \mathscr{X}(\Omega) \to \mathscr{X}(\Omega)$ is invertible for a suitable open neighborhood Ω of 0 in $\mathbf{R}^4$. The proof of this result goes as follows. From the result a) and the fact that $\mathrm{d}R \in \mathscr{O}(1)$ deduce that the coefficients in the matrix of A with respect to the basis $\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial p_x}, \frac{\partial}{\partial p_y}\}$ are flat at 0. For a suitable Ω deduce that $\sup_{z \in \Omega} \|A(z)\| < \frac{1}{2}$. Conclude that $\mathrm{id} - tA$ is invertible for all $t \in [0, 1]$.

c) (Proof of the focus-focus Morse lemma.) We use Moser's path method to construct the diffeomorphism Φ as the time 1 map of the flow φ_t^X of a time dependent vector field X with time variable $t \in [0, 1]$. Consider the interpolation

$\mathscr{H}_t = (1-t)Q + tH$, where $Q = (q_1, q_2)$ and $H = (h_1, h_2)$. We want the vector field X to satisfy

$$(\varphi_t^X)^* \mathscr{H} = Q. \tag{34}$$

Differentiating (34) with respect to t gives

$$\begin{aligned} 0 &= (\varphi_t^X)^* \big[\frac{\partial \mathscr{H}_t}{\partial t} + L_{X_t} \mathscr{H}_t \big], \text{ where } \mathscr{H}_t(x, y, p_x, p_y) = \mathscr{H}(x, y, p_x, p_y, t) \text{ and} \\ & \qquad\qquad\qquad\qquad\qquad\quad X_t(x, y, p_x, p_y) = X(x, y, p_x, p_y, t) \\ &= (\varphi_t^X)^* \big[-Q + H - X_t \lrcorner \, \mathrm{d}\mathscr{H}_t \big]. \end{aligned}$$

Thus we need to find an open neighborhood Ω of 0 in $\mathbf{R}^4$, where we can solve

$$\mathrm{d}\mathscr{H}_t(X_t) = Q - H = R \in \mathscr{O}(2). \tag{35}$$

Let $A = \Psi \circ \mathrm{d}R$, where $\Psi(u) = Y(u)$ is the vector field constructed in a). By b) we can shrink Ω so that the operator $\mathrm{id} - tA : \mathscr{X}(\Omega) \to \mathscr{X}(\Omega)$ is invertible for all $t \in [0, 1]$. For every $V \in \mathscr{X}(\Omega)$ we have

$$\mathrm{d}Q(A(V)) = \mathrm{d}Q(\Psi(\mathrm{d}R(V))) = \mathrm{d}R(V). \tag{36}$$

Let $X(x, \xi, t) = (\mathrm{id} - tA)^{-1} \Psi(R)$. Justify each step in the following calculation.

$$\begin{aligned} \mathrm{d}(\mathscr{H}_t)(X_t) &= \mathrm{d}Q(X_t) - t\, \mathrm{d}R(X_t) = \mathrm{d}Q(X_t) - t\, \mathrm{d}Q(A(X_t)) \\ &= \mathrm{d}Q\big((\mathrm{id} - tA)X\big) = \mathrm{d}Q\big(\Psi(R)\big) = R. \end{aligned}$$

So X_t solves (35). Since $X_t(0) = 0$ we can shrink Ω so that the flow φ_t^X is defined for every $t \in [0, 1]$. Thus $\Phi = \varphi_1^X = \mathrm{id} + \mathscr{O}(1)^2$ is a local diffeomorphism of $(\mathbf{R}^4, 0)$ into itself. Moreover, $Q = \Phi^* \mathscr{H}_1 = \Phi^* H$, as desired.

9. (Monodromy and Seifert manifolds.) Consider the circle S^1 defined by $j^2 + h^2 = \frac{1}{4}$ in the image of the integral map F of the particle in a champagne bottle potential, see exercise 2. Show that S^1 lies in the set of regular values of F.

a) Show that $F^{-1}(S^1)$ is a smooth compact connected orientable 3-manifold M^3.

b)* Let $\pi : V^2 \to M^3 \to S^1$ be a bundle with fiber V^2 a compact orientable surface of genus g, total space M^3 a compact orientable 3-manifold and base space a circle S^1. Let A be the $2g \times 2g$ matrix induced on $\mathrm{H}_1(M^3, \mathbf{Z})$ by the monodromy map of the bundle. Show that $\mathrm{H}_1(V^2, \mathbf{Z})$ is isomorphic to $\mathbf{Z} \oplus \mathrm{coker}\,(A - I)$. When π is the bundle $V^2 \to M^3 \xrightarrow{F} S^1$ with F the energy momentum map given in exercise 2, show that V^2 is diffeomorphic to a 2-torus.

c)* Show that the flow of the angular momentum vector field in exercise 2 allows one to realize $F^{-1}(S^1)$ as the total space M^3 of a principal circle bundle $\rho : S^1 \to M^3 \mapsto W^2$ with base space a smooth compact connected 2-manifold W^2. Show that $\mathrm{H}_1(W^2, \mathbf{Z})$ is isomorphic to $\mathbf{Z}/e\mathbf{Z} \oplus \mathbf{Z}$. Compute the Euler class e of the bundle ρ. Show that W^2 is diffeomorphic to a 2-torus. Obtain this result using invariant theory.

d) Identify M^3 as a Siefert manifold.

10. Give an example of a T^3-bundle π over S^2 which is not trivial. Is there an integrable Hamiltonian system on $(T\mathbf{R}^3, \omega)$, whose momentum mapping has $0 \in \mathbf{R}^3$ as an isolated critical value, which gives rise to the bundle π?

    a)* Describe the image of the energy momentum mapping.

    b)* Describe the topology of the fibers of the energy momentum mapping.

    c)* Do global action angle coordinates exist for this problem?

Chapter XI

Basic Morse theory

In this chapter we give an introduction to basic Morse theory. We define the notion of the Hessian of a smooth function at a critical point on a smooth manifold and show that if it is nondegenerate then there are local coordinates in which the function is equal to its second derivative. We also prove the Morse isotopy lemma which gives a criterion when two suitable level sets of a smooth function are diffeomorphic. We conclude the chapter by extending the notion of nondegenerate critical point to a nondegenerate critical submanifold.

1 Preliminaries

In this section we review the concepts of a smooth function and a smooth manifold in $\mathbf{R}^n$. We show that if 0 is a regular value of a smooth function then its zero level set is a smooth manifold. We prove the Lagrange multiplier criterion satisfied by the critical points of a real valued smooth function restricted to a manifold which is a level set of a smooth function.

Let $U \subseteq \mathbf{R}^m$ be an open set. The function $f : U \subseteq \mathbf{R}^m \rightarrow \mathbf{R}^k$ is *smooth* if for $x \in U$ and every $n \in \mathbf{Z}_{\geq 0}$, the n^{th} derivative $D^n f(x)$ of f at x exists. If $V \subseteq \mathbf{R}^m$ is a closed set, then $f : V \subseteq \mathbf{R}^m \rightarrow \mathbf{R}^k$ is *smooth* if there is an open set U containing V and a smooth function $F : U \subseteq \mathbf{R}^m \rightarrow \mathbf{R}^k$ such that $F|V = f$. A smooth function $f : V \subseteq \mathbf{R}^m \rightarrow \mathbf{R}^k$ is a *diffeomorphism* if it is one to one and its inverse is smooth on $f(V)$.

A set $M \subseteq \mathbf{R}^n$ is a k-dimensional *smooth manifold* if for every $m \in M$ there is an open set $U \subseteq \mathbf{R}^n$ containing m and a diffeomorphism $\varphi : U \cap M \rightarrow V \subseteq \mathbf{R}^k$. The pair (U, φ) is called a *coordinate chart at m* and the pair (V, φ^{-1}) is called a *local parametrization of M at m*.

Claim: Suppose that $F : \mathbf{R}^m \times \mathbf{R}^k \rightarrow \mathbf{R}^k$ is a smooth mapping with $F(0,0) = 0$ and $D_2F(0,0)$ is an isomorphism. Then there is an open set U in $\mathbf{R}^m$ containing 0, an open set V in $\mathbf{R}^k$ containing 0, and a smooth function $g : U \subseteq \mathbf{R}^m \rightarrow V \subseteq \mathbf{R}^k$ such that

$$F^{-1}(0) \cap (U \times V) = (\operatorname{gr} g)(U) = \big\{ (u, g(u)) \in \mathbf{R}^m \times \mathbf{R}^k \, \big| \, u \in U \big\}.$$

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Here $\operatorname{gr} g$ is the graph of g. Restated, the conclusion says that $F(u,g(u)) = 0$ for every $u \in U$, that is, locally $F^{-1}(0)$ is the graph of a smooth function.

(1.1) **Proof**: Define the mapping

$$H : \mathbf{R}^m \times \mathbf{R}^k \to \mathbf{R}^m \times \mathbf{R}^k : (z,w) \mapsto \big(z, F(z,w)\big) = (x,y).$$

Then $H(0,0) = (0,0)$ and $DH(x,y) = \begin{pmatrix} I & 0 \\ D_1F(x,y) & D_2F(x,y) \end{pmatrix}$. Since $D_2F(0,0)$ is a linear isomorphism, $DH(0,0)$ is a linear isomorphism of $\mathbf{R}^m \times \mathbf{R}^k$. By the inverse function theorem, H is a local diffeomorphism of $\mathbf{R}^m \times \mathbf{R}^k$, that is, there is an open set U' in $\mathbf{R}^m$ containing 0 and an open set V' in $\mathbf{R}^k$ containing 0 such that the open set $U' \times V'$ is mapped onto $U \times V$ by H and on $U \times V$ the function H has a smooth inverse. Let

$$\pi_2 : \mathbf{R}^m \times \mathbf{R}^k \to \mathbf{R}^k : (x,y) \mapsto y$$

be projection on the second factor. By definition of H we have $\pi_2 \circ H(z,w) = F(z,w)$. For $(x,y) \in U \times V$ the mapping H^{-1} is defined. Therefore $y = \pi_2(x,y) = F \circ H^{-1}(x,y)$, where $H^{-1}(x,y) = (z,w)$. In other words, after changing coordinates by H^{-1}, the function F is locally the projection π_2. Write $H^{-1}(x,y) = \big(h_1(x,y), h_2(x,y)\big)$. Using the definition of H it follows that

$$(x,y) = H \circ H^{-1}(x,y) = \big(h_1(x,y), F(h_1(x,y), h_2(x,y))\big). \tag{1}$$

Hence $h_1(x,y) = x \in U$. Setting $y = 0$ in (1) we obtain $0 = F\big(x, h_2(x,0)\big)$ for every $x \in U$. Let $g : U \subseteq \mathbf{R}^m \to V \subseteq \mathbf{R}^k : u \to h_2(u,0)$. Then for every $u \in U$ we have $F(u,g(u)) = 0$, that is, $F^{-1}(0) \cap (U \times V) = \operatorname{gr} g$. Since H^{-1} is differentiable, g is also. $\square$

Corollary 1: If $F : \mathbf{R}^{m+k} \to \mathbf{R}^k$ is smooth with $F(0) = 0$ and $DF(0) : \mathbf{R}^{m+k} \to \mathbf{R}^k$ is surjective, then near 0 the set $F^{-1}(0)$ is the graph of a smooth function.

Another way to state the conclusion of the above corollary is that the zero level set of F near 0 is an m-dimensional manifold, if $DF(0)$ is surjective.

Let $F : \mathbf{R}^{m+k} \to \mathbf{R}^k$ be a smooth mapping. If $DF(x)$ is not surjective, then x is a *critical point* of F. If for every $x \in F^{-1}(z)$, the derivative $DF(x)$ is surjective, then z is a *regular value* of F. Note that if z is not in the image of F, then z is a regular value. If z is *not* a regular value, then z is a *critical value* of F.

Corollary 2: If z is a regular value of the smooth mapping $F : \mathbf{R}^{m+k} \to \mathbf{R}^k$, then $F^{-1}(z)$ is an m-dimensional manifold.

Example 1: Let $f : \mathbf{R}^3 \to \mathbf{R} : (x,y,z) \to x^2 + y^2 + z^2 - 1$. Then $Df : \mathbf{R}^3 \to (\mathbf{R}^3)^* : (x,y,z) \mapsto 2(x,y,z)$ is not surjective if and only if $(x,y,z) = (0,0,0)$. But $(0,0,0) \notin f^{-1}(0)$. Thus 0 is a regular value of f. Hence $f^{-1}(0)$ is a 2-dimensional manifold called the unit *2-sphere*. $\square$

Example 2: Let $\mathrm{O}(n)$ be the set of all $n \times n$ real matrices A such that $AA^t = I$. Then $\mathrm{O}(n)$ is the set of all orthogonal matrices. We now show that $\mathrm{O}(n)$ is a smooth manifold. Let $F : \mathrm{gl}(n,\mathbf{R}) \to \mathrm{Sym}_n(\mathbf{R}) : A \mapsto AA^t - I$, where $\mathrm{Sym}_n(\mathbf{R})$ is the set of all $n \times n$ real *symmetric* matrices. Differentiating F gives $DF : \mathrm{gl}(n,\mathbf{R}) \rightarrow L\big(\mathrm{gl}(n,\mathbf{R}), \mathrm{Sym}_n(\mathbf{R})\big)$ with $DF(A)B = AB^t + BA^t$, where $B \in \mathrm{gl}(n,\mathbf{R})$. Suppose that $F(A) = 0$ and let $C \in \mathrm{Sym}_n(\mathbf{R})$. Then $B = \frac{1}{2}CA$ solves the equation $DF(A)B = C$. Therefore, for every $A \in F^{-1}(0)$ the derivative of F at A is surjective, that is, 0 is a regular value of the mapping F. Thus $\mathrm{O}(n)$ is a manifold of dimension $\frac{1}{2}n(n-1)$. □

Claim: Let $g : \mathbf{R}^m \times \mathbf{R}^k \rightarrow \mathbf{R}$ be a smooth function and let 0 be a regular value of the smooth function $f : \mathbf{R}^m \times \mathbf{R}^k \rightarrow \mathbf{R}^k$. If $x \in f^{-1}(0)$ is a critical point of $g|f^{-1}(0)$, then for some linear function $\lambda : \mathbf{R}^k \rightarrow \mathbf{R}$,

$$Dg(x) + \lambda Df(x) = 0. \tag{2}$$

(1.2) **Proof**: We assume that $D_2 f(x)$ is an isomorphism. Let $\varphi : \mathbf{R}^m \rightarrow \mathbf{R}^k$ be a smooth function such that $\mathrm{gr}\,\varphi : U \subseteq \mathbf{R}^m \rightarrow f^{-1}(0) \cap (U \times \varphi(U)) \subseteq \mathbf{R}^m \times \mathbf{R}^k$ with $\mathrm{gr}\,\varphi(0) = (0, \varphi(0)) = x$ is a local parametrization of $f^{-1}(0)$ at x. Look at the function $g \circ \mathrm{gr}\,\varphi : U \subseteq \mathbf{R}^m \rightarrow \mathbf{R} : y \mapsto g(y, \varphi(y))$. Because $\mathrm{gr}\,\varphi$ is a local diffeomorphism and $Dg(x)$ is not surjective, the function $g \circ \mathrm{gr}\,\varphi$ has a critical point at 0, that is,

$$0 = D(g \circ \mathrm{gr}\,\varphi)(0) = D_1 g(0, \varphi(0)) + D_2 g(0, \varphi(0)) D\varphi(0) = D_1 g(x) + D_2 g(x) D\varphi(0). \tag{3}$$

For every $y \in U$, we have $f(y, \varphi(y)) = 0$, which differentiated and evaluated at 0 gives $D_1 f(x) + D_2 f(x) D\varphi(0) = 0$. But $D_2 f(x)$ is an isomorphism. So we obtain $D\varphi(0) = -\big(D_2 f(x)\big)^{-1} \circ D_1 f(x)$, which substituted into (3) gives

$$0 = D_1 g(x) - \big(D_2 g(x) \circ (D_2 f(x))^{-1}\big) D_1 f(x). \tag{4}$$

Define a linear map $\lambda : \mathbf{R}^k \rightarrow \mathbf{R}$, by $\lambda = -D_2 g(x) \circ \big(D_2 f(x)\big)^{-1}$ called the *Lagrange multiplier*. Then (4) and the definition of λ may be written as $0 = Dg(x) + \lambda Df(x)$, because

$$Dg(x) + \lambda Df(x) = \big(D_1 g(x) + \lambda D_1 f(x), D_2 g(x) + \lambda D_2 f(x)\big) = (0,0). \qquad \square$$

Example 3. Let $(\,,\,)$ be the Euclidean inner product on $\mathbf{R}^{n+1}$ and let $S : \mathbf{R}^{n+1} \rightarrow \mathbf{R} : x \mapsto (x,x) - 1$. Because 0 is a regular value of S, the unit n-sphere S^n is the manifold $S^{-1}(0)$. Consider the function $Q : \mathbf{R}^{n+1} \rightarrow \mathbf{R} : x \mapsto (x, Ax)$, where $A : \mathbf{R}^{n+1} \rightarrow \mathbf{R}^{n+1}$ is linear and symmetric, that is, for every $y, z \in \mathbf{R}^{n+1}$, $(y, Az) = (Ay, z)$. Because $S^{-1}(0)$ is compact, $Q|S^n$ has a critical point, say x. Using Lagrange multipliers ((1.2)), we find that x satisfies

$$\begin{cases} 0 = DQ(x) - \lambda DS(x) \\ 0 = S(x). \end{cases}$$

Since $DQ(x)h = 2(x, Ah)$ and $DS(x)h = 2(x,h)$, we see that for every $h \in \mathbf{R}^{n+1}$

$$\begin{cases} 0 = (x, Ah) - \lambda(x,h) = ((A-\lambda)x, h) \\ 0 = (x,x) - 1, \end{cases}$$

which implies $(A - \lambda)x = 0$ and $(x,x) = 1$. Thus x is an eigenvector of A of unit length. If x is a unit length eigenvector of A corresponding to a simple real eigenvalue λ, $\pm x$ is a critical point of $Q|S^n$. More generally, if λ is an eigenvalue of A of geometric multiplicity $m+1 \leq n+1$, that is, the dimension of the space of eigenvectors V_λ corresponding to the eigenvalue λ is $m+1$, then $Q|S^n$ has a critical set which is a unit m-sphere $S^m \subseteq V_\lambda$. □

2 The Morse lemma

In this section, we prove the Morse lemma. We define the Hessian of a function on a manifold at a critical point and give a formula for computing it using Lagrange multipliers.

Suppose that x is a critical point of a smooth function $f : \mathbf{R}^m \rightarrow \mathbf{R}$. The Hessian of f at x is the symmetric bilinear mapping $D^2f(x)$ where $D^2f : \mathbf{R}^m \rightarrow \mathrm{Sym}_m(\mathbf{R}) : x \mapsto \big(\frac{\partial^2 f}{\partial x_i \partial x_j}\big)$. If the $m \times m$ matrix $D^2f(x)$ is invertible, then x is a *nondegenerate* critical point of f. If all the critical points of f are nondegenerate, then f is a *Morse function*. We now prove the *Morse lemma*.

Claim: Suppose that 0 is a nondegenerate critical point of a smooth function $f : \mathbf{R}^m \rightarrow \mathbf{R}$ with $f(0) = 0$. Then there is an open set $U \subseteq \mathbf{R}^m$ with $0 \in U$ and a diffeomorphism $\varphi : U \subseteq \mathbf{R}^m \rightarrow \mathbf{R}^m$ with $\varphi(0) = 0$ such that $f \circ \varphi^{-1}(y) = \frac{1}{2} D^2f(0)(y,y)$ for every $y \in \varphi(U)$.

(2.2) **Proof**: We start by computing the Taylor series of f at 0 to second order with integral remainder:

$$\begin{aligned} f(x) &= \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t} f(tx)\ \mathrm{d}t, \quad \text{by the fundamental theorem of calculus} \\ & \qquad\qquad\qquad\qquad\ \ \text{and the fact that } f(0) = 0 \\ &= -\frac{\mathrm{d}}{\mathrm{d}t} f(tx)(1-t)\Big|_0^1 + \int_0^1 (1-t) \frac{\mathrm{d}^2}{\mathrm{d}t^2} f(tx)\ \mathrm{d}t, \\ & \qquad\qquad \text{by partial integration} \\ &= Df(0)x + \Big(\int_0^1 (1-t) D^2 f(tx)\ \mathrm{d}t \Big)(x,x) \\ &= B(x)(x,x), \end{aligned}$$

where $B(x)$ is the quadratic function $\int_0^1 (1-t)D^2f(tx)\ \mathrm{d}t$. The last equality above follows because $Df(0) = 0$. Note that $B(0) = \frac{1}{2}D^2f(0)$ is invertible because 0 is a nondegenerate critical point of the function f.

Let $\varphi(x) = R(x)x$, where $R(x)$ is an $m \times m$ matrix depending smoothly on x such that $R(0) = I$. We must determine $R(x)$ so that

$$B(x)(x,x) = f(x) = B(0)\big(R(x)x, R(x)x\big) \tag{5}$$

for all x in some open neighborhood U of 0. Towards this goal let

$$F : \mathbf{R}^m \times \mathrm{gl}(m,\mathbf{R}) \rightarrow \mathrm{Sym}_m(\mathbf{R}) : (x,R) \mapsto R^t B(0) R - B(x). \tag{6}$$

Then $F(0,I) = 0$. Differentiating F with respect to R and evaluating at $(0,I)$ gives $D_2F(0,I)S = S^t B(0) + B(0)S$. $D_2F(0,I)$ is surjective because for any given $C \in \mathrm{Sym}_m(\mathbf{R})$, choose $S = \frac{1}{2}B(0)^{-1}C$. We see that $D_2F(0,I)S = C$. Therefore, by ((1.1)), there is an open neighborhood U of 0 in $\mathbf{R}^m$ and a differentiable function $R : U \subseteq \mathbf{R}^m \rightarrow \mathrm{gl}(m,\mathbf{R}) : x \mapsto R(x)$ with $R(0) = I$ such that $F(x,R(x)) = 0$ for every $x \in U$. Shrinking U, if needed, we may assume that $R(x)$ is invertible for every $x \in U$. Thus φ is a diffeomorphism of U. □

Let M be a smooth manifold and $F : M \rightarrow \mathbf{R}$ a smooth function. Suppose that $p \in M$ is a critical point of F. Define the *Hessian* of F at p as follows. Let $\varphi : U \subseteq \mathbf{R}^m \rightarrow M$ with

$\varphi(0) = p$ be a local parametrization of M. Then the Hessian of F at p is the symmetric bilinear form on the *tangent space* T_pM given by

$$(\mathrm{Hess}_p F)(v_p, w_p) = D^2(F\circ\varphi)(0)(v,w) \tag{7}$$

The curves $t \to \varphi(tv)$ and $t \to \varphi(tw)$ represent the *tangent vectors* v_p and w_p to M at p.

We now look at what happens to (7) when we change the local parametrization. Suppose that $\widetilde{\varphi} : U \subseteq \mathbf{R}^m \to M$ with $\widetilde{\varphi}(0) = p$ is another local parametrization for M. Let $\psi = \varphi^{-1}\circ\widetilde{\varphi}$. Then

$$\begin{aligned} D^2(F\circ\widetilde{\varphi})(0) &= D^2\big((F\circ\varphi)\circ\psi\big)(0) = D\big(D(F\circ\varphi)(\psi(0))\cdot D\psi(0)\big) \\ &= D^2(F\circ\varphi)(\psi(0))\big(D\psi(0), D\psi(0)\big) + D(F\circ\varphi)(\psi(0))\cdot D^2\psi(0) \\ &= D^2(F\circ\varphi)(0)\big(D\psi(0), D\psi(0)\big), \end{aligned} \tag{8}$$

where the second equality above follows by the chain rule, the third equality because $\cdot$ is bilinear and the last equality because $\psi(0) = 0$ and $D(F\circ\varphi)(\psi(0)) = 0$, since p is a critical point of F. Thus

$$D^2(F\circ\widetilde{\varphi})(0)(v,w) = D^2(F\circ\varphi)(0)\big(D\psi(0)v, D\psi(0)w\big). \tag{9}$$

Here the curves $t \to \varphi(tD\psi(0)v)$ and $t \to \varphi(tD\psi(0)w)$ represent the tangent vectors v_p and w_p in the local parametrization $\widetilde{\varphi}$. From (9) we see that the invariants of the Hessian such as its nondegeneracy (= its invertibility), its *Morse index* (= its number of negative eigenvalues) and *signature* (= the difference of its number of positive and negative eigenvalues), do not depend on the choice of parametrization.

Claim: Let 0 be a regular value of the smooth function $f : \mathbf{R}^n \to \mathbf{R}^k$ and suppose that $g : \mathbf{R}^n \to \mathbf{R}$ is a smooth function. If $x \in f^{-1}(0)$ is a critical point of $g|f^{-1}(0) : f^{-1}(0) \subseteq \mathbf{R}^n \to \mathbf{R}$ with Lagrange multiplier $\lambda : \mathbf{R}^k \to \mathbf{R}$, that is,

$$Dg(x) + \lambda\circ Df(x) = 0, \tag{10}$$

then the Hessian of $g|f^{-1}(0)$ at x is

$$\big(D^2g(x) + \lambda\circ D^2f(x)\big)\Big|T_xf^{-1}(0), \tag{11}$$

where $T_x\big(f^{-1}(0)\big) = \ker Df(x)$.

(2.3) **Proof**: Let $\varphi : U \subseteq \mathbf{R}^{n-k} \to f^{-1}(0)$ with $\varphi(0) = x$ be a local parametrization of $f^{-1}(0)$ at x. Then

$$D^2(g\circ\varphi)(0) = D^2g(x)\big(D\varphi(0), D\varphi(0)\big) + Dg(x)D^2\varphi(0), \tag{12}$$

using (8). Since φ is a local parametrization of $f^{-1}(0)$, for every $y \in U$ we have

$$\lambda\circ f(\varphi(y)) = 0. \tag{13}$$

Differentiating (13) twice and evaluating at 0 gives

$$\begin{aligned} 0 &= D^2((\lambda\circ f)\circ\varphi)(0) \\ &= D^2(\lambda\circ f)(x)\big(D\varphi(0),D\varphi(0)\big)+D(\lambda\circ f)(x)\,D^2\varphi(0), \\ &= D^2(\lambda\circ f)(x)\big(D\varphi(0),D\varphi(0)\big)-Dg(x)\,D^2\varphi(0). \end{aligned} \tag{14}$$

The second equality above follows from (8). Equation (14) follows because the map λ is linear, which implies that $D(\lambda\circ f)(x)=\lambda\circ Df(x)$, and because of (10). Adding (12) and (14) gives

$$D^2(g\circ\varphi)(0)(v,w)=D^2(g+\lambda\circ f)(x)(D\varphi(0)v,D\varphi(0)w).$$

Since the linear map $D\varphi(0): T_0\mathbf{R}^{n-k}\to T_x\big(f^{-1}(0)\big)$ is an isomorphism, we obtain

$$\mathrm{Hess}_x\, g|f^{-1}(0)=D^2(g+\lambda\circ f)(x)\Big|T_x\big(f^{-1}(0)\big). \qquad \square$$

Example 3 (continued): From (11) it follows that the Hessian of $Q|S^n$ at the critical point x is

$$\big(D^2Q(x)+\lambda D^2S(x)\big)\Big|\ker DS(x)=2(A-\lambda)|\,\langle x\rangle^{\perp},$$

where $\langle x\rangle^{\perp}=\{y\in\mathbf{R}^{n+1}\,|\,(y,x)=0\}$. Since $\ker(A-\lambda)|\,\langle x\rangle^{\perp}$ has dimension $\dim V_{\lambda}-1$, x is a nondegenerate critical point of $Q|S^n$ if and only if λ is a simple real eigenvalue of A. Suppose that all the eigenvalues λ_i for $i=1,\dots,n+1$ of A are real and simple and that $\lambda_{n+1}>\lambda_n>\cdots>\lambda_1$. Let x_i be unit eigenvectors corresponding to the eigenvalues λ_i. The Hessian of $Q|S^n$ at $\pm x_i$ with respect to the basis $\{e_1,e_2,\dots\widehat{e_i},\dots,e_{n+1}\}$ is the $n\times n$ matrix $\mathrm{diag}\,(\lambda_{n+1}-\lambda_i,\dots,\lambda_{i+1}-\lambda_i,\,\lambda_{i-1}-\lambda_i,\,\dots,\lambda_1-\lambda_i)$, which is invertible and has $i-1$ negative eigenvalues. In other words, $\pm x_i$ is a nondegenerate critical point of $Q|S^n$ of Morse index $i-1$. Hence $Q|S^n$ is a Morse function with $2(n+1)$ critical points. $\square$

Suppose that $f: M\subseteq\mathbf{R}^m\to\mathbf{R}$ is a smooth function with a critical set P which is a manifold. We say that P is a *nondegenerate critical manifold* of f if

1. $T_pP\subseteq\ker D^2f(p)$ for every $p\in P$.
2 $D^2f(p)|N_pP$ is nondegenerate for every $p\in P$, where N_pP is any subspace of T_pM such that $T_pM=T_pP\oplus N_pP$.

In other words, the tangent space to P at the point p is contained in the kernel of the Hessian of f at p for every $p\in P$ and the Hessian of f restricted to a normal space at $p\in P$ in M is nondegenerate. A smooth function whose critical set is made up of nondegenerate critical submanifolds is called a *Bott-Morse function*.

Example 3 (continued). Suppose that λ_i is an eigenvalue of A of multiplicity $m+1<n+1$. List the eigenvalues of A in a nonincreasing order with a mulitiple eigenvalue being listed as many times as its multiplicity. Then the m-sphere $\big\{x\in\mathbf{R}^{n+1}\,\big|\,(A-\lambda_i)x=0 \text{ and } (x,x)=1\big\}$ is a critical manifold of $Q|S^n$. For $x\in S^m$ let $N_xS^m=V_{\lambda_i}{}^{\perp}$. Then the Hessian of $Q|N_xS^m$ is the $(n-m)\times(n-m)$ matrix $\mathrm{diag}\,(\lambda_{n+1}-\lambda_i,\dots,\lambda_{i+1}-\lambda_i,\,\lambda_{i-1}-\lambda_i,\dots\lambda_1-\lambda_i)$, which is invertible and has Morse index $n-m-j$ where j is the first occurrence of the eigenvalue λ_i. Therefore S^m is a nondegenerate critical submanifold of Q. Hence Q is a Bott-Morse function on S^n. $\square$

3 The Morse isotopy lemma

In this section we prove the Morse isotopy lemma, which gives a useful criterion when two regular level sets of a smooth function are diffeomorphic.

Claim: Let $f : M \to \mathbf{R}$ be a proper smooth function. If $[a,b]$ is a nonempty closed interval contained in the set of regular values in the image of f, then $f^{-1}(a)$ is smoothly *isotopic* to $f^{-1}(b)$, that is, there is a smooth one parameter family of diffeomorphisms $\psi : [0,1] \times M \to M$ such that for every $t \in [0,1]$, $\psi_t : f^{-1}(a) \to f^{-1}(a+t(b-a))$ is a diffeomorphism.

(3.1) **Proof**: The idea of the proof is to push the level set $f^{-1}(a)$ to the level set $f^{-1}(b)$ along the integral curves of the gradient vector field associated to f.

First we define the gradient vector field associated to f. Using a partition of unity, we can define a smooth Riemannian metric $\langle \, , \rangle$ on the manifold M. Since the metric is nondegenerate, the mapping $\langle \, , \rangle^{\sharp} : T_mM \to T_m^*M : v_m \mapsto \langle v_m, \cdot \rangle$ is an isomorphism for every $m \in M$. Making use $\langle \, , \rangle^{\sharp}$ we convert the 1-form $\mathrm{d}f : M \to T^*M$ to a vector field X^f on M, called the *gradient vector field* associated to f. Explicitly, X^f is defined by $\mathrm{d}f(m)v_m = \langle X^f(m), v_m \rangle$ for every $m \in M$ and $v_m \in T_mM$. Suppose that a is a regular value of f. Then at every $m \in f^{-1}(a)$, we have $X^f(m) \in T_m\big(f^{-1}(a)\big)^{\perp}$. This follows because a is a regular value of f implies that $T_mf^{-1}(a) = \ker \mathrm{d}f(m)$. Hence $0 = \mathrm{d}f(m)v_m = \langle X^f(m), v_m \rangle$ for every $v_m \in T_mf^{-1}(a)$.
Because f is a proper mapping, $f^{-1}([a,b]) = K$ is a compact subset of M. Since $[a,b]$ contains no critical values of f, the gradient vector field X^f does not vanish on K. Let $\widetilde{X}(m) = \langle X^f(m), X^f(m) \rangle^{-1} X^f(m)$. Let U be an open subset of M containing K whose closure is compact. Using a partition of unity, there is a smooth function $\rho : M \to \mathbf{R}$, which is 1 on K and 0 outside U. Consider the vector field $X = \rho \widetilde{X}$. Since X is a smooth vector field on M which vanishes outside a compact subset, it is *complete*, that is, its flow $\varphi : \mathbf{R} \times M \to M$ is a one parameter group of diffeomorphisms of M. Now

$$\frac{\mathrm{d}}{\mathrm{d}t} f(\varphi_t(m)) = \mathrm{d}f(\varphi_t(m))X(\varphi_t(m)) = \mathrm{d}f(\varphi_t(m))\rho(\varphi_t(m))\widetilde{X}(\varphi_t(m)) = \rho(\varphi_t(m)). \quad (15)$$

Suppose that $\varphi_t(m) \in K$, that is, $f(\varphi_t(m)) \in [a,b]$. From the definition of ρ it follows that

$$\frac{\mathrm{d}}{\mathrm{d}t} f(\varphi_t(m)) = 1. \quad (16)$$

Integrating (16) gives $t \mapsto f(\varphi_t(m)) = t + f(m)$. Therefore $\varphi_{b-a} : M \to M$ is a diffeomorphism which maps $f^{-1}(a)$ onto $f^{-1}(b)$. Moreover

$$\psi : [0,1] \times M \to M : (t,m) \mapsto \varphi_{a+t(b-a)}(m)$$

is a smooth isotopy of $f^{-1}(a)$ onto $f^{-1}(b)$. □

We now apply the Morse lemma and the Morse isotopy lemma to prove a characterization of the n-sphere.

Claim: Let M be a compact smooth n-dimensional manifold without a boundary. Suppose that f is a Morse function on M with two critical points both of which are nondegenerate. Then M is homeomorphic to the n-sphere S^n.

(3.2) **Proof**: Let $p_\pm$ be the critical points of f. We may assume that f has a maximum value c_+ at the point p_+ and a minimum value c_- at the point p_-. By the Morse lemma, there are coordinates $(x_1,\ldots,x_n)$ in an open neighborhood U_+ of p_+ with 0 corresponding to p_+ such that

$$f|U_+(x_1,\ldots,x_n) = c_+ - x_1^2 - \cdots - x_n^2.$$

Thus there is a $b < c_+$ such that $f^{-1}([b,c_+]) = \overline{D}_+$ is a neighborhood of p_+ which is diffeomorphic to a closed n-disk. Similarly, there is an $a > c_-$ such that $f^{-1}([c_-,a]) = D_-$ is diffeomorphic to a closed n-disk containing p_-. We may assume that $a < b$. Note that $\partial\overline{D}_+$ and $\partial\overline{D}_-$ are diffeomorphic to an $(n-1)$-sphere S^{n-1}. Because the function f restricted to $f^{-1}([a,b])$ is proper and has no critical values, it follows from the Morse isotopy lemma that $f^{-1}([a,b])$ is diffeomorphic to $S^{n-1}\times[a,b]$.

Let $S^n \subseteq \mathbf{R}^{n+1}$ be the standard unit n-sphere in Euclidean $n+1$-space with $q_\pm$ its north and south poles respectively. Let $V_\pm$ be polar cap open neighborhoods of $q_\pm$ which are diffeomorphic to open n-disks $B_\pm$, respectively. Set $C = S^n - (B_+ \cup B_-)$. Then C is diffeomorphic to $S^{n-1}\times[0,1]$. Note that $\partial C = \partial\overline{B}_+ \cup \partial\overline{B}_-$. Let $h_0 : \overline{D}_+ \to \overline{B}_+$ be a diffeomorphism of the closed n-disk. Extend the diffeomorphism $h_0|\partial\overline{D}_+ : \partial\overline{D}_+ \to \partial\overline{B}_+$ of $(n-1)$-spheres to a diffeomorphism $\widetilde{h}_0 : \partial\overline{D}_+ \times [a,b] \to \partial\overline{B}_+ \times [0,1]$ of cylinders on S^{n-1}. This gives rise to the diffeomorphism $\widehat{h}_0 : f^{-1}([a,b]) \to C$, which extends to a diffeomorphism $h_1 : D_+ \cup f^{-1}([a,b]) \to B_+ \cup C$. Identify $f^{-1}(a)$ with S^{n-1}. Hence we are left with extending the diffeomorphism $\widetilde{h}_2 = h_1|f^{-1}(a) : f^{-1}(a) \to \partial\overline{B}_-$ of S^{n-1} into itself to a homeomorphism h_2 of the closed n-disk $\overline{D}^n$ to $\overline{B}_-$ where $\partial\overline{D}^n = S^{n-1}$. This is done by defining

$$h_2(x) = \begin{cases} |x|\widetilde{h}_2(x/|x|), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0. \end{cases}$$

The function h_2 maps the radial segment $[0,y]$ where $y \in S^{n-1}$ linearly onto the line segment $[0,\widetilde{h}_2(y)]$ in $\overline{D}^n$. The map h_2 extends the diffeomorphism h_1 to a *homeomorphism* between M and S^{n-1}. □

The example of an exotic 7-sphere shows that it is *not* always possible to extend a diffeomorphism of a 7-sphere to a *diffeomorphism* of the 8-disk.

4 Exercise

1. a) Find a Morse function f on real projective 2-space $\mathbf{RP}^2$ with three critical points, one of index 0, 1 and 2.

   b) How many orbits of the gradient vector field of f join critical points of index 0 or 2 with critical points of index 1? Take a sphere in the local unstable manifold of the critical point of index 1. Use the flow of the gradient vector field of f to transport this sphere until it intersects a sphere in the local stable manifold of the critical point of index 0. What is its intersection number? Answer a similar question for a sphere in the local stable manifold of the critical point of index 1.

   c)* Using ideas from b) find a decomposition of $\mathbf{RP}^2$ into cells together with the degrees of their attaching maps. From this compute the cohomology ring of $\mathbf{RP}^2$ with integer coefficients.

Notes

Foreword and introduction

page ix par. 3. *three body problem.* See for instance Xia [298] or [299].

page xv par. 5. *three critical points*. For a construction of a function on a two dimensional torus with three critical points see Banchoff and Takens [18].

Harmonic oscillator

page 3 par. 2 *energy momentum mapping*. This program for analyzing the energy momentum mapping was started by Smale [252] who suggested that studying the regular values was enough. The importance and physical significance of the critical values was missed. Indeed by studying popular texts such as Goldstein [108] one realizes that many physically interesting motions occur precisely at the critical values.

page 7 par. 1 *quadratic integrals*. Our treatment of the $\mathrm{U}(2)$-momentum mapping follows Cushman [55] and Cushman and Rod [68]. In [143] Jauch and Hill observed that in quantum mechanics the degeneracy of the spectrum of the harmonic oscillator Hamiltonian is due to the $\mathrm{U}(2)$ symmetry.

page 12 par. 2 *Hopf mapping*. Dulock and McIntosh in [89] observed that there was a mapping from integral curves of the harmonic oscillator of a fixed energy onto a 2-sphere, each point of which was on a unique such integral curve. They also noted that this orbit map was the Hopf fibration.

page 13 par. 1 *Hopf fibration*. The first paper on the Hopf fibration is that of Hopf [134]. Our definition of the Hopf fibration differs from that defined by Hopf.

page 15 par. 1 *linking number*. There are many equivalent definitions of linking number. For some, which are different from the one used in the text, see Milnor [200].

page 19 par. 2 *classifying map*. For the definition of the classifying map of a bundle with base space a sphere, see Steenrod [266].

page 20 par. 3 *invariant theory*. Arnol'd [10] seems to be the first one to have seen that a Hamiltonian invariant under the flow of the harmonic oscillator induces dynamics on the reduced space which are given by a generalized Euler equation. See Cushman and Rod [68] for a detailed treatment of reduction for such Hamiltonian functions. This result is basic to studying *perturbations* of the harmonic oscillator. See the papers [52] of Churchill, Kummer, and Rod, [130] of Henon and Heiles, [56] of Cushman, [154] of Kummer, and [54] of Cotter.

page 21 par. 3 *Claim* ((5.2)). The proof that there is only one relation follows that of Billera, Cushman and Sanders [40].

page 23 par. 3 *prime ideal*. For definitions of the terms from commutative algebra used in this and the next paragraph see Hungerford [137].

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page 23 par. 2 *Hilbert Nullstellensatz.* For a proof of the Nullstellensatz see Mumford [211].

page 23 par. 3 *Schwarz.* For a proof of Schwarz' theorem on smooth invariants see Schwarz [246] or Mather [195]. In Bierstone [39] one can find a transparent proof.

page 26 par. 0 *linking number*. This exercise is taken from Bott and Tu [45, p. 230].

page 27 par. 1 *Hopf invariant*. This exercise is taken from Bott and Tu [45, p. 228].

page 28 par. 0 *Hopf invariant of Hopf map*. This exercise is a modified version of one in Bott and Tu [45, p. 238] .

page 30 par. 1 *Fubini-Study*. See Arnol'd [12, p. 343] for a derivation of the Fubini-Study Hermitian metric on $\mathbf{CP}^{n-1}$.

Geodesics on $\mathbf{S}^3$

page 32 par. 3 *constrained.* Our discussion of constrained motion follows Moser [210] and Dirac [84].

page 36 par. 0 *dual pairs.* The term dual pairs was introduced by Howe [136]. See Cushman and Sanders [72] for an example of the use of dual pairs in studying perturbations of the geodesic flow on the 3-sphere. The subject of *perturbations* of the geodesic flow on the 3-sphere has a vast literature being essentially the main topic of *celestial mechanics*, see §4 of the text for an explanation of the relation between the Kepler problem and the geodesic flow. A nice example is given in [58] by Cushman. Some classic references are Delaunay [81], Hill [132], Lagrange [160], Poincaré [229], Levi-Civita [172] and Birkhoff [41]. For a survey of normalization techniques in the study of perturbations of the geodesic flow see Cushman [60].

page 40 par. 1 *smooth function.* This is the smooth analogue of the Hilbert Nullstellensatz of algebraic geometry.

page 41 par. 7 *Kepler's problem.* For Kepler's treatment see [148]. For more modern treatments see Guillemin and Sternberg [118] and Cordani [53]. A good historical study of Kepler's astronomical work may be found in Stephenson [267]. For Newton's treatment of the Kepler problem see his *Principia* [217] Book 1, paragraph 3, propositions XI, XII and XIII.

page 43 par. 1 *eccentricity vector.* The term eccentricity vector is due to Duistermaat. It is also called the Laplace-Runge-Lenz vector [162], but actually was discovered by J. Hermann [131] and J. Bernoulli [34]. See Goldstein [109] for a historical discussion. The eccentricity vector was first written down as a vector using quaternions by Hamilton [127] in 1845.

page 48 par. 6 *where on the orbit.* Our discussion of where the particle is at a given time on its elliptical orbit follows Pollard [232]. For another interesting treatment see Souriau [264] who uses an integrated version of the equation defining the eccentric anomaly. This reduces the integration of the Kepler problem to a four dimensional system of linear differential equations.

page 51 par. 1 *regularization.* The process of removing the incompleteness of the flow of the Kepler vector field has a long history. It begins with Levi-Civita [173, 174] in the context of removing binary collisions in the restricted three body problem and is followed by the *spinor* regularization of Kustaanheimo and Stiefel [158]. In the text we use Moser's mapping [209], which is a modification of the tangent of the stereographic projection map, to regularize the bounded Keplerian motion energy level by energy level. This works because of a theorem of Hamilton [126]. Moser's method is essentially the same as the one used in quantum mechanics by Fock [103], Bargmann [20] and Pauli [226], which explains the $\mathrm{SO}(4)$ symmetry in the spectrum of the *hydrogen atom*. See also Bander and Itzykson [19], Bacry [17], and Iosifescu and Scutaru [140]. See Kaplan [147] for an early discussion of the topology of the 2-body problem.

page 51 par. 2 *virial group.* A comprehensive discussion of the relation of the virial group and Kepler's third law may be found in Stephenson [267].

page 56 par. 2 *regularization.* Regularization of the Keplerian motion for all negative energies at once was first done by Souriau [263] and independently by Ligon and Schaaf [181]. See also Cushman and Duistermaat [62] and more recently Marle [190]. Our discussion leans heavily on the paper of Heckman and de Laat [128].

page 69 par. 3 *positive energy.* Some references relating the Kepler problem to the geodesic flow are Belbruno [33], Kummer [156], and Osipov [222].

page 70 par. 2 *Hamilton's theorem.* The original proof of Hamilton's theorem was given by Hamilton [126] in 1847. Other proofs may be found in Milnor [201], Anosov [4], and van Haandel and Heckman [279]. The observation [61] that the arc of the velocity circle traced out by a hyperbolic motion subtends an angle equal to the scattering angle of the hyperbola seems to be new.

page 74 par. 3 *coadjoint orbit.* We note that analogous formulæ (up to $\pm$ signs) hold for the positive energy case as well as the case of a repulsive potential. When you get stuck some good references to consult are Gyorgyi [123] and Souriau [264].

page 75 par. 3 *Kustaanheimo-Stiefel.* Our approach to Kustaanheimo-Stiefel regularization of the Kepler problem follows the treatment of Kummer [157]. See also Kustaanheimo and Stiefel [158] , Baumgarte [32], and van der Meer [278].

Euler top

page 79 par. 2 *Euler top*. Euler [100] was the first to treat the motion of a force free rigid body with one point fixed. We call this rigid body problem the Euler top. More modern treatments can be found in Whittaker [291, p.144–155] and Pars [225, p.216–224].

page 79 par. 2 *geodesics*. The model of the Euler top as a geodesic flow of a left invariant metric on the rotation group is due to Arnol'd [10]. This chapter can be considered as an attempt to make his ideas explicit and provide more detail than Iacob [139]. For generalizations of the Euler top to Lie groups other than the rotation group see Mischenko and Fomenko [205].

page 82 par. 5 *derivative of the exponential function*. The proof of the formula for the derivative of the exponential map is taken from Freudenthal and de Vries [104]. See also Tuynman [275].

page 83 par. 0 *solid ball model*. We know of no good reference for the solid ball model of the rotation group, even though every topologist knows it.

page 85 par. 3 *sphere bundle model*. The structure of the sphere bundle model of the rotation group as a Lie group is not the standard one of a closed subgroup of $\mathrm{Gl}(3)$.

page 88 par. 3 *double covered*. The fact that the bundle projection from the unit tangent sphere bundle of the 2-sphere to the 2-sphere is double covered by the Hopf fibration is vital for understanding the qualitative properties of the energy momentum mapping of the Euler top.

page 95 par. 2 *principal axes*. The Lie theoretic meaning of the principal axis transformation seems to be new.

page 96 par. 2 *Euler-Arnol'd*. The derivation of the Euler-Arnol'd equations in the sphere bundle model seems to be new.

page 112 par. 3 *level sets of H*. The picture in figure 5.2 of the foliation of a level set of angular momentum by energy level sets is new.

page 114 par. 4 *integrate*. The integration of the Euler-Arnol'd equations, which describe the motion of the Euler top in space, is usually done using Euler angles, see Whittaker [291] or Pars [225]. A discussion of the various conventions used for matrices, angular velocities, and Euler angles can be found in Synge [272]. When properly restricted, Euler angles form a chart for the rotation group. At least three such charts are needed to form an atlas for the rotation group. The trouble with the usual approach toward integrating the Euler-Arnol'd equations is that it is not obvious which motions of the top are left out when one uses only one chart. This difficulty is avoided when one uses the sphere bundle model.

page 123 par. 0 *herpolhode.* The projected integral curve was called the herpolhode by Poinsot [230]. It comes from the Greek word *herpes* meaning snake. Poinsot drew a picture of a snakelike herpolhode by which he meant coiled up like a snake. Routh [241, §9 p.472] gives a proof that the herpolhode has no inflection points for a physically realizable Euler top. Thus the herpolhode is actually snakelike. Routh says that Darboux [78] was the first to show this. Whittaker [291] leaves this as an exercise (# 29 on page 174) and refers to Lecornu [165] for a short proof. The data used to compute the herpolhode in figure 7.1 is: the moments of inertia of the body are $I_1 = 1$, $I_2 = 2$, and $I_3 = 2.9$; $|\ell| = 1$; the initial pair of orhonormal vectors $(x, y) = \big(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}, 0, 1, 0\big)$; the initial condition for Euler's equations is $(I_1^{-1}p_1, I_2^{-1}p_2, I_3^{-1}p_3) = \big(\frac{1}{\sqrt{2}}, 0, \frac{1}{2.9\sqrt{2}}\big)$ with energy $h = .344$; the Euler period is 17, and the rotation number is 1.1666.

page 123 par. 0 *second*. This geometric interpretation of the rotation number is basically that given by Arnol'd [10] and Iacob [139]. They both overlooked the fact that two isotropy orbits project onto a single boundary component of the annulus. Thus the rotation number is the angle between every *second* point of contact of the projected integral curve with a fixed boundary component of the annulus and not every point of contact.

page 125 par. 0 *relation.* This angle formula was first proved in Bates, Cushman, and Savev [25]. Another angle theorem can be found in Goodman and Robinson [107]. There is also an interesting account in Zhuravlev [302].

page 127 par. 1 *twisting.* This twisting phenomenon is well known to physicists, although its first published explanation seems to be Ashbaugh, Chicone, and Cushman [15]. Our exposition follows theirs quite closely, except we do not use Euler angles. The solid ball model pictures in figure 8.2 are taken from Cushman and Hoveijn [66]. For a movie of the twisting tennis racket see Murrell [212].

page 130 par. 3 *Jacobi elliptic functions*. This exercise is taken from Tricomi [274].

page 131 par. 0 *Euler's equations*. This exercise is taken from Lawden [164].

page 136 par. 0 *geometric formula.* The geometric formula for the rotation number has a long pedigree. MacCullagh gave a geometric interpretation in [186] with a refinement in [187]. The formula modulo 2π appears in Levi [175] and Montgomery [207]. A splitting similar to the dynamic and geometric phase appears in Poinsot [230]. The text follows Bates, Cushman, and Savev [25].

page 137 par. 0 *phenomenon*. This exercise is taken from Ashbaugh, Chicone, and Cushman [15].

Spherical pendulum

page 139 par. 139 *spherical pendulum.* According to Whittaker [291, p.104], the spherical pendulum was first treated by Lagrange [159] and was first integrated using elliptic functions by Jacobi [141] in 1839. This seems to overlook the fact that Huygens [138] understood the relative equilibria many years earlier than 1673, when he published his results about them. See the discussion in Yoder [300]. A modern analytic treatment of the spherical pendulum can be found in Whittaker [291, p.104–105]. Nice pictures of the orbits may be found in Webster [283].

page 142 par. 1 *singular reduction.* Arms, Cushman and Gotay [7] were the first to apply the technique of singular reduction to the spherical pendulum.

page 143 par. 4 *Claim* ((2.3)). This claim is a special case of the general fact, see Poénaru [228, p.20–21], that the orbit space of a linear action of a compact Lie group on some Euclidean space is homeomorphic to the image of its Hilbert map.

page 146 par. 0 *orbit space*. The model of the zero level set of angular momentum as an orbit space of a two element group on TS^1 may be found in Lerman, Montgomery, and Sjamaar [167].

page 146 par. 2 *smooth functions*. The use of subcartesian differential spaces in describing the smooth functions, the Poisson structure, and vector fields on the singular reduced space can be found in Śniatycki [255].

page 150 par. 1 *energy momentum mapping*. The geometry of the energy momentum map follows the treatment in Duistermaat [86] and Cushman [57]. The relative equilibria were found by Huygens [138]. The importance of their orientation on the different connected components of the set of relative equilibria was noted by Heckman [129]. For a description of the closed orbits on regular tori see Emch [99].

page 165 par. 1 *level set*. The fact that the 1-level set of the energy is a topological 3-sphere is seems to be new.

page 172 par. 0 *Hartog's theorem*. For a proof of Hartog's theorem see Gunning [120, p.15].

page 176 par. 2 *monodromy map*. We give three proofs for the existence of monodromy for the spherical pendulum. The first one due to Cushman [57] only shows that the monodromy is nontrivial. An analytic one due to Duistermaat [86] and a geometric one due to Heckman [129] compute the monodromy matrix. For another proof using *Picard Lefschetz* techniques see Zou [303]. Monodromy is a topological phenomenon which was classically unknown. Presumably this is a consequence of the linguistic curiosity that notions of global geometry require the language of modern geometry. Monodromy can *not* be observed experimentally by examining a single trajectory of the spherical pendulum, since it is a property of the way in which trajectories fit together. It can be observed in a series of experiments where the angular momentum and energy vary along a loop in the energy angular momentum plane which encloses the isolated critical value $(h, j) = (1,0)$. One sees the monodromy as the increase in the angle $\theta(h, j)$ by 2π. See Gavrilov [106] and Beukers and Cushman [37] for a complex analytic treatment of monodromy. In [119] Guillemin and Uribe give a spectral treatment of monodromy.

page 182 par. 2 *Weierstrass.* For more details about Weierstrass elliptic functions see Whittaker and Watson [292].

page 184 par. 2 *formula.* The text follows closely the unpublished manuscript [36] of Beukers.

page 186 par. 2 *Whittaker's formula.* Formula (15) differs from the formula given by Whittaker in [291, p.106] (where $\ell = g = h = 1$ and $a = \lambda$, $b = \mu$) because we have chosen a different complex square root and a different inverse for $\wp$.

page 186 par. 3 *rotation number*. The inequality $\frac{1}{2} < \theta(h,\ell)/2\pi < 1$ for the rotation number in the spherical pendulum was first proved by Puiseux in [234]. Our proof of $\theta(h,\ell) < 2\pi$ follows that of M.-A. de Saint-Germain [244], which in turn is based on an idea of Hadamard [124]. Our proof of $\theta(h,\ell) > \pi$ follows that of Alex Weinstein [284].

page 188 par. 2 *coordinates*. This theorem is due to Horozov [135] and was unknown classically. Our proof follows his with some modifications.

page 188 par. 3 *polar coordinates*. For a discussion of the global properties of Horozov's mapping, see Cushman and Sniatycki [75].

page 190 par. 0 *properties*. For an idea how to prove i) see Cushman and Sanders [71].

Lagrange top

page 193 par. 0 *Lagrange top.* The Lagrange top was studied by Lagrange [159, p.261]. Since then studies of the top are a legion. Some standard works are Golubev [110], Leimanis [166], Routh [241], and Klein and Sommerfeld [150]. Whittaker [291, p.155–63] gives a very analytic treatment. Goldstein [108] has a nice (but incomplete) qualitative treatment. Pars [225, p.113–117;152–57] has a thorough discussion which is a mixture of analysis and qualitative argument. See also [235] by Ratiu and van Moerbeke.

page 202 par. 3 *magnetic spherical pendulum.* The fact that reducing the right S^1 action for the Lagrange top leads to the magnetic spherical pendulum was first shown by Novikov [219, p.30–31]. The identification of the symplectic form of the reduced Lagrange top as the standard symplectic form on TS^2 plus a magnetic term is a special case of the cotangent reduction theorem of Kummer [155]. This theorem does *not* provide an explicit expression for the equivalence mapping.

page 205 par. 3 *Hamiltonian system.* Our treatment of the reduction of the Lagrange top to a system with one degree of freedom follows the treatment of reduction of the S^1 symmetry of the magnetic spherical pendulum given by Bates and Cushman [23].

page 210 par. 3 *tangent cone.* For the definition of the tangent cone to an algebraic variety see Brieskorn and Knörrer [47].

page 235 par. 5 *motion of the axis.* A proof that the classification of the possible motions of the axis is complete does not seem to be available in the classical literature on tops and gyroscopes. Many authors such as Goldstein [108] draw pictures sans supporting arguments. Goldstein's pictures do not give all the possibilities as Pars [225, p.152–57] shows in figures 9b) - f) on page 157. See also Webster [283]. Deimel [80, p. 84] cites Hadamard [124] for the proof of the nonexistence of a downward loop. Deimel's physical argument for the nonexistence of downward cusps is too soft for us.

page 239 par. 2 *blow up map.* For more details on the geometry of the blow up map see Brieskorn and Knörrer [47, p. 486–89].

page 244 par. 4 *rotation number.* Our proof that the rotation number is positive follows that of Hadamard [124].

page 248 par. 3 *reconstructing the topology.* The reconstruction of the topology of the singular energy momentum level sets is new. It completes the analysis of Iacob [139] and Tatarinov [273].

page 260 par. 4 *discriminant of cubic and quartic.* For the proofs of these facts about discriminants see Brieskorn and Knörrer [47, p.186–190].

page 260 par. 8 *swallowtail surface.* For more information about the swallowtail surface see Poston and Stewart [233]. Our discriminant locus consists of the real points of the complex discriminant and is *not* the locus of multiple real roots of the real quartic. This means that our swallowtail surface has a whisker of multiple purely imaginary roots whereas the locus of multiple real roots does not.

page 260 par. 8 *monodromy.* The fact that the energy momentum map of the Lagrange top has monodromy was first proved in Cushman and Knörrer [67].

page 268 par. 3 *gyroscopically stabilized.* See Klein and Sommerfeld [150] for a discussion of gyroscopic stabilization.

page 269 par. 5 *Linearizing.* A complete linear analysis of gyroscopic stability can not be found in such classical sources as Routh [241] or Webster [283], because the necessary symplectic linear algebra was not known until 1937, see Williamson [293].

page 270 par. 3 *normal form.* For more details on normal forms for real infinitesimally symplectic linear maps see Burgoyne and Cushman [49] and [50].

page 271 par. 3 *smooth normal form.* Arnol'd in [11] introduced the concept of smooth (versal) normal form for a smooth family of matrices.

page 273 par. 2 *Hamiltonian Hopf bifurcation.* Guillemin [115] observed that there was a Hamiltonian Hopf bifurcation in the Lagrange top. His unpublished notes inspired the treatment in Cushman and van der Meer [76]. For a full treatment of the Hamiltonian Hopf bifurcation see van der Meer [276].

page 275 par. 4 *Poincaré lemma.* Our proof of the Poincaré lemma follows Duistermaat [85].

page 279 par. 3 *explanation.* For a complete explanation of how the cusp becomes a swallowtail see van der Meer [277].

page 279 par. 6 *Lift.* This exercise is due to E. Lerman [168].

page 279 par. 9 *goal.* This exercise is due to E. Lerman [169].

Fundamental Concepts

page 284 par. 3 *Witt*. In [296] Witt proved the Witt decomposition for symmetric bilinear forms. Artin [14, p. 120-1] noted that Witt's proof also works for symplectic forms.

page 285 par. 3 *symplectic manifold.* It is a vexing historical question trying to answer who was the first to recognize explicitly that symplectic geometry was the basic underlying mathematical structure in Hamiltonian mechanics. For linear Hamiltonian systems the credit seems due to Wintner [295]. For nonlinear Hamiltonian systems the question is not so easily answered. The earliest paper we know of, which gives a modern definition of a Hamiltonian vector field as a differential operator, is by Slebodzinski [251]. In [51] Cartan gives a modern treatment of Hamilton's equations from the variational point of view. Even though he used exterior differential calculus, there is no mention of the symplectic structure of phase space. Later, symplectic structures were studied by Ehresmann and his school: Libermann [97, 176], Gallissot [105], Lichnerowicz [178], Souriau [258] and Reeb [238], [239], [240]. The term "symplectic geometry" seems to have been coined by Souriau [259] . All this activity did not bring symplectic geometry and its relation to mechanics to the attention of the working mathematician. Mackey [188] makes it clear that the symplectic formulation of mechanics was "understood" by 1963. This formulation was used extensively by Sternberg in his 1964 book [269]. In a widely circulated but unpublished letter in 1965 Palais [224] explained the symplectic formulation of Hamiltonian mechanics. Shortly after this the classic books by Abraham and Marsden [1], Arnol'd [12] and Souriau [262] appeared popularizing the symplectic geometric treatment of mechanics. See also Moser's memoir [208].

page 285 par. 7 *symplectic structure*. Lagrange [159] was the first to use the symplectic form in mechanics. In his case, the form was pulled back to the tangent bundle. He also proved its invariance under the flow of the Hamiltonian vector field. This was pointed out to us by J. Duistermaat. The symplectic form on a coadjoint orbit is often called the Kostant-Kirillov-Souriau form after Kostant [153], Kirillov [149] and Souriau [262]. However, it was discovered by Lie [180].

page 286 par. 1 *Let* X. The proof in the text of the nondegeneracy of the canonical 2-form is due to J. Rawnsley [236].

page 288 par. 2 *symplectic form.* The symplectic form on the left trivialization of the cotangent bundle of a Lie group is a basic formula of Hamiltonian mechanics.

page 290 par. 4 *Hamiltonian vector field.* Lagrange [159] was the first one to write down Hamilton's equations for Hamiltonians of the form kinetic plus potential. He also wrote the equations of motion for perturbations of the Kepler problem in Hamiltonian form. However, he did not write down Hamilton's equations for a general Hamiltonian. For an interesting discussion of this see Weinstein [286]. Poisson [231, p.343] wrote down the action corresponding to a Lagrangian and then obtained one half of Hamilton's equations. Hamilton [125] wrote the equations of motion in Hamiltonian form.

page 291 par. 1 *geodesic vector field.* Our derivation of the equations satisfied by the integral curves of the geodesic vector field follows Lang [161]. See also Besse [35].

page 292 par. 1 *Christoffel symbol.* For more details on the Christoffel symbols see Spivak [265].

page 297 par. 1 *Poisson bracket.* Poisson brackets were first written down by Poisson in [231, p.266]. Again it was Lie [180] who discovered that they gave a Lie algebra structure on the space of smooth functions.

page 298 par. 1 *because.* Our proof follows that of Duistermaat [85].

page 298 par. 2 *structure tensor.* The fact that the closedness of the 2-form ω is equivalent to the Jacobi identity was observed by Jost [144].

page 298 par. 5 *Poisson manifold.* The term "Poisson manifold" is due to Lichnerowicz [179]. Weinstein proved most of their basic properties in [287] and [289]. The concept of a Poisson manifold is due to Lie [180, section 62 volume 2]. For a discussion of Lie's contributions to symplectic geometry see Weinstein [288] and Bloch and Ratiu [42].

page 300 par. 1 *continue the argument.* The purely local character of this result was established by Gromov [114].

page 300 par. 3 *Claim ((4.8)).* This result is called Darboux's theorem after Darboux [77], even though it was known to Liouville. For a discussion of this see Lützen [185]. Our proof follows that of Arnol'd [12] and Weinstein [287].

page 303 par. 3 *identity.* Actually the requirement that c is a regular value of the constraint mapping $\mathscr{C}$ is not needed to show that the Dirac bracket satisfies the Jacobi identity. It is only necessary that the matrix C is invertible for every $n \in \mathscr{C}^{-1}(0)$. For details we refer to Dirac [84].

Systems with symmetry

page 305 par. 5 *closed subgroup.* For a proof of the fact that a closed subgroup of a Lie group is itself a Lie group, see Adams [2].

page 306 par. 7 *linearizability.* The local linearizablity of a proper action is called Bochner's lemma, see Bochner [43].

page 307 par. 3 *slice.* In [223] Palais proved the slice theorem. Our definition of a slice is slightly different from Palais'. Our proof follows that of Duistermaat and Kolk [88].

page 309 par. 4 *local finiteness.* Our proof of the local finiteness of the orbit type decomposition follows Palais [223].

page 311 par. 7 *partition of unity.* The proof of the existence of a G-invariant partition of unity subordinate to a G-invariant open covering follows that given in Palais [223].

page 313 par. 5 *free and proper.* The proof that the orbit space of a proper free action is a smooth manifold follows along the lines of Duistermaat and Kolk [88].

page 315 par. 3 *principal G-bundle.* For another definition see Steenrod [266].

page 318 par. 2 *differential space.* The concept of a differential space is due to Sikorski [248]. See also Śniatycki [255].

page 318 par. 5 *differential structure.* Sikorski [248] defined the differential structure $C^\infty(P)$ as a family of functions satisfying point 2 of the definition. He used point 1 to define a topology on P and imposed point 3 as a compatibility condition.

page 320 par. 2 *Whitney extension theorem.* This theorem was proved by Whitney in [290].

page 321 par. 4 *submanifold*. This result can be found in [64].

page 321 par. 11 *orbit space*. This result is due to Cushman and Śniatycki [74].

page 323 par. 1 *diffeomorphic*. This result is due to Cushman and Śniatycki [74].

page 327 par. 2 *theorem of Schwarz.* This result is due to Schwarz [246].

page 329 par. 5 *primary stratification.* This result is due to Bierstone [39].

page 330 par. 3 *minimal.* In [38] Bierstone shows that the stratification of the orbit space of a proper action is minimal.

page 331 par. 1 *vector field.* The notion of a vector field on a locally compact differential space is due to Śniatycki [254].

page 357 par. 8 *stratified vector field.* Schwarz [247] has shown that equality holds in claim ((4.2)).

page 336 par. 1 *momentum map.* The notion of a momentum mapping goes back at least to Lie [180]. In volume 2 he shows that the momentum map is canonical (page 300), equivariant with respect to some linear action (page 329), and that its image is invariant under the coadjoint action, provided that the rank of the momentum mapping is constant (page 338). The momentum map was formally defined by Souriau in [260] who studied its equivariance properties in [261]. Souriau discussed the momentum mapping at length in [262] as did Kostant in [153, p.187]. For a historical discussion see Bloch and Ratiu [42]. The French term of Souriau for momentum map is "moment". Hence the alternative English term "moment map". The term "momentum mapping" seems to have been first used by Cushman [55].

page 337 par. 6 *integral.* The fact that the G-momentum mapping of a G-invariant Hamiltonian is invariant under the flow of the Hamiltonian vector field is often called Noether's theorem because of [218].

page 338 par. 1 *coadjoint equivariance.* The coadjoint equivariance of the momentum mapping was known to Lie [180, volume 2, chapter 20] but its importance for understanding the topological structure of integral manifolds of mechanical systems was emphasized by Smale [252].

page 341 par. 2 *normal form.* Our proof of the normal form of a momentum mapping follows that given by Arms, Gotay and Jennings [8], which in turn is based on that of Arms, Marsden, and Moncrief [9]. For other proofs see Marle [190] and Guillemin and Sternberg [117].

page 344 par. 6 *shifting trick.* The proof of the global shifting trick for differential spaces is due to Śniatycki [255].

page 345 par. 1 *locally connected.* The proof of the local shifting trick is standard and seems to have first appeared in print in Guillemin and Sternberg [117]. However, it was certainly "known" to experts before that. See the discussion at the end of Arms [5].

page 346 par. 4 *regular reduction.* The regular reduction theorem has a long history going back to at least Jacobi [142] when he eliminated the node in the three body problem. The modern global formulation is due to Meyer [197] and Marsden and Weinstein [194]. For a sampling of other applications of regular reduction see Marsden and Ratiu [193] and Marsden, Motgomery, and Ratiu [192].

page 346 par. 2 *regular value.* For another proof that if the isotropy group at μ acts locally freely on $J^{-1}(\mu)$ then μ is a regular value see Guillemin and Sternberg [117].

page 354 par. 4 *nondegenerate.* Our proof of the nondegeneracy of the Poisson bracket follows Arms, Cushman and Gotay [7] and uses clarifications suggested by Lerman.

page 356 par. 6 *singular reduction.* The first paper on singular reduction was Arms, Marsden and Moncrief [9] who considered the singular reduced space as a *set* and studied its decomposition into connected symplectic manifolds. Early papers and books on singular reduction using only the fact that the group action is proper and Hamiltonian can be found in Ratiu and Ortega [221], Śniatycki, Schwarz and Bates [256], and Bates and Lerman [28]. A modern treatment using Sussmann's theorem can be found in Śniatycki [255]. A discussion of singular reduction for a Hamiltonian action of a *compact* group can be found in Sjamaar and Lerman [250]. The construction of a *smoothness structure* for the singular reduced space and a Poisson bracket on its smooth functions was first given by Arms, Cushman, and Gotay [7]. Our argument showing that the singular reduced space is a locally compact subcartestian differential space follows Śniatycki [255]. Examples using the technique of singular reduction (but with no proofs) had been given *before* the appearance of [7] by Cushman [59] and van der Meer [276]. Recent articles with additional examples are Cushman [60], Cushman and Sjamaar [73] and Lerman, Montgomery, and Sjamaar [167]. A rather complete survey and comparison of early theories of singular reduction may be found in Arms, Gotay, and Jennings [8]. Of these early papers we note the following: Śniatycki [253], Śniatycki and Weinstein [257], Gotay [111], Gotay and Bos [113] Arms [6], Gotay [112], and Wilbour and Arms [294].

page 357 par. 8 *reduced vector field.* Our proof that the derivation corresponding to the reduced Hamiltonian is a vector field uses the definition of a vector field on a locally compact differential space due to Śniatycki [255].

page 366 par. 2 *encoded.* The result that the stratification of the singular reduced space is determined by its Poisson structure is due to Sjamaar and Lerman [250]. Another proof using Sussmann's theorem [271] can be found in Śniatycki [255].

Ehresmann connections

page 373 par. 2 *Ehresmann connection.* For the idea of a fibration see Ehresmann's paper [95]. The basic notion of Ehresmann connection was defined by Ehresmann in [96]. We follow the exposition of Zou [303]. It is amazing that no detailed arguments establishing the basic properties of an Ehresmann connection are available in texts on differential geometry.

page 373 par. 5 *local horizontal lift.* Our proof of the existence of a local horizontal lift follows that indicated by Wolf in [297].

page 375 par. 5 *parallel translation.* Local parallel translation need not exist for every Ehresmann connection. Some authors finesse this by making it part of the definition.

page 379 par. 2 *trivialization theorem.* The Ehresmann trivialization theorem is a special case of the basic property of bundles called *covering homotopy theorem*, which is due to Steenrod [266].

page 381 par. 5 *holonomy.* This formula is due to Schlesinger [245] although it is often credited to Ambrose and Singer [3].

Action angle coordinates

page 385 par. 2 *Liouville integrable.* The earliest version of this theorem appears in Jacobi [141, p. 252] in the sense that n independent commuting integrals provide a complete solution to the Hamilton-Jacobi equation and thus would reduce the integration of Hamilton's equations to quadratures. This is a purely local procedure. Early authors did not discuss tori, although ones sees quasiperiodic motions in the special cases they treated. Jacobi's result was rediscovered by Liouville [182]. For a more detailed discussion of this see Lützen [185]. Mineur [202, 203] seems to be the first to have given a modern statement of the action angle coordinate theorem, which formulated the conclusion that in every integrable system, where the integral map has compact fibers, all nearby solutions are quasiperiodic. Until Vu Ngoc pointed it out in [281], Mineur's work had been forgotten, see Miranda and Zung [204]. In 2009 J. Duistermaat noted that there were some gaps in Mineur's proofs. The first well known modern statement and proof of the action angle coordinates theorem is due to Arnol'd and Avez [13]. Some hypotheses in the theorem, equivalent to the openness of the set of compact components, the functional independence of the actions and the exactness of the symplectic form were later removed by Jost [145] as well as Markus and Meyer [189]. The proof of Markus and Meyer was streamlined by Duistermaat [86] whose proof in turn was shortened by Bates and Śniatycki [29]. Other proofs of varying degrees of completeness appear in Abraham and Marsden [1], Audin [16], Guillemin and Sternberg [117] and Liebermann and Marle [177]. For generalizations to integrable systems on Poisson manifolds, see [163] by Laurent-Gengoux, Miranda, and Vanhaecke, to singular integrable systems may be found in Eliasson [98], Knörrer [151, 152], and Vu Ngoc and Wacheux [282], and to partially integrable systems see Nekhoroshev [213].

page 386 par. 2 *Proof.* Our proof of the existence of local action angle coordinates follows Bates and Śniatycki [29] with a bit more detail.

page 388 par. 3 *de Rham's theorem.* For a proof see de Rham [79].

page 390 par. 2 *action-angle theorem.* A proof of the theorem in this exercise can be found in the paper [184] by Lukina, Takens, and Broer.

Monodromy

page 391 par. 2 *obstructions.* All the obstructions for finding global action angle coordinates are described by Duistermaat in [86]. An example of the *Chern class obstruction* is given in Bates [21].

page 391 par. 2 *monodromy.* Our treatment of the *monodromy obstruction* follows the period lattice construction of Duistermaat [86]. A Čech approach has been given by Bates [21]. There are many interesting Liouville integrable systems which have been proved to have monodromy, namely, the spherical pendulum by Duistermaat [86] and Cushman [57]; the Lagrange top by Cushman and Knörrer [67]; the Hamiltonian Hopf bifurcation by van der Meer [276, p.83] and Duistermaat [87]; the champagne bottle by Bates [22]; the magnetic spherical pendulum [23] by Bates and Cushman; the hydrogen atom in orthogonal electric and magnetic fields by Cushman and Sadovskii [69, 70] and by Efstathiou and Sadovskii [93]; the swing spring by Cushman, Dullin and Giacobbe

[65]; a system of coupled angular momenta by Sadovskii and Zhilinskii [242]; and a spin oscillator by Vu Ngoc and Pelayo [227]. The confluence of two focus-focus points has been studied by Bates and Zou [31], see also Zou [303], and by Efstathiou in the quadratic spherical pendulum [91]. The papers [82] by Delos, Dhont, Sadovskii, and Zhilinskii and [102] by Fitch, Weidner, Parazzoli, Dullin, and Lewandowski, report experimental verification of monodromy in certain specific Hamiltonian systems.

page 392 par. 1 *bundle of period lattices.* Because the transitition functions for the charts of the bundle of period lattices are elements of $\mathrm{Sl}(n, \mathbf{Z})$ on the base B, we say that B has an *integer affine structure*. This structure was studied by Zung in [304] and Kantonistova in [146]. For an integrable system with monodromy the affine structure was interpreted as a lattice defect by Zhilinskii in [301]. An affine model of the space of actions for a Liouville integrable system was given by Bates and Fassò in [26].

page 393 par. 3 *Geometric monodromy.* The first proofs of the geometric monodromy theorem given by Lerman and Umanskii [170, 171] and also by Matveev [196] did not leave much of an impression. However, the one of Zung [305, 306] did. The argument in the text follows that of Cushman and Duistermaat [63] except that it does not use the hyperbolic linearization result [268] of Sternberg or the linearization lemma of Bochner [43]. Instead it uses the focus-focus Morse lemma of Vu Ngoc and Wacheux [282] to construct a Hamiltonian S^1-action near the focus-focus point. This idea, due to Zung [305], is crucial to our proof.

page 402 par. 3 *hyperbolic billiard.* The treatment of the hyperbolic billiard in the text follows the article [82] of Delos, Dhont, Sadovskii, and Zhilinskii.

page 410 par. 2 *champagne bottle.* See Bates [22] for more details about the motion in the champagne bottle potential.

page 410 par. 4 *accumulation of tori.* See [27] by Bates and Fassò for more details. For more information about the definition and geometry of a superintegrable Hamiltonian system see Fassò [101] and Dazord and Delzant [83].

page 411 par. 1 *fractional monodromy.* The first paper on fractional monodromy is [214] by Nekhoroshev,, Sadovskii and Zhilinskii. Fuller details are given in [215]. An analytic treatment of fractional monodromy for an integrable perturbation of the $1 : -2$ resonant oscillator can be found in [92] by Efstathiou, Cushman and Sadovskii. In [216] Nekhoroshev described fractional monodromy for an arbitrary resonance. A treatment of fractional monodromy by Gauss-Manin connections is given in [270] by Sugny, Mardešić, Pelletier, Jehrane, and Jauslin. In the paper [48] by Broer, Efstathiou, and Lukina one can find the first statement and proof of the geometric fractional monodromy theorem.

page 414 par. 1 *quadratic spherical pendulum.* For a detailed treatment of the *quadratic spherical pendulum* see Efstathiou [91, p.87–111]. The notion of bidromy is due to Sadovskii and Zhilinskii [243]. Our exercise follows the treatment in [94] of Efstathiou and Sugny. Other relevant papers are [280] by de Verdière and Vu Ngoc and [304] by Zung.

page 416 par. 1 *scattering monodromy.* This exercise is taken from the paper [24] of Bates and Cushman.

page 416 par. 3 *scattering angle.* Our definition of the *scattering angle* differs from the usual one given in physics, namely, the angle as measured from the incoming asymptote, thought of as a ray from the origin, to the outgoing asymptote, see Synge [272, figure 18, p.73]. The problem with the usual definition of scattering angle is that it does *not* vary continuously as a function of h and ℓ; whereas using our definition it does.

page 418 par. 1 *scattering Kepler problem.* The fact that the two degree of freedom scattering Kepler problem has *scattering monodromy* was observed by Dullin and Waalkens in [90] who seemed to be unaware of [24].

page 419 par. 1 *focus-focus Morse lemma.* Except for section a, the proof in the text of the focus-focus Morse lemma follows that of Vu Ngoc and Wacheux [282].

page 420 par. 3 *Seifert manifold.* See Orlik [220] and Montesinos [206] for more details.

page 420 par. 4 *Euler class.* See the paper [249] by Shastri and Zvengrowski.

Morse theory

page 424 par. 2 *inverse function.* For a proof of the inverse function theorem see Loomis and Sternberg [183].

page 425 par. 9 *claim ((1.2)).* This claim is usually referred to as the *Lagrange multiplier theorem*.

page 426 par. 3 *Morse lemma.* The proof we give here is attributed to Hörmander. For other proofs see Milnor [199] and Weinstein [285].

page 428 par. 2 *nondegenerate critical manifold.* The definition of a *nondegenerate critical manifold* was given by Bott [44].

page 429 par. 6 *Morse isotopy lemma.* The proof of the *Morse isotopy lemma* follows that given by Milnor in [199, p.12].

page 429 par. 4 *one parameter group.* For a proof of the existence of a flow for a vector field see Loomis and Sternberg [183].

page 429 par. 7 *two nondegenerate critical points.* This result is attributed by Milnor [199] to Reeb [237]. Our proof follows Hirsch [133, p.154–55].

page 430 par. 3 *exotic 7-sphere.* J. Milnor [198] gave an example of a 7-sphere which was not diffeomorphic to the standard 7-sphere. E. Brieskorn [46] gives explicit polynomials which give rise to every exotic 7-sphere.

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