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Marco Alberto Javarone

Statistical Physics and Computational Methods for Evolutionary Game Theory

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Disinterested love for all living creatures,
the most noble attribute of man.

Charles Darwin

Preface

The idea to write this book originated during the Conference on Complex Systems held in Amsterdam on September 2016, where the first satellite on evolutionary game theory was proposed and obtained the attention of several participants, thanks also to two great Keynote speakers: Yamir Moreno and Francisco C. Santos. The evolutionary game theory represents a vibrant and growing research field. So far, many interesting books have been written on this topic or have introduced at least in part its main elements. So, the reader might wonder why a further book on this topic should attract the attention. Well, actually, aside from providing a brief introduction to the main concepts, the aim of this book is to frame evolutionary game theory in the context of the science of complexity providing, at the same time, an overview on some computational strategies for dealing with the related models. As discussed in the following chapters, nowadays the science of complexity deals with different topics, which can be viewed as a very big basket including a number of challenging topics, spanning from biology to social science. Here, the evolutionary game theory constitutes a valuable framework for representing and studying different complex phenomena. In addition, as mentioned, the second goal is to highlight the connections with statistical physics issues, e.g., the phenomenon of phase transitions, and to show how to use its tools, combined with other computational methods, for studying the dynamics of evolutionary game theory models. Accordingly, we hope to stimulate the interest of readers with some relevant contributions to this field, as scientific papers and other books, found in the list of references. Unfortunately, due to the limited space, we have not been able to include all the valuable references. The structure of the book reflects the need to condensate relevant issues, exposing them in a clear and simple way. Therefore, we decided to begin with a simple introductory chapter, followed by a second one focusing on some statistical physics methods. Moreover, the second chapter illustrates some computational methods for generating and analyzing complex networks, being a "tool" of great interest also in this area. Then, Chaps. 3 and 4 illustrate practical cases, i.e., the structure of two famous games (Chap. 3), the prisoner's dilemma and the public goods game, and two applications (Chap. 4), one framed in the context

of social dynamics and the other in that of combinatorial optimization. Finally, a conclusive chapter summarizes some important concepts exposed in the previous chapters and provides a short overview on further developments. To conclude, we hope that students and researchers will find the book useful for starting their journey in this exciting field.

Acknowledgments There is a long list of people I wish to thank, and most of them are friends and colleagues. So far, I have been very lucky to have the opportunity to receive scientific mentorship, and useful suggestions, from researchers and scientists of very high level, as Adriano Barra, Matjaz Perc, and Attila Szolnoki. Nowadays, I have the fortune to work in the group of Daniel Polani, in the School of Computer Science at the University of Hertfordshire, where I completed this book in summer 2017. I want to thank Daniel for being strongly supportive and for sharing his ideas, which always stimulate exciting discussions. In this book, two chapters contain some material I published in collaboration with three friends: Alberto Antonioni, Federico Battiston, and Francesco Caravelli. I am very glad to have had the privilege to work with them. Furthermore, some ideas originated by discussing with Giuliano Armano, Serge Galam, Daniele Marinazzo, Salvatore Mignemi, Andrea Tagarelli, and Matteo Tamponi. All of them have been, and often are, my mentors. Finally, I want to thank also my first student, Antonio Emanuele Atzeni, who graduated in theoretical physics a few years ago and who collaborated with me during my first steps in this great scientific area. To conclude, this book is dedicated to my parents for their continuous support and to Elisa for making my days better.

August 2017 Marco Alberto Javarone

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Chapter 1
Introduction

In the last years, we are witnessing the emergence of a new scientific field, i.e., the Science of Complexity. Actually, due to its ubiquitness among different scientific communities, e.g., physicists, biologists, sociologists, and so on, this modern field is benefiting from a positive increasing trend. Here, a first interesting point is trying to define the meaning of the term "complexity." In fact, if on the one hand its ubiquitness is positive for its spreading and development, on the other hand it might generate a bit of confusion. Notably, talking to scientists belonging to different areas, the word complexity can lead to receive various definitions, even when asking to people belonging to the same area (e.g., physicists). As result, in few words, answering the question "what is a complex system?" might lead to different possibilities. While that is not in principle a big problem, it is important to be aware about, in particular, when one is interested in starting to investigate the related topics. For instance, if we are talking about Quantum Mechanics, we can have different visions (see interpretations) but, in this case, we are all referring to systems whose evolution can be described by the Schröedinger equation, using operators and vectors in the Hilbert space, and so on and so forth. Instead, in the case of complex systems, one might think about social networks, cognitive processes, biological information, spin glasses, and many more topics that can be described by the following various approaches. Now the good news is that, beyond the differences among the involved fields, with some efforts interdisciplinary collaborations are becoming very fruitful. Given this brief premise, it is worth to clarify what we mean here when referring to a complex system. In particular, we are indicating a system composed by a huge amount of simple elements, interacting in some way, whose global behavior cannot be directly referred to the local one. As result, this definition, inspired by a thermodynamic overview of systems, suggests that our aim is to develop an opportune Statistical Physics approach for linking the local behavior of a system with the global one, resulting from its interactions. Following such "philosophy," the goal of the book is to provide a brief introduction to the Evolutionary Game Theory, i.e., a vibrant field that is attracting the interest of an

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increasing amount of scientists, spanning from Physics to Computer Science and from Mathematics to Theoretical Biology. The main idea is to offer an overview of Evolutionary Game Theory showing, at the same time, both its "complex" side and how to deal with it from a computational point of view. In doing so, the reader might learn the general concepts, and be ready in few time to implement her/his own early models. Obviously, readers interested in further details can find a number of very good books and articles (only a very short list of them is reported among references in the following chapters). Furthermore, for the sake of completeness, we provide four practical use cases related to models of Evolutionary Games, related to different scenarios and motivations, and strongly linked with the statistical physics approach to complex systems we develop in what follows. In addition, it is important to remind that also from the point of view of statistical physics and computational methods, we are able to offer only a very limited view, hoping to attract the interest of readers and to make them able to appreciate how to use it in practical problems.

1.1 General Background

Evolutionary Game Theory (hereinafter EGT) represents the attempt to study the dynamics of a population, combining the principles of Game Theory with those of the Darwinian theory of evolution. At the first glance, these two fields have very few things in common. In very few words, Game Theory can be referred as an approach/framework aimed at finding the optimal strategy in a "competition," as well as the way for finding a kind of equilibrium among the competitors. Instead, the Darwinian Theory deals with the evolution of life, considering both its competitive aspect and the idea of "transmission and optimization of information." However, as later discussed, a closer look to these theories shows different points of connection. It is worth to remind that the early developments of the Game Theory, resulting from the works of John von Neumann and Oskar Morgenstern, were mainly focused on the modeling of the human behavior, with a clear reference to economic contexts. Later on, this area of Mathematics was strongly influenced by the Nobel laureate John Nash who introduced, among his relevant contributions to science, the so-called Nash equilibrium. The latter can be viewed as a particular state of a game, involving a number of "rational" agents that have to take an action without prior communications. On the other hand, the Darwinian theory of evolution, that constitutes one of the most important breakthroughs in Biology and more in general in science, after more than a century still constitutes a living theory, although yet debated in particular outside academia. His father, Charles Darwin, proposed it after a challenging journey around the world, which gave the opportunity to perform direct observations on a wild and unexplored Nature. So, after this experience, the young scientist envisioned a general theory for explaining natural evolution of life. As result, this fascinating combination of Game Theory and Darwinian Theory constitutes a powerful framework for modeling several scenarios, spanning from social systems to biological phenomena, and representing specific

mechanisms (e.g., reproduction, imitation), interaction patterns, and behaviors. Therefore, a number of "complex phenomena" can be characterized by EGT models. Then, in full agreement with the conclusions reported in the famous Anderson's work "More is different," this modern and interdisciplinary area can be actually classified as one of the components of the field of complexity. For instance, emergent behaviors that can be referred as a kind of phase transition can be observed studying an agent population whose local interactions are simple, e.g., based on a game like the Prisoner's Dilemma. Notably, from a state of disorder one can observe an evolution toward an ordered equilibrium, and the latter can be even opposite to the expected one (i.e., the Nash equilibrium). The last observation deserves particular attention, since one may experience some confusion. Notably, while the mathematical framework of Game Theory has very solid roots and deep basis, direct observations on the real world suggest that the Nash equilibrium is not always respected. For example, cooperation is a common phenomenon and takes different forms also in Nature, like mutualism. Here, one of the goals of EGT is to find out the motivations, and the mechanisms, that can lead to the phenomena we observe and that cannot be predicted using only the Game Theory. At the same time, also the Darwinian Theory takes profit by its combination with Game Theory. For instance, the altruistic behaviors detected among animals cannot be explained referring only to the Darwin's Theory. So, in this case as well as in others, the contribution of the Game Theory allows to cover this important lack. A fundamental key point of EGT is given by the adaptive behavior introduced in the dynamics of a population. Notably, this behavior is driven by a "rational mindset," obtained by implementing agents that take actions for optimizing their own gain. The latter, usually defined as "payoff", in EGT is equivalent to the concept of "fitness." This correspondence (i.e., payoff-fitness) is quite important since it represents one of the most relevant connections between the Game Theory and the Darwinian Theory, where the best individual is not the strongest but the fittest one or, in general, that who better adapts herself/himself to new environments. Thus, the equivalence payoff-fitness has a deep meaning that allows to use the "rationality" of Game Theory and the "evolutionary mechanisms" of the Darwinian Theory. In particular, as discussed later with more detail, "rationality" and "evolutionary mechanisms" are implemented in the process defined "strategy revision phase." The latter allows agents to revise their strategy, e.g., imitating their neighbors, and to support the replication of successful strategies. Accordingly, this process is fundamental for the actual evolution of a population and allows to introduce different mechanisms that can influence the way the system moves toward an equilibrium. Finally, even if EGT models can be based on different kinds of games, in this book we refer to the *two*-strategy games, i.e., games characterized by only two strategies. Even if this choice can be perceived as too limiting for modeling real scenarios, in our view it offers two main advantages. First, it allows to introduce in a very easy way the field of EGT; second, it facilitates the connection with models well-known in Statistical Physics. We can now go on with the next sections of this chapter, reporting some important concepts useful for proceeding through the presentation of more interesting models (i.e., Chap. 3).

1.2 Few Notes on Game Theory

In this section, we briefly summarize some relevant concepts we need from Game Theory for the purposes of this book. In particular, we present the Nash equilibrium and structure of two-strategy games, referring in particular to the Prisoner's Dilemma and Public Goods Game. This choice is motivated by the fact that the cases shown in Chap. 3 are related to these two games.

1.2.1 Nash Equilibrium

Developed by John Forbes Nash Jr., the Nash equilibrium constitutes a milestone in Game Theory. For the sake of simplicity, let us consider two agents playing a generic game that are not allowed to communicate before to take an action (i.e., a move). Here, the Nash equilibrium represents the particular situation where neither of them has something to gain from being the only one to change strategy. The lack of opportunities in doing prior agreements, i.e., preliminary communications, has deep consequences. First of all, if both agents want to increase their gain, some form of "blind coordination" is required, which in turn entails that reaching the maximum gain might need to take a very risky strategy. In the next section, introducing real games like the Prisoner's Dilemma, the "risky" side of some strategies will become more clear. Beyond its general description, the Nash equilibrium has obviously a formal mathematical definition. Let us consider a game described by a set S of strategy profiles and a payoff function f, i.e., (S,f). This game might involve n players, so that each one adopts a strategy. Here, a strategy profile S is defined as $S = S_1 \times S_2 \times \ldots \times S_n$, and the payoff function computes that gain for a specific $x \in S$ as $f(x) = (f_1(x), f_2(x), \ldots, f_n(x))$. Then, a set of strategies $x^* \in S$ is a Nash equilibrium if $\forall_i, x_i \in S_i : f_i(x_i^*, x_{-i}^*) \geq f_i(x_i, x_{-i}^*)$. Summarizing, reaching the Nash equilibrium can be viewed as a tradeoff, which ensures the potentiality of a good payoff while reducing possible risks. For this reason, if we consider a population whose agents interact according to a dilemma game, the expected equilibrium should correspond to the Nash equilibrium, i.e., all agents take the most convenient strategy, following a selfish behavior. However, in doing so, our population, as a whole, risks to be unable to produce a common wellness. The latter, in this context, has a very general meaning, as for instance might refer to people paying taxes for receiving public services or to people who vaccinate themselves for avoiding viral spreading, and so on. Now, beyond the need to identify strategies for the common wellness, a challenging point in EGT is the understanding of the mechanisms that lead toward non-expected equilibria, in particular because, often, the real world shows the emergence of equilibria different from the Nash equilibrium.

1.2.2 Two-Strategy Games

In this section, we describe two simple *two*-strategy games, i.e., the Prisoner's Dilemma and the Public Goods Game (also defined as *n*-person Prisoner's Dilemma). In both games, the agents can behave (i.e., act) as "Cooperators" (C) or as "Defectors" (D). Accordingly, the set of strategies is $\Sigma = \{C, D\}$.

1.2.2.1 The Prisoner's Dilemma

The Prisoner's Dilemma (PD hereinafter) is one of the most famous games who shows why rational approaches might lead people to avoid cooperation. Its general dynamics can be told using different stories; however, here, we report the classical one who gave the name to this game, i.e., the story of two criminals captured by police, and undergone to a questioning. Notably, after being captured, our criminals are separated to avoid potential agreements. In addition, in order to ensure at least one guilty, they are offered a bargain: each criminal has two options, i.e., betray (i.e., to defect) or remaining silent (i.e., to cooperate). Thus, despite what that the common sense might suggest, in this case being a cooperator indicates to collaborate with the "partner," not with the police. So, if the two criminals cooperate, they are sentenced to 1 year in prison, while if both betray the conviction is 2 years in prison. Instead, in the third case, i.e., one betrays and the other remains silent, the cooperator is sentenced to 5 years in prison, whereas the defector is set free. Recalling that they cannot communicate before to take an action (i.e., remaining silent or betray), it appears quite clear the motivation that leads our criminals to defect. The Nash equilibrium of this game is "defection," entailing criminals will spend 2 years in prison. At the same time, if both cooperate they can save 1 year, so that the highest "common" benefit can be achieved only by a coordinate action of cooperation, although very risky. Anyway, beyond this story or similar ones, the PD can be characterized by the following payoff matrix:

$$\begin{array}{cc} & \begin{array}{cc} C & D \end{array} \\ \begin{array}{c} C \\ D \end{array} & \begin{pmatrix} 1 & S \\ T & 0 \end{pmatrix} \end{array} \tag{1.1}$$

with T representing the *Temptation*, i.e., the payoff an agent gains defecting when the other cooperates, and S representing the *Sucker's payoff*, i.e., the gain achieved by a cooperator when its opponent defects. In principle, the general structure of the payoff matrix (3.1) can be adapted also for describing other games (e.g., the Hawk-Dove game). Notably, on varying the range of T and S, one can refer to different scenarios. In our case, i.e., in the PD, the values of T and S are $1 \leq T \leq 2$ and $-1 \leq S \leq 0$. Accordingly, we can draw the $T-S$ plane and analyze the outcomes of

the game from point to point, among the possible combinations (T, S). For instance, low values of T entail defectors receive a small payoff when they meet cooperators, while high value of S entail cooperators have small losses when meet defectors.

1.2.2.2 The Public Goods Game

The Public Goods Game (PGG hereinafter) considers a set of individuals that have to secretly decide if to contribute to the wellness of their own community by offering a token. Like for the PD, cooperators are those that aim to the "common" wellness, while defectors are those that follow a selfish behavior. In addition, being the choice "secret," prior communications are avoided also in this game. The token, or coin, provided by cooperators represents a very general form of contribution. For instance, in an economical context, a coin can be a kind of tax; in online platforms can be the sharing of knowledge (e.g., in forums, blogs, etc.). Thus, the contribution actually refers to an effort made by an individual for improving the services of her/his society. Then, the total amount of coins is enhanced by a numerical parameter, named synergy factor, that promotes collaborative efforts, and its final value is equally divided among all individuals, no matter their action. Therefore, defectors, i.e., those whose contribution is null (or smaller than the average value), can be considered as free riders. At the same time, since both defectors and cooperators receive an equal fraction of the total pot (i.e., the enhanced summation of coins), the most rational (and convenient) strategy is defection. In addition, the latter constitutes the Nash equilibrium of the PGG. According to the described dynamics, and in a more formal way, we can defined the payoff received by cooperators (i.e., π^c) and by defectors (i.e., π^d):

$$\begin{cases} \pi^c = r\dfrac{N^c}{G} - c \\ \pi^d = r\dfrac{N^c}{G} \end{cases} \tag{1.2}$$

where N^c indicates the number of cooperators among the G agents involved in the game, r indicates the synergy factor, and c represents the agents' contribution. Without loss of generality, usually c is set to 1. It is worth to highlight that the value of G strongly depends on agent topology, i.e., the way they interact. For instance, when they are arranged in a square lattice, G is equal to 5. This last point will be clarified in the next chapters, where some practical cases are presented. Finally, we deem interesting to emphasize that like for the PD, the "common wellness" requires a "blind" coordinate effort, otherwise following the Nash equilibrium we cannot observe the improvement of a society (no matter of what the contribution and the payoff represent).

1.3 Evolutionary Game Theory

After the very brief introduction to some concepts of Game Theory (the reader interested in knowing more is invited to consult the references, as well as to read one the many books on this field), we can start to move toward the modeling of evolutionary games. First of all, we want to pay attention to the understanding of mechanisms that lead to cooperation in dilemma games, with particular emphasis for those dilemmas whose Nash equilibrium is defection. The underlying motivation is born from observations on the real world where, fortunately, we can find clear examples of cooperation. In addition, now we move from the local level before discussed, i.e., the dynamics of a single game, to the global level of an agent population. Notably, here, agent interactions take the form of a game, and, being the system adaptive, we can study the evolution of strategies over time. This approach allows to obtain a thermodynamic view of our population and, at the same time, to study the local mechanisms that lead toward a particular equilibrium (or steady state), i.e., a particular distribution of strategies. As result, being particularly interested in defection-based games (i.e., games whose Nash equilibrium is defection), we pay a special attention for those mechanisms/conditions that allow to reach a state of full cooperation. At this point, one might begin to understand why Statistical Physics can constitute the optimal framework for analyzing the dynamics of EGT models. Notably, as we will see later, agent populations playing evolutionary games show critical behaviors, e.g., order-disorder phase transitions (well known in Statistical Physics). For this reason, Chap. 2 is devoted to summarize some mathematical methods and tools for studying these phenomena, as the Ising model. Giving a quick look to the literature, we can find several works focused on the connections between EGT and Physics, as the early works of Hauert and Szabo, or the more recent works of Perc, Szolnoki, and their colleagues. Actually, even considering the classical Game Theory, we can find physicists interested in defining a link with Physics, as shown in some works of Galam. As before mentioned, an agent population whose interactions are based on simple games like the PD constitutes an adaptive system. Due to its relevance, this point deserves attention. Notably, being adaptive means that some forms of adaptation/evolution can be detected in the system. In our case, the evolution refers to the strategies adopted by the agents and, in most agent based models, the mechanism responsible for this evolution is a process usually defined "strategy revision phase." The latter allows agents to change their strategy according to a particular rule, where usually "rationality" constitutes the main ingredient. In addition, further approaches can be used for modeling the dynamics of evolutionary games. For instance, without considering physical agents, a famous class of analytical methods is the "replicator dynamics." The latter, proposed by Taylor and Jonker, uses differential equations. This approach, better discussed in Chap. 2, is based on the following conditions: given a strategy i, used with a frequency x_i (in a population), the frequency rate reads

$$\dot{x}_i = x_i(f_i - \phi) \tag{1.3}$$

with f_i expected payoff associated to the strategy i and ϕ average payoff. It is worth to remark the correspondence, before introduced, between the payoff (Game Theory) and the fitness (Darwinian Theory). When considering agent populations, a further relevant aspect in EGT is given by the geometrical space, since the latter is directly related to the interaction topology. For instance, we can consider continuous spaces, where agents randomly move and play with their neighbors, and discrete spaces that usually are represented as graphs. Since agents interact with their neighbors, the way on "how to compute" neighbors in the two kinds of spaces constitutes one of the main practical differences between them. Usually, in continuous spaces, agents are provided with an interaction radius, so that all the others that are inside the related circles are considered as their neighbors. Instead, in the discrete case, the definition of neighbor is more natural, as we have to select all the directly connected agents (i.e., the nearest neighbor according to the graph structure). Now, it is worth to remind some of the recent advances in graph theory, resulting in the modern field of Complex Networks. Given the relevance of this topic, both for the Science of Complexity and for EGT, Chap. 2 describes some computational methods for dealing with them. So, according to the historical developments of graph theory and to the wide utilization of regular structures in Statistical Physics, early results of EGT on discrete spaces have been reached considering agents arranged on regular lattices (e.g., square lattices, rings, etc.). Later on, more complex structures, as scale-free networks, have been adopted. It is worth to recall that, when we study games whose agents interact with neighbors in some spatial array, EGT models are defined as "spatial games," and such condition has deep consequences in the dynamics of a population. For instance, a spatial structure (e.g., a network) implies agents interact with the same opponents many times. This occurrence as highlighted by Martin A. Nowak, one of the fathers of EGT, can strongly facilitate the emergence of cooperative behaviors, thanks to the effect defined "network reciprocity." Accordingly, in the case of the PGG, Matjaz Perc and Attila Szolnoki computed the critical threshold of the synergy factor for achieving cooperation. In particular, they found the minimum value of this numerical parameter to ensure the survival (and even the success) of cooperators, when agents are arranged on regular square lattices. Later on, other games and more complex topologies have been investigated. Just to cite few, Yamir Moreno and colleagues showed that heterogeneous networks do not promote cooperation when humans play the PD. Francisco Santos and colleagues studied the evolution of cooperation in heterogeneous structured populations, demonstrating that the cooperation increases as the heterogeneity, of the network structure, increases. Again, Yamir Moreno and colleagues found that as the network increases, the level of cooperation is weaker than that obtained in a static network. Actually, the list of results in EGT, achieved on complex networks, is much longer than that we are providing here. However, we deem fundamental recall that Francisco C. Santos and Jorge M. Pacheco found that scale-free networks support the emergence of cooperation and many other interesting results related to spatial and complex structures. Finally, we remind further works developed by implementing interdependent networks and multiplex networks, being the latter the one of the current frontiers in the modern

network science. Again, it is quite difficult to summarize in few lines all the relevant contributions achieved in this field, in the recent years, even only considering those that used complex networks. For this reason, the reader is strongly invited to consult the references (at the end of each chapter) and other useful resources online. Before concluding this part, it is important to emphasize the practical implications related to the utilization of graph structures. In particular, moving toward regular and random structures, the analytical tractability of evolutionary games is easily lost. Therefore, as described in Chap. 2, the investigations must be performed by means of numerical simulations (i.e., by using Monte Carlo methods). Then, once an investigation is performed on a particular structure, before concluding that some result can be related to the adopted topology, it is important to evaluate if the same can be achieved in a more regular topology. Actually, comparing results achieved in different network topologies is a task often required also in other problems where networks are adopted (e.g., opinion dynamics). Finally, as later discussed, on using networked populations one need to reproduce (or to approximate) when possible the thermodynamic limit, in order to avoid finite size effects.

1.3.1 Strategy Revision Phase

Let us now go back to the process before introducing the evolution of an agent population, i.e., the "strategy revision phase." The latter can be implemented according to different methods, usually related to the analysis of the payoff of the involved agent. In addition, further methods can consider different behaviors, as conformity (see Chap. 4), and pure imitation (see Chap. 3). In general, methods based on the payoff analysis can be divided in the following categories:

- Comparison
- Self-evaluation
- Imitation

The first one, i.e., the payoff comparison, is often implemented as a stochastic rule by a Fermi-like function. The latter allows to compute the probability an agent y takes the strategy of an agent x and reads

$$W(s_y \leftarrow s_x) = \left(1 + \exp\left[\frac{\pi_y - \pi_x}{K_y}\right]\right)^{-1} \tag{1.4}$$

where π_x and π_y correspond to the payoffs of two agents, and s_x and s_y indicate their strategy. $K_y > 0$ is an agent-dependent parameter whose role will be described in the Chap. 3. The Fermi-like function actually is adopted in a wide number of contexts and applications. A fast inspection to its shape—see Fig. 1.1 clarifies why it can be efficiently used for implementing stochastic and rational processes. Notably, its "stochastic" behavior comes from the opportunity to use it as a weighted distribution, where even inconvenient choices can be performed (e.g., imitating

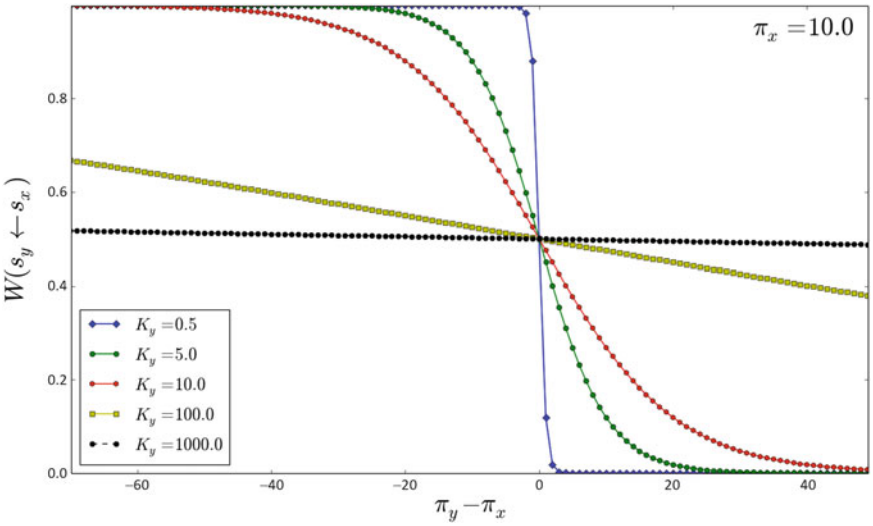

Fig. 1.1 Fermi-like function used for implementing the strategy revision phase. The x axis represents the payoff of the y-th agent, i.e., the one that has to decide whether to imitate the x-th agent, whose payoff has been set to $\pi_x = 10.0$. As reported in the legend, each curve refers to a different temperature

a poorer agent, even if with a very low probability), while its "rationality" is represented by the temperature K (or K_y if referred to a specific agent). The second part of Chap. 3 focuses on a complete analysis on the role of the temperature (indicated also as "noise") in the PGG. Then, the second category in the list, i.e., self-evaluation methods, entails agents decide to change their strategy whether the current payoff is smaller than the previous one. This approach can be viewed as a kind of evaluation on the own performance and entails that agents have some memory (we recall that usually the agent payoff is reset after each iteration). Last, methods based on imitative mechanisms (considering the payoff as reference) usually lead agents to imitate a richer opponent in their neighborhood. However, it is also possible (as shown in the application presented in the Chap. 4) to provide agents with behaviors not related to the payoff (e.g., conformity).

1.3.2 Cooperation: Motivations and Mechanisms

Cooperation is one of the most interesting phenomena in nature and in societies. Cooperation leads to forms of organization and to the growth of a system. However, at least according to the Game Theory, often is a target very difficult to be reached. Beyond the underlying motivations, and the potential risks, results coming from cooperation require joint efforts. For this reason, cooperation can be viewed as an

emergent phenomenon, where an increasing amount of agents becomes cooperator. Martin A. Nowak wrote a very important work, in the field, highlighting and explaining the famous five rules of cooperation, related to the concept of natural selection: kin selection, direct reciprocity, indirect reciprocity, network reciprocity, and group selection. Here, we just limit to mention and to briefly summarize each rule. The kin selection is a principle based on the similarity between the donor and the recipient of an altruistic act. For instance, in case of a parental relation between two individuals, it is very likely to observe cooperation. The direct reciprocity results from the observation that when a game involves many times always the same individuals, cooperation can actually become a promising option. The indirect reciprocity is a mechanism that explains why individuals act as donors, even if they know that the one receiving the benefit is not (will not be) in the condition to exchange the favor. Notably, especially in the human society, we can observe forms of cooperation related to indirect reciprocity, mainly because the donor has the opportunity to gain the respect of other individuals (that, obviously, must see the action). Accordingly, this action might allow to achieve, indirectly, some benefits. The network reciprocity is similar to the direct reciprocity and can be observed in spatially structured populations, where the individuals interact always with the same neighbors. This mechanism is then responsible for the emergence of clusters of cooperators. Then, the group selection indicates forms of cooperation observed within community of people, i.e., among individuals belonging to the same group. Notably, in this case, groups of cooperators can obtain more benefits than groups of defectors. Finally, further mechanisms responsible for the emergence of cooperation have been described in complex networks, in continuous spaces (e.g., random motion, also discussed in the Chap. 3) and in many other conditions.

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Chapter 2
Modeling Complex Systems

Statistical Physics deals with a number of topics of absolute relevance in Physics, as phase transitions. Notably, it aims to connect the macroscopic behavior of a system with the local mechanisms of its constituents, e.g., one aims to connect the thermodynamic view of a gas with its mechanical laws (i.e., the kinetic theory). As a result, this approach becomes strongly valuable when dealing with complex systems, also in those cases where the subject of investigation is a nonphysical system, like a social network or a socioeconomic system. Modern Network Theory represents one of the most successful frameworks for dealing with this kind of topics, and its link with Statistical Physics has deep roots uncovered in the early works of A.L. Barabasi, M. Newman, Y. Moreno, S. Boccaletti, A. Arenas, R. Albert, G. Caldarelli, A. Barrat, V. Latora, D. Krioukov, G. Bianconi and many other scientists, now forming the growing community of complex systems (i.e., the Complex Systems Society). Therefore, the scope of this chapter is to provide a very brief presentation of some mathematical and physical method for dealing with Evolutionary Games, focusing both on the mathematical description and on the computational strategies for implementing models and studying their behavior. The reader interested in further details is invited to consult the huge amount of texts on the specific topic (a brief list of reference can be found at the end of the chapter). Here, the material is organized as follows: we start with models related to population dynamics, then we move to a general discussion of phase transitions, introducing the Ising model, the Curie-Weiss model, and the Mean-field approach. Eventually, a section on complex networks ends the chapter.

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2.1 Population Dynamics

Population dynamics is an area that sinks its roots in the field of Mathematical Biology, adopted for representing processes like population growth, competitions, aging, and so on and so forth. Beyond the classical models introduced by Malthus, Lotka–Volterra, Verhulst, Ginzburg, and many more who contributed to the early developments of this field, EGT constitutes a further framework for studying the behavior and the dynamics of a population. Here, we present some basic concepts that can be adopted for defining new models both in the area of EGT and in contexts that might benefit from this mathematical approach (e.g., social dynamics). Let us begin with a simple continuous growth, considering a population composed of N individuals living in a system without competitors:

$$\frac{dN}{dt} = rN \tag{2.1}$$

with r defined as growth rate, or Malthusian parameter. From a mathematical point of view, computing the analytical solution of Eq. (2.1) is quite simple. In particular, we have $N(t) = N_0 e^{rt}$, with N_0 initial condition, indicating the population size at $t = 0$. As we can observe, Eq. (2.1) does not take under consideration further aspects that can be found in ecological contexts, e.g., processes/mechanisms that can reduce the growth of a population. For instance, we can be interested in analyzing the behavior of a system with two competing populations/species. Obviously, in order to model this occurrence, we have to know the rules underlying the interactions between individuals of the two species. One of the first proposals for representing these scenarios is the Lotka-Volterra model, also named predator-prey model. Notably, it aims to describe the dynamics of interactions between two species, i.e., predators (say A) and preys (say B). The mathematical definition of this model reads

$$\begin{cases} \dfrac{dA}{dt} = \alpha AB - \beta A \\ \dfrac{dB}{dt} = \gamma A - \delta AB \end{cases} \tag{2.2}$$

with α and γ representing internal processes within the single species (e.g., growth) and β and δ parameters that quantify the interactions between the two species. The values of these parameters can be modified for considering different scenarios.

2.1.1 Replicator Dynamics

In the classical cases above mentioned, individuals may belong only to one species, e.g., a predator cannot become a prey. In doing so, a competition process can

lead to the extinction of a species, i.e., to the complete removal of its individuals, without to affect the amount of individuals belonging to the winning species (whose value might depend on other parameters, like the growth factor, etc.). Instead, in EGT, individuals might change their group of belonging, and the total size of the population is conserved over time. Actually, as we will see, in EGT we do not focus on the agents from a physical point of view (e.g., growth mechanisms), but on their strategies that constitute the parameter of the system that varies over time. Here, the benefits (or gains) deriving from interactions can be interpreted as fitness. So, again, what that reproduces/extincts in these dynamics are not the individuals but their strategies. The equivalence payoff-fitness, and the relation between the payoff of an individual and its strategy (and those of its neighbors), allows to introduce an analytical description of the system. For instance, in two-strategy games, i.e., with individuals that can adopt the strategy C and the strategy D, we can write

$$\begin{cases} \dfrac{dC}{dt} = C(\pi_C - \phi) \\ \dfrac{dD}{dt} = D(\pi_D - \phi) \end{cases} \tag{2.3}$$

with $\phi = C\pi_C + D\pi_D$, and $C + D = 1$. The solution of this system can lead to different equilibria, as the extinction of a strategy, as well as the coexistence of both. This approach leads to the so-called replicator dynamics, which considers a population with n strategies, and a $n \times n$-matrix, named "payoff matrix," with elements a_{ij} representing the gain individuals receive according to their actions (i.e., strategies). The general differential equation, i.e., the replicator equation is defined as

$$\frac{dx_i}{dt} = x_i(\pi_i - \phi) \tag{2.4}$$

with i going from 1 to n, x_i representing the density of the i-th strategy in the population, π_i payoff (or fitness) of the i-th strategy computed as $\pi_i = \sum_{j=1}^{n} a_{ij}x_j$, and ϕ average payoff equal to $\phi = \sum_i \pi_i x_i$. The Eq. (2.4) shows a deterministic dynamics, and, as reported in several previous studies, the behavior of the system strongly depends on the value of n, i.e., the number of strategies. Notably, in the most simple case with $n = 2$, we can observe the prevalence of one strategy, their coexistence, and bistable behaviors. While for $n > 2$, different behaviors can be observed as limit cycles and chaos.

2.2 The Ising Model

Phase transitions are critical collective phenomena constituting one of the most important topics of Physics. Despite their underlying complexity, a number of models, for describing their dynamics, has been proposed. Among them, the Ising

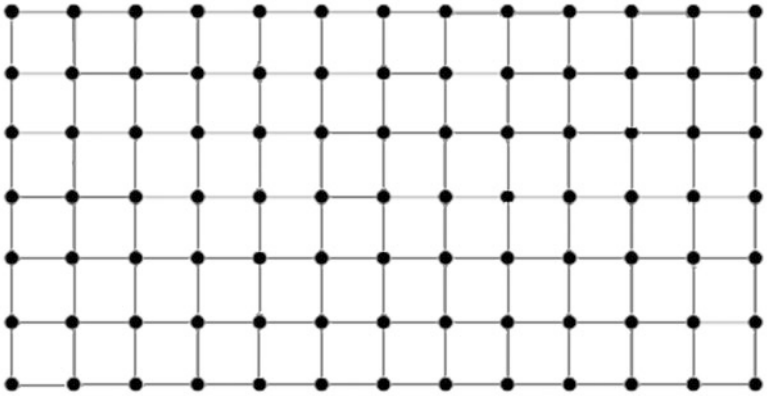

Fig. 2.1 A simple square lattice, i.e. with dimension $D = 2$

model covers a special relevance, being at the same time both simple, powerful, and, in addition, useful for investigating also the behavior of nonphysical systems (e.g., see the area named Sociophysics). The Ising model considers a lattice of dimension D composed of N cells (see Fig. 2.1), each one provided with a spin $\sigma = \pm 1$. Since a cell may, in general, represent different kinds of objects (e.g., atoms, neurons, etc.), this simple model can be adopted in a wide variety of contexts. Accordingly, the variable σ takes a meaning whose value depends on the related scenario, e.g., a magnetic moment, an opinion, an agent state, and so on. Here, it is worth to emphasize that the restricted range of σ, i.e., ± 1 in the majority of cases, is reflected in the descriptive power of the model. However, a number of problems can be successfully faced by using this very simple modelization. Then, in the defined lattice, a pair of cells (e.g., (i, j)) forms a bond J, which represents their interaction. The whole set of bonds can be denoted as B, and for each element of the set, we have an energy of value $-J\sigma_i\sigma_j$. In doing so, the interaction energy is equal to $-J$ for $\sigma_i = \sigma_j$ and to J in the opposite case. If J is positive, the case $\sigma_i = \sigma_j$ has an energy smaller than the case $\sigma_i = -\sigma_j$, so the former is more stable. Positive interactions (i.e., $J > 0$) are defined as "ferromagnetic," while negative interactions as "antiferromagnetic." In addition, some sites of the lattice can have an own energy of value $-h\sigma_i$ (here h may represent an external field). So, the Hamiltonian of the Ising model reads

$$H = -J \sum_{(ij) \in B} \sigma_i \sigma_j - h \sum_{i=1}^{N} \sigma_i \tag{2.5}$$

Once defined the Hamiltonian function, it is possible to compute the expected value of a physical quantity, of the system under consideration, by using the

Gibbs-Boltzmann distribution. For instance, it is interesting to compute the average spin configuration Σ at a given temperature T. To this end, we compute the distribution

$$P(\Sigma) = \frac{e^{-\beta H(\Sigma)}}{Z} \tag{2.6}$$

with Z representing the partition function, and $\beta = \frac{1}{k_b T}$, i.e., inverse of the product between the Boltzmann constant k_b and the system temperature T. As for the partition function Z, its role is normalizing the distribution (i.e., Eq. (2.6)) and, in general, takes the following form

$$Z = \sum_i e^{-\beta H(\Sigma_i)} \tag{2.7}$$

with the summation extended to all possible spin configurations. It is important to emphasize that, unfortunately, the explicit definition of Z is not always trivial. Like previously mentioned, when spins σ have values different from ± 1, other models can be considered (e.g., the XY model). Among the quantities that can be measured in the Ising model, the parameter called magnetization is often particularly useful, allowing to have a high level overview on the system. In particular, the magnetization is defined as

$$m = \frac{1}{N} \langle \sum_{i=1}^{N} \sigma_i \rangle \tag{2.8}$$

In the thermodynamic limit, i.e., for $N \to \infty$, this parameter (i.e., m) measures the order of a system. Notably, the magnetization vanishes when the amount of positive spins is equal to that of negative ones, i.e., full disorder, and it is maximized when all spins are aligned in the same direction. It is then interesting to evaluate how the temperature affects the state of order of a system. Notably, at low temperatures (i.e., for $\beta \gg 1$), Eq. (2.6) suggests that low-energy configurations have a probability higher than high-energy configurations. Moreover, in absence of external fields (i.e., for $h = 0$), low-energy states of the Ising model have all spins pointing in the same direction, so that the magnetization m has a (absolute) value close to 1. Now, increasing the temperature T, spin configurations with various energies emerge with equal probabilities. Accordingly, the macroscopic state of the Ising model becomes disordered, and its magnetization goes to zero. Therefore, it is possible to identify a relation between m and T and, most importantly, a critical temperature T_c. The latter entails that for $T < T_c$ the magnetization is greater than zero, while for $T > T_c$ the magnetization reduces until its value goes to zero. The phenomenon here briefly described is known as "order-disorder phase transition," and it has a deep relevance both in Physics and in the related applications to complex systems. Eventually, we

remind that spin configurations achieved at $T < T_c$ correspond to a ferromagnetic phase, while those achieved at $T > T_c$ correspond to a paramagnetic phase.

2.2.1 Mean Field

In principle, the Gibbs-Boltzmann distribution defined in Eq. (2.6) indicates that it is possible to compute the expected value of any physical quantity. However, due to the huge amount of sums over 2^N terms in the partition function (i.e., Eq. (2.7)), sometimes this task is actually almost impossible (in a limited amount of time). Thus, in such cases, the utilization of opportune methods of approximation becomes mandatory, like, for instance, the mean-field approach now described. In few words, the mean-field approach neglects fluctuations of variables around their mean values. Notably, we assume $m = \frac{\sum_i \langle \sigma_i \rangle}{N} = \sum_i \langle \sigma_i \rangle$, and the deviation $\delta\sigma_i = \sigma_i - m$, in addition the second-order term with respect to the fluctuation $\delta\sigma_i$ is assumed to be small enough to be neglected. Accordingly, the Hamiltonian can be rewritten as

$$\begin{aligned} H &= -J \sum_{(ij)\in B} (m + \delta\sigma_i)(m + \delta\sigma_j) - h \sum_{i=1}^{N} \sigma_i \\ &\sim -Jm^2 N_B - Jm \sum_{(ij)\in B} (\delta\sigma_i + \delta\sigma_j) - h \sum_{i=1}^{N} \sigma_i \end{aligned} \tag{2.9}$$

with N_B number of elements in the set B. Here $\delta\sigma_i$ and $\delta\sigma_j$, which refer to the extrema of each bond, are summed up z times, with z number of bonds starting from a site. In doing so, the Hamiltonian can be finally reduced to the following form:

$$H = N_B J m^2 - (Jmz + h) \sum_{i=1}^{N} \sigma_i \tag{2.10}$$

Becoming much more easy for analytical calculations.

2.2.2 Curie-Weiss Model

An important aspect of the Ising model is given by its dimension D. Notably, for $D = 1$, the Ising model has no phase transitions at finite temperature. For $D = 2$, according to the Onsager's solution, there is a phase transition (at a finite temperature). Then, in higher dimensions, although a phase transition can be observed, the definition of an analytical solution still constitutes an open problem. In particular, for $D = 3$, the problem has been solved only by a numerical approach,

while for $D > 3$ a solution is still required. Here, we briefly present a toy model that allows to describe the behavior of ferromagnetic transitions at infinite dimension, i.e., the Curie-Weiss (CW hereinafter) model. Remarkably the latter, despite being a toy model, has been proven to have a great relevance both in statistical mechanics and in information theory. The infinite dimension of the system entails that, in the CW model, each spin is connected with all the others. Accordingly, its Hamiltonian reads

$$H(\sigma_1, \ldots \sigma_n) = -\frac{1}{N} \sum_{(i<j)} \sigma_i \sigma_j - h \sum_{i=1}^{N} \sigma_i \tag{2.11}$$

As result, using the jargon of graph theory, the CW model defines a complete graph of N nodes and $N(N-1)/2$ links. Then, using the magnetization m before introduced, and without considering the contribution of external fields, the Hamiltonian of this model takes following form

$$H(\sigma_1, \ldots \sigma_n) = -\frac{N}{2} m^2 + O(1) \tag{2.12}$$

As above reported, for computing the expected values of physical quantities of systems like those we are considering, we need to compute the partition function of the system that in this case is equal to

$$Z = \sum_{\{\sigma_i\}} e^{-\beta H(\sigma_1, \ldots, \sigma_N)} \tag{2.13}$$

with H defined in Eq. (2.11). Now, further calculations are required for solving the equation, as for instance, for computing the summation over the spin variables appearing in Eq. (2.13). However, without to show the whole mathematical derivation, we only report the final equation of state of the CW model

$$m = \tanh(\beta J m + \beta h) \tag{2.14}$$

2.2.3 *Landau Theory of Phase Transitions*

The approaches here presented for studying the phenomenology of phase transitions by analytical methods allow to compute the partition function of a system. Therefore, at least in principle, a number of quantities can be computed according to Eq. (2.6). For a reason that will be soon explained, it is now worth to introduce an important thermodynamic potential, named "free energy," that allows to study the state of equilibrium of a system. In particular, the so-called Helmholtz free energy is defined as $F = U - TS$, with U internal energy, T temperature, and S entropy. So, since the second law of thermodynamics states that a system evolves toward

the state that maximizes its entropy, this law can be re-paraphrased stating that the state of equilibrium of a system corresponds to one that minimizes its free energy F. Now, we can motivate why we moved from the brief remark on the role of the partition function to the introduction of the free energy. Notably, a very important relation links the two quantities

$$-k_b T \ln Z = U - TS \tag{2.15}$$

As a result, we have $Z = e^{-\beta F}$. Here, the mean-field theory allows to obtain an approximated phase diagram of the system. However, when the latter is close to the critical point (e.g., the critical temperature), its behavior can be analyzed by using the formulation introduced by Landau, named "Landau theory of phase transitions." The underlying assumption of this theory is that a system close to the critical point has a small order parameter (i.e., m), which leads to the expression of the free energy as the following summation of power series:

$$F(T;m) = f(T;0) + \frac{1}{2}a(T)m^2 + \frac{1}{4!}b(T)m^4 + \ldots \tag{2.16}$$

with $a(T)$ and $b(T)$ coefficients that can be computed analytically. For instance, Fig. 2.2 shows the free energy of the CW model (with $h = 0$). In particular, for $T > T_c$ there is only one minimum of free energy ($m = 0$), corresponding to the state

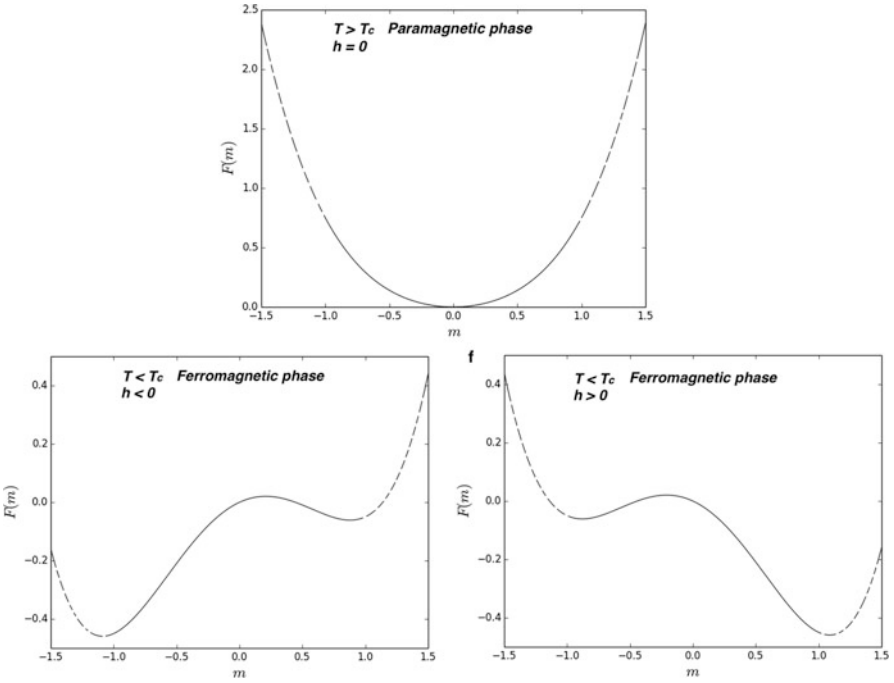

Fig. 2.2 Free energy in functions of the order parameter m, in absence of an external field

defined "paramagnetic phase." Instead, for $T < T_c$ there are two possible minima of free energy, and the symmetry $m \rightarrow -m$ is spontaneously broken (phenomenon known as "symmetry breaking").

2.2.4 Numerical Simulations

After introducing the Ising model, and some methods of approximation for studying special cases (e.g., $D \geq 4$), we can move toward a quick presentation of computational methods for performing the related numerical simulations. This part is of particular interest for achieving a preliminary idea on how to implement numerical simulations using networked agents, i.e., agents arranged in graphs, no matter if regular (as in the case of the Ising model) or random (as in the case of Complex Networks below discussed). In general, if we want to investigate the properties of a system using, for example, the formalism of the canonical ensemble, we have to deal with a system described by the macrostate (N, V, T) (i.e., number of particles, volume, and temperature). Given a microstate σ, a generic observable can be indicated as $O(\sigma)$, and its average value at equilibrium is

$$\langle O(\sigma) \rangle = \frac{1}{Z} \sum_{\sigma} e^{-\beta H(\sigma)} \tag{2.17}$$

thus, without the knowledge of the partition function Z, we cannot compute the average value of our observable $O(\sigma)$. A further example, previously mentioned, is given by the Ising model, where numerical simulations become mandatory for studying its behavior for $D \geq 3$. So, in order to overcome this limit, we can adopt Monte Carlo (MC hereinafter) methods for computing the value of the quantities we are interested in. The underlying idea of MC methods is to generate subset configurations (from the whole phase space), with a weight given by the Boltzmann statistics, that are representative for the entire ensemble. So, generating, for example, M configurations, we can have an estimate of the observable computing its average value, i.e.,

$$\langle O(\sigma) \rangle = \frac{1}{M} \sum_{i=1}^{M} O_i(\sigma) \tag{2.18}$$

Therefore, we are able to compute the average value of a physical quantity avoiding to deal with the partition function Z of the system (as in Eq. (2.17)). Now, before to proceed, it is worth to remind that the analytical solutions of a system (e.g., the Ising model) usually are computed considering the thermodynamic limit (i.e., $N \rightarrow \infty$). So, from a computational point of view, the first problem is how to approximate such limit/condition. In the case of the Ising model, a viable solution is given by the implementation of lattices which size is sufficiently big, removing the finite

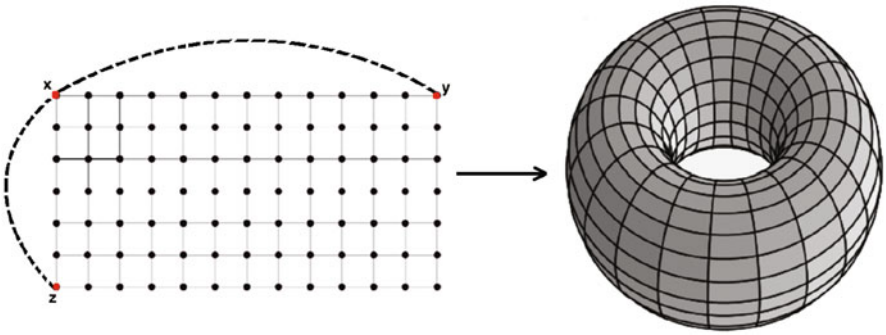

Fig. 2.3 Generating a 2D lattice with continuous boundary conditions, i.e. a toroid as shown on the right. The three red sites are connected by a dotted bond, showing how to generate the resulting toroid, i.e. the site x is connected to the site y and to the site z. As result, x has four bonds

size effect using a simple trick, i.e., generating lattices with continuous boundary conditions. Actually, under this condition, a bidimensional lattice takes the form of a toroid. In particular, this transformation can be easily performed by connecting the sites at the edges of the lattice. For instance, Fig. 2.3 shows an example focusing on the site named x, that is, connected to the sites y and z, increasing its amount of bonds up to four. Going back to the problem of performing a numerical simulation of the Ising model, we can implement different algorithms. Here, we refer to one of the most famous, i.e., the Metropolis algorithm. We remind that our aim is measuring the value of parameters like the magnetization at equilibrium. As we know from theory, at low temperatures we expect a ferromagnetic phase, i.e., a system close to the order, while at high temperatures a paramagnetic phase, i.e., a disordered system. The Metropolis algorithm is very simple, and its steps are:

1. Randomly select a site i and compute the local ΔE associated to its spin flip
2. IF ($\Delta E \leq 0$): accept the flip;
   ELSE: accept the flip with probability $e^{\frac{-\Delta E}{k_b T}}$

repeated until an equilibrium state is reached. The ΔE indicates a local difference in energy, i.e., computed considering only the selected site and its nearest neighbors.

2.3 Complex Networks: A Very Short Overview

Nowadays, Complex Networks represent a vibrant and independent research field that attracted the attention of scientists coming from different areas. The underlying reason is that many natural and man-made complex systems, as biological neural networks, social networks, and infrastructural networks, have a nontrivial topology that strongly influences the dynamics among the related agents (i.e., users of social networks, neurons of neural networks, and so on). An increasing amount of

investigations is demonstrating the relevance of the interaction structure in a wide amount of systems, and, even in the case of EGT, complex networks allow to obtain very interesting results. For instance, as recalled in Chap. 1, Santos and Pacheco showed the role of heterogeneity in the emergence of cooperation, modeling their system with scale-free networks. The latter, as well as others famous models, often is used as toy model both in EGT and in many other contexts as social dynamics, ecological networks, etc. Thus, in this section, we provide a very short overview on the main network properties, and on three different models that can be used for generating a complex network with a known topology. Readers interested in this topic are warmly encouraged to read the wide literature on complex networks. So, first of all, modern network theory has its basis in the classical theory of graphs. In particular, a preliminary definition of complex network can be "a graph with a nontrivial topology." In general, a graph is a mathematical object that allows to represent relations among a collection of items, named nodes. More formally, a graph G is defined as $G = (N, E)$, with N set of nodes/vertices and E set of edges/links (or bonds). Nodes can be described by a label and represent the elements of a system, e.g., users of a social network, websites of the WEB, and so on. In turn, the edges represent the connections among nodes, and map relations as friendship, physical links, etc. A graph can be "directed" or "undirected," i.e., the relation can be symmetrical (e.g., friendship) or not (e.g., a one way road), and can be "weighted" or "unweighted." The former allows to introduce some coarseness in the relations, e.g., in a transportation network the weights might refer to the actual geographical distance between two locations. The information related to the connections in a network is saved in a $N \times N$ matrix, with N number of nodes, defined "adjacency matrix." Numerical analysis on the adjacency matrix allow to investigate the properties of a network. For instance, the adjacency matrix A of an unweighted graph can have the following form:

$$a_{ij} = \begin{cases} 1 & \text{if } e_{ij} \text{ is defined} \\ 0 & \text{if } e_{ij} \text{ is not defined} \end{cases} \tag{2.19}$$

On the other hand, in the case of weighted networks, the inner values of the adjacency matrix are real. Among the properties of a complex network, the degree distribution is one of the most relevant. Notably, this "centrality measure" constitutes a kind of signature for classifying the nature of a network (e.g., scale-free), where the term "degree" means amount of connections (i.e., edges) of a node. So, indicating with k the degree of nodes, the distribution $P(k)$ of a network represents the probability to randomly select a node with a degree equal to k, i.e., a node with k connections. A second network property is called clustering coefficient, and it allows to know if nodes of a network tend to cluster together. Actually, this phenomenon is common in many real networks as social networks, where it is possible to identify circles of friends, or acquaintances in which every person knows all the others. For the sake of clarity, considering a social network, if the user

a is connected to the user b, and the latter is connected to the user c, there is a high probability that a be connected to c. The clustering coefficient can be computed as

$$C = \frac{3 \times Tn}{Tp} \tag{2.20}$$

with Tn number of triangles in a network, and Tp number of connected triples of nodes. A connected triple is a single node with links running to an unordered pair of others. This coefficient has a range that spans the interval $0 \leq C \leq 1$. A further mathematical definition of the clustering coefficient reads

$$C_i = \frac{Tn_i}{Tp_i} \tag{2.21}$$

with Tn_i number of triangles connected to node i, and Tp_i number of triples centered on node i. The main difference between the two definitions is that the second one is local, so that to obtain a global value one has to compute the following parameter

$$C = \frac{1}{n} \sum_i C_i \tag{2.22}$$

In doing so, one measures the density of triangles in a network, no matter if the structure is directed or undirected. Readers familiar with literature in Sociology might find the same parameter defined as "network density." Another measure is the betweenness centrality that quantifies the centrality of a node (or of an edge) in a network. Notably, this parameter refers to the number of geodetics from all nodes to all others that pass through a given node and can be computed as

$$B_i = \sum_{x \neq i \neq y} \frac{\sigma_{xy}(i)}{\sigma_{xy}} \tag{2.23}$$

with σ_{xy} total number of geodesics from the x-th node to the y-th node, $\sigma_{xy}(i)$ total number of geodesics from x-th node to y-th node passing through the i-th node. A similar approach can be implemented also for computing the centrality of an edge, i.e., considering the relevance of a particular edge in a network. Finally, we recall the parameter named assortativity that allows to evaluate if nodes of a network tend to attach to other nodes that are (not) similar. This property actually affects the whole structure of a network, e.g., social networks can be divided into communities of users speaking the same language or having same hobbies. Often indicated as r, the assortativity reads

$$r = \frac{\sum_i e_{ii} - \sum_i a_i b_i}{1 - \sum_i a_i b_i} \tag{2.24}$$

with e_{ij} fraction of edges in a network that connects a node of type i to one of type j, $a_i = \sum_j e_{ij}$ and $b_j = \sum_i e_{ij}$. A network is assortative when r is positive, while, if negative, is defined disassortative. It is worth to remind that the similarity can refer to different features, i.e., not only to the amount of links. Eventually, before to show the algorithms to implement some toy models, we remind that complex networks often show a community structure, i.e., nodes appear to be strongly connected, forming well-defined groups. Often, finding communities is not trivial, and this task requires the implementation of the so-called community detection algorithms.

2.3.1 Network Structures

Here, we briefly provide the algorithms for implementing three famous models of complex networks: classical random networks, scale-free networks, and small-world networks.

2.3.1.1 Classical Random Networks

One of the early works on random networks has been developed by Paul Erdös and Alfred Renyi. Their model, usually called E-R model/graph, considers a graph with N nodes and a probability p to generate each edge. Accordingly, an E-R graph contains about $p \cdot \frac{N(N-1)}{2}$ edges, and it has a binomial degree distribution

$$P(k) = \binom{N-1}{k} p^k (1-p)^{n-1-k} \tag{2.25}$$

for $N \to \inf$ and $np = \text{const}$, the degree distribution converges to a Poissonian distribution

$$P(k) \sim e^{-pn} \cdot \frac{(pn)^k}{k!} \tag{2.26}$$

To generate this kind of networks, one can implement the following simple algorithm:

1. Define the number of N of nodes and the probability p for each edge
2. Draw each potential-link with probability p

Figure 2.4 illustrates the $P(k)$ for an E-R graph with $N = 25,000$ and $p = 4 \cdot 10^{-4}$.

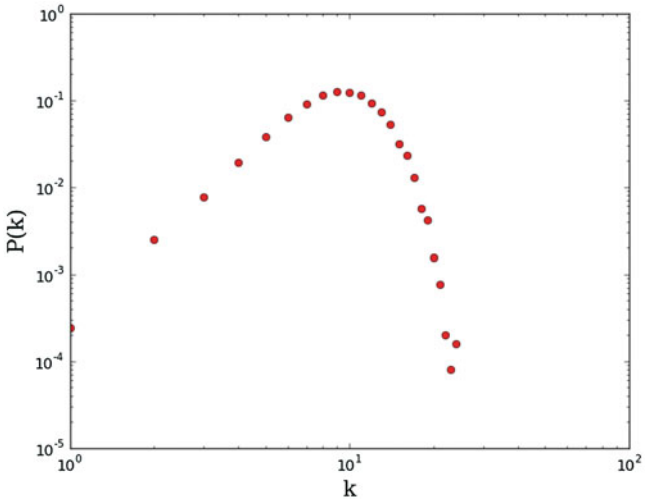

Fig. 2.4 $P(k)$ (in loglog scale) of an E-R graph with $N = 25,000$ and $p = 4 \cdot 10^{-4}$

2.3.1.2 Scale-Free Networks

Scale-free networks are characterized by the presence of few nodes (called hubs) that have many connections (i.e., a high degree), while the majority of nodes has a low degree. Therefore, these networks constitute a classical example of heterogeneous networks. The related degree distribution follows a power-law function

$$P(k) \sim c \cdot k^{-\gamma} \tag{2.27}$$

with c normalizing constant and γ scaling parameter of the distribution. A famous model for generating scale-free networks is the Barabasi-Albert model (BA model hereinafter) that considers two parameters: N nodes and m minimum number of edges drawn for each node. The BA model can be summarized as follows:

1. Define N number of nodes and m minimum number of edges drawn for each node
2. Add a new node and link it with other m pre-existing nodes. Pre-existing nodes are selected according to the following equation:

$$\Pi(k_i) = \frac{k_i}{\sum_j k_j} \tag{2.28}$$

with $\Pi(k_i)$ probability that the new node generates a link with the i-th node (having a k_i degree).

Figure 2.5 illustrates the $P(k)$ for a scale-free network with $N = 25,000$ and $m = 5$.

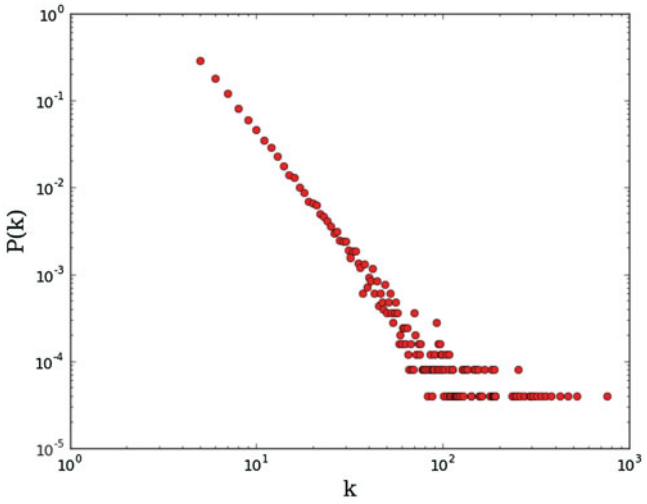

Fig. 2.5 $P(k)$ (in loglog scale) of a scale-free network with $N = 25,000$ and $m = 5$

2.3.1.3 Small-World Networks

A small-world network is characterized by the fact that every node can be reached from any other in a small number of hops. More formally, small-world networks have a distance L, between two randomly chosen nodes, equal to $L \propto \ln N$. Two main properties allow to evaluate if a network has a small-world structure, i.e., a shortest average path length and a relatively high clustering coefficient. In particular, the clustering coefficient of a small-world network is higher than that of its related E-R graph, i.e., the classical random network generated with the same set of nodes. Watts and Strogatz developed a very famous algorithm, i.e., the Watts-Strogatz model (WS hereinafter), for implementing small-world networks:

1. Define a regular ring lattice with N nodes, each connected to k neighbors ($k/2$ on each side)
2. For every node i take every edge (i, j) with $i \leq j$ and rewire it with probability β. Rewiring is done by replacing the edge (i, j) with (i, k) with k chosen with uniform probability from all nodes avoiding loop and edge duplication

The WS model shows an interesting behavior studying the effect of the rewiring probability β. In particular, we can start with a regular (ring) lattice setting $\beta = 0$, and we can obtain a completely disordered network by increasing the value of β up to 1. So, at intermediate values of β, the WS model generates networks that consist of a mixture of random and regular connections, providing the network with the small-world structure. This behavior is illustrated in Fig. 2.6. To conclude, the reader can use these algorithms for generating structured populations, whose agents play an evolutionary game. In doing so, it is possible to compare the outcomes on varying the underlying topology.

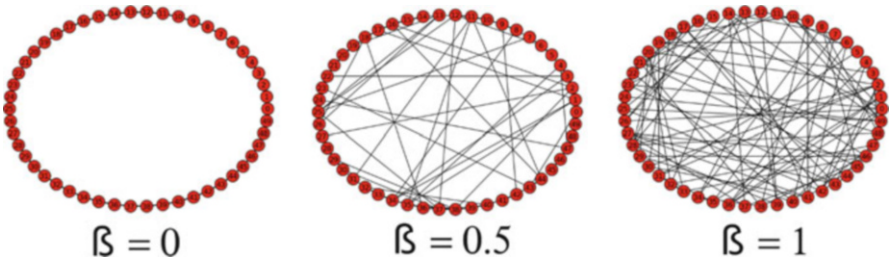

Fig. 2.6 Small-world networks generated by the WS model. From left to right: network generated with $\beta = 0$, $\beta = 0.5$ and $\beta = 1$

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Chapter 3
Evolutionary Games I: Statistical Physics

This chapter aims to illustrate some Statistical Physics approaches to EGT. In particular, the first part presents a model for studying the emergence of cooperation, in the Prisoner's Dilemma, mapping agents to particles of a gas, then using the kinetic theory of gases. The second part analyzes the dynamics of the Public Goods Game by varying the parameter named 'Temperature' (known also as "Noise") appearing in the equation adopted for performing the Strategy Revision Phase. Notably, results show how to link this game with the Voter Model. The reader interested in further details can find a brief list of manuscript at the end of the chapter. In particular, the content of this chapter is based on the first two manuscripts cited at the end of the conclusions.

3.1 Phase Transitions in the Prisoner's Dilemma

Now, we introduce an analytical model for studying the evolution toward equilibrium in spatial games, with "memory-aware" agents, i.e., agents that accumulate their payoff over time. In particular, we focus our attention on the PD, since as previously mentioned it constitutes an emblematic example of a game whose Nash equilibrium is defection. Previous investigations showed that, under opportune conditions, in this game, it is possible to reach an equilibrium of cooperation. In particular, it has been proved that some mechanisms, as random motion, can support an agent population to become cooperative. In the proposed model, we map agents to particles of a gas so that their motion can be related to the system temperature. In doing so, we can identify a relation between the temperature and the final equilibrium of our population, explaining how it is possible to break the classical Nash equilibrium in this game. It is worth to emphasize that the underlying condition, adopted in this investigation, is that agents are able to increase their payoff over time (thus named "memory-aware" agents). Remarkably, this condition

M.A. Javarone, *Statistical Physics and Computational Methods for Evolutionary Game Theory*, SpringerBriefs in Complexity,
https://doi.org/10.1007/978-3-319-70205-6_3

represents the major difference with most of the evolutionary game models studied by computational approaches. On the other hand, considering "memory-aware" agents makes the problem more tractable from an analytical perspective. Finally, we introduce a formalism for studying order-disorder phase transitions in these dynamics. We remind that beyond trying to understand why the random motion supports cooperation (in this game), an important goal of this investigation is to strengthen the link between EGT and statistical physics (see also Chap. 1).

3.1.1 Model

Here, we are interested in studying the prisoner's dilemma by an analytical approach, for the reasons above mentioned. Before introducing our model, let us remind the general form of a payoff matrix:

$$\begin{array}{cc} & \begin{array}{cc} C & D \end{array} \\ \begin{array}{c} C \\ D \end{array} & \begin{pmatrix} R & S \\ T & P \end{pmatrix} \end{array} \tag{3.1}$$

where the set of strategies is $\Sigma = \{C, D\}$: C stands for Cooperation and D for Defection. In the matrix (3.1), R is the gain achieved by two interacting cooperators, T represents the *Temptation*, i.e., the payoff that an agent receives whether it defects while its opponent cooperates, S the *Sucker's payoff*, i.e., the gain received by a cooperator while the opponent defects, eventually P the payoff of two interacting defectors. In the case of the PD, we can set the matrix elements of (3.1) to the following values: $R = 1$, $0 \leq S \leq -1$, $1 \leq T \leq 2$, and $P = 0$. As stated before, during the evolution of the system, agents can change their strategy from C to D, and vice versa, following an updating rule, as, for instance, the one named "imitation of the best," where they imitate the strategy of their richest (i.e., fittest) neighbor.

3.1.1.1 Mean Field Approach

Let us consider a mixed population composed of N agents, with an initial uniform starting distribution of strategies (i.e., cooperation and defection). As condition all agents can interact together so that, at each time step, the payoff gained by cooperators and defectors can be computed as follows:

$$\begin{cases} \pi_c = (\rho_c \cdot N - 1) + (\rho_d \cdot N)S \\ \pi_d = (\rho_c \cdot N)T \end{cases} \tag{3.2}$$

with $\rho_c + \rho_d = 1$, ρ_c density of cooperators and ρ_d density of defectors. We recall that, in the PD, defection is the dominant strategy, and, even setting $S = 0$ and

$T = 1$, it corresponds to the final equilibrium because π_d is always greater than π_c. We recall that usually these investigations are performed by using "memoryless" agents (i.e., agents unable to accumulate a payoff over time) whose interactions are defined only with their neighbors and focusing only on one agent (and its opponents) at a time. These conditions strongly influence the dynamics of the population. For instance, if at each time step we randomly select one agent, which interacts only with its neighbors, in principle it may occur that a series of random selections picks consecutively a number of close cooperators; therefore, in this case, we can observe the emergence of very rich cooperators, able to prevail on defectors, even without introducing mechanisms like motion. In addition, when $P = 0$, a homogeneous population of defectors does not increase its overall payoff. Instead, according to the matrix (3.1), a cooperative population continuously increases its payoff over time.

Now, we consider a population divided, by a wall, into two groups: a group G^a composed of cooperators and a mixed group G^b (i.e., composed of cooperators and defectors). Agents interact only with members of the same group, then the group G^a never changes, and, accordingly, it strongly increases its payoff over time. The opposite occurs in the group G^b, as it converges to an ordered phase of defection, limiting its final payoff once cooperators disappear. In this scenario, we can introduce a strategy to modify the equilibria of the two groups. In particular, we can turn to cooperation, the equilibrium of G^b, and to defection that of G^a. In the first case, we have to wait a while, and to move one or few cooperators to G^b, so that defectors increase their payoff, but during the revision phase, they become cooperators, since the newcomers are richer than them. In the second case, if we move, after few time steps, a small group of defectors from G^b to G^a, the latter converges to a final defection phase. These preliminary and theoretical observations let emerge an important property of the "memory-aware" PD: considering the two different groups, cooperators may succeed when act after a long time and individually. Instead, defectors can prevail by a fast group action. Notably, rich cooperators have to move individually; otherwise many of them risk to feed defectors, i.e., to increase too much their payoff, so avoiding that they change strategy. The opposite holds for defectors that, acting in group, may strongly reduce the payoff of a community of cooperators (for $S < 0$).

3.1.1.2 Mapping Agents to Gas Particles

We hypothesize that the PD, with moving agents, can be successfully studied by the framework of the kinetic theory of gases. Therefore, the idea is mapping agents to particles of a gas. In doing so, the average speed of particles can be computed as $\langle v \rangle = \sqrt{\frac{3T_s k_b}{m_p}}$, with T_s system temperature, k_b Boltzmann constant, and m_p particle mass. Particles are divided into two groups by a permeable wall. Thus, the latter can be crossed but, at the same time, avoids interactions among particles staying in the opposite sides (i.e., belonging to different groups). In doing so, we can provide a dual description of our system: one in the "physical" domain of particles

and the other in the "information" domain of agents. Notably, for analyzing the system in the "information" domain, strategies are mapped to spins. Summarizing, we map agents to gas particles in order to represent their "physical" property (i.e., random motion), and we map the strategies used by agents to spins for representing their "information" property (i.e., the strategy). These two mappings can be viewed as two different layers for studying how the agent population evolves over time. Although the physical property (i.e., the motion) affects the agent strategy (i.e., its spin), the equilibrium can be reached in both layers/domains independently. This last observation is quite important, since we are interested in evaluating only the final equilibrium reached in the "information" domain. Then, as stated before, agents interact only with those belonging to the same group, and the evolution of the mixed group G^b can be described by the following equations:

$$\begin{cases} \dfrac{d\rho_c^b(t)}{dt} = p_c^b(t) \cdot \rho_c^b(t) \cdot \rho_d^b(t) - p_d^b(t) \cdot \rho_d^b(t) \cdot \rho_c^b(t) \\ \dfrac{d\rho_d^b(t)}{dt} = p_d^b(t) \cdot \rho_d^b(t) \cdot \rho_c^b(t) - p_c^b(t) \cdot \rho_c^b(t) \cdot \rho_d^b(t) \\ \rho_c^b(t) + \rho_d^b(t) = 1 \end{cases} \tag{3.3}$$

with $p_c^b(t)$ probability that cooperators prevail on defectors (at time t) and $p_d^b(t)$ probability that defectors prevail on cooperators (at time t). These probabilities are computed according to the payoffs obtained, at each time step, by cooperators and defectors:

$$\begin{cases} p_c^b(t) = \dfrac{\pi_c^b(t)}{\pi_c^b(t) + \pi_d^b(t)} \\ p_d^b(t) = 1 - p_c^b(t) \end{cases} \tag{3.4}$$

System (3.3) can be analytically solved provided that, at each time step, values of $p_c^b(t)$ and $p_d^b(t)$ be updated. Accordingly, the density of cooperators reads

$$\rho_c^b(t) = \frac{\rho_c^b(0)}{\rho_c^b(0) - [(\rho_c^b(0) - 1) \cdot e^{\frac{\tau t}{N^b}}]} \tag{3.5}$$

with $\rho_c^b(0)$ initial density of cooperators in G^b, $\tau = p_d^b(t) - p_c^b(t)$, and N^b number of agents in G^b. We recall that setting $T_s = 0$, not allowed in a thermodynamic system, corresponds to a motionless case, leading to the Nash equilibrium in G^b. Instead, for $T_s > 0$, we can find more interesting scenarios. Now we suppose that, at time $t = 0$, particles of G^a are much closer to the wall than those of G^b (later we shall relax this constraint); for instance, let us consider a particle of G^a that, during its random path, follows a trajectory of length d (in the n-dimensional physical space) toward the wall. Assuming that this particle is moving with a speed equal to $\langle v \rangle$, we

can compute the instant of crossing $t_c = \frac{d}{\langle v \rangle}$, i.e., the instant when it moves from G^a to G^b. Thus, on varying the temperature T_s, we can vary t_c.

Looking at the two groups, we observe that each cooperator in G^a gains

$$\pi_c^a = (\rho_c^a \cdot N^a - 1) \cdot t \tag{3.6}$$

while cooperators in G^b, according to the Nash equilibrium, rapidly decrease over time. Focusing on the variation of the payoff, of the last cooperator survived in G^b, we have

$$\pi_c^b = \sum_{i=0}^{t} [(\rho_c^b \cdot N^b - 1) + (\rho_d^b \cdot N^b)S]_i \tag{3.7}$$

moreover, $\pi_c^b \to 0$ as $\rho_c^b \to 0$. At $t = t_c$, a new cooperator reaches G^b, with a payoff computed with Eq. (3.6).

3.1.2 Results

The analytical solution (3.5) allows to analyze the evolution of the system and to evaluate how initial conditions affect the outcomes of the proposed model. It is worth observing that, if $\pi_c^a(t_c)$ is "enough big," the new cooperator may modify the equilibrium of G^b, turning defectors to cooperators. Notably, the payoff considered to compute p_c^b, after t_c, corresponds to $\pi_c^a(t_c)$, as the newcomer is the richest cooperator in G^b. Furthermore, we note that $\pi_c^a(t_c)$ depends on N^a; hence, we analyze the evolution of the system on varying the parameter $\epsilon = \frac{N^a}{N^b}$, i.e., the ratio between particles in the two groups. Finally, for numerical convenience, we set $k_b = 1 \cdot 10^{-8}$, $m_p = 1$, and $d = 1$.

Figure 3.1 shows the evolution of G^b, for $\epsilon = 1$ on varying T_s and, depicted in the inner insets, the variation of system magnetization over time (always inside G^b). As discussed before, in the physical domain of particles, heating the system entails to increase the average speed of particles. Thus, under the assumption that two agents play together if they remain in the same group for a long enough time, we hypothesize that there exists a maximum allowed speed for observing interactions in the form of game (i.e., if the speed is higher than this limit, agents are not able to play the game). This hypothesis requires a critical temperature T_c, above which no "effective" interactions, in the "information" domain, are possible. As shown in plot (f) of Fig. 3.1, for temperatures in the range $0 < T_s < T_{\max}$, the system converges to a cooperation phase (i.e., $M = +1$), for $T_{\max} < T_s < T_c$, the system follows the Nash equilibrium (i.e., $M = -1$), and for $T > T_c$, a disordered phase emerges at equilibrium. Thus, results of this model suggest that it is always possible to compute a range of temperatures to obtain an equilibrium of full cooperation—see Fig. 3.2. Furthermore, we study the variation of $T_{\max}$ on varying ϵ (see Fig. 3.3)

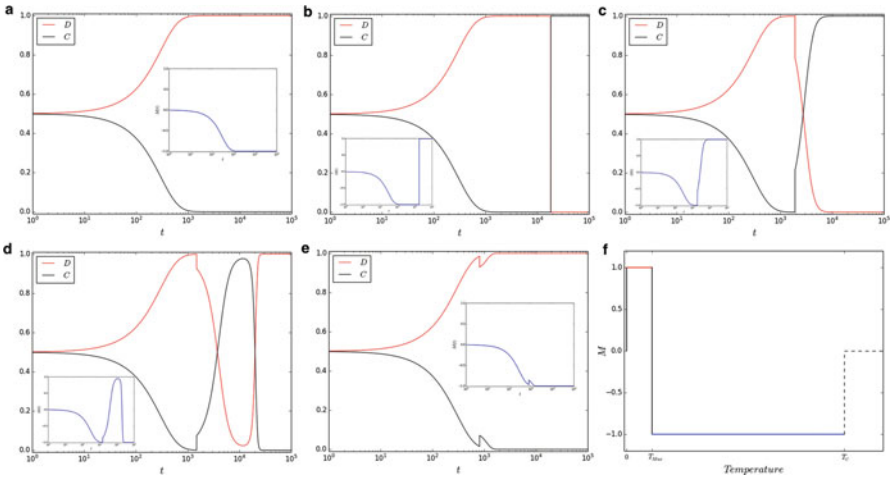

Fig. 3.1 From (**a**) to (**e**): Evolution of the group G^b, with $N = 100$ and $\epsilon = 1$, on varying the temperature: (**a**) $T_s = 0$. (**b**) $T_s = 0.1$. (**c**) $T_s = 9$. (**d**) $T_s = 15$. (**e**) $T_s = 50$. Insets show the system magnetization over time. The instant $t = t_c$ can be detected in plots (**c**–**e**) as a discontinuity of the two lines (i.e., red and black). (**f**) Final magnetization M, of G^b, for different temperatures (T_c indicates the "critical temperature")

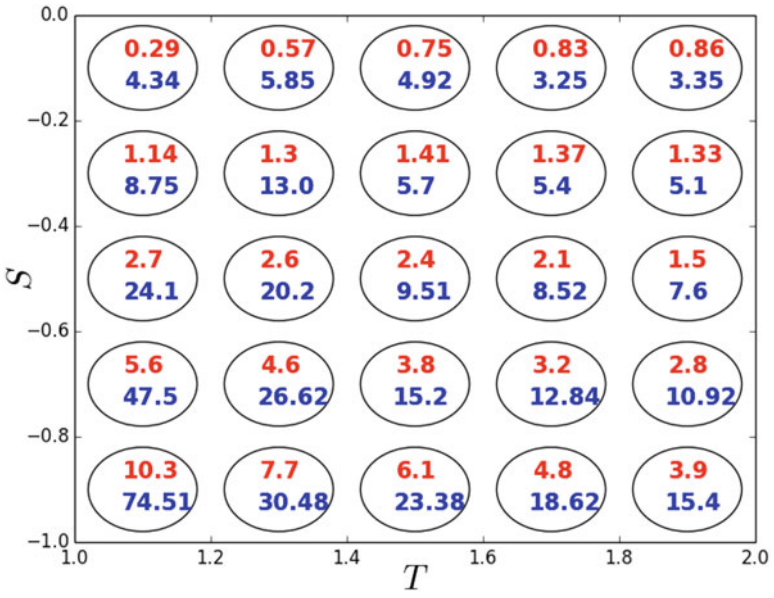

Fig. 3.2 Maximum values of temperature T_s that allow the group G^b to converge toward cooperation. Red values correspond to results computed with $\epsilon = 0.5$, while blue values to those computed with $\epsilon = 1$. Circles are placed in the TS diagram indicating values of T and S, of the payoff matrix, used for each case. Even for high values of T, and small values of S, it is possible to achieve cooperation

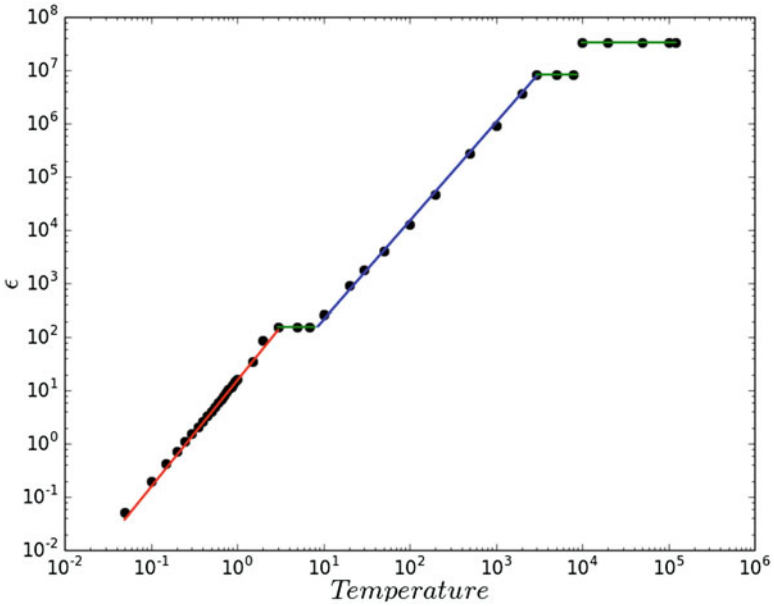

Fig. 3.3 Maximum value of system temperature that allows to achieve cooperation at equilibrium versus ϵ (i.e., the ratio between particles in the two groups). Different colors identify different trends, fitted by power-law functions. After the final green plateau, temperatures are too high to play the PD

showing that, even for low ϵ, it is possible to obtain a time t_c that allows the system to become cooperative. Eventually, we investigate the relation between the maximum value of T_s that allows a population to become cooperative and its size N (i.e., the amount of agents). As shown in Fig. 3.4, the maximum T_s scales with N following a power-law function characterized by a scaling parameter $\gamma \sim 2$. The value of γ has been computed by considering values of T_s shown in Fig. 3.2 for the case $\epsilon = 2$. Finally, it is worth to highlight that all analytical results let emerge a link between the system temperature and its final equilibrium. Recalling that we are not considering the equilibrium of the gas, i.e., it does not thermalize in the proposed model, we emphasize that the equilibrium is considered only in the "information domain."

3.1.3 Order-Disorder Phase Transitions in the Agent Population

As discussed before, in the "information domain," we can study the system by mapping strategies to spins. In addition, we can map the difference between winning

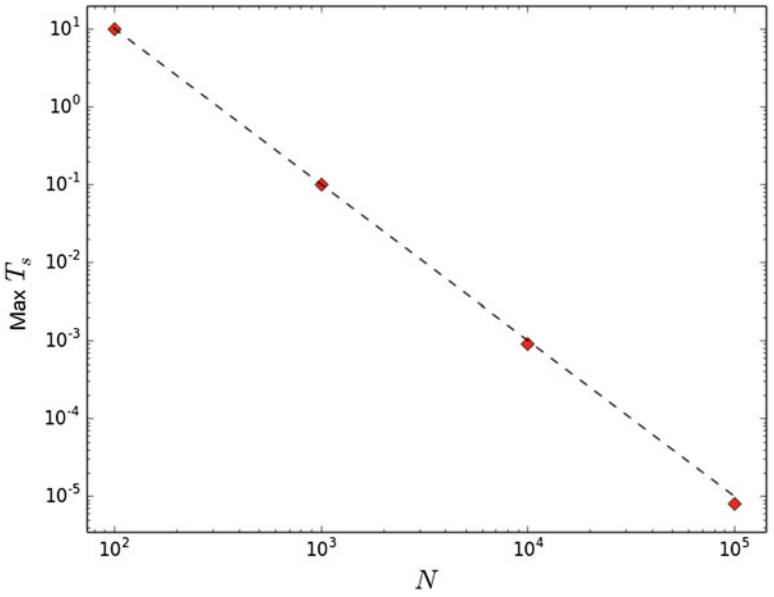

Fig. 3.4 Maximum value of T_s to achieve full cooperation at equilibrium in function of N, i.e., the size of the population. The fitting function (dotted line) is a power-law characterized by a scaling parameter equal to 2

probabilities, of cooperators and defectors, to an external magnetic field: $h = p_c^b - p_d^b$. In doing so, by the Landau theory, we can analytically identify an order-disorder phase transition. Notably, we analyze the free energy F of the spin system on varying the control parameter m (corresponding to the magnetization M)

$$F(m) = -hm \pm \frac{m^2}{2} + \frac{m^4}{4} \tag{3.8}$$

where the sign of the second term depends on the temperature, i.e., positive for $T_s > T_c$ and negative for $T_s < T_c$; we remind that T_c represents the temperature beyond which it is not possible to play the PD due to the high particle speed (according to our assumption). For the sake of clarity, we want to emphasize that the free energy is introduced in order to evaluate the nature of the final equilibrium achieved by the system. In particular, looking for the minima of F allows to investigate if our population reaches the Nash equilibrium, or different configurations (e.g., full cooperation). Figure 3.5 shows a pictorial representation of the phase transitions that can occur in our system, on varying T_s and the external field h. Then, the constraints related to the average speed of particles, and to the distance between each group and the permeable wall, can be in principle relaxed, as we can imagine to extend this description to a wider system with several groups, where agents are uniformly distributed in the whole space. Now, it is worth to highlight that our results are

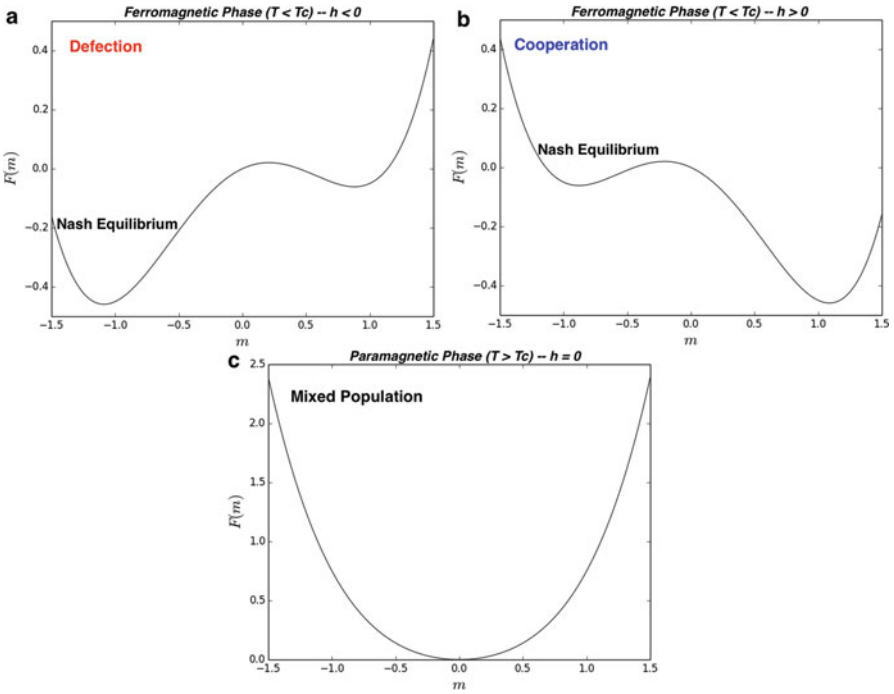

Fig. 3.5 Order-disorder phase transitions in the population. For $T_s < T_c$, the population is in a ferromagnetic phase: (**a**) Applying an external negative field, the system converges to the Nash equilibrium, corresponding to $m = -1$ (as $\sigma = -1$ represents defection); (**b**) Applying an external positive field, the population converges to cooperation ($\sigma = +1$), corresponding to $m = +1$. (**c**) For temperatures higher than T_c, a disordered paramagnetic phase emerges

completely in agreement with those achieved by authors who studied the role of motion in the PD and in addition are able to explain why clusters of cooperators emerge in these conditions. At the same time, we remind that, in this model, agents are "memory-aware," while usually investigations consider agents that reset their payoff at each step.

3.2 The Role of the Temperature in the Spatial Public Goods Game

In this section, we aim to analyze the role of the temperature in the spatial PGG. Before to proceed, it is important to remind the reader that, in this section, the terms "temperature" and "noise" refer to the same concept. As discussed in Chap. 1, the dynamics of this game are affected by a number of parameters and processes, namely, the topology of interactions among the agents, the synergy factor, and the

strategy revision phase. We remind that the latter is a process that allows agents to change their strategy. Notably, rational agents tend to imitate richer neighbors, in order to increase the probability to maximize their payoff. By implementing a stochastic revision process, it is possible to control the level of noise in the system, so that even irrational updates may be observed. In particular, we study the effect of noise on the macroscopic behavior of a finite structured population. We consider both the case of a homogeneous population, where the noise in the system is controlled by tuning a parameter representing the level of stochasticity in the strategy revision phase, and a heterogeneous population composed of a variable proportion of rational and irrational agents. In both cases numerical investigations show that the PGG has a very rich behavior, which strongly depends on the amount of noise in the system and on the value of the synergy factor. In doing so, we aim to provide a description of the PGG by the lens of statistical physics, focusing in particular on the impact of noise in the population dynamics. Saying that rational agents are those that tend to imitate their richer neighbors, we can state that irrational agents are those that randomly change their strategy. In the case of a homogeneous population, the intensity of noise in the system is controlled by tuning the level of stochasticity of all agents during the SRP, by means of a global parameter (indicated by K) that represents the noise/temperature. Instead, in the case of a heterogeneous population, the noise is controlled by tuning the density of irrational agents in the population. Results indicate that tuning the level of noise to interpolate between configurations where agents fully utilize payoff information (low noise) to those where they behave at random (high noise) strongly affects the macroscopic behavior of a population.

3.2.1 Model

In the case of well-mixed populations of infinite size, the behavior of the system can be predicted as a function of the synergy factor r by studying the related Nash equilibria. In particular, when agents play in groups of G players, two different absorbing states appear separated at a critical point $r_{\mathrm{wm}} = G$. The population falls into full defection for $r < r_{\mathrm{wm}}$ and into full cooperation for $r > r_{\mathrm{wm}}$. Conversely, when agents are arranged in the nodes of a network, surprisingly some cooperators can survive for values of r lower than r_{wm}. This effect, discussed in Chap. 1, is known as *network reciprocity*. At the same time, the network structure allows a limited number of defectors to survive also beyond $r = r_{\mathrm{wm}}$. We refer to the two critical values of r at which cooperators first appear and defectors eventually disappear from the population, respectively, as r_{c1} and r_{c2}. It is worth mentioning that most investigations in EGT are performed by numerical simulations, and an analytical definition of the critical thresholds (i.e., r_{c1} and r_{c2}) identified in networked topologies is missing. As a result, when studying EGT models by arranging agents in different spaces, the values of critical thresholds are achieved by Monte Carlo simulations (see Chap. 2). In a networked population, depending

on the values of r and on how agents are allowed to update their strategy, it is possible to observe different regimes: two ordered equilibrium absorbing phases, where only one strategy survives (either cooperation or defection), and an active but macroscopically stable disordered phase corresponding to the coexistence between the two species/strategies.

3.2.1.1 The Noise and the General Setup of the Model

As reported in Chap. 1, a crucial parameter appearing in Eq. (1.4) is K_y, which plays the role of noise and then parametrizes the uncertainty in adopting a strategy. Notably, a low noise entails agents to strongly consider the difference in payoff $\Delta p = \pi_y - \pi_x$ when deciding their next strategy, whereas increasing the noise the payoff difference plays a more marginal role. In the case of a homogeneous population, K is equal for all individuals, such that by tuning the value of this global variable, we are able to control the level of noise in the system. In the limit of $K = 0$, the y-th agent will imitate the strategy of the x-th agent with probability $W = 1$ if $\pi_x > \pi_y$, and $W = 0$ otherwise. Conversely, in the limit $K \to \infty$, the SRP becomes a coin flip, and the imitation occurs with probability $W = 1/2$ no matter the value of the synergy factor. In the latter case, the behavior of the PGG is analogous to that of a classical voter model where imitation between a pair of selected agents takes place with probability $W = 1/2$. We highlight that the aim of this analysis is to confirm computationally results reported in previous works, and to evaluate the relation between r and K, in order to provide a complete description of the PGG, from the microscopic dynamics to the global behavior of the population (i.e., following the Statistical Physics philosophy in the studying of a physical system). According to previous investigations, setting $K = 0.5$ is often considered a good choice to describe a rational population with a moderate level of noise and where only a limited number of irrational updates may occur. In the case of bidimensional lattices with periodic boundary conditions—see Fig. 3.6—since each agent has *four* neighbors, group interactions involve $G = 5$ players at a time (see Chap. 2). It has been shown in previous investigations that for such K, the values of r_{c1}, at which cooperators emerge, and r_{c2}, where defectors completely disappear from the population, are, respectively, equal to 3.75 and 5.5. Instead, a coexistence between cooperators and defectors occurs for intermediate values of r between the two thresholds (i.e., r_{c1} and r_{c2}). Conversely, in the corresponding PGG on well-mixed populations, where games are organized in groups of the same size $G = 5$, the full defection and full cooperation regimes are separated at the critical point $r_{\text{wm}} = 5$. It is also possible to consider the case of heterogenous populations where agents are characterized by different values K. In such scenario, the simplest setup is the one where only two different sets of agents exist: one endowed with K^1 and one with $K^2 > K^1$. Recalling that a higher value of K implies lower rationality in the SRP, by varying the density f of one species, with $0 \leq f \leq 1$, it is possible to control the level of noise in the system and to study the outcomes of the model in different conditions. For instance, setting $K^1 = 0.5$ and

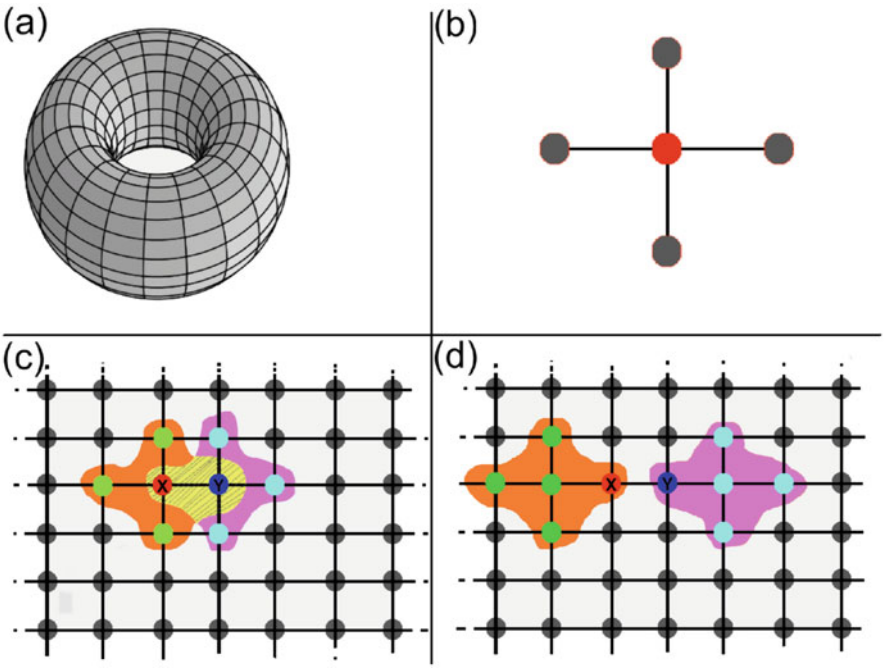

Fig. 3.6 Pictorial representation of the PGG on the considered topology. Agents are arranged in bidimensional square lattice with continuous boundary conditions, forming a toroid as shown in (**a**). Each agent belongs to five groups of size $G = 5$: one where he is the central player, in red, and four as a peripheral node, in black (**b**). At each time step, two agents x and y are randomly selected, and they play the PGG with all the players in their groups for all groups of belonging. In (**c**), we show the group where x and y are central: the green nodes are neighbors of node x, and this group has an orange shadow, while neighbors of agent y are cyan and this group has a violet shadow. The dotted lines in the area between x and y indicate the intersection between the groups formed by x and y. Notably, y belongs to the group formed by x and vice versa. In (**d**) we show for both x and y one of the possible groups where they are peripheral

$K^2 = \infty$, we can evaluate the influence of a density f of rational agents in driving the population toward a particular state. This is particularly useful to analyze the behavior of a population whose agents have a different sensibility to their payoff, and, from a social point of view, it allows to study the influence of rationality in driving the population toward an equilibrium. As shown in related works from the Sociophysics literature, random imitation is not the only relevant nonrational behavior able to impact the way in which agents choose their next strategy. Just to cite a few, investigations driven on simple principles such as social conformity or nonconformity, extremism, stubbornness, or multiplexity showed how simple changes in the microscopic dynamics of the agents can significantly affect the social dynamics of a given population. Eventually, we note that in a heterogeneous population it could be interesting to consider more complicated cases where agents

are characterized by a broad distribution of values of K and can possibly change their degree of rationality, for instance, by thermalization-like processes (i.e., when two agents play, they modify their degree of rationality taking the average value of their current K). Since in the PGG, the strategy of the x-th agent can be described by a binary spin variable $s_x = \pm 1$, and being interested only in the nature of the final equilibrium (i.e., paramagnetic or ferromagnetic), we can consider the absolute value of the magnetization $|M|$ (see Chap. 2). In addition, we can derive the density ρ of cooperators in the population as follows:

$$\rho = \frac{M+1}{2} \tag{3.9}$$

so that we can identify the two (ordered, i.e., $|M| = 1$) absorbing states corresponding to $\rho = 1$ (i.e., full cooperation) and $\rho = 0$ (i.e., full defection). At last, another interesting order parameter useful to detect fluctuations in the system's behavior is the standard deviation of the fraction of cooperator $\sigma(\rho)$ obtained over the different runs. In the following section, we describe the macroscopic state of the system by reporting the average value of ρ, $\sigma(\rho)$, $|M|$, and T averaged over 100 simulations for all the considered configurations.

3.2.2 Results

We performed several numerical simulations of the PGG, for different values of the synergy factor r and the noise (measured either in terms of K or density of irrational agents $1 - f$), in a population of $N = 10^4$ agents distributed on a bidimensional lattice with periodic boundary conditions.

Homogeneous Populations Here we show results for the homogeneous case, where the level of noise in the system is controlled by the global variable K used in the SRP. We first analyze the strategy distribution diagram, which reports the average density of cooperators $\langle\rho\rangle$ as a function of r and K—Fig. 3.7. We observe that the PGG has a very rich behavior. For instance, plot (a) of Fig. 3.7 shows five different regions (below described) of interest when studying the density of cooperators at equilibrium. Notably, low values of K (i.e., $K < 10$) let emerge three phases as a function of r in the considered range (i.e., from 3.4 to 6.0): two ordered phases (i.e., full defection and full cooperation) for low and high values of r and a mixed phase (i.e., coexistence) for intermediate values of r. Therefore, at a first glance, an order-disorder phase transition of a second kind emerges crossing the region labeled (1) in the first strategy distribution diagram (i.e., (a) of Fig. 3.7). For higher values of K, next to $K = 10$, the active phase vanishes and the population always reaches an ordered phase. A more abrupt phase transition between the two ordered phases, separating region (2A) (full defection) and (2B) (full cooperation), appears, resembling analytical results obtained for the well-mixed approximation,

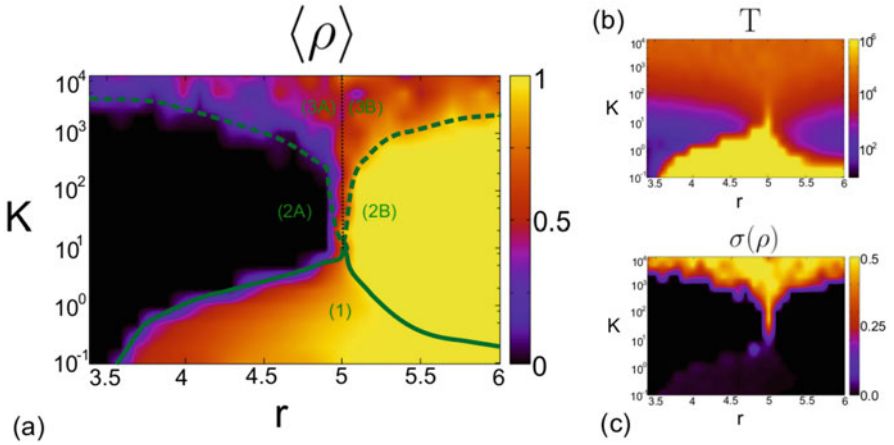

Fig. 3.7 Strategy distribution diagram showing the average density of cooperators $\langle\rho\rangle$ at the steady state (**a**), time to reach the absorbing state T (**b**) and standard deviation of the density of cooperators $\langle\rho\rangle$ at the steady state (**c**) as a function of the synergy factor r and the rationality K. Different regions are highlighted. In region (1) the system is stuck in a metastable active phase, macroscopically at the equilibrium, with coexistence of cooperators and defectors due to network reciprocity (the simulations have been stopped after $T = 10^6$ updates per agent). In region (2) the system always reaches the absorbing state predicted by the well-mixed population approximation, i.e., full defection for $r < r_{\text{wm}} = 5$ and full cooperation for $r > r_{\text{wm}}$. In region (3) both steady states become accessible with different probability, as in a biased voter model. Results are averaged over 100 simulation runs

even if fluctuations are possible near the critical point $r = 5$. For greater values of K, the region of r around $r = 5$ such that both ordered states are attainable increases. In such range of values, the system behaves as a biased voter model, where the absorbing states of cooperation (defection) is favored for $r > 5$ ($r < 5$). In the limit $K \to \infty$, the behavior of the system approaches that of a classical unbiased voter model, no matter the value of the adopted synergy factor. Plots (b) and (c) of Fig. 3.7 confirm the main differences among the five regions of plot (a). The former shows that the population does not reach an ordered phase for intermediate values of r around $r = 5$ and low K, and the simulation is only stopped once the average number of updates is equal to the considered maximum number $T = 10^6$ with the system macroscopically at the steady state. Conversely, for different parameters the population reaches an absorbing state (i.e., full defection or cooperation) relatively quickly. Instead, plot (c) shows that the variance reaches a maximum value (as expected), $\sigma(\rho) = 1/2$, when the PGG behaves like a voter model, while smaller non-null values are also obtained for the active phase, due to the existence of fluctuations. In order to obtain a deeper characterization of the phase transitions occurring in the PGG, we study the average absolute value of the magnetization $|M|$, as a function of the synergy factor for different K values. As shown in plot (a) of Fig. 3.8, only for values of $K < 10$, there are values of the synergy factors r such that $\langle|M|\rangle \neq 1$, since at $K \approx 10$ a more abrupt phase transition between full

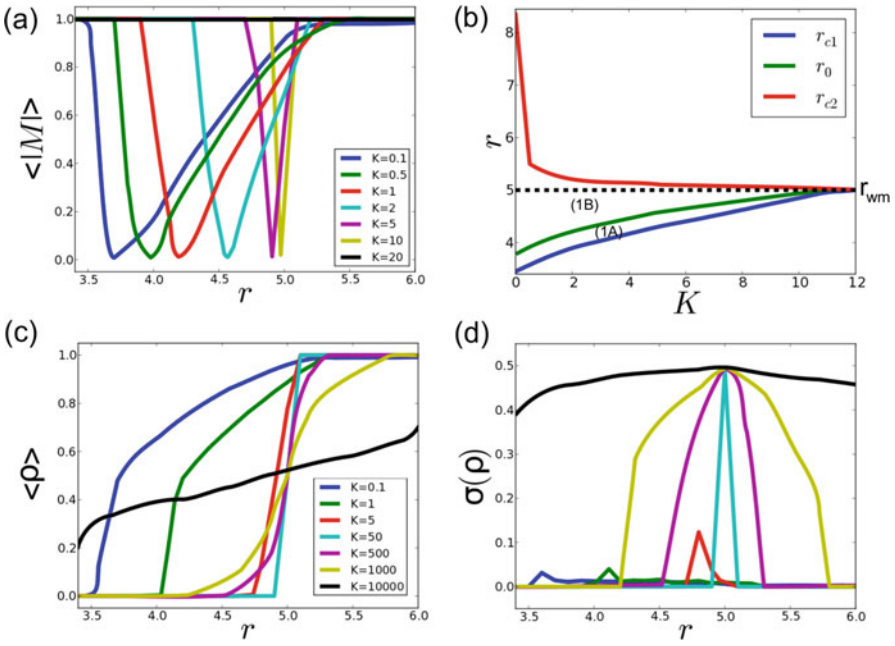

Fig. 3.8 In the top panels we focus on the transition from the active phase toward the ordered phase. In (**a**) we show the average absolute value of the magnetization $\langle |M| \rangle$ as a function of the synergy factor r for different K. As the temperature K increases, the range of r giving rise to an active phase shrinks around $r = r_{\text{wm}} = 5$ up to a critical value beyond which network reciprocity disappears. The scaling as a function of K of the two extreme points of the active range, r_{c1} and r_{c2}, as well as the value of r_0 for which $|M| = 0$, are shown in (**b**). In the bottom panels we show the average density of cooperators $\langle \rho \rangle$ (**c**) and the standard deviation $\sigma(\rho)$ for selected values of K (**d**). For the three smallest temperatures the system crosses region (1), marked by a second order transition in $\langle \rho \rangle$ and small values of σ. For $K = 50$, on each single run the system always reaches one of the two absorbing states. $\langle \rho \rangle$ is equal to 0 (1) for low (high) values of r, but takes intermediate values around $r = 5$. The transition is quite steep and $\sigma(\rho) = 0$ unless around $r = 5$. For higher values of K, for even a greater range of values of r around $r = 5$ both full defection or cooperation are achievable, $0 < \langle \rho \rangle < 1$ and $\sigma > 0$. In such regime the system behaves as a biased voter model under the external field $r - r_{\text{wm}}$. As K increases, the behavior of an unbiased voter model, no matter the value of r, is approached. Results are averaged over 100 simulation runs

defection and full cooperation emerges, resembling the first-order first transition predicted analytically in the case of well-mixed population of infinite size. Then, we note that for all K values in the range $[0 \leq K \leq 10]$, it is possible to find a synergy factor r such that $|M(r)| = 0$. Notably, as K increases, the difference between the two critical thresholds r_{c1} and r_{c2} goes to zero as both converge quickly toward $r_{\text{wm}} = 5$, eventually hitting such value at $K \approx 10$—see plot (b) of Fig. 3.8. Furthermore, we can also observe that the value of r_0, for which cooperators and defectors coexist in equal number in the active phase, is always smaller than r_{wm}—see plot (b) of Fig. 3.8. Here, we remark that r_0 separates the active phase in two

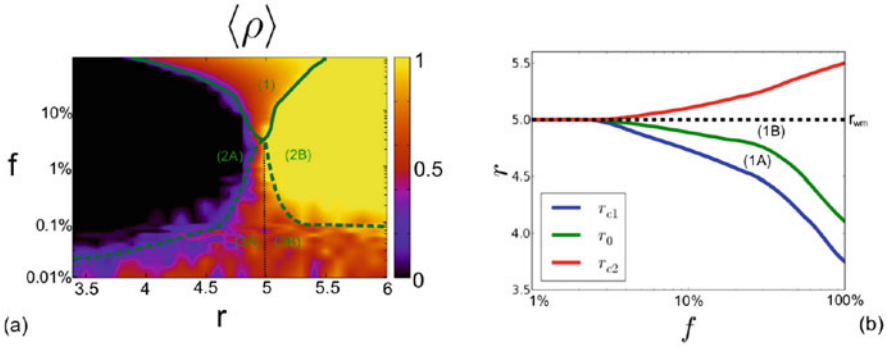

Fig. 3.9 (**a**) Strategy distribution diagram showing the average density of cooperators $\langle \rho \rangle$ as a function of the synergy factor r and the fraction of rational agents in the population f, i.e., those provided with $K^1 = 0.5$. (**b**) Critical thresholds r_{c1}, r_{c2} of synergy factors, and the value r_0 for which cooperators and defectors coexist in equal number as function of the fraction f of rational agents. Results are averaged over 100 simulation runs

regions, where either defectors (1A) or cooperators (1B) are predominant. Finally, both plots (c) and (d) of Fig. 3.8 clearly confirm the previous investigations. For instance, for $K = 10,000$ the density of cooperators becomes almost flat as in a Voter model (see plot (c) of Fig. 3.8).

Heterogeneous Populations We now report the analysis related to the second case. Notably, now we have a heterogenous population, where a density of agents f ($0 \leq f \leq 1$) with $K^1 = 0.5$ is inserted, spatially at random, in a population of irrational individuals which perform coin flips to decide their strategy. In this configuration, the level of noise is controlled by the variable f, and the lower its value, the higher the stochasticity in the population. As shown in Fig. 3.9, the strategy distribution diagram obtained as a function of the different values of noise is qualitatively comparable to the previously considered case. As f goes to 1, the PGG turns its behavior to the expected one for a population composed of only rational individuals (i.e., $r_{c1} = 3.75$ and $r_{c2} = 5.5$). The outcomes shown in Fig. 3.9 suggest that for values as small as $f \sim 3\%$, the PGG shows an active phase (i.e., the network reciprocity still holds). Thus, very few rational agents are able to provide the population an overall rational behavior at equilibrium. See plot (b) of Fig. 3.9 to observe the scaling for the critical values of the synergy factor: r_{c1} (at which cooperators first appear), r_{c2} (at which defectors disappears), and r_0 (where cooperators and defectors coexist in equal amount). Finally, considering both the homogenous and heterogenous cases here presented, it is important to highlight that these analyses do not study the evolution of a population over time, but focus only on the final equilibria.

3.2.3 Conclusion

To conclude, in this chapter we analyzed two different evolutionary games, i.e., the PD and the PGG, trying to understand the relation between their physical parameters/behaviors and the emergence of cooperation. In relation to the first part, we provided an analytical description of the PD by using the framework of Statistical Physics, studying the particular case of agents provided with memory (defined "memory-aware" agents). This condition entails that their payoff is not reset at each time step, so that in principle, they can increase their gain over time without limits. In particular, the proposed model is based on the kinetic theory of gases, showing how motion may lead a population toward an equilibrium far from the expected one (i.e., the Nash equilibrium). Being the final equilibrium dependent on the system temperature, it is possible to identify a range of temperatures that triggers cooperation for all values of the payoff matrix. In addition, it has been highlighted an interesting relation between the maximum temperature that fosters cooperation and the size of the system. Furthermore, the dynamics of the resulting model have been also described in terms of order-disorder phase transitions, showing a direct application of the Landau theory of phase transitions to EGT. In relation to the second part of the chapter, the proposed model provides a detailed study of the role of noise in the PGG, always by the lens of Statistical Physics. Notably, this investigation allows to define a clear relation between the noise introduced in the microscopic individual behavior and the macroscopic properties of a population. To achieve this goal, one starts from the theoretical considerations on the behavior of the population playing this game, then considering a richer scenario and controlling the noise in two different cases: a homogeneous population (i.e., with all agents having the same degree of rationality) and a heterogeneous one, where more degrees of rationality are considered. The phase diagram resulting from numerical simulations shows the influence of the synergy factor r and of the noise on the macroscopic behavior of the population. From the analysis of the heterogenous population case, we can note that even a very small density f of rational agents, $f \approx 3\%$, allows to observe a network "reciprocity effect." In such sense, beyond the physical interpretation of these results, it is important to highlight that, from the perspective of EGT and from that of Sociophysics, the PGG is a system that "correctly works" even in the presence of few rational players. Here, saying that the system "correctly works" means that the equilibrium predicted for a given r, by the analysis of the Nash equilibria of the system in the well-mixed approximation, is achieved. To conclude, both investigations clearly highlight the potential of Statistical Physics in analyzing the dynamics of EGT, finding also connections between different models as the PGG and the Voter model.

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Chapter 4
Evolutionary Games II: Applications

This chapter presents a couple of applications of EGT, one devised for investigating the role of conformity in the PGG and the other for solving combinatorial optimization problems. The former is of interest both for EGT and for the modern area of Social Dynamics (or Sociophysics), where the influence of conformist (and nonconformist) behaviors has been widely investigated, e.g., in processes of opinion dynamics. The second application, beyond to propose a new heuristic for solving the Traveling Salesman Problem, aims to stimulate new ideas in the field of EGT. In particular, considering the dynamics of the model here presented, one can see how the transitions toward ordered states can be exploited for devising different kinds of algorithms based on the utilization of “rational agents." The reader interested in further details can find a brief list of manuscripts at the end of the chapter. In particular, the content of this chapter is based on the first two manuscripts cited at the end of the conclusions.

4.1 The Role of Conformity in the Public Goods Game

In this section, we propose an investigation on the dynamics of the spatial PGG considering a population composed of “rational" agents and “conformist" agents. Notably, in the PGG, agents can be defined as fitness-driven agents when modify their strategy according to payoff-based rules. At the same time, representing strategies in terms of spins allows to investigate the role of conformity (as well as other characters) during the process of “strategy revision phase." In particular, some agents can be driven by conformity (i.e., conformity-driven), while others are susceptible to their payoff (i.e., fitness-driven). Notably, as discussed in Chap. 1, the fitness is mapped to the agents’ payoff, so that richer agents are those most imitated by fitness-driven agents, while conformity-driven agents tend to imitate the strategy assumed by the majority of their neighbors. Numerical simulations aim to

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identify the nature of the final equilibrium, reached by the population, on varying the amount of conformity-driven agents. Results show that conformist behaviors generally foster ordered cooperative phases but, in special conditions, may also lead to bistable phenomena. In physical terms, conformity-driven agents correspond to agents susceptible to the influence of a local field, i.e., that generated by their nearest neighbors. Notably, a field results from a number of spins, corresponding to the strategy of the selected agents. As discussed before, although the PGG exhibits a theoretically predicted Nash equilibrium of defection, several investigations identified a number of ingredients to support cooperation, spanning from awarding mechanisms to optimal game settings. For instance, the synergy factor (usually indicated as r), adopted to compute the agents' payoff, can be opportunely tuned in order to support cooperation on different topologies. This result is very important as it entails that if the payoff of cooperators reaches, or overtakes, a minimum value, all agents turn their strategy to cooperation. As below, the minimum threshold of the synergy factor depends on the topology of the population (i.e., the way agents are arranged). Therefore, adding a social influence in the PGG implies to deal with two degrees of freedom: the synergy factor r (whose individual effect is known on some topologies like regular square lattices) and the density of conformist agents ρ_c. The proposed model is studied by means of numerical simulations performed by arranging agents that play a spatial PGG on a bidimensional regular lattice with periodic boundary conditions.

4.1.1 Model

We remind that the PGG considers a population of N agents that can adopt two different strategies: cooperation and defection; so, at each time step, cooperators provide a unitary contribution to a common pool, whereas defectors do the opposite, i.e., not contribute. After all the agents have made a decision and accumulated their corresponding payoff, they undergo a round of strategy revision phase. In doing so, the population evolves until it reaches a final equilibrium (or steady state). From a Statistical Physics perspective, we can identify two different phases (or equilibria): a paramagnetic equilibrium in which we observe the coexistence of cooperators and defectors, and a ferromagnetic equilibrium, implying that one species prevails. In this model, we aim to investigate the outcomes of the PGG in heterogeneous populations, i.e., composed of fitness-driven agents (FDAs) and conformity-driven agents (CDAs). As result our population is composed of $N = N_f + N_c$ agents, with N_f is the number of FDAs and N_c that of CDAs. Thus, we can introduce $\rho_f = N_f/N$ and $\rho_c = N_c/N$ to identify the density of FDAs and CDAs, respectively. For the sake of clarity, we use the convention in which upper indices refer to the strategy (i.e., cooperation and defection) while lower indices to the agent's nature (i.e., conformity-driven and fitness-driven). Both FDAs and CDAs change strategy by a stochastic rule. In particular, we implement a Fermi rule (see Chap. 1) to compute the transition probability between two different

strategies for FDAs. CDAs adopt a simple majority voting (see, for instance, the works of S. Galam reported in references) rule to decide their next strategy: an agent computes the transition probability according to the density of neighbors having the strategy of majority. In doing so, FDAs act rationally, while CDAs follow a social behavior (i.e., conformism). Following the prescription of previous investigations, we arrange agents in a bidimensional regular lattice of degree 4 with periodic boundary conditions. Summarizing, our population evolves according to the following steps:

1. At $t = 0$, set an equal number of cooperators and defectors and the density of conformists $\rho_c \in [0, 1]$.
2. Select randomly one agent x, and select randomly one of its neighbors y.
3. Each selected agent plays the PGG with all its five communities and then computes its payoff.
4. Agent y performs the strategy revision phase according to its nature.
5. Repeat from (2) until an ordered phase is reached, or up to a limited number of time steps elapsed.

We remark that the neighborhood for each agent has always four agents. Therefore, one agent plays in five different groups at a time, all composed of five members. Finally, we remind that agents may change strategy, i.e., from cooperation to defection (and vice versa), but they cannot change their nature (i.e., fitness-driven and conformity-driven). Although in real social systems individuals might change also their behavior (e.g., from CDA to FDA), in this work we aim to analyze the relation between the density of CDAs and the outcomes of the PGG. Therefore, we need to assume agents keep constant their behavior.

4.1.2 Results

We investigate the behavior of the proposed model for different values of ρ_c, from 0 to 1, and of r. The latter assumes values in the range $[3, 6]$ since, in this topology, the two thresholds for different equilibria r_m and r_M are known (see Chap. 3). The threshold r_m indicates that lower values of r lead the population toward a phase of full defection at equilibrium. For intermediate values of r, i.e., $r_m \leq r \leq r_M$, the population reaches a disordered phase, i.e., a mixed phase characterized by the coexistence of both species at equilibrium; eventually, for values of $r > r_M$, cooperators succeed. In order to investigate the proposed model, we perform numerical simulations with populations of different size, from $N = 10^2$ to $N = 10^4$. The first analysis is related to the distribution of strategies, at equilibrium, on varying the synergy factor r and the density of conformists ρ_c-see Fig. 4.1. It is worth noting that the disordered phase becomes narrower as ρ_c increases. Notably, we observe that r_m and r_M are strongly affected by ρ_c. At a first glance, conformism fosters cooperation, as r_M strongly reduces while increasing ρ_c. On the other hand, for $\rho_c = 1$ a bistable behavior is expected as agents change strategy without

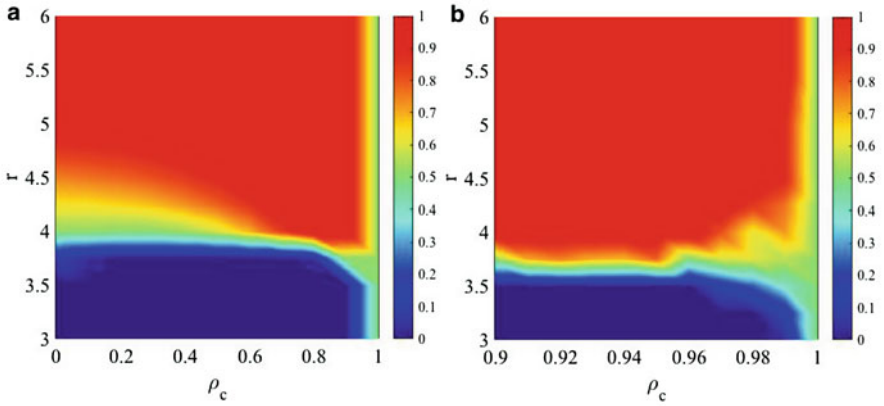

Fig. 4.1 Cooperation diagram on varying ρ_c in a population with $N = 10^4$. (**a**) ρ_c in range $\in [0.0, 1.0]$. (**b**) ρ_c in range $\in [0.9, 1.0]$. Red corresponds to areas of cooperation, while blue to those of defection. Results are averaged over 50 simulation runs and have been computed using 11×11 parameter values

considering the payoff. In particular, the minimal threshold of synergy factor to avoid cooperators disappear reduces to values smaller than $r = 3.75$ when ρ_c is greater than 0.85. Moreover, considering the higher threshold r_M (i.e., that to obtain full cooperation for $\rho_c = 0.0$), we observe that even with low density of conformist agents, r_M decreases, up to reach a value slightly smaller than 4.0. In the range $\rho_c \in [0.9, 1.0]$, a closer look allows to note a richer behavior of the model-see plot (b) of Fig. 4.1. We notice that defectors succeed only for values of r smaller than 3.6, while cooperators succeed for values of r greater than 3.78. As result, the mixed phase is obtained only in a narrow range between the two listed values (i.e., $3.6 \leq r \leq 3.78$). For values of $\rho_c \geq 0.97$, a bistable behavior can be observed: sometimes cooperators succeed, while other times fail (i.e., defectors succeed). Thus, since our results are computed as average values of different simulation runs, the colors represented in both plots of Fig. 4.1 in some cases reflect the probability to find the final population in a given status starting with those initial conditions (i.e., r and ρ_c).

In order to characterize the transition at fixed ρ_c, since we observe qualitatively different phases, we tentatively try to identify the transition lines by studying the behavior of the variance as a function of r, which plays the role of inverse "temperature." Here, the variance σ_M is referred to the magnetization of the system, which we identify as our order parameter (as discussed in Chap. 2). Hence, the variance σ_M is computed numerically but can be easily identified as the susceptibility of the order parameter χ,

$$\sigma_M = \frac{1}{Z} \sum_{i=1}^{Z} (M_i - \langle M \rangle)^2 \equiv \chi \tag{4.1}$$

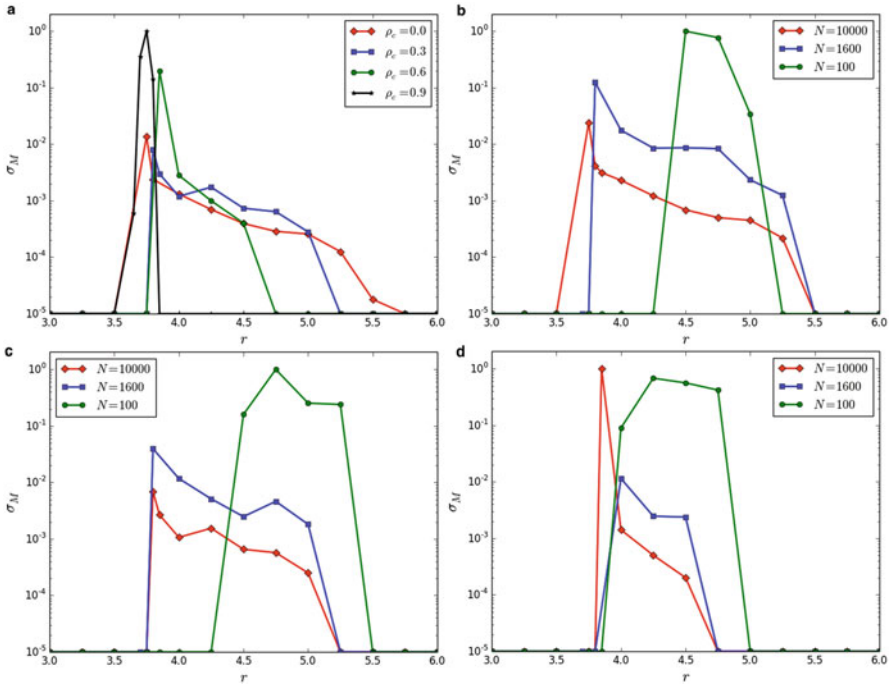

Fig. 4.2 Variance (σ_M) of the order parameter M as a function of the synergy factor r, for different configurations: (**a**) $N = 10^4$. (**b**) $\rho_c = 0.0$. (**c**) $\rho_c = 0.3$. (**d**) $\rho_c = 0.6$. Since we adopted a logarithmic scale for the y-axis, we highlight that all values equal to 10^{-5} correspond to 0

with Z number of simulations performed under the same conditions (i.e., fixed r and ρ_c) and $\langle M \rangle$ average magnetization (computed under the same conditions). Plot (a) of Fig. 4.2 shows the variance σ_M for different values of ρ_c: $0, 0.3, 0.6, and 0.9$, as a function of the synergy factor r. As expected, we found that for $\rho_c = 0.0$ the variance is maximum at $r_m \sim 3.75$. Plots (b-d) of Fig. 4.2 illustrate how these curves scale as we increase the number of agents for $\rho_c = 0.0$, $\rho_c = 0.3$ and $\rho_c = 0.6$, respectively. We observe that in the case $\rho_c = 0.6$, the limit $N \to \infty$ is critical, i.e., we find that there seems to exist a r_{crit} for which $\lim_{N\to\infty} \chi_N \equiv \chi \approx (r - r_{\text{crit}})^{-\alpha}$ for some exponent $\alpha > 0$. Then, in order to characterize the bistable behavior shown in Fig. 4.1, we study the probability for the system of being in the defecting or in the cooperating phase at the end of the simulation, as a function of r (see Fig. 4.3) and of ρ_c (see Fig. 4.4). In Fig. 4.3 the two dotted lines refer to the winning probabilities of defectors (i.e., blue) and of cooperators (i.e., red). Therefore, for $\rho_c = 0.0$, the two curves are zero in the intermediate range of r, i.e., $3.75 \leq r \leq r_M$, as none is expected to completely succeed. Remarkably, increasing ρ_c we found a decreasing paramagnetic range of r, disappearing for values $\rho_c \geq 0.8$. As shown in plots (b) and (c) of Fig. 4.3, at least one curve is always greater than zero. Although we are dealing with success probabilities, it is worth noting that the summation

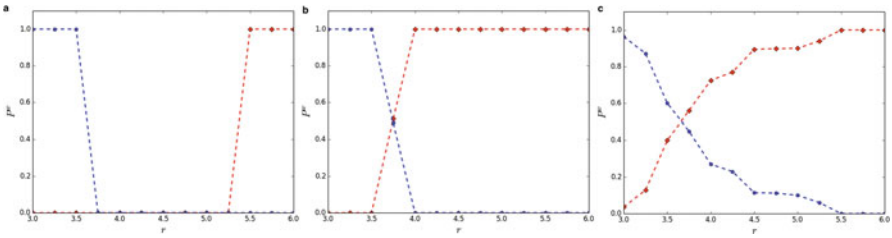

Fig. 4.3 Probability to succeed as a function of the synergy factor r, in a population with $N = 10^4$, for the two species: cooperators, i.e., red dotted line (diamonds $\diamond$), and defectors, i.e., blue dotted line (circles $\circ$). (**a**) $\rho_c = 0.0$. (**b**) $\rho_c = 0.9$. (**c**) $\rho_c = 0.99$

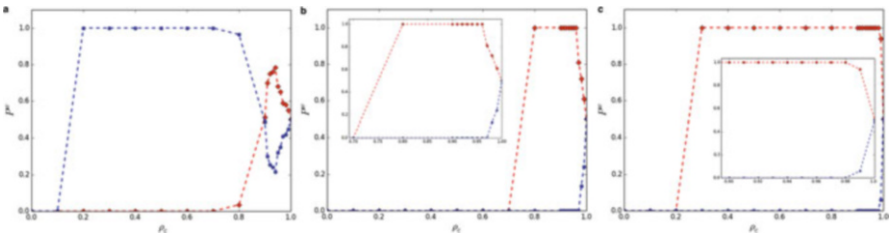

Fig. 4.4 Probability to succeed as a function of the density of conformists ρ_c, in a population with $N = 10^4$, for the two species: cooperators, i.e., red dotted line (diamonds $\diamond$), and defectors, i.e., blue dotted line (circles $\circ$). (**a**) $r = 3.75$. (**b**) $r = 4.0$. (**c**) $r = 5.25$

of values taken by the two curves has to be ≤ 1, thus even zero as it means that none succeeds once the disordered phase is reached. Moreover, Fig. 4.3 allows to observe the emergence of a bistable behavior, e.g., for $\rho_c = 0.9$ at $r = 3.75$ we have both curves having the same P^w, i.e., about 50% of cases defectors prevail, while in the remaining cases, cooperators succeed. Figure 4.4 aims to characterize the same bistable behavior on varying ρ_c and keeping fixed r. Plot (a) of Fig. 4.4 refers to $r = 3.75$, and it lets emerge an interesting result: in the range $0.2 \leq \rho_c \leq 0.7$ defectors prevail. This indicates that in this region conformism promotes defection, being 0 the expected value of P^w for both species. Moreover, the bistable behavior emerges as $\rho_c \geq 0.8$. Plot (b) of Fig. 4.4 refers to $r = 4.0$ and shows that the upper bound of the paramagnetic phase (i.e., r_M) is reduced to 4.0 as $\rho_c \geq 0.8$. Then, a bistable behavior emerges for $\rho_c \geq 0.92$. Eventually, in plot (c) of Fig. 4.4 referred to $r = 5.25$, we see that even for lower values of ρ_c cooperators succeed, and the bistable behavior emerges for $\rho_c \geq 0.93$. In the light of these results, we can state that when r is close to the lower bound of the paramagnetic phase, i.e., r_m, conformism supports defection until the emergence of a bistable behavior. While, for higher values of r, conformism supports cooperation, and only for high values of ρ_c the system becomes bistable.

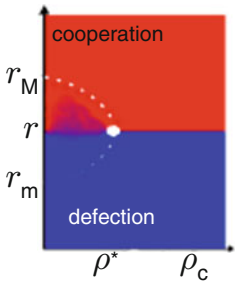

Fig. 4.5 For $r_m < r < r_M$ and $\rho_c < \rho^*$, we observe a phase where defection and cooperation coexist, represented in the dashed white line, where the variance is continuous. For $\rho_c > \rho^*$, the transition from defection to cooperation is sharper at r_c. This picture gives the idea of the existence of a triple point at $r = r_m$, $\rho_c = \rho^*$ where three different behaviors coexist

Finally, we construct an approximate phase diagram of our system-see Fig. 4.5. In the top part, we have the domination of cooperation (i.e., red) and in the lower one that of defection (i.e., blue). Along the line separating the two parts above identified (at fixed r), we find an important point indicated as ρ^* below described. In an area of the left diagram between defection and cooperation, for $\rho_c < \rho^*$ and $r_m < r < r_M(\rho_c)$, defectors and cooperators coexist, with the prevalence of the former. In this region, it is possible to change the parameters to reach smoothly the cooperation region. For $\rho_c > \rho^*$ we have the coexistence of cooperation and defection on the transition line $r = r_m$, due to the fact that r_M approaches r_m as an increasing function of ρ_c. The point in which $r_M = r_m$ is a triple point.

4.1.3 Conclusion

Summarizing, the proposed model shows the behavior of the spatial PGG in presence of conformist agents, i.e., agents susceptible to local fields generated by their nearest neighbors. Accordingly, the population is composed of conformity-driven agents and fitness-driven agents. In both cases, CDAs and FDAs update their strategy by considering only their neighborhood. It is worth to remind that previous studies reported that social influences strongly affect evolutionary games, and in the presented model, some results are similar to those achieved by other authors. However, other results here presented further extend previous findings. Now, we can highlight the prominent role of conformism in the spatial PGG: it seems that this social influence may lead the population toward different phases and behaviors, as full cooperation and bistable equilibria. In particular, conformism promotes the population to reach an ordered phase, even when a disordered one is expected. For intermediate densities of conformists (e.g., 0.5), the final equilibrium is that

closer to that one would expect considering only FDAs, at a given r. Therefore, our investigations suggest that conformism drives the system toward ordered states, with a prevalence for cooperative equilibria. To conclude, we found that the spatial PGG under social influences has a very rich behavior, characterized by different final states.

4.2 Solving Optimization Problems by the Public Goods Game

In this section, we introduce a method based on the PGG for solving optimization tasks. Beyond to propose a new heuristic for combinatorial optimization problems, this investigation aims to highlight the potentiality of EGT beyond its current horizons. In the last years, many evolutionary algorithms have been proposed for solving optimization problems, as, for instance, genetic algorithms and ant colonies heuristics. Remarkably, optimization problems have been widely investigated also within the realm of Statistical Physics, where Theoretical Physics and Information Theory meet forming a powerful framework for studying complex systems. For instance, a Statistical Physics mindset approach in combinatorial optimization problems emerges when the set of feasible solutions, of a problem like the Traveling Salesman Problem (TSP hereinafter), is represented in terms of an energetic landscape. In doing so, the searching of a solution corresponds to the searching of a minimum of free energy, in a landscape whose global minimum, i.e., the deepest valley, corresponds to the optimal solution of the problem. As before discussed, several models as the CW and spin glasses have an energy that can be studied by the Landau formulation of phase transitions. These models are successfully adopted for facing different issues as opinion dynamics, information retrieval, optimization tasks, and learning processes. Using the metaphor of the energy, heuristics like genetic algorithms and swarm logics implement strategies as genetic recombination, mutation, and collective motions, for surfing the energetic landscape with the aim to reach one of the deepest valleys in short time interval. Therefore, parameters as the mutation rate used in genetic algorithms can be compared to physical parameters as the system temperature. In the proposed model, we adopt a mechanism based on partial imitation: when an agent interacts with another one having a higher fitness, the former imitates a part of the latter's solution. For example, in the TSP, the weaker agent imitates only a part of the path traveled by a stronger opponent. In doing so, agents are able to generate solutions over time, with the aim to achieve the optimal one. In physical terms, a partial imitation can be interpreted as a slow cooling process of a spin particle system, where the slowness comes from an imitative dynamics that is only "partial" (i.e., only few entries of a solution array are imitated). The presented model considers an agent population, whose interactions are based on the PGG. As we know from EGT (see Chap. 1), the outcomes of the PGG are affected by a parameter defined synergy factor r, used for supporting cooperators.

Here, as shown below, this parameter (i.e., r) has a marginal interest, however what is relevant for our investigation is that an ordered phase (i.e., the prevalence of a species in the population) can be reached by an opportune tuning of its value. Usually, in EGT models, a species indicates a set of agents with the same strategy, e.g., cooperation, whereas in the proposed model a species corresponds to a set of agents having the same solution of a TSP. In general, ordered phases entail all agents have the same state (or strategy in EGT), i.e., in physical terms all spins are aligned in the same direction. We remind that the system magnetization allows to measure the state of order of a system, and its value equals to ± 1 in the ordered cases. Dealing with neural networks, and in general with spin glasses, it is possible to introduce a gauge for the magnetization so that its value goes to ± 1 when the spin alignments (i.e., agent states) follow particular patterns. For instance, in the case of the TSP, a pattern can be a specific sequence of cities. The mentioned gauge is defined Mattis magnetization, and it reads $M_m = \frac{1}{n}\sum_i \epsilon_i s_i$ with ϵ_i value in the i-th position of the pattern and s_i value of the spin in the same position of a signal S of length n. As we can observe, when spins are perfectly aligned with a pattern ϵ, the Mattis magnetization is 1. Here, we introduce a similar approach. In particular, each agent is provided with a random solution of the TSP (i.e., an array of cities representing a possible solution), and the order is reached when all agents hold the same solution. Therefore, in our case, the value of M_m is computed assigning the value of $+1$ when a city has the same position both in the pattern of reference (i.e., the known optimal solution of a TSP problem), and in the solution array computed by an agent, otherwise the value is -1. It is worth to recall that the utilization of the Mattis magnetization, as measure for the performance of our model, can be adopted only when the optimal solution of a TSP is known in advance. Since our agents interact by the PGG, the modification of their solution occurs during the phase of the game defined as “strategy revision phase" (previously described) that in our case is renamed “solution revision phase." Furthermore, our agents use their fitness as currency of the game, so that their payoff depends on the quality of their solution and on those of their opponents. Then, numerical simulations allow to evaluate the quality of this method, considering the TSP as reference (i.e., a famous NP-hard problem). Results show that the PGG can be successfully adopted for developing new heuristics, opening the way to investigations that cross the current fences of EGT.

4.2.1 Model

Before introducing the proposed model, we recall the basic dynamics of the PGG. The latter considers a population with N agents and two possible strategies: cooperation and defection. Cooperators contribute to a common pool with a coin (usually of unitary value), while defectors contribute nothing or, as in our case, provide a partial contribution (i.e., a coin whose value is lower than that of coins

provided by cooperators). Then, the total amount of coins is enhanced by a synergy factor r (whose value is greater than 1), and the resulting value is equally divided among all agents (no matter their strategy). As the definition of the payoff suggests, defection is more convenient than cooperation, and it also represents the Nash equilibrium of this game (see Chap. 1). The role of the synergy factor r is promoting cooperation, and, as demonstrated in previous investigations, its value may strongly affect the evolution of a population. As previously mentioned, the evolution of a population results from the process defined as "strategy revision phase." Notably, after each iteration, an agent has the opportunity to change its strategy by imitating that of a richer opponent (considering the gained payoff). In the proposed model, we consider a well-mixed population (in Appendix 2 the model is briefly analyzed on structured populations), so that agents may freely interact with their opponents. Moreover, agents are provided with a random solution of a TSP (i.e., an array of cities). Notably, each solution is evaluated by a fitness η computed as follows:

$$\eta = \frac{Z-1}{D} \tag{4.2}$$

with Z number of cities and D, total distance of a path. In doing so, its range is $\eta \in [0, 1]$. At each time step, one agent is randomly selected (say the xth) and plays the PGG with *four* (randomly chosen) opponents, forming a group with $G = 5$ agents. Now, every agent of the group contributes with its fitness; then, as in the PGG before summarized, the total summation of contributions is enhanced by a synergy factor r and eventually equally distributed among all agents of the group. It is worth noting that, in the proposed model, all agents always contribute. However, some agents provide a contribution higher/smaller than that of others. Therefore, "below-average contributors" (i.e., those having a low-quality solution) can be considered as defectors. According to this setting, the payoff reduces to one equation:

$$\pi_x = r\frac{\sum_{i=1}^{5}\eta_i}{G} - \eta_x \tag{4.3}$$

with π_x indicating the payoff of the xth agent, and η_x its fitness (i.e., that corresponding to its solution). Finally, the "solution revision phase" is based on the following process: the randomly selected agent computes the probability Π_s to modify each entry of its solution by imitating that of its best opponent (if exists):

$$\Pi_s = \frac{1}{1+e^{\frac{\eta_x-\pi_x}{K}}} \tag{4.4}$$

As in the PGG, K represents the uncertainty in imitating an opponent (i.e., plays the role of temperature/noise-see Chap. 3). Hence, setting $K = 0.5$ we implement a rational approach. In doing so, the x-th agent imitates with probability Π_s each entry of the solution of its best opponent, if the latter has a greater or, at least, equal fitness

(otherwise the x-th agent does not revise its solution). Summarizing, given a TSP, we define a population whose agents at the beginning receive a random solution of the problem. Then, local interactions, based on the PGG, allow the population to converge toward a shared solution. From a local point of view, at each time step, a randomly selected agent (say x) plays the PGG with *four* (randomly chosen) opponents and computes its payoff (i.e., by Eq. (4.3)). So, according to its fitness η_x and to the gained payoff π_x, the x-th agent computes the probability Π_s to imitate the solution of its best opponent (say y, if exists). In particular, if $\eta_y \geq \eta_x$, the x-th agent revises its solution, i.e., it imitates each entry of the solution of the y-th agent with probability Π_s (i.e., each entry is modified according to Π_s). The whole process is repeated until the population reaches an ordered phase (i.e., all agent share the same solution), or up to a limited number of time steps elapsed. It is worth observing that as Π_s goes to 1, the imitation process tends to become full (not partial) as each entry can be imitated, provided that the best agent has a greater (or an equal) fitness. Eventually, we remark that when an agent performs a "partial imitation," to modifying, for instance, one city along its path, the same city is never visited twice. In order to clarify this point, we provide a simple example. Let us consider an agent having the following solution: (Paris, New York, London, Miami, Rome, Madrid) that has to put in the third cell (now containing London) the city of Rome. Since currently Rome is in the fifth cell, the algorithm swaps the values for the third and fifth cells so that, after the whole process, the resulting array is (Paris, New York, Rome, Miami, London, Madrid). Thus, repetitions are completely avoided, and all solutions generated according to the proposed heuristic are suitable solutions.

4.2.2 Results

Numerical simulations have been performed considering a number of cities up to $Z = 50$ for defining the TSP. Agents know the starting city and the landing one, so, since each city can be visited only once, the number of feasible solutions is $(Z-2)!$. Moreover, without loss of generality, we consider that the distance between two close cities is always equal to one-see Fig. 4.6. Eventually, we set the synergy factor to $r = 2$. We remind that in the present work we are not interested in studying phenomena as the evolution of cooperation, but we aim to evaluate if agents are able to converge toward an ordered phase, characterized by the existence of only one shared solution of a TSP problem. Thus, the choice of setting $r = 2$ reflects this requirements, i.e., to use a value that in the PGG leads to an ordered phase (i.e., full defection in the specific case). As illustrated in Fig. 4.7, the ergodicity of the process always allows agents to converge to one common solution. Moreover, we are able to verify the quality of solutions both considering the related fitness and the Mattis magnetization (see the inset of Fig. 4.7). In particular, the latter can be used when the solution of a problem is known in advance, as in our case. An important relation to be considered is the one defined between the final average fitness and the

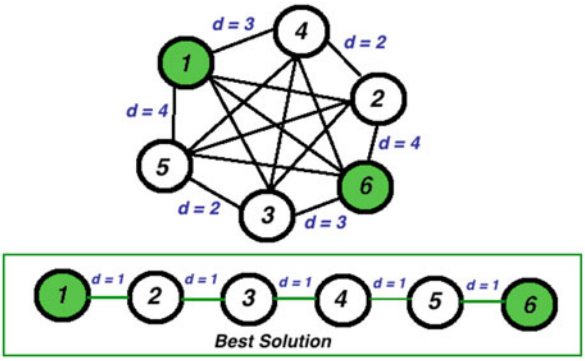

Fig. 4.6 General setting of the TSP considering $Z = 6$ cities forming a complete graph. Each node represents a city, and some distances are reported in blue, close to the related link. Then, the best solution is shown. Green nodes represent the starting and the landing ones

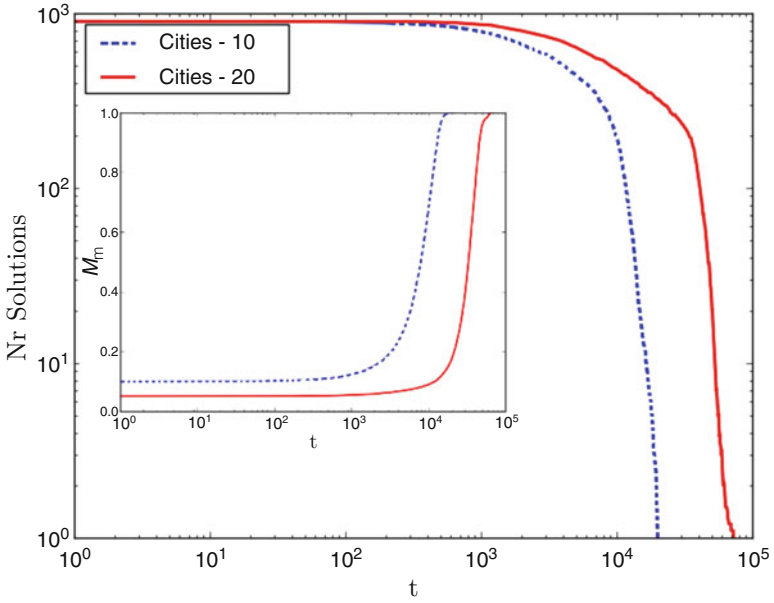

Fig. 4.7 Number of solutions over time in a population of $N = 900$ agents while solving a TSP with 10 cities (blue dotted line) and 20 cities (red line). The inset shows the related Mattis magnetization for the two cases (both successful). Results are averaged over different simulation runs

size of the population N, studied on varying the amount of cities Z-see plot (a) of Fig. 4.8. Moreover, as shown in plot (b) of Fig. 4.8, it is worth noting that also good suboptimal solutions may be computed using a number of agents N smaller than that required to compute the optimal one. As expected, increasing Z the average value of η reduces (keeping fixed the number of agents N). On the other hand, as

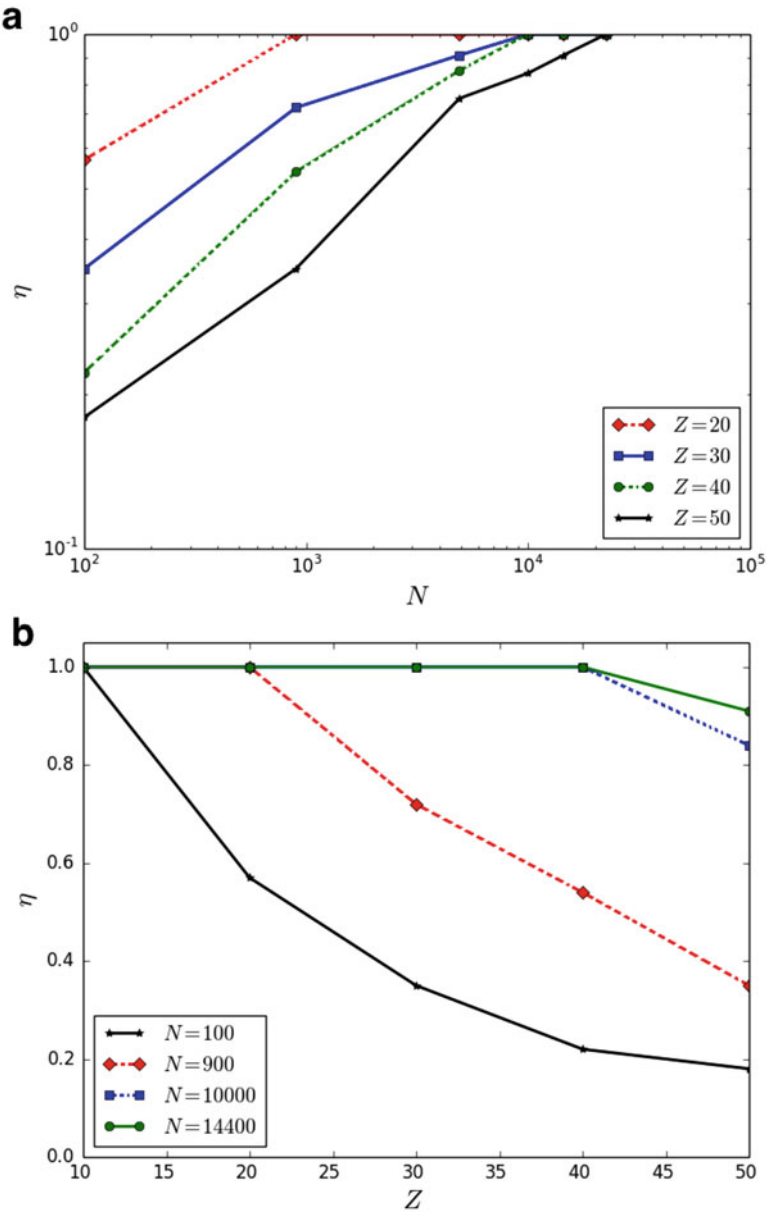

Fig. 4.8 (**a**) Average fitness of final solution in function of N (i.e., the number of agents), for different values of Z (i.e., the number of cities). (**b**) Average fitness of the final solution on varying the number of cities, for different agents N. Results are averaged over different simulation runs

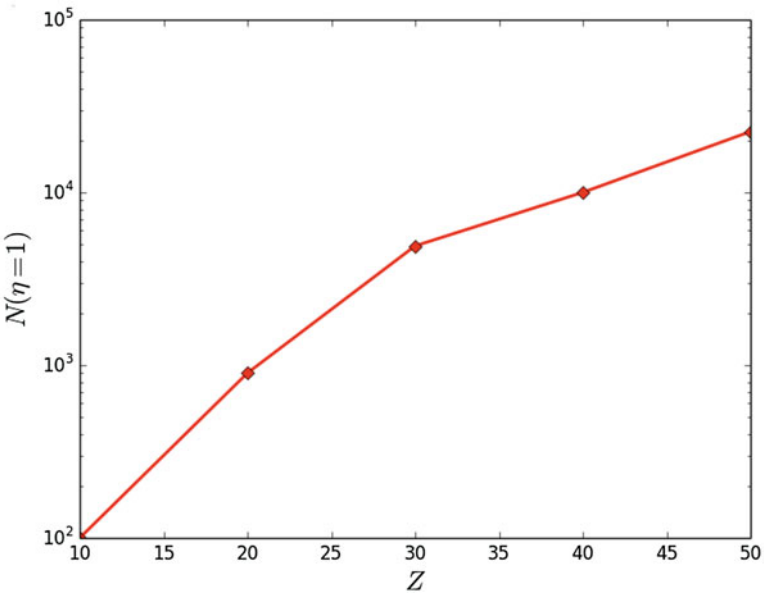

Fig. 4.9 Minimum number of agents to compute the optimal solution of a TSP on varying the number of cities Z. Results are averaged over different simulation runs

shown in Fig. 4.9, it is worth highlighting that it is possible to find an opportune N for each considered Z in order to achieve the highest fitness (i.e., $\eta = 1$). We deem relevant to note that the number of agents to compute the best solution, i.e., $N(\eta = 1)$, is much smaller than the number of feasible solutions for each problem; therefore our method can be considered a viable heuristic for facing combinatorial optimization problems. Eventually, we focused on the number of time steps to let the population converge, considering in particular the successful cases, i.e., those leading to the optimal solution-see Fig. 4.10. As expected, wide search spaces (e.g., $Z = 50$) require more time steps to let the population converge to the same final (and optimal) solution. Moreover, increasing N the number of time steps T increases accordingly for the same problem (i.e., keeping fixed Z). These results are in full agreement with converging processes that can be observed in generic agent-based models, e.g., increasing the size of a population the number of time steps, required to let agents converge toward the same state, increases.

4.2.3 Conclusion

In this work we show that evolutionary games as the PGG can be, in principle, applied also for solving combinatorial optimization problems. In particular, the order–disorder phase transition occurring in a population interacting by the PGG

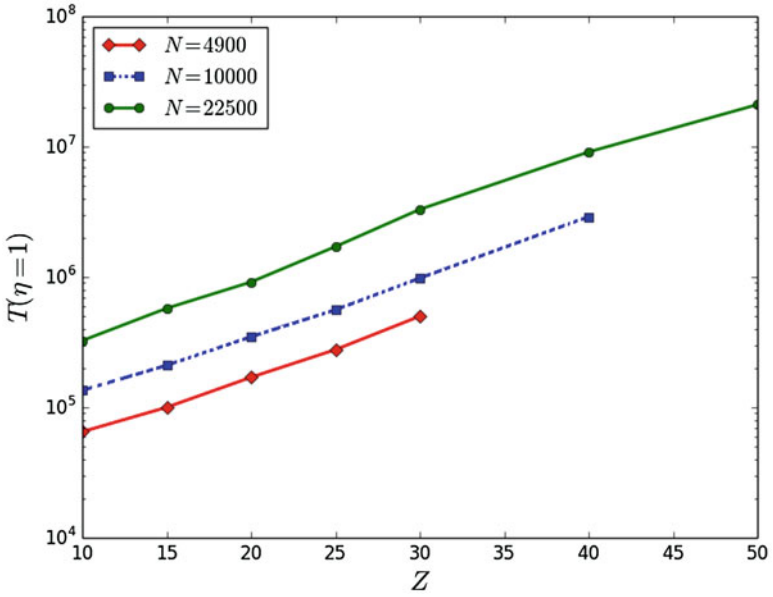

Fig. 4.10 Number time steps required for converging to the final (optimal) state on varying Z, for different population sizes N. Results are averaged over different simulation runs

can be adopted for letting the population converge toward a common solution of a given problem. Notably, the solution plays the same role of the strategy in the classical PGG, and the order is reached by implementing a mechanism of "partial imitation" (see references for further details). The latter allows agents with a weak solution to partially imitate stronger (i.e., richer) opponents. From a physical perspective, this mechanism corresponds to a slow cooling process that triggers the emergence of solutions over time, whereas the ergodicity of the process allows the population to reach an absorbing state of full order. In doing so, an ordered phase entails all agents share the same solution. Under the hypothesis that an evolutionary dynamics driven by the payoff, i.e., rational, may constitute the base for solving difficult problems as the TSP, we performed several numerical simulations by considering a well-mixed population. Although we implemented a simplified version of the TSP, with a limited number of cities, it is worth highlighting that results indicate that the proposed model allows to compute the optimal solution in all considered search spaces. Moreover, even using a reduced number of agents, it is possible to compute a good suboptimal solution. Furthermore, we note that even introducing spatial constraints in the TSP definition, the algorithm is able to face the problem, once the driveability of the graph is known (as shown in Fig. 4.6). Therefore, in the light of the achieved outcomes, we deem relevant to further investigate the potential of evolutionary games in optimization problems, then enlarging the domain of applications of EGT. However, it is important to emphasize that in order to really appreciate the quality of the proposed model as

algorithm for solving the TSP, further investigations are required, in particular, those for comparing the performances with other heuristics, as genetic algorithms (see Appendix 1). On the other hand, we remark that our results indicate a clear relation between the size of a population and the complexity of the faced problem. This last observation constitutes a first, even if theoretical, advantage of the proposed method with respect to others because, as far as we know, similar relations are not available for other strategies. Now, from the point of view of EGT, there are two important observations. First, the synergy factor has a marginal role in the proposed model. We recall that, for the aims of our work, we are interested in allowing the population to converge toward an ordered state. On studying the PGG, the synergy factor is fundamental because, as before mentioned, some values may lead a population toward a steady state of coexistence between cooperators and defectors. Therefore, since here we have to avoid similar outcomes, in principle, every value of the synergy factor that supports a generic state of full order can be adopted. At the same time, we think that the synergy factor should not be too high, otherwise it might generate problems when computing transition probabilities during the "solution revision phase." In particular, as indicated in Eq. (4.4), the fitness and the payoff are compared when evaluating whether one agent has to change its strategy. Thus, we suggest to use small values, like the one we adopted (i.e., $r = 2$). The second observation is related to the identification of defectors. Notably, here we refer to the PGG, i.e., a simple game with two strategies: cooperation and defection. In the classical version, cooperators contribute with a coin, while defectors do not contribute. However, we remind that when the amount of contributions is not set to a specific value (e.g., a coin of unitary value), those agents that contribute with a below-average contribution can be considered as defectors. To conclude, the proposed heuristic shows that cooperative dynamics, leading from disordered to ordered states, may constitute the basic mechanism for implementing optimization algorithms.

Appendix 1

Here, we report results of a comparative analysis between the proposed method and two heuristics: a genetic algorithm (GA hereinafter) and a strategy based on social imitation (SI hereinafter). Notably, although GAs have been proposed several years ago, they currently constitute one of more interesting methods in optimization. In addition, comparing the outcomes of the proposed model with those achieved by the SI method allows to evaluate the influence of the game dynamics (i.e., of the PGG). Before showing a comparative table, we briefly summarize how the GA has been implemented:

1. Define a population with N genes, assign each one a random solution for the considered TSP, and define a maximum number of iterations I.

2. While the best fitness in the population is smaller than 1, or the number of iterations is smaller than I:
   - Compute the fitness η of each gene (i.e., the goodness of its solution)
   - Select the best half of the population according to fitness
   - Generate two new solutions for each couple of genes, defined among the set computed at the previous step
   - Apply the random mutation, to each gene, with probability p_m;

We set to 0.1 the probability p_m (i.e., the random mutation) and to $30k$ the maximum number of iterations I. In addition, we emphasize that the crossover operator has been defined by cutting each gene parent (i.e., solution) in two different points so generating an offspring by using the central part of one parent and the side parts of the other parent. In the case this process does not generate viable solutions (e.g., in the presence of repetitions), the duplicates are removed for adding the missing cities. In addition, we briefly describe the SI method: given a TSP, start with a population composed of agents having a random solution (i.e., an array of cities). At each time step, randomly select two agents: the agent having the lower fitness imitates one entry of the solution of the other selected agent. Then, repeat this process until the population converges toward a shared solution (or a maximum number of time steps elapses). Further details are described in the Javarone's paper (see the reference related to the Partial Imitation mechanism for optimization problems). Table 4.1 shows the number of agents (or genes for the GA) for computing the optimal solution on varying the number of cities, the average number of time steps required to complete a simulation (computed on 20 different attempts) and, when smaller than 1, the average fitness.

According to these results, we observe that the proposed method requires the highest number of agents to solve a TSP. However, if compared to the SI algorithm, the proposed approach is much more faster (see the average number of time steps $\langle T \rangle$). Instead, the GA requires a smaller amount of agents than PGG, and

Table 4.1 Performance comparison, on varying the number of cities (Z), between the proposed method (PGG) and two heuristics: GA and SI

Z	PGG	SI	GA
10	$N = 100 \mid \langle T \rangle = 1K$	$N = 60 \mid \langle T \rangle = 8K$	$N = 100 \mid \langle T \rangle = 27$
20	$N = 900 \mid \langle T \rangle = 29K$	$N = 270 \mid \langle T \rangle = 500K$	$N = 100 \mid \langle T \rangle = 1.3k$
30	$N = 4900 \mid \langle T \rangle = 500K$	$N = 700 \mid \langle T \rangle = 5.5M$	$N = 100 \mid \langle T \rangle = 13.2k$
40	$N = 10,000 \mid \langle T \rangle = 3M$	$N = 1200 \mid \langle T \rangle = 40M$	$N = 200 \mid \langle T \rangle = 23k \mid \langle \eta \rangle = 0.76$
50	$N = 22,500 \mid \langle T \rangle = 21M$	$N = 1600 \mid \langle T \rangle = 360M$	$N = 200 \mid \langle T \rangle = 28.5k \mid \langle \eta \rangle = 0.61$

N indicates the minimum number of agents (genes for GA) used to solve the problem, and $\langle T \rangle$ indicates the average number of time steps required. The average fitness $\langle \eta \rangle$ is indicated only when smaller than 1, although the best value computed considering all attempts is 1 (i.e., the optimal solution has not been always computed)

it is also faster. At the same time, it is important to observe that the GA has a synchronous dynamics (while our method is asynchronous), i.e., during the same time step, all agents are involved for generating offsprings and updating their solution (according to the random mutation mechanism). Therefore, further analyses are required for a complete time comparison. Nevertheless we found that, considering 20 different simulation runs, the average fitness of the best solution (found in the gene population) is smaller than 1 when $Z \geq 40$. Hence, the GA must be run several times for each task, saving the best solution. To conclude, according to this analysis, we report that a GA constitutes the best choice for solving simple problems (i.e., with few cities) or for computing a good suboptimal solution in a short time. On the other hand, when the number of cities increases, the proposed method allows to reach a higher fitness in a smaller number of attempts than that required by a GA.

Appendix 2

Here, we present a brief analysis of the proposed model performed by using a structured population, i.e., agents arranged on a network. Notably, we considered regular square lattices (with periodic boundary conditions), and small-world networks implemented according to the Watts-Strogatz model. In particular, small-world networks have been defined starting with a two-dimensional ring with eight neighbors per node, and then rewiring with probability β each edge at random. Thus, using values of β higher than 0, we obtained small-world networks. Figure 4.11 shows results of the comparative analysis. For each type of network, we considered different realizations. So, we observe that the amount of agents to solve a TSP increases using structured populations, in particular in small-world networks. Therefore, the most convenient choice for solving a TSP remains the well-mixed population. Now, we discuss a possible explanation of this result. Notably, small-world networks contain few nodes with a number of connections (i.e., degree) higher than the average value. These nodes are usually defined hubs. On one hand, their role is fundamental in spreading processes, since they make them faster than those implemented by using regular topologies. On the other hand, when hubs are provided with a solution having a fitness higher than that of their neighbors, they may constitute a limit during the definition of new solutions. Notably, in this case all neighbors tend to imitate the solution of hubs thus, given a TSP with defined conditions (e.g., number of cities), small-world networks require a number of agents higher than that required in regular networks for solving the same problem. In few words, hubs are able to affect the solution of too many opponents, reducing the innovative potential of the whole population. Eventually, our observation is corroborated by comparing results obtained in small-world networks generated with different β. In particular, increasing β the number of hubs increases, and networks

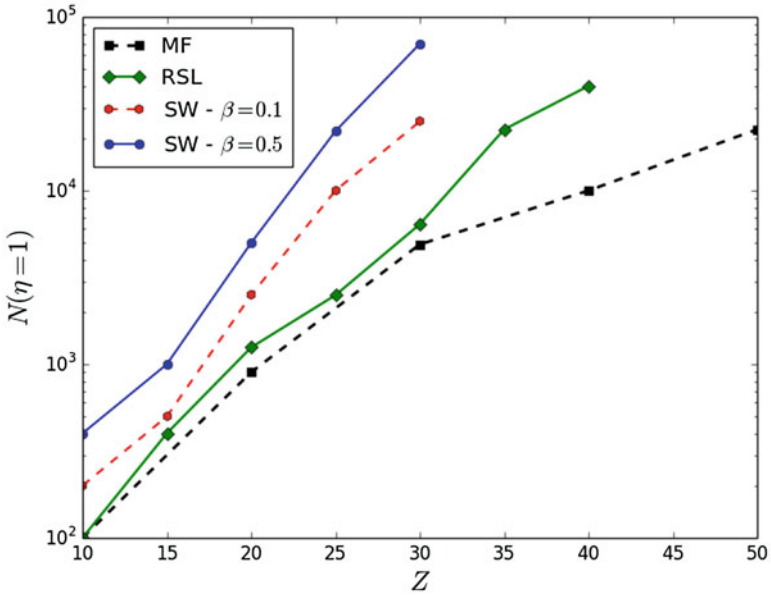

Fig. 4.11 Minimum number of agents to compute the optimal solution of a TSP on varying the number of cities Z. As indicated in the legend, the (dotted) black line refers to results obtained in the well-mixed population. The (continuous) green line refers to the regular square lattice, with periodic boundary conditions. The (dotted) red line refers to small-word networks achieved with $\beta = 0.1$, and the blue (continuous) line to those obtained in small-world networks achieved with $\beta = 0.5$. Results are averaged over different simulation runs

generated with $\beta = 0.5$ resulted less convenient than those generated with $\beta = 0.1$. To conclude, in the light of results, we deem that topologies containing hubs may reduce the computational power and the innovative potential of an agent population.

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Chapter 5
Conclusions

This last part of the book is devoted to summarize the most important concepts exposed in the previous chapters, and to offer a quick overview on some future perspectives in EGT that, hopefully, will stimulate the curiosity of the reader for this vibrant research field.

5.1 Summary

Nowadays, the area of Complex systems offers a number of exciting problems and questions, most of them strongly interdisciplinary. Here, the framework of EGT constitutes a new tool for facing very interesting challenges. So, first of all, one should keep in mind what actually is EGT, i.e., as described in the general background (see Chap. 1) an “alchemic combination" of two relevant fields: Game Theory and Evolutionary Theory. It is important to be aware about that in order to achieve a full comprehension of the related results, and to envision further developments. However, as suggested by the second application presented in Chap. 4, the applications of EGT can go beyond the natural scopes of this field. Second, EGT can be an efficient tool for studying and representing complex systems and their dynamics. At the same time, one should remember that some kind of complexity can be found in EGT models, since they show nonlinear and collective phenomena. For this reason, the Statistical Physics approach for studying the evolution of a population can be the right tool for connecting the macroscopic behavior of a population, with its local mechanisms (e.g., game dynamics, topology, etc.). Accordingly, Chap. 2 presented a number of mathematical strategies for dealing with these problems. Among them, we find the Ising model, a milestone of Statistical Physics. Actually, although it is not used for modeling EGT, the approach for studying games like the spatial PGG (see Chap. 3) is similar to that adopted for studying the dynamics of the Ising model. Tracing a route from the Ising model

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to the dynamics of a spatial game (on a square lattice) can be quite helpful for looking at the evolution of strategies as a phase transition. Notably, in systems composed of simple spins, one can only implement an updating rule that depends on the "local energy," while in the case of agents, despite their strategy can be codified as a spin, the updating rules can be a bit more complicated. Obviously, the utilization of agents provided with particular behaviors strongly reduces the analytical tractability of these models (so we need to adopt Monte Carlo methods). Then, models presented in Chaps. 3 and 4 highlight the nature of Cooperation as an emergent phenomenon, whose relevance can be observed in a number of contexts. For this reason, uncovering the mechanisms that lead a population to adopt/select this strategy (i.e., cooperation) is a really prominent issue. Moreover, from this point of view, the emergence of cooperation is clearly an issue that can be framed in the modern Science of Complexity. Finally, we wish to further emphasize that evolutionary games can be adopted for studying topics beyond the fences of EGT, as reported in the chapter of applications (i.e., Chap. 4).

5.2 Perspectives

In this concluding section, we provide a short analysis of potential developments of EGT, in particular considering some recent results. To this end, we look at five different areas: social dynamics, economic systems, ecology, biology, and complex networks. Notably, while the latter is currently adopted in the other mentioned areas, as a modeling tool, recent results suggest that a lot of attention must be still paid to the relation between the topology and the emergence of cooperation. For this reason, we think be appropriate to consider networks as an independent topic for EGT models. Considering social dynamics, further results on EGT can be obtained modeling human behaviors in order to understand the relation between the emergence of cooperation and specific characters. Actually, this approach follows the trend of Sociophysics, where behaviors are codified into physical variables, and then the dynamics of a population are studied. Here, another topic of interest can be the studying of the emergence of social norms, as well as that of the emergence of new societal structures (both among humans and animals). Notably, in these cases, some evolutionary mechanisms can be easily recognized. In the context of economy, recent results indicate in increasing interest for the so-called experimental economics, i.e., the area that aims to apply experimental methods for studying economic problems. Here, experiments usually involve groups of individuals that are asked to perform simple tasks/games, so that results can be analyzed under the lens of EGT. In addition, it is worth to remind that this area sometimes overlaps with the previous one, i.e., social dynamics, so that we might refer to socioeconomic systems, where the human behavior has a prominent role. Ecology is a very big scientific field, framed in the area of complex systems due to its focus on the nontrivial interactions among organisms/individuals and their environment. Accordingly, a number of evolutionary phenomena can be observed

in these systems, and EGT constitutes a powerful tool for modeling the related dynamics. In particular, self-organization and other adaptive behaviors are of strong interest, as well as those problems related to the climate governance. Actually, as we know, the latter represents one of the major challenges in the human societies. Therefore, all the studies that bring a valid contribution to this topic (i.e., climate governance), as those achieved by EGT, can be strongly valuable, not only from a scientific point of view. In the field of biology, numerous results suggest that EGT can be an effective and promising tool for uncovering the complex dynamics underlying several biological phenomena. Just to cite few, recent achievements can be found in theoretical biology, cancer evolution, cellular mechanisms, bacterial dynamics, dynamics of ant colonies, and so on. Therefore, although the nature of dilemma games might appear suitable only for modeling human behaviors, these dynamics can actually help to represent also further scenarios, where “rationality" can be replaced by other terms. To conclude, we deem worth to indicate two interesting and recent results at the edge between network theory and EGT. One related to the understanding of the “topological frustration" in the dynamics of the evolutionary games, where the authors found that the level of cooperation constitutes a direct indicator of the level of topological frustration. The other result is related to the implementation of EGT models on multilayer and on multiplex networks. In particular, authors of this second investigation focused on the PGG implemented on multiplex networks. Their results show that cooperation emerges only when a significant overlap of edges is combined with at least one layer able to support some cooperation by a sufficiently high synergy factor. Otherwise, the evolution of cooperation in this kind of networks is given by the bounds of the classical network reciprocity (see Chap. 3). It is worth to emphasize that multiplex networks constitute an advanced topic in the network theory; therefore, it is not so surprising that by their application exciting developments can be achieved, both in EGT and in many other fields. Finally, beyond the topic(s) one decides to investigate in EGT, we are strongly confident that the Statistical Physics approach, here described by simple examples, be the most promising and appropriate way for achieving interesting results and obtaining new insights.

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