Abdul-Majid Wazwaz

Linear and Nonlinear Integral Equations

Methods and Applications

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With 4 figures

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THIS BOOK IS DEDICATED TO
my wife, our son, and our three daughters
for supporting me in all my endeavors

Preface

Many remarkable advances have been made in the field of integral equations, but these remarkable developments have remained scattered in a variety of specialized journals. These new ideas and approaches have rarely been brought together in textbook form. If these ideas merely remain in scholarly journals and never get discussed in textbooks, then specialists and students will not be able to benefit from the results of the valuable research achievements.

The explosive growth in industry and technology requires constructive adjustments in mathematics textbooks. The valuable achievements in research work are not found in many of today's textbooks, but they are worthy of addition and study. The technology is moving rapidly, which is pushing for valuable insights into some substantial applications and developed approaches. The mathematics taught in the classroom should come to resemble the mathematics used in varied applications of nonlinear science models and engineering applications. This book was written with these thoughts in mind.

Linear and Nonlinear Integral Equations: Methods and Applications is designed to serve as a text and a reference. The book is designed to be accessible to advanced undergraduate and graduate students as well as a research monograph to researchers in applied mathematics, physical sciences, and engineering. This text differs from other similar texts in a number of ways. First, it explains the classical methods in a comprehensible, non-abstract approach. Furthermore, it introduces and explains the modern developed mathematical methods in such a fashion that shows how the new methods can complement the traditional methods. These approaches further improve the understanding of the material.

The book avoids approaching the subject through the compact and classical methods that make the material difficult to be grasped, especially by students who do not have the background in these abstract concepts. The aim of this book is to offer practical treatment of linear and nonlinear integral equations emphasizing the need to problem solving rather than theorem proving.

The book was developed as a result of many years of experiences in teaching integral equations and conducting research work in this field. The author has taken account of his teaching experience, research work as well as valuable suggestions received from students and scholars from a wide variety of audience. Numerous examples and exercises, ranging in level from easy to difficult, but consistent with the material, are given in each section to give the reader the knowledge, practice and skill in linear and nonlinear integral equations. There is plenty of material in this text to be covered in two semesters for senior undergraduates and beginning graduates of mathematics, physical science, and engineering.

The content of the book is divided into two distinct parts, and each part is self-contained and practical. Part I contains twelve chapters that handle the linear integral and nonlinear integro-differential equations by using the modern mathematical methods, and some of the powerful traditional methods. Since the book's readership is a diverse and interdisciplinary audience of applied mathematics, physical science, and engineering, attempts are made so that part I presents both analytical and numerical approaches in a clear and systematic fashion to make this book accessible to those who work in these fields.

Part II contains the remaining six chapters devoted to thoroughly examining the nonlinear integral equations and its applications. The potential theory contributed more than any field to give rise to nonlinear integral equations. Mathematical physics models, such as diffraction problems, scattering in quantum mechanics, conformal mapping, and water waves also contributed to the creation of nonlinear integral equations. Because it is not always possible to find exact solutions to problems of physical science that are posed, much work is devoted to obtaining qualitative approximations that highlight the structure of the solution.

Chapter 1 provides the basic definitions and introductory concepts. The Taylor series, Leibnitz rule, and Laplace transform method are presented and reviewed. This discussion will provide the reader with a strong basis to understand the thoroughly-examined material in the following chapters. In Chapter 2, the classifications of integral and integro-differential equations are presented and illustrated. In addition, the linearity and the homogeneity concepts of integral equations are clearly addressed. The conversion process of IVP and BVP to Volterra integral equation and Fredholm integral equation respectively are described. Chapters 3 and 5 deal with the linear Volterra integral equations and the linear Volterra integro-differential equations, of the first and the second kind, respectively. Each kind is approached by a variety of methods that are described in details. Chapters 3 and 5 provide the reader with a comprehensive discussion of both types of equations. The two chapters emphasize the power of the proposed methods in handling these equations. Chapters 4 and 6 are entirely devoted to Fredholm integral equations and Fredholm integro-differential equations, of the first and the second kind, respectively. The ill-posed Fredholm integral equation of the first kind is handled by the powerful method of regularization combined with other methods. The two kinds of equations are approached by many appropriate algorithms that are illustrated in details. A comprehensive study is introduced where a variety of reliable methods is applied independently and supported by many illustrative examples. Chapter 7 is devoted to studying the Abel's integral equations, generalized Abel's integral equations, and the weakly singular integral equations. The chapter also stresses the significant features of these types of singular equations with full explanations and many illustrative examples and exercises. Chapters 8 and 9 introduce a valuable study on Volterra-Fredholm integral equations and Volterra-Fredholm integro-differential equations respectively in one and two variables. The mixed Volterra-Fredholm integral and the mixed Volterra-Fredholm integro-differential equations in two variables are also examined with illustrative examples. The proposed methods introduce a powerful tool for handling these two types of equations. Examples are provided with a substantial amount of explanation. The reader will find a wealth of well-known models with one and two variables. A detailed and clear explanation of every application is introduced and supported by fully explained examples and exercises of every type.

Chapters 10, 11, and 12 are concerned with the systems of Volterra integral and integro-differential equations, systems of Fredholm integral and integro-differential equations, and systems of singular integral equations and systems of weakly singular integral equations respectively. Systems of integral equations that are important, are handled by using very constructive methods. A discussion of the basic theory and illustrations of the solutions to the systems are presented to introduce the material in a clear and useful fashion. Singular systems in one, two, and three variables are thoroughly investigated. The systems are supported by a variety of useful methods that are well explained and illustrated.

Part II is titled "Nonlinear Integral Equations". Part II of this book gives a self-contained, practical and realistic approach to nonlinear integral equations, where scientists and engineers are paying great attention to the effects caused by the nonlinearity of dynamical equations in nonlinear science. The potential theory contributed more than any field to give rise to nonlinear integral equations. Mathematical physics models, such as diffraction problems, scattering in quantum mechanics, conformal mapping, and water waves also contributed to the creation of nonlinear integral equations. The nonlinearity of these models may give more than one solution and this is the nature of nonlinear problems. Moreover, ill-posed Fredholm integral equations of the first kind may also give more than one solution even if it is linear.

Chapter 13 presents discussions about nonlinear Volterra integral equations and systems of Volterra integral equations, of the first and the second kind. More emphasis on the existence of solutions is proved and emphasized. A variety of methods are employed, introduced and explained in a clear and useful manner. Chapter 14 is devoted to giving a comprehensive study on nonlinear Volterra integro-differential equations and systems of nonlinear Volterra integro-differential equations, of the first and the second kind.

A variety of methods are introduced, and numerous practical examples are explained in a practical way. Chapter 15 investigates thoroughly the existence theorem, bifurcation points and singular points that may arise from nonlinear Fredholm integral equations. The study presents a variety of powerful methods to handle nonlinear Fredholm integral equations of the first and the second kind. Systems of these equations are examined with illustrated examples. Chapter 16 is entirely devoted to studying a family of nonlinear Fredholm integro-differential equations of the second kind and the systems of these equations. The approach we followed is identical to our approach in the previous chapters to make the discussion accessible for interdisciplinary audience. Chapter 17 provides the reader with a comprehensive discussion of the nonlinear singular integral equations, nonlinear weakly singular integral equations, and systems of these equations. Most of these equations are characterized by the singularity behavior where the proposed methods should overcome the difficulty of this singular behavior. The power of the employed methods is confirmed here by determining solutions that may not be unique. Chapter 18 presents a comprehensive study on five scientific applications that we selected because of its wide applicability for several other models. Because it is not always possible to find exact solutions to models of physical sciences, much work is devoted to obtaining qualitative approximations that highlight the structure of the solution. The powerful Padé approximants are used to give insight into the structure of the solution. This chapter closes Part II of this text.

The book concludes with seven useful appendices. Moreover, the book introduces the traditional methods in the same amount of concern to provide the reader with the knowledge needed to make a comparison.

I deeply acknowledge Professor Albert Luo for many helpful discussions, encouragement, and useful remarks. I am also indebted to Ms. Liping Wang, the Publishing Editor of the Higher Education Press for her effective cooperation and important suggestions. The staff of HEP deserve my thanks for their support to this project. I owe them all my deepest thanks.

I also deeply acknowledge Professor Louis Pennisi who made very valuable suggestions that helped a great deal in directing this book towards its main goal.

I am deeply indebted to my wife, my son and my daughters who provided me with their continued encouragement, patience and support during the long days of preparing this book.

The author would highly appreciate any notes concerning any constructive suggestions.

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April 20, 2011

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Part I
Linear Integral Equations

Chapter 1
Preliminaries

An *integral equation* is an equation in which the unknown function $u(x)$ appears under an integral sign [1–7]. A standard integral equation in $u(x)$ is of the form:

$$u(x) = f(x) + \lambda \int_{g(x)}^{h(x)} K(x,t)u(t)dt, \tag{1.1}$$

where $g(x)$ and $h(x)$ are the limits of integration, λ is a constant parameter, and $K(x,t)$ is a function of two variables x and t called the *kernel* or the *nucleus* of the integral equation. The function $u(x)$ that will be determined appears under the integral sign, and it appears inside the integral sign and outside the integral sign as well. The functions $f(x)$ and $K(x,t)$ are given in advance. It is to be noted that the limits of integration $g(x)$ and $h(x)$ may be both variables, constants, or mixed.

An *integro-differential equation* is an equation in which the unknown function $u(x)$ appears under an integral sign and contains an ordinary derivative $u^{(n)}(x)$ as well. A standard integro-differential equation is of the form:

$$u^{(n)}(x) = f(x) + \lambda \int_{g(x)}^{h(x)} K(x,t)u(t)dt, \tag{1.2}$$

where $g(x), h(x), f(x), \lambda$ and the kernel $K(x,t)$ are as prescribed before.

Integral equations and integro-differential equations will be classified into distinct types according to the limits of integration and the kernel $K(x,t)$. All types of integral equations and integro-differential equations will be classified and investigated in the forthcoming chapters.

In this chapter, we will review the most important concepts needed to study integral equations. The traditional methods, such as Taylor series method and the Laplace transform method, will be used in this text. Moreover, the recently developed methods, that will be used thoroughly in this text, will determine the solution in a power series that will converge to an exact solution if such a solution exists. However, if exact solution does not exist, we use as many terms of the obtained series for numerical purposes to approximate the solution. The more terms we determine the higher numerical accuracy we can achieve. Furthermore, we will review the basic concepts for solving ordinary differential equations. Other mathematical concepts, such as Leibnitz rule will be presented.

1.1 Taylor Series

Let $f(x)$ be a function with derivatives of all orders in an interval $[x_0, x_1]$ that contains an interior point a. The Taylor series of $f(x)$ generated at $x = a$ is

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n, \tag{1.3}$$

or equivalently

$$\begin{aligned} f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots \\ + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots . \end{aligned} \tag{1.4}$$

The Taylor series generated by $f(x)$ at $a = 0$ is called the Maclaurin series and given by

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n, \tag{1.5}$$

that is equivalent to

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots . \tag{1.6}$$

In what follows, we will discuss a few examples for the determination of the Taylor series at $x = 0$.

Example 1.1

Find the Taylor series generated by $f(x) = e^x$ at $x = 0$.
We list the exponential function and its derivatives as follows:

$f^{(n)}(x)$	$f(x) = e^x$	$f'(x) = e^x$	$f''(x) = e^x$	$f'''(x) = e^x$
$f^{(n)}(0)$	$f(0) = 1,$	$f'(0) = 1,$	$f''(0) = 1,$	$f'''(0) = 1,$
$\vdots$				

and so on. This gives the Taylor series for e^x by

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \tag{1.7}$$

and in a compact form by

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}. \tag{1.8}$$

Similarly, we can easily show that

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \tag{1.9}$$

and

$$e^{ax} = 1 + ax + \frac{(ax)^2}{2!} + \frac{(ax)^3}{3!} + \frac{(ax)^4}{4!} + \cdots \tag{1.10}$$

Example 1.2

Find the Taylor series generated by $f(x) = \cos x$ at $x = 0$.
Following the discussions presented before we find

$$\begin{array}{llllll} f^{(n)}(x) & f(x) = \cos x, & f'(x) = -\sin x, & f''(x) = -\cos x, & f'''(x) = \sin x, & f^{(\text{iv})}(x) = \cos x \\ f^{(n)}(0) & f(0) = 1, & f'(0) = 0, & f''(0) = -1, & f'''(0) = 0, & f^{(\text{iv})}(0) = 1, \\ \vdots & & & & & \end{array}$$

and so on. This gives the Taylor series for $\cos x$ by

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots \tag{1.11}$$

and in a compact form by

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}. \tag{1.12}$$

In a similar way we can derive the following

$$\cos(ax) = 1 - \frac{(ax)^2}{2!} + \frac{(ax)^4}{4!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} (ax)^{2n}. \tag{1.13}$$

For $f(x) = \sin x$ and $f(x) = \sin(ax)$, we can show that

$$\begin{aligned} \sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}, \\ \sin(ax) &= (ax) - \frac{(ax)^3}{3!} + \frac{(ax)^5}{5!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} (ax)^{2n+1}. \end{aligned} \tag{1.14}$$

In Appendix C, the Taylor series for many well known functions generated at $x = 0$ are given.

As stated before, the newly developed methods for solving integral equations determine the solution in a series form. Unlike calculus where we determine the Taylor series for functions and the radius of convergence for each series, it is required here that we determine the exact solution of the integral equation if we determine its series solution. In what follows, we will discuss some examples that will show how exact solution is obtained if the series solution is given. Recall that the Taylor series exists for analytic functions only.

Example 1.3

Find the closed form function for the following series:

$$f(x) = 1 + 2x + 2x^2 + \frac{4}{3}x^3 + \frac{2}{3}x^4 + \cdots \tag{1.15}$$

It is obvious that this series can be rewritten in the form:

$$f(x) = 1 + 2x + \frac{(2x)^2}{2!} + \frac{(2x)^3}{3!} + \frac{(2x)^4}{4!} + \cdots \tag{1.16}$$

that will converge to the exact form:

$$f(x) = e^{2x}. \tag{1.17}$$

Example 1.4

Find the closed form function for the following series:

$$f(x) = 1 + \frac{9}{2}x^2 + \frac{27}{8}x^4 + \frac{81}{80}x^6 + \cdots \tag{1.18}$$

Notice that the second term can be written as $\frac{(3x)^2}{2!}$, therefore the series can be rewritten as

$$f(x) = 1 + \frac{(3x)^2}{2!} + \frac{(3x)^4}{4!} + \frac{(3x)^6}{6!} + \cdots \tag{1.19}$$

that will converge to

$$f(x) = \cosh(3x). \tag{1.20}$$

Example 1.5

Find the closed form function for the following series:

$$f(x) = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \cdots \tag{1.21}$$

This series will converge to

$$f(x) = \tan x. \tag{1.22}$$

Example 1.6

Find the closed form function for the following series:

$$f(x) = 1 + x - \frac{1}{2!}x^2 - \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \cdots \tag{1.23}$$

The signs of the terms are positive for the first two terms then negative for the next two terms and so on. The series should be grouped as

$$f(x) = (1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \cdots) + (x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \cdots), \tag{1.24}$$

that will converge to

$$f(x) = \cos x + \sin x. \tag{1.25}$$

Exercises 1.1

Find the closed form function for the following Taylor series:

1. $f(x) = 2x + 2x^2 + \frac{4}{3}x^3 + \frac{2}{3}x^4 + \cdots$ 2. $f(x) = 1 - 3x + \frac{9}{2}x^2 - \frac{9}{2}x^3 + \frac{27}{8}x^4 + \cdots$

3. $f(x) = x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots$
4. $f(x) = 1 - x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 - \frac{1}{5!}x^5 - \frac{1}{6!}x^6 + \cdots$
5. $f(x) = 3x - \frac{9}{2}x^3 + \frac{81}{40}x^5 - \frac{243}{560}x^7 + \cdots$
6. $f(x) = 2x + \frac{4}{3}x^3 + \frac{4}{15}x^5 + \frac{8}{315}x^7 + \cdots$
7. $f(x) = 1 + 2x^2 + \frac{2}{3}x^4 + \frac{4}{45}x^6 + \cdots$
8. $f(x) = \frac{9}{2}x^2 + \frac{27}{8}x^4 + \frac{81}{80}x^6 + \cdots$
9. $f(x) = 2 - 2\,x^2 + \frac{2}{3}x^4 - \frac{4}{45}x^6 + \cdots$
10. $f(x) = 1 + x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{1}{5040}x^7 + \cdots$
11. $f(x) = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \frac{17}{315}x^7 + \cdots$
12. $f(x) = x - \frac{1}{3}x^3 + \frac{2}{15}x^5 - \frac{17}{315}x^7 + \cdots$
13. $f(x) = 2 + 2x + \frac{1}{2!}x^2 - \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \cdots.$
14. $2 + x - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots$

1.2 Ordinary Differential Equations

In this section we will review some of the linear ordinary differential equations that we will use for solving integral equations. For proofs, existence and uniqueness of solutions, and other details, the reader is advised to use ordinary differential equations texts.

1.2.1 First Order Linear Differential Equations

The standard form of first order linear ordinary differential equation is

$$u' + p(x)u = q(x), \tag{1.26}$$

where $p(x)$ and $q(x)$ are given continuous functions on $x_0 < x < x_1$. We first determine an integrating factor $\mu(x)$ by using the formula:

$$\mu(x) = e^{\int^x p(t)dt}. \tag{1.27}$$

Recall that an integrating factor $\mu(x)$ is a function of x that is used to facilitate the solving of a differential equation. The solution of (1.26) is obtained by using the formula:

$$u(x) = \frac{1}{\mu(x)}\left[\int^x \mu(t)q(t)dt + c\right], \tag{1.28}$$

where c is an arbitrary constant that can be determined by using a given initial condition.

Example 1.7

Solve the following first order ODE:

$$u' - 3u = 3x^2e^{3x}, \quad u(0) = 1. \tag{1.29}$$

Notice that $p(x) = -3$ and $q(x) = 3x^2e^{3x}$. The integrating factor $\mu(x)$ is obtained by

$$\mu(x) = e^{\int_0^x -3dt} = e^{-3x}. \tag{1.30}$$

Consequently, $u(x)$ is obtained by using

$$\begin{aligned} u(x) &= \frac{1}{\mu(x)}\left[\int^x \mu(t)q(t)dt + c\right] = e^{3x}\left(\int^x 3t^2dt + c\right) \\ &= e^{3x}\left(x^3 + c\right) = e^{3x}(x^3+1), \end{aligned} \tag{1.31}$$

obtained upon using the given initial condition.

Example 1.8

Solve the following first order ODE:

$$xu' + 3u = \frac{\cos x}{x}, \quad u(\pi) = 0, x > 0. \tag{1.32}$$

We first divide the equation by x to convert it to the standard form (1.26). As a result, $p(x) = \frac{3}{x}$ and $q(x) = \frac{\cos x}{x^2}$. The integrating factor $\mu(x)$ is

$$\mu(x) = e^{\int^x \frac{3}{t}dt} = e^{3\ln x} = x^3. \tag{1.33}$$

Consequently, $u(x)$ is

$$\begin{aligned} u(x) &= \frac{1}{\mu(x)}\left[\int^x \mu(t)q(t)dt + c\right] = \frac{1}{x^3}\left(\int^x t\cos t dt + c\right) \\ &= \frac{1}{x^3}\left(\cos x + x\sin x + c\right) = \frac{1}{x^3}(\cos x + x\sin x + 1), \end{aligned} \tag{1.34}$$

obtained upon using the given initial condition.

Exercises 1.2.1

Find the general solution for each of the following first order ODEs:

1. $u' + u = e^{-x}, x > 0$
2. $xu' - 4u = x^5e^x, x > 0$
3. $(x^2+9)u' + 2xu = 0, x > 0$
4. $xu' - 4u = 2x^6 + x^5, x > 0$
5. $xu' + u = 2x, x > 0$
6. $xu' - u = x^2\sin x, x > 0$

Find the particular solution for each of the following initial value problems:

7. $u' - u = 2xe^x,\ u(0) = 0$
8. $xu' + u = 2x,\ u(1) = 1$
9. $(\tan x)u' + (\sec^2 x)u = 2e^{2x},\ u\left(\frac{\pi}{4}\right) = e^{\frac{\pi}{2}}$
10. $u' - 3u = 4x^3e^{3x},\ u(0) = 1$

11. $(1 + x^3)u' + 3x^2 u = 1,\ u(0) = 0$

12. $u' + (\tan x)u = \cos x,\ u(0) = 1$

1.2.2 Second Order Linear Differential Equations

As stated before, we will review some second order linear ordinary differential equations. The focus will be on second order equations, homogeneous and inhomogeneous as well.

Homogeneous Equations with Constant Coefficients

The standard form of the second order homogeneous ordinary differential equations with constant coefficients is

$$au'' + bu' + cu = 0, a \neq 0, \tag{1.35}$$

where a, b, and c are constants. The solution of this equation is assumed to be of the form:

$$u(x) = e^{rx}. \tag{1.36}$$

Substituting this assumption into Eq. (1.35) gives the equation:

$$e^{rx}(ar^2 + br + c) = 0. \tag{1.37}$$

Since e^{rx} is not zero, then we have the characteristic or the auxiliary equation:

$$ar^2 + br + c = 0. \tag{1.38}$$

Solving this quadratic equation leads to one of the following three cases:

(i) If the roots r_1 and r_2 are real and $r_1 \neq r_2$, then the general solution of the homogeneous equation is

$$u(x) = Ae^{r_1 x} + Be^{r_2 x}, \tag{1.39}$$

where A and B are constants.

(ii) If the roots r_1 and r_2 are real and $r_1 = r_2 = r$, then the general solution of the homogeneous equation is

$$u(x) = Ae^{rx} + Bxe^{rx}, \tag{1.40}$$

where A and B are constants.

(iii) If the roots r_1 and r_2 are complex and $r_1 = \lambda + i\mu, r_2 = \lambda - i\mu$, then the general solution of the homogeneous equation is given by

$$u(x) = e^{\lambda x}\left(A\cos(\mu x) + B\sin(\mu x)\right), \tag{1.41}$$

where A and B are constants.

Inhomogeneous Equations with Constant Coefficients

The standard form of the second order inhomogeneous ordinary differential equations with constant coefficients is

$$au'' + bu' + cu = g(x), a \neq 0, \tag{1.42}$$

where a, b, and c are constants. The general solution consists of two parts, namely, *complementary solution* u_c, and a *particular solution* u_p where the general solution is of the form:

$$u(x) = u_c(x) + u_p(x), \tag{1.43}$$

where u_c is the solution of the related homogeneous equation:

$$au'' + bu' + cu = 0, a \neq 0, \tag{1.44}$$

and this is obtained as presented before. A particular solution u_p arises from the inhomogeneous part $g(x)$. It is called a particular solution because it justifies the inhomogeneous equation (1.42), but it is not the particular solution of the equation that is obtained from (1.43) upon using the given initial equations as will be discussed later. To obtain $u_p(x)$, we use the method of *undetermined coefficients*. To apply this method, we consider the following three types of $g(x)$:

(i) If $g(x)$ is a polynomial given by

$$g(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n, \tag{1.45}$$

then u_p should be assumed as

$$u_p = x^r(b_0x^n + b_1x^{n-1} + \cdots + b_n), \quad r = 0, 1, 2, \ldots \tag{1.46}$$

(ii) If $g(x)$ is an exponential function of the form:

$$g(x) = a_0e^{\alpha x}, \tag{1.47}$$

then u_p should be assumed as

$$u_p = b_0x^re^{\alpha x}, \quad r = 0, 1, 2, \ldots \tag{1.48}$$

(iii) If $g(x)$ is a trigonometric function of the form:

$$g(x) = a_0 \sin(\alpha x) + b_0 \cos(\beta x), \tag{1.49}$$

then u_p should be assumed as

$$u_p = x^r(A_0 \sin(\alpha x) + B_0 \cos(\beta x)), \quad r = 0, 1, 2, \ldots \tag{1.50}$$

For other forms of $g(x)$ such as $\tan x$ and $\sec x$, we usually use the *variation of parameters* method that will not be reviewed in this text. Notice that r is the smallest nonnegative integer that will guarantee no term in $u_p(x)$ is a solution of the corresponding homogeneous equation. The values of r are $0, 1$ and 2.

Example 1.9

Solve the following second order ODE:

$$u'' - u = 0. \tag{1.51}$$

The auxiliary equation is given by

$$r^2 - 1 = 0, \tag{1.52}$$

and this gives

$$r = \pm 1. \tag{1.53}$$

Accordingly, the general solution is given by

$$u(x) = Ae^{x} + Be^{-x}. \tag{1.54}$$

It is interesting to point out that the normal form ODE:

$$u'' + u = 0, \tag{1.55}$$

leads to the auxiliary equation:

$$r^2 + 1 = 0, \tag{1.56}$$

and this gives

$$r = \pm i. \tag{1.57}$$

The general solution is given by

$$u(x) = A\cos x + B\sin x. \tag{1.58}$$

Example 1.10

Solve the following second order ODE:

$$u'' - 7u' + 6u = 0. \tag{1.59}$$

The auxiliary equation is given by

$$r^2 - 7r + 6 = 0, \tag{1.60}$$

with roots given by

$$r = 1, 6. \tag{1.61}$$

The general solution is given by

$$u(x) = Ae^{x} + Be^{6x}. \tag{1.62}$$

Example 1.11

Solve the following second order ODE:

$$u'' - 5u' + 6u = 6x + 7. \tag{1.63}$$

We first find u_c. The auxiliary equation for the related homogeneous equation is given by

$$r^2 - 5r + 6 = 0, \tag{1.64}$$

with roots given by

$$r = 2, 3. \tag{1.65}$$

The general solution is given by

$$u(x) = \alpha e^{2x} + \beta e^{3x}. \tag{1.66}$$

Noting that $g(x) = 6x + 7$, then a particular solution is assumed to be of the form

$$u_p = Ax + B. \tag{1.67}$$

Since this is a particular solution, then substituting u_p into the inhomogeneous equation leads to

$$6Ax + (6B - 5A) = 6x + 7. \tag{1.68}$$

Equating the coefficients of like terms from both sides gives

$$A = 1, \quad B = 2. \tag{1.69}$$

This in turn gives

$$u(x) = u_c + u_p = \alpha e^{2x} + \beta e^{3x} + x + 2, \tag{1.70}$$

where α and β are arbitrary constants.

Example 1.12

Solve the following initial value problem

$$u'' + 9u = 20e^x, \quad u(0) = 3, \quad y'(0) = 5. \tag{1.71}$$

We first find u_c. The auxiliary equation for the related homogeneous equation is given by

$$r^2 + 9 = 0, \tag{1.72}$$

where we find

$$r = \pm 3i, \quad i^2 = -1. \tag{1.73}$$

The general solution is given by

$$u(x) = \alpha \cos(3x) + \beta \sin(3x). \tag{1.74}$$

Noting that $g(x) = 20e^x$, then a particular solution is assumed to be of the form:

$$u_p = Ae^x. \tag{1.75}$$

Since this is a particular solution, then substituting u_p into the inhomogeneous equation leads to

$$10Ae^x = 20e^x, \tag{1.76}$$

so that

$$A = 2. \tag{1.77}$$

This in turn gives the general solution

$$u(x) = u_c + u_p = \alpha \cos(3x) + \beta \sin(3x) + 2e^x. \tag{1.78}$$

Since the initial conditions are given, the numerical values for α and β should be determined. Substituting the initial values into the general solution we find

$$\alpha + 2 = 3, \quad 3\beta + 2 = 5, \tag{1.79}$$

where we find

$$\alpha = 1, \quad \beta = 1. \tag{1.80}$$

Accordingly, the particular solution is given by

$$u(x) = \cos(3x) + \sin(3x) + 2e^x. \tag{1.81}$$

Exercises 1.2.2

Find the general solution for the following second order ODEs:

1. $u'' - 4u' + 4u = 0$
2. $u'' - 2u' - 3u = 0$
3. $u'' - u' - 2u = 0$
4. $u'' - 2u' = 0$
5. $u'' - 6u' + 9u = 0$
6. $u'' + 4u = 0$

Find the general solution for the following initial value problems:

7. $u'' - 2u' + 2u = 0, u(0) = 1, u'(0) = 1$
8. $u'' - 6u' + 9u = 0, u(0) = 1, u'(0) = 4$
9. $u'' - 3u' - 10u = 0, u(0) = 2, u'(0) = 3$
10. $u'' + 9u = 0, u(0) = 1, u'(0) = 0$
11. $u'' - 9u' = 0, u(0) = 3, u'(0) = 9$
12. $u'' - 9u = 0, u(0) = 1, u'(0) = 0$

Use the method of undetermined coefficients to find the general solution for the following second order ODEs:

13. $u'' - u' = 1$
14. $u'' + u = 3$
15. $u'' - u = 3x$
16. $u'' - u = 2\cos x$

Use the method of undetermined coefficients to find the particular solution for the following initial value problems:

17. $u'' - u' = 6, u(0) = 3, u'(0) = 2$
18. $u'' + u = 6e^x, u(0) = 3, u'(0) = 2$
19. $u'' - u = -2\sin x, u(0) = 1, u'(0) = 2$
20. $u'' - 5u' + 4u = -1 + 4x, u(0) = 3, u'(0) = 9$

1.2.3 The Series Solution Method

For differential equations of any order, with constant coefficients or with variable coefficients, with $x = 0$ is an ordinary point, we can use the series solution method to determine the series solution of the differential equation. The obtained series solution may converge the exact solution if such a closed form solution exists. If an exact solution is not obtainable, we may use a truncated number of terms of the obtained series for numerical purposes.

Although the series solution can be used for equations with constant coefficients or with variable coefficients, where $x = 0$ is an ordinary point, but this method is commonly used for ordinary differential equations with variable coefficients where $x = 0$ is an ordinary point.

The series solution method assumes that the solution near an ordinary point $x = 0$ is given by

$$u(x) = \sum_{n=0}^{\infty} a_n x^n, \tag{1.82}$$

or by using few terms of the series

$$u(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + \cdots \tag{1.83}$$

Differentiating term by term gives

$$\begin{aligned} u'(x) &= a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + 5a_5 x^4 + 6a_6 x^5 + \cdots \\ u''(x) &= 2a_2 + 6a_3 x + 12a_4 x^2 + 20a_5 x^3 + 30a_6 x^4 + \cdots \\ u'''(x) &= 6a_3 + 24a_4 x + 60a_5 x^2 + 120a_6 x^3 + \cdots \end{aligned} \tag{1.84}$$

and so on. Substituting $u(x)$ and its derivatives in the given differential equation, and equating coefficients of like powers of x gives a recurrence relation that can be solved to determine the coefficients $a_n, n \geqslant 0$. Substituting the obtained values of $a_n, n \geqslant 0$ in the series assumption (1.82) gives the series solution. As stated before, the series may converge to the exact solution. Otherwise, the obtained series can be truncated to any finite number of terms to be used for numerical calculations. The more terms we use will enhance the level of accuracy of the numerical approximation.

It is interesting to point out that the series solution method can be used for homogeneous and inhomogeneous equations as well when $x = 0$ is an ordinary point. However, if $x = 0$ is a regular singular point of an ODE, then solution can be obtained by Frobenius method that will not be reviewed in this text. Moreover, the Taylor series of any elementary function involved in the differential equation should be used for equating the coefficients.

The series solution method will be illustrated by examining the following ordinary differential equations where $x = 0$ is an ordinary point. Some examples will give exact solutions, whereas others will provide series solutions that can be used for numerical purposes.

Example 1.13

Find a series solution for the following second order ODE:

$$u'' + u = 0. \tag{1.85}$$

Substituting the series assumption for $u(x)$ and $u''(x)$ gives

$$\begin{aligned} &2a_2 + 6a_3x + 12a_4x^2 + 20a_5x^3 + 30a_6x^4 + \cdots \\ &+a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \cdots = 0, \end{aligned} \tag{1.86}$$

that can be rewritten by

$$\begin{aligned} &(a_0 + 2a_2) + (a_1 + 6a_3)x + (a_2 + 12a_4)x^2 + (a_3 + 20a_5)x^3 \\ &+(a_4 + 30a_6)x^4 + \cdots = 0. \end{aligned} \tag{1.87}$$

This equation is satisfied only if the coefficient of each power of x vanishes. This in turn gives the recurrence relation

$$\begin{aligned} a_0 + 2a_2 = 0, &\quad a_1 + 6a_3 = 0, \\ a_2 + 12a_4 = 0, &\quad a_3 + 20a_5 = 0, \\ \vdots & \end{aligned} \tag{1.88}$$

By solving this recurrence relation, we obtain

$$\begin{aligned} &a_2 = -\frac{1}{2!}a_0, a_3 = -\frac{1}{3!}a_1, \\ &a_4 = -\frac{1}{12}a_2 = \frac{1}{4!}a_0, \quad a_5 = -\frac{1}{20}a_3 = \frac{1}{5!}a_1, \cdots \end{aligned} \tag{1.89}$$

The solution in a series form is given by

$$u(x) = a_0\left(1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \cdots\right) + a_1\left(x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \cdots\right), \tag{1.90}$$

and in closed form by

$$u(x) = a_0 \cos x + a_1 \sin x, \tag{1.91}$$

where a_0 and a_1 are constants that will be determined for particular solution if initial conditions are given.

Example 1.14

Find a series solution for the following second order ODE:

$$u'' - xu' - u = 0. \tag{1.92}$$

Substituting the series assumption for $u(x), u'(x)$ and $u''(x)$ gives

$$\begin{aligned} &2a_2 + 6a_3x + 12a_4x^2 + 20a_5x^3 + 30a_6x^4 + 42a_7x^5 + \cdots \\ &-a_1x - 2a_2x^2 - 3a_3x^3 - 4a_4x^4 - 5a_5x^5 - \cdots \\ &-a_0 - a_1x - a_2x^2 - a_3x^3 - a_4x^4 - a_5x^5 - \cdots = 0, \end{aligned} \tag{1.93}$$

that can be rewritten by

$$\begin{aligned} &(-a_0 + 2a_2) + (-2a_1 + 6a_3)x + (-3a_2 + 12a_4)x^2 + (-4a_3 + 20a_5)x^3 \\ &+(30a_6 - 5a_4)x^4 + (42a_7 - 6a_5)x^5 + \cdots = 0. \end{aligned} \tag{1.94}$$

This equation is satisfied only if the coefficient of each power of x is zero. This gives the recurrence relation

$$\begin{aligned} -a_0 + 2a_2 = 0, \quad -2a_1 + 6a_3 = 0, \quad -3a_2 + 12a_4 = 0, \\ -4a_3 + 20a_5 = 0, \quad -5a_4 + 30a_6 = 0, \quad -6a_5 + 42a_7 = 0, \cdots \end{aligned} \tag{1.95}$$

where by solving this recurrence relation we find

$$\begin{aligned} &a_2 = \frac{1}{2}a_0, \quad a_3 = \frac{1}{3}a_1, \quad a_4 = -\frac{1}{4}a_2 = \frac{1}{8}a_0, \\ &a_5 = \frac{1}{5}a_3 = \frac{1}{15}a_1, \quad a_6 = \frac{1}{6}a_4 = \frac{1}{48}a_0, \quad a_7 = \frac{1}{7}a_5 = \frac{1}{105}a_1, \cdots \end{aligned} \tag{1.96}$$

The solution in a series form is given by

$$\begin{aligned} u(x) = a_0 &\left(1 - \frac{1}{2}x^2 + \frac{1}{8}x^4 + \frac{1}{48}x^6 + \cdots\right) \\ &+a_1\left(x + \frac{1}{3}x^3 + \frac{1}{15}x^5 + \frac{1}{105}x^7 + \cdots\right), \end{aligned} \tag{1.97}$$

where a_0 and a_1 are constants, where $a_0 = u(0)$ and $a_1 = u'(0)$. It is clear that a closed form solution is not obtainable. If a particular solution is required, then initial conditions $u(0)$ and $u'(0)$ should be specified to determine the coefficients a_0 and a_1.

Example 1.15

Find a series solution for the following second order ODE:

$$u'' - xu' + u = -x \cos x. \tag{1.98}$$

Substituting the series assumption for $u(x), u'$, and $u''(x)$, and using the Taylor series for $\cos x$, we find

$$\begin{aligned}
&2a_2 + 6a_3x + 12a_4x^2 + 20a_5x^3 + 30a_6x^4 + 42a_7x^5 + \cdots \\
&-a_1x - 2a_2x^2 - 3a_3x^3 - 4a_4x^4 - 5a_5x^5 - \cdots \\
&+a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \cdots \\
&= -x + \frac{1}{2!}x^3 - \frac{1}{4!}x^5 + \cdots ,
\end{aligned} \tag{1.99}$$

that can be rewritten by

$$\begin{aligned}
&(a_0 + 2a_2) + 6a_3x + (-a_2 + 12a_4)x^2 + (-2a_3 + 20a_5)x^3 \\
&+(30a_6 - 3a_4)x^4 + (42a_7 - 4a_5)x^5 + \cdots = -x + \frac{1}{2!}x^3 - \frac{1}{4!}x^5 + \cdots .
\end{aligned} \tag{1.100}$$

Equating the coefficient of each power of x be zero and solving the recurrence relation we obtain

$$\begin{aligned}
a_2 &= -\frac{1}{2}a_0, & a_3 &= -\frac{1}{3!}, \\
a_4 &= -\frac{1}{12}a_2 = -\frac{1}{24}a_0, & a_5 &= \frac{1}{10}a_3 + \frac{1}{40} = \frac{1}{120}, \\
a_6 &= \frac{1}{10}a_4 = -\frac{1}{240}a_0, & a_7 &= \frac{2}{21}a_5 - \frac{1}{1008} = -\frac{1}{5040}, \cdots
\end{aligned} \tag{1.101}$$

The solution in a series form is given by

$$\begin{aligned}
u(x) = a_0 &\left(1 - \frac{1}{2}x^2 - \frac{1}{24}x^4 - \frac{1}{240}x^6 + \cdots\right) \\
&+a_1x + \left(-\frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots\right),
\end{aligned} \tag{1.102}$$

that can be rewritten as

$$\begin{aligned}
u(x) = a_0 &\left(1 - \frac{1}{2}x^2 - \frac{1}{24}x^4 - \frac{1}{240}x^6 + \cdots\right) \\
&+Bx + \left(x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots\right),
\end{aligned} \tag{1.103}$$

by setting $B = a_1 - 1$. Consequently, the solution is given by

$$u(x) = a_0\left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{240}x^6 + \cdots\right) + Bx + \sin x. \tag{1.104}$$

Notice that $\sin x$ gives the particular solution that arises as an effect of the inhomogeneous part.

Exercises 1.2.3

Find the series solution for the following homogeneous second order ODEs:

1. $u'' + xu' + u = 0$
2. $u'' - xu' + xu = 0$
3. $u'' - (1+x)u' + u = 0$
4. $u'' - u' + xu = 0$

Find the series solution for the following inhomogeneous second order ODEs:

5. $u'' - u' + xu = \sin x$
6. $u'' - xu' + xu = e^x$
7. $u'' - xu = \cos x$
8. $u'' - x^2u = \ln(1-x)$

1.3 Leibnitz Rule for Differentiation of Integrals

One of the methods that will be used to solve integral equations is the conversion of the integral equation to an equivalent differential equation. The conversion is achieved by using the well-known *Leibnitz rule* [4,6,7] for differentiation of integrals.

Let $f(x,t)$ be continuous and $\frac{\partial f}{\partial t}$ be continuous in a domain of the x-t plane that includes the rectangle $a \leqslant x \leqslant b$, $t_0 \leqslant t \leqslant t_1$, and let

$$F(x) = \int_{g(x)}^{h(x)} f(x,t)dt, \tag{1.105}$$

then differentiation of the integral in (1.105) exists and is given by

$$F'(x) = \frac{dF}{dx} = f(x,h(x))\frac{dh(x)}{dx} - f(x,g(x))\frac{dg(x)}{dx} + \int_{g(x)}^{h(x)} \frac{\partial f(x,t)}{\partial x}dt. \tag{1.106}$$

If $g(x) = a$ and $h(x) = b$ where a and b are constants, then the Leibnitz rule (1.106) reduces to

$$F'(x) = \frac{dF}{dx} = \int_a^b \frac{\partial f(x,t)}{\partial x}dt, \tag{1.107}$$

which means that differentiation and integration can be interchanged such as

$$\frac{d}{dx}\int_a^b e^{xt}dt = \int_a^b te^{xt}dt. \tag{1.108}$$

It is interested to notice that Leibnitz rule is not applicable for the Abel's singular integral equation:

$$F(x) = \int_0^x \frac{u(t)}{(x-t)^\alpha}dt, 0 < \alpha < 1. \tag{1.109}$$

The integrand in this equation does not satisfy the conditions that $f(x,t)$ be continuous and $\frac{\partial f}{\partial t}$ be continuous, because it is unbounded at $x = t$. We illustrate the Leibnitz rule by the following examples.

Example 1.16

Find $F'(x)$ for the following:

$$F(x) = \int_{\sin x}^{\cos x} \sqrt{1+t^3}dt. \tag{1.110}$$

We can set $g(x) = \sin x$ and $h(x) = \cos x$. It is also clear that $f(x,t)$ is a function of t only. Using Leibnitz rule (1.106) we find that

$$F'(x) = -\sin x\sqrt{1+\cos^3 x} - \cos x\sqrt{1+\sin^3 x}. \tag{1.111}$$

Example 1.17

Find $F'(x)$ for the following:

$$F(x) = \int_x^{x^2} (x-t)\cos t dt. \tag{1.112}$$

We can set $g(x) = x$, $h(x) = x^2$, and $f(x,t) = (x-t)\cos t$ is a function of x and t. Using Leibnitz rule (1.106) we find that

$$F'(x) = 2x(x - x^2)\cos x^2 + \int_x^{x^2} \cos t dt, \tag{1.113}$$

or equivalently

$$F'(x) = 2x(x - x^2)\cos x^2 + \sin x^2 - \sin x. \tag{1.114}$$

Remarks

In this text of integral equations, we will concern ourselves in differentiation of integrals of the form:

$$F(x) = \int_0^x K(x,t)u(t)dt. \tag{1.115}$$

In this case, Leibnitz rule (1.106) reduces to the form:

$$F'(x) = K(x,x)u(x) + \int_0^x \frac{\partial K(x,t)}{\partial x} u(t)dt. \tag{1.116}$$

This will be illustrated by the following examples.

Example 1.18

Find $F'(x)$ for the following:

$$F(x) = \int_0^x (x-t)u(t)dt. \tag{1.117}$$

Applying the reduced Leibnitz rule (1.116) yields

$$F'(x) = \int_0^x u(t)dt. \tag{1.118}$$

Example 1.19

Find $F'(x)$ and $F''(x)$ for the following:

$$F(x) = \int_0^x xtu(t)dt. \tag{1.119}$$

Applying the reduced Leibnitz rule (1.116) gives

$$\begin{aligned} F'(x) &= x^2u(x) + \int_0^x tu(t)dt, \\ F''(x) &= x^2u'(x) + 3xu(x). \end{aligned} \tag{1.120}$$

Example 1.20

Find $F'(x), F''(x)$, $F'''(x)$ and $F^{(\text{iv})}(x)$ for the following integral

$$F(x) = \int_0^x (x-t)^3u(t)dt. \tag{1.121}$$

Applying the reduced Leibnitz rule (1.116) four times gives

$$
\begin{aligned}
F'(x) &= \int_0^x 3(x-t)^2 u(t)dt, \quad & F''(x) &= \int_0^x 6(x-t)u(t)dt, \\
F'''(x) &= \int_0^x 6u(t)dt, & F^{(\mathrm{iv})}(x) &= 6u(x).
\end{aligned} \tag{1.122}
$$

Exercises 1.3

Find $F'(x)$ for the following integrals:

1. $F(x) = \int_0^x e^{-x^2t^2} dt$

2. $F(x) = \int_x^{x^2} \ln(1+xt)dt$

3. $F(x) = \int_0^x \sin(x^2+t^2)dt$

4. $F(x) = \int_0^x \cosh(x^3+t^3)dt$

Find $F'(x)$ for the following integrals:

5. $F(x) = \int_0^x (x-t)u(t)dt$

6. $F(x) = \int_0^x (x-t)^2 u(t)dt$

7. $F(x) = \int_0^x (x-t)^3 u(t)dt$

8. $F(x) = \int_0^x (x-t)^4 u(t)dt$

Differentiate both sides of the following equations:

9. $x^3 + \frac{1}{6}x^6 = \int_0^x (4+x-t)u(t)dt$

10. $1 + xe^x = \int_0^x e^{x-t}u(t)dt$

11. $2x^2 + 3x^3 = \int_0^x (6+5x-5t)u(t)dt$

12. $\sinh x + \ln(\sin x) = \int_0^x (3+x-t)u(t)dt, \ 0 < x < \frac{\pi}{2}$

Differentiate the following $F(x)$ as many times as you need to get rid of the integral sign:

13. $F(x) = x + \int_0^x (x-t)u(t)dt$

14. $F(x) = x^2 + \int_0^x (x-t)^2 u(t)dt$

15. $F(x) = 1 + \int_0^x (x-t)^3 u(t)dt$

16. $F(x) = e^x + \int_0^x (x-t)^4 u(t)dt$

Use Leibnitz rule to prove the following identities:

17. If $F(x) = \int_0^x (x-t)^n u(t)dt$, show that $F^{(n+1)} = n!u(x), \ n \geqslant 0$.

18. $F(x) = \int_0^x t^n (x-t)^m dt$, show that $F^{(m)} = \frac{m!}{n+1}x^{n+1}$,

    m and n are positive integers.

1.4 Reducing Multiple Integrals to Single Integrals

It will be seen later that we can convert initial value problems and other problems to integral equations. It is normal to outline the formula that will reduce multiple integrals to single integrals. We will first show that the double integral can be reduced to a single integral by using the formula:

$$\int_0^x \int_0^{x_1} F(t)dtdx_1 = \int_0^x (x-t)F(t)dt. \tag{1.123}$$

This can be easily proved by two ways. The first way is to set

$$G(x) = \int_0^x (x-t)F(t)dt, \tag{1.124}$$

where $G(0) = 0$. Differentiating both sides of (1.124) gives

$$G'(x) = \int_0^x F(t)dt, \tag{1.125}$$

obtained by using the reduced Leibnitz rule. Now by integrating both sides of the last equation from 0 to x yields

$$G(x) = \int_0^x \int_0^{x_1} F(t)dtdx_1. \tag{1.126}$$

Consequently, the right side of the two equations (1.124) and (1.126) are equivalent. This completes the proof.

For the second method we will use the concept of integration by parts. Recall that

$$\begin{aligned} \int u dv &= uv - \int v du, \\ u(x_1) &= \int_0^{x_1} F(t)dt, \end{aligned} \tag{1.127}$$

then we find

$$\begin{aligned} \int_0^x \int_0^{x_1} F(t)dtdx_1 &= x_1 \int_0^{x_1} F(t)dt \Big|_0^x - \int_0^x x_1 F(x_1)dx_1 \\ &= x \int_0^x F(t)dt - \int_0^x x_1 F(x_1)dx_1 \\ &= \int_0^x (x-t)F(t)dt, \end{aligned} \tag{1.128}$$

obtained upon setting $x_1 = t$.

The general formula that converts multiple integrals to a single integral is given by

$$\int_0^x \int_0^{x_1} \cdots \int_0^{x_{n-1}} u(x_n)dx_n dx_{n-1} \cdots dx_1 = \frac{1}{(n-1)!} \int_0^x (x-t)^{n-1} u(t)dt. \tag{1.129}$$

The conversion formula (1.129) is very useful and facilitates the calculation works. Moreover, this formula will be used to convert initial value problems to Volterra integral equations.

Corollary

As a result to (1.129) we can easily show the following corollary

$$\underbrace{\int_0^x \int_0^x \cdots \int_0^x (x-t)u(t)dtdt\cdots dt}_{n \text{ integrals}} = \frac{1}{n!}\int_0^x (x-t)^n u(t)dt \qquad (1.130)$$

Example 1.21

Convert the following multiple integral to a single integral:

$$\int_0^x \int_0^{x_1} u(x_2)dx_1dx. \qquad (1.131)$$

Using the formula (1.129) we obtain

$$\int_0^x \int_0^{x_1} u(x_2)dx_1dx = \int_0^x (x-t)u(t)dt. \qquad (1.132)$$

Example 1.22

Convert the following multiple integral to a single integral:

$$\int_0^x \int_0^{x_1} \int_0^{x_2} u(x_3)dx_2dx_1dx. \qquad (1.133)$$

Using the formula (1.129) we obtain

$$\int_0^x \int_0^{x_1} \int_0^{x_2} u(x_3)dx_2dx_1dx = \frac{1}{2!}\int_0^x (x-t)^2u(t)dt. \qquad (1.134)$$

Example 1.23

Convert the following multiple integral to a single integral:

$$\int_0^x \int_0^x \int_0^x (x-t)dtdtdt. \qquad (1.135)$$

Using the corollary (1.130) we obtain

$$\int_0^x \int_0^x \int_0^x (x-t)dtdtdt = \frac{1}{3!}\int_0^x (x-t)^3u(t)dt. \qquad (1.136)$$

Example 1.24

Convert the following multiple integral to a single integral:

$$\int_0^x \int_0^{x_1} (x-t)u(x_1)dtdx_1. \qquad (1.137)$$

Using the corollary (1.130) we obtain

$$\int_0^x \int_0^{x_1} (x-t)u(x_1)dtdx_1 = \frac{1}{2}\int_0^x (x-t)^2u(t)dt. \qquad (1.138)$$

Example 1.25

Convert the following multiple integral to a single integral:

$$\int_0^x \int_0^{x_1} (x-t)^2u(x_1)dtdx_1. \qquad (1.139)$$

Using the corollary (1.130) we obtain

$$\int_0^x \int_0^{x_1} (x-t)^2 u(x_1) dt dx_1 = \frac{1}{3} \int_0^x (x-t)^3 u(t) dt. \tag{1.140}$$

Exercises 1.4

Prove the following:

1. $\int_0^x \int_0^{x_1} (x-t)^3 u(x_1) dt dx_1 = \frac{1}{4} \int_0^x (x-t)^4 u(t) dt$
2. $\int_0^x \int_0^{x_1} (x-t)^4 u(x_1) dt dx_1 = \frac{1}{5} \int_0^x (x-t)^5 u(t) dt$
3. $\int_0^x \int_0^{x_1} (x-t) u(x_1) dt dx_1 + \int_0^x \int_0^{x_1} (x-t)^2 u(x_1) dt dx_1$
   $= \frac{1}{6} \int_0^x (x-t)^2 (3 + 2(x-t)) u(t) dt$
4. $\int_0^x \int_0^{x_1} u(x_1) dt dx_1 + \int_0^x \int_0^{x_1} (x-t) u(x_1) dt dx_1 + \int_0^x \int_0^{x_1} (x-t)^3 u(x_1) dt dx_1$
   $= \frac{1}{4} \int_0^x (x-t)^2 \left(4 + 2(x-t) + (x-t)^3\right) u(t) dt$

1.5 Laplace Transform

In this section we will review only the basic concepts of the Laplace transform method. The details can be found in any text of ordinary differential equations. The Laplace transform method is a powerful tool used for solving differential and integral equations. The Laplace transform [1–4,7] changes differential equations and integral equations to polynomial equations that can be easily solved, and hence by using the inverse Laplace transform gives the solution of the examined equation.

The Laplace transform of a function $f(x)$, defined for $x \geqslant 0$, is defined by

$$F(s) = \mathcal{L}\{f(x)\} = \int_0^\infty e^{-sx} f(x) dx, \tag{1.141}$$

where s is real, and $\mathcal{L}$ is called the Laplace transform operator. The Laplace transform $F(s)$ may fail to exist. If $f(x)$ has infinite discontinuities or if it grows up rapidly, then $F(s)$ does not exist. Moreover, an important necessary condition for the existence of the Laplace transform $F(s)$ is that $F(s)$ must vanish as s approaches infinity. This means that

$$\lim_{s \to \infty} F(s) = 0. \tag{1.142}$$

In other words, the conditions for the existence of a Laplace transform $F(s)$ of any function $f(x)$ are:

1. $f(x)$ is piecewise continuous on the interval of integration $0 \leqslant x < A$ for any positive A,

2. $f(x)$ is of exponential order e^{ax} as $x \to \infty$, i.e. $|f(x)| \leqslant Ke^{ax}, x \geqslant M$, where a is real constant, and K and M are positive constants. Accordingly, the Laplace transform $F(s)$ exists and must satisfy

$$\lim_{s\to\infty} F(s) = 0. \tag{1.143}$$

1.5.1 Properties of Laplace Transforms

From the definition of the Laplace transform given in (1.141), we can easily derive the following properties of the Laplace transforms:

1). **Constant Multiple:**

$$\mathcal{L}\{af(x)\} = a\mathcal{L}\{f(x)\}, a \text{ is a constant.} \tag{1.144}$$

For example:

$$\mathcal{L}\{4e^x\} = 4\mathcal{L}\{e^x\} = \frac{4}{s-1}. \tag{1.145}$$

2). **Linearity Property:**

$$\mathcal{L}\{af(x) + bg(x)\} = a\mathcal{L}\{f(x)\} + b\mathcal{L}\{g(x)\}, a, b \text{ are constants.} \tag{1.146}$$

For example:

$$\mathcal{L}\{4x + 3x^2\} = 4\mathcal{L}\{x\} + 3\mathcal{L}\{x^2\} = \frac{4}{s^2} + \frac{6}{s^3}. \tag{1.147}$$

3). **Multiplication by x:**

$$\mathcal{L}\{xf(x)\} = -\frac{d}{ds}\mathcal{L}\{f(x)\} = -F'(s). \tag{1.148}$$

For example:

$$\mathcal{L}\{x \sin x\} = -\frac{d}{ds}\mathcal{L}\{\sin x\} = -\frac{d}{ds}\left(\frac{1}{s^2+1}\right) = \frac{2s}{(s^2+1)^2}. \tag{1.149}$$

To use the Laplace transform $\mathcal{L}$ for solving initial value problems or integral equations, we use the following table of elementary Laplace transforms as shown below:

Table 1.1 Elementary Laplace Transforms

$f(x)$	$F(s) = \mathcal{L}\{f(x)\} = \int_0^\infty e^{-sx} f(x)dx$
c	$\frac{c}{s},\ s > 0$
x	$\frac{1}{s^2},\ s > 0$
x^n	$\frac{n!}{s^{n+1}} = \frac{\Gamma(n+1)}{s^{n+1}},\ s > 0,\ \text{Re}\ n > -1$
e^{ax}	$\frac{1}{s-a},\ s > a$

Continued

$f(x)$	$F(s) = \mathcal{L}\{f(x)\} = \int_0^\infty e^{-sx} f(x)dx$
$\sin ax$	$\dfrac{a}{s^2+a^2}$
$\cos ax$	$\dfrac{s}{s^2+a^2}$
$\sin^2 ax$	$\dfrac{2a^2}{s(s^2+4a^2)},\ \mathrm{Re}(s) > \lvert\mathrm{Im}(a)\rvert$
$\cos^2 ax$	$\dfrac{s^2+2a^2}{s(s^2+4a^2)},\ \mathrm{Re}(s) > \lvert\mathrm{Im}(a)\rvert$
$x\sin ax$	$\dfrac{2as}{(s^2+a^2)^2}$
$x\cos ax$	$\dfrac{s^2-a^2}{(s^2+a^2)^2}$
$\sinh ax$	$\dfrac{a}{s^2-a^2},\ s > \lvert a\rvert$
$\cosh ax$	$\dfrac{s}{s^2-a^2},\ s > \lvert a\rvert$
$\sinh^2 ax$	$\dfrac{2a^2}{s(s^2-4a^2)},\ \mathrm{Re}(s) > \lvert\mathrm{Im}(a)\rvert$
$\cosh^2 ax$	$\dfrac{s^2-2a^2}{s(s^2-4a^2)},\ \mathrm{Re}(s) > \lvert\mathrm{Im}(a)\rvert$
$x\sinh ax$	$\dfrac{2as}{(s^2-a^2)^2},\ s > \lvert a\rvert$
$x\cosh ax$	$\dfrac{s^2+a^2}{(s^2-a^2)^2},\ s > \lvert a\rvert$
$x^n e^{ax}$	$\dfrac{n!}{(s-a)^{n+1}}, s > a,\ n$ is a positive integer
$e^{ax}\sin bx$	$\dfrac{b}{(s-a)^2+b^2},\ s > a$
$e^{ax}\cos bx$	$\dfrac{s-a}{(s-a)^2+b^2},\ s > a$
$e^{ax}\sinh bx$	$\dfrac{b}{(s-a)^2-b^2},\ s > a$
$e^{ax}\cosh bx$	$\dfrac{s-a}{(s-a)^2-b^2},\ s > a$
$H(x-a)$	$s^{-1}e^{-as},\ a \geqslant 0$
$\delta(x)$	1
$\delta(x-a)$	$e^{-as},\ a \geqslant 0$
$\delta'(x-a)$	$se^{-as},\ a \geqslant 0$

4). **Laplace Transforms of Derivatives:**

$$\begin{aligned}
\mathcal{L}\{f'(x)\} &= s\mathcal{L}\{f(x)\} - f(0),\\
\mathcal{L}\{f''(x)\} &= s^2\mathcal{L}\{f(x)\} - sf(0) - f'(0),\\
\mathcal{L}\{f'''(x)\} &= s^3\mathcal{L}\{f(x)\} - s^2f(0) - sf'(0) - f''(0),\\
&\ \vdots\\
\mathcal{L}\{f^{(n)}(x)\} &= s^n\mathcal{L}\{f(x)\} - s^{n-1}f(0) - \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0).
\end{aligned} \tag{1.150}$$

5). **Inverse Laplace Transform**

If the Laplace transform of $f(x)$ is $F(s)$, then we say that the inverse Laplace transform of $F(s)$ is $f(x)$. In other words, we write

$$\mathcal{L}^{-1}\{F(s)\} = f(x), \tag{1.151}$$

where $\mathcal{L}^{-1}$ is the operator of the inverse Laplace transform. The linearity property holds also for the inverse Laplace transform. This means that

$$\begin{aligned}
\mathcal{L}^{-1}\{aF(s) + bG(s)\} &= a\mathcal{L}^{-1}\{F(s)\} + b\mathcal{L}^{-1}\{G(s)\}\\
&= af(x) + bg(x).
\end{aligned} \tag{1.152}$$

Notice that the computer symbolic systems such as Maple and Mathematica can be used to find the Laplace transform and the inverse Laplace transform.

The Laplace transform method and the inverse Laplace transform method will be illustrated by using the following examples.

Example 1.26

Solve the following initial value problem:

$$y'' + y = 0, \quad y(0) = 1, \quad y'(0) = 1. \tag{1.153}$$

By taking Laplace transforms of both sides of the ODE, we use

$$\begin{aligned}
\mathcal{L}\{y(x)\} &= Y(s),\\
\mathcal{L}\{y''(x)\} &= s^2\mathcal{L}\{y(x)\} - sy(0) - y'(0)\\
&= s^2Y(s) - s - 1,
\end{aligned} \tag{1.154}$$

obtained upon using the initial conditions. Substituting this into the ODE gives

$$Y(s) = \frac{s}{s^2+1} + \frac{1}{s^2+1}. \tag{1.155}$$

To determine the solution $y(x)$ we then take the inverse Laplace transform L^{-1} to both sides of the last equation to find

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}. \tag{1.156}$$

This in turn gives the solution by

$$y(x) = \cos x + \sin x, \tag{1.157}$$

obtained upon using the table of Laplace transforms. Notice that we obtained the particular solution for the differential equation. This is due to the fact that the initial conditions were used in the solution.

Example 1.27

Solve the following initial value problem

$$y'' - y' = 0, \quad y(0) = 2, \quad y'(0) = 1. \tag{1.158}$$

By taking Laplace transforms of both sides of the ODE, we set

$$\begin{aligned} \mathcal{L}\{y(x)\} &= Y(s), \\ \mathcal{L}\{y'(x)\} &= sY(s) - 2, \\ \mathcal{L}\{y''(x)\} &= s^2 L\{y(x)\} - sy(0) - y'(0) \\ &= s^2 Y(s) - 2s - 1, \end{aligned} \tag{1.159}$$

obtained by using the initial conditions. Substituting this into the ODE gives

$$Y(s) = \frac{1}{s-1} + \frac{1}{s}. \tag{1.160}$$

To determine the solution $y(x)$ we then the take inverse Laplace transform L^{-1} of both sides of the last equation to find

$$\mathcal{L}^{-1}\{Y(s)\} = L^{-1}\left\{\frac{1}{s-1}\right\} + L^{-1}\left\{\frac{1}{s}\right\}. \tag{1.161}$$

This in turn gives the solution by

$$y(x) = e^x + 1, \tag{1.162}$$

obtained upon using the table of Laplace transforms.

6). **The Convolution Theorem for Laplace Transform**

This is an important theorem that will be used in solving integral equations. The kernel $K(x,t)$ of the integral equation:

$$u(x) = f(x) + \lambda \int_{g(x)}^{h(x)} K(x,t)u(t)dt, \tag{1.163}$$

is termed *difference kernel* if it depends on the difference $x - t$. Examples of the difference kernels are $e^{x-t}, \sin(x-t)$, and $\cosh(x-t)$. The integral equation (1.163) can be expressed as

$$u(x) = f(x) + \lambda \int_{g(x)}^{h(x)} K(x-t)u(t)dt. \tag{1.164}$$

Consider two functions $f_1(x)$ and $f_2(x)$ that possess the conditions needed for the existence of Laplace transform for each. Let the Laplace transforms for the functions $f_1(x)$ and $f_2(x)$ be given by

$$\begin{aligned} \mathcal{L}\{f_1(x)\} &= F_1(s), \\ \mathcal{L}\{f_2(x)\} &= F_2(s). \end{aligned} \tag{1.165}$$

The *Laplace convolution product* of these two functions is defined by

$$(f_1 * f_2)(x) = \int_0^x f_1(x-t)f_2(t)dt, \tag{1.166}$$

or

$$(f_2 * f_1)(x) = \int_0^x f_2(x-t)f_1(t)dt. \tag{1.167}$$

Recall that

$$(f_1 * f_2)(x) = (f_2 * f_1)(x). \tag{1.168}$$

We can easily show that the Laplace transform of the convolution product $(f_1 * f_2)(x)$ is given by

$$\mathcal{L}\{(f_1 * f_2)(x)\} = \mathcal{L}\left\{\int_0^x f_1(x-t)f_2(t)dt\right\} = F_1(s)F_2(s). \tag{1.169}$$

It was stated before, that this theorem will be used in the coming chapters. To illustrate the use of this theorem we examine the following example.

Example 1.28

Find the Laplace transform of

$$x + \int_0^x (x-t)y(t)dt. \tag{1.170}$$

Notice that the kernel depends on the difference $x - t$. The integral includes $f_1(x) = x$ and $f_2(x) = y(x)$. The integral is the convolution product $(f_1 * f_2)(x)$. This means that if we take Laplace transform of each term we obtain

$$\mathcal{L}\{x\} + \mathcal{L}\left\{\int_0^x (x-t)y(t)dt\right\} = \mathcal{L}\{x\} + \mathcal{L}\{x\}\mathcal{L}\{y(t)\}. \tag{1.171}$$

Using the table of Laplace transforms gives

$$\frac{1}{s^2} + \frac{1}{s^2}Y(s). \tag{1.172}$$

Example 1.29

Solve the following integral equation

$$xe^x = \int_0^x e^{x-t}y(t)dt. \tag{1.173}$$

Notice that $f_1(x) = e^x$ and $f_2(x) = y(x)$. The right hand side is the convolution product $(f_1 * f_2)(x)$. This means that if we take Laplace transforms of both sides we obtain

$$\mathcal{L}\{xe^x\} = \mathcal{L}\left\{\int_0^x e^{x-t}y(t)dt\right\} = \mathcal{L}\{e^x\}\mathcal{L}\{y(t)\}. \tag{1.174}$$

Using the table of Laplace transforms gives

$$\frac{1}{(s-1)^2} = \frac{1}{s-1}Y(s), \tag{1.175}$$

that gives

$$Y(s) = \frac{1}{s-1}. \tag{1.176}$$

From this we find the solution is

$$y(x) = \mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} = e^x. \tag{1.177}$$

The Laplace transform method will be used for solving Volterra integral equations of the first and the second type in coming chapters.

Exercises 1.5

Solve the given ODEs:

1. $y'' + 4y = 0,\ y(0) = 0,\ y'(0) = 1$
2. $y'' + 4y = 4x,\ y(0) = 0,\ y'(0) = 1$
3. $y'' - y = -2x,\ y(0) = 0,\ y'(0) = 1$
4. $y'' - 3y' + 2y = 0,\ y(0) = 2,\ y'(0) = 3$

Find the Laplace transform of the following expressions:

5. $x + \sin x$ 6. $e^x - \cos x$ 7. $1 + xe^x$ 8. $\sin x + \sinh x$

Find the Laplace transform of the following expressions that include convolution products:

9. $\int_0^x \sinh(x-t)y(t)dt$
10. $x^2 + \int_0^x e^{x-t}y(t)dt$
11. $\int_0^x (x-t)e^{x-t}y(t)dt$
12. $1 + x - \int_0^x (x-t)y(t)dt$

Find the inverse Laplace transform of the following:

13. $F(s) = \dfrac{1}{s^2+1} + \dfrac{4}{s^3}$
14. $F(s) = \dfrac{1}{s^2-1} + \dfrac{3}{s^2+9}$
15. $F(s) = \dfrac{1}{s^2+1} + \dfrac{s}{s^2-4}$
16. $F(s) = \dfrac{1}{s(s^2+1)} - \dfrac{1}{s(s^2-1)}$

1.6 Infinite Geometric Series

A *geometric sequence* is a sequence of numbers where each term after the first is obtained by multiplying the previous term by a non-zero number r called the common ratio. In other words, a sequence is geometric if there is a fixed non-zero number r such that

$$a_{n+1} = a_n r, \quad n \geqslant 1. \tag{1.178}$$

This means that the geometric sequence can be written in a general form as

$$a_1, a_1 r, a_1 r^2, \ldots, a_1 r^{n-1}, \ldots, \tag{1.179}$$

where a_1 is the starting value of the sequence and r is the common ratio.

The associated *geometric series* is obtained as the sum of the terms of the geometric series, and therefore given by

$$S_n = \sum_{k=0}^{n} a_1 r^k = a_1 + a_1 r + a_1 r^2 + a_1 r^3 + a_1 r^4 + \cdots + a_1 r^n. \tag{1.180}$$

The sum of the first n terms of a geometric sequence is given by

$$S_n = \frac{a_1(1-r^n)}{1-r}, \quad r \neq 1. \tag{1.181}$$

An *infinite geometric series* converges if and only if $|r| < 1$. Otherwise it diverges. The sum of infinite geometric series, for $|r| < 1$, is given by

$$\lim_{n\to\infty} S_n = \frac{a_1}{1-r}, \tag{1.182}$$

obtained from (1.181) by noting that

$$\lim_{n\to\infty} r^n = 0, \quad |r| < 1. \tag{1.183}$$

Some of the methods that will be used in this text may give the solutions in an infinite series. Some of the obtained series include infinite geometric series. For this reason we will study examples of infinite geometric series.

Example 1.30

Find the sum of the infinite geometric series

$$1 + \frac{2}{3} + \frac{4}{9} + \frac{8}{27} + \cdots \tag{1.184}$$

The first value of the sequence and the common ratio are given by $a_1 = 1$ and $r = \frac{2}{3}$ respectively. The sum is therefore given by

$$S = \frac{1}{1-\frac{2}{3}} = 3. \tag{1.185}$$

Example 1.31

Find the sum of the infinite geometric series:

$$e^{-1} + e^{-2} + e^{-3} + e^{-4} + \cdots \tag{1.186}$$

The first term and the common ratio are $a_1 = e^{-1}$ and $r = e^{-1} < 1$. The sum is therefore given by

$$S = \frac{e^{-1}}{1-e^{-1}} = \frac{1}{e-1}. \tag{1.187}$$

Example 1.32

Find the sum of the infinite geometric series:

$$x + \frac{n}{4}x + \frac{n^2}{16}x + \frac{n^3}{64}x + \cdots, \quad 0 < n < 4. \tag{1.188}$$

It is obvious that $a_1 = x$ and $r = \frac{n}{4} < 1$. Consequently, the sum of this infinite series is given by

$$S = \frac{x}{1-\frac{n}{4}} = \frac{4}{4-n}x, \quad 0 < n < 4. \tag{1.189}$$

Example 1.33

Find the sum of the infinite geometric series:

$$\frac{1}{x}+\frac{1}{x^2}+\frac{1}{x^3}+\cdots, \quad x>1. \tag{1.190}$$

It is obvious that $a_1=\frac{1}{x}$ and $r=\frac{1}{x}<1$. The sum is therefore given by

$$S=\frac{\frac{1}{x}}{1-\frac{1}{x}}=\frac{1}{x-1}, \quad x>1. \tag{1.191}$$

Example 1.34

Find the sum of the infinite geometric series:

$$1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\cdots \tag{1.192}$$

It is obvious that $a_1=1$ and $r=-\frac{1}{2}, |r|<1$. The sum is therefore given by

$$S=\frac{1}{1+\frac{1}{2}}=\frac{2}{3}. \tag{1.193}$$

Exercises 1.6

Find the sum of the following infinite series:

1. $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots$
2. $\frac{1}{2}-\frac{1}{4}+\frac{1}{8}-\frac{1}{16}+\cdots$
3. $\frac{5}{6}x+\frac{5}{36}x+\frac{5}{216}x+\frac{5}{1296}x+\cdots$
4. $\frac{6}{7}\sin x+\frac{6}{49}\sin x+\frac{6}{343}\sin x+\frac{6}{2401}\sin x+\cdots$
5. $e^{-2}+e^{-4}+e^{-6}+e^{-8}+\cdots$
6. $x+\frac{n}{9}x+\frac{n^2}{81}x+\frac{n^3}{729}x+\cdots, \ 0<n<9$
7. $\pi x+\frac{\pi}{2}x+\frac{\pi}{4}x+\frac{\pi}{8}x+\cdots$
8. $\frac{e}{\pi}+\frac{e^2}{\pi^2}+\frac{e^3}{\pi^3}+\frac{e^4}{\pi^4}+\cdots$
9. Show that $\frac{d}{dx}\left(\ln x+\frac{\ln x}{2}+\frac{\ln x}{4}+\cdots\right)=\frac{2}{x}, \ x>0$
10. Show that $\int_2^{e+1}\left(\frac{1}{x}+\frac{1}{x^2}+\frac{1}{x^3}+\cdots\right)dx=1, \ x>1$

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Chapter 2
Introductory Concepts of Integral Equations

As stated in the previous chapter, an *integral equation* is the equation in which the unknown function $u(x)$ appears inside an integral sign [1–5]. The most standard type of integral equation in $u(x)$ is of the form

$$u(x) = f(x) + \lambda \int_{g(x)}^{h(x)} K(x,t)u(t)dt, \tag{2.1}$$

where $g(x)$ and $h(x)$ are the limits of integration, λ is a constant parameter, and $K(x,t)$ is a known function, of two variables x and t, called the *kernel* or the *nucleus* of the integral equation. The unknown function $u(x)$ that will be determined appears inside the integral sign. In many other cases, the unknown function $u(x)$ appears inside and outside the integral sign. The functions $f(x)$ and $K(x,t)$ are given in advance. It is to be noted that the limits of integration $g(x)$ and $h(x)$ may be both variables, constants, or mixed.

Integral equations appear in many forms. Two distinct ways that depend on the limits of integration are used to characterize integral equations, namely:

1. If the limits of integration are fixed, the integral equation is called a *Fredholm integral equation* given in the form:

$$u(x) = f(x) + \lambda \int_{a}^{b} K(x,t)u(t)dt, \tag{2.2}$$

where a and b are constants.

2. If at least one limit is a variable, the equation is called a *Volterra integral equation* given in the form:

$$u(x) = f(x) + \lambda \int_{a}^{x} K(x,t)u(t)dt. \tag{2.3}$$

Moreover, two other distinct kinds, that depend on the appearance of the unknown function $u(x)$, are defined as follows:

1. If the unknown function $u(x)$ appears only under the integral sign of Fredholm or Volterra equation, the integral equation is called *a first kind* Fredholm or Volterra integral equation respectively.

2. If the unknown function $u(x)$ appears both inside and outside the integral sign of Fredholm or Volterra equation, the integral equation is called *a second kind* Fredholm or Volterra equation integral equation respectively.

In all Fredholm or Volterra integral equations presented above, if $f(x)$ is identically zero, the resulting equation:

$$u(x) = \lambda \int_a^b K(x,t)u(t)dt \tag{2.4}$$

or

$$u(x) = \lambda \int_a^x K(x,t)u(t)dt \tag{2.5}$$

is called *homogeneous* Fredholm or *homogeneous* Volterra integral equation respectively.

It is interesting to point out that any equation that includes both integrals and derivatives of the unknown function $u(x)$ is called *integro-differential equation.* The Fredholm integro-differential equation is of the form:

$$u^{(k)}(x) = f(x) + \lambda \int_a^b K(x,t)u(t)dt, \quad u^{(k)} = \frac{d^k u}{dx^k}. \tag{2.6}$$

However, the Volterra integro-differential equation is of the form:

$$u^{(k)}(x) = f(x) + \lambda \int_a^x K(x,t)u(t)dt, \quad u^{(k)} = \frac{d^k u}{dx^k}. \tag{2.7}$$

The integro-differential equations [6] will be defined and classified in this text.

2.1 Classification of Integral Equations

Integral equations appear in many types. The types depend mainly on the limits of integration and the kernel of the equation. In this text we will be concerned on the following types of integral equations.

2.1.1 Fredholm Integral Equations

For Fredholm integral equations, the limits of integration are fixed. Moreover, the unknown function $u(x)$ may appear only inside integral equation in the form:

$$f(x) = \int_a^b K(x,t)u(t)dt. \tag{2.8}$$

This is called Fredholm integral equation of the *first kind*. However, for Fredholm integral equations of the *second kind*, the unknown function $u(x)$ appears inside and outside the integral sign. The second kind is represented by the form:

$$u(x) = f(x) + \lambda \int_a^b K(x,t)u(t)dt. \tag{2.9}$$

Examples of the two kinds are given by

$$\frac{\sin x - x\cos x}{x^2} = \int_0^1 \sin(xt)u(t)dt, \tag{2.10}$$

and

$$u(x) = x + \frac{1}{2}\int_{-1}^1 (x-t)u(t)dt, \tag{2.11}$$

respectively.

2.1.2 Volterra Integral Equations

In Volterra integral equations, at least one of the limits of integration is a variable. For the *first kind* Volterra integral equations, the unknown function $u(x)$ appears only inside integral sign in the form:

$$f(x) = \int_0^x K(x,t)u(t)dt. \tag{2.12}$$

However, Volterra integral equations of the *second kind*, the unknown function $u(x)$ appears inside and outside the integral sign. The second kind is represented by the form:

$$u(x) = f(x) + \lambda\int_0^x K(x,t)u(t)dt. \tag{2.13}$$

Examples of the Volterra integral equations of the first kind are

$$xe^{-x} = \int_0^x e^{t-x}u(t)dt, \tag{2.14}$$

and

$$5x^2 + x^3 = \int_0^x (5+3x-3t)u(t)dt. \tag{2.15}$$

However, examples of the Volterra integral equations of the second kind are

$$u(x) = 1 - \int_0^x u(t)dt, \tag{2.16}$$

and

$$u(x) = x + \int_0^x (x-t)u(t)dt. \tag{2.17}$$

2.1.3 Volterra-Fredholm Integral Equations

The Volterra-Fredholm integral equations [6,7] arise from parabolic boundary value problems, from the mathematical modelling of the spatio-temporal development of an epidemic, and from various physical and biological models. The Volterra-Fredholm integral equations appear in the literature in two forms, namely

$$u(x) = f(x) + \lambda_1 \int_a^x K_1(x,t)u(t)dt + \lambda_2 \int_a^b K_2(x,t)u(t)dt, \tag{2.18}$$

and

$$u(x,t) = f(x,t) + \lambda \int_0^t \int_\Omega F(x,t,\xi,\tau,u(\xi,\tau))d\xi d\tau, (x,t) \in \Omega \times [0,T], \tag{2.19}$$

where $f(x,t)$ and $F(x,t,\xi,\tau,u(\xi,\tau))$ are analytic functions on $D = \Omega \times [0,T]$, and Ω is a closed subset of $\mathbb{R}^n, n = 1,2,3$. It is interesting to note that (2.18) contains disjoint Volterra and Fredholm integral equations, whereas (2.19) contains mixed Volterra and Fredholm integral equations. Moreover, the unknown functions $u(x)$ and $u(x,t)$ appear inside and outside the integral signs. This is a characteristic feature of a second kind integral equation. If the unknown functions appear only inside the integral signs, the resulting equations are of first kind, but will not be examined in this text. Examples of the two types are given by

$$u(x) = 6x + 3x^2 + 2 - \int_0^x xu(t)dt - \int_0^1 tu(t)dt, \tag{2.20}$$

and

$$u(x,t) = x + t^3 + \frac{1}{2}t^2 - \frac{1}{2}t - \int_0^t \int_0^1 (\tau - \xi)d\xi d\tau. \tag{2.21}$$

2.1.4 Singular Integral Equations

Volterra integral equations of the first kind [4,7]

$$f(x) = \lambda \int_{g(x)}^{h(x)} K(x,t)u(t)dt \tag{2.22}$$

or of the second kind

$$u(x) = f(x) + \int_{g(x)}^{h(x)} K(x,t)u(t)dt \tag{2.23}$$

are called *singular* if one of the limits of integration $g(x)$, $h(x)$ or both are infinite. Moreover, the previous two equations are called singular if the kernel $K(x,t)$ becomes unbounded at one or more points in the interval of integration. In this text we will focus our concern on equations of the form:

$$f(x) = \int_0^x \frac{1}{(x-t)^\alpha}u(t)dt, \quad 0 < \alpha < 1, \tag{2.24}$$

or of the second kind:

$$u(x) = f(x) + \int_0^x \frac{1}{(x-t)^\alpha}u(t)dt, \quad 0 < \alpha < 1. \tag{2.25}$$

The last two standard forms are called *generalized Abel's integral equation* and *weakly singular integral equations* respectively. For $\alpha = \frac{1}{2}$, the equation:

$$f(x) = \int_0^x \frac{1}{\sqrt{x-t}} u(t)dt \tag{2.26}$$

is called the Abel's singular integral equation. It is to be noted that the kernel in each equation becomes infinity at the upper limit $t = x$. Examples of Abel's integral equation, generalized Abel's integral equation, and the weakly singular integral equation are given by

$$\sqrt{x} = \int_0^x \frac{1}{\sqrt{x-t}} u(t)dt, \tag{2.27}$$

$$x^3 = \int_0^x \frac{1}{(x-t)^{\frac{1}{3}}} u(t)dt, \tag{2.28}$$

and

$$u(x) = 1 + \sqrt{x} + \int_0^x \frac{1}{(x-t)^{\frac{1}{3}}} u(t)dt, \tag{2.29}$$

respectively.

Exercises 2.1

For each of the following integral equations, classify as Fredholm, Volterra, or Volterra-Fredholm integral equation and find its kind. Classify the equation as singular or not.

1. $u(x) = 1 + \int_0^x u(t)dt$
2. $x = \int_0^x (1 + x - t)u(t)dt$
3. $u(x) = e^x + e - 1 - \int_0^1 u(t)dt$
4. $x + 1 - \frac{\pi}{2} = \int_0^{\frac{\pi}{2}} (x - t)u(t)dt$
5. $u(x) = \frac{3}{2}x - \frac{1}{3} - \int_0^1 (x - t)u(t)dt$
6. $u(x) = x + \frac{1}{6}x^3 - \int_0^x (x - t)u(t)dt$
7. $\frac{1}{6}x^3 = \int_0^x (x - t)u(t)dt$
8. $\frac{1}{2}x^2 - \frac{2}{3}x + \frac{1}{4} = \int_0^1 (x - t)u(t)dt$
9. $u(x) = \frac{3}{2}x + \frac{1}{6}x^3 - \int_0^x (x - t)u(t)dt - \int_0^1 xu(t)dt$
10. $u(x, t) = x + t^3 + \frac{1}{2}t^2 - \frac{1}{2}t - \int_0^t \int_0^1 (\tau - \xi)d\xi d\tau$
11. $x^3 + \sqrt{x} = \int_0^x \frac{1}{(x-t)^{\frac{5}{6}}} u(t)dt$
12. $u(x) = 1 + x^2 + \int_0^x \frac{1}{\sqrt{x-t}} u(t)dt$

2.2 Classification of Integro-Differential Equations

Integro-differential equations appear in many scientific applications, especially when we convert initial value problems or boundary value problems to integral equations. The integro-differential equations contain both integral and differential operators. The derivatives of the unknown functions may appear to any order. In classifying integro-differential equations, we will follow the same category used before.

2.2.1 Fredholm Integro-Differential Equations

Fredholm integro-differential equations appear when we convert differential equations to integral equations. The Fredholm integro-differential equation contains the unknown function $u(x)$ and one of its derivatives $u^{(n)}(x), n \geqslant 1$ inside and outside the integral sign respectively. The limits of integration in this case are fixed as in the Fredholm integral equations. The equation is labeled as integro-differential because it contains differential and integral operators in the same equation. It is important to note that initial conditions should be given for Fredholm integro-differential equations to obtain the particular solutions. The Fredholm integro-differential equation appears in the form:

$$u^{(n)}(x) = f(x) + \lambda \int_a^b K(x,t)u(t)dt, \tag{2.30}$$

where $u^{(n)}$ indicates the nth derivative of $u(x)$. Other derivatives of less order may appear with $u^{(n)}$ at the left side. Examples of the Fredholm integro-differential equations are given by

$$u'(x) = 1 - \frac{1}{3}x + \int_0^1 xu(t)dt, \quad u(0) = 0, \tag{2.31}$$

and

$$u''(x) + u'(x) = x - \sin x - \int_0^{\frac{\pi}{2}} xtu(t)dt, \quad u(0) = 0, \quad u'(0) = 1. \tag{2.32}$$

2.2.2 Volterra Integro-Differential Equations

Volterra integro-differential equations appear when we convert initial value problems to integral equations. The Volterra integro-differential equation contains the unknown function $u(x)$ and one of its derivatives $u^{(n)}(x), n \geqslant 1$ inside and outside the integral sign. At least one of the limits of integration in this case is a variable as in the Volterra integral equations. The equation is called integro-differential because differential and integral operators are involved in the same equation. It is important to note that initial conditions should be given for Volterra integro-differential equations to determine the particular solutions. The Volterra integro-differential equation appears in the form:

$$u^{(n)}(x) = f(x) + \lambda \int_0^x K(x,t)u(t)dt, \tag{2.33}$$

where $u^{(n)}$ indicates the nth derivative of $u(x)$. Other derivatives of less order may appear with $u^{(n)}$ at the left side. Examples of the Volterra integro-differential equations are given by

$$u'(x) = -1 + \frac{1}{2}x^2 - xe^x - \int_0^x tu(t)dt,\ u(0) = 0, \tag{2.34}$$

and

$$u''(x)+u'(x) = 1-x(\sin x+\cos x)-\int_0^x tu(t)dt,\ u(0) = -1,\ u'(0) = 1. \tag{2.35}$$

2.2.3 Volterra-Fredholm Integro-Differential Equations

The Volterra-Fredholm integro-differential equations arise in the same manner as Volterra-Fredholm integral equations with one or more of ordinary derivatives in addition to the integral operators. The Volterra-Fredholm integro-differential equations appear in the literature in two forms, namely

$$u^{(n)}(x) = f(x) + \lambda_1 \int_a^x K_1(x,t)u(t)dt + \lambda_2 \int_a^b K_2(x,t)u(t)dt, \tag{2.36}$$

and

$$u^{(n)}(x,t) = f(x,t) + \lambda \int_0^t \int_\Omega F(x,t,\xi,\tau,u(\xi,\tau))d\xi d\tau, (x,t) \in \Omega \times [0,T], \tag{2.37}$$

where $f(x,t)$ and $F(x,t,\xi,\tau,u(\xi,\tau))$ are analytic functions on $D = \Omega \times [0,T]$, and Ω is a closed subset of $\mathbb{R}^n, n = 1,2,3$. It is interesting to note that (2.36) contains disjoint Volterra and Fredholm integral equations, whereas (2.37) contains mixed integrals. Other derivatives of less order may appear as well. Moreover, the unknown functions $u(x)$ and $u(x,t)$ appear inside and outside the integral signs. This is a characteristic feature of a second kind integral equation. If the unknown functions appear only inside the integral signs, the resulting equations are of first kind. Initial conditions should be given to determine the particular solution. Examples of the two types are given by

$$u'(x) = 24x + x^4 + 3 - \int_0^x (x-t)u(t)dt - \int_0^1 tu(t)dt, \quad u(0) = 0, \tag{2.38}$$

and

$$u'(x,t) = 1 + t^3 + \frac{1}{2}t^2 - \frac{1}{2}t - \int_0^t \int_0^1 (\tau - \xi)d\xi d\tau, \quad u(0,t) = t^3. \tag{2.39}$$

Exercises 2.2

For each of the following integro-differential equations, classify as Fredholm, Volterra, or Volterra-Fredholm integro-equation

1. $u'(x) = 1 + \int_0^x xu(t)dt,\ u(0) = 0$

2. $u''(x) = x + \int_0^1 (1 + x - t)u(t)dt,\ u(0) = 1,\ u'(0) = 0$

3. $u''(x) + u(x) = x + \int_0^x tu(t)dt + \int_0^1 u(t)dt,\ u(0) = 0,\ u'(0) = 1$

4. $u'''(x) + u'(x) = x + \int_0^x tu(t)dt + \int_0^1 u(t)dt,\ u(0) = 0, u'(0) = 1,\ u''(0) = 1$

5. $u'(x) + u(x) = x + \int_0^1 (x - t)u(t)dt,\ u(0) = 1$

6. $u''(x) = 1 + \int_0^x tu(t)dt,\ u(0) = 0,\ u'(0) = 1$

2.3 Linearity and Homogeneity

Integral equations and integro-differential equations fall into two other types of classifications according to *linearity* and *homogeneity* concepts. These two concepts play a major role in the structure of the solutions. In what follows we highlight the definitions of these concepts.

2.3.1 Linearity Concept

If the exponent of the unknown function $u(x)$ inside the integral sign is one, the integral equation or the integro-differential equation is called *linear* [6]. If the unknown function $u(x)$ has exponent other than one, or if the equation contains nonlinear functions of $u(x)$, such as $e^u, \sinh u, \cos u, \ln(1 + u)$, the integral equation or the integro-differential equation is called *nonlinear*. To explain this concept, we consider the equations:

$$u(x) = 1 - \int_0^x (x - t)u(t)dt, \tag{2.40}$$

$$u(x) = 1 - \int_0^1 (x - t)u(t)dt, \tag{2.41}$$

$$u(x) = 1 + \int_0^x (1 + x - t)u^4(t)dt, \tag{2.42}$$

and

$$u'(x) = 1 + \int_0^1 xte^{u(t)}dt, \quad u(0) = 1. \tag{2.43}$$

The first two examples are linear Volterra and Fredholm integral equations respectively, whereas the last two are nonlinear Volterra integral equation and nonlinear Fredholm integro-differential equation respectively.

It is important to point out that linear equations, except Fredholm integral equations of the first kind, give a unique solution if such a solution exists. However, solution of nonlinear equation may not be unique. Nonlinear equations usually give more than one solution and it is not usually easy to handle. Both linear and nonlinear integral equations of any kind will be investigated in this text by using traditional and new methods.

2.3.2 Homogeneity Concept

Integral equations and integro-differential equations of the second kind are classified as *homogeneous* or *inhomogeneous*, if the function $f(x)$ in the second kind of Volterra or Fredholm integral equations or integro-differential equations is identically zero, the equation is called homogeneous. Otherwise it is called inhomogeneous. Notice that this property holds for equations of the second kind only. To clarify this concept we consider the following equations:

$$u(x) = \sin x + \int_0^x xtu(t)dt, \tag{2.44}$$

$$u(x) = x + \int_0^1 (x-t)^2 u(t)dt, \tag{2.45}$$

$$u(x) = \int_0^x (1+x-t)u^4(t)dt, \tag{2.46}$$

and

$$u''(x) = \int_0^x xtu(t)dt, \quad u(0) = 1, \quad u'(0) = 0. \tag{2.47}$$

The first two equations are inhomogeneous because $f(x) = \sin x$ and $f(x) = x$, whereas the last two equations are homogeneous because $f(x) = 0$ for each equation. We usually use specific approaches for homogeneous equations, and other methods are used for inhomogeneous equations.

Exercises 2.3

Classify the following equations as Fredholm, or Volterra, linear or nonlinear, and homogeneous or inhomogeneous

1. $u(x) = 1 + \int_0^x (x-t)^2 u(t)dt$
2. $u(x) = \cosh x + \int_0^1 (x-t)u(t)dt$
3. $u(x) = \int_0^x (2+x-t)u(t)dt$
4. $u(x) = \lambda \int_{-1}^1 t^2 u(t)dt$

5. $u(x) = 1 + x + \int_0^x (x-t)\frac{1}{1+u^2}dt$

6. $u(x) = 1 + \int_0^1 u^3(t)dt$

7. $u'(x) = 1 + \int_0^1 (x-t)u(t)dt,\ u(0) = 1$

8. $u'(x) = \int_0^x (x-t)u(t)dt,\ u(0) = 0$

2.4 Origins of Integral Equations

Integral and integro-differential equations arise in many scientific and engineering applications. Volterra integral equations and Volterra integro-differential equations can be obtained from converting initial value problems with prescribed initial values. However, Fredholm integral equations and Fredholm integro-differential equations can be derived from boundary value problems with given boundary conditions.

It is important to point out that converting initial value problems to Volterra integral equations, and converting Volterra integral equations to initial value problems are commonly used in the literature. This will be explained in detail in the coming section. However, converting boundary value problems to Fredholm integral equations, and converting Fredholm integral equations to equivalent boundary value problems are rarely used. The conversion techniques will be examined and illustrated examples will be presented.

In what follows we will examine the steps that we will use to obtain these integral and integro-differential equations.

2.5 Converting IVP to Volterra Integral Equation

In this section, we will study the technique that will convert an initial value problem (IVP) to an equivalent Volterra integral equation and Volterra integro-differential equation as well [3]. For simplicity reasons, we will apply this process to a second order initial value problem given by

$$y''(x) + p(x)y'(x) + q(x)y(x) = g(x) \tag{2.48}$$

subject to the initial conditions:

$$y(0) = \alpha, \quad y'(0) = \beta, \tag{2.49}$$

where α and β are constants. The functions $p(x)$ and $q(x)$ are analytic functions, and $g(x)$ is continuous through the interval of discussion. To achieve our goal we first set

$$y''(x) = u(x), \tag{2.50}$$

where $u(x)$ is a continuous function. Integrating both sides of (2.50) from 0 to x yields

$$y'(x) - y'(0) = \int_0^x u(t)dt, \tag{2.51}$$

or equivalently

$$y'(x) = \beta + \int_0^x u(t)dt. \tag{2.52}$$

Integrating both sides of (2.52) from 0 to x yields

$$y(x) - y(0) = \beta x + \int_0^x \int_0^x u(t)dtdt, \tag{2.53}$$

or equivalently

$$y(x) = \alpha + \beta x + \int_0^x (x-t)u(t)dt, \tag{2.54}$$

obtained upon using the formula that reduce double integral to a single integral that was discussed in the previous chapter. Substituting (2.50), (2.52), and (2.54) into the initial value problem (2.48) yields the Volterra integral equation:

$$u(x) + p(x)\left[\beta + \int_0^x u(t)dt\right] + q(x)\left[\alpha + \beta x + \int_0^x (x-t)u(t)dt\right] = g(x). \tag{2.55}$$

The last equation can be written in the standard Volterra integral equation form:

$$u(x) = f(x) - \int_0^x K(x,t)u(t)dt, \tag{2.56}$$

where

$$K(x,t) = p(x) + q(x)(x-t), \tag{2.57}$$

and

$$f(x) = g(x) - \left[\beta p(x) + \alpha q(x) + \beta x q(x)\right]. \tag{2.58}$$

It is interesting to point out that by differentiating Volterra equation (2.56) with respect to x, using Leibnitz rule, we obtain an equivalent Volterra integro-differential equation in the form:

$$u'(x) + K(x,x)u(x) = f'(x) - \int_0^x \frac{\partial K(x,t)}{\partial x}u(t)dt, \quad u(0) = f(0). \tag{2.59}$$

The technique presented above to convert initial value problems to equivalent Volterra integral equations can be generalized by considering the general initial value problem:

$$y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_{n-1}(x)y' + a_n(x)y = g(x), \tag{2.60}$$

subject to the initial conditions

$$y(0) = c_0, y'(0) = c_1, y''(0) = c_2, \ldots, y^{(n-1)}(0) = c_{n-1}. \tag{2.61}$$

We assume that the functions $a_i(x), 1 \leqslant i \leqslant n$ are analytic at the origin, and the function $g(x)$ is continuous through the interval of discussion. Let $u(x)$ be a continuous function on the interval of discussion, and we consider the transformation:

$$y^{(n)}(x) = u(x). \tag{2.62}$$

Integrating both sides with respect to x gives

$$y^{(n-1)}(x) = c_{n-1} + \int_0^x u(t)dt. \tag{2.63}$$

Integrating again both sides with respect to x yields

$$\begin{aligned} y^{(n-2)}(x) &= c_{n-2} + c_{n-1}x + \int_0^x \int_0^x u(t)dtdt \\ &= c_{n-2} + c_{n-1}x + \int_0^x (x-t)u(t)dt, \end{aligned} \tag{2.64}$$

obtained by reducing the double integral to a single integral. Proceeding as before we find

$$\begin{aligned} y^{(n-3)}(x) &= c_{n-3} + c_{n-2}x + \frac{1}{2}c_{n-1}x^2 + \int_0^x \int_0^x \int_0^x u(t)dtdtdt \\ &= c_{n-3} + c_{n-2}x + \frac{1}{2}c_{n-1}x^2 + \frac{1}{2}\int_0^x (x-t)^2 u(t)dt. \end{aligned} \tag{2.65}$$

Continuing the integration process leads to

$$y(x) = \sum_{k=0}^{n-1} \frac{c_k}{k!}x^k + \frac{1}{(n-1)!}\int_0^x (x-t)^{n-1}u(t)dt. \tag{2.66}$$

Substituting (2.62)–(2.66) into (2.60) gives

$$u(x) = f(x) - \int_0^x K(x,t)u(t)dt, \tag{2.67}$$

where

$$K(x,t) = \sum_{k=1}^{n} \frac{a_n}{(k-1)!}(x-t)^{k-1}, \tag{2.68}$$

and

$$f(x) = g(x) - \sum_{j=1}^{n} a_j \left(\sum_{k=1}^{j} \frac{c_{n-k}}{(j-k)!} x^{j-k} \right). \tag{2.69}$$

Notice that the Volterra integro-differential equation can be obtained by differentiating (2.67) as many times as we like, and by obtaining the initial conditions of each resulting equation. The following examples will highlight the process to convert initial value problem to an equivalent Volterra integral equation.

Example 2.1

Convert the following initial value problem to an equivalent Volterra integral equation:

$$y'(x) - 2xy(x) = e^{x^2}, \quad y(0) = 1. \tag{2.70}$$

We first set

$$y'(x) = u(x). \tag{2.71}$$

Integrating both sides of (2.71), using the initial condition $y(0) = 1$ gives

$$y(x) - y(0) = \int_0^x u(t)dt, \tag{2.72}$$

or equivalently

$$y(x) = 1 + \int_0^x u(t)dt, \tag{2.73}$$

Substituting (2.71) and (2.73) into (2.70) gives the equivalent Volterra integral equation:

$$u(x) = 2x + e^{x^2} + 2x \int_0^x u(t)dt. \tag{2.74}$$

Example 2.2

Convert the following initial value problem to an equivalent Volterra integral equation:

$$y''(x) - y(x) = \sin x, \quad y(0) = 0, \quad y'(0) = 0. \tag{2.75}$$

Proceeding as before, we set

$$y''(x) = u(x). \tag{2.76}$$

Integrating both sides of (2.76), using the initial condition $y'(0) = 0$ gives

$$y'(x) = \int_0^x u(t)dt. \tag{2.77}$$

Integrating (2.77) again, using the initial condition $y(0) = 0$ yields

$$y(x) = \int_0^x \int_0^x u(t)dtdt = \int_0^x (x-t)u(t)dt, \tag{2.78}$$

obtained upon using the rule to convert double integral to a single integral. Inserting (2.76)–(2.78) into (2.70) leads to the following Volterra integral equation:

$$u(x) = \sin x + \int_0^x (x-t)u(t)dt. \tag{2.79}$$

Example 2.3

Convert the following initial value problem to an equivalent Volterra integral equation:

$$y''' - y'' - y' + y = 0, \quad y(0) = 1, \quad y'(0) = 2, \quad y''(0) = 3. \tag{2.80}$$

We first set

$$y'''(x) = u(x), \tag{2.81}$$

where by integrating both sides of (2.81) and using the initial condition $y''(0) = 3$ we obtain

$$y'' = 3 + \int_0^x u(t)dt. \tag{2.82}$$

Integrating again and using the initial condition $y'(0) = 2$ we find

$$y'(x) = 2 + 3x + \int_0^x \int_0^x u(t)dtdt = 2 + 3x + \int_0^x (x-t)u(t)dt. \tag{2.83}$$

Integrating again and using $y(0) = 1$ we obtain

$$\begin{aligned} y(x) &= 1+2x+\frac{3}{2}x^2+\int_0^x\int_0^x\int_0^x u(t)dtdtdt \\ &= 1+2x+\frac{3}{2}x^2+\frac{1}{2}\int_0^x (x-t)^2u(t)dt. \end{aligned} \tag{2.84}$$

Notice that in (2.83) and (2.84) the multiple integrals were reduced to single integrals as used before. Substituting (2.81) – (2.84) into (2.80) leads to the Volterra integral equation:

$$u(x) = 4+x+\frac{3}{2}x^2+\int_0^x [1+(x-t)-\frac{1}{2}(x-t)^2]u(t)dt. \tag{2.85}$$

Remark

We can also show that if $y^{(\text{iv})}(x)=u(x)$, then

$$\begin{aligned} y'''(x) &= y'''(0)+\int_0^x u(t)dt \\ y''(x) &= y''(0)+xy'''(0)+\int_0^x (x-t)u(t)dt \\ y'(x) &= y'(0)+xy''(0)+\frac{1}{2}x^2y'''(0)+\frac{1}{2}\int_0^x (x-t)^2u(t)dt \\ y(x) &= y(0)+xy'(0)+\frac{1}{2}x^2y''(0)+\frac{1}{6}x^3y'''(0)+\frac{1}{6}\int_0^x (x-t)^3u(t)dt. \end{aligned} \tag{2.86}$$

This process can be generalized to any derivative of a higher order.

In what follows we summarize the relation between derivatives of $y(x)$ and $u(x)$:

Table 2.1 The relation between derivatives of $y(x)$ and $u(x)$

$y^{(n)}(x)$	Integral Equations
$y'(x)=u(x)$	$y(x)=y(0)+\int_0^x u(t)dt$
$y''(x)=u(x)$	$y'(x)=y'(0)+\int_0^x u(t)dt$
	$y(x)=y(0)+xy'(0)+\int_0^x (x-t)u(t)dt$
	$y''(x)=y''(0)+\int_0^x u(t)dt$
$y'''(x)=u(x)$	$y'(x)=y'(0)+xy''(0)+\int_0^x (x-t)u(t)dt$
	$y(x)=y(0)+xy'(0)+\frac{1}{2}x^2y''(0)+\frac{1}{2}\int_0^x (x-t)^2u(t)dt$

2.5.1 Converting Volterra Integral Equation to IVP

A well-known method for solving Volterra integral and Volterra integro-differential equation, that we will use in the forthcoming chapters, converts these equations to equivalent initial value problems. The method is achieved simply by differentiating both sides of Volterra equations [6] with respect to x as many times as we need to get rid of the integral sign and come out with a differential equation. The conversion of Volterra equations requires the use of Leibnitz rule for differentiating the integral at the right hand side. The initial conditions can be obtained by substituting $x = 0$ into $u(x)$ and its derivatives. The resulting initial value problems can be solved easily by using ODEs methods that were summarized in Chapter 1. The conversion process will be illustrated by discussing the following examples.

Example 2.4

Find the initial value problem equivalent to the Volterra integral equation:

$$u(x) = e^x + \int_0^x u(t)dt. \tag{2.87}$$

Differentiating both sides of (2.87) and using Leibnitz rule we find

$$u'(x) = e^x + u(x). \tag{2.88}$$

It is clear that there is no need for differentiating again because we got rid of the integral sign. To determine the initial condition, we substitute $x = 0$ into both sides of (2.87) to find $u(0) = 1$. This in turn gives the initial value problem:

$$u'(x) - u(x) = e^x, \quad u(0) = 1. \tag{2.89}$$

Notice that the resulting ODE is a linear inhomogeneous equation of first order.

Example 2.5

Find the initial value problem equivalent to the Volterra integral equation:

$$u(x) = x^2 + \int_0^x (x-t)u(t)dt. \tag{2.90}$$

Differentiating both sides of (2.90) and using Leibnitz rule we find

$$u'(x) = 2x + \int_0^x u(t)dt. \tag{2.91}$$

To get rid of the integral sign we should differentiate (2.91) and by using Leibnitz rule we obtain the second order ODE:

$$u''(x) = 2 + u(x). \tag{2.92}$$

To determine the initial conditions, we substitute $x = 0$ into both sides of (2.90) and (2.91) to find $u(0) = 0$ and $u'(0) = 0$ respectively. This in turn gives the initial value problem:

$$u''(x) - u(x) = 2, \quad u(0) = 0, \quad u'(0) = 0. \tag{2.93}$$

Notice that the resulting ODE is a second order inhomogeneous equation.

Example 2.6

Find the initial value problem equivalent to the Volterra integral equation:

$$u(x) = \sin x - \frac{1}{2}\int_0^x (x-t)^2 u(t)dt. \tag{2.94}$$

Differentiating both sides of the integral equation three times to get rid of the integral sign to find

$$\begin{aligned} u'(x) &= \cos x - \int_0^x (x-t)u(t)dt, \\ u''(x) &= -\sin x - \int_0^x u(t)dt, \\ u'''(x) &= -\cos x - u(x). \end{aligned} \tag{2.95}$$

Substituting $x = 0$ into (2.94) and into the first two integro-differential equations in (2.95) gives the initial conditions:

$$u(0) = 0, \quad u'(0) = 1, \quad u''(0) = 0. \tag{2.96}$$

In view of the last results, the initial value problem equivalent to the Volterra integral equation (2.94) is a third order inhomogeneous ODE given by

$$u'''(x) + u(x) = -\cos x, \quad u(0) = 0, \quad u'(0) = 1, \quad u''(0) = 0. \tag{2.97}$$

Exercises 2.5

Convert each of the following IVPs in 1–8 to an equivalent Volterra integral equation:

1. $y' - 4y = 0,\ y(0) = 1$
2. $y' + 4xy = e^{-2x^2},\ y(0) = 0$
3. $y'' + 4y = 0,\ y(0) = 0,\ y'(0) = 1$
4. $y'' - 6y' + 8y = 1,\ y(0) = 1,\ y'(0) = 1$
5. $y''' - y' = 0,\ y(0) = 2,\ y'(0) = y''(0) = 1$
6. $y''' - 2y'' + y = x,\ y(0) = 1, y'(0) = 0,\ y''(0) = 1$
7. $y^{(\mathrm{iv})} - y'' = 1,\ y(0) = y'(0) = 0,\ y''(0) = y'''(0) = 1$
8. $y^{(\mathrm{iv})} + y'' + y = x,\ y(0) = y'(0) = 1,\ y''(0) = y'''(0) = 0$

Convert each of the following Volterra integral equation in 9–16 to an equivalent IVP:

9. $u(x) = x + 2\int_0^x u(t)dt$
10. $u(x) = 1 + e^x - \int_0^x u(t)dt$
11. $u(x) = 1 + x^2 + \int_0^x (x-t)u(t)dt$
12. $u(x) = \sin x - \int_0^x (x-t)u(t)dt$
13. $u(x) = 1 - \cos x + 2\int_0^x (x-t)^2 u(t)dt$
14. $u(x) = 2 + \sinh x + \int_0^x (x-t)^2 u(t)dt$
15. $u(x) = 1 + 2\int_0^x (x-t)^3 u(t)dt$
16. $u(x) = 1 + e^x + \int_0^x (1+x-t)^3 u(t)dt$

2.6 Converting BVP to Fredholm Integral Equation

In this section, we will present a method that will convert a boundary value problem to an equivalent Fredholm integral equation. The method is similar to the method that was presented in the previous section for converting Volterra equation to IVP, with the exception that boundary conditions will be used instead of initial values. In this case we will determine another initial condition that is not given in the problem. The technique requires more work if compared with the initial value problems when converted to Volterra integral equations. For this reason, the technique that will be presented is rarely used. Without loss of generality, we will present two specific distinct boundary value problems (BVPs) to derive two distinct formulas that can be used for converting BVP to an equivalent Fredholm integral equation.

Type I

We first consider the following boundary value problem:

$$y''(x) + g(x)y(x) = h(x), \quad 0 < x < 1, \tag{2.98}$$

with the boundary conditions:

$$y(0) = \alpha, \quad y(1) = \beta. \tag{2.99}$$

We start as in the previous section and set

$$y''(x) = u(x). \tag{2.100}$$

Integrating both sides of (2.100) from 0 to x we obtain

$$\int_0^x y''(t)dt = \int_0^x u(t)dt, \tag{2.101}$$

that gives

$$y'(x) = y'(0) + \int_0^x u(t)dt, \tag{2.102}$$

where the initial condition $y'(0)$ is not given in a boundary value problem. The condition $y'(0)$ will be determined later by using the boundary condition at $x = 1$. Integrating both sides of (2.102) from 0 to x gives

$$y(x) = y(0) + xy'(0) + \int_0^x \int_0^x u(t)dt, \tag{2.103}$$

or equivalently

$$y(x) = \alpha + xy'(0) + \int_0^x (x - t)u(t)dt, \tag{2.104}$$

obtained upon using the condition $y(0) = \alpha$ and by reducing double integral to a single integral. To determine $y'(0)$, we substitute $x = 1$ into both sides of (2.104) and using the boundary condition at $y(1) = \beta$ we find

$$y(1) = \alpha + y'(0) + \int_0^1 (1 - t)u(t)dt, \tag{2.105}$$

that gives

$$\beta = \alpha + y'(0) + \int_0^1 (1-t)u(t)dt. \tag{2.106}$$

This in turn gives

$$y'(0) = (\beta - \alpha) - \int_0^1 (1-t)u(t)dt. \tag{2.107}$$

Substituting (2.107) into (2.104) gives

$$y(x) = \alpha + (\beta - \alpha)x - \int_0^1 x(1-t)u(t)dt + \int_0^x (x-t)u(t)dt. \tag{2.108}$$

Substituting (2.100) and (2.108) into (2.98) yields

$$\begin{aligned} u(x) + \alpha g(x) + (\beta - \alpha)xg(x) - \int_0^1 xg(x)(1-t)u(t)dt \\ + \int_0^x g(x)(x-t)u(t)dt = h(x). \end{aligned} \tag{2.109}$$

From calculus we can use the formula:

$$\int_0^1 (\cdot) = \int_0^x (\cdot) + \int_x^1 (\cdot), \tag{2.110}$$

to carry Eq. (2.109) to

$$\begin{aligned} u(x) = h(x) - \alpha g(x) - (\beta - \alpha)xg(x) - g(x)\int_0^x (x-t)u(t)dt \\ +xg(x)\left[\int_0^x (1-t)u(t)dt + \int_x^1 (1-t)u(t)dt\right], \end{aligned} \tag{2.111}$$

that gives

$$u(x) = f(x) + \int_0^x t(1-x)g(x)u(t)dt + \int_x^1 x(1-t)g(x)u(t)dt, \tag{2.112}$$

that leads to the Fredholm integral equation:

$$u(x) = f(x) + \int_0^1 K(x,t)u(t)dt, \tag{2.113}$$

where

$$f(x) = h(x) - \alpha g(x) - (\beta - \alpha)xg(x), \tag{2.114}$$

and the kernel $K(x,t)$ is given by

$$K(x,t) = \begin{cases} t(1-x)g(x), & \text{for } \ 0 \leqslant t \leqslant x, \\ x(1-t)g(x), & \text{for } \ x \leqslant t \leqslant 1. \end{cases} \tag{2.115}$$

An important conclusion can be made here. For the specific case where $y(0) = y(1) = 0$ which means that $\alpha = \beta = 0$, i.e. the two boundaries of a moving string are fixed, it is clear that $f(x) = h(x)$ in this case. This means that the resulting Fredholm equation in (2.113) is homogeneous or inhomogeneous if the boundary value problem in (2.98) is homogeneous or inhomogeneous respectively when $\alpha = \beta = 0$. The techniques presented above will be illustrated by the following two examples.

Example 2.7

Convert the following BVP to an equivalent Fredholm integral equation:

$$y''(x) + 9y(x) = \cos x, \quad y(0) = y(1) = 0. \tag{2.116}$$

We can easily observe that $\alpha = \beta = 0, g(x) = 9$ and $h(x) = \cos x$. This in turn gives

$$f(x) = \cos x. \tag{2.117}$$

Substituting this into (2.113) gives the Fredholm integral equation:

$$u(x) = \cos x + \int_0^1 K(x,t)u(t)dt, \tag{2.118}$$

where the kernel $K(x,t)$ is given by

$$K(x,t) = \begin{cases} 9t(1-x), & \text{for } \ 0 \leqslant t \leqslant x, \\ 9x(1-t), & \text{for } \ x \leqslant t \leqslant 1. \end{cases} \tag{2.119}$$

Example 2.8

Convert the following BVP to an equivalent Fredholm integral equation:

$$y''(x) + xy(x) = 0, \quad y(0) = 0, \quad y(1) = 2. \tag{2.120}$$

Recall that this is a boundary value problem because the conditions are given at the boundaries $x = 0$ and $x = 1$. Moreover, the coefficient of $y(x)$ is a variable and not a constant.

We can easily observe that $\alpha = 0, \beta = 2, g(x) = x$ and $h(x) = 0$. This in turn gives

$$f(x) = -2x^2. \tag{2.121}$$

Substituting this into (2.113) gives the Fredholm integral equation:

$$u(x) = -2x^2 + \int_0^1 K(x,t)u(t)dt, \tag{2.122}$$

where the kernel $K(x,t)$ is given by

$$K(x,t) = \begin{cases} tx(1-x), & \text{for } \ 0 \leqslant t \leqslant x, \\ x^2(1-t), & \text{for } \ x \leqslant t \leqslant 1. \end{cases} \tag{2.123}$$

Type II

We next consider the following boundary value problem:

$$y''(x) + g(x)y(x) = h(x), \quad 0 < x < 1, \tag{2.124}$$

with the boundary conditions:

$$y(0) = \alpha_1, \quad y'(1) = \beta_1. \tag{2.125}$$

We again set

$$y''(x) = u(x). \tag{2.126}$$

Integrating both sides of (2.126) from 0 to x we obtain

$$\int_0^x y''(t)dt = \int_0^x u(t)dt, \tag{2.127}$$

that gives

$$y'(x) = y'(0) + \int_0^x u(t)dt, \tag{2.128}$$

where the initial condition $y'(0)$ is not given. The condition $y'(0)$ will be derived later by using the boundary condition at $y'(1) = \beta_1$. Integrating both sides of (2.128) from 0 to x gives

$$y(x) = y(0) + xy'(0) + \int_0^x \int_0^x u(t)dtdt, \tag{2.129}$$

or equivalently

$$y(x) = \alpha_1 + xy'(0) + \int_0^x (x-t)u(t)dt, \tag{2.130}$$

obtained upon using the condition $y(0) = \alpha_1$ and by reducing double integral to a single integral. To determine $y'(0)$, we first differentiate (2.130) with respect to x to get

$$y'(x) = y'(0) + \int_0^x u(t)dt, \tag{2.131}$$

where by substituting $x = 1$ into both sides of (2.131) and using the boundary condition at $y'(1) = \beta_1$ we find

$$y'(1) = y'(0) + \int_0^1 u(t)dt, \tag{2.132}$$

that gives

$$y'(0) = \beta_1 - \int_0^1 u(t)dt. \tag{2.133}$$

Using (2.133) into (2.130) gives

$$y(x) = \alpha_1 + x\left[\beta_1 - \int_0^1 u(t)dt\right] + \int_0^x (x-t)u(t)dt. \tag{2.134}$$

Substituting (2.126) and (2.134) into (2.124) yields

$$u(x) + \alpha_1 g(x) + \beta_1 xg(x) - \int_0^1 xg(x)u(t)dt + \int_0^x g(x)(x-t)u(t)dt = h(x). \tag{2.135}$$

From calculus we can use the formula:

$$\int_0^1 (\cdot) = \int_0^x (\cdot) + \int_x^1 (\cdot), \tag{2.136}$$

to carry Eq. (2.135) to

$$\begin{aligned} u(x) &= h(x) - (\alpha_1 + \beta_1 x)g(x) \\ &\quad + xg(x)\left[\int_0^x u(t)dt + \int_x^1 u(t)dt\right] - g(x)\int_0^x (x-t)u(t)dt. \end{aligned} \tag{2.137}$$

The last equation can be written as

$$u(x) = f(x) + \int_0^x tg(x)u(t)dt + \int_x^1 xg(x)u(t)dt, \tag{2.138}$$

that leads to the Fredholm integral equation:

$$u(x) = f(x) + \int_0^1 K(x,t)u(t)dt, \tag{2.139}$$

where

$$f(x) = h(x) - (\alpha_1 + \beta_1 x)g(x), \tag{2.140}$$

and the kernel $K(x,t)$ is given by

$$K(x,t) = \begin{cases} tg(x), & \text{for } 0 \leqslant t \leqslant x, \\ xg(x), & \text{for } x \leqslant t \leqslant 1. \end{cases} \tag{2.141}$$

An important conclusion can be made here. For the specific case where $y(0) = y'(1) = 0$ which means that $\alpha_1 = \beta_1 = 0$, it is clear that $f(x) = h(x)$ in this case. This means that the resulting Fredholm equation in (2.139) is homogeneous or inhomogeneous if the boundary value problem in (2.124) is homogeneous or inhomogeneous respectively.

The second type of conversion that was presented above will be illustrated by the following two examples.

Example 2.9

Convert the following BVP to an equivalent Fredholm integral equation:

$$y''(x) + y(x) = 0, \quad y(0) = y'(1) = 0. \tag{2.142}$$

We can easily observe that $\alpha_1 = \beta_1 = 0, g(x) = 1$ and $h(x) = 0$. This in turn gives

$$f(x) = 0. \tag{2.143}$$

Substituting this into (2.139) gives the homogeneous Fredholm integral equation:

$$u(x) = \int_0^1 K(x,t)u(t)dt, \tag{2.144}$$

where the kernel $K(x,t)$ is given by

$$K(x,t) = \begin{cases} t, & \text{for } 0 \leqslant t \leqslant x, \\ x, & \text{for } x \leqslant t \leqslant 1. \end{cases} \tag{2.145}$$

Example 2.10

Convert the following BVP to an equivalent Fredholm integral equation:

$$y''(x) + 2y(x) = 4, \quad y(0) = 0, \quad y'(1) = 1. \tag{2.146}$$

We can easily observe that $\alpha_1 = 0, \beta_1 = 1, g(x) = 2$ and $h(x) = 4$. This in turn gives

$$f(x) = 4 - 2x. \tag{2.147}$$

Substituting this into (2.139) gives the inhomogeneous Fredholm integral equation:

$$u(x) = 4 - 2x + \int_0^1 K(x,t)u(t)dt, \tag{2.148}$$

where the kernel $K(x,t)$ is given by

$$K(x,t) = \begin{cases} 2t, & \text{for } 0 \leqslant t \leqslant x, \\ 2x, & \text{for } x \leqslant t \leqslant 1. \end{cases} \tag{2.149}$$

2.6.1 Converting Fredholm Integral Equation to BVP

In a previous section, we presented a technique to convert Volterra integral equation to an equivalent initial value problem. In a similar manner, we will present another technique that will convert Fredholm integral equation to an equivalent boundary value problem (BVP). In what follows we will examine two types of problems:

Type I

We first consider the Fredholm integral equation given by

$$u(x) = f(x) + \int_0^1 K(x,t)u(t)dt, \tag{2.150}$$

where $f(x)$ is a given function, and the kernel $K(x,t)$ is given by

$$K(x,t) = \begin{cases} t(1-x)g(x), & \text{for } 0 \leqslant t \leqslant x, \\ x(1-t)g(x), & \text{for } x \leqslant t \leqslant 1. \end{cases} \tag{2.151}$$

For simplicity reasons, we may consider $g(x) = \lambda$ where λ is constant. Equation (2.150) can be written as

$$u(x) = f(x) + \lambda \int_0^x t(1-x)u(t)dt + \lambda \int_x^1 x(1-t)u(t)dt, \tag{2.152}$$

or equivalently

$$u(x) = f(x) + \lambda(1-x) \int_0^x tu(t)dt + \lambda x \int_x^1 (1-t)u(t)dt. \tag{2.153}$$

Each term of the last two terms at the right side of (2.153) is a product of two functions of x. Differentiating both sides of (2.153), using the product rule of differentiation and using Leibnitz rule we obtain

$$\begin{aligned} u'(x) &= f'(x) + \lambda x(1-x)u(x) - \lambda \int_0^x tu(t) \\ &\quad -\lambda x(1-x)u(x) + \lambda \int_x^1 (1-t)u(t)dt \\ &= f'(x) - \lambda \int_0^x tu(t) + \lambda \int_x^1 (1-t)u(t)dt. \end{aligned} \tag{2.154}$$

To get rid of integral signs, we differentiate both sides of (2.154) again with respect to x to find that

$$u''(x) = f''(x) - \lambda x u(x) - \lambda(1-x)u(x), \tag{2.155}$$

that gives the ordinary differential equations:

$$u''(x) + \lambda u(x) = f''(x). \tag{2.156}$$

The related boundary conditions can be obtained by substituting $x = 0$ and $x = 1$ in (2.153) to find that

$$u(0) = f(0), \quad u(1) = f(1). \tag{2.157}$$

Combining (2.156) and (2.157) gives the boundary value problem equivalent to the Fredholm equation (2.150).

Recall that $y''(x) = u(x)$. Moreover, if $g(x)$ is not a constant, we can proceed in a manner similar to the discussion presented above to obtain the boundary value problem. The technique above for type I will be explained by studying the following examples.

Example 2.11

Convert the Fredholm integral equation

$$u(x) = e^x + \int_0^1 K(x,t)u(t)dt, \tag{2.158}$$

where the kernel $K(x,t)$ is given by

$$K(x,t) = \begin{cases} 9t(1-x), & \text{for } \ 0 \leqslant t \leqslant x, \\ 9x(1-t), & \text{for } \ x \leqslant t \leqslant 1, \end{cases} \tag{2.159}$$

to an equivalent boundary value problem.

The Fredholm integral equation can be written as

$$u(x) = e^x + 9(1-x)\int_0^x tu(t)dt + 9x\int_x^1 (1-t)u(t)dt. \tag{2.160}$$

Differentiating (2.160) twice with respect to x gives

$$u'(x) = e^x - 9\int_0^x tu(t)dt + 9\int_x^1 (1-t)u(t)dt, \tag{2.161}$$

and

$$u''(x) = e^x - 9u(x). \tag{2.162}$$

This in turn gives the ODE:

$$u''(x) + 9u(x) = e^x. \tag{2.163}$$

The related boundary conditions are given by

$$u(0) = f(0) = 1, \quad u(1) = f(1) = e, \tag{2.164}$$

obtained upon substituting $x = 0$ and $x = 1$ into (2.160).

Example 2.12

Convert the Fredholm integral equation

$$u(x) = x^3 + \int_0^1 K(x,t)u(t)dt, \tag{2.165}$$

where the kernel $K(x,t)$ is given by

$$K(x,t) = \begin{cases} 4t(1-x), & \text{for } 0 \leqslant t \leqslant x, \\ 4x(1-t), & \text{for } x \leqslant t \leqslant 1, \end{cases} \tag{2.166}$$

to an equivalent boundary value problem.

The Fredholm integral equation can be written as

$$u(x) = x^3 + 4(1-x)\int_0^x tu(t)dt + 4x\int_x^1 (1-t)u(t)dt. \tag{2.167}$$

Proceeding as before we find

$$u''(x) = 6x - 4u(x). \tag{2.168}$$

This in turn gives the ODE:

$$u''(x) + 4u(x) = 6x, \tag{2.169}$$

with the related boundary conditions:

$$u(0) = f(0) = 0, \quad u(1) = f(1) = 1. \tag{2.170}$$

Type II

We next consider the Fredholm integral equation given by

$$u(x) = f(x) + \int_0^1 K(x,t)u(t)dt, \tag{2.171}$$

where $f(x)$ is a given function, and the kernel $K(x,t)$ is given by

$$K(x,t) = \begin{cases} tg(x), & \text{for } 0 \leqslant t \leqslant x, \\ xg(x), & \text{for } x \leqslant t \leqslant 1. \end{cases} \tag{2.172}$$

For simplicity reasons, we again consider $g(x) = \lambda$ where λ is constant. Equation (2.171) can be written as

$$u(x) = f(x) + \lambda\int_0^x tu(t)dt + \lambda x\int_x^1 u(t)dt. \tag{2.173}$$

Each integral at the right side of (2.173) is a product of two functions of x. Differentiating both sides of (2.173), using the product rule of differentiation and using Leibnitz rule we obtain

$$u'(x) = f'(x) + \lambda\int_x^1 u(t)dt. \tag{2.174}$$

To get rid of integral signs, we differentiate again with respect to x to find that

$$u''(x) = f''(x) - \lambda u(x), \tag{2.175}$$

that gives the ordinary differential equations. Also change equations to equation

$$u''(x) + \lambda u(x) = f''(x). \tag{2.176}$$

Notice that the boundary condition $u(1)$ in this case cannot be obtained from (2.173). Therefore, the related boundary conditions can be obtained by substituting $x = 0$ and $x = 1$ in (2.173) and (2.174) respectively to find that

$$u(0) = f(0), \quad u'(1) = f'(1). \tag{2.177}$$

Combining (2.176) and (2.177) gives the boundary value problem equivalent to the Fredholm equation (2.171).

Recall that $y''(x) = u(x)$. Moreover, if $g(x)$ is not a constant, we can proceed in a manner similar to the discussion presented above to obtain the boundary value problem. The approach presented above for type II will be illustrated by studying the following examples.

Example 2.13

Convert the Fredholm integral equation:

$$u(x) = e^x + \int_0^1 K(x,t)u(t)dt, \tag{2.178}$$

where the kernel $K(x,t)$ is given by

$$K(x,t) = \begin{cases} 4t, & \text{for} \;\; 0 \leqslant t \leqslant x, \\ 4x, & \text{for} \;\; x \leqslant t \leqslant 1, \end{cases} \tag{2.179}$$

to an equivalent boundary value problem.

The Fredholm integral equation can be written as

$$u(x) = e^x + 4\int_0^x tu(t)dt + 4x\int_x^1 u(t)dt. \tag{2.180}$$

Differentiating (2.180) twice with respect to x gives

$$u'(x) = e^x + 4\int_x^1 u(t)dt, \tag{2.181}$$

and

$$u''(x) = e^x - 4u(x). \tag{2.182}$$

This in turn gives the ODE:

$$u''(x) + 4u(x) = e^x. \tag{2.183}$$

The related boundary conditions are given by

$$u(0) = f(0) = 1, \quad u'(1) = f'(1) = e, \tag{2.184}$$

obtained upon substituting $x = 0$ and $x = 1$ into (2.180) and (2.181) respectively. Recall that the boundary condition $u(1)$ cannot obtained in this case.

Example 2.14

Convert the Fredholm integral equation

$$u(x) = x^2 + \int_0^1 K(x,t)u(t)dt, \tag{2.185}$$

where the kernel $K(x,t)$ is given by

$$K(x,t) = \begin{cases} 6t, & \text{for } 0 \leqslant t \leqslant x, \\ 6x, & \text{for } x \leqslant t \leqslant 1 \end{cases} \tag{2.186}$$

to an equivalent boundary value problem.

The Fredholm integral equation can be written as

$$u(x) = x^2 + 6\int_0^x tu(t)dt + 6x\int_x^1 u(t)dt. \tag{2.187}$$

Proceeding as before we find

$$u'(x) = 2x + 6\int_x^1 u(t)dt. \tag{2.188}$$

and

$$u''(x) + 6u(x) = 2, \tag{2.189}$$

with the related boundary conditions

$$u(0) = f(0) = 0, \quad u'(1) = f'(1) = 2. \tag{2.190}$$

Exercises 2.6

Convert each of the following BVPs in 1–8 to an equivalent Fredholm integral equation:

1. $y'' + 4y = 0,\ 0 < x < 1,\ y(0) = y(1) = 0$
2. $y'' + xy = 0,\ y(0) = y(1) = 0$
3. $y'' + 2y = x,\ 0 < x < 1,\ y(0) = 1, y(1) = 0$
4. $y'' + 3xy = 4,\ 0 < x < 1,\ y(0) = 0, y(1) = 0$
5. $y'' + 4y = 0,\ 0 < x < 1,\ y(0) = 0, y'(1) = 0$
6. $y'' + xy = 0,\ y(0) = 0,\ y'(1) = 0$
7. $y'' + 4y = x,\ 0 < x < 1,\ y(0) = 1, y'(1) = 0$
8. $y'' + 4xy = 2,\ 0 < x < 1,\ y(0) = 0, y'(1) = 1$

Convert each of the following Fredholm integral equation in 9–16 to an equivalent BVP:

9. $u(x) = e^{2x} + \int_0^1 K(x,t)u(t)dt,\ K(x,t) = \begin{cases} 3t(1-x), & \text{for } 0 \leqslant t \leqslant x \\ 3x(1-t), & \text{for } x \leqslant t \leqslant 1 \end{cases}$
10. $u(x) = 3x^2 + \int_0^1 K(x,t)u(t)dt,\ K(x,t) = \begin{cases} t(1-x), & \text{for } 0 \leqslant t \leqslant x \\ x(1-t), & \text{for } x \leqslant t \leqslant 1 \end{cases}$
11. $u(x) = \cos x + \int_0^1 K(x,t)u(t)dt,\ K(x,t) = \begin{cases} 6t(1-x), & \text{for } 0 \leqslant t \leqslant x \\ 6x(1-t), & \text{for } x \leqslant t \leqslant 1 \end{cases}$
12. $u(x) = \sinh x + \int_0^1 K(x,t)u(t)dt,\ K(x,t) = \begin{cases} 4t(1-x), & \text{for } 0 \leqslant t \leqslant x \\ 4x(1-t), & \text{for } x \leqslant t \leqslant 1 \end{cases}$

13. $u(x) = e^{3x} + \int_0^1 K(x,t)u(t)dt,\ K(x,t) = \begin{cases} t, & \text{for } 0 \leqslant t \leqslant x \\ x, & \text{for } x \leqslant t \leqslant 1 \end{cases}$

14. $u(x) = x^4 + \int_0^1 K(x,t)u(t)dt,\ K(x,t) = \begin{cases} 6t, & \text{for } 0 \leqslant t \leqslant x \\ 6x, & \text{for } x \leqslant t \leqslant 1 \end{cases}$

15. $u(x) = 2x^2 + 3 + \int_0^1 K(x,t)u(t)dt,\ K(x,t) = \begin{cases} 4t, & \text{for } 0 \leqslant t \leqslant x \\ 4x, & \text{for } x \leqslant t \leqslant 1 \end{cases}$

16. $u(x) = e^x + 1 + \int_0^1 K(x,t)u(t)dt,\ K(x,t) = \begin{cases} 2t, & \text{for } 0 \leqslant t \leqslant x \\ 2x, & \text{for } x \leqslant t \leqslant 1 \end{cases}$

2.7 Solution of an Integral Equation

A solution of a differential or an integral equation arises in any of the following two types:

1). **Exact solution**:
The solution is called exact if it can be expressed in a closed form, such as a polynomial, exponential function, trigonometric function or the combination of two or more of these elementary functions. Examples of exact solutions are as follows:

$$\begin{aligned} u(x) &= x + e^x, \\ u(x) &= \sin x + e^{2x}, \\ u(x) &= 1 + \cosh x + \tan x, \end{aligned} \tag{2.191}$$

and many others.

2). **Series solution**:
For concrete problems, sometimes we cannot obtain exact solutions. In this case we determine the solution in a series form that may converge to exact soliton if such a solution exists. Other series may not give exact solution, and in this case the obtained series can be used for numerical purposes. The more terms that we determine the higher accuracy level that we can achieve.

A solution of an integral equation or integro-differential equation is a function $u(x)$ that satisfies the given equation. In other words, the obtained solution $u(x)$ must satisfy both sides of the examined equation. The following examples will be examined to explain the meaning of a solution.

Example 2.15

Show that $u(x) = \sinh x$ is a solution of the Volterra integral equation:

$$u(x) = x + \int_0^x (x-t)u(t)dt. \tag{2.192}$$

Substituting $u(x) = \sinh x$ in the right hand side (RHS) of (2.192) yields

$$\begin{aligned} \text{RHS} &= x + \int_0^x (x-t)\sinh t dt \\ &= x + (\sinh t - t)|_0^x \\ &= \sinh x = u(x) = \text{LHS}. \end{aligned} \tag{2.193}$$

Example 2.16

Show that $u(x) = \sec^2 x$ is a solution of the Fredholm integral equation

$$u(x) = -\frac{1}{2} + \sec^2 x + \frac{1}{2}\int_0^{\frac{\pi}{4}} u(t)dt. \tag{2.194}$$

Substituting $u(x) = \sec^2 x$ in the right hand side of (2.194) gives

$$\begin{aligned} \text{RHS} &= -\frac{1}{2} + \sec^2 x + \frac{1}{2}\int_0^{\frac{\pi}{4}} u(t)dt \\ &= -\frac{1}{2} + \sec^2 x + \frac{1}{2}(\tan t)\Big|_0^{\frac{\pi}{4}} \\ &= \sec^2 x = u(x) = \text{LHS}. \end{aligned} \tag{2.195}$$

Example 2.17

Show that $u(x) = \sin x$ is a solution of the Volterra integro-differential equation:

$$u'(x) = 1 - \int_0^x u(t)dt. \tag{2.196}$$

Proceeding as before, and using $u(x) = \sin x$ into both sides of (2.196) we find

$$\begin{aligned} \text{LHS} &= u'(x) = \cos x, \\ \text{RHS} &= 1 - \int_0^x \sin t dt = 1 - (-\cos t)\Big|_0^x = \cos x. \end{aligned} \tag{2.197}$$

Example 2.18

Show that $u(x) = x + e^x$ is a solution of the Fredholm integro-differential equation:

$$u''(x) = e^x - \frac{4}{3}x + \int_0^1 xtu(t)dt. \tag{2.198}$$

Substituting $u(x) = x + e^x$ into both sides of (2.198) we find

$$\begin{aligned} \text{LHS} &= u''(x) = e^x, \\ \text{RHS} &= e^x - \frac{4}{3}x + x\int_0^1 t(t+e^t)dt \\ &= e^x - \frac{4}{3}x + x\left(\frac{1}{3}t^3 + te^t - e^t\right)\Bigg|_0^1 = e^x. \end{aligned} \tag{2.199}$$

Example 2.19

Show that $u(x) = \cos x$ is a solution of the Volterra-Fredholm integral equation:

$$u(x) = \cos x - x + \int_0^x \int_0^{\frac{\pi}{2}} u(t)dtdt. \quad (2.200)$$

Proceeding as before, and using $u(x) = \cos x$ into both sides of (2.200) we find

$$\begin{aligned} \text{LHS} &= \cos x, \\ \text{RHS} &= \cos x - x + \int_0^x \int_0^{\frac{\pi}{2}} \cos t dt = \cos x. \end{aligned} \quad (2.201)$$

Example 2.20

Show that $u(x) = e^x$ is a solution of the Fredholm integral equation of the first kind:

$$\frac{e^{x^2+1} - 1}{x^2 + 1} = \int_0^1 e^{x^2 t} u(t)dt. \quad (2.202)$$

Proceeding as before, and using $u(x) = e^x$ into the right side of (2.202) we find

$$\text{RHS} = \int_0^1 e^{(x^2+1)t} dt = \left. \frac{e^{(x^2+1)t}}{x^2+1} \right|_{t=0}^{t=1} = \frac{e^{x^2+1} - 1}{x^2+1} = \text{LHS}. \quad (2.203)$$

Example 2.21

Show that $u(x) = x$ is a solution of the nonlinear Fredholm integral equation

$$u(x) = x - \frac{\pi}{12} + \frac{1}{3} \int_0^1 \frac{1}{1 + u^2(t)} dt. \quad (2.204)$$

Using $u(x) = x$ into the right side of (2.204) we find

$$\begin{aligned} \text{RHS} &= x - \frac{\pi}{12} + \frac{1}{3} \int_0^1 \frac{1}{1+t^2} dt = x - \frac{\pi}{12} + \frac{1}{3} \tan^{-1} t \Big|_{t=0}^{t=1} \\ &= x - \frac{\pi}{12} + \frac{1}{3} \left(\frac{\pi}{4} - 0 \right) = x = \text{LHS}. \end{aligned} \quad (2.205)$$

Example 2.22

Find $f(x)$ if $u(x) = x^2 + x^3$ is a solution of the Fredholm integral equation

$$u(x) = f(x) + \frac{5}{2} \int_{-1}^1 (xt^2 + x^2 t) u(t)dt. \quad (2.206)$$

Using $u(x) = x^2 + x^3$ into both sides of (2.206) we find

$$\begin{aligned} \text{LHS} &= x^2 + x^3, \\ \text{RHS} &= f(x) + \frac{5}{2} \int_{-1}^1 (xt^2 + x^2 t) u(t)dt = f(x) + x^2 + x. \end{aligned} \quad (2.207)$$

Equating the left and right sides gives

$$f(x) = x^3 - x. \tag{2.208}$$

Exercises 2.7

In Exercises 1–4, show that the given function $u(x)$ is a solution of the corresponding Fredholm integral equation:

1. $u(x) = \cos x + \frac{1}{2}\int_0^{\frac{\pi}{2}} \sin x u(t)dt,\ u(x) = \sin x + \cos x$

2. $u(x) = e^{2x+\frac{1}{3}} - \frac{1}{3}\int_0^1 e^{2x-\frac{5}{3}t}u(t)dt,\ u(x) = e^{2x}$

3. $u(x) = x + \int_{-1}^1 (x^4 - t^4)u(t)dt,\ \ -1 \leqslant x \leqslant 1,\ u(x) = x$

4. $u(x) = x + (1-x)e^x + \int_0^1 x^2 e^{t(x-1)}u(t)dt,\ u(x) = e^x$

In Exercises 5–8, show that the given function $u(x)$ is a solution of the corresponding Volterra integral equation:

5. $u(x) = 1 + \frac{1}{2}\int_0^x u(t)dt,\ u(x) = e^{2x}$

6. $u(x) = 4x + \sin x + 2x^2 - \cos x + 1 - \int_0^x u(t)dt,\ u(x) = 4x + \sin x$

7. $u(x) = 1 - \frac{1}{2}x^2 - \int_0^x (x-t)u(t)dt,\ u(x) = 2\cos x - 1$

8. $u(x) = 1 + 2x + \sin x + x^2 - \cos x - \int_0^x u(t)dt,\ u(x) = 2x + \sin x$

In Exercises 9–12, show that the given function $u(x)$ is a solution of the corresponding Fredholm integro-differential equation:

9. $u'(x) = xe^x + e^x - x + \frac{1}{2}\int_0^1 xu(t)dt, u(0) = 0,\ u(x) = xe^x$

10. $u'(x) = e^x + (e-1) - \int_0^1 u(t)dt,\ u(0) = 1,\ u(x) = e^x$

11. $u''(x) = 1 - \sin x - \int_0^{\frac{\pi}{2}} tu(t)dt,\ u(0) = 0,\ u'(0) = 1,\ u(x) = \sin x$

12. $u'''(x) = 1 + \sin x - \int_0^{\frac{\pi}{2}} (x-t)u(t)dt,$

    $u(0) = 1,\ u'(0) = 0,\ u''(0) = -1,\ u(x) = \cos x$

In Exercises 13–16, show that the given function $u(x)$ is a solution of the corresponding Volterra integro-differential equation:

13. $u'(x) = 2 + x + x^2 - \int_0^x u(t)dt,\ u(0) = 1,\ u(x) = 1 + 2x$

14. $u''(x) = x\cos x - 2\sin x + \int_0^x tu(t)dt,\ u(0) = 0,\ u'(0) = 1,\ u(x) = \sin x$

15. $u''(x) = 1 + \int_0^x (x-t)u(t)dt,\ u(0) = 1,\ u'(0) = 0,\ u(x) = \cosh x$

16. $u''(x) = 1 - xe^{-x} - \int_0^x tu(t)dt,\ u(0) = 1,\ u'(0) = -1,\ u(x) = e^{-x}$

In Exercises 17–24, find the unknown if the solution of each equation is given:

17. If $u(x) = e^{4x}$ is a solution of $u(x) = f(x) + 16\int_0^x (x-t)u(t)dt$, find $f(x)$

18. If $u(x) = e^{2x}$ is a solution of $u(x) = e^{2x} - \alpha(e^2+1)x + \int_0^1 xtu(t)dt$, find α

19. If $u(x) = \sin x$ is a solution of $u(x) = f(x) + \sin x - \int_0^{\frac{\pi}{2}} xu(t)dt$, find $f(x)$

20. If $u(x) = e^{-x^2}$ is a solution of $u(x) = 1 - \alpha\int_0^x tu(t)dt$, find α

21. If $u(x) = e^x$ is a solution of $u(x) = f(x) + \int_0^x (2u^2(t) + u(t))dt$, find $f(x)$

22. If $u(x) = \sin x$ is a solution of $u(x) = f(x) + \frac{4}{\pi}\int_0^x\int_0^{\frac{\pi}{2}} u^2(t)dtdt$, find $f(x)$

23. If $u(x) = 2 + 12x^2$ is a solution of $u'(x) = f(x) + 20x - \int_0^x\int_0^1 (x-t)u(t)dtdt$,

    find $f(x)$

24. If $u(x) = 6x$ is a solution of $u(x) = f(x) + \int_0^x (1-t)u(t)dt -$

    $x\int_0^1 (x-t)u(t)dtdt$, find $f(x)$

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Chapter 3
Volterra Integral Equations

3.1 Introduction

It was stated in Chapter 2 that Volterra integral equations arise in many scientific applications such as the population dynamics, spread of epidemics, and semi-conductor devices. It was also shown that Volterra integral equations can be derived from initial value problems. Volterra started working on integral equations in 1884, but his serious study began in 1896. The name integral equation was given by du Bois-Reymond in 1888. However, the name Volterra integral equation was first coined by Lalesco in 1908.

Abel considered the problem of determining the equation of a curve in a vertical plane. In this problem, the time taken by a mass point to slide under the influence of gravity along this curve, from a given positive height, to the horizontal axis is equal to a prescribed function of the height. Abel derived the singular Abel's integral equation, a specific kind of Volterra integral equation, that will be studied in a forthcoming chapter.

Volterra integral equations, of the first kind or the second kind, are characterized by a variable upper limit of integration [1]. For the **first kind** Volterra integral equations, the unknown function $u(x)$ occurs only under the integral sign in the form:

$$f(x) = \int_0^x K(x,t)u(t)dt. \tag{3.1}$$

However, Volterra integral equations of the **second kind**, the unknown function $u(x)$ occurs inside and outside the integral sign. The second kind is represented in the form:

$$u(x) = f(x) + \lambda \int_0^x K(x,t)u(t)dt. \tag{3.2}$$

The kernel $K(x,t)$ and the function $f(x)$ are given real-valued functions, and λ is a parameter [2–4].

A variety of analytic and numerical methods, such as successive approximations method, Laplace transform method, spline collocation method, Runge-Kutta method, and others have been used to handle Volterra integral equations. In this text we will apply the recently developed methods, namely, the Adomian decomposition method (ADM), the modified decomposition method (mADM), and the variational iteration method (VIM) to handle Volterra integral equations. Some of the traditional methods, namely, successive approximations method, series solution method, and the Laplace transform method will be employed as well. The emphasis in this text will be on the use of these methods and approaches rather than proving theoretical concepts of convergence and existence. The theorems of uniqueness, existence, and convergence are important and can be found in the literature. The concern will be on the determination of the solution $u(x)$ of the Volterra integral equation of first and second kind.

3.2 Volterra Integral Equations of the Second Kind

We will first study Volterra integral equations of the second kind given by

$$u(x) = f(x) + \lambda \int_0^x K(x,t)u(t)dt. \tag{3.3}$$

The unknown function $u(x)$, that will be determined, occurs inside and outside the integral sign. The kernel $K(x,t)$ and the function $f(x)$ are given real-valued functions, and λ is a parameter. In what follows we will present the methods, new and traditional, that will be used.

3.2.1 The Adomian Decomposition Method

The Adomian decomposition method (ADM) was introduced and developed by George Adomian in [5–7] and is well addressed in many references. A considerable amount of research work has been invested recently in applying this method to a wide class of linear and nonlinear ordinary differential equations, partial differential equations and integral equations as well.

The Adomian decomposition method consists of decomposing the unknown function $u(x)$ of any equation into a sum of an infinite number of components defined by the decomposition series

$$u(x) = \sum_{n=0}^{\infty} u_n(x), \tag{3.4}$$

or equivalently

$$u(x) = u_0(x) + u_1(x) + u_2(x) + \cdots, \tag{3.5}$$

where the components $u_n(x), n \geqslant 0$ are to be determined in a recursive manner. The decomposition method concerns itself with finding the components

$u_0, u_1, u_2, \ldots$ individually. As will be seen through the text, the determination of these components can be achieved in an easy way through a recurrence relation that usually involves simple integrals that can be easily evaluated.

To establish the recurrence relation, we substitute (3.4) into the Volterra integral equation (3.3) to obtain

$$\sum_{n=0}^{\infty} u_n(x) = f(x) + \lambda \int_0^x K(x,t) \left(\sum_{n=0}^{\infty} u_n(t) \right) dt, \tag{3.6}$$

or equivalently

$$u_0(x) + u_1(x) + u_2(x) + \cdots = f(x) + \lambda \int_0^x K(x,t) \left[u_0(t) + u_1(t) + \cdots \right] dt. \tag{3.7}$$

The zeroth component $u_0(x)$ is identified by all terms that are not included under the integral sign. Consequently, the components $u_j(x), j \geqslant 1$ of the unknown function $u(x)$ are completely determined by setting the recurrence relation:

$$\begin{aligned} u_0(x) &= f(x), \\ u_{n+1}(x) &= \lambda \int_0^x K(x,t) u_n(t) dt, \quad n \geqslant 0, \end{aligned} \tag{3.8}$$

that is equivalent to

$$\begin{aligned} u_0(x) &= f(x), & u_1(x) &= \lambda \int_0^x K(x,t) u_0(t) dt, \\ u_2(x) &= \lambda \int_0^x K(x,t) u_1(t) dt, & u_3(x) &= \lambda \int_0^x K(x,t) u_2(t) dt, \end{aligned} \tag{3.9}$$

and so on for other components.

In view of (3.9), the components $u_0(x), u_1(x), u_2(x), u_3(x), \ldots$ are completely determined. As a result, the solution $u(x)$ of the Volterra integral equation (3.3) in a series form is readily obtained by using the series assumption in (3.4).

It is clearly seen that the decomposition method converted the integral equation into an elegant determination of computable components. It was formally shown by many researchers that if an exact solution exists for the problem, then the obtained series converges very rapidly to that solution. The convergence concept of the decomposition series was thoroughly investigated by many researchers to confirm the rapid convergence of the resulting series. However, for concrete problems, where a closed form solution is not obtainable, a truncated number of terms is usually used for numerical purposes. The more components we use the higher accuracy we obtain.

Example 3.1

Solve the following Volterra integral equation:

$$u(x) = 1 - \int_0^x u(t) dt. \tag{3.10}$$

We notice that $f(x) = 1$, $\lambda = -1, K(x,t) = 1$. Recall that the solution $u(x)$ is assumed to have a series form given in (3.4). Substituting the decomposition series (3.4) into both sides of (3.10) gives

$$\sum_{n=0}^{\infty} u_n(x) = 1 - \int_0^x \sum_{n=0}^{\infty} u_n(t)dt, \tag{3.11}$$

or equivalently

$$u_0(x) + u_1(x) + u_2(x) + \cdots = 1 - \int_0^x [u_0(t) + u_1(t) + u_2(t) + \cdots]\, dt. \tag{3.12}$$

We identify the zeroth component by all terms that are not included under the integral sign. Therefore, we obtain the following recurrence relation:

$$\begin{aligned} u_0(x) &= 1, \\ u_{k+1}(x) &= -\int_0^x u_k(t)dt, k \geqslant 0, \end{aligned} \tag{3.13}$$

so that

$$\begin{aligned} u_0(x) &= 1, \\ u_1(x) &= -\int_0^x u_0(t)dt = -\int_0^x 1dt = -x, \\ u_2(x) &= -\int_0^x u_1(t)dt = -\int_0^x (-t)dt = \frac{1}{2!}x^2, \\ u_3(x) &= -\int_0^x u_2(t)dt = -\int_0^x \frac{1}{2!}t^2 dt = -\frac{1}{3!}x^3, \\ u_4(x) &= -\int_0^x u_3(t)dt = -\int_0^x -\frac{1}{3!}t^3 dt = -\frac{1}{4!}x^4, \end{aligned} \tag{3.14}$$

and so on. Using (3.4) gives the series solution:

$$u(x) = 1 - x + \frac{1}{2!}x^2 - \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots, \tag{3.15}$$

that converges to the closed form solution:

$$u(x) = e^{-x}. \tag{3.16}$$

Example 3.2

Solve the following Volterra integral equation:

$$u(x) = 1 + \int_0^x (t-x)u(t)dt. \tag{3.17}$$

We notice that $f(x) = 1$, $\lambda = 1, K(x,t) = t - x$. Substituting the decomposition series (3.4) into both sides of (3.17) gives

$$\sum_{n=0}^{\infty} u_n(x) = 1 + \int_0^x \sum_{n=0}^{\infty} (t-x)u_n(t)dt, \tag{3.18}$$

or equivalently

$$u_0(x)+u_1(x)+u_2(x)+\cdots=1+\int_0^x (t-x)\left[u_0(t)+u_1(t)+u_2(t)+\cdots\right]dt. \tag{3.19}$$

Proceeding as before we set the following recurrence relation:

$$\begin{aligned} u_0(x) &= 1,\\ u_{k+1}(x) &= \int_0^x (t-x)u_k(t)dt, k\geqslant 0, \end{aligned} \tag{3.20}$$

that gives

$$\begin{aligned} u_0(x) &= 1,\\ u_1(x) &= \int_0^x (t-x)u_0(t)dt=\int_0^x (t-x)dt=-\frac{1}{2!}x^2,\\ u_2(x) &= \int_0^x (t-x)u_1(t)dt=-\frac{1}{2!}\int_0^x (t-x)t^2dt=\frac{1}{4!}x^4,\\ u_3(x) &= \int_0^x (t-x)u_2(t)dt=\frac{1}{4!}\int_0^x (t-x)t^4dt=-\frac{1}{6!}x^6,\\ u_4(x) &= \int_0^x (t-x)u_3(t)dt=-\frac{1}{6!}\int_0^x (t-x)t^6dt=\frac{1}{8!}x^8, \end{aligned} \tag{3.21}$$

and so on. The solution in a series form is given by

$$u(x)=1-\frac{1}{2!}x^2+\frac{1}{4!}x^4-\frac{1}{6!}x^6+\frac{1}{8!}x^8+\cdots, \tag{3.22}$$

and in a closed form by

$$u(x)=\cos x, \tag{3.23}$$

obtained upon using the Taylor expansion for $\cos x$.

Example 3.3

Solve the following Volterra integral equation:

$$u(x)=1-x-\frac{1}{2}x^2-\int_0^x (t-x)u(t)dt. \tag{3.24}$$

Notice that $f(x)=1-x-\frac{1}{2}x^2, \lambda=-1, K(x,t)=t-x$. Substituting the decomposition series (3.4) into both sides of (3.24) gives

$$\sum_{n=0}^{\infty} u_n(x)=1-x-\frac{1}{2}x^2-\int_0^x \sum_{n=0}^{\infty}(t-x)u_n(t)dt, \tag{3.25}$$

or equivalently

$$u_0(x)+u_1(x)+u_2(x)+\cdots=1-x-\frac{1}{2}x^2-\int_0^x (t-x)\left[u_0(t)+u_1(t)+\cdots\right]dt. \tag{3.26}$$

This allows us to set the following recurrence relation:

$$\begin{aligned} u_0(x) &= 1-x-\frac{1}{2}x^2,\\ u_{k+1}(x) &= -\int_0^x (t-x)u_k(t)dt, k\geqslant 0, \end{aligned} \tag{3.27}$$

that gives

$$\begin{aligned} u_0(x) &= 1 - x - \frac{1}{2}x^2, \\ u_1(x) &= -\int_0^x (t-x)u_0(t)dt = \frac{1}{2!}x^2 - \frac{1}{3!}x^3 - \frac{1}{4!}x^4, \\ u_2(x) &= -\int_0^x (t-x)u_1(t)dt = \frac{1}{4!}x^4 - \frac{1}{5!}x^5 - \frac{1}{6!}x^6, \\ u_3(x) &= -\int_0^x (t-x)u_2(t)dt = \frac{1}{6!}x^6 - \frac{1}{7!}x^7 - \frac{1}{8!}x^8, \end{aligned} \tag{3.28}$$

and so on. The solution in a series form is given by

$$u(x) = 1 - (x + \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \frac{1}{7!}x^7 + \cdots), \tag{3.29}$$

and in a closed form by

$$u(x) = 1 - \sinh x, \tag{3.30}$$

obtained upon using the Taylor expansion for $\sinh x$.

Example 3.4

We consider here the Volterra integral equation:

$$u(x) = 5x^3 - x^5 + \int_0^x tu(t)dt. \tag{3.31}$$

Identifying the zeroth component $u_0(x)$ by the first two terms that are not included under the integral sign, and using the ADM we set the recurrence relation as

$$\begin{aligned} u_0(x) &= 5x^3 - x^5, \\ u_{k+1}(x) &= \int_0^x tu_k(t)dt, k \geqslant 0. \end{aligned} \tag{3.32}$$

This in turn gives

$$\begin{aligned} u_0(x) &= 5x^3 - x^5, \\ u_1(x) &= \int_0^x tu_0(t)dt = x^5 - \frac{1}{7}x^7, \\ u_2(x) &= \int_0^x tu_1(t)dt = \frac{1}{7}x^7 - \frac{1}{63}x^9, \\ u_3(x) &= \int_0^x tu_1(t)dt = \frac{1}{63}x^9 - \frac{1}{693}x^{11}, \end{aligned} \tag{3.33}$$

The solution in a series form is given by

$$u(x) = \left(5x^3 - x^5\right) + \left(x^5 - \frac{1}{7}x^7\right) + \left(\frac{1}{7}x^7 - \frac{1}{63}x^9\right) + (\frac{1}{63}x^9 - \frac{1}{693}x^{11}) + \cdots. \tag{3.34}$$

We can easily notice the appearance of identical terms with opposite signs. Such terms are called **noise terms** that will be discussed later. Canceling the identical terms with opposite signs gives the exact solution

$$u(x) = 5x^3, \tag{3.35}$$

that satisfies the Volterra integral equation (3.31).

Example 3.5

We now consider the Volterra integral equation:

$$u(x) = x + x^4 + \frac{1}{2}x^2 + \frac{1}{5}x^5 - \int_0^x u(t)dt. \tag{3.36}$$

Identifying the zeroth component $u_0(x)$ by the first four terms that are not included under the integral sign, and using the ADM we set the recurrence relation as

$$\begin{aligned} u_0(x) &= x + x^4 + \frac{1}{2}x^2 + \frac{1}{5}x^5, \\ u_{k+1}(x) &= -\int_0^x u_k(t)dt, k \geqslant 0. \end{aligned} \tag{3.37}$$

This in turn gives

$$\begin{aligned} u_0(x) &= x + x^4 + \frac{1}{2}x^2 + \frac{1}{5}x^5, \\ u_1(x) &= -\int_0^x u_0(t)dt = -\frac{1}{2}x^2 - \frac{1}{5}x^5 - \frac{1}{6}x^3 - \frac{1}{30}x^6, \\ u_2(x) &= -\int_0^x u_1(t)dt = \frac{1}{6}x^3 + \frac{1}{30}x^6 + \frac{1}{24}x^4 + \frac{1}{210}x^7, \\ u_3(x) &= -\int_0^x u_2(t)dt = -\frac{1}{24}x^4 - \frac{1}{210}x^7 + \cdots. \end{aligned} \tag{3.38}$$

The solution in a series form is given by

$$\begin{aligned} u(x) = &\left(x + x^4 + \frac{1}{2}x^2 + \frac{1}{5}x^5\right) - \left(\frac{1}{2}x^2 + \frac{1}{5}x^5 + \frac{1}{6}x^3 + \frac{1}{30}x^6\right) \\ &+ \left(\frac{1}{6}x^3 + \frac{1}{30}x^6 + \frac{1}{24}x^4 + \frac{1}{210}x^7\right) - \left(\frac{1}{24}x^4 + \frac{1}{210}x^7 + \cdots\right) + \cdots. \end{aligned} \tag{3.39}$$

We can easily notice the appearance of identical terms with opposite signs. This phenomenon of such terms is called *noise terms* phenomenon that will be presented later. Canceling the identical terms with opposite terms gives the exact solution

$$u(x) = x + x^4. \tag{3.40}$$

Example 3.6

We finally solve the Volterra integral equation:

$$u(x) = 2 + \frac{1}{3}\int_0^x xt^3 u(t)dt. \tag{3.41}$$

Proceeding as before we set the recurrence relation

$$u_0(x) = 2, \quad u_{k+1}(x) = \frac{1}{3}\int_0^x xt^3 u_k(t)dt, \quad k \geqslant 0. \tag{3.42}$$

This in turn gives

$$\begin{aligned} u_0(x) &= 2, \\ u_1(x) &= \frac{1}{3}\int_0^x xt^3 u_0(t)dt = \frac{1}{6}x^5, \\ u_2(x) &= \frac{1}{3}\int_0^x xt^3 u_1(t)dt = \frac{1}{162}x^{10}, \\ u_3(x) &= \frac{1}{3}\int_0^x xt^3 u_2(t)dt = \frac{1}{6804}x^{15}, \\ u_4(x) &= \frac{1}{3}\int_0^x xt^3 u_3(t)dt = \frac{1}{387828}x^{20}, \end{aligned} \tag{3.43}$$

and so on. The solution in a series form is given by

$$u(x) = 2 + \frac{1}{6}x^5 + \frac{1}{6\cdot 3^3}x^{10} + \frac{1}{6\cdot 3^4\cdot 14}x^{15} + \frac{1}{6\cdot 3^5\cdot 14\cdot 19}x^{20} + \cdots. \tag{3.44}$$

It seems that an exact solution is not obtainable. The obtained series solution can be used for numerical purposes. The more components that we determine the higher accuracy level that we can achieve.

Exercises 3.2.1

In Exercises 1–26, solve the following Volterra integral equations by using the *Adomian decomposition method*:

1. $u(x) = 6x - 3x^2 + \int_0^x u(t)dt$
2. $u(x) = 6x - x^3 + \int_0^x (x-t)u(t)dt$
3. $u(x) = 1 - \frac{1}{2}x^2 + \int_0^x u(t)dt$
4. $u(x) = x - \frac{2}{3}x^3 - 2\int_0^x u(t)dt$
5. $u(x) = 1 + x + \int_0^x (x-t)u(t)dt$
6. $u(x) = 1 - x + \int_0^x (x-t)u(t)dt$
7. $u(x) = 1 + x - \int_0^x (x-t)u(t)dt$
8. $u(x) = 1 - x - \int_0^x (x-t)u(t)dt$
9. $u(x) = 1 - \int_0^x (x-t)u(t)dt$
10. $u(x) = 1 + \int_0^x (x-t)u(t)dt$
11. $u(x) = x - \int_0^x (x-t)u(t)dt$
12. $u(x) = x + \int_0^x (x-t)u(t)dt$
13. $u(x) = 1 + \int_0^x u(t)dt$
14. $u(x) = 1 - \int_0^x u(t)dt$
15. $u(x) = 1 + 2\int_0^x tu(t)dt$
16. $u(x) = 1 - 2\int_0^x tu(t)dt$
17. $u(x) = 1 - x^2 - \int_0^x (x-t)u(t)dt$
18. $u(x) = -2 + 3x - x^2 - \int_0^x (x-t)u(t)dt$

19. $u(x) = x^2 + \int_0^x (x-t)u(t)dt$

20. $u(x) = -2 + 2x + x^2 + \int_0^x (x-t)u(t)dt$

21. $u(x) = 1 + 2x + 4\int_0^x (x-t)u(t)dt$

22. $u(x) = 5 + 2x^2 - \int_0^x (x-t)u(t)dt$

23. $u(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{2}\int_0^x (x-t)^2u(t)dt$

24. $u(x) = 1 - \frac{1}{2}x^2 + \frac{1}{6}\int_0^x (x-t)^3u(t)dt$

25. $u(x) = 1 + \frac{1}{2}x + \frac{1}{2}\int_0^x (x-t+1)u(t)dt$

26. $u(x) = 1 + x^2 - \int_0^x (x-t+1)^2u(t)dt$

In Exercises 27–30, use the *Adomian decomposition method* to find the series solution

27. $u(x) = 3 + \frac{1}{4}\int_0^x xt^2u(t)dt$

28. $u(x) = 3 + \frac{1}{4}\int_0^x (x+t^2)u(t)dt$

29. $u(x) = 1 + \frac{1}{2}\int_0^x (x^2-t^2)u(t)dt$

30. $u(x) = 1 + \frac{1}{2}\int_0^x x^2u(t)dt$

3.2.2 The Modified Decomposition Method

As shown before, the Adomian decomposition method provides the solution in an infinite series of components. The components $u_j, j \geqslant 0$ are easily computed if the inhomogeneous term $f(x)$ in the Volterra integral equation:

$$u(x) = f(x) + \lambda\int_0^x K(x,t)u(t)dt \tag{3.45}$$

consists of a polynomial. However, if the function $f(x)$ consists of a combination of two or more of polynomials, trigonometric functions, hyperbolic functions, and others, the evaluation of the components $u_j, j \geqslant 0$ requires cumbersome work. A reliable modification of the Adomian decomposition method was developed by Wazwaz and presented in [7–9]. The modified decomposition method will facilitate the computational process and further accelerate the convergence of the series solution. The modified decomposition method will be applied, wherever it is appropriate, to all integral equations and differential equations of any order. It is interesting to note that the modified decomposition method depends mainly on splitting the function $f(x)$ into two parts, therefore it cannot be used if the function $f(x)$ consists of only one term. The modified decomposition method will be outlined and employed in this section and in other chapters as well.

To give a clear description of the technique, we recall that the standard Adomian decomposition method admits the use of the recurrence relation:

$$\begin{aligned} u_0(x) &= f(x), \\ u_{k+1}(x) &= \lambda \int_0^x K(x,t)u_k(t)dt, k \geqslant 0, \end{aligned} \tag{3.46}$$

where the solution $u(x)$ is expressed by an infinite sum of components defined before by

$$u(x) = \sum_{n=0}^{\infty} u_n(x). \tag{3.47}$$

In view of (3.46), the components $u_n(x)$, $n \geqslant 0$ can be easily evaluated.

The modified decomposition method [7–9] introduces a slight variation to the recurrence relation (3.46) that will lead to the determination of the components of $u(x)$ in an easier and faster manner. For many cases, the function $f(x)$ can be set as the sum of two partial functions, namely $f_1(x)$ and $f_2(x)$. In other words, we can set

$$f(x) = f_1(x) + f_2(x). \tag{3.48}$$

In view of (3.48), we introduce a qualitative change in the formation of the recurrence relation (3.46). To minimize the size of calculations, we identify the zeroth component $u_0(x)$ by one part of $f(x)$, namely $f_1(x)$ or $f_2(x)$. The other part of $f(x)$ can be added to the component $u_1(x)$ among other terms. In other words, the modified decomposition method introduces the modified recurrence relation:

$$\begin{aligned} u_0(x) &= f_1(x), \\ u_1(x) &= f_2(x) + \lambda \int_0^x K(x,t)u_0(t)dt, \\ u_{k+1}(x) &= \lambda \int_0^x K(x,t)u_k(t)dt, \quad k \geqslant 1. \end{aligned} \tag{3.49}$$

This shows that the difference between the standard recurrence relation (3.46) and the modified recurrence relation (3.49) rests only in the formation of the first two components $u_0(x)$ and $u_1(x)$ only. The other components $u_j, j \geqslant 2$ remain the same in the two recurrence relations. Although this variation in the formation of $u_0(x)$ and $u_1(x)$ is slight, however it plays a major role in accelerating the convergence of the solution and in minimizing the size of computational work. Moreover, reducing the number of terms in $f_1(x)$ affects not only the component $u_1(x)$, but also the other components as well. This result was confirmed by several research works.

Two important remarks related to the modified method [7–9] can be made here. First, by proper selection of the functions $f_1(x)$ and $f_2(x)$, the exact solution $u(x)$ may be obtained by using very few iterations, and sometimes by evaluating only two components. The success of this modification depends only on the proper choice of $f_1(x)$ and $f_2(x)$, and this can be made through trials only. A rule that may help for the proper choice of $f_1(x)$ and $f_2(x)$ could not be found yet. Second, if $f(x)$ consists of one term only, the standard decomposition method can be used in this case.

It is worth mentioning that the modified decomposition method will be used for Volterra and Fredholm integral equations, linear and nonlinear equations. The modified decomposition method will be illustrated by discussing the following examples.

Example 3.7

Solve the Volterra integral equation by using the modified decomposition method:

$$u(x) = \sin x + (e - e^{\cos x}) - \int_0^x e^{\cos t} u(t)dt. \tag{3.50}$$

We first split $f(x)$ given by

$$f(x) = \sin x + (e - e^{\cos x}), \tag{3.51}$$

into two parts, namely

$$\begin{aligned} f_1(x) &= \sin x, \\ f_2(x) &= e - e^{\cos x}. \end{aligned} \tag{3.52}$$

We next use the modified recurrence formula (3.49) to obtain

$$\begin{aligned} u_0(x) &= f_1(x) = \sin x, \\ u_1(x) &= (e - e^{\cos x}) - \int_0^x e^{\cos t} u_0(t)dt = 0, \\ u_{k+1}(x) &= -\int_0^x K(x,t)u_k(t)dt = 0, \quad k \geqslant 1. \end{aligned} \tag{3.53}$$

It is obvious that each component of $u_j, j \geqslant 1$ is zero. This in turn gives the exact solution by

$$u(x) = \sin x. \tag{3.54}$$

Example 3.8

Solve the Volterra integral equation by using the modified decomposition method:

$$u(x) = \sec x \tan x + (e^{\sec x} - e) - \int_0^x e^{\sec t} u(t)dt, x < \frac{\pi}{2}. \tag{3.55}$$

Proceeding as before we split $f(x)$ into two parts

$$f_1(x) = \sec x \tan x, \quad f_2(x) = e^{\sec x} - e. \tag{3.56}$$

We next use the modified recurrence formula (3.49) to obtain

$$\begin{aligned} u_0(x) &= f_1(x) = \sec x \tan x, \\ u_1(x) &= (e^{\sec x} - e) - \int_0^x e^{\sec t} u_0(t)dt = 0, \\ u_{k+1}(x) &= -\int_0^x K(x,t)u_k(t)dt = 0, \quad k \geqslant 1. \end{aligned} \tag{3.57}$$

It is obvious that each component of $u_j, j \geqslant 1$ is zero. This in turn gives the exact solution by

$$u(x) = \sec x \tan x. \tag{3.58}$$

Example 3.9

Solve the Volterra integral equation by using the modified decomposition method:

$$u(x) = 2x + \sin x + x^2 - \cos x + 1 - \int_0^x u(t)dt. \tag{3.59}$$

The function $f(x)$ consists of five terms. By trial we divide $f(x)$ given by

$$f(x) = 2x + \sin x + x^2 - \cos x + 1, \tag{3.60}$$

into two parts, first two terms and the next three terms to find

$$\begin{aligned} f_1(x) &= 2x + \sin x, \\ f_2(x) &= x^2 - \cos x + 1. \end{aligned} \tag{3.61}$$

We next use the modified recurrence formula (3.49) to obtain

$$\begin{aligned} u_0(x) &= 2x + \sin x, \\ u_1(x) &= x^2 - \cos x + 1 - \int_0^x u_0(t)dt = 0, \\ u_{k+1}(x) &= -\int_0^x K(x,t)u_k(t)dt = 0, \quad k \geqslant 1. \end{aligned} \tag{3.62}$$

It is obvious that each component of $u_j, j \geqslant 1$ is zero. The exact solution is given by

$$u(x) = 2x + \sin x. \tag{3.63}$$

Example 3.10

Solve the Volterra integral equation by using the modified decomposition method:

$$u(x) = 1 + x^2 + \cos x - x - \frac{1}{3}x^3 - \sin x + \int_0^x u(t)dt. \tag{3.64}$$

The function $f(x)$ consists of six terms. By trial we split $f(x)$ given by

$$f(x) = 1 + x^2 + \cos x - x - \frac{1}{3}x^3 - \sin x, \tag{3.65}$$

into two parts, the first three terms and the next three terms, hence we set

$$\begin{aligned} f_1(x) &= 1 + x^2 + \cos x, \\ f_2(x) &= -(x + \frac{1}{3}x^3 + \sin x). \end{aligned} \tag{3.66}$$

Using the modified recurrence formula (3.49) gives

$$\begin{aligned} u_0(x) &= 1 + x^2 + \cos x, \\ u_1(x) &= -(x + \frac{1}{3}x^3 + \sin x) + \int_0^x u_0(t)dt = 0, \\ u_{k+1}(x) &= \int_0^x K(x,t)u_k(t)dt = 0, \quad k \geqslant 1. \end{aligned} \tag{3.67}$$

As a result, the exact solution is given by

$$u(x) = 1 + x^2 + \cos x. \tag{3.68}$$

Exercises 3.2.2

Use the *modified decomposition method* to solve the following Volterra integral equations:

1. $u(x) = \cos x + \sin x - \int_0^x u(t)dt$

2. $u(x) = \sinh x + \cosh x - 1 - \int_0^x u(t)dt$

3. $u(x) = 2x = 3x^2 + (e^{x^2+x^3} - 1) - \int_0^x e^{t^2+t^3} u(t)dt$

4. $u(x) = 3x^2 + (1 - e^{-x^3}) - \int_0^x e^{-x^3+t^3} u(t)dt$

5. $u(x) = 2x - (1 - e^{-x^2}) + \int_0^x e^{-x^2+t^2} u(t)dt$

6. $u(x) = e^{-x^2} - \frac{x}{2}(1 - e^{-x^2}) - \int_0^x xtu(t)dt$

7. $u(x) = \cosh x + x \sinh x - \int_0^x xu(t)dt$

8. $u(x) = e^x + xe^x - x - \int_0^x xu(t)dt$

9. $u(x) = 1 + \sin x + x + x^2 - x \cos x - \int_0^x xu(t)dt$

10. $u(x) = e^x - xe^x + \sin x + x \cos x + \int_0^x xu(t)dt$

11. $u(x) = 1 + x + x^2 + \frac{1}{2}x^3 + \cosh x + x \sinh x - \int_0^x xu(t)dt$

12. $u(x) = \cos x - \left(1 - e^{\sin x}\right) x - x \int_0^x e^{\sin t} u(t)dt$

13. $u(x) = \sec x^2 - \left(1 - e^{\tan x}\right) x - x \int_0^x e^{\tan t} u(t)dt$

14. $u(x) = \cosh x + \frac{x}{2} \left(1 - e^{\sinh x}\right) + \frac{x}{2} \int_0^x e^{\sinh t} u(t)dt$

15. $u(x) = \sinh x + \frac{1}{10} \left(e - e^{\cosh x}\right) + \frac{1}{10} \int_0^x e^{\cosh t} u(t)dt$

16. $u(x) = x^3 - x^5 + 5\frac{1}{10} \int_0^x tu(t)dt$

3.2.3 The Noise Terms Phenomenon

It was shown before that the modified decomposition method presents a reliable tool for accelerating the computational work. However, a proper selection of $f_1(x)$ and $f_2(x)$ is essential for a successful use of this technique.

A useful tool that will accelerate the convergence of the Adomian decomposition method is developed. The new technique depends mainly on the so-called *noise terms phenomenon* that demonstrates a fast convergence of the solution. The noise terms phenomenon can be used for all differential and integral equations. The noise terms, if existed between the components $u_0(x)$ and $u_1(x)$, will provide the exact solution by using only the first two iterations.

In what follows, we outline the main concepts of the noise terms :

1. The *noise terms* are defined as the identical terms with opposite signs that arise in the components $u_0(x)$ and $u_1(x)$. Other noise terms may appear between other components. As stated above, these identical terms with opposite signs may exist for some equations, and may not appear for other equations.

2. By canceling the noise terms between $u_0(x)$ and $u_1(x)$, even though $u_1(x)$ contains further terms, the remaining non-canceled terms of $u_0(x)$ may give the exact solution of the integral equation. The appearance of the noise terms between $u_0(x)$ and $u_1(x)$ is not always sufficient to obtain the exact solution by canceling these noise terms. Therefore, it is necessary to show that the non-canceled terms of $u_0(x)$ satisfy the given integral equation.

On the other hand, if the non-canceled terms of $u_0(x)$ did not satisfy the given integral equation, or the noise terms did not appear between $u_0(x)$ and $u_1(x)$, then it is necessary to determine more components of $u(x)$ to determine the solution in a series form as presented before.

3.It was formally shown that the noise terms appear for specific cases of inhomogeneous differential and integral equations, whereas homogeneous equations do not give rise to noise terms. The conclusion about the self-canceling noise terms was based on solving several specific differential and integral models. However, a proof for this conclusion was not given. For further readings about the noise terms phenomenon, see [7,10].

4. It was formally proved in [7,10] that the appearance of the noise terms is governed by a necessary condition. The conclusion made in [7,10] is that the zeroth component $u_0(x)$ must contain the exact solution $u(x)$ among other terms. In addition, it was proved that the inhomogeneity condition of the equation does not always guarantee the appearance of the noise terms as examined in [10].

A useful summary about the noise terms phenomenon can be drawn as follows:

1. The noise terms are defined as the identical terms with opposite signs that may appear in the components $u_0(x)$ and $u_1(x)$ and in the other components as well.

2. The noise terms appear only for specific types of inhomogeneous equations whereas noise terms do not appear for homogeneous equations.

3. Noise terms may appear if the exact solution of the equation is part of the zeroth component $u_0(x)$.

4. Verification that the remaining non-canceled terms satisfy the integral equation is necessary and essential.

The phenomenon of the useful noise terms will be explained by the following illustrative examples.

Example 3.11

Solve the Volterra integral equation by using noise terms phenomenon:

$$u(x) = 8x + x^3 - \frac{3}{8}\int_0^x tu(t)dt. \tag{3.69}$$

Following the standard Adomian method we set the recurrence relation:

$$\begin{aligned} u_0(x) &= 8x + x^3, \\ u_{k+1}(x) &= -\frac{3}{8}\int_0^x tu_k(t)dt, \quad k \geqslant 0. \end{aligned} \tag{3.70}$$

This gives

$$\begin{aligned} u_0(x) &= 8x + x^3, \\ u_1(x) &= -\frac{3}{8}\int_0^x tu_0(t)dt = -\frac{3}{40}x^5 - x^3. \end{aligned} \tag{3.71}$$

The noise terms $\pm x^3$ appear in $u_0(x)$ and $u_1(x)$. Canceling this term from the zeroth component $u_0(x)$ gives the exact solution:

$$u(x) = 8x, \tag{3.72}$$

that satisfies the integral equation. Notice that if the modified method is used, we select $u_0(x) = 8x$. As a result, we find that $u_1(x) = 0$. This in turn gives the same result.

Example 3.12

Solve the Volterra integral equation by using noise terms phenomenon:

$$u(x) = -2 + x + x^2 + \frac{1}{12}x^4 + \sin x + 2\cos x - \int_0^x (x-t)^2 u(t)dt. \tag{3.73}$$

Following the standard Adomian method we set the recurrence relation:

$$\begin{aligned} u_0(x) &= -2 + x + x^2 + \frac{1}{12}x^4 + \sin x + 2\cos x, \\ u_{k+1}(x) &= -\int_0^x (x-t)^2 u_k(t)dt, \quad k \geqslant 0. \end{aligned} \tag{3.74}$$

This gives

$$\begin{aligned} u_0(x) &= -2 + x + x^2 + \frac{1}{12}x^4 + \sin x + 2\cos x, \\ u_1(x) &= -\frac{3}{8}\int_0^x tu_0(t)dt \\ &= 2 - x^2 - \frac{1}{12}x^4 - 2\cos x + 4\sin x - \frac{1}{30}x^5 + \frac{2}{3}x^3 - \frac{1}{1260}x^7 - 4x. \end{aligned} \tag{3.75}$$

The noise terms $\mp 2, \pm x^2, \pm\frac{1}{12}x^4$ and $\pm 2\cos x$ appear in $u_0(x)$ and $u_1(x)$. Canceling these terms from the zeroth component $u_0(x)$ gives the exact solution

$$u(x) = x + \sin x, \tag{3.76}$$

that satisfies the integral equation. It is to be noted that the other terms of $u_1(x)$ vanish in the limit with other terms of the other components.

Example 3.13

Solve the Volterra integral equation by using noise terms phenomenon

$$u(x) = \frac{1}{2}x - \frac{1}{4}\sinh(2x) + \sinh^2 x + \int_0^x u(t)dt. \tag{3.77}$$

We set the recurrence relation:

$$\begin{aligned} u_0(x) &= \frac{1}{2}x - \frac{1}{4}\sinh(2x) + \sinh^2 x, \\ u_{k+1}(x) &= \int_0^x u_k(t)dt, \quad k \geqslant 0. \end{aligned} \tag{3.78}$$

This gives

$$\begin{aligned} u_0(x) &= \frac{1}{2}x - \frac{1}{4}\sinh(2x) + \sinh^2 x, \\ u_1(x) &= \int_0^x u_0(t)dt = -\frac{1}{2}x + \frac{1}{4}\sinh(2x) + \frac{1}{4} + \frac{1}{4}x^2 - \frac{1}{4}\cosh^2 x. \end{aligned} \tag{3.79}$$

The noise terms $\pm\frac{1}{2}x$ and $\mp\frac{1}{4}\sinh(2x)$ appear in $u_0(x)$ and $u_1(x)$. Canceling these terms from the zeroth component $u_0(x)$ gives the exact solution:

$$u(x) = \sinh^2 x, \tag{3.80}$$

that satisfies the integral equation.

Example 3.14

Show that the exact solution for the Volterra integral equation:

$$u(x) = -1 + x + \frac{1}{2}x^2 + 2e^x - \int_0^x u(t)dt, \tag{3.81}$$

cannot be obtained by using the noise terms phenomenon.

We set the recurrence relation:

$$\begin{aligned} u_0(x) &= -1 + x + \frac{1}{2}x^2 + 2e^x, \\ u_{k+1}(x) &= -\int_0^x u_k(t)dt, \quad k \geqslant 0. \end{aligned} \tag{3.82}$$

This gives

$$u_0(x) = -1 + x + \frac{1}{2}x^2 + 2e^x,$$
$$u_1(x) = -\int_0^x u_0(t)dt = -\frac{1}{2}x^2 - 2e^x + 2 + x - \frac{1}{6}x^3. \quad (3.83)$$

The noise terms $\pm\frac{1}{2}x^2$ and $\pm 2e^x$ appear in $u_0(x)$ and $u_1(x)$. Canceling these terms from the zeroth component $u_0(x)$ gives

$$\tilde{u}(x) = x - 1, \quad (3.84)$$

that does not satisfy the integral equation. This confirms our belief that the non-canceled terms of $u_0(x)$ do not always give the exact solution, and therefore justification is necessary. The exact solution is given by

$$u(x) = x + e^x, \quad (3.85)$$

that can be easily obtained by using the modified decomposition method by setting $f_1(x) = x + e^x$.

Exercises 3.2.3

Use the *noise terms phenomenon* to solve the following Volterra integral equations:

1. $u(x) = 6x + 3x^2 - \int_0^x u(t)dt$
2. $u(x) = 6x + 3x^2 - \int_0^x xu(t)dt$
3. $u(x) = 6x + 2x^3 - \int_0^x tu(t)dt$
4. $u(x) = x + x^2 - 2x^3 - x^4 + 12\int_0^x (x-t)u(t)dt$
5. $u(x) = -2 + x^2 + \sin x + 2\cos x - \int_0^x (x-t)^2 u(t)dt$
6. $u(x) = 2x - 2\sin x + \cos x - \int_0^x (x-t)^2 u(t)dt$
7. $u(x) = \sinh x + x\sinh x - x^2\cosh x + \int_0^x xtu(t)dt$
8. $u(x) = x + \cosh x + x^2\sinh x - x\cosh x - \int_0^x xtu(t)dt$
9. $u(x) = \sec^2 x - \tan x + \int_0^x u(t)dt$
10. $u(x) = -\frac{1}{2}x - \frac{1}{4}\sin(2x) + \cos^2 x + \int_0^x u(t)dt$
11. $u(x) = -\frac{1}{2}x + \frac{1}{4}\sin(2x) + \sin^2 x + \int_0^x u(t)dt$
12. $u(x) = -x + \tan x + \tan^2 x - \int_0^x u(t)dt$

3.2.4 The Variational Iteration Method

In this section we will study the newly developed *variational iteration method* that proved to be effective and reliable for analytic and numerical purposes. The variational iteration method (VIM) established by Ji-Huan He [11–12] is now used to handle a wide variety of linear and nonlinear, homogeneous and inhomogeneous equations. The method provides rapidly convergent successive approximations of the exact solution if such a closed form solution exists, and not components as in Adomian decomposition method. The variational iteration method handles linear and nonlinear problems in the same manner without any need to specific restrictions such as the so called Adomian polynomials that we need for nonlinear problems. Moreover, the method gives the solution in a series form that converges to the closed form solution if an exact solution exists. The obtained series can be employed for numerical purposes if exact solution is not obtainable. In what follows, we present the main steps of the method.

Consider the differential equation:

$$Lu + Nu = g(t), \tag{3.86}$$

where L and N are linear and nonlinear operators respectively, and $g(t)$ is the source inhomogeneous term.

The variational iteration method presents a correction functional for equation (3.86) in the form:

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(\xi)\left(Lu_n(\xi) + N\,\tilde{u}_n(\xi) - g(\xi)\right) d\xi, \tag{3.87}$$

where λ is a general Lagrange's multiplier, noting that in this method λ may be a constant or a function, and $\tilde{u}_n$ is a restricted value that means it behaves as a constant, hence $\delta\tilde{u}_n = 0$, where δ is the variational derivative. The Lagrange multiplier λ can be identified optimally via the variational theory as will be seen later.

For a complete use of the variational iteration method, we should follow two steps, namely:

1. the determination of the Lagrange multiplier $\lambda(\xi)$ that will be identified optimally, and
2. with λ determined, we substitute the result into (3.87) where the restrictions should be omitted.

Taking the variation of (3.87) with respect to the independent variable u_n we find

$$\frac{\delta u_{n+1}}{\delta u_n} = 1 + \frac{\delta}{\delta u_n}\left(\int_0^x \lambda(\xi)\left(Lu_n(\xi) + N\tilde{u}_n(\xi) - g(\xi)\right) d\xi\right), \tag{3.88}$$

or equivalently

$$\delta u_{n+1} = \delta u_n + \delta\left(\int_0^x \lambda(\xi)\left(Lu_n(\xi)\right) d\xi\right). \tag{3.89}$$

Integration by parts is usually used for the determination of the Lagrange multiplier $\lambda(\xi)$. In other words we can use

$$\int_0^x \lambda(\xi)u_n'(\xi)d\xi = \lambda(\xi)u_n(\xi) - \int_0^x \lambda'(\xi)u_n(\xi)d\xi,$$

$$\int_0^x \lambda(\xi)u_n''(\xi)d\xi = \lambda(\xi)u_n'(\xi) - \lambda'(\xi)u_n(\xi) + \int_0^x \lambda''(\xi)u_n(\xi)d\xi,$$

$$\begin{aligned}\int_0^x \lambda(\xi)u_n'''(\xi)d\xi &= \lambda(\xi)u_n''(\xi) - \lambda'(\xi)u_n'(\xi) + \lambda''(\xi)u_n(\xi)\\ &\quad - \int_0^x \lambda'''(\xi)u_n(\xi)d\xi,\end{aligned} \tag{3.90}$$

$$\begin{aligned}\int_0^x \lambda(\xi)u_n^{(\text{iv})}(\xi)d\xi &= \lambda(\xi)u_n'''(\xi) - \lambda'(\xi)u_n''(\xi) + \lambda''(\xi)u_n'(\xi) - \lambda'''u_n(\xi)\\ &\quad + \int_0^x \lambda^{(\text{iv})}(\xi)u_n(\xi)d\xi,\end{aligned}$$

and so on. These identities are obtained by integrating by parts.

For example, if $Lu_n(\xi) = u_n'(\xi)$ in (3.89), then (3.89) becomes

$$\delta u_{n+1} = \delta u_n + \delta\left(\int_0^x \lambda(\xi)\left(Lu_n(\xi)\right)d\xi\right). \tag{3.91}$$

Integrating the integral of (3.91) by parts using (3.90) we obtain

$$\delta u_{n+1} = \delta u_n + \delta\lambda(\xi)u_n(\xi) - \int_0^x \lambda'(\xi)\delta u_n(\xi)d\xi, \tag{3.92}$$

or equivalently

$$\delta u_{n+1} = \delta u_n(\xi)(1 + \lambda|_{\xi=x}) - \int_0^x \lambda'\delta u_n d\xi. \tag{3.93}$$

The extremum condition of u_{n+1} requires that $\delta u_{n+1} = 0$. This means that the left hand side of (3.93) is zero, and as a result the right hand side should be 0 as well. This yields the stationary conditions:

$$1 + \lambda|_{\xi=x} = 0, \quad \lambda'|_{\xi=x} = 0. \tag{3.94}$$

This in turn gives

$$\lambda = -1. \tag{3.95}$$

As a second example, if $Lu_n(\xi) = u_n''(\xi)$ in (3.89), then (3.89) becomes

$$\delta u_{n+1} = \delta u_n + \delta\left(\int_0^x \lambda(\xi)\left(Lu_n(\xi)\right)d\xi\right). \tag{3.96}$$

Integrating the integral of (3.96) by parts using (3.90) we obtain

$$\delta u_{n+1} = \delta u_n + \delta\lambda((u_n)')_0^x - (\lambda'\delta u_n)_0^x + \int_0^x \lambda''\delta u_n d\xi, \tag{3.97}$$

or equivalently

$$\delta u_{n+1} = \delta u_n(\xi)(1 - \lambda'|_{\xi=x}) + \delta\lambda((u_n)'|_{\xi=x}) + \int_0^x \lambda''\delta u_n d\xi, \tag{3.98}$$

The extremum condition of u_{n+1} requires that $\delta u_{n+1} = 0$. This means that the left hand side of (3.98) is zero, and as a result the right hand side should be 0 as well. This yields the stationary conditions:

$$1 - \lambda'|_{\xi=x} = 0, \quad \lambda|_{\xi=x} = 0, \quad \lambda''|_{\xi=x} = 0. \tag{3.99}$$

This in turn gives

$$\lambda = \xi - x. \tag{3.100}$$

Having determined the Lagrange multiplier $\lambda(\xi)$, the successive approximations $u_{n+1}, n \geqslant 0$, of the solution $u(x)$ will be readily obtained upon using selective function $u_0(x)$. However, for fast convergence, the function $u_0(x)$ should be selected by using the initial conditions as follows:

$$\begin{aligned} u_0(x) &= u(0), \quad \text{for first order } u_n', \\ u_0(x) &= u(0) + xu'(0), \quad \text{for second order } u_n'', \\ u_0(x) &= u(0) + xu'(0) + \frac{1}{2!}x^2u''(0), \quad \text{for third order } u_n''', \\ &\ \vdots \end{aligned} \tag{3.101}$$

and so on. Consequently, the solution

$$u(x) = \lim_{n\to\infty} u_n(x). \tag{3.102}$$

In other words, the correction functional (3.87) will give several approximations, and therefore the exact solution is obtained as the limit of the resulting successive approximations.

The determination of the Lagrange multiplier plays a major role in the determination of the solution of the problem. In what follows, we summarize some iteration formulae that show ODE, its corresponding Lagrange multipliers, and its correction functional respectively:

$$\text{(i)} \begin{cases} u' + f(u(\xi), u'(\xi)) = 0, \lambda = -1, \\ u_{n+1} = u_n - \displaystyle\int_0^x [u_n' + f(u_n, u_n')]\, d\xi, \end{cases}$$

$$\text{(ii)} \begin{cases} u'' + f(u(\xi), u'(\xi), u''(\xi)) = 0,\ \lambda = (\xi - x), \\ u_{n+1} = u_n + \displaystyle\int_0^x (\xi - x)\, [u_n'' + f(u_n, u_n', u_n'')]\, d\xi, \end{cases}$$

$$\text{(iii)} \begin{cases} u''' + f(u(\xi), u'(\xi), u''(\xi), u'''(\xi)) = 0, \lambda = -\dfrac{1}{2!}(\xi - x)^2, \\ u_{n+1} = u_n - \displaystyle\int_0^x \frac{1}{2!}(\xi - x)^2\, [u_n''' + f(u_n, \ldots, u_n''')]\, d\xi, \end{cases}$$

$$\text{(iv)} \begin{cases} u^{(\text{iv})} + f(u(\xi), u'(\xi), u''(\xi), u'''(\xi), u^{(\text{iv})}(\xi)) = 0, \lambda = \dfrac{1}{3!}(\xi - x)^3, \\ u_{n+1} = u_n + \displaystyle\int_0^x \frac{1}{3!}(\xi - x)^3 \left[u_n''' + f(u_n, u_n', \ldots, u_n^{(\text{iv})})\right] d\xi, \end{cases}$$

and generally

$$\text{(v)}\begin{cases} u^{(n)} + f(u(\xi), u'(\xi), \ldots, u^{(n)}(\xi)) = 0, \lambda = (-1)^n \dfrac{1}{(n-1)!}(\xi - x)^{(n-1)}, \\ u_{n+1} = u_n + (-1)^n \displaystyle\int_0^x \frac{1}{(n-1)!}(\xi - x)^{(n-1)} \left[u_n''' + f(u_n, \ldots, u_n^{(n)})\right] d\xi, \end{cases}$$

for $n \geqslant 1$.

To use the variational iteration method for solving Volterra integral equations, it is necessary to convert the integral equation to an equivalent initial value problem or to an equivalent integro-differential equation. As defined before, integro-differential equation is an equation that contains differential and integral operators in the same equation. The integro-differential equations will be studied in details in Chapter 5. The conversion process is presented in Section 2.5.1. However, for comparison reasons, we will examine the obtained initial value problem by two methods, namely, standard methods used for solving ODEs, and by using the variational iteration method as will be seen by the following examples.

Example 3.15

Solve the Volterra integral equation by using the variational iteration method

$$u(x) = 1 + \int_0^x u(t)dt. \tag{3.103}$$

Using Leibnitz rule to differentiate both sides of (3.103) gives

$$u'(x) - u(x) = 0. \tag{3.104}$$

Substituting $x = 0$ into (3.103) gives the initial condition $u(0) = 1$.

Using the variational iteration method

The correction functional for equation (3.104) is

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(\xi)(u_n'(\xi) - \tilde{u}_n(\xi))d\xi. \tag{3.105}$$

Using the formula (i) given above leads to

$$\lambda = -1. \tag{3.106}$$

Substituting this value of the Lagrange multiplier $\lambda = -1$ into the functional (3.105) gives the iteration formula:

$$u_{n+1}(x) = u_n(x) - \int_0^x (u_n'(\xi) - u_n(\xi))\, d\xi. \tag{3.107}$$

As stated before, we can use the initial condition to select $u_0(x) = u(0) = 1$. Using this selection into (3.105) gives the following successive approximations:

$$\begin{aligned} u_0(x) &= 1, \\ u_1(x) &= 1 - \int_0^x (u_0'(\xi) - u_0(\xi))\, d\xi = 1 + x, \\ u_2(x) &= 1 + x - \int_0^x (u_1'(\xi) - u_1(\xi))\, d\xi = 1 + x + \frac{1}{2!}x^2, \end{aligned}$$

$$u_3(x) = 1 + x + \frac{1}{2!}x^2 - \int_0^x (u_2'(\xi) - u_2(\xi))\, d\xi = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3, \tag{3.108}$$

and so on. The VIM admits the use of

$$\begin{aligned} u(x) &= \lim_{n\to\infty} u_n(x) \\ &= \lim_{n\to\infty}(1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots + \frac{1}{n!}x^n), \end{aligned} \tag{3.109}$$

that gives the exact solution by

$$u(x) = e^x. \tag{3.110}$$

Using ODEs method

The ODE (3.104) is of first order, therefore the integrating factor $\mu(x)$ is given by

$$\mu(x) = e^{\int^x (-1)dx} = e^{-x}. \tag{3.111}$$

For first order ODE, we use the formula:

$$u(x) = \frac{1}{\mu}\left[\int^x \mu q(x)dx + C\right] = Ce^x. \tag{3.112}$$

To obtain the particular solution, we use the initial condition $u(0) = 1$ to find that $C = 1$. This gives the particular solution:

$$u(x) = e^x. \tag{3.113}$$

Example 3.16

Solve the Volterra integral equation by using the variational iteration method

$$u(x) = x + \int_0^x (x - t)\, u(t)dt. \tag{3.114}$$

Using Leibnitz rule to differentiate both sides of (3.114) once with respect to x gives the integro-differential equation:

$$u'(x) = 1 + \int_0^x u(t)dt, u(0) = 0. \tag{3.115}$$

However, by differentiating (3.115) with respect to x we obtain the differential equation:

$$u''(x) = u(x). \tag{3.116}$$

Substituting $x = 0$ into (3.114) and (3.115) gives the initial conditions $u(0) = 0$ and $u'(0) = 1$. The resulting initial value problem, that consists of a second order ODE and initial conditions is given by

$$u''(x) - u(x) = 0, \quad u(0) = 0, \quad u'(0) = 1. \tag{3.117}$$

The integro-differential equation (3.115) and the initial value problem (3.116) will be handled independently by using the variational iteration method.

Using the variational iteration method

(i) We first start using the variational iteration method to handle the integro-differential equation (3.115) given by

$$u'(x) = 1 + \int_0^x u(t)dt, u(0) = 0. \tag{3.118}$$

The correction functional for Eq. (3.118) is

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(\xi)\left(u_n'(\xi) - 1 - \int_0^\xi \tilde{u}_n(r)dr\right)d\xi. \tag{3.119}$$

Using the formula (i) for λ we find that

$$\lambda = -1. \tag{3.120}$$

Substituting this value of the Lagrange multiplier $\lambda = -1$ into the functional (3.119) gives the iteration formula:

$$u_{n+1}(x) = u_n(x) - \int_0^x \left(u_n'(\xi) - 1 - \int_0^\xi u_n(r)dr\right)d\xi. \tag{3.121}$$

We can use the initial conditions to select $u_0(x) = u(0) = 0$. Using this selection into (3.121) gives the following successive approximations:

$$\begin{aligned}
u_0(x) &= 0,\\
u_1(x) &= x - \int_0^x \left(u_0'(\xi) - 1 - \int_0^\xi u_0(r)dr\right)d\xi = x,\\
u_2(x) &= x - \int_0^x \left(u_1'(\xi) - 1 - \int_0^\xi u_1(r)dr\right)d\xi = x + \frac{1}{3!}x^3,\\
u_3(x) &= x - \int_0^x \left(u_2'(\xi) - 1 - \int_0^\xi u_2(r)dr\right)d\xi = x + \frac{1}{3!}x^3 + \frac{1}{5!}x^5,\\
&\ \ \vdots\\
u_n(x) &= x + \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \frac{1}{7!}x^7 + \cdots + \frac{1}{(2n+1)!}x^{2n+1}.
\end{aligned} \tag{3.122}$$

The VIM admits the use of

$$u(x) = \lim_{n\to\infty} u_n(x), \tag{3.123}$$

that gives the exact solution by

$$u(x) = \sinh x. \tag{3.124}$$

(ii) We can obtain the same result by applying the variational iteration method to handle the initial value problem (3.117) given by

$$u''(x) - u(x) = 0, \quad u(0) = 0, \quad u'(0) = 1. \tag{3.125}$$

The correction functional for Eq. (3.117) is

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(\xi)\left(u_n''(\xi) - \tilde{u}_n(\xi)\right)d\xi. \tag{3.126}$$

Using the formula (ii) given above leads to

$$\lambda = \xi - x. \tag{3.127}$$

Substituting this value of the Lagrange multiplier $\lambda = \xi - x$ into the functional (3.126) gives the iteration formula:

$$u_{n+1}(x) = u_n(x) + \int_0^x (\xi - x)\left(u_n''(\xi) - u_n(\xi)\right) d\xi. \tag{3.128}$$

We can use the initial conditions to select $u_0(x) = u(0) + xu'(0) = x$. Using this selection into (3.128) gives the following successive approximations

$$\begin{aligned}
u_0(x) &= x,\\
u_1(x) &= x + \int_0^x (\xi - x)\left(u_0''(\xi) - u_0(\xi)\right) d\xi = x + \frac{1}{3!}x^3,\\
u_2(x) &= x + \frac{1}{3!}x^3 + \int_0^x (\xi - x)\left(u_1''(\xi) - u_1(\xi)\right) d\xi = x + \frac{1}{3!}x^3 + \frac{1}{5!}x^5,\\
u_3(x) &= x + \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \int_0^x (\xi - x)\left(u_2''(\xi) - u_2(\xi)\right) d\xi\\
&= x + \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \frac{1}{7!}x^7,\\
&\ \vdots\\
u_n(x) &= x + \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \frac{1}{7!}x^7 + \cdots + \frac{1}{(2n+1)!}x^{2n+1}.
\end{aligned} \tag{3.129}$$

The VIM admits the use of

$$u(x) = \lim_{n\to\infty} u_n(x), \tag{3.130}$$

that gives the exact solution by

$$u(x) = \sinh x. \tag{3.131}$$

Standard methods for solving ODEs

The initial value problem (3.125) is of second order, therefore the auxiliary equation is of the form

$$r^2 - 1 = 0, \tag{3.132}$$

that gives $r = \pm 1$. This in turn gives the general solution by

$$u(x) = A\sinh x + B\cosh x. \tag{3.133}$$

To obtain the particular solution, we use the initial conditions $u(0) = 0, u'(0) = 1$ to find that the particular solution is given by

$$u(x) = \sinh x. \tag{3.134}$$

Example 3.17

Solve the Volterra integral equation by using the variational iteration method

$$u(x) = 1 + x + \frac{1}{3!}x^3 - \int_0^x (x - t)u(t)dt. \tag{3.135}$$

Using Leibnitz rule to differentiate both sides of (3.135) once with respect to x gives the integro-differential equation:

$$u'(x) = 1 + \frac{1}{2!}x^2 - \int_0^x u(t)dt, u(0) = 1, \tag{3.136}$$

and by differentiating again we obtain the initial value problem

$$u''(x) + u(x) = x, u(0) = 1, u'(0) = 1. \tag{3.137}$$

Using the variational iteration method

(i) We first start using the variational iteration method to handle the integro-differential equation (3.136) given by

$$u'(x) = 1 + \frac{1}{2!}x^2 - \int_0^x u(t)dt, u(0) = 1, \tag{3.138}$$

The correction functional for Eq. (3.138) is

$$u_{n+1}(x) = u_n(x) - \int_0^x \lambda(\xi)\left(u_n'(\xi) - 1 - \frac{1}{2}\xi^2 + \int_0^\xi \tilde{u}_n(r)dr\right)d\xi. \tag{3.139}$$

Using the formula (i) for λ we find that

$$\lambda = -1. \tag{3.140}$$

Substituting this value of the Lagrange multiplier $\lambda = -1$ into the functional (3.139) gives the iteration formula

$$u_{n+1}(x) = u_n(x) - \int_0^x \left(u_n'(\xi) - 1 - \frac{1}{2}\xi^2 + \int_0^\xi u_n(r)dr\right)d\xi. \tag{3.141}$$

We can use the initial conditions to select $u_0(x) = u(0) = 1$. Using this selection into (3.141) gives the following successive approximations

$$\begin{aligned}
u_0(x) &= 1,\\
u_1(x) &= x - \int_0^x \left(u_0'(\xi) - 1 - \frac{1}{2}\xi^2 + \int_0^\xi u_0(r)dr\right)d\xi\\
&= 1 + x - \frac{1}{2!}x^2 + \frac{1}{3!}x^3,\\
u_2(x) &= x - \int_0^x \left(u_1'(\xi) - 1 - \frac{1}{2}\xi^2 + \int_0^\xi u_1(r)dr\right)d\xi\\
&= 1 + x - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{5!}x^5,\\
u_3(x) &= x - \int_0^x \left(u_2'(\xi) - 1 - \frac{1}{2}\xi^2 + \int_0^\xi u_2(r)dr\right)d\xi\\
&= 1 + x - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6,\\
&\vdots\\
u_n(x) &= x + \left(1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots + \frac{(-1)^n}{(2n)!}x^{2n}\right).
\end{aligned} \tag{3.142}$$

The VIM admits the use of

$$u(x) = \lim_{n\to\infty} u_n(x), \tag{3.143}$$

that gives the exact solution by

$$u(x) = x + \cos x. \tag{3.144}$$

(ii) We next use the variational iteration method for solving the initial value problem

$$u''(x) + u(x) = x, \quad u(0) = 1, \quad u'(0) = 1. \tag{3.145}$$

The correction functional for Eq. (3.145) is

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(\xi)\left(u_n''(\xi) + \tilde{u}_n(\xi) - \xi\right) d\xi. \tag{3.146}$$

Using the formula (ii) given above leads to

$$\lambda = \xi - x. \tag{3.147}$$

Substituting this value of the Lagrange multiplier $\lambda = \xi - x$ into the functional (3.146) gives the iteration formula

$$u_{n+1}(x) = u_n(x) + \int_0^x (\xi - x)\left(u_n''(\xi) + u_n(\xi) - \xi\right) d\xi. \tag{3.148}$$

We can use the initial conditions to select $u_0(x) = u(0) + xu'(0) = 1 + x$. Using this selection into (3.148) gives the following successive approximations

$$\begin{aligned}
u_0(x) &= 1 + x,\\
u_1(x) &= 1 + x + \int_0^x (\xi - x)\left(u_0''(\xi) + u_0(\xi) - \xi\right) d\xi\\
&= 1 + x - \frac{1}{2!}x^2,\\
u_2(x) &= 1 + x - \frac{1}{2!}x^2 + \int_0^x (\xi - x)\left(u_1''(\xi) + u_1(\xi) - \xi\right) d\xi\\
&= 1 + x - \frac{1}{2!}x^2 + \frac{1}{4!}x^4, \qquad (3.149)\\
u_3(x) &= 1 + x - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \int_0^x (\xi - x)\left(u_2''(\xi) + u_2(\xi) - \xi\right) d\xi\\
&= 1 + x - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6\\
&\ \ \vdots\\
u_n(x) &= x + \left(1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 - \cdots + \frac{(-1)^n}{(2n)!}x^{2n}\right).
\end{aligned}$$

Proceeding as before, the VIM gives the exact solution by

$$u(x) = x + \cos x. \tag{3.150}$$

Standard methods for solving ODEs

The ODE (3.169) is of second order and nonhomogeneous. The auxiliary equation for the homogeneous part is of the form

$$r^2 + 1 = 0, \tag{3.151}$$

that gives $r = \pm i, i^2 = -1$. The general solution is given by

$$\begin{aligned} u(x) &= u_c + u_p, \\ u(x) &= A\cos x + B\sin x + a + bx, \end{aligned} \tag{3.152}$$

where u_c is the complementary solution, and u_p is a particular solution. Using ODE methods and initial conditions we find that $B = a = 0$, and $A = b = 1$. The particular solution is given by

$$u(x) = x + \cos x. \tag{3.153}$$

Example 3.18

Solve the Volterra integral equation by using the variational iteration method

$$u(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{2}\int_0^x (x-t)^2\, u(t)dt. \tag{3.154}$$

Using Leibnitz rule to differentiate both sides of (3.154) three times with respect to x gives the two integro-differential equations

$$\begin{aligned} u'(x) &= 1 + x + \int_0^x (x-t)u(t)dt, \quad u(0) = 1 \\ u''(x) &= 1 + \int_0^x u(t)dt, u(0) = 1, \quad u'(0) = 1. \end{aligned} \tag{3.155}$$

and the third order initial value problem

$$u'''(x) = u(x), \quad u(0) = u'(0) = u''(0) = 1. \tag{3.156}$$

Using the variational iteration method VIM

(i) We first note that we obtained two equivalent integro-differential equations (3.155). We will apply the VIM to these two equations. We first start using the VIM to handle the integro-differential equation

$$u'(x) = 1 + x + \int_0^x (x-t)u(t)dt, u(0) = 1. \tag{3.157}$$

The correction functional for Eq. (3.157) is

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(\xi)\left(u_n'(\xi) - 1 - \xi - \int_0^\xi (\xi - r)\tilde{u}_n(r)dr\right)d\xi. \tag{3.158}$$

Proceeding as before we find

$$\lambda = -1, \tag{3.159}$$

that gives the iteration formula

$$u_{n+1}(x) = u_n(x) - \int_0^x \left(u_n'(\xi) - 1 - \xi - \int_0^\xi (\xi - r)u_n(r)dr\right)d\xi. \tag{3.160}$$

We can use the initial conditions to select $u_0(x) = u(0) = 1$. Using this selection into (3.160) gives the following successive approximations

$$\begin{aligned}
u_0(x) &= 1,\\
u_1(x) &= 1 - \int_0^x \left(u_0'(\xi) - 1 - \xi - \int_0^{\xi} (\xi - r) u_0(r) dr \right) d\xi\\
&= 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3,\\
u_2(x) &= 1 - \int_0^x \left(u_1'(\xi) - 1 - \xi - \int_0^{\xi} (\xi - r) u_1(r) dr \right) d\xi\\
&= 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \frac{1}{6!}x^6,\\
&\vdots\\
u_n(x) &= 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \frac{1}{6!}x^6 + \cdots + \frac{1}{n!}x^n.
\end{aligned} \tag{3.161}$$

This in turn gives the exact solution by

$$u(x) = e^x. \tag{3.162}$$

(ii) We next consider the integro-differential equation

$$u''(x) = 1 + \int_0^x u(t)dt, \quad u(0) = 1, \quad u'(0) = 1. \tag{3.163}$$

The correction functional for Eq. (3.163) is

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(\xi) \left(u_n''(\xi) - 1 - \int_0^{\xi} \tilde{u}_n(r) dr \right) d\xi. \tag{3.164}$$

Notice that the integro-differential equation is of second order. Therefore, we can show that

$$\lambda = \xi - x, \tag{3.165}$$

that gives the iteration formula

$$u_{n+1}(x) = u_n(x) + \int_0^x \left((\xi - x)(u_n''(\xi) - 1 - \int_0^{\xi} (\xi - r) u_n(r) dr) \right) d\xi. \tag{3.166}$$

We can use the initial conditions to select $u_0(x) = 1 + x$. Using this selection into (3.166) gives the following successive approximations

$$\begin{aligned}
u_0(x) &= 1 + x,\\
u_1(x) &= 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4,\\
u_2(x) &= 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \frac{1}{6!}x^6 + \frac{1}{7!}x^7,\\
&\vdots\\
u_n(x) &= 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \frac{1}{6!}x^6 + \cdots + \frac{1}{n!}x^n.
\end{aligned} \tag{3.167}$$

This in turn gives the exact solution by

$$u(x) = e^x. \tag{3.168}$$

(iii) We next use the variational iteration method for solving the initial value problem

$$u'''(x) - u(x) = 0, \quad u(0) = u'(0) = u''(0) = 1. \tag{3.169}$$

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(\xi) \left(u_n'''(\xi) - \tilde{u}_n(\xi)\right) d\xi. \tag{3.170}$$

Using the formula (iii) given above for λ leads to

$$\lambda = -\frac{1}{2!}(\xi - x)^2. \tag{3.171}$$

Substituting this value of the Lagrange multiplier into the functional (3.170) gives the iteration formula

$$u_{n+1}(x) = u_n(x) - \frac{1}{2!}\int_0^x (\xi - x)^2 \left(u_n'''(\xi) - u_n(\xi)\right) d\xi. \tag{3.172}$$

As stated before, we can use the initial conditions to select

$$u_0(x) = u(0) + xu'(0) + \frac{1}{2}u''(0) = 1 + x + \frac{1}{2!}x^2. \tag{3.173}$$

Using this selection into (3.172) gives the following successive approximations

$$\begin{aligned}
u_0(x) &= 1 + x + \frac{1}{2!}x^2,\\
u_1(x) &= 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5,\\
u_2(x) &= 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \frac{1}{6!}x^6 + \frac{1}{7!}x^7 + \frac{1}{8!}x^8, \quad (3.174)\\
&\vdots\\
u_n(x) &= 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \frac{1}{6!}x^6 + \frac{1}{7!}x^7 + \frac{1}{8!}x^8 + \cdots
\end{aligned}$$

The VIM admits the use of

$$u(x) = \lim_{n\to\infty} u_n(x), \tag{3.175}$$

that gives the exact solution by

$$u(x) = e^x. \tag{3.176}$$

Using standard methods for solving ODEs

The ODE (3.169) is of third order, therefore the auxiliary equation is of the form

$$r^3 - 1 = 0, \tag{3.177}$$

that gives $r = 1, -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i, i^2 = -1$. The general solution is given by

$$u(x) = Ae^x + e^{-\frac{1}{2}x}\left[B\cos\frac{\sqrt{3}}{2}x + C\sin\frac{\sqrt{3}}{2}x\right]. \tag{3.178}$$

To obtain the particular solution, we use the initial conditions to find that the particular solution

$$u(x) = e^x. \tag{3.179}$$

It is interesting to point out that we need to use different approaches to solve ODEs by standard methods, whereas the variational iteration method attacks all problems directly and in a straightforward manner.

Exercises 3.2.4

Use the *variational iteration method* to solve the following Volterra integral equations by converting the equation to initial value problem or to an equivalent integro-differential equation:

1. $u(x) = 1 - \int_0^x u(t)dt$
2. $u(x) = x + x^4 + \frac{1}{2}x^2 + \frac{1}{5}x^5 - \int_0^x u(t)dt$
3. $u(x) = 1 - \frac{1}{2}x^2 + \int_0^x u(t)dt$
4. $u(x) = 1 - x - \frac{1}{2}x^2 + \int_0^x (x-t)u(t)dt$
5. $u(x) = 1 - \int_0^x (x-t)u(t)dt$
6. $u(x) = x + \int_0^x (x-t)u(t)dt$
7. $u(x) = 1 + 2x + 4\int_0^x (x-t)u(t)dt$
8. $u(x) = 5 + 2x^2 - \int_0^x (x-t)u(t)dt$
9. $u(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{2}\int_0^x (x-t)^2u(t)dt$
10. $u(x) = 1 + \frac{1}{2}x + \frac{1}{2}\int_0^x (x-t+1)u(t)dt$
11. $u(x) = 1 + x^2 - \int_0^x (x-t+1)^2u(t)dt$
12. $u(x) = 1 - \frac{1}{2}x^2 + \frac{1}{6}\int_0^x (x-t)^3u(t)dt$
13. $u(x) = 2 + x - 2\cos x - \int_0^x (x-t+2)u(t)dt$
14. $u(x) = 1 - 2\sinh x + \int_0^x (x-t+2)u(t)dt$
15. $u(x) = 1 + \frac{3}{5}x + \frac{1}{10}x^2 + \frac{1}{10}\int_0^x (x-t+2)^2u(t)dt$
16. $u(x) = 2 + 3x + \frac{3}{2}x^2 + \frac{1}{6}x^3 - \frac{1}{2}\int_0^x (x-t+2)^2u(t)dt$
17. $u(x) = 1 - x\sin x + \int_0^x tu(t)dt$
18. $u(x) = x\cosh x - \int_0^x tu(t)dt$
19. $u(x) = -1 + e^x + \frac{1}{2}x^2e^x - \frac{1}{2}\int_0^x tu(t)dt$
20. $u(x) = 1 - x\sin x + x\cos x + \int_0^x tu(t)dt$

3.2.5 The Successive Approximations Method

The *successive approximations method*, also called the *Picard iteration method* provides a scheme that can be used for solving initial value problems or integral equations. This method solves any problem by finding successive approximations to the solution by starting with an initial guess, called the zeroth approximation. As will be seen, the zeroth approximation is any selective real-valued function that will be used in a recurrence relation to determine the other approximations.

Given the linear Volterra integral equation of the second kind

$$u(x) = f(x) + \lambda \int_0^x K(x,t)u(t)dt, \tag{3.180}$$

where $u(x)$ is the unknown function to be determined, $K(x,t)$ is the kernel, and λ is a parameter. The successive approximations method introduces the recurrence relation

$$u_n(x) = f(x) + \lambda \int_0^x K(x,t)u_{n-1}(t)dt, n \geqslant 1, \tag{3.181}$$

where the zeroth approximation $u_0(x)$ can be any selective real valued function. We always start with an initial guess for $u_0(x)$, mostly we select $0, 1, x$ for $u_0(x)$, and by using (3.181), several successive approximations $u_k, k \geqslant 1$ will be determined as

$$\begin{aligned}
u_1(x) &= f(x) + \lambda \int_0^x K(x,t)u_0(t)dt, \\
u_2(x) &= f(x) + \lambda \int_0^x K(x,t)u_1(t)dt, \\
u_3(x) &= f(x) + \lambda \int_0^x K(x,t)u_2(t)dt, \\
&\vdots \\
u_n(x) &= f(x) + \lambda \int_0^x K(x,t)u_{n-1}(t)dt.
\end{aligned} \tag{3.182}$$

The question of convergence of $u_n(x)$ is justified by noting the following theorem.

Theorem 3.1 *If $f(x)$ in (3.181) is continuous for the interval $0 \leqslant x \leqslant a$, and the kernel $K(x,t)$ is also continuous in the triangle $0 \leqslant x \leqslant a, 0 \leqslant t \leqslant x$, the sequence of successive approximations $u_n(x), n \geqslant 0$ converges to the solution $u(x)$ of the integral equation under discussion.*

It is interesting to point out that the variational iteration method admits the use of the iteration formula:

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(\xi) \left(\frac{\partial u_n(\xi)}{\partial \xi} - \tilde{u}_n(\xi) \right) d\xi. \tag{3.183}$$

whereas the successive approximations method uses the iteration formula

$$u_n(x) = f(x) + \lambda \int_0^x K(x,t)u_{n-1}(t)dt, n \geqslant 1. \quad (3.184)$$

The difference between the two formulas can be summarized as follows:

1. The first formula contains the Lagrange multiplier λ that should be determined first before applying the formula. The successive approximations formula does not require the use of λ.

2. The first variational iteration formula allows the use of the restriction $\tilde{u}_n(\xi)$ where $\delta\tilde{u}_n(\xi) = 0$. The second formula does require this restriction.

3. The first formula is applied to an equivalent ODE of the integral equation, whereas the second formula is applied directly to the iteration formula of the integral equation itself.

The successive approximations method, or the Picard iteration method will be illustrated by the following examples.

Example 3.19

Solve the Volterra integral equation by using the successive approximations method

$$u(x) = 1 - \int_0^x (x-t)u(t)dt. \quad (3.185)$$

For the zeroth approximation $u_0(x)$, we can select

$$u_0(x) = 1. \quad (3.186)$$

The method of successive approximations admits the use of the iteration formula

$$u_{n+1}(x) = 1 - \int_0^x (x-t)u_n(t)dt, n \geqslant 0. \quad (3.187)$$

Substituting (3.186) into (3.187) we obtain

$$\begin{aligned}
u_1(x) &= 1 - \int_0^x (x-t)u_0(t)dt = 1 - \frac{1}{2!}x^2,\\
u_2(x) &= 1 - \int_0^x (x-t)u_1(t)dt = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4,\\
u_3(x) &= 1 - \int_0^x (x-t)u_2(t)dt = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6, \qquad (3.188)\\
u_4(x) &= 1 - \int_0^x (x-t)u_3(t)dt = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8,\\
&\vdots
\end{aligned}$$

Consequently, we obtain

$$u_{n+1}(x) = \sum_{k=0}^{n} (-1)^k \frac{x^{2k}}{(2k)!}. \quad (3.189)$$

The solution $u(x)$ of (3.185)

$$u(x) = \lim_{n\to\infty} u_{n+1}(x) = \cos x. \quad (3.190)$$

Example 3.20

Solve the Volterra integral equation by using the successive approximations method

$$u(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{2}\int_0^x (x-t)^2 u(t)dt. \tag{3.191}$$

For the zeroth approximation $u_0(x)$, we can select

$$u_0(x) = 0. \tag{3.192}$$

The method of successive approximations admits the use of the iteration formula

$$u_{n+1}(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{2}\int_0^x (x-t)^2 u_n(t)dt, n \geqslant 0. \tag{3.193}$$

Substituting (3.192) into (3.193) we obtain

$$\begin{aligned}
u_1(x) &= 1 + x + \frac{1}{2!}x^2,\\
u_2(x) &= 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5, \qquad (3.194)\\
u_3(x) &= 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \frac{1}{6!}x^6 + \frac{1}{7!}x^7 + \frac{1}{8!}x^8,\\
&\ \ \vdots
\end{aligned}$$

and so on. The solution $u(x)$ of (3.191) is given by

$$u(x) = \lim_{n\to\infty} u_{n+1}(x) = e^x. \tag{3.195}$$

Example 3.21

Solve the Volterra integral equation by using the successive approximations method

$$u(x) = -1 + e^x + \frac{1}{2}x^2e^x - \frac{1}{2}\int_0^x tu(t)dt. \tag{3.196}$$

For the zeroth approximation $u_0(x)$, we select

$$u_0(x) = 0. \tag{3.197}$$

We next use the iteration formula

$$u_{n+1}(x) = -1 + e^x + \frac{1}{2}x^2e^x - \frac{1}{2}\int_0^x tu_n(t)dt, n \geqslant 0. \tag{3.198}$$

Substituting (3.197) into (3.198) we obtain

$$\begin{aligned}
u_1(x) &= -1 + e^x + \frac{1}{2!}x^2e^x,\\
u_2(x) &= -3 + \frac{1}{4}x^2 + e^x(3 - 2x + \frac{5}{4}x^2 - \frac{1}{4}x^3),
\end{aligned}$$

$$u_3(x) = x\left(1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \cdots\right), \tag{3.199}$$

$$\vdots$$

$$u_{n+1}(x) = x\left(1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \frac{1}{7!}x^7 + \cdots\right).$$

Notice that we used the Taylor expansion for e^x to determine $u_3(x), u_4(x), \ldots$ The solution $u(x)$ of (3.196)

$$u(x) = \lim_{n\to\infty} u_{n+1}(x) = xe^x. \tag{3.200}$$

Example 3.22

Solve the Volterra integral equation by using the successive approximations method

$$u(x) = 1 - x\sin x + x\cos x + \int_0^x tu(t)dt. \tag{3.201}$$

For the zeroth approximation $u_0(x)$, we may select

$$u_0(x) = x. \tag{3.202}$$

We next use the iteration formula

$$u_{n+1}(x) = 1 - x\sin x + x\cos x + \int_0^x tu_n(t)dt, n \geqslant 0. \tag{3.203}$$

Substituting (3.202) into (3.203) gives

$$\begin{aligned}
u_1(x) &= 1 + \frac{1}{3}x^3 - x\sin x + x\cos x,\\
u_2(x) &= 3 + \frac{1}{2}x^2 + \frac{1}{15}x^3 - (2 + 3x - x^2)\sin x - (2 - 3x - x^2)\cos x,\\
u_3(x) &= \left(x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7\right) + \left(1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6\right),\\
&\vdots\\
u_{n+1}(x) &= \sum_{k=0}^{n}(-1)^k\frac{x^{2k+1}}{(2k+1)!} + \sum_{k=0}^{n}(-1)^k\frac{x^{2k}}{(2k)!}.
\end{aligned} \tag{3.204}$$

Notice that we used the Taylor expansion for $\sin x$ and $\cos x$ to determine the approximations $u_3(x), u_4(x), \ldots$. The solution $u(x)$ of (3.201) is given by

$$u(x) = \lim_{n\to\infty} u_{n+1}(x) = \sin x + \cos x. \tag{3.205}$$

Exercises 3.2.5

Use the *successive approximations method* to solve the following Volterra integral equations:

1. $u(x) = x + \int_0^x u(t)dt$
2. $u(x) = x + \int_0^x (x-t)u(t)dt$

3. $u(x) = \frac{1}{6}x^3 - \int_0^x (x-t)u(t)dt$
4. $u(x) = 1 + 2x + 4\int_0^x (x-t)u(t)dt$
5. $u(x) = \frac{1}{6}x^3 + \int_0^x (x-t)u(t)dt$
6. $u(x) = 1 + x^2 - \int_0^x (x-t+1)^2u(t)dt$
7. $u(x) = \frac{1}{2}x^2 - \int_0^x (x-t)u(t)dt$
8. $u(x) = 1 - \frac{1}{2}x^2 + \frac{1}{6}\int_0^x (x-t)^3u(t)dt$
9. $u(x) = 1 + 3\int_0^x u(t)dt$
10. $u(x) = 1 - 2\sinh x + \int_0^x (x-t+2)u(t)dt$
11. $u(x) = 3 + x^2 - \int_0^x (x-t)u(t)dt$
12. $u(x) = 1 - x\sin x + \int_0^x tu(t)dt$
13. $u(x) = x\cosh x - \int_0^x tu(t)dt$
14. $u(x) = 1 - x - \int_0^x (x-t)u(t)dt$
15. $u(x) = 1 - \int_0^x 3t^2u(t)dt$
16. $u(x) = 2x\cosh x - 4\int_0^x tu(t)dt$
17. $u(x) = 1 + \sinh x - \sin x + \cos x - \cosh x + \int_0^x u(t)dt$
18. $u(x) = 1 + \sinh x + \sin x - \cos x + \cosh x - \int_0^x u(t)dt$
19. $u(x) = 2 - 2\cos x - \int_0^x (x-t)u(t)dt$
20. $u(x) = -x + 2\sinh x + \int_0^x (x-t)u(t)dt$

3.2.6 The Laplace Transform Method

The *Laplace transform method* is a powerful technique that can be used for solving initial value problems and integral equations as well. We assume that the reader has used the Laplace transform method, and the inverse Laplace transform, for solving ordinary differential equations. The details and properties of the Laplace method can be found in ordinary differential equations texts.

Before we start applying this method, we summarize some of the concepts presented in Section 1.5. In the convolution theorem for the Laplace transform, it was stated that if the kernel $K(x,t)$ of the integral equation:

$$u(x) = f(x) + \lambda\int_0^x K(x,t)u(t)dt, \tag{3.206}$$

depends on the difference $x-t$, then it is called a *difference kernel*. Examples of the difference kernel are $e^{x-t}, \cos(x-t)$, and $x-t$. The integral equation can thus be expressed as

$$u(x) = f(x) + \lambda\int_0^x K(x-t)u(t)dt. \tag{3.207}$$

Consider two functions $f_1(x)$ and $f_2(x)$ that possess the conditions needed for the existence of Laplace transform for each. Let the Laplace transforms for the functions $f_1(x)$ and $f_2(x)$ be given by

$$\begin{aligned}\mathcal{L}\{f_1(x)\} &= F_1(s),\\ \mathcal{L}\{f_2(x)\} &= F_2(s).\end{aligned} \tag{3.208}$$

The *Laplace convolution product* of these two functions is defined by

$$(f_1 * f_2)(x) = \int_0^x f_1(x-t)f_2(t)dt, \tag{3.209}$$

or

$$(f_2 * f_1)(x) = \int_0^x f_2(x-t)f_1(t)dt. \tag{3.210}$$

Recall that

$$(f_1 * f_2)(x) = (f_2 * f_1)(x). \tag{3.211}$$

We can easily show that the Laplace transform of the convolution product $(f_1 * f_2)(x)$ is given by

$$\mathcal{L}\{(f_1 * f_2)(x)\} = \mathcal{L}\left\{\int_0^x f_1(x-t)f_2(t)dt\right\} = F_1(s)F_2(s). \tag{3.212}$$

Based on this summary, we will examine specific Volterra integral equations where the kernel is a difference kernel. Recall that we will apply the Laplace transform method and the inverse of the Laplace transform using Table 1.1 in Section 1.5.

By taking Laplace transform of both sides of (3.207) we find

$$U(s) = F(s) + \lambda\mathcal{K}(s)U(s), \tag{3.213}$$

where

$$U(s) = \mathcal{L}\{u(x)\}, \quad \mathcal{K}(s) = \mathcal{L}\{K(x)\}, \quad F(s) = \mathcal{L}\{f(x)\}. \tag{3.214}$$

Solving (3.213) for $U(s)$ gives

$$U(s) = \frac{F(s)}{1-\lambda\mathcal{K}(s)}, \quad \lambda\mathcal{K}(s) \neq 1. \tag{3.215}$$

The solution $u(x)$ is obtained by taking the inverse Laplace transform of both sides of (3.215) where we find

$$u(x) = \mathcal{L}^{-1}\left\{\frac{F(s)}{1-\lambda\mathcal{K}(s)}\right\}. \tag{3.216}$$

Recall that the right side of (3.216) can be evaluated by using Table 1.1 in Section 1.5. The Laplace transform method for solving Volterra integral equations will be illustrated by studying the following examples.

Example 3.23

Solve the Volterra integral equation by using the Laplace transform method

$$u(x) = 1 + \int_0^x u(t)dt. \tag{3.217}$$

Notice that the kernel $K(x-t) = 1, \lambda = 1$. Taking Laplace transform of both sides (3.217) gives

$$\mathcal{L}\{u(x)\} = \mathcal{L}\{1\} + \mathcal{L}\{1 * u(x)\}, \tag{3.218}$$

so that

$$U(s) = \frac{1}{s} + \frac{1}{s}U(s), \tag{3.219}$$

or equivalently

$$U(s) = \frac{1}{s-1}. \tag{3.220}$$

By taking the inverse Laplace transform of both sides of (3.220), the exact solution is therefore given by

$$u(x) = e^x. \tag{3.221}$$

Example 3.24

Solve the Volterra integral equation by using the Laplace transform method

$$u(x) = 1 - \int_0^x (x-t)u(t)dt. \tag{3.222}$$

Notice that the kernel $K(x-t) = (x-t), \lambda = -1$. Taking Laplace transform of both sides (3.222) gives

$$\mathcal{L}\{u(x)\} = \mathcal{L}\{1\} - \mathcal{L}\{(x-t) * u(x)\}, \tag{3.223}$$

so that

$$U(s) = \frac{1}{s} - \frac{1}{s^2}U(s), \tag{3.224}$$

or equivalently

$$U(s) = \frac{s}{s^2+1}. \tag{3.225}$$

By taking the inverse Laplace transform of both sides of (3.225), the exact solution

$$u(x) = \cos x, \tag{3.226}$$

is readily obtained.

Example 3.25

Solve the Volterra integral equation by using the Laplace transform method

$$u(x) = \frac{1}{3!}x^3 - \int_0^x (x-t)u(t)dt. \tag{3.227}$$

Taking Laplace transform of both sides (3.227) gives

$$\mathcal{L}\{u(x)\} = \frac{1}{3!}\mathcal{L}\{x^3\} - \mathcal{L}\{(x-t) * u(x)\}. \tag{3.228}$$

This gives

$$U(s) = \frac{1}{3!} \times \frac{3!}{s^4} - \frac{1}{s^2}U(s), \tag{3.229}$$

so that

$$U(s)=\frac{1}{s^2(s^2+1)}=\frac{1}{s^2}-\frac{1}{s^2+1}. \tag{3.230}$$

Taking the inverse Laplace transform of both sides of (3.230) gives the exact solution

$$u(x)=x-\sin x. \tag{3.231}$$

Example 3.26

Solve the Volterra integral equation by using the Laplace transform method

$$u(x)=\sin x+\cos x+2\int_0^x \sin(x-t)u(t)dt. \tag{3.232}$$

Recall that we should use the linear property of the Laplace transforms. Taking Laplace transform of both sides (3.232) gives

$$\mathcal{L}\{u(x)\}=\mathcal{L}\{\sin x+\cos x\}+2\mathcal{L}\{\sin(x-t)*u(x)\}, \tag{3.233}$$

so that

$$U(s)=\frac{1}{s^2+1}+\frac{s}{s^2+1}+\frac{2}{s^2+1}U(s), \tag{3.234}$$

or equivalently

$$U(s)=\frac{1}{s-1}. \tag{3.235}$$

Taking the inverse Laplace transform of both sides of (3.235) gives the exact solution

$$u(x)=e^x. \tag{3.236}$$

Exercises 3.2.6

Use the *Laplace transform method* to solve the Volterra integral equations:

1. $u(x)=x+\int_0^x (x-t)u(t)dt$
2. $u(x)=1-x-\int_0^x (x-t)u(t)dt$
3. $u(x)=1-\frac{1}{2}x^2+\frac{1}{6}\int_0^x (x-t)^3u(t)dt$
4. $u(x)=1+3\int_0^x (x-t)u(t)dt$
5. $u(x)=x-1+\int_0^x (x-t)u(t)dt$
6. $u(x)=\cos x-\sin x+2\int_0^x \cos(x-t)u(t)dt$
7. $u(x)=e^x-\cos x-2\int_0^x e^{x-t}u(t)dt$
8. $u(x)=1-\int_0^x \left((x-t)^2-1\right)u(t)dt$
9. $u(x)=\sin x+\sinh x+\cosh x-2\int_0^x \cos(x-t)u(t)dt$
10. $u(x)=\sinh x+\cosh x-\cos x-2\int_0^x \cos(x-t)u(t)dt$
11. $u(x)=\sin x-\cos x+\cosh x-2\int_0^x \cosh(x-t)u(t)dt$

12. $u(x) = \sin x + \cos x + \sinh x - 2\int_0^x \cosh(x-t)u(t)dt$

13. $u(x) = 2e^x - 2 - x + \int_0^x (x-t)u(t)dt$ 14. $u(x) = 2\cosh x - 2 + \int_0^x (x-t)u(t)dt$

15. $u(x) = 2 - 2\cos x - \int_0^x (x-t)u(t)dt$ 16. $u(x) = 1 + \int_0^x \sin(x-t)u(t)dt$

3.2.7 The Series Solution Method

A real function $u(x)$ is called analytic if it has derivatives of all orders such that the Taylor series at any point b in its domain

$$u(x) = \sum_{k=0}^{n} \frac{f^{(k)}(b)}{k!}(x-b)^k, \tag{3.237}$$

converges to $f(x)$ in a neighborhood of b. For simplicity, the generic form of Taylor series at $x = 0$ can be written as

$$u(x) = \sum_{n=0}^{\infty} a_n x^n. \tag{3.238}$$

In this section we will present a useful method, that stems mainly from the Taylor series for analytic functions, for solving Volterra integral equations. We will assume that the solution $u(x)$ of the Volterra integral equation

$$u(x) = f(x) + \lambda \int_0^x K(x,t)u(t)dt, \tag{3.239}$$

is analytic, and therefore possesses a Taylor series of the form given in (3.238), where the coefficients a_n will be determined recurrently. Substituting (3.238) into both sides of (3.239) gives

$$\sum_{n=0}^{\infty} a_n x^n = T(f(x)) + \lambda \int_0^x K(x,t)\left(\sum_{n=0}^{\infty} a_n t^n\right) dt, \tag{3.240}$$

or for simplicity we use

$$a_0 + a_1 x + a_2 x^2 + \cdots = T(f(x)) + \lambda \int_0^x K(x,t)\left(a_0 + a_1 t + a_2 t^2 + \cdots\right) dt, \tag{3.241}$$

where $T(f(x))$ is the Taylor series for $f(x)$. The integral equation (3.239) will be converted to a traditional integral in (3.240) or (3.241) where instead of integrating the unknown function $u(x)$, terms of the form t^n, $n \geqslant 0$ will be integrated. Notice that because we are seeking series solution, then if $f(x)$ includes elementary functions such as trigonometric functions, exponential functions, etc., then Taylor expansions for functions involved in $f(x)$ should be used.

We first integrate the right side of the integral in (3.240) or (3.241), and collect the coefficients of like powers of x. We next equate the coefficients of like powers of x in both sides of the resulting equation to obtain a recurrence relation in $a_j, j \geqslant 0$. Solving the recurrence relation will lead to a complete determination of the coefficients $a_j, j \geqslant 0$. Having determined the coefficients $a_j, j \geqslant 0$, the series solution follows immediately upon substituting the derived coefficients into (3.238). The exact solution may be obtained if such an exact solution exists. If an exact solution is not obtainable, then the obtained series can be used for numerical purposes. In this case, the more terms we evaluate, the higher accuracy level we achieve.

Example 3.27

Solve the Volterra integral equation by using the series solution method

$$u(x) = 1 + \int_0^x u(t)dt. \tag{3.242}$$

Substituting $u(x)$ by the series

$$u(x) = \sum_{n=0}^{\infty} a_n x^n, \tag{3.243}$$

into both sides of Eq. (3.242) leads to

$$\sum_{n=0}^{\infty} a_n x^n = 1 + \int_0^x \left(\sum_{n=0}^{\infty} a_n t^n\right) dt. \tag{3.244}$$

Evaluating the integral at the right side gives

$$\sum_{n=0}^{\infty} a_n x^n = 1 + \sum_{n=0}^{\infty} \frac{1}{n+1} a_n x^{n+1}, \tag{3.245}$$

that can be rewritten as

$$a_0 + \sum_{n=1}^{\infty} a_n x^n = 1 + \sum_{n=1}^{\infty} \frac{1}{n} a_{n-1} x^n, \tag{3.246}$$

or equivalently

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots = 1 + a_0 x + \frac{1}{2} a_1 x^2 + \frac{1}{3} a_2 x^2 + \cdots . \tag{3.247}$$

In (3.245), the powers of x of both sides are different, therefore, we make them the same by changing the index of the second sum to obtain (3.246). Equating the coefficients of like powers of x in both sides of (3.246) gives the recurrence relation

$$\begin{aligned} a_0 &= 1, \\ a_n &= \frac{1}{n} a_{n-1}, n \geqslant 1. \end{aligned} \tag{3.248}$$

where this result gives

$$a_n = \frac{1}{n!}, n \geqslant 0. \tag{3.249}$$

Substituting this result into (3.243) gives the series solution:

$$u(x) = \sum_{n=0}^{\infty} \frac{1}{n!} x^n, \tag{3.250}$$

that converges to the exact solution $u(x) = e^x$.

It is interesting to point out that this result can be obtained by equating coefficients of like terms in both sides of (3.247), where we find

$$\begin{aligned} a_0 &= 1, \quad a_1 = a_0 = 1, \\ a_2 &= \frac{1}{2} a_1 = \frac{1}{2!}, \\ &\vdots \\ a_n &= \frac{1}{n} a_{n-1} = \frac{1}{n!}. \end{aligned} \tag{3.251}$$

This leads to the same result obtained before by solving the recurrence relation.

Example 3.28

Solve the Volterra integral equation by using the series solution method

$$u(x) = x + \int_0^x (x - t) u(t) dt. \tag{3.252}$$

Substituting $u(x)$ by the series

$$u(x) = \sum_{n=0}^{\infty} a_n x^n, \tag{3.253}$$

into both sides of Eq. (3.252) leads to

$$\sum_{n=0}^{\infty} a_n x^n = x + \int_0^x \left(\sum_{n=0}^{\infty} x a_n t^n - \sum_{n=0}^{\infty} a_n t^{n+1} \right) dt. \tag{3.254}$$

Evaluating the right side leads to

$$\sum_{n=0}^{\infty} a_n x^n = x + \sum_{n=0}^{\infty} \frac{1}{(n+2)(n+1)} a_n x^{n+2}, \tag{3.255}$$

that can be rewritten as

$$a_0 + a_1 x + \sum_{n=2}^{\infty} a_n x^n = x + \sum_{n=2}^{\infty} \frac{1}{n(n-1)} a_{n-2} x^n, \tag{3.256}$$

or equivalently

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots = x + \frac{1}{2} a_0 x^2 + \frac{1}{6} a_1 x^3 + \frac{1}{12} a_2 x^4 + \cdots \tag{3.257}$$

Equating the coefficients of like powers of x in both sides of (3.256) gives the recurrence relation

$$\begin{aligned} a_0 &= 0, \qquad\qquad a_1 = 1, \\ &\vdots \\ a_n &= \frac{1}{n(n-1)} a_{n-2}, \qquad n \geqslant 2. \end{aligned} \tag{3.258}$$

This result can be combined to obtain

$$a_n = \frac{1}{(2n+1)!}, n \geqslant 0. \tag{3.259}$$

Substituting this result into (3.253) gives the series solution

$$u(x) = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} x^{2n+1}, \tag{3.260}$$

that converges to the exact solution

$$u(x) = \sinh x. \tag{3.261}$$

It is interesting to point out that this result can also be obtained by equating coefficients of like terms in both sides of (3.257), where we find

$$\begin{aligned} &a_0 = 0, \quad &a_1 = 1, \quad &a_2 = \frac{1}{2}a_0 = 0, \\ &a_3 = \frac{1}{6}a_1 = \frac{1}{3!}, \quad &a_4 = \frac{1}{12}a_2 = 0. \end{aligned} \tag{3.262}$$

This leads to the same result obtained before by solving the recurrence relation.

Example 3.29

Solve the Volterra integral equation by using the series solution method

$$u(x) = 1 - x\sin x + \int_0^x tu(t)dt. \tag{3.263}$$

For simplicity reasons, we will use few terms of the Taylor series for $\sin x$ and for the solution $u(x)$ in (3.263) to find

$$\begin{aligned} &a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots \\ &= 1 - x\left(x - \frac{x^3}{3!} + \cdots\right) + \int_0^x t(a_0 + a_1 t + a_2 t^2 + \cdots)dt. \end{aligned} \tag{3.264}$$

Integrating the right side and collecting the like terms of x we find

$$a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+\cdots = 1+(\frac{1}{2}a_0-1)x^2+\frac{1}{3}a_1x^3+(\frac{1}{6}+\frac{1}{4}a_2)x^4+\cdots. \tag{3.265}$$

Equating the coefficients of like powers of x in (3.265) yields

$$\begin{aligned} &a_0 = 1, \quad &a_1 = 0, \\ &a_2 = \frac{1}{2}a_0 - 1 = -\frac{1}{2!}, \quad &a_3 = \frac{1}{3}a_1 = 0, \\ &a_4 = \frac{1}{6} + \frac{1}{4}a_2 = \frac{1}{4!}, \\ &\quad\vdots \end{aligned} \tag{3.266}$$

and generally

$$a_{2n+1} = 0, \quad a_{2n} = \frac{(-1)^n}{(2n)!}, \quad n \geqslant 0. \tag{3.267}$$

The solution in a series form is given by

$$u(x) = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots \tag{3.268}$$

that gives the exact solution by

$$u(x) = \cos x. \tag{3.269}$$

Example 3.30

Solve the Volterra integral equation by using the series solution method

$$u(x) = 2e^x - 2 - x + \int_0^x (x-t)u(t)dt. \tag{3.270}$$

Proceeding as before, we will use few terms of the Taylor series for e^x and for the solution $u(x)$ in (3.270) to find

$$\begin{aligned} &a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \cdots \\ &= x + x^2 + \frac{1}{3}x^3 + \frac{1}{12}x^4 + \cdots + \int_0^x (x-t)(a_0 + a_1t + a_2t^2 + \cdots)dt. \end{aligned} \tag{3.271}$$

Integrating the right side and collecting the like terms of x we find

$$\begin{aligned} &a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \cdots \\ &= x + \left(1 + \frac{1}{2}a_0\right)x^2 + \left(\frac{1}{3} + \frac{1}{6}a_1\right)x^3 + \left(\frac{1}{12} + \frac{1}{12}a_2\right)x^4 + \cdots . \end{aligned} \tag{3.272}$$

Equating the coefficients of like powers of x in (3.272) yields

$$a_0 = 0,\ a_1 = 1,\ a_2 = 1,\ a_3 = \frac{1}{2!},\ a_4 = \frac{1}{3!}, \ldots \tag{3.273}$$

and generally

$$a_n = \frac{1}{n!}, \quad n \geqslant 1, a_0 = 0. \tag{3.274}$$

The solution in a series form is given by

$$u(x) = x(1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots), \tag{3.275}$$

that converges to the exact solution

$$u(x) = xe^x. \tag{3.276}$$

Exercises 3.2.7

Use the *series solution method* to solve the Volterra integral equations:

1. $u(x) = 1 - \int_0^x u(t)dt$
2. $u(x) = 1 - \int_0^x (x-t)u(t)dt$
3. $u(x) = x + \int_0^x (x-t)u(t)dt$
4. $u(x) = 1 + \frac{1}{2}x + \frac{1}{2}\int_0^x (x-t+1)u(t)dt$
5. $u(x) = 1 + xe^x - \int_0^x tu(t)dt$
6. $u(x) = 1 + 2x + 4\int_0^x (x-t)u(t)dt$

7. $u(x) = 3 + x^2 - \int_0^x (x-t)u(t)dt$
8. $u(x) = 1 + 2\sin x - \int_0^x u(t)dt$
9. $u(x) = x\cos x + \int_0^x tu(t)dt$
10. $u(x) = x\cosh x - \int_0^x tu(t)dt$
11. $u(x) = 2\cosh x - 2 + \int_0^x (x-t)u(t)dt$
12. $u(x) = 1 - x - \int_0^x (x-t)u(t)dt$
13. $u(x) = x - x\ln(1+x) + \int_0^x u(t)dt$
14. $u(x) = x^2 - \frac{1}{2}x^3 + x^3\ln(1+x) - \int_0^x 2xu(t)dt$
15. $u(x) = \sec x + \tan x - \int_0^x \sec t u(t)dt$
16. $u(x) = x + \int_0^x \tan t u(t)dt$

3.3 Volterra Integral Equations of the First Kind

The standard form of the Volterra integral equations of the first kind is given by

$$f(x) = \int_0^x K(x,t)u(t)dt, \tag{3.277}$$

where the kernel $K(x,t)$ and the function $f(x)$ are given real-valued functions, and $u(x)$ is the function to be determined. Recall that the unknown function $u(x)$ appears inside and outside the integral sign for the Volterra integral equations of the second kind, whereas it occurs only inside the integral sign for the Volterra integral equations of the first kind. This equation of the first kind motivated mathematicians to develop reliable methods for solving it. In this section we will discuss three main methods that are commonly used for handling the Volterra integral equations of the first kind. Other methods are available in the literature but will not be presented in this text.

3.3.1 The Series Solution Method

As in the previous section, we will consider the solution $u(x)$ to be analytic, where it has derivatives of all orders, and it possesses Taylor series at $x = 0$ of the form

$$u(x) = \sum_{n=0}^{\infty} a_n x^n, \tag{3.278}$$

where the coefficients a_n will be determined recurrently. Substituting (3.278) into (3.277) gives

$$T(f(x)) = \int_0^x K(x,t)\left(\sum_{n=0}^{\infty} a_n t^n\right) dt, \tag{3.279}$$

or for simplicity we can use

$$T(f(x)) = \int_0^x K(x,t)\left(a_0 + a_1 t + a_2 t^2 + \cdots\right) dt, \tag{3.280}$$

where $T(f(x))$ is the Taylor series for $f(x)$.

The integral equation (3.277) will be converted to a traditional integral in (3.279) or (3.280) where instead of integrating the unknown function $u(x)$, terms of the form t^n, $n \geqslant 0$ will be integrated. Notice that because we are seeking series solution, then if $f(x)$ includes elementary functions such as trigonometric functions, exponential functions, etc., then Taylor expansions for functions involved in $f(x)$ should be used.

The method is identical to that presented before for the Volterra integral equations of the second kind. We first integrate the right side of the integral in (3.279) or (3.280), and collect the coefficients of like powers of x. We next equate the coefficients of like powers of x in both sides of the resulting equation to obtain a recurrence relation in $a_j, j \geqslant 0$. Solving the recurrence relation will lead to a complete determination of the coefficients $a_j, j \geqslant 0$. Having determined the coefficients $a_j, j \geqslant 0$, the series solution follows immediately upon substituting the derived coefficients into (3.278). The exact solution may be obtained if such an exact solution exists. If an exact solution is not obtainable, then the obtained series can be used for numerical purposes. In this case, the more terms we evaluate, the higher accuracy level we achieve. This method will be illustrated by discussing the following examples.

Example 3.31

Solve the Volterra integral equation by using the series solution method

$$\sin x - x\cos x = \int_0^x t u(t) dt. \tag{3.281}$$

Proceeding as before, only few terms of the Taylor series for $\sin x - x\cos x$ and for the solution $u(x)$ in (3.281) will be used. Integrating the right side we obtain

$$\begin{aligned} \frac{1}{3}x^3 - \frac{1}{30}x^5 + \frac{1}{840}x^7 + \cdots &= \int_0^x t(a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \cdots)dt, \\ &= \frac{1}{2}a_0 x^2 + \frac{1}{3}a_1 x^3 + \frac{1}{4}a_2 x^4 + \frac{1}{5}a_3 x^5 \\ &\quad + \frac{1}{6}a_4 x^6 + \frac{1}{7}a_5 x^7 + \cdots \end{aligned} \tag{3.282}$$

Equating the coefficients of like powers of x in (3.282) yields

$$a_0 = 0, \quad a_1 = 1, \quad a_2 = 0, \quad a_3 = -\frac{1}{3!}, \quad a_4 = 0, a_5 = \frac{1}{5!} \cdots \tag{3.283}$$

The solution in a series form is given by

$$u(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \cdots \tag{3.284}$$

that converges to the exact solution

$$u(x) = \sin x. \tag{3.285}$$

Example 3.32

Solve the Volterra integral equation by using the series solution method

$$2 + x - 2e^x + xe^x = \int_0^x (x-t)u(t)dt. \tag{3.286}$$

Using few terms of the Taylor series for $2+x-2e^x+xe^x$ and for the solution $u(x)$ in (3.286), and by integrating the right side we find

$$\begin{aligned}\frac{1}{6}x^3 + \frac{1}{12}x^4 + \frac{1}{40}x^5 + \cdots &= \int_0^x (x-t)(a_0 + a_1 t + a_2 t^2 + \cdots)dt \\ &= \frac{1}{2}a_0x^2 + \frac{1}{6}a_1x^3 + \frac{1}{12}a_2x^4 + \frac{1}{20}a_3x^5 \\ &\quad + \frac{1}{30}a_4x^6 + \cdots\end{aligned} \tag{3.287}$$

Equating the coefficients of like powers of x in (3.287) yields

$$a_0 = 0, \quad a_1 = 1, \quad a_2 = 1, \quad a_3 = \frac{1}{2!}, \quad a_4 = \frac{1}{3!}, \cdots \tag{3.288}$$

The solution in a series form is given by

$$u(x) = x(1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots), \tag{3.289}$$

that converges to the exact solution

$$u(x) = xe^x. \tag{3.290}$$

Example 3.33

Solve the Volterra integral equation by using the series solution method

$$x - \frac{1}{2}x^2 - \ln(1+x) + x^2\ln(1+x) = \int_0^x 2tu(t)dt. \tag{3.291}$$

Using the Taylor series for $x - \frac{1}{2}x^2 - \ln(1+x) + x^2\ln(1+x)$ and proceeding as before we find

$$\begin{aligned}&\frac{2}{3}x^3 - \frac{1}{4}x^4 + \frac{2}{15}x^5 - \frac{1}{12}x^6 + \cdots \\ &= \int_0^x 2t(a_0 + a_1t + a_2t^2 + a_3t^3 + \cdots)dt \\ &= a_0x^2 + \frac{2}{3}a_1x^3 + \frac{1}{2}a_2x^4 + \frac{2}{5}a_3x^5 + \frac{1}{3}a_4x^6 + \cdots\end{aligned} \tag{3.292}$$

Equating the coefficients of like powers of x in (3.292) yields

$$a_0 = 0,\ a_1 = 1,\ a_2 = -\frac{1}{2},\ a_3 = \frac{1}{3},\ a_4 = -\frac{1}{4}, \ldots \tag{3.293}$$

The solution in a series form is given by

$$u(x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots \tag{3.294}$$

that converges to the exact solution

$$u(x) = \ln(1 + x). \tag{3.295}$$

Exercises 3.3.1

Use the *series solution method* to solve the Volterra integral equations of the first kind:

1. $e^x - 1 - x = \int_0^x (x - t + 1)u(t)dt$
2. $x\cosh x - \sinh x = \int_0^x u(t)dt$
3. $1 + xe^x - e^x = \int_0^x tu(t)dt$
4. $1 + \frac{1}{3}x^3 + xe^x - e^x = \int_0^x tu(t)dt$
5. $-1 - x + \frac{1}{6}x^3 + e^x = \int_0^x (x - t)u(t)dt$
6. $-x + 2\sin x - x\cos x = \int_0^x (x - t)u(t)dt$
7. $-1 + \cosh x = \int_0^x (x - t)u(t)dt$
8. $x - 2\sinh x + x\cosh x = \int_0^x (x - t)u(t)dt$
9. $1 + x - \sin x - \cos x = \int_0^x (x - t)u(t)dt$
10. $1 - x - e^{-x} = \int_0^x (x - t)u(t)dt$
11. $-x + \frac{1}{2}x^2 + \ln(1 + x) + x\ln(1 + x) = \int_0^x u(t)dt$
12. $\frac{1}{2}x^2 e^x = \int_0^x e^{x-t}u(t)dt$

3.3.2 The Laplace Transform Method

The *Laplace transform method* is a powerful technique that we used before for solving initial value problems and Volterra integral equations of the second kind. In the convolution theorem for the Laplace transform method, it was stated that if the kernel $K(x, t)$ of the integral equation

$$f(x) = \int_0^x K(x, t)u(t)dt, \tag{3.296}$$

depends on the difference $x - t$, then it is called a *difference kernel*. The Volterra integral equation of the first kind can thus be expressed as

$$f(x) = \int_0^x K(x - t)u(t)dt. \tag{3.297}$$

Consider two functions $f_1(x)$ and $f_2(x)$ that possess the conditions needed for the existence of Laplace transform for each. Let the Laplace transforms for the functions $f_1(x)$ and $f_2(x)$ be given by

$$\mathcal{L}\{f_1(x)\} = F_1(s),$$
$$\mathcal{L}\{f_2(x)\} = F_2(s). \tag{3.298}$$

The *Laplace convolution product* of these two functions is defined by

$$(f_1 * f_2)(x) = \int_0^x f_1(x-t)f_2(t)dt, \tag{3.299}$$

or

$$(f_2 * f_1)(x) = \int_0^x f_2(x-t)f_1(t)dt. \tag{3.300}$$

Recall that

$$(f_1 * f_2)(x) = (f_2 * f_1)(x). \tag{3.301}$$

We can easily show that the Laplace transform of the convolution product $(f_1 * f_2)(x)$ is given by

$$\mathcal{L}\{(f_1 * f_2)(x)\} = \mathcal{L}\left\{\int_0^x f_1(x-t)f_2(t)dt\right\} = F_1(s)F_2(s). \tag{3.302}$$

Based on this summary, we will examine specific Volterra integral equations of the first kind where the kernel is a difference kernel. Recall that we will apply the Laplace transform method and the inverse of the Laplace transform using Table 1.1 in Section 1.5.

By taking Laplace transform of both sides of (3.297) we find

$$F(s) = \mathcal{K}(s)U(s), \tag{3.303}$$

where

$$U(s) = \mathcal{L}\{u(x)\}, \quad \mathcal{K}(s) = \mathcal{L}\{K(x)\}, \quad F(s) = \mathcal{L}\{f(x)\}. \tag{3.304}$$

Solving (3.303) gives

$$U(s) = \frac{F(s)}{\mathcal{K}(s)}, \tag{3.305}$$

where

$$\mathcal{K}(s) \neq 0. \tag{3.306}$$

The solution $u(x)$ is obtained by taking the inverse Laplace transform of both sides of (3.305) where we find

$$u(x) = \mathcal{L}^{-1}\left\{\frac{F(s)}{\mathcal{K}(s)}\right\}. \tag{3.307}$$

Recall that the right side of (3.307) can be evaluated by using Table 1.1 in Section 1.5. The Laplace transform method for solving Volterra integral equations of the first kind will be illustrated by studying the following examples.

Example 3.34

Solve the Volterra integral equation of the first kind by using the Laplace transform method

$$e^x - \sin x - \cos x = \int_0^x 2e^{x-t}u(t)dt. \tag{3.308}$$

Taking the Laplace transform of both sides of (3.308) yields

$$\frac{1}{s-1} - \frac{1}{s^2+1} - \frac{s}{s^2+1} = \frac{2}{s-1}U(s), \tag{3.309}$$

or equivalently

$$\frac{2}{(s-1)(s^2+1)} = \frac{2}{s-1}U(s). \tag{3.310}$$

This in turn gives

$$U(s) = \frac{1}{s^2+1}. \tag{3.311}$$

Taking the inverse Laplace transform of both sides gives the exact solution

$$u(x) = \sin x. \tag{3.312}$$

Example 3.35

Solve the Volterra integral equation of the first kind by using the Laplace transform method

$$1 + x - e^x = \int_0^x (t-x)u(t)dt. \tag{3.313}$$

Notice that the kernel is $(t-x) = -(x-t)$.

Taking the Laplace transform of both sides of (3.313) yields

$$\frac{1}{s} + \frac{1}{s^2} - \frac{1}{s-1} = -\frac{1}{s^2}U(s), \tag{3.314}$$

so that

$$\frac{s(s-1)+s-1-s^2}{s^2(s-1)} = -\frac{1}{s^2}U(s). \tag{3.315}$$

Solving for $U(s)$ we find

$$U(s) = \frac{1}{s-1}. \tag{3.316}$$

Taking the inverse Laplace transform of both sides gives the exact solution

$$u(x) = e^x. \tag{3.317}$$

Example 3.36

Solve the Volterra integral equation of the first kind by using the Laplace transform method

$$-1 + x^2 + \frac{1}{6}x^3 + 2\sinh x + \cosh x = \int_0^x (x-t+2)u(t)dt. \tag{3.318}$$

Taking the Laplace transform of both sides of (3.318) yields

$$\frac{s^3+s^2-1}{s^2(s^2-1)} = \left(\frac{1}{s^2} + \frac{2}{s}\right)U(s), \tag{3.319}$$

or equivalently

$$U(s) = \frac{1}{s^2} + \frac{s}{s^2-1}. \tag{3.320}$$

Taking the inverse Laplace transform of both sides gives the exact solution

$$u(x) = x + \cosh x. \tag{3.321}$$

Exercises 3.3.2

Use the *Laplace transform method* to solve the Volterra integral equations of the first kind:

1. $x - \sin x = \int_0^x (x-t)u(t)dt$
2. $e^x + \sin x - \cos x = \int_0^x 2e^{x-t}u(t)dt$
3. $1 + \frac{1}{3!}x^3 - \cos x = \int_0^x (x-t)u(t)dt$
4. $1 + x - \sin x - \cos x = \int_0^x (x-t)u(t)dt$
5. $x = \int_0^x (1 + 2(x-t))u(t)dt$
6. $\sinh x = \int_0^x e^{x-t}u(t)dt$
7. $x = \int_0^x (x-t+1)u(t)dt$
8. $1 - x - e^{-x} = \int_0^x (t-x)u(t)dt$
9. $1 + x - \frac{1}{3!}x^3 - e^x = \int_0^x (t-x)u(t)dt$
10. $1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 - \sin x - \cos x = \int_0^x (x-t+1)u(t)dt$
11. $3 - 7x + x^2 + \sinh x - 3\cosh x = \int_0^x (x-t-3)u(t)dt$
12. $1 - \cos x = \int_0^x \cos(x-t)u(t)dt$

3.3.3 Conversion to a Volterra Equation of the Second Kind

In this section we will present a method that will convert Volterra integral equations of the first kind to Volterra integral equations of the second kind. The conversion technique works effectively only if $K(x,x) \neq 0$. Differentiating both sides of the Volterra integral equation of the first kind

$$f(x) = \int_0^x K(x,t)u(t)dt, \tag{3.322}$$

with respect to x, and using Leibnitz rule, we find

$$f'(x) = K(x,x)u(x) + \int_0^x K_x(x,t)u(t)dt. \tag{3.323}$$

Solving for $u(x)$, provided that $K(x,x) \neq 0$, we obtain the Volterra integral equation of the second kind given by

$$u(x) = \frac{f'(x)}{K(x,x)} - \int_0^x \frac{1}{K(x,x)}K_x(x,t)u(t)dt. \tag{3.324}$$

Notice that the non-homogeneous term and the kernel have changed to

$$g(x) = \frac{f'(x)}{K(x,x)}, \tag{3.325}$$

and

$$G(x,t) = -\frac{1}{K(x,x)} K_x(x,t), \tag{3.326}$$

respectively.

Having converted the Volterra integral equation of the first kind to the Volterra integral equation of the second kind, we then can use any method that was presented before. Because we solved the Volterra integral equations of the second kind by many methods, we will select distinct methods for solving the Volterra integral equation of the first kind after reducing it to a second kind Volterra integral equation.

Example 3.37

Convert the Volterra integral equation of the first kind to the second kind and solve the resulting equation

$$\sinh x = \int_0^x e^{x-t} u(t) dt. \tag{3.327}$$

Differentiating both sides of (3.327) and using Leibnitz rule we obtain

$$\cosh x = u(x) + \int_0^x e^{x-t} u(t) dt, \tag{3.328}$$

that gives the Volterra integral equation of the second kind

$$u(x) = \cosh x - \int_0^x e^{x-t} u(t) dt. \tag{3.329}$$

We select the Laplace transform method for solving this problem. Taking Laplace transform of both sides gives

$$U(s) = \frac{s}{s^2-1} - \frac{1}{s-1} U(s), \tag{3.330}$$

that leads to

$$U(s) = \frac{1}{s+1}. \tag{3.331}$$

Taking the inverse Laplace transform of both sides gives the exact solution

$$u(x) = e^{-x}. \tag{3.332}$$

Example 3.38

Convert the Volterra integral equation of the first kind to the second kind and solve the resulting equation

$$1 + \sin x - \cos x = \int_0^x (x - t + 1) u(t) dt. \tag{3.333}$$

Differentiating both sides of (3.333) and using Leibnitz rule we obtain the Volterra integral equation of the second kind

$$u(x) = \cos x + \sin x - \int_0^x u(t) dt. \tag{3.334}$$

We select the modified decomposition method for solving this problem. Therefore we set the modified recurrence relation

$$\begin{aligned} u_0(x) &= \cos x, \\ u_1(x) &= \sin x - \int_0^x u_0(t)dt = 0, \\ u_{k+1}(x) &= -\int_0^x u_k(t)dt = 0, k \geqslant 1. \end{aligned} \tag{3.335}$$

This gives the exact solution by

$$u(x) = \cos x. \tag{3.336}$$

Example 3.39

Convert the Volterra integral equation of the first kind to the second kind and solve the resulting equation

$$9x^2 + 5x^3 = \int_0^x (10x - 10t + 6)u(t)dt. \tag{3.337}$$

Differentiating both sides of (3.337) and using Leibnitz rule we obtain

$$18x + 15x^2 = 6u(x) + \int_0^x 10u(t)dt, \tag{3.338}$$

that gives the Volterra integral equation of the second kind

$$u(x) = 3x + \frac{5}{2}x^2 - \frac{5}{3}\int_0^x u(t)dt. \tag{3.339}$$

We select the Adomian decomposition method combined with the noise terms phenomenon for solving this problem. Therefore we set the recurrence relation

$$u_0(x) = 3x + \frac{5}{2}x^2, \quad u_{k+1}(x) = -\frac{5}{3}\int_0^x u_k(t)dt, \quad k \geqslant 0, \tag{3.340}$$

that gives

$$u_0(x) = 3x + \frac{5}{2}x^2, \quad u_1(x) = -\frac{5}{2}x^2 - \frac{25}{18}x^3. \tag{3.341}$$

Canceling the noise term $\frac{5}{2}x^2$, that appear in $u_0(x)$ and $u_1(x)$, from $u_0(x)$ gives the exact solution by

$$u(x) = 3x. \tag{3.342}$$

Remarks

1. It was stated before that if $K(x,x) = 0$, then the conversion of the first kind to the second kind fails. However, if $K(x,x) = 0$ and $K_x(x,x) \neq 0$, then by differentiating the Volterra integral equation of the first kind as many times as needed, provided that $K(x,t)$ is differentiable, then the equation will be reduced to the Volterra integral equation of the second kind.

2. The function $f(x)$ must satisfy specific conditions to guarantee a unique continuous solution for $u(x)$. The determination of these special conditions will be left as an exercise.

However, for the first remark, where $K(x,x)=0$ but $K_x(x,x)\neq 0$, we will differentiate twice, by using Leibnitz rule, as will be shown by the following illustrative example.

Example 3.40

Convert the Volterra integral equation of the first kind to the second kind and solve the resulting equation

$$x\sinh x = 2\int_0^x \sinh(x-t)u(t)dt. \tag{3.343}$$

Differentiating both sides of (3.343) and using Leibnitz rule we obtain

$$x\cosh x + \sinh x = 2\int_0^x \cosh(x-t)u(t)dt, \tag{3.344}$$

which is still a Volterra integral equation of the first kind. However, because $K_x(x,x)\neq 0$, we differentiate again to obtain the Volterra integral equation of the second kind

$$u(x) = \cosh x + \frac{1}{2}x\sinh x - \int_0^x \sinh(x-t)u(t)dt. \tag{3.345}$$

We select the modified decomposition method for solving this problem. Therefore we set the modified recurrence relation

$$\begin{aligned} u_0(x) &= \cosh x,\\ u_1(x) &= \frac{1}{2}x\sinh x - \int_0^x \sinh(x-t)u_0(t)dt = 0,\\ u_{k+1}(x) &= -\int_0^x \sinh(x-t)u_k(t)dt = 0, \quad k\geqslant 1. \end{aligned} \tag{3.346}$$

The exact solution is given by

$$u(x) = \cosh x. \tag{3.347}$$

Exercises 3.3.3

In Exercises 1–12, use Leibnitz rule to convert the Volterra integral equation of the first kind to a second kind and solve the resulting equation:

1. $e^x + \sin x - \cos x = \int_0^x 2e^{x-t}u(t)dt$
2. $e^x - \cos x = \int_0^x e^{x-t}u(t)dt$
3. $x = \int_0^x (x-t+1)u(t)dt$
4. $e^x + \sin x - \cos x = \int_0^x 2\cos(x-t)u(t)dt$
5. $e^x - x - 1 = \int_0^x (x-t+1)u(t)dt$
6. $\frac{1}{2}x^2e^x = \int_0^x e^{x-t}u(t)dt$
7. $e^x - 1 = \int_0^x (x-t+1)u(t)dt$
8. $\sin x - \cos x + 1 = \int_0^x (x-t+1)u(t)dt$
9. $5x^4 + x^5 = \int_0^x (x-t+1)u(t)dt$

10. $4 + x - 4e^x + 3xe^x = \int_0^x (x - t + 2)u(t)dt$

11. $-3 - x + x^2 + \frac{1}{3!}x^3 + 3e^x = \int_0^x (x - t + 2)u(t)dt$

12. $\tan x - \ln\cos x = \int_0^x (x - t + 1)u(t)dt, x < \frac{\pi}{2}$

In Exercises 13–16, use Leibnitz rule twice to convert the Volterra integral equation of the first kind to the second kind and solve the resulting equation:

13. $x\sin x = \int_0^x 2\sin(x - t)u(t)dt$

14. $e^x - \sin x - \cos x = \int_0^x 2\sin(x - t)u(t)dt$

15. $\sin x - \cos x + e^{-x} = \int_0^x 2\sin(x - t)u(t)dt$

16. $\sin x - x\cos x = \int_0^x 2\sinh(x - t)u(t)dt$

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Chapter 4
Fredholm Integral Equations

4.1 Introduction

It was stated in Chapter 2 that Fredholm integral equations arise in many scientific applications. It was also shown that Fredholm integral equations can be derived from boundary value problems. Erik Ivar Fredholm (1866–1927) is best remembered for his work on integral equations and spectral theory. Fredholm was a Swedish mathematician who established the theory of integral equations and his 1903 paper in *Acta Mathematica* played a major role in the establishment of operator theory.

As stated before, in Fredholm integral equations, the integral containing the unknown function $u(x)$ is characterized by fixed limits of integration in the form

$$u(x) = f(x) + \lambda \int_a^b K(x,t)u(t)dt, \tag{4.1}$$

where a and b are constants. For the *first kind* Fredholm integral equations, the unknown function $u(x)$ occurs only under the integral sign in the form

$$f(x) = \int_a^b K(x,t)u(t)dt. \tag{4.2}$$

However, Fredholm integral equations of the *second kind*, the unknown function $u(x)$ occurs inside and outside the integral sign. The second kind is represented by the form

$$u(x) = f(x) + \lambda \int_a^b K(x,t)u(t)dt. \tag{4.3}$$

The kernel $K(x,t)$ and the function $f(x)$ are given real-valued functions [9], and λ is a parameter. When $f(x) = 0$, the equation is said to be *homogeneous*.

In this chapter, we will mostly use *degenerate or separable kernels*. A degenerate or a separable kernel is a function that can be expressed as the sum of the product of two functions each depends only on one variable. Such a kernel can be expressed in the form

$$K(x,t) = \sum_{i=1}^{n} f_i(x)g_i(t). \tag{4.4}$$

Examples of separable kernels are $x - t, (x - t)^2, 4xt$, etc. In what follows we state, without proof, the Fredholm alternative theorem.

Theorem 4.1 (Fredholm Alternative Theorem) *If the homogeneous Fredholm integral equation*

$$u(x) = \lambda \int_a^b K(x,t)u(t)dt \tag{4.5}$$

has only the trivial solution $u(x) = 0$, then the corresponding nonhomogeneous Fredholm equation

$$u(x) = f(x) + \lambda \int_a^b K(x,t)u(t)dt \tag{4.6}$$

has always a unique solution. This theorem is known by the Fredholm alternative theorem [1].

Theorem 4.2 (Unique Solution) *If the kernel $K(x,t)$ in Fredholm integral equation* (4.1) *is continuous, real valued function, bounded in the square $a \leqslant x \leqslant b$ and $a \leqslant t \leqslant b$, and if $f(x)$ is a continuous real valued function, then a necessary condition for the existence of a unique solution for Fredholm integral equation* (4.1) *is given by*

$$|\lambda| M(b-a) < 1, \tag{4.7}$$

where

$$|K(x,t)| \leqslant M \in \mathbb{R}. \tag{4.8}$$

On the contrary, if the necessary condition (4.7) does not hold, then a continuous solution may exist for Fredholm integral equation. To illustrate this, we consider the Fredholm integral equation

$$u(x) = -2 - 3x + \int_0^1 (3x + t)u(t)dt. \tag{4.9}$$

It is clear that $\lambda = 1, |K(x,t)| \leqslant 4$ and $(b - a) = 1$. This gives

$$|\lambda| M(b-a) = 4 \nless 1. \tag{4.10}$$

However, the Fredholm equation (4.9) has an exact solution given by

$$u(x) = 6x. \tag{4.11}$$

A variety of analytic and numerical methods have been used to handle Fredholm integral equations. The direct computation method, the successive approximations method, and converting Fredholm equation to an equivalent boundary value problem are among many traditional methods that were commonly used. However, in this text we will apply the recently developed methods, namely, the Adomian decomposition method (ADM), the modified decomposition method (mADM), and the variational iteration method (VIM) to handle the Fredholm integral equations. Some of the traditional methods, namely, successive approximations method, and the direct computation method will be employed as well. The emphasis in this text will be on the use of these methods rather than proving theoretical concepts of convergence and existence. The theorems of uniqueness, existence, and convergence are important and can be found in the literature. The concern will be on the determination of the solution $u(x)$ of the Fredholm integral equations of the first kind and the second kind.

4.2 Fredholm Integral Equations of the Second Kind

We will first study Fredholm integral equations of the second kind given by

$$u(x) = f(x) + \lambda \int_a^b K(x,t)u(t)dt. \tag{4.12}$$

The unknown function $u(x)$, that will be determined, occurs inside and outside the integral sign. The kernel $K(x,t)$ and the function $f(x)$ are given real-valued functions, and λ is a parameter. In what follows we will present the methods, new and traditional, that will be used to handle the Fredholm integral equations (4.12).

4.2.1 The Adomian Decomposition Method

The Adomian decomposition method (ADM) was introduced and developed by George Adomian in [2–5] and was used before in Chapter 3. The Adomian method will be briefly outlined.

The Adomian decomposition method consists of decomposing the unknown function $u(x)$ of any equation into a sum of an infinite number of components defined by the decomposition series

$$u(x) = \sum_{n=0}^{\infty} u_n(x), \tag{4.13}$$

or equivalently

$$u(x) = u_0(x) + u_1(x) + u_2(x) + \cdots \tag{4.14}$$

where the components $u_n(x), n \geqslant 0$ will be determined recurrently. The Adomian decomposition method concerns itself with finding the components $u_0, u_1, u_2, \ldots$ individually. As we have seen before, the determination of these components can be achieved in an easy way through a recurrence relation that usually involves simple integrals that can be easily evaluated.

To establish the recurrence relation, we substitute (4.13) into the Fredholm integral equation (4.12) to obtain

$$\sum_{n=0}^{\infty} u_n(x) = f(x) + \lambda \int_a^b K(x,t) \left(\sum_{n=0}^{\infty} u_n(t) \right) dt, \tag{4.15}$$

or equivalently

$$u_0(x) + u_1(x) + u_2(x) + \cdots = f(x) + \lambda \int_a^b K(x,t) \left[u_0(t) + u_1(t) + \cdots \right] dt. \tag{4.16}$$

The zeroth component $u_0(x)$ is identified by all terms that are not included under the integral sign. This means that the components $u_j(x), j \geqslant 0$ of the unknown function $u(x)$ are completely determined by setting the recurrence relation

$$u_0(x) = f(x), \quad u_{n+1}(x) = \lambda \int_a^b K(x,t) u_n(t) dt, \quad n \geqslant 0, \tag{4.17}$$

or equivalently

$$\begin{aligned} u_0(x) &= f(x), \\ u_1(x) &= \lambda \int_a^b K(x,t) u_0(t) dt, \\ u_2(x) &= \lambda \int_a^b K(x,t) u_1(t) dt, \\ u_3(x) &= \lambda \int_a^b K(x,t) u_2(t) dt, \end{aligned} \tag{4.18}$$

and so on for other components.

In view of (4.18), the components $u_0(x), u_1(x), u_2(x), u_3(x), \ldots$ are completely determined. As a result, the solution $u(x)$ of the Fredholm integral equation (4.12) is readily obtained in a series form by using the series assumption in (4.13).

It is clearly seen that the decomposition method converted the integral equation into an elegant determination of computable components. It was formally shown that if an exact solution exists for the problem, then the obtained series converges very rapidly to that exact solution. The convergence concept of the decomposition series was thoroughly investigated by many researchers to confirm the rapid convergence of the resulting series. However, for concrete problems, where a closed form solution is not obtainable, a truncated number of terms is usually used for numerical purposes. The more components we use the higher accuracy we obtain.

Example 4.1

Solve the following Fredholm integral equation

$$u(x) = e^x - x + x \int_0^1 t u(t) dt. \tag{4.19}$$

The Adomian decomposition method assumes that the solution $u(x)$ has a series form given in (4.13). Substituting the decomposition series (4.13) into both sides of (4.19) gives

$$\sum_{n=0}^{\infty} u_n(x) = e^x - x + x\int_0^1 t\sum_{n=0}^{\infty} u_n(t)dt, \tag{4.20}$$

or equivalently

$$u_0(x) + u_1(x) + u_2(x) + \cdots = e^x - x + x\int_0^1 t\,[u_0(t) + u_1(t) + u_2(t) + \cdots]\,dt. \tag{4.21}$$

We identify the zeroth component by all terms that are not included under the integral sign. Therefore, we obtain the following recurrence relation

$$u_0(x) = e^x - x, \quad u_{k+1}(x) = x\int_0^1 tu_k(t)dt, \quad k \geqslant 0. \tag{4.22}$$

Consequently, we obtain

$$\begin{aligned}
u_0(x) &= e^x - x,\\
u_1(x) &= x\int_0^1 tu_0(t)dt = x\int_0^1 t(e^t - t)dt = \frac{2}{3}x,\\
u_2(x) &= x\int_0^1 tu_1(t)dt = x\int_0^1 \frac{2}{3}t^2dt = \frac{2}{9}x,\\
u_3(x) &= x\int_0^1 tu_2(t)dt = x\int_0^1 \frac{2}{9}t^2dt = \frac{2}{27}x,\\
u_4(x) &= x\int_0^1 tu_3(t)dt = x\int_0^1 \frac{2}{27}t^2dt = \frac{2}{81}x,
\end{aligned} \tag{4.23}$$

and so on. Using (4.13) gives the series solution

$$u(x) = e^x - x + \frac{2}{3}x\left(1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \cdots\right). \tag{4.24}$$

Notice that the infinite geometric series at the right side has $a_1 = 1$, and the ratio $r = \frac{1}{3}$. The sum of the infinite series is therefore given by

$$S = \frac{1}{1 - \frac{1}{3}} = \frac{3}{2}. \tag{4.25}$$

The series solution (4.24) converges to the closed form solution

$$u(x) = e^x, \tag{4.26}$$

obtained upon using (4.25) into (4.24).

Example 4.2

Solve the following Fredholm integral equation

$$u(x) = \sin x - x + x\int_0^{\frac{\pi}{2}} u(t)dt. \tag{4.27}$$

Proceeding as before, we substitute the decomposition series (4.13) into both sides of (4.27) to find

$$\sum_{n=0}^{\infty} u_n(x) = \sin x - x + x\int_0^{\frac{\pi}{2}} \sum_{n=0}^{\infty} u_n(t)dt, \tag{4.28}$$

or equivalently

$$u_0(x)+u_1(x)+u_2(x)+\cdots = \sin x - x + x\int_0^{\frac{\pi}{2}} \left[u_0(t)+u_1(t)+\cdots\right]dt. \tag{4.29}$$

We identify the zeroth component by all terms that are not included under the integral sign. Therefore, we obtain the following recurrence relation:

$$u_0(x) = \sin x - x, \quad u_{k+1}(x) = x\int_0^{\frac{\pi}{2}} u_k(t)dt, \quad k \geqslant 0. \tag{4.30}$$

Consequently, we obtain

$$\begin{aligned}
u_0(x) &= \sin x - x,\\
u_1(x) &= x\int_0^{\frac{\pi}{2}} u_0(t)dt = x - \frac{\pi^2}{8}x,\\
u_2(x) &= x\int_0^{\frac{\pi}{2}} u_1(t)dt = \frac{\pi^2}{8}x - \frac{\pi^4}{64}x,\\
u_3(x) &= x\int_0^{\frac{\pi}{2}} u_2(t)dt = \frac{\pi^4}{64}x - \frac{\pi^6}{512}x,\\
u_4(x) &= x\int_0^{\frac{\pi}{2}} u_3(t)dt = \frac{\pi^6}{512}x - \frac{\pi^8}{4096}x,
\end{aligned} \tag{4.31}$$

and so on. Using (4.13) gives the series solution

$$\begin{aligned}
u(x) &= \sin x - x + \left(1 - \frac{\pi^2}{8}\right)x + \left(\frac{\pi^2}{8} - \frac{\pi^4}{64}\right)x\\
&\quad + \left(\frac{\pi^4}{64} - \frac{\pi^6}{512}\right)x + \left(\frac{\pi^6}{512} - \frac{\pi^8}{4096}\right)x + \cdots .
\end{aligned} \tag{4.32}$$

We can easily observe the appearance of the noise terms, i.e the identical terms with opposite signs. Canceling these noise terms in (4.32) gives the exact solution

$$u(x) = \sin x. \tag{4.33}$$

Example 4.3

Solve the following Fredholm integral equation

$$u(x) = x + e^x - \frac{4}{3} + \int_0^1 tu(t)dt. \tag{4.34}$$

Substituting the decomposition series (4.13) into both sides of (4.34) gives

$$\sum_{n=0}^{\infty} u_n(x) = x + e^x - \frac{4}{3} + \int_0^1 t\sum_{n=0}^{\infty} u_n(t)dt, \tag{4.35}$$

or equivalently

$$u_0(x)+u_1(x)+u_2(x)+\cdots = x+e^x-\frac{4}{3}+\int_0^1 t\,[u_0(t)+u_1(t)+\cdots]\,dt. \quad (4.36)$$

Proceeding as before, we set the following recurrence relation

$$u_0(x) = x+e^x-\frac{4}{3}, \quad u_{k+1}(x) = \int_0^1 tu_k(t)dt, \quad k \geqslant 0. \quad (4.37)$$

Consequently, we obtain

$$\begin{aligned} u_0(x) &= x+e^x-\frac{4}{3}, & u_1(x) &= \int_0^1 tu_0(t)dt = \frac{2}{3}, \\ u_2(x) &= \int_0^1 tu_1(t)dt = \frac{1}{3}, & u_3(x) &= \int_0^1 tu_2(t)dt = \frac{1}{6}, \\ u_4(x) &= \int_0^1 tu_3(t)dt = \frac{1}{12}, \end{aligned} \quad (4.38)$$

and so on. Using (4.13) gives the series solution

$$u(x) = x+e^x-\frac{4}{3}+\frac{2}{3}\left(1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots\right). \quad (4.39)$$

Notice that the infinite geometric series at the right side has $a_1 = 1$, and the ratio $r = \frac{1}{2}$. The sum of the infinite series is therefore given by

$$S = \frac{1}{1-\frac{1}{2}} = 2. \quad (4.40)$$

The series solution (4.39) converges to the closed form solution

$$u(x) = x+e^x. \quad (4.41)$$

Example 4.4

Solve the following Fredholm integral equation

$$u(x) = 2+\cos x+\int_0^\pi tu(t)dt. \quad (4.42)$$

Proceeding as before we find

$$\sum_{n=0}^{\infty} u_n(x) = 2+\cos x+\int_0^\pi t\sum_{n=0}^{\infty} u_n(t)dt, \quad (4.43)$$

or equivalently

$$u_0(x)+u_1(x)+u_2(x)+\cdots = 2+\cos x+\int_0^\pi t\,[u_0(t)+u_1(t)+\cdots]\,dt. \quad (4.44)$$

We next set the following recurrence relation

$$u_0(x) = 2+\cos x, \quad u_{k+1}(x) = \int_0^\pi tu_k(t)dt, \quad k \geqslant 0. \quad (4.45)$$

This in turn gives

$$\begin{aligned} u_0(x) &= 2 + \cos x, \\ u_1(x) &= \int_0^{\pi} u_0(t)dt = -2 + \pi^2, \\ u_2(x) &= \int_0^{\pi} u_1(t)dt = -\pi^2 + \frac{1}{2}\pi^4, \\ u_3(x) &= \int_0^{\pi} u_2(t)dt = -\frac{1}{2}\pi^4 + \frac{1}{4}\pi^6, \\ u_4(x) &= \int_0^{\pi} u_3(t)dt = -\frac{1}{4}\pi^6 + \frac{1}{8}\pi^8, \end{aligned} \tag{4.46}$$

and so on. Using (4.13) gives the series solution

$$\begin{aligned} u(x) = 2 + \cos x + (-2 + \pi^2) + \left(-\pi^2 + \frac{1}{2}\pi^4\right) \\ + \left(-\frac{1}{2}\pi^4 + \frac{1}{4}\pi^6\right) + \left(-\frac{1}{4}\pi^6 + \frac{1}{8}\pi^8\right) + \cdots . \end{aligned} \tag{4.47}$$

We can easily observe the appearance of the noise terms, i.e the identical terms with opposite signs. Canceling these noise terms in (4.47) gives the exact solution

$$u(x) = \cos x. \tag{4.48}$$

Example 4.5

Solve the following Fredholm integral equation

$$u(x) = 1 + \frac{1}{2}\int_0^{\frac{\pi}{4}} \sec^2 x u(t)dt. \tag{4.49}$$

Substituting the decomposition series (4.13) into both sides of (4.49) gives

$$\sum_{n=0}^{\infty} u_n(x) = 1 + \frac{1}{2}\sec^2 x \int_0^{\frac{\pi}{4}} \sum_{n=0}^{\infty} u_n(t)dt, \tag{4.50}$$

or equivalently

$$u_0(x)+u_1(x)+u_2(x)+\cdots = 1+\frac{1}{2}\sec^2 x \int_0^{\frac{\pi}{4}} \left[u_0(t) + u_1(t) + \cdots\right] dt. \tag{4.51}$$

Proceeding as before, we set the recurrence relation

$$u_0(x) = 1, \quad u_{k+1}(x) = \frac{1}{2}\sec^2 x \int_0^{\frac{\pi}{4}} u_k(t)dt, \quad k \geqslant 0. \tag{4.52}$$

Consequently, we obtain

$$\begin{aligned} u_0(x) &= 1, \\ u_1(x) &= \frac{1}{2}\sec^2 x \int_0^{\frac{\pi}{4}} u_0(t)dt = \frac{\pi}{8}\sec^2 x, \end{aligned}$$

$$u_2(x) = \frac{1}{2}\sec^2 x \int_0^{\frac{\pi}{4}} u_1(t)dt = \frac{\pi}{16}\sec^2 x,$$
$$u_3(x) = \frac{1}{2}\sec^2 x \int_0^{\frac{\pi}{4}} u_2(t)dt = \frac{\pi}{32}\sec^2 x, \tag{4.53}$$
$$u_4(x) = \frac{1}{2}\sec^2 x \int_0^{\frac{\pi}{4}} u_3(t)dt = \frac{\pi}{64}\sec^2 x,$$

and so on. Using (4.13) gives the series solution

$$u(x) = 1 + \frac{\pi}{8}\sec^2 x\left(1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots\right). \tag{4.54}$$

The sum of the infinite series at the right side is $S = 2$. The series solution (4.54) converges to the closed form solution

$$u(x) = 1 + \frac{\pi}{4}\sec^2 x. \tag{4.55}$$

Example 4.6

Solve the following Fredholm integral equation

$$u(x) = \pi x + \sin 2x + x\int_{-\pi}^{\pi} tu(t)dt. \tag{4.56}$$

Proceeding as before we find

$$\sum_{n=0}^{\infty} u_n(x) = \pi x + \sin 2x + x\int_{-\pi}^{\pi} t\sum_{n=0}^{\infty} u_n(t)dt. \tag{4.57}$$

To determine the components of $u(x)$, we use the recurrence relation

$$u_0(x) = \pi x + \sin 2x, \quad u_{k+1}(x) = x\int_{-\pi}^{\pi} tu_k(t)dt, \quad k \geqslant 0. \tag{4.58}$$

This in turn gives

$$u_0(x) = \pi x + \sin 2x,$$
$$u_1(x) = x\int_0^{\pi} u_0(t)dt = -\pi x + \frac{2}{3}\pi^4 x,$$
$$u_2(x) = x\int_0^{\pi} u_1(t)dt = -\frac{2}{3}\pi^4 x + \frac{4}{9}\pi^7 x, \tag{4.59}$$
$$u_3(x) = x\int_0^{\pi} u_2(t)dt = -\frac{4}{9}\pi^7 x + \frac{8}{27}\pi^{10} x,$$

and so on. Using (4.13) gives the series solution

$$u(x) = \pi x + \sin 2x + \left(-\pi + \frac{2}{3}\pi^4\right)x + \left(-\frac{2}{3}\pi^4 + \frac{4}{9}\pi^7\right)x$$
$$+ \left(-\frac{4}{9}\pi^7 + \frac{8}{27}\pi^{10}\right)x + \cdots. \tag{4.60}$$

Canceling the noise terms in (4.60) gives the exact solution

$$u(x) = \sin 2x. \tag{4.61}$$

Exercises 4.2.1

In Exercises 1–20, solve the following Fredholm integral equations by using the *Adomian decomposition method*

1. $u(x) = e^x + 1 - e + \int_0^1 u(t)dt$
2. $u(x) = e^x + e^{-1}\int_0^1 u(t)dt$
3. $u(x) = \cos x + 2x + \int_0^{\pi} xtu(t)dt$
4. $u(x) = \sin x - x + \int_0^{\frac{\pi}{2}} xtu(t)dt$
5. $u(x) = e^{x+2} - 2\int_0^1 e^{x+t}u(t)dt$
6. $u(x) = e^x + \dfrac{e^{x+1}-1}{x+1} - \int_0^1 e^{xt}u(t)dt$
7. $u(x) = x + (1-x)e^x + \int_0^1 x^2 e^{t(x-1)}u(t)dt$
8. $u(x) = 1 + \frac{1}{2}\sin^2 x\int_0^{\frac{\pi}{2}} u(t)dt$
9. $u(x) = xe^x - \frac{1}{2} + \frac{1}{2}\int_0^1 u(t)dt$
10. $u(x) = x\sin x - \frac{1}{2} + \frac{1}{2}\int_0^{\frac{\pi}{2}} u(t)dt$
11. $u(x) = x\cos x + 1 + \frac{1}{2}\int_0^{\pi} u(t)dt$
12. $u(x) = \sin x + \int_0^{\frac{\pi}{2}} \sin x\cos tu(t)dt$
13. $u(x) = x + \sin x - \int_0^{\frac{\pi}{2}} xu(t)dt$
14. $u(x) = 1 - \frac{1}{15}x^2 + \int_{-1}^1 (xt + x^2t^2)u(t)dt$
15. $u(x) = 1 - \frac{19}{15}x^2 + \int_{-1}^1 (xt + x^2t^2)u(t)dt$
16. $u(x) = -x + \sin x + \int_0^{\frac{\pi}{2}} (1 + x - t)u(t)dt$
17. $u(x) = e^{-x} + \dfrac{e^{-(x+1)}-1}{x+1} + \int_0^1 e^{-xt}u(t)dt$
18. $u(x) = \frac{3}{2}e^x - \frac{1}{2}e^{x+2} + \int_0^1 e^{x+t}u(t)dt$
19. $u(x) = \frac{1}{2}\cos x + \int_0^{\frac{\pi}{2}} \cos x\sin tu(t)dt$
20. $u(x) = \frac{\pi}{4} - \sec^2 x - \int_0^{\frac{\pi}{4}} u(t)dt$

4.2.2 The Modified Decomposition Method

As stated before, the Adomian decomposition method provides the solutions in an infinite series of components. The components $u_j, j \geqslant 0$ are easily computed if the inhomogeneous term $f(x)$ in the Fredholm integral equation

$$u(x) = f(x) + \lambda\int_a^b K(x,t)u(t)dt, \tag{4.62}$$

consists of a polynomial of one or two terms. However, if the function $f(x)$ consists of a combination of two or more of polynomials, trigonometric functions, hyperbolic functions, and others, the evaluation of the components $u_j, j \geqslant 0$ requires more work. A reliable modification of the Adomian decomposition method was presented and used in Chapter 3, and it was shown that this modification facilitates the computational work and accelerates the convergence of the series solution. As presented before, the modified decomposition method depends mainly on splitting the function $f(x)$ into two parts, therefore it cannot be used if the $f(x)$ consists of only one term. The modified decomposition method will be briefly outlined here, but will be used in this section and in other chapters as well.

The standard Adomian decomposition method employs the recurrence relation

$$\begin{aligned} u_0(x) &= f(x), \\ u_{k+1}(x) &= \lambda \int_a^b K(x,t)u_k(t)dt, \quad k \geqslant 0, \end{aligned} \tag{4.63}$$

where the solution $u(x)$ is expressed by an infinite sum of components defined by

$$u(x) = \sum_{n=0}^{\infty} u_n(x). \tag{4.64}$$

In view of (4.63), the components $u_n(x)$, $n \geqslant 0$ are readily obtained.

The modified decomposition method presents a slight variation to the recurrence relation (4.63) to determine the components of $u(x)$ in an easier and faster manner. For many cases, the function $f(x)$ can be set as the sum of two partial functions, namely $f_1(x)$ and $f_2(x)$. In other words, we can set

$$f(x) = f_1(x) + f_2(x). \tag{4.65}$$

In view of (4.65), we introduce a qualitative change in the formation of the recurrence relation (4.63). The modified decomposition method identifies the zeroth component $u_0(x)$ by one part of $f(x)$, namely $f_1(x)$ or $f_2(x)$. The other part of $f(x)$ can be added to the component $u_1(x)$ that exists in the standard recurrence relation. The modified decomposition method admits the use of the modified recurrence relation:

$$\begin{aligned} u_0(x) &= f_1(x), \\ u_1(x) &= f_2(x) + \lambda \int_a^b K(x,t)u_0(t)dt, \\ u_{k+1}(x) &= \lambda \int_a^b K(x,t)u_k(t)dt, \quad k \geqslant 1. \end{aligned} \tag{4.66}$$

It is obvious that the difference between the standard recurrence relation (4.63) and the modified recurrence relation (4.66) rests only in the formation of the first two components $u_0(x)$ and $u_1(x)$ only. The other components $u_j, j \geqslant 2$ remain the same in the two recurrence relations. Although this variation in the formation of $u_0(x)$ and $u_1(x)$ is slight, however it has been shown that it accelerates the convergence of the solution and minimizes the size of calculations. Moreover, reducing the number of terms in $f_1(x)$ affects not only the component $u_1(x)$, but also the other components as well.

We here emphasize on the two important remarks made in Chapter 3. First, by proper selection of the functions $f_1(x)$ and $f_2(x)$, the exact solution $u(x)$ may be obtained by using very few iterations, and sometimes by evaluating only two components. The success of this modification depends only on the proper choice of $f_1(x)$ and $f_2(x)$, and this can be made through trials only. A rule that may help for the proper choice of $f_1(x)$ and $f_2(x)$ could not be found yet. Second, if $f(x)$ consists of one term only, the modified decomposition method cannot be used in this case.

Example 4.7

Solve the Fredholm integral equation by using the modified decomposition method

$$u(x) = 3x + e^{4x} - \frac{1}{16}(17 + 3e^4) + \int_0^1 tu(t)dt. \tag{4.67}$$

We first decompose $f(x)$ given by

$$f(x) = 3x + e^{4x} - \frac{1}{16}(17 + 3e^4), \tag{4.68}$$

into two parts, namely

$$f_1(x) = 3x + e^{4x}, \quad f_2(x) = -\frac{1}{16}(17 + 3e^4). \tag{4.69}$$

We next use the modified recurrence formula (4.66) to obtain

$$\begin{aligned} u_0(x) &= f_1(x) = 3x + e^{4x}, \\ u_1(x) &= -\frac{1}{16}(17 + 3e^4) + \int_0^1 tu_0(t)dt = 0, \\ u_{k+1}(x) &= \int_0^x K(x,t)u_k(t)dt = 0, \quad k \geqslant 1. \end{aligned} \tag{4.70}$$

It is obvious that each component of $u_j, j \geqslant 1$ is zero. This in turn gives the exact solution by

$$u(x) = 3x + e^{4x}. \tag{4.71}$$

Example 4.8

Solve the Fredholm integral equation by using the modified decomposition method

$$u(x) = \frac{1}{1+x^2} - 2\sinh\frac{\pi}{4} + \int_{-1}^1 e^{\arctan t}u(t)dt. \tag{4.72}$$

Proceeding as before we split $f(x)$ given by

$$f(x) = \frac{1}{1+x^2} - 2\sinh\frac{\pi}{4}, \tag{4.73}$$

into two parts, namely

$$f_1(x) = \frac{1}{1+x^2}, \quad f_2(x) = -2\sinh\frac{\pi}{4}. \tag{4.74}$$

We next use the modified recurrence formula (4.66) to obtain

$$\begin{aligned} u_0(x) &= f_1(x) = \frac{1}{1+x^2}, \\ u_1(x) &= -2\sinh\frac{\pi}{4} + \int_{-1}^{1} e^{\arctan t} u_0(t)dt = 0, \\ u_{k+1}(x) &= \int_{-1}^{1} e^{\arctan t} u_k(t)dt = 0, \quad k \geqslant 1. \end{aligned} \tag{4.75}$$

It is obvious that each component of $u_j, j \geqslant 1$ is zero. This in turn gives the exact solution by

$$u(x) = \frac{1}{1+x^2}. \tag{4.76}$$

Example 4.9

Solve the Fredholm integral equation by using the modified decomposition method

$$u(x) = x + \sin^{-1}\frac{x+1}{2} + \frac{2-\pi}{2}x^2 + \frac{1}{2}x^2\int_{-1}^{1} u(t)dt. \tag{4.77}$$

We decompose $f(x)$ given by

$$f(x) = x + \sin^{-1}\frac{x+1}{2} + \frac{2-\pi}{2}x^2, \tag{4.78}$$

into two parts given by

$$f_1(x) = x + \sin^{-1}\frac{x+1}{2}, \quad f_2(x) = \frac{2-\pi}{2}x^2. \tag{4.79}$$

We next use the modified recurrence formula (4.66) to obtain

$$\begin{aligned} u_0(x) &= x + \sin^{-1}\frac{x+1}{2}, \\ u_1(x) &= \frac{2-\pi}{2}x^2 + \frac{1}{2}x^2\int_{-1}^{1} u_0(t)dt = 0, \\ u_{k+1}(x) &= -\int_{-1}^{1} u_k(t)dt = 0, \quad k \geqslant 1. \end{aligned} \tag{4.80}$$

It is obvious that each component of $u_j, j \geqslant 1$ is zero. The exact solution is therefore given by

$$u(x) = x + \sin^{-1}\frac{x+1}{2}. \tag{4.81}$$

Example 4.10

Solve the Fredholm integral equation by using the modified decomposition method

$$u(x) = \sec^2 x + x^2 + x - \int_0^{\frac{\pi}{4}} \left(\frac{4}{\pi}x^2 + x\,u(t)\right) dt. \tag{4.82}$$

The function $f(x)$ consists of three terms. By trial we split $f(x)$ given by

$$f(x) = \sec^2 x + x^2 + x, \tag{4.83}$$

into two parts

$$f_1(x) = \sec^2 x, \quad f_2(x) = x^2 + x. \tag{4.84}$$

Using the modified recurrence formula (4.66) gives

$$\begin{aligned} u_0(x) &= f_1(x) = \sec^2 x, \\ u_1(x) &= x^2 + x - \int_0^{\frac{\pi}{4}} \left(\frac{4}{\pi}x^2 + xu_0(t)\right) dt = 0, \\ u_{k+1}(x) &= \int_0^{\frac{\pi}{4}} K(x,t)u_k(t)dt = 0, \quad k \geqslant 1. \end{aligned} \tag{4.85}$$

As a result, the exact solution is given by

$$u(x) = \sec^2 x. \tag{4.86}$$

Exercises 4.2.2

Use the *modified decomposition method* to solve the following Fredholm integral equations:

1. $u(x) = \sin x - x + x\int_0^{\frac{\pi}{2}} tu(t)dt$
2. $u(x) = (\pi + 2)x + \sin x - \cos x - x\int_0^{\pi} tu(t)dt$
3. $u(x) = e^x + 12x^2 + (3 + e)x - 4 - \int_0^1 (x - t)u(t)dt$
4. $u(x) = (\pi - 2)x + \sin^{-1}\dfrac{x+1}{2} - \sin^{-1}\dfrac{x-1}{2} - \int_0^1 xu(t)dt$
5. $u(x) = -6 + 14x^4 + 21x^2 + x - \int_{-1}^1 (x^4 - t^4)u(t)dt$
6. $u(x) = x + x^4 + 9e^{x+1} - 23e^x - \int_0^1 e^{x+t}u(t)dt$
7. $u(x) = x + e^x - 2e^{x-1} + 2e^x - \int_0^1 e^{x-t}u(t)dt$
8. $u(x) = xe^x + \dfrac{xe^{x+1}+1}{(x+1)^2} - \int_0^1 e^{xt}u(t)dt$
9. $u(x) = e^{x+1} + e^{x-1} + (e^2 + 1)e^{x-1} - \int_0^1 e^{x-t}u(t)dt$
10. $u(x) = \dfrac{2}{15} + \dfrac{7}{12}x + x^2 + x^3 - \int_0^1 (1 + x - t)u(t)dt$

11. $u(x) = e^{2x} - \frac{1}{4}(e^2+1)x + \int_0^1 xtu(t)dt$

12. $u(x) = e^x(e^{\frac{1}{3}}-1) + e^{2x} - \frac{1}{3}\int_0^1 e^{x-\frac{5}{3}t}u(t)dt$

13. $u(x) = x + e^x - xe^x + \int_0^1 x^2 e^{t(x-1)}u(t)dt$

14. $u(x) = (\pi^2+2\pi-4) - (\pi+2)x + x(\sin x - \cos x) + \int_0^{\pi}(x-t)u(t)dt$

15. $u(x) = \frac{(\pi-2)(x+1)}{2} + x\tan^{-1}x - \int_{-1}^1 (1+x-t)u(t)dt$

16. $u(x) = \ln\left(\frac{1+e}{2}\right)x + \frac{e^x}{1+e^x} - \int_0^1 xu(t)dt$

4.2.3 The Noise Terms Phenomenon

It was shown that a proper selection of $f_1(x)$ and $f_2(x)$ is essential to use the modified decomposition method. However, the *noise terms phenomenon*, that was introduced in Chapter 3, demonstrated a fast convergence of the solution. This phenomenon was presented before, therefore we present here the main steps for using this effect concept. The noise terms as defined before are the identical terms with opposite signs that may appear between components $u_0(x)$ and $u_1(x)$. Other noise terms may appear between other components. By canceling the noise terms between $u_0(x)$ and $u_1(x)$, even though $u_1(x)$ contains further terms, the remaining non-canceled terms of $u_0(x)$ may give the exact solution of the integral equation. The appearance of the noise terms between $u_0(x)$ and $u_1(x)$ is not always sufficient to obtain the exact solution by canceling these noise terms. Therefore, it is necessary to show that the non-canceled terms of $u_0(x)$ satisfy the given integral equation.

It was formally proved in [6] that a necessary condition for the appearance of the noise terms is required. The conclusion made in [6] is that the zeroth component $u_0(x)$ must contain the exact solution $u(x)$ among other terms.

The phenomenon of the useful noise terms will be explained by the following illustrative examples.

Example 4.11

Solve the Fredholm integral equation by using the noise terms phenomenon:

$$u(x) = x\sin x - x + \int_0^{\frac{\pi}{2}} xu(t)dt. \tag{4.87}$$

Following the standard Adomian method we set the recurrence relation

$$u_0(x) = x\sin x - x,$$
$$u_{k+1}(x) = \int_0^{\frac{\pi}{2}} xu_k(t)dt, \quad k \geqslant 0. \tag{4.88}$$

This gives

$$u_0(x) = x\sin x - x,$$
$$u_1(x) = \int_0^{\frac{\pi}{2}} xu_0(t)dt = x - \frac{\pi^2}{8}x. \tag{4.89}$$

The noise terms $\mp x$ appear in $u_0(x)$ and $u_1(x)$. Canceling this term from the zeroth component $u_0(x)$ gives the exact solution

$$u(x) = x\sin x, \tag{4.90}$$

that justifies the integral equation. The other terms of $u_1(x)$ vanish in the limit with other terms of the other components.

Example 4.12

Solve the Fredholm integral equation by using the noise terms phenomenon:

$$u(x) = \sin x + \cos x - \frac{\pi}{2}x + \int_0^{\frac{\pi}{2}} xtu(t)dt. \tag{4.91}$$

The standard Adomian method gives the recurrence relation

$$u_0(x) = \sin x + \cos x - \frac{\pi}{2}x,$$
$$u_{k+1}(x) = \int_0^{\frac{\pi}{2}} xtu_k(t)dt, \quad k \geqslant 0. \tag{4.92}$$

This gives

$$u_0(x) = \sin x + \cos x - \frac{\pi}{2}x,$$
$$u_1(x) = \int_0^{\frac{\pi}{2}} xtu_0(t)dt = \frac{\pi}{2}x - \frac{\pi^4}{48}x. \tag{4.93}$$

The noise terms $\mp\frac{\pi}{2}x$ appear in $u_0(x)$ and $u_1(x)$. Canceling this term from the zeroth component $u_0(x)$ gives the exact solution

$$u(x) = \sin x + \cos x, \tag{4.94}$$

that justifies the integral equation. It is to be noted that the other terms of $u_1(x)$ vanish in the limit with other terms of the other components.

Example 4.13

Solve the Fredholm integral equation by using the noise terms phenomenon:

$$u(x) = x^2 + \frac{\sin x}{1+\cos x} + \frac{\pi^3}{24}x + x\ln 2 - x\int_0^{\frac{\pi}{2}} u(t)dt. \tag{4.95}$$

The standard Adomian method gives the recurrence relation:

$$u_0(x) = x^2 + \frac{\sin x}{1+\cos x} + \frac{\pi^3}{24}x + x\ln 2,$$
$$u_{k+1}(x) = -x\int_0^{\frac{\pi}{2}} u(t)dt, \quad k \geqslant 0. \tag{4.96}$$

This gives

$$u_0(x) = x^2 + \frac{\sin x}{1+\cos x} + \frac{\pi^3}{24}x + x\ln 2,$$
$$u_1(x) = -\int_0^{\frac{\pi}{2}} xu_0(t)dt = -\frac{\pi^3}{24}x - x\ln 2 - \frac{\pi^2\ln 2}{8}x - \frac{\pi^5}{192}x. \tag{4.97}$$

The noise terms $\pm\frac{\pi^3}{24}x, \pm x\ln 2$ appear in $u_0(x)$ and $u_1(x)$. Canceling these terms from the zeroth component $u_0(x)$ gives the exact solution

$$u(x) = x^2 + \frac{\sin x}{1+\cos x}, \tag{4.98}$$

that justifies the integral equation. The other terms of $u_1(x)$ vanish in the limit with other terms of the other components.

Example 4.14

Solve the Fredholm integral equation by using the noise terms phenomenon

$$u(x) = x^2 + x\cos x + \frac{\pi^3}{3}x - 2x - x\int_0^{\pi} u(t)dt. \tag{4.99}$$

Proceeding as before, we set the recurrence relation

$$u_0(x) = x^2 + x\cos x + \frac{\pi^3}{3}x - 2x,$$
$$u_{k+1}(x) = -x\int_0^{\pi} u_k(t)dt, \quad k \geqslant 0. \tag{4.100}$$

This gives

$$u_0(x) = x^2 + x\cos x + \frac{\pi^3}{3}x - 2x,$$
$$u_1(x) = -x\int_0^{\pi} u_0(t)dt = -x\left(-2 - \pi^2 + \frac{\pi^3}{3} + \frac{\pi^5}{6}\right). \tag{4.101}$$

The noise terms $\pm\frac{\pi^3}{3}x, \mp 2x$ appear in $u_0(x)$ and $u_1(x)$. Canceling these terms from the zeroth component $u_0(x)$ gives the exact solution

$$u(x) = x^2 + x\cos x, \tag{4.102}$$

that justifies the integral equation (4.99).

Exercises 4.2.3

Use the *noise terms phenomenon* to solve the following Fredholm integral equations:

1. $u(x) = 1 + \left(\frac{\pi}{2} + \ln 2\right)x + \frac{\cos x}{1+\sin x} - \int_0^{\frac{\pi}{2}} xu(t)dt$

2. $u(x) = x\left(1+\pi+\frac{\pi^2}{2}\right) + x\sin x - \int_0^{\pi} xu(t)dt$

3. $u(x) = x\left(1+\frac{\pi^3}{192}\right) + x^2 + \sec^2 x - \int_0^{\frac{\pi}{4}} xu(t)dt$

4. $u(x) = x(\pi - 2 + \sin x + \cos x) - \int_0^{\pi} xu(t)dt$

5. $u(x) = x\left(1+\frac{\pi^2}{8}+\ln 2\right) + \frac{\sin x}{1+\cos x} - \int_0^{\frac{\pi}{2}} xu(t)dt$

6. $u(x) = x(1+\ln 2) + \sin x + \frac{\cos x}{1+\sin x} - \int_0^{\frac{\pi}{2}} xu(t)dt$

7. $u(x) = \frac{\pi}{2}x + \frac{\sin x(2+\sin x)}{1+\sin x} - \int_0^{\frac{\pi}{2}} xu(t)dt$

8. $u(x) = \left(\frac{\pi}{2}-1\right)x + \frac{\sin x}{1+\sin x} - \int_0^{\frac{\pi}{2}} xu(t)dt$

9. $u(x) = \left(\frac{\pi-\sqrt{3}}{3}\right)x + \frac{\cos x}{1+\sin x} - \int_0^{\frac{\pi}{3}} xu(t)dt$

10. $u(x) = \left(4-\frac{\pi^2}{2}\right)x + x^2(\sin x + \cos x) + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} xu(t)dt$

11. $u(x) = x\left(\frac{1}{4}+\sin(2x)\right) - \int_0^{\frac{\pi}{4}} xu(t)dt$

12. $u(x) = \left(\frac{\pi-2}{8}\right)x + x\cos(2x) - \int_0^{\frac{\pi}{4}} xu(t)dt$

13. $u(x) = \frac{1}{1+x^2} - \frac{\pi^2}{32}x + \int_0^1 x\tan^{-1} tu(t)dt$

14. $u(x) = \pi x + \cos^{-1} x - \int_{-1}^{1} xu(t)dt$

15. $u(x) = -\frac{\pi}{4}x + x\cos^{-1} x - \int_{-1}^{1} xu(t)dt$

16. $u(x) = \left(\frac{\pi-2}{2}\right)x + x\tan^{-1} x - \int_{-1}^{1} xu(t)dt$

4.2.4 The Variational Iteration Method

In Chapter 3, the *variational iteration method* was used to handle Volterra integral equations, where the Volterra integral equation was converted to an initial value problem or to an equivalent integro-differential equation. The method provides rapidly convergent successive approximations to the exact solution if such a closed form solution exists.

In this section, we will apply the variational iteration method to handle Fredholm integral equation. The method works effectively if the kernel

$K(x,t)$ is separable and can be written in the form $K(x,t) = g(x)h(t)$. The approach to be used here is identical to the approach used in the previous chapter. This means that we should differentiate both sides of the Fredholm integral equation to convert it to an identical Fredholm integro-differential equation. It is important to note that integro-differential equation needs an initial condition that should be defined. In view of this fact, we will study only the cases where $g(x) = x^n, n \geqslant 1$. Solving Fredholm integro-differential equation by the variational iteration method will be studied again in details in Chapter 6.

The standard Fredholm integral equation is of the form

$$u(x) = f(x) + \int_a^b K(x,t)u(t)dt, \tag{4.103}$$

or equivalently

$$u(x) = f(x) + g(x)\int_a^b h(t)u(t)dt. \tag{4.104}$$

Recall that the integral at the right side represents a constant value. Differentiating both sides of (4.104) with respect to x gives

$$u'(x) = f'(x) + g'(x)\int_a^b h(t)u(t)dt. \tag{4.105}$$

The correction functional for the integro-differential equation (4.105) is

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(\xi)\left(u_n'(\xi) - f'(\xi) - g'(\xi)\int_a^b h(r)\tilde{u}_n(r)dr\right)d\xi. \tag{4.106}$$

As presented before, the variational iteration method is used by applying two essential steps. It is required first to determine the Lagrange multiplier $\lambda(\xi)$ that can be identified optimally via integration by parts and by using a restricted variation. However, $\lambda(\xi) = -1$ for first order integro-differential equations. Having determined λ, an iteration formula, without restricted variation, given by

$$u_{n+1}(x) = u_n(x) - \int_0^x \left(u_n'(\xi) - f'(\xi) - g'(\xi)\int_a^b h(r)u_n(r)dr\right)d\xi, \tag{4.107}$$

is used for the determination of the successive approximations $u_{n+1}(x), n \geqslant 0$ of the solution $u(x)$. The zeroth approximation u_0 can be any selective function. However, using the given initial value $u(0)$ is preferably used for the selective zeroth approximation u_0 as will be seen later. Consequently, the solution is given by

$$u(x) = \lim_{n\to\infty} u_n(x). \tag{4.108}$$

The variational iteration method will be illustrated by studying the following Fredholm integral equations.

Example 4.15

Use the variational iteration method to solve the Fredholm integral equation

$$u(x) = e^x - x + x\int_0^1 tu(t)dt. \qquad (4.109)$$

Differentiating both sides of this equation with respect to x yields

$$u'(x) = e^x - 1 + \int_0^1 tu(t)dt. \qquad (4.110)$$

The correction functional for this equation is given by

$$u_{n+1}(x) = u_n(x) - \int_0^x \left(u_n'(\xi) - e^\xi + 1 - \int_0^1 ru_n(r)dr\right) d\xi, \qquad (4.111)$$

where we used $\lambda = -1$ for first-order integro-differential equations. Notice that the initial condition $u(0) = 1$ is obtained by substituting $x = 0$ into (4.109).

We can use the initial condition to select $u_0(x) = u(0) = 1$. Using this selection into the correction functional gives the following successive approximations

$$\begin{aligned}
u_0(x) &= 1,\\
u_1(x) &= u_0(x) - \int_0^x \left(u_0'(\xi) - e^\xi + 1 - \int_0^1 ru_0(r)dr\right) d\xi = e^x - \frac{1}{2}x,\\
u_2(x) &= u_1(x) - \int_0^x \left(u_1'(\xi) - e^\xi + 1 - \int_0^1 ru_1(r)dr\right) d\xi = e^x - \frac{1}{2\times 3}x,\\
u_3(x) &= u_2(x) - \int_0^x \left(u_2'(\xi) - e^\xi + 1 - \int_0^1 ru_2(r)dr\right) d\xi = e^x - \frac{1}{2\times 3^2}x,\\
u_4(x) &= u_3(x) - \int_0^x \left(u_3'(\xi) - e^\xi + 1 - \int_0^1 ru_3(r)dr\right) d\xi = e^x - \frac{1}{2\times 3^3}x,\\
&\ \ \vdots\\
u_{n+1} &= e^x - \frac{1}{2\times 3^n}x, n \geqslant 0. \qquad (4.112)
\end{aligned}$$

The VIM admits the use of

$$u(x) = \lim_{n\to\infty} u_n(x) = e^x. \qquad (4.113)$$

Example 4.16

Use the variational iteration method to solve the Fredholm integral equation

$$u(x) = \sin x - x + x\int_0^{\frac{\pi}{2}} u(t)dt. \qquad (4.114)$$

Differentiating both sides of this equation with respect to x gives

$$u'(x) = \cos x - 1 + \int_0^{\frac{\pi}{2}} u(t)dt. \tag{4.115}$$

The correction functional for this equation is given by

$$u_{n+1}(x) = u_n(x) - \int_0^x \left(u_n'(\xi) - \cos\xi + 1 - \int_0^{\frac{\pi}{2}} u_n(r)dr \right) d\xi, \tag{4.116}$$

where we used $\lambda = -1$ for first-order integro-differential equations. The initial condition $u(0) = 0$ is obtained by substituting $x = 0$ into (4.114).

We can use the initial condition to select $u_0(x) = u(0) = 0$. Using this selection into the correction functional gives the following successive approximations

$$\begin{aligned} u_0(x) &= 0, \\ u_1(x) &= u_0(x) - \int_0^x \left(u_0'(\xi) - \cos\xi + 1 - \int_0^{\frac{\pi}{2}} u_0(r)dr \right) d\xi \\ &= \sin x - x, \\ u_2(x) &= u_1(x) - \int_0^x \left(u_1'(\xi) - \cos\xi + 1 - \int_0^{\frac{\pi}{2}} u_1(r)dr \right) d\xi \\ &= (\sin x - x) + \left(x - \frac{\pi^2}{8}x \right), \\ u_3(x) &= u_2(x) - \int_0^x \left(u_2'(\xi) - \cos\xi + 1 - \int_0^{\frac{\pi}{2}} u_2(r)dr \right) d\xi \\ &= (\sin x - x) + \left(x - \frac{\pi^2}{8}x \right) + \left(\frac{\pi^2}{8}x - \frac{\pi^4}{64}x \right), \end{aligned} \tag{4.117}$$

and so on. Canceling the noise terms, the exact solution is given by

$$u(x) = \sin x. \tag{4.118}$$

Example 4.17

Use the variational iteration method to solve the Fredholm integral equation

$$u(x) = -2x + \sin x + \cos x + \int_0^{\pi} xu(t)dt. \tag{4.119}$$

Differentiating both sides of this equation with respect to x gives

$$u'(x) = -2 + \cos x - \sin x + \int_0^{\pi} u(t)dt. \tag{4.120}$$

The correction functional for this equation is given by

$$u_{n+1}(x) = u_n(x) - \int_0^x \left(u_n'(\xi) + 2 - \cos\xi + \sin\xi - \int_0^{\pi} u_n(r)dr \right) d\xi. \tag{4.121}$$

The initial condition $u(0) = 1$ is obtained by substituting $x = 0$ into (4.119). Using this selection into the correction functional gives the following successive approximations

$$\begin{aligned}
u_0(x) &= 1,\\
u_1(x) &= u_0(x) - \int_0^x \left(u_0'(\xi) + 2 - \cos\xi + \sin\xi - \int_0^\pi u_0(r)dr\right) d\xi\\
&= \sin x + \cos x + (\pi x - 2x),\\
u_2(x) &= u_1(x) - \int_0^x \left(u_1'(\xi) + 2 - \cos\xi + \sin\xi - \int_0^\pi u_1(r)dr\right) d\xi\\
&= \sin x + \cos x + (\pi x - 2x) + \left(-\pi x + 2x - \pi^2 x + \frac{\pi^3}{2}x\right),\\
u_3(x) &= u_2(x) - \int_0^x \left(u_2'(\xi) + 2 - \cos\xi + \sin\xi - \int_0^\pi u_2(r)dr\right) d\xi\\
&= \sin x + \cos x + (\pi x - 2x) + \left(-\pi x + 2x - \pi^2 x + \frac{\pi^3}{2}x\right)\\
&\quad + \left(\pi^2 x - \frac{\pi^3}{2}x + \cdots\right),
\end{aligned} \tag{4.122}$$

and so on. Canceling the noise terms, the exact solution is given by

$$u(x) = \sin x + \cos x. \tag{4.123}$$

Example 4.18

Use the variational iteration method to solve the Fredholm integral equation

$$u(x) = -x^3 + \cos x + \int_0^{\frac{\pi}{2}} x^3 u(t)dt. \tag{4.124}$$

Differentiating both sides of this equation with respect to x gives

$$u'(x) = -3x^2 - \sin x + 3x^2 \int_0^{\frac{\pi}{2}} u(t)dt. \tag{4.125}$$

The correction functional for this equation is given by

$$u_{n+1}(x) = u_n(x) - \int_0^x \left(u_n'(\xi) + 3\xi^2 + \sin\xi - 3\xi^2 \int_0^{\frac{\pi}{2}} u_n(r)dr\right) d\xi. \tag{4.126}$$

The initial condition is $u(0) = 1$. Using this selection into the correction functional gives the following successive approximations

$$\begin{aligned}
u_0(x) &= 1,\\
u_1(x) &= u_0(x) - \int_0^x \left(u_0'(\xi) + 3\xi^2 + \sin\xi - 3\xi^2 \int_0^{\frac{\pi}{2}} u_0(r)dr\right) d\xi\\
&= \cos x + \left(\frac{\pi}{2}x^3 - x^3\right),\\
u_2(x) &= u_1(x) - \int_0^x \left(u_1'(\xi) + 3\xi^2 + \sin\xi - 3\xi^2 \int_0^{\frac{\pi}{2}} u_1(r)dr\right) d\xi\\
&= \cos x + \left(\frac{\pi}{2}x^3 - x^3\right) + \left(-\frac{\pi}{2}x^3 + x^3 - \frac{\pi^4}{64}x^3 + \frac{\pi^5}{128}x^3\right),
\end{aligned} \tag{4.127}$$

and so on. Canceling the noise terms, the exact solution is given by

$$u(x) = \cos x. \tag{4.128}$$

Exercises 4.2.4

Solve the following Fredholm integral equations by using the *the variational iteration method*

1. $u(x) = \cos x + 2x + \int_0^{\pi} xtu(t)dt$
2. $u(x) = \sin x - x + \int_0^{\frac{\pi}{2}} xtu(t)dt$
3. $u(x) = 1 - \frac{1}{15}x^2 + \int_{-1}^{1} (xt + x^2t^2)u(t)dt$
4. $u(x) = 1 - \frac{19}{15}x^2 + \int_{-1}^{1} (xt + x^2t^2)u(t)dt$
5. $u(x) = (\pi + 2)x + \sin x - \cos x - x\int_0^{\pi} tu(t)dt$
6. $u(x) = e^{2x} - \frac{1}{4}(e^2 + 1)x + \int_0^1 xtu(t)dt$
7. $u(x) = 1 + 9x + 2x^2 + x^3 - \int_0^1 (20xt + 10x^2t^2)u(t)dt$
8. $u(x) = \frac{\pi}{2}x + \sin x - \cos x + \int_0^{\pi} x\cos tu(t)dt$
9. $u(x) = 1 + x + e^x - \frac{2}{3}\int_0^1 xtu(t)dt$
10. $u(x) = 2x + e^x - \frac{3}{4}\int_0^1 xtu(t)dt$
11. $u(x) = x\left(\frac{1}{4} + \sin(2x)\right) - \int_0^{\frac{\pi}{4}} xu(t)dt$
12. $u(x) = e^x + 2xe^{-1} - \int_{-1}^{1} xtu(t)dt$
13. $u(x) = \left(\frac{\pi^3}{3} + 3\right)x - \cos x - \int_0^{\pi} xtu(t)dt$
14. $u(x) = \frac{\pi^3}{24}x - \sin x - \int_0^{\frac{\pi}{2}} xtu(t)dt$
15. $u(x) = \sqrt{2}\,x + \sec x + \tan x - \int_0^{\frac{\pi}{4}} x\sec tu(t)dt$
16. $u(x) = \frac{\pi}{8}x + \sin x + \cos x - \int_0^{\frac{\pi}{4}} x\sin tu(t)dt$

4.2.5 The Direct Computation Method

In this section, the *direct computation method* will be applied to solve the Fredholm integral equations. The method approaches Fredholm integral equations in a direct manner and gives the solution in an exact form and not in a series form. It is important to point out that this method will be applied for the degenerate or separable kernels of the form

$$K(x,t) = \sum_{k=1}^{n} g_k(x)h_k(t). \tag{4.129}$$

Examples of separable kernels are $x-t, xt, x^2-t^2, xt^2+x^2t$, etc.

The direct computation method can be applied as follows:

1. We first substitute (4.129) into the Fredholm integral equation the form

$$u(x) = f(x) + \int_a^b K(x,t)\,u(t)dt. \tag{4.130}$$

2. This substitution gives

$$\begin{aligned} u(x) = f(x) + g_1(x)\int_a^b h_1(t)u(t)dt + g_2(x)\int_a^b h_2(t)u(t)dt + \cdots \\ +g_n(x)\int_a^b h_n(t)u(t)dt. \end{aligned} \tag{4.131}$$

3. Each integral at the right side depends only on the variable t with constant limits of integration for t. This means that each integral is equivalent to a constant. Based on this, Equation (4.131) becomes

$$u(x) = f(x) + \lambda\alpha_1 g_1(x) + \lambda\alpha_2 g_2(x) + \cdots + \lambda\alpha_n g_n(x), \tag{4.132}$$

where

$$\alpha_i = \int_a^b h_i(t)u(t)dt, \quad 1 \leqslant i \leqslant n. \tag{4.133}$$

4. Substituting (4.132) into (4.133) gives a system of n algebraic equations that can be solved to determine the constants $\alpha_i, 1 \leqslant i \leqslant n$. Using the obtained numerical values of α_i into (4.132), the solution $u(x)$ of the Fredholm integral equation (4.130) is readily obtained.

Example 4.19

Solve the Fredholm integral equation by using the *direct computation method*

$$u(x) = 3x + 3x^2 + \frac{1}{2}\int_0^1 x^2 t u(t)dt. \tag{4.134}$$

The kernel $K(x,t) = x^2 t$ is separable. Consequently, we rewrite (4.134) as

$$u(x) = 3x + 3x^2 + \frac{1}{2}x^2\int_0^1 tu(t)dt. \tag{4.135}$$

The integral at the right side is equivalent to a constant because it depends only on functions of the variable t with constant limits of integration. Consequently, Equation (4.135) can be rewritten as

$$u(x) = 3x + 3x^2 + \frac{1}{2}\alpha x^2, \tag{4.136}$$

where

$$\alpha = \int_0^1 tu(t)dt. \tag{4.137}$$

To determine α, we substitute (4.136) into (4.137) to obtain

$$\alpha = \int_0^1 t\left(3t + 3t^2 + \frac{1}{2}\alpha t^2\right)dt. \tag{4.138}$$

Integrating the right side of (4.138) yields

$$\alpha = \frac{7}{4} + \frac{1}{8}\alpha, \tag{4.139}$$

that gives

$$\alpha = 2. \tag{4.140}$$

Substituting (4.140) into (4.136) leads to the exact solution

$$u(x) = 3x + 4x^2, \tag{4.141}$$

obtained by substituting $\alpha = 2$ in (4.136).

Example 4.20

Solve the Fredholm integral equation by using the *direct computation method*

$$u(x) = \frac{1}{3}x + \sec x \tan x - \frac{1}{3}x \int_0^{\frac{\pi}{3}} u(t)dt. \tag{4.142}$$

The integral at the right side is equivalent to a constant because it depends only on functions of the variable t with constant limits of integration. Consequently, Equation (4.142) can be rewritten as

$$u(x) = \frac{1}{3}x + \sec x \tan x - \frac{1}{3}\alpha x, \tag{4.143}$$

where

$$\alpha = \int_0^{\frac{\pi}{3}} u(t)dt. \tag{4.144}$$

To determine α, we substitute (4.143) into (4.144) to obtain

$$\alpha = \int_0^{\frac{\pi}{3}} \left(\frac{1}{3}t + \sec t \tan t - \frac{1}{3}\alpha t\right) dt. \tag{4.145}$$

Integrating the right side of (4.145) yields

$$\alpha = 1 + \frac{1}{54}\pi^2 - \frac{1}{54}\alpha\pi^2, \tag{4.146}$$

that gives

$$\alpha = 1. \tag{4.147}$$

Substituting (4.147) into (4.143) gives the exact solution

$$u(x) = \sec x \tan x. \tag{4.148}$$

Example 4.21

Solve the Fredholm integral equation by using the *direct computation method*

$$u(x) = 11x + 10x^2 + x^3 - \int_0^1 (30xt^2 + 20x^2t)u(t)dt. \tag{4.149}$$

The kernel $K(x,t) = 30xt^2 + 20x^2t$ is separable. Consequently, we rewrite (4.149) as

$$u(x) = 11x + 10x^2 + x^3 - 30x \int_0^1 t^2u(t)dt - 20x^2 \int_0^1 tu(t)dt. \tag{4.150}$$

Each integral at the right side is equivalent to a constant because it depends only on functions of the variable t with constant limits of integration. Consequently, Equation (4.150) can be rewritten as

$$u(x) = (11 - 30\alpha)x + (10 - 20\beta)x^2 + x^3, \tag{4.151}$$

where

$$\begin{aligned} \alpha &= \int_0^1 t^2 u(t)dt, \\ \beta &= \int_0^1 tu(t)dt. \end{aligned} \tag{4.152}$$

To determine the constants α and β, we substitute (4.151) into (4.152) to obtain

$$\begin{aligned} \alpha &= \int_0^1 t^2((11 - 30\alpha)x + (10 - 20\beta)x^2 + x^3)dt = \frac{59}{12} - \frac{15}{2}\alpha - 4\beta, \\ \beta &= \int_0^1 t((11 - 30\alpha)t + (10 - 20\beta)t^2 + x^3)dt = \frac{191}{30} - 10\alpha - 5\beta. \end{aligned} \tag{4.153}$$

Solving this system of algebraic equations gives

$$\alpha = \frac{11}{30}, \quad \beta = \frac{9}{20}. \tag{4.154}$$

Substituting (4.154) into (4.151) gives the exact solution

$$u(x) = x^2 + x^3. \tag{4.155}$$

Example 4.22

Solve the Fredholm integral equation by using the *direct computation method*

$$u(x) = 4 + 45x + 26x^2 - \int_0^1 (1 + 30xt^2 + 12x^2t)u(t)dt. \tag{4.156}$$

The kernel $K(x,t) = 1 + 30xt^2 + 12x^2t$ is separable. Consequently, we rewrite (4.156) as

$$u(x) = 4 + 45x + 26x^2 - \int_0^1 u(t)dt - 30x\int_0^1 t^2u(t)dt - 12x^2\int_0^1 tu(t)dt. \tag{4.157}$$

Each integral at the right side is equivalent to a constant because it depends only on functions of the variable t with constant limits of integration. Consequently, Equation (4.157) can be rewritten as

$$u(x) = (4 - \alpha) + (45 - 30\beta)x + (26 - 12\gamma)x^2, \tag{4.158}$$

where

$$\alpha = \int_0^1 u(t)dt, \quad \beta = \int_0^1 t^2u(t)dt, \quad \gamma = \int_0^1 tu(t)dt. \tag{4.159}$$

To determine the constants α, β and γ, we substitute (4.158) into each equation of (4.159) to obtain

$$\begin{aligned}
\alpha &= \int_0^1 \left((4-\alpha)+(45-30\beta)t+(26-12\gamma)t^2\right) dt \\
&= \frac{211}{6} - \alpha - 15\beta - 4\gamma, \\
\beta &= \int_0^1 t^2 \left((4-\alpha)+(45-30\beta)t+(26-12\gamma)t^2\right) dt \\
&= \frac{1067}{60} - \frac{1}{3}\alpha - \frac{15}{2}\beta - \frac{12}{5}\gamma, \\
\gamma &= \int_0^1 t \left((4-\alpha)+(45-30\beta)t+(26-12\gamma)t^2\right) dt \\
&= \frac{47}{2} - \frac{1}{2}\alpha - 10\beta - 3\gamma.
\end{aligned} \tag{4.160}$$

Unlike the previous examples, we obtain a system of three equations in three unknowns α, β, and γ. Solving this system of algebraic equations gives

$$\alpha = 3, \quad \beta = \frac{43}{30}, \quad \gamma = \frac{23}{12}. \tag{4.161}$$

Substituting (4.161) into (4.158), the exact solution is given by

$$u(x) = 1 + 2x + 3x^2. \tag{4.162}$$

Exercises 4.2.5

Use the *direct computation method* to solve the following Fredholm integral equations:

1. $u(x) = 1 + 9x + 2x^2 + x^3 - \int_0^1 (20xt + 10x^2t^2)u(t)dt$
2. $u(x) = -8 + 11x - x^2 + x^3 - \int_0^1 (12x - 20t)u(t)dt$
3. $u(x) = -11 + 9x + x^3 + x^4 - \int_0^1 (20x - 30t)u(t)dt$
4. $u(x) = -15 + 10x^3 + x^4 - \int_0^1 (20x^3 - 56t^3)u(t)dt$
5. $u(x) = 1 + 7x + 20x^2 + x^3 - \int_0^1 (10xt^2 + 20x^2t)u(t)dt$
6. $u(x) = \left(\frac{2}{\sqrt{3}} - 1\right) x + \sec x \tan x - \int_0^{\frac{\pi}{6}} xu(t)dt$
7. $u(x) = \left(\frac{2\pi}{3} - \ln(2+\sqrt{3})\right) x + \sec x \tan x - \int_0^{\frac{\pi}{3}} xtu(t)dt$
8. $u(x) = 1 + \int_{0^+}^1 \ln(xt)u(t)dt, 0 < x \leqslant 1$
9. $u(x) = 1 + \ln x - \int_{0^+}^1 \ln(xt^2)u(t)dt, 0 < x \leqslant 1$

10. $u(x) = 1 + \ln x - \int_{0^+}^{1} \ln \frac{x}{t^2} u(t)dt, 0 < x \leqslant 1$

11. $u(x) = \sin x + (\pi - 1)\cos x - \cos x \int_0^{\pi} tu(t)dt$

12. $u(x) = \frac{\pi}{2}x + \sin x - \cos x + \int_0^{\pi} x \cos tu(t)dt$

13. $u(x) = 1 + \frac{\pi}{2}\sec^2 x - \int_0^{\frac{\pi}{4}} \sec^2 xu(t)dt$

14. $u(x) = 1 - \int_0^{\frac{\pi}{4}} \sec^2 xu(t)dt$

15. $u(x) = 1 + x + e^x - \frac{2}{3}\int_0^1 xtu(t)dt$

16. $u(x) = 2x + e^x - \frac{3}{4}\int_0^1 xtu(t)dt$

4.2.6 The Successive Approximations Method

The *successive approximations method*, or the *Picard iteration method* was introduced before in Chapter 3. The method provides a scheme that can be used for solving initial value problems or integral equations. This method solves any problem by finding successive approximations to the solution by starting with an initial guess as $u_0(x)$, called the zeroth approximation. As will be seen, the zeroth approximation is any selective real-valued function that will be used in a recurrence relation to determine the other approximations. The most commonly used values for the zeroth approximations are $0, 1$, or x. Of course, other real values can be selected as well.

Given Fredholm integral equation of the second kind

$$u(x) = f(x) + \lambda \int_a^b K(x,t)u(t)dt, \tag{4.163}$$

where $u(x)$ is the unknown function to be determined, $K(x,t)$ is the kernel, and λ is a parameter. The successive approximations method introduces the recurrence relation

$$\begin{aligned} u_0(x) &= \text{any selective real valued function}, \\ u_{n+1}(x) &= f(x) + \lambda \int_a^b K(x,t)u_n(t)dt, \quad n \geqslant 0. \end{aligned} \tag{4.164}$$

The question of convergence of $u_n(x)$ is justified by Theorem 3.1. At the limit, the solution is determined by using the limit

$$u(x) = \lim_{n\to\infty} u_{n+1}(x). \tag{4.165}$$

It is interesting to point out that the Adomian decomposition method admits the use of an iteration formula of the form

$$\begin{aligned} u_0(x) &= \text{all terms not included inside the integral sign}, \\ u_1(x) &= \lambda \int_a^b K(x,t)u_0(t)dt, \end{aligned}$$

$$u_2(x) = \lambda \int_a^b K(x,t)u_1(t)dt, \tag{4.166}$$

$$\vdots$$

$$u_{n+1}(x) = u_n(x) + \lambda \int_a^b K(x,t)u_n(t)dt.$$

The difference between the two formulas (4.164) and (4.166) can be summarized as follows:

1. The successive approximations method gives successive approximations of the solution $u(x)$, whereas the Adomian method gives successive components of the solution $u(x)$.

2. The successive approximations method admits the use of a selective real-valued function for the zeroth approximation u_0, whereas the Adomian decomposition method assigns all terms that are not inside the integral sign for the zeroth component $u_0(x)$. Recall that this assignment was modified when using the modified decomposition method.

3. The successive approximations method gives the exact solution, if it exists, by

$$u(x) = \lim_{n\to\infty} u_{n+1}(x). \tag{4.167}$$

However, the Adomian decomposition method gives the solution as infinite series of components by

$$u(x) = \sum_{n=0}^{\infty} u_n(x). \tag{4.168}$$

This series solution converges rapidly to the exact solution if such a solution exists.

The successive approximations method, or the iteration method will be illustrated by studying the following examples.

Example 4.23

Solve the Fredholm integral equation by using the successive approximations method

$$u(x) = x + e^x - \int_0^1 xtu(t)dt. \tag{4.169}$$

For the zeroth approximation $u_0(x)$, we can select

$$u_0(x) = 0. \tag{4.170}$$

The method of successive approximations admits the use of the iteration formula

$$u_{n+1}(x) = x + e^x - \int_0^1 xtu_n(t)dt, n \geqslant 0. \tag{4.171}$$

Substituting (4.170) into (4.171) we obtain

$$\begin{aligned}
u_1(x) &= x + e^x - \int_0^1 xtu_0(t)dt = e^x + x,\\
u_2(x) &= x + e^x - \int_0^1 xtu_1(t)dt = e^x - \frac{1}{3}x,\\
u_3(x) &= x + e^x - \int_0^1 xtu_2(t)dt = e^x + \frac{1}{9}x,\\
&\vdots\\
u_{n+1} &= x + e^x - \int_0^1 xtu_n(t)dt = e^x + \frac{(-1)^n}{3^n}x.
\end{aligned} \tag{4.172}$$

Consequently, the solution $u(x)$ of (4.169) is given by

$$u(x) = \lim_{n\to\infty} u_{n+1}(x) = e^x. \tag{4.173}$$

Example 4.24

Solve the Fredholm integral equation by using the successive approximations method

$$u(x) = x + \lambda \int_{-1}^1 xtu(t)dt. \tag{4.174}$$

For the zeroth approximation $u_0(x)$, we can select

$$u_0(x) = x. \tag{4.175}$$

The method of successive approximations admits the use of the iteration formula

$$u_{n+1}(x) = x + \lambda \int_{-1}^1 xtu_n(t)dt, n \geqslant 0. \tag{4.176}$$

Substituting (4.175) into (4.176) we obtain

$$\begin{aligned}
u_1(x) &= x + \frac{2}{3}\lambda x,\\
u_2(x) &= x + \frac{2}{3}\lambda x + \left(\frac{2}{3}\right)^2 \lambda^2 x,\\
u_3(x) &= x + \frac{2}{3}\lambda x + \left(\frac{2}{3}\right)^2 \lambda^2 x + \left(\frac{2}{3}\right)^3 \lambda^3 x,\\
&\vdots\\
u_{n+1}(x) &= x + \frac{2}{3}\lambda x + \left(\frac{2}{3}\right)^2 \lambda^2 x + \left(\frac{2}{3}\right)^3 \lambda^3 x + \cdots + \left(\frac{2}{3}\right)^{n+1} \lambda^{n+1} x.
\end{aligned} \tag{4.177}$$

The solution $u(x)$ of (4.174) is given by

$$u(x) = \lim_{n\to\infty} u_{n+1}(x) = \frac{3x}{3-2\lambda}, \quad 0 < \lambda < \frac{3}{2}, \tag{4.178}$$

obtained upon using the infinite geometric series for the right side of (4.177).

Example 4.25

Solve the Fredholm integral equation by using the successive approximations method

$$u(x) = \sin x + \sin x \int_0^{\frac{\pi}{2}} \cos t u(t) dt. \tag{4.179}$$

For the zeroth approximation $u_0(x)$, we select

$$u_0(x) = 0. \tag{4.180}$$

We next use the iteration formula

$$u_{n+1}(x) = \sin x + \sin x \int_0^{\frac{\pi}{2}} \cos t u_n(t) dt, \quad n \geqslant 0. \tag{4.181}$$

Substituting (4.180) into (4.181) we obtain

$$\begin{aligned} u_1(x) &= \sin x, \qquad u_2(x) = \frac{3}{2}\sin x, \\ u_3(x) &= \frac{7}{4}\sin x, \quad u_4(x) = \frac{15}{8}\sin x, \\ &\vdots \\ u_{n+1}(x) &= \frac{2^{n+1}-1}{2^n}\sin x = \left(2 - \frac{1}{2^n}\right)\sin x. \end{aligned} \tag{4.182}$$

The solution $u(x)$ of (4.179) is given by

$$u(x) = \lim_{n\to\infty} u_{n+1}(x) = 2\sin x. \tag{4.183}$$

Example 4.26

Solve the Fredholm integral equation by using the successive approximations method

$$u(x) = x + \sec^2 x - \int_0^{\frac{\pi}{4}} x u(t) dt. \tag{4.184}$$

For the zeroth approximation $u_0(x)$, we may select

$$u_0(x) = 0. \tag{4.185}$$

We next use the iteration formula

$$u_{n+1}(x) = x + \sec^2 x - \int_0^{\frac{\pi}{4}} x u_n(t) dt, \quad n \geqslant 0. \tag{4.186}$$

This in turn gives

$$\begin{aligned} u_1(x) &= \sec^2 x + x, \qquad u_2(x) = \sec^2 x - \frac{\pi^2}{32}x, \\ u_3(x) &= \sec^2 x + \frac{\pi^4}{1024}x, \quad u_4(x) = \sec^2 x - \frac{\pi^6}{32768}x, \\ &\vdots \end{aligned} \tag{4.187}$$

$$u_{n+1}(x) = \sec^2 x + (-1)^n \left(\frac{\pi^2}{32}\right)^n x.$$

Notice that

$$\lim_{n\to\infty} \left(\frac{\pi^2}{32}\right)^n = 0. \tag{4.188}$$

Consequently, the solution $u(x)$ of (4.184) is given by

$$u(x) = \lim_{n\to\infty} u_{n+1}(x) = \sec^2 x. \tag{4.189}$$

Exercises 4.2.6

Use the *successive approximations method* to solve the following Fredholm integral equations:

1. $u(x) = x + \lambda \int_{-1}^{1} xtu(t)dt$
2. $u(x) = 1 + x^3 + \lambda \int_{-1}^{1} xtu(t)dt$
3. $u(x) = e^x + 2xe^{-1} - \int_{-1}^{1} xtu(t)dt$
4. $u(x) = 2 + 2x + e^x - \int_{0}^{1} xtu(t)dt$
5. $u(x) = 2x + \sec^2 x - \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} xu(t)dt$
6. $u(x) = 1 + \left(1 + \frac{\pi}{4}\right) x + \sec^2 x - \int_{0}^{\frac{\pi}{4}} xu(t)dt$
7. $u(x) = \left(\frac{\pi}{2} - 1\right) x + \cos x - \int_{0}^{\frac{\pi}{2}} xtu(t)dt$
8. $u(x) = \left(\frac{\pi^3}{3} + 3\right) x - \cos x - \int_{0}^{\pi} xtu(t)dt$
9. $u(x) = \frac{\pi^3}{24}x - \sin x - \int_{0}^{\frac{\pi}{2}} xtu(t)dt$
10. $u(x) = x + \sin x - \int_{0}^{\frac{\pi}{2}} xtu(t)dt$
11. $u(x) = \left(1 + \frac{\pi}{3}\right) x + \sec x \tan x - \int_{0}^{\frac{\pi}{3}} x(1 + u(t))dt$
12. $u(x) = \frac{3}{2}x + \sec x \tan x - \int_{0}^{\frac{\pi}{3}} x \sec t u(t)dt$
13. $u(x) = \sqrt{2}\, x + \sec x + \tan x - \int_{0}^{\frac{\pi}{4}} x \sec t u(t)dt$
14. $u(x) = \frac{\pi}{8}x + \sin x + \cos x - \int_{0}^{\frac{\pi}{4}} x \sin t u(t)dt$
15. $u(x) = (\pi + 2)x + \sin x - \cos x - \int_{0}^{\pi} x(1 + u(t))dt$
16. $u(x) = x + \ln(xt) - \int_{0^+}^{1} x(3 + u(t))dt$

4.2.7 The Series Solution Method

A real function $u(x)$ is called analytic if it has derivatives of all orders such that the Taylor series at any point b in its domain

$$u(x) = \sum_{n=0}^{k} \frac{u^{(n)}(b)}{n!}(x-b)^n, \tag{4.190}$$

converges to $f(x)$ in a neighborhood of b. For simplicity, the generic form of Taylor series at $x = 0$ can be written as

$$u(x) = \sum_{n=0}^{\infty} a_n x^n. \tag{4.191}$$

Following the discussion presented before in Chapter 3, the series solution method that stems mainly from the Taylor series for analytic functions, will be used for solving Fredholm integral equations. We will assume that the solution $u(x)$ of the Fredholm integral equations

$$u(x) = f(x) + \lambda \int_a^b K(x,t)u(t)dt \tag{4.192}$$

is analytic, and therefore possesses a Taylor series of the form given in (4.191), where the coefficients a_n will be determined recurrently. Substituting (4.191) into both sides of (4.192) gives

$$\sum_{n=0}^{\infty} a_n x^n = T(f(x)) + \lambda \int_a^b K(x,t) \left(\sum_{n=0}^{\infty} a_n t^n \right) dt, \tag{4.193}$$

or for simplicity we use

$$a_0 + a_1 x + a_2 x^2 + \cdots = T(f(x)) + \lambda \int_a^b K(x,t) \left(a_0 + a_1 t + a_2 t^2 + \cdots \right) dt, \tag{4.194}$$

where $T(f(x))$ is the Taylor series for $f(x)$. The integral equation (4.192) will be converted to a traditional integral in (4.193) or (4.194) where instead of integrating the unknown function $u(x)$, terms of the form t^n, $n \geqslant 0$ will be integrated. Notice that because we are seeking series solution, then if $f(x)$ includes elementary functions such as trigonometric functions, exponential functions, etc., then Taylor expansions for functions involved in $f(x)$ should be used.

We first integrate the right side of the integral in (4.193) or (4.194), and collect the coefficients of like powers of x. We next equate the coefficients of like powers of x in both sides of the resulting equation to obtain a recurrence relation in $a_j, j \geqslant 0$. Solving the recurrence relation will lead to a complete determination of the coefficients $a_j, j \geqslant 0$. Having determined the coefficients $a_j, j \geqslant 0$, the series solution follows immediately upon substituting the derived coefficients into (4.191). The exact solution may be obtained if such an exact solution exists. If an exact solution is not obtainable, then the obtained series can be used for numerical purposes. In this case, the more terms we evaluate, the higher accuracy level we achieve.

It is worth noting that using the series solution method for solving Fredholm integral equations gives exact solutions if the solution $u(x)$ is a polynomial. However, if the solution is any other elementary function such as $\sin x, e^x$, etc, the series method gives the exact solution after rounding few of the coefficients $a_j, j \geqslant 0$. This will be illustrated by studying the following examples.

Example 4.27

Solve the Fredholm integral equation by using the series solution method

$$u(x) = (x+1)^2 + \int_{-1}^{1} (xt + x^2t^2)u(t)dt. \tag{4.195}$$

Substituting $u(x)$ by the series

$$u(x) = \sum_{n=0}^{\infty} a_n x^n \tag{4.196}$$

into both sides of Eq. (4.195) leads to

$$\sum_{n=0}^{\infty} a_n x^n = (x+1)^2 + \int_{-1}^{1} \left((xt + x^2t^2) \sum_{n=0}^{\infty} (a_n t^n) \right) dt. \tag{4.197}$$

Evaluating the integral at the right side gives

$$\begin{aligned} a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots &= 1 + \left(2 + \frac{2}{3}a_1 + \frac{2}{5}a_3 + \frac{2}{7}a_5 + \frac{2}{9}a_7\right)x \\ &\quad + \left(1 + \frac{2}{3}a_0 + \frac{2}{5}a_2 + \frac{2}{7}a_4 + \frac{2}{9}a_6 + \frac{2}{11}a_8\right)x^2. \end{aligned} \tag{4.198}$$

Equating the coefficients of like powers of x in both sides of (4.198) gives

$$a_0 = 1, \quad a_1 = 6, \quad a_2 = \frac{25}{9}, \quad a_n = 0, \quad n \geqslant 3. \tag{4.199}$$

The exact solution is given by

$$u(x) = 1 + 6x + \frac{25}{9}x^2, \tag{4.200}$$

obtained upon substituting (4.199) into (4.196).

Example 4.28

Solve the Fredholm integral equation by using the series solution method

$$u(x) = x^2 - x^3 + \int_0^1 (1 + xt)u(t)dt. \tag{4.201}$$

Substituting $u(x)$ by the series

$$u(x) = \sum_{n=0}^{\infty} a_n x^n, \tag{4.202}$$

into both sides of Eq. (4.201) leads to

$$\sum_{n=0}^{\infty} a_n x^n = x^2 - x^3 + \int_0^1 \left((1 + xt) \sum_{n=0}^{\infty} (a_n t^n) \right) dt. \tag{4.203}$$

Evaluating the integral at the right side, and equating the coefficients of like powers of x in both sides of the resulting equation we find

$$a_0 = -\frac{29}{60}, \quad a_1 = -\frac{1}{6}, \quad a_2 = 1, \quad a_3 = -1, \quad a_n = 0, \quad n \geqslant 4. \tag{4.204}$$

Consequently, the exact solution is given by

$$u(x) = -\frac{29}{60} - \frac{1}{6}x + x^2 - x^3. \tag{4.205}$$

Example 4.29

Solve the Fredholm integral equation by using the series solution method

$$u(x) = -x^4 + \int_{-1}^{1} (xt^2 - x^2 t) u(t) dt. \tag{4.206}$$

Substituting $u(x)$ by the series

$$u(x) = \sum_{n=0}^{\infty} a_n x^n, \tag{4.207}$$

into both sides of Eq. (4.206) leads to

$$\sum_{n=0}^{\infty} a_n x^n = -x^4 + \int_{-1}^{1} \left((xt^2 - x^2 t) \sum_{n=0}^{\infty} a_n t^n \right) dt. \tag{4.208}$$

Evaluating the integral at the right side, and equating the coefficients of like powers of x in both sides of the resulting equation we find

$$\begin{aligned} &a_0 = 0, \quad a_1 = -\frac{30}{133}, \quad a_2 = \frac{20}{133}, \quad a_3 = 0, \quad a_4 = -1 \\ &a_n = 0, \quad n \geqslant 5. \end{aligned} \tag{4.209}$$

Consequently, the exact solution is given by

$$u(x) = -\frac{30}{133}x + \frac{20}{133}x^2 - x^4. \tag{4.210}$$

Example 4.30

Solve the Fredholm integral equation by using the series solution method

$$u(x) = -1 + \cos x + \int_0^{\frac{\pi}{2}} u(t) dt. \tag{4.211}$$

Substituting $u(x)$ by the series

$$u(x) = \sum_{n=0}^{\infty} a_n x^n, \tag{4.212}$$

into both sides of Eq. (4.211) gives

$$\sum_{n=0}^{\infty} a_n x^n = -1 + \cos x + \int_0^{\frac{\pi}{2}} \left(\sum_{n=0}^{\infty} (a_n t^n) \right) dt. \tag{4.213}$$

Evaluating the integral at the right side, using the Taylor series of $\cos x$, and proceeding as before we find

$$a_0 = 1,\ a_{2j+1} = 0,\ a_{2j} = \frac{(-1)^j}{(2j)!}, j \geqslant 0. \tag{4.214}$$

Consequently, the exact solution is given by

$$u(x) = \cos x. \tag{4.215}$$

Exercises 4.2.7

Use the *series solution method* to solve the following Fredholm integral equations:

1. $u(x) = 1 + \int_0^1 (1 - 3xt)u(t)dt$
2. $u(x) = 6x + 4x^2 + \int_{-1}^1 (xt^2 - x^2 t)u(t)dt$
3. $u(x) = 5x - 2x^2 + \int_{-1}^1 (x^2 t^3 - x^3 t^2)u(t)dt$
4. $u(x) = 5x + \int_{-1}^1 (1 - xt)u(t)dt$
5. $u(x) = 2 + 5x - 3x^2 + \int_{-1}^1 (1 - xt)u(t)dt$
6. $u(x) = 3x - 5x^3 + \int_{-1}^1 (1 - xt)u(t)dt$
7. $u(x) = 2 - 2x + 5x^4 + 7x^5 + \int_{-1}^1 (x - t)u(t)dt$
8. $u(x) = 3x^2 - 5x^3 + \int_{-1}^1 x^4 t u(t)dt$
9. $u(x) = -2 + \sin x + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} tu(t)dt$
10. $u(x) = -2 + x^2 + \sin x + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} tu(t)dt$
11. $u(x) = \sec^2 x - 1 + \int_0^{\frac{\pi}{4}} u(t)dt$
12. $u(x) = -1 + \ln(1 + x) + \int_0^{e-1} u(t)dt$

4.3 Homogeneous Fredholm Integral Equation

Substituting $f(x) = 0$ into the Fredholm integral equation of the second kind

$$u(x) = f(x) + \lambda \int_a^b K(x,t)u(t)dt, \tag{4.216}$$

the *homogeneous Fredholm integral equation* of the second kind is given by

$$u(x) = \lambda \int_a^b K(x,t)u(t)dt. \tag{4.217}$$

In this section we will focus our study on the homogeneous Fredholm integral equation (4.217) for separable kernel $K(x,t)$ only. The main goal for studying the homogeneous Fredholm equation is to find nontrivial solution, because the trivial solution $u(x) = 0$ is a solution of this equation. Moreover, the Adomian decomposition method is not applicable here because it depends mainly on assigning a non-zero value for the zeroth component $u_0(x)$, and in this kind of equations $f(x) = 0$. Based on this, the direct computation method will be employed here to handle this kind of equations.

4.3.1 The Direct Computation Method

The direct computation method was used before in this chapter. This method replaces the homogeneous Fredholm integral equations by a single algebraic equation or by a system of simultaneous algebraic equations depending on the number of terms of the separable kernel $K(x,t)$.

As stated before, the direct computation method handles Fredholm integral equations in a direct manner and gives the solution in an exact form but not in a series form as Adomian method or the successive approximations method. It is important to point out that this method will be applied for the degenerate or separable kernels of the form

$$K(x,t) = \sum_{k=1}^{n} g_k(x)h_k(t). \tag{4.218}$$

The direct computation method can be applied as follows:

1. We first substitute (4.218) into the homogeneous Fredholm integral equation the form:

$$u(x) = \lambda \int_a^b K(x,t)\, u(t)dt. \tag{4.219}$$

2. This substitution leads to

$$\begin{aligned} u(x) = {} & \lambda g_1(x) \int_a^b h_1(t)u(t)dt + \lambda g_2(x) \int_a^b h_2(t)u(t)dt + \cdots \\ & + \lambda g_n(x) \int_a^b h_n(t)u(t)dt. \end{aligned} \tag{4.220}$$

3. Each integral at the right side depends only on the variable t with constant limits of integration for t. This means that each integral is equivalent to a constant. Based on this, Equation (4.220) becomes

$$u(x) = \lambda\alpha_1 g_1(x) + \lambda\alpha_2 g_2(x) + \cdots + \lambda\alpha_n g_n(x), \tag{4.221}$$

where

$$\alpha_i = \int_a^b h_i(t)\, u(t)dt, 1 \leqslant i \leqslant n. \tag{4.222}$$

4. Substituting (4.221) into (4.222) gives a system of n simultaneous algebraic equations that can be solved to determine the constants $\alpha_i, 1 \leqslant i \leqslant n$. Using the obtained numerical values of α_i into (4.221), the solution $u(x)$ of the homogeneous Fredholm integral equation (4.217) follows immediately.

Example 4.31

Solve the homogeneous Fredholm integral equation by using the direct computation method

$$u(x) = \lambda \int_0^{\frac{\pi}{2}} \cos x \sin t\, u(t)dt. \tag{4.223}$$

This equation can be rewritten as

$$u(x) = \alpha\lambda \cos x, \tag{4.224}$$

where

$$\alpha = \int_0^{\frac{\pi}{2}} \sin t\, u(t)dt. \tag{4.225}$$

Substituting (4.224) into (4.225) gives

$$\alpha = \alpha\lambda \int_0^{\frac{\pi}{2}} \cos t \sin t dt, \tag{4.226}$$

that gives

$$\alpha = \frac{1}{2}\alpha\lambda. \tag{4.227}$$

Recall that $\alpha = 0$ gives the trivial solution. For $\alpha \neq 0$, we find that the eigenvalue λ is given by

$$\lambda = 2. \tag{4.228}$$

This in turn gives the eigenfunction $u(x)$ by

$$u(x) = A \cos x, \tag{4.229}$$

where A is a non zero arbitrary constant, with $A = 2\alpha$.

Example 4.32

Solve the homogeneous Fredholm integral equation by using the direct computation method

$$u(x) = \lambda \int_0^1 2e^{x+t}\, u(t)dt. \tag{4.230}$$

This equation can be rewritten as

$$u(x) = 2\alpha\lambda e^x, \tag{4.231}$$

where

$$\alpha = \int_0^1 e^t\, u(t)dt. \tag{4.232}$$

Substituting (4.231) into (4.232) gives

$$\alpha = 2\alpha\lambda \int_0^1 e^{2t} dt, \tag{4.233}$$

that gives

$$\alpha = \alpha\lambda(e^2 - 1). \tag{4.234}$$

Recall that $\alpha = 0$ gives the trivial solution. For $\alpha \neq 0$, we find that the eigenvalue λ is given by

$$\lambda = \frac{1}{e^2 - 1}. \tag{4.235}$$

This in turn gives the eigenfunction $u(x)$ by

$$u(x) = \frac{A}{e^2 - 1} e^x, \tag{4.236}$$

where A is a non zero arbitrary constant, with $A = 2\alpha$.

Example 4.33

Solve the homogeneous Fredholm integral equation by using the direct computation method

$$u(x) = \lambda \int_0^\pi \sin(x + t)\, u(t) dt. \tag{4.237}$$

Notice that the kernel $\sin(x+t) = \sin x \cos t + \cos x \sin t$ is separable. Equation (4.237) can be rewritten as

$$u(x) = \alpha\lambda \sin x + \beta\lambda \cos x, \tag{4.238}$$

where

$$\alpha = \int_0^\pi \cos t u(t) dt, \quad \beta = \int_0^\pi \sin t u(t) dt. \tag{4.239}$$

Substituting (4.238) into (4.239) gives

$$\begin{aligned} \alpha &= \int_0^\pi \cos t(\alpha\lambda \sin t + \beta\lambda \cos t) dt, \\ \beta &= \int_0^\pi \sin t(\alpha\lambda \sin t + \beta\lambda \cos t) dt, \end{aligned} \tag{4.240}$$

that gives

$$\alpha = \frac{1}{2}\beta\lambda\pi,\ \beta = \frac{1}{2}\alpha\lambda\pi. \tag{4.241}$$

For $\alpha \neq 0, \beta \neq 0$, we find that the eigenvalue λ is given by

$$\lambda = \pm\frac{2}{\pi}, \alpha = \beta. \tag{4.242}$$

This in turn gives the eigenfunction $u(x)$ by

$$u(x) = \pm\frac{A}{\pi}(\sin x + \cos x), \tag{4.243}$$

where $A = 2\alpha$.

Example 4.34

Solve the homogeneous Fredholm integral equation by using the direct computation method

$$u(x) = \lambda \int_{-1}^{1} (12x + 2t)\, u(t)dt. \qquad (4.244)$$

Equation (4.244) can be rewritten as

$$u(x) = 12\alpha\lambda x + 2\beta\lambda, \qquad (4.245)$$

where

$$\alpha = \int_{-1}^{1} u(t)dt, \quad \beta = \int_{-1}^{1} tu(t)dt. \qquad (4.246)$$

Substituting (4.245) into (4.246) gives

$$\begin{aligned} \alpha &= \int_{-1}^{1} (12\alpha\lambda t + 2\beta\lambda)dt = 4\beta\lambda, \\ \beta &= \int_{-1}^{1} t(12\alpha\lambda t + 2\beta\lambda)dt = 8\alpha\lambda. \end{aligned} \qquad (4.247)$$

Recall that $\alpha = 0$ and $\beta = 0$ give the trivial solution. For $\alpha \neq 0, \beta \neq 0$, we find that the eigenvalue λ is given by

$$\lambda = \pm\frac{1}{4\sqrt{2}}, \beta = \sqrt{2}\alpha. \qquad (4.248)$$

This in turn gives the eigenfunction $u(x)$ by

$$u(x) = \pm\frac{\alpha}{2\sqrt{2}}(6x + \sqrt{2}). \qquad (4.249)$$

Example 4.35

Solve the homogeneous Fredholm integral equation by using the direct computation method

$$u(x) = \lambda \int_{0}^{1} 10(x^2 - 2xt - t^2)\, u(t)dt. \qquad (4.250)$$

Equation (4.250) can be rewritten as

$$u(x) = 10\alpha\lambda x^2 - 20\beta\lambda x - 10\gamma\lambda, \qquad (4.251)$$

where

$$\alpha = \int_{0}^{1} u(t)dt, \quad \beta = \int_{0}^{1} tu(t)dt, \quad \gamma = \int_{0}^{1} t^2u(t)dt. \qquad (4.252)$$

Substituting (4.251) into (4.252) gives

$$\begin{aligned} \alpha &= \int_{0}^{1} (10\alpha\lambda t^2 - 20\beta\lambda t - 10\gamma\lambda)dt = \frac{10}{3}\alpha\lambda - 10\beta\lambda - 10\lambda\gamma, \\ \beta &= \int_{0}^{1} t(10\alpha\lambda t^2 - 20\beta\lambda t - 10\gamma\lambda)dt = \frac{5}{2}\alpha\lambda - \frac{20}{3}\beta\lambda - 5\lambda\gamma, \end{aligned} \qquad (4.253)$$

$$\gamma = \int_0^1 t^2(10\alpha\lambda t^2 - 20\beta\lambda t - 10\gamma\lambda)dt = 2\alpha\lambda - 5\beta\lambda - \frac{10}{3}\lambda\gamma.$$

Recall that $\alpha = 0, \beta = 0$ and $\gamma = 0$ give the trivial solution. For $\alpha \neq 0, \beta \neq 0$ and $\gamma \neq 0$, and by solving the system of equations we find

$$\lambda = -\frac{3}{5}, \quad \alpha = \frac{20}{3}\gamma, \quad \beta = \frac{7}{3}\gamma, \tag{4.254}$$

and γ is left as a free parameter. This in turn gives the eigenfunction $u(x)$ by

$$u(x) = \gamma(-40x^2 + 28x + 6), \tag{4.255}$$

where γ is a non-zero arbitrary constant.

Exercises 4.3

Use the direct computation method to solve the homogeneous Fredholm integral equations:

1. $u(x) = \lambda \int_0^{\pi} \sin^2 x u(t)dt$
2. $u(x) = \lambda \int_0^{\frac{\pi}{3}} \tan x \sec t u(t)dt$
3. $u(x) = \lambda \int_0^{\frac{\pi}{4}} 10 \sec^2 x u(t)dt$
4. $u(x) = \lambda \int_0^{\pi} \sin x u(t)dt$
5. $u(x) = \lambda \int_0^{2} xt u(t)dt$
6. $u(x) = \lambda \int_0^{1} xe^t u(t)dt$
7. $u(x) = \lambda \int_0^{1} 8 \sin^{-1} xt u(t)dt$
8. $u(x) = \lambda \int_0^{1} 8 \cos^{-1} xt u(t)dt$
9. $u(x) = \lambda \int_{-1}^{1} (x + t)u(t)dt$
10. $u(x) = \lambda \int_{-1}^{1} (x - 10t^2)u(t)dt$
11. $u(x) = \lambda \int_0^{\pi} \frac{1}{\pi} \cos(x - t)u(t)dt$
12. $u(x) = \lambda \int_0^{1} (3 - 6x + 9t)u(t)dt$
13. $u(x) = \lambda \int_{-1}^{1} (2 - 3x - 3t)u(t)dt$
14. $u(x) = \lambda \int_0^{1} (-12x^2 + 24xt + 18t^2)u(t)dt$

4.4 Fredholm Integral Equations of the First Kind

We close this chapter by studying Fredholm integral equations of the first kind given by

$$f(x) = \lambda \int_a^b K(x, t)u(t)dt, x \in D, \tag{4.256}$$

where D is a closed and bounded set in real numbers, and $f(x)$ is the data. The range of x does not necessarily coincide with the range of integration [7]. The unknown function $u(x)$, that will be determined, occurs only inside the integral sign and this causes special difficulties. The kernel $K(x, t)$ and the function $f(x)$ are given real-valued functions, and λ is a parameter that is often omitted. However, the parameter λ plays an important role in the singular cases and in the bifurcation points as will be seen later in the text.

An important remark has been reported in [7] and other references concerning the data function $f(x)$. The function $f(x)$ must lie in the range of the kernel $K(x,t)$ [7]. For example, if we set the kernel by

$$K(x,t) = \sin x \sin t. \tag{4.257}$$

Then if we substitute any integrable function $u(x)$ in (4.256), and we evaluate the integral, the resulting $f(x)$ must clearly be a multiple of $\sin x$ [7]. This means that if $f(x)$ is not a multiple of the x component of the kernel, then a solution for (4.256) does not exist. This necessary condition on $f(x)$ can be generalized. In other words, the data function $f(x)$ must contain components which are matched by the corresponding x components of the kernel $K(x,t)$

Fredholm integral equation of the first kind is considered ill-posed problem. Hadamard [8] postulated the following three properties:

1. Existence of a solution.
2. Uniqueness of a solution.
3. Continuous dependence of the solution $u(x)$ on the data $f(x)$. This property means that small errors in the data $f(x)$ should cause small errors [9] in the solution $u(x)$.

A problem is called a *well-posed* problem if it satisfies the three aforementioned properties. Problems that are not well-posed are called ill-posed problems such as inverse problems. Inverse problems are ill-posed problems that might not have a solution in the true sense, if a solution exists it may not be unique, and the obtained solution might not depend continuously on the observed data. If the kernel $K(x,t)$ is smooth, then the Fredholm integral equation (4.256) is very often ill-posed and the solution $u(x)$ is very sensitive to any change in the data $f(x)$. In other words, a very small change on the data $f(x)$ can give a large change in the solution $u(x)$. For all these reasons, the Fredholm integral equations of the first kind is ill-posed that may have no solution, or if a solution exists it is not unique and may not depend continuously on the data.

The Fredholm integral equations of the first kind (4.256) appear in many physical models such as radiography, stereology, spectroscopy, cosmic radiation, image processing and electromagnetic fields. Fredholm integral equations of the first kind arise naturally in the theory of signal processing. Many inverse problems in science and engineering lead to the Fredholm integral equations of the first kind. An inverse problem is a process where the solution $u(x)$ can be obtained by solving (4.256) from the observed data $f(x)$ at various values of x. Most inverse problems are ill-posed problems. This means that the Fredholm integral equations of the first kind is aill-posed problem, and solving this equation may lead to a lot of difficulties.

Several methods have been used to handle the Fredholm integral equations of the first kind. The Legendre wavelets, the augmented Galerkin method, and the collocation method are examples of the methods used to handle this equation. The methods that we used so far in this text cannot handle this kind of equations independently if it is expressed in its standard form (4.256).

However, in this text, we will first apply the *method of regularization* that received a considerable amount of interest, especially in solving first order integral equations. The method transforms first kind equation to second kind equation. We will second apply the homotopy perturbation method [10] to handle specific cases of the Fredholm integral equations where the kernel $K(x,t)$ is separable.

In what follows we will present a brief summary of the method of regularization and the homotopy perturbation method that will be used to handle the Fredholm integral equations of the first kind.

4.4.1 The Method of Regularization

The method of regularization was established independently by Phillips [11] and Tikhonov [12]. The method of regularization consists of replacing ill-posed problem by well-posed problem. The method of regularization transforms the linear Fredholm integral equation of the first kind

$$f(x) = \int_a^b K(x,t)u(t)dt, x \in D, \tag{4.258}$$

to the approximation Fredholm integral equation

$$\mu u_\mu(x) = f(x) - \int_a^b K(x,t)u_\mu(t)dt, x \in D, \tag{4.259}$$

where μ is a small positive parameter. It is clear that (4.259) is a Fredholm integral equation of the second kind that can be rewritten

$$u_\mu(x) = \frac{1}{\mu}f(x) - \frac{1}{\mu}\int_a^b K(x,t)u_\mu(t)dt, x \in D. \tag{4.260}$$

Moreover, it was proved in [7,13] that the solution u_μ of equation (4.260) converges to the solution $u(x)$ of (4.258) as $\mu \to 0$ according to the following lemma [14]:

Lemma 4.1

Suppose that the integral operator of (4.258) *is continuous and coercive in the Hilbert space where* $f(x), u(x)$, *and* $u_\mu(x)$ *are defined, then:*

1. $|u_\mu|$ *is bounded independently of* μ, *and*
2. $|u_\mu(x) - u(x)| \to 0$ *when* $\mu \to 0$.

The proof of this lemma can be found in [7,13].

In summary, by combining the method of regulariztion with any of the methods used before for solving Fredholm integral equation of the second kind, we can solve Fredholm integral equation of the first kind (4.258). The method of regulariztion transforms the first kind to a second kind. The resulting integral equation (4.260) can be solved by any of the methods that were presented before in this chapter. The exact solution $u(x)$ of (4.258) can thus be obtained by

$$u(x) = \lim_{\mu \to 0} u_\mu(x). \tag{4.261}$$

In what follows we will present five illustrative examples where we will use the method of regulariztion to transform the first kind equation to a second kind equation. The resulting equation will be solved by any appropriate method that we used before.

Example 4.36

Combine the method of regulariztion and the direct computation method to solve the Fredholm integral equation of the first kind

$$\frac{1}{4}e^x = \int_0^{\frac{1}{4}} e^{x-t}\, u(t)dt. \tag{4.262}$$

Using the method of regularization, Equation (4.262) can be transformed to

$$u_\mu(x) = \frac{1}{4\mu}e^x - \frac{1}{\mu}\int_0^{\frac{1}{4}} e^{x-t}u_\mu(t)dt. \tag{4.263}$$

The resulting Fredholm integral equation of the second kind will be solved by the direct computation method. Equation (4.263) can be written as

$$u_\mu(x) = \left(\frac{1}{4\mu} - \frac{\alpha}{\mu}\right)e^x, \tag{4.264}$$

where

$$\alpha = \int_0^{\frac{1}{4}} e^{-t}u_\mu(t)dt. \tag{4.265}$$

To determine α, we substitute (4.264) into (4.265), integrate the resulting integral and solve to find that

$$\alpha = \frac{1}{1+4\mu}. \tag{4.266}$$

This in turn gives

$$u_\mu(x) = \frac{e^x}{1+4\mu}. \tag{4.267}$$

The exact solution $u(x)$ of (4.262) can be obtained by

$$u(x) = \lim_{\mu \to 0} u_\mu(x) = e^x. \tag{4.268}$$

Example 4.37

Combine the method of regulariztion and the direct computation method to solve the Fredholm integral equation of the first kind

$$e^x + 1 = \int_0^1 (4te^x + 3)\, u(t)dt. \tag{4.269}$$

Notice that the data function $f(x) = e^x + 1$ contains components which are matched by the corresponding x components of the kernel $K(x,t) = 4te^x + 3$. This is a necessary condition to guarantee a solution.

Using the method of regularization, Equation (4.269) can be transformed to

$$u_\mu(x) = \frac{1}{\mu}e^x + \frac{1}{\mu} - \frac{1}{\mu}\int_0^1 (4te^x + 3)u_\mu(t)dt. \tag{4.270}$$

The resulting Fredholm integral equation of the second kind will be solved by the direct computation method. Equation (4.270) can be written as

$$u_\mu(x) = \left(\frac{1}{\mu} - \frac{4\alpha}{\mu}\right)e^x + \left(\frac{1}{\mu} - \frac{3\beta}{\mu}\right), \tag{4.271}$$

where

$$\alpha = \int_0^1 tu_\mu(t)dt, \quad \beta = \int_0^1 u_\mu(t)dt. \tag{4.272}$$

To determine α and β, we substitute (4.271) into (4.272), integrate the resulting integrals and solve to find that

$$\alpha = \frac{3(e - 3 - \mu)}{2(6e - 18 - 7\mu - \mu^2)}, \quad \beta = -\frac{-2(e + 6 + \mu e)}{6e - 18 - 7\mu - \mu^2}. \tag{4.273}$$

Substituting this result into (4.271) gives the approximate solution

$$u_\mu(x) = \frac{(1+\mu)e^x + (7 - 3e + \mu)}{6(3-e) + (7\mu + \mu^2)}. \tag{4.274}$$

The exact solution $u(x)$ of (4.269) can be obtained by

$$u(x) = \lim_{\mu \to 0} u_\mu(x) = \frac{1}{6(3-e)}e^x + \frac{7 - 3e}{6(3-e)}. \tag{4.275}$$

It is interesting to point out that another solution to this equation is given by

$$u(x) = x^2. \tag{4.276}$$

As stated before, the Fredholm integral equation of the first kind is ill-posed problem. For ill-posed problems, the solution might not exist, and if it exists, the solution may not be unique.

Example 4.38

Combine the method of regulariztion and the direct computation method to solve the Fredholm integral equation of the first kind

$$\frac{\pi}{2}\sin x = \int_0^\pi \sin(x - t)\, u(t)dt. \tag{4.277}$$

Notice that the data function $f(x) = \frac{\pi}{2}\sin x$ contains component which is matched by the corresponding x component of the kernel $K(x,t)$. This is a necessary condition to guarantee a solution.

Using the method of regularization, Equation (4.277) can be transformed to

$$u_\mu(x) = \frac{\pi}{2\mu}\sin x - \frac{1}{\mu}\int_0^\pi \sin(x-t)u_\mu(t)dt, \tag{4.278}$$

that can be written as

$$u_\mu(x) = \left(\frac{\pi}{2\mu} - \frac{\alpha}{\mu}\right)\sin x + \frac{\beta}{\mu}\cos x, \tag{4.279}$$

where

$$\alpha = \int_0^\pi \cos t u_\mu(t)dt, \quad \beta = \int_0^\pi \sin t u_\mu(t)dt. \tag{4.280}$$

To determine α and β, we substitute (4.279) into (4.280), integrate the resulting integrals and solve to find that

$$\alpha = \frac{\pi^3}{2(\pi^2+4\mu^2)}, \quad \beta = \frac{\pi^2\mu}{\pi^2+4\mu^2}. \tag{4.281}$$

Substituting this result into (4.279) gives the approximate solution

$$u_\mu(x) = \frac{2\pi\mu}{\pi^2+4\mu^2}\sin x + \frac{\pi^2}{\pi^2+4\mu^2}\cos x. \tag{4.282}$$

The exact solution $u(x)$ of (4.277) can be obtained by

$$u(x) = \lim_{\mu\to 0} u_\mu(x) = \cos x. \tag{4.283}$$

Example 4.39

Combine the method of regulariztion and the Adomian decomposition method to solve the Fredholm integral equation of the first kind

$$\frac{1}{3}e^{-x} = \int_0^{\frac{1}{3}} e^{t-x}\, u(t)dt. \tag{4.284}$$

Using the method of regularization, Equation (4.284) can be transformed to

$$u_\mu(x) = \frac{1}{3\mu}e^{-x} - \frac{1}{\mu}\int_0^{\frac{1}{3}} e^{t-x}u_\mu(t)dt. \tag{4.285}$$

The Adomian decomposition method admits the use of

$$u_\mu(x) = \sum_{n=0}^{\infty} u_{\mu_n}(x), \tag{4.286}$$

and the recurrence relation

$$\begin{aligned} u_{\mu_0}(x) &= \frac{1}{3\mu}e^{-x}, \\ u_{\mu_{k+1}}(x) &= -\frac{1}{\mu}\int_0^{\frac{1}{3}} e^{t-x}u_{\mu_k}(t)dt, \quad k \geqslant 0. \end{aligned} \tag{4.287}$$

This in turn gives the components

$$\begin{aligned} u_{\mu_0}(x) &= \frac{1}{3\mu}e^{-x}, \qquad u_{\mu_1}(x) = -\frac{1}{9\mu^2}e^{-x}, \\ u_{\mu_2}(x) &= \frac{1}{27\mu^3}e^{-x}, \quad u_{\mu_3}(x) = -\frac{1}{81\mu^2}e^{-x}, \end{aligned} \tag{4.288}$$

and so on. Substituting this result into (4.286) gives the approximate solution

$$u_\mu(x) = \frac{1}{1+3\mu}e^{-x}. \tag{4.289}$$

The exact solution $u(x)$ of (4.284) can be obtained by

$$u(x) = \lim_{\mu \to 0} u_\mu(x) = e^{-x}. \tag{4.290}$$

Example 4.40

Combine the method of regulariztion and the successive approximations method to solve the Fredholm integral equation of the first kind

$$\frac{1}{4}x = \int_0^1 xt\, u(t)dt. \tag{4.291}$$

Using the method of regularization, Equation (4.291) can be transformed to

$$u_\mu(x) = \frac{1}{4\mu}x - \frac{1}{\mu}\int_0^1 xtu_\mu(t)dt. \tag{4.292}$$

To use the successive approximations method, we first select $u_{\mu_0}(x) = 0$. Consequently, we obtain the following approximations

$$\begin{aligned} u_{\mu_0}(x) &= 0, \\ u_{\mu_1}(x) &= \frac{1}{4\mu}x, \\ u_{\mu_2}(x) &= \frac{1}{4\mu}x - \frac{1}{12\mu^2}x, \\ u_{\mu_3}(x) &= \frac{1}{4\mu}x - \frac{1}{12\mu^2}x + \frac{1}{36\mu^3}x, \\ u_{\mu_4}(x) &= \frac{1}{4\mu}x - \frac{1}{12\mu^2}x + \frac{1}{36\mu^3}x - \frac{1}{108\mu^4}x, \end{aligned} \tag{4.293}$$

and so on. Based on this we obtain the approximate solution

$$u_\mu(x) = \frac{3}{4(1+3\mu)}x. \tag{4.294}$$

The exact solution $u(x)$ of (4.291) can be obtained by

$$u(x) = \lim_{\mu \to 0} u_\mu(x) = \frac{3}{4}x. \tag{4.295}$$

It is interesting to point out that another solution to this equation is given by

$$u(x) = x^2. \tag{4.296}$$

As stated before, the Fredholm integral equation of the first kind is ill-posed problem and the solution may not be unique.

Exercises 4.4.1

Combine the regularization method with any other method to solve the Fredholm integral equations of the first kind

1. $\frac{1}{2}(1-e^{-2})e^{3x} = \int_0^1 e^{3x-4t}u(t)dt$

2. $\frac{1}{2}e^{3x} = \int_0^{\frac{1}{2}} e^{3x-3t}u(t)dt$

3. $\frac{3}{4}x = \int_0^1 xt^2u(t)dt$

4. $\frac{6}{5}x^2 = \int_0^1 x^2t^2u(t)dt$

5. $\frac{2}{5}x^2 = \int_{-1}^1 x^2t^2u(t)dt$

6. $\frac{1}{5}x = \int_0^1 xtu(t)dt$

7. $\frac{1}{6}x^2 = \int_0^1 x^2t^2u(t)dt$

8. $\frac{2}{3}x^2 = \int_{-1}^1 x^2t^2u(t)dt$

9. $-\frac{1}{4}x = \int_0^1 xtu(t)dt$

10. $\frac{1}{4}x = \int_0^1 xtu(t)dt$

11. $\frac{1}{12}x = \int_0^1 xtu(t)dt$

12. $\frac{7}{12}x = \int_0^1 xtu(t)dt$

13. $\frac{\pi}{2}\sin x = \int_0^\pi \cos(x-t)u(t)dt$

14. $\frac{\pi}{2}\cos x = \int_0^\pi \cos(x-t)u(t)dt$

15. $2-\pi+2x = \int_0^\pi (x-t)u(t)dt$

16. $2+\pi-2x = \int_0^\pi (x-t)u(t)dt$

4.4.2 The Homotopy Perturbation Method

The homotopy perturbation method was introduced and developed by Ji-Huan He in [10] and was used recently in the literature for solving linear and nonlinear problems. The homotopy perturbation method couples a homotopy technique of topology and a perturbation technique. A homotopy with an embedding parameter $p \in [0,1]$ is constructed, and the impeding parameter p is considered a small parameter. The method was derived and illustrated in [10], and several differential equations were examined. The coupling of the perturbation method and the homotopy method has eliminated the limitations of the traditional perturbation technique [10]. In what follows we illustrate the homotopy perturbation method to handle Fredholm integral equations of the second kind and the first kind.

The HPM for Fredholm Integral Equation of the Second Kind

In what follows we present the homotopy perturbation method for handling the Fredholm integral equations of the second kind. We first consider the Fredholm integral equation of the second kind

$$v(x) = f(x) + \int_a^b K(x,t)v(t)dt. \tag{4.297}$$

We now define the operator

$$L(u) = u(x) - f(x) - \int_a^b K(x,t)u(t)dt = 0, \tag{4.298}$$

where $u(x) = v(x)$. Next we define the homotopy $H(u,p), p \in [0,1]$ by

$$H(u,0) = F(u), \quad H(u,1) = L(u), \tag{4.299}$$

where $F(u)$ is a functional operator. We construct a convex homotopy of the form

$$H(u,p) = (1-p)F(u) + pL(u) = 0. \tag{4.300}$$

This homotopy satisfies (4.299) for $p = 0$ and $p = 1$ respectively. The embedding parameter p monotonically increases from 0 to 1 as the trivial problem $F(u) = 0$ continuously deformed [10] to the original problem $L(u) = 0$. The homotopy perturbation method admits the use of the expansion

$$u = \sum_{n=0}^{\infty} p^n u_n, \tag{4.301}$$

and consequently

$$v = \lim_{p \to 1} \sum_{n=0}^{\infty} p^n u_n. \tag{4.302}$$

The series (4.302) converges to the exact solution if such a solution exists.

Substituting (4.301) into (4.300), using $F(x) = u(x) - f(x)$, and equating the terms with like powers of the embedding parameter p we obtain the recurrence relation

$$p^0 : u_0(x) = f(x), \quad p^{n+1} : u_{n+1} = \int_a^b K(x,t)u_n(t)dt, n \geqslant 0. \tag{4.303}$$

Notice that the recurrence relation (4.303) is the same standard Adomian decomposition method as presented before in this chapter. This proves the following theorem:

Theorem 4.3 *The Adomian decomposition method is a homotopy perturbation method with a convex homotopy given by*

$$H(u,p) = u(x) - f(x) - p\int_a^b K(x,t)u_n(t)dt = 0. \tag{4.304}$$

HPM for Fredholm Integral Equation of the First Kind

In what follows we present the homotopy perturbation method for handling the Fredholm integral equations of the first kind of the form

$$f(x) = \int_a^b K(x,t)v(t)dt. \tag{4.305}$$

We now define the operator

$$L(u) = f(x) - \int_a^b K(x,t)u(t)dt = 0. \tag{4.306}$$

We construct a convex homotopy of the form

$$H(u,p) = (1-p)u(x) + pL(u)(x) = 0. \tag{4.307}$$

The embedding parameter p monotonically increases from 0 to 1. The homotopy perturbation method admits the use of the expansion

$$u = \sum_{n=0}^{\infty} p^n u_n, \tag{4.308}$$

and consequently

$$v(x) = \lim_{p \to 1} \sum_{n=0}^{\infty} p^n u_n(x). \tag{4.309}$$

The series (4.309) converges to the exact solution if such a solution exists.

Substituting (4.308) into (4.307), and proceeding as before we obtain the recurrence relation

$$\begin{aligned} u_0(x) &= 0, \quad u_1(x) = f(x), \\ u_{n+1}(x) &= u_n(x) - \int_a^b K(x,t)u_n(t)dt, \quad n \geqslant 1. \end{aligned} \tag{4.310}$$

If the kernel is separable, i.e. $K(x,t) = g(x)h(t)$, then the following condition

$$\left| 1 - \int_a^b K(t,t)dt \right| < 1, \tag{4.311}$$

must be justified for convergence. The proof of this condition is left to the reader.

We will concern ourselves only on the case where $K(x,t) = g(x)h(t)$. The HPM will be used to solve the following Fredholm integral equations of the first kind.

Example 4.41

Use the homotopy perturbation method to solve the Fredholm integral equation of the first kind

$$\frac{1}{3}e^x = \int_0^{\frac{1}{3}} e^{x-t}\, u(t)dt. \tag{4.312}$$

Notice that

$$\left| 1 - \int_0^{\frac{1}{3}} K(t,t)dt \right| = \frac{2}{3} < 1. \tag{4.313}$$

Using the recurrence relation (4.310) we find

$$u_0(x) = 0, \quad u_1(x) = \frac{1}{3}e^x,$$
$$u_{n+1}(x) = u_n(x) - \int_0^{\frac{1}{3}} e^{x-t} u_n(t)dt, \quad n \geqslant 1. \tag{4.314}$$

This in turn gives

$$\begin{aligned}
u_0(x) &= 0, \quad u_1(x) = \frac{1}{3}e^x,\\
u_2(x) &= u_1(x) - \int_0^{\frac{1}{3}} e^{x-t} u_1(t)dt = \frac{2}{9}e^x,\\
u_3(x) &= u_2(x) - \int_0^{\frac{1}{3}} e^{x-t} u_2(t)dt = \frac{4}{27}e^x,\\
u_4(x) &= u_3(x) - \int_0^{\frac{1}{3}} e^{x-t} u_3(t)dt = \frac{8}{81}e^x,
\end{aligned} \tag{4.315}$$

and so on. Consequently, the approximate solution is given by

$$u(x) = e^x \left(\frac{1}{3} + \frac{2}{9} + \frac{4}{27} + \frac{8}{81} + \cdots \right), \tag{4.316}$$

that converges to the exact solution

$$u(x) = e^x. \tag{4.317}$$

Example 4.42

Use the homotopy perturbation method to solve the Fredholm integral equation of the first kind

$$\frac{1}{4}e^{-x} = \int_0^{\frac{1}{4}} e^{t-x}\, u(t)dt. \tag{4.318}$$

Notice that

$$\left| 1 - \int_0^{\frac{1}{4}} K(t,t)dt \right| = \frac{3}{4} < 1. \tag{4.319}$$

Using the recurrence relation (4.310) we find

$$u_0(x) = 0, \quad u_1(x) = \frac{1}{4}e^{-x},$$
$$u_{n+1}(x) = u_n(x) - \int_0^{\frac{1}{4}} e^{t-x} u_n(t)dt, \quad n \geqslant 1. \tag{4.320}$$

This in turn gives

$$\begin{aligned}
u_0(x) &= 0, \quad u_1(x) = \frac{1}{4}e^{-x},\\
u_2(x) &= u_1(x) - \int_0^{\frac{1}{4}} e^{t-x} u_1(t)dt = \frac{3}{16}e^{-x},\\
u_3(x) &= u_2(x) - \int_0^{\frac{1}{4}} e^{t-x} u_2(t)dt = \frac{9}{64}e^{-x},
\end{aligned} \tag{4.321}$$

$$u_4(x) = u_3(x) - \int_0^{\frac{1}{4}} e^{t-x} u_3(t)dt = \frac{27}{256}e^{-x},$$

and so on. Consequently, the approximate solution is given by

$$u(x) = e^{-x}\left(\frac{1}{4} + \frac{3}{16} + \frac{9}{64} + \frac{27}{256} + \cdots\right), \tag{4.322}$$

that converges to the exact solution

$$u(x) = e^{-x}, \tag{4.323}$$

obtained by evaluating the sum of the infinite geometric series.

Example 4.43

Use the homotopy perturbation method to solve the Fredholm integral equation of the first kind

$$x = \int_0^1 xt\, u(t)dt. \tag{4.324}$$

Notice that

$$\left|1 - \int_0^1 K(t,t)dt\right| = \frac{2}{3} < 1. \tag{4.325}$$

Using the recurrence relation (4.310) we find

$$\begin{aligned} u_0(x) &= 0, \quad u_1(x) = x, \\ u_{n+1}(x) &= u_n(x) - \int_0^1 xtu_n(t)dt, \quad n \geqslant 1. \end{aligned} \tag{4.326}$$

This in turn gives

$$\begin{aligned} u_0(x) &= 0, \quad u_1(x) = x, \\ u_2(x) &= u_1(x) - \int_0^1 xtu_1(t)dt = \frac{2}{3}x, \\ u_3(x) &= u_2(x) - \int_0^1 xtu_2(t)dt = \frac{4}{9}x, \\ u_4(x) &= u_3(x) - \int_0^1 xtu_3(t)dt = \frac{8}{27}x, \end{aligned} \tag{4.327}$$

and so on. Consequently, the approximate solution is given by

$$u(x) = x\left(1 + \frac{2}{3} + \frac{4}{9} + \frac{8}{27} + \cdots\right), \tag{4.328}$$

that converges to the exact solution

$$u(x) = 3x. \tag{4.329}$$

Example 4.44

Use the homotopy perturbation method to solve the Fredholm integral equation of the first kind

$$\frac{5}{6}x = \int_0^1 xt\, u(t)dt. \tag{4.330}$$

Notice that

$$\left|1-\int_0^1 K(t,t)dt\right| = \frac{2}{3} < 1. \tag{4.331}$$

Using the recurrence relation (4.310) we find

$$\begin{aligned} u_0(x) &= 0, \quad u_1(x) = \frac{5}{6}x, \\ u_{n+1}(x) &= u_n(x) - \int_0^1 xt u_n(t)dt, \quad n \geqslant 1. \end{aligned} \tag{4.332}$$

This in turn gives

$$\begin{aligned} u_0(x) &= 0, \quad u_1(x) = \frac{5}{6}x, \\ u_2(x) &= u_1(x) - \int_0^1 xt u_1(t)dt = \frac{5}{9}x, \\ u_3(x) &= u_2(x) - \int_0^1 xt u_2(t)dt = \frac{10}{27}x, \\ u_4(x) &= u_3(x) - \int_0^1 xt u_3(t)dt = \frac{20}{81}x, \end{aligned} \tag{4.333}$$

and so on. Consequently, the approximate solution is given by

$$u(x) = \frac{5}{6}x + \frac{5}{9}x\left(1+\frac{2}{3}+\frac{4}{9}+\frac{8}{27}+\cdots\right), \tag{4.334}$$

that converges to the exact solution

$$u(x) = \frac{5}{2}x, \tag{4.335}$$

obtained by evaluating the sum of the infinite geometric series.

It is interesting to point out that another solution to this equation is given by

$$u(x) = 1 + x. \tag{4.336}$$

As stated before, the Fredholm integral equation of the first kind is ill-posed problem. For ill-posed problems, the solution might not exist, and if it exists, the solution may not be unique.

Example 4.45

Use the homotopy perturbation method to solve the Fredholm integral equation of the first kind

$$\frac{1}{4}x = \int_0^1 xt\, u(t)dt. \tag{4.337}$$

Notice that

$$\left|1-\int_0^1 K(t,t)dt\right| = \frac{2}{3} < 1. \tag{4.338}$$

Using the recurrence relation (4.310) we find

$$\begin{aligned} u_0(x) &= 0, \quad u_1(x) = \frac{1}{4}x, \\ u_{n+1}(x) &= u_n(x) - \int_0^1 xtu_n(t)dt, \quad n \geqslant 1. \end{aligned} \tag{4.339}$$

This in turn gives

$$\begin{aligned} u_0(x) &= 0, \quad u_1(x) = \frac{1}{4}x, \\ u_2(x) &= u_1(x) - \int_0^1 xtu_1(t)dt = \frac{1}{6}x, \\ u_3(x) &= u_2(x) - \int_0^1 xtu_2(t)dt = \frac{1}{9}x, \\ u_4(x) &= u_3(x) - \int_0^1 xtu_3(t)dt = \frac{2}{27}x, \end{aligned} \tag{4.340}$$

and so on. Consequently, the approximate solution is given by

$$u(x) = \frac{1}{4}x\left(1 + \frac{2}{3} + \frac{4}{9} + \frac{8}{27} + \cdots\right), \tag{4.341}$$

that converges to the exact solution

$$u(x) = \frac{3}{4}x. \tag{4.342}$$

It is interesting to point out that another solution to this equation is given by

$$u(x) = x^2. \tag{4.343}$$

As stated before, the Fredholm integral equation of the first kind is ill-posed problem and the solution may not be unique. Notice that the ill-posed Fredholm problem is linear.

Exercises 4.4.2

Use the homotopy perturbation method to solve the Fredholm integral equations of the first kind

1. $\frac{1}{2}(1-e^{-2})e^{3x} = \int_0^1 e^{3x-4t}u(t)dt$
2. $\frac{1}{2}e^{3x} = \int_0^{\frac{1}{2}} e^{3x-3t}u(t)dt$
3. $\frac{3}{4}x = \int_0^1 xt^2u(t)dt$
4. $\frac{6}{5}x^2 = \int_0^1 x^2t^2u(t)dt$
5. $\frac{2}{5}x^2 = \int_{-1}^1 x^2t^2u(t)dt$
6. $\frac{1}{5}x = \int_0^1 xtu(t)dt$
7. $\frac{1}{6}x^2 = \int_0^1 x^2t^2u(t)dt$
8. $\frac{2}{3}x^2 = \int_{-1}^1 x^2t^2u(t)dt$
9. $-\frac{1}{4}x = \int_0^1 xtu(t)dt$
10. $\frac{1}{4}x = \int_0^1 xtu(t)dt$

11. $\frac{1}{12}x = \int_0^1 xtu(t)dt$

12. $\frac{7}{12}x = \int_0^1 xtu(t)dt$

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Chapter 5
Volterra Integro-Differential Equations

5.1 Introduction

Volterra studied the hereditary influences when he was examining a population growth model. The research work resulted in a specific topic, where both differential and integral operators appeared together in the same equation. This new type of equations was termed as Volterra integro-differential equations [1–4], given in the form

$$u^{(n)}(x) = f(x) + \lambda \int_0^x K(x,t)u(t)dt, \tag{5.1}$$

where $u^{(n)}(x) = \frac{d^n u}{dx^n}$. Because the resulted equation in (5.1) combines the differential operator and the integral operator, then it is necessary to define initial conditions $u(0), u'(0), \ldots, u^{(n-1)}(0)$ for the determination of the particular solution $u(x)$ of the Volterra integro-differential equation (5.1). Any Volterra integro-differential equation is characterized by the existence of one or more of the derivatives $u'(x)$, $u''(x)$,... outside the integral sign. The Volterra integro-differential equations may be observed when we convert an initial value problem to an integral equation by using Leibnitz rule.

The Volterra integro-differential equation appeared after its establishment by Volterra. It then appeared in many physical applications such as glass-forming process, nanohydrodynamics, heat transfer, diffusion process in general, neutron diffusion and biological species coexisting together with increasing and decreasing rates of generating, and wind ripple in the desert. More details about the sources where these equations arise can be found in physics, biology and engineering applications books.

To determine a solution for the integro-differential equation, the initial conditions should be given, and this may be clearly seen as a result of involving $u(x)$ and its derivatives. The initial conditions are needed to determine the exact solution.

5.2 Volterra Integro-Differential Equations of the Second Kind

In what follows we will present the recently developed methods, namely the *Adomian decomposition method* (ADM) and the *variational iteration method* (VIM) that will be used to handle the Volterra integro-differential equations of the second kind. Moreover, some of the traditional methods, namely the Laplace transform method, the series solution method, converting Volterra integro-differential equations to equivalent Volterra integral equations and converting Volterra integro-differential equations to equivalent initial value problem, will be studied as well.

However, the Volterra integro-differential equations of the first kind will be examined. The Laplace transform method and the variational iteration method will be used to handle the first kind equations.

5.2.1 The Adomian Decomposition Method

The Adomian decomposition method [5–9] gives the solution in an infinite series of components that can be recurrently determined. The obtained series may give the exact solution if such a solution exists. Otherwise, the series gives an approximation for the solution that gives high accuracy level.

Without loss of generality, we may assume a Volterra integro-differential equation of the second kind given by

$$u''(x) = f(x) + \int_0^x K(x,t)u(t)dt, \quad u(0) = a_0, \quad u'(0) = a_1. \tag{5.2}$$

Integrating both sides of (5.2) from 0 to x twice leads to

$$u(x) = a_0 + a_1 x + L^{-1}(f(x)) + L^{-1}\left(\int_0^x K(x,t)u(t)dt\right), \tag{5.3}$$

where the initial conditions $u(0)$ and $u'(0)$ are used, and L^{-1} is a two-fold integral operator. We then use the decomposition series

$$u(x) = \sum_{n=0}^{\infty} u_n(x), \tag{5.4}$$

into both sides of (5.3) to obtain

$$\sum_{n=0}^{\infty} u_n(x) = a_0 + a_1 x + L^{-1}\left(f(x)\right) + L^{-1}\left(\int_0^x K(x,t)\left(\sum_{n=0}^{\infty} u_n(t)\right)dt\right), \tag{5.5}$$

or equivalently

$$\begin{aligned}
&u_0(x)+u_1(x)+u_2(x)+u_3(x)+\cdots=a_0+a_1x+L^{-1}\left(f(x)\right)\\
&+L^{-1}\left(\int_0^x K(x,t)u_0(t)dt\right)+L^{-1}\left(\int_0^x K(x,t)u_1(t)dt\right)\\
&+L^{-1}\left(\int_0^x K(x,t)u_2(t)dt\right)+\cdots. \qquad (5.6)
\end{aligned}$$

To determine the components $u_0(x), u_1(x), u_2(x), u_3(x), \ldots$ of the solution $u(x)$, we set the recurrence relation

$$\begin{aligned}
u_0(x)&=a_0+a_1x+L^{-1}\left(f(x)\right),\\
u_{k+1}(x)&=L^{-1}\left(\int_0^x K(x,t)u_k(t)dt\right),\quad k\geqslant 0,
\end{aligned} \qquad (5.7)$$

where the zeroth component $u_0(x)$ is defined by all terms not included inside the integral sign of (5.6). Having determined the components $u_i(x), i \geqslant 0$, the solution $u(x)$ of (5.2) is then obtained in a series form. Using (5.4), the obtained series converges to the exact solution if such a solution exists. However, for concrete problems, a truncated series $\sum_{k=0}^{n} u_k(x)$ is usually used to approximate the solution $u(x)$ that can be used for numerical purposes.

Remarks

1. The Adomian decomposition method was presented before to handle a second order Volterra integro-differential equations. For other orders, we can follow the same approach as presented. This will be explained later in details by discussing the illustrative examples, where first-order, second-order, third-order, and fourth-order Volterra integro-differential equations will be studied.
2. The modified decomposition method that we used before can be used for handling Volterra integro-differential equations of any order.
3. The phenomenon of the noise terms that was applied before can be used here if noise terms appear.

The Adomian decomposition method for solving the second kind Volterra integro-differential equations will be illustrated by studying the following examples. The selected equations are of orders 1, 2, 3, and 4. Other equations of higher orders can be treated in a like manner.

Example 5.1

Use the Adomian method to solve the Volterra integro-differential equation

$$u'(x)=1-\int_0^x u(t)dt,\quad u(0)=0. \qquad (5.8)$$

Applying the integral operator L^{-1} defined by

$$L^{-1}(\cdot)=\int_0^x (\cdot)dx, \qquad (5.9)$$

to both sides of (5.8), i.e. integrating both sides of (5.8) once from 0 to x, and using the given initial condition we obtain

$$u(x) = x - L^{-1}\left(\int_0^x u(t)dt\right). \tag{5.10}$$

Using the decomposition series (5.4), and using the recurrence relation (5.7) we obtain

$$\begin{aligned}
u_0(x) &= x,\\
u_1(x) &= -L^{-1}\left(\int_0^x u_0(t)dt\right) = -\frac{1}{3!}x^3,\\
u_2(x) &= L^{-1}\left(\int_0^x u_1(t)dt\right) = \frac{1}{5!}x^5,\\
u_3(x) &= L^{-1}\left(\int_0^x u_2(t)dt\right) = -\frac{1}{7!}x^7,
\end{aligned} \tag{5.11}$$

and so on. This gives the solution in a series form

$$u(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots, \tag{5.12}$$

and hence the exact solution is given by

$$u(x) = \sin x. \tag{5.13}$$

Example 5.2

Use the Adomian method to solve the Volterra integro-differential equation

$$u''(x) = 1 + x + \int_0^x (x-t)u(t)dt, \quad u(0) = 1, \quad u'(0) = 1. \tag{5.14}$$

Applying the two-fold integral operator L^{-1} defined by

$$L^{-1}(\cdot) = \int_0^x \int_0^x (\cdot)dxdx, \tag{5.15}$$

to both sides of (5.14), i.e. integrating both sides of (5.14) twice from 0 to x, and using the given initial conditions we obtain

$$u(x) = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + L^{-1}\left(\int_0^x (x-t)u(t)dt\right). \tag{5.16}$$

Using the decomposition series (5.4), and using the recurrence relation (5.7) we obtain

$$\begin{aligned}
u_0(x) &= 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3,\\
u_1(x) &= L^{-1}\left(\int_0^x (x-t)u_0(t)dt\right)\\
&= \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \frac{1}{6!}x^6 + \frac{1}{7!}x^7,
\end{aligned} \tag{5.17}$$

and so on. This gives the solution in a series form

$$u(x) = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \frac{1}{6!}x^6 + \frac{1}{7!}x^7 + \cdots, \tag{5.18}$$

and this converges to the exact solution

$$u(x) = e^x. \tag{5.19}$$

Example 5.3

Use the Adomian method to solve the Volterra integro-differential equation

$$u'''(x) = -1 + x - \int_0^x (x-t)u(t)dt, \quad u(0) = 1, \quad u'(0) = -1, \quad u''(0) = 1. \tag{5.20}$$

Applying the three-fold integral operator L^{-1} defined by

$$L^{-1}(\cdot) = \int_0^x \int_0^x \int_0^x (\cdot) dxdxdx, \tag{5.21}$$

to both sides of (5.20), and using the given initial conditions we obtain

$$u(x) = 1 - x + \frac{1}{2!}x^2 - \frac{1}{3!}x^3 + \frac{1}{4!}x^4 - L^{-1}\left(\int_0^x (x-t)u(t)dt\right). \tag{5.22}$$

Using the decomposition series (5.4), and using the recurrence relation (5.7) we obtain

$$\begin{aligned} u_0(x) &= 1 - x + \frac{1}{2!}x^2 - \frac{1}{3!}x^3 + \frac{1}{4!}x^4, \\ u_1(x) &= -L^{-1}\left(\int_0^x (x-t)u_0(t)dt\right) \\ &= -\frac{1}{5!}x^5 + \frac{1}{6!}x^6 - \frac{1}{7!}x^7 + \frac{1}{8!}x^8 - \frac{1}{9!}x^9, \end{aligned} \tag{5.23}$$

and so on. This gives the solution in a series form

$$u(x) = 1 - x + \frac{1}{2!}x^2 - \frac{1}{3!}x^3 + \frac{1}{4!}x^4 - \frac{1}{5!}x^5 + \frac{1}{6!}x^6 - \frac{1}{7!}x^7 + \cdots, \tag{5.24}$$

and this converges to the exact solution

$$u(x) = e^{-x}. \tag{5.25}$$

Example 5.4

Use the Adomian method to solve the Volterra integro-differential equation

$$\begin{gathered} u^{(\text{iv})}(x) = -1 + x - \int_0^x (x-t)u(t)dt, \\ u(0) = -1, \quad u'(0) = 1, \quad u''(0) = 1, \quad u'''(0) = -1. \end{gathered} \tag{5.26}$$

Applying the four-fold integral operator L^{-1} to both sides of (5.26) and using the given initial condition we obtain

$$u(x) = x - 1 - L^{-1}\left(\int_0^x (x-t)u(t)dt\right). \tag{5.27}$$

Using the decomposition series (5.4), and using the recurrence relation (5.7) we obtain

$$
\begin{aligned}
u_0(x) &= -1 + x + \frac{1}{2!}x^2 - \frac{1}{3!}x^3 - \frac{1}{4!}x^4 + \frac{1}{5!}x^5, \\
u_1(x) &= L^{-1}\left(\int_0^x (x-t)u_0(t)dt\right) = \frac{1}{6!}x^6 - \frac{1}{7!}x^7 + \frac{1}{8!}x^8 + \cdots,
\end{aligned}
\tag{5.28}
$$

and so on. The series solution is given by

$$
u(x) = \left(x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots\right) - \left(1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots\right),
\tag{5.29}
$$

so that the exact solution is given by

$$
u(x) = \sin x - \cos x. \tag{5.30}
$$

Exercises 5.2.1

Solve the following Volterra integro-differential equations by using the Adomian decomposition method:

1. $u'(x) = 1 + \int_0^x u(t)dt,\ u(0) = 0$

2. $u'(x) = 1 + x + \int_0^x (x-t)u(t)dt,\ u(0) = 1$

3. $u'(x) = 2 + 4x + 8\int_0^x (x-t)u(t)dt,\ u(0) = 1$

4. $u''(x) = 1 + \int_0^x (x-t)u(t)dt,\ u(0) = 1,\ u'(0) = 0$

5. $u''(x) = -1 + \int_0^x (x-t)u(t)dt,\ u(0) = 1,\ u'(0) = 0$

6. $u''(x) = 1 + x + \int_0^x (x-t)u(t)dt,\ u(0) = 1,\ u'(0) = 1$

7. $u''(x) = -1 - x + \int_0^x (x-t)u(t)dt,\ u(0) = 1,\ u'(0) = 1$

8. $u''(x) = 2 - 2x\sin x - \int_0^x (x-t)u(t)dt,\ u(0) = 0,\ u'(0) = 0$

9. $u''(x) = -x - \frac{1}{6}x^3 + \int_0^x (x-t)u(t)dt,\ u(0) = 0,\ u'(0) = 2$

10. $u''(x) = 1 + x - \frac{1}{31}x^3 + \int_0^x (x-t)u(t)dt,\ u(0) = 1,\ u'(0) = 2$

11. $u'''(x) = 1 - x + 2\sin x - \int_0^x (x-t)u(t)dt,\ u(0) = 1,\ u'(0) = -1,\ u''(0) = -1$

12. $u'''(x) = 1 + x + \frac{1}{6}x^3 + \int_0^x (x-t)u(t)dt,\ u(0) = 1,\ u'(0) = 0,\ u''(0) = 1$

13. $u'''(x) = 1 + x - \frac{1}{12}x^4 + \int_0^x (x-t)u(t)dt,\ u(0) = 1,\ u'(0) = 1,\ u''(0) = 3$

14. $u^{(\text{iv})}(x) = 1 + x - \int_0^x (x-t)u(t)dt,\ u(0) = u'(0) = 1,\ u''(0) = u'''(0) = -1$

15. $u^{(\text{iv})}(x) = 1 + x - \frac{1}{20}x^5 + \int_0^x (x-t)u(t)dt,\ u(0) = u'(0) = u''(0) = 1,$
$u'''(0) = 7$

16. $u^{(\text{iv})}(x) = 1 + x - \frac{1}{2!}x^2 - \frac{1}{3!}x^3 + \int_0^x (x-t)u(t)dt,$
$u(0) = u'(0) = 2,\ u''(0) = u'''(0) = 1$

5.2.2 The Variational Iteration Method

In Chapter 3, the *variational iteration method* was used to handle Volterra integral equations by converting it to an initial value problem or by converting it to an equivalent integro-differential equation. The method provides rapidly convergent successive approximations of the exact solution if such a closed form solution exists, and not components as in Adomian decomposition method. The variational iteration method handles linear and nonlinear problems in the same manner without any need to specific restrictions such as the so called Adomian polynomials that we need for nonlinear problems.

The standard ith order integro-differential equation is of the form

$$u^{(i)}(x) = f(x) + \int_0^x K(x,t)u(t)dt, \tag{5.31}$$

where $u^{(i)}(x) = \frac{d^i u}{dx^i}$, and $u(0), u'(0), \ldots, u^{(i-1)}(0)$ are the initial conditions.

The correction functional for the integro-differential equation (5.31) is

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(\xi)\left(u_n^{(i)}(\xi) - f(\xi) - \int_0^{\xi} K(\xi,r)\tilde{u}_n(r)\,dr\right)d\xi. \tag{5.32}$$

The variational iteration method [8,10] is used by applying two essential steps. It is required first to determine the Lagrange multiplier λ that can be identified optimally via integration by parts and by using a restricted variation. Having λ determined, an iteration formula, without restricted variation, should be used for the determination of the successive approximations $u_{n+1}(x), n \geqslant 0$ of the solution $u(x)$. The zeroth approximation $u_0(x)$ can be any selective function. However, the initial values $u(0), u'(0), \cdots$ are preferably used for the selective zeroth approximation $u_0(x)$ as will be seen later. Consequently, the solution is given by

$$u(x) = \lim_{n\to\infty} u_n(x). \tag{5.33}$$

It is useful to summarize the Lagrange multipliers as derived in Chapter 3:

$$\begin{aligned}
&u' + f(u(\xi), u'(\xi)) = 0, \lambda = -1,\\
&u'' + f(u(\xi), u'(\xi), u''(\xi)) = 0, \lambda = \xi - x,\\
&u''' + f(u(\xi), u'(\xi), u''(\xi), u'''(\xi)) = 0, \lambda = -\frac{1}{2!}(\xi - x)^2,\\
&u^{(n)} + f(u(\xi),\ u'(\xi),\ u''(\xi), \cdots, u^{(n)}(\xi)) = 0,\\
&\lambda = (-1)^n \frac{1}{(n-1)!}(\xi - x)^{(n-1)}.
\end{aligned} \tag{5.34}$$

The VIM will be illustrated by studying the following examples.

Example 5.5

Use the variational iteration method to solve the Volterra integro-differential equation

$$u'(x) = 1 + \int_0^x u(t)dt, u(0) = 1. \tag{5.35}$$

The correction functional for this equation is given by

$$u_{n+1}(x) = u_n(x) - \int_0^x \left(u_n'(\xi) - 1 - \int_0^\xi u_n(r)dr \right) d\xi, \tag{5.36}$$

where we used $\lambda = -1$ for first-order integro-differential equations as shown in (5.34).

We can use the initial condition to select $u_0(x) = u(0) = 1$. Using this selection into the correction functional gives the following successive approximations

$$\begin{aligned}
u_0(x) &= 1,\\
u_1(x) &= u_0(x) - \int_0^x \left(u_0'(\xi) - 1 - \int_0^\xi u_0(r)dr \right) d\xi\\
&= 1 + x + \frac{1}{2!}x^2,\\
u_2(x) &= u_1(x) - \int_0^x \left(u_1'(\xi) - 1 - \int_0^\xi u_1(r)dr \right) d\xi\\
&= 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4,\\
u_3(x) &= u_2(x) - \int_0^x \left(u_2'(\xi) - 1 - \int_0^\xi u_2(r)dr \right) d\xi\\
&= 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \frac{1}{6!}x^6,
\end{aligned} \tag{5.37}$$

and so on. The VIM admits the use of

$$u(x) = \lim_{n \to \infty} u_n(x), \tag{5.38}$$

that gives the exact solution

$$u(x) = e^x. \tag{5.39}$$

Example 5.6

Use the variational iteration method to solve the Volterra integro-differential equation

$$u''(x) = 1 + \int_0^x (x-t)u(t)dt, u(0) = 1, u'(0) = 0. \tag{5.40}$$

The correction functional for this equation is given by

$$u_{n+1}(x) = u_n(x) + \int_0^x (\xi - x)\left(u_n''(\xi) - 1 - \int_0^\xi (\xi - r)u_n(r)dr\right)d\xi, \tag{5.41}$$

where $\lambda = \xi - x$ for second-order integro-differential equations as shown in (5.34).

We can use the initial conditions to select $u_0(x) = u(0) + xu'(0) = 1$. Using this selection into the correction functional gives the following successive approximations

$$\begin{aligned} u_0(x) &= 1,\\ u_1(x) &= u_0(x) + \int_0^x (\xi - x)\left(u_0''(\xi) - 1 - \int_0^\xi (\xi - r)u_0(r)dr\right)d\xi\\ &= 1 + \frac{1}{2!}x^2 + \frac{1}{4!}x^4,\\ u_2(x) &= u_1(x) + \int_0^x (\xi - x)\left(u_1''(\xi) - 1 - \int_0^\xi (\xi - r)u_1(r)dr\right)d\xi\\ &= 1 + \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \frac{1}{6!}x^6 + \frac{1}{8!}x^8,\\ u_3(x) &= u_2(x) + \int_0^x (\xi - x)\left(u_2''(\xi) - 1 - \int_0^\xi (\xi - r)u_2(r)dr\right)d\xi\\ &= 1 + \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \frac{1}{6!}x^6 + \frac{1}{8!}x^8 + \frac{1}{10!}x^{10} + \frac{1}{12!}x^{12}, \end{aligned} \tag{5.42}$$

and so on.

The VIM admits the use of

$$u(x) = \lim_{n\to\infty} u_n(x), \tag{5.43}$$

that gives the exact solution

$$u(x) = \cosh x. \tag{5.44}$$

Example 5.7

Use the variational iteration method to solve the Volterra integro-differential equation

$$u'''(x) = 1 + x + \frac{1}{3!}x^3 + \int_0^x (x-t)u(t)dt,\ u(0) = 1,\ u'(0) = 0,\ u''(0) = 1. \tag{5.45}$$

The correction functional for this equation is given by

$$u_{n+1}(x) = u_n(x) \\ -\frac{1}{2}\int_0^x (\xi - x)^2 \left(u_n'''(\xi) - 1 - \xi - \frac{1}{3!}\xi^3 - \int_0^\xi (\xi - r)u_n(r)dr \right) d\xi, \tag{5.46}$$

where $\lambda = -\frac{1}{2}(\xi - x)^2$ for third-order ODEs and for second-order integro-differential equations as shown in (5.34).

The zeroth approximation $u_0(x)$ can be selected by using the initial conditions, hence we set $u_0(x) = u(0) + xu'(0) + \frac{1}{2!}x^2 u''(0) = 1 + \frac{1}{2!}x^2$. Using this selection into the correction functional gives the following successive approximations

$$\begin{aligned} u_0(x) &= 1 + \frac{1}{2!}x^2, \\ u_1(x) &= u_0(x) \\ &\quad -\frac{1}{2}\int_0^x (\xi - x)^2 \left(u_0'''(\xi) - 1 - \xi - \frac{1}{3!}\xi^3 - \int_0^\xi (\xi - r)u_0(r)dr \right) d\xi \\ &= 1 + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \frac{1}{6!}x^6 + \frac{1}{7!}x^7, \\ u_2(x) &= u_1(x) \\ &\quad -\frac{1}{2}\int_0^x (\xi - x)^2 \left(u_1'''(\xi) - 1 - \xi - \frac{1}{3!}\xi^3 - \int_0^\xi (\xi - r)u_1(r)dr \right) d\xi \\ &= 1 + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \frac{1}{6!}x^6 + \frac{1}{7!}x^7 + \frac{1}{8!}x^8 + \cdots, \end{aligned} \tag{5.47}$$

and so on. The VIM admits the use of

$$u(x) = \lim_{n\to\infty} u_n(x), \tag{5.48}$$

that gives the exact solution

$$u(x) = e^x - x. \tag{5.49}$$

Example 5.8

Use the variational iteration method to solve the Volterra integro-differential equation

$$u^{(\text{iv})}(x) = 24 + x + \frac{1}{30}x^6 - \int_0^x (x - t)u(t)dt, \\ u(0) = 0, \quad u'(0) = 1, \quad u''(0) = 0, \quad u'''(0) = -1. \tag{5.50}$$

The correction functional for this equation is given by

$$u_{n+1}(x) = u_n(x) \\ +\frac{1}{6}\int_0^x (\xi - x)^3 \left(u_n^{(\text{iv})}(\xi) - 24 - \xi - \frac{1}{30}\xi^6 + \int_0^\xi (\xi - r)u_n(r)dr \right) d\xi, \tag{5.51}$$

where $\lambda = \frac{1}{6}(\xi - x)^3$ for fourth-order ODEs integro-differential equations as shown in (5.34).

Using $u_0(x) = x - \frac{1}{3!}x^3$ into the correction functional gives the following successive approximations

$$
\begin{aligned}
u_0(x) &= x - \frac{1}{3!}x^3, \\
u_1(x) &= u_0(x) \\
&\quad + \frac{1}{6}\int_0^x (\xi - x)^3 \left(u_0^{(\mathrm{iv})}(\xi) - 24 - \xi - \frac{1}{30}\xi^6 + \int_0^\xi (\xi - r)u_0(r)dr \right) d\xi \\
&= x^4 + x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \frac{1}{9!}x^9 + \frac{1}{151200}x^{10}, \\
u_2(x) &= u_1(x) \\
&\quad + \frac{1}{6}\int_0^x (\xi - x)^3 \left(u_1^{(\mathrm{iv})}(\xi) - 24 - \xi - \frac{1}{30}\xi^6 + \int_0^\xi (\xi - r)u_1(r)dr \right) d\xi \\
&= x^4 + x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \frac{1}{9!}x^9 - \frac{1}{11!}x^{11} + \cdots ,
\end{aligned}
\tag{5.52}
$$

and so on. Notice that x^{10} in $u_1(x)$ has vanished from $u_2(x)$. Similarly other noise terms vanish in the limit. The VIM admits the use of

$$u(x) = \lim_{n\to\infty} u_n(x), \tag{5.53}$$

that gives the exact solution by

$$u(x) = x^4 + \sin x. \tag{5.54}$$

Exercises 5.2.2

Solve the following Volterra integro-differential equations by using the variational iteration method:

1. $u'(x) = x - 1 - \int_0^x (x - t)u(t)dt,\ u(0) = 1$
2. $u'(x) = 1 + x + \int_0^x (x - t)u(t)dt,\ u(0) = 1$
3. $u'(x) = 3e^x - 2 - x + \int_0^x (x - t)u(t)dt,\ u(0) = 0$
4. $u''(x) = x + \int_0^x (x - t)u(t)dt,\ u(0) = 0,\ u'(0) = 1$
5. $u''(x) = x - \frac{1}{3!}x^3 + \int_0^x (x - t)u(t)dt,\ u(0) = 0,\ u'(0) = 2$
6. $u''(x) = 1 - \frac{1}{3!}x^3 + \int_0^x (x - t)u(t)dt,\ u(0) = 1,\ u'(0) = 1$
7. $u''(x) = 1 + x + \int_0^x (x - t)u(t)dt,\ u(0) = 1,\ u'(0) = 1$

8. $u''(x) = -1 - x + \int_0^x (x-t)u(t)dt,\ u(0) = 1,\ u'(0) = 1$

9. $u''(x) = -1 - \frac{1}{3!}x^3 + \int_0^x (x-t)u(t)dt,\ u(0) = 1,\ u'(0) = 1$

10. $u''(x) = 2 - x - \frac{1}{12}x^4 + \int_0^x (x-t)u(t)dt,\ u(0) = 0,\ u'(0) = 1$

11. $u'''(x) = 1 - x + 2\sin x - \int_0^x (x-t)u(t)dt,\ u(0) = 1,\ u'(0) = -1,\ u''(0) = -1$

12. $u'''(x) = 1 + x + \frac{1}{2!}x^2 - \frac{1}{4!}x^4 + \frac{1}{2}\int_0^x (x-t)^2u(t)dt,$
$u(0) = 1,\ u'(0) = 2,\ u''(0) = 1$

13. $u'''(x) = -3 - 2x - \frac{1}{2}x^2 + 6e^x + \frac{1}{2}\int_0^x (x-t)^2u(t)dt,$
$u(0) = 0,\ u'(0) = 1,\ u''(0) = 2$

14. $u^{(\text{iv})}(x) = 1 + x - \int_0^x (x-t)u(t)dt,\ u(0) = u'(0) = 1,\ u''(0) = u'''(0) = -1$

15. $u^{(\text{iv})}(x) = 1 + x + \frac{1}{2!}x^2 - \frac{1}{60}x^5 + \frac{1}{2}\int_0^x (x-t)^2u(t)dt,\ u(0) = u'(0) = u'''(0) = 1,$
$u''(0) = 3$

16. $u^{(\text{iv})}(x) = 1 + x + \frac{1}{2!}x^2 - \frac{1}{3}x^3 + \frac{1}{2}\int_0^x (x-t)^2u(t)dt,$
$u(0) = 3,\ u'(0) = u''(0) = u'''(0) = 1$

5.2.3 The Laplace Transform Method

The *Laplace transform method* was used before for solving Volterra integral equations of the first and the second kind in Chapter 3. The details and properties of the Laplace transform method can be found in ordinary differential equations texts.

Before we start applying this method, we summarize some of the concepts presented in Section 1.5. In the Laplace transform convolution theorem, it was stated that if the kernel $K(x,t)$ of the integral equation

$$u^{(n)}(x) = f(x) + \lambda \int_0^x K(x,t)u(t)dt, \tag{5.55}$$

depends on the difference $x - t$, then it is called a *difference kernel*. The integro-differential equation can thus be expressed as

$$u^{(n)}(x) = f(x) + \lambda \int_0^x K(x-t)u(t)dt. \tag{5.56}$$

Consider two functions $f_1(x)$ and $f_2(x)$ that possess the conditions needed for the existence of Laplace transform for each. Let the Laplace transforms for the functions $f_1(x)$ and $f_2(x)$ be given by

$$\mathcal{L}\{f_1(x)\} = F_1(s), \quad \mathcal{L}\{f_2(x)\} = F_2(s). \tag{5.57}$$

The *Laplace convolution product* of these two functions is defined by

$$(f_1 * f_2)(x) = \int_0^x f_1(x-t)f_2(t)dt, \tag{5.58}$$

or

$$(f_2 * f_1)(x) = \int_0^x f_2(x-t)f_1(t)dt. \tag{5.59}$$

Recall that

$$(f_1 * f_2)(x) = (f_2 * f_1)(x). \tag{5.60}$$

We can easily show that the Laplace transform of the convolution product $(f_1 * f_2)(x)$ is given by

$$\mathcal{L}\{(f_1 * f_2)(x)\} = \mathcal{L}\left\{\int_0^x f_1(x-t)f_2(t)dt\right\} = F_1(s)F_2(s). \tag{5.61}$$

To solve Volterra integro-differential equations by using the Laplace transform method, it is essential to use the Laplace transforms of the derivatives of $u(x)$. We can easily show that

$$\mathcal{L}\{u^{(n)}(x)\} = s^n\mathcal{L}\{u(x)\} - s^{n-1}u(0) - s^{n-2}u'(0) - \cdots - u^{(n-1)}(0). \tag{5.62}$$

This simply gives

$$\begin{aligned}
\mathcal{L}\{u'(x)\} &= s\mathcal{L}\{u(x)\} - u(0)\\
&= sU(s) - u(0),\\
\mathcal{L}\{u''(x)\} &= s^2\mathcal{L}\{u(x)\} - su(0) - u'(0)\\
&= s^2U(s) - su(0) - u'(0),\\
\mathcal{L}\{u'''(x)\} &= s^3\mathcal{L}\{u(x)\} - s^2u(0) - su'(0) - u''(0)\\
&= s^3U(s) - s^2u(0) - su'(0) - u''(0),\\
\mathcal{L}\{u^{(iv)}(x)\} &= s^4\mathcal{L}\{u(x)\} - s^3u(0) - s^2u'(0) - su''(0) - u'''(0)\\
&= s^4U(s) - s^3u(0) - s^2u'(0) - su''(0) - u'''(0),
\end{aligned} \tag{5.63}$$

and so on for derivatives of higher order.

The Laplace transform method can be applied in a similar manner to the approach used before in Chapter 3. We first apply the Laplace transform to both sides of (5.56), use the proper Laplace transform for the derivative of $u(x)$, and then solve for $U(s)$. We next use the inverse Laplace transform of both sides of the resulting equation to obtain the solution $u(x)$ of the equation. Recall that Table 1.2 in Chapter 1 should be used. The Laplace transform method for solving Volterra integro-differential equations will be illustrated by studying the following examples.

Example 5.9

Use the Laplace transform method to solve the Volterra integro-differential equation

$$u'(x) = 1 + \int_0^x u(t)dt, u(0) = 1. \tag{5.64}$$

Notice that the kernel $K(x-t)=1$. Taking Laplace transform of both sides of (5.64) gives

$$\mathcal{L}(u'(x)) = \mathcal{L}(1) + \mathcal{L}(1 * u(x)), \tag{5.65}$$

so that

$$sU(s) - u(0) = \frac{1}{s} + \frac{1}{s}U(s), \tag{5.66}$$

obtained upon using (5.63). Using the given initial condition and solving for $U(s)$ we find

$$U(s) = \frac{1}{s-1}. \tag{5.67}$$

By taking the inverse Laplace transform of both sides of (5.67), the exact solution is given by

$$u(x) = e^x. \tag{5.68}$$

Example 5.10

Use the Laplace transform method to solve the Volterra integro-differential equation

$$u''(x) = -1 - x + \int_0^x (x-t)u(t)dt, \quad u(0)=1, \quad u'(0)=1. \tag{5.69}$$

Notice that the kernel $K(x-t)=(x-t)$. Taking Laplace transform of both sides of (5.69) gives

$$\mathcal{L}(u''(x)) = -\mathcal{L}(1) - \mathcal{L}(x) + \mathcal{L}((x-t) * u(x)), \tag{5.70}$$

so that

$$s^2U(s) - su(0) - u'(0) = -\frac{1}{s} - \frac{1}{s^2} + \frac{1}{s^2}U(s), \tag{5.71}$$

obtained upon using (5.63). Using the given initial condition and solving for $U(s)$ we find

$$U(s) = \frac{1}{s^2+1} + \frac{s}{s^2+1}. \tag{5.72}$$

By taking the inverse Laplace transform of both sides of (5.72), the exact solution is given by

$$u(x) = \sin x + \cos x. \tag{5.73}$$

Example 5.11

Use the Laplace transform method to solve the Volterra integro-differential equation

$$u'''(x) = 1+x+\frac{1}{2!}x^2-\frac{1}{4!}x^4+\frac{1}{2}\int_0^x (x-t)^2u(t)dt, u(0)=1, u'(0)=2, u''=1. \tag{5.74}$$

Taking Laplace transform of both sides of (5.74) gives

$$\mathcal{L}\{u'''(x)\} = \mathcal{L}\{1\}+\mathcal{L}\{x\}+\frac{1}{2!}\mathcal{L}\{x^2\}-\frac{1}{4!}\mathcal{L}\{x^4\}+\frac{1}{2}\mathcal{L}\{(x-t)^2 * u(x)\}, \tag{5.75}$$

so that

$$s^3U(s) - s^2 - 2s - 1 = \frac{1}{s} + \frac{1}{s^2} + \frac{1}{s^3} - \frac{1}{s^5} + \frac{1}{s^3}U(s). \tag{5.76}$$

Using the given initial condition and solving for $U(s)$ we find

$$U(s) = \frac{1}{s^2} + \frac{1}{s-1}. \tag{5.77}$$

Using the inverse Laplace transform, the exact solution is given by

$$u(x) = x + e^x. \tag{5.78}$$

Example 5.12

Solve the Volterra integral equation by using by the Laplace transform method

$$\begin{gathered} u^{(\mathrm{iv})}(x) = \sin x + \cos x + 2\int_0^x \sin(x-t)u(t)dt, \\ u(0) = u'(0) = u''(0) = u'''(0) = 1. \end{gathered} \tag{5.79}$$

Taking Laplace transform of both sides of (5.79) gives

$$\mathcal{L}\{u^{(\mathrm{iv})}(x)\} = \mathcal{L}\{\sin x\} + \mathcal{L}\{\cos x\} + 2\mathcal{L}\{\sin(x-t) * u(x)\}, \tag{5.80}$$

so that

$$s^4U(s) - s^3 - s^2 - s - 1 = \frac{1}{s^2+1} + \frac{s}{s^2+1} + \frac{2}{s^2+1}U(s). \tag{5.81}$$

Using the given initial condition and solving for $U(s)$ we find

$$U(s) = \frac{1}{s-1}. \tag{5.82}$$

Using the inverse Laplace transform, the exact solution is given by

$$u(x) = e^x. \tag{5.83}$$

Exercises 5.2.3

Solve the following Volterra integro-differential equations by using the Laplace transform method

1. $u'(x) = 1 + x - x^2 + \int_0^x (x-t)u(t)dt,\ u(0) = 3$

2. $u'(x) = 2 + x - \frac{1}{3!}x^3 + \int_0^x (x-t)u(t)dt,\ u(0) = 1$

3. $u'(x) = -(a+b)u(x) - ab\int_0^x u(t)dt,\ u(0) = a - b$

4. $u''(x) = -x + \int_0^x (x-t)u(t)dt,\ u(0) = 0,\ u'(0) = 1$

5. $u''(x) = -x - \frac{1}{3!}x^3 + \int_0^x (x-t)u(t)dt,\ u(0) = 0,\ u'(0) = 2$

6. $u''(x) = \cosh x + \int_0^x e^{-(x-t)}u(t)dt,\ u(0) = 1, u'(0) = 1$

7. $u''(x) = \cosh x - \int_0^x e^{(x-t)} u(t)dt,\ u(0) = 1, u'(0) = -1$

8. $u''(x) = 4e^x + 4\int_0^x e^{(x-t)} u(t)dt,\ u(0) = 1, u'(0) = 2$

9. $u''(x) = -x + \frac{1}{3!}x^3 - \frac{1}{6}\int_0^x (x-t)^3 u(t)dt,\ u(0) = 0,\ u'(0) = 1$

10. $u''(x) = 1 + \frac{1}{2}x^2 + \frac{1}{6}\int_0^x (x-t)^3 u(t)dt,\ u(0) = 1,\ u'(0) = 0$

11. $u'''(x) = 1 + x - 2x^2 + \int_0^x (x-t)u(t)dt,\ u(0) = 5, u'(0) = u''(0) = 1$

12. $u'''(x) = 1 + \frac{1}{2!}x^2 + \frac{1}{2}\int_0^x (x-t)^2 u(t)dt,\ u(0) = u''(0) = 0, u'(0) = 1$

13. $u'''(x) = x - \frac{1}{4!}x^4 + \frac{1}{2}\int_0^x (x-t)^2 u(t)dt,\ u(0) = u'(0) = u''(0) = 1$

14. $u^{(\text{iv})}(x) = e^{2x} - \int_0^x e^{2(x-t)} u(t)dt,\ u(0) = u'(0) = u''(0) = u'''(0) = 1$

15. $u^{(\text{iv})}(x) = 3e^x + e^{2x} - \int_0^x e^{2(x-t)} u(t)dt,\ u(0) = 0,$
$u'(0) = 1,\ u''(0) = 2,\ u'''(0) = 3$

16. $u^{(\text{iv})}(x) = -\frac{1}{4} - \frac{1}{2}x + \frac{5}{4}e^{2x} - \int_0^x e^{2(x-t)} u(t)dt,$
$u(0) = 1,\ u'(0) = 2,\ u''(0) = u'''(0) = 1$

5.2.4 The Series Solution Method

It was stated before in Chapter 3 that a real function $u(x)$ is called analytic if it has derivatives of all orders such that the Taylor series at any point b in its domain

$$u(x) = \sum_{n=0}^{\infty} \frac{u^{(n)}(b)}{n!}(x-b)^n, \tag{5.84}$$

converges to $u(x)$ in a neighborhood of b. For simplicity, the generic form of Taylor series at $x = 0$ can be written as

$$u(x) = \sum_{n=0}^{\infty} a_n x^n. \tag{5.85}$$

In this section we will apply the series solution method for solving Volterra integro-differential equations of the second kind. We will assume that the solution $u(x)$ of the Volterra integro-differential equation

$$u^{(n)}(x) = f(x) + \lambda\int_0^x K(x,t)u(t)dt,\ u^{(k)}(0) = k!a_k,\ 0 \leqslant k \leqslant (n-1), \tag{5.86}$$

is analytic, and therefore possesses a Taylor series of the form given in (5.85), where the coefficients a_n will be determined recurrently.

The first few coefficients a_k can be determined by using the initial conditions so that

$$a_0 = u(0), \quad a_1 = u'(0), \quad a_2 = \frac{1}{2!}u''(0), \quad a_3 = \frac{1}{3!}u'''(0), \tag{5.87}$$

and so on. The remaining coefficients a_k of (5.85) will be determined by applying the series solution method to the Volterra integro-differential equation (5.86). Substituting (5.85) into both sides of (5.86) gives

$$\left(\sum_{k=0}^{\infty} a_k x^k\right)^{(n)} = T(f(x)) + \int_0^x K(x,t)\left(\sum_{k=0}^{\infty} a_k t^k\right) dt, \tag{5.88}$$

or for simplicity we use

$$(a_0+a_1x+a_2x^2+\cdots)^{(n)} = T(f(x))+\lambda\int_0^x K(x,t)\left(a_0 + a_1t + a_2t^2 + \cdots\right) dt, \tag{5.89}$$

where $T(f(x))$ is the Taylor series for $f(x)$. The integro-differential equation (5.86) will be converted to a traditional integral in (5.88) or (5.89) where instead of integrating the unknown function $u(x)$, terms of the form t^n, $n \geqslant 0$ will be integrated. Notice that because we are seeking series solution, then if $f(x)$ includes elementary functions such as trigonometric functions, exponential functions, etc., then Taylor expansions for functions involved in $f(x)$ should be used.

We first integrate the right side of the integral in (5.88) or (5.89), and collect the coefficients of like powers of x. We next equate the coefficients of like powers of x into both sides of the resulting equation to determine a recurrence relation in $a_j, j \geqslant 0$. Solving the recurrence relation will lead to a complete determination of the coefficients $a_j, j \geqslant 0$, where some of these coefficients will be used from the initial conditions. Having determined the coefficients $a_j, j \geqslant 0$, the series solution follows immediately upon substituting the derived coefficients into (5.85). The exact solution may be obtained if such an exact solution exists. If an exact solution is not obtainable, then the obtained series can be used for numerical purposes. In this case, the more terms we evaluate, the higher accuracy level we achieve.

Example 5.13

Use the series solution method to solve the Volterra integro-differential equation

$$u'(x) = 1 + \int_0^x u(t)dt. \tag{5.90}$$

Substituting $u(x)$ by the series

$$u(x) = \sum_{n=0}^{\infty} a_n x^n, \tag{5.91}$$

into both sides of Eq. (5.90) leads to

$$\left(\sum_{n=0}^{\infty} a_n x^n\right)' = 1 + \int_0^x \left(\sum_{n=0}^{\infty} a_n t^n\right) dt. \tag{5.92}$$

Differentiating the left side once with respect to x, and evaluating the integral at the right side we find

$$\sum_{n=1}^{\infty} n a_n x^{n-1} = 1 + \sum_{n=1}^{\infty} \frac{1}{n} a_{n-1} x^n, \tag{5.93}$$

that can be rewritten as

$$a_1 + \sum_{n=1}^{\infty} (n+1) a_{n+1} x^n = 1 + \sum_{n=1}^{\infty} \frac{1}{n} a_{n-1} x^n, \tag{5.94}$$

where we unified the exponent of x in both sides, and used $a_0 = 0$ from the given initial condition. Equating the coefficients of like powers of x in both sides of (5.94) gives the recurrence relation

$$a_0 = 0, \quad a_1 = 1, \quad a_{n+1} = \frac{1}{n(n+1)} a_{n-1}, \quad n \geqslant 1. \tag{5.95}$$

where this result gives

$$a_{2n} = 0, \quad a_{2n+1} = \frac{1}{(2n+1)!}, \tag{5.96}$$

for $n \geqslant 0$. Substituting this result into (5.91) gives the series solution

$$u(x) = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} x^{2n+1}, \tag{5.97}$$

that converges to the exact solution

$$u(x) = \sinh x. \tag{5.98}$$

Example 5.14

Use the series solution method to solve the Volterra integro-differential equation

$$u''(x) = 1 + x + \int_0^x (x-t) u(t) dt, \quad u(0) = u'(0) = 1. \tag{5.99}$$

Substituting $u(x)$ by the series

$$u(x) = \sum_{n=0}^{\infty} a_n x^n, \tag{5.100}$$

into both sides of Eq. (5.99) leads to

$$\left(\sum_{n=0}^{\infty} a_n x^n\right)'' = 1 + x + \int_0^x \left((x-t) \sum_{n=0}^{\infty} a_n t^n\right) dt. \tag{5.101}$$

Differentiating the left side twice, and by evaluating the integral at the right side we find

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = 1 + x + \sum_{n=0}^{\infty} \frac{1}{(n+2)(n+1)} a_n x^{n+2}, \tag{5.102}$$

or equivalently

$$2a_2+6a_3x+\sum_{n=2}^{\infty}(n+2)(n+1)a_{n+2}x^n = 1+x+\sum_{n=2}^{\infty}\frac{1}{n(n-1)}a_{n-2}x^n. \quad (5.103)$$

Using the initial conditions and equating the coefficients of like powers of x in both sides of (5.103) gives the recurrence relation

$$\begin{aligned} a_0 &= 1, \quad a_1 = 1, \quad a_2 = \frac{1}{2!}, \quad a_3 = \frac{1}{3!}, \\ a_{n+2} &= \frac{1}{(n+2)(n+1)n(n-1)}a_{n-2}, \quad n \geqslant 2. \end{aligned} \quad (5.104)$$

where this result gives

$$a_n = \frac{1}{n!}, \quad (5.105)$$

for $n \geqslant 0$. Substituting this result into (5.100) gives the series solution

$$u(x) = \sum_{n=0}^{\infty}\frac{1}{n!}x^n, \quad (5.106)$$

that converges to the exact solution

$$u(x) = e^x. \quad (5.107)$$

Example 5.15

Use the series solution method to solve the Volterra integro-differential equation

$$u'''(x) = 1 - x + 2\sin x - \int_0^x (x-t)u(t)dt, u(0) = 1, u'(0) = u''(0) = -1. \quad (5.108)$$

Substituting $u(x)$ by the series

$$u(x) = \sum_{n=0}^{\infty} a_n x^n, \quad (5.109)$$

into both sides of the equation (5.108), and using Taylor expansion for $\sin x$ we obtain

$$\left(\sum_{n=0}^{\infty} a_n x^n\right)''' = 1 - x + 2\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}x^{2n+1} - \int_0^x \left((x-t)\sum_{n=0}^{\infty} a_n t^n\right) dt. \quad (5.110)$$

Differentiating the left side three times, and by evaluating the integral at the right side we find

$$6a_3 + 24a_4x + 60a_5x^2 + 120a_6x^3 + \cdots = 1 + x - \frac{1}{2}a_0x^2 - \left(\frac{1}{3} + \frac{1}{6}a_1\right)x^3 + \cdots, \quad (5.111)$$

where we used few terms for simplicity reasons. Using the initial conditions and equating the coefficients of like powers of x in both sides of (5.111) we find

$$\begin{array}{lll} a_0 = 1, & a_1 = -1, & a_2 = -\dfrac{1}{2!}, \\ a_3 = \dfrac{1}{3!}, & a_4 = \dfrac{1}{4!}, & a_5 = -\dfrac{1}{5!}. \end{array} \tag{5.112}$$

Consequently, the series solution is given by

$$u(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} - \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}, \tag{5.113}$$

that converges to the exact solution

$$u(x) = \cos x - \sin x. \tag{5.114}$$

Example 5.16

Use the series solution method to solve the Volterra integro-differential equation

$$u^{(\text{iv})}(x) = 1 + x - \int_0^x (x-t)u(t)dt,\ u(0) = u'(0) = 1,\ u''(0) = u'''(0) = -1. \tag{5.115}$$

Proceeding as before we set

$$\left(\sum_{n=0}^{\infty} a_n x^n\right)^{(\text{iv})} = 1 + x - \int_0^x \left((x-t)\sum_{n=0}^{\infty} a_n t^n\right) dt, \tag{5.116}$$

Differentiating the left side, integrating the right side, and using few terms for simplicity we find

$$24a_4 + 120a_5 x + 360a_6 x^2 + 840a_7 x^3 + \cdots = 1 + x - \frac{1}{2}a_0 x^2 - \frac{1}{6}a_1 x^3 + \cdots. \tag{5.117}$$

Using the initial conditions and equating the coefficients of like powers of x in both sides of (5.117) gives the recurrence relation

$$\begin{array}{llll} a_0 = 1, & a_1 = 1, & a_2 = -\dfrac{1}{2!}, & \\ a_3 = -\dfrac{1}{3!}, & a_4 = \dfrac{1}{4!}, & a_5 = \dfrac{1}{5!}, & a_6 = -\dfrac{1}{6!}, \cdots \end{array} \tag{5.118}$$

Consequently, the series solution is given by

$$u(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} + \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}, \tag{5.119}$$

that converges to the exact solution

$$u(x) = \cos x + \sin x. \tag{5.120}$$

Exercises 5.2.4

Solve the following Volterra integro-differential equations by using the series solution method:

1. $u'(x) = 1 + \int_0^x u(t)dt,\ u(0) = 0$

2. $u'(x) = 1 + x + \int_0^x (x-t)u(t)dt,\ u(0) = 1$
3. $u'(x) = 2 - x + \frac{1}{3!}x^3 - \int_0^x (x-t)u(t)dt,\ u(0) = -1$
4. $u''(x) = 1 + \int_0^x (x-t)u(t)dt,\ u(0) = 1,\ u'(0) = 0$
5. $u''(x) = x - \frac{1}{3!}x^3 + \int_0^x (x-t)u(t)dt,\ u(0) = 0,\ u'(0) = 2$
6. $u''(x) = \cosh x + \int_0^x e^{-(x-t)}u(t)dt,\ u(0) = 1,\ u'(0) = 1$
7. $u''(x) = -1 - x + \int_0^x (x-t)u(t)dt,\ u(0) = 1, u'(0) = 1$
8. $u''(x) = 2 - x - \frac{1}{12}x^4 + \int_0^x (x-t)u(t)dt,\ u(0) = 0,\ u'(0) = 1$
9. $u''(x) = -x + \frac{1}{3!}x^3 - \frac{1}{6}\int_0^x (x-t)^3u(t)dt,\ u(0) = 0,\ u'(0) = 1$
10. $u''(x) = 1 + x - \frac{1}{3!}x^3 + \int_0^x (x-t)u(t)dt,\ u(0) = 1,\ u'(0) = 2$
11. $u'''(x) = 1 + x + \frac{1}{3!}x^3 + \int_0^x (x-t)u(t)dt,\ u(0) = 1, u'(0) = 0, u''(0) = 1$
12. $u'''(x) = 1 + \frac{1}{2!}x^2 + \frac{1}{2}\int_0^x (x-t)^2u(t)dt,\ u(0) = u''(0) = 0,\ u'(0) = 1$
13. $u'''(x) = 1 + x - \frac{1}{12}x^4 + \int_0^x (x-t)u(t)dt,\ u(0) = 1,\ u'(0) = 1,\ u''(0) = 3$
14. $u^{(\text{iv})}(x) = 1 + x - \frac{1}{20}x^5 + \int_0^x (x-t)u(t)dt,$
    $u(0) = u'(0) = u''(0) = 1,\ u'''(0) = 7$
15. $u^{(\text{iv})}(x) = 3e^x + e^{2x} - \int_0^x e^{2(x-t)}u(t)dt,\ u(0) = 0,$
    $u'(0) = 1,\ u''(0) = 2,\ u'''(0) = 3$
16. $u^{(\text{iv})}(x) = 1 + x - \frac{1}{2!}x^2 - \frac{1}{3!}x^3 + \int_0^x (x-t)u(t)dt,$
    $u(0) = u'(0) = 2,\ u''(0) = u'''(0) = 1$

5.2.5 Converting Volterra Integro-Differential Equations to Initial Value Problems

The Volterra integro-differential equation

$$u^{(n)}(x) = f(x) + \lambda \int_0^x K(x,t)u(t)dt, u^{(k)}(0) = k!a_k, 0 \leqslant k \leqslant (n-1), \quad (5.121)$$

can be solved by converting it to an equivalent initial value problem. The study will be concerned on the Volterra integro-differential equations where the kernel is a difference kernel of the form $K(x,t) = K(x-t)$, such as $x-t$. The reason for this selection is that we seek ODEs with constant coefficients. Having converted the integro-differential equation to an initial value problem, we then can use any standard method for solving ODEs. It is worth noting that the conversion process can be easily used, but it requires more works if compared with the integro-differential equations methods.

The conversion process is obtained by differentiating both sides of the Volterra integro-differential equation as many times until we get rid of the integral sign. To perform the differentiation for the integral at the right side, the Leibnitz rule, that was introduced in Chapter 1, should be used. The initial conditions should be determined by using a variety of integral equations that we will obtain in the process of differentiation. To give a clear overview of this method we discuss the following illustrative examples.

Example 5.17

Solve the Volterra integro-differential equation by converting it to an initial value problem

$$u'(x) = 1 + \int_0^x u(t)dt,\ u(0) = 1. \tag{5.122}$$

Differentiating both sides of (5.122) with respect to x and using the Leibnitz rule we find

$$u''(x) = u(x), \tag{5.123}$$

with initial conditions given by

$$u(0) = 1,\ u'(0) = 1. \tag{5.124}$$

The second initial condition is obtained by substituting $x = 0$ in both sides of (5.122). The characteristic equation for the ODE (5.122) is

$$r^2 - 1 = 0, \tag{5.125}$$

which gives the roots

$$r = \pm 1 \tag{5.126}$$

so that the general solution is given by

$$u(x) = Ae^x + Be^{-x}. \tag{5.127}$$

The constants A and B can be determined by using the initial conditions, where we find $A = 1, B = 0$. The exact solution is

$$u(x) = e^x. \tag{5.128}$$

Example 5.18

Solve the Volterra integro-differential equation by converting it to an initial value problem

$$u'(x) = 6 - 3x^2 + \int_0^x u(t)dt,\ u(0) = 0. \tag{5.129}$$

Differentiating both sides of (5.129) with respect to x and using the Leibnitz rule we find

$$u''(x) - u(x) = -6x, \tag{5.130}$$

with initial conditions given by

$$u(0) = 0,\ u'(0) = 6. \tag{5.131}$$

The second initial condition is obtained by substituting $x = 0$ in both sides of (5.129). The characteristic equation for the related homogeneous ODE of (5.129) is

$$r^2 - 1 = 0, \tag{5.132}$$

which gives the roots

$$r = \pm 1 \tag{5.133}$$

so that the complementary solution is given by

$$u_c(x) = Ae^x + Be^{-x}. \tag{5.134}$$

To find a particular solution $u_p(x)$, we substitute

$$u_p(x) = C + Dx, \tag{5.135}$$

into (5.130) and equate coefficients of like powers of x from both side to find that $C = 0, D = 6$. This gives the general solution

$$u(x) = u_c(x) + u_p(x) = Ae^x + Be^{-x} + 6x. \tag{5.136}$$

The constants A and B can be determined by using the initial conditions, where we find $A = 0, B = 0$. The exact solution is given by

$$u(x) = 6x. \tag{5.137}$$

Example 5.19

Solve the following Volterra integro-differential equation by converting it to an initial value problem

$$u''(x) = 1 + x + \int_0^x (x-t)u(t)dt, \quad u(0) = 1, \quad u'(0) = 1. \tag{5.138}$$

Differentiating both sides and proceeding as before we find

$$u'''(x) = 1 + \int_0^x u(t)dt, \tag{5.139}$$

and by differentiating again to remove the integral sign we find

$$u^{(iv)}(x) = u(x). \tag{5.140}$$

Two more initial conditions can be obtained by substituting $x = 0$ in (5.138) and (5.139), hence the four initial conditions are

$$u(0) = 1, \quad u'(0) = 1, \quad u''(0) = 1, \quad u'''(0) = 1. \tag{5.141}$$

The characteristic equation of (5.140) is

$$r^4 - 1 = 0, \tag{5.142}$$

which gives the roots

$$r = \pm 1, \pm i, i = \sqrt{-1}, \tag{5.143}$$

so that the general solution is given by

$$u(x) = Ae^x + Be^{-x} + C\cos x + D\sin x. \tag{5.144}$$

The constants A, B, C, and D can be obtained by using the initial conditions where we found $A = 1$ and $B = C = D = 0$. The exact solution is given by

$$u(x) = e^x. \tag{5.145}$$

Example 5.20

Solve the following Volterra integro-differential equation by converting it to an initial value problem

$$u''(x) = -1 - x + \int_0^x (x-t)u(t)dt, \quad u(0) = 1, \quad u'(0) = 1. \tag{5.146}$$

Differentiating both sides and proceeding as before we obtain

$$u'''(x) = -1 + \int_0^x u(t)dt, \tag{5.147}$$

and by differentiating again to remove the integral sign we find

$$u^{(\mathrm{iv})}(x) = u(x). \tag{5.148}$$

Two more initial conditions can be obtained by substituting $x = 0$ in (5.146) and (5.147), hence the four initial conditions are

$$u(0) = 1, \quad u'(0) = 1, \quad u''(0) = -1, \quad u'''(0) = -1. \tag{5.149}$$

The characteristic equation of (5.148) is

$$r^4 - 1 = 0, \tag{5.150}$$

which gives the roots

$$r = \pm 1, \pm i, i = \sqrt{-1}, \tag{5.151}$$

so that the general solution is given by

$$u(x) = Ae^x + Be^{-x} + C\cos x + D\sin x. \tag{5.152}$$

The constants A, B, C, and D can be obtained by using the initial conditions where we found $A = B = 0$ and $C = D = 1$. The exact solution is given by

$$u(x) = \sin x + \cos x. \tag{5.153}$$

Exercises 5.2.5

Solve the following Volterra integro-differential equations by converting the problem to *an initial value problem*:

1. $u'(x) = 1 + \int_0^x u(t)dt,\ u(0) = 0$
2. $u'(x) = -1 + x - \int_0^x (x-t)u(t)dt,\ u(0) = 1$
3. $u''(x) = 1 + \int_0^x (x-t)u(t)dt,\ u(0) = 1,\ u'(0) = 0$

4. $u''(x) = x + \int_0^x (x-t)u(t)dt,\ u(0) = 0,\ u'(0) = 1$

5. $u''(x) = -1 - \frac{1}{3!}x^3 + \int_0^x (x-t)u(t)dt,\ u(0) = 1,\ u'(0) = 1$

6. $u''(x) = -x + \int_0^x (x-t)u(t)dt,\ u(0) = 0,\ u'(0) = 1$

7. $u''(x) = 1 + x - \frac{1}{3!}x^3 + \int_0^x (x-t)u(t)dt,\ u(0) = 1,\ u'(0) = 2$

8. $u''(x) = -\frac{1}{2}x^2 - \frac{2}{3}x^3 + \int_0^x (x-t)u(t)dt,\ u(0) = 1,\ u'(0) = 4$

5.2.6 Converting Volterra Integro-Differential Equation to Volterra Integral Equation

The Volterra integro-differential equation

$$u^{(n)}(x) = f(x) + \lambda \int_0^x K(x,t)u(t)dt, \tag{5.154}$$

can also be solved by converting it to an equivalent Volterra integral equation. Recall that in integro-differential equations, the initial conditions are usually prescribed. The study will be focused on the Volterra integro-differential equations where the kernel is a difference kernel. Having converted the integro-differential equation to an equivalent integral equation, the latter can be solved by using any of the methods presented in Chapter 3, such as Adomian method, series solution method, and Laplace transform method.

It is obvious that the Volterra integro-differential equation (5.154) involves derivatives at the left side, and integral at the right side. To perform the conversion process, we need to integrate both sides n times to convert it to a standard Volterra integral equation. It is therefore useful to summarize some of formulas to support the conversion process.

We point out that the first set of formulas is usually studied in calculus. However, the second set is given in Section 1.4, where reducing multiple integrals to a single integral is presented.

I. Integration of derivatives: from calculus we observe the following:

$$\begin{aligned} \int_0^x u'(t)dt &= u(x) - u(0), \\ \int_0^x \int_0^{x_1} u''(t)dtdx_1 &= u(x) - xu'(0) - u(0), \\ \int_0^x \int_0^{x_1} \int_0^{x_2} u'''(t)dtdx_2dx_1 &= u(x) - \frac{1}{2!}x^2u''(0) - xu'(0) - u(0), \end{aligned} \tag{5.155}$$

and so on for other derivatives.

II. Reducing multiple integrals to a single integral: from Chapter 1, we studied the following:

$$\begin{aligned}\int_0^x\int_0^{x_1} u(t)dtdx_1 &= \int_0^x (x-t)u(t)dt,\\ \int_0^x\int_0^{x_1} (x-t)u(t)dtdx_1 &= \frac{1}{2}\int_0^x (x-t)^2u(t)dt,\\ \int_0^x\int_0^{x_1} (x-t)^2u(t)dtdx_1 &= \frac{1}{3}\int_0^x (x-t)^3u(t)dt,\\ \int_0^x\int_0^{x_1} (x-t)^3u(t)dtdx_1 &= \frac{1}{4}\int_0^x (x-t)^4u(t)dt,\end{aligned} \tag{5.156}$$

and so on. This can be generalized in the form

$$\int_0^x\int_0^{x_1}\int_0^{x_2}\cdots\int_0^{x_{n-1}} (x-t)u(t)dtdx_{n-1}\cdots dx_1 = \frac{1}{n!}\int_0^x (x-t)^n u(t)dt. \tag{5.157}$$

The conversion to an equivalent Volterra integral equation will be illustrated by studying the following examples.

Example 5.21

Solve the following Volterra integro-differential equation by converting it to Volterra integral equation

$$u'(x) = 1 + \int_0^x u(t)dt, \quad u(0) = 0. \tag{5.158}$$

Integrating both sides from 0 to x, and using the aforementioned formulas we find

$$u(x) - u(0) = x + \int_0^x\int_0^x u(t)dt. \tag{5.159}$$

Using the initial condition gives the Volterra integral equation

$$u(x) = x + \int_0^x (x-t)u(t)dt. \tag{5.160}$$

We can select any of the proposed methods. The Adomian decomposition method will be used to solve this problem. Using the series assumption

$$u(x) = \sum_{n=0}^{\infty} u_n(x), \tag{5.161}$$

into both sides of the Volterra integral equation gives the recursion relation

$$u_0(x) = x, \quad u_1(x) = \frac{1}{3!}x^3, \quad u_2(x) = \frac{1}{5!}x^5, \quad u_3(x) = \frac{1}{7!}x^7, \tag{5.162}$$

so that the series solution is given by

$$u(x) = x + \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \frac{1}{7!}x^7 + \cdots, \tag{5.163}$$

that converges to the exact solution

$$u(x) = \sinh x. \tag{5.164}$$

Example 5.22

Solve the following Volterra integro-differential equation by converting it to Volterra integral equation

$$u'(x) = 1 + x - x^2 + \int_0^x (x-t)u(t)dt,\ u(0) = 3. \tag{5.165}$$

Integrating both sides once from 0 to x and using the formulas given above we obtain

$$u(x) - u(0) = x + \frac{1}{2}x^2 - \frac{1}{3}x^3 + \frac{1}{2!}\int_0^x (x-t)^2 u(t)dt. \tag{5.166}$$

Notice that when we integrate the integral at the right side, we should use the formula (5.157) for the resulting double integral. By using the initial condition we obtain the Volterra integral equation

$$u(x) = 3 + x + \frac{1}{2}x^2 - \frac{1}{3}x^3 + \frac{1}{2!}\int_0^x (x-t)^2 u(t)dt. \tag{5.167}$$

We proceed as before and use the Adomian decomposition method to find

$$\begin{aligned} u_0(x) &= 3 + x + \frac{1}{2}x^2 - \frac{1}{3}x^3, \\ u_1(x) &= \frac{1}{2!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 - \frac{1}{360}x^6, \\ u_2(x) &= \frac{1}{240}x^6 + \frac{1}{7!}x^7 + \frac{1}{8!}x^8 + \cdots . \end{aligned} \tag{5.168}$$

Consequently, the series solution is given by

$$u(x) = 2 + \left(1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \cdots\right), \tag{5.169}$$

that converges to the exact solution

$$u(x) = 2 + e^x. \tag{5.170}$$

Example 5.23

Solve the following Volterra integro-differential equation by converting it to Volterra integral equation

$$u''(x) = -x + \int_0^x (x-t)u(t)dt,\ u(0) = 0, u'(0) = 1. \tag{5.171}$$

Integrating both sides twice from 0 to x and using the formulas given above we find

$$u(x) - xu'(0) - u(0) = -\frac{1}{3!}x^3 + \frac{1}{3!}\int_0^x (x-t)^3 u(t)dt. \tag{5.172}$$

Notice that when we integrate the integral at the right side, we should use the formula (5.157) for the resulting triple integral. By using the initial conditions we obtain the Volterra integral equation

$$u(x) = x - \frac{1}{3!}x^3 + \frac{1}{3!}\int_0^x (x-t)^3 u(t)dt. \tag{5.173}$$

We now select the successive approximations method and set the recurrence relation

$$u_n(x) = x - \frac{1}{3!}x^3 + \frac{1}{3!}\int_0^x (x-t)^3 u_{n-1}(t)dt, n \geqslant 1. \tag{5.174}$$

Selecting the zeroth approximation $u_0(x) = x$ gives the approximations

$$\begin{aligned} u_0(x) &= x, \\ u_1(x) &= x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5, \\ u_2(x) &= x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \frac{1}{9!}x^9. \end{aligned} \tag{5.175}$$

Consequently, the series solution is given by

$$u(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \frac{1}{9!}x^9 + \cdots, \tag{5.176}$$

that converges to the exact solution

$$u(x) = \sin x. \tag{5.177}$$

Example 5.24

Solve the following Volterra integro-differential equation by converting it to Volterra integral equation

$$u'''(x) = 1 - x + 2\sin x - \int_0^x (x-t)u(t)dt, \; u(0) = 1, u'(0) = -1, u''(0) = -1. \tag{5.178}$$

Integrating both sides three times from 0 to x and using the formulas given above we find

$$u(x) + \frac{1}{2!}x^2 + x - 1 = \frac{1}{3!}x^3 - \frac{1}{4!}x^4 + x^2 + 2\cos x - 2 + \frac{1}{4!}\int_0^x (x-t)^4 u(t)dt. \tag{5.179}$$

By using the initial conditions we obtain the Volterra integral equation

$$u(x) = -1 - x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 - \frac{1}{4!}x^4 + 2\cos x + \frac{1}{4!}\int_0^x (x-t)^4 u(t)dt. \tag{5.180}$$

Using the Adomian decomposition method gives the recurrence relation

$$\begin{aligned} u_0(x) &= -1 - x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 - \frac{1}{4!}x^4 + 2\cos x, \\ u_1(x) &= 2x - \frac{1}{3}x^3 + \frac{1}{5!}x^5 + \frac{1}{6!}x^6 - 2\sin x + \cdots, \\ u_2(x) &= 2 - x^2 + \frac{1}{12}x^4 - \frac{1}{360}x^6 - 2\cos x + \cdots, \end{aligned} \tag{5.181}$$

and so on. Consequently, the series solution is given by

$$u(x) = \left(1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \cdots\right) + \left(x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots\right) - 2\sin x, \tag{5.182}$$

that converges to the exact solution

$$u(x) = \cos x - \sin x. \tag{5.183}$$

Exercises 5.2.6

Solve the following Volterra integro-differential equations by converting the problem to *Volterra integral equation*:

1. $u'(x) = 1 - \int_0^x u(t)dt,\ u(0) = 0$
2. $u'(x) = 1 - \int_0^x u(t)dt,\ u(0) = 1$
3. $u''(x) = x + \int_0^x (x-t)u(t)dt,\ u(0) = 0, u'(0) = 1$
4. $u''(x) = 1 + x + \int_0^x (x-t)u(t)dt,\ u(0) = u'(0) = 1$
5. $u''(x) = -1 - \frac{1}{3!}x^3 + \int_0^x (x-t)u(t)dt,\ u(0) = 0,\ u'(0) = 2$
6. $u''(x) = 1 + x - \frac{1}{3!}x^3 + \int_0^x (x-t)u(t)dt,\ u(0) = 1,\ u'(0) = 2$
7. $u'''(x) = 1 + x + \frac{1}{3!}x^3 + \int_0^x (x-t)u(t)dt,\ u(0) = u''(0) = 1,\ u'(0) = 0$
8. $u'''(x) = 1 - e^{-x} + \frac{1}{3!}x^3 + \int_0^x (x-t)u(t)dt,\ u(0) = u''(0) = 1,\ u'(0) = 0$

5.3 Volterra Integro-Differential Equations of the First Kind

The standard form of the Volterra integro-differential equation [11–12] of the first kind is given by

$$\int_0^x K_1(x,t)u(t)dt + \int_0^x K_2(x,t)u^{(n)}(t)dt = f(x), \quad K_2(x,t) \neq 0, \tag{5.184}$$

where initial conditions are prescribed. The Volterra integro-differential equation of the first kind (5.184) can be converted into a Volterra integral equation of the second kind, for $n = 1$, by integrating the second integral in (5.184) by parts. The Volterra integro-differential equations of the first kind will be handled in this section by Laplace transform method and the variational iteration method.

5.3.1 Laplace Transform Method

The *Laplace transform method* was used before for solving Volterra integral equations of the first and the second kind in Chapter 3. It was also used in this chapter for solving integro-differential equations of the second kind.

The analysis will be focused on equations where the kernels $K_1(x,t)$ and $K_2(x,t)$ of (5.184) are *difference kernels*. This means that each kernel depends on the difference $(x-t)$.

Taking Laplace transform of both sides of (5.184) gives

$$\mathcal{L}(K_1(x-t)*u(x)) + \mathcal{L}(K_2(x-t)*u^{(n)}(x)) = \mathcal{L}(f(x)), \tag{5.185}$$

so that

$$\mathcal{K}_1(s)U(s)+\mathcal{K}_2(s)\left(s^nU(s)-s^{n-1}u(0)-s^{n-2}u'(0)-\cdots-u^{(n-1)}(0)\right)=F(s), \tag{5.186}$$

where

$$U(s)=\mathcal{L}(u(x)), \mathcal{K}_1(s)=\mathcal{L}(K_1(x)),\quad \mathcal{K}_2(s)=\mathcal{L}(K_2(x)),\quad F(s)=\mathcal{L}(f(x)). \tag{5.187}$$

Using the given initial conditions and solving for $U(s)$ we find

$$U(s)=\frac{F(s)+\mathcal{K}_2(s)\left(s^{n-1}u(0)+s^{n-2}u'(0)+\cdots-u^{(n-1)}(0)\right)}{\mathcal{K}_1(s)+s^n\mathcal{K}_2(s)}, \tag{5.188}$$

provided that

$$\mathcal{K}_1(s)+s^n\mathcal{K}_2(s)\neq 0. \tag{5.189}$$

By taking the inverse Laplace transform of both sides of (5.188), the exact solution is readily obtained. The analysis presented above can be explained by using the following illustrative examples.

Example 5.25

Solve the following Volterra integro-differential equation of the first kind

$$\int_0^x (x-t)u(t)dt+\int_0^x (x-t)^2u'(t)dt = 3x-3\sin x,\quad u(0)=0. \tag{5.190}$$

Taking Laplace transforms of both sides gives

$$\frac{1}{s^2}U(s)+\frac{2}{s^3}(sU(s)-u(0)) = \frac{3}{s^2}-\frac{3}{1+s^2}, \tag{5.191}$$

where by using the given initial condition and solving for $U(s)$ we obtain

$$U(s)=\frac{1}{1+s^2}. \tag{5.192}$$

Taking the inverse Laplace transform of both sides we find

$$u(x)=\sin x. \tag{5.193}$$

Example 5.26

Solve the following Volterra integro-differential equation of the first kind

$$\int_0^x (x-t+1)u''(t)dt = 2xe^x + e^x - x - 1, \quad u(0)=0, \quad u'(0)=1. \quad (5.194)$$

Notice in this equation that $K_1(x,t)=0$, and $K_2(x,t)=(x-t+1)$. Taking Laplace transforms of both sides gives

$$\left(\frac{1}{s^2}+\frac{1}{s}\right)(s^2U(s)-su(0)-u'(0)) = \frac{2}{(s-1)^2}+\frac{1}{s-1}-\frac{1}{s^2}-\frac{1}{s}, \quad (5.195)$$

where by using the given initial conditions and solving for $U(s)$ we obtain

$$U(s) = \frac{1}{(s-1)^2}. \quad (5.196)$$

Taking the inverse Laplace transform of both sides we find

$$u(x) = xe^x. \quad (5.197)$$

Example 5.27

Solve the following Volterra integro-differential equation of the first kind

$$\int_0^x \cos(x-t)u(t)dt + \int_0^x \sin(x-t)u'''(t)dt = 1+\sin x - \cos x,$$
$$u(0)=1, \quad u'(0)=1, \quad u''(0)=-1. \quad (5.198)$$

Notice in this equation that $K_1(x,t)=\cos(x-t)$, and $K_2(x,t)=\sin(x-t)$. Taking Laplace transforms of both sides gives

$$\frac{s}{1+s^2}U(s)+\frac{1}{1+s^2}(s^3U(s)-s^2u(0)-su'(0)-u''(0)) = \frac{1}{s}+\frac{1}{1+s^2}-\frac{s}{1+s^2}, \quad (5.199)$$

where by using the given initial conditions and solving for $U(s)$ we obtain

$$U(s) = \frac{1}{s^2}+\frac{s}{1+s^2}. \quad (5.200)$$

Taking the inverse Laplace transform of both sides we find

$$u(x) = x+\cos x. \quad (5.201)$$

Example 5.28

Solve the following Volterra integro-differential equation of the first kind

$$\int_0^x (x-t)u(t)dt + \frac{1}{4}\int_0^x (x-t-1)u''(t)dt = \frac{1}{2}\sin 2x, \; u(0)=1, \; u'(0)=0. \quad (5.202)$$

Taking Laplace transforms of both sides gives

$$\frac{1}{s^2}U(s)+\frac{1}{4}\left(\frac{1}{s^2}-\frac{1}{s}\right)\left(s^2U(s)-su(0)-u'(0)\right) = \frac{1}{s^2+4}, \quad (5.203)$$

where by using the given initial conditions and solving for $U(s)$ we obtain

$$U(s) = \frac{s}{4+s^2}. \quad (5.204)$$

Taking the inverse Laplace transform of both sides we find

$$u(x) = \cos 2x. \quad (5.205)$$

Exercises 5.3.1

Solve the following Volterra integro-differential equations of the first kind by using *Laplace transform method*:

1. $\int_0^x (x-t)u(t)dt + \int_0^x (x-t)^2 u'(t)dt = 3x + \frac{1}{2}x^2 - 3\sin x,\ u(0)=0$
2. $\int_0^x (x-t)u(t)dt + \int_0^x (x-t)^2 u'(t)dt = 6 - 6\cos x - 3x\sin x,\ u(0)=0$
3. $\int_0^x (x-t)u(t)dt - \frac{1}{2}\int_0^x (x-t+1)u'(t)dt = 2x^2 - \frac{1}{2}x,\ u(0)=5$
4. $\int_0^x (x-t+1)u'(t)dt = e^x + \frac{1}{2}x^2 - 1,\ u(0)=1$
5. $\int_0^x (x-t+1)u''(t)dt = 2\sin x - x,\ u(0)=-1, u'(0)=1$
6. $\int_0^x (x-t)u(t)dt + \int_0^x (x-t+1)u''(t)dt = \frac{1}{6}x^3 + \sin x,\ u(0)=-1,\ u'(0)=1$
7. $\int_0^x \sin(x-t)u(t)dt - \frac{1}{2}\int_0^x (x-t)u''(t)dt = \frac{1}{2}x - \frac{1}{2}x\cos x,\ u(0)=0,\ u'(0)=1$
8. $\int_0^x \cosh(x-t)u(t)dt + \int_0^x \sinh(x-t)u''(t)dt = \frac{1}{2}xe^x - \frac{1}{2}\sinh x,$
   $u(0)=0,\ u'(0)=1$
9. $\int_0^x \cosh(x-t)u(t)dt + \int_0^x \sinh(x-t)u'''(t)dt = xe^x,\ u(0)=u'(0)=u''(0)=1$
10. $\int_0^x (x-t)u(t)dt + \frac{1}{2}\int_0^x (x-t)^2 u'''(t)dt = \frac{1}{2}x^2,\ u(0)=u'(0)=1,\ u''(0)=-1$
11. $\int_0^x (x-t)u(t)dt - \frac{1}{24}\int_0^x (x-t)^4 u'''(t)dt = \frac{1}{3}x^3,\ u(0)=u''(0)=0,\ u'(0)=2,$
12. $\int_0^x (x-t)^2 u(t)dt - \frac{1}{12}\int_0^x (x-t)^3 u'''(t)dt = \frac{1}{12}x^4,\ u(0)=u''(0)=0,\ u'(0)=3$

5.3.2 The Variational Iteration Method

The standard form of the Volterra integro-differential equation [11–12] of the first kind is given by

$$\int_0^x K_1(x,t)u(t)dt + \int_0^x K_2(x,t)u^{(n)}(t)dt = f(x), \quad K_2(x,t) \neq 0, \quad (5.206)$$

where initial conditions are prescribed. We first differentiate both sides of (5.206) to convert it to an equivalent Volterra integro-differential equation of the second kind, where Leibnitz rule should be used for differentiating the integrals at the left side, hence we obtain

$$u^{(n)}(x) = \frac{f'(x)}{K_2(x,x)} - \frac{K_1(x,x)}{K_2(x,x)}u(x) - \frac{1}{K_2(x,x)}\int_0^x \frac{\partial(K_1(x,t))}{\partial x}u^{(n)}(t)dt$$
$$-\frac{1}{K_2(x,x)}\int_0^x \frac{\partial(K_2(x,t))}{\partial x}u^{(n)}(t)dt, \quad K_2(x,x) \neq 0. \tag{5.207}$$

As stated before, to use the variational iteration method, we should first determine the Lagrange multiplier λ. The Lagrange multiplier λ can be determined based on the resulting integro-differential equations, where the following rule for λ

$$u^{(n)} + f(u(t), u'(t), u''(t), \cdots, u^{(n)}(t)) = 0, \ \lambda = (-1)^n \frac{1}{(n-1)!}(t-x)^{(n-1)}, \tag{5.208}$$

was derived. Having λ determined, an iteration formula, should be used for the determination of the successive approximations $u_{n+1}(x), n \geqslant 0$ of the solution $u(x)$. The zeroth approximation u_0 can be any selective function. However, using the initial values $u(0), u'(0), \ldots$ are preferably used for the selective zeroth approximation u_0 as will be seen later. Consequently, the solution is given by

$$u(x) = \lim_{n\to\infty} u_n(x). \tag{5.209}$$

The VIM will be illustrated by studying the following examples.

Example 5.29

Solve the Volterra integro-differential equation of the first kind

$$\int_0^x (x-t+1)u'(t)dt = e^x + \frac{1}{2}x^2 - 1, u(0) = 1. \tag{5.210}$$

Differentiating both sides of (5.210) once with respect to x gives the Volterra integro-differential equation of the second kind

$$u'(x) = e^x + x - \int_0^x u'(t)dt. \tag{5.211}$$

The correction functional for Eq. (5.211) is given by

$$u_{n+1}(x) = u_n(x) - \int_0^x \left(u_n'(t) - e^t - t + \int_0^t u_n'(r)dr \right) dt. \tag{5.212}$$

where we used $\lambda(t) = -1$. The zeroth approximation $u_0(x)$ can be selected by $u_0(x) = 1$. This gives the successive approximations

$$\begin{aligned} u_0(x) &= 1, \\ u_1(x) &= e^x + \frac{1}{2!}x^2, \\ u_2(x) &= 1 + x + \frac{1}{2!}x^2 - \frac{1}{3!}x^3, \\ u_3(x) &= e^x - \frac{1}{3!}x^3 + \frac{1}{4!}x^4, \end{aligned} \tag{5.213}$$

$$\begin{aligned} u_4(x) &= 1 + x + \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{5!}x^5, \\ u_5(x) &= 1 + x + \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \frac{1}{6!}x^6 \\ &\ \vdots\ . \end{aligned}$$

This gives

$$u_n(x) = x + \left(1 + \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \frac{1}{6!}x^6 + \cdots\right). \tag{5.214}$$

The exact solution is therefore given by

$$u(x) = x + \cosh x. \tag{5.215}$$

Example 5.30

Use the variational iteration method to solve the Volterra integro-differential equation of the first kind

$$\int_0^x (x-t)u(t)dt - \frac{1}{2}\int_0^x (x-t+1)u'(t)dt = 2x^2 - \frac{1}{2}x, u(0) = 5. \tag{5.216}$$

Differentiating both sides of (5.216) once with respect to x gives the Volterra integro-differential equation of the second kind

$$u'(x) = -8x + 1 + \int_0^x (2u(t) - u'(t))dt. \tag{5.217}$$

The correction functional for equation this equation is given by

$$u_{n+1}(x) = u_n(x) - \int_0^x \left(u_n'(t) + 8t - 1 - \int_0^t (2u_n(r) - u_n'(r))\, dr\right) dt. \tag{5.218}$$

where we used $\lambda(t) = -1$. The zeroth approximation $u_0(x)$ can be selected by $u_0(x) = 5$. This gives the successive approximations

$$\begin{aligned} u_0(x) &= 5, \\ u_1(x) &= 5 + x + x^2, \\ u_2(x) &= 5 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^4, \\ u_3(x) &= 5 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{12}x^4 - \frac{1}{30}x^5 + \cdots, \\ u_4(x) &= 5 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{90}x^6 + \cdots, \\ u_5(x) &= 5 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \cdots, \\ &\ \vdots\ . \end{aligned} \tag{5.219}$$

This gives

$$u_n(x) = 4 + \left(1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \cdots\right). \tag{5.220}$$

The exact solution is therefore given by

$$u(x) = 4 + e^x. \tag{5.221}$$

Example 5.31

Use the variational iteration method to solve the Volterra integro-differential equation of the first kind

$$\int_0^x (x-t)u(t)dt + \int_0^x (x-t+1)u''(t)dt = \sin x + \frac{1}{3!}x^3, u(0) = -1, u'(0) = 1. \tag{5.222}$$

Differentiating both sides of this equation once with respect to x gives the Volterra integro-differential equation of the second kind

$$u''(x) = \cos x + \frac{1}{2}x^2 - \int_0^x u(t)dt - \int_0^x u''(t)dt. \tag{5.223}$$

The correction functional for equation this equation is given by

$$u_{n+1}(x) = u_n(x) + \int_0^x \left((t-x)\Gamma(t)\right) dt. \tag{5.224}$$

where

$$\Gamma(t) = \left(u_n''(t) - \cos t - \frac{1}{2}t^2 + \int_0^t \left(u_n(r) + u_n''(r)\right) dr\right). \tag{5.225}$$

The zeroth approximation $u_0(x)$ can be selected by $u_0(x) = -1 + x$ by using the initial conditions. This gives the successive approximations

$$\begin{aligned}
u_0(x) &= -1 + x,\\
u_1(x) &= x + \frac{1}{3!}x^3 - \cos x,\\
u_2(x) &= -1 + x + \frac{1}{2!}x^2 - \frac{1}{12}x^4 + \cdots,\\
u_3(x) &= -1 + x + \frac{1}{2!}x^2 - \frac{1}{4!}x^4 + \cdots,\\
u_4(x) &= -1 + x + \frac{1}{2!}x^2 - \frac{1}{4!}x^4 + \cdots,\\
u_5(x) &= -1 + x + \frac{1}{2!}x^2 - \frac{1}{4!}x^4 + \frac{1}{6!}x^6 - \frac{1}{8!}x^8 + \cdots,\\
&\vdots .
\end{aligned} \tag{5.226}$$

This gives

$$u_n(x) = x - \left(1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 + \cdots\right). \tag{5.227}$$

The exact solution is therefore given by

$$u(x) = x - \cos x. \tag{5.228}$$

Example 5.32

Use the variational iteration method to solve the Volterra integro-differential equation of the first kind

$$\begin{gathered}\int_0^x \sinh(x-t)u(t)dt + \int_0^x \cosh(x-t)u'''(t)dt = xe^x, \\ u(0) = u'(0) = u''(0) = 1.\end{gathered} \tag{5.229}$$

Differentiating both sides gives the Volterra integro-differential equation of the second kind

$$u'''(x) = xe^x + e^x - \int_0^x \cosh(x-t)u(t)dt - \int_0^x \sinh(x-t)u'''(t)dt. \tag{5.230}$$

The correction functional for equation this equation is given by

$$u_{n+1}(x) = u_n(x) - \frac{1}{2}\int_0^x \left((t-x)^2\Gamma_1(t)\right) dt, \tag{5.231}$$

where

$$\Gamma_1(t) = \left(u_n'''(t) - te^t - e^t + \int_0^t \cosh(t-r)u_n(r)dr + \int_0^t u_n'''(r)\,dr\right). \tag{5.232}$$

The zeroth approximation can be selected by $u_0(x) = 1 + x + \frac{1}{2}x^2$ by using the initial conditions. This gives the successive approximations

$$\begin{aligned} u_0(x) &= 1 + x + \frac{1}{2!}x^2, \\ u_1(x) &= 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{60}x^5 + \cdots, \\ u_2(x) &= 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \frac{1}{6!}x^6 + \cdots, \\ &\ \vdots\ . \end{aligned} \tag{5.233}$$

This gives

$$u_n(x) = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \frac{1}{6!}x^6 + \cdots. \tag{5.234}$$

The exact solution is therefore given by

$$u(x) = e^x. \tag{5.235}$$

Exercises 5.3.2

Solve the following Volterra integro-differential equations of the first kind by using *the variational iteration method*:

1. $\int_0^x (x-t)u(t)dt - \frac{1}{2}\int_0^x (x-t+1)u'(t)dt = \frac{3}{2} + x - \frac{3}{2}e^x,\ u(0) = 0$
2. $\int_0^x (x-t)u(t)dt + \int_0^x (x-t+1)u'(t)dt = 1 + x - \cos x,\ u(0) = 0$

3. $\int_0^x (x-t+1)u'(t)dt = 1+\sin x - \cos x,\ u(0)=0$

4. $\int_0^x (x-t)u(t)dt + \int_0^x (x-t+1)u'(t)dt = \frac{1}{2}x^2 + \frac{1}{6}x^3 + \sin x,\ u(0)=1$

5. $\int_0^x (x-t+1)u''(t)dt = 2\sin x - x,\ u(0)=-1,\ u'(0)=1$

6. $\int_0^x (x-t)u(t)dt - \int_0^x (x-t+1)u''(t)dt = \frac{1}{6}x^3 - \sinh x,\ u(0)=1,\ u'(0)=1$

7. $\int_0^x \sinh(x-t)u(t)dt + \int_0^x \cosh(x-t)u''(t)dt = \frac{1}{2}xe^x + \frac{1}{2}\sinh x,$
$u(0)=1, u'(0)=-1$

8. $\int_0^x (x-t)u(t)dt + \int_0^x (x-t+2)u''(t)dt = 2x, u(0)=0, u'(0)=1$

9. $\int_0^x (x-t-1)u(t)dt + \int_0^x (x-t+1)u'''(t)dt = 2x - 4\sin x,$
$u(0)=u'(0)=1, u''(0)=-1$

10. $\int_0^x (x-t+1)u(t)dt - \int_0^x (x-t-1)u'''(t)dt = 2x - 4\sin x,$
$u(0)=-1, u'(0)=u''(0)=1$

11. $\int_0^x \sinh(x-t)u(t)dt - \int_0^x \cosh(x-t)u'''(t)dt = -x + \sinh x,$
$u(0)=u''(0)=u''(0)=1$

12. $\int_0^x (x-t)u(t)dt - \frac{1}{2}\int_0^x (x-t+1)u'''(t)dt = -\frac{1}{2}x + \frac{1}{6}x^3,$
$u(0)=u''(0)=1, u'(0)=2$

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Chapter 6
Fredholm Integro-Differential Equations

6.1 Introduction

In Chapter 2, the conversion of boundary value problems to Fredholm integral equations was presented. However, the research work in this field resulted in a new specific topic, where both differential and integral operators appeared together in the same equation. This new type of equations, with constant limits of integration, was termed as Fredholm integro-differential equations, given in the form

$$u^{(n)}(x) = f(x) + \int_a^b K(x,t)u(t)dt, u^{(k)}(0) = b_k, 0 \leqslant k \leqslant n-1, \qquad (6.1)$$

where $u^{(n)}(x) = \frac{d^n u}{dx^n}$. Because the resulted equation in (6.1) combines the differential operator and the integral operator, then it is necessary to define initial conditions $u(0), u'(0), \ldots, u^{(n-1)}(0)$ for the determination of the particular solution $u(x)$ of equation (6.1). Any Fredholm integro-differential equation is characterized by the existence of one or more of the derivatives $u'(x)$, $u''(x)$,... outside the integral sign. The Fredholm integro-differential equations of the second kind appear in a variety of scientific applications such as the theory of signal processing and neural networks [1–3].

A variety of numerical and analytical methods are used in the literature to solve the Fredholm integro-differential equations. In this chapter, these equations will be handled by using the traditional methods, namely, the direct computation method and the Taylor series method. Moreover, these equations will be handled by the newly developed methods, namely the variational iteration method, and the Adomian decomposition method.

6.2 Fredholm Integro-Differential Equations of the Second Kind

In this chapter, we will focus our study on the equations that involve separable kernels where the kernel $K(x, t)$ can be expressed as a finite sum of the form

$$K(x,t) = \sum_{k=1}^{n} g_k(x)\, h_k(t). \tag{6.2}$$

Without loss of generality, we will make our analysis on a one term kernel $K(x, t)$ of the form

$$K(x,t) = g(x)\, h(t). \tag{6.3}$$

Other cases can be examined in a like manner. The non-separable kernel can be reduced to separable kernel by using the Taylor expansion for the kernel involved. However, the separable kernels only will be presented in this text. The methods that will be used were presented before in details, but here we will outline the main steps of each method to illustrate its use.

6.2.1 The Direct Computation Method

The direct computation method has been introduced in Chapter 4. The standard form of the Fredholm integro-differential equation is given by

$$u^{(n)}(x) = f(x) + \int_a^b K(x,t)u(t)dt, \quad u^{(k)}(0) = b_k, \quad 0 \leqslant k \leqslant n-1 \tag{6.4}$$

where $u^{(n)}(x)$ indicates the nth derivative of $u(x)$ with respect to x and b_k are the initial conditions. Substituting (6.3) into (6.4) gives

$$u^{(n)}(x) = f(x) + g(x)\int_a^b h(t)u(t)dt, \quad u^{(k)}(0) = b_k, \quad 0 \leqslant k \leqslant n-1. \tag{6.5}$$

We can easily observe that the definite integral in the integro-differential equation (6.5) involves an integrand that depends completely on the variable t. This means that the definite integral at the right side of (6.5) is equivalent to a constant α. In other words, we set

$$\alpha = \int_a^b h(t)u(t)dt. \tag{6.6}$$

Consequently, Equation. (6.5) becomes

$$u^{(n)}(x) = f(x) + \alpha\, g(x). \tag{6.7}$$

Integrating both sides of (6.7) n times from 0 to x, and using the prescribed initial conditions, we can find an expression for $u(x)$ that involves the constant α in addition to the variable x. This means we can write

$$u(x) = v(x;\alpha). \tag{6.8}$$

Substituting (6.8) into the right side of (6.6), evaluating the integral, and solving the resulting equation, we determine a numerical value for the constant α. This leads to the exact solution $u(x)$ obtained upon substituting the resulting value of α into (6.8). It is important to recall that this method leads always to the exact solution and not to series components. The method was used before in Chapter 4 for handling Fredholm integral equations. The method will be illustrated by studying the following examples.

Example 6.1

Solve the following Fredholm integro-differential equation

$$u'(x) = 3 + 6x + x\int_0^1 tu(t)dt, \quad u(0) = 0. \tag{6.9}$$

This equation may be written as

$$u'(x) = 3 + 6x + \alpha x, \quad u(0) = 0, \tag{6.10}$$

obtained by setting

$$\alpha = \int_0^1 tu(t)dt. \tag{6.11}$$

Integrating both sides of (6.10) from 0 to x, and by using the given initial condition we obtain

$$u(x) = 3x + 3x^2 + \frac{1}{2}\alpha x^2. \tag{6.12}$$

Substituting (6.12) into (6.11) and evaluating the integral yield

$$\alpha = \int_0^1 tu(t)dt = \frac{7}{4} + \frac{1}{8}\alpha, \tag{6.13}$$

hence we find

$$\alpha = 2. \tag{6.14}$$

The exact solution is given by

$$u(x) = 3x + 4x^2. \tag{6.15}$$

Example 6.2

Solve the following Fredholm integro-differential equation

$$u''(x) = 1 - e + e^x + \int_0^1 u(t)dt, \quad u(0) = u'(0) = 1. \tag{6.16}$$

This equation may be written as

$$u''(x) = 1 - e + e^x + \alpha, \quad u(0) = 1, \quad u'(0) = 1, \tag{6.17}$$

obtained by setting

$$\alpha = \int_0^1 u(t)dt. \tag{6.18}$$

Integrating both sides of (6.17) twice from 0 to x, and by using the given initial conditions we obtain

$$u(x) - x - 1 = \frac{1-e+\alpha}{2}x^2 + e^x - 1 - x, \tag{6.19}$$

or equivalently

$$u(x) = \frac{1-e+\alpha}{2}x^2 + e^x. \tag{6.20}$$

Substituting (6.20) into (6.18) and evaluating the integral we obtain

$$\alpha = \int_0^1 u(t)dt = \frac{1-e+\alpha}{6} + e - 1, \tag{6.21}$$

hence we find

$$\alpha = e - 1. \tag{6.22}$$

The exact solution is given by

$$u(x) = e^x. \tag{6.23}$$

Example 6.3

Solve the following Fredholm integro-differential equation

$$u'''(x) = 2+\sin x - \int_0^{\pi} (x-t)u(t)dt,\ u(0) = 1,\ u'(0) = 0,\ u''(0) = -1. \tag{6.24}$$

This equation may be written as

$$u'''(x) = 2 + \beta - \alpha x + \sin x,\ u(0) = 1,\ u'(0) = 0,\ u''(0) = -1, \tag{6.25}$$

obtained by setting

$$\alpha = \int_0^{\pi} u(t)dt, \quad \beta = \int_0^{\pi} tu(t)dt. \tag{6.26}$$

Integrating both sides of (6.25) three times from 0 to x, and by using the given initial conditions we obtain

$$u(x) = \frac{1}{3!}(2+\beta)x^3 - \frac{\alpha}{4!}x^4 + \cos x. \tag{6.27}$$

Substituting (6.27) into (6.26) and evaluating the integrals we find

$$\alpha = \frac{\pi^4}{4!}(2+\beta) - \frac{\alpha}{5!}\pi^5, \quad \beta = \frac{\pi^5}{30}(2+\beta) - \frac{\alpha}{144}\pi^6 - 2. \tag{6.28}$$

Solving this system of equations gives

$$\alpha = 0, \quad \beta = -2. \tag{6.29}$$

The exact solution is given by

$$u(x) = \cos x. \tag{6.30}$$

Example 6.4

Solve the following Fredholm integro-differential equation

$$\begin{gathered} u^{(\mathrm{iv})}(x) = (2x-\pi) + \sin x + \cos x - \int_0^{\frac{\pi}{2}} (x-2t)u(t)dt, \\ u(0) = u'(0) = 1, \quad u''(0) = u'''(0) = -1. \end{gathered} \tag{6.31}$$

This equation may be written as

$$u^{(\text{iv})}(x) = (2-\alpha)x + (2\beta - \pi) + \sin x + \cos x,$$
$$u(0) = u'(0) = 1, \quad u''(0) = u'''(0) = -1, \tag{6.32}$$

obtained by setting

$$\alpha = \int_0^{\frac{\pi}{2}} u(t)dt, \quad \beta = \int_0^{\frac{\pi}{2}} tu(t)dt. \tag{6.33}$$

Integrating both sides of (6.32) four times from 0 to x, and by using the given initial conditions we obtain

$$u(x) = \sin x + \cos x + \left(\frac{1}{60} - \frac{1}{120}\alpha\right)x^5 + \left(\frac{\beta}{12} - \frac{\pi}{24}\right)x^4. \tag{6.34}$$

Substituting (6.34) into (6.33) and evaluating the integrals we find

$$\alpha = -\frac{\pi^6}{46080}(10+\alpha) + (2 + \frac{\pi^5}{1920}\beta),$$
$$\beta = -\frac{\pi^7}{322560}(29+3\alpha) + (\frac{\pi}{2} + \frac{\pi^6}{4608}\beta). \tag{6.35}$$

Solving this system of equations gives

$$\alpha = 2, \beta = \frac{\pi}{2}. \tag{6.36}$$

The exact solution is therefore given by

$$u(x) = \sin x + \cos x, \tag{6.37}$$

obtained upon substituting (6.36) into (6.34).

Exercises 6.2.1

Solve the following Fredholm integro-differential equations by using the direct computation method

1. $u'(x) = 12x + \int_0^1 u(t)dt,\ u(0) = 0$
2. $u'(x) = 36x^2 + \int_0^1 u(t)dt,\ u(0) = 1$
3. $u'(x) = \sec^2 x - \ln 2 + \int_0^{\frac{\pi}{3}} u(t)dt,\ u(0) = 0$
4. $u'(x) = -10x + \int_{-1}^1 (x-t)u(t)dt,\ u(0) = 1$
5. $u'(x) = 2\sec^2 x \tan x - 1 + \int_0^{\pi/4} u(t)dt,\ u(0) = 1$
6. $u'(x) = -1 + \cos x + \int_0^{\frac{\pi}{2}} tu(t)dt,\ u(0) = 0$
7. $u''(x) = -1 - \sin x + \int_0^{\frac{\pi}{2}} tu(t)dt,\ u(0) = 0,\ u'(0) = 1$
8. $u''(x) = 2 - \cos x + \int_0^{\pi} tu(t)dt,\ u(0) = 1, u'(0) = 0$

9. $u''(x) = -2x - \sin x + \cos x + \int_0^{\pi} xu(t)dt,\ u(0) = -1,\ u'(0) = 1$

10. $u''(x) = 1 - 3\ln 2 + 4\cosh x + \int_0^{\ln 2} tu(t)dt,\ u(0) = 4,\ u'(0) = 0$

11. $u''(x) = 2x - x\sin x + 2\cos x + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (t-x)u(t)dt,\ u(0) = 0,\ u'(0) = 0$

12. $u''(x) = 4x - \sin x + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (x-t)^2u(t)dt,\ u(0) = 0,\ u'(0) = 1$

13. $u'''(x) = 5\ln 2 - 3 - x + 4\cosh x + \int_0^{\ln 2} (x-t)u(t)dt,$

$u(0) = u''(0) = 0,\ u'(0) = 4$

14. $u'''(x) = e - 2 - x + e^x(3+x) + \int_0^1 (x-t)u(t)dt,$

$u(0) = 0, u'(0) = 1,\ u''(0) = 2$

15. $u^{(\text{iv})}(x) = -1 + \sin x + \int_0^{\frac{\pi}{2}} tu(t)dt,\ u(0) = u''(0) = 0, u'(0) = 1, u'''(0) = -1$

16. $u^{(\text{iv})}(x) = 4x + \sin x + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (x-t)^2u(t)dt,\ u(0) = u''(0) = 0,$

$u'(0) = 1,\ u'''(0) = -1$

6.2.2 The Variational Iteration Method

The variational iteration method [4] was used in Chapters 3, 4 and 5. The method provides rapidly convergent successive approximations of the exact solution if such a closed form solution exists.

The standard ith order Fredholm integro-differential equation is of the form

$$u^{(i)}(x) = f(x) + \int_a^b K(x,t)u(t)dt, \tag{6.38}$$

where $u^{(i)}(x) = \frac{d^iu}{dx^i}$, and $u(0), u'(0), \ldots, u^{(i-1)}(0)$ are the initial conditions.

The correction functional for the integro-differential equation (6.38) is

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(\xi)\left(u_n^{(i)}(\xi) - f(\xi) - \int_a^b K(\xi,r)\tilde{u}_n(r)dr\right)d\xi. \tag{6.39}$$

As presented before, the variational iteration method is used by applying two essential steps. It is required first to determine the Lagrange multiplier $\lambda(\xi)$ that can be identified optimally via integration by parts and by using a restricted variation. Having $\lambda(\xi)$ determined, an iteration formula, without restricted variation, should be used for the determination of the successive approximations $u_{n+1}(x), n \geqslant 0$ of the solution $u(x)$. The zeroth approximation u_0 can be any selective function. However, using the given initial values $u(0), u'(0), \ldots$ are preferably used for the selective zeroth approximation u_0 as will be seen later. Consequently, the solution is given by

$$u(x) = \lim_{n\to\infty} u_n(x). \tag{6.40}$$

The VIM will be illustrated by studying the following examples.

Example 6.5

Use the variational iteration method to solve the Fredholm integro-differential equation

$$u'(x) = -1 + \cos x + \int_0^{\frac{\pi}{2}} tu(t)dt, u(0) = 0. \tag{6.41}$$

The correction functional for this equation is given by

$$u_{n+1}(x) = u_n(x) - \int_0^x \left(u_n'(\xi) + 1 - \cos\xi - \int_0^{\frac{\pi}{2}} ru_n(r)\,dr \right) d\xi, \tag{6.42}$$

where we used $\lambda = -1$ for first-order integro-differential equations. Notice that the correction functional is the so-called Volterra-Fredholm integral equation, because it involves both types of integral equations.

We can use the initial condition to select $u_0(x) = u(0) = 0$. Using this selection into the correction functional gives the following successive approximations

$$\begin{aligned}
u_0(x) &= 0,\\
u_1(x) &= u_0(x) - \int_0^x \left(u_0'(\xi) + 1 - \cos\xi - \int_0^{\frac{\pi}{2}} ru_0(r)dr \right) d\xi\\
&= \sin x - x,\\
u_2(x) &= u_1(x) - \int_0^x \left(u_1'(\xi) + 1 - \cos\xi - \int_0^{\frac{\pi}{2}} ru_1(r)dr \right) d\xi\\
&= (\sin x - x) + \left(x - \frac{\pi^3}{24}x \right),\\
u_3(x) &= u_2(x) - \int_0^x \left(u_2'(\xi) + 1 - \cos\xi - \int_0^{\frac{\pi}{2}} ru_2(r)dr \right) d\xi\\
&= (\sin x - x) + \left(x - \frac{\pi^3}{24}x \right) + \left(\frac{\pi^3}{24}x - \frac{\pi^6}{576}x \right),\\
u_4(x) &= u_3(x) - \int_0^x \left(u_3'(\xi) + 1 - \cos\xi - \int_0^{\frac{\pi}{2}} ru_3(r)dr \right) d\xi\\
&= (\sin x - x) + \left(x - \frac{\pi^3}{24}x \right) + \left(\frac{\pi^3}{24}x - \frac{\pi^6}{576}x \right) + \left(\frac{\pi^6}{576}x + \cdots \right),
\end{aligned} \tag{6.43}$$

and so on. The VIM admits the use of

$$u(x) = \lim_{n\to\infty} u_n(x). \tag{6.44}$$

It is obvious that noise terms appear in the successive approximations, where by canceling these noise terms, we obtain the exact solution

$$u(x) = \sin x. \tag{6.45}$$

Example 6.6

Use the variational iteration method to solve the Fredholm integro-differential equation

$$u''(x) = 2 - \cos x + \int_0^{\pi} tu(t)dt, \quad u(0) = 1, \quad u'(0) = 0. \tag{6.46}$$

The correction functional for this equation is given by

$$u_{n+1}(x) = u_n(x) + \int_0^x (\xi - x)\left(u_n''(\xi) - 2 + \cos\xi - \int_0^{\pi} ru_n(r)\, dr\right) d\xi, \tag{6.47}$$

where $\lambda = \xi - x$ for second-order integro-differential equations.

We can use the initial conditions $u(0) = 1, u'(0) = 0$ to select the zeroth approximation by

$$u_0(x) = u(0) + xu'(0) = 1. \tag{6.48}$$

Using this selection into the correction functional gives the following successive approximations

$$\begin{aligned}
u_0(x) &= 1,\\
u_1(x) &= u_0(x) + \int_0^x (\xi - x)\left(u_0''(\xi) - 2 + \cos\xi - \int_0^{\pi} ru_0(r)dr\right) d\xi,\\
&= \cos x + x^2\left(1 + \frac{\pi^2}{4}\right),\\
u_2(x) &= u_1(x) + \int_0^x (\xi - x)\left(u_1''(\xi) - 2 + \cos\xi - \int_0^{\pi} ru_1(r)dr\right) d\xi,\\
&= \cos x + x^2\left(1 + \frac{\pi^2}{4}\right) - x^2\left(1 + \frac{\pi^2}{4}\right) + x^2\left(\frac{\pi^4}{8} + \frac{\pi^6}{32}\right),\\
u_3(x) &= u_2(x) + \int_0^x (\xi - x)\left(u_2''(\xi) - 2 + \cos\xi - \int_0^{\pi} ru_2(r)dr\right) d\xi,\\
&= \cos x + x^2\left(1 + \frac{\pi^2}{4}\right) - x^2\left(1 + \frac{\pi^2}{4}\right) + x^2\left(\frac{\pi^4}{8} + \frac{\pi^6}{32}\right)\\
&\quad -x^2\left(\frac{\pi^4}{8} + \frac{\pi^6}{32}\right) + \cdots,
\end{aligned} \tag{6.49}$$

and so on. The VIM admits the use of

$$u(x) = \lim_{n\to\infty} u_n(x). \tag{6.50}$$

Canceling the noise terms gives the exact solution

$$u(x) = \cos x. \tag{6.51}$$

Example 6.7

Use the variational iteration method to solve the second-order Fredholm integro-differential equation

$$u''(x) = 2x - x\sin x + 2\cos x - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (x-t)u(t)dt,\ u(0) = u'(0) = 0. \quad (6.52)$$

The correction functional for this equation is given by

$$u_{n+1}(x) = u_n(x) + \int_0^x (\xi - x)\Gamma_n(\xi)d\xi, \quad n \geqslant 0, \quad (6.53)$$

where

$$\Gamma_n(\xi) = u_n''(\xi) - 2\xi + \xi\sin\xi - 2\cos\xi + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\xi - r)u_n(r)\,dr. \quad (6.54)$$

We can use the initial conditions $u(0) = 0, u'(0) = 0$ to select the zeroth approximation by

$$u_0(x) = u(0) + xu'(0) = 0. \quad (6.55)$$

Using this selection into the correction functional gives the following successive approximations

$$\begin{aligned}
u_0(x) &= 0,\\
u_1(x) &= u_0(x) + \int_0^x (\xi - x)\Gamma_0(\xi)d\xi = x\sin x + \frac{1}{3}x^3,\\
u_2(x) &= u_1(x) + \int_0^x (\xi - x)\Gamma_1(\xi)d\xi\\
&= \left(x\sin x + \frac{1}{3}x^3\right) + \left(-\frac{1}{3}x^3 + \frac{\pi^5}{480}x^2\right),\\
u_3(x) &= u_2(x) + \int_0^x (\xi - x)\Gamma_2(\xi)d\xi\\
&= \left(x\sin x + \frac{1}{3}x^3\right) + \left(-\frac{1}{3}x^3 + \frac{\pi^5}{480}x^2\right) + \left(-\frac{\pi^5}{480}x^2 + \cdots\right),
\end{aligned} \quad (6.56)$$

and so on,where

$$\Gamma_i(\xi) = u_i''(\xi) - 2\xi + \xi\sin\xi - 2\cos\xi + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\xi - r)u_i(r)\,dr, i \geqslant 0. \quad (6.57)$$

Canceling the noise terms gives the exact solution

$$u(x) = x\sin x. \quad (6.58)$$

Example 6.8

Use the variational iteration method to solve the third-order Fredholm integro-differential equation

$$u'''(x) = e^x - 1 + \int_0^1 tu(t)dt,\ u(0) = 1,\ u'(0) = 1,\ u''(0) = 1. \quad (6.59)$$

The correction functional for this equation is given by

$$u_{n+1}(x) = u_n(x) - \frac{1}{2}\int_0^x (\xi - x)^2 \left(u_n'''(\xi) - e^{\xi} + 1 - \int_0^1 r u_n(r)\, dr \right) d\xi, \tag{6.60}$$

where $\lambda = -\frac{1}{2}(\xi - x)^2$ for third-order integro-differential equations.

The zeroth approximation $u_0(x)$ can be selected by

$$u_0(x) = u(0) + x u'(0) + \frac{1}{2!}x^2 u''(0) = 1 + x + \frac{1}{2!}x^2. \tag{6.61}$$

Using this selection into the correction functional gives the following successive approximations

$$\begin{aligned}
u_0(x) &= 1 + x + \frac{1}{2!}x^2,\\
u_1(x) &= u_0(x) - \frac{1}{2}\int_0^x (\xi - x)^2 \left(u_0'''(\xi) - e^{\xi} + 1 - \int_0^1 r u_0(r) dr \right) d\xi\\
&= e^x - \frac{1}{144}x^3,\\
u_2(x) &= u_1(x) - \frac{1}{2}\int_0^x (\xi - x)^2 \left(u_1'''(\xi) - e^{\xi} + 1 - \int_0^1 r u_1(r) dr \right) d\xi\\
&= \left(e^x - \frac{1}{144}x^3\right) + \left(\frac{1}{144}x^3 - \frac{29}{4320}x^3\right),\\
u_3(x) &= u_2(x) - \frac{1}{2}\int_0^x (\xi - x)^2 \left(u_2'''(\xi) - e^{\xi} + 1 - \int_0^1 r u_2(r) dr \right) d\xi\\
&= \left(e^x - \frac{1}{144}x^3\right) + \left(\frac{1}{144}x^3 - \frac{29}{4320}x^3\right) + \left(\frac{29}{4320}x^3 + \cdots\right),
\end{aligned} \tag{6.62}$$

and so on. Canceling the noise terms gives the exact solution

$$u(x) = e^x. \tag{6.63}$$

Exercises 6.2.2

Solve the following Fredholm integro-differential equations by using the variational iteration method

1. $u'(x) = 12x + \int_0^1 u(t)dt,\ u(0) = 0$
2. $u'(x) = -4 + 6x + \int_0^1 u(t)dt,\ u(0) = 2$
3. $u'(x) = \sec^2 x - \ln 2 + \int_0^{\frac{\pi}{3}} u(t)dt,\ u(0) = 0$
4. $u'(x) = -8x + \int_{-1}^1 (x - t)u(t)dt,\ u(0) = 2$

5. $u'(x) = 2\sec^2 x \tan x - 1 + \int_0^{\pi/4} u(t)dt,\ u(0) = 1$

6. $u'(x) = -1 + \cos x + \int_0^{\frac{\pi}{2}} (2-t)u(t)dt,\ u(0) = 0$

7. $u''(x) = -1 - \sin x + \int_0^{\frac{\pi}{2}} tu(t)dt,\ u(0) = 0, u'(0) = 1$

8. $u''(x) = -2 - \cos x + \int_0^{\pi} (1-t)u(t)dt,\ u(0) = 1, u'(0) = 0$

9. $u''(x) = -2x - \sin x + \cos x + \int_0^{\pi} xu(t)dt,\ u(0) = -1, u'(0) = 1$

10. $u''(x) = -4 + 3\ln 2 + 4\cosh x + \int_0^{\ln 2} (1-t)u(t)dt,\ u(0) = 4, u'(0) = 0$

11. $u''(x) = 2x - x\sin x + 2\cos x + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (t-x)u(t)dt,\ u(0) = 0,\ u'(0) = 0$

12. $u''(x) = 4\pi - \cos x + \int_{-\pi}^{\pi} (x-t)^2 u(t)dt,\ u(0) = 1,\ u'(0) = 0$

13. $u'''(x) = 5\ln 2 - 3 - x + 4\cosh x + \int_0^{\ln 2} (x-t)u(t)dt,$

$u(0) = u''(0) = 0,\ u'(0) = 4$

14. $u'''(x) = -1 + e^x + \int_0^1 tu(t)dt,\ u(0) = u'(0) = u''(0) = 1$

15. $u^{(\text{iv})}(x) = -1 + \sin x + \int_0^{\frac{\pi}{2}} tu(t)dt,\ u(0) = u''(0) = 0,\ u'(0) = 1,\ u'''(0) = -1$

16. $u^{(\text{iv})}(x) = 2 + \cos x + \int_0^{\pi} tu(t)dt,\ u(0) = 1,\ u'(0) = u'''(0) = 0,\ u''(0) = -1$

6.2.3 The Adomian Decomposition Method

The Adomian decomposition method was presented before and used thoroughly in previous chapters. This method was used in its standard form to obtain the solution in an infinite series of components that can be recurrently determined. In addition, it was used jointly with the noise terms phenomenon. Moreover, a modified form was developed to facilitate the computational work. The obtained series may give the exact solution if such a solution exists. Otherwise, the series gives an approximation for the solution that gives high accuracy level.

The main focus in this section will be directed to converting a Fredholm integro-differential equation to an equivalent Fredholm integral equation. The converted Fredholm integral equation can then be solved by any of the methods presented before in Chapter 4, such as the direct computation method, successive approximations method, and others. However, in this section we will use the Adomian decomposition method for solving the Fredholm integro-differential equation by converting it first to an integral equation. The Adomian decomposition method was presented in details in previous chapters, only a summary will be outlined here.

Without loss of generality, we may assume a second order Fredholm integro-differential equation given by

$$u''(x) = f(x) + \int_a^b K(x,t)u(t)dt, \quad u(0) = a_0, \quad u'(0) = a_1. \tag{6.64}$$

Integrating both sides of (6.64) from 0 to x twice gives

$$u(x) = a_0 + a_1 x + L^{-1}(f(x)) + L^{-1}\left(\int_a^a K(x,t) \quad u(t)dt\right), \tag{6.65}$$

where the initial conditions $u(0) = a_0$ and $u'(0) = a_1$ are used, and L^{-1} is a two-fold integral operator. The Adomian decomposition method admits the use of the decomposition series

$$u(x) = \sum_{n=0}^{\infty} u_n(x), \tag{6.66}$$

into both sides of (6.65) to obtain

$$\sum_{n=0}^{\infty} u_n(x) = a_0 + a_1 x + L^{-1}(f(x)) + L^{-1}\left(\int_0^x K(x,t) \sum_{n=0}^{\infty} u_n(t)dt\right), \tag{6.67}$$

or equivalently

$$\begin{aligned} u_0(x) + u_1(x) + u_2(x) + u_3(x) + \cdots &= a_0 + a_1 x + L^{-1}(f(x)) \\ &+L^{-1}\left(\int_a^b K(x,t)u_0(t)dt\right) + L^{-1}\left(\int_a^b K(x,t)u_1(t)dt\right) \\ &+L^{-1}\left(\int_a^b K(x,t)u_2(t)dt\right) + \cdots . \end{aligned} \tag{6.68}$$

Consequently, to determine the components $u_0(x), u_1(x), u_2(x), u_3(x), \ldots$ of the solution $u(x)$, we set the recurrence relation

$$\begin{aligned} u_0(x) &= a_0 + a_1 x + L^{-1}(f(x)), \\ u_{k+1}(x) &= L^{-1}\left(\int_a^b K(x,t)u_k(t)dt\right), \quad k \geqslant 0, \end{aligned} \tag{6.69}$$

where the zeroth component $u_0(x)$ is defined by all terms not included inside the integral sign of (6.68). Having determined the components $u_i(x), i \geqslant 0$, the solution $u(x)$ of (6.64) is then obtained in a series form. Using (6.66), the obtained series converges to the exact solution if such a solution exists.

We point out that some modified forms of the Adomian method was developed in the literature. The most commonly used form is that one developed in [5–6] and was employed in Chapter 3. In what follows, we summarize the main steps for this modified form.

The Modified Decomposition Method

The Adomian decomposition method provides the solutions in an infinite series of components. The method substitutes the decomposition series of $u(x)$, given by

$$u(x) = \sum_{n=0}^{\infty} u_n(x), \tag{6.70}$$

into both sides of the Fredholm integral equation

$$u(x) = f(x) + \lambda \int_a^b K(x,t)u(t)dt. \tag{6.71}$$

The standard Adomian decomposition method introduces the recurrence relation

$$\begin{aligned} u_0(x) &= f(x), \\ u_{k+1}(x) &= \lambda \int_a^b K(x,t)u_k(t)dt, \quad k \geqslant 0. \end{aligned} \tag{6.72}$$

In view of (6.72), the components $u_n(x)$, $n \geqslant 0$ are readily obtained.

The modified decomposition method presents a slight variation to the recurrence relation (6.72) to determine the components of $u(x)$ in an easier and faster manner. For many cases, the function $f(x)$ can be set as the sum of two partial functions, namely $f_1(x)$ and $f_2(x)$. In other words, we can set

$$f(x) = f_1(x) + f_2(x). \tag{6.73}$$

In view of (6.73), we introduce a qualitative change in the formation of the recurrence relation (6.72). The modified decomposition method identifies the zeroth component $u_0(x)$ by one part of $f(x)$, namely $f_1(x)$ or $f_2(x)$. The other part of $f(x)$ can be added to the component $u_1(x)$ that exists in the standard recurrence relation. The modified decomposition method admits the use of the modified recurrence relation

$$\begin{aligned} u_0(x) &= f_1(x), \\ u_1(x) &= f_2(x) + \lambda \int_a^b K(x,t)u_0(t)dt, \\ u_{k+1}(x) &= \lambda \int_a^b K(x,t)u_k(t)dt, \quad k \geqslant 1. \end{aligned} \tag{6.74}$$

It is obvious that the difference between the standard recurrence relation (6.72) and the modified recurrence relation (6.74) rests only in the formation of the first two components $u_0(x)$ and $u_1(x)$ only. The other components $u_j, j \geqslant 2$ remain the same in the two recurrence relations.

It is interesting to recall that the noise terms phenomenon can be used here. This phenomenon was employed before, but it is useful to summarize the main steps of this phenomenon.

The Noise Terms Phenomenon

It was shown before that if the noise terms appear between components of $u(x)$, then the exact solution can be obtained only by considering only the first two components $u_0(x)$ and $u_1(x)$. The noise terms are defined as the identical terms, with opposite signs, that may appear between the components of the solution $u(x)$. The conclusion made by [5–8] suggests that if we observe the appearance of identical terms in both components with opposite signs, then by canceling these terms, the remaining non-canceled terms of u_0 may in some cases provide the exact solution, that should be justified through substitution. It was formally proved that other terms in other components will vanish in the limit if the noise terms occurred in $u_0(x)$ and $u_1(x)$. However, if the exact solution was not attainable by using this phenomenon, then we should continue determining other components of $u(x)$ in a standard way.

The Adomian decomposition method, the noise terms phenomenon, and the modified decomposition method for solving Fredholm integro-differential equations will be illustrated by studying the following examples. The selected equations are of orders 1, 2, 3, and 4. Other equations of higher orders can be treated in a like manner.

Example 6.9

Use the Adomian decomposition method to solve the Fredholm integro-differential equation

$$u'(x) = 36x^2 + \int_0^1 u(t)dt, \quad u(0) = 1. \tag{6.75}$$

Recall that the integral at the right side is equivalent to a constant because it depends only on the variable t with constant limits of integration for t. Integrating both sides of Eq. (6.75) from 0 to x gives

$$u(x) - u(0) = 12x^3 + x\left(\int_0^1 u(t)dt\right), \tag{6.76}$$

which gives upon using the initial condition

$$u(x) = 1 + 12x^3 + x\left(\int_0^1 u(t)dt\right). \tag{6.77}$$

Substituting the series assumption

$$u(x) = \sum_{n=0}^{\infty} u_n(x), \tag{6.78}$$

into both sides of (6.77) gives

$$\sum_{n=0}^{\infty} u_n(x) = 1 + 12x^3 + x\left(\int_0^1 \sum_{n=0}^{\infty} u_n(t)dt\right). \tag{6.79}$$

The components $u_j(x), j \geqslant 0$ of $u(x)$ can be determined by using the recurrence relation

$$u_0(x) = 1 + 12x^3, \quad u_{k+1}(x) = x\int_0^1 u_k(t)dt, \quad k \geqslant 0. \tag{6.80}$$

This in turn gives

$$\begin{aligned} u_0(x) &= 1 + 12x^3, & u_1(x) &= x\int_0^1 u_0(t)dt = 4x, \\ u_2(x) &= x\int_0^1 u_1(t)dt = 2x, & u_3(x) &= x\int_0^1 u_2(t)dt = x, \\ &\vdots \end{aligned} \tag{6.81}$$

The solution in a series form is given by

$$u(x) = 1 + 12x^3 + 4x\left(1 + \frac{1}{2} + \frac{1}{4} + \cdots\right), \tag{6.82}$$

which gives the exact solution

$$u(x) = 1 + 8x + 12x^3, \tag{6.83}$$

obtained upon evaluating the sum of the infinite geometric series.

Example 6.10

Use the Adomian decomposition method to solve the Fredholm integro-differential equation

$$u''(x) = 2 - \cos x + \int_0^\pi tu(t)dt, \quad u(0) = 1, \quad u'(0) = 0. \tag{6.84}$$

Integrating both sides of Eq. (6.84) twice from 0 to x gives

$$u(x) - u(0) - xu'(0) = -1 + \cos x + x^2 + \frac{1}{2}x^2\left(\int_0^\pi tu(t)dt\right), \tag{6.85}$$

which gives upon using the initial conditions

$$u(x) = \cos x + x^2 + \frac{1}{2}x^2\left(\int_0^\pi tu(t)dt\right). \tag{6.86}$$

Substituting the series assumption

$$u(x) = \sum_{n=0}^{\infty} u_n(x), \tag{6.87}$$

into both sides of (6.86) gives

$$\sum_{n=0}^{\infty} u_n(x) = \cos x + x^2 + \frac{1}{2}x^2\left(\int_0^\pi t\sum_{n=0}^{\infty} u_n(t)dt\right). \tag{6.88}$$

Proceeding as before, we set the recurrence relation

$$u_0(x) = \cos x + x^2, \quad u_{k+1}(x) = \frac{1}{2}x^2\int_0^\pi tu_k(t)dt, \quad k \geqslant 0. \tag{6.89}$$

This in turn gives

$$u_0(x) = \cos x + x^2, \quad u_1(x) = \frac{1}{2}x^2\int_0^\pi u_0(t)dt = -x^2 + \frac{\pi^4}{8}x^2. \tag{6.90}$$

The noise terms $\pm x^2$ appear between $u_0(x)$ and $u_1(x)$. Canceling the noise term from $u_0(x)$ gives the exact solution

$$u(x) = \cos x, \tag{6.91}$$

that justifies the integro-differential equation.

Example 6.11

Use the Adomian decomposition method to solve the Fredholm integro-differential equation

$$u'''(x) = e^x - x + \int_0^1 xtu(t)dt, \quad u(0) = u'(0) = u''(0) = 1. \tag{6.92}$$

Integrating both sides of Eq. (6.92) three times from 0 to x, and using the initial conditions we obtain

$$u(x) = e^x - \frac{1}{3!}x^3 + \frac{1}{3!}x^3 \left(\int_0^1 tu(t)dt \right). \tag{6.93}$$

Proceeding as before, we set the recurrence relation

$$u_0(x) = e^x - \frac{1}{3!}x^3, \quad u_{k+1}(x) = \frac{1}{3!}x^3 \int_0^1 tu_k(t)dt, \quad k \geqslant 0. \tag{6.94}$$

This in turn gives

$$\begin{aligned} u_0(x) &= e^x - \frac{1}{3!}x^3, \\ u_1(x) &= \frac{1}{3!}x^3 \int_0^1 tu_0(t)dt = \frac{29}{180}x^3, \\ u_2(x) &= \frac{1}{3!}x^3 \int_0^1 tu_1(t)dt = \frac{29}{5400}x^3, \\ u_3(x) &= \frac{1}{3!}x^3 \int_0^1 tu_2(t)dt = \frac{29}{162000}x^3, \\ &\vdots \end{aligned} \tag{6.95}$$

The solution in a series form is given by

$$u(x) = e^x - \frac{1}{3!}x^3 + \frac{29}{180}x^3 \left(1 + \frac{1}{30} + \frac{1}{900} + \cdots \right). \tag{6.96}$$

The infinite geometric series has $a_1 = 1, r = \frac{1}{3}$. The sum of the infinite geometric series is given by

$$S = \frac{1}{1 - \frac{1}{30}} = \frac{30}{29}. \tag{6.97}$$

Using this result gives the exact solution

$$u(x) = e^x \tag{6.98}$$

Example 6.12

Use the Adomian decomposition method to solve the Fredholm integro-differential equation

$$u^{(\mathrm{iv})}(x) = 1 - x + \sin x + \int_0^{\frac{\pi}{2}} (x-t)u(t)dt, \qquad (6.99)$$
$$u(0) = u''(0) = 0, \quad u'(0) = -u'''(0) = 1.$$

Integrating both sides of Eq. (6.99) four times from 0 to x and using the initial conditions we find

$$u(x) = \sin x + \frac{1}{3!}x^3 - \frac{1}{4!}x^4 + \frac{1}{4!}x^4 \int_0^{\frac{\pi}{2}} u(t)dt - \frac{1}{3!}x^3 \int_0^{\frac{\pi}{2}} tu(t)dt. \quad (6.100)$$

Substituting the series assumption and proceeding as before we obtain the recurrence relation

$$u_0(x) = \sin x + \frac{1}{3!}x^3 - \frac{1}{4!}x^4,$$
$$u_{k+1} = \frac{1}{4!}x^4 \int_0^{\frac{\pi}{2}} u_k(t)dt - \frac{1}{3!}x^3 \int_0^{\frac{\pi}{2}} tu_k(t)dt, \quad k \geqslant 0. \qquad (6.101)$$

This in turn gives

$$u_0(x) = \sin x + \frac{1}{3!}x^3 - \frac{1}{4!}x^4, \quad u_1(x) = -\frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots, \qquad (6.102)$$
$$\vdots$$

Canceling the noise terms $\frac{1}{3!}x^3, -\frac{1}{4!}x^4$ from $u_0(x)$ gives the exact solution

$$u(x) = \sin x. \qquad (6.103)$$

Exercises 6.2.3

Solve the following Fredholm integro-differential equations by using the Adomian decomposition method

1. $u'(x) = \sec^2 x - \ln 2 + \int_0^{\frac{\pi}{3}} u(t)dt,\ u(0) = 0$

2. $u'(x) = -1 + 24x + \int_0^1 u(t)dt,\ u(0) = 0$

3. $u'(x) = 6 + 17x + \int_0^1 xu(t)dt,\ u(0) = 0$

4. $u'(x) = -\sqrt{3} + 2\sec^2 x \tan x + \int_0^{\frac{\pi}{3}} u(t)dt,\ u(0) = 1$

5. $u'(x) = -\frac{\pi}{2}x + \sin 2x + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} u(t)dt,\ u(0) = 0$

6. $u'(x) = 1 - x + \cos x + \int_0^{\frac{\pi}{2}} (x-t)u(t)dt,\ u(0) = 0$

7. $u''(x) = -\frac{\pi}{4} - 2\cos 2x + \int_0^{\frac{\pi}{2}} u(t)dt,\ u(0) = 1,\ u'(0) = 0$

8. $u''(x) = 2x - \cos x + \int_0^{\pi} xtu(t)dt,\ u(0) = 1,\ u'(0) = 0$

9. $u''(x) = -2x - \sin x + \cos x + \int_0^{\pi} xu(t)dt,\ u(0) = -1,\ u'(0) = 1$

10. $u''(x) = -2 - \cos x + \int_0^{\pi} (x-t)u(t)dt,\ u(0) = 1,\ u'(0) = 0$

11. $u''(x) = 3\ln 2 - 1 - 3x + 4\cosh x + \int_0^{\ln 2} (x-t)u(t)dt,\ u(0) = 4, u'(0) = 0$

12. $u''(x) = \frac{1}{4} + (1 - 2\ln 2)x - \frac{1}{(1+x)^2} + \int_0^1 (x-t)u(t)dt,\ u(0) = 0, u'(0) = 1$

13. $u'''(x) = 2e^{-1} - e^{-x} + \int_{-1}^1 tu(t)dt,\ u(0) = u''(0) = 1, u'(0) = -1$

14. $u'''(x) = e - 2 - x + (3+x)e^x + \int_0^1 (x-t)u(t)dt,\ u(0) = u'(0) = 1, u''(0) = 2$

15. $u^{(\mathrm{iv})}(x) = -2x + \sin x + \cos x + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} xtu(t)dt,$

$u(0) = u'(0) = 1,\ u''(0) = u'''(0) = -1$

16. $u^{(\mathrm{iv})}(x) = \frac{1}{4} + (1 - 2\ln 2)x - \frac{6}{(1+x)^4} + \int_0^1 (x-t)u(t)dt,$

$u(0) = 0,\ u'(0) = 1,\ u''(0) = -1,\ u'''(0) = 2$

6.2.4 The Series Solution Method

The series solution method was used before, where the generic form of Taylor series for an analytic solution $u(x)$ at $x = 0$

$$u(x) = \sum_{n=0}^{\infty} a_n x^n \tag{6.104}$$

is used. Following the discussion presented before in Chapters 3 and 4, the series solution method will be used for solving Fredholm integro-differential equations. We will assume that the solution $u(x)$ of the Fredholm integro-differential equation

$$u^{(k)}(x) = f(x) + \lambda \int_a^b K(x,t)u(t)dt,\ u^{(j)}(0) = a_j,\ 0 \leqslant j \leqslant (k-1), \tag{6.105}$$

is analytic, and therefore possesses a Taylor series of the form given in (6.104), where the coefficients a_n will be determined recurrently. Substituting (6.104) into both sides of (6.105) gives

$$\left(\sum_{n=0}^{\infty} a_n x^n\right)^{(k)} = T(f(x)) + \lambda \int_a^b K(x,t) \left(\sum_{n=0}^{\infty} a_n t^n\right) dt, \tag{6.106}$$

or for simplicity we use

$$\left(a_0 + a_1 x + a_2 x^2 + \cdots\right)^{(k)} = T(f(x)) + \lambda \int_a^b K(x,t) \left(a_0 + a_1 t + a_2 t^2 + \cdots\right) dt, \tag{6.107}$$

where $T(f(x))$ is the Taylor series for $f(x)$. The integral equation (6.105) will be converted to a traditional integral in (6.106) or (6.107) where instead of integrating the unknown function $u(x)$, terms of the form t^n, $n \geqslant 0$ will be integrated. Notice that because we are seeking series solution, then if $f(x)$ includes elementary functions such as trigonometric functions, exponential functions, etc., then Taylor expansions for functions involved in $f(x)$ should be used. Moreover, the given initial equations should be used in the series assumption (6.104).

We first integrate the right side of the integral in (6.106) or (6.107), and collect the coefficients of like powers of x. We next equate the coefficients of like powers of x in both sides of the resulting equation to obtain a recurrence relation in $a_j, j \geqslant 0$. Solving the recurrence relation will lead to a complete determination of the coefficients $a_j, j \geqslant 0$. Having determined the coefficients $a_j, j \geqslant 0$, the series solution follows immediately upon substituting the derived coefficients into (6.104). The exact solution may be obtained if such an exact solution exists. If an exact solution is not obtainable, then the obtained series can be used for numerical purposes. In this case, the more terms we evaluate, the higher accuracy level we achieve.

It is worth noting that using the series solution method for solving Fredholm integro-differential equations gives exact solutions if the solution $u(x)$ is a polynomial. However, if the solution is any other elementary function such as $\sin x, e^x$, etc, the series method gives the exact solution after rounding few of the coefficients $a_j, j \geqslant 0$. This will be illustrated by studying the following examples.

Example 6.13

Solve the Fredholm integro-differential equation by using the series solution method

$$u'(x) = 4 + 4x + \int_{-1}^{1} (1 - xt)u(t)dt, u(0) = 1. \tag{6.108}$$

Substituting $u(x)$ by the series

$$u'(x) = \left(\sum_{n=0}^{\infty} a_n x^n\right)', \tag{6.109}$$

into both sides of Eq. (6.108) leads to

$$\sum_{n=1}^{\infty} n a_n x^{n-1} = 4 + 4x + \int_{-1}^{1} \left((1 - xt)\sum_{n=0}^{\infty} a_n t^n\right) dt. \tag{6.110}$$

Notice that $a_0 = 1$ from the given initial conditions. Evaluating the integral at the right side gives

$$\begin{aligned} & a_1 + 2a_2x + 3a_3x^2 + \cdots \\ & = 6 + \frac{2}{3}a_2 + \frac{2}{5}a_4 + \frac{2}{7}a_6 + \frac{2}{9}a_8 + (4 - \frac{2}{3}a_1 - \frac{2}{5}a_3 - \frac{2}{7}a_5 - \frac{2}{9}a_7)x. \end{aligned} \tag{6.111}$$

Equating the coefficients of like powers of x in both sides of (6.111) gives

$$a_1 = 6,\ a_n = 0, n \geqslant 2. \tag{6.112}$$

The exact solution is given by

$$u(x) = 1 + 6x, \tag{6.113}$$

where we used $a_0 = 1$ from the initial condition.

Example 6.14

Solve the Fredholm integro-differential equation by using the series solution method

$$u'(x) = 7 - 2x + 15x^2 + \int_{-1}^{1} (x - t)u(t)dt, u(0) = 1. \tag{6.114}$$

Substituting the series

$$u'(x) = \left(\sum_{n=0}^{\infty} a_n x^n\right)', \tag{6.115}$$

into both sides of Eq. (6.114) leads to

$$\sum_{n=1}^{\infty} n a_n x^{n-1} = 7 - 2x + 15x^2 + \int_{-1}^{1} \left((x - t)\sum_{n=0}^{\infty} a_n t^n\right) dt. \tag{6.116}$$

Notice that $a_0 = 1$ from the given initial conditions. Evaluating the integral at the right side gives

$$\begin{aligned} a_1 + 2a_2x + 3a_3x^2 + \cdots &= 7 - \frac{2}{3}a_1 - \frac{2}{5}a_3 - \frac{2}{7}a_5 - \frac{2}{9}a_7 \\ &\quad + \left(\frac{2}{3}a_2 + \frac{2}{5}a_4 + \frac{2}{7}a_6 + \frac{2}{9}a_8\right) x + 15x^2. \end{aligned} \tag{6.117}$$

Equating the coefficients of like powers of x in both sides of (6.117) gives

$$a_1 = 3, \quad a_2 = 0, \quad a_3 = 5, \quad a_n = 0, \quad n \geqslant 4. \tag{6.118}$$

The exact solution is given by

$$u(x) = 1 + 3x + 5x^3, \tag{6.119}$$

where we used $a_0 = 1$ from the initial condition.

Example 6.15

Solve the Fredholm integro-differential equation by using the series solution method

$$u''(x) = \frac{5}{3} - 11x + \int_{-1}^{1} (xt^2 - x^2t)u(t)dt, \quad u(0) = 1, u'(0) = 1. \tag{6.120}$$

Substituting the series

$$u''(x) = \left(\sum_{n=0}^{\infty} a_n x^n\right)'', \tag{6.121}$$

into both sides of Eq. (6.120) leads to

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \frac{5}{3} - 11x + \int_{-1}^{1}\left((xt^2 - x^2t)\sum_{n=0}^{\infty} a_n t^n\right) dt. \tag{6.122}$$

Notice that $a_0 = a_1 = 1$ from the given initial conditions. Evaluating the integral at the right side gives

$$\begin{aligned} 2a_2 + 6a_3x + 12a_4x^2 + \cdots = {} & \frac{5}{3} + \left(-\frac{31}{7} + \frac{2}{5}a_2 + \frac{2}{7}a_4 + \frac{2}{9}a_6 + \frac{2}{11}a_8\right) x \\ & - \left(\frac{2}{3} + \frac{2}{5}a_3 + \frac{2}{7}a_5 + \frac{2}{9}a_7\right) x^2. \end{aligned} \tag{6.123}$$

Equating the coefficients of like powers of x in both sides of (6.123) gives

$$a_2 = \frac{5}{6}, \quad a_3 = -\frac{5}{3}, \quad a_n = 0, \quad n \geqslant 4. \tag{6.124}$$

The exact solution is given by

$$u(x) = 1 + x + \frac{5}{6}x^2 - \frac{5}{3}x^3, \tag{6.125}$$

where we used $a_0 = a_1 = 1$ from the initial condition.

Example 6.16

Solve the Fredholm integro-differential equation by using the series solution method

$$u'''(x) = 1 - x - \cos x + \int_0^{\pi} xtu(t)dt, u(0) = 0, u'(0) = 1, u'(0) = 0. \tag{6.126}$$

Substituting the series

$$u'''(x) = \left(\sum_{n=0}^{\infty} a_n x^n\right)''', \tag{6.127}$$

into both sides of Eq. (6.126) leads to

$$\sum_{n=3}^{\infty} n(n-1)(n-2)a_n x^{n-3} = 1 - x - \cos x + \int_0^{\pi}\left(xt\sum_{n=0}^{\infty} a_n t^n\right) dt. \tag{6.128}$$

Evaluating the integral at the right side and proceeding as before we obtain

$$a_{2n+1} = \frac{(-1)^n}{(2n+1)!}, a_{2n+2} = 0, n \geqslant 1. \tag{6.129}$$

Recall that the initial conditions give $a_0 = 0, a_1 = 1, a_2 = 0$. Combining the initial conditions with the last results for the coefficients we find

$$u(x) = \sin x. \tag{6.130}$$

It is to be noted that we used terms up to $O(x^{12})$ in the series to obtain this result.

Exercises 6.2.4

Use the *series solution method* to solve the following Fredholm integro-differential equations:

1. $u'(x) = 4x + \int_{-1}^{1} (x-t)u(t)dt,\ u(0) = 2$
2. $u'(x) = 4 + \int_{-1}^{1} (1-x^2t^2)u(t)dt,\ u(0) = -2$
3. $u'(x) = 5 + 4x + \int_{-1}^{1} (x-t)u(t)dt,\ u(0) = 0$
4. $u'(x) = 10x - \frac{10}{3}x^2 + \int_{-1}^{1} (x^2-t^2)u(t)dt,\ u(0) = 0$
5. $u''(x) = \frac{92}{5} + \int_{-1}^{1} (x-t)u(t)dt,\ u(0) = u'(0) = 0$
6. $u''(x) = -18 + \int_{-1}^{1} (x-t)u(t)dt,\ u(0) = 3,\ u'(0) = 0$
7. $u''(x) = -18 + \int_{-1}^{1} (1-xt)u(t)dt,\ u(0) = 3,\ u'(0) = 0$
8. $u''(x) = 32x + \int_{-1}^{1} (1-xt)u(t)dt,\ u(0) = 1,\ u'(0) = 0$
9. $u'''(x) = -2x + \sin x - \cos x + \int_{0}^{\pi} xu(t)dt,\ u(0) = u'(0) = 1,\ u''(0) = -1$
10. $u'''(x) = 2 + \sin x + \int_{0}^{\pi} tu(t)dt,\ u(0) = 1,\ u'(0) = 0,\ u''(0) = -1$
11. $u'''(x) = -1 + \sin x + \int_{0}^{\frac{\pi}{2}} u(t)dt,\ u(0) = 1,\ u'(0) = 0,\ u''(0) = -1$
12. $u'''(x) = 1 - e + e^x + \int_{0}^{1} u(t)dt,\ u(0) = u'(0) = u''(0) = 1$

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Chapter 7
Abel's Integral Equation and Singular Integral Equations

7.1 Introduction

Abel's integral equation occurs in many branches of scientific fields [1], such as microscopy, seismology, radio astronomy, electron emission, atomic scattering, radar ranging, plasma diagnostics, X-ray radiography, and optical fiber evaluation. Abel's integral equation is the earliest example of an integral equation [2]. In Chapter 2, Abel's integral equation was defined as a singular integral equation. Volterra integral equations of the first kind

$$f(x) = \lambda \int_{g(x)}^{h(x)} K(x,t)u(t)dt, \tag{7.1}$$

or of the second kind

$$u(x) = f(x) + \lambda \int_{g(x)}^{h(x)} K(x,t)u(t)dt, \tag{7.2}$$

are called *singular* [3–4] if:

1. one of the limits of integration $g(x)$, $h(x)$ or both are *infinite*, or
2. if the kernel $K(x,t)$ becomes *infinite* at one or more points at the range of integration.

Examples of the first type are given by Fourier transform and Laplace transform of the function $u(x)$

$$F(\lambda) = \int_{-\infty}^{\infty} e^{-\imath\lambda x} u(x)dx, \tag{7.3}$$

$$\mathcal{L}\{u(x)\}(s) = \int_{0}^{\infty} e^{-sx} u(x)dx. \tag{7.4}$$

respectively. It is obvious that the range of integration is infinite for the Fourier transform and Laplace transform respectively.

Examples of the second type are the Abel's integral equation, generalized Abel's integral equation, and weakly singular integral equation are given by

$$f(x) = \int_0^x \frac{1}{\sqrt{(x-t)}} u(t)dt, \tag{7.5}$$

$$f(x) = \int_0^x \frac{1}{(x-t)^\alpha} u(t)dt, 0 < \alpha < 1, \tag{7.6}$$

and

$$u(x) = f(x) + \int_0^x \frac{1}{(x-t)^\alpha} u(t)dt, 0 < \alpha < 1, \tag{7.7}$$

respectively. It is clear that the kernel in each equation becomes infinite at the upper limit $t = x$. The last three singular integral equations are among the earliest integral equations established by the Norwegian mathematician Niles Abel in 1823.

In this chapter we will focus our study on the second style of singular integral equations, namely the equations where the kernel $K(x, t)$ becomes unbounded at one or more points of singularities in its domain of definition. The equations that will be investigated are Abel's problem, generalized Abel integral equations and the weakly-singular second-kind Volterra type integral equations. In a manner parallel to the approach used in previous chapters, we will focus our study on the techniques that will guarantee the existence of a unique solution to any singular integral equation. We point out here that singular integral equations are in general very difficult to handle.

7.2 Abel's Integral Equation

Abel in 1823 investigated the motion of a particle that slides down along a smooth unknown curve, in a vertical plane, under the influence of the gravity. The particle takes the time $f(x)$ to move from the highest point of vertical height x to the lowest point 0 on the curve. The Abel's problem is derived to find the equation of that curve.

Abel derived the equation of motion of the sliding particle along a smooth curve by the singular integral equation

$$f(x) = \int_0^x \frac{1}{\sqrt{x-t}} u(t)dt, \tag{7.8}$$

where $f(x)$ is a predetermined data function, and $u(x)$ is the solution that will be determined. It is to be noted that Abel's integral equation (7.8) is also called Volterra integral equation of the first kind. Besides the kernel $K(x, t)$ in Abel's integral equation (7.8) is

$$K(x, t) = \frac{1}{\sqrt{x-t}}, \tag{7.9}$$

where

$$K(x, t) \to \infty, \quad \text{as} \quad t \to x. \tag{7.10}$$

It is interesting to point out that although Abel's integral equation is a Volterra integral equation of the first kind, but two of the methods used before in Section 3.3, namely the series solution method and the conversion to a second kind Volterra equation, are not applicable here. The series solution cannot be used in this case especially if $u(x)$ is not analytic. Moreover, converting Abel's integral equation to a second kind Volterra equation is not obtainable because we cannot use Leibnitz rule due to the singularity behavior of the kernel in (7.8).

7.2.1 The Laplace Transform Method

Although the Laplace transform method was presented before, but a brief summary will be helpful. In the convolution theorem for the Laplace transform method, it was stated that if the kernel $K(x,t)$ of the integral equation

$$f(x) = \int_0^x K(x,t)u(t)dt, \tag{7.11}$$

depends on the difference $x-t$, then it is called a *difference kernel*. The Abel's integral equation can thus be expressed as

$$f(x) = \int_0^x K(x-t)u(t)dt. \tag{7.12}$$

Consider two functions $f_1(x)$ and $f_2(x)$ that possess the conditions needed for the existence of Laplace transform for each. Let the Laplace transforms for the functions $f_1(x)$ and $f_2(x)$ be given by

$$\begin{aligned} \mathcal{L}\{f_1(x)\} &= F_1(s), \\ \mathcal{L}\{f_2(x)\} &= F_2(s). \end{aligned} \tag{7.13}$$

The *Laplace convolution product* of these two functions is defined by

$$(f_1 * f_2)(x) = \int_0^x f_1(x-t)f_2(t)dt, \tag{7.14}$$

or

$$(f_2 * f_1)(x) = \int_0^x f_2(x-t)f_1(t)dt. \tag{7.15}$$

Recall that

$$(f_1 * f_2)(x) = (f_2 * f_1)(x). \tag{7.16}$$

We can easily show that the Laplace transform of the convolution product $(f_1 * f_2)(x)$ is given by

$$\mathcal{L}\{(f_1 * f_2)(x)\} = F_1(s)F_2(s). \tag{7.17}$$

Based on this summary, we will examine Abel's integral equation where the kernel is a difference kernel. Recall that we will apply the Laplace transform method and the inverse of the Laplace transform using Table 2 in section 1.5.

Taking Laplace transforms of both sides of (7.8) leads to

$$\mathcal{L}\{f(x)\} = \mathcal{L}\{u(x)\}\mathcal{L}\{x^{-\frac{1}{2}}\}, \tag{7.18}$$

or equivalently

$$F(s) = U(s)\frac{\Gamma(1/2)}{s^{1/2}} = U(s)\frac{\sqrt{\pi}}{s^{1/2}}, \tag{7.19}$$

that gives

$$U(s) = \frac{s^{1/2}}{\sqrt{\pi}}F(s), \tag{7.20}$$

where Γ is the gamma function, and $\Gamma(\frac{1}{2}) = \sqrt{\pi}$. In Appendix D, the definition of the gamma function and some of the relations related to it are given. The last equation (7.20) can be rewritten as

$$U(s) = \frac{s}{\pi}(\sqrt{\pi}s^{-\frac{1}{2}}F(s)), \tag{7.21}$$

which can be rewritten by

$$\mathcal{L}\{u(x)\} = \frac{s}{\pi}\mathcal{L}\{y(x)\}, \tag{7.22}$$

where

$$y(x) = \int_0^x (x-t)^{-\frac{1}{2}}f(t)dt. \tag{7.23}$$

Using the fact

$$\mathcal{L}\{y'(x)\} = s\mathcal{L}\{y(x)\} - y(0), \tag{7.24}$$

into (7.22) we obtain

$$\mathcal{L}\{u(x)\} = \frac{1}{\pi}\mathcal{L}\{y'(x)\}. \tag{7.25}$$

Applying L^{-1} to both sides of (7.25) gives the formula

$$u(x) = \frac{1}{\pi}\frac{d}{dx}\int_0^x \frac{f(t)}{\sqrt{x-t}}dt, \tag{7.26}$$

that will be used for the determination of the solution $u(x)$. Notice that the formula (7.26) will be used for solving Abel's integral equation, and it is not necessary to use Laplace transform method for each problem. Abel's problem given by (7.8) can be solved directly by using the formula (7.26) where the unknown function $u(x)$ has been replaced by the given function $f(x)$. For $f(x) = x^n, n$ is a positive integer, Table 7.1 can be used for evaluating the integral in Abel's problem.

Table 7.1 For integrals of the form $u(x) = \int_0^x \frac{t^n}{\sqrt{x-t}}dt, n \geqslant 0$.

$u(x)$	$2c\sqrt{x}$	$\frac{4}{3}x^{\frac{3}{2}}$	$\frac{16}{15}x^{\frac{5}{2}}$	$\frac{32}{35}x^{\frac{7}{2}}$	$\vdots$	$\frac{2^{n+1}\Gamma(n+1)x^{n+\frac{1}{2}}}{1\cdot 3\cdot 5\cdots(2n+1)}$
$f(t)$	c	t	t^2	t^3	$\vdots$	t^n

However, for $f(x) = x^{\frac{n}{2}}, n$ is an odd positive integer, Table 7.2 can be used for evaluating the integral in Abel's problem.

Table 7.2 For integrals of the form $u(x) = \int_0^x \frac{t^{\frac{n}{2}}}{\sqrt{x-t}}dt$.

$u(x)$	$2c\sqrt{x}$	$\frac{1}{2}\pi x$	$\frac{3}{8}\pi x^2$	$\frac{5}{16}\pi x^3$	$\vdots$	$\frac{\Gamma(\frac{n+2}{2})}{\Gamma(\frac{n+3}{2})}\sqrt{\pi}x^{\frac{n+1}{2}}$
$f(t)$	c	$t^{\frac{1}{2}}$	$t^{\frac{3}{2}}$	$t^{\frac{5}{2}}$	$\vdots$	$t^{\frac{n}{2}}$

More details can be found in Appendix B. Using formula (7.26) to determine the solution of Abel's problem (7.8) will be illustrated by the following examples.

Example 7.1

Solve the following Abel's integral equation

$$2\pi\sqrt{x} = \int_0^x \frac{1}{\sqrt{x-t}}u(t)dt. \tag{7.27}$$

Substituting $f(x) = 2\pi\sqrt{x}$ in (7.26) gives

$$u(x) = \frac{1}{\pi}\frac{d}{dx}\int_0^x \frac{2\pi\sqrt{t}}{\sqrt{x-t}}dt = \frac{d}{dx}(\pi x) = \pi. \tag{7.28}$$

Example 7.2

Solve the following Abel's integral equation

$$\frac{4}{3}x^{\frac{3}{2}} = \int_0^x \frac{1}{\sqrt{x-t}}u(t)dt. \tag{7.29}$$

Substituting $f(x) = \frac{4}{3}x^{\frac{3}{2}}$ in (7.26) gives

$$u(x) = \frac{1}{\pi}\frac{d}{dx}\int_0^x \frac{\frac{4}{3}t^{\frac{3}{2}}}{\sqrt{x-t}}dt = \frac{1}{2}\frac{d}{dx}(x^2) = x \tag{7.30}$$

Example 7.3

Solve the following Abel's integral equation

$$\frac{8}{3}x^{\frac{3}{2}} + \frac{16}{5}x^{\frac{5}{2}} = \int_0^x \frac{1}{\sqrt{x-t}}u(t)dt. \tag{7.31}$$

Using (7.26) gives

$$\begin{aligned} u(x) &= \frac{1}{\pi}\frac{d}{dx}\int_0^x \frac{\frac{8}{3}t^{\frac{3}{2}} + \frac{16}{5}t^{\frac{5}{2}}}{\sqrt{x-t}}dt, \\ &= \frac{1}{\pi}\frac{d}{dx}(\pi x^2 + \pi x^3) = 2x + 3x^2. \end{aligned} \tag{7.32}$$

Example 7.4

Find two-terms approximation of the solution of the following Abel's integral equation

$$\sin x = \int_0^x \frac{1}{\sqrt{x-t}} u(t)dt, \quad x \in [0,1]. \tag{7.33}$$

Using (7.26) gives

$$\begin{aligned} u(x) &= \frac{1}{\pi}\frac{d}{dx}\int_0^x \frac{\sin t}{\sqrt{x-t}}dt = \frac{1}{\pi}\frac{d}{dx}\int_0^x \frac{t-\frac{1}{3!}t^3}{\sqrt{x-t}}dt, \\ &= \frac{1}{\pi}\frac{d}{dx}\left(\frac{4}{3}x^{\frac{3}{2}} - \frac{16}{105}x^{\frac{7}{2}}\right) = \frac{1}{\pi}\left(\frac{1}{2}\sqrt{x} - \frac{8}{15}x^{\frac{5}{2}}\right). \end{aligned} \tag{7.34}$$

Exercises 7.2.1

In Exercises 1–12, solve the following Abel's integral equations where $x \in [0, 2]$

1. $\pi = \int_0^x \frac{1}{\sqrt{x-t}} u(t)dt$
2. $\frac{\pi}{2}x = \int_0^x \frac{1}{\sqrt{x-t}} u(t)dt$
3. $2\sqrt{x} = \int_0^x \frac{1}{\sqrt{x-t}} u(t)dt$
4. $2\sqrt{x} + \frac{8}{3}x^{\frac{3}{2}} = \int_0^x \frac{1}{\sqrt{x-t}} u(t)dt$
5. $\frac{\pi}{2}(x^2 - x) = \int_0^x \frac{1}{\sqrt{x-t}} u(t)dt$
6. $\frac{3}{8}\pi x^2 = \int_0^x \frac{1}{\sqrt{x-t}} u(t)dt$
7. $\frac{4}{3}x^{\frac{3}{2}} = \int_0^x \frac{1}{\sqrt{x-t}} u(t)dt$
8. $\frac{1}{2}\pi x + \frac{4}{3}x^{\frac{3}{2}} = \int_0^x \frac{1}{\sqrt{x-t}} u(t)dt$
9. $2\sqrt{x} - \frac{1}{2}\pi x = \int_0^x \frac{1}{\sqrt{x-t}} u(t)dt$
10. $x^3 = \int_0^x \frac{1}{\sqrt{x-t}} u(t)dt$
11. $4x^{\frac{3}{2}} - x = \int_0^x \frac{1}{\sqrt{x-t}} u(t)dt$
12. $3x^2 = \int_0^x \frac{1}{\sqrt{x-t}} u(t)dt$

In Exercises 13–16, find two-terms approximation for the solution of the following Abel's integral equations

13. $\pi \sinh x = \int_0^x \frac{1}{\sqrt{x-t}} u(t)dt$
14. $\pi \sinh x \ln(1+x) = \int_0^x \frac{1}{\sqrt{x-t}} u(t)dt$
15. $\pi \cosh x \ln(1+x) = \int_0^x \frac{1}{\sqrt{x-t}} u(t)dt$
16. $\pi \sin x \tan x = \int_0^x \frac{1}{\sqrt{x-t}} u(t)dt, x$

7.3 The Generalized Abel's Integral Equation

Abel generalized his original problem by introducing the singular integral equation

$$f(x) = \int_0^x \frac{1}{(x-t)^\alpha} u(t)dt, \quad 0 < \alpha < 1, \tag{7.35}$$

known as the *Generalized Abel's integral equation* where α are known constants such that $0 < \alpha < 1$, $f(x)$ is a predetermined data function, and $u(x)$ is the solution that will be determined. The Abel's problem discussed above is a special case of the generalized equation where $\alpha = \frac{1}{2}$. The expression $(x-t)^{-\alpha}$ is called the kernel of the Abel integral equation, or simply Abel's kernel.

7.3.1 The Laplace Transform Method

To determine a formula that will be used for solving the generalized Abel's integral equation (7.35), we will apply the Laplace transform method in a parallel manner to the approach followed before. Taking Laplace transforms of both sides of (7.35) leads to

$$\mathcal{L}\{f(x)\} = \mathcal{L}\{u(x)\}\mathcal{L}\{x^{-\alpha}\}, \tag{7.36}$$

or equivalently

$$F(s) = U(s)\frac{\Gamma(1-\alpha)}{s^{1-\alpha}}, \tag{7.37}$$

that gives

$$U(s) = \frac{s^{1-\alpha}}{\Gamma(1-\alpha)}F(s), \tag{7.38}$$

where Γ is the gamma function. The last equation (7.38) can be rewritten as

$$\mathcal{L}\{u(x)\} = \frac{s}{\Gamma(\alpha)\Gamma(1-\alpha)}\mathcal{L}(y(x)), \tag{7.39}$$

where

$$y(x) = \int_0^x \frac{1}{(x-t)^{\alpha-1}}f(t)dt. \tag{7.40}$$

Using the facts

$$\mathcal{L}\{y'(x)\} = s\mathcal{L}\{y(x)\} - y(0), \tag{7.41}$$

and

$$\Gamma(\alpha)\Gamma(1-\alpha) = \frac{\pi}{\sin\alpha\pi}, \tag{7.42}$$

into (7.39) we obtain

$$\mathcal{L}\{u(x)\} = \frac{\sin\alpha\pi}{\pi}\mathcal{L}(y'(x)). \tag{7.43}$$

Applying L^{-1} to both sides of (7.43) gives the formula

$$u(x) = \frac{\sin\alpha\pi}{\pi}\frac{d}{dx}\int_0^x \frac{f(t)}{(x-t)^{1-\alpha}}dt. \tag{7.44}$$

Integrating the integral at the right side of (7.44) and differentiating the result we obtain the more suitable formula

$$u(x) = \frac{\sin\alpha\pi}{\pi}\left(\frac{f(0)}{x^{1-\alpha}} + \int_0^x \frac{f'(t)}{(x-t)^{1-\alpha}}dt\right), \quad 0 < \alpha < 1. \tag{7.45}$$

Four remarks can be made here:

1. The kernel is called weakly singular as the singularity may be transformed away by a change of variable [5].

2. The exponent of the kernel of the generalized Abel's integral equation is $-\alpha$, but the exponent of the kernel of the two formulae (7.44) and (7.45) is $\alpha - 1$.

3. The unknown function in (7.35) has been replaced by $f(t)$ and $f'(t)$ in (7.44) and (7.45) respectively.

4. In (7.44), differentiation is used after integrating the integral at the right side, whereas in (7.45), integration is only required.

Example 7.5

Solve the following generalized Abel's integral equation

$$\frac{128}{231}x^{\frac{11}{4}} = \int_0^x \frac{1}{(x-t)^{\frac{1}{4}}}u(t)dt. \tag{7.46}$$

Notice that $\alpha = \frac{1}{4}, f(x) = \frac{128}{231}x^{\frac{11}{4}}$. Using (7.44) gives

$$u(x) = \frac{1}{\sqrt{2}\pi}\frac{d}{dx}\int_0^x \frac{\frac{128}{231}t^{\frac{11}{4}}}{(x-t)^{\frac{3}{4}}}dt = \frac{1}{\sqrt{2}\pi}\frac{d}{dx}\left(\frac{\sqrt{2}}{3}\pi x^3\right) = x^2. \tag{7.47}$$

Example 7.6

Solve the following generalized Abel's integral equation

$$\pi x = \int_0^x \frac{1}{(x-t)^{\frac{2}{3}}}u(t)dt. \tag{7.48}$$

Notice that $\alpha = \frac{2}{3}, f(x) = \pi x$. Using (7.44) gives

$$u(x) = \frac{\sqrt{3}}{2\pi}\frac{d}{dx}\int_0^x \frac{\pi t}{(x-t)^{\frac{1}{3}}}dt = \frac{\sqrt{3}}{2}\frac{d}{dx}\left(\frac{9}{10}x^{\frac{5}{3}}\right) = \frac{3\sqrt{3}}{4}x^{\frac{2}{3}}. \tag{7.49}$$

Example 7.7

Solve the following generalized Abel's integral equation

$$\frac{36}{55}x^{\frac{11}{6}} = \int_0^x \frac{1}{(x-t)^{\frac{1}{6}}}u(t)dt. \tag{7.50}$$

Notice that $\alpha = \frac{1}{6}, f(x) = \frac{36}{55}x^{\frac{11}{6}}$. Using (7.44) gives

$$u(x) = \frac{1}{2\pi}\frac{d}{dx}\int_0^x \frac{\frac{36}{55}t^{\frac{11}{6}}}{(x-t)^{\frac{5}{6}}}dt = \frac{1}{2\pi}\frac{d}{dx}(\pi x^2) = x. \tag{7.51}$$

Example 7.8

Solve the following generalized Abel's integral equation

$$\frac{512}{1155}x^{\frac{15}{4}} = \int_0^x \frac{1}{(x-t)^{\frac{1}{4}}}u(t)dt. \tag{7.52}$$

Notice that $\alpha = \frac{1}{4}, f(x) = \frac{512}{1155}x^{\frac{15}{4}}$. Using (7.44) gives

$$u(x) = \frac{1}{\sqrt{2}\pi}\frac{d}{dx}\int_0^x \frac{\frac{512}{1155}t^{\frac{15}{4}}}{(x-t)^{\frac{3}{4}}}dt = x^3. \tag{7.53}$$

7.3.2 The Main Generalized Abel Equation

It is useful to introduce a further generalization to Abel's equation by considering a generalized singular kernel instead of $K(x,t) = \frac{1}{\sqrt{x-t}}$. The generalized kernel will be of the form

$$K(x,t) = \frac{1}{[g(x)-g(t)]^{\alpha}}, 0 < \alpha < 1. \tag{7.54}$$

The main generalized Abel's integral equation is given by

$$f(x) = \int_0^x \frac{1}{[g(x)-g(t)]^{\alpha}}u(t)dt, 0 < \alpha < 1, \tag{7.55}$$

where $g(t)$ is strictly monotonically increasing and differentiable function in some interval $0 < t < b$, and $g'(t) \neq 0$ for every t in the interval. The solution $u(x)$ of (7.55) is given by

$$u(x) = \frac{\sin\alpha\pi}{\pi}\frac{d}{dx}\int_0^x \frac{g'(t)f(t)}{[g(x)-g(t)]^{1-\alpha}}dt, \quad 0 < \alpha < 1. \tag{7.56}$$

To prove this formula, we follow [6–7] and consider the integral

$$\int_0^x \frac{g'(y)f(y)}{[g(x)-g(y)]^{1-\alpha}}dy,$$

and substitute for $f(u)$ from (7.55) to obtain

$$\int_0^x\int_0^y \frac{u(t)g'(y)}{[g(y)-g(t)]^{\alpha}[g(x)-g(y)]^{1-\alpha}}dtdy,$$

where by changing the order of integration we find

$$\int_0^x u(t)dt\int_0^y \frac{g'(y)}{[g(y)-g(t)]^{\alpha}[g(x)-g(y)]^{1-\alpha}}dy.$$

We can prove that

$$\int_0^y \frac{g'(y)}{[g(y)-g(t)]^{\alpha}[g(x)-g(y)]^{1-\alpha}}dy = \beta(\alpha, 1-\alpha) = \frac{\pi}{\sin\alpha\pi}, \tag{7.57}$$

where $\beta(\alpha, 1-\alpha)$ is the beta function. This means that

$$\int_0^x \frac{g'(y)f(y)}{[g(x)-g(y)]^{1-\alpha}}dy = \frac{\pi}{\sin\alpha\pi}\int_0^x u(t)dt. \tag{7.58}$$

Differentiating both sides of (7.58) gives

$$u(x) = \frac{\sin\alpha\pi}{\pi}\frac{d}{dx}\int_0^x \frac{g'(y)f(y)}{[g(x)-g(y)]^{1-\alpha}}dy, 0 < \alpha < 1. \tag{7.59}$$

It is interesting to point out that the Abel's integral equation and the generalized Abel's integral equation are special case of (7.55) by substituting $g(x) = x$ respectively.

The following illustrative examples explain how we can use the formula (7.56) in solving the further generalized Abel's equation.

Example 7.9

Solve the following generalized Abel's integral equation

$$\frac{4}{3}(\sin x)^{\frac{3}{4}} = \int_0^x \frac{u(t)}{(\sin x - \sin t)^{\frac{1}{4}}} dt, \tag{7.60}$$

where $0 < x < \frac{\pi}{2}$. Notice that $\alpha = \frac{1}{4}, f(x) = \frac{4}{3}(\sin x)^{\frac{3}{4}}$. Besides, $g(x) = \sin x$ is strictly monotonically increasing in $0 < x < \frac{\pi}{2}$, and $g'(x) = \cos x \neq 0$ for every x in $0 < x < \frac{\pi}{2}$. Using (7.56) gives

$$\begin{aligned} u(x) &= \frac{1}{\sqrt{2}\pi}\frac{d}{dx}\int_0^x \frac{\frac{4}{3}\cos t(\sin t)^{\frac{3}{4}}}{(\sin x - \sin t)^{\frac{3}{4}}} dt \\ &= \frac{4}{3\sqrt{2}\pi}\frac{d}{dx}\left(\frac{3\sqrt{2}\pi}{4}\sin x\right) = \cos x. \end{aligned} \tag{7.61}$$

Example 7.10

Solve the following generalized Abel's integral equation

$$\frac{2}{3}\pi x^3 = \int_0^x \frac{u(t)}{\sqrt{x^2 - t^2}} dt, \tag{7.62}$$

where $0 < x < 2$. Notice that $\alpha = \frac{1}{2}, f(x) = \frac{2}{3}\pi x^3$. Besides, $g(x) = x^2$ is strictly monotonically increasing in $0 < x < 2$, and $g'(x) = 2x \neq 0$ for every x in $0 < x < 2$. Using (7.56) gives

$$u(x) = \frac{1}{\pi}\frac{d}{dx}\int_0^x \frac{\frac{4}{3}\pi t^4}{\sqrt{x^2 - t^2}} dt = \frac{1}{\pi}\frac{d}{dx}\left(\frac{\pi^2}{4}x^4\right) = \pi x^3. \tag{7.63}$$

Example 7.11

Solve the following generalized Abel's integral equation

$$x^2 = \int_0^x \frac{u(t)}{\sqrt{x^2 - t^2}} dt, \tag{7.64}$$

where $0 < x < 2$. Notice that $\alpha = \frac{1}{2}, f(x) = x^2$. Besides, $g(x) = x^2$ is strictly monotonically increasing in $0 < x < 2$, and $g'(x) = 2x \neq 0$ for every x in $0 < x < 2$. Using (7.56) gives

$$u(x) = \frac{1}{\pi}\frac{d}{dx}\int_0^x \frac{2t^3}{\sqrt{x^2 - t^2}} dt = \frac{1}{\pi}\frac{d}{dx}\left(\frac{4}{3}x^3\right) = \frac{4}{\pi}x^2. \tag{7.65}$$

Example 7.12

Solve the following generalized Abel's integral equation

$$\frac{6}{25}x^{\frac{25}{6}} = \int_0^x \frac{u(t)}{(x^5 - t^5)^{\frac{1}{6}}}dt, \tag{7.66}$$

where $0 < x < 2$. Notice that $\alpha = \frac{1}{6}, f(x) = \frac{6}{25}x^{\frac{25}{6}}$. Besides, because $g(x)$ is strictly monotonically increasing in $0 < x < 2$, and $g'(x) = 5x^4 \neq 0$ for every x in $0 < x < 2$. Using (7.56) gives

$$u(x) = \frac{1}{2\pi}\frac{d}{dx}\int_0^x \frac{\frac{6}{5}t^{\frac{49}{6}}}{(x^5 - t^5)^{\frac{5}{6}}}dt = \frac{1}{2\pi}\frac{d}{dx}\left(\frac{2}{5}\pi x^5\right) = x^4. \tag{7.67}$$

Exercises 7.3

In Exercises 1–8, solve the following generalized Abel's integral equations

1. $\frac{9}{4}x^{\frac{4}{3}} = \int_0^x \frac{1}{(x-t)^{\frac{2}{3}}}u(t)dt$

2. $\frac{432}{935}x^{\frac{17}{6}} = \int_0^x \frac{1}{(x-t)^{\frac{1}{6}}}u(t)dt$

3. $\frac{24}{5}x^{\frac{5}{6}} = \int_0^x \frac{1}{(x-t)^{\frac{1}{6}}}u(t)dt$

4. $\frac{3}{2}x^{\frac{2}{3}} + \frac{9}{10}x^{\frac{5}{3}} = \int_0^x \frac{1}{(x-t)^{\frac{1}{3}}}u(t)dt$

5. $\frac{243}{440}x^{\frac{11}{3}} = \int_0^x \frac{1}{(x-t)^{\frac{1}{3}}}u(t)dt$

6. $2\pi x^{\frac{1}{3}} - \frac{27}{14}x^{\frac{7}{3}} = \int_0^x \frac{1}{(x-t)^{\frac{2}{3}}}u(t)dt$

7. $\frac{16}{21}x^{\frac{7}{4}} + \frac{128}{231}x^{\frac{11}{4}} = \int_0^x \frac{1}{(x-t)^{\frac{1}{4}}}u(t)dt$

8. $\frac{25}{36}x^{\frac{9}{5}} = \int_0^x \frac{1}{(x-t)^{\frac{1}{5}}}u(t)dt$

In Exercises 9–16, use the formula (7.56) to solve the main generalized Abel's integral equations

9. $2x^{\frac{3}{2}} = \int_0^x \frac{1}{(x^3-t^3)^{\frac{1}{2}}}u(t)dt$

10. $\frac{3}{2}x^2 = \int_0^x \frac{1}{(x^3-t^3)^{\frac{1}{3}}}u(t)dt$

11. $\frac{3}{4}x^{\frac{4}{3}} = \int_0^x \frac{1}{(x^2-t^2)^{\frac{1}{3}}}u(t)dt$

12. $\frac{3}{2}(\sin x)^{\frac{2}{3}} = \int_0^x \frac{1}{(\sin x - \sin t)^{\frac{1}{3}}}u(t)dt$

13. $\frac{6}{5}(e^x - 1)^{\frac{5}{6}} = \int_0^x \frac{1}{(e^x - e^t)^{\frac{1}{6}}}u(t)dt$

14. $\frac{2}{3}\sqrt{e^x - 1}(2e^x + 1) = \int_0^x \frac{1}{(e^x - e^t)^{\frac{1}{2}}}u(t)dt$

15. $2\sqrt{\sin x} = \int_0^x \frac{1}{(\sin x - \sin t)^{\frac{1}{2}}}u(t)dt$

16. $\frac{64}{231}x^{\frac{11}{2}} = \int_0^x \frac{1}{(x^2-t^2)^{\frac{1}{4}}}u(t)dt$

7.4 The Weakly Singular Volterra Equations

The weakly-singular Volterra-type integral equations of the second kind are given by

$$u(x) = f(x) + \int_0^x \frac{\beta}{\sqrt{x-t}}\,u(t)dt,\ x \in [0, T], \tag{7.68}$$

and

$$u(x) = f(x) + \int_0^x \frac{\beta}{[g(x) - g(t)]^\alpha} u(t)dt, 0 < \alpha < 1, \, x \in [0, T], \tag{7.69}$$

where β is a constant. Equation. (7.69) is known as the generalized weakly singular Volterra equation. These equations arise in many mathematical physics and chemistry applications such as stereology, heat conduction, crystal growth and the radiation of heat from a semi-infinite solid. It is also assumed that the function $f(x)$ is sufficiently smooth so that it guarantees a unique solution to (7.68) and to (7.69). The weakly-singular and the generalized weakly-singular equations (7.68) and (7.69) fall under the category of singular equations with singular kernels

$$\begin{aligned} K(x,t) &= \frac{1}{\sqrt{x-t}}, \\ K(x,t) &= \frac{1}{[g(x) - g(t)]^\alpha}, \end{aligned} \tag{7.70}$$

respectively. Notice that the kernel is called weakly singular as the singularity may be transformed away by a change of variable [5].

The weakly-singular Volterra equations (7.68) and (7.69) were investigated by many analytic and numerical methods. In this text we will use only three methods that were used before for handling Volterra integral equations in Chapter 3.

7.4.1 The Adomian Decomposition Method

The Adomian decomposition method has been discussed extensively in this text. We will focus our study on the generalized weakly singular Volterra equation (7.69), because Eq. (7.68) is a special case of the generalized equation with $\alpha = \frac{1}{2}, g(x) = x$. In the following we outline a brief framework of the method. To determine the solution $u(x)$ of (7.69) we substitute the decomposition series

$$u(x) = \sum_{n=0}^{\infty} u_n(x), \tag{7.71}$$

into both sides of (7.69) to obtain

$$\sum_{n=0}^{\infty} u_n(x) = f(x) + \int_0^x \frac{\beta}{[g(x) - g(t)]^\alpha} \left(\sum_{n=0}^{\infty} u_n(t) \right) dt, 0 < \alpha < 1. \tag{7.72}$$

The components $u_0(x), u_1(x), u_2(x), \cdots$ are usually determined by using the recurrence relation

$$\begin{aligned} u_0(x) &= f(x) \\ u_1(x) &= \int_0^x \frac{\beta}{[g(x) - g(t)]^\alpha} u_0(t)dt, \end{aligned}$$

$$u_2(x) = \int_0^x \frac{\beta}{[g(x)-g(t)]^\alpha} u_1(t)dt, \quad (7.73)$$
$$u_3(x) = \int_0^x \frac{\beta}{[g(x)-g(t)]^\alpha} u_2(t)dt,$$
$$\vdots$$

Having determined the components $u_0(x), u_1(x), u_2(x), \ldots$, the solution $u(x)$ of (7.69) will be determined in the form of a rapid convergent power series [8] by substituting the derived components in (7.71). The determination of the previous components can be obtained by using Appendix B, calculator, or any computer symbolic systems such as Maple or Mathematica. The series solution converges to the exact solution if such a solution exists. For concrete problems, we use as many terms as we need for numerical purposes. The method has been proved to be effective in handling this kind of integral equations.

The Modified Decomposition Method

The modified decomposition method has been used effectively in this text. Recall that this method splits the source term $f(x)$ into two parts $f_1(x)$ and $f_2(x)$, where the first part is $f_1(x)$ assigned to the zeroth component $u_0(x)$ and the other part $f_2(x)$ to the first component $u_1(x)$. Based on this decomposition of $f(x)$, we use the modified recurrence relation as follows:

$$u_0(x) = f_1(x),$$
$$u_1(x) = f_2(x) + \int_0^x \frac{\beta}{[g(x)-g(t)]^\alpha} u_0(t)dt,$$
$$u_2(x) = \int_0^x \frac{\beta}{[g(x)-g(t)]^\alpha} u_1(t)dt, \quad (7.74)$$
$$u_3(x) = \int_0^x \frac{\beta}{[g(x)-g(t)]^\alpha} u_2(t)dt,$$
$$\vdots$$

It was proved before that the modified method accelerates the convergence of the solution.

The Noise Terms Phenomenon

The noise terms that may appear between various components of $u(x)$ are defined as the identical terms with opposite signs. By canceling these noise terms between the components $u_0(x)$ and $u_1(x)$ may give the exact solution that should be justified through substitution. The noise terms, if appeared between components of $u(x)$, accelerate the convergence of the solution and thus minimize the size of the calculations. The noise terms will be used in the following examples.

Example 7.13

Solve the weakly singular second kind Volterra integral equation

$$u(x) = x^2 + \frac{16}{15}x^{\frac{5}{2}} - \int_0^x \frac{1}{\sqrt{x-t}}u(t)dt. \tag{7.75}$$

Using the recurrence relation we set

$$\begin{aligned} u_0(x) &= x^2 + \frac{16}{15}x^{\frac{5}{2}}, \\ u_1(x) &= -\int_0^x \frac{t^2 + \frac{16}{15}t^{\frac{5}{2}}}{\sqrt{x-t}}dt = -\frac{16}{15}x^{\frac{5}{2}} - \frac{\pi}{3}x^3. \end{aligned} \tag{7.76}$$

The noise terms $\pm\frac{16}{15}x^{\frac{5}{2}}$ appear between $u_0(x)$ and $u_1(x)$. By canceling the noise term $\frac{16}{15}x^{\frac{5}{2}}$ from $u_0(x)$ and verifying that the non-canceled term in $u_0(x)$ justifies the equation (7.75), the exact solution is therefore given by

$$u(x) = x^2. \tag{7.77}$$

Moreover, we can easily obtain the exact solution by using the modified decomposition method. This can be done by splitting the nonhomogeneous part $f(x)$ into two parts $f_1(x) = x^2$ and $f_2(x) = \frac{16}{15}x^{\frac{5}{2}}$. Accordingly, we set the modified recurrence relation

$$\begin{aligned} u_0(x) &= x^2, \\ u_1(x) &= \frac{16}{15}x^{\frac{5}{2}} - \int_0^x \frac{t^2}{\sqrt{x-t}}dt = 0. \end{aligned} \tag{7.78}$$

Example 7.14

Solve the weakly singular second kind Volterra integral equation

$$u(x) = 1 - 2x - \frac{32}{21}x^{\frac{7}{4}} + \frac{4}{3}x^{\frac{3}{4}} - \int_0^x \frac{1}{(x-t)^{\frac{1}{4}}}u(t)dt. \tag{7.79}$$

Using the recurrence relation we set

$$\begin{aligned} u_0(x) &= 1 - 2x - \frac{32}{21}x^{\frac{7}{4}} + \frac{4}{3}x^{\frac{3}{4}}, \\ u_1(x) &= -\int_0^x \frac{1 - 2t - \frac{32}{21}t^{\frac{7}{4}} + \frac{4}{3}t^{\frac{3}{4}}}{(x-t)^{\frac{1}{4}}}dt \\ &= \frac{32}{21}x^{\frac{7}{4}} - \frac{4}{3}x^{\frac{3}{4}} + \frac{128}{63}x^{\frac{5}{2}} - \frac{16}{9}x^{\frac{3}{2}}. \end{aligned} \tag{7.80}$$

The noise terms $\mp\frac{32}{21}x^{\frac{7}{4}}$ and $\pm\frac{4}{3}x^{\frac{3}{4}}$ appear between $u_0(x)$ and $u_1(x)$. By canceling the noise terms from $u_0(x)$ and verifying that the non-canceled term in $u_0(x)$ justifies the equation (7.79), the exact solution is therefore given by

$$u(x) = 1 - 2x. \tag{7.81}$$

Moreover, we can easily obtain the exact solution by using the modified decomposition method. This can be done by splitting the nonhomogeneous part $f(x)$ into two parts $f_1(x) = 1 - 2x$ and $f_2(x) = -\frac{32}{21}x^{\frac{7}{4}} + \frac{4}{3}x^{\frac{3}{4}}$. The use of the modified method is left as an exercise to the reader.

Example 7.15

Solve the weakly singular second kind Volterra integral equation

$$u(x) = e^x - 3(e^x - 1)^{\frac{1}{3}} + \int_0^x \frac{2}{(e^x - e^t)^{\frac{1}{3}}} u(t)dt. \tag{7.82}$$

In this example, the modified decomposition method will be used. We first set

$$f_1(x) = e^x, \quad f_2(x) = -3(e^x - 1)^{\frac{1}{3}}. \tag{7.83}$$

In view of this, we use the modified recurrence relation

$$u_0(x) = e^x, \quad u_1(x) = -3(e^x - 1)^{\frac{1}{3}} + \int_0^x \frac{2e^t}{(e^x - e^t)^{\frac{1}{3}}} dt = 0. \tag{7.84}$$

The exact solution is therefore given by

$$u(x) = e^x. \tag{7.85}$$

Example 7.16

Solve the weakly singular second-type Volterra integral equation

$$u(x) = \sin x + (\cos x - 1)^{\frac{1}{6}} + \frac{1}{6}\int_0^x \frac{1}{(\cos x - \cos t)^{\frac{5}{6}}} u(t)dt. \tag{7.86}$$

In this example, we will also use the modified decomposition method. We first set

$$f_1(x) = \sin x, \quad f_2(x) = (\cos x - 1)^{\frac{1}{6}}. \tag{7.87}$$

In view of this, we use the modified recurrence relation

$$\begin{aligned} u_0(x) &= \sin x, \\ u_1(x) &= (\cos x - 1)^{\frac{1}{6}} + \frac{1}{6}\int_0^x \frac{\sin t}{(\cos x - \cos t)^{\frac{5}{6}}} dt = 0. \end{aligned} \tag{7.88}$$

The exact solution is therefore given by

$$u(x) = \sin x. \tag{7.89}$$

Example 7.17

Solve the weakly singular second kind Volterra integral equation

$$u(x) = x + x^3 - \frac{3}{4}x^{\frac{4}{3}} - \frac{9}{20}x^{\frac{10}{3}} + \int_0^x \frac{1}{(x^2 - t^2)^{\frac{1}{3}}} u(t)dt. \tag{7.90}$$

In this example, we will also use the modified decomposition method. We first set

$$f_1(x) = x + x^3, \quad f_2(x) = -\frac{3}{4}x^{\frac{4}{3}} - \frac{9}{20}x^{\frac{10}{3}}. \tag{7.91}$$

In view of this, we use the modified recurrence relation

$$u_0(x) = x + x^3, \quad u_1(x) = -\frac{3}{4}x^{\frac{4}{3}} - \frac{9}{20}x^{\frac{10}{3}} + \int_0^x \frac{t+t^3}{(x^2-t^2)^{\frac{1}{3}}}dt = 0. \quad (7.92)$$

The exact solution is therefore given by

$$u(x) = x + x^3. \quad (7.93)$$

Example 7.18

As a final example we consider the weakly singular second kind Volterra integral equation

$$u(x) = 2\sqrt{x} - \int_0^x \frac{1}{\sqrt{x-t}}u(t)dt. \quad (7.94)$$

Following the discussion in the previous examples we use the recurrence relation

$$\begin{aligned} u_0(x) &= 2\sqrt{x}, \\ u_1(x) &= -2\int_0^x \frac{\sqrt{t}}{\sqrt{x-t}}dt = -\pi x, \\ u_2(x) &= \int_0^x \frac{\pi t}{\sqrt{x-t}}dt = \frac{4}{3}\pi x^{3/2}, \\ u_3(x) &= -\frac{4}{3}\int_0^x \frac{\pi t^{3/2}}{\sqrt{x-t}}dt = -\frac{1}{2}\pi^2 x^2, \\ &\vdots \end{aligned} \quad (7.95)$$

Because self-canceling noise terms did not appear in the components $u_0(x)$ and $u_1(x)$, we obtained more components. Consequently, the solution $u(x)$ in a series form is given by

$$u(x) = 2\sqrt{x} - \pi x + \frac{4}{3}\pi x^{3/2} - \frac{1}{2}\pi^2 x^2 + \cdots, \quad (7.96)$$

and in a closed form by

$$u(x) = 1 - e^{\pi x}\mathrm{erfc}(\sqrt{\pi x}), \quad (7.97)$$

where erfc is the complementary error function normally used in probability topics. The definitions of the error function and the complementary error function can be found in Appendix D.

Exercises 7.4.1

Solve the following weakly singular Volterra equations, where $x \in [0, T]$.

1. $u(x) = x - \frac{9}{4}x^{\frac{4}{3}} + \int_0^x \frac{1}{(x-t)^{\frac{2}{3}}}u(t)dt$
2. $u(x) = x - \frac{9}{10}x^{\frac{5}{3}} + \int_0^x \frac{1}{(x-t)^{\frac{1}{3}}}u(t)dt$

3. $u(x) = \sin x + \frac{3}{2}(\cos x - 1)^{\frac{2}{3}} + \int_0^x \frac{1}{(\cos x - \cos t)^{\frac{1}{3}}} u(t)dt$

4. $u(x) = \cos x - \frac{2}{3}(\sin x)^{\frac{2}{3}} + \int_0^x \frac{1}{(\sin x - \sin t)^{\frac{1}{3}}} u(t)dt$

5. $u(x) = e^{-x} + \frac{3}{2}(e^{-x} - 1)^{\frac{2}{3}} + \int_0^x \frac{1}{(e^{-x} - e^{-t})^{\frac{1}{3}}} u(t)dt$

6. $u(x) = x - \frac{3}{4}x^{\frac{4}{3}} + \int_0^x \frac{1}{(x^2 - t^2)^{\frac{1}{3}}} u(t)dt$

7. $u(x) = x^3 - \frac{8}{21}x^{\frac{7}{2}} + \int_0^x \frac{1}{(x^2 - t^2)^{\frac{1}{4}}} u(t)dt$

8. $u(x) = \frac{2}{3}x^3 + \int_0^x \frac{1}{(x^4 - t^4)^{\frac{1}{4}}} u(t)dt$

9. $u(x) = x^5 - \frac{16}{63}x^{\frac{21}{4}} + \int_0^x \frac{1}{(x^3 - t^3)^{\frac{1}{4}}} u(t)dt$

10. $u(x) = \frac{7}{10}x^5 + \int_0^x \frac{1}{(x^3 - t^3)^{\frac{1}{3}}} u(t)dt$

11. $u(x) = e^x - \frac{7}{6}(e^x - 1)^{\frac{6}{7}} + \int_0^x \frac{1}{(e^x - e^t)^{\frac{1}{7}}} u(t)dt$

12. $u(x) = e^x(e^x + 1) - \frac{3}{10}(e^x - 1)^{\frac{2}{3}}(3e^x + 7) + \int_0^x \frac{1}{(e^x - e^t)^{\frac{1}{3}}} u(t)dt$

13. $u(x) = \sin 2x + \frac{3}{5}(3\cos x + 2)(\cos x - 1)^{\frac{2}{3}} + \int_0^x \frac{1}{(\cos x - \cos t)^{\frac{1}{3}}} u(t)dt$

14. $u(x) = \sinh x - \frac{3}{2}(\cosh x - 1)^{\frac{2}{3}} + \int_0^x \frac{1}{(\cosh x - \cosh t)^{\frac{1}{3}}} u(t)dt$

15. $u(x) = 1 - \frac{\pi}{2} + \int_0^x \frac{1}{(x^2 - t^2)^{\frac{1}{2}}} u(t)dt$

16. $u(x) = \left(\pi - \frac{\pi^2}{2}\right) + \left(1 - \frac{\pi}{4}\right)x^2 + \int_0^x \frac{1}{(x^2 - t^2)^{\frac{1}{2}}} u(t)dt$

7.4.2 The Successive Approximations Method

The *successive approximations method*, or *Picard iteration method* provides a scheme that can be used for solving initial value problem or integral equation. The method was used before in Chapter 3. This method solves any problem by finding successive approximations to the solution by starting with an initial guess, called the zeroth approximation. As will be seen, the zeroth approximation is any selective real-valued function that will be used in a recurrence relation to determine the other approximations.

In this section, we will study the weakly singular Volterra integral equation of the second kind

$$u(x) = f(x) + \int_0^x \frac{1}{\sqrt{x-t}} u(t)dt. \tag{7.98}$$

The successive approximations method introduces the recurrence relation

$$u_{n+1}(x) = f(x) + \int_0^x \frac{1}{\sqrt{x-t}} u_n(t)dt, n \geqslant 0, \tag{7.99}$$

where the zeroth approximation $u_0(x)$ can be any selective real valued function. We always start with an initial guess for $u_0(x)$, mostly we select $0, 1, x$ for $u_0(x)$, and by using (7.99), several successive approximations $u_k, k \geqslant 1$ will be determined as

$$\begin{aligned} u_1(x) &= f(x) + \int_0^x \frac{1}{\sqrt{x-t}} u_0(t)dt, \\ u_2(x) &= f(x) + \int_0^x \frac{1}{\sqrt{x-t}} u_1(t)dt, \\ u_3(x) &= f(x) + \int_0^x \frac{1}{\sqrt{x-t}} u_2(t)dt, \end{aligned} \tag{7.100}$$

and so on. As stated before in Chapter 3, the solution $u(x)$ is given by

$$u(x) = \lim_{n\to\infty} u_{n+1}(x). \tag{7.101}$$

The successive approximations method will be explained by studying the following illustrative questions.

Example 7.19

Solve the weakly singular integral equation by using the successive approximations method

$$u(x) = \sqrt{x} + \frac{\pi}{2}x - \int_0^x \frac{1}{\sqrt{x-t}} u(t)dt. \tag{7.102}$$

For the zeroth approximation $u_0(x)$, we can select

$$u_0(x) = 0. \tag{7.103}$$

The method of successive approximations admits the use of the iteration formula

$$u_{n+1}(x) = \sqrt{x} + \frac{\pi}{2}x - \int_0^x \frac{1}{\sqrt{x-t}} u_n(t)dt, n \geqslant 0. \tag{7.104}$$

Substituting (7.103) into (7.104) we obtain

$$\begin{aligned} u_1(x) &= \sqrt{x} + \frac{\pi}{2}x, \\ u_2(x) &= \left(\sqrt{x} + \frac{\pi}{2}x\right) - \left(\frac{\pi}{2}x + \frac{2}{3}x^{\frac{3}{2}}\right), \\ &\vdots \\ u_{n+1}(x) &= \left(\sqrt{x} + \frac{\pi}{2}x\right) - \left(\frac{\pi}{2}x + \frac{2}{3}x^{\frac{3}{2}}\right) + \left(\frac{2}{3}x^{\frac{3}{2}} + \frac{\pi^2}{4}x^2\right) - \cdots . \end{aligned} \tag{7.105}$$

The solution $u(x)$ is given by

$$u(x) = \lim_{n\to\infty} u_{n+1}(x) = \sqrt{x}. \tag{7.106}$$

Example 7.20

Solve the weakly singular integral equation by using the successive approximations method

$$u(x) = x - \frac{4}{3}x^{\frac{3}{2}} + \int_0^x \frac{1}{\sqrt{x-t}}u(t)dt. \tag{7.107}$$

For the zeroth approximation $u_0(x)$, we can select

$$u_0(x) = 0. \tag{7.108}$$

The method of successive approximations admits the use of the iteration formula

$$u_{n+1}(x) = x - \frac{4}{3}x^{\frac{3}{2}} + \int_0^x \frac{1}{\sqrt{x-t}}u_n(t)dt, n \geqslant 0. \tag{7.109}$$

Proceeding as before we set

$$\begin{aligned}
u_1(x) &= x - \frac{4}{3}x^{\frac{3}{2}},\\
u_2(x) &= \left(x - \frac{4}{3}x^{\frac{3}{2}}\right) + \left(\frac{4}{3}x^{\frac{3}{2}} - \frac{\pi}{2}x^2\right),\\
u_3(x) &= \left(x - \frac{4}{3}x^{\frac{3}{2}}\right) + \left(\frac{4}{3}x^{\frac{3}{2}} - \frac{\pi}{2}x^2\right) + \left(\frac{\pi}{2}x^2 - \frac{8\pi}{15}x^{\frac{5}{2}}\right),\\
&\ \vdots\\
u_{n+1}(x) &= \left(x - \frac{4}{3}x^{\frac{3}{2}}\right) + \left(\frac{4}{3}x^{\frac{3}{2}} - \frac{\pi}{2}x^2\right) + \left(\frac{\pi}{2}x^2 - \frac{8\pi}{15}x^{\frac{5}{2}}\right) + \cdots.
\end{aligned} \tag{7.110}$$

The solution $u(x)$ is given by

$$u(x) = \lim_{n\to\infty} u_{n+1}(x) = x. \tag{7.111}$$

Example 7.21

Solve the weakly singular integral equation

$$u(x) = 1 + x - 2\sqrt{x} - \frac{4}{3}x^{\frac{3}{2}} + \int_0^x \frac{1}{\sqrt{x-t}}u(t)dt. \tag{7.112}$$

For the zeroth approximation $u_0(x)$, we can select $u_0(x) = 0$.

Proceeding as before, we use of the iteration formula

$$u_{n+1}(x) = 1 + x - 2\sqrt{x} - \frac{4}{3}x^{\frac{3}{2}} + \int_0^x \frac{1}{\sqrt{x-t}}u_n(t)dt, n \geqslant 0. \tag{7.113}$$

This in turn gives

$$\begin{aligned}
u_1(x) &= (1+x) - \left(2\sqrt{x} + \frac{4}{3}x^{\frac{3}{2}}\right), \\
u_2(x) &= (1+x) - \left(2\sqrt{x} + \frac{4}{3}x^{\frac{3}{2}}\right) + \left(2\sqrt{x} + \frac{4}{3}x^{\frac{3}{2}}\right) - \left(\pi x + \frac{\pi}{2}x^2\right), \\
&\vdots \\
u_{n+1}(x) &= (1+x) - \left(\pi x + \frac{\pi}{2}x^2\right) + \left(\pi x + \frac{\pi}{2}x^2\right) + \cdots .
\end{aligned} \tag{7.114}$$

The solution $u(x)$ is given by

$$u(x) = \lim_{n\to\infty} u_{n+1}(x) = 1 + x. \tag{7.115}$$

Example 7.22

We finally consider the weakly singular Volterra integral equation

$$u(x) = 2\sqrt{x} - \int_0^x \frac{1}{\sqrt{x-t}} u(t)dt. \tag{7.116}$$

We select $u_0(x) = 0$, hence we use the recurrence relation

$$\begin{aligned}
u_1(x) &= 2\sqrt{x}, \\
u_2(x) &= 2\sqrt{x} - \int_0^x \frac{u_0(t)}{\sqrt{x-t}} dt = 2\sqrt{x} - \pi x, \\
&\vdots \\
u_{n+1}(x) &= 2\sqrt{x} - \pi x + \frac{4}{3}x^{\frac{3}{2}} - \frac{1}{2}\pi^2 x^2 + \cdots .
\end{aligned} \tag{7.117}$$

This in turn gives

$$u(x) = 1 - e^{\pi x}\operatorname{erfc}(\sqrt{\pi x}). \tag{7.118}$$

Exercises 7.4.2

Use the successive approximations method to solve the following weakly singular Volterra equations:

1. $u(x) = \sqrt{x} - 2\pi x + 4\int_0^x \frac{1}{\sqrt{x-t}} u(t)dt$
2. $u(x) = x^{\frac{5}{2}} - \frac{5}{16}\pi x^3 + \int_0^x \frac{1}{\sqrt{x-t}} u(t)dt$
3. $u(x) = 3 - 6\sqrt{x} + \int_0^x \frac{1}{\sqrt{x-t}} u(t)dt$
4. $u(x) = 1 - \sqrt{x} - \frac{1}{2}\pi x + \int_0^x \frac{1}{\sqrt{x-t}} u(t)dt$
5. $u(x) = x^{\frac{3}{2}} - \frac{3}{8}\pi x^2 + \int_0^x \frac{1}{\sqrt{x-t}} u(t)dt$
6. $u(x) = x^3 - \frac{32}{33}x^{\frac{7}{2}} + \int_0^x \frac{1}{\sqrt{x-t}} u(t)dt$

7. $u(x) = 1 + x^2 - 2\sqrt{x} - \frac{16}{15}x^{\frac{5}{2}} + \int_0^x \frac{1}{\sqrt{x-t}} u(t)dt$

8. $u(x) = 1 - x - 2\sqrt{x} + \frac{4}{3}x^{\frac{3}{2}} + \int_0^x \frac{1}{\sqrt{x-t}} u(t)dt$

9. $u(x) = x^2 - \frac{16}{15}x^{\frac{5}{2}} + \int_0^x \frac{1}{\sqrt{x-t}} u(t)dt$

10. $u(x) = \left(1 - \frac{\pi}{2}\right) x + \sqrt{x}(1 - \frac{4}{3}x) + \int_0^x \frac{1}{\sqrt{x-t}} u(t)dt$

11. $u(x) = \pi(1 - 2\sqrt{x}) + \int_0^x \frac{1}{\sqrt{x-t}} u(t)dt$

12. $u(x) = x(1 + x) - \frac{4}{3}x^{\frac{3}{2}} \left(1 + \frac{4}{5}x\right) + \int_0^x \frac{1}{\sqrt{x-t}} u(t)dt$

7.4.3 The Laplace Transform Method

The Laplace transform method is used before, and the properties were presented in details in Chapters 1, 3, 5, and in the previous section. Recall that the Laplace transform of the convolution product $(f_1 * f_2)(x)$ is given by

$$\mathcal{L}\{(f_1 * f_2)(x)\} = F_1(s)F_2(s). \tag{7.119}$$

We will focus our study on the generalized weakly singular Volterra equation of the form

$$u(x) = f(x) + \int_0^x \frac{1}{(x-t)^\alpha} u(t)dt, \quad 0 < \alpha < 1. \tag{7.120}$$

Taking Laplace transforms of both sides of (7.120) leads to

$$\mathcal{L}(u(x)) = \mathcal{L}(f(x)) + \mathcal{L}(x^{-\alpha})\mathcal{L}(u(x)), \tag{7.121}$$

or equivalently

$$U(s) = F(s) + \frac{\Gamma(1-\alpha)}{s^{1-\alpha}} U(s), \tag{7.122}$$

that gives

$$U(s) = \frac{s^{1-\alpha}F(s)}{s^{1-\alpha} - \Gamma(1-\alpha)}, \tag{7.123}$$

where Γ is the gamma function, $U(s) = \mathcal{L}\{u(x)\}$, and $F(s) = \mathcal{L}\{f(x)\}$. In Appendix D, the definition of the gamma function and some of the relations related to it are given.

Applying $\mathcal{L}^{-1}$ to both sides of (7.123) gives the formula

$$u(x) = \mathcal{L}^{-1}\left(\frac{s^{1-\alpha}F(s)}{s^{1-\alpha} - \Gamma(1-\alpha)}\right), \tag{7.124}$$

that will be used for the determination of the solution $u(x)$. Notice that the formula (7.124) will be used for solving the weakly singular integral equation, and it is not necessary to use Laplace transform method for each problem.

The following weakly singular Volterra equations will be examined by using the Laplace transform method.

Example 7.23

Use the Laplace transform method to solve the weakly singular Volterra integral equation

$$u(x) = 1 - 2\sqrt{x} + \int_0^x \frac{1}{\sqrt{x-t}} u(t)dt. \tag{7.125}$$

Notice that $f(x) = 1 - 2\sqrt{x}$ and $\alpha = \frac{1}{2}$. This means that

$$F(s) = \frac{1}{s} - 2\frac{\Gamma(\frac{3}{2})}{s^{\frac{3}{2}}} = \frac{1}{s} - \frac{\sqrt{\pi}}{s^{\frac{3}{2}}}, \tag{7.126}$$

where we used $\Gamma(\frac{3}{2}) = \frac{1}{2}\Gamma(\frac{1}{2}) = \frac{\sqrt{\pi}}{2}$. Substituting (7.126) into the formula given in (7.124) gives

$$u(x) = \mathcal{L}^{-1}\left(\frac{s^{\frac{1}{2}}\left(\frac{1}{s} - \frac{\sqrt{\pi}}{s^{\frac{3}{2}}}\right)}{s^{\frac{1}{2}} - \sqrt{\pi}}\right) = \mathcal{L}^{-1}\left(\frac{1}{s}\right) = 1. \tag{7.127}$$

Example 7.24

Use the Laplace transform method to solve the weakly singular Volterra integral equation

$$u(x) = x - \frac{4}{3}x^{\frac{3}{2}} + \int_0^x \frac{1}{\sqrt{x-t}} u(t)dt. \tag{7.128}$$

Notice that $f(x) = x - \frac{4}{3}x^{\frac{3}{2}}$ and $\alpha = \frac{1}{2}$. This means that

$$F(s) = \frac{1}{s^2} - \frac{4}{3}\frac{\Gamma(\frac{5}{2})}{s^{\frac{5}{2}}} = \frac{1}{s^2} - \frac{\sqrt{\pi}}{s^{\frac{5}{2}}}, \tag{7.129}$$

where we used $\Gamma(\frac{5}{2}) = \frac{3}{4}\Gamma(\frac{1}{2}) = \frac{3}{4}\sqrt{\pi}$. Substituting (7.129) into (7.124) gives

$$u(x) = \mathcal{L}^{-1}\left(\frac{s^{\frac{1}{2}}\left(\frac{1}{s^2} - \frac{\sqrt{\pi}}{s^{\frac{5}{2}}}\right)}{s^{\frac{1}{2}} - \sqrt{\pi}}\right) = \mathcal{L}^{-1}\left(\frac{1}{s^2}\right) = x. \tag{7.130}$$

Example 7.25

Use the Laplace transform method to solve the weakly singular Volterra integral equation

$$u(x) = x - \frac{9}{10}x^{\frac{5}{3}} + \int_0^x \frac{1}{(x-t)^{\frac{1}{3}}} u(t)dt. \tag{7.131}$$

Notice that $f(x) = x - \frac{9}{10}x^{\frac{5}{3}}$ and $\alpha = \frac{1}{3}$. This means that

$$F(s) = \frac{1}{s^2} - \frac{9}{10}\frac{\Gamma(\frac{8}{3})}{s^{\frac{8}{3}}} = \frac{1}{s^2} - \frac{\Gamma(\frac{2}{3})}{s^{\frac{8}{3}}}, \tag{7.132}$$

where we used $\Gamma(\frac{8}{3}) = \frac{10}{9}\Gamma(\frac{2}{3})$. Substituting (7.132) into (7.124) gives

$$u(x) = \mathcal{L}^{-1}\left(\frac{s^{\frac{2}{3}}\left(\frac{1}{s^2} - \frac{\Gamma(\frac{2}{3})}{s^{\frac{8}{3}}}\right)}{s^{\frac{2}{3}} - \Gamma(\frac{2}{3})}\right) = \mathcal{L}^{-1}\left(\frac{1}{s^2}\right) = x. \tag{7.133}$$

Exercises 7.4.3

In Exercises 1–8, use the Laplace transform method to solve the following weakly singular Volterra equations:

1. $u(x) = \sqrt{x} - 2\pi x + 4\int_0^x \frac{1}{\sqrt{x-t}}u(t)dt$
2. $u(x) = x^2 - \frac{16}{15}x^{\frac{5}{2}} + \int_0^x \frac{1}{\sqrt{x-t}}u(t)dt$
3. $u(x) = 4 - 8\sqrt{x} + \int_0^x \frac{1}{\sqrt{x-t}}u(t)dt$
4. $u(x) = x^3 - \frac{32}{33}x^{\frac{7}{2}} + \int_0^x \frac{1}{\sqrt{x-t}}u(t)dt$
5. $u(x) = 1 - \sqrt{x} - \frac{1}{2}\pi x + \int_0^x \frac{1}{\sqrt{x-t}}u(t)dt$
6. $u(x) = 1 + x - 2\sqrt{x} - \frac{4}{3}x^{\frac{3}{2}} + \int_0^x \frac{1}{\sqrt{x-t}}u(t)dt$
7. $u(x) = x^{\frac{3}{2}} - \frac{3}{8}\pi x^2 + \int_0^x \frac{1}{\sqrt{x-t}}u(t)dt$
8. $u(x) = (1 - \frac{\pi}{2})x + \sqrt{x}(1 - \frac{4}{3}x) + \int_0^x \frac{1}{\sqrt{x-t}}u(t)dt$

In Exercises 9–16, use the Laplace transform method to solve the generalized weakly singular Volterra equations:

9. $u(x) = x - \frac{9}{4}x^{\frac{4}{3}} + \int_0^x \frac{1}{(x-t)^{\frac{2}{3}}}u(t)dt$
10. $u(x) = x^2 - \frac{27}{40}x^{\frac{8}{3}} + \int_0^x \frac{1}{(x-t)^{\frac{1}{3}}}u(t)dt$
11. $u(x) = x - \frac{16}{21}x^{\frac{7}{4}} + \int_0^x \frac{1}{(x-t)^{\frac{1}{4}}}u(t)dt$
12. $u(x) = x^3 - \frac{243}{440}x^{\frac{11}{3}} + \int_0^x \frac{1}{(x-t)^{\frac{1}{3}}}u(t)dt$
13. $u(x) = 1 + x - \frac{9}{10}x^{\frac{5}{3}} - \frac{3}{2}x^{\frac{2}{3}} + \int_0^x \frac{1}{(x-t)^{\frac{1}{4}}}u(t)dt$
14. $u(x) = x - \frac{36}{55}x^{\frac{11}{6}} + \int_0^x \frac{1}{(x-t)^{\frac{1}{6}}}u(t)dt$
15. $u(x) = x - \frac{25}{14}x^{\frac{7}{5}} + \int_0^x \frac{1}{(x-t)^{\frac{3}{5}}}u(t)dt$
16. $u(x) = x^2 - \frac{128}{45}x^{\frac{9}{4}} + \int_0^x \frac{1}{(x-t)^{\frac{3}{4}}}u(t)dt$

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Chapter 8
Volterra-Fredholm Integral Equations

8.1 Introduction

The Volterra-Fredholm integral equations [1–2] arise from parabolic boundary value problems, from the mathematical modelling of the spatio-temporal development of an epidemic, and from various physical and biological models. The Volterra-Fredholm integral equations appear in the literature in two forms, namely

$$u(x) = f(x) + \lambda_1 \int_0^x K_1(x,t)u(t)dt + \lambda_2 \int_a^b K_2(x,t)u(t)dt, \tag{8.1}$$

and the mixed form

$$u(x) = f(x) + \lambda \int_0^x \int_a^b K(r,t)u(t)dtdr, \tag{8.2}$$

where $f(x)$ and $K(x,t)$ are analytic functions. It is interesting to note that (8.1) contains disjoint Volterra and Fredholm integrals, whereas (8.2) contains mixed Volterra and Fredholm integrals. Moreover, the unknown functions $u(x)$ appears inside and outside the integral signs. This is a characteristic feature of a second kind integral equation. If the unknown functions appear only inside the integral signs, the resulting equations are of first kind. Examples of the two types of the Volterra-Fredholm integral equations of the second kind are given by

$$u(x) = 6x + 3x^2 + 2 - \int_0^x xu(t)dt - \int_0^1 tu(t)dt, \tag{8.3}$$

and

$$u(x) = x + \frac{17}{2}x^2 - \int_0^x \int_0^1 (r-t)u(t)drdt. \tag{8.4}$$

However, the Volterra-Fredholm integro-differential equations that appear in scientific applications will be presented in the next chapter.

8.2 The Volterra-Fredholm Integral Equations

In this section, we will study some of the reliable methods that will be used for analytic treatment of the Volterra-Fredholm integral equations [1–2] of the form

$$u(x) = f(x) + \int_0^x K_1(x,t)u(t)dt + \int_a^b K_2(x,t)u(t)dt. \tag{8.5}$$

This type of equations will be handled by using the Taylor series method and the Adomian decomposition method combined with the noise terms phenomenon or the modified decomposition method. Other approaches exist in the literature but will not be presented in this text.

8.2.1 The Series Solution Method

The series solution method was examined before in this text . A real function $u(x)$ is called analytic if it has derivatives of all orders such that the generic form of Taylor series at $x = 0$ can be written as

$$u(x) = \sum_{n=0}^{\infty} a_n x^n. \tag{8.6}$$

In this section we will apply the series solution method [3–4], that stems mainly from the Taylor series for analytic functions, for solving Volterra-Fredholm integral equations. We will assume that the solution $u(x)$ of the Volterra-Fredholm integral equation

$$u(x) = f(x) + \int_0^x K_1(x,t)u(t)dt + \int_a^b K_2(x,t)u(t)dt, \tag{8.7}$$

is analytic, and therefore possesses a Taylor series of the form given in (8.6), where the coefficients a_n will be determined recurrently. In this method, we usually substitute the Taylor series (8.6) into both sides of (8.7) to obtain

$$\begin{aligned}\sum_{k=0}^{\infty} a_k x^k = T(f(x)) &+ \int_0^x K_1(x,t)\left(\sum_{k=0}^{\infty} a_k t^k\right) dt \\ &+ \int_a^b K_2(x,t)\left(\sum_{k=0}^{\infty} a_k t^k\right) dt,\end{aligned} \tag{8.8}$$

or for simplicity we use

$$\begin{aligned}a_0 + a_1 x + a_2 x^2 + \cdots = T(f(x)) &+ \int_0^x K_1(x,t)\left(a_0 + a_1 t + a_2 t^2 + \cdots\right) dt \\ &+ \int_a^b K_2(x,t)\left(a_0 + a_1 t + a_2 t^2 + \cdots\right) dt,\end{aligned} \tag{8.9}$$

where $T(f(x))$ is the Taylor series for $f(x)$. The Volterra-Fredholm integral equation (8.7) will be converted to a regular integral in (8.8) or (8.9) where instead of integrating the unknown function $u(x)$, terms of the form $t^n, n \geqslant 0$ will be integrated. Notice that because we are seeking series solution, then if $f(x)$ includes elementary functions such as trigonometric functions, exponential functions, etc., then Taylor expansions for functions involved in $f(x)$ should be used.

We first integrate the right side of the integrals in (8.8) or (8.9), and collect the coefficients of like powers of x. We next equate the coefficients of like powers of x into both sides of the resulting equation to determine a recurrence relation in $a_j, j \geqslant 0$. Solving the recurrence relation will lead to a complete determination of the coefficients $a_j, j \geqslant 0$. Having determined the coefficients $a_j, j \geqslant 0$, the series solution follows immediately upon substituting the derived coefficients into (8.6). The exact solution may be obtained if such an exact solution exists. If an exact solution is not obtainable, then the obtained series can be used for numerical purposes. In this case, the more terms we evaluate, the higher accuracy level we achieve.

Example 8.1

Solve the Volterra-Fredholm integral equation by using the series solution method

$$u(x) = -5 - x + 12x^2 - x^3 - x^4 + \int_0^x (x-t)u(t)dt + \int_0^1 (x+t)u(t)dt. \quad (8.10)$$

Substituting $u(x)$ by the series

$$u(x) = \sum_{n=0}^{\infty} a_n x^n, \quad (8.11)$$

into both sides of Eq. (8.10) leads to

$$\begin{aligned} \sum_{n=0}^{\infty} a_n x^n = & -5 - x + 12x^2 - x^3 - x^4 + \int_0^x \left((x-t) \sum_{n=0}^{\infty} a_n t^n \right) dt \\ & + \int_0^1 \left((x+t) \sum_{n=0}^{\infty} a_n t^n \right) dt. \end{aligned} \quad (8.12)$$

Evaluating the integrals at the right side, using few terms from both sides, and collecting the coefficients of like powers of x, we find

$$\begin{aligned} & a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots \\ = & -5 + \frac{1}{2}a_0 + \frac{1}{3}a_1 + \frac{1}{4}a_2 + \frac{1}{5}a_3 + \frac{1}{6}a_4 \\ & + \left(-1 + a_0 + \frac{1}{2}a_1 + \frac{1}{3}a_2 + \frac{1}{4}a_3 + \frac{1}{5}a_4 \right) x \\ & + \left(12 + \frac{1}{2}a_0 \right) x^2 + \left(-1 + \frac{1}{6}a_1 \right) x^3 \end{aligned}$$

$$+\left(-1+\frac{1}{12}a_2\right)x^4+\cdots. \tag{8.13}$$

Equating the coefficients of like powers of x in both sides of (8.13), and solving the resulting system of equations, we obtain

$$\begin{aligned} &a_0=0,\ a_1=6,\\ &a_2=12,\ a_3=a_4=a_5=\cdots=0. \end{aligned} \tag{8.14}$$

The exact solution is therefore given by

$$u(x)=6x+12x^2. \tag{8.15}$$

Example 8.2

Use the series solution method to solve the Volterra-Fredholm integral equation

$$u(x)=2-x-x^2-6x^3+x^5+\int_0^x tu(t)dt+\int_{-1}^1 (x+t)u(t)dt. \tag{8.16}$$

Substituting $u(x)$ by the series

$$u(x)=\sum_{n=0}^{\infty} a_n x^n, \tag{8.17}$$

into both sides of Eq. (8.16) leads to

$$\begin{aligned} \sum_{n=0}^{\infty} a_n x^n &= 2-x-x^2-6x^3+x^5+\int_0^x \left(t\sum_{n=0}^{\infty} a_n t^n\right)dt\\ &+\int_{-1}^1 \left((x+t)\sum_{n=0}^{\infty} a_n t^n\right)dt. \end{aligned} \tag{8.18}$$

Evaluating the integrals at the right side, using few terms from both sides, and collecting the coefficients of like powers of x, we find

$$\begin{aligned} &a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+\cdots\\ &=2+\frac{2}{3}a_1+\frac{2}{5}a_3\\ &+\left(-1+2a_0+\frac{2}{3}a_2+\frac{2}{5}a_4\right)x\\ &+\left(-1+\frac{1}{2}a_0\right)x^2+\left(-6+\frac{1}{3}a_1\right)x^3\\ &+\frac{1}{4}a_2x^4+\left(1+\frac{1}{5}a_3\right)x^5+\cdots. \end{aligned} \tag{8.19}$$

Equating the coefficients of like powers of x in both sides of (8.19), and solving the resulting system of equations, we obtain

$$\begin{aligned} &a_0=2, \quad a_1=3,\\ &a_2=0, \quad a_3=-5,\\ &a_4=a_5=\cdots=0. \end{aligned} \tag{8.20}$$

The exact solution is therefore given by

$$u(x) = 2 + 3x - 5x^3. \tag{8.21}$$

Example 8.3

Solve the Volterra-Fredholm integral equation by using the series solution method

$$u(x) = e^x - 1 - x + \int_0^x u(t)dt + \int_0^1 xu(t)dt. \tag{8.22}$$

Using the Taylor polynomial for e^x up to x^7, substituting $u(x)$ by the Taylor polynomial

$$u(x) = \sum_{n=0}^{7} a_n x^n, \tag{8.23}$$

and proceeding as before leads to

$$\begin{aligned} \sum_{n=0}^{7} a_n x^n = & \left(2a_0 + \sum_{n=1}^{7} \frac{1}{n+1} a_n\right) x \\ & + \frac{1+a_1}{2!}x^2 + \frac{1+2!a_2}{3!}x^3 + \frac{1+3!a_3}{4!}x^4 \\ & + \frac{1+4!a_4}{5!}x^5 + \frac{1+5!a_5}{6!}x^6 + \frac{1+6!a_6}{7!}x^7 + \cdots . \end{aligned} \tag{8.24}$$

Equating the coefficients of like powers of x in both sides of (8.19), and proceeding as before, we obtain

$$\begin{aligned} & a_0 = 0, \quad a_1 = 1, \quad a_2 = 1, \quad a_3 = \frac{1}{2!}, \\ & a_4 = \frac{1}{3!}, \quad a_5 = \frac{1}{4!}, \quad a_6 = \frac{1}{5!}. \end{aligned} \tag{8.25}$$

The exact solution is given by

$$u(x) = xe^x. \tag{8.26}$$

Example 8.4

Solve the Volterra-Fredholm integral equation by using the series solution method

$$u(x) = 1 - \int_0^x (x-t)u(t)dt + \int_0^\pi u(t)dt. \tag{8.27}$$

Substituting $u(x)$ by the Taylor polynomial

$$u(x) = \sum_{n=0}^{8} a_n x^n, \tag{8.28}$$

and proceeding as before, we obtain

$$\begin{aligned} & a_0 = 1, \quad a_1 = a_3 = a_5 = a_7 = 0, \\ & a_2 = -\frac{1}{21}, \quad a_4 = \frac{1}{4!}, \end{aligned} \tag{8.29}$$

$$a_6 = -\frac{1}{6!}, \quad a_8 = \frac{1}{8!}.$$

Consequently, the exact solution is given by

$$u(x) = \cos x. \tag{8.30}$$

Exercises 8.2.1

Use the series solution method to solve the following Volterra-Fredholm integral equations

1. $u(x) = -4 - 3x + 11x^2 - x^3 - x^4 + \int_0^x (x-t)u(t)dt + \int_0^1 (x+t)u(t)dt$
2. $u(x) = -1 - 3x^2 - 2x^3 + \int_0^x (x-t)u(t)dt + \int_0^1 (x+t)u(t)dt$
3. $u(x) = 4 - x - 4x^2 - x^3 + \int_0^x (x-t+1)u(t)dt + \int_0^1 (x+t-1)u(t)dt$
4. $u(x) = -8 - 6x + 11x^2 - x^4 + \int_0^x (x-t)u(t)dt + \int_0^1 (x+t+1)u(t)dt$
5. $u(x) = -6 - 2x + 19x^3 - x^5 + \int_0^x (x-t)u(t)dt + \int_0^1 (x+t)u(t)dt$
6. $u(x) = -x^2 - x^4 - x^6 + \int_0^x (3xt + 4x^2t^2)u(t)dt + \int_0^1 (4xt^2 + 3x^2t)u(t)dt$
7. $u(x) = e^x - xe^x + x - 1 + \int_0^x xu(t)dt + \int_0^1 tu(t)dt$
8. $u(x) = -2x + x\cos x + \int_0^x tu(t)dt + \int_0^\pi xu(t)dt$
9. $u(x) = 2x - \int_0^x (x-t)u(t)dt - \int_0^{\frac{\pi}{2}} xu(t)dt$
10. $u(x) = 2 + x - 2\cos x - \int_0^x (x-t)u(t)dt - \int_0^{\frac{\pi}{2}} xu(t)dt$
11. $u(x) = -x\ln(1+x) + \int_0^x u(t)dt + \int_0^{e-1} xu(t)dt$
12. $u(x) = -2 - 2x + 2e^x + \int_0^x (x-t)u(t)dt + \int_0^1 xu(t)dt$

8.2.2 The Adomian Decomposition Method

The Adomian decomposition method [5–8] (ADM) was introduced thoroughly in this text for handling independently Volterra and Fredholm integral equations. The method consists of decomposing the unknown function $u(x)$ of any equation into a sum of an infinite number of components defined by the decomposition series

$$u(x) = \sum_{n=0}^{\infty} u_n(x), \tag{8.31}$$

where the components $u_n(x), n \geqslant 0$ are to be determined in a recursive manner. To establish the recurrence relation, we substitute the decomposition series into the Volterra-Fredholm integral equation (8.5) to obtain

$$\sum_{n=0}^{\infty} u_n(x) = f(x) + \int_0^x K_1(x,t)\left(\sum_{n=0}^{\infty} u_n(t)\right) dt + \int_a^b K_2(x,t)\left(\sum_{n=0}^{\infty} u_n(t)\right) dt. \tag{8.32}$$

The zeroth component $u_0(x)$ is identified by all terms that are not included under the integral sign. Consequently, we set the recurrence relation

$$\begin{aligned} u_0(x) &= f(x), \\ u_{n+1}(x) &= \int_0^x K_1(x,t)u_n(t)dt + \int_a^b K_2(x,t)u_n(t)dt, \quad n \geqslant 0. \end{aligned} \tag{8.33}$$

Having determined the components $u_0(x), u_1(x), u_2(x), \ldots$, the solution in a series form is readily obtained upon using (8.31). The series solution converges to the exact solution if such a solution exists. We point here that the noise terms phenomenon and the modified decomposition method will be employed in this section. This will be illustrated by using the following examples.

Example 8.5

Use the Adomian decomposition method to solve the following Volterra-Fredholm integral equation

$$u(x) = e^x + 1 + x + \int_0^x (x-t)u(t)dt - \int_0^1 e^{x-t}u(t)dt. \tag{8.34}$$

Using the decomposition series (8.31), and using the recurrence relation (8.33) we obtain

$$\begin{aligned} u_0(x) &= e^x + 1 + x, \\ u_1(x) &= \int_0^x (x-t)u_0(t)dt + \int_0^1 e^{x-t}u_0(t)dt \\ &= -x - 1 + \frac{1}{2}x^2 + \cdots, \end{aligned} \tag{8.35}$$

and so on. We notice the appearance of the noise terms ± 1 and $\pm x$ between the components $u_0(x)$ and $u_1(x)$. By canceling these noise terms from $u_0(x)$, the non-canceled term of $u_0(x)$ gives the exact solution

$$u(x) = e^x, \tag{8.36}$$

that satisfies the given equation.

It is to be noted that the modified decomposition method can be applied here. Using the modified recurrence relation

$$\begin{aligned} u_0(x) &= e^x, \\ u_1(x) &= 1 + x + \int_0^x (x-t)u_0(t)dt - \int_0^1 e^{x-t}u_0(t)dt = 0. \end{aligned} \tag{8.37}$$

The exact solution $u(x) = e^x$ follows immediately.

Example 8.6

Use the modified Adomian decomposition method to solve the following Volterra-Fredholm integral equation

$$u(x) = x^2 - \frac{1}{12}x^4 - \frac{1}{4} - \frac{1}{3}x + \int_0^x (x-t)u(t)dt + \int_0^1 (x+t)u(t)dt. \quad (8.38)$$

Using the modified decomposition method gives the recurrence relation

$$\begin{aligned} u_0(x) &= x^2 - \frac{1}{12}x^4, \\ u_1(x) &= -\frac{1}{4} - \frac{1}{3}x + \int_0^x (x-t)u(t)dt + \int_0^1 (x+t)u(t)dt \\ &= \frac{1}{12}x^4 - \frac{1}{360}x^6 - \frac{1}{60}x - \frac{1}{72}. \end{aligned} \quad (8.39)$$

and so on. We notice the appearance of the noise terms $\pm\frac{1}{12}x^4$ between the components $u_0(x)$ and $u_1(x)$. By canceling the noise term from the $u_0(x)$, the non-canceled term gives the exact solution

$$u(x) = x^2, \quad (8.40)$$

that satisfies the given equation.

Example 8.7

Use the modified Adomian decomposition method to solve the following Volterra-Fredholm integral equation

$$u(x) = \cos x - \sin x - 2 + \int_0^x u(t)dt + \int_0^\pi (x-t)u(t)dt. \quad (8.41)$$

Using the modified decomposition method gives the recurrence relation

$$\begin{aligned} u_0(x) &= \cos x, \\ u_1(x) &= -\sin x - 2 + \int_0^x u(t)dt + \int_0^\pi (x-t)u(t)dt = 0. \end{aligned} \quad (8.42)$$

Consequently, the exact solution is given by

$$u(x) = \cos x. \quad (8.43)$$

Example 8.8

Use the modified Adomian decomposition method to solve the following Volterra-Fredholm integral equation

$$u(x) = 3x + 4x^2 - x^3 - x^4 - 2 + \int_0^x tu(t)dt + \int_{-1}^1 tu(t)dt. \quad (8.44)$$

Using the modified decomposition method gives the recurrence relation

$$\begin{aligned} u_0(x) &= 3x + 4x^2 - x^3, \\ u_1(x) &= -x^4 - 2 + \int_0^x tu(t)dt + \int_{-1}^1 tu(t)dt \end{aligned} \quad (8.45)$$

$$= -\frac{2}{5} - \frac{1}{5}x^5 + x^3.$$

Canceling the noise term $-x^3$ from $u_0(x)$ gives the exact solution

$$u(x) = 3x + 4x^2. \tag{8.46}$$

Exercises 8.2.2

Use the modified decomposition method to solve the following Volterra-Fredholm integral equations

1. $u(x) = x - \frac{1}{3}x^3 + \int_0^x tu(t)dt + \int_{-1}^1 t^2 u(t)dt$

2. $u(x) = \sec^2 x - \tan x - 1 + \int_0^x u(t)dt + \int_0^{\frac{\pi}{4}} u(t)dt$

3. $u(x) = \sin x - \cos x - \int_0^x u(t)dt + \int_0^{\frac{\pi}{2}} u(t)dt$

4. $u(x) = x^3 - \frac{1}{20}x^5 - \frac{1}{4}x - \frac{1}{5} + \int_0^x (x-t)u(t)dt + \int_0^1 (x+t)u(t)dt$

5. $u(x) = x^3 - \frac{9}{20}x^5 - \frac{1}{4}x + \frac{1}{5} + \int_0^x (x+t)u(t)dt + \int_0^1 (x-t)u(t)dt$

6. $u(x) = \sin x + \cos x - x + \int_0^x u(t)dt + \int_0^{\frac{\pi}{2}} (x-t)u(t)dt$

7. $u(x) = \cos x - \sin x - 2 + \int_0^x u(t)dt + \int_0^{\pi} (x-t)u(t)dt$

8. $u(x) = \cos x + (1-x)\sin x - 2 + \int_0^x (x-1)u(t)dt + \int_0^{\pi} (x-t)u(t)dt$

9. $u(x) = \tan x - \ln(\cos x) + \int_0^x u(t)dt + \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} xu(t)dt$

10. $u(x) = \cot x - \ln(\sin x) - \frac{1}{2}x\ln(2) + \int_{\frac{\pi}{2}}^x u(t)dt + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} xu(t)dt$

11. $u(x) = x + \sin x + \cos x - \frac{1}{2}x^2 - \frac{1}{2} + \int_{\frac{\pi}{2}}^x u(t)dt + \int_0^{\pi} \frac{1}{4}u(t)dt$

12. $u(x) = x + \cos x - \sin x - \frac{1}{2}x^2 + 1 + \int_{\frac{\pi}{2}}^x u(t)dt + \int_0^{\pi} \frac{1}{4}u(t)dt$

8.3 The Mixed Volterra-Fredholm Integral Equations

In a parallel manner to our analysis in the previous section, we will study the mixed Volterra-Fredholm integral equations [2], given by

$$u(x) = f(x) + \int_0^x \int_a^b K(r,t)u(t)dtdr, \tag{8.47}$$

where $f(x)$ and $K(x,t)$ are analytic functions. It is interesting to note that (8.47) contains mixed Volterra and Fredholm integral equations. The Fredholm integral is the interior integral, whereas the Volterra integral is the exterior one. Moreover, the unknown function $u(x)$ appears inside and outside the integral signs. This is a characteristic feature of a second kind integral equation. This type of equations will be handled by using the series solution method and the Adomian decomposition method, including the use of the noise terms phenomenon or the modified decomposition method. Other approaches exist in the literature but will not be presented in this text.

8.3.1 The Series Solution Method

The series solution method was examined before in this chapter. A real function $u(x)$ is called analytic if it has derivatives of all orders such that the generic form of Taylor series at $x = 0$ can be written as

$$u(x) = \sum_{n=0}^{\infty} a_n x^n. \tag{8.48}$$

We will assume that the solution $u(x)$ of the mixed Volterra-Fredholm integral equation

$$u(x) = f(x) + \int_0^x \int_a^b K(r,t)u(t)dtdr, \tag{8.49}$$

is analytic, and therefore possesses a Taylor series of the form given in (8.48), where the coefficients a_n will be determined in an algebraic manner.

In this method, we usually substitute the Taylor series (8.48) into both sides of (8.49) to obtain

$$\sum_{k=0}^{\infty} a_k x^k = T(f(x)) + \int_0^x \int_a^b K(r,t) \left(\sum_{k=0}^{\infty} a_k t^k \right) dtdr, \tag{8.50}$$

or for simplicity we use

$$a_0 + a_1 x + a_2 x^2 + \cdots = T(f(x)) + \int_0^x \int_a^b K(r,t) \left(a_0 + a_1 t + a_2 t^2 + \cdots \right) dtdr, \tag{8.51}$$

where $T(f(x))$ is the Taylor series for $f(x)$. The mixed Volterra-Fredholm integral equation (8.49) will be converted to a traditional integral in (8.50) or (8.51) where instead of integrating the unknown function $u(x)$, terms of the form t^n, $n \geqslant 0$ will be integrated. Notice that because we are seeking series solution, then if $f(x)$ includes elementary functions such as trigonometric functions, exponential functions, etc., then Taylor expansions for functions involved in $f(x)$ should be used.

We first integrate the inner integral, and then we integrate the outer integral in (8.50) or (8.51), and collect the coefficients of like powers of x. We next equate the coefficients of like powers of x into both sides of the resulting equation to determine a system of algebraic equations in $a_j, j \geqslant 0$. Solving this system of equations will lead to a complete determination of the coefficients $a_j, j \geqslant 0$. Having determined the coefficients $a_j, j \geqslant 0$, the series solution follows immediately upon substituting the derived coefficients into (8.48). The exact solution may be obtained if such an exact solution exists, otherwise a truncated series is used for numerical approximations.

Example 8.9

Solve the mixed Volterra-Fredholm integral equation by using the Taylor series solution method

$$u(x) = 11x + \frac{17}{2}x^2 + \int_0^x \int_0^1 (r-t)u(t)dtdr. \tag{8.52}$$

Substituting $u(x)$ by the series

$$u(x) = \sum_{n=0}^{\infty} a_n x^n, \tag{8.53}$$

into both sides of Eq. (8.52) leads to

$$\sum_{n=0}^{\infty} a_n x^n = 11x + \frac{17}{2}x^2 + \int_0^x \int_0^1 \left((r-t) \sum_{n=0}^{\infty} a_n t^n \right) dtdr. \tag{8.54}$$

Evaluating the integrals at the right side, using few terms from both sides, collecting the coefficients of like powers of x, and equating the coefficients of like powers of x in both sides, we obtain

$$\begin{aligned} &a_0 = 0, \quad a_1 = 6, \\ &a_2 = 12, \quad a_3 = a_4 = a_5 = \cdots = 0. \end{aligned} \tag{8.55}$$

The exact solution is therefore given by

$$u(x) = 6x + 12x^2. \tag{8.56}$$

Example 8.10

Solve the mixed Volterra-Fredholm integral equation by using the series solution method

$$u(x) = 2 + 4x - \frac{9}{8}x^2 - 5x^3 + \int_0^x \int_0^1 (r-t)u(t)dtdr. \tag{8.57}$$

Substituting $u(x)$ by the series

$$u(x) = \sum_{n=0}^{\infty} a_n x^n, \tag{8.58}$$

into both sides of Eq. (8.57) leads to

$$\sum_{n=0}^{\infty} a_n x^n = 2 + 4x - \frac{9}{8}x^2 - 5x^3 + \int_0^x \int_0^1 \left((r-t) \sum_{n=0}^{\infty} a_n t^n \right) dt\, dr. \tag{8.59}$$

Evaluating the integrals at the right side, using few terms from both sides, collecting the coefficients of like powers of x, and equating the coefficients of like powers of x in both sides, we obtain

$$\begin{array}{ll} a_0 = 2, & a_1 = 3, \\ a_2 = 0, & a_3 = -5, \\ a_4 = a_5 = \cdots = 0. & \end{array} \tag{8.60}$$

The exact solution is therefore given by

$$u(x) = 2 + 3x - 5x^3. \tag{8.61}$$

Example 8.11

Solve the mixed Volterra-Fredholm integral equation by using the series solution method

$$u(x) = xe^x - \frac{1}{2}x^2 + \int_0^x \int_0^1 ru(t)dtdr. \tag{8.62}$$

Using the Taylor polynomial for xe^x up to x^9, substituting $u(x)$ by the Taylor polynomial

$$u(x) = \sum_{n=0}^{9} a_n x^n, \tag{8.63}$$

and proceeding as before, we obtain

$$\begin{array}{lll} a_0 = 0, a_1 = 1, & a_2 = 1, & \\ a_3 = \frac{1}{2!}, & a_4 = \frac{1}{3!}, & a_5 = \frac{1}{4!}, \\ \vdots & & \end{array} \tag{8.64}$$

The exact solution is therefore given by

$$u(x) = xe^x. \tag{8.65}$$

Example 8.12

Solve the mixed Volterra-Fredholm integral equation by using the series solution method

$$u(x) = \cos x + \sin x - x^2 + \frac{\pi}{2}x + \int_0^x \int_0^{\frac{\pi}{2}} (r-t)u(t)dt\,dr. \tag{8.66}$$

Proceeding as before we find

$$\begin{array}{lll} a_0 = 1, & a_1 = 1, & a_2 = -\frac{1}{2!}, \\ a_3 = -\frac{1}{3!}, & a_4 = \frac{1}{4!}, & a_5 = \frac{1}{5!}, \\ a_6 = -\frac{1}{6!}, & a_7 = -\frac{1}{7!}, & a_8 = \frac{1}{8!} \end{array} \tag{8.67}$$

and so on. The series solution is given by

$$u(x) = \left(1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \cdots\right) + \left(x - \frac{1}{3!} + \frac{1}{5!}x^5 + \cdots\right), \tag{8.68}$$

that converge to the exact solution

$$u(x) = \cos x + \sin x. \tag{8.69}$$

Exercises 8.3.1

Use the series solution method to solve the following mixed Volterra-Fredholm integral equations

1. $u(x) = 2 + 12x + \frac{15}{2}x^2 + \int_0^x \int_0^1 (r-t)u(t)dtdr$
2. $u(x) = 6 + 19x - 6x^2 + \int_0^x \int_0^1 (r-t)u(t)dtdr$
3. $u(x) = 2 + 10x - 2x^2 + \int_0^x \int_{-1}^1 (r-t)u(t)dtdr$
4. $u(x) = 2 + 6x^2 + \int_0^x \int_{-1}^1 (r-t)u(t)dtdr$
5. $u(x) = 7x + \frac{31}{3}x^2 + \int_0^x \int_0^1 (r^2-t^2)u(t)dtdr$
6. $u(x) = 4 + 14x - 2x^2 + \int_0^x \int_{-1}^1 (r^3-t^3)u(t)dtdr$
7. $u(x) = 6 + 9x + 2x^2 - 2x^3 + \int_0^x \int_{-1}^1 (rt^2+r^2t)u(t)dtdr$
8. $u(x) = \sin x + x - \frac{1}{2}x^2 + \int_0^x \int_0^{\frac{\pi}{2}} (r-t)u(t)dtdr$
9. $u(x) = \ln(1+x) - x + \int_0^x \int_0^{e-1} u(t)dtdr$
10. $u(x) = \ln(1+x) - \frac{1}{2}x^2 + \int_0^x \int_0^{e-1} ru(t)dtdr$
11. $u(x) = \cos x + \frac{2}{\pi^2}\int_0^x \int_0^{\pi} u(t)dtdr$
12. $u(x) = \tan x - x + \frac{1}{\ln 2}\int_0^x \int_0^{\pi} u(t)dtdr$

8.3.2 The Adomian Decomposition Method

The Adomian decomposition method (ADM) was introduced thoroughly in this text for handling independently Volterra integral equations, Fredholm integral equations, and the Volterra-Fredholm integral equations. The method consists of decomposing the unknown function $u(x)$ of any equation into a sum of an infinite number of components defined by the decomposition series

$$u(x) = \sum_{n=0}^{\infty} u_n(x). \tag{8.70}$$

To establish the recurrence relation, we substitute decomposition series into the mixed Volterra-Fredholm integral equation (8.49) to obtain

$$\sum_{n=0}^{\infty} u_n(x) = f(x) + \int_0^x \int_a^b K(r,t) \left(\sum_{n=0}^{\infty} u_n(t) \right) dt\, dr. \tag{8.71}$$

Consequently, we set the recurrence relation

$$\begin{aligned} u_0(x) &= f(x), \\ u_{n+1}(x) &= \int_0^x \int_a^b K(r,t) u_n(t) dt dr, \quad n \geqslant 0. \end{aligned} \tag{8.72}$$

The solution in a series form is readily obtained upon using (8.70). The series may converge to the exact solution if such a solution exists. We point out here that the noise terms phenomenon and the modified decomposition method will be employed heavily.

Example 8.13

Solve the mixed Volterra-Fredholm integral equation by using the Adomian decomposition method

$$u(x) = xe^x - \frac{1}{2}x^2 + \int_0^x \int_0^1 ru(t) dt dr. \tag{8.73}$$

Using the Adomian decomposition method we set the recurrence relation

$$\begin{aligned} u_0(x) &= xe^x - \frac{1}{2}x^2, \\ u_{k+1}(x) &= \int_0^x \int_0^1 ru_k(t) dt dr, \quad k \geqslant 0. \end{aligned} \tag{8.74}$$

This in turn gives

$$\begin{aligned} u_0(x) &= xe^x - \frac{1}{2}x^2, \\ u_1(x) &= \int_0^x \int_0^1 ru_0(t) dt dr = \frac{5}{12}x^2, \\ u_2(x) &= \int_0^x \int_0^1 ru_1(t) dt dr = \frac{5}{72}x^2, \\ u_3(x) &= \int_0^x \int_0^1 ru_2(t) dt dr = \frac{5}{432}x^2, \end{aligned} \tag{8.75}$$

and so on. Using (8.70) gives the series solution

$$u(x) = xe^x - \frac{1}{2}x^2 + \frac{5}{12}x^2 \left(1 + \frac{1}{6} + \frac{1}{36} + \cdots \right), \tag{8.76}$$

that converges to the exact solution

$$u(x) = xe^x, \tag{8.77}$$

obtained upon finding the sum of the infinite geometric series.

However, we can obtain the exact solution by using the modified decomposition method. This can be achieved by setting

$$\begin{aligned} u_0(x) &= xe^x, \\ u_1(x) &= -\frac{1}{2}x^2 + \int_0^x \int_0^1 ru_0(t)dtdr = 0. \end{aligned} \tag{8.78}$$

Accordingly the other components $u_j(x), j \geqslant 2 = 0$. This gives the exact solution by

$$u(x) = xe^x. \tag{8.79}$$

Example 8.14

Solve the mixed Volterra-Fredholm integral equation by using the Adomian decomposition method

$$u(x) = \sin x - \frac{1}{2}x^2 + \int_0^x \int_0^{\frac{\pi}{2}} ru(t)dtdr. \tag{8.80}$$

Using the Adomian decomposition method we set the recurrence relation

$$\begin{aligned} u_0(x) &= \sin x - \frac{1}{2}x^2, \\ u_{k+1}(x) &= \int_0^x \int_0^{\frac{\pi}{2}} ru_k(t)dtdr, \quad k \geqslant 0. \end{aligned} \tag{8.81}$$

This in turn gives

$$\begin{aligned} u_0(x) &= \sin x - \frac{1}{2}x^2, \\ u_1(x) &= \int_0^x \int_0^{\frac{\pi}{2}} ru_0(t)dtdr = \sin x + \frac{1}{2}x^2 - \frac{\pi^3}{96}x^2. \end{aligned} \tag{8.82}$$

By canceling the noise term $-\frac{1}{2}x^2$ from u_0, the exact solution is given by

$$u(x) = \sin x. \tag{8.83}$$

This result satisfies the given equation.

However, we can obtain the exact solution by using the modified decomposition method. This can be achieved by setting

$$\begin{aligned} u_0(x) &= \sin x, \\ u_1(x) &= -\frac{1}{2}x^2 + \int_0^x \int_0^{\frac{\pi}{2}} ru_0(t)dtdr = 0. \end{aligned} \tag{8.84}$$

Accordingly the other components $u_j(x), j \geqslant 2 = 0$. This gives the same solution obtained above.

Example 8.15

Solve the mixed Volterra-Fredholm integral equation by using the Adomian decomposition method

$$u(x) = \cos x - 2x + \int_0^x \int_0^{\pi} (r - t)u(t)dtdr. \tag{8.85}$$

Using the modified decomposition method we set

$$\begin{aligned} u_0(x) &= \cos x, \\ u_1(x) &= -2x + \int_0^x \int_0^{\pi} (r-t)u_0(t)dtdr = 0. \end{aligned} \tag{8.86}$$

Accordingly the other components $u_j(x), j \geqslant 2 = 0$. This gives the exact solution by

$$u(x) = \cos x. \tag{8.87}$$

Example 8.16

Solve the mixed Volterra-Fredholm integral equation by using the Adomian decomposition method

$$u(x) = \ln(1+x) - \frac{1}{2}x^2 + \int_0^x \int_0^{e-1} ru(t)dtdr. \tag{8.88}$$

Using the modified decomposition method we set

$$\begin{aligned} u_0(x) &= \ln(1+x), \\ u_1(x) &= -\frac{1}{2}x^2 + \int_0^x \int_0^{e-1} ru_0(t)dtdr = 0. \end{aligned} \tag{8.89}$$

Accordingly the other components $u_j(x), j \geqslant 2 = 0$. This gives the exact solution by

$$u(x) = \ln(1+x). \tag{8.90}$$

Exercises 8.3.2

In Exercises 1–6, use the Adomian decomposition method to solve the mixed Volterra-Fredholm integral equations

1. $u(x) = \frac{7}{5}x + \int_0^x \int_{-1}^1 (r^3 - t^3)u(t)dtdr$

2. $u(x) = x + \frac{2}{9}x^3 + \int_0^x \int_{-1}^1 (rt^2 - r^2t)u(t)dtdr$

3. $u(x) = \frac{4}{5}x^2 + \int_0^x \int_{-1}^1 (rt^2 - r^2t)u(t)dtdr$

4. $u(x) = \frac{2}{15}x^3 + \int_0^x \int_{-1}^1 (rt^2 + r^2t)u(t)dtdr$

5. $u(x) = x^3 - \frac{1}{5}x^2 + \int_0^x \int_{-1}^1 (1 + rt)u(t)dtdr$

6. $u(x) = x^2 - \frac{2}{3}x + \int_0^x \int_{-1}^1 (1 + rt)u(t)dtdr$

In Exercises 7–12, use the modified decomposition method to solve the following Volterra-Fredholm integral equations

7. $u(x) = x\sin x - \frac{1}{2}x^2 + (\pi - 2)x + \int_0^x \int_0^{\frac{\pi}{2}} (r - t)u(t)dtdr$

8. $u(x) = x\cos x - \frac{\pi^2}{4}x^2 + \int_0^x \int_0^\pi ru(t)dtdr$

9. $u(x) = \ln(1+x) - x + \int_0^x \int_0^{e-1} u(t)dtdr$

10. $u(x) = \tan x - x^2 \ln 2 + \int_0^x \int_0^{\frac{\pi}{3}} ru(t)dtdr$

11. $u(x) = \sec^2 x - (\frac{\pi}{4} - \ln 2)x - \frac{1}{2}x^2 + \int_0^x \int_0^{\frac{\pi}{4}} (r+t)u(t)dtdr$

12. $u(x) = 2xe^x - (e^2+1)x^2 + \int_0^x \int_0^2 ru(t)dtdr$

8.4 The Mixed Volterra-Fredholm Integral Equations in Two Variables

In this section we present a reliable strategy for solving the linear mixed Volterra-Fredholm integral equation of the form

$$u(x,t) = f(x,t) + \int_0^t \int_\Omega F(x,t,r,s)u(r,s)drds,\ (x,t) \in \Omega \times [0,T], \quad (8.91)$$

where $u(x,t)$ is an unknown function, and $f(x,t)$ and $F(x,t,r,s)$ are analytic functions on $D = \Omega \times [0,T]$, where Ω is a closed subset of $\mathbb{R}^n, n = 1,2,3$.

The mixed Volterra-Fredholm integral equation (8.91) in two variables arises from parabolic boundary value problems, the mathematical modeling of the spatio-temporal development of an epidemic, and in various physical and biological models. In the literature, some methods, such as the projection method, time collocation method, the trapezoidal Nystrom method, and Adomian method were used to handle this problem. It was found that these techniques encountered difficulties in terms of computational work used, and therefore, approximate solutions were obtained for numerical purposes. In particular, it was found that a complicated term $f(x,t)$ can cause difficult integrations and proliferation of terms in Adomian recursive scheme.

To overcome the tedious work of the existing strategies, the modified decomposition method will form a useful basis for studying the mixed Volterra-Fredholm integral equation (8.91). The size of the computational work can be dramatically reduced by using the modified decomposition method. Moreover, the convergence can be accelerated by combining the modified method with the noise terms phenomenon. The noise terms were defined as the identical terms with opposite signs that may appear in the first two components of the series solution of $u(x)$. The modified decomposition method and the noise terms phenomenon were presented in details in Chapters 3, 4, and 5.

8.4.1 The Modified Decomposition Method

In what follows we give a brief review of the modified decomposition method. For many cases, the function $f(x,t)$ can be set as the sum of two partial functions, namely $f_1(x,t)$ and $f_2(x,t)$. In other words, we can set

$$f(x,t) = f_1(x,t) + f_2(x,t). \tag{8.92}$$

To minimize the size of calculations, we identify the zeroth component $u_0(x,t)$ by one part of $f(x,t)$, namely $f_1(x,t)$ or $f_2(x,t)$. The other part of $f(x,t)$ can be added to the component $u_1(x,t)$ among other terms. In other words, the modified decomposition method introduces the modified recurrence relation

$$\begin{aligned} u_0(x,t) &= f_1(x,t), \\ u_1(x,t) &= f_2(x,t) + \int_0^t \int_\Omega F(x,t,r,s)u_0(r,s)drds, \\ u_{k+1}(x,t) &= \int_0^t \int_\Omega F(x,t,r,s)u_k(r,s)drds, \quad k \geqslant 1. \end{aligned} \tag{8.93}$$

If noise terms appear between $u_0(x,t)$ and $u_1(x,t)$, then by canceling these terms from $u_0(x,t)$, the remaining non-canceled terms of $u_0(x,t)$ may give the exact solution. This can be satisfied by direct substitution. In what follows, we study some illustrative examples.

Example 8.17

Use the modified decomposition method to solve the mixed Volterra-Fredholm integral equation

$$u(x,t) = f(x,t) + \int_0^t \int_\Omega F(x,t,r,s)u(r,s)drds,\ (x,t) \in \Omega \times [0,T], \tag{8.94}$$

where

$$\begin{aligned} f(x,t) &= e^{-t}\left[\cos(x) + t\cos(x) + \frac{1}{2}t\cos(x-1)\sin 1\right], \\ F(x,t,r,s) &= -\cos(x-r)e^{s-t}, \end{aligned} \tag{8.95}$$

with $\Omega = [0,1]$.

Substituting the decomposition series

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t), \tag{8.96}$$

into (8.94) gives

$$\begin{aligned} \sum_{n=0}^{\infty} u_n(x,t) = e^{-t}&\left[\cos(x) + t\cos(x) + \frac{1}{2}t\cos(x-1)\sin 1\right] \\ &- \int_0^t \int_\Omega \cos(x-r)e^{s-t}\left(\sum_{n=0}^{\infty} u_n(r,s)\right) drds. \end{aligned} \tag{8.97}$$

As stated before we decompose $f(x,t)$ into two parts as follows:

$$\begin{aligned} f_0(x,t) &= e^{-t}[\cos(x) + t\cos(x)], \\ f_1(x,t) &= \frac{1}{2}te^{-t}\cos(x-1)\sin 1. \end{aligned} \tag{8.98}$$

The modified decomposition technique admits the use of the recursive relation

$$\begin{aligned} u_0(x,t) &= e^{-t}[\cos(x) + t\cos(x)], \\ u_1(x,t) &= \frac{1}{2}te^{-t}\cos(x-1)\sin 1 \\ &\quad - \int_0^t \int_\Omega \cos(x-r)e^{s-t}u_0(r,s)drds \\ u_{k+1}(x,t) &= -\int_0^t \int_\Omega \cos(x-r)e^{s-t}u_k(r,s)drds, \quad k \geqslant 1. \end{aligned} \tag{8.99}$$

This gives

$$\begin{aligned} u_0(x,t) &= e^{-t}[\cos(x) + t\cos(x)], \\ u_1(x,t) &= -te^{-t}\cos(x) + \frac{1}{2}te^{-t}\left(\frac{1}{2}\sin(x+4) + \sin^2 2\sin x\right) + \cdots \end{aligned} \tag{8.100}$$

The self-canceling noise terms $te^{-t}\cos(x)$ and $-te^{-t}\cos(x)$ appear between the components $u_0(x,t)$ and $u_1(x,t)$ respectively. By canceling this term from $u_0(x,t)$, and showing that the remaining non-canceled term of $u_0(x,t)$ satisfies the equation (8.94), the exact solution is given by

$$u(x,t) = e^{-t}\cos(x). \tag{8.101}$$

Example 8.18

Use the modified decomposition method to solve the mixed Volterra-Fredholm integral equation

$$u(x,t) = f(x,t) + \int_0^t \int_\Omega F(x,t,r,s)u(r,s)\,dr\,ds,\ (x,t) \in \Omega \times [0,T], \tag{8.102}$$

where

$$\begin{aligned} f(x,t) &= \cos x \sin t + \frac{\pi}{4}\cos x(t\cos t - \sin t), \\ F(x,t,r,s) &= \cos(x-r)\sin(t-s), \end{aligned} \tag{8.103}$$

with $\Omega = [0,\pi]$.

Substituting the decomposition series

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t), \tag{8.104}$$

into (8.102) gives

$$\begin{aligned} \sum_{n=0}^{\infty} u_n(x,t) &= \cos x \sin t + \frac{\pi}{4}\cos x(t\cos t - \sin t) \\ &\quad + \int_0^t \int_\Omega \cos(x-r)\sin(t-s)\left(\sum_{n=0}^{\infty} u_n(r,s)\right)drds. \end{aligned} \tag{8.105}$$

As stated before we decompose $f(x,t)$ into two parts as follows:

$$\begin{aligned} f_0(x,t) &= \cos x \sin t + \frac{\pi}{4} t \cos x \cos t, \\ f_1(x,t) &= -\frac{\pi}{4} \cos x \sin t. \end{aligned} \tag{8.106}$$

The modified decomposition technique admits the use of the recursive relation

$$\begin{aligned} u_0(x,t) &= \cos x \sin t + \frac{\pi}{4} t \cos x \cos t, \\ u_1(x,t) &= -\frac{\pi}{4} \cos x \sin t + \int_0^t \int_\Omega \cos(x-r) \sin(t-s) u_0(r,s) drds \\ u_{k+1}(x,t) &= \int_0^t \int_\Omega \cos(x-r) \sin(t-s) u_k(r,s) drds, \quad k \geqslant 1. \end{aligned} \tag{8.107}$$

This gives

$$\begin{aligned} u_0(x,t) &= \cos x \sin t + \frac{\pi}{4} t \cos x \cos t, \\ u_1(x,t) &= -\frac{\pi}{4} t \cos x \cos t + \cdots. \end{aligned} \tag{8.108}$$

The self-canceling noise terms $\frac{\pi}{4} t \cos x \cos t$ and $-\frac{\pi}{4} t \cos x \cos t$ appear between the components $u_0(x,t)$ and $u_1(x,t)$ respectively. By canceling this term from $u_0(x,t)$, and showing that the remaining non-canceled term of $u_0(x,t)$ satisfies the equation (8.102), the exact solution

$$u(x,t) = \cos x \sin t, \tag{8.109}$$

follows immediately.

Example 8.19

Use the modified decomposition method to solve the mixed Volterra-Fredholm integral equation

$$u(x,t) = f(x,t) + \int_0^t \int_\Omega F(x,t,r,s) u(r,s)\, dr\, ds, \ (x,t) \in \Omega \times [0,T], \tag{8.110}$$

where

$$\begin{aligned} f(x,t) &= \sin(x+t) - 4\cos t - \pi \sin t - 2t \sin t + 4, \\ F(x,t,r,s) &= r + s, \end{aligned} \tag{8.111}$$

with $\Omega = [0, \pi]$.

Substituting the decomposition series

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t), \tag{8.112}$$

into (8.110) gives

$$\begin{aligned} \sum_{n=0}^{\infty} u_n(x,t) &= \sin(x+t) - 4\cos t - \pi \sin t - 2t \sin t + 4 \\ &\quad + \int_0^t \int_\Omega (r+s) \left(\sum_{n=0}^{\infty} u_n(r,s) \right) drds. \end{aligned} \tag{8.113}$$

As stated before we decompose $f(x,t)$ into two parts as follows:

$$\begin{aligned} f_0(x,t) &= \sin(x+t) - 4\cos t - \pi \sin t, \\ f_1(x,t) &= -2t\sin t + 4. \end{aligned} \tag{8.114}$$

The modified decomposition technique admits the use of the recursive relation

$$\begin{aligned} u_0(x,t) &= \sin(x+t) - 4\cos t - \pi\sin t, \\ u_1(x,t) &= -2t\sin t + 4 + \int_0^t \int_\Omega (r+s)\left(u_0(r,s)\right) drds \\ u_{k+1}(x,t) &= \int_0^t \int_\Omega (r+s) u_k(r,s) drds, \quad k \geqslant 1. \end{aligned} \tag{8.115}$$

This gives

$$\begin{aligned} u_0(x,t) &= \sin(x+t) - 4\cos t - \pi\sin t, \\ u_1(x,t) &= 4\cos t + \pi \sin t + \cdots . \end{aligned} \tag{8.116}$$

By canceling the two noise terms from $u_0(x,t)$, and showing that the remaining non-canceled term of $u_0(x,t)$ satisfies the equation (8.110), the exact solution

$$u(x,t) = \sin(x+t), \tag{8.117}$$

is readily obtained.

Example 8.20

Use the modified decomposition method to solve the mixed Volterra-Fredholm integral equation

$$u(x,t) = f(x,t) + \int_0^t \int_\Omega F(x,t,r,s)u(r,s)drds,\ (x,t) \in \Omega \times [0,T], \tag{8.118}$$

where

$$\begin{aligned} f(x,t) &= x^2 - \frac{3+4x}{12}t + t^2 - \frac{1+2x}{6}t^3 + \frac{1}{6}t^4, \\ F(x,t,r,s) &= x + t + r - 2s, \end{aligned} \tag{8.119}$$

with $\Omega = [0,1]$.

Proceeding as before we set

$$\begin{aligned} \sum_{n=0}^{\infty} u_n(x,t) &= x^2 - \frac{3+4x}{12}t + t^2 - \frac{1+2x}{6}t^3 + \frac{1}{6}t^4 \\ &+ \int_0^t \int_\Omega (x+t+r-2s)\left(\sum_{n=0}^{\infty} u_n(r,s)\right) drds. \end{aligned} \tag{8.120}$$

Decomposing $f(x,t)$ into two parts as follows:

$$f_0(x,t) = x^2 - \frac{3+4x}{12}t + t^2, \quad f_1(x,t) = -\frac{1+2x}{6}t^3 + \frac{1}{6}t^4, \tag{8.121}$$

and using recursive relation

$$\begin{aligned} u_0(x,t) &= x^2 - \frac{3+4x}{12}t + t^2, \\ u_1(x,t) &= v + \int_0^t \int_\Omega (x+t+r-2s)u_0(r,s)drds \\ u_{k+1}(x,t) &= \int_0^t \int_\Omega (x+t+r-2s)u_k(r,s)drds, \quad k \geqslant 1. \end{aligned} \tag{8.122}$$

This gives

$$\begin{aligned} u_0(x,t) &= x^2 - \frac{3+4x}{12}t + t^2, \\ u_1(x,t) &= \frac{3+4x}{12}t + \cdots . \end{aligned} \tag{8.123}$$

By canceling the noise term from $u_0(x,t)$, and showing that the remaining non-canceled term of $u_0(x,t)$ satisfies the equation (8.118), the exact solution

$$u(x,t) = x^2 + t^2, \tag{8.124}$$

is readily obtained.

Exercises 8.4.1

Use the modified decomposition method to solve the mixed Volterra-Fredholm integral equations in two variables

$$u(x,t) = f(x,t) + \int_0^t \int_\Omega F(x,t,r,s)u(r,s)drds, \ (x,t) \in \Omega \times [0,T], \tag{8.125}$$

where $f(x,t), F(x,t,r,s)$, and Ω are given by:

1. $f = xt - \frac{1}{12}xt^3 + \frac{1}{18}t^3, \ F = (x-r)(t-s), \ \Omega = [0,1]$
2. $f = x + t - 3t^2 - 4t^3, \ F = 12s, \ \Omega = [0,1]$
3. $f = x^2 - t^2 + \cos t\left(\frac{\pi^2}{4} - t^2\right) + 2t\sin t - \frac{\pi^2}{4}, \ F = \cos r \sin s, \ \Omega = \left[0, \frac{\pi}{2}\right]$
4. $f = x^2 - t^2 - \sin t\left(\frac{\pi^2}{4} - t^2\right) + 2t\cos t, \ F = \cos r \cos s, \ \Omega = \left[0, \frac{\pi}{2}\right]$
5. $f = 1 + x^3 + \frac{1}{5}t^2 + t^3 - \left(t^2 + \frac{1}{10}t^5\right)x, \ F = (x-r)(t-s), \ \Omega = [-1,1]$
6. $f = x^2t - xt^2 + \frac{4}{9}t^3 + t^2, \ F = r - s, \ \Omega = [-1,1]$
7. $f = e^{-t}\cos x + \frac{\pi}{2}te^{-t}\cos x, \ F = -\cos(x-r)e^{s-t}, \ \Omega = [0,\pi]$
8. $f = \cos(x-t) + 4\sin t + \cos t(\pi - 2t) - \pi, \ F = r - s, \ \Omega = [0,\pi]$
9. $f = xe^t - \frac{5}{6}e^t + \frac{1}{2}te^t + \frac{5}{6}, \ F = r - s, \ \Omega = [0,1]$
10. $f = \cos(x+t) + 2\pi\sin t - 2\pi t\cos t, \ F = rs, \ \Omega = [-\pi,\pi]$
11. $f = \cos x\cos t + 2\cos t + 2t\sin t - 2, \ F = rs, \ \Omega = [0,\pi]$

12. $f = e^{-t}\sin x + \frac{\pi}{2} t e^{-t}\sin x,\ F = -\cos(x-r)e^{s-t},\ \Omega = [0,\pi]$

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Chapter 9
Volterra-Fredholm Integro-Differential Equations

9.1 Introduction

The Volterra-Fredholm integro-differential equations [1–4] appear in two types, namely:

$$u^{(k)}(x) = f(x) + \lambda_1 \int_a^x K_1(x,t)u(t)dt + \lambda_2 \int_a^b K_2(x,t)u(t)dt, \qquad (9.1)$$

and the mixed form

$$u^{(k)}(x) = f(x) + \lambda \int_0^x \int_a^b K(r,t)u(t)dtdr, \qquad (9.2)$$

where $u^{(k)}(x) = \frac{d^k u(x)}{dx^k}$. The first type contains disjoint integrals and the second type contains mixed integrals such that the Fredholm integral is the interior one, and Volterra is the exterior integral.

9.2 The Volterra-Fredholm Integro-Differential Equation

In this section we will first study the Volterra-Fredholm integro-differential equation (9.1). The Adomian decomposition method can be used after integrating both sides of any equation k times and using the initial conditions. This type of equations will be handled by using the Taylor series solution method and the variational iteration method only.

9.2.1 The Series Solution Method

The series solution method [2,5] was examined before in this chapter and in Chapters 3, 4, and 5. The generic form of Taylor series at $x = 0$ can be written as

$$u(x) = \sum_{n=0}^{\infty} a_n x^n. \tag{9.3}$$

We will assume that the solution $u(x)$ of the Volterra-Fredholm integro-differential equation

$$u^{(k)}(x) = f(x) + \int_0^x K_1(x,t)u(t)dt + \int_a^b K_2(x,t)u(t)dt, \tag{9.4}$$

is analytic, and therefore possesses a Taylor series of the form given in (9.3), where the coefficients a_n will be determined algebraically.

In this method, we usually substitute the Taylor series (9.3) into both sides of (9.4) to obtain

$$\begin{aligned}\left(\sum_{n=0}^{\infty} a_n x^n\right)^{(k)} &= T(f(x)) + \int_0^x K_1(x,t)\left(\sum_{n=0}^{\infty} a_n t^n\right) dt \\ &+ \int_a^b K_2(x,t)\left(\sum_{k=0}^{\infty} a_n t^n\right) dt,\end{aligned} \tag{9.5}$$

or for simplicity we use

$$\begin{aligned}(a_0 + a_1 x + a_2 x^2 + \cdots)^{(k)} = T(f(x)) \\ + \int_0^x K_1(x,t)\left(a_0 + a_1 t + a_2 t^2 + \cdots\right) dt \\ + \int_a^b K_2(x,t)\left(a_0 + a_1 t + a_2 t^2 + \cdots\right) dt,\end{aligned} \tag{9.6}$$

where $T(f(x))$ is the Taylor series for $f(x)$. The integro-differential equation (9.4) will be converted to a traditional integral, where terms of the form t^n, $n \geqslant 0$ will be integrated.

We first integrate the right side of the integrals in (9.5) or (9.6), and collect the coefficients of like powers of x. We next equate the coefficients of like powers of x into both sides of the resulting equation to determine the coefficients $a_j, j \geqslant 0$. Solving the resulting equations will lead to a complete determination of the coefficients $a_j, j \geqslant 0$. Having determined the coefficients $a_j, j \geqslant 0$, the series solution follows immediately upon substituting the derived coefficients into (9.3). The exact solution may be obtained if such an exact solution exists.

It is to be noted that the series method works effectively if the solution $u(x)$ is a polynomial. However, if $u(x)$ is any elementary function, more terms of the series are needed to obtain an approximation of high degree of accuracy. This analysis will be tested by discussing the following examples.

Example 9.1

Solve the Volterra-Fredholm integro-differential equation by using the series solution method

$$u'(x) = 11 + 17x - 2x^3 - 3x^4 + \int_0^x tu(t)dt + \int_0^1 (x-t)u(t)dt, u(0) = 0. \quad (9.7)$$

Substituting $u(x)$ by the series

$$u(x) = \sum_{n=0}^{\infty} a_n x^n, \quad (9.8)$$

into both sides of the equation (9.7), and using the initial condition $a_0 = 0$, we find

$$\left(\sum_{n=1}^{\infty} n a_n x^n\right)' = 11 + 17x - 2x^3 - 3x^4 + \int_0^x \left(t \sum_{n=0}^{\infty} a_n t^n\right) dt \quad (9.9)$$

$$+ \int_0^1 \left((x-t) \sum_{n=0}^{\infty} a_n t^n\right) dt.$$

Evaluating the integrals at the right side, using few terms from both sides, and collecting the coefficients of like powers of x, and equating the coefficients of like powers of x in both sides of the resulting equation, we obtain

$$a_0 = 0, \quad a_1 = 6, \quad a_2 = 12, \quad a_j = 0, \quad j \geqslant 3. \quad (9.10)$$

The exact solution is therefore given by

$$u(x) = 6x + 12x^2. \quad (9.11)$$

Example 9.2

Solve the Volterra-Fredholm integro-differential equation by using the series solution method

$$u'(x) = 2 - 5x - 3x^2 - 20x^3 - x^5 + \int_0^x u(t)dt + \int_{-1}^1 (1+xt)u(t)dt, u(0) = 1. \quad (9.12)$$

Substituting $u(x)$ by the series

$$u(x) = \sum_{n=0}^{\infty} a_n x^n, \quad (9.13)$$

into both sides of Eq. (9.12), and using the initial condition $a_0 = 1$, we find

$$\left(\sum_{n=1}^{\infty} n a_n x^n\right)' = 2 - 5x - 3x^2 - 20x^3 - x^5 + \int_0^x \left(\sum_{n=0}^{\infty} a_n t^n\right) dt$$
$$+ \int_{-1}^1 (1+xt) \sum_{n=0}^{\infty} a_n t^n dt. \quad (9.14)$$

Evaluating the integrals at the right side, using few terms from both sides, and collecting the coefficients of like powers of x, and equating the coefficients of like powers of x in both sides of the resulting equation, we obtain

$$a_0 = 1, \quad a_1 = 6, \quad a_2 = a_3 = 0, \quad a_4 = 5, \quad a_j = 0, \quad j \geqslant 5. \quad (9.15)$$

The exact solution is therefore given by

$$u(x) = 1 + 6x + 5x^4. \tag{9.16}$$

Example 9.3

Solve the Volterra-Fredholm integro-differential equation by using the series solution method

$$u'(x) = 2e^x - 2 + \int_0^x u(t)dt + \int_0^1 u(t)dt, u(0) = 0. \tag{9.17}$$

Substituting $u(x)$ by the Taylor polynomial

$$u(x) = \sum_{n=0}^{10} a_n x^n, \tag{9.18}$$

and proceeding as before, we obtain

$$a_0 = 0, \quad a_1 = 1, \quad a_2 = 1, \quad a_3 = \frac{1}{2!}, \quad a_4 = \frac{1}{3!}, \quad a_5 = \frac{1}{4!}. \tag{9.19}$$

The exact solution is given by

$$u(x) = xe^x. \tag{9.20}$$

Example 9.4

Solve the Volterra-Fredholm integro-differential equation by using the series solution method

$$\begin{gathered} u'''(x) = 2\sin x - x - 3\int_0^x (x-t)u(t)dt + \int_0^{\frac{\pi}{2}} u(t)dt, \\ u(0) = u'(0) = 1, \quad u''(0) = -1. \end{gathered} \tag{9.21}$$

Using the series assumption for $u(x)$ and proceeding as before, we obtain

$$\begin{gathered} a_0 = 1, \quad a_1 = 1, \quad a_2 = -\frac{1}{2!}, \\ a_3 = -\frac{1}{3!}, \quad a_4 = \frac{1}{4!}, \quad a_5 = \frac{1}{5!}, \end{gathered} \tag{9.22}$$

and so on. the exact solution is given by

$$u(x) = \sin x + \cos x. \tag{9.23}$$

Exercises 9.2.1

Solve the Volterra-Fredholm integro-differential equations by using the series solution method

1. $u'(x) = 6 + 4x - x^3 + \int_0^x (x-t)u(t)dt + \int_{-1}^1 (1-xt)u(t)dt, \ u(0) = 0$
2. $u'(x) = 4 + 6x + \frac{11}{2}x^2 - 2x^3 + \int_0^x (t-2)u(t)dt + \int_{-1}^1 (1-xt)u(t)dt, \ u(0) = 1$
3. $u'(x) = -4 + 6x - x^2 - x^4 + \int_0^x xu(t)dt + \int_{-1}^1 (1-xt)u(t)dt, \ u(0) = 1$

4. $u'(x) = -4 + 6x + \frac{1}{6}x^4 + \frac{3}{20}x^6 + \int_0^x (xt^2 - x^2t)u(t)dt + \int_{-1}^1 (1 - xt)u(t)dt,$

   $u(0) = 1$

5. $u''(x) = -8 - \frac{1}{2}x^2 + x^3 + \frac{3}{4}x^4 + \int_0^x tu(t)dt + \int_{-1}^1 (x - t)u(t)dt,$

   $u(0) = 1, \; u'(0) = -3$

6. $u''(x) = -500 + \frac{250}{3}x^3 + \int_0^x u(t)dt + \int_0^1 (xt^2 - x^2t)u(t)dt,$

   $u(0) = 69, \; u'(0) = -168$

7. $u''(x) = 2\cos x - 1 + \int_0^x u(t)dt + \int_0^{\frac{\pi}{2}} u(t)dt, \; u(0) = -1, \; u'(0) = 1$

8. $u''(x) = \int_0^x u(t)dt + \int_0^1 tu(t)dt, \; u(0) = 1, \; u'(0) = 1$

9. $u'''(x) = -6 - 2x - 3x^2 - 4x^3 + \int_0^x u(t)dt + \int_0^1 tu(t)dt,$

   $u(0) = 2, \; u'(0) = 6, \; u''(0) = -24$

10. $u'''(x) = x - 2x^3 + 3x^4 + \int_0^x tu(t)dt + \int_0^1 xu(t)dt,$

    $u(0) = 0, \; u'(0) = 6, \; u''(0) = -24$

11. $u'''(x) = 3 - 4\cos x + \int_0^x u(t)dt + \int_0^{\pi} u(t)dt, \; u(0) = 0, \; u'(0) = 1, \; u''(0) = 0$

12. $u'''(x) = -1 - 4\sin x + \int_0^x u(t)dt + \int_0^{\frac{\pi}{2}} u(t)dt, \; u(0) = 0, \; u'(0) = 0, \; u''(0) = 2$

9.2.2 The Variational Iteration Method

The variational iteration method was used before to handle Volterra and Fredholm integral equations. It was discussed before, that the method provides rapidly convergent successive approximations of the exact solution if such a closed form solution exists. The variational iteration method will also be used in this section to study the Volterra-Fredholm integro-differential equations.

The standard ith order Volterra-Fredholm integro-differential equation is of the form

$$u^{(i)}(x) = f(x) + \int_0^x K_1(x,t)u(t)dt + \int_a^b K_2(x,t)u(t)dt, \tag{9.24}$$

where $u^{(i)}(x) = \frac{d^i u}{dx^i}$, and $u(0), u'(0), \ldots, u^{(i-1)}(0)$ are the initial conditions. The right side contains two disjoint integrals. Concerning the kernel $K_2(x,t)$, we will discuss $K_2(x,t)$ being separable and given by

$$K_2(x,t) = g(x)h(t), \tag{9.25}$$

or may be difference kernel and given by

$$K_2(x,t) = K_2(x-t) = g(x) - h(t). \tag{9.26}$$

Consequently, the second integral at the right side of (9.24) becomes

$$\int_a^b K_2(x,t)u(t)dt = \alpha g(x), \tag{9.27}$$

and

$$\int_a^b K_2(x,t)u(t)dt = \beta g(x) - \alpha, \tag{9.28}$$

by using (9.25) and (9.26) respectively, where

$$\alpha = \int_a^b h(t)u(t)dt, \quad \beta = \int_a^b u(t)dt. \tag{9.29}$$

The correction functional for the Volterra-Fredholm integro-differential equation (9.24) is

$$\begin{aligned} u_{n+1}(x) &= u_n(x) \\ &+ \int_0^x \lambda(\xi)\left(u_n^{(i)}(\xi) - f(\xi) - \int_0^t K_1(\xi,r)\tilde{u}_n(r)dr - \alpha g(\xi)\right)d\xi, \end{aligned} \tag{9.30}$$

or

$$\begin{aligned} u_{n+1}(x) &= u_n(x) \\ &+ \int_0^x \lambda(\xi)\left(u_n^{(i)}(\xi) - f(\xi) - \int_0^t K_1(\xi,r)\tilde{u}_n(r)dr - \beta g(\xi) + \alpha\right)d\xi, \end{aligned} \tag{9.31}$$

by using (9.27) and (9.28) respectively.

The variational iteration method is used by applying two essential steps. It is required first to determine the Lagrange multiplier $\lambda(\xi)$ that can be identified optimally via integration by parts and by using a restricted variation. Having $\lambda(\xi)$ determined, an iteration formula, without restricted variation, should be used for the determination of the successive approximations $u_{n+1}(x), n \geqslant 0$ of the solution $u(x)$. The zeroth approximation u_0 can be any selective function. However, the given initial values $u(0), u'(0), \ldots$ are preferably used for selecting the zeroth approximation u_0 as will be seen later. Consequently, the solution is given by

$$u(x) = \lim_{n\to\infty} u_n(x). \tag{9.32}$$

It is worth noting to summarize the Lagrange multipliers $\lambda(\xi)$ for a variety of ODEs as formally derived in Chapter 3:

$$\begin{aligned} &u' + f(u(\xi), u'(\xi)) = 0, \lambda = -1, \\ &u'' + f(u(\xi), u'(\xi), u''(\xi)) = 0, \lambda = \xi - x, \\ &u''' + f(u(\xi), u'(\xi), u''(\xi), u'''(\xi)) = 0, \lambda = -\frac{1}{2!}(\xi - x)^2, \\ &u^{(\mathrm{iv})} + f(u(\xi), u'(\xi), u''(\xi), u'''(\xi), u^{(\mathrm{iv})}(\xi)) = 0, \lambda = \frac{1}{3!}(\xi - x)^3, \end{aligned}$$

$$u^{(n)} + f(u(\xi), u'(\xi), u''(\xi), \ldots, u^{(n)}(\xi)) = 0, \lambda = (-1)^n \frac{1}{(n-1)!}(\xi - x)^{(n-1)}. \tag{9.33}$$

The VIM will be illustrated by studying the following examples.

Example 9.5

Solve the following Volterra-Fredholm integro-differential equation by using the variational iteration method

$$u'(x) = 1 + \int_0^x (x-t)u(t)dt + \int_0^1 xtu(t), \quad u(0) = 1. \tag{9.34}$$

The correction functional for this equation is given by

$$u_{n+1}(x) = u_n(x) - \int_0^x \left(u_n'(t) - 1 - \int_0^t (t-r)u_n(r)\,dr - \alpha t \right) dt, \tag{9.35}$$

where we used $\lambda = -1$ for first-order integro-differential equation, and

$$\alpha = \int_0^1 tu(t)dt. \tag{9.36}$$

We can use the initial condition to select $u_0(x) = u(0) = 1$. Using this selection into the correction functional gives the following successive approximations

$$\begin{aligned}
u_0(x) &= 1, \\
u_1(x) &= u_0(x) - \int_0^x \left(u_0'(t) - 1 - \int_0^t (t-r)u_0(r)dr - \alpha t \right) dt \\
&= 1 + x + \frac{1}{2}\alpha x^2 + \frac{1}{3!}x^3, \\
u_2(x) &= u_1(x) - \int_0^x \left(u_1'(t) - 1 - \int_0^t (t-r)u_1(r)dr - \alpha t \right) dt \\
&= 1 + x + \frac{1}{2}\alpha x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}\alpha x^5 + \frac{1}{6!}x^6, \\
u_3(x) &= u_2(x) - \int_0^x \left(u_2'(t) - 1 - \int_0^t (t-r)u_2(r)dr - \alpha t \right) dt \\
&= 1 + x + \frac{1}{2}\alpha x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}\alpha x^5 + \frac{1}{6!}x^6 \\
&\quad + \frac{1}{7!}x^7 + \frac{1}{8!}\alpha x^8 + \frac{1}{9!}x^9, \\
u_4(x) &= u_3(x) - \int_0^x \left(u_3'(t) - 1 - \int_0^t (t-r)u_3(r)dr - \alpha t \right) dt \\
&= 1 + x + \frac{1}{2}\alpha x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}\alpha x^5 + \frac{1}{6!}x^6 \\
&\quad + \frac{1}{7!}x^7 + \frac{1}{8!}\alpha x^8 + \frac{1}{9!}x^9 + \frac{1}{10!}x^{10} + \frac{1}{11!}\alpha x^{11} + \frac{1}{12!}x^{12},
\end{aligned} \tag{9.37}$$

and so on. To determine α, we substitute $u_4(x)$ into (9.36) to find that

$$\alpha = 1. \tag{9.38}$$

Substituting $\alpha = 1$ into $u_4(x)$, and using

$$u(x) = \lim_{n\to\infty} u_n(x), \tag{9.39}$$

we find that the exact solution is

$$u(x) = e^x, \tag{9.40}$$

obtained upon using the Taylor series for e^x.

Example 9.6

Solve the following Volterra-Fredholm integro-differential equation by using the variational iteration method

$$u'(x) = 9 - 5x - x^2 - x^3 + \int_0^x (x-t)u(t)dt + \int_0^1 (x-t)u(t), u(0) = 2, \tag{9.41}$$

that can be written as

$$u'(x) = 9 - 5x - x^2 - x^3 + \int_0^x (x-t)u(t)dt + \alpha x - \beta, u(0) = 2, \tag{9.42}$$

where

$$\alpha = \int_0^1 u(t)dt, \quad \beta = \int_0^1 tu(t)dt. \tag{9.43}$$

The correction functional for the Volterra-Fredholm integro-differential equation is given by

$$\begin{aligned} u_{n+1}(x) &= u_n(x) \\ &- \int_0^x \left(u_n'(t) - 9 + 5t + t^2 + t^3 - \int_0^t (t-r)u_n(r)dr - \alpha t + \beta \right) dt, \end{aligned} \tag{9.44}$$

where we used $\lambda = -1$ for first-order integro-differential equations. We can use the initial condition to select $u_0(x) = u(0) = 2$. Using this selection we obtain

$$\begin{aligned} u_0(x) &= 2, \\ u_1(x) &= u_0(x) \\ &\quad - \int_0^x \left(u_0'(t) - 9 + 5t + t^2 + t^3 - \int_0^t (t-r)u_0(r)dr - \alpha t + \beta \right) dt \\ &= 2 + (9-\beta)x - \frac{5-\alpha}{2}x^2 - \frac{1}{4}x^4, \\ u_2(x) &= u_1(x) \\ &\quad - \int_0^x \left(u_1'(t) - 9 + 5t + t^2 + t^3 - \int_0^t (t-r)u_1(r)dr - \alpha t + \beta \right) dt \\ &= 2 + (9-\beta)x - \frac{5-\alpha}{2!}x^2 + \frac{3-\beta}{4!}x^4 - \frac{5-\alpha}{5!}x^5 - \frac{1}{840}x^7, \\ u_3(x) &= u_2(x) \\ &\quad - \int_0^x \left(u_2'(t) - 9 + 5t + t^2 + t^3 - \int_0^t (t-r)u_2(r)dr - \alpha t + \beta \right) dt, \\ &= 2 + (9-\beta)x - \frac{5-\alpha}{2!}x^2 + \frac{3-\beta}{4!}x^4 \end{aligned}$$

$$-\frac{5-\alpha}{5!}x^5 + \frac{3-\beta}{7!}x^7 - \frac{5-\alpha}{8!}x^8 + \cdots, \tag{9.45}$$

and so on. To determine α and β, we substitute $u_3(x)$ into (9.43), and solve the resulting equations to find that

$$\alpha = 5, \quad \beta = 3. \tag{9.46}$$

This in turn gives the exact solution

$$u(x) = 2 + 6x, \tag{9.47}$$

obtained upon substituting $\alpha = 5$, and $\beta = 3$ into $u_n(x)$.

Example 9.7

Solve the following Volterra-Fredholm integro-differential equation by using the variational iteration method

$$u''(x) = -8+6x-3x^2+x^3+\int_0^x u(t)dt+\int_{-1}^1 (1-2xt)u(t),\ u(0) = 2,\ u'(0) = 6, \tag{9.48}$$

that can be written as

$$u''(x) = -8+6x-3x^2+x^3+\int_0^x u(t)dt+\alpha-\beta x,\ u(0) = 2,\ u'(0) = 6, \tag{9.49}$$

where

$$\alpha = \int_0^1 u(t)dt, \quad \beta = \int_0^1 2tu(t)dt. \tag{9.50}$$

The correction functional for the Volterra-Fredholm integro-differential equation is given by

$$\begin{aligned} u_{n+1}(x) &= u_n(x) \\ &+ \int_0^x (t-x)\left(u_n''(t) + 8 - 6t + 3t^2 - t^3 - \int_0^t u_n(r)dr - \alpha + \beta t\right) dt, \end{aligned} \tag{9.51}$$

where we used $\lambda = (t-x)$ for second-order integro-differential equations. We can select $u_0(x) = u(0) + u'(0)x = 2 + 6x$ to determine the successive approximations

$$\begin{aligned} u_0(x) &= 2 + 6x, \\ u_1(x) &= 2 + 6x + \left(\frac{1}{2}\alpha - 4\right)x^2 + \left(\frac{4}{3} - \frac{1}{6}\beta\right)x^3 + \frac{1}{20}x^5, \\ u_2(x) &= 2 + 6x + \left(\frac{1}{2}\alpha - 4\right)x^2 + \left(\frac{4}{3} - \frac{1}{6}\beta\right)x^3 + \left(\frac{1}{120}\alpha - \frac{1}{60}\right)x^5 \\ &\quad + \left(\frac{1}{90} - \frac{1}{720}\beta\right)x^6 + \frac{1}{6720}x^8, \qquad (9.52) \\ u_3(x) &= 2 + 6x + \left(\frac{1}{2}\alpha - 4\right)x^2 + \left(\frac{4}{3} - \frac{1}{6}\beta\right)x^3 + \left(\frac{1}{120}\alpha - \frac{1}{60}\right)x^5 \\ &\quad + \left(\frac{1}{90} - \frac{1}{720}\beta\right)x^6 + \left(\frac{1}{40320}\alpha - \frac{1}{20160}\right)x^8 \end{aligned}$$

$$+\left(\frac{1}{45360}-\frac{1}{362880}\beta\right)x^9+\cdots,$$

and so on. To determine α and β, we substitute $u_3(x)$ into (9.50), and solve the resulting equations to find that

$$\alpha=2, \beta=8. \tag{9.53}$$

This in turn gives the exact solution

$$u(x)=2+6x-3x^2. \tag{9.54}$$

Example 9.8

Solve the following Volterra-Fredholm integro-differential equation by using the variational iteration method

$$u'''(x)=-\frac{1}{2}x^2+\int_0^x u(t)dt+\int_{-\pi}^{\pi} xu(t), u(0)=u'(0)=-u''(0)=1, \tag{9.55}$$

that can be written as

$$u'''(x)=-\frac{1}{2}x^2+\int_0^x u(t)dt+\alpha x, u(0)=u'(0)=-u''(0)=1, \tag{9.56}$$

where

$$\alpha=\int_{-\pi}^{\pi} u(t)dt. \tag{9.57}$$

The correction functional for the Volterra-Fredholm integro-differential equation is given by

$$u_{n+1}(x)=u_n(x)-\frac{1}{2}\int_0^x (t-x)^2\left(u_n'''(t)+\frac{1}{2}t^2-\int_0^t u_n(r)\,dr-\alpha t\right)dt, \tag{9.58}$$

where we used

$$\lambda=-\frac{1}{2}(t-x)^2, \tag{9.59}$$

for third-order integro-differential equations. We can use the initial conditions to select

$$\begin{aligned} u_0(x)&=u(0)+u'(0)x+\frac{1}{2}u''(0)x^2,\\ &=1+x-\frac{1}{2}x^2. \end{aligned} \tag{9.60}$$

Using this selection into the correction functional gives the following successive approximations

$$\begin{aligned} u_0(x)&=1+x-\frac{1}{2!}x^2,\\ u_1(x)&=1+x-\frac{1}{2}x^2+\frac{1}{4!}(1+\alpha)x^4-\frac{1}{6!}x^6,\\ u_2(x)&=1+x-\frac{1}{2}x^2+\frac{1}{4!}(1+\alpha)x^4-\frac{1}{6!}x^6+\frac{1}{8!}(1+\alpha)x^8-\frac{1}{10!}x^{10}, \end{aligned}$$

$$u_3(x) = 1 + x - \frac{1}{2}x^2 + \frac{1}{4!}(1+\alpha)x^4 - \frac{1}{6!}x^6$$
$$+\frac{1}{8!}(1+\alpha)x^8 - \frac{1}{10!}x^{10} + \frac{1}{12!}(1+\alpha)x^{12} - \frac{1}{14!}x^{14}, \quad (9.61)$$

where other approximations are obtained up to $u_8(x)$, but not listed. To determine α, we substitute $u_8(x)$ into (9.57), and solve the resulting equations to find that

$$\alpha = 0. \quad (9.62)$$

This in turn gives the series solution

$$u(x) = x + \left(1 - \frac{1}{2}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 + \frac{1}{10!}x^{10} + \cdots\right), \quad (9.63)$$

that converges to the exact solution

$$u(x) = x + \cos x. \quad (9.64)$$

Exercises 9.2.2

Solve the following Volterra-Fredholm integro-differential equations by using the variational iteration method

1. $u'(x) = 6 + 4x - x^3 + \int_0^x (x-t)u(t)dt + \int_{-1}^1 (1-xt)u(t)dt,\ u(0) = 0$

2. $u'(x) = 6 - 2x - \frac{1}{2}x^2 - x^3 + \int_0^x (x-t)u(t)dt + \int_{-1}^1 xu(t)dt,\ u(0) = 1$

3. $u'(x) = -4 + 6x - x^2 - x^4 + \int_0^x xu(t)dt + \int_{-1}^1 (1-xt)u(t)dt,\ u(0) = 1$

4. $u'(x) = 5x - \frac{1}{2}x^3 + \frac{3}{4}x^5 + \int_0^x (1-xt)u(t)dt + \int_{-1}^1 xtu(t)dt,\ u(0) = 1$

5. $u''(x) = -8 - \frac{1}{2}x^2 + x^3 + \frac{3}{4}x^4 + \int_0^x tu(t)dt + \int_{-1}^1 (x-t)u(t)dt,$

   $u(0) = 1,\ u'(0) = -3$

6. $u''(x) = \frac{2}{5} - 3x^2 - \frac{1}{4}x^5 + \int_0^x xu(t)dt + \int_0^1 (x-t)u(t)dt,\ u(0) = 3,\ u'(0) = 0$

7. $u''(x) = -x - \frac{1}{6}x^3 + \int_0^x (x-t)u(t)dt + \int_{-\pi}^{\pi} xu(t)dt,\ u(0) = 0,\ u'(0) = 2$

8. $u''(x) = -1 - x + \int_0^x (x-t)u(t)dt + \int_{-\pi}^{\pi} xu(t)dt,\ u(0) = 1,\ u'(0) = 1$

9. $u'''(x) = -\frac{1}{3} - \frac{1}{2}x^2 + \int_0^x u(t)dt + \int_0^1 tu(t)dt,\ u(0) = 1,\ u'(0) = 2, u''(0) = 1$

10. $u'''(x) = 5 - \frac{1}{4}x^4 + \int_0^x u(t)dt + \int_{-\pi}^{\pi} xu(t)dt, u(0) = 0,\ u'(0) = 1,\ u''(0) = 0$

11. $u'''(x) = 3 - 4\cos x + \int_0^x u(t)dt + \int_0^{\pi} u(t)dt,\ u(0) = 0,\ u'(0) = 1,\ u''(0) = 0$

12. $u'''(x) = -1 - 4\sin x + \int_0^x u(t)dt + \int_0^{\frac{\pi}{2}} u(t)dt,\ u(0) = 0,\ u'(0) = 0,\ u''(0) = 2$

9.3 The Mixed Volterra-Fredholm Integro-Differential Equations

In a parallel manner to our analysis in the previous section, we will study the mixed Volterra-Fredholm integro-differential equation of the form

$$u^{(i)}(x) = f(x) + \lambda \int_0^x \int_a^b K(r,t)u(t)dtdr, \tag{9.65}$$

where $f(x)$, $K(x,t)$ are analytic functions, and $u^{(i)}(x) = \frac{d^i u}{dx^i}$. It is interesting to note that (9.65) contains mixed Volterra and Fredholm integral equations, where the Fredholm integral is the interior integral, whereas the Volterra integral is the exterior one. Moreover, the unknown function $u(x)$ appears inside the integral, whereas the derivative $u^{(i)}(x)$ appears outside the integral. This type of equations will be handled by using the direct computation method and the series solution method. Other methods exist in the literature but will not be presented in this text.

9.3.1 The Direct Computation Method

The standard ith order mixed Volterra-Fredholm integro-differential equation is of the form

$$u^{(i)}(x) = f(x) + \int_0^x \int_a^b K(r,t)u(t)dtdr, \tag{9.66}$$

where $f(x)$, $K(x,t)$ are analytic functions, and $u^{(i)}(x) = \frac{d^i u}{dx^i}$. The initial conditions should be prescribed. The unknown function $u(x)$ appears inside the integral, whereas the ith derivative $u^{(i)}(x)$ appears outside the integral.

The *direct computation method* [6] was used before to handle Fredholm integral equation in Chapter 4. In this section, the direct computation method will be used to solve the mixed Volterra-Fredholm integro-differential equations. The method gives the solution in an exact form and not in a series form. It is important to point out that this method will be applied for the degenerate or separable kernels of the form

$$K(x,t) = \sum_{k=1}^{n} g_k(x)h_k(t). \tag{9.67}$$

Examples of separable kernels are $x - t, xt, x^2 - t^2, xt^2 + x^2t$, etc.

We will focus our study on $K(x,t)$ being separable of the form

$$K(r,t) = g(r)h(t), \tag{9.68}$$

or being difference kernel of the form

$$K(r,t) = K(r-t) = g(r) - h(t). \tag{9.69}$$

Substituting (9.68) or (9.69) into (9.66), the integro-differential equation becomes

$$u^{(i)}(x) = f(x) + \int_0^x \beta g(r)\, dr, \tag{9.70}$$

or

$$u^{(i)}(x) = f(x) + \int_0^x (\alpha g(r) - \beta)\, dr, \tag{9.71}$$

respectively, where

$$\alpha = \int_a^b u(t)dt, \quad \beta = \int_a^b h(t)u(t)dt. \tag{9.72}$$

Integrating (9.70) or (9.71) i times for 0 to x gives the unknown solution $u(x)$, where the constants α and β are to be determined. This can be achieved by substituting the resulting value of $u(x)$ into (9.72). Using the obtained numerical values of α and β, the solution $u(x)$ of the mixed Volterra-Fredholm integro-differential equation (9.70) or (9.71) is readily obtained.

The direct computation method will be illustrated by studying the following mixed Volterra-Fredholm integro-differential examples.

Example 9.9

Solve the mixed Volterra-Fredholm integro-differential equation by using the direct computation method

$$u'(x) = e^x(1+x) - \frac{1}{2}x^2 + \int_0^x \int_0^1 ru(t)dtdr, \quad u(0) = 0. \tag{9.73}$$

This equation can be written as

$$u'(x) = e^x(1+x) - \frac{1}{2}x^2 + \int_0^x \alpha r dr, \quad u(0) = 0. \tag{9.74}$$

where we used

$$\alpha = \int_0^1 u(t)dt. \tag{9.75}$$

Integrating both sides once from 0 to x, and using the initial condition we find

$$u(x) = xe^x + \frac{\alpha - 1}{6}x^3. \tag{9.76}$$

To determine α, we substitute $u(x)$ from (9.76) into (9.75) to find that

$$\alpha = 1. \tag{9.77}$$

This in turn gives the exact solution by

$$u(x) = xe^x. \tag{9.78}$$

Example 9.10

Solve the following Volterra-Fredholm integro-differential equation by using the direct computation method

$$u'(x) = 6 + 29x - \frac{7}{2}x^2 + \int_0^x \int_0^1 (r-t)u(t)dtdr, \quad u(0) = 0. \tag{9.79}$$

This equation can be written as

$$u'(x) = 6 + 29x - \frac{7}{2}x^2 + \int_0^x (\alpha r - \beta)dr, \tag{9.80}$$

where

$$\alpha = \int_0^1 u(t)dt, \quad \beta = \int_0^1 tu(t)dt. \tag{9.81}$$

Integrating both sides once from 0 to x, and using the initial condition we find

$$u(x) = 6x + \frac{29-\beta}{2}x^2 + \frac{\alpha - 1}{6}x^3. \tag{9.82}$$

To determine α and β, we substitute $u(x)$ from (9.82) into (9.81) to find that

$$\alpha = 1, \beta = 5. \tag{9.83}$$

Substituting $\alpha = 7$ and $\beta = 5$ into $u(x)$ gives the exact solution by

$$u(x) = 6x + 12x^2. \tag{9.84}$$

Example 9.11

Solve the following Volterra-Fredholm integro-differential equation by using the direct computation method

$$u''(x) = -x^2 - \sin x - \cos x + \int_0^x \int_0^\pi ru(t)dt\, dr, u(0) = 1, u'(0) = 1. \tag{9.85}$$

This equation can be written as

$$u''(x) = -x^2 - \sin x - \cos x + \int_0^x \alpha r\, dr, \tag{9.86}$$

where

$$\alpha = \int_0^\pi u(t)dt. \tag{9.87}$$

Integrating both sides from 0 to x twice, and using the initial conditions we obtain

$$u(x) = \sin x + \cos x + \frac{\alpha - 2}{24}x^4. \tag{9.88}$$

To determine α, we substitute $u(x)$ into (9.87) to find that

$$\alpha = 2. \tag{9.89}$$

Substituting $\alpha = 2$ into $u(x)$ gives the exact solution by

$$u(x) = \sin x + \cos x. \tag{9.90}$$

Example 9.12

Solve the following Volterra-Fredholm integro-differential equation by using the direct computation method

$$u''(x) = -\frac{10}{3} + 2x^3 + \int_0^x \int_{-1}^1 (rt^2 - r^2 t)u(t)dt\,dr, u(0) = 1, u'(0) = 9. \quad (9.91)$$

Proceeding as before, we set

$$u''(x) = -\frac{10}{3} + 2x^3 + \int_0^x (\alpha r - \beta r^2)\,dr, \quad (9.92)$$

where

$$\alpha = \int_{-1}^1 t^2 u(t)dt, \quad \beta = \int_{-1}^1 tu(t)dt. \quad (9.93)$$

Integrating both sides from 0 to x twice, and using the initial conditions we obtain

$$u(x) = 1 + 9x - \frac{5}{3}x^2 + \frac{\alpha}{24}x^4 + \frac{6-\beta}{60}x^5. \quad (9.94)$$

To determine α and β, we substitute $u(x)$ into (9.93) to find that

$$\alpha = 0, \beta = 6. \quad (9.95)$$

Substituting $\alpha = 0$ and $\beta = 6$ into $u(x)$gives the exact solution by

$$u(x) = 1 + 9x - \frac{5}{3}x^2. \quad (9.96)$$

Exercises 9.3.1

Solve the following Volterra-Fredholm integro-differential equations by using the direct computation method

1. $u'(x) = 8x + \frac{5}{4}x^2 + \int_0^x \int_0^1 (1 - rt)u(t)dtdr,\ u(0) = 2$

2. $u'(x) = 1 + 2x - \frac{8}{15}x^2 + \int_0^x \int_{-1}^1 rt^2 u(t)dtdr,\ u(0) = 1$

3. $u'(x) = 1 - 2x - \frac{43}{15}x^2 + \int_0^x \int_{-1}^1 (1 - rt)u(t)dtdr,\ u(0) = 1$

4. $u'(x) = 1 - 3x^2 - \frac{4}{45}x^3 + \int_0^x \int_0^1 r^2 tu(t)dtdr,\ u(0) = 1$

5. $u'(x) = 1 - 2x - \sin x + \int_0^x \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} u(t)dtdr,\ u(0) = 1$

6. $u'(x) = 1 - \frac{4}{3}x + e^x + \int_0^x \int_0^1 tu(t)dtdr,\ u(0) = 1$

7. $u'(x) = \frac{1}{4}x^2 - e^x + \int_0^x \int_0^1 rtu(t)dtdr,\ u(0) = 0$

8. $u'(x) = \sin x + \cos x + \int_0^x \int_{-\pi}^{\pi} u(t)dtdr,\ u(0) = -1$

9. $u''(x) = 2x^3 - \frac{1}{3}x^2 + \int_0^x \int_{-1}^1 (rt^2 - r^2t)u(t)dtdr,\ u(0) = 1,\ u'(0) = 9$

10. $u''(x) = -2\sin x - x\cos x + \int_0^x \int_{-\pi}^{\pi} u(t)dtdr,\ u(0) = 0,\ u'(0) = 1$

11. $u''(x) = e^x - 3x + \int_0^x \int_0^1 tu(t)dtdr,\ u(0) = 2,\ u'(0) = \frac{11}{2}$

12. $u''(x) = -4\sin 2x + \int_0^x \int_0^{\pi} (1+r)u(t)dtdr,\ u(0) = 0, u'(0) = 2$

9.3.2 The Series Solution Method

The series solution method was examined before in this chapter. A real function $u(x)$ is called analytic if it has derivatives of all orders such that the generic form of Taylor series at $x = 0$ can be written as

$$u(x) = \sum_{n=0}^{\infty} a_n x^n. \tag{9.97}$$

In this section we will present the series solution method, that stems mainly from the Taylor series for analytic functions, for solving the mixed Volterra-Fredholm integro-differential equations. We will assume that the solution $u(x)$ of the mixed Volterra-Fredholm integro-differential equation

$$u^{(i)}(x) = f(x) + \int_0^x \int_a^b K(r,t)u(t)dtdr, \tag{9.98}$$

is analytic, and therefore possesses a Taylor series of the form given in (9.97), where the coefficients a_n will be determined recurrently.

In this method, we usually substitute the Taylor series (9.97) into both sides of (9.98) to obtain

$$\left(\sum_{k=0}^{\infty} a_k x^k\right)^{(i)} = T(f(x)) + \int_0^x \int_a^b K(r,t) \left(\sum_{k=0}^{\infty} a_k t^k\right) dt\, dr, \tag{9.99}$$

or for simplicity we use

$$(a_0 + a_1x + a_2x^2 + \cdots)^{(i)} = T(f(x)) + \int_0^x \int_a^b K(r,t)\,(a_0 + a_1t + \cdots)\,dt\,dr, \tag{9.100}$$

where $T(f(x))$ is the Taylor series for $f(x)$.

We integrate the inner then the outer integral in (9.99) or (9.100), and collect the coefficients of like powers of x. We next equate the coefficients of like powers of x into both sides of the resulting equation to determine a system of equations in $a_j, j \geqslant 0$. Solving this system will lead to a complete determination of the coefficients $a_j, j \geqslant 0$. Having determined the coefficients $a_j, j \geqslant 0$, the series solution follows immediately upon substituting the derived coefficients into (9.97). The exact solution may be obtained if such an exact solution exists. If an exact solution is not obtainable, then the obtained series can be used for numerical purposes. In this case, the more terms we evaluate, the higher accuracy level we achieve.

Example 9.13

Solve the mixed Volterra-Fredholm integro-differential equation by using the Taylor series solution method

$$u'(x) = 6 + 29x - \frac{7}{2}x^2 + \int_0^x \int_0^1 ru(t)dtdr, \quad u(0) = 0. \tag{9.101}$$

Substituting $u(x)$ by the series

$$u(x) = \sum_{n=0}^{\infty} a_n x^n, \tag{9.102}$$

into both sides of (9.101) leads to

$$\left(\sum_{n=0}^{\infty} a_n x^n\right)' = 6 + 29x - \frac{27}{2}x^2 + \int_0^x \int_0^1 \left((r-t)\sum_{n=0}^{\infty} a_n t^n\right) dt\, dr. \tag{9.103}$$

Using the initial condition gives $a_0 = 0$. Evaluating the integrals at the right side, using few terms from both sides, collecting the coefficients of like powers of x, and equating the coefficients of like powers of x in both sides, we obtain

$$a_0 = 0, \quad a_1 = 6, \quad a_2 = 12, \quad a_j = 0, \quad j \geqslant 3. \tag{9.104}$$

The exact solution is therefore given by

$$u(x) = 6x + 12x^2. \tag{9.105}$$

Example 9.14

Solve the Volterra-Fredholm integro-differential equation by using the series solution method

$$u'(x) = e^x - x + \int_0^x \int_0^1 tu(t)dtdr, \quad u(0) = 1. \tag{9.106}$$

Substituting $u(x)$ by the series

$$u(x) = \sum_{n=0}^{\infty} a_n x^n, \tag{9.107}$$

into both sides of (9.106) leads to

$$\left(\sum_{n=0}^{\infty} a_n x^n\right)' = e^x - x + \int_0^x \int_0^1 \left(t\sum_{n=0}^{\infty} a_n t^n\right) dt\, dr. \tag{9.108}$$

Using the given initial condition gives $a_0 = 1$. Evaluating the integrals at the right side, using few terms from both sides, collecting the coefficients of like powers of x, and equating the coefficients of like powers of x in both sides, we obtain

$$a_n = \frac{1}{n!}, n \geqslant 0. \tag{9.109}$$

The exact solution is therefore given by

$$u(x) = e^x. \tag{9.110}$$

Example 9.15

Solve the Volterra-Fredholm integro-differential equation by using the series solution method

$$u''(x) = -2x - \cos x + \int_0^x \int_0^1 (1 - rt)u(t)dt\,dr, u(0) = 1, u'(0) = 0. \tag{9.111}$$

Proceeding as before, we obtain

$$a_{2n} = \frac{(-1)^n}{(2n)!}, n \geqslant 0, \tag{9.112}$$

and zero otherwise. This gives the exact solution by

$$u(x) = \cos x. \tag{9.113}$$

Example 9.16

Solve the mixed Volterra-Fredholm integro-differential equation by using the series solution method

$$u''(x) = -\frac{20}{3} + \frac{2}{3}x^3 + \int_0^x \int_{-1}^1 (rt^2 - r^2 t)u(t)dt\,dr, u(0) = 2, u'(0) = 3. \tag{9.114}$$

Using the initial conditions, and proceeding as before we find

$$a_0 = 2, \quad a_1 = 3, \quad a_2 = -\frac{10}{3}, \quad a_j = 0, \quad j \geqslant 4. \tag{9.115}$$

The exact solution is given by

$$u(x) = 2 + 3x - \frac{10}{3}x^2. \tag{9.116}$$

Exercises 9.3.2

Use the series solution method to solve the following mixed Volterra-Fredholm integro-differential equations

1. $u'(x) = 8x + \frac{5}{4}x^2 + \int_0^x \int_0^1 (1 - rt)u(t)dtdr,\ u(0) = 2$
2. $u'(x) = 1 - \frac{127}{8}x + \int_0^x \int_0^1 (1 - rt)u(t)dtdr,\ u(0) = 1$
3. $u'(x) = 1 - 2x - \frac{43}{15}x^2 + \int_0^x \int_{-1}^1 (1 - rt)u(t)dtdr,\ u(0) = 1$
4. $u'(x) = 1 - 2x - \frac{77}{15}x^2 + \int_0^x \int_{-1}^1 (rt^2 - r^2 t)u(t)dtdr, u(0) = 1$
5. $u'(x) = 1 - 2x - \sin x + \int_0^x \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} u(t)dtdr,\ u(0) = 1$

6. $u'(x) = e^x - \frac{3}{2}x + \int_0^x \int_0^1 tu(t)dtdr, \; u(0) = 2$

7. $u'(x) = \frac{1}{4}x^2 - e^x + \int_0^x \int_0^1 rtu(t)dtdr, \; u(0) = 0$

8. $u'(x) = \frac{1}{2}x^2 - e^{-x} + \int_0^x \int_{-1}^0 rtu(t)dtdr, \; u(0) = 1$

9. $u''(x) = 2x^3 - \frac{1}{3}x^2 + \int_0^x \int_{-1}^1 (rt^2 - r^2t)u(t)dtdr, \; u(0) = 1, \; u'(0) = 9$

10. $u''(x) = -\frac{10}{3} + \frac{2}{9}x^3 + \int_0^x \int_{-1}^1 (rt^2 - r^2t)u(t)dtdr, \; u(0) = 1, \; u'(0) = 1$

11. $u''(x) = 2 + 6x - \frac{77}{120}x^2 + \int_0^x \int_0^1 rtu(t)dtdr, \; u(0) = 1, \; u'(0) = 1$

12. $u''(x) = -15x + \int_0^x \int_0^1 rtu(t)dtdr, \; u(0) = 1, \; u'(0) = 0$

9.4 The Mixed Volterra-Fredholm Integro-Differential Equations in Two Variables

In this section we will study the linear mixed Volterra-Fredholm integro-differential equations in two variables given by

$$u'(x,t) = \tilde{f}(x,t) + \int_0^t \int_\Omega F(x,t,r,s)u(r,s)drds, \; (x,t) \in \Omega \times [0,T], \quad (9.117)$$

where $u(0,t) = u_0$. The functions $\tilde{f}(x)$, and $F(x,t,r,s)$ are analytic functions on $D = \Omega \times [0,T]$, and Ω is a closed subset of $\mathrm{R}^n, n = 1,2,3$. It is interesting to note that (9.117) contains mixed Volterra and Fredholm integral equations, where the Fredholm integral is the interior integral, whereas the Volterra integral is the exterior one. Moreover, the unknown function $u(x,t)$ appears inside the integral, whereas the derivative $u'(x)$ appears outside the integral.

The mixed Volterra-Fredholm integro-differential equation (9.117) arises from parabolic boundary value problems, and in various physical and biological models. In the literature, some methods, such as the projection method, time collocation method, the trapezoidal Nystrom method, Adomian method, and other analytical or numerical techniques were used to handle this equation. It was found that these techniques encountered difficulties in terms of computational work used, and therefore, approximate solutions were obtained for numerical purposes. In particular, it was found that a complicated term $f(x,t)$ can cause difficult integrations and proliferation of terms in Adomian recursive scheme.

To overcome the tedious work of the existing strategies, the modified decomposition method, combined sometimes with the noise terms phenomenon, will form a useful basis for studying the mixed Volterra-Fredholm integro-differential equation (9.117). The size of the computational work can be dramatically reduced by using the modified decomposition method. The noise terms were defined as the identical terms with opposite signs that may appear in the first two components of the series solution of $u(x,t)$. The modified decomposition method and the noise terms phenomenon were presented in details in Chapters 3, 4 and in this chapter.

9.4.1 The Modified Decomposition Method

We first integrate both sides of (9.117) from 0 to x to obtain

$$u(x,t) - u(0,t) = f(x,t) + \int_0^x \int_0^t \int_\Omega F(w,t,r,s)u(r,s)\,dr\,ds\,dw, \quad (9.118)$$

where

$$f(x,t) = \int_0^x \tilde{f}(w,t)dw, \quad (9.119)$$

and $(x,t) \in \Omega \times [0,T], u(0,t) = u_0$. For mixed integro-differential equations of higher orders, we can integrate the equation as many times as the order of the given equation. However, in this section we will focus our study only on the first order equations.

The modified decomposition method expresses the function $f(x,t)$ as the sum of two partial functions, namely $f_1(x,t)$ and $f_2(x,t)$. In other words, we can set

$$f(x,t) = f_1(x,t) + f_2(x,t). \quad (9.120)$$

To minimize the size of calculations, we identify the zeroth component $u_0(x,t)$ by one part of $f(x,t)$, namely $f_1(x,t)$ or $f_2(x,t)$. The other part of $f(x,t)$ can be added to the component $u_1(x,t)$ among other terms. In other words, the modified decomposition method [7–8] introduces the modified recurrence relation

$$\begin{aligned} u_0(x,t) &= f_1(x,t), \\ u_1(x,t) &= f_2(x,t) + \int_0^x \int_0^t \int_\Omega F(w,t,r,s)u_0(r,s)drdsdw, \\ u_{k+1}(x,t) &= \int_0^x \int_0^t \int_\Omega F(w,t,r,s)u_k(r,s)drdsdw, \quad k \geqslant 1. \end{aligned} \quad (9.121)$$

If noise terms appear between $u_0(x,t)$ and $u_1(x,t)$, then by canceling these terms from $u_0(x,t)$, the remaining non-canceled terms of $u_0(x,t)$ may give the exact solution. This can be satisfied by direct substitution. In what follows, we study some illustrative examples.

Example 9.17

Solve the mixed Volterra-Fredholm integro-differential equation by using the modified decomposition method

$$u'(x,t) = t + \frac{1}{18}t^3 - \frac{1}{12}xt^3 + \int_0^t \int_\Omega (x-r)(t-s)u(r,s)drds, u(0,t) = 0, \tag{9.122}$$

with $\Omega = [0,1]$. Substituting the decomposition series

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t), \tag{9.123}$$

into (9.122) gives

$$\begin{aligned} \left(\sum_{n=0}^{\infty} u_n(x,t)\right)' &= t + \frac{1}{18}t^3 - \frac{1}{12}xt^3 \\ &+ \int_0^t \int_\Omega (x-r)(t-s)\left(\sum_{n=0}^{\infty} u_n(r,s)\right) drds. \end{aligned} \tag{9.124}$$

Integrating both sides from 0 to x and using the initial condition we find

$$\begin{aligned} \sum_{n=0}^{\infty} u_n(x,t) &= xt + \frac{1}{18}xt^3 - \frac{1}{24}x^2t^3 \\ &+ \int_0^x \int_0^t \int_\Omega (w-t)(r-s)\left(\sum_{n=0}^{\infty} u_n(r,s)\right) drdsdw. \end{aligned} \tag{9.125}$$

We then decompose $f(x,t)$ into two parts as follows:

$$f_0(x,t) = xt + \frac{1}{18}xt^3, \quad f_1(x,t) = -\frac{1}{24}x^2t^3. \tag{9.126}$$

The modified decomposition technique admits the use of the recursive relation

$$\begin{aligned} u_0(x,t) &= xt + \frac{1}{18}xt^3, \\ u_1(x,t) &= -\frac{1}{24}x^2t^3 + \int_0^x \int_0^t \int_\Omega (w-t)(r-s)u_0(r,s)drdsdw, \\ u_{k+1}(x,t) &= \int_0^x \int_0^t \int_\Omega (w-t)(r-s)u_k(r,s)drdsdw, \quad k \geqslant 1. \end{aligned} \tag{9.127}$$

This gives

$$\begin{aligned} u_0(x,t) &= xt + \frac{1}{18}xt^3, \\ u_1(x,t) &= -\frac{1}{18}xt^3 - \frac{1}{1080}xt^5 + \frac{1}{1440}x^2t^5. \end{aligned} \tag{9.128}$$

The self-canceling noise terms $\frac{1}{18}xt^3$ and $-\frac{1}{18}xt^3$ appear between the components $u_0(x,t)$ and $u_1(x,t)$ respectively. By canceling this term from $u_0(x,t)$, and showing that the remaining non-canceled term of $u_0(x,t)$ satisfies the equation (9.122), the exact solution

$$u(x,t) = xt, \tag{9.129}$$

is readily obtained.

Example 9.18

Solve the mixed Volterra-Fredholm integro-differential equation by using the modified decomposition method

$$u'(x,t) = 2x - \frac{1}{4}t^2 + \frac{1}{6}t^4 + \int_0^t \int_\Omega rtu(r,s)drds, u(0,t) = -t^2, \tag{9.130}$$

with $\Omega = [0,1]$. Substituting the decomposition series

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t), \tag{9.131}$$

into the last equation gives

$$\left(\sum_{n=0}^{\infty} u_n(x,t)\right)' = 2x - \frac{1}{4}t^2 + \frac{1}{6}t^4 + \int_0^t \int_\Omega rt\left(\sum_{n=0}^{\infty} u_n(r,s)\right)drds. \tag{9.132}$$

Integrating both sides from 0 to x and using the initial condition we find

$$\begin{aligned}\sum_{n=0}^{\infty} u_n(x,t) &= x^2 - t^2 - \frac{1}{4}xt^2 + \frac{1}{6}xt^4 \\ &\quad + \int_0^x \int_0^t \int_\Omega rt\left(\sum_{n=0}^{\infty} u_n(r,s)\right)drdsdw.\end{aligned} \tag{9.133}$$

We decompose $f(x,t)$ into two parts as follows:

$$f_0(x,t) = x^2 - t^2 - \frac{1}{4}xt^2, \quad f_1(x,t) = \frac{1}{6}xt^4. \tag{9.134}$$

The modified decomposition technique admits the use of the recursive relation

$$\begin{aligned}u_0(x,t) &= x^2 - t^2 - \frac{1}{4}xt^2, \\ u_1(x,t) &= \frac{1}{6}xt^4 + \int_0^x \int_0^t \int_\Omega rtu_0(r,s)drdsdw, \\ u_{k+1}(x,t) &= \int_0^x \int_0^t \int_\Omega rtu_k(r,s)drdsdw, \quad k \geqslant 1.\end{aligned} \tag{9.135}$$

This gives

$$\begin{aligned}u_0(x,t) &= x^2 - t^2 - \frac{1}{4}xt^2, \\ u_1(x,t) &= \frac{1}{4}xt^2 - \frac{1}{36}xt^4.\end{aligned} \tag{9.136}$$

The self-canceling noise terms $-\frac{1}{4}xt^2$ and $\frac{1}{4}xt^2$ appear between the components $u_0(x,t)$ and $u_1(x,t)$ respectively. By canceling this term from $u_0(x,t)$, and showing that the remaining non-canceled term of $u_0(x,t)$ satisfies the given equation, the exact solution is given by

$$u(x,t) = x^2 - t^2. \tag{9.137}$$

Example 9.19

Solve the mixed Volterra-Fredholm integro-differential equation by using the modified decomposition method

$$u'(x,t) = e^x - \frac{1}{2} + \frac{1}{2}e^t(1+t) - \frac{1}{2}t^2 + \int_0^t \int_\Omega rsu(r,s)drds, u(0,t) = 1 + e^t, \tag{9.138}$$

with $\Omega = [0,1]$. Proceeding as before, we find

$$\begin{aligned}\sum_{n=0}^{\infty} u_n(x,t) &= e^x + e^t - \frac{1}{2}x + \frac{1}{2}xe^t(1+t) - \frac{1}{2}xt^2 \\ &\quad + \int_0^x \int_0^t \int_\Omega rs\left(\sum_{n=0}^{\infty} u_n(r,s)\right) drdsdw.\end{aligned} \tag{9.139}$$

We decompose $f(x,t)$ into two parts as follows:

$$\begin{aligned}f_0(x,t) &= e^x + e^t - \frac{1}{2}x, \\ f_1(x,t) &= \frac{1}{2}xe^t(1+t) - \frac{1}{2}xt^2.\end{aligned} \tag{9.140}$$

The modified decomposition technique admits the use of the recursive relation

$$\begin{aligned}u_0(x,t) &= e^x + e^t - \frac{1}{2}x, \\ u_1(x,t) &= \frac{1}{2}xe^t(1+t) - \frac{1}{2}xt^2 + \int_0^x \int_0^t \int_\Omega rsu_0(r,s)drdsdw, \\ u_{k+1}(x,t) &= \int_0^x \int_0^t \int_\Omega rsu_k(r,s)drdsdw, \quad k \geqslant 1.\end{aligned} \tag{9.141}$$

This in turn gives

$$\begin{aligned}u_0(x,t) &= e^x + e^t - \frac{1}{2}x, \\ u_1(x,t) &= \frac{1}{2}x - \frac{1}{12}xt^2.\end{aligned} \tag{9.142}$$

The self-canceling noise terms $-\frac{1}{2}x$ and $\frac{1}{2}x$ appear between the components $u_0(x,t)$ and $u_1(x,t)$ respectively. By canceling this term from $u_0(x,t)$, and showing that the remaining non-canceled term of $u_0(x,t)$ satisfies the given equation, the exact solution is therefore given by

$$u(x,t) = e^x + e^t. \tag{9.143}$$

Example 9.20

Solve the mixed Volterra-Fredholm integro-differential equation by using the modified decomposition method

$$u'(x,t) = t\cos x - \frac{1}{3}t^3 - \frac{1}{2}xt^3 + \int_0^t \int_\Omega (xt+rs)u(r,s)drds, u(0,t) = 0, \tag{9.144}$$

with $\Omega = [0, \frac{\pi}{2}]$. Proceeding as before, we find

$$\sum_{n=0}^{\infty} u_n(x,t) = t\sin x - \frac{1}{3}xt^3 - \frac{1}{4}x^2t^3 + \int_0^x \int_0^t \int_\Omega (wt+rs)\left(\sum_{n=0}^{\infty} u_n(r,s)\right) drdsdw. \quad (9.145)$$

We decompose $f(x,t)$ into two parts as follows:

$$f_0(x,t) = t\sin x - \frac{1}{3}xt^3,$$
$$f_1(x,t) = -\frac{1}{4}x^2t^3. \quad (9.146)$$

The modified decomposition technique admits the use of the recursive relation

$$u_0(x,t) = t\sin x - \frac{1}{3}xt^3,$$
$$u_1(x,t) = -\frac{1}{4}x^2t^3 + \int_0^x \int_0^t \int_\Omega (wt+rs)u_0(r,s)drdsdw, \quad (9.147)$$
$$u_{k+1}(x,t) = \int_0^x \int_0^t \int_\Omega (wt+rs)u_k(r,s)drdsdw, \quad k \geqslant 1.$$

This in turn gives

$$u_0(x,t) = t\sin x - \frac{1}{3}xt^3,$$
$$u_1(x,t) = \frac{1}{3}xt^3 + \cdots. \quad (9.148)$$

The self-canceling noise terms $-\frac{1}{3}xt^3$ and $\frac{1}{3}xt^3$ appear between the components $u_0(x,t)$ and $u_1(x,t)$ respectively. By canceling this term from $u_0(x,t)$, and showing that the remaining non-canceled term of $u_0(x,t)$ satisfies the given equation, the exact solution is therefore given by

$$u(x,t) = t\sin x. \quad (9.149)$$

Exercises 9.4.1

Use the modified decomposition method to solve the mixed Volterra-Fredholm integro-differential equations in two variables

$$u'(x,t) = f(x,t) + \int_0^t \int_\Omega F(x,t,r,s)u(r,s)drds, (x,t) \in \Omega \times [0,T], \quad (9.150)$$

where $f(x,t), F(x,t,r,s), u(0,t)$ and Ω are given by:

1. $f = 1 - \frac{1}{6}t^2 - \frac{1}{6}t^3, F = rs, u(0,t) = t, \Omega = [0,1]$
2. $f = 2x - \frac{1}{8}t^2 - \frac{1}{8}t^4, F = rs, u(0,t) = t^2, \Omega = [0,1]$
3. $f = 2t - \frac{2}{9}t^3 - \frac{1}{4}t^3, F = rs, u(0,t) = 1, \Omega = [0,1]$

4. $f = 2x - \frac{1}{4}t - \frac{5}{12}t^2 - \frac{1}{3}t^3, F = r + s, u(0,t) = t, \Omega = [0,1]$

5. $f = 2xt^2 - \frac{7}{18}t^4 - \frac{2}{9}xt^3, F = x + t + r + s, u(0,t) = 0, \Omega = [-1,1]$

6. $f = 3x^2 + \frac{2}{5}t^5 + \frac{1}{2}xt^4, F = x + s, u(0,t) = -t^3, \Omega = [-1,1]$

7. $f = -t\sin x + t^2, F = r + s, u(0,t) = 0, \Omega = [0,\pi]$

8. $f = \cos x - t^2, F = rs, u(0,t) = t, \Omega = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

9. $f = -\sin x, F = rs, u(0,t) = 1 + \sin t, \Omega = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

10. $f = e^x - \frac{1}{2}t^2 - \frac{1}{6}t^3, F = rs, u(0,t) = 1 + t, \Omega = [0,1]$

11. $f = e^t + \frac{1}{2}xt - \frac{1}{2}xte^t, F = xt, u(0,t) = 0, \Omega = [0,1]$

12. $f = te^x - \frac{1}{3}t^3, F = rs, u(0,t) = t, \Omega = [0,1]$

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Chapter 10
Systems of Volterra Integral Equations

10.1 Introduction

Systems of integral equations, linear or nonlinear, appear in scientific applications in engineering, physics, chemistry and populations growth models [1–4]. Studies of systems of integral equations have attracted much concern in applied sciences. The general ideas and the essential features of these systems are of wide applicability.

The systems of Volterra integral equations appear in two kinds. For systems of Volterra integral equations of the first kind, the unknown functions appear only under the integral sign in the form:

$$\begin{aligned} f_1(x) &= \int_0^x \Big(K_1(x,t)u(t) + \widetilde{K}_1(x,t)v(t) + \cdots\Big)\,dt, \\ f_2(x) &= \int_0^x \Big(K_2(x,t)u(t) + \widetilde{K}_2(x,t)v(t) + \cdots\Big)\,dt, \\ &\vdots \end{aligned} \tag{10.1}$$

However, systems of Volterra integral equations of the second kind, the unknown functions appear inside and outside the integral sign of the form:

$$\begin{aligned} u(x) &= f_1(x) + \int_0^x \Big(K_1(x,t)u(t) + \widetilde{K}_1(x,t)v(t) + \cdots\Big)\,dt, \\ v(x) &= f_2(x) + \int_0^x \Big(K_2(x,t)u(t) + \widetilde{K}_2(x,t)v(t) + \cdots\Big)\,dt, \\ &\vdots \end{aligned} \tag{10.2}$$

The kernels $K_i(x,t)$ and $\widetilde{K}_i(x,t)$, and the functions $f_i(x), i = 1, 2, \ldots, n$ are given real-valued functions.

A variety of analytical and numerical methods are used to handle systems of Volterra integral equations. The existing techniques encountered some difficulties in terms of the size of computational work, especially when the system involves several integral equations. To avoid the difficulties that usually arise from the traditional methods, we will use some of the methods presented in this text. The Adomian decomposition method, the Variational iteration method, and the Laplace transform method will form a reasonable basis for studying systems of integral equations. The emphasis in this text will be on the use of these methods rather than proving theoretical concepts of convergence and existence that can be found in other texts.

10.2 Systems of Volterra Integral Equations of the Second Kind

We will first study systems of Volterra integral equations of the second kind given by

$$\begin{aligned} u(x) &= f_1(x) + \int_0^x \left(K_1(x,t)u(t) + \widetilde{K}_1(x,t)v(t) + \cdots\right) dt, \\ v(x) &= f_2(x) + \int_0^x \left(K_2(x,t)u(t) + \widetilde{K}_2(x,t)v(t) + \cdots\right) dt. \end{aligned} \tag{10.3}$$

The unknown functions $u(x), v(x), \ldots$, that will be determined, appear inside and outside the integral sign. The kernels $K_i(x,t)$ and $\widetilde{K}_i(x,t)$, and the function $f_i(x)$ are given real-valued functions. In what follows we will present the methods, new and traditional, that will be used to handle these systems.

10.2.1 The Adomian Decomposition Method

The Adomian decomposition method [5–7] was presented before. The method decomposes each solution as an infinite sum of components, where these components are determined recurrently. This method can be used in its standard form, or combined with the noise terms phenomenon. Moreover, the modified decomposition method will be used wherever it is appropriate. It is interesting to point out that the VIM method can also be used, but we need to transform the system of integral equations to a system of integro-differential equations that will be presented later in this chapter.

Example 10.1

Use the Adomian decomposition method to solve the following system of Volterra integral equations

$$\begin{aligned} u(x) &= x - \frac{1}{6}x^4 + \int_0^x \left((x-t)^2 u(t) + (x-t)v(t)\right) dt, \\ v(x) &= x^2 - \frac{1}{12}x^5 + \int_0^x \left((x-t)^3 u(t) + (x-t)^2 v(t)\right) dt. \end{aligned} \tag{10.4}$$

The Adomian decomposition method suggests that the linear terms $u(x)$ and $v(x)$ be decomposed by an infinite series of components

$$u(x) = \sum_{n=0}^{\infty} u_n(x), \quad v(x) = \sum_{n=0}^{\infty} v_n(x), \tag{10.5}$$

where $u_n(x)$ and $v_n(x), n \geqslant 0$ are the components of $u(x)$ and $v(x)$ that will be elegantly determined in a recursive manner.

Substituting (10.5) into (10.4) gives

$$\begin{aligned} \sum_{n=0}^{\infty} u_n(x) &= x - \frac{1}{6}x^4 + \int_0^x \left((x-t)^2 \sum_{n=0}^{\infty} u_n(t) + (x-t) \sum_{n=0}^{\infty} v_n(t)\right) dt, \\ \sum_{n=0}^{\infty} v_n(x) &= x^2 - \frac{1}{12}x^5 + \int_0^x \left((x-t)^3 \sum_{n=0}^{\infty} u_n(t) + (x-t)^2 \sum_{n=0}^{\infty} v_n(t)\right) dt. \end{aligned} \tag{10.6}$$

The zeroth components $u_0(x)$ and $v_0(x)$ are defined by all terms that are not included under the integral sign. Following Adomian analysis, the system (10.6) is transformed into a set of recursive relations given by

$$\begin{aligned} u_0(x) &= x - \frac{1}{6}x^4, \\ u_{k+1}(x) &= \int_0^x \left((x-t)^2 u_k(t) + (x-t)v_k(t)\right) dt, \quad k \geqslant 0, \end{aligned} \tag{10.7}$$

and

$$\begin{aligned} v_0(x) &= x^2 - \frac{1}{12}x^5, \\ v_{k+1}(x) &= \int_0^x \left((x-t)^3 u_k(t) + (x-t)^2 v_k(t)\right) dt, \quad k \geqslant 0. \end{aligned} \tag{10.8}$$

This in turn gives

$$u_0(x) = x - \frac{1}{6}x^4, \quad u_1(x) = \frac{1}{6}x^4 - \frac{1}{280}x^7, \tag{10.9}$$

and

$$v_0(x) = x^2 - \frac{1}{12}x^5, \quad v_1(x) = \frac{1}{12}x^5 - \frac{11}{10080}x^8. \tag{10.10}$$

It is obvious that the noise terms $\mp\frac{1}{6}x^4$ appear between $u_0(x)$ and $u_1(x)$. Moreover, the noise terms $\mp\frac{1}{12}x^5$ appear between $v_0(x)$ and $v_1(x)$. By canceling these noise terms from $u_0(x)$ and $v_0(x)$, the non-canceled terms of $u_0(x)$ and $v_0(x)$ give the exact solutions

$$(u(x), v(x)) = (x, x^2). \tag{10.11}$$

Example 10.2

Use the Adomian decomposition method to solve the following system of Volterra integral equations

$$\begin{aligned} u(x) &= \cos x - x\sin x + \int_0^x (\sin(x-t)u(t) + \cos(x-t)v(t))\,dt, \\ v(x) &= \sin x - x\cos x + \int_0^x (\cos(x-t)u(t) - \sin(x-t)v(t))\,dt. \end{aligned} \tag{10.12}$$

We first decompose the linear terms $u(x)$ and $v(x)$ by an infinite series of components

$$u(x) = \sum_{n=0}^{\infty} u_n(x), \quad v(x) = \sum_{n=0}^{\infty} v_n(x), \tag{10.13}$$

where $u_n(x)$ and $v_n(x), n \geqslant 0$ are the components of $u(x)$ and $v(x)$ that will be elegantly determined in a recursive manner.

Substituting (10.13) into (10.12) gives

$$\begin{aligned} \sum_{n=0}^{\infty} u_n(x) &= \cos x - x\sin x \\ &\quad + \int_0^x \left(\sin(x-t)\sum_{n=0}^{\infty} u_n(t) + \cos(x-t)\sum_{n=0}^{\infty} v_n(t)\right) dt, \\ \sum_{n=0}^{\infty} v_n(x) &= \sin x - x\cos x \\ &\quad + \int_0^x \left(\cos(x-t)\sum_{n=0}^{\infty} u_n(t) - \sin(x-t)\sum_{n=0}^{\infty} v_n(t)\right) dt. \end{aligned} \tag{10.14}$$

The zeroth components $u_0(x)$ and $v_0(x)$ are defined by all terms that are not included under the integral sign. For this example, we will use the modified decomposition method, therefore we set the recursive relation

$$\begin{aligned} u_0(x) &= \cos x, \\ u_{k+1}(x) &= -x\sin x + \int_0^x (\sin(x-t)u_k(t) + \cos(x-t)v_k(t))\,dt, \quad k \geqslant 0, \end{aligned} \tag{10.15}$$

and

$$\begin{aligned} v_0(x) &= \sin x, \\ v_{k+1}(x) &= -x\cos x + \int_0^x (\cos(x-t)u_k(t) - \sin(x-t)v_k(t))\,dt, \quad k \geqslant 0. \end{aligned} \tag{10.16}$$

This in turn gives

$$u_0(x) = \cos x, \quad u_1(x) = 0, \quad u_{k+1}(x) = 0, \quad k \geqslant 1, \tag{10.17}$$

and

$$v_0(x) = \sin x, \quad v_1(x) = 0, \quad v_{k+1}(x) = 0, \quad k \geqslant 1. \tag{10.18}$$

This gives the exact solutions

$$(u(x), v(x)) = (\cos x, \sin x), \tag{10.19}$$

that satisfy the system (10.12).

Example 10.3

Use the Adomian decomposition method to solve the following system of Volterra integral equations

$$\begin{aligned} u(x) &= 1 + x^2 - \frac{1}{3}x^3 - \frac{1}{3}x^4 + \int_0^x \left((x-t)^3 u(t) + (x-t)^2 v(t)\right) dt, \\ v(x) &= 1 + x - x^3 - \frac{1}{4}x^4 - \frac{1}{4}x^5 + \int_0^x \left((x-t)^4 u(t) + (x-t)^3 v(t)\right) dt. \end{aligned} \tag{10.20}$$

Proceeding as before we find

$$\begin{aligned} \sum_{n=0}^{\infty} u_n(x) &= 1 + x^2 - \frac{1}{3}x^3 - \frac{1}{3}x^4 \\ &\quad + \int_0^x \left((x-t)^3 \sum_{n=0}^{\infty} u_n(t) + (x-t)^2 \sum_{n=0}^{\infty} v_n(t)\right) dt, \\ \sum_{n=0}^{\infty} v_n(x) &= 1 + x - x^3 - \frac{1}{4}x^4 - \frac{1}{4}x^5 \\ &\quad + \int_0^x \left((x-t)^4 \sum_{n=0}^{\infty} u_n(t) + (x-t)^3 \sum_{n=0}^{\infty} v_n(t)\right) dt. \end{aligned} \tag{10.21}$$

We next set the recursive relations

$$\begin{aligned} u_0(x) &= 1 + x^2 - \frac{1}{3}x^3 - \frac{1}{3}x^4, \\ u_{k+1}(x) &= \int_0^x \left((x-t)^3 u_k(t) + (x-t)^2 v_k(t)\right) dt, \quad k \geqslant 0, \end{aligned} \tag{10.22}$$

and

$$\begin{aligned} v_0(x) &= 1 + x - x^3 - \frac{1}{4}x^4 - \frac{1}{4}x^5, \\ v_{k+1}(x) &= \int_0^x \left((x-t)^4 u_k(t) + (x-t)^3 v_k(t)\right) dt, \quad k \geqslant 0. \end{aligned} \tag{10.23}$$

This in turn gives

$$\begin{aligned} u_0(x) &= 1 + x^2 - \frac{1}{3}x^3 - \frac{1}{3}x^4, \\ u_1(x) &= \frac{1}{3}x^3 + \frac{1}{3}x^4 + \cdots, \end{aligned} \tag{10.24}$$

and

$$\begin{aligned} v_0(x) &= 1 + x - x^3 - \frac{1}{4}x^4 - \frac{1}{4}x^5, \\ v_1(x) &= \frac{1}{4}x^4 + \frac{1}{4}x^5 + \cdots. \end{aligned} \tag{10.25}$$

It is obvious that the noise terms $\mp\frac{1}{3}x^3$ and $\mp\frac{1}{3}x^4$ appear between $u_0(x)$ and $u_1(x)$. Moreover, the noise terms $\mp\frac{1}{4}x^4$ and $\mp\frac{1}{4}x^5$ appear between $v_0(x)$ and $v_1(x)$. By canceling these noise terms from $u_0(x)$ and $v_0(x)$, the non-canceled terms of $u_0(x)$ and $v_0(x)$ give the exact solutions

$$(u(x), v(x)) = (1+x^2, 1+x-x^3), \tag{10.26}$$

that satisfy the given system (10.20).

Example 10.4

Use the Adomian decomposition method to solve the following system of Volterra integral equations

$$\begin{aligned} u(x) &= e^x - 2x + \int_0^x \left(e^{-t}u(t) + e^t v(t)\right) dt, \\ v(x) &= e^{-x} + \sinh 2x + \int_0^x \left(e^t u(t) + e^{-t} v(t)\right) dt. \end{aligned} \tag{10.27}$$

Proceeding as before we find

$$\begin{aligned} \sum_{n=0}^{\infty} u_n(x) &= e^x - 2x + \int_0^x \left(e^{-t}\sum_{n=0}^{\infty} u_n(t) + e^t \sum_{n=0}^{\infty} v_n(t)\right) dt, \\ \sum_{n=0}^{\infty} v_n(x) &= e^{-x} + \sinh 2x + \int_0^x \left(e^{t}\sum_{n=0}^{\infty} u_n(t) + e^{-t} \sum_{n=0}^{\infty} v_n(t)\right) dt. \end{aligned} \tag{10.28}$$

Proceeding as before we set

$$u_0(x) = e^x - 2x, \quad u_1(x) = 2x + \cdots, \tag{10.29}$$

and

$$v_0(x) = e^{-x} + \sinh 2x, \quad v_1(x) = -\sinh 2x + \cdots. \tag{10.30}$$

By canceling the noise terms from $u_0(x)$ and $v_0(x)$, the non-canceled terms of $u_0(x)$ and $v_0(x)$ give the exact solutions

$$(u(x), v(x)) = (e^x, e^{-x}), \tag{10.31}$$

that satisfy the given system (10.27).

Exercises 10.2.1

Use the Adomian decomposition method to solve the following systems of Volterra integral equations

1. $$\begin{cases} u(x) = x^2 - \frac{1}{12}x^5 + \int_0^x \left((x-t)^2 u(t) + (x-t)v(t)\right) dt \\ v(x) = x^3 - \frac{1}{30}x^6 + \int_0^x \left((x-t)^3 u(t) + (x-t)^2 v(t)\right) dt \end{cases}$$

2. $$\begin{cases} u(x) = 1 + x + x^2 - x^3 + \int_0^x (xtu(t) + xtv(t))\,dt \\ v(x) = 1 - x - x^2 - \frac{1}{3}x^3 - \frac{1}{6}x^4 - \int_0^x ((x-t)u(t) + (x-t)v(t))\,dt \end{cases}$$

3. $$\begin{cases} u(x) = 1 - x^2 + x^3 + \int_0^x ((x-t)u(t) + (x-t)v(t))\,dt \\ v(x) = 1 - x^3 - \frac{1}{10}x^5 + \int_0^x ((x-t)u(t) - (x-t)v(t))\,dt \end{cases}$$

4. $$\begin{cases} u(x) = x^2 + x^3 - \frac{1}{30}x^6 + \int_0^x \left((x-t)^2u(t) - (x-t)^2v(t)\right)dt \\ v(x) = x^2 - x^3 - \frac{1}{6}x^4 + \int_0^x ((x-t)u(t) + (x-t)v(t))\,dt \end{cases}$$

5. $$\begin{cases} u(x) = \cos x - v + \int_0^x (\cos(x-t)u(t) + \sin(x-t)v(t))\,dt \\ v(x) = \sin x - xv + \int_0^x (\sin(x-t)u(t) + \cos(x-t)v(t))\,dt \end{cases}$$

6. $$\begin{cases} u(x) = \sin x - xu + \int_0^x (\cos(x-t)u(t) + \sin(x-t)v(t))\,dt \\ v(x) = \cos x - u + \int_0^x (\sin(x-t)u(t) + \cos(x-t)v(t))\,dt \end{cases}$$

7. $$\begin{cases} u(x) = \sec^2 x - 2\tan x + x + \int_0^x (u(t) + v(t))\,dt \\ v(x) = \tan^2 x - x + \int_0^x (u(t) - v(t))\,dt \end{cases}$$

8. $$\begin{cases} u(x) = \sin^2 x - \frac{1}{2}x^2 + \int_0^x ((x-t)u(t) + (x-t)v(t))\,dt \\ v(x) = \cos^2 x + \frac{1}{2}x - \frac{1}{2}\sin 2x + \int_0^x \left((x-t)^2u(t) - (x-t)^2v(t)\right)dt \end{cases}$$

9. $$\begin{cases} u(x) = e^{-x} - \sinh 2x + \int_0^x \left(e^{-t}u(t) + e^{t}v(t)\right)dt \\ v(x) = e^{x} - 2x + \int_0^x \left(e^{t}u(t) + e^{-t}v(t)\right)dt \end{cases}$$

10. $$\begin{cases} u(x) = 1 - 2x + e^x + \int_0^x (u(t) + v(t))\,dt \\ v(x) = 1 - x + \frac{1}{2}x^2 - (x+1)e^x + \int_0^x (u(t) - tv(t))\,dt \end{cases}$$

11. $$\begin{cases} u(x) = 1 - 2x + \sin x + \int_0^x (u(t) + v(t))\,dt \\ v(x) = 1 - x^2 - \sin x + \int_0^x (tu(t) + tv(t))\,dt \end{cases}$$

12. $$\begin{cases} u(x) = x - x^2 + \cos x + \int_0^x (u(t) + v(t))\, dt \\ v(x) = x - \frac{2}{3}x^3 - \cos x + \int_0^x (tu(t) + tv(t))\, dt \end{cases}$$

10.2.2 The Laplace Transform Method

The Laplace transform method is a powerful technique that can be used for solving initial value problems and integral equations as well. The Laplace transform method was presented in Chapter 1, and has been used in other chapters in this text. Before we start applying this method, we summarize some of the concepts presented in Section 1.5. In the convolution theorem for the Laplace transform, it was stated that if the kernel $K(x,t)$ of the integral equation

$$u(x) = f(x) + \lambda \int_0^x K(x,t)u(t)dt, \tag{10.32}$$

depends on the difference $x - t$, then it is called a *difference kernel*. Examples of the difference kernel are $e^{x-t}, \cos(x-t)$, and $x - t$. The integral equation can thus be expressed as

$$u(x) = f(x) + \lambda \int_0^x K(x-t)u(t)dt. \tag{10.33}$$

Consider two functions $f_1(x)$ and $f_2(x)$ that possess the conditions needed for the existence of Laplace transform for each. Let the Laplace transforms for the functions $f_1(x)$ and $f_2(x)$ be given by

$$\mathcal{L}\{f_1(x)\} = F_1(s), \quad \mathcal{L}\{f_2(x)\} = F_2(s). \tag{10.34}$$

The Laplace convolution product of these two functions is defined by

$$(f_1 * f_2)(x) = \int_0^x f_1(x-t)f_2(t)dt, \tag{10.35}$$

or

$$(f_2 * f_1)(x) = \int_0^x f_2(x-t)f_1(t)dt. \tag{10.36}$$

Recall that

$$(f_1 * f_2)(x) = (f_2 * f_1)(x). \tag{10.37}$$

We can easily show that the Laplace transform of the convolution product $(f_1 * f_2)(x)$ is given by

$$\mathcal{L}\{(f_1 * f_2)(x)\} = \mathcal{L}\{\int_0^x f_1(x-t)f_2(t)dt\} = F_1(s)F_2(s). \tag{10.38}$$

Based on this summary, we will examine specific Volterra integral equations where the kernel is a difference kernel. Recall that we will apply the Laplace transform method and the inverse of the Laplace transform using Table 1.1

in Section 1.5. The Laplace transform method for solving systems of Volterra integral equations will be illustrated by studying the following examples.

Example 10.5

Solve the system of Volterra integral equations by using the Laplace transform method

$$\begin{aligned} u(x) &= 1 - x^2 + x^3 + \int_0^x ((x-t)u(t) + (x-t)v(t))\,dt, \\ v(x) &= 1 - x^3 - \tfrac{1}{10}x^5 + \int_0^x ((x-t)u(t) - (x-t)v(t))\,dt. \end{aligned} \tag{10.39}$$

Notice that the kernels $K_1(x-t) = K_2(x-t) = x-t$. Taking Laplace transform of both sides of each equation in (10.39) gives

$$\begin{aligned} U(s) &= \mathcal{L}\{u(x)\} = \mathcal{L}\{1 - x^2 + x^3\} + \mathcal{L}\{(x-t) * u(x) + (x-t) * v(x)\}, \\ V(s) &= \mathcal{L}\{v(x)\} = \mathcal{L}\left\{1 - x^3 - \frac{1}{10}x^5\right\} + \mathcal{L}\{(x-t) * u(x) - (x-t) * v(x)\}. \end{aligned} \tag{10.40}$$

This in turn gives

$$\begin{aligned} U(s) &= \frac{1}{s} - \frac{2}{s^3} + \frac{6}{s^4} + \frac{1}{s^2}U(s) + \frac{1}{s^2}V(s), \\ V(s) &= \frac{1}{s} - \frac{6}{s^4} - \frac{12}{s^6} + \frac{1}{s^2}U(s) - \frac{1}{s^2}V(s), \end{aligned} \tag{10.41}$$

or equivalently

$$\begin{aligned} \left(1 - \frac{1}{s^2}\right)U(s) - \frac{1}{s^2}V(s) &= \frac{1}{s} - \frac{2}{s^3} + \frac{6}{s^4}, \\ \left(1 + \frac{1}{s^2}\right)V(s) - \frac{1}{s^2}U(s) &= \frac{1}{s} - \frac{6}{s^4} - \frac{12}{s^6}. \end{aligned} \tag{10.42}$$

Solving this system of equations for $U(s)$ and $V(s)$ gives

$$\begin{aligned} U(s) &= \frac{1}{s} + \frac{3!}{s^4}, \\ V(s) &= \frac{1}{s} - \frac{3!}{s^4}. \end{aligned} \tag{10.43}$$

By taking the inverse Laplace transform of both sides of each equation in (10.43), the exact solutions are given by

$$(u(x), v(x)) = (1 + x^3, 1 - x^3). \tag{10.44}$$

Example 10.6

Solve the system of Volterra integral equations by using the Laplace transform method

$$\begin{aligned} u(x) &= \cos x - \sin x + \int_0^x (\cos(x-t)u(t) + \sin(x-t)v(t))\,dt, \\ v(x) &= \sin x - x\sin x + \int_0^x (\sin(x-t)u(t) + \cos(x-t)v(t))\,dt. \end{aligned} \tag{10.45}$$

Taking Laplace transform of both sides of each equation in (10.45) gives

$$\begin{aligned}
U(s) = \mathcal{L}\{u(x)\} &= \mathcal{L}\{\cos x - \sin x\} + \mathcal{L}\{\cos(x-t) * u(x) \\
&\quad + \sin(x-t) * v(x)\}, \\
V(s) = \mathcal{L}\{v(x)\} &= \mathcal{L}\{\sin x - x\sin x\} + \mathcal{L}\{\sin(x-t) * u(x) \\
&\quad + \cos(x-t) * v(x)\}.
\end{aligned} \tag{10.46}$$

This in turn gives

$$\begin{aligned}
U(s) &= \frac{s}{1+s^2} - \frac{1}{1+s^2} + \frac{s}{1+s^2}U(s) + \frac{1}{1+s^2}V(s), \\
V(s) &= \frac{1}{1+s^2} - \frac{2s}{(1+s^2)^2} + \frac{1}{1+s^2}U(s) + \frac{s}{1+s^2}V(s),
\end{aligned} \tag{10.47}$$

or equivalently

$$\begin{aligned}
\left(1 - \frac{s}{1+s^2}\right)U(s) - \frac{1}{1+s^2}V(s) &= \frac{s}{1+s^2} - \frac{1}{1+s^2}, \\
\left(1 - \frac{s}{1+s^2}\right)V(s) - \frac{1}{1+s^2}U(s) &= \frac{1}{1+s^2} - \frac{2s}{(1+s^2)^2}.
\end{aligned} \tag{10.48}$$

Solving this system of equations for $U(s)$ and $V(s)$ gives

$$U(s) = \frac{s}{1+s^2}, \quad V(s) = \frac{1}{1+s^2}. \tag{10.49}$$

By taking the inverse Laplace transform of both sides of each equation in (10.49), the exact solutions are given by

$$(u(x), v(x)) = (\cos x, \sin x). \tag{10.50}$$

Example 10.7

Solve the system of Volterra integral equations by using the Laplace transform method

$$\begin{aligned}
u(x) &= 2 - e^{-x} + \int_0^x ((x-t)u(t) + (x-t)v(t))\,dt, \\
v(x) &= 2x - e^{x} + 2e^{-x} + \int_0^x ((x-t)u(t) - (x-t)v(t))\,dt.
\end{aligned} \tag{10.51}$$

Taking Laplace transform of both sides of each equation in (10.51) gives

$$\begin{aligned}
U(s) &= \mathcal{L}\{2 - e^{-x}\} + \mathcal{L}\{(x-t) * u(x) + (x-t) * v(x)\}, \\
V(s) &= \mathcal{L}\{2x - e^{x} + 2e^{-x}\} + \mathcal{L}\{(x-t) * u(x) - (x-t) * v(x)\}.
\end{aligned} \tag{10.52}$$

This in turn gives

$$\begin{aligned}
U(s) &= \frac{2}{s} - \frac{1}{s+1} + \frac{1}{s^2}U(s) + \frac{1}{s^2}V(s), \\
V(s) &= \frac{2}{s^2} - \frac{1}{s-1} + \frac{2}{s+1} + \frac{1}{s^2}U(s) - \frac{1}{s^2}V(s),
\end{aligned} \tag{10.53}$$

or equivalently

$$\begin{aligned}\left(1-\frac{1}{s^2}\right)U(s)-\frac{1}{s^2}V(s)&=\frac{2}{s}-\frac{1}{s+1},\\ \left(1+\frac{1}{s^2}\right)V(s)-\frac{1}{s^2}U(s)&=\frac{2}{s^2}-\frac{1}{s-1}+\frac{2}{s+1}.\end{aligned}\tag{10.54}$$

Solving this system of equations for $U(s)$ and $V(s)$ gives

$$U(s)=\frac{1}{s-1},\quad V(s)=\frac{1}{s+1}.\tag{10.55}$$

Taking the inverse Laplace transform of (10.55) gives the exact solutions by

$$(u(x),v(x))=(e^x,e^{-x}).\tag{10.56}$$

Example 10.8

Solve the system of Volterra integral equations by using the Laplace transform method

$$\begin{aligned}u(x)&=x-\frac{1}{12}x^4-\frac{1}{20}x^5+\int_0^x((x-t)v(t)+(x-t)w(t))\,dt,\\ v(x)&=x^2-\frac{1}{6}x^3-\frac{1}{20}x^5+\int_0^x((x-t)u(t)+(x-t)w(t))\,dt,\\ w(x)&=\frac{5}{6}x^3-\frac{1}{12}x^4+\int_0^x((x-t)u(t)+(x-t)v(t))\,dt.\end{aligned}\tag{10.57}$$

Taking Laplace transform of both sides of each equation in (10.57) gives

$$\begin{aligned}U(s)&=\mathcal{L}\left\{x-\frac{1}{12}x^4-\frac{1}{20}x^5\right\}+\mathcal{L}\left\{(x-t)*v(x)+(x-t)*w(x)\right\},\\ V(s)&=\mathcal{L}\left\{x^2-\frac{1}{6}x^3-\frac{1}{20}x^5\right\}+\mathcal{L}\left\{(x-t)*u(x)+(x-t)*w(x)\right\},\\ W(s)&=\mathcal{L}\left\{\frac{5}{6}x^3-\frac{1}{12}x^4\right\}+\mathcal{L}\left\{(x-t)*u(x)+(x-t)*v(x)\right\}\end{aligned}\tag{10.58}$$

Proceeding as before we find

$$\begin{aligned}U(s)-\frac{1}{s^2}V(s)-\frac{1}{s^2}W(s)&=\frac{1}{s^2}-\frac{2}{s^5}-\frac{6}{s^6},\\ V(s)-\frac{1}{s^2}U(s)-\frac{1}{s^2}W(s)&=\frac{2}{s^3}-\frac{1}{s^4}-\frac{6}{s^6},\\ W(s)-\frac{1}{s^2}U(s)-\frac{1}{s^2}V(s)&=\frac{5}{s^4}-\frac{10}{s^5}.\end{aligned}\tag{10.59}$$

Solving this system of equations for $U(s),V(s)$ and $W(s)$ gives

$$U(s)=\frac{1}{s^2},\quad V(s)=\frac{2!}{s^3},\quad W(s)=\frac{3!}{s^4}.\tag{10.60}$$

By taking the inverse Laplace transform of both sides of each equation in (10.60), the exact solutions are given by

$$(u(x),\quad v(x),\quad w(x))=(x,x^2,x^3).\tag{10.61}$$

Exercises 10.2.2

Use the Laplace transform method to solve the following systems of Volterra integral equations

1. $$\begin{cases} u(x) = x^2 - \dfrac{1}{12}x^5 + \displaystyle\int_0^x \big((x-t)^2u(t) + (x-t)v(t)\big)\,dt \\ v(x) = x^3 - \dfrac{1}{30}x^6 + \displaystyle\int_0^x \big((x-t)^3u(t) + (x-t)^2v(t)\big)\,dt \end{cases}$$

2. $$\begin{cases} u(x) = 1 + x + \displaystyle\int_0^x ((x-t)u(t) + (x-t)v(t))\,dt \\ v(x) = 1 - x - 2x^2 - \dfrac{2}{3}x^3 + \displaystyle\int_0^x (u(t) - v(t))\,dt \end{cases}$$

3. $$\begin{cases} u(x) = 1 + x^3 - \dfrac{1}{10}x^5 + \displaystyle\int_0^x ((x-t)u(t) - (x-t)v(t))\,dt \\ v(x) = 1 - x^3 - \dfrac{1}{2}x^4 + \displaystyle\int_0^x (u(t) - v(t))\,dt \end{cases}$$

4. $$\begin{cases} u(x) = x^2 + x^3 - \dfrac{1}{30}x^6 + \displaystyle\int_0^x \big((x-t)^2u(t) - (x-t)^2v(t)\big)\,dt \\ v(x) = x^2 - x^3 - \dfrac{1}{6}x^4 + \displaystyle\int_0^x ((x-t)u(t) + (x-t)v(t))\,dt \end{cases}$$

5. $$\begin{cases} u(x) = \cos x + 2\sin x - 1 + \displaystyle\int_0^x (\cos(x-t)u(t) + \sin(x-t)v(t))\,dt \\ v(x) = \sin x + x\cos x - 1 + \displaystyle\int_0^x (\sin(x-t)u(t) + \cos(x-t)v(t))\,dt \end{cases}$$

6. $$\begin{cases} u(x) = x - x\sin x + \displaystyle\int_0^x (\cos(x-t)u(t) + \sin(x-t)v(t))\,dt \\ v(x) = \cos x - \sin x + \displaystyle\int_0^x (\sin(x-t)u(t) + \cos(x-t)v(t))\,dt \end{cases}$$

7. $$\begin{cases} u(x) = 2 - \cosh x + \displaystyle\int_0^x (\cosh(x-t)u(t) + \sinh(x-t)v(t))\,dt \\ v(x) = 2 - 2\cosh x + \displaystyle\int_0^x (\sinh(x-t)u(t) + \cosh(x-t)v(t))\,dt \end{cases}$$

8. $$\begin{cases} u(x) = e^x - \sinh x - x\sinh x + \displaystyle\int_0^x (\cosh(x-t)u(t) + \sinh(x-t)v(t))\,dt \\ v(x) = e^{-x} - x\cosh x + \displaystyle\int_0^x (\sinh(x-t)u(t) + \cosh(x-t)v(t))\,dt \end{cases}$$

9. $$\begin{cases} u(x) = \cos x + 3\sin x - 2x + \displaystyle\int_0^x ((x-t)u(t) + (x-t)v(t))\,dt \\ v(x) = \cos x + \sin x - 2 + \displaystyle\int_0^x ((x-t)u(t) - (x-t)v(t))\,dt \end{cases}$$

10. $\begin{cases} u(x) = \dfrac{1}{2}e^x(\sin x + \cos x) + \displaystyle\int_0^x ((x-t)u(t) + (x-t)v(t))\,dt \\ v(x) = e^x(\cos x - \sin x) + \displaystyle\int_0^x (u(t) + v(t))\,dt \end{cases}$

11. $\begin{cases} u(x) = \cos x - \sin x - 1 + \displaystyle\int_0^x (v(t) + w(t))\,dt \\ v(x) = 3\cos x - \sin x - 2 + \displaystyle\int_0^x (u(t) + w(t))\,dt \\ w(x) = 2\cos x - 1 + \displaystyle\int_0^x (u(t) + v(t))\,dt \end{cases}$

12. $\begin{cases} u(x) = 1 - x^2 + \dfrac{1}{4}x^4 + \displaystyle\int_0^x (u(t) + v(t) - w(t))\,dt \\ v(x) = 2x + x^2 - \dfrac{2}{3}x^3 - \dfrac{1}{4}x^4 + \displaystyle\int_0^x (v(t) + w(t) - u(t))\,dt \\ w(x) = -x + x^2 + x^3 - \dfrac{1}{4}x^4 + \displaystyle\int_0^x (w(t) + u(t) - v(t))\,dt \end{cases}$

10.3 Systems of Volterra Integral Equations of the First Kind

The standard form of the systems of Volterra integral equations of the first kind is given by

$$\begin{aligned} f_1(x) &= \int_0^x \left(K_1(x,t)u(t) + \widetilde{K}_1(x,t)v(t) + \cdots\right) dt, \\ f_2(x) &= \int_0^x \left(K_2(x,t)u(t) + \widetilde{K}_2(x,t)v(t) + \cdots\right) dt, \\ &\ \ \vdots \end{aligned} \tag{10.62}$$

where the kernels $K_i(x,t)$ and $\widetilde{K}_i(x,t)$, and the functions $f_i(x)$ are given real-valued functions, and $u(x), v(x), \ldots$ are the unknown functions that will be determined. Recall that the unknown functions appear inside the integral sign for the Volterra integral equations of the first kind.

In this section we will discuss two main methods that are commonly used for handling the Volterra integral equations of the first kind. Other methods are available in the literature but will not be presented in this text.

10.3.1 The Laplace Transform Method

We first begin by using the Laplace transform method. Recall that the Laplace transform of the convolution product $(f_1 * f_2)(x)$ is given by

$$\mathcal{L}\{(f_1 * f_2)(x)\} = \mathcal{L}\left\{\int_0^x f_1(x-t)f_2(t)dt\right\} = F_1(s)F_2(s). \tag{10.63}$$

proceeding as in the previous section, we will examine specific systems of Volterra integral equations where the kernel is a difference kernel. We will apply the Laplace transform method and the inverse of the Laplace transform using Table 1.1 in Section 1.5. The Laplace transform method for solving systems of Volterra integral equations will be illustrated by studying the following examples.

Example 10.9

Solve the system of Volterra integral equations of the first kind by using the Laplace transform method

$$\begin{aligned} \frac{1}{2}x^2 + \frac{1}{2}x^3 + \frac{1}{12}x^4 &= \int_0^x ((x-t-1)u(t) + (x-t+1)v(t))\,dt, \\ \frac{3}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{12}x^4 &= \int_0^x ((x-t+1)u(t) + (x-t-1)v(t))\,dt. \end{aligned} \tag{10.64}$$

Taking Laplace transform of both sides of each equation in (10.64) gives

$$\begin{aligned} \mathcal{L}\left\{\frac{1}{2}x^2 + \frac{1}{2}x^3 + \frac{1}{12}x^4\right\} &= \mathcal{L}\left\{(x-t-1) * u(x) + (x-t+1) * v(x)\right\}, \\ \mathcal{L}\left\{\frac{3}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{12}x^4\right\} &= \mathcal{L}\left\{(x-t+1) * u(x) + (x-t-1) * v(x)\right\}. \end{aligned} \tag{10.65}$$

This in turn gives

$$\begin{aligned} \left(\frac{1}{s^2} - \frac{1}{s}\right)U(s) + \left(\frac{1}{s^2} + \frac{1}{s}\right)V(s) &= \frac{1}{s^3} + \frac{3}{s^4} + \frac{2}{s^5}, \\ \left(\frac{1}{s^2} + \frac{1}{s}\right)U(s) + \left(\frac{1}{s^2} - \frac{1}{s}\right)V(s) &= \frac{3}{s^3} - \frac{1}{s^4} + \frac{2}{s^5} \end{aligned} \tag{10.66}$$

Solving this system of equations for $U(s)$ and $V(s)$ gives

$$U(s) = \frac{1}{s} + \frac{1}{s^2}, \quad V(s) = \frac{1}{s} + \frac{2}{s^3}. \tag{10.67}$$

By taking the inverse Laplace transform of both sides of each equation in (10.67), the exact solutions are given by

$$(u(x), v(x)) = (1 + x, 1 + x^2). \tag{10.68}$$

Example 10.10

Solve the system of Volterra integral equations of the first kind by using the Laplace transform method

$$\begin{aligned} e^x - 1 &= \int_0^x ((x-t)u(t) + (x-t+1)v(t))\,dt, \\ e^{-x} - 1 &= \int_0^x ((x-t-1)u(t) + (x-t)v(t))\,dt. \end{aligned} \tag{10.69}$$

Taking Laplace transform of both sides of each equation in (10.69) gives

$$\begin{aligned}\mathcal{L}\{e^{x}-1\} &= \mathcal{L}\{(x-t)*u(x)+(x-t+1)*v(x)\},\\ \mathcal{L}\{e^{-x}-1\} &= \mathcal{L}\{(x-t-1)*u(x)+(x-t)*v(x)\}.\end{aligned} \tag{10.70}$$

This in turn gives

$$\begin{aligned}\frac{1}{s^2}U(s)+\left(\frac{1}{s^2}+\frac{1}{s}\right)V(s) &= \frac{1}{s-1}-\frac{1}{s},\\ \left(\frac{1}{s^2}-\frac{1}{s}\right)U(s)+\frac{1}{s^2}V(s) &= \frac{1}{s+1}-\frac{1}{s}.\end{aligned} \tag{10.71}$$

Solving this system of equations for $U(s)$ and $V(s)$ gives

$$U(s)=\frac{1}{s-1},\quad V(s)=\frac{1}{s+1}. \tag{10.72}$$

By taking the inverse Laplace transform of both sides of each equation in (10.72), the exact solutions are given by

$$(u(x),v(x))=(e^{x},e^{-x}). \tag{10.73}$$

Example 10.11

Solve the system of Volterra integral equations of the first kind by using the Laplace transform method

$$\begin{aligned}1-\sin x-\cos x &= \int_0^x ((x-t-1)u(t)-(x-t)v(t))\,dt,\\ 3-\sin x-3\cos x &= \int_0^x ((x-t)u(t)-(x-t-1)v(t))\,dt.\end{aligned} \tag{10.74}$$

Taking Laplace transform of both sides of each equation in (10.74) gives

$$\begin{aligned}\left(\frac{1}{s^2}-\frac{1}{s}\right)U(s)-\frac{1}{s^2}V(s) &= \frac{1}{s}-\frac{s+1}{s^2+1},\\ \frac{1}{s^2}U(s)-\left(\frac{1}{s^2}-\frac{1}{s}\right)V(s) &= \frac{3}{s}-\frac{3s+1}{s^2+1}.\end{aligned} \tag{10.75}$$

Solving this system of equations for $U(s)$ and $V(s)$ gives

$$U(s)=\frac{1}{s^2+1}+\frac{s}{s^2+1},\quad V(s)=\frac{1}{s^2+1}-\frac{s}{s^2+1}. \tag{10.76}$$

Taking the inverse Laplace transform of (10.76) gives the exact solutions by

$$(u(x),v(x))=(\sin x+\cos x,\sin x-\cos x). \tag{10.77}$$

Example 10.12

Solve the system of Volterra integral equations of the first kind by using the Laplace transform method

$$
\begin{aligned}
-2+2\cosh x &= \int_0^x (u(t)-v(t))\,dt,\\
-1+x+\frac{1}{2}x^2+e^x &= \int_0^x ((x-t+1)v(t)-(x-t-1)w(t))\,dt, \qquad (10.78)\\
x+\frac{1}{2}x^2+xe^x &= \int_0^x ((x-t)w(t)+(x-t+1)u(t))\,dt.
\end{aligned}
$$

Taking Laplace transform of both sides of each equation in (10.78) and solving the system of equations for $U(s), V(s)$ and $W(s)$ we find

$$U(s)=\frac{1}{s}+\frac{1}{s-1},\quad V(s)=\frac{1}{s}+\frac{s}{s+1},\quad W(s)=\frac{1}{(s-1)^2} \qquad (10.79)$$

Taking the inverse Laplace transform of (10.79) gives the exact solutions by

$$(u(x),v(x),w(x))=(1+e^x,1+e^{-x},xe^x). \qquad (10.80)$$

Exercises 10.3.1

Use the Laplace transform method to solve the following systems of Volterra integral equations of the first kind

1. $$\begin{cases} x^2+\frac{2}{3}x^3 = \int_0^x ((x-t+1)u(t)+(x-t-1)v(t))\,dt\\ x^2-\frac{2}{3}x^3 = \int_0^x ((x-t-1)u(t)+(x-t+1)v(t))\,dt \end{cases}$$

2. $$\begin{cases} x^3 = \int_0^x ((x-t+1)u(t)+(x-t-1)v(t))\,dt\\ -\frac{1}{3}x^3 = \int_0^x ((x-t-1)u(t)+(x-t+1)v(t))\,dt \end{cases}$$

3. $$\begin{cases} 2x^2+\frac{1}{2}x^4 = \int_0^x ((x-t+1)u(t)+(x-t-1)v(t))\,dt\\ -\frac{1}{2}x^4 = \int_0^x ((x-t-1)u(t)+(x-t+1)v(t))\,dt \end{cases}$$

4. $$\begin{cases} 2+x-\sin x-2\cos x = \int_0^x ((x-t+1)u(t)+(x-t)v(t))\,dt\\ 1+x-\cos x = \int_0^x ((x-t)u(t)+(x-t+1)v(t))\,dt \end{cases}$$

5. $$\begin{cases} x+\frac{1}{2}x^2+\frac{1}{3}x^3-\sin x = \int_0^x ((x-t+1)u(t)+(x-t)v(t))\,dt\\ x-x^2 = \int_0^x ((x-t-1)u(t)-(x-t+1)v(t))\,dt \end{cases}$$

6. $$\begin{cases} 2x+x^2+2\sin x = \int_0^x ((x-t+2)u(t)+(x-t-2)v(t))\,dt\\ -2x = \int_0^x ((x-t-1)u(t)-(x-t+1)v(t))\,dt \end{cases}$$

7. $$\begin{cases} -4+x^2+4e^x = \displaystyle\int_0^x ((x-t+2)u(t)+(x-t-2)v(t))\,dt \\ -2-4x+2e^x = \displaystyle\int_0^x ((x-t+1)u(t)-(x-t+1)v(t))\,dt \end{cases}$$

8. $$\begin{cases} 6-6e^x+4xe^x = \displaystyle\int_0^x ((x-t)u(t)-(x-t+2)v(t))\,dt \\ 4-4e^x+2xe^x = \displaystyle\int_0^x ((x-t-1)u(t)-(x-t+1)v(t))\,dt \end{cases}$$

9. $$\begin{cases} -2-2x+2e^x = \displaystyle\int_0^x ((x-t-1)u(t)+(x-t+1)v(t))\,dt \\ -2-2x+2x^2+2e^x = \displaystyle\int_0^x ((x-t+1)u(t)+(x-t-1)v(t))\,dt \end{cases}$$

10. $$\begin{cases} x^2+\frac{1}{6}x^3+\frac{1}{12}x^4 = \displaystyle\int_0^x ((x-t)u(t)+(x-t)v(t))\,dt \\ x+\frac{1}{2}x^2+\frac{1}{3}x^4 = \displaystyle\int_0^x ((x-t)v(t)+w(t))\,dt \\ x+x^2+\frac{1}{20}x^3 = \displaystyle\int_0^x ((x-t)w(t)+u(t))\,dt \end{cases}$$

11. $$\begin{cases} x+\frac{1}{2}x^2-\sin x = \displaystyle\int_0^x ((x-t)u(t)+(x-t)v(t))\,dt \\ x = \displaystyle\int_0^x ((x-t)v(t)+w(t))\,dt \\ 1+x-\cos x = \displaystyle\int_0^x ((x-t)w(t)+u(t))\,dt \end{cases}$$

12. $$\begin{cases} 2-x-2\cos x = \displaystyle\int_0^x ((x-t)u(t)-(x-t)v(t)+w(t))\,dt \\ 1-\frac{1}{2}x^2+\sin x-\cos x = \displaystyle\int_0^x (u(t)-(x-t)v(t)+w(t))\,dt \\ -2+x+2\cos x = \displaystyle\int_0^x (u(t)-v(t)+(x-t)w(t))\,dt \end{cases}$$

10.3.2 Conversion to a Volterra System of the Second Kind

The conversion technique works effectively by using Leibnitz rule that was presented in Section 1.3. Differentiating both sides of each equation in (10.62), and using Leibnitz rule, we obtain

$$
\begin{aligned}
f_1'(x) &= K_1(x,x)u(x) + \widetilde{K}_1(x,x)v(x)\\
&\quad + \int_0^x \Big(K_{1_x}(x,t)u(t) + \widetilde{K}_{1_x}(x,t)v(t) + \cdots\Big)\,dt,\\
f_2'(x) &= K_2(x,x)u(x) + \widetilde{K}_2(x,x)v(x)\\
&\quad + \int_0^x \Big(K_{2_x}(x,t)u(t) + \widetilde{K}_{2_x}(x,t)v(t) + \cdots\Big)\,dt,\\
&\ \ \vdots
\end{aligned}
\tag{10.81}
$$

Three remarks can be made here:

1. If at least one of $K_i(x,x)$ and $\widetilde{K}_i(x,x), i = 1, 2, \ldots, n$ in each of the above equations does not vanish, then the system is reduced to a system of Volterra integral equations of the second kind. In this case, we can use any method that we studied before.
2. If $K_i(x,x) = 0$ and $\widetilde{K}_i(x,x) = 0, i = 1, 2, \ldots, n$, for any equation, and if $K_{i_x}(x,x) \neq 0$ and $\widetilde{K}_{i_x}(x,x) \neq 0$, then we differentiate again that equation.
3. The functions $f_i(x)$ must satisfy specific conditions to guarantee a unique continuous solution for each of the unknown solutions. The determination of these special conditions will be left as an exercise.

10.4 Systems of Volterra Integro-Differential Equations

Volterra studied the hereditary influences when he was examining a population growth model. The research work resulted in a specific topic, where both differential and integral operators appeared together in the same equation. This new type of equations was termed as Volterra integro-differential equations, given in the form

$$
u^{(i)}(x) = f(x) + \int_0^x K(x,t)u(t)dt, \tag{10.82}
$$

where $u^{(i)}(x) = \frac{d^n u}{dx^i}$. Because the resulted equation combines the differential operator and the integral operator, then it is necessary to define initial conditions $u(0), u'(0), \ldots, u^{(i-1)}(0)$ for the determination of the particular solution $u(x)$ of the Volterra integro-differential equation. The integro-differential equations were investigated in Chapter 5.

In this section, we will study systems of Volterra integro-differential equations of the second kind given by

$$
\begin{aligned}
u^{(i)}(x) &= f_1(x) + \int_0^x \Big(K_1(x,t)u(t) + \widetilde{K}_1(x,t)v(t) + \cdots\Big)\,dt,\\
v^{(i)}(x) &= f_2(x) + \int_0^x \Big(K_2(x,t)u(t) + \widetilde{K}_2(x,t)v(t) + \cdots\Big)\,dt.
\end{aligned}
\tag{10.83}
$$

The unknown functions $u(x), v(x), \ldots$, that will be determined, occur inside the integral sign whereas the derivatives of $u(x), v(x), \ldots$ appear mostly outside the integral sign. The kernels $K_i(x,t)$ and $\widetilde{K}_i(x,t)$, and the function $f_i(x)$ are given real-valued functions.

There is a variety of numerical and analytical methods that will be used for solving the system of integro-differential equations. However, in this section, we will present only two methods, new and traditional, that will be used for this study.

10.4.1 The Variational Iteration Method

In Chapter 3, the variational iteration method (VIM) was used before in this text. The method provides rapidly convergent successive approximations of the exact solution if such a closed form solution exists, and not components as in Adomian decomposition method. The variational iteration method [8] handles linear and nonlinear problems in the same manner without any need to specific restrictions such as the so called Adomian polynomials that we need for nonlinear terms.

The correction functionals for the Volterra system of integro-differential equations (10.83) are given by

$$\begin{aligned} u_{n+1}(x) &= u_n(x) + \int_0^x \lambda(t)\left(u_n^{(i)}(t) - f_1(t) - \int_0^t K(t,r)\tilde{u}_n(r)dr\right)dt, \\ v_{n+1}(x) &= v_n(x) + \int_0^x \lambda(t)\left(v_n^{(i)}(t) - f_2(t) - \int_0^t K(t,r)\tilde{v}_n(r)dr\right)dt. \end{aligned} \tag{10.84}$$

As presented before, the variational iteration method is used by applying two essential steps. It is required first to determine the Lagrange multiplier λ that can be identified optimally via integration by parts and by using a restricted variation. Having λ determined, an iteration formula, without restricted variation, should be used for the determination of the successive approximations $u_{n+1}(x), n \geqslant 0$ and $v_{n+1}(x), n \geqslant 0$ of the solutions $u(x)$ and $v(x)$. The zeroth approximations $u_0(x)$ and $v_0(x)$ can be any selective functions. However, the initial conditions are preferably used to select these approximations u_0 and $v_0(x)$ as will be seen later. Consequently, the solutions are given by

$$u(x) = \lim_{n\to\infty} u_n(x), \quad v(x) = \lim_{n\to\infty} v_n(x). \tag{10.85}$$

The VIM will be illustrated by studying the following examples.

Example 10.13

Use the VIM to solve the system of Volterra integro-differential equations

$$u'(x) = 1 + x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \int_0^x ((x-t)u(t) + (x-t+1)v(t))dt,$$
$$v'(x) = -1 - 3x - \frac{3}{2}x^2 - \frac{1}{3}x^3 + \int_0^x ((x-t+1)u(t) + (x-t)v(t))dt, \tag{10.86}$$

where $u(0) = 1, v(0) = 1$. The correction functionals for this system are given by

$$u_{n+1}(x) = u_n(x) - \int_0^x \left(u_n'(t) - 1 - t + \frac{1}{2}t^2 - \frac{1}{3}t^3 - I_1(t)\right) dt,$$
$$v_{n+1}(x) = v_n(x) - \int_0^x \left(v_n'(t) + 1 + 3t + \frac{3}{2}t^2 + \frac{1}{3}t^3 - I_2(t)\right) dt, \tag{10.87}$$

where

$$I_1(t) = \int_0^t ((t-r)u_n(r) + (t-r+1)v_n(r))dr,$$
$$I_2(t) = \int_0^t ((t-r+1)u_n(r) + (t-r)v_n(r))dr, \tag{10.88}$$

and $\lambda = -1$ for first order integro-differential equation. We can use the initial conditions to select $u_0(x) = u(0) = 1$ and $v_0(x) = v(0) = 1$. Using this selection into the correction functionals gives the following successive approximations

$$\begin{cases} u_0(x) = 1, \\ v_0(x) = 1, \end{cases}$$

$$\begin{cases} u_1(x) = 1 + x + x^2 + \frac{1}{6}x^3 + \frac{1}{12}x^4, \\ v_1(x) = 1 - x - x^2 - \frac{1}{6}x^3 - \frac{1}{12}x^4, \end{cases}$$

$$\begin{cases} u_2(x) = 1 + x + x^2 + \left(\frac{1}{6}x^3 - \frac{1}{6}x^3\right) + \left(\frac{1}{12}x^4 - \frac{1}{12}x^4\right) - \frac{1}{120}x^5 - \frac{1}{360}x^6, \\ v_2(x) = 1 - x - x^2 + \left(\frac{1}{6}x^3 - \frac{1}{6}x^3\right) + \left(\frac{1}{12}x^4 - \frac{1}{12}x^4\right) + \frac{1}{120}x^5 + \frac{1}{360}x^6, \end{cases}$$

$$\begin{cases} u_3(x) = 1 + x + x^2 + \left(\frac{1}{120}x^5 - \frac{1}{120}x^5\right) + \left(\frac{1}{360}x^6 - \frac{1}{360}x^6\right) + \cdots, \\ v_3(x) = 1 - x - x^2 + \left(\frac{1}{120}x^5 - \frac{1}{120}x^5\right) + \left(\frac{1}{360}x^6 - \frac{1}{360}x^6\right) + \cdots, \end{cases}$$

and so on. By canceling the noise terms, the exact solutions are given by

$$(u(x), v(x)) = (1 + x + x^2, 1 - x - x^2). \tag{10.89}$$

Example 10.14

Use the VIM to solve the system of Volterra integro-differential equations

$$u'(x) = 1 - x^2 + e^x + \int_0^x (u(t)+v(t))dt,\ u(0)=1,\ v(0)=-1,$$
$$v'(x) = 3 - 3e^x + \int_0^x (u(t)-v(t))dt. \tag{10.90}$$

The correction functionals for this system are given by

$$u_{n+1}(x) = u_n(x) - \int_0^x \left(u_n'(t) - 1 + t^2 - e^t - \int_0^t (u_n(r)+v_n(r))dr\right) dt,$$
$$v_{n+1}(x) = v_n(x) - \int_0^x \left(v_n'(t) - 3 + 3e^t - \int_0^t (u_n(r)-v_n(r))dr\right) dt. \tag{10.91}$$

We use the given initial conditions to select the zeroth approximations $u_0(x) = u(0) = 1$ and $v_0(x) = v(0) = -1$. Using this selection into the correction functionals gives the following successive approximations

$$\begin{cases} u_0(x) = 1 \\ v_0(x) = -1 \end{cases}$$

$$\begin{cases} u_1(x) = u_0(x) - \int_0^x \left(u_0'(t) - 1 + t^2 - e^t - \int_0^t (u_0(r)+v_0(r))dr\right) dt \\ \qquad\;\; = 1 - \frac{1}{3}x^3 \\ v_1(x) = v_0(x) - \int_0^x \left(v_0'(t) - 3 + 3e^t - \int_0^t (u_0(r)-v_0(r))dr\right) dt \\ \qquad\;\; = 2 + 3x - 3e^x + x^2 \end{cases}$$

$$\begin{cases} u_2(x) = u_1(x) - \int_0^x \left(u_1'(t) - 1 + t^2 - e^t - \int_0^t (u_1(r)+v_1(r))dr\right) dt \\ \qquad\;\; = x + \left(1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots\right) \\ v_2(x) = v_1(x) - \int_0^x \left(v_1'(t) - 3 + 3e^t - \int_0^t (u_1(r)-v_1(r))dr\right) dt \\ \qquad\;\; = x - \left(1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \cdots\right) \end{cases}$$

$$\begin{cases} u_3(x) = x + \left(1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \cdots\right) \\ v_3(x) = x - \left(1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \frac{1}{6!}x^6 + \cdots\right) \end{cases}$$

and so on. The exact solutions are therefore given by

$$(u(x), v(x)) = (x + e^x, x - e^x). \tag{10.92}$$

Example 10.15

Use the VIM to solve the system of Volterra integro-differential equations

$$u''(x) = -1 - x^2 - \sin x + \int_0^x (u(t) + v(t))dt, u(0) = 1, u'(0) = 1,$$
$$v''(x) = 1 - 2\sin x - \cos x + \int_0^x (u(t) - v(t))dt, v(0) = 0, v'(0) = 2. \tag{10.93}$$

The correction functionals for this system are given by

$$u_{n+1}(x) = u_n(x) + \int_0^x \left((t-x)(u_n''(t) + 1 + t^2 + \sin t - \int_0^t (u_n(r) + v_n(r))dr\right) dt,$$
$$v_{n+1}(x) = v_n(x) + \int_0^x \left((t-x)(v_n''(t) - 1 + 2\sin t + \cos t - \int_0^t (u_n(r) - v_n(r))dr\right) dt, \tag{10.94}$$

where we used $\lambda = t - x$ for second-order integro-differential equation.

We use the initial conditions to select $u_0(x) = 1 + x$ and $v_0(x) = 2x$. Using this selection into the correction functionals gives the following successive approximations

$$\begin{cases} u_0(x) = 1 + x, \\ v_0(x) = 2x, \end{cases}$$
$$\begin{cases} u_1(x) = 1 - \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \sin x, \\ v_1(x) = \cos x + 2\sin x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 - \frac{1}{4!}x^4 - 1, \end{cases}$$
$$\begin{cases} u_2(x) = x + \left(1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots\right), \\ v_2(x) = x + \left(x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \frac{1}{7!}x^7 + \cdots\right), \end{cases}$$

and so on. The exact solutions are therefore given by

$$(u(x), v(x)) = (x + \cos x, x + \sin x). \tag{10.95}$$

Example 10.16

Use the variational iteration method to solve the system of Volterra integro-differential equations

$$u'(x) = 2 + e^x - 3e^{2x} + e^{3x} + \int_0^x (6v(t) - 3w(t))dt, u(0) = 1,$$
$$v'(x) = e^x + 2e^{2x} - e^{3x} + \int_0^x (3w(t) - u(t))dt, v(0) = 1, \tag{10.96}$$
$$w'(x) = -e^x + e^{2x} + 3e^{3x} + \int_0^x (u(t) - 2v(t))dt, w(0) = 1.$$

The correction functionals for this system are given by

$$\begin{aligned}
u_{n+1}(x) &= u_n(x) \\
&\quad - \int_0^x \left(u_n'(t) - 2 - e^t + 3e^{2t} - e^{3t} - \int_0^t (6v_n(r) - 3w_n(r)dr) \right) dt, \\
v_{n+1}(x) &= v_n(x) \\
&\quad - \int_0^x \left(v_n'(t) - e^t - 2e^{2t} + e^{3t} - \int_0^t (3w_n(r) - u_n(r))dr \right) dt, \\
w_{n+1}(x) &= w_n(x) \\
&\quad - \int_0^x \left(w_n'(t) + e^t - e^{2t} - 3e^{3t} - \int_0^t (u_n(r) - 2v_n(r))dr \right) dt.
\end{aligned} \tag{10.97}$$

The initial conditions can be used to select the zeroth approximations as $u_0(x) = 1, v_0(x) = 1$ and $w_0(x) = 1$. Using this selection into the correction functionals gives the following successive approximations

$$\begin{cases} u_0(x) = 1 \\ v_0(x) = 1 \\ w_0(x) = 1 \end{cases}$$

$$\begin{cases} u_1(x) = \frac{7}{6} + 2x + \frac{3}{2}x^2 + e^x - \frac{3}{2}e^{2x} + \frac{1}{3}e^{3x} \\ v_1(x) = -\frac{2}{3} + x^2 + e^x + e^{2x} - \frac{1}{3}e^{3x} \\ w_1(x) = \frac{1}{2} - \frac{1}{2}x^2 - e^x + \frac{1}{2}e^{2x} + e^{3x} \end{cases}$$

$$\begin{cases} u_2(x) = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots \\ v_2(x) = 1 + 2x + \frac{1}{2!}(2x)^2 + \frac{1}{3!}(2x)^3 + \frac{1}{4!}(2x)^4 + \cdots \\ w_2(x) = 1 + 3x + \frac{1}{2!}(3x)^2 + \frac{1}{3!}(3x)^3 + \frac{1}{4!}(3x)^4 + \cdots \end{cases}$$

and so on. The exact solutions are therefore given by

$$(u(x), v(x), w(x)) = (e^x, e^{2x}, e^{3x}). \tag{10.98}$$

Exercises 10.4.1

Use the variational iteration method to solve the following systems of Volterra integro-differential equations

1. $$\begin{cases} u'(x) = 2x + \frac{1}{3}x^3 + \int_0^x ((1 - xt)u(t) - (1 + xt)v(t))\, dt, u(0) = 1 \\ v'(x) = -4x - \frac{1}{2}x^5 + \int_0^x ((1 + xt)u(t) + (1 - xt)v(t))\, dt, v(0) = 1 \end{cases}$$

2. $$\begin{cases} u'(x) = 3 - 3x + \frac{3}{2}x^3 + \int_0^x ((1-xt)u(t) + (1-xt)v(t))\,dt \\ v'(x) = -3 - 3x - \frac{3}{2}x^3 + \int_0^x ((1+xt)u(t) + (1+xt)v(t))\,dt \\ u(0) = 1, v(0) = 2 \end{cases}$$

3. $$\begin{cases} u'(x) = 1 - 4x + x^3 + \int_0^x ((1-xt)u(t) + (1-xt)v(t))\,dt, u(0) = 1 \\ v'(x) = -1 - x^3 + \int_0^x ((1+xt)u(t) + (1+xt)v(t))\,dt, v(0) = 1 \end{cases}$$

4. $$\begin{cases} u'(x) = -2x + x^3 + \cos x + \int_0^x ((1-xt)u(t) + (1-xt)v(t))\,dt \\ v'(x) = -2x - x^3 - \cos x + \int_0^x ((1+xt)u(t) + (1+xt)v(t))\,dt \\ u(0) = 1,\ v(0) = 1 \end{cases}$$

5. $$\begin{cases} u'(x) = 1 - x + \sin x + \int_0^x ((x-t)u(t) - (x-t)v(t))\,dt,\ u(0) = 1 \\ v'(x) = -1 + \cos x + \int_0^x (u(t) - v(t))\,dt,\ v(0) = 2 \end{cases}$$

6. $$\begin{cases} u'(x) = 1 - x^2 - \sin x + \int_0^x (u(t) + v(t))\,dt \\ v'(x) = 1 + \frac{2}{3}x^4 - \sin x + \int_0^x ((1-xt)u(t) - (1+xt)v(t))\,dt \\ u(0) = 1,\ v(0) = 1 \end{cases}$$

7. $$\begin{cases} u'(x) = 2 - e^{2x} + \int_0^x (u(t) + v(t))\,dt \\ v'(x) = -\frac{3}{2} + \frac{11}{2}e^{2x} + \int_0^x ((x-t-1)u(t) - (x-t+1)v(t))\,dt \\ u(0) = 1,\ v(0) = 2 \end{cases}$$

8. $$\begin{cases} u'(x) = -3x + e^x + \int_0^x (u(t) + v(t))\,dt,\ u(0) = 2 \\ v'(x) = 2 + x^2 - 3e^x + \int_0^x (u(t) - (x-t)v(t))\,dt,\ v(0) = 1 \end{cases}$$

9. $$\begin{cases} u'(x) = 3 + \frac{2}{3}x^4 - e^x + \int_0^x ((1-xt)u(t) - (1+xt)v(t))\,dt \\ v'(x) = 3 - \frac{2}{3}x^4 - 3e^x + \int_0^x ((1+xt)u(t) - (1-xt)v(t))\,dt \\ u(0) = 1, v(0) = -1 \end{cases}$$

10. $$\begin{cases} u'(x) = -x - \frac{1}{2}x^2 + e^x + \int_0^x (v(t) + w(t))\,dt,\ u(0) = 2 \\ v'(x) = x - \frac{1}{2}x^2 - e^x + \int_0^x (w(t) - u(t))\,dt,\ v(0) = 0 \\ w'(x) = 3 - e^x + \int_0^x (u(t) - v(t))\,dt,\ w(0) = 1 \end{cases}$$

11. $$\begin{cases} u'(x) = -x - \frac{1}{2}x^2 - \sin x + \int_0^x (v(t) + w(t))\,dt, \;\; u(0) = 2 \\ v'(x) = x - \frac{1}{2}x^2 + \sin x + \int_0^x (w(t) - u(t))\,dt, \;\; v(0) = 0 \\ w'(x) = 1 - 3\sin x + \int_0^x (u(t) - v(t))\,dt, \;\; w(0) = 1 \end{cases}$$

12. $$\begin{cases} u'(x) = \cos x - 2\sin x - e^x + \int_0^x (u(t) - v(t) + w(t))\,dt, \;\; u(0) = 2 \\ v'(x) = -2 - \sin x + e^x + \int_0^x (u(t) - v(t) - w(t))\,dt, \;\; v(0) = 1 \\ w'(x) = 2 + \sin x - \cos x + \int_0^x (v(t) + w(t) \; - u(t))\,dt, \;\; w(0) = 1 \end{cases}$$

10.4.2 The Laplace Transform Method

The Laplace transform method was presented in Chapter 1, and was used in other chapters and in this chapter as well. Before we start applying this method, we summarize some of the concepts presented in this text. The Laplace transform of the convolution product $(f_1 * f_2)(x)$ is given by

$$\mathcal{L}\{(f_1 * f_2)(x)\} = \mathcal{L}\left\{\int_0^x f_1(x-t)f_2(t)dt\right\} = F_1(s)F_2(s). \tag{10.99}$$

Moreover, the Laplace Transforms of Derivatives can be summarized as follows:

$$\begin{aligned} \mathcal{L}\{f'(x)\} &= s\mathcal{L}\{f(x)\} - f(0), \\ \mathcal{L}\{f''(x)\} &= s^2\mathcal{L}\{f(x)\} - sf(0) - f'(0), \\ \mathcal{L}\{f''(x)\} &= s^3\mathcal{L}\{f(x)\} - s^2f(0) - sf'(0) - f''(0), \end{aligned} \tag{10.100}$$

and so on. Based on this summary, we will examine specific Volterra integro-differential equations where the kernel is a difference kernel. Recall that we will apply the Laplace transform method and the inverse of the Laplace transform using Table 1.1 in section 1.5. The Laplace transform method for solving systems of Volterra integro-differential equations will be illustrated by studying the following examples.

Example 10.17

Solve the system of Volterra integro-differential equations by using the Laplace transform method

$$\begin{aligned} u'(x) &= 2x^2 + \int_0^x ((x-t)u(t) + (x-t)v(t))\,dt, u(0) = 1 \\ v'(x) &= -3x^2 - \frac{1}{10}x^5 + \int_0^x ((x-t)u(t) - (x-t)v(t))\,dt, \;\; v(0) = 1. \end{aligned} \tag{10.101}$$

Notice that the kernels $K_1(x-t) = K_2(x-t) = x-t$. Taking Laplace transform of both sides of each equation gives

$$\begin{aligned} sU(s) - 1 &= \frac{4}{s^3} + \frac{1}{s^2}U(s) + \frac{1}{s^2}V(s), \\ sV(s) - 1 &= -\frac{6}{s^3} - \frac{12}{s^6} + \frac{1}{s^2}U(s) - \frac{1}{s^2}V(s), \end{aligned} \tag{10.102}$$

or equivalently

$$\begin{aligned} \left(s - \frac{1}{s^2}\right)U(s) - \frac{1}{s^2}V(s) &= 1 + \frac{4}{s^3}, \\ \left(s + \frac{1}{s^2}\right)V(s) - \frac{1}{s^2}U(s) &= 1 - \frac{6}{s^3} - \frac{12}{s^6}. \end{aligned} \tag{10.103}$$

Solving this system of equations for $U(s)$ and $V(s)$ gives

$$U(s) = \frac{1}{s} + \frac{3!}{s^4}, \quad V(s) = \frac{1}{s} - \frac{3!}{s^4}. \tag{10.104}$$

By taking the inverse Laplace transform of both sides of each equation, the exact solutions are given by

$$(u(x), v(x)) = (1 + x^3, 1 - x^3). \tag{10.105}$$

Example 10.18

Solve the system of Volterra integro-differential equations by using the Laplace transform method

$$\begin{aligned} u'(x) &= 1 - 2\sin x + \int_0^x (\cos(x-t)u(t) + \cos(x-t)v(t))\, dt, \ u(0) = 1 \\ v'(x) &= -3 + 2\cos x + \int_0^x (\sin(x-t)u(t) + \sin(x-t)v(t))\, dt, \ v(0) = 1. \end{aligned} \tag{10.106}$$

Taking Laplace transform of both sides of each equation gives

$$\begin{aligned} sU(s) - 1 &= \frac{1}{s} - \frac{2}{s^2+1} + \frac{s}{s^2+1}U(s) + \frac{s}{s^2+1}V(s), \\ sV(s) - 1 &= -\frac{3}{s} + \frac{2s}{s^2+1} + \frac{1}{s^2+1}U(s) + \frac{1}{s^2+1}V(s). \end{aligned} \tag{10.107}$$

Solving this system of equations for $U(s)$ and $V(s)$ gives

$$U(s) = \frac{1}{s} + \frac{1}{s^2}, \quad V(s) = \frac{1}{s} - \frac{1}{s^2}. \tag{10.108}$$

By taking the inverse Laplace transform of both sides of each equation, the exact solutions are given by

$$(u(x), v(x)) = (1 + x, 1 - x). \tag{10.109}$$

Example 10.19

Solve the system of Volterra integro-differential equations by using the Laplace transform method

$$u'(x) = \cos x - 2\sin x + \int_0^x (\cos(x-t)u(t) + \sin(x-t)v(t))\,dt,$$
$$v'(x) = \cos x + x\cos x + \int_0^x (\sin(x-t)u(t) + \sin(x-t)v(t))\,dt, \quad (10.110)$$
$$u(0) = 1, v(0) = -1.$$

Taking Laplace transform of both sides of each equation gives

$$sU(s) - 1 = \frac{s}{s^2+1} - \frac{2}{s^2+1} + \frac{s}{s^2+1}U(s) + \frac{1}{s^2+1}V(s),$$
$$sV(s) + 1 = \frac{s}{s^2+1} + \frac{s^2-1}{(s^2+1)^2} + \frac{1}{s^2+1}U(s) + \frac{1}{s^2+1}V(s). \quad (10.111)$$

Solving this system of equations for $U(s)$ and $V(s)$ gives

$$U(s) = \frac{1}{s^2+1} + \frac{s}{s^2+1}, \quad V(s) = \frac{1}{s^2+1} - \frac{s}{s^2+1}. \quad (10.112)$$

By taking the inverse Laplace transform of both sides of each equation, the exact solutions are given by

$$(u(x), v(x)) = (\sin x + \cos x, \sin x - \cos x). \quad (10.113)$$

Example 10.20

Solve the system of Volterra integro-differential equations by using the Laplace transform method

$$u'(x) = e^x - e^{2x} + e^{4x} + \int_0^x (2v(t) - 4w(t))\,dt,\ u(0) = 1,\ u'(0) = 1,$$
$$v'(x) = e^x + 4e^{2x} - e^{4x} + \int_0^x (4w(t) - u(t))\,dt,\ v(0) = 1,\ v'(0) = 2,$$
$$w'(x) = -e^x + e^{2x} + 16e^{4x} + \int_0^x (u(t) - 2v(t))\,dt,\ w(0) = 1,\ w'(0) = 4.$$
$$(10.114)$$

Taking Laplace transform of both sides of each equation gives

$$s^2U(s) - s - 1 = \frac{1}{s-1} - \frac{1}{s-2} + \frac{1}{s-4} + \frac{2}{s}V(s) - \frac{4}{s}W(s),$$
$$s^2V(s) - s - 2 = \frac{1}{s-1} + \frac{4}{s-2} - \frac{1}{s-4} + \frac{4}{s}W(s) - \frac{1}{s}U(s), \quad (10.115)$$
$$s^2W(s) - s - 4 = -\frac{1}{s-1} + \frac{1}{s-2} + \frac{16}{s-4} + \frac{1}{s}U(s) - \frac{2}{s}V(s).$$

Solving this system of equations for $U(s)$ and $V(s)$ gives

$$U(s) = \frac{1}{s-1}, \quad V(s) = \frac{1}{s-2}, \quad W(s) = \frac{1}{s-4}. \quad (10.116)$$

By taking the inverse Laplace transform of both sides of each equation, the exact solutions are given by

$$(u(x), v(x), w(x)) = (e^x, e^{2x}, e^{4x}). \quad (10.117)$$

Exercises 10.4.2

Use the Laplace transform method to solve the following systems of Volterra integro-differential equations

1. $$\begin{cases} u'(x) = -x^2+\frac{1}{6}x^3+\int_0^x ((x-t)u(t)+(x-t+1)v(t))\,dt, \ u(0)=1 \\ v'(x) = -x-\frac{1}{12}x^4+\int_0^x ((x-t)u(t)-(x-t)v(t))\,dt, \ v(0)=1 \end{cases}$$

2. $$\begin{cases} u'(x) = 1+x-\frac{1}{3}x^3+\int_0^x ((x-t)u(t)+(x-t)v(t))\,dt, \ u(0)=0 \\ v'(x) = 1-x-\frac{1}{12}x^4+\int_0^x ((x-t)u(t)-(x-t)v(t))\,dt, \ v(0)=0 \end{cases}$$

3. $$\begin{cases} u'(x) = 2-2\sin x+\int_0^x (\cos(x-t)u(t)+\cos(x-t)v(t))\,dt \\ v'(x) = -4+2\cos x+\int_0^x (\sin(x-t)u(t)+\sin(x-t)v(t))\,dt \\ u(0)=1, \ v(0)=1 \end{cases}$$

4. $$\begin{cases} u'(x) = 3+3x-2e^x+\int_0^x \left(e^{x-t}u(t)+e^{x-t}v(t)\right)dt, u(0)=0 \\ v'(x) = 3+x+x^2-2e^x+\int_0^x \left(e^{x-t}u(t)-e^{x-t}v(t)\right)dt, v(0)=0 \end{cases}$$

5. $$\begin{cases} u'(x) = 2\cos x-e^x+\int_0^x \left(e^{x-t}u(t)+e^{x-t}v(t)\right)dt, u(0)=0 \\ v'(x) = 1-x-\cos x+\int_0^x ((x-t)u(t)-(x-t)v(t))\,dt, v(0)=1 \end{cases}$$

6. $$\begin{cases} u'(x) = \cos x-x\sin x+\int_0^x (\cos(x-t)u(t)+\sin(x-t)v(t))\,dt \\ v'(x) = -2\sin x+\int_0^x (\sin(x-t)u(t)+\cos(x-t)v(t))\,dt \\ u(0)=0, \ v(0)=1 \end{cases}$$

7. $$\begin{cases} u'(x) = -2x-\frac{3}{2}x^2+\cos x+\int_0^x (u(t)+(x-t)v(t))\,dt, u(0)=2 \\ v'(x) = -4x-x^2+3\sin x+\int_0^x ((x-t)u(t)+v(t))\,dt, \ v(0)=2 \end{cases}$$

8. $$\begin{cases} u'(x) = 5-4e^x+\int_0^x \left(e^{x-t}u(t)+e^{x-t}v(t)\right)dt, \ u(0)=3 \\ v'(x) = -1-2xe^x+\int_0^x \left(e^{x-t}u(t)-e^{x-t}v(t)\right)dt, \ v(0)=2 \end{cases}$$

9. $$\begin{cases} u'(x) = 3+2x-3e^x+\int_0^x \left(e^{x-t}u(t)+e^{x-t}v(t)\right)dt, \ u(0)=-1 \\ v'(x) = 1+e^x+2xe^x+\int_0^x \left(e^{x-t}u(t)-e^{x-t}v(t)\right)dt, \ v(0)=1 \end{cases}$$

10. $$\begin{cases} u'(x) = -x - \dfrac{1}{2}x^2 - \sin x + \displaystyle\int_0^x (v(t) + w(t))\,dt,\ u(0) = 2 \\ v'(x) = x - \dfrac{1}{2}x^2 + \sin x + \displaystyle\int_0^x (w(t) - u(t))\,dt,\ v(0) = 0 \\ w'(x) = 1 - 3\sin x + \displaystyle\int_0^x (u(t) - v(t))\,dt, w(0) = 1 \end{cases}$$

11. $$\begin{cases} u'(x) = \cos x - 2\sin x - e^x + \displaystyle\int_0^x (u(t) - v(t) + w(t))\,dt,\ u(0) = 2 \\ v'(x) = -2 - \sin x + e^x + \displaystyle\int_0^x (u(t) - v(t) - w(t))\,dt,\ v(0) = 1 \\ w'(x) = 2 + \sin x - \cos x + \displaystyle\int_0^x (v(t) + w(t) - u(t))\,dt,\ w(0) = 1 \end{cases}$$

12. $$\begin{cases} u''(x) = -x^3 - x^4 + \displaystyle\int_0^x (3v(t) + 4w(t))\,dt,\ u(0) = 0,\ u'(0) = 1 \\ v''(x) = 2 + x^2 - x^4 + \displaystyle\int_0^x (4w(t) - 2u(t))\,dt,\ v(0) = 0,\ v'(0) = 0 \\ w''(x) = 6x - x^2 + x^3 + \displaystyle\int_0^x (2u(t) - 3v(t))\,dt,\ w(0) = 0,\ w'(0) = 0 \end{cases}$$

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Chapter 11
Systems of Fredholm Integral Equations

11.1 Introduction

Systems of Volterra and Fredholm integral equations have attracted much concern in applied sciences. The systems of Fredholm integral equations appear in two kinds. The system of Fredholm integral equations of the first kind [1–5] reads

$$\begin{aligned} f_1(x) &= \int_a^b \Big(K_1(x,t)u(t) + \tilde{K}_1(x,t)v(t)\Big)\,dt, \\ f_2(x) &= \int_a^b \Big(K_2(x,t)u(t) + \tilde{K}_2(x,t)v(t)\Big)\,dt, \end{aligned} \tag{11.1}$$

where the unknown functions $u(x)$ and $v(x)$ appear only under the integral sign, and a and b are constants. However, for systems of Fredholm integral equations of the second kind, the unknown functions $u(x)$ and $v(x)$ appear inside and outside the integral sign. The second kind is represented by the form

$$\begin{aligned} u(x) &= f_1(x) + \int_a^b \Big(K_1(x,t)u(t) + \tilde{K}_1(x,t)v(t)\Big)\,dt, \\ v(x) &= f_2(x) + \int_a^b \Big(K_2(x,t)u(t) + \tilde{K}_2(x,t)v(t)\Big)\,dt. \end{aligned} \tag{11.2}$$

The systems of Fredholm integro-differential equations have also attracted a considerable size of interest. These systems are given by

$$\begin{aligned} u^{(i)}(x) &= f_1(x) + \int_a^b \Big(K_1(x,t)u(t) + \tilde{K}_1(x,t)v(t)\Big)\,dt, \\ v^{(i)}(x) &= f_2(x) + \int_a^b \Big(K_2(x,t)u(t) + \tilde{K}_2(x,t)v(t)\Big)\,dt, \end{aligned} \tag{11.3}$$

where the initial conditions for the last system should be prescribed.

11.2 Systems of Fredholm Integral Equations

In this section we will study systems of Fredholm integral equations of the second kind given by

$$\begin{aligned} u(x) &= f_1(x) + \int_a^b \left(K_1(x,t)u(t) + \tilde{K}_1(x,t)v(t) + \cdots\right) dt, \\ v(x) &= f_2(x) + \int_a^b \left(K_2(x,t)u(t) + \tilde{K}_2(x,t)v(t) + \cdots\right) dt, \\ &\vdots \end{aligned} \tag{11.4}$$

The unknown functions $u(x), v(x), \ldots$ that will be determined, appear inside and outside the integral sign. The kernels $K_i(x,t)$ and $\tilde{K}_i(x,t)$, and the function $f_i(x)$ are given real-valued functions. In what follows we will present the methods, new and traditional, that will be used. Recall that the Fredholm integral equations were presented in Chapter 4 where a variety of methods, new and traditional, were applied. In this section, we will focus our study only on two methods, namely the Adomian decomposition method and the direct computation method.

11.2.1 The Adomian Decomposition Method

The Adomian decomposition method [6–7], as presented before, decomposes each solution as an infinite sum of components, where these components are determined recurrently. This method can be used in its standard form, or combined with the noise terms phenomenon. Moreover, the modified decomposition method will be used wherever it is appropriate.

Example 11.1

Use the Adomian decomposition method to solve the following system of Fredholm integral equations

$$\begin{aligned} u(x) &= \sin x - 2 - 2x - \pi x + \int_0^{\pi} \left((1+xt)u(t) + (1-xt)v(t)\right) dt, \\ v(x) &= \cos x - 2 - 2x + \pi x + \int_0^{\pi} \left((1-xt)u(t) - (1+xt)v(t)\right) dt. \end{aligned} \tag{11.5}$$

The Adomian decomposition method suggests that the linear terms $u(x)$ and $v(x)$ be decomposed by an infinite series of components

$$u(x) = \sum_{n=0}^{\infty} u_n(x), \quad v(x) = \sum_{n=0}^{\infty} v_n(x), \tag{11.6}$$

where $u_n(x)$ and $v_n(x), n \geqslant 0$ are the components of $u(x)$ and $v(x)$ that will be elegantly determined in a recursive manner.

Substituting (11.6) into (11.5) gives

$$\begin{aligned}\sum_{n=0}^{\infty} u_n(x) &= \sin x - 2 - 2x - \pi x \\ &\quad + \int_0^{\pi} \left((1+xt)\sum_{n=0}^{\infty} u_n(t) + (1-xt)\sum_{n=0}^{\infty} v_n(t)\right) dt, \\ \sum_{n=0}^{\infty} v_n(x) &= \cos x - 2 - 2x + \pi x \\ &\quad + \int_0^{\pi} \left((1-xt)\sum_{n=0}^{\infty} u_n(t) - (1+xt)\sum_{n=0}^{\infty} v_n(t)\right) dt.\end{aligned} \tag{11.7}$$

The modified decomposition method will be used here, hence we set the recursive relation

$$\begin{aligned} u_0(x) &= \sin x - 2, \\ v_0(x) &= \cos x - 2, \\ u_1(x) &= -2x - \pi x + \int_0^{\pi} ((1+xt)u_0(t) + (1-xt)v_0(t))\, dt, \\ v_1(x) &= -2x + \pi x + \int_0^{\pi} ((1-xt)u_0(t) - (1+xt)v_0(t))\, dt, \\ u_{k+1}(x) &= \int_0^{\pi} ((1-xt)u_k(t) - (1+xt)v_k(t))\, dt, \quad k \geqslant 1, \\ v_{k+1}(x) &= \int_0^{x} ((1-xt)u_k(t) - (1+xt)v_k(t))\, dt, \quad k \geqslant 1. \end{aligned} \tag{11.8}$$

This in turn gives

$$u_0(x) = \sin x - 2, \quad u_1(x) = 2 - 4\pi, \tag{11.9}$$

and

$$v_0(x) = \cos x - 2, \quad v_1(x) = 2 + 2\pi^2 x. \tag{11.10}$$

By canceling the noise terms ∓ 2 from $u_0(x)$ and from $v_0(x)$ we obtain the exact solutions

$$(u(x), v(x)) = (\sin x, \cos x). \tag{11.11}$$

Example 11.2

Use the Adomian decomposition method to solve the following system of Fredholm integral equations

$$\begin{aligned} u(x) &= x + \sec^2 x - \frac{\pi^3}{96} + \int_0^{\frac{\pi}{4}} (tu(t) + tv(t))\, dt, \\ v(x) &= x - \sec^2 x - \frac{\pi^2}{16} + \int_0^{\frac{\pi}{4}} (u(t) + v(t))\, dt. \end{aligned} \tag{11.12}$$

Proceeding as before we obtain

$$\sum_{n=0}^{\infty} u_n(x) = x + \sec^2 x - \frac{\pi^3}{96} + \int_0^{\frac{\pi}{4}} \left(t\sum_{n=0}^{\infty} u_n(t) + t\sum_{n=0}^{\infty} v_n(t) \right) dt,$$
$$\sum_{n=0}^{\infty} v_n(x) = x - \sec^2 x - \frac{\pi^2}{16} + \int_0^{\frac{\pi}{4}} \left(\sum_{n=0}^{\infty} u_n(t) + \sum_{n=0}^{\infty} v_n(t) \right) dt. \quad (11.13)$$

The modified decomposition method will be used here, hence we set the recursive relation

$$\begin{aligned} u_0(x) &= x + \sec^2 x, \quad v_0(x) = x - \sec^2 x, \\ u_1(x) &= -\frac{\pi^3}{96} + \int_0^{\frac{\pi}{4}} (tu_0(t) + tv_0(t))\, dt, \\ v_1(x) &= -\frac{\pi^2}{16} + \int_0^{\frac{\pi}{4}} (u_0(t) + v_0(t))\, dt, \\ u_{k+1}(x) &= \int_0^{\frac{\pi}{4}} (tu_k(t) + tv_k(t))\, dt, \quad k \geqslant 1, \\ v_{k+1}(x) &= \int_0^{\frac{\pi}{4}} (u_k(t) + v_k(t))\, dt, \quad k \geqslant 1. \end{aligned} \quad (11.14)$$

This in turn gives

$$u_0(x) = x + \sec^2 x, \quad u_1(x) = 0, \quad (11.15)$$

and

$$v_0(x) = x - \sec^2 x, \quad v_1(x) = 0. \quad (11.16)$$

As a result, the remaining components $u_k(x)$ and $v_k(x)$ for $k \geqslant 2$ vanish. The exact solutions are given by

$$(u(x), v(x)) = (x + \sec^2 x, x - \sec^2 x). \quad (11.17)$$

Example 11.3

Use the Adomian decomposition method to solve the following system of Fredholm integral equations

$$u(x) = e^x - \frac{2}{1+x}\sinh(1+x) + \int_0^1 \left(e^{xt}u(t) + e^{-xt}v(t)\right) dt,$$
$$v(x) = e^{-x} - \frac{2}{1-x}\sinh(1-x) + \int_0^1 \left(e^{-xt}u(t) + e^{xt}v(t)\right) dt. \quad (11.18)$$

Proceeding as before and using the modified decomposition method, we find the recursive relation

$$
\begin{aligned}
u_0(x) &= e^x, \quad v_0(x) = e^{-x},\\
u_1(x) &= \frac{2}{1+x}\sinh(1+x) + \int_0^1 \left(e^{xt}u_0(t) + e^{-xt}v_0(t)\right) dt = 0,\\
v_1(x) &= -\frac{2}{1-x}\sinh(1-x) + \int_0^1 \left(e^{-xt}u_0(t) + e^{xt}v_0(t)\right) dt = 0,\\
u_{k+1}(x) &= \int_0^1 \left(e^{xt}u_k(t) + e^{-xt}v_k(t)\right) dt = 0, \quad k \geqslant 1,\\
v_{k+1}(x) &= \int_0^1 \left(e^{-xt}u_k(t) + e^{xt}v_k(t)\right) dt = 0, \quad k \geqslant 1.
\end{aligned}
\tag{11.19}
$$

The exact solutions are given by

$$
(u(x), v(x)) = (e^x, e^{-x}). \tag{11.20}
$$

Example 11.4

Use the Adomian decomposition method to solve the following system of Fredholm integral equations

$$
\begin{aligned}
u(x) &= x - \frac{4}{3} + \int_{-1}^1 (v(t) + w(t))\, dt,\\
v(x) &= x + x^2 - \frac{2}{3} + \int_{-1}^1 (w(t) + u(t))\, dt,\\
w(x) &= x^2 + x^3 - \frac{2}{3} + \int_{-1}^1 (u(t) + v(t))\, dt.
\end{aligned}
\tag{11.21}
$$

Proceeding as before and using the modified decomposition method, we find the recursive relation

$$
\begin{aligned}
u_0(x) &= x,\\
v_0(x) &= x + x^2,\\
w_0(x) &= x^2 + x^3,\\
u_1(x) &= -\frac{4}{3} + \int_{-1}^1 (v_0(t) + w_0(t))\, dt = 0,\\
v_1(x) &= -\frac{2}{3} + \int_{-1}^1 (w_0(t) + u_0(t))\, dt = 0,\\
w_1(x) &= -\frac{2}{3} + \int_{-1}^1 (u_0(t) + v_0(t))\, dt = 0,\\
u_{k+1}(x) &= 0, \quad v_{k+1} = 0, \quad w_{k+1} = 0, \quad k \geqslant 1.
\end{aligned}
\tag{11.22}
$$

The exact solutions are given by

$$
(u(x), v(x), w(x)) = (x, x + x^2, x^2 + x^3). \tag{11.23}
$$

Exercises 11.2.1

Solve the following systems of Fredholm integral equations

1. $$\begin{cases} u(x) = x - \dfrac{16}{15} + \displaystyle\int_{-1}^{1} (tu(t) + tv(t))\, dt \\ v(x) = x^2 + x^3 - \dfrac{2}{3}x + \displaystyle\int_{-1}^{1} (xu(t) + xv(t))\, dt \end{cases}$$

2. $$\begin{cases} u(x) = x - \dfrac{2}{3}x^2 + \dfrac{2}{5} + \displaystyle\int_{-1}^{1} \big((x^2 - t^2)u(t) + (x^2 - t^2)v(t)\big)\, dt \\ v(x) = x^2 + x^3 + \dfrac{2}{5}x + \displaystyle\int_{-1}^{1} \big((x^2 - t^2)u(t) - xtv(t)\big)\, dt \end{cases}$$

3. $$\begin{cases} u(x) = x + x^2 - \dfrac{16}{15} + \displaystyle\int_{-1}^{1} (u(t) + v(t))\, dt \\ v(x) = x^3 + x^4 - \dfrac{16}{15}x + \dfrac{2}{3} + \displaystyle\int_{-1}^{1} ((x - t)u(t) + xtv(t))\, dt \end{cases}$$

4. $$\begin{cases} u(x) = \sin x + \cos x - 4 + \displaystyle\int_{0}^{\pi} (u(t) + v(t))\, dt \\ v(x) = \sin x - \cos x - 4x + 2\pi + \displaystyle\int_{0}^{\pi} ((x - t)u(t) + (x - t)v(t))\, dt \end{cases}$$

5. $$\begin{cases} u(x) = x + \sin^2 x - \pi^2 + \displaystyle\int_{0}^{\pi} (u(t) + v(t))\, dt \\ v(x) = x - \cos^2 x - \pi^2 x + \dfrac{\pi^2}{2} + \displaystyle\int_{0}^{\pi} ((x - t)u(t) + (x + t)v(t))\, dt \end{cases}$$

6. $$\begin{cases} u(x) = x^2 + \sin x + 2\pi + \displaystyle\int_{-\pi}^{\pi} ((x - t)u(t) - (x + t)v(t))\, dt \\ v(x) = x^2 + \cos x - 2\pi + \displaystyle\int_{-\pi}^{\pi} ((x + t)u(t) - (x - t)v(t))\, dt \end{cases}$$

7. $$\begin{cases} u(x) = \left(\dfrac{2 - \pi}{2}\right) x + x\tan^{-1} x + \displaystyle\int_{-1}^{1} (xu(t) + xv(t))\, dt \\ v(x) = \dfrac{3\pi - 2}{6} + x + \tan^{-1} x + \displaystyle\int_{-1}^{1} (tu(t) - tv(t))\, dt \end{cases}$$

8. $$\begin{cases} u(x) = -x + \dfrac{e^x}{1 + e^x} + \displaystyle\int_{0}^{1} (xu(t) + xv(t))\, dt \\ v(x) = -1 + \dfrac{1}{1 + e^x} + \displaystyle\int_{0}^{1} (u(t) + v(t))\, dt \end{cases}$$

9. $$\begin{cases} u(x) = (2 - \pi)x + \dfrac{\sin x}{1 + \sin x} + \displaystyle\int_{0}^{\frac{\pi}{2}} (xu(t) + xv(t))\, dt \\ v(x) = (2 - \pi) + \dfrac{\cos x}{1 + \cos x} + \displaystyle\int_{0}^{\frac{\pi}{2}} (u(t) + v(t))\, dt \end{cases}$$

10. $$\begin{cases} u(x) = \dfrac{3-\sqrt{3}}{3} + \sec x \tan x + \displaystyle\int_0^{\frac{\pi}{6}} (u(t) - v(t))\, dt \\ v(x) = (1-\sqrt{3})x + \sec^2 x + \displaystyle\int_0^{\frac{\pi}{6}} (xu(t) + xv(t))\, dt \end{cases}$$

11. $$\begin{cases} u(x) = x - \dfrac{1}{12} + \displaystyle\int_{-1}^{1} (v(t) - w(t))\, dt \\ v(x) = x^2 + \dfrac{1}{4} + \displaystyle\int_{-1}^{1} (w(t) - u(t))\, dt \\ w(x) = x^2 - \dfrac{1}{6} + \displaystyle\int_{-1}^{1} (u(t) - v(t))\, dt \end{cases}$$

12. $$\begin{cases} u(x) = 1 + \dfrac{5\pi}{8} + \dfrac{\pi}{8}\sec^2 x + \displaystyle\int_0^{\frac{\pi}{4}} (v(t) - w(t))\, dt \\ v(x) = 1 - \dfrac{3\pi}{8} - \dfrac{\pi}{8}\sec^2 x + \displaystyle\int_0^{\frac{\pi}{4}} (w(t) - u(t))\, dt \\ w(x) = 1 - \dfrac{\pi}{4} + \dfrac{\pi}{2}\sec^2 x + \displaystyle\int_0^{\frac{\pi}{4}} (u(t) - v(t))\, dt \end{cases}$$

11.2.2 The Direct Computation Method

It was stated before that all methods that we used in Chapter 4 to handle Fredholm integral equations can be used here to solve systems of Fredholm integral equations. In the previous section we selected the Adomian decomposition method. In this section we will employ the direct computational method. The other methods such as the variational iteration method, successive substitution method, and others can be used as well.

The direct computational method will be applied to solve the systems of Fredholm integral equations of the second kind. The method was used before in this text, therefore we summarize the necessary steps. The method will be applied for the degenerate or separable kernels of the form

$$K_1(x,t) = \sum_{k=1}^{n} g_k(x)h_k(t), \quad K_2(x,t) = \sum_{k=1}^{n} r_k(x)s_k(t). \tag{11.24}$$

The direct computation method can be applied as follows:

1. We first substitute (11.24) into the system of Fredholm integral equations the form

$$\begin{aligned} u(x) &= f_1(x) + \int_a^b \Big(K_1(x,t)u(t) + \tilde{K}_1(x,t)v(t)\Big)\, dt, \\ v(x) &= f_2(x) + \int_a^b \Big(K_2(x,t)u(t) + \tilde{K}_2(x,t)v(t)\Big)\, dt. \end{aligned} \tag{11.25}$$

2. This substitution gives

$$u(x) = f_1(x) + \sum_{k=1}^{n} g_k(x) \int_a^b h_k(t)u(t)dt + \sum_{k=1}^{n} \tilde{g}_k(x) \int_a^b \tilde{h}_k(t)v(t)dt,$$
$$v(x) = f_2(x) + \sum_{k=1}^{n} r_k(x) \int_a^b s_k(t)u(t)dt + \sum_{k=1}^{n} \tilde{r}_k(x) \int_a^b \tilde{s}_k(t)v(t)dt. \qquad (11.26)$$

3. Each integral at the right side depends only on the variable t with constant limits of integration for t. This means that each integral is equivalent to a constant. Based on this, Equation (11.26) becomes

$$u(x) = f_1(x) + \alpha_1 g_1(x) + \cdots + \alpha_n g_n(x) + \beta_1 \tilde{g}_1(x) + \cdots + \beta_n \tilde{g}_n(x),$$
$$v(x) = f_2(x) + \gamma_1 r_1(x) + \cdots + \gamma_n r_n(x) + \delta_1 \tilde{r}_1(x) + \cdots + \delta_n \tilde{r}_n(x), \qquad (11.27)$$

where

$$\alpha_i = \int_a^b h_i(t)u(t)dt,\ 1 \leqslant i \leqslant n,$$
$$\beta_i = \int_a^b \tilde{h}_i(t)v(t)dt, 1 \leqslant i \leqslant n,$$
$$\gamma_i = \int_a^b s_i(t)u(t)dt,\ 1 \leqslant i \leqslant n,$$
$$\delta_i = \int_a^b \tilde{s}_i(t)v(t)dt,\ 1 \leqslant i \leqslant n. \qquad (11.28)$$

4. Substituting (11.27) into (11.28) gives a system of n algebraic equations that can be solved to determine the constants $\alpha_i, \beta_i, \gamma_i$, and δ_i. To facilitate the computational work, we can use the computer symbolic systems such as Maple and Mathematica. Using the obtained numerical values of these constants into (11.27), the solutions $u(x)$ and $v(x)$ of the system of Fredholm integral equations (11.25) follow immediately.

Example 11.5

Solve the following system of Fredholm integral equations by using the direct computation method

$$u(x) = \sin x + \cos x - 4x + \int_0^{\pi} (xu(t) + xv(t))dt,$$
$$v(x) = \sin x - \cos x + \int_0^{\pi} (u(t) - v(t))dt. \qquad (11.29)$$

Following the analysis presented above this system can be rewritten as

$$u(x) = \sin x + \cos x + (\alpha - 4)x,$$
$$v(x) = \sin x - \cos x + \beta, \qquad (11.30)$$

where

$$\alpha = \int_0^{\pi} (u(t) + v(t))dt, \quad \beta = \int_0^{\pi} (u(t) - v(t))dt. \qquad (11.31)$$

To determine α, and β, we substitute (11.30) into (11.31) to obtain

$$\alpha = (4 - 2\pi^2) + \frac{\pi^2}{2}\alpha + \pi\beta, \quad \beta = -2\pi^2 + \frac{\pi^2}{2}\alpha - \pi\beta. \tag{11.32}$$

Solving this system gives

$$\alpha = 4, \quad \beta = 0. \tag{11.33}$$

Substituting (11.33) into (11.30) leads to the exact solutions

$$(u(x), v(x)) = (\sin x + \cos x, \sin x - \cos x). \tag{11.34}$$

Example 11.6

Solve the following system of Fredholm integral equations by using the direct computation method

$$\begin{aligned} u(x) &= \sec x - 2 + \int_0^{\frac{\pi}{3}} (\tan t u(t) + \sec t v(t))dt, \\ v(x) &= -(1 + \sqrt{3}) + \tan x + \int_0^{\frac{\pi}{3}} (\sec t u(t) + \sec t v(t))dt. \end{aligned} \tag{11.35}$$

Following the analysis presented above this system can be rewritten as

$$\begin{aligned} u(x) &= \sec x - 2 + \alpha_1 + \beta_1, \\ v(x) &= \tan x - (1 + \sqrt{3}) + \alpha_2 + \beta_1, \end{aligned} \tag{11.36}$$

where

$$\alpha_1 = \int_0^{\frac{\pi}{3}} \tan t u(t)dt, \quad \alpha_2 = \int_0^{\frac{\pi}{3}} \sec t u(t)dt, \quad \beta_1 = \int_0^{\frac{\pi}{3}} \sec t v(t)dt. \tag{11.37}$$

To determine α_1, α_2, and β_1, we substitute (11.36) into (11.37) and solve the resulting system we find

$$\alpha_1 = 1, \quad \alpha_2 = \sqrt{3}, \quad \beta_1 = 1. \tag{11.38}$$

Substituting (11.38) into (11.36) leads to the exact solutions

$$(u(x), v(x)) = (\sec x, \tan x). \tag{11.39}$$

Example 11.7

Solve the following system of Fredholm integral equations by using the direct computation method

$$\begin{aligned} u(x) &= 6 - \ln x + \int_{0^+}^{1} (\ln(xt)u(t) + \ln(xt^2)v(t))dt, \\ v(x) &= -4 + \ln x + \int_{0^+}^{1} (\ln(xt^2)u(t) - \ln(xt)v(t))dt. \end{aligned} \tag{11.40}$$

Proceeding as before, this system can be rewritten as

$$\begin{aligned} u(x) &= 6 + (\alpha_1 + \beta_1 - 1)\ln x + \alpha_2 + 2\beta_2, \\ v(x) &= -4 + (\alpha_1 - \beta_1 + 1)\ln x + 2\alpha_2 - \beta_2, \end{aligned} \tag{11.41}$$

where

$$\alpha_1 = \int_{0^+}^{1} u(t)dt, \quad \alpha_2 = \int_{0^+}^{1} \ln(t)u(t)dt,$$
$$\beta_1 = \int_{0^+}^{1} v(t)dt, \quad \beta_2 = \int_{0^+}^{1} \ln(t)v(t)dt. \tag{11.42}$$

To determine α_i, and $\beta_i, 1 \leqslant i \leqslant 2$, we substitute (11.41) into (11.42) to obtain

$$\alpha_1 = 7 - \alpha_1 + \alpha_2 - \beta_1 + 2\beta_2, \quad \alpha_2 = -8 + 2\alpha_1 - \alpha_2 + 2\beta_1 - 2\beta_2,$$
$$\beta_1 = -5 - \alpha_1 + 2\alpha_2 + \beta_1 - \beta_2, \quad \beta_2 = 6 + 2\alpha_1 - 2\alpha_2 - 2\beta_1 + \beta_2. \tag{11.43}$$

Solving this system gives

$$\alpha_1 = 0, \quad \alpha_2 = 1, \quad \beta_1 = 2, \quad \beta_2 = -3. \tag{11.44}$$

These results lead to the exact solutions

$$(u(x), v(x)) = (1 + \ln x, 1 - \ln x). \tag{11.45}$$

Example 11.8

Solve the following system of Fredholm integral equations by using the direct computation method

$$u(x) = \frac{2}{3} + \sec^2 x + \int_0^{\frac{\pi}{4}} (v(t) - w(t))dt,$$
$$v(x) = \frac{10}{3} - \sec^2 x + \int_0^{\frac{\pi}{4}} (w(t) - u(t))dt, \tag{11.46}$$
$$w(x) = -1 - \sec^4 x + \int_0^{\frac{\pi}{4}} (u(t) - v(t))dt.$$

Proceeding as before, this system can be rewritten as

$$u(x) = \frac{2}{3} + \sec^2 x + \beta - \gamma,$$
$$v(x) = \frac{10}{3} - \sec^2 x + \gamma - \alpha, \tag{11.47}$$
$$w(x) = -1 - \sec^4 x + \alpha - \beta,$$

where

$$\alpha = \int_0^{\frac{\pi}{4}} u(t)dt, \quad \beta = \int_0^{\frac{\pi}{4}} v(t)dt, \quad \gamma = \int_0^{\frac{\pi}{4}} w(t)dt. \tag{11.48}$$

To determine α, β, and γ, we substitute (11.47) into (11.48) and by solving the resulting system we find

$$\alpha = \frac{\pi}{4} + 1, \quad \beta = \frac{\pi}{4} - 1, \quad \gamma = \frac{\pi}{4} - \frac{4}{3}. \tag{11.49}$$

These results lead to the exact solutions

$$(u(x), v(x), w(x)) = (1 + \sec^2 x, 1 - \sec^2 x, 1 - \sec^4 x). \tag{11.50}$$

Exercises 11.2.2

Use the direct computation method to solve the following systems of Fredholm integral equations

1. $$\begin{cases} u(x) = x - \frac{16}{15} + \int_{-1}^{1} (tu(t) + tv(t))\, dt \\ v(x) = x^2 + x^3 - \frac{2}{3}x + \int_{-1}^{1} (xu(t) + xv(t))\, dt \end{cases}$$

2. $$\begin{cases} u(x) = -6 + 4\ln x + \int_{0^+}^{1} \left(\ln tu(t) - \ln(xt^2)v(t)\right) dt \\ v(x) = -2 - 2\ln x + \int_{0^+}^{1} \left(\ln(xt^2)u(t) - \ln tv(t)\right) dt \end{cases}$$

3. $$\begin{cases} u(x) = x + x^2 - \frac{16}{15} + \int_{-1}^{1} (u(t) + v(t))\, dt \\ v(x) = x^3 + x^4 - \frac{16}{15}x + \frac{2}{3} + \int_{-1}^{1} ((x-t)u(t) + xtv(t))\, dt \end{cases}$$

4. $$\begin{cases} u(x) = \sin x + \cos x - 2\pi + \int_{0}^{\pi} (tu(t) + tv(t))\, dt \\ v(x) = \sin x - \cos x + \pi x + \int_{0}^{\pi} (xu(t) - xtv(t))\, dt \end{cases}$$

5. $$\begin{cases} u(x) = x + \sin^2 x - \pi^2 + \int_{0}^{\pi} (u(t) + v(t))\, dt \\ v(x) = x - \cos^2 x - \pi^2 x + \frac{\pi^2}{2} + \int_{0}^{\pi} ((x-t)u(t) + (x+t)v(t))\, dt \end{cases}$$

6. $$\begin{cases} u(x) = -2 + \tan x + \int_{0}^{\frac{\pi}{3}} (\sec tu(t) + \tan tv(t))\, dt \\ v(x) = -\frac{\pi}{3} + \sec x + \int_{0}^{\frac{\pi}{3}} (\tan tu(t) - \sec tv(t))\, dt \end{cases}$$

7. $$\begin{cases} u(x) = x^2 + \sin x + 2\pi + \int_{-\pi}^{\pi} ((x-t)u(t) - (x+t)v(t))\, dt \\ v(x) = x^2 + \cos x - 2\pi + \int_{-\pi}^{\pi} ((x+t)u(t) - (x-t)v(t))\, dt \end{cases}$$

8. $$\begin{cases} u(x) = -x + \frac{e^x}{1+e^x} + \int_{0}^{1} (xu(t) + xv(t))\, dt \\ v(x) = -1 + \frac{1}{1+e^x} + \int_{0}^{1} (u(t) + v(t))\, dt \end{cases}$$

9. $$\begin{cases} u(x) = (2-\pi)x + \frac{\sin x}{1+\sin x} + \int_{0}^{\frac{\pi}{2}} (xu(t) + xv(t))\, dt \\ v(x) = (2-\pi) + \frac{\cos x}{1+\cos x} + \int_{0}^{\frac{\pi}{2}} (u(t) + v(t))\, dt \end{cases}$$

10. $$\begin{cases} u(x) = \dfrac{3-\sqrt{3}}{3} + \sec x \tan x + \displaystyle\int_0^{\frac{\pi}{6}} (u(t) - v(t))\, dt \\ v(x) = (1-\sqrt{3})x + \sec^2 x + \displaystyle\int_0^{\frac{\pi}{6}} (xu(t) + xv(t))\, dt \end{cases}$$

11. $$\begin{cases} u(x) = x - \dfrac{1}{12} + \displaystyle\int_{-1}^{1} (v(t) - w(t))\, dt \\ v(x) = x^2 + \dfrac{1}{4} + \displaystyle\int_{-1}^{1} (w(t) - u(t))\, dt \\ w(x) = x^2 - \dfrac{1}{6} + \displaystyle\int_{-1}^{1} (u(t) - v(t))\, dt \end{cases}$$

12. $$\begin{cases} u(x) = 1 + \sec^2 x + \sqrt{2} - \dfrac{\pi}{4} + \displaystyle\int_0^{\frac{\pi}{4}} (v(t) - w(t))\, dt \\ v(x) = 1 - \sec^2 x - \sqrt{2} - \dfrac{\pi}{4} + \displaystyle\int_0^{\frac{\pi}{4}} (w(t) + u(t))\, dt \\ w(x) = \sec x \tan x - \dfrac{\pi}{2} + \displaystyle\int_0^{\frac{\pi}{4}} (u(t) + v(t))\, dt \end{cases}$$

11.3 Systems of Fredholm Integro-Differential Equations

In Chapter 6, the Fredholm integro-differential equations were studied. The Fredholm integro-differential equations, were given in the form

$$u^{(i)}(x) = f(x) + \lambda \int_0^x K(x,t)u(t)dt, \tag{11.51}$$

where $u^{(i)}(x) = \frac{d^i u}{dx^i}$. Because the resulted equation combines the differential operator and the integral operator, then it is necessary to define initial conditions $u(0), u'(0), \ldots, u^{(i-1)}(0)$ for the determination of the particular solution $u(x)$ of the Fredholm integro-differential equation (11.51).

In this section, we will study systems of Fredholm integro-differential equations of the second kind given by

$$\begin{aligned} u^{(i)}(x) &= f_1(x) + \int_a^b \left(K_1(x,t)u(t) + \tilde{K}_1(x,t)v(t) \right) dt, \\ v^{(i)}(x) &= f_2(x) + \int_a^b \left(K_2(x,t)u(t) + \tilde{K}_2(x,t)v(t) \right) dt. \end{aligned} \tag{11.52}$$

The unknown functions $u(x), v(x), \ldots$, that will be determined, occur inside the integral sign whereas the derivatives of $u(x), v(x), \ldots$ appear mostly outside the integral sign. The kernels $K_i(x,t)$ and $\tilde{K}_i(x,t)$, and the function $f_i(x)$ are given real-valued functions.

In Chapter 6, four analytical methods were used for solving the Fredholm integro-differential equations. These methods are the variational iteration method (VIM), the Adomian decomposition method (ADM), the direct computation method, and the series solution method. The aforementioned methods can effectively handle the systems of Fredholm integro-differential equations (11.52). However, in this section, we will use only two of these methods, namely the direct computation method and the variational iteration method. The reader can apply the other two methods to show that it can be used to handle the system (11.52).

11.3.1 The Direct Computation Method

The direct computation method is a reliable technique that was used before to handle Fredholm integral equations in Chapter 4, Fredholm integro-differential equations in Chapter 6, and systems of Fredholm integral equations in this chapter. The direct computation method will be applied to solve the systems of Fredholm integro-differential equations of the second kind in a parallel manner to our treatment that was used before. The method approaches any Fredholm equation in a direct manner and gives the solution in an exact form and not in a series form. The method will be applied for the degenerate or separable kernels of the form

$$\begin{aligned} K_1(x,t) &= \sum_{k=1}^{n} g_k(x)h_k(t), \quad \tilde{K}_1(x,t) = \sum_{k=1}^{n} \tilde{g}_k(x)\tilde{h}_k(t),\\ K_2(x,t) &= \sum_{k=1}^{n} r_k(x)s_k(t), \quad \tilde{K}_2(x,t) = \sum_{k=1}^{n} \tilde{r}_k(x)\tilde{s}_k(t). \end{aligned} \tag{11.53}$$

The direct computation method can be applied as follows:

1. We first substitute (11.53) into the system of Fredholm integro-differential equations (11.52) to obtain

$$\begin{aligned} u^{(i)}(x) &= f_1(x) + \sum_{k=1}^{n} g_k(x)\int_a^b h_k(t)u(t)dt + \sum_{k=1}^{n} \tilde{g}_k(x)\int_a^b \tilde{h}_k(t)v(t)dt,\\ v^{(i)}(x) &= f_2(x) + \sum_{k=1}^{n} r_k(x)\int_a^b s_k(t)u(t)dt + \sum_{k=1}^{n} \tilde{r}_k(x)\int_a^b \tilde{s}_k(t)v(t)dt. \end{aligned} \tag{11.54}$$

2. Each integral at the right side depends only on the variable t with constant limits of integration for t. This means that each integral is equivalent to a constant. Based on this, Equation (11.54) becomes

$$\begin{aligned} u^{(i)}(x) &= f_1(x) + \alpha_1 g_1(x) + \cdots + \alpha_n g_n(x) + \beta_1 \tilde{g}_1(x) + \cdots + \beta_n \tilde{g}_n(x),\\ v^{(i)}(x) &= f_2(x) + \gamma_1 r_1(x) + \cdots + \gamma_n r_n(x) + \delta_1 \tilde{r}_1(x) + \cdots + \delta_n \tilde{r}_n(x), \end{aligned} \tag{11.55}$$

where

$$\alpha_i = \int_a^b h_i(t)\, u(t)dt, 1 \leqslant i \leqslant n, \quad \beta_i = \int_a^b \tilde{h}_i(t)\, v(t)dt, 1 \leqslant i \leqslant n,$$
$$\gamma_i = \int_a^b s_i(t)\, u(t)dt, 1 \leqslant i \leqslant n, \quad \delta_i = \int_a^b \tilde{s}_i(t)\, v(t)dt, 1 \leqslant i \leqslant n. \tag{11.56}$$

3. Integrating both sides of (11.55) i times from 0 to x, using the given initial conditions, and substituting the resulting equations for $u(x)$ and $v(x)$ into (11.56) gives a system of algebraic equations that can be solved to determine the constants $\alpha_i, \beta_i, \gamma_i$, and δ_i. Using the obtained numerical values of these constants into the obtained equations for $u(x)$ and $v(x)$, the solutions $u(x)$ and $v(x)$ of the system of Fredholm integro-differential equations (11.52) follow immediately.

Example 11.9

Solve the following system of Fredholm integro-differential equations by using the direct computation method

$$u'(x) = \sin x + x\cos x + (2-\pi^2) + \int_0^{\pi} (tu(t) - v(t))\, dt, u(0) = 0,$$
$$v'(x) = \cos x - x\sin x - 3\pi + \int_0^{\pi} (u(t) - tv(t))\, dt, v(0) = 0. \tag{11.57}$$

Following the analysis presented above, this system can be rewritten as

$$u'(x) = \sin x + x\cos x + (2-\pi^2+\alpha),$$
$$v'(x) = \cos x - x\sin x + (\beta - 3\pi), \tag{11.58}$$

where

$$\alpha = \int_0^{\pi} (tu(t) - v(t))dt, \quad \beta = \int_0^{\pi} (u(t) - tv(t))dt. \tag{11.59}$$

Integrating both sides of (11.58) once from 0 to x, and using the initial conditions we find

$$u(x) = x\sin x + (2-\pi^2+\alpha)x,$$
$$v(x) = x\cos x + (\beta - 3\pi)x. \tag{11.60}$$

To determine α, and β, we substitute (11.60) into (11.59) and solving the resulting system we obtain

$$\alpha = \pi^2 - 2, \quad \beta = 3\pi. \tag{11.61}$$

Substituting (11.61) into (11.60) leads to the exact solutions

$$(u(x), v(x)) = (x\sin x, x\cos x). \tag{11.62}$$

Example 11.10

Solve the following system of Fredholm integro-differential equations by using the direct computation method

$$
\begin{aligned}
u'(x) &= -1 + \sinh x + \int_0^{\ln 2} (u(t)+v(t))dt, u(0) = 1,\\
v'(x) &= 1 - 2\ln 2 + \cosh x + \int_0^{\ln 2} (tu(t)+tv(t))dt, v(0) = 0.
\end{aligned} \tag{11.63}
$$

This system can be rewritten as

$$
\begin{aligned}
u'(x) &= \sinh x + \alpha - 1,\\
v'(x) &= \cosh x + 1 - 2\ln 2 + \beta,
\end{aligned} \tag{11.64}
$$

where

$$
\alpha = \int_0^{\ln 2} (u(t)+v(t))dt, \quad \beta = \int_0^{\ln 2} (tu(t)+tv(t))dt. \tag{11.65}
$$

Integrating both sides of (11.64) once from 0 to x, and using the initial conditions we find

$$
\begin{aligned}
u(x) &= \cosh x + (\alpha - 1)x,\\
v(x) &= \sinh x + (1 - 2\ln 2 + \beta)x.
\end{aligned} \tag{11.66}
$$

To determine α, and β, we substitute (11.66) into (11.65) and solving the resulting system we obtain

$$
\alpha = 1, \quad \beta = 2\ln 2 - 1. \tag{11.67}
$$

Substituting (11.67) into (11.66) leads to the exact solutions

$$
(u(x), v(x)) = (\cosh x, \sinh x). \tag{11.68}
$$

Example 11.11

Solve the following system of Fredholm integro-differential equations by using the direct computation method

$$
\begin{aligned}
u'(x) &= e^x - \frac{5}{2} + \int_0^{\ln 2} (u(t)+v(t))dt, u(0) = 1,\\
v'(x) &= 2e^{2x} + \frac{1}{4} + \int_0^{\ln 2} (tu(t)-tv(t))dt, v(0) = 1.
\end{aligned} \tag{11.69}
$$

This system can be rewritten as

$$
u'(x) = e^x + \alpha - \frac{5}{2}, \quad v'(x) = 2e^{2x} + \frac{1}{4} + \beta, \tag{11.70}
$$

where

$$
\alpha = \int_0^{\ln 2} (u(t)+v(t))dt, \quad \beta = \int_0^{\ln 2} (tu(t)-tv(t))dt. \tag{11.71}
$$

Integrating both sides of (11.70) once from 0 to x, and using the initial conditions we find

$$
u(x) = e^x + \left(\alpha - \frac{5}{2}\right)x, \quad v(x) = e^{2x} + \left(\frac{1}{4} + \beta\right)x. \tag{11.72}
$$

To determine α, and β, we substitute (11.72) into (11.71) and solving the resulting system we obtain

$$\alpha = \frac{5}{2}, \quad \beta = -\frac{1}{4}. \tag{11.73}$$

Substituting (11.73) into (11.72) leads to the exact solutions

$$(u(x), v(x)) = (e^x, e^{2x}). \tag{11.74}$$

Example 11.12

Solve the following system of Fredholm integro-differential equations by using the direct computation method

$$\begin{aligned} u''(x) &= -\cos x - \left(2 - \frac{\pi}{2}\right) + \int_0^{\frac{\pi}{2}} ((x-t)u(t) - (x-t)v(t))\,dt, \\ &\quad u(0) = 1,\ u'(0) = 0 \\ v''(x) &= -\sin x + \left(2 - \frac{\pi}{2}\right) + \int_0^{\frac{\pi}{2}} ((x+t)u(t) - (x+t)v(t))\,dt, \\ &\quad v(0) = 0, v'(0) = 1. \end{aligned} \tag{11.75}$$

This system can be rewritten as

$$\begin{aligned} u''(x) &= -\cos x - \left(2 - \frac{\pi}{2}\right) + \alpha x - \beta, \\ v''(x) &= -\sin x + \left(2 - \frac{\pi}{2}\right) + \alpha x + \beta, \end{aligned} \tag{11.76}$$

where

$$\alpha = \int_0^{\frac{\pi}{2}} (u(t) - v(t))dt, \quad \beta = \int_0^{\frac{\pi}{2}} (tu(t) - tv(t))dt. \tag{11.77}$$

Integrating both sides of (11.76) twice from 0 to x, and using the initial conditions and proceeding as before we obtain

$$\alpha = 0, \quad \beta = \frac{\pi}{2} - 2. \tag{11.78}$$

This in turn gives the exact solutions

$$(u(x), v(x)) = (\cos x, \sin x). \tag{11.79}$$

Exercises 11.3.1

Use the direct computation method to solve the following systems of Fredholm integro-differential equations

1. $\begin{cases} u'(x) = \cos x - 4 + \int_0^{\pi} (u(t) - tv(t))\,dt,\ u(0) = 0 \\ v'(x) = -\sin x - \pi + \int_0^{\pi} (tu(t) - v(t))\,dt,\ v(0) = 1 \end{cases}$

2. $$\begin{cases} u'(x) = -\sin x - 2x + \dfrac{\pi}{2} + \displaystyle\int_0^{\frac{\pi}{2}} ((x-t)u(t) - (x-t)v(t))\,dt \\ v'(x) = -\cos x - 2x - \dfrac{\pi}{2} + \displaystyle\int_0^{\frac{\pi}{2}} ((x+t)u(t) - (x+t)v(t))\,dt \\ u(0) = 2,\ v(0) = 1 \end{cases}$$

3. $$\begin{cases} u'(x) = -2\sin(2x) - \dfrac{\pi}{2} + \displaystyle\int_0^{\pi} ((x-t)u(t) - (x+t)v(t))\,dt \\ v'(x) = 2\cos(2x) + \dfrac{\pi}{2} + \displaystyle\int_0^{\pi} ((x+t)u(t) - (x-t)v(t))\,dt \\ u(0) = 1,\ v(0) = 0 \end{cases}$$

4. $$\begin{cases} u'(x) = \sinh 2x + \dfrac{1}{2}(\ln 2)^2 + \displaystyle\int_0^{\ln 2} (tu(t) - tv(t))\,dt \\ v'(x) = \sinh 2x - \dfrac{15}{16} - 2\ln 2 + \displaystyle\int_0^{\ln 2} (u(t) + v(t))\,dt \\ u(0) = 1,\ v(0) = 2 \end{cases}$$

5. $$\begin{cases} u'(x) = \sinh 2x - (\ln 2)^2 + \displaystyle\int_0^{\ln 2} (tu(t) + tv(t))\,dt,\ u(0) = 2 \\ v'(x) = -2\sinh 2x - 2\ln 2 + \displaystyle\int_0^{\ln 2} (u(t) + v(t))\,dt,\ v(0) = 0 \end{cases}$$

6. $$\begin{cases} u'(x) = \cosh x + \dfrac{3}{2} - \dfrac{1}{2}\ln 2 + \displaystyle\int_0^{\ln 2} (tu(t) - tv(t))\,dt,\ u(0) = 0 \\ v'(x) = \sinh x - (\ln 2)^2 + \displaystyle\int_0^{\ln 2} (u(t) + v(t))\,dt,\ v(0) = 1 \end{cases}$$

7. $$\begin{cases} u'(x) = 1 + e^x - \dfrac{2}{3}(\ln 2)^3 + \displaystyle\int_0^{\ln 2} (tu(t) + tv(t))\,dt,\ u(0) = 1 \\ v'(x) = 1 - e^x - (\ln 2)^2 + \displaystyle\int_0^{\ln 2} (u(t) + v(t))\,dt,\ v(0) = -1 \end{cases}$$

8. $$\begin{cases} u'(x) = (1+x)e^x + \dfrac{1}{2}(1 - 3\ln 2) + \displaystyle\int_0^{\ln 2} (u(t) + v(t))\,dt \\ v'(x) = (1-x)e^{-x} + \dfrac{1}{2}(3 - 5\ln 2) + \displaystyle\int_0^{\ln 2} (u(t) - v(t))\,dt \\ u(0) = 0, v(0) = 0 \end{cases}$$

9. $$\begin{cases} u'(x) = e^x - 2\ln 2 - \dfrac{1}{2} + \displaystyle\int_0^{\ln 2} (tu(t) + v(t))\,dt,\ u(0) = 1 \\ v'(x) = 2e^{2x} - 2\ln 2 - \dfrac{1}{4} + \displaystyle\int_0^{\ln 2} (u(t) + tv(t))\,dt,\ v(0) = 1 \end{cases}$$

10. $$\begin{cases} u''(x) = 2\cos 2x + \dfrac{4}{3}\cos x \\ \qquad\qquad + \displaystyle\int_0^{\frac{\pi}{2}} (\sin(x-t)u(t) - \cos(x-t)v(t))\,dt,\ u(0)=0,\ u'(0)=0 \\ v''(x) = -2\cos 2x - \dfrac{2}{3}\cos x \\ \qquad\qquad + \displaystyle\int_0^{\frac{\pi}{2}} (\cos(x-t)u(t) - \sin(x-t)v(t))\,dt,\ v(0)=1,\ v'(0)=0 \end{cases}$$

11. $$\begin{cases} u''(x) = -\sin x + \pi\cos x + \displaystyle\int_0^{\pi} (\sin(x-t)u(t) - \cos(x-t)v(t))\,dt, \\ u(0)=0,\ u'(0)=1 \\ v''(x) = -\cos x - \pi\sin x + \displaystyle\int_0^{\pi} (\cos(x-t)u(t) + \sin(x-t)v(t))\,dt, \\ v(0)=1,\ v'(0)=0 \end{cases}$$

12. $$\begin{cases} u''(x) = -(1+\pi)\sin x - \cos x \\ \qquad\qquad + \displaystyle\int_0^{\pi} (\cos(x-t)u(t) + \cos(x-t)v(t))\,dt,\ u(0)=1,\ u'(0)=1 \\ v''(x) = (1-\pi)\cos x - \sin x \\ \qquad\qquad + \displaystyle\int_0^{\pi} (\cos(x-t)u(t) - \cos(x-t)v(t))\,dt,\ v(0)=-1,\ v'(0)=1 \end{cases}$$

11.3.2 The Variational Iteration Method

In Chapters 4 and 6, the variational iteration method was used to handle the Fredholm integral equations and the Fredholm integro-differential equations respectively. The method provides rapidly convergent successive approximations of the exact solution if such a closed form solution exists, and not components as in Adomian decomposition method. The variational iteration method [8] handles linear and nonlinear problems in the same manner without any need to specific restrictions such as the so called Adomian polynomials that we need for nonlinear problems.

The correction functionals for the system integro-differential equations (11.52) are given by

$$\begin{aligned} u_{n+1}(x) &= u_n(x) \\ &+ \int_0^x \lambda(t)\left(u_n^{(i)}(t) - f_1(t) - \int_a^b \left(K_1(t,r)\tilde{u}_n(r)dr + \tilde{K}_1(t,r)\tilde{v}_n(r)dr\right)\right) dt, \\ v_{n+1}(x) &= v_n(x) \\ &+ \int_0^x \lambda(t)\left(u_n^{(i)}(t) - f_2(t) - \int_a^b \left(K_2(t,r)\tilde{u}_n(r)dr + \tilde{K}_2(t,r)\tilde{v}_n(r)dr\right)\right) dt. \end{aligned} \tag{11.80}$$

It is required first to determine the Lagrange multiplier λ that can be identified optimally. Having λ determined, an iteration formula, without restricted variation, should be used for the determination of the successive approximations $u_{n+1}(x), v_{n+1}(x), n \geqslant 0$ of the solutions $u(x)$ and $v(x)$. The zeroth approximations $u_0(x)$ and $v_0(x)$ can be any selective functions. However, we can use the initial conditions to select the zeroth approximations u_0 and $v_0(x)$. The VIM will be illustrated by studying the following examples.

Example 11.13

Solve the system of Fredholm integro-differential equations by using the variational iteration method

$$\begin{aligned} u'(x) &= -2 - \sin x + \int_0^{\pi} (u(t) + v(t))dt, u(0) = 1, \\ v'(x) &= 2 - \pi + \cos x + \int_0^{\pi} (tu(t) + tv(t))dt, v(0) = 0. \end{aligned} \tag{11.81}$$

The correction functionals for this system are given by

$$\begin{aligned} u_{n+1}(x) &= u_n(x) - \int_0^x (u'_n(t) + 2 + \sin t - \rho_1)\, dt, \\ v_{n+1}(x) &= v_n(x) - \int_0^x (v'_n(t) + \pi - 2 - \cos t - \rho_2)\, dt, \end{aligned} \tag{11.82}$$

where

$$\rho_1 = \int_0^{\pi} (u_n(r) + v_n(r))dr, \quad \rho_2 = \int_0^{\pi} (ru_n(r) + rv_n(r))dr. \tag{11.83}$$

Selecting $u_0(x) = 1$ and $v_0(x) = 0$, the correction functionals gives the following successive approximations

$$\begin{aligned} u_0(x) &= 1, \\ v_0(x) &= 0, \\ u_1(x) &= u_0(x) - \int_0^x \left(u'_0(t) + 2 + \sin t - \int_0^{\pi} (u_0(r) + v_0(r))dr \right) dt, \\ &= \cos x - 2x + \pi x, \\ v_1(x) &= v_0(x) - \int_0^x \left(v'_0(t) + \pi - 2 - \cos t - \int_0^{\pi} (ru_0(r) + rv_0(r))dr \right) dt, \\ &= \sin x + 2x - \pi x + \frac{\pi^2}{2} x, \\ u_2(x) &= u_1(x) - \int_0^x \left(u'_1(t) + 2 + \sin t - \int_0^{\pi} (u_1(r) + v_1(r))dr \right) dt, \\ &= \cos x + (2x - 2x) + (\pi x - \pi x) + \cdots, \\ v_2(x) &= v_1(x) - \int_0^x \left(v'_1(t) + \pi - 2 - \cos t - \int_0^{\pi} (ru_1(r) + rv_1(r))dr \right) dt, \end{aligned} \tag{11.84}$$

$$= \sin x + (2x - 2x) + (\pi x - \pi x) + \left(\frac{\pi^2}{2}x - \frac{\pi^2}{2}x\right) + \cdots,$$

and so on. It is obvious that noise terms appear in each component. By canceling the noise terms, the exact solutions are therefore given by

$$(u(x), v(x)) = (\cos x, \sin x). \tag{11.85}$$

Example 11.14

Solve the system of Fredholm integro-differential equations by using the variational iteration method

$$\begin{aligned} u'(x) &= -1 + \cosh x + \int_0^{\ln 2} (u(t) + v(t))dt,\ u(0) = 0, \\ v'(x) &= 1 - 2\ln 2 + \sinh x + \int_0^{\ln 2} (tu(t) + tv(t))dt,\ v(0) = 1. \end{aligned} \tag{11.86}$$

The correction functionals for this system are given by

$$\begin{aligned} u_{n+1}(x) &= u_n(x) - \int_0^x (u_n'(t) + 1 - \cosh t - \rho_3)\, dt, \\ v_{n+1}(x) &= v_n(x) - \int_0^x (v_n'(t) + 2\ln 2 - 1 - \sinh t - \rho_4)\, dt. \end{aligned} \tag{11.87}$$

where

$$\begin{aligned} \rho_3 &= \int_0^{\ln 2} (u_n(r) + v_n(r))dr, \\ \rho_4 &= \int_0^{\ln 2} (ru_n(r) + rv_n(r))dr. \end{aligned} \tag{11.88}$$

We can use the initial conditions to select $u_0(x) = 0$ and $v_0(x) = 1$. Using this selection into the correction functionals gives the following successive approximations

$$\begin{aligned} u_0(x) &= 0, \\ v_0(x) &= 1, \\ u_1(x) &= \sinh x + x\ln 2 - x, \\ v_1(x) &= \cosh x + x - 2x\ln 2 + \frac{x}{2}(\ln 2)^2, \\ u_2(x) &= \sinh x + (x - x) + (2x\ln 2 - 2x\ln 2) + \left(\frac{x}{2}(\ln 2)^2 - \frac{x}{2}(\ln 2)^2\right) + \cdots, \\ v_2(x) &= \cosh x + (x - x) + (2x\ln 2 - 2x\ln 2) + \left(\frac{x}{2}(\ln 2)^2 - \frac{x}{2}(\ln 2)^2\right) + \cdots, \\ u_3(x) &= \sinh x + \left(\frac{x}{4}(\ln 2)^3 - \frac{x}{4}(\ln 2)^3\right) + \cdots, \\ v_3(x) &= \cosh x + \left(\frac{x}{3}(\ln 2)^4 - \frac{x}{3}(\ln 2)^4\right) + \cdots, \end{aligned} \tag{11.89}$$

and so on. It is obvious that noise terms appear in each component. By canceling the noise terms, the exact solutions are therefore given by

$$(u(x), v(x)) = (\sinh x, \cosh x). \tag{11.90}$$

Example 11.15

Solve the system of Fredholm integro-differential equations by using the variational iteration method

$$\begin{aligned} u'(x) &= e^x - 6 + \int_0^{\ln 3} (u(t) + v(t))dt,\ u(0) = 1, \\ v'(x) &= 2e^{2x} + 2 + \int_0^{\ln 3} (u(t) - v(t))dt,\ v(0) = 1. \end{aligned} \tag{11.91}$$

The correction functionals for this system are given by

$$\begin{aligned} u_{n+1}(x) &= u_n(x) - \int_0^x \left(u_n'(t) - e^t + 6 - \int_0^{\ln 3} (u_n(r) + v_n(r))dr \right) dt, \\ v_{n+1}(x) &= v_n(x) - \int_0^x \left(v_n'(t) - 2e^{2t} - 2 - \int_0^{\ln 3} (u_n(r) - v_n(r))dr \right) dt, \end{aligned} \tag{11.92}$$

where we used the Lagrange multiplier $\lambda = -1$ for the each of first order Fredholm integro-differential equations.

Selecting $u_0(x) = 1$ and $v_0(x) = 1$gives the following approximations

$$\begin{aligned} u_0(x) &= 1, \quad v_0(x) = 1, \\ u_1(x) &= e^x - 6x + 2x\ln 3, \\ v_1(x) &= e^{2x} + 2x, \\ u_2(x) &= e^x + (6x - 6x) + (2x\ln 3 - 2x\ln 3) - 2x(\ln 3)^2 + x(\ln 3)^3, \\ v_2(x) &= e^{2x} + (2x - 2x) - 4x(\ln 3)^2 + x(\ln 3)^3, \\ u_3(x) &= e^x + (2x(\ln 3)^2 - 2x(\ln 3)^2) + (x(\ln 3)^3 - x(\ln 3)^3) + \cdots, \\ v_3(x) &= e^{2x} + (4x(\ln 3)^2 - 4x(\ln 3)^2) + (x(\ln 3)^3 - x(\ln 3)^3) + \cdots, \end{aligned} \tag{11.93}$$

and so on. It is obvious that noise terms appear in each component. By canceling the noise terms, the exact solutions are therefore given by

$$(u(x), v(x)) = (e^x, e^{2x}). \tag{11.94}$$

Example 11.16

Solve the system of Fredholm integro-differential equations by using the variational iteration method

$$\begin{aligned} u''(x) &= 2\cos 2x - \frac{3\pi}{4}(2 + \pi) + \int_0^{\pi} (u(t) + tv(t))dt,\ u(0) = 1,\ u'(0) = 0 \\ v''(x) &= -2\cos 2x + \frac{3\pi}{4}(2 - \pi) + \int_0^{\pi} (tu(t) - v(t))dt,\ v(0) = 2,\ v'(0) = 0. \end{aligned} \tag{11.95}$$

The correction functionals for this system are given by

$$u_{n+1}(x) = u_n(x) + \int_0^x \left((t-x)(u_n''(t) - 2\cos 2t + \frac{3\pi}{4}(2+\pi) - \rho_5)\right) dt,$$
$$v_{n+1}(x) = v_n(x) + \int_0^x \left((t-x)(v_n''(t) + 2\cos 2t - \frac{3\pi}{4}(2-\pi) - \rho_6)\right) dt. \tag{11.96}$$

where

$$\rho_5 = \int_0^\pi (u_n(r) + rv_n(r))dr, \quad \rho_6 = \int_0^\pi (ru_n(r) - v_n(r))dr. \tag{11.97}$$

Notice that the Lagrange multiplier $\lambda = (t-x)$ because each equation is of second order. We can use the initial conditions to select $u_0(x) = 1$ and $v_0(x) = 2$. Using this selection into the correction functionals and proceeding as before we obtain the following successive approximations

$$\begin{aligned}
u_0(x) &= 1,\\
v_0(x) &= 2,\\
u_1(x) &= 1 + \sin^2 x - \frac{\pi}{8}(2-\pi)x^2,\\
v_1(x) &= 1 + \cos^2 x - \frac{\pi}{8}(2+\pi)x^2,\\
u_2(x) &= 1 + \sin^2 x + \left(\frac{\pi}{8}(2-\pi)x^2 - \frac{\pi}{8}(2-\pi)x^2\right) + \cdots,\\
v_2(x) &= 1 + \cos^2 x + \left(\frac{\pi}{8}(2+\pi)x^2 - \frac{\pi}{8}(2+\pi)x^2\right) + \cdots,
\end{aligned} \tag{11.98}$$

and so on. It is obvious that noise terms appear as explained in the previous examples. By canceling the noise terms, the exact solutions are therefore given by

$$(u(x), v(x)) = (1 + \sin^2 x, 1 + \cos^2 x). \tag{11.99}$$

Exercises 11.3.2

Use the variational iteration method to solve the following systems of Fredholm integro-differential equations

1. $$\begin{cases} u'(x) = \cos x - 4 + \int_0^\pi (u(t) - tv(t))\, dt, \ u(0) = 0 \\ v'(x) = -\sin x - \pi + \int_0^\pi (tu(t) - v(t))\, dt, \ v(0) = 1 \end{cases}$$

2. $$\begin{cases} u'(x) = \cos x - 2x + \frac{\pi}{2} + \int_0^{\frac{\pi}{2}} ((x-t)u(t) + (x-t)v(t))\, dt, \ u(0) = 0 \\ v'(x) = -\sin x - 2 + \frac{\pi}{2} + \int_0^{\frac{\pi}{2}} ((x+t)u(t) - (x+t)v(t))\, dt, \ v(0) = 1 \end{cases}$$

3. $\begin{cases} u'(x) = \sin x + x\cos x + (2-\pi^2) + \displaystyle\int_0^{\pi} (tu(t) - v(t))\,dt, \;\; u(0) = 0 \\ v'(x) = \cos x - x\sin x - 3\pi + \displaystyle\int_0^{\pi} (u(t) - tv(t))\,dt, \;\; v(0) = 0 \end{cases}$

4. $\begin{cases} u'(x) = \sinh 2x + \dfrac{1}{2}(\ln 2)^2 + \displaystyle\int_0^{\ln 2} (tu(t) - tv(t))\,dt, \;\; u(0) = 1 \\ v'(x) = \sinh 2x - \dfrac{15}{16} - 2\ln 2 + \displaystyle\int_0^{\ln 2} (u(t) + v(t))\,dt, \;\; v(0) = 2 \end{cases}$

5. $\begin{cases} u'(x) = \sinh 2x - (\ln 2)^2 + \displaystyle\int_0^{\ln 2} (tu(t) + tv(t))\,dt, \;\; u(0) = 1 \\ v'(x) = -\sinh 2x - 2\ln 2 + \displaystyle\int_0^{\ln 2} (u(t) + v(t))\,dt, \;\; v(0) = 1 \end{cases}$

6. $\begin{cases} u'(x) = \cosh x + \dfrac{3}{2} - \dfrac{1}{2}\ln 2 + \displaystyle\int_0^{\ln 2} (tu(t) - tv(t))\,dt, \;\; u(0) = 0 \\ v'(x) = \sinh x - (\ln 2)^2 + \displaystyle\int_0^{\ln 2} (u(t) + v(t))\,dt, \;\; v(0) = 1 \end{cases}$

7. $\begin{cases} u'(x) = e^x + \dfrac{3}{2} - \dfrac{5}{2}\ln 2 + \displaystyle\int_0^{\ln 2} (tu(t) - tv(t))\,dt, \;\; u(0) = 1 \\ v'(x) = -e^{-x} - \dfrac{3}{2} + \displaystyle\int_0^{\ln 2} (u(t) + v(t))\,dt, \;\; v(0) = 1 \end{cases}$

8. $\begin{cases} u'(x) = (1+x)e^x + \dfrac{1}{2}(1 - 3\ln 2) + \displaystyle\int_0^{\ln 2} (u(t) + v(t))\,dt, \;\; u(0) = 0 \\ v'(x) = (1-x)e^{-x} + \dfrac{1}{2}(3 - 5\ln 2) + \displaystyle\int_0^{\ln 2} (u(t) - v(t))\,dt, \;\; v(0) = 0 \end{cases}$

9. $\begin{cases} u'(x) = e^x - \dfrac{10}{3} + \displaystyle\int_0^{\ln 2} (u(t) + v(t))\,dt, \;\; u(0) = 1 \\ v'(x) = 3e^{3x} + \dfrac{4}{3} + \displaystyle\int_0^{\ln 2} (u(t) - v(t))\,dt, \;\; v(0) = 1 \end{cases}$

10. $\begin{cases} u''(x) = e^x - \dfrac{10}{3} + \displaystyle\int_0^{\ln 2} (u(t) + v(t))\,dt, \;\; u(0) = 1, \;\; u'(0) = 1 \\ v''(x) = 9e^{3x} + \dfrac{4}{3} + \displaystyle\int_0^{\ln 2} (u(t) - v(t))\,dt, \;\; v(0) = 1, \;\; v'(0) = 3 \end{cases}$

11. $\begin{cases} u''(x) = -\sin x + \pi\cos x + \displaystyle\int_0^{\pi} (\sin(x-t)u(t) - \cos(x-t)v(t))\,dt, \\ u(0) = 0, \;\; u'(0) = 1 \\ v''(x) = -\cos x - \pi\sin x + \displaystyle\int_0^{\pi} (\cos(x-t)u(t) + \sin(x-t)v(t))\,dt, \\ v(0) = 1, \;\; v'(0) = 0 \end{cases}$

12. $$\begin{cases} u''(x) = -(1+\pi)\sin x - \cos x + \displaystyle\int_0^{\pi} (\cos(x-t)u(t) + \cos(x-t)v(t))\,dt, \\ u(0) = 1,\ u'(0) = 1 \\ v''(x) = (1-\pi)\cos x - \sin x + \displaystyle\int_0^{\pi} (\cos(x-t)u(t) - \cos(x-t)v(t))\,dt, \\ v(0) = -1,\ v'(0) = 1 \end{cases}$$

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Chapter 12
Systems of Singular Integral Equations

12.1 Introduction

Systems of singular integral equations appear in many branches of scientific fields [1–6], such as microscopy, seismology, radio astronomy, electron emission, atomic scattering, radar ranging, plasma diagnostics, X-ray radiography, and optical fiber evaluation. Studies of systems of singular integral equations have attracted much concern in applied sciences. The use of computer symbolic systems such as Maple and Mathematica facilitates the tedious work of computation. The general ideas and the essential features of these systems are of wide applicability.

The well known systems of singular integral equations [7] are given by

$$\begin{aligned} f_1(x) &= \int_0^x (K_{11}(x,t)u(t) + K_{12}(x,t)v(t))\,dt, \\ f_2(x) &= \int_0^x (K_{21}(x,t)u(t) + K_{22}(x,t)v(t))\,dt. \end{aligned} \tag{12.1}$$

and

$$\begin{aligned} u(x) &= f_1(x) + \int_0^x (K_{11}(x,t)u(t) + K_{12}(x,t)v(t))\,dt, \\ v(x) &= f_2(x) + \int_0^x (K_{21}(x,t)u(t) + K_{22}(x,t)v(t))\,dt, \end{aligned} \tag{12.2}$$

where the kernels K_{ij} are singular kernels given by

$$K_{ij} = \frac{1}{(x-t)^{\alpha_{ij}}}, \quad 1 \leqslant i, \quad j \leqslant 2. \tag{12.3}$$

The system (12.1) and the system (12.2) are called the system of the generalized Abel singular integral equations and the system of the weakly generalized singular integral equations respectively. For $\alpha_{ij} = \frac{1}{2}$, the system (12.1) is called the system of Abel singular integral equations.

12.2 Systems of Generalized Abel Integral Equations

In Chapter 7, the singular integral equations were presented. The Abel's singular integral equation and its generalized form, and the weakly singular integral equations were handled by a variety of methods. In this section, the systems of Abel's generalized singular integral equations will be examined by using only the Laplace transform method that we used before in Chapter 7 among other places. In this section, first we will study systems of two unknown functions $u(x)$ and $v(x)$, and then we will study systems of three unknown functions $u(x), v(x)$ and $w(x)$ as well. Generalization to any number of unknowns can be easily developed.

12.2.1 Systems of Generalized Abel Integral Equations in Two Unknowns

The system of generalized Abel integral equation in two unknowns is of the form

$$\begin{aligned} f_1(x) &= \int_0^x (K_{11}(x,t)u(t) + K_{12}(x,t)v(t))\,dt, \\ f_2(x) &= \int_0^x (K_{21}(x,t)u(t) + K_{22}(x,t)v(t))\,dt. \end{aligned} \tag{12.4}$$

The kernels $K_{ij}(x,t), 1 \leqslant i,j \leqslant 2$ and the functions $f_i(x), i = 1,2$ are given real-valued functions. Recall that the kernels K_{ij} are singular kernels given by

$$K_{ij} = \frac{1}{(x-t)^{\alpha_{ij}}}, 1 \leqslant i,j \leqslant 2. \tag{12.5}$$

For $\alpha_{ij} = \frac{1}{2}, 1 \leqslant i,j \leqslant 2$, the system is called system of Abel integral equations. Abel's systems of three equations in three unknowns will be examined in details in the next section.

The system of Abel's generalized singular integral equations (12.4) gives a solution if

$$\det\left(\begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix}\right) \neq 0. \tag{12.6}$$

In what follows we will apply the Laplace transform method to handle the system (12.4).

Taking Laplace transform of both sides of the system (12.4) gives the linear system in $U(s)$ and $V(s)$

$$\begin{aligned} F_1(s) &= \mathcal{K}_{11}(s)U(s) + \mathcal{K}_{12}(s)V(s), \\ F_2(s) &= \mathcal{K}_{21}(s)U(s) + \mathcal{K}_{22}(s)V(s), \end{aligned} \tag{12.7}$$

where

$$\begin{aligned} U(s) &= \mathcal{L}\{u(x)\}, & V(s) &= \mathcal{L}\{v(x)\}, \\ F_i(s) &= \mathcal{L}\{f_i(x)\}, & 1 &\leqslant i \leqslant 2, \\ \mathcal{K}_{ij}(s) &= \mathcal{L}\{K_{ij}(x)\}, & 1 &\leqslant i, \quad j \leqslant 2. \end{aligned} \tag{12.8}$$

Solving (12.7) for $U(s)$ and $V(s)$ by using Cramer's rule gives

$$U(s) = \frac{\begin{vmatrix} F_1(s) & \mathcal{K}_{12}(s) \\ F_2(s) & \mathcal{K}_{22}(s) \end{vmatrix}}{\begin{vmatrix} \mathcal{K}_{11}(s) & \mathcal{K}_{12}(s) \\ \mathcal{K}_{12}(s) & \mathcal{K}_{22}(s) \end{vmatrix}}, \quad V(s) = \frac{\begin{vmatrix} \mathcal{K}_{11}(s) & F_1(s) \\ \mathcal{K}_{21}(s) & F_2(s) \end{vmatrix}}{\begin{vmatrix} \mathcal{K}_{11}(s) & \mathcal{K}_{12}(s) \\ \mathcal{K}_{21}(s) & \mathcal{K}_{22}(s) \end{vmatrix}}. \tag{12.9}$$

Having determined $U(s)$ and $V(s)$, the unique solution for $u(x)$ and $v(x)$ can be determined by using the inverse Laplace transform method. Table 1 in Section 1.5 shows the Laplace transforms and the inverse Laplace transforms for a variety of functions. It is interesting to point out that the computer symbolic systems such as Maple and Mathematica allow us to perform complicated and tedious calculations. This will be used here to facilitate the computational work. Moreover, recall that the linear system (12.7) has a unique solution for $u(x)$ and $v(x)$ if and only if

$$\begin{vmatrix} \mathcal{K}_{11}(s) & \mathcal{K}_{12}(s) \\ \mathcal{K}_{12}(s) & \mathcal{K}_{22}(s) \end{vmatrix} \neq 0. \tag{12.10}$$

It is worth noting that for simplicity reasons we will focus our study only on the specific case where

$$\alpha_{11} = \alpha_{22}, \quad \alpha_{12} = \alpha_{21}. \tag{12.11}$$

The other cases of α_{ij} can be handled by a like manner. In this case, the use of the computer symbolic systems is necessary to perform tedious calculations.

Example 12.1

Solve the system of singular integral equations by using the Laplace transform method

$$\begin{aligned} 2\sqrt{x}\left(1+\frac{2}{3}x\right)+\frac{3}{2}x^{\frac{2}{3}}\left(1-\frac{3}{5}x\right) &= \int_0^x \left(\frac{1}{(x-t)^{\frac{1}{2}}}u(t)+\frac{1}{(x-t)^{\frac{1}{3}}}v(t)\right)dt, \\ 2\sqrt{x}\left(1-\frac{2}{3}x\right)+\frac{3}{2}x^{\frac{2}{3}}\left(1+\frac{3}{5}x\right) &= \int_0^x \left(\frac{1}{(x-t)^{\frac{1}{3}}}u(t)+\frac{1}{(x-t)^{\frac{1}{2}}}v(t)\right)dt. \end{aligned} \tag{12.12}$$

Taking Laplace transform of both sides of each equation in (12.12) gives

$$\begin{aligned} \sqrt{\pi}s^{-\frac{5}{2}}(1+s)-\Gamma\left(\frac{2}{3}\right)s^{-\frac{8}{3}}(1-s) &= \sqrt{\pi}s^{-\frac{1}{2}}U(s)+\Gamma\left(\frac{2}{3}\right)s^{-\frac{2}{3}}V(s), \\ -\sqrt{\pi}s^{-\frac{5}{2}}(1-s)+\Gamma\left(\frac{2}{3}\right)s^{-\frac{8}{3}}(1+s) &= \Gamma\left(\frac{2}{3}\right)s^{-\frac{2}{3}}U(s)+\sqrt{\pi}s^{-\frac{1}{2}}V(s). \end{aligned} \tag{12.13}$$

Solving this systems of equations for $U(s)$ and $V(s)$, by using any computer symbolic system, such as Maple or Mathematica, gives

$$U(s) = \frac{1}{s} + \frac{1}{s^2}, \quad V(s) = \frac{1}{s} - \frac{1}{s^2}. \tag{12.14}$$

By taking the inverse Laplace transform of both sides of each equation in (12.14), the exact solutions are given by

$$(u(x), v(x)) = (1 + x, 1 - x). \tag{12.15}$$

Example 12.2

Solve the system of singular integral equations by using the Laplace transform method

$$\begin{aligned}
&\frac{16}{195}x^{\frac{5}{4}}(32x^2+39) + \frac{25}{4788}x^{\frac{9}{5}}(57x^2-133) \\
&= \int_0^x \left(\frac{1}{(x-t)^{\frac{3}{4}}}u(t) + \frac{1}{(x-t)^{\frac{1}{5}}}v(t)\right) dt, \\
&\frac{16}{195}x^{\frac{5}{4}}(32x^2-39) + \frac{25}{4788}x^{\frac{9}{5}}(57x^2+133) \\
&= \int_0^x \left(\frac{1}{(x-t)^{\frac{1}{5}}}u(t) + \frac{1}{(x-t)^{\frac{3}{4}}}v(t)\right) dt.
\end{aligned} \tag{12.16}$$

Taking Laplace transform of both sides of each equation in (12.16) gives

$$\begin{aligned}
\frac{\pi\sqrt{2}}{s^{\frac{9}{4}}\Gamma\left(\frac{3}{4}\right)}\left(\frac{6}{s^2}+1\right) + \frac{\Gamma\left(\frac{4}{5}\right)}{s^{\frac{14}{5}}}\left(\frac{6}{s^2}-1\right) &= \frac{\pi\sqrt{2}}{s^{\frac{1}{4}}\Gamma\left(\frac{3}{4}\right)}U(s) + \frac{\Gamma\left(\frac{4}{5}\right)}{s^{\frac{4}{5}}}V(s), \\
\frac{\pi\sqrt{2}}{s^{\frac{9}{4}}\Gamma\left(\frac{3}{4}\right)}\left(\frac{6}{s^2}-1\right) + \frac{\Gamma\left(\frac{4}{5}\right)}{s^{\frac{14}{5}}}\left(\frac{6}{s^2}+1\right) &= \frac{\Gamma\left(\frac{4}{5}\right)}{s^{\frac{4}{5}}}U(s) + \frac{\pi\sqrt{2}}{s^{\frac{1}{4}}\Gamma\left(\frac{3}{4}\right)}V(s).
\end{aligned} \tag{12.17}$$

Solving this systems of equations for $U(s)$ and $V(s)$, by using any computer symbolic system, such as Maple or Mathematica, gives

$$U(s) = \frac{6}{s^4} + \frac{1}{s^2}, \quad V(s) = \frac{6}{s^4} - \frac{1}{s^2}. \tag{12.18}$$

By taking the inverse Laplace transform of both sides of each equation in (12.18), the exact solutions are given by

$$(u(x), v(x)) = (x^3 + x, x^3 - x). \tag{12.19}$$

Exercises 12.2.1

Solve the system of singular integral equations by using the Laplace transform method

1. $$\begin{cases} \frac{8}{5}x^{\frac{1}{4}}(6x+5) + \frac{5}{36}x^{\frac{4}{5}}(20x+27) = \int_0^x \left(\frac{1}{(x-t)^{\frac{3}{4}}}u(t) + \frac{1}{(x-t)^{\frac{1}{5}}}v(t)\right) dt \\ \frac{4}{5}x^{\frac{1}{4}}(16x+15) + \frac{5}{12}x^{\frac{4}{5}}(5x+6) = \int_0^x \left(\frac{1}{(x-t)^{\frac{1}{5}}}u(t) + \frac{1}{(x-t)^{\frac{3}{4}}}v(t)\right) dt \end{cases}$$

2. $$\begin{cases} \dfrac{5}{36}x^{\frac{4}{5}}(5x+9\pi)+\dfrac{3}{10}x^{\frac{2}{3}}(-3x+5\pi) = \displaystyle\int_0^x \left(\frac{1}{(x-t)^{\frac{1}{5}}}u(t)+\frac{1}{(x-t)^{\frac{1}{3}}}v(t)\right)dt \\ \dfrac{5}{36}x^{\frac{4}{5}}(-5x+9\pi)+\dfrac{3}{10}x^{\frac{2}{3}}(3x+5\pi) = \displaystyle\int_0^x \left(\frac{1}{(x-t)^{\frac{1}{3}}}u(t)+\frac{1}{(x-t)^{\frac{1}{5}}}v(t)\right)dt \end{cases}$$

3. $$\begin{cases} \dfrac{4}{3}x^{\frac{1}{2}}(x+9)+\dfrac{5}{36}x^{\frac{4}{5}}(5x-54) = \displaystyle\int_0^x \left(\frac{1}{(x-t)^{\frac{1}{2}}}u(t)+\frac{1}{(x-t)^{\frac{1}{5}}}v(t)\right)dt \\ \dfrac{4}{3}x^{\frac{1}{2}}(x-9)+\dfrac{5}{36}x^{\frac{4}{5}}(5x+54) = \displaystyle\int_0^x \left(\frac{1}{(x-t)^{\frac{1}{5}}}u(t)+\frac{1}{(x-t)^{\frac{1}{2}}}v(t)\right)dt \end{cases}$$

4. $$\begin{cases} \dfrac{16}{3}x^{\frac{3}{4}}+\dfrac{256}{315}x^{\frac{9}{2}} = \displaystyle\int_0^x \left(\frac{1}{(x-t)^{\frac{1}{2}}}u(t)+\frac{1}{(x-t)^{\frac{1}{4}}}v(t)\right)dt \\ \dfrac{8192}{21945}x^{\frac{19}{4}}+8x^{\frac{1}{2}} = \displaystyle\int_0^x \left(\frac{1}{(x-t)^{\frac{1}{4}}}u(t)+\frac{1}{(x-t)^{\frac{1}{2}}}v(t)\right)dt \end{cases}$$

5. $$\begin{cases} \dfrac{2}{15}x^{\frac{1}{2}}(8x^2+15)-\dfrac{3}{40}x^{\frac{2}{3}}(9x^2-20) = \displaystyle\int_0^x \left(\frac{1}{(x-t)^{\frac{1}{2}}}u(t)+\frac{1}{(x-t)^{\frac{1}{3}}}v(t)\right)dt \\ -\dfrac{2}{15}x^{\frac{1}{2}}(8x^2-15)+\dfrac{3}{40}x^{\frac{2}{3}}(9x^2+20) = \displaystyle\int_0^x \left(\frac{1}{(x-t)^{\frac{1}{3}}}u(t)+\frac{1}{(x-t)^{\frac{1}{2}}}v(t)\right)dt \end{cases}$$

6. $$\begin{cases} \dfrac{2}{35}x^{\frac{1}{2}}(16x^3+35)+\dfrac{3}{440}x^{\frac{2}{3}}(81x^3-220) = \displaystyle\int_0^x \left(\frac{1}{(x-t)^{\frac{1}{2}}}u(t)+\frac{1}{(x-t)^{\frac{1}{3}}}v(t)\right)dt \\ \dfrac{2}{35}x^{\frac{1}{2}}(16x^3-35)+\dfrac{3}{440}x^{\frac{2}{3}}(81x^3+220) = \displaystyle\int_0^x \left(\frac{1}{(x-t)^{\frac{1}{3}}}u(t)+\frac{1}{(x-t)^{\frac{1}{2}}}v(t)\right)dt \end{cases}$$

7. $$\begin{cases} -\dfrac{2}{15}x^{\frac{1}{2}}(8x^2-10x-15)+\dfrac{3}{40}x^{\frac{2}{3}}(9x^2-12x+20) \\ = \displaystyle\int_0^x \left(\frac{1}{(x-t)^{\frac{1}{2}}}u(t)+\frac{1}{(x-t)^{\frac{1}{3}}}v(t)\right)dt \\ \dfrac{2}{15}x^{\frac{1}{2}}(8x^2-10x+15)-\dfrac{3}{40}x^{\frac{2}{3}}(9x^2-12x-20) \\ = \displaystyle\int_0^x \left(\frac{1}{(x-t)^{\frac{1}{3}}}u(t)+\frac{1}{(x-t)^{\frac{1}{2}}}v(t)\right)dt \end{cases}$$

8. $$\begin{cases} -\dfrac{4}{1155}x^{\frac{3}{4}}(128x^3+220x-385)+\dfrac{3}{440}x^{\frac{2}{3}}(81x^3+132x+220) \\ = \displaystyle\int_0^x \left(\frac{1}{(x-t)^{\frac{1}{3}}}u(t)+\frac{1}{(x-t)^{\frac{1}{4}}}v(t)\right)dt \\ \dfrac{4}{1155}x^{\frac{3}{4}}(128x^3+220x+385)-\dfrac{3}{440}x^{\frac{2}{3}}(81x^3+132x-220) \\ = \displaystyle\int_0^x \left(\frac{1}{(x-t)^{\frac{1}{4}}}u(t)+\frac{1}{(x-t)^{\frac{1}{3}}}v(t)\right)dt \end{cases}$$

12.2.2 Systems of Generalized Abel Integral Equations in Three Unknowns

The system of Abel's generalized singular integral equations in three unknowns is of the form

$$\begin{aligned} f_1(x) &= \int_0^x (K_{11}(x,t)u(t) + K_{12}(x,t)v(t) + K_{13}(x,t)w(t))\,dt, \\ f_2(x) &= \int_0^x (K_{21}(x,t)u(t) + K_{22}(x,t)v(t) + K_{23}(x,t)w(t))\,dt, \\ f_3(x) &= \int_0^x (K_{31}(x,t)u(t) + K_{32}(x,t)v(t) + K_{33}(x,t)w(t))\,dt. \end{aligned} \tag{12.20}$$

The kernels $K_{ij}(x,t), 1 \leqslant i,j \leqslant 3$ and the functions $f_i(x), i = 1,2,3$ are given real-valued functions. Recall that the kernels K_{ij} are singular kernels given by

$$K_{ij} = \frac{1}{(x-t)^{\alpha_{ij}}}, \quad 1 \leqslant i, \quad j \leqslant 3. \tag{12.21}$$

For $\alpha_{ij} = \frac{1}{2}, 1 \leqslant i,j \leqslant 3$, the system is called system of Abel integral equations in three unknowns.

The system of Abel's generalized singular integral equations in three unknowns gives a solution if

$$\det\left(\begin{bmatrix} K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{bmatrix}\right) \neq 0. \tag{12.22}$$

In what follows we will apply the Laplace transform method to handle the system (12.20).

Taking Laplace transform of both sides of the system (12.20) gives the linear system in $U(s)$, $V(s)$, and $W(s)$

$$\begin{aligned} F_1(s) &= \mathcal{K}_{11}(s)U(s) + \mathcal{K}_{12}(s)V(s) + \mathcal{K}_{13}(s)W(s), \\ F_2(s) &= \mathcal{K}_{21}(s)U(s) + \mathcal{K}_{22}(s)V(s) + \mathcal{K}_{23}(s)W(s), \\ F_3(s) &= \mathcal{K}_{31}(s)U(s) + \mathcal{K}_{32}(s)V(s) + \mathcal{K}_{33}(s)W(s). \end{aligned} \tag{12.23}$$

where

$$\begin{gathered} U(s) = \mathcal{L}\{u(x)\},\ V(s) = \mathcal{L}\{v(x)\},\ W(s) = \mathcal{L}\{w(x)\}, \\ F_i(s) = \mathcal{L}\{f_i(x)\},\ 1 \leqslant i \leqslant 3, \quad \mathcal{K}_{ij}(s) = \mathcal{L}\{K_{ij}(x)\},\ 1 \leqslant i,j \leqslant 3, \end{gathered} \tag{12.24}$$

and the singular kernels K_{ij} are given above in (12.21).

Solving (12.23) for $U(s)$, $V(s)$, and $W(s)$ by using Cramer's rule gives

$$U(s) = \frac{\begin{vmatrix} F_1(s) & \mathcal{K}_{12}(s) & \mathcal{K}_{13}(s) \\ F_2(s) & \mathcal{K}_{22}(s) & \mathcal{K}_{23}(s) \\ F_3(s) & \mathcal{K}_{32}(s) & \mathcal{K}_{33}(s) \end{vmatrix}}{\begin{vmatrix} \mathcal{K}_{11}(s) & \mathcal{K}_{12}(s) & \mathcal{K}_{13}(s) \\ \mathcal{K}_{21}(s) & \mathcal{K}_{22}(s) & \mathcal{K}_{23}(s) \\ \mathcal{K}_{31}(s) & \mathcal{K}_{32}(s) & \mathcal{K}_{33}(s) \end{vmatrix}}, \tag{12.25}$$

$$V(s) = \frac{\begin{vmatrix} \mathcal{K}_{11}(s) & F_1(s) & \mathcal{K}_{13}(s) \\ \mathcal{K}_{21}(s) & F_2(s) & \mathcal{K}_{23}(s) \\ \mathcal{K}_{31}(s) & F_3(s) & \mathcal{K}_{33}(s) \end{vmatrix}}{\begin{vmatrix} \mathcal{K}_{11}(s) & \mathcal{K}_{12}(s) & \mathcal{K}_{13}(s) \\ \mathcal{K}_{21}(s) & \mathcal{K}_{22}(s) & \mathcal{K}_{23}(s) \\ \mathcal{K}_{31}(s) & \mathcal{K}_{32}(s) & \mathcal{K}_{33}(s) \end{vmatrix}}, \tag{12.26}$$

and

$$W(s) = \frac{\begin{vmatrix} \mathcal{K}_{11}(s) & \mathcal{K}_{12}(s) & F_1(s) \\ \mathcal{K}_{21}(s) & \mathcal{K}_{22}(s) & F_2(s) \\ \mathcal{K}_{31}(s) & \mathcal{K}_{23}(s) & F_3(s) \end{vmatrix}}{\begin{vmatrix} \mathcal{K}_{11}(s) & \mathcal{K}_{12}(s) & \mathcal{K}_{13}(s) \\ \mathcal{K}_{21}(s) & \mathcal{K}_{22}(s) & \mathcal{K}_{23}(s) \\ \mathcal{K}_{31}(s) & \mathcal{K}_{32}(s) & \mathcal{K}_{33}(s) \end{vmatrix}}. \tag{12.27}$$

The linear system (12.23) gives a unique solution for $u(x), v(x)$ and $w(x)$ if and only if

$$\begin{vmatrix} \mathcal{K}_{11}(s) & \mathcal{K}_{12}(s) & \mathcal{K}_{13}(s) \\ \mathcal{K}_{21}(s) & \mathcal{K}_{22}(s) & \mathcal{K}_{23}(s) \\ \mathcal{K}_{31}(s) & \mathcal{K}_{32}(s) & \mathcal{K}_{33}(s) \end{vmatrix} \neq 0. \tag{12.28}$$

It is worth noting that for simplicity reasons we will focus our study only on the specific case where

$$\alpha_{11} = \alpha_{22}, \quad \alpha_{12} = \alpha_{21}. \tag{12.29}$$

Having determined $U(s)$, $V(s)$, and $W(s)$, the unique solution for $u(x)$, $v(x)$, and $w(x)$ can be determined by using the inverse Laplace transform method. It is interesting to point out that the computer symbolic systems such as Maple and Mathematica allow us to perform complicated and tedious calculations. This will be used here to facilitate the computational work.

For simplicity reasons we will focus our study only on the specific case where

$$\begin{aligned} &\text{For the first equation in (12.20): } \alpha_{11} = \alpha_{13}, v(x) = 0, \\ &\text{For the second equation in (12.20): } \alpha_{22} = \alpha_{23}, u(x) = 0, \\ &\text{For the third equation in (12.20): } \alpha_{31} = \alpha_{32}, w(x) = 0. \end{aligned} \tag{12.30}$$

Example 12.3

Solve the system of singular integral equations by using the Laplace transform method

$$\begin{aligned} \frac{2}{15}\sqrt{x}(15+8x^2) &= \int_0^x \left(\frac{1}{(x-t)^{\frac{1}{2}}} u(t) + \frac{1}{(x-t)^{\frac{1}{2}}} w(t) \right) dt, \\ \frac{9}{28} x^{\frac{4}{3}}(7+6x) &= \int_0^x \left(\frac{1}{(x-t)^{\frac{2}{3}}} v(t) + \frac{1}{(x-t)^{\frac{2}{3}}} w(t) \right) dt, \\ \frac{4}{5} x^{\frac{1}{4}}(5+4x) &= \int_0^x \left(\frac{1}{(x-t)^{\frac{3}{4}}} u(t) + \frac{1}{(x-t)^{\frac{3}{4}}} v(t) \right) dt. \end{aligned} \tag{12.31}$$

Taking Laplace transform of both sides of each equation in (12.31) gives

$$\begin{aligned}
\sqrt{\pi}s^{-\frac{3}{2}}\left(1+\frac{2}{s^2}\right) &= \sqrt{\frac{\pi}{s}}(U(s)+W(s)),\\
\frac{2\pi}{\sqrt{3}\Gamma\left(\frac{2}{3}\right)}s^{-\frac{7}{3}}\left(1+\frac{2}{s}\right) &= \frac{2\pi}{\sqrt{3}\Gamma\left(\frac{2}{3}\right)}s^{-\frac{1}{3}}(V(s)+W(s)),\\
\frac{\sqrt{2}\pi}{\Gamma\left(\frac{3}{4}\right)}s^{-\frac{5}{4}}\left(1+\frac{1}{s}\right) &= \frac{\sqrt{2}\pi}{\Gamma\left(\frac{3}{4}\right)}s^{-\frac{1}{4}}(U(s)+V(s))
\end{aligned} \tag{12.32}$$

Solving this systems of equations for $U(s)$, $V(s)$ and $W(s)$ by using any computer symbolic system, such as Maple or Mathematica, gives

$$U(s)=\frac{1}{s},\quad V(s)=\frac{1}{s^2},\quad W(s)=\frac{2}{s^3}. \tag{12.33}$$

By taking the inverse Laplace transform of both sides of each equation in (12.33), the exact solutions are given by

$$(u(x),v(x),w(x))=(1,x,x^2). \tag{12.34}$$

Example 12.4

Solve the system of singular integral equations by using the Laplace transform method

$$\begin{aligned}
3x^{\frac{1}{3}}(2+3x) &= \int_0^x \left(\frac{1}{(x-t)^{\frac{2}{3}}}u(t)+\frac{1}{(x-t)^{\frac{2}{3}}}w(t)\right)dt,\\
\frac{8}{15}x^{\frac{1}{4}}(15-32x^2) &= \int_0^x \left(\frac{1}{(x-t)^{\frac{3}{4}}}v(t)+\frac{1}{(x-t)^{\frac{3}{4}}}w(t)\right)dt,\\
\frac{10}{3}x^{\frac{3}{5}} &= \int_0^x \left(\frac{1}{(x-t)^{\frac{2}{5}}}u(t)+\frac{1}{(x-t)^{\frac{2}{5}}}v(t)\right)dt.
\end{aligned} \tag{12.35}$$

Taking Laplace transform of both sides of each equation in (12.35) gives

$$\begin{aligned}
\frac{4\pi}{\sqrt{3}\Gamma\left(\frac{2}{3}\right)}s^{-\frac{4}{3}}\left(1+\frac{2}{s}\right) &= \frac{2\pi}{\sqrt{3}\Gamma(\frac{2}{3})}s^{-\frac{1}{3}}(U(s)+W(s)),\\
\frac{2\sqrt{2}\pi}{\Gamma\left(\frac{3}{4}\right)}s^{-\frac{5}{4}}\left(1-\frac{6}{s^2}\right) &= \frac{\sqrt{2}\pi}{\Gamma(\frac{3}{4})}s^{-\frac{1}{4}}(V(s)+W(s)),\\
2\Gamma\left(\tfrac{3}{5}\right)s^{-\frac{8}{5}} &= \Gamma\left(\tfrac{3}{5}\right)s^{-\frac{3}{5}}(U(s)+V(s))
\end{aligned} \tag{12.36}$$

Solving this systems of equations for $U(s)$, $V(s)$, and $W(s)$ by using any computer symbolic system, such as Maple or Mathematica, gives

$$U(s)=\frac{1}{s}+\frac{2}{s^2}+\frac{6}{s^3},\ V(s)=\frac{1}{s}-\frac{2}{s^2}-\frac{6}{s^3},\ W(s)=\frac{1}{s}+\frac{2}{s^2}-\frac{6}{s^3}. \tag{12.37}$$

By taking the inverse Laplace transform of both sides of each equation in (12.37), the exact solutions are given by

$$(u(x),v(x),w(x))=(1+2x+3x^2,1-2x-3x^2,1+2x-3x^2). \tag{12.38}$$

Exercises 12.2.2

Solve the system of singular integral equations by using the Laplace transform method

1. $$\begin{cases} \frac{4}{105}x^{\frac{3}{2}}(24x^2+35) = \int_0^x \frac{1}{(x-t)^{\frac{1}{2}}}(u(t)+w(t))\,dt \\ \frac{27}{440}x^{\frac{8}{3}}(9x+11) = \int_0^x \frac{1}{(x-t)^{\frac{1}{3}}}(v(t)+w(t))\,dt \\ \frac{16}{231}x^{\frac{7}{4}}(8x+11) = \int_0^x \frac{1}{(x-t)^{\frac{1}{4}}}(u(t)+v(t))\,dt \end{cases}$$

2. $$\begin{cases} \frac{4}{3}x^{\frac{1}{2}}(8x+9) = \int_0^x \frac{1}{(x-t)^{\frac{1}{2}}}(u(t)+w(t))\,dt \\ \frac{3}{10}x^{\frac{2}{3}}(27x+35) = \int_0^x \frac{1}{(x-t)^{\frac{1}{3}}}(v(t)+w(t))\,dt \\ \frac{4}{3}x^{\frac{3}{4}}(4x+5) = \int_0^x \frac{1}{(x-t)^{\frac{1}{4}}}(u(t)+v(t))\,dt \end{cases}$$

3. $$\begin{cases} \frac{3}{440}x^{\frac{2}{3}}(81x^3+99x^2+132x+220) = \int_0^x \frac{1}{(x-t)^{\frac{1}{3}}}(u(t)+w(t))\,dt \\ \frac{16}{1155}x^{\frac{7}{4}}(32x^2+80x+55) = \int_0^x \frac{1}{(x-t)^{\frac{1}{4}}}(v(t)+w(t))\,dt \\ \frac{5}{252}x^{\frac{4}{5}}(25x^2+70x+63) = \int_0^x \frac{1}{(x-t)^{\frac{1}{5}}}(u(t)+v(t))\,dt \end{cases}$$

4. $$\begin{cases} \frac{3}{140}x^{\frac{1}{3}}(81x^3+90x^2+105x+140) = \int_0^x \frac{1}{(x-t)^{\frac{2}{3}}}(u(t)+w(t))\,dt \\ \frac{16}{585}x^{\frac{5}{4}}(96x^2+208x+117) = \int_0^x \frac{1}{(x-t)^{\frac{3}{4}}}(v(t)+w(t))\,dt \\ \frac{5}{156}x^{\frac{3}{5}}(25x^2+65x+52) = \int_0^x \frac{1}{(x-t)^{\frac{2}{5}}}(u(t)+v(t))\,dt \end{cases}$$

5. $$\begin{cases} -\frac{4}{35}x^{\frac{1}{2}}(8x^3-70x-105) = \int_0^x \frac{1}{(x-t)^{\frac{1}{2}}}(u(t)+w(t))\,dt \\ -\frac{9}{140}x^{\frac{1}{3}}(27x^3-30x^2-560) = \int_0^x \frac{1}{(x-t)^{\frac{2}{3}}}(v(t)+w(t))\,dt \\ \frac{8}{231}x^{\frac{3}{4}}(16x^2+132x+231) = \int_0^x \frac{1}{(x-t)^{\frac{1}{4}}}(u(t)+v(t))\,dt \end{cases}$$

6. $$\begin{cases} -\frac{9}{40}x^{\frac{2}{3}}(3x^2-8x-20) = \int_0^x \frac{1}{(x-t)^{\frac{1}{3}}}(u(t)+w(t))\,dt \\ -\frac{5}{312}x^{\frac{3}{5}}(50x^2+65x-208) = \int_0^x \frac{1}{(x-t)^{\frac{2}{5}}}(v(t)+w(t))\,dt \\ \frac{4}{45}x^{\frac{1}{4}}(64x^2-36x+135) = \int_0^x \frac{1}{(x-t)^{\frac{3}{4}}}(u(t)+v(t))\,dt \end{cases}$$

7. $$\begin{cases} \frac{9}{220}x^{\frac{5}{3}}(81x^2+44) = \int_0^x \frac{1}{(x-t)^{\frac{1}{3}}}(u(t)+w(t))\,dt \\ \frac{25}{468}x^{\frac{8}{5}}(25x^2+30x+39) = \int_0^x \frac{1}{(x-t)^{\frac{2}{5}}}(v(t)+w(t))\,dt \\ \frac{32}{585}x^{\frac{5}{4}}(96x^2+208x+117) = \int_0^x \frac{1}{(x-t)^{\frac{3}{4}}}(u(t)+v(t))\,dt \end{cases}$$

8. $$\begin{cases} \frac{27}{440}x^{\frac{5}{3}}(45x^2+33x+44) = \int_0^x \frac{1}{(x-t)^{\frac{1}{3}}}(u(t)+v(t)+w(t))\,dt \\ \frac{25}{468}x^{\frac{8}{5}}(25x^2+30x+39) = \int_0^x \frac{1}{(x-t)^{\frac{2}{5}}}(v(t)+w(t))\,dt \\ \frac{32}{585}x^{\frac{5}{4}}(96x^2+208x+117) = \int_0^x \frac{1}{(x-t)^{\frac{3}{4}}}(u(t)+v(t))\,dt \end{cases}$$

12.3 Systems of the Weakly Singular Volterra Integral Equations

In this section we will study systems of the weakly singular integral equations in two unknowns $u(x)$ and $v(x)$. Generalization to any number of unknowns can be followed in a parallel manner. Recall that the kernel is called weakly singular as the singularity may be transformed away by a change of variable [2]. In this section we will use only the Laplace transform method and the Adomian decomposition method to handle this type of systems.

12.3.1 The Laplace Transform Method

The system of weakly singular Volterra integral equations of the convolution type in two unknowns is of the form

$$\begin{aligned} u(x) &= f_1(x) + \int_0^x (K_{11}(x,t)u(t) + K_{12}(x,t)v(t))\,dt, \\ v(x) &= f_2(x) + \int_0^x (K_{21}(x,t)u(t) + K_{22}(x,t)v(t))\,dt. \end{aligned} \tag{12.39}$$

The kernels $K_{ij}(x,t), 1 \leqslant i,j \leqslant 2$ and the functions $f_i(x), i = 1,2$ are given real-valued functions. The kernels K_{ij} are singular kernels given by

$$K_{ij} = \frac{1}{(x-t)^{\alpha_{ij}}}, \quad 1 \leqslant i, \quad j \leqslant 2. \tag{12.40}$$

Taking Laplace transforms of both sides of the system (12.39) gives the linear system in $U(s)$ and $V(s)$

$$\begin{aligned} U(s) &= F_1(s) + \mathcal{K}_{11}(s)U(s) + \mathcal{K}_{12}(s)V(s), \\ V(s) &= F_2(s) + \mathcal{K}_{21}(s)U(s) + \mathcal{K}_{22}(s)V(s), \end{aligned} \tag{12.41}$$

or equivalently

$$\begin{aligned}(1-\mathcal{K}_{11}(s))U(s)-\mathcal{K}_{12}(s)V(s)&=F_1(s),\\ -\mathcal{K}_{21}(s)U(s)+(1-\mathcal{K}_{22}(s))V(s)&=F_2(s),\end{aligned} \tag{12.42}$$

where

$$\begin{gathered}U(s)=\mathcal{L}\{u(x)\},\ V(s)=\mathcal{L}\{v(x)\},\\ F_i(s)=\mathcal{L}\{f_i(x)\},\ 1\leqslant i\leqslant 2,\ \mathcal{K}_{ij}(s)=\mathcal{L}\{K_{ij}(x)\},\ 1\leqslant i,j\leqslant 2.\end{gathered} \tag{12.43}$$

Solving (12.41) for $U(s)$ and $V(s)$ by using Cramer's rule gives

$$U(s)=\frac{\begin{vmatrix}F_1(s) & -\mathcal{K}_{12}(s)\\ F_2(s) & 1-\mathcal{K}_{22}(s)\end{vmatrix}}{\begin{vmatrix}1-\mathcal{K}_{11}(s) & -\mathcal{K}_{12}(s)\\ -\mathcal{K}_{21}(s) & 1-\mathcal{K}_{22}(s)\end{vmatrix}}, \tag{12.44}$$

and

$$V(s)=\frac{\begin{vmatrix}1-\mathcal{K}_{11}(s) & F_1(s)\\ -\mathcal{K}_{21}(s) & F_2(s)\end{vmatrix}}{\begin{vmatrix}1-\mathcal{K}_{11}(s) & -\mathcal{K}_{12}(s)\\ -\mathcal{K}_{21}(s) & 1-\mathcal{K}_{22}(s)\end{vmatrix}}. \tag{12.45}$$

Having determined $U(s)$ and $V(s)$, the unique solution for $u(x)$ and $v(x)$ can be determined by using the inverse Laplace transform method. It is interesting to point out that the computer symbolic systems such as Maple and Mathematica allow us to perform complicated and tedious calculations of this weakly singular equations. The linear system (12.41) has a unique solution for $u(x)$ and $v(x)$ if and only if

$$\begin{vmatrix}1-\mathcal{K}_{11}(s) & -\mathcal{K}_{12}(s)\\ -\mathcal{K}_{21}(s) & 1-\mathcal{K}_{22}(s)\end{vmatrix}\neq 0. \tag{12.46}$$

Example 12.5

Solve the system of weakly singular Volterra integral equations by using the Laplace transform method

$$\begin{aligned}u(x)&=x^4-\frac{64}{105}x^{\frac{7}{2}}(4x+3)+\int_0^x\left(\frac{3}{(x-t)^{\frac{1}{2}}}u(t)+\frac{2}{(x-t)^{\frac{1}{2}}}v(t)\right)dt,\\ v(x)&=x^3-\frac{32}{315}x^{\frac{7}{2}}(16x-27)+\int_0^x\left(\frac{2}{(x-t)^{\frac{1}{2}}}u(t)-\frac{3}{(x-t)^{\frac{1}{2}}}v(t)\right)dt.\end{aligned} \tag{12.47}$$

Taking Laplace transform of both sides of each equation in (12.47) gives

$$\begin{aligned}\left(1-3\sqrt{\frac{\pi}{s}}\right)U(s)-2\sqrt{\frac{\pi}{s}}V(s)&=\frac{4!}{s^5}-\frac{12\sqrt{\pi}}{s^{\frac{9}{2}}}\left(\frac{6}{s}+1\right),\\ -2\sqrt{\frac{\pi}{s}}U(s)+\left(1+3\sqrt{\frac{\pi}{s}}\right)V(s)&=\frac{3!}{s^4}-\frac{6\sqrt{\pi}}{s^{\frac{9}{2}}}\left(\frac{8}{s}-3\right).\end{aligned} \tag{12.48}$$

Solving this system of equations for $U(s)$ and $V(s)$, by using any computer symbolic system, such as Maple or Mathematica, gives

$$U(s) = \frac{4!}{s^5}, \quad V(s) = \frac{3!}{s^4}. \tag{12.49}$$

By taking the inverse Laplace transform of both sides of each equation in (12.49), the exact solutions are given by

$$(u(x), v(x)) = (x^4, x^3). \tag{12.50}$$

Example 12.6

Solve the system of weakly singular Volterra integral equations by using the Laplace transform method

$$\begin{aligned} u(x) &= 1 + 2x - \frac{9}{20}x^{\frac{2}{3}}(9x^2 + 6x + 10) \\ &\quad + \int_0^x \left(\frac{1}{(x-t)^{\frac{1}{3}}}u(t) + \frac{2}{(x-t)^{\frac{1}{3}}}v(t) \right) dt, \\ v(x) &= 1 + 3x^2 + \frac{3}{40}x^{\frac{2}{3}}(27x^2 - 72x - 20) \\ &\quad + \int_0^x \left(\frac{2}{(x-t)^{\frac{1}{3}}}u(t) - \frac{1}{(x-t)^{\frac{1}{3}}}v(t) \right) dt. \end{aligned} \tag{12.51}$$

Taking Laplace transforms of both sides of each equation in (12.51) gives

$$\begin{aligned} \left(1 - \frac{\Gamma\left(\frac{2}{3}\right)}{s^{\frac{2}{3}}}\right) U(s) - \frac{2\Gamma\left(\frac{2}{3}\right)}{s^{\frac{2}{3}}} V(s) &= \frac{1}{s} + \frac{3}{s^2} - \frac{3\Gamma\left(\frac{2}{3}\right)}{s^{\frac{5}{3}}} \left(1 + \frac{1}{s} + \frac{4}{s^2}\right), \\ -\frac{2\Gamma\left(\frac{2}{3}\right)}{s^{\frac{2}{3}}} U(s) + \left(1 + \frac{\Gamma\left(\frac{2}{3}\right)}{s^{\frac{2}{3}}}\right) V(s) &= \frac{1}{s} + \frac{6}{s^3} - \frac{\Gamma\left(\frac{2}{3}\right)}{s^{\frac{5}{3}}} \left(1 + \frac{6}{s} - \frac{6}{s^2}\right). \end{aligned} \tag{12.52}$$

Solving this system of equations for $U(s)$ and $V(s)$ we find

$$U(s) = \frac{1}{s} + \frac{3}{s^2}, \quad V(s) = \frac{1}{s} + \frac{6}{s^3}. \tag{12.53}$$

The inverse Laplace transform of (12.53) gives the exact solutions by

$$(u(x), v(x)) = (1 + 3x, 1 + 3x^2). \tag{12.54}$$

Example 12.7

Solve the system of weakly singular Volterra integral equations by using the Laplace transform method

$$\begin{aligned} u(x) &= x + x^2 - \frac{25}{6552}x^{\frac{8}{5}}(130x^{\frac{6}{5}} + 182x^{\frac{1}{5}} - 210x + 273) \\ &\quad + \int_0^x \left(\frac{1}{(x-t)^{\frac{1}{5}}}u(t) + \frac{1}{(x-t)^{\frac{2}{5}}}v(t) \right) dt, \\ v(x) &= x - x^2 - \frac{25}{924}x^{\frac{6}{5}}(55x^{\frac{6}{5}} + 66x^{\frac{1}{5}} - 140x + 154) \\ &\quad + \int_0^x \left(\frac{1}{(x-t)^{\frac{3}{5}}}u(t) + \frac{1}{(x-t)^{\frac{4}{5}}}v(t) \right) dt. \end{aligned} \tag{12.55}$$

Taking Laplace transforms of both sides of each equation in (12.55) and solving the resulting system of equations for $U(s)$ and $V(s)$ we obtain

$$U(s)=\frac{1}{s^2}+\frac{2}{s^3},\quad V(s)=\frac{1}{s^2}-\frac{2}{s^3}. \tag{12.56}$$

The inverse Laplace transform of (12.56) gives the exact solutions by

$$(u(x),v(x))=(x+x^2,x-x^2). \tag{12.57}$$

Example 12.8

Use the Laplace transform method to solve the system

$$\begin{aligned} u(x)&=1+x-x^2+\frac{3}{40}x^{\frac{2}{3}}(9x^2-12x-20)-\frac{8}{231}x^{\frac{3}{4}}(32x^2-44x+77)\\ &\quad+\int_0^x\left(\frac{1}{(x-t)^{\frac{1}{3}}}u(t)+\frac{2}{(x-t)^{\frac{1}{4}}}v(t)\right)dt,\\ v(x)&=1-x+x^2+\frac{5}{126}x^{\frac{4}{5}}(25x^2-35x-63)\\ &\quad+\frac{6}{935}x^{\frac{5}{6}}(72x^2-102x+187)\\ &\quad+\int_0^x\left(\frac{2}{(x-t)^{\frac{1}{5}}}u(t)-\frac{1}{(x-t)^{\frac{1}{6}}}v(t)\right)dt. \end{aligned} \tag{12.58}$$

Taking Laplace transform of both sides of each equation in (12.58), proceeding as before, and solving the resulting system of equations for $U(s)$ and $V(s)$ we obtain

$$U(s)=\frac{1}{s}+\frac{1}{s^2}-\frac{2}{s^3},\quad V(s)=\frac{1}{s}-\frac{1}{s^2}+\frac{2}{s^3}. \tag{12.59}$$

The inverse Laplace transform of (12.59) gives the exact solutions by

$$(u(x),v(x))=(1+x-x^2,1-x+x^2). \tag{12.60}$$

Exercises 12.3.1

Solve the following systems of weakly singular Volterra integral equations by using the Laplace transform method

1. $$\begin{cases} u(x)=x+x^2-\dfrac{25}{7}x^{\frac{7}{5}}+\displaystyle\int_0^x\left(\frac{1}{(x-t)^{\frac{3}{5}}}u(t)+\frac{1}{(x-t)^{\frac{3}{5}}}v(t)\right)dt\\ v(x)=x-x^2-\dfrac{25}{12}x^{\frac{8}{5}}+\displaystyle\int_0^x\left(\frac{1}{(x-t)^{\frac{2}{5}}}u(t)+\frac{1}{(x-t)^{\frac{2}{5}}}v(t)\right)dt \end{cases}$$

2. $$\begin{cases} u(x)=x^2-\dfrac{4}{15}x^{\frac{3}{2}}(4x+5)+\displaystyle\int_0^x\left(\frac{1}{(x-t)^{\frac{1}{2}}}u(t)+\frac{1}{(x-t)^{\frac{1}{2}}}v(t)\right)dt\\ v(x)=x-\dfrac{4}{15}x^{\frac{3}{2}}(4x-5)+\displaystyle\int_0^x\left(\frac{1}{(x-t)^{\frac{1}{2}}}u(t)-\frac{1}{(x-t)^{\frac{1}{2}}}v(t)\right)dt \end{cases}$$

3. $$\begin{cases} u(x) = 1 + x^2 - \frac{5}{2}x^{\frac{4}{5}} + \int_0^x \left(\frac{1}{(x-t)^{\frac{1}{5}}} u(t) + \frac{1}{(x-t)^{\frac{1}{5}}} v(t) \right) dt \\ v(x) = 1 - x^2 - \frac{5}{2}x^{\frac{4}{5}} + \int_0^x \left(\frac{1}{(x-t)^{\frac{1}{5}}} u(t) + \frac{1}{(x-t)^{\frac{1}{5}}} v(t) \right) dt \end{cases}$$

4. $$\begin{cases} u(x) = 1 + 3x - \frac{2\sqrt{x}}{5}(16x^2 + 10x + 15) + \int_0^x \left(\frac{1}{(x-t)^{\frac{1}{2}}} (u(t) + 2v(t)) \right) dt \\ v(x) = 1 + 3x^2 + \frac{2\sqrt{x}}{5}(8x^2 - 20x - 5) + \int_0^x \left(\frac{1}{(x-t)^{\frac{1}{2}}} (2u(t) - v(t)) \right) dt \end{cases}$$

5. $$\begin{cases} u(x) = 1 + x + x^2 - 4\sqrt{x} + \int_0^x \left(\frac{1}{(x-t)^{\frac{1}{2}}} (u(t) + v(t)) \right) dt \\ v(x) = 1 - x - x^2 - \frac{8}{15}x^{\frac{3}{2}}(4x + 5) + \int_0^x \left(\frac{1}{(x-t)^{\frac{1}{2}}} (u(t) - v(t)) \right) dt \end{cases}$$

6. $$\begin{cases} u(x) = 1 + x + 2x^2 + 2\sqrt{x}(2x - 3) + \int_0^x \left(\frac{1}{(x-t)^{\frac{1}{2}}} (u(t) + 2v(t)) \right) dt \\ v(x) = 1 - 2x - x^2 - \frac{2\sqrt{x}}{3}(8x^2 + 8x + 3) + \int_0^x \left(\frac{1}{(x-t)^{\frac{1}{2}}} (2u(t) - v(t)) \right) dt \end{cases}$$

7. $$\begin{cases} u(x) = 1 + x - x^2 - \frac{9}{40}x^{\frac{2}{3}}(3x^2 - 4x + 20) + \int_0^x \left(\frac{1}{(x-t)^{\frac{1}{3}}} (u(t) + 2v(t)) \right) dt \\ v(x) = 1 - x + x^2 + \frac{3}{40}x^{\frac{2}{3}}(27x^2 - 36x - 20) + \int_0^x \left(\frac{1}{(x-t)^{\frac{1}{3}}} (2u(t) - v(t)) \right) dt \end{cases}$$

8. $$\begin{cases} u(x) = 6 + x^2 - \frac{9}{280}x^{\frac{1}{3}}(21x^{\frac{7}{3}} + 280x^{\frac{1}{3}} - 60x^2 + 560) \\ \qquad + \int_0^x \left(\frac{1}{(x-t)^{\frac{1}{3}}} u(t) + \frac{1}{(x-t)^{\frac{2}{3}}} v(t) \right) dt \\ v(x) = 6 - x^2 - \frac{5}{924}x^{\frac{1}{5}}(275x^{\frac{11}{5}} + 2772x^{\frac{1}{5}} - 700x^2 + 5544) \\ \qquad + \int_0^x \left(\frac{1}{(x-t)^{\frac{3}{5}}} u(t) + \frac{1}{(x-t)^{\frac{4}{5}}} v(t) \right) dt \end{cases}$$

12.3.2 The Adomian Decomposition Method

The Adomian decomposition method, as presented before, decomposes each solution as an infinite sum of components, where these components are determined recurrently. This method can be used in its standard form, or combined with the noise terms phenomenon. It will be shown that the modified decomposition method is effective and reliable in handling the systems of weakly singular Volterra integral equations. In view of this, the modified decomposition method will be used extensively in this section.

We will focus our work on the system of the generalized weakly singular Volterra integral equations in two unknowns of the form

$$\begin{aligned} u(x) &= f_1(x) + \int_0^x (K_{11}(x,t)u(t) + K_{12}(x,t)v(t))\,dt, \\ v(x) &= f_2(x) + \int_0^x (K_{21}(x,t)u(t) + K_{22}(x,t)v(t))\,dt. \end{aligned} \tag{12.61}$$

The kernels $K_{ij}(x,t), 1 \leqslant i,j \leqslant 2$ and the functions $f_i(x), i = 1,2$ are given real-valued functions. The kernels K_{ij} are singular kernels of the generalized form given by

$$K_{ij} = \frac{1}{[g(x) - g(t)]^{\alpha_{ij}}}, \quad 1 \leqslant i, \quad j \leqslant 2. \tag{12.62}$$

This type of systems arise in many mathematical physics and chemistry applications such as stereology, heat conduction, crystal growth and the radiation of heat from a semi-infinite solid.

The Adomian decomposition method [8] has been discussed extensively in this text. As will be seen later, the modified decomposition method is reliable and effective in handling the system (12.61). In addition, the noise terms phenomenon will be used when noise terms appear.

For revision purposes, we give a brief review of the modified decomposition method. In this method we usually split each of the source terms $f_1(x)$ and $f_2(x)$ into two parts $f_{i1}(x)$ defined by

$$f_1(x) = f_{11} + f_{12}, \quad f_2(x) = f_{21} + f_{22}. \tag{12.63}$$

Based on this, we use the modified recurrence relation as follows:

$$\begin{aligned} u_0(x) &= f_{11}(x), \quad v_0(x) = f_{21}(x), \\ u_1(x) &= f_{12}(x) + \int_0^x (K_{11}(x,t)u_0(t) + K_{12}(x,t)v_0(t))\,dt, \\ v_1(x) &= f_{22}(x) + \int_0^x (K_{21}(x,t)u_0(t) + K_{22}(x,t)v_0(t))\,dt, \\ u_{k+1}(x) &= \int_0^x (K_{11}(x,t)u_k(t) + K_{12}(x,t)v_k(t))\,dt, \quad k \geqslant 1, \\ v_{k+1}(x) &= \int_0^x (K_{21}(x,t)u_k(t) + K_{22}(x,t)v_k(t))\,dt, \quad k \geqslant 1. \end{aligned} \tag{12.64}$$

The following examples will illustrate the analysis presented above.

Example 12.9

Use the modified method to solve the system of the generalized weakly singular integral equations

$$u(x) = e^x - \frac{3}{10}(3e^x + 7)(e^x - 1)^{\frac{2}{3}} + \int_0^x \left(\frac{1}{(e^x - e^t)^{\frac{1}{3}}} u(t) + \frac{1}{(e^x - e^t)^{\frac{1}{3}}} v(t) \right) dt,$$
$$v(x) = e^{2x} + \frac{9}{10}(e^x - 1)^{\frac{5}{3}} + \int_0^x \left(\frac{1}{(e^x - e^t)^{\frac{1}{3}}} u(t) - \frac{1}{(e^x - e^t)^{\frac{1}{3}}} v(t) \right) dt. \tag{12.65}$$

We set the modified recurrence relation

$$u_0(x) = e^x, \quad v_0(x) = e^{2x},$$
$$u_1(x) = -\frac{3}{10}(3e^x + 7)(e^x - 1)^{\frac{2}{3}} + \int_0^x \left(\frac{1}{(e^x - e^t)^{\frac{1}{3}}} u_0(t) + \frac{1}{(e^x - e^t)^{\frac{1}{3}}} v_0(t) \right) dt$$
$$= 0,$$
$$v_1(x) = \frac{9}{10}(e^x - 1)^{\frac{5}{3}} + \int_0^x \left(\frac{1}{(e^x - e^t)^{\frac{1}{3}}} u_0(t) - \frac{1}{(e^x - e^t)^{\frac{1}{3}}} v_0(t) \right) dt = 0. \tag{12.66}$$

The exact solutions are therefore given by

$$(u(x), v(x)) = (e^x, e^{2x}). \tag{12.67}$$

Example 12.10

Solve the system of the generalized weakly singular Volterra integral equations by using the modified decomposition method

$$\begin{aligned} u(x) &= \sin x + \frac{3}{2}(\cos x - 1)^{\frac{2}{3}} - \frac{3}{2}\sin^{\frac{2}{3}} x \\ &\quad + \int_0^x \left(\frac{1}{(\cos x - \cos t)^{\frac{1}{3}}} u(t) + \frac{1}{(\sin x - \sin t)^{\frac{1}{3}}} v(t) \right) dt, \\ v(x) &= \cos x + 3(\cos x - 1)^{\frac{1}{3}} - 3\sin^{\frac{1}{3}} x \\ &\quad + \int_0^x \left(\frac{1}{(\cos x - \cos t)^{\frac{2}{3}}} u(t) - \frac{1}{(\sin x - \sin t)^{\frac{2}{3}}} v(t) \right) dt. \end{aligned} \tag{12.68}$$

Proceeding as before, we set the modified recurrence relation

$$\begin{aligned} u_0(x) &= \sin x, \quad v_0(x) = \cos x, \\ u_1(x) &= \frac{3}{2}(\cos x - 1)^{\frac{2}{3}} - \frac{3}{2}\sin^{\frac{2}{3}} x \\ &\quad + \int_0^x \left(\frac{1}{(\cos x - \cos t)^{\frac{1}{3}}} u_0(t) + \frac{1}{(\sin x - \sin t)^{\frac{1}{3}}} v_0(t) \right) dt = 0, \\ v_1(x) &= 3(\cos x - 1)^{\frac{1}{3}} - 3\sin^{\frac{1}{3}} x \\ &\quad + \int_0^x \left(\frac{1}{(\cos x - \cos t)^{\frac{2}{3}}} u_0(t) - \frac{1}{(\sin x - \sin t)^{\frac{2}{3}}} v_0(t) \right) dt = 0, \\ u_{k+1}(x) &= 0, \quad v_{k+1}(x) = 0, \quad k \geqslant 1. \end{aligned} \tag{12.69}$$

The exact solutions are therefore given by

$$(u(x), v(x)) = (\sin x, \cos x). \tag{12.70}$$

Example 12.11

Solve the system of the generalized weakly singular Volterra integral equations by using the modified decomposition method

$$\begin{aligned} u(x) &= x^4 - 2x^{\frac{5}{2}} - x^4 + \int_0^x \left(\frac{5}{(x^5 - t^5)^{\frac{1}{2}}} u(t) + \frac{4}{(x^6 - t^6)^{\frac{1}{3}}} v(t) \right) dt, \\ v(x) &= x^5 - 4x^{\frac{15}{4}} + 5x^{\frac{24}{5}} + \int_0^x \left(\frac{15}{(x^5 - t^5)^{\frac{1}{4}}} u(t) - \frac{24}{(x^6 - t^6)^{\frac{1}{5}}} v(t) \right) dt. \end{aligned} \tag{12.71}$$

We next set the modified recurrence relation

$$\begin{aligned} &u_0(x) = x^4, \qquad v_0(x) = x^5, \\ &u_1(x) = 0, \qquad v_1(x) = 0, \\ &u_{k+1}(x) = 0, \quad v_{k+1}(x) = 0, \quad k \geqslant 1. \end{aligned} \tag{12.72}$$

The exact solutions are therefore given by

$$(u(x), v(x)) = (x^4, x^5). \tag{12.73}$$

Example 12.12

Solve the system of the generalized weakly singular Volterra integral equations by using the modified decomposition method

$$\begin{aligned} u(x) &= e^x - 2(e^x - 1)^{\frac{1}{2}}(1 + e^{-\frac{1}{2}x}) \\ &\quad + \int_0^x \left(\frac{1}{(e^x - e^t)^{\frac{1}{2}}} u(t) + \frac{1}{(e^{-t} - e^{-x})^{\frac{1}{2}}} v(t) \right) dt, \\ v(x) &= e^{-x} - \frac{3}{2}(e^x - 1)^{\frac{2}{3}}(1 + e^{-\frac{2}{3}x}) \\ &\quad + \int_0^x \left(\frac{1}{(e^x - e^t)^{\frac{1}{3}}} u(t) + \frac{1}{(e^{-t} - e^{-x})^{\frac{1}{3}}} v(t) \right) dt. \end{aligned} \tag{12.74}$$

Now we use the modified recurrence relation

$$\begin{aligned} &u_0(x) = e^x, \quad v_0(x) = e^{-x}, \\ &u_1(x) = 0, \quad v_1(x) = 0, \\ &u_{k+1}(x) = 0, \quad v_{k+1}(x) = 0, \quad k \geqslant 1. \end{aligned} \tag{12.75}$$

The exact solutions are therefore given by

$$(u(x), v(x)) = (e^x, e^{-x}). \tag{12.76}$$

Exercises 12.3.2

Solve the following systems of generalized weakly singular integral equations by using the modified decomposition method

1. $$\begin{cases} u(x) = 1 + x^2 - \frac{\pi}{4}(x^2 - 6) + \int_0^x \left(\frac{1}{(x^2 - t^2)^{\frac{1}{2}}} u(t) + \frac{2}{(x^2 - t^2)^{\frac{1}{2}}} v(t) \right) dt \\ v(x) = 1 - x^2 - \frac{9}{8} x^{\frac{2}{3}} (3x^2 + 4) + \int_0^x \left(\frac{4}{(x - t)^{\frac{1}{3}}} u(t) - \frac{1}{(x - t)^{\frac{1}{3}}} v(t) \right) dt \end{cases}$$

2. $$\begin{cases} u(x) = x - 4x^{\frac{1}{2}}(x + 1) + \int_0^x \left(\frac{1}{(x - t)^{\frac{1}{2}}} u(t) + \frac{2}{(x - t)^{\frac{1}{2}}} v(t) \right) dt \\ v(x) = 1 + x - \frac{3}{10} x^{\frac{2}{3}} (9x - 5) + \int_0^x \left(\frac{4}{(x - t)^{\frac{1}{3}}} u(t) - \frac{1}{(x - t)^{\frac{1}{3}}} v(t) \right) dt \end{cases}$$

3. $$\begin{cases} u(x) = x - \frac{4}{15} x^{\frac{3}{2}} (16x + 5) + \int_0^x \left(\frac{1}{(x - t)^{\frac{1}{2}}} u(t) + \frac{4}{(x - t)^{\frac{1}{2}}} v(t) \right) dt \\ v(x) = x^4 + \frac{9}{40} x^{\frac{5}{3}} (3x - 4) + \int_0^x \left(\frac{1}{(x - t)^{\frac{1}{3}}} u(t) - \frac{1}{(x - t)^{\frac{1}{3}}} v(t) \right) dt \end{cases}$$

4. $$\begin{cases} u(x) = \sin x + 2(\cos x - 1)^{\frac{1}{2}} - 2\sin^{\frac{1}{2}} x \\ \qquad + \int_0^x \left(\frac{1}{(\cos x - \cos t)^{\frac{1}{2}}} u(t) + \frac{1}{(\sin x - \sin t)^{\frac{1}{2}}} v(t) \right) dt, \\ v(x) = \cos x + \frac{3}{2}(\cos x - 1)^{\frac{2}{3}} + \frac{3}{2}\sin^{\frac{2}{3}} x \\ \qquad + \int_0^x \left(\frac{1}{(\cos x - \cos t)^{\frac{1}{3}}} u(t) - \frac{1}{(\sin x - \sin t)^{\frac{1}{3}}} v(t) \right) dt. \end{cases}$$

5. $$\begin{cases} u(x) = \cos x + \frac{3}{2}(\cos x - 1)^{\frac{2}{3}} - 2\sin^{\frac{1}{2}} x \\ \qquad + \int_0^x \left(\frac{1}{(\sin x - \sin t)^{\frac{1}{2}}} u(t) + \frac{1}{(\cos x - \cos t)^{\frac{1}{3}}} v(t) \right) dt, \\ v(x) = \sin x + \frac{5}{4}(\cos x - 1)^{\frac{4}{5}} - \frac{4}{3}\sin^{\frac{3}{4}} x \\ \qquad + \int_0^x \left(\frac{1}{(\sin x - \sin t)^{\frac{1}{4}}} u(t) + \frac{1}{(\cos x - \cos t)^{\frac{1}{5}}} v(t) \right) dt. \end{cases}$$

6. $$\begin{cases} u(x) = \sinh x - \frac{3}{2}(\cosh x - 1)^{\frac{2}{3}} - \frac{3}{2}\sinh^{\frac{2}{3}} x \\ \qquad + \int_0^x \left(\frac{1}{(\cosh x - \cosh t)^{\frac{1}{3}}} u(t) + \frac{1}{(\sinh x - \sinh t)^{\frac{1}{3}}} v(t) \right) dt, \\ v(x) = \cosh x - 3(\cosh x - 1)^{\frac{1}{3}} - 3\sinh^{\frac{1}{3}} x \\ \qquad + \int_0^x \left(\frac{1}{(\cosh x - \cosh t)^{\frac{2}{3}}} u(t) + \frac{1}{(\sinh x - \sinh t)^{\frac{2}{3}}} v(t) \right) dt. \end{cases}$$

7. $$\begin{cases} u(x) = \cosh x - \dfrac{3}{2}(\cosh x - 1)^{\frac{2}{3}} - 2\sinh^{\frac{1}{2}} x \\ \qquad\qquad + \displaystyle\int_0^x \left(\frac{1}{(\sinh x - \sinh t)^{\frac{1}{2}}} u(t) + \frac{1}{(\cosh x - \cosh t)^{\frac{1}{3}}} v(t)\right) dt, \\ v(x) = \sinh x - \dfrac{5}{4}(\cosh x - 1)^{\frac{4}{5}} - \dfrac{4}{3}\sinh^{\frac{3}{4}} x \\ \qquad\qquad + \displaystyle\int_0^x \left(\frac{1}{(\sinh x - \sinh t)^{\frac{1}{4}}} u(t) + \frac{1}{(\cosh x - \cosh t)^{\frac{1}{5}}} v(t)\right) dt. \end{cases}$$

8. $$\begin{cases} u(x) = e^x - \dfrac{3}{2}(e^x - 1)^{\frac{2}{3}}(1 + e^{-\frac{2}{3}x}) \\ \qquad\qquad + \displaystyle\int_0^x \left(\frac{1}{(e^x - e^t)^{\frac{1}{3}}} u(t) + \frac{1}{(e^{-t} - e^{-x})^{\frac{1}{3}}} v(t)\right) dt, \\ v(x) = e^{-x} - \dfrac{4}{3}(e^x - 1)^{\frac{3}{4}}(1 - e^{-\frac{3}{4}x}) \\ \qquad\qquad + \displaystyle\int_0^x \left(\frac{1}{(e^x - e^t)^{\frac{1}{4}}} u(t) - \frac{1}{(e^{-t} - e^{-x})^{\frac{1}{4}}} v(t)\right) dt \end{cases}$$

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Part II
Nonlinear Integral Equations

Chapter 13
Nonlinear Volterra Integral Equations

13.1 Introduction

It is well known that linear and nonlinear Volterra integral equations arise in many scientific fields such as the population dynamics, spread of epidemics, and semi-conductor devices. Volterra started working on integral equations in 1884, but his serious study began in 1896. The name integral equation was given by du Bois-Reymond in 1888. However, the name Volterra integral equation was first coined by Lalesco in 1908.

The linear Volterra integral equations and the linear Volterra integro-differential equations were presented in Chapters 3 and 5 respectively. It is our goal in this chapter to study the nonlinear Volterra integral equations of the first and the second kind. The nonlinear Volterra equations are characterized by at least one variable limit of integration. In the nonlinear Volterra integral equations of the *second kind*, the unknown function $u(x)$ appears inside and outside the integral sign. The nonlinear Volterra integral equation of the second kind is represented by the form

$$u(x) = f(x) + \int_0^x K(x,t)F(u(t))dt. \tag{13.1}$$

However, the nonlinear Volterra integral equations of the *first kind* contains the nonlinear function $F(u(x))$ inside the integral sign. The nonlinear Volterra integral equation of the first kind is expressed in the form

$$f(x) = \int_0^x K(x,t)F(u(t))dt. \tag{13.2}$$

For these two kinds of equations, the kernel $K(x,t)$ and the function $f(x)$ are given real-valued functions, and $F(u(x))$ is a nonlinear function of $u(x)$ such as $u^2(x), \sin(u(x))$, and $e^{u(x)}$.

13.2 Existence of the Solution for Nonlinear Volterra Integral Equations

In this section we will present an existence theorem for the solution of nonlinear Volterra integral equations. The complete proof of this theorem can be found in [1–6]. However, in what follows, we present a brief summary of the conditions under which a solution exists for this equation.

We first rewrite the nonlinear Volterra integral equation of the second kind by

$$u(x) = f(x) + \int_0^x G(x, t, u(t))dt. \tag{13.3}$$

The specific conditions under which a solution exists for the nonlinear Volterra integral equation are:

(i) The function $f(x)$ is integrable and bounded in $a \leqslant x \leqslant b$.

(ii) The function $f(x)$ must satisfy the Lipschitz condition in the interval (a, b). This means that

$$|f(x) - f(y)| < k|x - y|. \tag{13.4}$$

(iii) The function $G(x, t, u(t))$ is integrable and bounded $|G(x, t, u(t))| < K$ in $a \leqslant x, t \leqslant b$.

(iv) The function $G(x, t, u(t))$ must satisfy the Lipschitz condition

$$|G(x, t, z) - G(x, t, z')| < M|z - z'|. \tag{13.5}$$

The emphasis in this chapter will be on solving the nonlinear Volterra integral equations rather than proving theoretical concepts of convergence and existence. The theorems of uniqueness, existence, and convergence are important and can be found in the literature. The concern in this text will be on the determination of the solution $u(x)$ of the nonlinear Volterra integral equation of the first and the second kind.

13.3 Nonlinear Volterra Integral Equations of the Second Kind

We begin our study on nonlinear Volterra integral equations of the second kindgiven by

$$u(x) = f(x) + \int_0^x K(x, t)F(u(t))dt, \tag{13.6}$$

where the kernel $K(x, t)$ and the function $f(x)$ are given real-valued functions, and $F(u(x))$ is a nonlinear function of $u(x)$ such as $u^3(x), \cos(u(x))$, and$e^{u(x)}$. The unknown function $u(x)$, that will be determined, occurs inside and outside the integral sign.

The nonlinear Volterra equation (13.6) will be handled by using three distinct methods. The three methods are the successive approximations method, the series solution method, and the Adomian decomposition method (ADM). The latter will be combined with the modified decomposition method and the noise terms phenomenon.

13.3.1 The Successive Approximations Method

The successive approximations method [7], or the *Picard iteration method* was used before in Chapters 3 and 4. This method solves any problem by finding successive approximations to the solution by starting with an initial guess, called the zeroth approximation. As will be seen later, the zeroth approximation is any selective real-valued function that will be used in a recurrence relation to determine the other approximations.

Given the nonlinear Volterra integral equation of the second kind

$$u(x) = f(x) + \int_0^x K(x,t)F(u(t))dt, \tag{13.7}$$

where $u(x)$ is the unknown function to be determined and $K(x,t)$ is the kernel. The successive approximations method introduces the recurrence relation

$$u_{n+1}(x) = f(x) + \int_0^x K(x,t)F(u_n(t))dt, n \geqslant 0, \tag{13.8}$$

where the zeroth approximation $u_0(x)$ can be any selective real valued function. We always start with an initial guess for $u_0(x)$, mostly we select $0, 1$, or x for $u_0(x)$. Using this selection of $u_0(x)$ into (13.8), several successive approximations $u_k, k \geqslant 1$ will be determined as

$$\begin{aligned} u_1(x) &= f(x) + \int_0^x K(x,t)F(u_0(t))dt, \\ u_2(x) &= f(x) + \int_0^x K(x,t)F(u_1(t))dt, \\ u_3(x) &= f(x) + \int_0^x K(x,t)F(u_2(t))dt, \\ &\vdots \\ u_{n+1}(x) &= f(x) + \int_0^x K(x,t)F(u_n(t))dt. \end{aligned} \tag{13.9}$$

Consequently, the solution $u(x)$ is obtained by using

$$u(x) = \lim_{n\to\infty} u_{n+1}(x). \tag{13.10}$$

The question of convergence of $u_{n+1}(x)$ was examined in Chapter 3. The successive approximations method, or the Picard iteration method will be illustrated by the following examples.

Example 13.1

Use the successive approximations method to solve the nonlinear Volterra integral equation

$$u(x) = e^x + \frac{1}{3}x(1 - e^{3x}) + \int_0^x xu^3(t)dt. \tag{13.11}$$

For the zeroth approximation $u_0(x)$, we can select

$$u_0(x) = 1. \tag{13.12}$$

The method of successive approximations admits the use of the iteration formula

$$u_{n+1}(x) = e^x + \frac{1}{3}x(1 - e^{3x}) + \int_0^x xu_n^3(t)dt, \quad n \geqslant 0. \tag{13.13}$$

Substituting (13.12) into (13.13) we obtain the approximations

$$\begin{aligned}
u_0(x) &= 1,\\
u_1(x) &= e^x + \frac{1}{3}x(1 - e^{3x}) + \int_0^x xu_0^3(t)dt\\
&= 1 + x + \frac{1}{2!}x^2 - \frac{4}{3}x^3 - \frac{35}{24}x^4 - \frac{67}{60}x^5 + \cdots,\\
u_2(x) &= e^x + \frac{1}{3}x(1 - e^{3x}) + \int_0^x xu_1^3(t)dt\\
&= 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 - \frac{67}{60}x^5 + \cdots,\\
u_3(x) &= e^x + \frac{1}{3}x(1 - e^{3x}) + \int_0^x xu_2^3(t)dt\\
&= 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \frac{1}{6!}x^6 + \cdots,
\end{aligned} \tag{13.14}$$

and so on. Consequently, the solution $u(x)$ of (13.11) is given by

$$u(x) = \lim_{n\to\infty} u_n(x) = e^x. \tag{13.15}$$

Example 13.2

Use the successive approximations method to solve the nonlinear Volterra integral equation

$$u(x) = 4x - \frac{16}{3}x^3 - \frac{4}{3}x^4 + \int_0^x (x - t + 1)u^2(t)dt. \tag{13.16}$$

For the zeroth approximation $u_0(x)$, we can select

$$u_0(x) = 0. \tag{13.17}$$

The method of successive approximations admits the use of the iteration formula

$$u_{n+1}(x) = 4x - \frac{16}{3}x^3 - \frac{4}{3}x^4 + \int_0^x (x-t+1)u_n^2(t)dt, n \geqslant 0. \tag{13.18}$$

Substituting (13.17) into (13.18) we obtain the approximations

$$\begin{aligned}
u_0(x) &= 0,\\
u_1(x) &= 4x - \frac{16}{3}x^3 - \frac{4}{3}x^4 + \int_0^x (x-t+1)u_0^2(t)dt\\
&= 4x - \frac{16}{3}x^3 - \frac{4}{3}x^4,\\
u_2(x) &= 4x - \frac{16}{3}x^3 - \frac{4}{3}x^4 + \int_0^x (x-t+1)u_1^2(t)dt\\
&= 4x + \left(\frac{16}{3}x^3 - \frac{16}{3}x^3\right) + \left(\frac{4}{3}x^4 - \frac{4}{3}x^4\right) - \frac{128}{15}x^5 - \frac{16}{5}x^6 + \cdots,\\
u_3(x) &= 4x - \frac{16}{3}x^3 - \frac{4}{3}x^4 + \int_0^x (x-t+1)u_2^2(t)dt\\
&= 4x + \left(\frac{128}{15}x^5 - \frac{128}{15}x^5\right) + \left(\frac{16}{5}x^6 - \frac{16}{5}x^6\right) + \cdots .
\end{aligned} \tag{13.19}$$

By canceling the noise terms, the solution $u(x)$ of (13.16) is given by

$$u(x) = \lim_{n\to\infty} u_n(x) = 4x. \tag{13.20}$$

Example 13.3

Use the successive approximations method to solve the nonlinear Volterra integral equation

$$u(x) = \cos(x) + \frac{1}{8}\cos(2x) - \frac{1}{4}x^2 - \frac{1}{8} + \int_0^x (x-t)u^2(t)dt. \tag{13.21}$$

For the zeroth approximation $u_0(x)$, we can select

$$u_0(x) = 1. \tag{13.22}$$

The method of successive approximations admits the use of the iteration formula

$$u_{n+1}(x) = \cos(x) + \frac{1}{8}\cos(2x) - \frac{1}{4}x^2 - \frac{1}{8} + \int_0^x (x-t)u_n^2(t)dt, n \geqslant 0. \tag{13.23}$$

Substituting (13.22) into (13.23) we obtain the approximations

$$\begin{aligned}
u_0(x) &= 1,\\
u_1(x) &= 1 - \frac{1}{2!}x^2 + \frac{1}{8}x^4 - \frac{1}{80}x^6 + \cdots,\\
u_2(x) &= 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \frac{1}{240}x^6 + \cdots,\\
u_3(x) &= 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots .
\end{aligned} \tag{13.24}$$

Consequently, the solution $u(x)$ of (13.21) is given by

$$u(x) = \lim_{n\to\infty} u_n(x) = \cos x. \tag{13.25}$$

Example 13.4

Use the successive approximations method to solve the nonlinear Volterra integral equation

$$u(x) = 1 + \frac{1}{2}x^2 - \frac{1}{6}x^4 - \frac{1}{30}x^6 + \int_0^x (x-t)u^2(t)dt. \tag{13.26}$$

For the zeroth approximation $u_0(x)$, we can select

$$u_0(x) = 1. \tag{13.27}$$

The method of successive approximations admits the use of the iteration formula

$$u_{n+1}(x) = 1 + \frac{1}{2}x^2 - \frac{1}{6}x^4 - \frac{1}{30}x^6 + \int_0^x (x-t)u_n^2(t)dt, n \geqslant 0. \tag{13.28}$$

Substituting (13.27) into (13.28) we obtain the approximations

$$\begin{aligned}
u_0(x) &= 1,\\
u_1(x) &= 1 + x^2 - \frac{1}{6}x^4 - \frac{1}{30}x^6,\\
u_2(x) &= 1 + x^2 + \left(\frac{1}{6}x^4 - \frac{1}{6}x^4\right) - \frac{1}{90}x^6 + \cdots,\\
u_3(x) &= 1 + x^2 + \left(\frac{1}{6}x^4 - \frac{1}{6}x^4\right) + \left(\frac{1}{90}x^6 - \frac{1}{90}x^6\right) + \cdots,
\end{aligned} \tag{13.29}$$

and so on. By canceling the noise terms, the solution $u(x)$ of (13.26) is given by

$$u(x) = \lim_{n\to\infty} u_n(x) = 1 + x^2. \tag{13.30}$$

Exercises 13.3.1

Solve the following nonlinear Volterra integral equations by using the *successive approximations method*

1. $u(x) = 1 + \int_0^x u^2(t)dt, |x| < 1$
2. $u(x) = 1 + 3x - \frac{1}{2}x^2 - x^3 - \frac{3}{4}x^4 + \int_0^x (x-t)u^2(t)dt$
3. $u(x) = 1 + 2x - \frac{7}{2}x^2 - 4x^3 - \frac{3}{4}x^4 + \int_0^x (x-t+1)u^2(t)dt$
4. $u(x) = 1 + 2x + x^2 + \frac{1}{2}x^3 - \frac{1}{20}x^5 + \int_0^x (x-t-1)u^3(t)dt$
5. $u(x) = \sin x + \frac{1}{4}\sin^2 x - \frac{1}{4}x^2 + \int_0^x (x-t)u^2(t)dt$

6. $u(x) = \sin x + \cos x + \frac{1}{4}\sin 2x - \frac{1}{2}x - \frac{1}{2}x^2 + \int_0^x (x-t)u^2(t)dt$

7. $u(x) = \cos x - \sin x - \frac{1}{4}\sin 2x + \frac{1}{2}x - \frac{1}{2}x^2 + \int_0^x (x-t)u^2(t)dt$

8. $u(x) = 3\cos x + \frac{1}{4}\cos^2 x - \frac{5}{4} - \frac{3}{4}x^2 + \int_0^x (x-t)u^2(t)dt$

9. $u(x) = \sinh x - \frac{1}{4}\cosh^2 x + \frac{5}{4} - 2x - \frac{1}{4}x^2 + \int_0^x (x-t)u^2(t)dt$

10. $u(x) = e^x - \frac{1}{4}e^{2x} + \frac{1}{4} + \frac{1}{2}x + \int_0^x (x-t)u^2(t)dt$

11. $u(x) = e^x - \frac{1}{9}e^{3x} + \frac{1}{9} + \frac{1}{3}x + \int_0^x (x-t)u^3(t)dt$

12. $u(x) = e^x + \frac{1}{4}e^{2x} + \frac{3}{4} + \frac{7}{2}x - \frac{1}{2}x^2 + \int_0^x (x-t-1)u^2(t)dt$

13.3.2 The Series Solution Method

The series solution method was applied in Chapters 3, 4 and 5 to handle linear Volterra and Fredholm integral equations [1–2]. In this section, the series solution method will be applied in a similar manner to handle the nonlinear Volterra integral equations.

Recall that the generic form of Taylor series at $x = 0$ can be written as

$$u(x) = \sum_{n=0}^{\infty} a_n x^n. \tag{13.31}$$

We will assume that the solution $u(x)$ of the nonlinear Volterra integral equation

$$u(x) = f(x) + \int_0^x K(x,t)F(u(t))dt, \tag{13.32}$$

is analytic, and therefore possesses a Taylor series of the form given in (13.31), where the coefficients a_n will be determined recurrently. Substituting (13.31) into both sides of (13.32) gives

$$\sum_{n=0}^{\infty} a_n x^n = T(f(x)) + \int_0^x K(x,t)\left(F\left(\sum_{n=0}^{\infty} a_n t^n\right)\right)dt, \tag{13.33}$$

or for simplicity we use

$$a_0 + a_1 x + a_2 x^2 + \cdots = T(f(x)) + \int_0^x K(x,t)\left(F(a_0 + a_1 t + a_2 t^2 + \cdots)\right)dt, \tag{13.34}$$

where $T(f(x))$ is the Taylor series for $f(x)$. The integral equation (13.32) will be converted to a traditional integral in (13.33) or (13.34) where instead of integrating the nonlinear term $F(u(x))$, terms of the form t^n, $n \geqslant 0$ will be integrated. Notice that because we are seeking series solution, then if $f(x)$

includes elementary functions such as trigonometric functions, exponential functions, etc., then Taylor expansions for functions involved in $f(x)$ should be used.

We first integrate the right side of the integral in (13.33) or (13.34), and collect the coefficients of like powers of x. We next equate the coefficients of like powers of x in both sides of the resulting equation to obtain a recurrence relation in $a_j, j \geqslant 0$. Solving the recurrence relation will lead to a complete determination of the coefficients $a_j, j \geqslant 0$. Having determined the coefficients $a_j, j \geqslant 0$, the series solution follows immediately upon substituting the derived coefficients into (13.31). The exact solution may be obtained if such an exact solution exists. If an exact solution is not obtainable, then the obtained series can be used for numerical purposes. In this case, the more terms we determine, the higher accuracy level we achieve.

Example 13.5

Solve the following nonlinear Volterra integral equation by using the series solution method

$$u(x) = 1 + x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \frac{1}{12}x^4 + \int_0^x (x-t)u^2(t)dt. \tag{13.35}$$

Using the series form (13.31) into both sides of (13.35) gives

$$\begin{aligned} a_0 + a_1 x + a_2 x^2 + \cdots &= 1 + x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \frac{1}{12}x^4 \\ &\quad + \int_0^x (x-t)\left(a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \cdots\right)^2 dt, \end{aligned} \tag{13.36}$$

where by integrating the integral at the right side, and collecting like powers of x we obtain

$$\begin{aligned} & a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots \\ & = 1 + x + \frac{1}{2}(a_0^2 - 1)x^2 + \frac{1}{3}(a_0 a_1 - 1)x^3 + \frac{1}{12}(a_1^2 + 2a_0 a_2 - 1)x^4 + \cdots . \end{aligned} \tag{13.37}$$

Equating the coefficients of like powers of x in both sides yields

$$a_0 = 1, \quad a_1 = 1, \quad a_n = 0, \quad \text{for} \quad n \geqslant 2. \tag{13.38}$$

The exact solution is given by

$$u(x) = 1 + x. \tag{13.39}$$

Example 13.6

We next consider the nonlinear Volterra integral equation

$$u(x) = \frac{1}{9} + \frac{1}{3}x + e^x - \frac{1}{9}e^{3x} + \int_0^x (x-t)u^3(t)dt. \tag{13.40}$$

Substituting the series (13.31) into both sides of (13.40) noting that

$$u^3(x) = {a_0}^3 + 3{a_0}^2 a_1 x + (3a_0{a_1}^2 + 3{a_0}^2 a_2)x^2 + (a_1^3 + 6a_1 a_2 a_0 + 3a_3 a_0^2)x^3 + \cdots , \tag{13.41}$$

gives

$$\begin{aligned} a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \cdots = \frac{1}{9} + \frac{1}{3}x + e^x - \frac{1}{9}e^{3x} \\ + \int_0^x (x-t)\left(a_0 + a_1t + a_2t^2 + a_3t^3 + \cdots\right)^3 dt. \end{aligned} \tag{13.42}$$

Integrating the integral at the right hand side of (13.42), using the Taylor series of e^x and e^{3x}, and equating the coefficients of like powers of x we find

$$\begin{aligned} a_0 &= 1, \quad a_1 = 1, \\ a_2 &= \frac{1}{2!}a_0^2 = \frac{1}{2!}, \\ a_3 &= -\frac{1}{3} + \frac{1}{2}a_0^2a_1 = \frac{1}{3!}, \\ a_4 &= -\frac{1}{3} + \frac{1}{4}a_0(a_1^2 + a_0a_2) = \frac{1}{4!}, \\ &\ \vdots \\ a_n &= \frac{1}{n!} \end{aligned} \tag{13.43}$$

This gives the solution in a series form

$$u(x) = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots . \tag{13.44}$$

Consequently, the exact solution is given by

$$u(x) = e^x. \tag{13.45}$$

Example 13.7

We next consider the nonlinear Volterra integral equation

$$u(x) = \frac{1}{2}(x - x^2) + \cos x - \sin x - \frac{1}{4}\sin(2x) + \int_0^x (x-t)u^2(t)dt. \tag{13.46}$$

Substituting the series (13.31) into both sides of (13.46) gives

$$\begin{aligned} a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \cdots = \frac{1}{2}(x - x^2) + \cos x - \sin x \\ -\frac{1}{4}\sin 2x + \int_0^x (x-t)\left(a_0 + a_1t + a_2t^2 + a_3t^3 + \cdots\right)^2 dt. \end{aligned} \tag{13.47}$$

Integrating the integral at the right hand side of (13.47), using the Taylor series of $\sin x, \cos x$ and $\sin 2x$, and equating the coefficients of like powers of x we find

$$\begin{aligned} a_0 &= 1, \quad a_1 = -1, \\ a_2 &= -1 + \frac{1}{2!}a_0^2 = -\frac{1}{2!}, \\ a_3 &= \frac{1}{2} + \frac{1}{3}a_0a_1 = \frac{1}{3!}, \end{aligned} \tag{13.48}$$

$$a_4 = \frac{1}{4!} + \frac{1}{6}a_0a_2 + \frac{1}{12}a_1^2 = \frac{1}{4!},$$
$$a_5 = -\frac{3}{40} + \frac{1}{10}a_1a_2 + \frac{1}{10}a_0a_3 = -\frac{1}{5!},$$

and so on. The solution in a series form is given by

$$u(x) = \left(1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \cdots\right) - \left(x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \cdots\right), \quad (13.49)$$

that converges to the exact solution

$$u(x) = \cos x - \sin x. \quad (13.50)$$

Example 13.8

We finally consider the nonlinear Volterra integral equation

$$u(x) = \frac{9}{8} + 2x - \frac{1}{4}x^2 - \sinh x - \frac{1}{8}\cosh 2x + \int_0^x (x-t)u^2(t)dt. \quad (13.51)$$

Substituting the series (13.31) into both sides of (13.51) gives

$$\begin{aligned} a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \cdots &= \frac{9}{8} + 2x - \frac{1}{4}x^2 - \sinh x \\ -\frac{1}{8}\cosh 2x + \int_0^x (x-t)\left(a_0 + a_1t + a_2t^2 + a_3t^3 + \cdots\right)^2 dt. \end{aligned} \quad (13.52)$$

Integrating the integral at the right hand side of (13.52), using the Taylor series of $\sinh x$ and $\cosh 2x$, and equating the coefficients of like powers of x we find

$$\begin{aligned} a_0 &= 1, \quad a_1 = 1, \\ a_2 &= -\frac{1}{2} + \frac{1}{2!}a_0^2 = 0, \\ a_3 &= -\frac{1}{6} + \frac{1}{3}a_0a_1 = \frac{1}{3!}, \\ a_4 &= -\frac{1}{12} + \frac{1}{6}a_0a_2 + \frac{1}{12}a_1^2 = 0, \\ a_5 &= -\frac{1}{120} + \frac{1}{10}a_1a_2 + \frac{1}{10}a_0a_3 = \frac{1}{5!}, \end{aligned} \quad (13.53)$$

and so on. The solution in a series form is given by

$$u(x) = 1 + \left(x + \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \cdots\right), \quad (13.54)$$

that converges to the exact solution

$$u(x) = 1 + \sinh x. \quad (13.55)$$

Exercises 13.3.2

Solve the following nonlinear Volterra integral equations by using the *series solution method*

1. $u(x) = 1 + x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \frac{1}{12}x^4 + \int_0^x (x-t)u^2(t)dt$

2. $u(x) = 1 + \frac{1}{2}x^2 - \frac{1}{6}x^4 - \frac{1}{30}x^6 + \int_0^x (x-t)u^2(t)dt$

3. $u(x) = 1 + 2x - \frac{7}{2}x^2 - 4x^3 - \frac{3}{4}x^4 + \int_0^x (x-t+1)u^2(t)dt$

4. $u(x) = 1 + 2x - \frac{1}{2}x^2 - x^3 - x^4 - \frac{2}{5}x^5 + \int_0^x (x-t)u^3(t)dt$

5. $u(x) = \sin x + \frac{1}{4}\sin^2 x - \frac{1}{4}x^2 + \int_0^x (x-t)u^2(t)dt$

6. $u(x) = \sin x + \cos x + \frac{1}{4}\sin 2x - \frac{1}{2}x - \frac{1}{2}x^2 + \int_0^x (x-t)u^2(t)dt$

7. $u(x) = \frac{1}{8} - \frac{1}{4}x^2 + \cosh x - \frac{1}{8}\cosh 2x + \int_0^x (x-t)u^2(t)dt$

8. $u(x) = \frac{1}{4}x - \frac{1}{6}x^3 + \cosh x - \frac{1}{8}\sinh 2x + \int_0^x (x-t)^2u^2(t)dt$

9. $u(x) = \sinh x - \frac{1}{4}\cosh^2 x + \frac{5}{4} - 2x - \frac{1}{4}x^2 + \int_0^x (x-t)u^2(t)dt$

10. $u(x) = \frac{1}{4} + \frac{1}{2}x + \frac{1}{2}x^2 + e^x - \frac{1}{4}e^{2x} + \int_0^x (x-t)^2u^2(t)dt$

11. $u(x) = e^x - \frac{1}{9}e^{3x} + \frac{1}{9} + \frac{1}{3}x + \int_0^x (x-t)u^3(t)dt$

12. $u(x) = \frac{1}{4} - \frac{1}{2}x + e^{-x} - \frac{1}{4}e^{-2x} + \int_0^x (x-t)u^2(t)dt$

13.3.3 The Adomian Decomposition Method

The Adomian decomposition method has been outlined before in previous chapters and has been applied to a wide class of linear Volterra and Fredholm integral equations. The method usually decomposes the unknown function $u(x)$ into an infinite sum of components that will be determined recursively through iterations as discussed before. The Adomian decomposition method will be applied in this chapter and in the coming chapters to handle nonlinear integral equations.

Although the linear term $u(x)$ is represented by an infinite sum of components, the nonlinear terms such as $u^2, u^3, u^4, \sin u, e^u$, etc. that appear in the equation, should be expressed by a special representation, called the Adomian polynomials $A_n, n \geqslant 0$. Adomian introduced a formal algorithm to establish a reliable representation for all forms of nonlinear terms. Other techniques to evaluate Adomian polynomials were developed, but Adomian technique remains the commonly used one. In this text, we will use the Adomian algorithm to evaluate Adomian polynomials. The representation of the nonlinear terms by Adomian polynomials is necessary to handle the nonlinear integral equations in a reliable way.

In the following, the Adomian algorithm for calculating the so-called Adomian polynomials for representing nonlinear terms will be introduced in details. The algorithm will be explained by illustrative examples that will cover a wide variety of nonlinear forms.

Calculation of Adomian Polynomials

The Adomian decomposition method [8] assumes that the unknown linear function $u(x)$ may be represented by the infinite decomposition series

$$u(x) = \sum_{n=0}^{\infty} u_n(x), \tag{13.56}$$

where the components $u_n(x), n \geqslant 0$ will be computed in a recursive way. However, the nonlinear term $F(u(x))$, such as u^2, u^3, u^4, $\sin u, e^u$, etc. can be expressed by an infinite series of the so-called Adomian polynomials A_n given by

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[F\left(\sum_{i=0}^{n}\lambda^i\, u_i\right)\right]_{\lambda=0}, n = 0, 1, 2, \cdots, \tag{13.57}$$

where the so-called Adomian polynomials A_n can be evaluated for all forms of nonlinearity. The general formula (13.57) can be easily used as follows. Assuming that the nonlinear function is $F(u(x))$, therefore by using (13.57), Adomian polynomials are given by

$$\begin{aligned}
A_0 &= F(u_0),\\
A_1 &= u_1 F'(u_0),\\
A_2 &= u_2 F'(u_0) + \frac{1}{2!}u_1^2 F''(u_0),\\
A_3 &= u_3 F'(u_0) + u_1 u_2 F''(u_0) + \frac{1}{3!}u_1^3 F'''(u_0),\\
A_4 &= u_4 F'(u_0) + \left(\frac{1}{2!}u_2^2 + u_1 u_3\right) F''(u_0) + \frac{1}{2!}u_1^2 u_2 F'''(u_0)\\
&\quad + \frac{1}{4!}u_1^4 F^{(\text{iv})}(u_0).
\end{aligned} \tag{13.58}$$

Two important observations can be made here. First, A_0 depends only on u_0, A_1 depends only on u_0 and u_1, A_2 depends only on u_0, u_1, and u_2, and so on. Second, substituting (13.58) into (13.57) gives

$$\begin{aligned}
F(u) &= A_0 + A_1 + A_2 + A_3 + \cdots\\
&= F(u_0) + (u_1 + u_2 + u_3 + \cdots)F'(u_0)\\
&\quad + \frac{1}{2!}(u_1^2 + 2u_1u_2 + 2u_1u_3 + u_2^2 + \cdots)F''(u_0) + \cdots
\end{aligned}$$

$$+\frac{1}{3!}(u_1^3 + 3u_1^2 u_2 + 3u_1^2 u_3 + 6u_1 u_2 u_3 + \cdots)F'''(u_0) + \cdots$$
$$= F(u_0) + (u - u_0)F'(u_0) + \frac{1}{2!}(u - u_0)^2 F''(u_0) + \cdots . \qquad (13.59)$$

The last expansion confirms a fact that the series in A_n polynomials is a Taylor series about a function u_0 and not about a point as in the standard Taylor series. The few Adomian polynomials given above in (13.58) clearly show that the sum of the subscripts of the components of $u(x)$ of each term of A_n is equal to n.

In the following, we will calculate Adomian polynomials for several nonlinear terms that may arise in nonlinear integral equations.

Case 1.

The first four Adomian polynomials for $F(u) = u^2$ are given by

$$A_0 = F(u_0) = u_0^2,$$
$$A_1 = u_1 F'(u_0) = 2u_0 u_1,$$
$$A_2 = u_2 F'(u_0) + \frac{1}{2!}u_1^2 F''(u_0) = 2u_0 u_2 + u_1^2,$$
$$A_3 = u_3 F'(u_0) + u_1 u_2 F''(u_0) + \frac{1}{3!}u_1^3 F'''(u_0) = 2u_0 u_3 + 2u_1 u_2.$$

Case 2.

The first four Adomian polynomials for $F(u) = u^3$ are given by

$$A_0 = F(u_0) = u_0^3,$$
$$A_1 = u_1 F'(u_0) = 3u_0^2 u_1,$$
$$A_2 = u_2 F'(u_0) + \frac{1}{2!}u_1^2 F''(u_0) = 3u_0^2 u_2 + 3u_0 u_1^2,$$
$$A_3 = u_3 F'(u_0) + u_1 u_2 F''(u_0) + \frac{1}{3!}u_1^3 F'''(u_0) = 3u_0^2 u_3 + 6u_0 u_1 u_2 + u_1^3,$$
$$A_4 = u_4 F'(u_0) + \left(\frac{1}{2!}u_2^2 + u_1 u_3\right) F''(u_0) + \frac{1}{2!}u_1^2 u_2 F'''(u_0) + \frac{1}{4!}u_1^4 F^{(\text{iv})}(u_0).$$

Case 3.

The first four Adomian polynomials for $F(u) = u^4$ are given by

$$A_0 = u_0^4,$$
$$A_1 = 4u_0^3 u_1,$$
$$A_2 = 4u_0^3 u_2 + 6u_0^2 u_1^2,$$
$$A_3 = 4u_0^3 u_3 + 4u_1^3 u_0 + 12u_0^2 u_1 u_2.$$

Case 4.

The first four Adomian polynomials for $F(u) = \sin u$ are given by

$$A_0 = \sin u_0,$$

$$
\begin{aligned}
A_1 &= u_1 \cos u_0, \\
A_2 &= u_2 \cos u_0 - \frac{1}{2!} u_1^2 \sin u_0, \\
A_3 &= u_3 \cos u_0 - u_1 u_2 \sin u_0 - \frac{1}{3!} u_1^3 \cos u_0.
\end{aligned}
$$

Case 5.

The first four Adomian polynomials for $F(u) = \cos u$ are given by

$$
\begin{aligned}
A_0 &= \cos u_0, \\
A_1 &= -u_1 \sin u_0, \\
A_2 &= -u_2 \sin u_0 - \frac{1}{2!} u_1^2 \cos u_0, \\
A_3 &= -u_3 \sin u_0 - u_1 u_2 \cos u_0 + \frac{1}{3!} u_1^3 \sin u_0.
\end{aligned}
$$

Case 6.

The first four Adomian polynomials for $F(u) = e^u$ are given by

$$
\begin{aligned}
A_0 &= e^{u_0}, \\
A_1 &= u_1 e^{u_0}, \\
A_2 &= \left(u_2 + \frac{1}{2!} u_1^2 \right) e^{u_0}, \\
A_3 &= \left(u_3 + u_1 u_2 + \frac{1}{3!} u_1^3 \right) e^{u_0}.
\end{aligned}
$$

Applying the Adomian Decomposition Method

In what follows we present an outline for using the Adomian decomposition method for solving the nonlinear Volterra integral equation

$$u(x) = f(x) + \int_0^x K(x,t)\, F(u(t)) dt, \tag{13.60}$$

where $F(u(t))$ is a nonlinear function of $u(x)$. The nonlinear Volterra integral equation (13.60) contains the linear term $u(x)$ and the nonlinear function $F(u(x))$. The linear term $u(x)$ of (13.60) can be represented normally by the decomposition series

$$u(x) = \sum_{n=0}^{\infty} u_n(x), \tag{13.61}$$

where the components $u_n(x)$, $n \geqslant 0$ can be easily computed in a recursive manner as discussed before. However, the nonlinear function $F(u(x))$ of (13.60) should be represented by the so-called Adomian polynomials A_n by using the algorithm

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[F\left(\sum_{i=0}^{n}\lambda^i u_i\right)\right]_{\lambda=0}, n = 0, 1, 2, \cdots, \tag{13.62}$$

that was given above in (13.57). The standard Adomian method will be used for the first example. However, for the next three examples, we will use the modified decomposition method and the noise terms phenomenon to minimize the use of Adomian polynomials.

Example 13.9

Use the Adomian decomposition method to solve the nonlinear Volterra integral equation

$$u(x) = x + \int_0^x u^2(t)dt. \tag{13.63}$$

Substituting the series (13.61) and the Adomian polynomials (13.62) into the left side and the right side of (13.63) respectively gives

$$\sum_{n=0}^{\infty} u_n(x) = x + \int_0^x \sum_{n=0}^{\infty} A_n(t)dt, \tag{13.64}$$

where A_n are the Adomian polynomials for $u^2(x)$ as shown above. Using the Adomian decomposition method we set

$$u_0(x) = x, \quad u_{k+1}(x) = \int_0^x A_k(t)dt, \quad k \geqslant 0. \tag{13.65}$$

This in turn gives

$$\begin{aligned} u_0(x) &= x, \\ u_1(x) &= \int_0^x u_0^2(t)dt = \frac{1}{3}x^3, \\ u_2(x) &= \int_0^x (2u_0(t)u_1(t))dt = \frac{2}{15}x^5, \\ u_3(x) &= \int_0^x (2u_0(t)u_2(t) + u_1^2(t))dt = \frac{17}{315}x^7, \end{aligned} \tag{13.66}$$

and so on. The solution in a series form is given by

$$u(x) = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \frac{17}{315}x^7 + \cdots, \tag{13.67}$$

that converges to the exact solution

$$u(x) = \tan x. \tag{13.68}$$

Example 13.10

We consider the nonlinear Volterra integral equation

$$u(x) = 1 + 3x - \frac{1}{2}x^2 - x^3 - \frac{3}{4}x^4 + \int_0^x (x-t)u^2(t)dt. \tag{13.69}$$

Substituting the series (13.61) and the Adomian polynomials (13.62) into the left side and the right side of (13.69) respectively gives

$$\sum_{n=0}^{\infty} u_n(x) = 1 + 3x - \frac{1}{2}x^2 - x^3 - \frac{3}{4}x^4 + \int_0^x (x-t)\sum_{n=0}^{\infty} A_n(t)dt, \quad (13.70)$$

where A_n are the Adomian polynomials for $u^2(x)$ as shown above. Using the modified decomposition method we set

$$\begin{aligned} u_0(x) &= 1 + 3x - \frac{1}{2}x^2, \\ u_1(x) &= -x^3 - \frac{3}{4}x^4 + \int_0^x (x-t)A_0(t)dt \\ &= -x^3 - \frac{3}{4}x^4 + \int_0^x (x-t)u_0^2(t)dt = \frac{1}{2}x^2 - \frac{5}{6}x^4 - \frac{3}{20}x^5 + \frac{1}{120}x^6, \\ u_{k+1}(x) &= \int_0^x (x-t)A_k(t)dt, \quad k \geqslant 1. \end{aligned} \quad (13.71)$$

We can observe the appearance of the noise terms $-\frac{1}{2}x^2$ and $\frac{1}{2}x^2$ between $u_0(x)$ and $u_1(x)$. By canceling the noise term $-\frac{1}{2}x^2$ from $u_0(x)$, we can show that

$$u(x) = 1 + 3x, \quad (13.72)$$

is the exact solution that satisfies the integral equation. It is worth noting that we did not use the Adomian polynomials. This is due to the fact the modified decomposition method and the noise terms phenomenon accelerate the convergence of the solution.

Example 13.11

We consider the nonlinear Volterra integral equation

$$u(x) = \sin x + \frac{1}{4}\sin^2 x - \frac{1}{4}x^2 + \int_0^x (x-t)u^2(t)dt. \quad (13.73)$$

Substituting the series (13.61) and the Adomian polynomials (13.62) into the left side and the right side of (13.73) respectively we find

$$\sum_{n=0}^{\infty} u_n(x) = \sin x + \frac{1}{4}\sin^2 x - \frac{1}{4}x^2 + \int_0^x (x-t)\sum_{n=0}^{\infty} A_n(t)dt, \quad (13.74)$$

where A_n are the Adomian polynomials for $u^2(x)$ as shown above. Using the modified decomposition method we set

$$\begin{aligned} u_0(x) &= \sin x + \frac{1}{4}\sin^2 x, \\ u_1(x) &= -\frac{1}{4}x^2 + \int_0^x (x-t)A_0(t)dt \\ &= -\frac{1}{4}x^2 + \int_0^x (x-t)u_0^2(t)dt = -\frac{1}{4}\sin^2 x + \cdots, \\ u_{k+1}(x) &= \int_0^x (x-t)A_k(t)dt, \quad k \geqslant 1. \end{aligned} \quad (13.75)$$

We can observe the appearance of the noise terms $\frac{1}{4}\sin^2 x$ and $-\frac{1}{4}\sin^2 x$ between $u_0(x)$ and $u_1(x)$ respectively. By canceling the noise term $\frac{1}{4}\sin^2 x$ from $u_0(x)$, we can show that

$$u(x) = \sin x, \tag{13.76}$$

is the exact solution that satisfies the integral equation.

Example 13.12

We consider the nonlinear Volterra integral equation

$$u(x) = \sec x - \tan x - \frac{1}{2}x^2 + \int_0^x (x-t)u^2(t)dt. \tag{13.77}$$

Substituting the series (13.61) and the Adomian polynomials (13.62) into the left side and the right side of (13.77) respectively gives

$$\sum_{n=0}^{\infty} u_n(x) = \sec x - \tan x - \frac{1}{2}x^2 + \int_0^x (x-t)\sum_{n=0}^{\infty} A_n(t)dt, \tag{13.78}$$

where A_n are the Adomian polynomials for $u^2(x)$ as shown above. Using the modified decomposition method we set

$$\begin{aligned} u_0(x) &= \sec x, \\ u_1(x) &= -\tan x - \frac{1}{4}x^2 + \int_0^x (x-t)A_0(t)dt \\ &= -\tan x - \frac{1}{2}x^2 + \int_0^x (x-t)u_0^2(t)dt = 0. \end{aligned} \tag{13.79}$$

The other components $u_k, k \geqslant 2$ vanish as a result. Consequently, the exact solution is given by

$$u(x) = \sec x. \tag{13.80}$$

Exercises 13.3.3

Solve the following nonlinear Volterra integral equations of the second kind by using the Adomian decomposition method, the modified Adomian decomposition method, or by using the noise terms phenomenon

1. $u(x) = x - \int_0^x u^2(t)dt$
2. $u(x) = 1 + \frac{1}{2}x^2 - \frac{1}{6}x^4 - \frac{1}{30}x^6 + \int_0^x (x-t)u^2(t)dt$
3. $u(x) = 1 + x^2 + \frac{1}{6}x^4 + \frac{1}{10}x^6 + \frac{1}{42}x^8 + \int_0^x (xt^2 - x^2t)u^2(t)dt$
4. $u(x) = 1 + 2x - \frac{1}{2}x^2 - x^3 - x^4 - \frac{2}{5}x^5 + \int_0^x (x-t)u^3(t)dt$
5. $u(x) = x(1 - e^x) + \int_0^x e^{x-t}e^{u(t)}dt$

6. $u(x) = \sin x + \cos x + \frac{1}{4}\sin 2x - \frac{1}{2}x - \frac{1}{2}x^2 + \int_0^x (x-t)u^2(t)dt$

7. $u(x) = 1 + x^2 - xe^{x^2} + \int_0^x e^{x^2-t^2-1}e^{u(t)}dt$

8. $u(x) = e^x - xe^x + \int_0^x e^{x-2t}u^2(t)dt$

9. $u(x) = \sinh x - \frac{1}{4}\cosh^2 x + \frac{5}{4} - 2x - \frac{1}{4}x^2 + \int_0^x (x-t)u^2(t)dt$

10. $u(x) = \sin x + \frac{2}{3}\cos x - \frac{1}{3}\cos^2 x - \frac{1}{3} + \int_0^x \sin(x-t)u^2(t)dt$

11. $u(x) = \sin x - \frac{1}{5}\cos^2 x + \frac{1}{5}\sin 2x - \frac{2}{5}e^x + \frac{3}{5} + \int_0^x e^{x-t}u(t)dt$

12. $u(x) = \cos x - \frac{1}{5}\sin 2x - \frac{3}{5}\sinh x + \int_0^x \cosh(x-t)u^2(t)dt$

13.4 Nonlinear Volterra Integral Equations of the First Kind

The standard form of the nonlinear Volterra integral equation of the first kind is given by

$$f(x) = \int_0^x K(x,t)F(u(t))dt, \tag{13.81}$$

where the kernel $K(x,t)$ and the function $f(x)$ are given real-valued functions, and $F(u(x))$ is a nonlinear function of $u(x)$. Recall that the unknown function $u(x)$ occurs only inside the integral sign for the Volterra integral equation of the first kind. The linear Volterra integral equation of the first kind is presented in Section 3.3 where three main methods were used for handling this kind of equations.

To determine a solution for the nonlinear Volterra integral equation of the first kind (13.81), we first convert it to a linear Volterra integral equation of the first kind of the form

$$f(x) = \int_0^x K(x,t)v(t)dt, \tag{13.82}$$

by using the transformation

$$v(x) = F(u(x)). \tag{13.83}$$

This in turn means that

$$u(x) = F^{-1}(v(x)). \tag{13.84}$$

It is worth noting that the Volterra integral equation of the first kind (13.82) can be solved by any method that was studied in Section 3.3. However, in this section we will handle Eq. (13.82) by the Laplace transform method and the conversion to Volterra integral equation of the second kind. Other methods can be found in the literature and in this text as well.

13.4.1 The Laplace Transform Method

The Laplace transform method is a powerful technique that we used before for solving Volterra integral equations of the first and the second kinds. We assume that the kernel $K(x,t)$ is a difference kernel. Taking the Laplace transforms of both sides of (13.82) gives

$$\mathcal{L}\{f(x)\} = \mathcal{L}\{K(x-t)\} \times \mathcal{L}\{v(x)\}, \tag{13.85}$$

so that

$$V(s) = \frac{F(s)}{\mathcal{K}(s)}, \tag{13.86}$$

where

$$F(s) = \mathcal{L}\{f(x)\}, \mathcal{K}(s) = \mathcal{L}\{K(x)\}, V(s) = \mathcal{L}\{v(x)\}. \tag{13.87}$$

Taking the inverse Laplace transform of both sides of (13.86) gives $v(x)$. The solution $u(x)$ is obtained by using (13.84). It is obvious that the Laplace transform method works effectively provided that

$$\lim_{s\to\infty} \frac{F(s)}{\mathcal{K}(s)} = 0. \tag{13.88}$$

The Laplace transform method will be used for studying the following nonlinear Volterra integral equations of the first kind.

Example 13.13

Solve the nonlinear Volterra integral equation of the first kind by using the Laplace transform method

$$\frac{1}{4}e^{2x} - \frac{1}{2}x - \frac{1}{4} = \int_0^x (x-t)u^2(t)dt. \tag{13.89}$$

We first set

$$v(x) = u^2(x), u(x) = \pm\sqrt{v(x)}, \tag{13.90}$$

to carry out (13.89) into

$$\frac{1}{4}e^{2x} - \frac{1}{2}x - \frac{1}{4} = \int_0^x (x-t)v(t)dt. \tag{13.91}$$

Taking the Laplace transform of both sides of (13.91) yields

$$\frac{1}{4(s-2)} - \frac{1}{2s^2} - \frac{1}{4s} = \frac{1}{s^2}V(s), \tag{13.92}$$

or equivalently

$$V(s) = \frac{1}{s-2}, \tag{13.93}$$

where

$$V(s) = \mathcal{L}\{v(x)\}. \tag{13.94}$$

Taking the inverse Laplace transform of both sides of (13.93) gives

$$v(x) = e^{2x}. \tag{13.95}$$

The exact solutions are therefore given by

$$u(x) = \pm e^{x}. \tag{13.96}$$

It is worth noting that two solutions were obtained because Eq. (13.89) is a nonlinear equation, and the solution may not be unique.

Example 13.14

Solve the nonlinear Volterra integral equation of the first kind by using the Laplace transform method

$$\frac{1}{2}\sin x - \frac{1}{2}x\cos x = \int_0^x \sin(x-t)\sin u(t)dt. \tag{13.97}$$

We first set

$$v(x) = \sin u(x), \quad u(x) = \sin^{-1} v(x), \tag{13.98}$$

to carry out (13.97) into

$$\frac{1}{2}\sin x - \frac{1}{2}x\cos x = \int_0^x \sin(x-t)v(t)dt. \tag{13.99}$$

Taking the Laplace transform of both sides of (13.99) yields

$$\frac{1}{2(s^2+1)} - \frac{s^2-1}{2(s^2+1)^2} = \frac{1}{s^2+1}V(s), \tag{13.100}$$

or equivalently

$$V(s) = \frac{1}{s^2+1}. \tag{13.101}$$

Taking the inverse Laplace transform of both sides of the last equation gives

$$v(x) = \sin x. \tag{13.102}$$

The exact solution is therefore given by

$$u(x) = x. \tag{13.103}$$

Example 13.15

Solve the nonlinear Volterra integral equation of the first kind by using the Laplace transform method

$$\frac{1}{2} + \frac{1}{6}\cosh 2x - \frac{2}{3}\cosh x = \int_0^x \sinh(x-t)u^2(t)dt. \tag{13.104}$$

We first set

$$v(x) = u^2(x), u(x) = \pm\sqrt{v(x)}, \tag{13.105}$$

to carry out (13.104) into

$$\frac{1}{2} + \frac{1}{6}\cosh 2x - \frac{2}{3}\cosh x = \int_0^x \sinh(x-t)v(t)dt. \tag{13.106}$$

Taking the Laplace transform of both sides of (13.106) yields

$$\frac{1}{2s} + \frac{s}{6(s^2-4)} - \frac{2s}{3(s^2-1)} = \frac{1}{s^2-1}V(s), \tag{13.107}$$

or equivalently

$$V(s) = -\frac{1}{2s} + \frac{s}{2(s^2-4)}. \tag{13.108}$$

Taking the inverse Laplace transform of both sides of the last equation gives

$$v(x) = \sinh^2 x. \tag{13.109}$$

The exact solutions are therefore given by

$$u(x) = \pm \sinh x. \tag{13.110}$$

Example 13.16

Solve the nonlinear Volterra integral equation of the first kind by using the Laplace transform method

$$e^{2x} - e^x = \int_0^x e^{x-t}u^2(t)dt. \tag{13.111}$$

We first set

$$v(x) = u^2(x), \quad u(x) = \pm\sqrt{v(x)}, \tag{13.112}$$

to carry out (13.111) into

$$e^{2x} - e^x = \int_0^x e^{x-t}v(t)dt. \tag{13.113}$$

Taking the Laplace transform of both sides of (13.113) yields

$$V(s) = \frac{1}{s-2}. \tag{13.114}$$

Taking the inverse Laplace transform of both sides of the last equation gives

$$v(x) = e^{2x}. \tag{13.115}$$

The exact solutions are therefore given by

$$u(x) = \pm e^x. \tag{13.116}$$

Exercises 13.4.1

Use the *Laplace transform method* to solve the nonlinear Volterra integral equations of the first kind:

1. $\frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{12}x^4 = \int_0^x (x-t)u^2(t)dt$
2. $\sin x + \frac{2}{3}\cos x - \frac{2}{3}\cos(2x) = \int_0^x \cos(x-t)u^2(t)dt$
3. $\frac{1}{3}e^{4x} - \frac{1}{3}e^x = \int_0^x e^{x-t}u^2(t)dt$
4. $\frac{2}{3}\sin x - \frac{1}{3}\sin 2x = \int_0^x \cos(x-t)u^2(t)dt$

5. $\frac{1}{4}x^2 - \frac{1}{2}x - \frac{1}{4}\sin^2 x + \frac{1}{4}\sin 2x = \int_0^x (x-t-1)u^2(t)dt$

6. $\frac{1}{2}x^2 + \frac{1}{2}x^3 + \frac{1}{4}x^4 + \frac{1}{20}x^5 = \int_0^x (x-t)u^3(t)dt$

7. $\frac{1}{36}e^{6x} - \frac{1}{6}x - \frac{1}{36} = \int_0^x (x-t)e^{2u(t)}dt$

8. $\frac{1}{5}e^{6x} - \frac{1}{5}e^{x} = \int_0^x e^{x-t}e^{2u(t)}dt$

13.4.2 Conversion to a Volterra Equation of the Second Kind

Consider the nonlinear Volterra integral equation of the first kind

$$f(x) = \int_0^x K(x,t)F(u(t))dt, \tag{13.117}$$

where the kernel $K(x,t)$ and the function $f(x)$ are given real-valued functions, and $u(x)$ is the function to be determined. In a manner parallel to our discussion before, we convert (13.117) to a linear Volterra integral equation of the first kind of the form

$$f(x) = \int_0^x K(x,t)v(t)dt, \tag{13.118}$$

by using the transformation

$$v(x) = F(u(x)). \tag{13.119}$$

This in turn means that

$$u(x) = F^{-1}(v(x)). \tag{13.120}$$

We next convert the linear Volterra integral equations of the first kind (13.118) to a Volterra integral equations of the second kind. The conversion technique works effectively only if $K(x,x) \neq 0$. Differentiating both sides of (13.118) with respect to x, and using Leibnitz rule, we find

$$f'(x) = K(x,x)v(x) + \int_0^x K_x(x,t)v(t)dt. \tag{13.121}$$

Solving for $v(x)$, provided that $K(x,x) \neq 0$, we obtain the Volterra integral equation of the second kind given by

$$v(x) = \frac{f'(x)}{K(x,x)} - \int_0^x \frac{1}{K(x,x)} K_x(x,t)v(t)dt. \tag{13.122}$$

Having converted the Volterra integral equation of the first kind to the Volterra integral equation of the second kind, we then can use any method that was presented before. Because we solved the Volterra integral equations of the second kind by many methods, we will select distinct methods for solving the nonlinear Volterra integral equation of the first kind after converting it to a Volterra integral equation of the second kind.

Example 13.17

Convert the nonlinear Volterra integral equation of the first kind to the second kind and solve the resulting equation

$$\frac{1}{3}(e^x - e^{-2x}) = \int_0^x e^{x-t}u^2(t)dt. \tag{13.123}$$

We first set

$$v(x) = u^2(x), u(x) = \pm\sqrt{v(x)}, \tag{13.124}$$

to carry out (13.123) into

$$\frac{1}{3}(e^x - e^{-2x}) = \int_0^x e^{x-t}v(t)dt. \tag{13.125}$$

Differentiating both sides of (13.125) with respect to x by using Leibnitz rule we find the Volterra integral equation of the second kind

$$v(x) = \frac{1}{3}(e^x + 2e^{-2x}) - \int_0^x e^{x-t}v(t)dt. \tag{13.126}$$

This can be solved by many methods as presented in Chapter 3. We select the Laplace transform method to solve this equation. Taking Laplace transform of both sides of (13.126) yields

$$V(s) = \frac{1}{s+2}. \tag{13.127}$$

Taking the inverse Laplace transform of both sides gives

$$v(x) = e^{-2x}. \tag{13.128}$$

The exact solutions are therefore given by

$$u(x) = \pm e^{-x}. \tag{13.129}$$

Example 13.18

Convert the Volterra integral equation of the first kind to the second kind and solve the resulting equation

$$\frac{1}{3}x^3 + \frac{1}{12}x^4 = \int_0^x (x-t+1)u^2(t)dt. \tag{13.130}$$

We first set

$$v(x) = u^2(x), u(x) = \pm\sqrt{v(x)}, \tag{13.131}$$

to carry out (13.130) into

$$\frac{1}{3}x^3 + \frac{1}{12}x^4 = \int_0^x (x-t+1)v(t)dt. \tag{13.132}$$

Differentiating both sides of (13.132) with respect to x by using Leibnitz rule we find the Volterra integral equation of the second kind

$$v(x) = x^2 + \frac{1}{3}x^3 - \int_0^x v(t)dt. \tag{13.133}$$

We will select the modified decomposition method to solve this equation. Using the recursive relation

$$v_0(x) = x^2, \quad v_1(x) = \frac{1}{3}x^3 - \int_0^x v_0(t)dt = 0. \tag{13.134}$$

The exact solutions are therefore given by

$$u(x) = \pm x. \tag{13.135}$$

Example 13.19

Convert the Volterra integral equation of the first kind to the second kind and solve the resulting equation

$$\frac{3}{4}e^{2x} - \frac{1}{2}x - \frac{3}{4} = \int_0^x (x - t + 1)u^2(t)dt. \tag{13.136}$$

We first set

$$v(x) = u^2(x), \quad u(x) = \pm\sqrt{v(x)}, \tag{13.137}$$

to carry out (13.136) into

$$\frac{3}{4}e^{2x} - \frac{1}{2}x - \frac{3}{4} = \int_0^x (x - t + 1)v(t)dt. \tag{13.138}$$

Differentiating both sides of (13.138) and proceeding as before we find

$$v(x) = \frac{3}{2}e^{2x} - \frac{1}{2} - \int_0^x v(t)dt. \tag{13.139}$$

We will select the modified decomposition method to solve this equation. Using the recursive relation

$$v_0(x) = e^{2x}, \quad v_1(x) = \frac{1}{2}e^{2x} - \frac{1}{2} - \int_0^x v_0(t)dt = 0. \tag{13.140}$$

The exact solutions are therefore given by

$$u(x) = \pm e^x. \tag{13.141}$$

Example 13.20

Convert the Volterra integral equation of the first kind to the second kind and solve the resulting equation

$$\frac{1}{3}\sinh x + \frac{1}{3}\sinh 2x = \int_0^x \cosh(x - t)u^2(t)dt. \tag{13.142}$$

This equation is equivalent to

$$\frac{1}{3}\sinh x + \frac{1}{3}\sinh 2x = \int_0^x \cosh(x - t)v(t)dt, \tag{13.143}$$

upon setting $u^2(x) = v(x)$. Differentiating both sides of (13.143) and using Leibnitz rule we obtain

$$v(x) = \frac{1}{3}\cosh x + \frac{2}{3}\cosh(2x) - \int_0^x \sinh(x - t)v(t)dt. \tag{13.144}$$

For this problem, we select the successive approximations method. We select $v_0(x) = 1$. Consequently, we obtain

$$\begin{aligned}
v_0(x) &= 1,\\
v_1(x) &= -\frac{2}{3}\cosh x + \frac{2}{3}\cosh 2x + 1,\\
v_2(x) &= -\frac{4}{9}\cosh x + \frac{4}{9}\cosh 2x + \frac{1}{3}x\sinh x,\\
v_3(x) &= -\frac{14}{27}\cosh x + \frac{14}{27}\cosh 2x + \frac{11}{36}x\sinh x - \frac{1}{12}x^2\cosh x + 1\\
&= 1 + x^2 + \frac{1}{3}x^4 + \frac{2}{45}x^6 + \cdots .
\end{aligned} \tag{13.145}$$

The last approximation is a Taylor series for $\cosh^2 x$. This in turn gives

$$v(x) = \cosh^2 x. \tag{13.146}$$

Consequently, the exact solution of the integral equation is given by

$$u(x) = \pm\cosh x. \tag{13.147}$$

Exercises 13.4.2

Convert the nonlinear Volterra integral equations of the first kind to the second kind linear Volterra integral equation, and solve the resulting equation:

1. $-\frac{1}{2}x + \frac{1}{2}x^2 + \frac{1}{4}\sin 2x = \int_0^x (x-t)u^2(t)dt$
2. $\sin x + \frac{2}{3}\cos x - \frac{2}{3}\cos 2x = \int_0^x \cos(x-t)u^2(t)dt$
3. $\frac{2}{3}x - \frac{2}{3}\sin x - \frac{1}{9}\sin^3 x = \int_0^x (x-t)u^3(t)dt$
4. $\frac{2}{3}\sin x - \frac{1}{3}\sin 2x = \int_0^x \cos(x-t)u^2(t)dt$
5. $\frac{1}{3}x^3 - \frac{1}{3}x^4 + \frac{2}{15}x^5 = \int_0^x (x-t)^2u^2(t)dt$
6. $-\frac{1}{4} - \frac{1}{4}x^2 + \frac{1}{4}\cosh^2 x = \int_0^x (x-t)u^2(t)dt$
7. $\frac{1}{36}e^{6x} - \frac{1}{6}x - \frac{1}{36} = \int_0^x (x-t)e^{2u(t)}dt$
8. $\frac{1}{5}e^{6x} - \frac{1}{5}e^{x} = \int_0^x e^{x-t}e^{2u(t)}dt$

13.5 Systems of Nonlinear Volterra Integral Equations

In this section, systems of nonlinear Volterra integral equations of the first kind and the second kind will be studied. Many numerical and analytical methods are usually are used to handle the two kinds of systems. However, in this section we will concern ourselves with the Adomian decomposition method, and the successive approximations method to handle the nonlinear Volterra integral equations of the first kind and the second kind.

13.5.1 Systems of Nonlinear Volterra Integral Equations of the Second Kind

Systems of nonlinear Volterra integral equations of the second kind are given by

$$\begin{aligned} u(x) &= f_1(x) + \int_0^x \left(K_1(x,t)F_1(u(t)) + \widetilde{K}_1(x,t)\widetilde{F}_1(v(t))\right) dt, \\ v(x) &= f_2(x) + \int_0^x \left(K_2(x,t)F_2(u(t)) + \widetilde{K}_2(x,t)\widetilde{F}_2(v(t))\right) dt. \end{aligned} \tag{13.148}$$

The unknown functions $u(x)$ and $v(x)$, that will be determined, occur inside and outside the integral sign. The kernels $K_i(x,t)$ and $\widetilde{K}_i(x,t)$, and the function $f_i(x)$ are given real-valued functions, for $i = 1, 2$. The functions F_i and $\widetilde{F}_i$, for $i = 1, 2$ are nonlinear functions of $u(x)$ and $v(x)$.

The Adomian decomposition method, as presented before, decomposes each solution as an infinite sum of components, where these components are determined recurrently. This method can be used in its standard form, or combined with the noise terms phenomenon. Moreover, the modified decomposition method will be used wherever it is appropriate. It is interesting to point out that the nonlinear functions F_i and $\widetilde{F}_i$, for $i = 1, 2$, should be replaced by the Adomian polynomials A_n defined by

$$\begin{aligned} A_0 &= F(u_0), \\ A_1 &= u_1 F'(u_0), \\ A_2 &= u_2 F'(u_0) + \frac{1}{2!}u_1^2 F''(u_0), \\ A_3 &= u_3 F'(u_0) + u_1 u_2 F''(u_0) + \frac{1}{3!}u_1^3 F'''(u_0), \\ A_4 &= u_4 F'(u_0) + \left(\frac{1}{2!}u_2^2 + u_1 u_3\right) F''(u_0) + \frac{1}{2!}u_1^2 u_2 F'''(u_0) \\ &\quad + \frac{1}{4!}u_1^4 F^{(\text{iv})}(u_0). \end{aligned} \tag{13.149}$$

As stated before, the successive substitutions method, that was used in this chapter and in other chapters as well, will be also used to study the nonlinear systems of the two kinds.

Example 13.21

Use the Adomian decomposition method to solve the following system of nonlinear Volterra integral equations

$$\begin{aligned} u(x) &= x - \frac{2}{3}x^3 + \int_0^x \left(u^2(t) + v(t)\right) dt, \\ v(x) &= x^2 - \frac{1}{4}x^4 + \int_0^x u(t)v(t)dt. \end{aligned} \tag{13.150}$$

The Adomian decomposition method suggests that the linear terms $u(x)$ and $v(x)$ be expressed by an infinite series of components

$$u(x) = \sum_{n=0}^{\infty} u_n(x), \quad v(x) = \sum_{n=0}^{\infty} v_n(x), \tag{13.151}$$

where $u_n(x)$ and $v_n(x), n \geqslant 0$ are the components of $u(x)$ and $v(x)$ that will be elegantly determined in a recursive manner. The nonlinear terms $u^2(t)$ and $u(t)v(t)$ are given by the Adomian polynomials A_n and B_n:

$$A_0 = u_0^2, \quad A_1 = 2u_0u_1, \quad A_2 = 2u_0u_2 + u_1^2 \tag{13.152}$$

and

$$\begin{aligned} B_0 &= u_0v_0(t), \\ B_1 &= v_0(t)u_1(t) + u_0(t)v_1(t), \\ B_2 &= v_0(t)u_2(t) + u_0(t)v_2(t) + u_1(t)v_1(t), \end{aligned} \tag{13.153}$$

respectively. Substituting the previous assumptions for linear and nonlinear terms into (13.150), and following Adomian analysis, the system (13.150) is transformed into a set of recursive relations given by

$$u_0(x) = x - \frac{2}{3}x^3, \quad u_{k+1}(x) = \int_0^x \left(A_k(t) + v_k(t)\right) dt, \quad k \geqslant 0, \tag{13.154}$$

and

$$v_0(x) = x^2 - \frac{1}{4}x^4, \quad v_{k+1}(x) = \int_0^x B_k(t)dt, \quad k \geqslant 0. \tag{13.155}$$

This in turn gives

$$u_0(x) = x - \frac{2}{3}x^3, \quad u_1(x) = \frac{2}{3}x^3 - \frac{19}{60}x^5 + \frac{4}{63}x^7, \tag{13.156}$$

and

$$v_0(x) = x^2 - \frac{1}{4}x^4, \quad v_1(x) = \frac{1}{4}x^4 - \frac{11}{72}x^6 + \frac{1}{48}x^8. \tag{13.157}$$

It is obvious that the noise terms $\mp\frac{2}{3}x^3$ appear between $u_0(x)$ and $u_1(x)$. Moreover, the noise terms $\mp\frac{1}{4}x^4$ appear between $v_0(x)$ and $v_1(x)$. By canceling these noise terms from $u_0(x)$ and $v_0(x)$, the non-canceled terms of $u_0(x)$ and $v_0(x)$ give the exact solutions

$$(u(x), v(x)) = (x, x^2), \tag{13.158}$$

that satisfy the given system (13.150).

Example 13.22

Use the Adomian decomposition method to solve the following system of nonlinear Volterra integral equations

$$\begin{aligned} u(x) &= \cos x - \frac{1}{2}x^2 + \int_0^x (x-t)(u^2(t)+v^2(t))dt, \\ v(x) &= \sin x - \frac{1}{2}\sin^2 x + \int_0^x (x-t)(u^2(t)-v^2(t))dt. \end{aligned} \tag{13.159}$$

For this example, we will use the modified decomposition method, therefore we set the recursive relation

$$\begin{aligned} u_0(x) &= \cos x, \\ u_{k+1}(x) &= -\frac{1}{2}x^2 + \int_0^x (x-t)(A_k(t)+C_k(t))dt, \quad k \geqslant 0, \end{aligned} \tag{13.160}$$

and

$$\begin{aligned} v_0(x) &= \sin x, \\ v_{k+1}(x) &= -\frac{1}{2}\sin^2 x + \int_0^x (x-t)(A_k(t)-C_k(t))dt, \quad k \geqslant 0, \end{aligned} \tag{13.161}$$

where $A_n(t)$ and $C_n(t)$ are the Adomian polynomials for $u^2(t)$ and $v^2(t)$ respectively. This in turn gives

$$u_0(x) = \cos x, \quad u_1(x) = 0, \quad u_{k+1}(x) = 0, \quad k \geqslant 1, \tag{13.162}$$

and

$$v_0(x) = \sin x, \quad v_1(x) = 0, \quad v_{k+1}(x) = 0, \quad k \geqslant 1. \tag{13.163}$$

This gives the exact solutions

$$(u(x), v(x)) = (\cos x, \sin x). \tag{13.164}$$

Example 13.23

Use the successive approximations method to solve the following system of nonlinear Volterra integral equations

$$\begin{aligned} u(x) &= \cos x + \sin x + (1+x)\cos^2 x - (1+x^2) + \int_0^x \left(xu^2(t) - v^2(t)\right) dt, \\ v(x) &= \cos x - \sin x + (1-x)\cos^2 x - (1+x^2) + \int_0^x \left(u^2(t) + xv^2(t)\right) dt. \end{aligned} \tag{13.165}$$

To use the successive approximations method, we first select the zeroth approximations $u_0(x)$ and $v_0(x)$ by

$$u_0(x) = v_0(x) = 1. \tag{13.166}$$

The method of successive approximations admits the use of the iteration formulas

$$\begin{aligned} u_{n+1}(x) &= \cos x + \sin x + (1+x)\cos^2 x - (1+x^2) \\ &\quad + \int_0^x \left(xu_n^2(t) - v_n^2(t)\right) dt, \\ v_{n+1}(x) &= \cos x - \sin x + (1-x)\cos^2 x - (1+x^2) \\ &\quad + \int_0^x \left(u_n^2(t) + xv_n^2(t)\right) dt, \quad n \geqslant 0. \end{aligned} \tag{13.167}$$

Substituting (13.166) into (13.167) we obtain the series

$$\begin{aligned}
u_0(x) &= 1,\\
v_0(x) &= 1,\\
u_1(x) &= 1 + x - \frac{3}{2}x^2 - \frac{7}{6}x^3 + \frac{3}{8}x^4 + \frac{41}{120}x^5 + \cdots,\\
v_1(x) &= 1 - x - \frac{3}{2}x^2 + \frac{7}{6}x^3 + \frac{3}{8}x^4 - \frac{41}{120}x^5 + \cdots,\\
u_2(x) &= 1 + x - \frac{1}{2}x^2 - \frac{1}{2}x^3 - \frac{13}{8}x^4 - \frac{9}{8}x^5 + \cdots,\\
v_2(x) &= 1 - x - \frac{1}{2}x^2 - \frac{1}{2}x^3 - \frac{13}{8}x^4 + \frac{9}{8}x^5 + \cdots,\\
u_3(x) &= 1 + x - \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{3}{8}x^4 + \frac{89}{120}x^5 + \cdots,\\
v_3(x) &= 1 - x - \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{3}{8}x^4 - \frac{89}{120}x^5 + \cdots,\\
u_4(x) &= 1 + x - \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{24}x^4 - \frac{1}{8}x^5 + \cdots,\\
v_4(x) &= 1 - x - \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{8}x^5 + \cdots,\\
u_5(x) &= 1 + x - \frac{1}{2!}x^2 - \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \cdots,\\
v_5(x) &= 1 - x - \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 - \frac{1}{5!}x^5 + \cdots.
\end{aligned} \tag{13.168}$$

This means that the series solutions are given by

$$\begin{aligned}
u(x) &= \left(1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \cdots\right) + \left(x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \cdots\right),\\
v(x) &= \left(1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \cdots\right) - \left(x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \cdots\right).
\end{aligned} \tag{13.169}$$

Consequently, the closed form solutions $u(x)$ and $v(x)$ are given by

$$(u(x), v(x)) = (\cos x + \sin x, \cos x - \sin x), \tag{13.170}$$

obtained upon using the Taylor series for $\cos x$ and $\sin x$. Notice that this system can be solved easily by using the modified decomposition method, where we can select $u_0(x) = \cos x + \sin x$ and $v_0(x) = \cos x - \sin x$.

Example 13.24

Use the successive approximations method to solve the following system of nonlinear Volterra integral equations

$$\begin{aligned}
u(x) &= e^x + x - \frac{1}{2}\sinh 2x + \int_0^x (x-t)(u^2(t) - v^2(t))dt,\\
v(x) &= e^{-x} + x - xe^x + \int_0^x xu^2(t)v(t)dt.
\end{aligned} \tag{13.171}$$

To use the successive approximations method, we first select the zeroth approximations $u_0(x)$ and $v_0(x)$ by

$$u_0(x) = v_0(x) = 1. \tag{13.172}$$

The method of successive approximations admits the use of the iteration formulas

$$\begin{aligned} u_{n+1}(x) &= e^x + x - \frac{1}{2}\sinh(2x) + \int_0^x (x-t)(u_n^2(t) - v_n^2(t))dt, \\ v_{n+1}(x) &= e^{-x} + x - xe^x + \int_0^x xu_n^2(t)v_n(t)dt, \quad n \geqslant 0. \end{aligned} \tag{13.173}$$

Substituting (13.172) into (13.173) we obtain the series

$$\begin{aligned} u_0(x) &= 1, \\ v_0(x) &= 1, \\ u_1(x) &= 1 + x + \frac{1}{2}x^2 - \frac{1}{2}x^3 + \frac{1}{24}x^4 + \cdots, \\ v_1(x) &= 1 - x + \frac{1}{2}x^2 - \frac{2}{3}x^3 - \frac{1}{8}x^4 + \cdots, \\ u_2(x) &= 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots, \\ v_2(x) &= 1 - x + \frac{1}{2!}x^2 - \frac{1}{3!}x^3 + \frac{1}{4!}x^4 - \cdots, \end{aligned} \tag{13.174}$$

and so on. Consequently, the solutions $u(x)$ and $v(x)$ are given by

$$(u(x), v(x)) = (e^x, e^{-x}). \tag{13.175}$$

Exercises 13.5.1

In Exercises 1–4, use the Adomian decomposition method to solve the systems of nonlinear Volterra integral equations

1. $$\begin{cases} u(x) = x - \frac{1}{30}x^5 + \frac{1}{30}x^6 + \int_0^x \left((x-t)^2u^2(t) - (x-t)v^2(t)\right)dt \\ v(x) = x^2 - \frac{1}{60}x^6 - \frac{1}{105}x^7 + \int_0^x \left((x-t)^3u^2(t) + (x-t)^2v^2(t)\right)dt \end{cases}$$

2. $$\begin{cases} u(x) = 1 + x^2 - x^3 - \frac{1}{3}x^7 + \int_0^x \left(xtu^2(t) + xtv^2(t)\right)dt \\ v(x) = 1 - x^2 - \frac{1}{3}x^4 + \int_0^x \left((x-t)u^2(t) - (x-t)v^2(t)\right)dt \end{cases}$$

3. $$\begin{cases} u(x) = 1 + e^x - 4x(1 - e^x + xe^x) + \int_0^x \left(xtu^2(t) - xtv^2(t)\right)dt \\ v(x) = 1 - e^x - 4xe^x + \int_0^x \left(e^{x-t}u^2(t) - e^{x-t}v^2(t)\right)dt \end{cases}$$

4. $$\begin{cases} u(x) = e^{x} - \sinh(2x) + \displaystyle\int_0^x \left(u^2(t) + v^2(t)\right) dt \\ v(x) = e^{-x} + 1 - \cosh(2x) + \displaystyle\int_0^x \left(u^2(t) - v^2(t)\right) dt \end{cases}$$

In Exercises 5–8, use the successive approximations method to solve the systems of nonlinear Volterra integral equations

5. $$\begin{cases} u(x) = \cos x - \sin x + \displaystyle\int_0^x \cos(x-t)(u^2(t) + v^2(t))dt \\ v(x) = \sin x + \cos x - x + \displaystyle\int_0^x \sin(x-t)(u^2(t) + v^2(t))dt \end{cases}$$

6. $$\begin{cases} u(x) = 1 + \sin x - 3x + \frac{1}{2}\sin(2x) + \displaystyle\int_0^x (u^2(t) + v^2(t))dt \\ v(x) = 1 - \sin x - \frac{3}{2}x^2 + \frac{1}{2}\sin^2 x + \displaystyle\int_0^x (x-t)(u^2(t) + v^2(t))dt \end{cases}$$

7. $$\begin{cases} u(x) = \cosh x - x + \displaystyle\int_0^x (u^2(t) - v^2(t))dt \\ v(x) = \sinh x - \frac{1}{2}\sinh^2 x + \displaystyle\int_0^x (x-t)(u^2(t) + v^2(t))dt \end{cases}$$

8. $$\begin{cases} u(x) = 1 + \cosh x - 4\sinh x + \displaystyle\int_0^x (u^2(t) - v^2(t))dt \\ v(x) = 1 - \cosh x - \frac{1}{2}\sinh(2x) - 3x + \displaystyle\int_0^x (u^2(t) + v^2(t))dt \end{cases}$$

13.5.2 Systems of Nonlinear Volterra Integral Equations of the First Kind

In this section, we will study a specific case of the systems of nonlinear Volterra integral equations of the first kind given by

$$\begin{aligned} f_1(x) &= \int_0^x \left(K_1(x,t)u(t) + \widetilde{K}_1(x,t)\widetilde{F}_1(v(t))\right) dt, \\ f_2(x) &= \int_0^x \left(K_2(x,t)F_2(u(t)) + \widetilde{K}_2(x,t)v(t)\right) dt, \end{aligned} \tag{13.176}$$

where the kernels $K_i(x,t)$ and $\widetilde{K}_i(x,t)$, and the functions $f_i(x)$ are given real-valued functions, and $u(x)$, and $v(x)$ are the unknown functions that will be determined. Recall that the unknown functions $u(x)$ and $v(x)$ appear inside the integral sign for the Volterra integral equations of the first kind.

We first need to convert this system to a system of nonlinear Volterra integral equation of the second kind. This can be achieved by differentiating both sides of each part of the system. The conversion technique works effectively by using Leibnitz rule that was presented in section 1.3. Differentiating both sides of each equation in (13.176), and using Leibnitz rule, we obtain

$$
\begin{aligned}
f_1'(x) &= K_1(x,x)u(x) + \widetilde{K}_1(x,x)\widetilde{F}_1(v(x)) \\
&\quad + \int_0^x \Big(K_{1_x}(x,t)u(t) + \widetilde{K}_{1_x}(x,t)\widetilde{F}_1(v(t))\Big)\,dt, \\
f_2'(x) &= K_2(x,x)F_2(u(x)) + \widetilde{K}_2(x,x)v(x) \\
&\quad + \int_0^x \Big(K_{2_x}(x,t)F_2(u(t)) + \widetilde{K}_{2_x}(x,t)v(t)\Big)\,dt,
\end{aligned} \tag{13.177}
$$

that can be rewritten as

$$
\begin{aligned}
u(x) &= \frac{f_1'(x) - \widetilde{K}_1(x,x)\widetilde{F}_1(v(x))}{K_1(x,x)} \\
&\quad - \frac{1}{K_1(x,x)} \int_0^x \Big(K_{1_x}(x,t)u(t) + \widetilde{K}_{1_x}(x,t)\widetilde{F}_1(v(t))\Big)\,dt, \\
v(x) &= \frac{f_2'(x) - K_2(x,x)F_2(u(x))}{\widetilde{K}_2(x,x)} \\
&\quad - \frac{1}{\widetilde{K}_2(x,x)} \int_0^x \Big(K_{2_x}(x,t)F_2(u(t)) + \widetilde{K}_{2_x}(x,t)v(t)\Big)\,dt.
\end{aligned} \tag{13.178}
$$

It is obvious that the last system is a system of nonlinear Volterra integral equations of the second kind. This system can be handled by many distinct methods. However, in this section we will use the Adomian decomposition method and the successive approximations method for handling the resulting system of nonlinear Volterra integral equations of the second kind. Notice that other methods can be used as well. The Adomian decomposition method and the successive approximations method were introduced before, hence we skip details.

It is important to present the following two remarks:

1. It is necessary that $K_1(x,x) \neq 0$ and $\widetilde{K}_2(x,x) \neq 0$ for the system to be reduced to a system of Volterra integral equations of the second kind.
2. If $K_1(x,x) = 0$ and $\widetilde{K}_2(x,x) = 0$, then we differentiate again.

In the following, we will examine four examples, where Adomian method will be used in the first two examples, and the successive approximations method will be used for the other two examples.

Example 13.25

Solve the system of nonlinear Volterra integral equations of the first kind by using the Adomian method

$$
\begin{aligned}
\frac{1}{3}x^3 + \frac{1}{12}x^4 + \frac{1}{56}x^8 &= \int_0^x \big((x-t+1)u(t) + (x-t)v^2(t)\big)\,dt, \\
\frac{1}{4}x^4 + \frac{1}{20}x^5 + \frac{1}{30}x^6 &= \int_0^x \big((x-t)u^2(t) + (x-t+1)v(t)\big)\,dt.
\end{aligned} \tag{13.179}
$$

Differentiating both sides of each equation, and using Leibnitz rule, we find

$$\begin{aligned} x^2+\frac{1}{3}x^3+\frac{1}{7}x^7 &= u(x)+\int_0^x (u(t)+v^2(t))dt, \\ x^3+\frac{1}{4}x^4+\frac{1}{5}x^5 &= v(x)+\int_0^x \left(u^2(t)+v(t)\right)dt. \end{aligned} \tag{13.180}$$

This system can be rewritten as

$$\begin{aligned} u(x) &= x^2+\frac{1}{3}x^3+\frac{1}{7}x^7-\int_0^x \left(u(t)+v^2(t)\right)dt, \\ v(x) &= x^3+\frac{1}{4}x^4+\frac{1}{5}x^5-\int_0^x \left(u^2(t)+v(t)\right)dt. \end{aligned} \tag{13.181}$$

Using the standard Adomian decomposition method, we set the recurrence relations

$$\begin{aligned} u_0(x) &= x^2+\frac{1}{3}x^3+\frac{1}{7}x^7, \\ v_0(x) &= x^3+\frac{1}{4}x^4+\frac{1}{5}x^5, \\ u_1(x) &= -\int_0^x \left(u_0(t)+v_0^2(t)\right)dt = -\frac{1}{3}x^3-\frac{1}{7}x^7+\cdots, \\ v_1(x) &= -\int_0^x \left(u_0^2(t)+v_0(t)\right)dt = -\frac{1}{4}x^4-\frac{1}{5}x^5+\cdots. \end{aligned} \tag{13.182}$$

It is obvious that two noise terms appear between $u_0(x)$ and $u_1(x)$, and two other noise terms between $v_0(x)$ and $v_1(x)$. Canceling the noise terms in $u_0(x)$ and $v_0(x)$ gives the exact solutions by

$$(u(x),v(x)) = (x^2,x^3). \tag{13.183}$$

Example 13.26

Solve the system of nonlinear Volterra integral equations of the first kind by using the modified Adomian method

$$\begin{aligned} &\sin x+\frac{1}{4}\sin(2x)-\frac{1}{4}\sin^2 x+\frac{1}{4}x^2-\frac{1}{2}x-\frac{1}{4} \\ &\quad= \int_0^x \left(u(t)+(x-t-1)v^2(t)\right)dt, \\ &-\cos x-\sin x-\frac{1}{4}\cos^2 x+\frac{1}{4}x^2+x+\frac{5}{4} \\ &\quad= \int_0^x \left((x-t)u^2(t)+(x-t+1)v(t)\right)dt. \end{aligned} \tag{13.184}$$

Differentiating both sides of each equation, and using Leibnitz rule, we find

$$\begin{aligned} u(x) &= \cos x+\frac{1}{2}\cos(2x)-\frac{1}{4}\sin(2x)+\frac{1}{2}x-\frac{1}{2}-v^2(x)-\int_0^x v^2(t)dt, \\ v(x) &= \sin x-\cos x+\frac{1}{4}\sin(2x)+\frac{1}{2}x+1-\int_0^x \left(u^2(t)+v(t)\right)dt. \end{aligned} \tag{13.185}$$

To handle this example we select the modified decomposition method, hence we set the recurrence relations

$$
\begin{aligned}
u_0(x) &= \cos x, \quad v_0(x) = \sin x,\\
u_1(x) &= \frac{1}{2}\cos(2x) - \frac{1}{4}\sin(2x) + \frac{1}{2}x - \frac{1}{2} - v_0^2(x) - \int_0^x v_0^2(t)dt = 0,\\
v_1(x) &= -\cos x + \frac{1}{4}\sin(2x) + \frac{1}{2}x + 1 - \int_0^x \big(u_0^2(t) + v_0(t)\big)\,dt = 0.
\end{aligned} \tag{13.186}
$$

This in turn gives the exact solutions by

$$
(u(x), v(x)) = (\cos x, \sin x). \tag{13.187}
$$

Example 13.27

Solve the system of nonlinear Volterra integral equations of the first kind by using the successive approximations method

$$
\begin{aligned}
2e^x + \frac{1}{4}e^{-2x} - \frac{1}{2}x - \frac{9}{4} &= \int_0^x \big((x-t+1)u(t) + (x-t)v^2(t)\big)\,dt,\\
\frac{1}{4}e^{2x} + \frac{1}{2}x - \frac{1}{4} &= \int_0^x \big((x-t)u^2(t) + (x-t+1)v(t)\big)\,dt.
\end{aligned} \tag{13.188}
$$

Differentiating both sides of each equation, and using Leibnitz rule, we find

$$
\begin{aligned}
u(x) &= 2e^x - \frac{1}{2}e^{-2x} - \frac{1}{2} - \int_0^x \big(u(t) + v^2(t)\big)\,dt,\\
v(x) &= \frac{1}{2}e^{2x} + \frac{1}{2} - \int_0^x \big(u^2(t) + v(t)\big)\,dt.
\end{aligned} \tag{13.189}
$$

To use the successive approximations method, we first select the zeroth approximations $u_0(x)$ and $v_0(x)$ by

$$
u_0(x) = v_0(x) = 1. \tag{13.190}
$$

The method of successive approximations admits the use of the iteration formulas

$$
\begin{aligned}
u_{n+1}(x) &= 2e^x - \frac{1}{2}e^{-2x} - \frac{1}{2} - \int_0^x \big(u_n(t) + v_n^2(t)\big)\,dt,\\
v_{n+1}(x) &= \frac{1}{2}e^{2x} + \frac{1}{2} - \int_0^x \big(u_n^2(t) + v_n(t)\big)\,dt, \quad n \geqslant 0.
\end{aligned} \tag{13.191}
$$

Substituting (13.190) into (13.191) we obtain the series

$$
\begin{aligned}
u_0(x) &= 1, \quad v_0(x) = 1,\\
u_1(x) &= 1 + x + x^3 - \frac{1}{4}x^4 + \cdots,\\
v_1(x) &= 1 - x + x^2 + \frac{2}{3}x^3 + \frac{1}{3}x^4 + \cdots,\\
u_2(x) &= 1 + x + \frac{1}{2}x^2 - \frac{1}{3}x^4 + \cdots,
\end{aligned}
$$

$$v_2(x) = 1 - x + \frac{1}{2}x^2 - \frac{1}{3}x^4 + \cdots, \tag{13.192}$$

$$u_3(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \cdots,$$

$$v_3(x) = 1 - x + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{12}x^4 + \cdots,$$

$$u_4(x) = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots,$$

$$v_4(x) = 1 - x + \frac{1}{2!}x^2 - \frac{1}{3!}x^3 + \frac{1}{4!}x^4 - \cdots.$$

Consequently, the solutions $u(x)$ and $v(x)$ are given by

$$(u(x), v(x)) = (e^x, e^{-x}). \tag{13.193}$$

Example 13.28

Solve the system of nonlinear Volterra integral equations of the first kind by using the successive approximations method

$$\begin{aligned} &-2\cos x + \frac{1}{4}\sin(2x) + \frac{1}{2}x^2 + \frac{1}{2}x + 2 \\ &\quad = \int_0^x \big((x-t+1)u(t) + (x-t)v^2(t)\big)\,dt, \\ &2\sin x - \frac{1}{4}\sin(2x) + \frac{1}{2}x^2 - \frac{1}{2}x \\ &\quad = \int_0^x \big((x-t)u^2(t) + (x-t+1)v(t)\big)\,dt. \end{aligned} \tag{13.194}$$

Differentiating both sides of each equation, and using Leibnitz rule, we find

$$\begin{aligned} u(x) &= 2\sin x + \frac{1}{2}\cos(2x) + x + \frac{1}{2} - \int_0^x \big(u(t) + v^2(t)\big)\,dt, \\ v(x) &= 2\cos x - \frac{1}{2}\cos(2x) + x - \frac{1}{2} - \int_0^x \big(u^2(t) + v(t)\big)\,dt. \end{aligned} \tag{13.195}$$

We select the zeroth approximations $u_0(x)$ and $v_0(x)$ by $u_0(x) = v_0(x) = 1$. We next use the iteration formulas

$$\begin{aligned} u_{n+1}(x) &= 2\sin x + \frac{1}{2}\cos(2x) + x + \frac{1}{2} - \int_0^x \big(u_n(t) + v_n^2(t)\big)\,dt, \\ v_{n+1}(x) &= 2\cos x - \frac{1}{2}\cos(2x) + x - \frac{1}{2} - \int_0^x \big(u_n^2(t) + v_n(t)\big)\,dt, \quad n \geqslant 0. \end{aligned} \tag{13.196}$$

Substituting the zeroth selections into (13.196) we obtain the series

$$
\begin{aligned}
u_0(x) &= 1, \quad v_0(x) = 1,\\
u_1(x) &= 1 + x - x^2 - \frac{1}{3}x^3 + \frac{1}{3}x^4 + \frac{1}{60}x^5 + \cdots,\\
v_1(x) &= 1 - x - \frac{1}{4}x^4 + \cdots,\\
u_2(x) &= 1 + x - \frac{1}{2}x^2 - \frac{1}{3}x^3 + \frac{5}{12}x^4 + \frac{1}{20}x^5 + \cdots,\\
v_2(x) &= 1 - x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{5}{12}x^4 - \frac{3}{20}x^5 + \cdots,\\
u_3(x) &= 1 + x - \frac{1}{2}x^2 - \frac{1}{6}x^3 - \frac{3}{20}x^5 + \cdots,\\
v_3(x) &= 1 - x - \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{12}x^4 - \frac{1}{6}x^5 + \cdots,\\
u_4(x) &= 1 + x - \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{24}x^4 + \cdots,\\
v_4(x) &= 1 - x - \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \cdots,\\
u_5(x) &= 1 + x - \frac{1}{2!}x^2 - \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \cdots,\\
v_5(x) &= 1 - x - \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 - \frac{1}{5!}x^5 + \cdots.
\end{aligned}
\tag{13.197}
$$

This means that the series solutions are given by

$$
\begin{aligned}
u(x) &= \left(1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \cdots\right) + \left(x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \cdots\right),\\
v(x) &= \left(1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \cdots\right) - \left(x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \cdots\right).
\end{aligned}
\tag{13.198}
$$

Consequently, the closed form solutions $u(x)$ and $v(x)$ are given by

$$
(u(x), v(x)) = (\cos x + \sin x, \cos x - \sin x). \tag{13.199}
$$

Exercises 13.5.2

In Exercises 1–4, use the Adomian decomposition method to solve the following systems of Volterra integral equations of the first kind

1. $$\begin{cases} \frac{1}{2}x^2 + \frac{1}{5}x^5 + \frac{1}{30}x^6 = \int_0^x \big(u(t) + (x-t+1)v^2(t)\big)\,dt \\ \frac{1}{3}x^3 + \frac{1}{6}x^4 = \int_0^x \big((x-t)u^2(t) + (x-t+1)v(t)\big)\,dt \end{cases}$$

2. $$\begin{cases} \frac{1}{4}x^4 + \frac{1}{20}x^5 + \frac{1}{132}x^{12} = \int_0^x \big((x-t+1)u(t) + (x-t)v^2(t)\big)\,dt \\ \frac{1}{6}x^6 + \frac{1}{42}x^7 + \frac{1}{56}x^8 = \int_0^x \big((x-t)u^2(t) + (x-t+1)v(t)\big)\,dt \end{cases}$$

3. $$\begin{cases} -\cos x + \frac{1}{4}\cos^2 x - \frac{1}{4}x^2 + \frac{3}{4} = \int_0^x \big(u(t) - (x-t)v^2(t)\big)\,dt \\ \sin x - \cos x + \frac{1}{4}\cos^2 x + \frac{1}{4}x^2 + \frac{3}{4} = \int_0^x \big((x-t)u^2(t) + (x-t+1)v(t)\big)\,dt \end{cases}$$

4. $$\begin{cases} \sinh x - \frac{1}{4}\cosh^2 x + \frac{1}{4}x^2 + \frac{1}{4} = \int_0^x \big(u(t) - (x-t)v^2(t)\big)\,dt \\ \cosh x + \frac{1}{4}\cosh^2 x + \frac{1}{4}x^2 - \frac{5}{4} = \int_0^x \big((x-t)u^2(t)v(t)\big)\,dt \end{cases}$$

In Exercises 5–8, use the successive approximations method to solve the systems of Volterra integral equations of the first kind

5. $$\begin{cases} 1 - 2\cos x + \cos^2 x + 2x = \int_0^x \big((x-t+1)u(t) + v^2(t)\big)\,dt \\ 1 - 2\sin x - \cos^2 x + 2x = \int_0^x \big(u^2(t) + (x-t+1)v(t)\big)\,dt \end{cases}$$

6. $$\begin{cases} \frac{3}{4}e^{2x} - \frac{1}{4}e^{-4x} - \frac{1}{2}x - \frac{1}{2} = \int_0^x \big((x-t+1)u(t) + v^2(t)\big)\,dt \\ \frac{1}{4}e^{4x} - \frac{1}{4}e^{-4x} + \frac{1}{2}x = \int_0^x \big(u^2(t) + (x-t+1)v(t)\big)\,dt \end{cases}$$

7. $$\begin{cases} \frac{1}{2}e^{2x} + \frac{1}{2}x^2 + x - \frac{1}{2} = \int_0^x \big((x-t+1)u(t) + v^2(t)\big)\,dt \\ \frac{1}{2}e^{2x} + \frac{1}{2}x^2 + 3x - \frac{1}{2} = \int_0^x \big(u^2(t) + (x-t+1)v(t)\big)\,dt \end{cases}$$

8. $$\begin{cases} \sin x - \cos x + \frac{1}{4}\cos^2 x + \frac{5}{4}x^2 + \frac{3}{4} = \int_0^x \big((x-t+1)u(t) + (x-t)v^2(t)\big)\,dt \\ -\sin x + \cos x + \frac{1}{4}\cos^2 x + \frac{5}{4}x^2 + 2x - \frac{5}{4} = \int_0^x \big((x-t)u^2(t) + (x-t+1)v(t)\big)\,dt \end{cases}$$

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Chapter 14
Nonlinear Volterra Integro-Differential Equations

14.1 Introduction

It is well known that linear and nonlinear Volterra integral equations arise in many scientific fields such as the population dynamics, spread of epidemics, and semi-conductor devices. Volterra started working on integral equations in 1884, but his serious study began in 1896. The name integral equation was given by du Bois-Reymond in 1888.

The linear Volterra integro-differential equations were presented in Chapter 5. It is our goal in this chapter to study the nonlinear Volterra integro-differential equations of the first and the second kind. The nonlinear Volterra integro-differential equations are characterized by at least one variable limit of integration.

The nonlinear Volterra integro-differential equation of the second kind reads

$$u^{(n)}(x) = f(x) + \int_0^x K(x,t)F(u(t))\,dt, \qquad (14.1)$$

and the standard form of the nonlinear Volterra integro-differential equation [1–3] of the first kind is given by

$$\int_0^x K_1(x,t)F(u(t))dt + \int_0^x K_2(x,t)u^{(n)}(t)dt = f(x), K_2(x,x) \neq 0, \quad (14.2)$$

where $u^{(n)}(x)$ is the n th derivative of $u(x)$. For these equations, the kernels $K(x,t)$, $K_1(x,t)$ and $K_2(x,t)$, and the function $f(x)$ are given real-valued functions. The function $F(u(x))$ is a nonlinear function of $u(x)$ such as $u^2(x), \sin(u(x))$, and $e^{u(x)}$.

14.2 Nonlinear Volterra Integro-Differential Equations of the Second Kind

The linear Volterra integro-differential equation [4–6], where both differential and integral operators appear together in the same equation, has been studied in Chapter 5. In this section, we will extend the work presented in Chapter 5 to nonlinear Volterra integro-differential equation. The nonlinear Volterra integro-differential equation of the second kind reads

$$u^{(i)}(x) = f(x) + \int_0^x K(x,t)F(u(t))dt, \tag{14.3}$$

where $u^{(i)}(x) = \frac{d^i u}{dx^i}$, and $F(u(x))$ is a nonlinear function of $u(x)$. Because the equation in (14.3) combines the differential operator and the integral operator, then it is necessary to define initial conditions for the determination of the particular solution $u(x)$ of the nonlinear Volterra integro-differential equation.

The nonlinear Volterra integro-differential equation [7–10] appeared after its establishment by Volterra. It appears in many physical applications such as glass-forming process, heat transfer, diffusion process in general, neutron diffusion and biological species coexisting together with increasing and decreasing rates of generating. More details about the sources where these equations arise can be found in physics, biology and books of engineering applications.

In Chapter 5, we applied many methods to handle the linear Volterra integro-differential equations of the second kind. In this section we will use only some of these methods. However, the other methods presented in Chapter 5 can be used as well. In what follows we will apply the combined Laplace transform-Adomian decomposition method, the variational iteration method (VIM), and the series solution method to handle nonlinear Volterra integro-differential equations of the second kind (14.3).

14.2.1 The Combined Laplace Transform-Adomian Decomposition Method

In this section we will consider the kernel $K(x,t)$ of (14.3) as a *difference kernel* that depends on the difference $x-t$, such as $e^{x-t}, \cosh(x-t)$, and $\sin(x-t)$. The nonlinear Volterra integro-differential equation (14.3) can thus be expressed as

$$u^{(i)}(x) = f(x) + \int_0^x K(x-t)F(u(t))\,dt. \tag{14.4}$$

To solve the nonlinear Volterra integro-differential equations by using the Laplace transform method, it is essential to use the Laplace transforms of the derivatives of $u(x)$. We can easily show that

$$\mathcal{L}\{u^{(i)}(x)\} = s^i\mathcal{L}\{u(x)\} - s^{i-1}u(0) - s^{i-2}u'(0) - \cdots - u^{(i-1)}(0). \quad (14.5)$$

This simply gives

$$\begin{aligned}
\mathcal{L}\{u'(x)\} &= s\mathcal{L}\{u(x)\} - u(0) = sU(s) - u(0),\\
\mathcal{L}\{u''(x)\} &= s^2\mathcal{L}\{u(x)\} - su(0) - u'(0)\\
&= s^2U(s) - su(0) - u'(0),\\
\mathcal{L}\{u'''(x)\} &= s^3\mathcal{L}\{u(x)\} - s^2u(0) - su'(0) - u''(0)\\
&= s^3U(s) - s^2u(0) - su'(0) - u''(0),\\
\mathcal{L}\{u^{(\text{iv})}(x)\} &= s^4\mathcal{L}\{u(x)\} - s^3u(0) - s^2u'(0) - su''(0) - u'''(0)\\
&= s^4U(s) - s^3u(0) - s^2u'(0) - su''(0) - u'''(0),
\end{aligned} \quad (14.6)$$

and so on for derivatives of higher order, where $U(s) = \mathcal{L}\{u(x)\}$.

Applying the Laplace transform to both sides of (14.4) gives

$$\begin{aligned}
&s^i\mathcal{L}\{u(x)\} - s^{i-1}u(0) - s^{i-2}u'(0) - \cdots - u^{(i-1)}(0)\\
&= \mathcal{L}\{f(x)\} + \mathcal{L}\{K(x-t)\}\mathcal{L}\{F(u(t))\},
\end{aligned} \quad (14.7)$$

or equivalently

$$\begin{aligned}
\mathcal{L}\{u(x)\} = &\frac{1}{s}u(0) + \frac{1}{s^2}u'(0) + \cdots + \frac{1}{s^i}u^{(i-1)}(0)\\
&+\frac{1}{s^i}\mathcal{L}\{f(x)\} + \frac{1}{s^i}\mathcal{L}\{K(x-t)\}\mathcal{L}\{F(u(t))\}.
\end{aligned} \quad (14.8)$$

To overcome the difficulty of the nonlinear term $F(u(x))$, we apply the Adomian decomposition method for handling (14.8). To achieve this goal, we first represent the linear term $u(x)$ at the left side by an infinite series of components given by

$$u(x) = \sum_{n=0}^{\infty} u_n(x), \quad (14.9)$$

where the components $u_n(x), n \geqslant 0$ will be recursively determined. However, the nonlinear term $F(u(x))$ at the right side of (14.8) will be represented by an infinite series of the Adomian polynomials A_n in the form

$$F(u(x)) = \sum_{n=0}^{\infty} A_n(x), A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[F\left(\sum_{i=0}^{n} \lambda^i\, u_i\right)\right]_{\lambda=0}, n = 0, 1, 2, \cdots, \quad (14.10)$$

where $A_n, n \geqslant 0$ can be obtained for all forms of nonlinearity.

Substituting (14.9) and (14.10) into (14.8) leads to

$$\begin{aligned}
\mathcal{L}\left\{\sum_{n=0}^{\infty} u_n(x)\right\} = &\frac{1}{s}u(0) + \frac{1}{s^2}u'(0) + \cdots + \frac{1}{s^i}u^{(i-1)}(0)\\
&+\frac{1}{s^i}\mathcal{L}\{f(x)\} + \frac{1}{s^i}\mathcal{L}\{K(x-t)\}\mathcal{L}\left\{\sum_{n=0}^{\infty} A_n(x)\right\}.
\end{aligned} \quad (14.11)$$

The Adomian decomposition method admits the use of the following recursive relation

$$\begin{aligned}\mathcal{L}\{u_0(x)\} &= \frac{1}{s}u(0) + \frac{1}{s^2}u'(0) + \cdots + \frac{1}{s^i}u^{(i-1)}(0) + \frac{1}{s^i}\mathcal{L}\{f(x)\},\\ \mathcal{L}\{u_{k+1}(x)\} &= \frac{1}{s^i}\mathcal{L}\{K(x-t)\}\mathcal{L}\{A_k(x)\}, \quad k \geqslant 0.\end{aligned} \tag{14.12}$$

The necessary conditions presented in Chapter 1 for Laplace transform method concerning the limit as $s \to \infty$, should be satisfied here for a successful use of this method. Applying the inverse Laplace transform to the first part of (14.12) gives $u_0(x)$, that will define A_0. This in turn will lead to the complete determination of the components of $u_{k+1}, k \geqslant 0$ upon using the second part of (14.12).

The combined Laplace transform Adomian-decomposition method for solving nonlinear Volterra integro-differential equations of the second kind will be illustrated by studying the following examples.

Example 14.1

Solve the nonlinear Volterra integro-differential equation by using the combined Laplace transform-Adomian decomposition method

$$u'(x) = \frac{9}{4} - \frac{5}{2}x - \frac{1}{2}x^2 - 3e^{-x} - \frac{1}{4}e^{-2x} + \int_0^x (x-t)u^2(t)\,dt, u(0) = 2. \tag{14.13}$$

Notice that the kernel $K(x-t) = (x-t)$. Taking Laplace transform of both sides of (14.13) gives

$$\mathcal{L}\{u'(x)\} = \mathcal{L}\left\{\frac{9}{4} - \frac{5}{2}x - \frac{1}{2}x^2 - 3e^{-x} - \frac{1}{4}e^{-2x}\right\} + \mathcal{L}\{(x-t)*u^2(x)\}, \tag{14.14}$$

so that

$$sU(s) - u(0) = \frac{9}{4s} - \frac{5}{2s^2} - \frac{1}{s^3} - \frac{3}{s+1} - \frac{1}{4(s+2)} + \frac{1}{s^2}\mathcal{L}\{u^2(x)\}, \tag{14.15}$$

or equivalently

$$U(s) = \frac{2}{s} + \frac{9}{4s^2} - \frac{5}{2s^3} - \frac{1}{s^4} - \frac{3}{s(s+1)} - \frac{1}{4s(s+2)} + \frac{1}{s^3}\mathcal{L}\{u^2(x)\}. \tag{14.16}$$

Substituting the series assumption for $U(s)$ and the Adomian polynomials for $u^2(x)$ as given above in (14.9) and (14.10) respectively, and using the recursive relation (14.12) we obtain

$$\begin{aligned}U_0(s) &= \frac{2}{s} + \frac{9}{4s^2} - \frac{5}{2s^3} - \frac{1}{s^4} - \frac{3}{s(s+1)} - \frac{1}{4s(s+2)},\\ \mathcal{L}\{u_{k+1}(x)\} &= \frac{1}{s^3}\mathcal{L}\{A_k(x)\}, \quad k \geqslant 0.\end{aligned} \tag{14.17}$$

Recall that the Adomian polynomials for $F(u(x)) = u^2(x)$ are given by

$$\begin{aligned}A_0 &= u_0^2, \quad A_1 = 2u_0u_1,\\ A_2 &= 2u_0u_2 + u_1^2,\\ A_3 &= 2u_0u_3 + 2u_1u_2.\end{aligned} \tag{14.18}$$

Taking the inverse Laplace transform of both sides of the first part of (14.17), and using the recursive relation (14.17) gives

$$\begin{aligned} u_0(x) &= 2 - x + \frac{1}{2}x^2 - \frac{5}{6}x^3 + \frac{5}{24}x^4 - \frac{7}{120}x^5 + \cdots, \\ u_1(x) &= \frac{2}{3}x^3 - \frac{1}{6}x^4 + \frac{1}{20}x^5 + \cdots. \end{aligned} \tag{14.19}$$

Using (14.9), the series solution is therefore given by

$$u(x) = 2 - x + \frac{1}{2!}x^2 - \frac{1}{3!}x^3 + \frac{1}{4!}x^4 - \frac{1}{5!}x^5 + \cdots, \tag{14.20}$$

that converges to the exact solution

$$u(x) = 1 + e^{-x}. \tag{14.21}$$

Example 14.2

Solve the nonlinear Volterra integro-differential equation by using the combined Laplace transform-Adomian decomposition method

$$u'(x) = -\frac{2}{3}(2\sin x + \sin(2x)) + \int_0^x \cos(x-t)u^2(t)dt, \; u(0) = 1. \tag{14.22}$$

Notice that the kernel $K(x-t) = \cos(x-t)$. Taking Laplace transform of both sides of (14.22) gives

$$\mathcal{L}\{u'(x)\} = \mathcal{L}\left\{-\frac{2}{3}(2\sin x + \sin(2x))\right\} + \mathcal{L}\{\cos(x-t) * u^2(x)\}, \tag{14.23}$$

so that

$$sU(s) - u(0) = -\frac{4}{3(s^2+1)} - \frac{2}{3(s^2+4)} + \frac{s}{s^2+1}\mathcal{L}\{u^2(x)\}, \tag{14.24}$$

or equivalently

$$U(s) = \frac{1}{s} - \frac{4}{3s(s^2+1)} - \frac{2}{3s(s^2+4)} + \frac{1}{s^2+1}\mathcal{L}\{u^2(x)\}. \tag{14.25}$$

Substituting the series assumption for $U(s)$ and the Adomian polynomials for $u^2(x)$ as given above in (14.9) and (14.10) respectively, and using the recursive relation (14.12) we obtain

$$\begin{aligned} U_0(s) &= \frac{2}{s} + \frac{9}{4s^2} - \frac{5}{2s^3} - \frac{1}{s^4} - \frac{3}{s(s+1)} - \frac{1}{4s(s+2)}, \\ \mathcal{L}\{u_{k+1}(x)\} &= \frac{1}{s^3}\mathcal{L}\{A_k(x)\}, \quad k \geqslant 0. \end{aligned} \tag{14.26}$$

Taking the inverse Laplace transform of both sides of the first part of (14.26), and using the recursive relation (14.26) gives

$$\begin{aligned} u_0(x) &= 1 - x^2 + \frac{1}{6}x^4 - \frac{1}{60}x^6 + \cdots, \\ u_1(x) &= \frac{1}{2}x^2 - \frac{5}{24}x^4 + \frac{37}{720}x^6 + \cdots, \\ u_2(x) &= \frac{1}{22}x^4 - \frac{1}{20}x^6 + \cdots, \end{aligned} \tag{14.27}$$

$$u_3(x) = \frac{1}{72}x^6 + \cdots .$$

Using (14.9), the series solution is therefore given by

$$u(x) = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots , \tag{14.28}$$

that converges to the exact solution

$$u(x) = \cos x. \tag{14.29}$$

Example 14.3

Solve the nonlinear Volterra integro-differential equation by using the combined Laplace transform-Adomian decomposition method

$$u''(x) = 2 + 2x + x^2 - x^2 e^x - e^{2x} + \int_0^x e^{x-t} u^2(t)dt,\ u(0) = 1,\ u'(0) = 2. \tag{14.30}$$

Notice that the kernel $K(x - t) = e^{x-t}$. Taking Laplace transform of both sides of (14.30) gives

$$\mathcal{L}\{u''(x)\} = \mathcal{L}\{2 + 2x + x^2 - x^2 e^x - e^{2x}\} + \mathcal{L}\{e^{x-t} * u^2(x)\}, \tag{14.31}$$

so that

$$s^2 U(s) - su(0) - u'(0) = \frac{2}{s} + \frac{2}{s^2} + \frac{2}{s^3} - \frac{2}{(s-1)^3} - \frac{1}{s-2} + \frac{1}{s-1}\mathcal{L}\{u^2(x)\}, \tag{14.32}$$

or equivalently

$$U(s) = \frac{1}{s} + \frac{2}{s^2} + \frac{2}{s^3} + \frac{2}{s^4} + \frac{2}{s^5} - \frac{2}{s^2(s-1)^3} - \frac{1}{s^2(s-2)} + \frac{1}{s^2(s-1)}\mathcal{L}\{u^2(x)\}. \tag{14.33}$$

Proceeding as before we find

$$\begin{aligned} U_0(s) &= \frac{1}{s} + \frac{2}{s^2} + \frac{2}{s^3} + \frac{2}{s^4} + \frac{2}{s^5} - \frac{2}{s^2(s-1)^3} - \frac{1}{s^2(s-2)}, \\ \mathcal{L}\{u_{k+1}(x)\} &= \frac{1}{s^2(s-1)}\mathcal{L}\{A_k(x)\}, \quad k \geqslant 0. \end{aligned} \tag{14.34}$$

Taking the inverse Laplace transform of both sides of the first part of (14.34), and using the recursive relation (14.34) gives

$$\begin{aligned} u_0(x) &= 1 + 2x + \frac{1}{2}x^2 - \frac{1}{6}x^4 - \frac{7}{60}x^5 - \frac{7}{180}x^6 + \cdots , \\ u_1(x) &= \frac{1}{6}x^3 + \frac{5}{24}x^4 + \frac{1}{8}x^5 + \frac{3}{80}x^6 + \cdots , \\ u_2(x) &= \frac{1}{360}x^6 + \cdots . \end{aligned} \tag{14.35}$$

Using (14.9), the series solution is therefore given by

$$u(x) = x + \left(1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \frac{1}{6!}x^6 + \cdots\right), \tag{14.36}$$

that converges to the exact solution

$$u(x) = x + e^x. \tag{14.37}$$

Example 14.4

Solve the nonlinear Volterra integro-differential equation by using the combined Laplace transform-Adomian decomposition method

$$u''(x) = -1 - \frac{1}{3}(\sin x + \sin(2x)) + 2\cos x + \int_0^x \sin(x-t)u^2(t)dt, \tag{14.38}$$

where $u(0) = -1, u'(0) = 1$.

Notice that the kernel $K(x-t) = \sin(x-t)$. Taking Laplace transform of both sides of (14.38) gives

$$\mathcal{L}\{u''(x)\} = \mathcal{L}\left\{-1 - \frac{1}{3}(\sin x + \sin(2x)) + 2\cos x\right\} + \mathcal{L}\{\sin(x-t) * u^2(x)\}, \tag{14.39}$$

so that

$$s^2U(s) - su(0) - u'(0) = -\frac{1}{s} - \frac{1}{3(s^2+1)} - \frac{2}{3(s^2+4)} + \frac{2s}{s^2+1} + \frac{1}{s^2+1}\mathcal{L}\{u^2(x)\}, \tag{14.40}$$

so that

$$\begin{aligned} U(s) = &-\frac{1}{s} + \frac{1}{s^2} - \frac{1}{s^3} - \frac{1}{3s^2(s^2+1)} - \frac{2}{3s^2(s^2+4)} + \frac{2}{s(s^2+1)} \\ &+ \frac{1}{s^2(s^2+1)}\mathcal{L}\{u^2(x)\}. \end{aligned} \tag{14.41}$$

Proceeding as before we find

$$\begin{aligned} u_0(x) &= -1 + x + \frac{1}{2}x^2 - \frac{1}{6}x^3 - \frac{1}{12}x^4 + \frac{1}{40}x^5 + \frac{1}{360}x^6 - \frac{11}{5040}x^7 + \cdots, \\ u_1(x) &= \frac{1}{24}x^4 - \frac{1}{60}x^5 - \frac{1}{720}x^6 + \frac{1}{504}x^7 + \cdots. \end{aligned} \tag{14.42}$$

Using (14.9), the series solution is therefore given by

$$u(x) = \left(x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots\right) - \left(1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots\right), \tag{14.43}$$

that converges to the exact solution

$$u(x) = \sin x - \cos x. \tag{14.44}$$

Exercises 14.2.1

Solve the following nonlinear Volterra integro-differential equations by using the combined Laplace transform-Adomian decomposition method

1. $u'(x) = \frac{17}{4} + \frac{9}{2}x - 2x^2 - 3e^x - \frac{1}{4}e^{2x} + \int_0^x (x-t)u^2(t)dt,\ u(0) = 3$
2. $u'(x) = -\frac{11}{4} - \frac{3}{2}x - \frac{1}{12}x^4 + e^x(5-2x) - \frac{1}{4}e^{2x} + \int_0^x (x-t)u^2(t)dt,\ u(0) = 1$

3. $u'(x) = 1 - \frac{1}{3}e^{x} + \frac{1}{3}e^{-2x} + \int_0^x e^{x-t}u^2(t)dt,\ u(0) = 0$

4. $u'(x) = -\frac{1}{3} + \frac{5}{3}\cos x - \frac{1}{3}\cos^2 x + \int_0^x \sin(x-t)u^2(t)dt,\ u(0) = 0$

5. $u''(x) = -\frac{5}{3}\sin x + \frac{1}{3}\sin(2x) + \int_0^x \cos(x-t)u^2(t)dt,\ u(0) = 0,\ u'(0) = 1$

6. $u''(x) = \cosh x - \frac{1}{3}\sinh x - \frac{1}{3}\sinh(2x) + \int_0^x \cosh(x-t)u^2(t)dt,$
$u(0) = 1,\ u'(0) = 0$

7. $u'''(x) = \frac{3}{2}e^{x} - \frac{1}{2}e^{3x} + \int_0^x e^{x-t}u^3(t)dt,\ u(0) = u'(0) = u''(0) = 1$

8. $u'''(x) = -\frac{2}{3} - \frac{5}{3}\cos x + \frac{4}{3}\cos^2 x + \int_0^x \cos(x-t)u^2(t)dt,$
$u(0) = u'(0) = 1,\ u''(0) = -1$

14.2.2 The Variational Iteration Method

The variational iteration method was used before in other chapters. The method provides rapidly convergent successive approximations of the exact solution if such a closed form solution exists, and not components as in Adomian decomposition method. The variational iteration method handles linear and nonlinear problems in the same manner without any need to specific restrictions such as the Adomian polynomials.

The standard i th order nonlinear Volterra integro-differential equation is of the form

$$u^{(i)}(x) = f(x) + \int_0^x K(x,t)F(u(t))dt, \tag{14.45}$$

where $u^{(i)}(x) = \frac{d^i u}{dx^i}$, and $F(u(x))$ is a nonlinear function of $u(x)$. The initial conditions should be prescribed for the complete determination of the exact solution.

The correction functional for the nonlinear integro-differential equation (14.45) is

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(\xi)\left(u_n^{(i)}(\xi) - f(\xi) - \int_0^{\xi} K(\xi,r)F(\tilde{u}_n(r))\,dr\right)d\xi. \tag{14.46}$$

The variational iteration method is used by applying two essential steps. It is required first to determine the Lagrange multiplier λ that can be identified optimally via integration by parts and by using a restricted variation. The Lagrange multiplier λ may be a constant or a function. Having λ determined, an iteration formula, without restricted variation, should be used for the determination of the successive approximations $u_{n+1}(x), n \geqslant 0$ of the solution

$u(x)$. The zeroth approximation u_0 can be any selective function. However, the initial values $u(0), u'(0), \ldots$ are preferably used for the selective zeroth approximation u_0. In what follows we summarize the Lagrange multipliers as derived in Chapter 3, and the selective zeroth approximations:

$$\begin{aligned} &u' + f(u(\xi), u'(\xi)) = 0,\ \lambda = -1,\ u_0(x) = u(0), \\ &u'' + f(u(\xi), u'(\xi), u''(\xi)) = 0,\ \lambda = \xi - x,\ u_0(x) = u(0) + u'(0)x \\ &u''' + f(u(\xi),\ u'(\xi),\ u''(\xi),\ u'''(\xi)) = 0,\ \lambda = -\frac{1}{2!}(\xi - x)^2, \\ &\quad u_0(x) = u(0) + u'(0)x + \frac{1}{2!}u''(0)x^2, \end{aligned} \tag{14.47}$$

and so on. Consequently, the solution is given by

$$u(x) = \lim_{n\to\infty} u_n(x). \tag{14.48}$$

The VIM will be illustrated by studying the following examples.

Example 14.5

Use the variational iteration method to solve the nonlinear Volterra integro-differential equation

$$u'(x) = 1 + e^x - 2xe^x - e^{2x} + \int_0^x e^{x-t}u^2(t)dt,\ u(0) = 2. \tag{14.49}$$

The correction functional for this equation is given by

$$u_{n+1}(x) = u_n(x) - \int_0^x \left(u_n'(t) - 1 - e^t + 2te^t + e^{2t} - \int_0^t e^{t-r}u_n^2(r)\,dr \right) dt, \tag{14.50}$$

where we used $\lambda = -1$ for first-order integro-differential equation.

We can use the initial condition to select $u_0(x) = u(0) = 2$. Using this selection into the correction functional gives the following successive approximations

$$\begin{aligned} u_0(x) &= 2, \\ u_1(x) &= 2 + x + \frac{1}{2}x^2 - \frac{1}{2}x^3 - \frac{3}{8}x^4 - \frac{19}{120}x^5 + \cdots, \\ u_2(x) &= 2 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 - \frac{1}{8}x^5 + \cdots, \\ u_3(x) &= 2 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \cdots, \end{aligned} \tag{14.51}$$

and so on for other approximations.

The VIM admits the use of

$$u(x) = \lim_{n\to\infty} u_n(x), \tag{14.52}$$

that gives the exact solution by

$$u(x) = 1 + e^x. \tag{14.53}$$

Example 14.6

Use the variational iteration method to solve the nonlinear Volterra integro-differential equation

$$u'(x) = -x + \sec x \tan x - \tan x + \int_0^x \left(1 + u^2(t)\right) dt, u(0) = 1. \tag{14.54}$$

The correction functional for this equation is given by

$$\begin{aligned} u_{n+1}(x) &= u_n(x) \\ &- \int_0^x \left(u_n'(t) + t - \sec t \tan t + \tan t - \int_0^t (1 + u_n^2(r))dr\right) dt, \end{aligned} \tag{14.55}$$

where we used $\lambda = -1$ for first-order integro-differential equation.

We can use the initial condition to select $u_0(x) = u(0) = 1$. Using this selection into the correction functional gives the following successive approximations

$$\begin{aligned} u_0(x) &= 1, \\ u_1(x) &= 1 + \frac{1}{2}x^2 + \frac{1}{8}x^4 + \frac{1}{16}x^6 + \cdots, \\ u_2(x) &= 1 + \frac{1}{2}x^2 + \frac{5}{24}x^4 + \frac{19}{240}x^6 + \cdots, \\ u_3(x) &= 1 + \frac{1}{2}x^2 + \frac{5}{24}x^4 + \frac{61}{720}x^6 + \cdots, \end{aligned} \tag{14.56}$$

and so on. The VIM admits the use of

$$u(x) = \lim_{n\to\infty} u_n(x), \tag{14.57}$$

that gives the exact solution by

$$u(x) = \sec x. \tag{14.58}$$

Example 14.7

Use the variational iteration method to solve the nonlinear Volterra integro-differential equation

$$u'(x) = x + \cos x - \tan x + \tan^2 x + \int_0^x \left(\sin t + u^2(t)\right) dt, u(0) = 0. \tag{14.59}$$

The correction functional for this equation is given by

$$\begin{aligned} u_{n+1}(x) &= u_n(x) \\ &- \int_0^x \left(u_n'(t) - t - \cos t + \tan t - \tan^2 t - \int_0^t (\sin t + u_n^2(r))dr\right) dt. \end{aligned} \tag{14.60}$$

We can use the initial condition to select $u_0(x) = u(0) = 0$. Using this selection into the correction functional gives the following successive approximations

$$
\begin{aligned}
u_0(x) &= 0,\\
u_1(x) &= x + \frac{1}{3}x^3 - \frac{1}{12}x^4 + \frac{2}{15}x^5 - \frac{1}{45}x^6 + \frac{17}{315}x^7 + \cdots,\\
u_2(x) &= x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \frac{17}{315}x^7 + \cdots,
\end{aligned}
\tag{14.61}
$$

and so on. The VIM admits the use of

$$u(x) = \lim_{n\to\infty} u_n(x), \tag{14.62}$$

that gives the exact solution by

$$u(x) = \tan x. \tag{14.63}$$

Example 14.8

Use the variational iteration method to solve the nonlinear Volterra integro-differential equation

$$u''(x) = -\frac{5}{3}\sin x + \frac{1}{3}\sin(2x) + \int_0^x \cos(x-t)u^2(t)dt,\ u(0)=0,\ u'(0)=1. \tag{14.64}$$

The correction functional for this equation is given by

$$
\begin{aligned}
&u_{n+1}(x) = u_n(x)\\
&+ \int_0^x (t-x)\left(u_n''(t) + \frac{5}{3}\sin t - \frac{1}{3}\sin(2t) - \int_0^t \cos(t-r)u_n^2(r)dr\right)dt.
\end{aligned}
\tag{14.65}
$$

We can use the initial condition to select $u_0(x) = u(0) + xu'(0) = x$. Using this selection into the correction functional gives the following successive approximations

$$
\begin{aligned}
u_0(x) &= x,\\
u_1(x) &= x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \frac{1}{720}x^7 + \cdots,\\
u_2(x) &= x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots,
\end{aligned}
\tag{14.66}
$$

and so on. The VIM gives the exact solution by

$$u(x) = \sin x. \tag{14.67}$$

Exercises 14.2.2

Solve the following nonlinear Volterra integro-differential equations by using the variational iteration method

1. $u'(x) = \frac{1}{4}(1-x^2) - \sin x - \frac{1}{4}\cos^2 x + \int_0^x (x-t)(1-u^2(t))dt,\ u(0)=0$

2. $u'(x) = -\frac{1}{4}(1-2x+2x^2) - e^{-x} + \frac{1}{4}e^{-2x} + \int_0^x (x-t)(1-u^2(t))dt,\ u(0)=1$

3. $u'(x) = 1 + u - xe^{-x^2} - 2\int_0^x xte^{-u^2}(t)dt,\ u(0)=0$

4. $u'(x) = -\frac{4}{3}\sin x - \frac{1}{3}\sin(2x) + \int_0^x \cos(x-t)u^2(t)dt,\ u(0) = 1$

5. $u''(x) = -\frac{1}{2}x - \cos x + \frac{1}{4}\sin(2x) + \int_0^x (1-u^2(t))dt,\ u(0) = 1, u'(0) = 0$

6. $u''(x) = 1 + e^x - e^{2x} + \int_0^x e^{x-t}(1+u^2(t))dt,\ u(0) = 1, u'(0) = 1$

7. $u''(x) = e^x(2-x) - e^{2x} + \int_0^x e^{x-t}(u+u^2(t))dt,\ u(0) = 1, u'(0) = 1$

8. $u'''(x) = -x + \tan x(6\sec^3 x - \sec x + 1) + \int_0^x (1-u^2(t))dt,$

$u(0) = 1,\ u'(0) = 0,\ u''(0) = 1$

14.2.3 The Series Solution Method

The series solution method [5,7] was effectively used in this text to handle integral and integro-differential equations. The method stems mainly from the Taylor series for analytic functions. A real function $u(x)$ is called analytic if it has derivatives of all orders such that the Taylor series at any point b in its domain

$$u_k(x) = \sum_{n=0}^{k} \frac{u^{(n)}(b)}{n!}(x-b)^n, \tag{14.68}$$

converges to $u(x)$ in a neighborhood of b. For simplicity, the generic form of Taylor series at $x = 0$ can be written as

$$u(x) = \sum_{n=0}^{\infty} a_n x^n. \tag{14.69}$$

The Taylor series method, or simply the series solution method will be used in this section for solving nonlinear Volterra integro-differential equations. We will assume that the solution $u(x)$ of the nonlinear Volterra integro-differential equation

$$u^{(n)}(x) = f(x) + \lambda \int_0^x K(x,t)F(u(t))dt, u^{(k)}(0) = k!a_k, 0 \leqslant k \leqslant (n-1), \tag{14.70}$$

is analytic, and therefore possesses a Taylor series of the form given in (14.69), where the coefficients a_n will be determined recurrently.

The first few coefficients a_k can be determined by using the initial conditions so that

$$a_0 = u(0),\ a_1 = u'(0),\ a_2 = \frac{1}{2!}u''(0), a_3 = \frac{1}{3!}u'''(0), \tag{14.71}$$

and so on. The remaining coefficients a_k of (14.69) will be determined by applying the series solution method to the nonlinear Volterra integro-differential equation (14.70). Substituting (14.69) into both sides of (14.70) gives

$$\left(\sum_{k=0}^{\infty} a_k x^k\right)^{(n)} = T(f(x)) + \int_0^x K(x,t)F\left(\sum_{k=0}^{\infty} a_k t^k\right) dt, \tag{14.72}$$

where $T(f(x))$ is the Taylor series for $f(x)$. The integro-differential equation (14.70) will be converted to a traditional integral in (14.72) where instead of integrating the unknown function $F(u(x))$, terms of the form t^n, $n \geqslant 0$ will be integrated. Notice that because we are seeking series solution, then if $f(x)$ includes elementary functions such as trigonometric functions, exponential functions, etc., then Taylor expansions for functions involved in $f(x)$ should be used.

We first integrate the right side of the integral in (14.72), and collect the coefficients of like powers of x. We next equate the coefficients of like powers of x into both sides of the resulting equation to determine a recurrence relation in $a_j, j \geqslant 0$. Solving the recurrence relation will lead to a complete determination of the coefficients $a_j, j \geqslant 0$, where some of these coefficients will be used from the initial conditions. Having determined the coefficients $a_j, j \geqslant 0$, the series solution follows immediately upon substituting the derived coefficients into (14.69).

Example 14.9

Solve the nonlinear Volterra integro-differential equation by using the series solution method

$$u'(x) = 1 - e^x + e^{2x} + \int_0^x e^{x-t}(1-u^2(t))dt, \;\; u(0) = 1. \tag{14.73}$$

Substituting $u(x)$ by the series

$$u(x) = \sum_{n=0}^{\infty} a_n x^n, \tag{14.74}$$

into both sides of equation (14.73) leads to

$$\left(\sum_{n=0}^{\infty} a_n x^n\right)' = T_1(1 - e^x + e^{2x}) + \int_0^x \left(T_2(e^{x-t})\left(1 - \left(\sum_{n=0}^{\infty} a_n t^n\right)^2\right)\right) dt, \tag{14.75}$$

where T_1 and T_2 are the Taylor series about $x = 0$ and about $t = 0$ respectively. Evaluating the integral at the right side, using $a_0 = 1$, we find

$$\begin{aligned}
&a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + 5a_5x^4 + \cdots \\
&= 1 + x + \left(\frac{3}{2} - a_1\right)x^2 + \left(\frac{7}{6} - \frac{1}{3}a_1^2 - \frac{2}{3}a_2 - \frac{1}{3}a_1\right)x^3 \\
&+ \left(\frac{5}{8} - \frac{1}{12}a_1^2 - \frac{1}{6}a_2 - \frac{1}{12}a_1 - \frac{1}{2}a_3 - \frac{1}{2}a_1a_2\right)x^4 \\
&+ \left(\frac{31}{120} - \frac{1}{60}a_1 - \frac{1}{60}a_1^2 - \frac{1}{30}a_2 - \frac{1}{5}a_2^2 - \frac{2}{5}a_4 - \frac{2}{5}a_1a_3 - \frac{1}{10}a_3 - \frac{1}{10}a_1a_2\right)x^5 \\
&+ O(x^6).
\end{aligned} \tag{14.76}$$

Equating the coefficients of like powers of x in both sides, and solving the system of equations we obtain

$$a_n = \frac{1}{n!}, n \geqslant 0. \tag{14.77}$$

This gives the exact solution by

$$u(x) = e^x. \tag{14.78}$$

Example 14.10

Solve the nonlinear Volterra integro-differential equation by using the series solution method

$$u'(x) = \cos x - \frac{5}{3}\sin x + \frac{1}{3}\sin(2x) + \int_0^x \cos(x-t)(1+u^2(t))dt, \;\; u(0) = 0. \tag{14.79}$$

Substituting $u(x)$ by the series

$$u(x) = \sum_{n=0}^{\infty} a_n x^n, \tag{14.80}$$

into both sides of equation (14.79) leads to

$$\begin{aligned}\left(\sum_{n=0}^{\infty} a_n x^n\right)' &= T_1\left(\cos x - \frac{5}{3}\sin x + \frac{1}{3}\sin(2x)\right) \\ &\quad + \int_0^x (T_2(\cos(x-t))\left(1 + \left(\sum_{n=0}^{\infty} a_n t^n\right)^2\right) dt,\end{aligned} \tag{14.81}$$

where T_1 and T_2 are the Taylor series about $x = 0$ and about $t = 0$ respectively. Evaluating the integral at the right side, using $a_0 = 0$, and proceeding as before we find

$$a_{2n+1} = \frac{(-1)^n}{(2n+1)!}, \;\; a_{2n} = 0, \;\; n \geqslant 0. \tag{14.82}$$

This gives the exact solution by

$$u(x) = \sin x. \tag{14.83}$$

Example 14.11

Solve the nonlinear Volterra integro-differential equation by using the series solution method

$$u''(x) = \frac{1}{2}x - \sin x - \cos x - \frac{1}{4}\sin(2x) + \int_0^x (x-t)(1-u^2(t))dt, u(0) = u'(0) = 1. \tag{14.84}$$

Substituting $u(x)$ by the series

$$u(x) = \sum_{n=0}^{\infty} a_n x^n, \tag{14.85}$$

into both sides of equation (14.84) leads to

$$\left(\sum_{n=0}^{\infty} a_n x^n\right)'' = T_1\left(\frac{1}{2}x - \sin x - \cos x - \frac{1}{4}\sin(2x)\right) + \int_0^x \left((x-t)\left(1-\left(\sum_{n=0}^{\infty} a_n t^n\right)^2\right)\right) dt, \quad (14.86)$$

where T_1 is the Taylor series about $x = 0$. Evaluating the integral at the right side, using $a_0 = 1, a_1 = 1$, and proceeding as before we find

$$a_{2n} = \frac{(-1)^n}{(2n)!}, a_{2n+1} = \frac{(-1)^n}{(2n+1)!}, \; n \geqslant 0. \quad (14.87)$$

Consequently, the series solution is given by

$$u(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}x^{2n} + \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}x^{2n+1}, \quad (14.88)$$

that converges to the exact solution

$$u(x) = \cos x + \sin x. \quad (14.89)$$

Example 14.12

Solve the nonlinear Volterra integro-differential equation by using the series solution method

$$u'''(x) = 2\sec^2 x(1 + 3\tan^2 x) - \tan x + \int_0^x (1 + u^2(t))dt, \quad (14.90)$$

with initial conditions $u(0) = u''(0) = 0, u'(0) = 1$. Substituting $u(x)$ by the series

$$u(x) = \sum_{n=0}^{\infty} a_n x^n, \quad (14.91)$$

into both sides of equation (14.90), evaluating the integral at the right side, using $a_0 = a_2 = 0, a_1 = 1$, and proceeding as before we find

$$\begin{aligned} &a_1 = 1, \quad a_3 = \frac{1}{3}, \quad a_5 = \frac{2}{15}, \\ &a_7 = \frac{17}{315}, \quad a_{2k} = 0, \quad k \geqslant 0, \end{aligned} \quad (14.92)$$

Consequently, the series solution is given by

$$u(x) = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \frac{17}{315}x^7 + \cdots, \quad (14.93)$$

that converges to the exact solution

$$u(x) = \tan x. \quad (14.94)$$

Exercises 14.2.3

Solve the following nonlinear Volterra integro-differential equations by using the series solution method

1. $u'(x) = -x + \tanh x(1 - \operatorname{sech} x) + \int_0^x (1 - u^2(t))dt,\ u(0) = 1$

2. $u'(x) = -\frac{3}{2}x + \cosh x + \frac{1}{4}\sinh(2x) + \int_0^x (1 - u^2(t))dt,\ u(0) = 0$

3. $u'(x) = -\frac{1}{2}x + \sin x + \cos x + \frac{1}{4}\sin(2x) + \int_0^x (x-t)(1 - u^2(t))dt,\ u(0) = -1$

4. $u'(x) = 1 - 2\cosh x + \int_0^x e^{x+t}u^2(t)dt,\ u(0) = 1$

5. $u''(x) = 1 - 2\sinh x + \int_0^x e^{x+t}u^2(t)dt,\ u(0) = 1,\ u'(0) = -1$

6. $u''(x) = 1 + e^x(1-2x) - e^{2x} + \int_0^x e^{x-t}u^2(t)dt,\ u(0) = 2,\ u'(0) = 1$

7. $u'''(x) = 2xe^x + e^{2x} + \int_0^x e^{x-t}(1 - u^2(t))dt,\ u(0) = 2,\ u'(0) = u''(0) = 1$

8. $u'''(x) = -\frac{1}{2}x - 8\cos(2x) - \frac{1}{8}\sin(4x) + \int_0^x (1 - u^2(t))dt,$
   $u(0) = u''(0) = 0, u'(0) = 2$

14.3 Nonlinear Volterra Integro-Differential Equations of the First Kind

The standard form of the nonlinear Volterra integro-differential equation [1–3] of the first kind is given by

$$\int_0^x K_1(x,t)F(u(t))dt + \int_0^x K_2(x,t)u^{(i)}(t)dt = f(x), \tag{14.95}$$

where $u^{(i)}(x)$ is the i th derivative of $u(x)$. For this equation, the kernels $K_1(x,t)$ and $K_2(x,t)$, and the function $f(x)$ are given real-valued functions, and $F(u(x))$ is a nonlinear function of $u(x)$. For the determination of the exact solution, the initial conditions should be prescribed. The nonlinear Volterra integro-differential equation of the first kind (14.95) can be converted to a nonlinear Volterra integral equation of the second kind by integrating the second integral in (14.95) by parts. The nonlinear Volterra integro-differential equations of the first kind will be handled in this section by the combined Laplace transform-Adomian decomposition method and by converting it to a nonlinear Volterra integro-differential equation of the second kind.

14.3.1 The Combined Laplace Transform-Adomian Decomposition Method

The combined Laplace transform-Adomian decomposition method was used in the previous section for solving nonlinear Volterra integro-differential equations of the second kind. The analysis will be focused on equations where the kernels $K_1(x,t)$ and $K_2(x,t)$ of (14.95) are *difference kernels*. This means that each kernel depends on the difference $(x-t)$. Recall that the Laplace transform of the convolution product $(f_1 * f_2)(x)$ is given by

$$\mathcal{L}\{(f_1 * f_2)(x)\} = \mathcal{L}\left\{\int_0^x f_1(x-t)f_2(t)dt\right\} = F_1(s)F_2(s). \quad (14.96)$$

Recall that

$$\mathcal{L}\{u^{(n)}(x)\} = s^n\mathcal{L}\{u(x)\} - s^{n-1}u(0) - s^{n-2}u'(0) - \cdots u^{(n-1)}(0). \quad (14.97)$$

Taking Laplace transform of both sides of (14.95) gives

$$\mathcal{L}\{K_1(x-t) * F(u(x))\} + \mathcal{L}\{K_2(x-t) * u^{(i)}(x)\} = \mathcal{L}\{(f(x)\}, \quad (14.98)$$

so that

$$\mathcal{K}_1(s)\mathcal{L}\{F(u(x))\} + \mathcal{K}_2(s)\mathcal{L}\{u^{(i)}(x)\} = \phi(s), \quad (14.99)$$

where

$$\phi(s) = \mathcal{L}\{f(x)\}, \mathcal{K}_1(s) = \mathcal{L}\{K_1(x)\}, \mathcal{K}_2(s) = \mathcal{L}\{K_2(x)\}. \quad (14.100)$$

Using (14.97) and solving for $U(s)$ we find

$$U(s) = \frac{\phi(s) + \mathcal{K}_2(s)\Gamma(s) - \mathcal{K}_1(s)\mathcal{L}\{F(u(x))\}}{s^i\mathcal{K}_2(s)}, \quad (14.101)$$

where

$$\begin{aligned} \Gamma(s) &= s^{i-1}u(0) + s^{i-2}u'(0) + \cdots + u^{(i-1)}(0), \\ U(s) &= \mathcal{L}\{u(x)\}. \end{aligned} \quad (14.102)$$

The combined Laplace transform-Adomian decomposition method can be used effectively in (14.101) provided that

$$\lim_{s\to\infty} \frac{\mathcal{K}_1(s)}{s^i\mathcal{K}_2(s)} = 0. \quad (14.103)$$

To overcome the difficulty of the nonlinear term $F(u(x))$, we apply the Adomian decomposition method for handling (14.101). To achieve this goal, we first represent the linear term $u(x)$ at the left side by an infinite series of components given by

$$u(x) = \sum_{n=0}^{\infty} u_n(x), \quad (14.104)$$

where the components $u_n(x), n \geqslant 0$ will be recursively determined. However, the nonlinear term $F(u(x))$ at the right side of (14.101) will be represented by an infinite series of the Adomian polynomials A_n in the form

$$F(u(x)) = \sum_{n=0}^{\infty} A_n(x), \quad (14.105)$$

where the Adomian polynomials $A_n, n \geqslant 0$ are given by

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[F\left(\sum_{i=0}^{n}\lambda^i u_i\right)\right]_{\lambda=0}, n = 0, 1, 2, \cdots. \quad (14.106)$$

Substituting (14.104) and (14.105) into (14.101) leads to

$$\mathcal{L}\left\{\sum_{n=0}^{\infty} u_n(x)\right\} = \frac{1}{s}u(0) + \frac{1}{s^2}u'(0) + \cdots + \frac{1}{s^i}u^{(i-1)}(0) + \frac{1}{s^i\mathcal{K}_2(s)}\phi(s) - \frac{\mathcal{K}_1(s)}{s^i\mathcal{K}_2(s)}\mathcal{L}\left\{\sum_{n=0}^{\infty} A_n(x)\right\}. \quad (14.107)$$

The Adomian decomposition method admits the use of the following recursive relation

$$\begin{aligned} U_0(s) &= \frac{1}{s}u(0) + \frac{1}{s^2}u'(0) + \cdots + \frac{1}{s^i}u^{(i-1)}(0) + \frac{1}{s^i\mathcal{K}_2(s)}\phi(s), \\ \mathcal{L}\{u_{k+1}(x)\} &= -\frac{\mathcal{K}_1(s)}{s^i\mathcal{K}_2(s)}\mathcal{L}\{A_k(x)\}, \quad k \geqslant 0, \end{aligned} \quad (14.108)$$

provided that

$$\lim_{s\to\infty} \frac{\mathcal{K}_1(s)}{s^i\mathcal{K}_2(s)} = 0. \quad (14.109)$$

For example if $K_2(x,t) = \cosh(x-t)$, $K_1(x,t) = x - t$, and the equation includes $u'(x)$, then

$$\lim_{s\to\infty} \frac{s^2}{s^2-1} = 1. \quad (14.110)$$

In such a problem, the combined Laplace transform-Adomian decomposition method cannot be used. Instead another approach should be used to handle this case.

Applying the inverse Laplace transform to the first part of (14.108) gives $u_0(x)$, that will define A_0. This in turn will lead to the complete determination of the components of $u_k, k \geqslant 0$. The analysis presented above will be illustrated by the following examples.

Example 14.13

Solve the following nonlinear Volterra integro-differential equation of the first kind by the combined Laplace transform-Adomian decomposition method

$$\int_0^x (x-t)u^2(t)dt + \int_0^x (x-t)u'(t)dt = \frac{7}{8} + \frac{1}{4}x^2 - \cos x + \frac{1}{8}\cos(2x),\ u(0) = 0. \quad (14.111)$$

Taking Laplace transforms of both sides gives

$$\frac{1}{s^2}\mathcal{L}\{u^2\}(s) + \frac{1}{s^2}(sU(s) - u(0)) = \frac{7}{8s} + \frac{1}{2s^3} - \frac{s}{s^2+1} + \frac{s}{8(s^2+4)}, \quad (14.112)$$

where by using the given initial condition and solving for $U(s)$ we obtain

$$U(s) = \frac{7}{8} + \frac{1}{2s^2} - \frac{s^2}{s^2+1} + \frac{s^2}{8(s^2+4)} - \frac{1}{s}\mathcal{L}\{u^2\}(s). \quad (14.113)$$

Substituting the series assumption for $U(s)$ and the Adomian polynomials for $u^2(x)$, and using the recursive relation (14.108) we obtain

$$\begin{aligned} U_0(s) &= \frac{7}{8} + \frac{1}{2s^2} - \frac{s^2}{s^2+1} + \frac{s^2}{8(s^2+4)}, \\ \mathcal{L}\{u_{k+1}(x)\} &= -\frac{1}{s}\mathcal{L}\{A_k(x)\}, \quad k \geqslant 0. \end{aligned} \tag{14.114}$$

Taking the inverse Laplace transform of both sides of the first part of (14.114), and using the recursive relation (14.114) gives

$$\begin{aligned} u_0(x) &= x + \frac{1}{3!}x^3 - \frac{7}{120}x^5 + \cdots, \\ u_1(x) &= -\frac{1}{3}x^3 - \frac{1}{15}x^5 + \cdots, \\ u_2(x) &= \frac{2}{15}x^5 + \cdots. \end{aligned} \tag{14.115}$$

The series solution is therefore given by

$$u(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \cdots, \tag{14.116}$$

that converges to the exact solution

$$u(x) = \sin x. \tag{14.117}$$

Example 14.14

Solve the following nonlinear Volterra integro-differential equation of the first kind by the combined Laplace transform-Adomian decomposition method

$$\int_0^x (x-t)u^2(t)dt + \int_0^x e^{x-t}u'(t)dt = -\frac{1}{4} - \frac{1}{2}x + xe^x + \frac{1}{4}e^{2x}, \ u(0) = 1. \tag{14.118}$$

Taking Laplace transforms of both sides gives

$$\frac{1}{s^2}\mathcal{L}\{u^2\}(s) + \frac{1}{s-1}(sU(s) - u(0)) = -\frac{1}{4s} - \frac{1}{2s^2} + \frac{s}{(s-1)^2} + \frac{1}{4(s-2)}, \tag{14.119}$$

where by using the given initial condition and solving for $U(s)$ we obtain

$$U(s) = \frac{1}{s} + \frac{s-1}{s}\left(-\frac{1}{4s} - \frac{1}{2s^2} + \frac{s}{(s-1)^2} + \frac{1}{4(s-2)}\right) - \frac{s-1}{s^3}\mathcal{L}\{u^2\}(s). \tag{14.120}$$

Substituting the series assumption for $U(s)$ and the Adomian polynomials for $u^2(x)$, and using the recursive relation (14.108) we obtain

$$\begin{aligned} U_0(s) &= \frac{1}{s} + \frac{s-1}{s}\left(-\frac{1}{4s} - \frac{1}{2s^2} + \frac{s}{(s-1)^2} + \frac{1}{4(s-2)}\right), \\ \mathcal{L}\{u_{k+1}(x)\} &= -\frac{s-1}{s^3}\mathcal{L}\{A_k(x)\}, \quad k \geqslant 0. \end{aligned} \tag{14.121}$$

Taking the inverse Laplace transform of both sides of the first part of (14.121), and using the recursive relation (14.121) gives

$$\begin{aligned} u_0(x) &= 1 + x + x^2 + \frac{1}{3}x^3 + \frac{1}{8}x^4 + \frac{1}{24}x^5 + \cdots, \\ u_1(x) &= -\frac{1}{2}x^2 - \frac{1}{6}x^3 - \frac{1}{6}x^4 - \frac{1}{12}x^5 + \cdots, \\ u_2(x) &= \frac{1}{12}x^4 + \frac{1}{20}x^5 + \cdots. \end{aligned} \tag{14.122}$$

The series solution is therefore given by

$$u(x) = 1 + x + x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \cdots, \tag{14.123}$$

that converges to the exact solution

$$u(x) = e^x. \tag{14.124}$$

Example 14.15

Solve the following nonlinear Volterra integro-differential equation of the first kind by the combined Laplace transform-Adomian decomposition method

$$\int_0^x (x-t)u^2(t)dt + \int_0^x (x-t)u''(t)dt = -\frac{15}{16} + \frac{1}{4}x^2 + \cos(2x) - \frac{1}{16}\cos^2(2x), \tag{14.125}$$

where $u(0) = 1, u'(0) = 0$. Taking Laplace transforms of both sides and using the initial conditions we obtain

$$U(s) = \frac{1}{s} + \mathcal{L}\left\{-\frac{15}{16} + \frac{1}{4}x^2 + \cos(2x) - \frac{1}{16}\cos^2(2x)\right\} - \frac{1}{s^2}\mathcal{L}\{u^2\}(s). \tag{14.126}$$

Proceeding as before leads to

$$\begin{aligned} u_0(x) &= 1 - \frac{3}{2}x^2 + \frac{1}{3}x^4 + \frac{4}{45}x^6 + \cdots, \\ u_1(x) &= -\frac{1}{2}x^2 + \frac{1}{4}x^4 - \frac{7}{72}x^6 + \cdots, \\ u_2(x) &= \frac{1}{12}x^4 + \frac{1}{15}x^6 + \cdots, \\ u_3(x) &= -\frac{1}{72}x^6 + \cdots, \end{aligned} \tag{14.127}$$

The series solution is therefore given by

$$u(x) = 1 - \frac{1}{2!}(2x)^2 + \frac{1}{4!}(2x)^4 - \frac{1}{6!}(2x)^6 + \cdots, \tag{14.128}$$

that converges to the exact solution

$$u(x) = \cos(2x). \tag{14.129}$$

Example 14.16

Solve the following nonlinear Volterra integro-differential equation of the first kind by the combined Laplace transform-Adomian decomposition method

$$\int_0^x (x-t)u^2(t)dt + \int_0^x (x-t)u''(t)dt = -\frac{1}{4} - 3x + \frac{1}{4}x^2 + 3\sinh x + \frac{1}{4}\cosh^2 x, \tag{14.130}$$

where $u(0) = 1, u'(0) = 1$. Taking Laplace transforms of both sides and using the initial conditions we obtain

$$U(s) = \frac{1}{s} + \frac{1}{s^2} + \mathcal{L}\{-\frac{1}{4} - 3x + \frac{1}{4}x^2 + 3\sinh x + \frac{1}{4}\cosh^2 x\} - \frac{1}{s^2}\mathcal{L}\{u^2\}(s). \tag{14.131}$$

Proceeding as before leads to

$$\begin{aligned} u_0(x) &= 1 + x + \frac{1}{2}x^2 + \frac{1}{2}x^3 + \frac{1}{12}x^4 + \frac{1}{40}x^5 + \cdots, \\ u_1(x) &= -\frac{1}{2}x^2 - \frac{1}{3}x^3 - \frac{1}{6}x^4 - \frac{1}{10}x^5 + \cdots, \\ u_2(x) &= \frac{1}{12}x^4 + \frac{1}{12}x^5 + \cdots, \end{aligned} \tag{14.132}$$

The series solution is therefore given by

$$u(x) = 1 + x + \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \cdots, \tag{14.133}$$

that converges to the exact solution

$$u(x) = 1 + \sinh x. \tag{14.134}$$

Exercises 14.3.1

Solve the following nonlinear Volterra integro-differential equations of the first kind by using the combined Laplace transform-Adomian decomposition method

1. $\int_0^x (x-t)u^2(t)dt + \int_0^x (x-t)u'(t)dt = \frac{1}{8} - x + \frac{1}{4}x^2 + \sin x - \frac{1}{8}\cos(2x),\ u(0) = 1$
2. $\int_0^x (x-t)u^2(t)dt + \int_0^x (x-t)u'(t)dt = -\frac{1}{8} - x + \frac{1}{4}x^2 + \sinh x + \frac{1}{8}\cosh(2x),$
   $u(0) = 1$
3. $\int_0^x (x-t)u^2(t)dt + \int_0^x (x-t)u'(t)dt = 1 - \frac{1}{2}x + \frac{1}{2}x^2 + \sin x - \cos x - \frac{1}{4}\sin(2x),$
   $u(0) = 1$
4. $\int_0^x e^{x-t}u^2(t)dt + \int_0^x e^{x-t}u'(t)dt = -1 + 3xe^x + e^{2x},\ u(0) = 2$
5. $\int_0^x e^{x-t}u^2(t)dt + \int_0^x e^{x-t}u''(t)dt = -4 + 3e^x + 5xe^x + e^{2x},\ u(0) = 3,\ u'(0) = 1$
6. $\int_0^x (x-t)u^2(t)dt + \int_0^x (x-t)u''(t)dt = -\frac{1}{16} - 2x + \frac{1}{4}x^2 + \sin(2x)$
   $+\frac{1}{16}\cos^2 x,\ u(0) = 0,\ u'(0) = 2$
7. $\int_0^x (x-t)u^2(t)dt + \int_0^x (x-t)u''(t)dt = x^2 - \frac{1}{36}x^6,\ u(0) = 0,\ u'(0) = 0$
8. $\int_0^x (x-t)u^3(t)dt + \int_0^x (x-t)u''(t)dt = -\frac{10}{9} - \frac{4}{3}x + e^x + \frac{1}{9}e^{3x},$
   $u(0) = 1,\ u'(0) = 1$

14.3.2 Conversion to Nonlinear Volterra Equation of the Second Kind

In this section we will convert the nonlinear Volterra integro-differential equation [1–3] of the first kind

$$\int_0^x K_1(x,t)F(u(t))dt + \int_0^x K_2(x,t)u^{(n)}(t)dt = f(x),\ \ K_2(x,x) \neq 0, \tag{14.135}$$

to a nonlinear Volterra integral equation of the second kind or nonlinear Volterra integro-differential equation of the second kind. Without loss of generality, we will study the cases of the first and second order derivatives of the form

$$\int_0^x K_1(x,t)F(u(t))dt + \int_0^x K_2(x,t)u'(t)dt = f(x),\ \ K_2(x,x) \neq 0, \tag{14.136}$$

and

$$\int_0^x K_1(x,t)F(u(t))dt + \int_0^x K_2(x,t)u''(t)dt = f(x),\ \ K_2(x,x) \neq 0. \tag{14.137}$$

However, equations of higher order can be handled in a similar manner.

Integrating the second integral in (14.136) by parts gives the nonlinear Volterra integral equation

$$\begin{aligned}&\int_0^x K_1(x,t)F(u(t))dt + K_2(x,x)u(x) - K_2(x,0)u(0) - \int_0^x \frac{\partial K_2(x,t)}{\partial t}u(t)dt\\ &= f(x),\end{aligned} \tag{14.138}$$

or equivalently

$$\begin{aligned}u(x) = \frac{f(x)}{K_2(x,x)} + \frac{K_2(x,0)}{K_2(x,x)}u(0) + \frac{1}{K_2(x,x)}\int_0^x \frac{\partial(K_2(x,t))}{\partial t}u(t)dt\\ -\frac{1}{K_2(x,x)}\int_0^x K_1(x,t)F(u(t))dt, \quad K_2(x,x) \neq 0.\end{aligned} \tag{14.139}$$

Equation (14.139) is the nonlinear Volterra integral equation of the second kind that was handled in this chapter by distinct methods.

In a like manner, we integrate the second integral in (14.137) by parts to obtain the nonlinear Volterra integro-differential equation of the second kind

$$\begin{aligned}\int_0^x K_1(x,t)F(u(t))dt + K_2(x,x)u'(x) - K_2(x,0)u'(0)\\ -\int_0^x \frac{\partial K_2(x,t)}{\partial t}u'(t)dt = f(x), \quad K_2(x,x) \neq 0.\end{aligned} \tag{14.140}$$

or equivalently

$$u'(x) = \frac{f(x)}{K_2(x,x)} + \frac{K_2(x,0)}{K_2(x,x)}u'(0) + \frac{1}{K_2(x,x)}\int_0^x \frac{\partial(K_2(x,t))}{\partial t}u'(t)dt - \frac{1}{K_2(x,x)}\int_0^x K_1(x,t)F(u(t))dt, \quad K_2(x,x) \neq 0. \tag{14.141}$$

It is important to notice that if the nonlinear Volterra integral equation of the first kind contains the first derivative of $u(x)$, then the conversion process will give a nonlinear Volterra integral equation of the second kind as shown by (14.139). However, if the nonlinear Volterra integral equation of the first kind contains $u^{(i)}(x), i \geqslant 2$, then integrating the second integral once will give a nonlinear Volterra integro-differential equation of the second kind as shown by (14.141). Both types of equations were examined before.

In what follows, the first two illustrative examples will be handled by the modified decomposition method and the other two examples will be handled by using the variational iteration method. Other methods can be used as well.

Example 14.17

Convert the nonlinear Volterra integro-differential equation of the first kind

$$\int_0^x (x-t)u^2(t)dt + \int_0^x e^{x-t}u'(t)dt = xe^x + \frac{1}{4}e^{2x} - \frac{1}{4} - \frac{1}{2}x, \ u(0) = 1, \tag{14.142}$$

to a nonlinear Volterra integral equation of the second kind and solve it.

Integrating the second integral by parts, using the initial condition, and solving for $u(x)$ we find

$$u(x) = e^x + xe^x + \frac{1}{4}e^{2x} - \frac{1}{4} - \frac{1}{2}x - \int_0^x (x-t)u^2(t)\, dt - \int_0^x e^{x-t}u(t)\, dt. \tag{14.143}$$

We select the modified decomposition method, therefore we use the recurrence relation approximations

$$\begin{aligned} u_0(x) &= e^x + xe^x, \\ u_1(x) &= \frac{1}{4}e^{2x} - \frac{1}{4} - \frac{1}{2}x - \int_0^x (x-t)u_0^2(t)dt - \int_0^x e^{x-t}u_0(t)dt, \\ &= -xe^x + \cdots . \end{aligned} \tag{14.144}$$

Canceling the noise term xe^x from $u_0(x)$ gives the exact solution

$$u(x) = e^x. \tag{14.145}$$

Example 14.18

Convert the nonlinear Volterra integro-differential equation of the first kind

$$\int_0^x u^2(t)\, dt + \int_0^x (x-t+1)\, u'(t)\, dt = \sin x + \cos x + \frac{1}{4}\sin(2x) - 1 - \frac{1}{2}x, u(0) = 1, \tag{14.146}$$

to a nonlinear Volterra integral equation of the second kind and solve it.

Integrating the second integral by parts, using the initial condition, and solving for $u(x)$ we find

$$u(x) = \sin x + \cos x + \frac{1}{4}\sin(2x) + \frac{1}{2}x - \int_0^x u^2(t)\,dt - \int_0^x u(t)\,dt. \quad (14.147)$$

We again select the modified decomposition method, therefore we use the recurrence relation approximations

$$\begin{aligned} u_0(x) &= \sin x + \cos x, \\ u_1(x) &= \frac{1}{4}\sin(2x) + \frac{1}{2}x - \int_0^x u_0^2(t)dt - \int_0^x u_0(t)dt, \\ &= -\sin x + \cdots . \end{aligned} \quad (14.148)$$

Canceling the noise term $\sin x$ from $u_0(x)$ gives the exact solution

$$u(x) = \cos x. \quad (14.149)$$

Example 14.19

Convert the nonlinear Volterra integro-differential equation of the first kind

$$\int_0^x u^2(t)dt + \int_0^x (x-t+1)u''(t)dt = \sin x - \cos x - \frac{1}{4}\sin(2x) + 1 + \frac{1}{2}x, \quad (14.150)$$

with $u(0) = u'(0) = 1$, to a nonlinear Volterra integro-differential equation of the second kind and solve it.

Integrating the second integral by parts, using the initial conditions, and solving for $u(x)$ we find

$$u'(x) = 2 + \frac{3}{2}x + \sin x - \cos x - \frac{1}{4}\sin(2x) - \int_0^x u^2(t)dt - \int_0^x u'(t)dt. \quad (14.151)$$

This equation will be solved by using the variational iteration method. The correction functional for this equation is given by

$$\begin{aligned} u_{n+1}(x) &= u_n(x) \\ &- \int_0^x \left(u_n'(t) - 2 - \frac{3}{2}t - \sin t + \cos t + \frac{1}{4}\sin(2t) + \int_0^t (u_n'(r) + u_n^2(r))dr \right) dt. \end{aligned} \quad (14.152)$$

We can use the initial conditions to select $u_0(x) = 1+x$. Using this selection into the correction functional gives the following successive approximations

$$\begin{aligned} u_0(x) &= 1 + x, \\ u_1(x) &= 1 + x - \frac{1}{3!}x^3 - \frac{1}{4!}x^4 - \frac{1}{5!}x^5 + \cdots , \\ u_2(x) &= 1 + x - \frac{1}{3!}x^3 + \frac{1}{60}x^5 + \cdots , \\ u_3(x) &= 1 + x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \cdots , \end{aligned} \quad (14.153)$$

and so on. The VIM gives the exact solution by

$$u(x) = 1 + \sin x. \quad (14.154)$$

Example 14.20

Convert the nonlinear Volterra integro-differential equation of the first kind

$$\int_0^x (t^2 - u^2(t))\,dt + \int_0^x e^{x-t}\,u''(t)\,dt = -\frac{3}{2} + e^x(2-x) - \frac{1}{2}e^{2x}, \quad (14.155)$$

with $u(0) = 1, u'(0) = 2$, to a nonlinear Volterra integro-differential equation of the second kind and solve it.

Integrating the second integral by parts, using the initial equation, and solving for $u(x)$ we find

$$u'(x) = -\frac{3}{2} + e^x(4-x) - \frac{1}{2}e^{2x} - \int_0^x (t^2 - u^2(t))\,dt - \int_0^x e^{x-t}u'(t)\,dt. \quad (14.156)$$

This equation will be solved by using the variational iteration method. The correction functional for this equation is given by

$$\begin{aligned} u_{n+1}(x) &= u_n(x) \\ &- \int_0^x \left(u_n'(t) + \frac{3}{2} - e^t(4-t) + \frac{1}{2}e^{2t} + \int_0^t (r^2 - u_n^2(r) + e^{t-r}u_n'(r))dr \right) dt. \end{aligned} \quad (14.157)$$

Using the initial conditions to select $u_0(x) = u(0) + xu'(0) = 1 + 2x$, we obtain the following successive approximations

$$\begin{aligned} u_0(x) &= 1 + 2x, \\ u_1(x) &= 1 + 2x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots, \\ u_2(x) &= 1 + 2x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{60}x^5 + \cdots, \\ u_3(x) &= 1 + 2x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots, \\ u_4(x) &= 1 + 2x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \cdots, \end{aligned} \quad (14.158)$$

and so on. The VIM gives the exact solution by

$$u(x) = x + e^x. \quad (14.159)$$

Exercises 14.3.2

In Exercises 1–4, convert nonlinear Volterra integro-differential equations of the first kind to nonlinear Volterra integral equations of the second kind and solve the resulting equations by any method

1. $\int_0^x (\cosh^2 t - u^2(t))dt + \int_0^x \cosh(x-t)u'(t)dt = -x - 2\sinh x + \frac{1}{2}x\sinh x,\ u(0) = 2$
2. $\int_0^x (\cos^2 t - u^2(t))dt + \int_0^x \cos(x-t)u'(t)dt = -x - 2\sin x - \frac{1}{2}x\sin x,\ u(0) = 2$
3. $\int_0^x (t^2 - u^2(t))dt + \int_0^x e^{x-t}u'(t)dt = -\frac{5}{2} + e^x(3-x) - \frac{1}{2}e^{2x},\ u(0) = 1$
4. $\int_0^x (1 - u^2(t))dt + \int_0^x \cos(x-t)u'(t)dt = \frac{1}{2}\sin x(2\sin x + x + 1) + \frac{1}{2}x\cos x,$
   $u(0) = -1$

In Exercises 5–8, convert nonlinear Volterra integro-differential equations of the first kind to a nonlinear Volterra integro-differential equations of the second kind and solve the resulting equations by any method

5. $\int_0^x (u^2(t)-1)dt + \int_0^x \cos(x-t)u''(t)dt = \frac{1}{2}\sin x(2\sin x - x - 1) - \frac{1}{2}x\cos x,$
$u(0)=1,\ u'(0)=1$

6. $\int_0^x (\sin^2 t - u^2(t))dt + \int_0^x \cos(x-t)u''(t)dt = -\frac{1}{3}x^3 - \frac{1}{2}(x+4)\sin x$
$+2x\cos x,\ u(0)=0,\ u'(0)=2$

7. $\int_0^x e^{x-4t}u^2(t)dt + \int_0^x e^{x-2t}u''(t)dt = 5xe^x,\ u(0)=1,\ u'(0)=2$

8. $\int_0^x (x-t)u^2(t)dt + \int_0^x \cosh(x-t)u''(t)dt = -\frac{1}{4}(1+x^2)$
$+\frac{1}{2}x\sinh x + \frac{1}{4}\cosh^2 x,\ u(0)=0,\ u'(0)=1$

14.4 Systems of Nonlinear Volterra Integro-Differential Equations

In this section, we will study systems of nonlinear Volterra integro-differential equations of the second kind given by

$$\begin{aligned} u^{(i)}(x) &= f_1(x) + \int_0^x \Big(K_1(x,t)F_1(u(t)) + \widetilde{K}_1(x,t)\widetilde{F}_1(v(t))\Big)\,dt, \\ v^{(i)}(x) &= f_2(x) + \int_0^x \Big(K_2(x,t)F_2(u(t)) + \widetilde{K}_2(x,t)\widetilde{F}_2(v(t))\Big)\,dt. \end{aligned} \tag{14.160}$$

The nonlinear functions $F_i, \widetilde{F}_i, i = 1, 2$ of the unknown functions $u(x), v(x)$ occur inside the integral sign whereas the derivatives of $u(x), v(x)$ appear mostly outside the integral sign. The kernels $K_i(x,t)$ and $\widetilde{K}_i(x,t)$, and the functions $f_i(x), i = 1, 2$ are given real-valued functions. To determine the exact solutions for the system (14.160), the initial conditions $u^{(j-1)}(0)$ and $v^{(j-1)}(0), 1 \leqslant j \leqslant i$ should be prescribed.

There is a variety of numerical and analytical methods that are usually used for solving the systems of nonlinear Volterra integro-differential equations (14.160). However, in this section, we will concern ourselves with two methods, namely the variational iteration method and the combined Laplace transform-Adomian method.

14.4.1 The Variational Iteration Method

The variational iteration method (VIM) was used to handle Volterra integral equations and Volterra integro-differential equations. The method provides rapidly convergent successive approximations of the exact solution if such a closed form solution exists, and not components as in Adomian decomposition method. The variational iteration method handles linear and nonlinear problems in the same manner without any need to specific restrictions such as the so called Adomian polynomials that we need to express nonlinear terms.

The correction functionals for the Volterra system of integro-differential equations (14.160) are given by

$$\begin{aligned} u_{n+1}(x) &= u_n(x) + \int_0^x \lambda(t)\left(u_n^{(i)}(t) - f_1(t) - \int_0^t \gamma_1(t,r)dr\right)dt, \\ v_{n+1}(x) &= v_n(x) + \int_0^x \lambda(t)\left(v_n^{(i)}(t) - f_2(t) - \int_0^t \gamma_2(t,r)dr\right)dt. \end{aligned} \tag{14.161}$$

where

$$\begin{aligned} \gamma_1(t,r) &= K_1(t,r)F_1(\tilde{u}_n(r)) + \widetilde{K}_1(t,r)\widetilde{F}_1(\tilde{v}_n(r)), \\ \gamma_2(t,r) &= K_2(t,r)F_2(\tilde{u}_n(r)) + \widetilde{K}_2(t,r)\widetilde{F}_2(\tilde{v}_n(r)). \end{aligned} \tag{14.162}$$

The variational iteration method is used by applying two essential steps. It is required first to determine the Lagrange multiplier λ that can be identified optimally via integration by parts and by using a restricted variation. Having λ determined, an iteration formula, without restricted variation, should be used for the determination of the successive approximations $u_{n+1}(x), n \geqslant 0$ and $v_{n+1}(x), n \geqslant 0$ of the solutions $u(x)$ and $v(x)$. The zeroth approximations $u_0(x)$ and $v_0(x)$ can be any selective functions. However, using the initial conditions are preferably used for the selective zeroth approximations u_0 and $v_0(x)$ as will be seen later. Consequently, the solutions are given by

$$u(x) = \lim_{n\to\infty} u_n(x), \quad v(x) = \lim_{n\to\infty} v_n(x). \tag{14.163}$$

The VIM will be illustrated by studying the following systems of nonlinear Volterra integro-differential equations of the second kind.

Example 14.21

Use the VIM to solve the system of nonlinear Volterra integro-differential equations

$$\begin{aligned} u'(x) &= 1 - x + \frac{1}{2}x^2 - \frac{1}{12}x^4 + \int_0^x ((x-t)u^2(t) + v^2(t))dt, \\ v'(x) &= -1 - x - \frac{3}{2}x^2 - \frac{1}{12}x^4 + \int_0^x (u^2(t) + (x-t)v^2(t))dt, \end{aligned} \tag{14.164}$$

where $u(0) = 1, v(0) = 1$. The correction functionals for this system are

$$
\begin{aligned}
u_{n+1}(x) &= u_n(x) - \int_0^x \left(u_n'(t) - 1 + t - \frac{1}{2}t^2 + \frac{1}{12}t^4 - I_1(t)\right) dt, \\
v_{n+1}(x) &= v_n(x) - \int_0^x \left(v_n'(t) + 1 + t + \frac{3}{2}t^2 + \frac{1}{12}t^4 - I_2(t)\right) dt,
\end{aligned} \tag{14.165}
$$

where

$$
\begin{aligned}
I_1(t) &= \int_0^t ((t-r)u_n^2(r) + v_n^2(r))dr, \\
I_2(t) &= \int_0^t (u_n^2(r) + (t-r)v_n^2(r))dr,
\end{aligned} \tag{14.166}
$$

and $\lambda = -1$ for first order integro-differential equation.

Selecting $u_0(x) = u(0) = 1$ and $v_0(x) = v(0) = 1$ gives the successive approximations

$$
\begin{aligned}
u_0(x) &= 1, \quad v_0(x) = 1, \\
u_1(x) &= 1 + x + \frac{1}{3}x^3 - \frac{1}{6}x^4 - \frac{1}{60}x^5, \\
v_1(x) &= 1 - x - \frac{1}{3}x^3 - \frac{1}{60}x^5, \\
u_2(x) &= 1 + x + \left(\frac{1}{3}x^3 - \frac{1}{3}x^3\right) + \left(\frac{1}{6}x^4 - \frac{1}{6}x^4\right) - \frac{1}{30}x^5 + \cdots, \\
v_2(x) &= 1 - x + \left(\frac{1}{3}x^3 - \frac{1}{3}x^3\right) + \left(\frac{1}{6}x^4 - \frac{1}{6}x^4\right) + \frac{1}{30}x^5 + \cdots, \\
u_3(x) &= 1 + x + \left(\frac{1}{30}x^5 - \frac{1}{30}x^5\right) + \cdots, \\
v_3(x) &= 1 - x + \left(\frac{1}{30}x^5 - \frac{1}{30}x^5\right) + \cdots,
\end{aligned}
$$

and so on. It is obvious that the noise terms appear in each approximation, hence the exact solutions are given by

$$
(u(x), v(x)) = (1 + x, 1 - x). \tag{14.167}
$$

Example 14.22

Use the VIM to solve the system of Volterra integro-differential equations

$$
\begin{aligned}
u'(x) &= e^x - \frac{1}{2}e^{2x} - \frac{1}{6}x^4 + x + \frac{3}{2} + \int_0^x ((x-t)u^2(t) + (x-t)v^2(t))dt, \\
v'(x) &= 7e^x - 4xe^x - 4x - 7 + \int_0^x ((x-t)u^2(t) - (x-t)v^2(t))dt,
\end{aligned} \tag{14.168}
$$

where $u(0) = 1, v(0) = -1$. The correction functionals for this system are given by

$$u_{n+1}(x) = u_n(x) - \int_0^x \left(u_n'(t) - e^t + \frac{1}{2}e^{2t} + \frac{1}{6}t^4 - t - \frac{3}{2} - I_1(t) \right) dt,$$
$$v_{n+1}(x) = v_n(x) - \int_0^x \left(v_n'(t) - 7e^t + 4te^t + 4t + 7 - I_2(t) \right) dt, \qquad (14.169)$$

where

$$I_1(t) = \int_0^t ((t-r)u_n^2(r) + (t-r)v_n^2(r))dr,$$
$$I_2(t) = \int_0^t ((t-r)u_n^2(r) - (t-r)v_n^2(r))dr, \qquad (14.170)$$

and $\lambda = -1$ for first-order integro-differential equations.

We can use the initial conditions to select $u_0(x) = u(0) = 1$ and $v_0(x) = v(0) = -1$. Using this selection into the correction functionals gives the following successive approximations

$$u_0(x) = 1, \quad v_0(x) = -1,$$
$$u_1(x) = 1 + 2x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 - \frac{1}{8}x^4 - \frac{11}{120}x^5 + \cdots,$$
$$v_1(x) = -1 - \frac{1}{2!}x^2 - \frac{1}{3!}x^3 - \frac{5}{24}x^4 - \frac{3}{40}x^5 + \cdots,$$
$$u_2(x) = 1 + 2x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \cdots,$$
$$v_2(x) = -(1 + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \cdots),$$

and so on. The exact solutions are therefore given by

$$(u(x), v(x)) = (x + e^x, x - e^x). \qquad (14.171)$$

Example 14.23

Use the VIM to solve the system of nonlinear Volterra integro-differential equations

$$u''(x) = \cosh x - \frac{1}{2}\sinh^2 x - \frac{1}{6}x^4 - \frac{1}{2}x^2$$
$$+ \int_0^x ((x-t)u^2(t) + (x-t)v^2(t))dt, \; u(0) = 1, \; u'(0) = 1,$$
$$v''(x) = -(1 + 4x)\cosh x + 8\sinh x - 4x$$
$$+ \int_0^x ((x-t)u^2(t) - (x-t)v^2(t))dt, \; v(0) = -1, \; v'(0) = 1. \qquad (14.172)$$

The correction functionals for this system are given by

$$
\begin{aligned}
&u_{n+1}(x) = u_n(x)\\
&+\int_0^x \left((t-x)\left(u_n''(t)-\cosh t+\frac{1}{2}\sinh^2 t+\frac{1}{6}t^4+\frac{1}{2}t^2-\int_0^t I_1(t,r)dr\right)\right)dt,\\
&v_{n+1}(x) = v_n(x) \qquad (14.173)\\
&+\int_0^x \left((t-x)\left(v_n''(t)+(1+4t)\cosh t-8\sinh t+4t-\int_0^t I_2(t,r)dr\right)\right)dt,
\end{aligned}
$$

where

$$
\begin{aligned}
I_1(t,r) &= (t-r)u^2(r)+(t-r)v^2(r),\\
I_2(t,r) &= (t-r)u^2(r)-(t-r)v^2(r).
\end{aligned} \qquad (14.174)
$$

and $\lambda = t - x$ for second-order integro-differential equation.

Using $u_0(x) = 1+x$ and $v_0(x) = -1+x$ gives the successive approximations

$$
\begin{aligned}
u_0(x) &= 1+x, \quad v_0(x) = -1+x,\\
u_1(x) &= 1+x+\frac{1}{2!}x^2+\frac{1}{4!}x^4-\frac{1}{240}x^6+\cdots,\\
v_1(x) &= -1+x-\frac{1}{2!}x^2-\frac{1}{4!}x^4-\frac{1}{720}x^6+\cdots,\\
u_2(x) &= x+\left(1+\frac{1}{2!}x^2+\frac{1}{4!}x^4+\frac{1}{6!}x^6+\cdots\right),\\
v_2(x) &= x-\left(1+\frac{1}{2!}x^2+\frac{1}{4!}x^4+\frac{1}{6!}x^6+\cdots\right),
\end{aligned}
$$

and so on. The exact solutions are therefore given by

$$
(u(x), v(x)) = (x+\cosh x, x-\cosh x). \qquad (14.175)
$$

Example 14.24

Use the variational iteration method to solve the following system

$$
\begin{aligned}
u''(x) &= e^x+\frac{e^{2x}}{2}(x-1)+\frac{e^{4x}}{4}(3x-1)+\frac{3}{4}(x+1)\\
&\quad+\int_0^x ((x-2t)u^2(t)+(x-4t)v^2(t))dt, \quad u(0)=1,\ u'(0)=1,\\
v''(x) &= 4e^{2x}+\frac{e^{4x}}{4}(3x-1)+\frac{e^{6x}}{6}(5x-1)+\frac{5}{12}(x+1)\\
&\quad+\int_0^x ((x-4t)v^2(t)+(x-6t)w^2(t))dt, \quad v(0)=1,\ v'(0)=2,\\
w''(x) &= 9e^{3x}+\frac{e^{2x}}{2}(x-1)+\frac{e^{6x}}{6}(5x-1)+\frac{2}{3}(x+1)\\
&\quad+\int_0^x ((x-6t)w^2(t)+(x-2t)u^2(t))dt, \quad w(0)=1,\ w'(0)=3.
\end{aligned} \qquad (14.176)
$$

We select the zeroth approximations as $u_0(x) = 1+x, v_0(x) = 1+2x$ and $w_0(x) = 1+3x$. Proceeding as before, we obtain

$$u_0(x) = 1 + x, \quad v_0(x) = 1 + 2x, \quad w_0(x) = 1 + 3x,$$
$$u_1(x) = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \cdots,$$
$$v_1(x) = 1 + 2x + \frac{1}{2!}(2x)^2 + \frac{1}{3!}(2x)^3 + \frac{1}{4!}(2x)^4 + \frac{1}{5!}(2x)^5 + \cdots,$$
$$w_1(x) = 1 + 3x + \frac{1}{2!}(3x)^2 + \frac{1}{3!}(3x)^3 + \frac{1}{4!}(3x)^4 + \frac{1}{5!}(3x)^5 + \cdots,$$

The exact solutions are therefore given by

$$(u(x), v(x), w(x)) = (e^x, e^{2x}, e^{3x}). \tag{14.177}$$

Exercises 14.4.1

Use the variational iteration method to solve the following systems of nonlinear Volterra integro-differential equations

1. $$\begin{cases} u'(x) = 2x + \frac{1}{6}x^4 + \frac{2}{15}x^6 + \int_0^x (x-2t)(u^2(t) + v(t))dt \\ v'(x) = -2x - \frac{1}{6}x^4 + \frac{2}{15}x^6 + \int_0^x (x-2t)(u(t) + v^2(t))dt \\ u(0) = 1,\ v(0) = 1 \end{cases}$$

2. $$\begin{cases} u'(x) = 1 + 3x^2 + \frac{1}{3}x^4 + \frac{3}{14}x^8 + \int_0^x (x-2t)(u^2(t) + v^2(t))\ dt \\ v'(x) = 1 - 3x^2 + \frac{8}{15}x^6 + \int_0^x (x-2t)(u^2(t) - v^2(t))dt \\ u(0) = 0,\ v(0) = 0 \end{cases}$$

3. $$\begin{cases} u'(x) = e^x - 2e^{2x} + 2 + \int_0^x e^{x-t}(u^2(t) + v^2(t))dt \\ v'(x) = -e^x - 4xe^x + \int_0^x e^{x-t}(u^2(t) - v^2(t))dt \\ u(0) = 2\ v(0) = 0 \end{cases}$$

4. $$\begin{cases} u'(x) = 1 + \frac{7}{3}\sin x - \frac{2}{3}\sin(2x) - 4x + \int_0^x \cos(x-t)(u^2(t) + v^2(t))dt \\ v'(x) = 1 + (2 - x^2)\sin x - x\cos x + \int_0^x \sin(x-t)(u^2(t) - v^2(t))dt \\ u(0) = 1,\ v(0) = -1 \end{cases}$$

5. $$\begin{cases} u''(x) = 2 - 2x^2 - 2x^3 - \frac{1}{3}x^4 + \int_0^x (x-t+1)(u^2(t) - v^2(t))dt \\ v''(x) = -2 + \frac{2}{3}x^3 + \frac{2}{3}x^4 + \int_0^x (x-2t)(u^2(t) - v^2(t))dt \\ u(0) = u'(0) = 1,\ v(0) = 1,\ v'(0) = -1 \end{cases}$$

6. $$\begin{cases} u''(x) = \frac{5}{3}\sin x + \frac{2}{3}\sin(2x) - 4x + \int_0^x \cos(x-t)(u^2(t)+v^2(t))dt \\ v''(x) = (1-x)\sin x + x^2\cos x + \int_0^x \sin(x-t)(u^2(t)-v^2(t))dt \\ u(0)=0,\ u'(0)=2,\ v(0)=v'(0)=0 \end{cases}$$

7. $$\begin{cases} u''(x) = (1-2x)e^x + \int_0^x e^{x-2t}(u^2(t)+v(t))dt \\ v''(x) = 4e^{2x} - 2xe^x + \int_0^x \left(e^{x-4t}v^2(t) + e^{x-3t}w(t)\right)dt \\ w''(x) = 9e^{3x} - 2xe^x + \int_0^x \left(e^{x-6t}w^2(t) + e^{x-t}u(t)\right)dt \\ u(0)=u'(0)=1,\ v(0)=1,\ v'(0)=2,\ w(0)=1,\ w'(0)=3 \end{cases}$$

8. $$\begin{cases} u''(x) = e^x + \frac{e^{2x}}{2}(x-1) + e^{4x}(3x-1) + \frac{3}{2}(x+1) \\ \qquad + \int_0^x \left((x-2t)u^2(t) + (x-4t)v^2(t)\right)dt \\ v''(x) = 8e^{2x} + e^{4x}(3x-1) + \frac{3e^{6x}}{2}(5x-1) + \frac{5}{2}(x+1) \\ \qquad + \int_0^x \left((x-4t)v^2(t) + (x-6t)w^2(t)\right)dt \\ w''(x) = 27e^{3x} + \frac{e^{2x}}{2}(x-1) + \frac{3e^{6x}}{2}(5x-1) + 2(x+1) \\ \qquad + \int_0^x \left((x-6t)w^2(t) + (x-2t)u^2(t)\right)dt \\ u(0)=u'(0)=1,\ v(0)=1,\ v'(0)=4,\ w(0)=1,\ w'(0)=9 \end{cases}$$

14.4.2 The Combined Laplace Transform-Adomian Decomposition Method

The combined Laplace transform-Adomian decomposition method was used in this chapter to handle nonlinear Volterra integral equations where it worked effectively. We will use the combined Laplace transform-Adomian decomposition method to study systems of nonlinear Volterra integro-differential equations of the second kind

$$\begin{aligned} u^{(i)}(x) &= f_1(x) + \int_0^x (K_1(x,t)F_1(u(t)) + \widetilde{K}_1(x,t)\widetilde{F}_1(v(t)))dt, \\ v^{(i)}(x) &= f_2(x) + \int_0^x (K_2(x,t)F_2(u(t)) + \widetilde{K}_2(x,t)\widetilde{F}_2(v(t)))dt. \end{aligned} \tag{14.178}$$

The nonlinear functions $F_i, \widetilde{F}_i, i = 1,2$ of the unknown functions $u(x), v(x)$ occur inside the integral sign whereas the derivatives of $u(x)$ and $v(x)$ appear mostly outside the integral sign. The kernels $K_i(x,t)$ and $\widetilde{K}_i(x,t)$, and the functions $f_i(x), i=1,2$ are given real-valued functions.

We will consider the kernels $K_i(x,t)$ and $\widetilde{K}_i(x,t)$ as *difference kernels* where each kernel depends on the difference $x-t$. To determine the exact solutions for the system (14.178), the initial conditions $u^{(j-1)}(0)$ and $v^{(j-1)}(0), 1 \leqslant j \leqslant i$ should be prescribed.

To use the combined-Laplace transform-Adomian decomposition method, recall that the Laplace transforms of the derivatives of $u(x)$ are given by

$$\mathcal{L}\{u^{(i)}(x)\} = s^i\mathcal{L}\{u(x)\} - s^{i-1}u(0) - s^{i-2}u'(0) - \cdots - u^{(i-1)}(0). \quad (14.179)$$

Applying the Laplace transforms to both sides of (14.178) gives

$$\begin{aligned} &s^i\mathcal{L}\{u(x)\} - s^{i-1}u(0) - s^{i-2}u'(0) - \cdots - u^{(i-1)}(0)\\ &\quad = \mathcal{L}\{f_1(x)\} + \mathcal{L}\{K_1(x-t) * F_1(u(t)) + \widetilde{K}_1(x-t) * \widetilde{F}_1(v(t))\},\\ &s^i\mathcal{L}\{v(x)\} - s^{i-1}v(0) - s^{i-2}v'(0) - \cdots - v^{(i-1)}(0)\\ &\quad = \mathcal{L}\{f_2(x)\} + \mathcal{L}\{K_2(x-t) * F_2(u(t)) + \widetilde{K}_2(x-t) * \widetilde{F}_2(v(t))\}, \end{aligned} \quad (14.180)$$

or equivalently

$$\begin{aligned} \mathcal{L}\{u(x)\} &= \frac{1}{s}u(0) + \frac{1}{s^2}u'(0) + \cdots + \frac{1}{s^i}u^{(i-1)}(0) + \frac{1}{s^i}\mathcal{L}\{f_1(x)\}\\ &\quad + \frac{1}{s^i}\mathcal{L}\{K_1(x-t) * F_1(u(t)) + \widetilde{K}_1(x-t) * \widetilde{F}_1(v(t))\},\\ \mathcal{L}\{v(x)\} &= \frac{1}{s}v(0) + \frac{1}{s^2}v'(0) + \cdots + \frac{1}{s^i}v^{(i-1)}(0) + \frac{1}{s^i}\mathcal{L}\{f_2(x)\}\\ &\quad + \frac{1}{s^i}\mathcal{L}\{K_2(x-t) * F_2(u(t)) + \widetilde{K}_2(x-t) * \widetilde{F}_2(v(t))\}. \end{aligned} \quad (14.181)$$

To overcome the difficulty of the nonlinear terms $F_i(u(x)), i=1,2$, we apply the Adomian decomposition method for handling (14.181). To achieve this goal, we first represent the linear terms $u(x)$ and $v(x)$ at the left side by an infinite series of components given by

$$u(x) = \sum_{n=0}^{\infty} u_n(x), \quad v(x) = \sum_{n=0}^{\infty} v_n(x), \quad (14.182)$$

and the nonlinear terms $F_i(u(x))$ at the right side of (14.181) by

$$F(u(x)) = \sum_{n=0}^{\infty} A_n(x), \quad A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[F\left(\sum_{i=0}^{n}\lambda^i u_i\right)\right]_{\lambda=0}, n=0,1,2,\cdots, \quad (14.183)$$

where the Adomian polynomials $A_n, n \geqslant 0$ can be obtained for all forms of nonlinearity.

Substituting (14.182) and (14.183) into (14.181) leads to

$$\begin{aligned} \mathcal{L}\left\{\sum_{n=0}^{\infty} u_n(x)\right\} &= \frac{1}{s}u(0) + \frac{1}{s^2}u'(0) + \cdots + \frac{1}{s^i}u^{(i-1)}(0) + \frac{1}{s^i}\mathcal{L}\{f_1(x)\}\\ &\quad + \frac{1}{s^i}\mathcal{L}\{K_1(x-t)\}\mathcal{L}\left\{\sum_{n=0}^{\infty} A_n(x)\right\} \end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{s^i}\mathcal{L}\{\widetilde{K}_1(x-t)\}\mathcal{L}\left\{\sum_{n=0}^{\infty}\widetilde{A}_n(x)\right\},\\
\mathcal{L}\left\{\sum_{n=0}^{\infty}v_n(x)\right\} &= \frac{1}{s}v(0)+\frac{1}{s^2}v'(0)+\cdots+\frac{1}{s^i}v^{(i-1)}(0)+\frac{1}{s^i}\mathcal{L}\{f_2(x)\}\\
&+\frac{1}{s^i}\mathcal{L}\{K_2(x-t)\}\mathcal{L}\left\{\sum_{n=0}^{\infty}B_n(x)\right\}\\
&+\frac{1}{s^i}\mathcal{L}\{\widetilde{K}_2(x-t)\}\mathcal{L}\left\{\sum_{n=0}^{\infty}\tilde{B}_n(x)\right\}.
\end{aligned}
\tag{14.184}
$$

The Adomian decomposition method admits the use of the following recursive relations

$$
\begin{aligned}
\mathcal{L}\{u_0(x)\} &= \frac{1}{s}u(0)+\frac{1}{s^2}u'(0)+\cdots+\frac{1}{s^i}u^{(i-1)}(0)+\frac{1}{s^i}\mathcal{L}\{f_1(x)\},\\
\mathcal{L}\{u_{k+1}(x)\} &= \frac{1}{s^i}\mathcal{L}\{K_1(x-t)\}\mathcal{L}\{A_k(x)\}+\frac{1}{s^i}\mathcal{L}\{\widetilde{K}_1(x-t)\}\mathcal{L}\{\widetilde{A}_k(x)\},
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{L}\{v_0(x)\} &= \frac{1}{s}v(0)+\frac{1}{s^2}v'(0)+\cdots+\frac{1}{s^i}v^{(i-1)}(0)+\frac{1}{s^i}\mathcal{L}\{f_2(x)\},\\
\mathcal{L}\{v_{k+1}(x)\} &= \frac{1}{s^i}\mathcal{L}\{K_2(x-t)\}\mathcal{L}\{B_k(x)\}+\frac{1}{s^i}\mathcal{L}\{\widetilde{K}_2(x-t)\}\mathcal{L}\{\widetilde{B}_k(x)\},
\end{aligned}
\tag{14.185}
$$

for $k \geqslant 0$. The necessary conditions presented in Chapter 1 for Laplace transform method concerning the limit as $s \to \infty$, should be satisfied here for a successful use of this method. Applying the inverse Laplace transform to the first part of (14.185) gives $u_0(x)$ and $v_0(x)$, that will define $A_0, \tilde{A}_0, B_0, \tilde{B}_0$. This in turn will lead to the complete determination of the components of $u_{k+1}, v_{k+1}(x, k \geqslant 0$ upon using the second part of (14.185).

The combined Laplace transform Adomian-decomposition method for solving systems of nonlinear Volterra integro-differential equations of the second kind will be illustrated by studying the following examples.

Example 14.25

Solve the system of nonlinear Volterra integro-differential equation by using the combined Laplace transform-Adomian decomposition method

$$
\begin{aligned}
u'(x) &= e^x - 2e^{2x} + 2 + \int_0^x e^{x-t}\left(u^2(t)+v^2(t)\right)dt,\ u(0)=2,\\
v'(x) &= -e^x - 4xe^x + 2 + \int_0^x e^{x-t}\left(u^2(t)-v^2(t)\right)dt,\ v(0)=0.
\end{aligned}
\tag{14.186}
$$

Notice that the four kernels are $K(x-t)=e^{x-t}$. Taking Laplace transforms of both sides of (14.186) gives

$$\begin{aligned}\mathcal{L}\{u'(x)\} &= \mathcal{L}\{e^x - 2e^{2x} + 2\} + \mathcal{L}\{e^{x-t} * (u^2(x) + v^2(x))\},\\ \mathcal{L}\{v'(x)\} &= \mathcal{L}\{-e^x - 4xe^x\} + \mathcal{L}\{e^{x-t} * (u^2(x) - v^2(x))\},\end{aligned} \quad (14.187)$$

so that

$$\begin{aligned}sU(s) - u(0) &= \frac{1}{s-1} - \frac{2}{s-2} + \frac{2}{s} + \frac{1}{s-1}\mathcal{L}\{u^2(x) + v^2(x)\},\\ sV(s) - v(0) &= -\frac{1}{s-1} - \frac{4}{(s-1)^2} + \frac{1}{s-1}\mathcal{L}\{u^2(x) - v^2(x)\},\end{aligned} \quad (14.188)$$

or equivalently

$$\begin{aligned}U(s) &= \frac{2}{s} + \frac{1}{s(s-1)} - \frac{2}{s(s-2)} + \frac{2}{s^2} + \frac{1}{s(s-1)}\mathcal{L}\{u^2(x) + v^2(x)\},\\ V(s) &= -\frac{1}{s(s-1)} - \frac{4}{s(s-1)^2} + \frac{1}{s(s-1)}\mathcal{L}\{u^2(x) - v^2(x)\}.\end{aligned} \quad (14.189)$$

Substituting the series assumption for $U(s)$ and $V(s)$, and the Adomian polynomials for $u^2(x)$ and $v^2(x)$ as given above in (14.182) and (14.183) respectively, and using the recursive relation (14.185) we obtain

$$\begin{aligned}U_0(s) &= \frac{2}{s} + \frac{1}{s(s-1)} + \frac{2}{s^2} - \frac{2}{s(s-2)},\\ U_{k+1}(s) &= \frac{1}{s(s-1)}\mathcal{L}\{A_k(x) + B_k(x)\},\end{aligned}$$

and

$$\begin{aligned}V_0(s) &= -\frac{1}{s(s-1)} - \frac{4}{s(s-1)^2},\\ V_{k+1}(s) &= \frac{1}{s(s-1)}\mathcal{L}\{A_k(x) - B_k(x)\}.\end{aligned} \quad (14.190)$$

Recall that the Adomian polynomials for $u^2(x)$ and $v^2(x)$ are given by

$$\begin{aligned}A_0(x) &= u_0^2, & B_0(x) &= v_0^2,\\ A_1(x) &= 2u_0u_1, & B_1(x) &= 2v_0v_1,\\ A_2(x) &= 2u_0u_2 + u_1^2, & B_2(x) &= 2v_0v_2 + v_1^2,\\ A_3(x) &= 2u_0u_3 + 2u_1u_2, & B_3(x) &= 2v_0v_3 + 2v_1v_2.\end{aligned} \quad (14.191)$$

Taking the inverse Laplace transform of both sides of (14.190), and using the recursive relation (14.190) gives

$$\begin{aligned}u_0(x) &= 2 + x - \frac{3}{2}x^2 - \frac{7}{6}x^3 - \frac{5}{8}x^4 - \frac{31}{120}x^5 + \cdots,\\ u_1(x) &= 2x^2 + \frac{4}{3}x^3 - \frac{2}{15}x^5 + \cdots,\\ u_2(x) &= \frac{2}{3}x^4 + \frac{2}{5}x^5 + \cdots,\end{aligned} \quad (14.192)$$

and

$$\begin{aligned} v_0(x) &= -x - \frac{5}{2}x^2 - \frac{3}{2}x^3 - \frac{13}{24}x^4 - \frac{17}{120}x^5 + \cdots, \\ v_1(x) &= 2x^2 + \frac{4}{3}x^3 - \frac{1}{6}x^4 - \frac{2}{3}x^5, \cdots, \\ v_2(x) &= \frac{2}{3}x^4 + \frac{4}{5}x^5 + \cdots. \end{aligned} \tag{14.193}$$

Using (14.182) gives the series solutions by

$$\begin{aligned} u(x) &= 1 + \left(1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \cdots\right), \\ v(x) &= 1 - \left(1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \cdots\right). \end{aligned} \tag{14.194}$$

Consequently, the exact solutions are given by

$$(u(x), v(x)) = (1 + e^x, 1 - e^x). \tag{14.195}$$

Example 14.26

Solve the system of nonlinear Volterra integro-differential equation by using the combined Laplace transform-Adomian decomposition method

$$\begin{aligned} u'(x) &= 1 + \frac{7}{3}\sin x - \frac{2}{3}\sin(2x) - 4x + \int_0^x \cos(x-t)(u^2(t) + v^2(t))dt, \\ v'(x) &= 1 - 7\sin x + 4x\cos x + 4x + \int_0^x (x-t)(u^2(t) - v^2(t))dt, \\ & u(0) = 1,\ v(0) = -1. \end{aligned} \tag{14.196}$$

Taking Laplace transforms of both sides of (14.196) gives

$$\begin{aligned} \mathcal{L}\{u'(x)\} &= \mathcal{L}\left\{1 + \frac{7}{3}\sin x - \frac{2}{3}\sin(2x) - 4x\right\} \\ &\quad + \mathcal{L}\{\cos(x-t) * (u^2(x) + v^2(x))\}, \\ \mathcal{L}\{v'(x)\} &= \mathcal{L}\{1 - 7\sin x + 4x\cos x + 4x\} \\ &\quad + \mathcal{L}\{(x-t) * (u^2(x) - v^2(x))\}, \end{aligned} \tag{14.197}$$

so that

$$\begin{aligned} sU(s) - u(0) &= \frac{1}{s} + \frac{7}{3(s^2+1)} - \frac{4}{3(s^2+4)} - \frac{4}{s^2} + \frac{s}{s^2+1}\mathcal{L}\{u^2(x) + v^2(x)\}, \\ sV(s) - v(0) &= \frac{1}{s} - \frac{7}{s^2+1} + \frac{4(s^2-1)}{(s^2+1)^2} + \frac{4}{s^2} + \frac{1}{s^2}\mathcal{L}\{u^2(x) - v^2(x)\}, \end{aligned} \tag{14.198}$$

or equivalently

$$\begin{aligned} U(s) &= \frac{1}{s} + \frac{1}{s^2} + \frac{7}{3s(s^2+1)} - \frac{4}{3s(s^2+4)} - \frac{4}{s^3} + \frac{1}{s^2+1}\mathcal{L}\{u^2(x) + v^2(x)\}, \\ V(s) &= -\frac{1}{s} + \frac{1}{s^2} - \frac{7}{s(s^2+1)} + \frac{4s(s^2-1)}{(s^2+1)^2} + \frac{4}{s^3} + \frac{1}{s^3}\mathcal{L}\{u^2(x) - v^2(x)\}. \end{aligned} \tag{14.199}$$

Substituting the series assumption for $U(s)$ and $V(s)$, and the Adomian polynomials for $u^2(x)$ and $v^2(x)$ as given above in (14.182) and (14.183) respectively, and using the recursive relation (14.185) we obtain

$$\begin{aligned} U_0(s) &= \frac{1}{s} + \frac{1}{s^2} + \frac{7}{3s(s^2+1)} - \frac{4}{3s(s^2+4)} - \frac{4}{s^3}, \\ U_{k+1}(s) &= \frac{1}{s^2+1}\mathcal{L}\{A_k(x) + B_k\}, \end{aligned}$$

and

$$\begin{aligned} V_0(s) &= -\frac{1}{s} + \frac{1}{s^2} - \frac{7}{s(s^2+1)} + \frac{4s(s^2-1)}{(s^2+1)^2} + \frac{4}{s^3}, \\ V_{k+1}(s) &= \frac{1}{s^3}\mathcal{L}\{A_k(x) - B_k\}. \end{aligned} \tag{14.200}$$

Using the Adomian polynomials, and taking the inverse Laplace transform of both sides of the first part of (14.190), and using the recursive relation (14.200) gives

$$\begin{aligned} u_0(x) &= 1 + x - \frac{3}{2}x^2 + \frac{1}{8}x^4 - \frac{19}{720}x^6 + \cdots, \\ u_1(x) &= x^2 - \frac{1}{4}x^4 - \frac{1}{10}x^5 + \frac{41}{360}x^6 + \cdots, \\ u_2(x) &= \frac{1}{6}x^4 + \frac{1}{10}x^5 - \frac{2}{15}x^6 + \cdots, \\ u_3(x) &= \frac{2}{45}x^6 + \cdots, \end{aligned} \tag{14.201}$$

and

$$\begin{aligned} v_0(x) &= -1 + x + \frac{1}{2}x^2 - \frac{5}{24}x^4 + \frac{13}{720}x^6 + \cdots, \\ v_1(x) &= \frac{1}{6}x^4 - \frac{1}{30}x^5 - \frac{1}{30}x^6, \cdots, \\ v_2(x) &= \frac{1}{30}x^5 + \frac{1}{60}x^6. \end{aligned} \tag{14.202}$$

This in turn gives the series solutions by

$$\begin{aligned} u(x) &= x + \left(1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots\right), \\ v(x) &= x - \left(1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots\right). \end{aligned} \tag{14.203}$$

Consequently, the exact solutions are given by

$$(u(x), v(x)) = (x + \cos x, x - \cos x). \tag{14.204}$$

Example 14.27

Solve the system of nonlinear Volterra integro-differential equation by using the combined Laplace transform-Adomian decomposition method

$$
\begin{aligned}
u'(x) &= \cos x - \sin x - 2x + \int_0^x (u^2(t) + v^2(t))dt,\ u(0) = 1,\ v(0) = 1, \\
v'(x) &= -\cos x - \sin x + 2\cos^2 x - 2 + \int_0^x (u^2(t) - v^2(t))dt.
\end{aligned} \tag{14.205}
$$

Taking Laplace transforms of both sides of (14.205) gives

$$
\begin{aligned}
\mathcal{L}\{u'(x)\} &= \mathcal{L}\{\cos x - \sin x - 2x\} + \mathcal{L}\{1 * (u^2(x) + v^2(x))\}, \\
\mathcal{L}\{v'(x)\} &= \mathcal{L}\{-\cos x - \sin x + 2\cos^2 x - 2\} + \mathcal{L}\{1 * (u^2(x) - v^2(x))\},
\end{aligned} \tag{14.206}
$$

so that

$$
\begin{aligned}
U(s) &= \frac{1}{s} + \frac{s-1}{s(s^2+1)} - \frac{2}{s^3} + \frac{1}{s^2}\mathcal{L}\{u^2(x) + v^2(x)\}, \\
V(s) &= \frac{1}{s} - \frac{s+1}{s(s^2+1)} + \frac{2(s^2+2)}{s^2(s^2+4)} - \frac{2}{s^2} + \frac{1}{s^2}\mathcal{L}\{u^2(x) - v^2(x)\}.
\end{aligned} \tag{14.207}
$$

Proceeding as in the previous two examples we find

$$
\begin{aligned}
u_0(x) &= 1 + x - \frac{3}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 + \cdots, \\
u_1(x) &= x^2 - \frac{1}{6}x^4 - \frac{1}{6}x^5 + \cdots, \\
u_2(x) &= \frac{1}{6}x^4 + \frac{1}{6}x^5 + \cdots,
\end{aligned}
$$

and

$$
\begin{aligned}
v_0(x) &= 1 - x - \frac{1}{2}x^2 - \frac{1}{2}x^3 + \frac{1}{24}x^4 + \frac{1}{8}x^5 + \cdots, \\
v_1(x) &= \frac{2}{3}x^3 - \frac{1}{6}x^4 - \frac{1}{6}x^5, \cdots, \\
v_2(x) &= \frac{1}{6}x^4 + \frac{1}{30}x^5 + \cdots.
\end{aligned} \tag{14.208}
$$

Using (14.182) we obtain the series solutions

$$
\begin{aligned}
u(x) &= \left(1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \cdots\right) + \left(x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \cdots\right), \\
v(x) &= \left(1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \cdots\right) - \left(x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \cdots\right).
\end{aligned} \tag{14.209}
$$

Consequently, the exact solutions are given by

$$
(u(x), v(x)) = (\cos x + \sin x, \cos x - \sin x). \tag{14.210}
$$

Example 14.28

Solve the system of nonlinear Volterra integro-differential equation by using the combined Laplace transform-Adomian decomposition method

$$\begin{aligned} u''(x) &= \frac{7}{3}e^x - e^{2x} - \frac{1}{3}e^{4x} + \int_0^x e^{x-t}\left(u^2(t)+v^2(t)\right)dt, \\ v''(x) &= \frac{2}{3}e^x + 3e^{2x} + \frac{1}{3}e^{4x} + \int_0^x e^{x-t}\left(u^2(t)-v^2(t)\right)dt, \\ &u(0)=1,\ u'(0)=1,\ v(0)=1,\ v'(0)=2. \end{aligned} \tag{14.211}$$

Taking Laplace transforms of both sides of (14.211) gives

$$\begin{aligned} \mathcal{L}\{u''(x)\} &= \mathcal{L}\left\{\frac{7}{3}e^x - e^{2x} - \frac{1}{3}e^{4x}\right\} + \mathcal{L}\{e^{x-t} * (u^2(x)+v^2(x))\}, \\ \mathcal{L}\{v''(x)\} &= \mathcal{L}\left\{\frac{2}{3}e^x + 3e^{2x} + \frac{1}{3}e^{4x}\right\} + \mathcal{L}\{e^{x-t} * (u^2(x)-v^2(x))\}, \end{aligned} \tag{14.212}$$

so that

$$\begin{aligned} s^2U(s) - su(0) - u'(0) &= \frac{7}{3(s-1)} - \frac{1}{s-2} \\ &\quad - \frac{1}{3(s-4)} + \frac{1}{s-1}\mathcal{L}\{u^2(x)+v^2(x)\}, \\ s^2V(s) - sv(0) - v'(0) &= \frac{2}{3(s-1)} + \frac{3}{s-2} \\ &\quad + \frac{1}{3(s-4)} + \frac{1}{s-1}\mathcal{L}\{u^2(x)-v^2(x)\}, \end{aligned} \tag{14.213}$$

or equivalently

$$\begin{aligned} U(s) &= \frac{1}{s^2} + \frac{1}{s} + \frac{7}{3s^2(s-1)} - \frac{1}{s^2(s-2)} - \frac{1}{3s^2(s-4)} \\ &\quad + \frac{1}{s^2(s-1)}\mathcal{L}\{u^2(x)+v^2(x)\}, \\ V(s) &= \frac{2}{s^2} + \frac{1}{s} + \frac{2}{3s^2(s-1)} + \frac{3}{s^2(s-2)} + \frac{1}{3s^2(s-4)} \\ &\quad + \frac{1}{s^2(s-1)}\mathcal{L}\{u^2(x)-v^2(x)\}. \end{aligned} \tag{14.214}$$

Proceeding as before we obtain

$$\begin{aligned} u_0(x) &= 1 + x + \frac{1}{2}x^2 - \frac{1}{6}x^3 - \frac{7}{24}x^4 - \frac{9}{40}x^5 + \cdots, \\ u_1(x) &= \frac{1}{3}x^3 + \frac{1}{3}x^4 + \frac{7}{30}x^5 + \cdots, \end{aligned}$$

and

$$\begin{aligned} v_0(x) &= 1 + 2x + 2x^2 + \frac{4}{3}x^3 + \frac{3}{4}x^4 + \frac{23}{60}x^5 + \cdots, \\ v_1(x) &= -\frac{1}{12}x^4 - \frac{7}{60}x^5, \cdots. \end{aligned} \tag{14.215}$$

Using (14.182) gives the series solutions by

$$\begin{aligned} u(x) &= 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \cdots, \\ v(x) &= 1 + 2x + \frac{1}{2!}(2x)^2 + \frac{1}{3!}(2x)^3 + \frac{1}{4!}(2x)^4 + \frac{1}{5!}(2x)^5 + \cdots. \end{aligned} \tag{14.216}$$

Consequently, the exact solutions are given by

$$(u(x), v(x)) = (e^x, e^{2x}). \tag{14.217}$$

Exercises 14.4.2

Use the combined Laplace transform-Adomian decomposition method to solve the following systems of nonlinear Volterra integro-differential equations by

1. $\begin{cases} u'(x) = 2x - x^2 - \frac{1}{15}x^6 + \int_0^x (x-t)(u^2(t) + v^2(t))dt \\ v'(x) = \frac{1}{3}x^4 + \int_0^x (x-t)(u^2(t) - v^2(t))dt \\ u(0) = 1,\ v(0) = 1 \end{cases}$

2. $\begin{cases} u'(x) = 3x^2 - \frac{2}{3}x^3 - \frac{1}{126}x^9 + \int_0^x (x-t)^2(u^2(t) + v^2(t))dt \\ v'(x) = -3x^2 - \frac{1}{35}x^7 + \int_0^x (x-t)^2(u^2(t) - v^2(t))dt \\ u(0) = 1,\ v(0) = 1 \end{cases}$

3. $\begin{cases} u'(x) = -2 - 3x - 2\sin x + 3\cos x + \int_0^x (u^2(t) + v^2(t))dt \\ v'(x) = -2 + \sin x + 2\cos x + \frac{1}{2}\sin(2x) + \int_0^x (u^2(t) - v^2(t))dt \\ u(0) = 1, v(0) = 2 \end{cases}$

4. $\begin{cases} u'(x) = \cos x - 3\sin x + \int_0^x \cos(x-t)(u^2(t) + v^2(t))dt \\ v'(x) = \sin x + \cos x + \frac{1}{2}\sin(2x) - x + \int_0^x (x-t)(u^2(t) - v^2(t))dt \\ u(0) = 1,\ v(0) = -1 \end{cases}$

5. $\begin{cases} u'(x) = \frac{8}{15}e^x + 2e^{2x} - \frac{1}{3}e^{4x} - \frac{1}{5}e^{6x} + \int_0^x e^{x-t}(u^2(t) + v^2(t))dt \\ v'(x) = \frac{2}{15}e^x + 3e^{3x} - \frac{1}{3}e^{4x} + \frac{1}{5}e^{6x} + \int_0^x e^{x-t}(u^2(t) - v^2(t))dt \\ u(0) = 1,\ v(0) = 1 \end{cases}$

6. $\begin{cases} u''(x) = -\sin x - \cos x + \int_0^x \cos(x+t)(u^2(t) + v^2(t))dt \\ v''(x) = -\sin x - \frac{1}{2}\sin(2x) + \int_0^x (u^2(t) - v^2(t))dt \\ u(0) = 1,\ u'(0) = 0,\ v(0) = 0,\ v'(0) = 1 \end{cases}$

7. $$\begin{cases} u''(x) = \frac{3}{2}e^{x} + \frac{1}{2}e^{-x} - e^{2x} + \int_0^x e^{x-t}(u^2(t) + v(t))dt \\ v''(x) = -\frac{1}{3}(1+3x)e^{x} + e^{-x} + \frac{1}{3}e^{-2x} + \int_0^x e^{x-t}(u(t) + v^2(t))dt \\ \quad u(0) = 1,\ u'(0) = 1,\ v(0) = 1,\ v'(0) = -1 \end{cases}$$

8. $$\begin{cases} u'''(x) = -2x - 2x^3 - \frac{2}{5}x^5 + \int_0^x (u^2(t) + v^2(t))dt \\ v'''(x) = -\frac{2}{3}x^3 - \frac{1}{5}x^5 + \int_0^x (x-t)(u^2(t) - v^2(t))dt \\ \quad u(0) = u'(0) = 1,\ u''(0) = 2,\ v(0) = -v'(0) = 1,\ v''(0) = 2 \end{cases}$$

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Chapter 15
Nonlinear Fredholm Integral Equations

15.1 Introduction

It was stated in Chapter 4 that Fredholm integral equations arise in many scientific applications. It was also shown that Fredholm integral equations can be derived from boundary value problems. Erik Ivar Fredholm (1866–1927) is best remembered for his work on integral equations and spectral theory. Fredholm was a Swedish mathematician who established the theory of integral equations and his 1903 paper in *Acta Mathematica* played a major role in the establishment of operator theory. The linear Fredholm integral equations and the linear Fredholm integro-differential equations were presented in Chapters 4 and 6 respectively. It is our goal in this chapter to study the nonlinear Fredholm integral equations of the second kind and systems of nonlinear Fredholm integral equations of the second kind.

The nonlinear Fredholm integral equations of the *second kind* are characterized by fixed limits of integration of the form

$$u(x) = f(x) + \lambda \int_a^b K(x,t)F(u(t))dt. \tag{15.1}$$

where the unknown function $u(x)$ occurs inside and outside the integral sign, λ is a parameter, and a and b are constants. For this type of equations, the kernel $K(x,t)$ and the function $f(x)$ are given real-valued functions, and $F(u(x))$ is a nonlinear function of $u(x)$ such as $u^2(x), \sin(u(x))$, and $e^{u(x)}$.

In this chapter, we will mostly use *degenerate or separable kernels*. A degenerate or a separable kernel is a function that can be expressed as the sum of product of two functions each depends only on one variable. Such a kernel can be expressed in the form

$$K(x,t) = \sum_{i=1}^{n} g_i(x)\, f_i(t). \tag{15.2}$$

Several analytic and numerical methods have been used to handle the nonlinear Fredholm integral equations. In this text we will use the direct computation method, the Adomian decomposition method (ADM) combined with the modified decomposition method (mADM), and the successive substitution method to handle these equations. Systems of nonlinear Fredholm integral equations will be examined as well. For each type of equations we will select the proper methods that facilitate the computational work. The emphasis in this text will be on the use of these methods rather than proving theoretical concepts of convergence and existence. The theorems of existence, uniqueness, and convergence are important and can be found in the literature. The concern will be on the determination of the solutions $u(x)$ of the nonlinear Fredholm integral equations and systems of these equations.

15.2 Existence of the Solution for Nonlinear Fredholm Integral Equations

In this section we will present an existence theorem for the solution of nonlinear Fredholm integral equations. The proof of this theorem can be found in [1–2] among other references. The criteria is similar to the criteria presented in Chapter 13 for nonlinear Volterra integral equations. We first rewrite the nonlinear Fredholm integral equation of the second kind by

$$u(x) = f(x) + \lambda \int_a^b G(x, t, u(t))\, dt. \tag{15.3}$$

The specific conditions under which a solution exists for the nonlinear Fredholm integral equation are:

(i) The function $f(x)$ is bounded, $|f(x)| < R$, in $a \leqslant x \leqslant b$.
(ii) The function $G(x, t, u(t))$ is integrable and bounded where

$$|G(x, t, u(t))| < K, \tag{15.4}$$

in $a \leqslant x, t \leqslant b$.
(iii) The function $G(x, t, u(t))$ satisfies the Lipschitz condition

$$|G(x, t, z) - G(x, t, z')| < M|z - z'|. \tag{15.5}$$

Using the successive approximations method, it is proved in [1] that the series obtained by this method converges uniformly for all values of λ for

$$\lambda < \frac{1}{k(b-a)}, \tag{15.6}$$

where k is the larger of the two numbers $K\left(1 + \frac{R}{|\lambda| K(b-a)}\right)$ and M.

15.2.1 Bifurcation Points and Singular Points

When the nonlinear Fredholm integral equation includes a parameter λ, it is then obvious that the solution of the equation depends on λ. It is possible that λ has a *bifurcation point*. The bifurcation point is a value of the parameter λ, say λ_0, such that when λ changes through λ_0, then the number of real solutions will be changed. To illustrates this phenomenon, we consider the nonlinear Fredholm integral equation

$$u(x) = 3 + \lambda \int_0^1 u^2(t)\,dt. \tag{15.7}$$

The solution of this equation is

$$u(x) = \frac{(1-2\lambda) \pm \sqrt{1-12\lambda}}{2\lambda}. \tag{15.8}$$

The bifurcation point in this problem is $\lambda_0 = \frac{1}{12}$. For $\lambda \leqslant \frac{1}{12}$, the nonlinear Fredholm equation has two real solutions, but has no real solutions for $\lambda > \frac{1}{12}$. This change of λ through the bifurcation point $\lambda_0 = \frac{1}{12}$ caused a change in the number and structure of solutions.

However, for $\lambda = 0$, we obtain $u(x) = 3$ by substituting this value of λ in the integral equation itself, whereas $u(x)$ is undefined by substituting $\lambda = 0$ into (15.8). Accordingly, the point $\lambda = 0$ is called a *singular point*. More examples will be presented in the next section to address the two phenomena.

The following conclusions can be made for nonlinear Fredholm integral equations:

(i) The solution of the nonlinear equation may not be unique, there may be more than one solution.

(ii) Concerning the bifurcation point, there may be one ore more bifurcation points. This will be seen in the forthcoming examples.

15.3 Nonlinear Fredholm Integral Equations of the Second Kind

We begin our study on nonlinear Fredholm integral equations of the second kind of the form

$$u(x) = f(x) + \lambda \int_a^b K(x,t)F(u(t))dt, \tag{15.9}$$

where the kernel $K(x,t)$ and the function $f(x)$ are given real-valued functions, and $F(u(x))$ is a nonlinear function of $u(x)$. The unknown function $u(x)$, that will be determined, occurs inside and outside the integral sign. In what follows we will employ four distinct methods, namely the direct computation method, the series solution method, the Adomian decomposition method, and the successive approximations method to handle Eq. (15.9). Other methods can be found in this text and in the literature.

15.3.1 The Direct Computation Method

In this section, the direct computational method will be applied to solve the nonlinear Fredholm integral equations. The method was used before in Chapters 4 and 6. It approaches nonlinear Fredholm integral equations in a direct manner and gives the solution in an exact form and not in a series form. It is important to point out that this method will be applied for equations where the kernels are degenerate or separable of the form

$$K(x,t) = \sum_{k=1}^{n} g_k(x)h_k(t). \tag{15.10}$$

Examples of separable kernels are $x - t, xt^2, x^3 - t^3, xt^4 + x^4t$, etc.

The direct computation method can be applied as follows:

1. We first substitute (15.10) into the nonlinear Fredholm integral equation

$$u(x) = f(x) + \lambda \int_a^b K(x,t)F(u(t))dt. \tag{15.11}$$

2. This substitution gives

$$\begin{aligned} u(x) &= f(x) \\ &+\lambda g_1(x) \int_a^b h_1(t)\, F(u(t))dt + \lambda g_2(x) \int_a^b h_2(t)\, F(u(t))dt + \cdots \\ &+\lambda g_n(x) \int_a^b h_n(t)\, F(u(t))dt. \end{aligned} \tag{15.12}$$

3. Each integral at the right side of (15.12) depends only on the variable t with constant limits of integration for t. This means that each integral is equivalent to a constant. Based on this, Equation (15.12) becomes

$$u(x) = f(x) + \lambda\alpha_1 g_1(x) + \lambda\alpha_2 g_2(x) + \cdots + \lambda\alpha_n g_n(x), \tag{15.13}$$

where

$$\alpha_i = \int_a^b h_i(t)\, u(t)dt, 1 \leqslant i \leqslant n. \tag{15.14}$$

4. Substituting (15.13) into (15.14) gives a system of n algebraic equations that can be solved to determine the constants $\alpha_i, 1 \leqslant i \leqslant n$. Using the obtained numerical values of α_i into (15.13), the solution $u(x)$ of the nonlinear Fredholm integral equation (15.11) follows immediately.

It is interesting to point out that we may get more than one value for one or more of $\alpha_i, 1 \leqslant i \leqslant n$. This is normal because the equation is nonlinear and the solution $u(x)$ is not necessarily unique for nonlinear problems. Linear Fredholm integral equations give unique solutions under the existence conditions presented above. In what follows we present some examples to illustrate the use of this method.

Example 15.1

Use the direct computation method to solve the nonlinear Fredholm integral equation

$$u(x) = a + \lambda \int_0^1 u^2(t)dt, a > 0. \tag{15.15}$$

The integral at the right side of (15.15) is equivalent to a constant because it depends only on a function of the variable t with constant limits of integration. Consequently, we rewrite (15.15) as

$$u(x) = a + \lambda\alpha, \tag{15.16}$$

where

$$\alpha = \int_0^1 u^2(t)dt. \tag{15.17}$$

To determine α, we substitute (15.16) into (15.17) to obtain

$$\alpha = \int_0^1 (a + \lambda\alpha)^2 dt, \tag{15.18}$$

where by integrating the right side we find

$$\lambda^2\alpha^2 - (1 - 2\lambda a)\alpha + a^2 = 0. \tag{15.19}$$

Solving the quadratic equation (15.19) for α gives

$$\alpha = \frac{(1 - 2a\lambda) \pm \sqrt{1 - 4a\lambda}}{2\lambda^2}. \tag{15.20}$$

Substituting (15.20) into (15.16) leads to the exact solutions:

$$u(x) = \frac{1 \pm \sqrt{1 - 4a\lambda}}{2\lambda}. \tag{15.21}$$

The following conclusions can be made here:

1. Using $\lambda = 0$ into (15.15) gives the exact solution $u(x) = a$. However, $u(x)$ is undefined by using $\lambda = 0$ into (15.21). The point $\lambda = 0$ is called the singular point of Equation (15.15).

2. For $\lambda = \frac{1}{4a}$, Equation (15.21) gives only one solution $u(x) = 2a$. The point $\lambda = \frac{1}{4a}$ is called the *bifurcation point* of the equation. This shows that for $\lambda < \frac{1}{4a}$, then Equation (15.15) gives two real solutions, but has no real solutions for $\lambda > \frac{1}{4a}$.

3. For $\lambda < \frac{1}{4a}$, Equation (15.21) gives two exact real solutions.

Example 15.2

Use the direct computation method to solve the nonlinear Fredholm integral equation

$$u(x) = x + \lambda \int_0^1 xtu^2(t)dt. \tag{15.22}$$

The integral at the right side of (15.22) is equivalent to a constant because it depends only on a function of the variable t with constant limits of integration. Consequently, we rewrite (15.22) as

$$u(x) = (1 + \lambda\alpha)x, \tag{15.23}$$

where

$$\alpha = \int_0^1 tu^2(t)dt. \tag{15.24}$$

To determine α, we substitute (15.23) into (15.24) to obtain

$$\alpha = \int_0^1 (1 + \lambda\alpha)^2 t^3 dt, \tag{15.25}$$

where by integrating the right side and solving the resulting equation for α we obtain

$$\alpha = \frac{(4 - 2\lambda) \pm 4\sqrt{1 - \lambda}}{2\lambda^2}. \tag{15.26}$$

Substituting (15.26) into (15.23) leads to the exact solutions

$$u(x) = \left(\frac{2 \pm 2\sqrt{1 - \lambda}}{\lambda}\right) x. \tag{15.27}$$

We next consider the following three cases:

1. Using $\lambda = 0$ into (15.22) gives the exact solution $u(x) = x$. However, $u(x)$ is undefined by using $\lambda = 0$ into (15.27). Hence, $\lambda = 0$ is called the singular point of Equation (15.22).

2. For $\lambda = 1$, Equation (15.27) gives only one solution $u(x) = 2x$. Therefore, the point $\lambda = 1$ is called the *bifurcation point* of the equation.

3. For $\lambda < 1$, Equation (15.27) gives two exact real solutions. This is normal for nonlinear equations.

Example 15.3

Use the direct computation method to solve the nonlinear Fredholm integral equation

$$u(x) = \frac{\sqrt{3}}{12} + \lambda \int_{-1}^1 xu^2(t)dt. \tag{15.28}$$

Proceeding as before, we rewrite (15.28) as

$$u(x) = \frac{\sqrt{3}}{12} + \lambda\alpha x, \tag{15.29}$$

where

$$\alpha = \int_{-1}^1 u^2(t)dt. \tag{15.30}$$

To determine α, we substitute (15.29) into (15.30) to obtain

$$\alpha = \int_{-1}^{1} \left(\frac{\sqrt{3}}{12} + \lambda\alpha t\right)^2 dt, \tag{15.31}$$

where by integrating the right side and solving the resulting equation for α we obtain

$$\alpha = \frac{6 \pm 2\sqrt{9-\lambda^2}}{8\lambda^2}. \tag{15.32}$$

Substituting (15.32) into (15.29) leads to the exact solutions

$$u(x) = \frac{\sqrt{3}}{12} + \left(\frac{6 \pm 2\sqrt{9-\lambda^2}}{8\lambda}\right) x. \tag{15.33}$$

We consider the following three cases:

1. Using $\lambda = 0$ into (15.28) gives the exact solution $u(x) = \frac{\sqrt{3}}{12}$. However, $u(x)$ is undefined by using $\lambda = 0$ into (15.33). Hence, $\lambda = 0$ is called the singular point of equation (15.28).

2. For $\lambda = \pm 3$, Equation (15.33) gives two solutions $u(x) = \frac{\sqrt{3}}{12} \pm \frac{1}{4}x$. Consequently, there are two *bifurcation points*, namely ± 3 for this equation. This shows that for $-3 < \lambda < 3$, then equation (15.15) gives two real solutions, but has no real solutions for $\lambda > 3$ or $\lambda < -3$.

3. For $-3 < \lambda < 3$, Equation (15.33) gives two exact real solutions.

Example 15.4

Use the direct computation method to solve the nonlinear Fredholm integral equation

$$u(x) = \frac{9}{5}x + \frac{1}{3}\int_{-1}^{1} xt^2u^2(t)dt. \tag{15.34}$$

Proceeding as before, we rewrite (15.34) as

$$u(x) = \left(\frac{9}{5} + \frac{1}{3}\alpha\right) x, \tag{15.35}$$

where

$$\alpha = \int_{-1}^{1} t^2u^2(t)dt. \tag{15.36}$$

To determine α, we substitute (15.35) into (15.36) to obtain

$$\alpha = \int_{-1}^{1} \left(\frac{9}{5} + \frac{1}{3}\alpha\right)^2 t^4 dt, \tag{15.37}$$

where by integrating the right side and solving the resulting equation for α we obtain

$$\alpha = \frac{18}{5}, \frac{81}{10}. \tag{15.38}$$

Substituting (15.38) into (15.35) leads to the two exact solutions

$$u(x) = 3x, \frac{9}{2}x. \tag{15.39}$$

Example 15.5

Use the direct computation method to solve the nonlinear Fredholm integral equation

$$u(x) = \frac{5}{6}x + x\int_0^1 t^2u^3(t)dt. \tag{15.40}$$

Proceeding as before, we rewrite (15.40) as

$$u(x) = \left(\frac{5}{6} + \alpha\right)x, \tag{15.41}$$

where

$$\alpha = \int_0^1 t^2u^3(t)dt. \tag{15.42}$$

To determine α, we substitute (15.41) into (15.42) to obtain

$$\alpha = \int_0^1 \left(\frac{5}{6} + \alpha\right)^3 t^5dt, \tag{15.43}$$

where by integrating the right side and solving the resulting equation for α we obtain

$$\alpha = \frac{1}{6}, -\frac{4}{3} \pm \frac{\sqrt{21}}{2}. \tag{15.44}$$

Substituting (15.44) into (15.41) leads to the three exact solutions

$$u(x) = x, -\frac{1}{2}(1 - \sqrt{21})x, -\frac{1}{2}(1 + \sqrt{21})x. \tag{15.45}$$

Example 15.6

Use the direct computation method to solve the nonlinear Fredholm integral equation

$$u(x) = \frac{11}{35}x + \frac{1}{5}x^2 + \int_{-1}^1 (xt^2 + x^2t)u^2(t)dt. \tag{15.46}$$

Proceeding as before, we rewrite (15.46) as

$$u(x) = \left(\frac{11}{35} + \alpha\right)x + \left(\frac{1}{5} + \beta\right)x^2, \tag{15.47}$$

where

$$\alpha = \int_{-1}^1 t^2u^2(t)dt, \quad \beta = \int_{-1}^1 tu^2(t)dt. \tag{15.48}$$

To determine α and β, we substitute (15.47) into (15.48) and integrate the right side to find

$$\alpha = \frac{2}{5}\left(\frac{11}{35} + \alpha\right)^2 + \frac{2}{7}\left(\frac{1}{5} + \beta\right)^2, \quad \beta = \frac{4}{5}\left(\frac{11}{35} + \alpha\right)\left(\frac{1}{5} + \beta\right). \tag{15.49}$$

Solving (15.49) for α and β we obtain

$$(\alpha, \beta) = \left(\frac{24}{35}, \frac{4}{5}\right), \left(\frac{83}{70}, -\frac{6}{5}\right), \left(\frac{131}{140} \mp \frac{\sqrt{35}}{7}, -\frac{1}{5} \pm \frac{\sqrt{35}}{20}\right). \tag{15.50}$$

Substituting (15.50) into (15.47) leads to the four exact solutions

$$u(x) = x + x^2, \frac{3}{2}x - x^2, \left(\frac{5}{4} \mp \frac{\sqrt{35}}{7}\right) x \pm \frac{\sqrt{35}}{20}x^2. \tag{15.51}$$

Example 15.7

Use the direct computation method to solve the nonlinear Fredholm integral equation

$$u(x) = -\frac{2}{15} - \frac{83}{30}x + \frac{50}{21}x^2 + \int_0^1 (1 + xt + x^2t^2)u^2(t)\,dt. \tag{15.52}$$

We rewrite (15.52) as

$$u(x) = \left(-\frac{2}{15} + \alpha\right) - \left(\frac{83}{30} - \beta\right) x + \left(\frac{50}{21} + \gamma\right) x^2, \tag{15.53}$$

where

$$\alpha = \int_0^1 u^2(t)dt, \quad \beta = \int_0^1 tu^2(t)dt, \quad \gamma = \int_0^1 t^2u^2(t)dt. \tag{15.54}$$

To determine α,β and γ, we substitute (15.53) into (15.54), integrate the right side of the resulting equations, and by solving the resulting system we obtain

$$(\alpha, \beta, \gamma) = \left(\frac{17}{15}, \frac{23}{30}, \frac{13}{21}\right). \tag{15.55}$$

Substituting (15.55) into (15.53) leads to the exact solutions

$$u(x) = 1 - 2x + 3x^2. \tag{15.56}$$

Example 15.8

Use the direct computation method to solve the nonlinear Fredholm integral equation

$$u(x) = \frac{131}{210} - \frac{691}{630}x - \frac{17}{120}x^2 - x^3 + \int_0^1 (1 + x^2t + xt^2)u^2(t)\,dt. \tag{15.57}$$

We rewrite (15.57) as

$$u(x) = \left(\frac{131}{210} + \alpha\right) - \left(\frac{691}{630} - \gamma\right) x - \left(\frac{17}{120} - \beta\right) x^2 - x^3, \tag{15.58}$$

where

$$\alpha = \int_0^1 u^2(t)dt, \quad \beta = \int_0^1 tu^2(t)dt, \quad \gamma = \int_0^1 t^2u^2(t)dt. \tag{15.59}$$

To determine α,β and γ, we substitute (15.58) into (15.59), integrate the right side of the resulting equations, and by solving the resulting system we obtain

$$(\alpha, \beta, \gamma) = \left(\frac{79}{210}, \frac{17}{120}, \frac{61}{630}\right). \tag{15.60}$$

Substituting (15.60) into (15.58) leads to the exact solutions

$$u(x) = 1 - x - x^3. \tag{15.61}$$

Exercises 15.3.1

In Exercises 1–6, use the direct computation method to solve the following nonlinear Fredholm integral equations. Find the *singular point* and the *bifurcation point* for each equation:

1. $u(x) = 4 + \lambda \int_0^1 tu^2(t)dt$
2. $u(x) = 2 + \lambda \int_0^1 t^2u^2(t)dt$
3. $u(x) = 1 + \lambda \int_{-1}^1 t^2u^2(t)dt$
4. $u(x) = \frac{\sqrt{3}}{4} + \lambda \int_{-1}^1 xu^2(t)dt$
5. $u(x) = 3 - \lambda \int_0^1 tu^2(t)dt$
6. $u(x) = \frac{5}{4}x + \lambda \int_0^1 t^2u^2(t)dt$

In Exercises 7–12, use the direct computation method so solve the following nonlinear Fredholm integral equations:

7. $u(x) = 1 - \frac{51}{35}x^2 + \int_0^1 x^2u^3(t)dt$
8. $u(x) = 1 - \frac{17}{3}x + \int_{-1}^1 x(u^2(t) + u^3(t))dt$
9. $u(x) = \frac{14}{15}x - \frac{437}{420}x^2 + \int_0^1 (xt + x^2t^2)(u(t) - u^2(t))dt$
10. $u(x) = \frac{7}{3}\sin x + \int_0^{\pi} \cos(x+t)u^2(t)dt$
11. $u(x) = -\frac{11}{15} + \frac{7}{15}x - \frac{163}{105}x^2 + \int_{-1}^1 (1 + xt + x^2t^2)u^2(t)dt$
12. $u(x) = \frac{1}{3}\cos x - \frac{\pi}{2} + \int_0^{\pi} (1 + \sin(x+t))u^2(t)dt$

15.3.2 The Series Solution Method

In this section, the series solution method will be applied to handle nonlinear Fredholm integral equations. Recall that the generic form of Taylor series at $x = 0$ can be written as

$$u(x) = \sum_{n=0}^{\infty} a_n x^n. \tag{15.62}$$

We will assume that the solution $u(x)$ of the nonlinear Fredholm integral equation

$$u(x) = f(x) + \lambda \int_0^1 K(x,t)F(u(t))dt, \tag{15.63}$$

exists and is analytic, and therefore possesses a Taylor series of the form given in (15.62), where the coefficients a_n will be determined recurrently. Substituting (15.62) into both sides of (15.63) gives

$$\sum_{n=0}^{\infty} a_n x^n = T(f(x)) + \int_0^1 K(x,t)F\left(\sum_{n=0}^{\infty} a_n t^n\right) dt, \qquad (15.64)$$

or for simplicity we use

$$a_0+a_1x+a_2x^2+\cdots = T(f(x))+\int_0^1 K(x,t)F(a_0+a_1t+a_2t^2+\cdots)dt, \quad (15.65)$$

where $T(f(x))$ is the Taylor series for $f(x)$. The integral equation (15.63) will be converted to a traditional integral in (15.64) or (15.65), where instead of integrating the nonlinear term $F(u(x))$, terms of the form t^n, $n \geqslant 0$ will be integrated. Notice that because we are seeking series solution, then if $f(x)$ includes elementary functions such as trigonometric functions, exponential functions, etc., then Taylor expansions for functions involved in $f(x)$ should be used.

Proceeding as in previous chapters, we first integrate the right side of the integral in (15.64) or (15.65), and collect the coefficients of like powers of x. We next equate the coefficients of like powers of x in both sides of the resulting equation to obtain a recurrence relation in $a_j, j \geqslant 0$. Solving the recurrence relation will lead to a complete determination of the coefficients $a_j, j \geqslant 0$. Having determined the coefficients $a_j, j \geqslant 0$, the series solution follows immediately upon substituting the derived coefficients into (15.62). The exact solution may be obtained if such an exact solution exists. If an exact solution is not obtainable, then the obtained series can be used for numerical purposes. In this case, the more terms we determine, the higher accuracy level we achieve. Recall that the series solution method works effectively if the solution $u(x)$ is a polynomial. However, if $u(x)$ is not a polynomial, then approximations to the coefficients $a_j, j \geqslant 0$ will be used.

Example 15.9

Solve the nonlinear Fredholm integral equation by using the series solution method

$$u(x) = 1 + \frac{7159}{7560}x + \frac{2309}{2160}x^2 + \frac{1}{36}\int_0^1 (xt^2 - x^2t)u^2(t)dt. \qquad (15.66)$$

Using the series form (15.62) into both sides of (15.66) gives

$$\begin{aligned} a_0 + a_1x + a_2x^2 + \cdots = 1 + \frac{7159}{7560}x + \frac{2309}{2160}x^2 \\ +\frac{1}{36}\int_0^1 (xt^2 - x^2t)\left(a_0 + a_1t + a_2t^2 + \cdots\right)^2 dt, \end{aligned} \qquad (15.67)$$

where by integrating the right side, collecting the like powers of x, and equating the coefficients of like powers of x in both sides yields

$$\begin{array}{ll} a_0 = 1, 1, & a_1 = 1, 3611.190273, \\ a_2 = 1, -4848.332424, & a_n = 0, \text{ for } n \geqslant 3. \end{array} \qquad (15.68)$$

The exact solutions are given by

$$u(x) = 1 + x + x^2, 1 + 3611.190273x - 4848.332424x^2. \tag{15.69}$$

Example 15.10

Solve the nonlinear Fredholm integral equation by using the series solution method

$$u(x) = 2 - \frac{41}{45}x + \frac{76}{945}x^2 - x^3 + \frac{1}{48}\int_{-1}^{1}(xt - x^2t^2)u^2(t)dt. \tag{15.70}$$

Using the series form (15.62) into both sides of (15.70) gives

$$\begin{aligned} a_0 + a_1x + a_2x^2 + \cdots &= 2 - \frac{41}{45}x + \frac{76}{945}x^2 - x^3 \\ &+ \frac{1}{48}\int_{-1}^{1}(xt - x^2t^2)\left(a_0 + a_1t + a_2t^2 + \cdots\right)^2 dt, \end{aligned} \tag{15.71}$$

where by integrating the right side, collecting the like powers of x, and equating the coefficients of like powers of x in both sides yields

$$\begin{array}{ll} a_0 = 2, 2, & a_1 = -1, 0.2924553739, \\ a_2 = 0, -173.6222619, & a_3 = -1, -1, \\ a_n = 0, \text{ for } n \geqslant 4. & \end{array} \tag{15.72}$$

The exact solutions are given by

$$u(x) = 2 - x - x^2, 2 + 0.2924553739x - 173.6222619x^2 - x^3. \tag{15.73}$$

Example 15.11

Solve the nonlinear Fredholm integral equation by using the series solution method

$$u(x) = e^x + \frac{1}{16}(3 - e^2) + \frac{1}{4}\int_0^1 (x - t)u^2(t)dt. \tag{15.74}$$

Substituting the series (15.62) into both sides of (15.74) gives

$$\begin{aligned} & a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots \\ &= e^x + \frac{1}{16}(3 - e^2) + \frac{1}{4}\int_0^1 (x - t)\left(a_0 + a_1t + a_2t^2 + \cdots\right)^2 dt. \end{aligned} \tag{15.75}$$

Proceeding as before we find

$$\begin{array}{l} a_0 = 1, \quad a_1 = 1, \quad a_2 = \dfrac{1}{2!}, \\ a_3 = \dfrac{1}{3!}, a_4 = \dfrac{1}{4!}, \quad \cdots, \quad a_n = \dfrac{1}{n!}. \end{array} \tag{15.76}$$

This gives the solution in a series form

$$u(x) = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots. \tag{15.77}$$

Consequently, the exact solution is given by

$$u(x) = e^x. \tag{15.78}$$

Example 15.12

We next consider the nonlinear Fredholm integral equation

$$u(x) = \cos x - \sin x - \frac{\pi}{60} + \frac{\pi}{120}(1+\pi) + \frac{1}{60}\int_0^{\pi}(x-t)u^2(t)dt. \quad (15.79)$$

Substituting the series (15.62) into both sides of (15.79) gives

$$\begin{aligned} & a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots \\ &= \cos x - \sin x - \frac{\pi}{60} + \frac{\pi}{120}(1+\pi) \\ & +\frac{1}{60}\int_0^{\pi}(x-t)\left(a_0 + a_1 t + a_2 t^2 + \cdots\right)^2 dt. \end{aligned} \quad (15.80)$$

Proceeding as before we find

$$\begin{aligned} & a_0 = 1, \quad a_1 = -1, \, a_2 = -\frac{1}{2!}, \\ & a_3 = \frac{1}{3!}, \, a_4 = \frac{1}{4!}, \quad a_5 = -\frac{1}{5!}, \cdots , \end{aligned} \quad (15.81)$$

and so on. The solution in a series form is given by

$$u(x) = \left(1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \cdots\right) - \left(x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \cdots\right), \quad (15.82)$$

that converges to the exact solution

$$u(x) = \cos x - \sin x. \quad (15.83)$$

Exercises 15.3.2

Solve the following nonlinear Fredholm integral equations by using the series solution method

1. $u(x) = \frac{241}{240} - \frac{1091}{1080}x - x^2 + \frac{1}{36}\int_0^1 (x-t)u^2(t)dt$

2. $u(x) = \frac{7537}{7560} - \frac{119}{120}x - \frac{1091}{1080}x^2 + \frac{1}{36}\int_0^1 (x-t)^2 u^2(t)dt$

3. $u(x) = 1 - \frac{23}{945}x + \frac{223}{216}x^2 + \frac{1}{36}\int_0^1 (xt^2 - x^2 t)u^2(t)dt$

4. $u(x) = 1 - \frac{23}{630}x + x^2 + \frac{1}{48}\int_{-1}^1 (xt^2 - x^2 t)u^2(t)dt$

5. $u(x) = \frac{2357}{2310} + \frac{892}{945}x - x^2 - x^3 + \frac{1}{18}\int_0^1 (x-t)u^3(t)dt$

6. $u(x) = e^x + \frac{1}{16}(1-e^2)x + \frac{1}{32}(1+e^2) + \frac{1}{8}\int_0^1 (x-t)u^2(t)dt$

7. $u(x) = e^x - \frac{1}{32}(1+e^2)x + \frac{1}{8}\int_0^1 xtu^2(t)dt$

8. $u(x) = \cos x - \frac{\pi}{112} + \frac{1}{56}\int_0^1 u^2(t)dt$

15.3.3 The Adomian Decomposition Method

The Adomian decomposition method has been outlined before in previous chapters and has been applied to a wide class of linear Fredholm integral equations and linear Fredholm integro-differential equations. The method usually decomposes the unknown function $u(x)$ into an infinite sum of components that will be determined recursively through iterations as discussed before. The Adomian decomposition method will be applied in this chapter to handle nonlinear Fredholm integral equations.

Although the linear term $u(x)$ is represented by an infinite sum of components, the nonlinear terms such as $u^2, u^3, u^4, \sin u, e^u$, etc. that appear in the equation, should be expressed by a special representation, called the Adomian polynomials $A_n, n \geqslant 0$. Adomian introduced a formal algorithm to establish a reliable representation for all forms of nonlinear terms. The Adomian polynomials were introduced in Chapter 13.

In what follows we present a brief outline for using the Adomian decomposition method for solving the nonlinear Fredholm integral equation

$$u(x) = f(x) + \lambda \int_a^b K(x,t)F(u(t))dt, \tag{15.84}$$

where $F(u(t))$ is a nonlinear function of $u(x)$. The nonlinear Fredholm integral equation (15.84) contains the linear term $u(x)$ and the nonlinear function $F(u(x))$. The linear term $u(x)$ of (15.84) can be represented normally by the decomposition series

$$u(x) = \sum_{n=0}^{\infty} u_n(x), \tag{15.85}$$

where the components $u_n(x)$, $n \geqslant 0$ can be easily computed in a recursive manner as discussed before. However, the nonlinear term $F(u(x))$ of (15.84) should be represented by the so-called Adomian polynomials A_n by using the algorithm

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[F\left(\sum_{i=0}^{n} \lambda^i u_i\right)\right]_{\lambda=0}, n = 0, 1, 2, \cdots. \tag{15.86}$$

Substituting (15.85) and (15.86) into (15.84) gives

$$\sum_{n=0}^{\infty} u_n(x) = f(x) + \lambda \int_a^b K(x,t)\left(\sum_{n=0}^{\infty} A_n(t)\right) dt. \tag{15.87}$$

To determine the components $u_0(x), u_1(x), \ldots$, we use the following recurrence relation

$$\begin{aligned} u_0(x) &= f(x), \\ u_1(x) &= \lambda \int_a^b K(x,t) A_0(t) dt, \\ u_2(x) &= \lambda \int_a^b K(x,t) A_1(t) dt, \\ &\vdots \\ u_{n+1}(x) &= \lambda \int_a^b K(x,t) A_n(t) dt, n \geqslant 0. \end{aligned} \tag{15.88}$$

Recall that in the modified decomposition method, a modified recurrence relation is usually used, where $f(x)$ is decomposed into two components $f_1(x)$ and $f_2(x)$, such that $f(x) = f_1(x) + f_2(x)$. In this case the modified recurrence relation becomes in the form

$$\begin{aligned} u_0(x) &= f_1(x), \\ u_1(x) &= f_2(x) + \lambda \int_a^b K(x,t) A_0(t) dt, \\ u_2(x) &= \lambda \int_a^b K(x,t) A_1(t) dt, \\ &\vdots \\ u_{n+1}(x) &= \lambda \int_a^b K(x,t) A_n(t) dt, n \geqslant 0. \end{aligned} \tag{15.89}$$

Having determined the components, the solution in a series form is readily obtained. The obtained series solution may converge to the exact solution if such a solution exists, otherwise the series can be used for numerical purposes.

The following remarks can be observed:

(i) The convergence of the decomposition method has been examined in the literature by many authors.

(ii) The decomposition method always gives one solution, although the solution of the nonlinear Fredholm equation is not unique. The decomposition method does not address the existence and uniqueness concepts.

(iii) The modified decomposition method and the noise terms phenomenon can be used to accelerate the convergence of the solution.

Generally speaking, the Adomian decomposition method is reliable and effective to handle differential and integral equations. This will be illustrated by using the following examples.

Example 15.13

Use the Adomian decomposition method to solve the nonlinear Fredholm integral equation

$$u(x) = a + \lambda \int_0^1 u^2(t)dt, a > 0. \tag{15.90}$$

The Adomian polynomials for the nonlinear term $u^2(x)$ are given by

$$\begin{aligned} A_0(x) &= u_0^2(x), \\ A_1(x) &= 2u_0(x)u_1(x), \\ A_2(x) &= 2u_0(x)u_2(x) + u_1^2(x), \end{aligned} \tag{15.91}$$

and so on. Substituting the series (15.85) and the Adomian polynomials (15.91) into the left side and the right side of (15.90) respectively we find

$$\sum_{n=0}^{\infty} u_n(x) = a + \lambda \int_0^1 \sum_{n=0}^{\infty} A_n(t)dt. \tag{15.92}$$

Using the Adomian decomposition method we set

$$u_0(x) = a, \quad u_{k+1}(x) = \lambda \int_0^1 A_k(t)dt, k \geqslant 0. \tag{15.93}$$

This in turn gives

$$\begin{aligned} u_0(x) &= a, \\ u_1(x) &= \lambda \int_0^1 u_0^2(t)dt = \lambda a^2, \\ u_2(x) &= \lambda \int_0^1 (2u_0(t)u_1(t))dt = 2\lambda^2 a^3, \\ u_3(x) &= \lambda \int_0^1 (2u_0(t)u_2(t) + u_1^2(t))dt = 5\lambda^3 a^4, \\ u_4(x) &= \lambda \int_0^1 (2u_0(t)u_3(t) + 2u_1(t)u_2(t))dt = 14\lambda^4 a^5, \\ &\vdots \end{aligned} \tag{15.94}$$

The solution in a series form is given by

$$u(x) = a + \lambda a^2 + 2\lambda^2 a^3 + 5\lambda^3 a^4 + 14\lambda^4 a^5 + \cdots, \tag{15.95}$$

that converges to the exact solution

$$u(x) = \frac{1 - \sqrt{1 - 4a\lambda}}{2\lambda}, 0 < \lambda \leqslant \frac{1}{4a}. \tag{15.96}$$

It is clear that only one solution was obtained by using the Adomian decomposition method. However, by using the direct computation method we obtained two solutions for this nonlinear problem as shown in Example 1 that was presented before.

Example 15.14

Use the Adomian decomposition method to solve the nonlinear Fredholm integral equation

$$u(x) = 1 + \lambda \int_0^1 (1 - u(t) + u^2(t))dt. \tag{15.97}$$

This equation can be rewritten as

$$u(x) = 1 + \lambda + \lambda \int_0^1 (-u(t) + u^2(t))dt. \tag{15.98}$$

Substituting the series assumption for $u(x)$ and the Adomian polynomials for $u^2(x)$ into the left side and the right side of (15.98) we find

$$\sum_{n=0}^{\infty} u_n(x) = 1 + \lambda + \lambda \int_0^1 \left(-\sum_{n=0}^{\infty} u_n(t) + \sum_{n=0}^{\infty} A_n(t) \right) dt. \tag{15.99}$$

Using the Adomian decomposition method we set

$$u_0(x) = 1 + \lambda, \quad u_{k+1}(x) = \lambda \int_0^1 (-u_k(t) + A_k(t))dt, k \geqslant 0. \tag{15.100}$$

This in turn gives

$$\begin{aligned}
u_0(x) &= 1 + \lambda, \\
u_1(x) &= \lambda \int_0^1 (-u_0(t) + u_0^2(t))dt = \lambda^2 + \lambda^3, \\
u_2(x) &= \lambda \int_0^1 (-u_1(t) + 2u_0(t)u_1(t))dt = \lambda^3 + 3\lambda^4 + 2\lambda^5, \\
u_3(x) &= \lambda \int_0^1 (-u_2(t) + 2u_0(t)u_2(t) + u_1^2(t))dt = \lambda^4 + 6\lambda^5 + 10\lambda^6 + 5\lambda^7, \\
u_4(x) &= \lambda \int_0^1 (-u_3(t) + 2u_0(t)u_3(t) + 2u_1(t)u_2(t))dt = \lambda^5 + 10\lambda^6 + \cdots, \\
&\vdots
\end{aligned} \tag{15.101}$$

The solution in a series form is given by

$$u(x) = 1 + \lambda + \lambda^2 + 2\lambda^3 + 4\lambda^4 + 9\lambda^5 + \cdots, \tag{15.102}$$

that converges to the exact solution

$$u(x) = \frac{1 + \lambda - \sqrt{(1-3\lambda)(1+\lambda)}}{2\lambda}. \tag{15.103}$$

It is clear again that only one solution was obtained by using the Adomian decomposition method. However, by using the direct computation method we can easily derive two solutions given by

$$u(x) = \frac{1 + \lambda \pm \sqrt{(1-3\lambda)(1+\lambda)}}{2\lambda}. \tag{15.104}$$

It is worth noting that two bifurcation points appear at $\lambda = -1$ and $\lambda = \frac{1}{3}$.

Example 15.15

Use the modified Adomian decomposition method to solve the nonlinear Fredholm integral equation

$$u(x) = e^x + \frac{1 - e^{x+3}}{x+3} + \int_0^1 e^{xt} u^3(t)dt. \tag{15.105}$$

Substituting the series (15.85) and the Adomian polynomials (15.86) into the left side and the right side of (15.105) respectively we find

$$\sum_{n=0}^{\infty} u_n(x) = e^x + \frac{1-e^{x+3}}{x+3} + \int_0^1 e^{xt} \sum_{n=0}^{\infty} A_n(t)dt, \tag{15.106}$$

where A_n are the Adomian polynomials for $u^2(x)$ as shown above. Using the modified decomposition method we set

$$\begin{aligned} u_0(x) &= e^x, \\ u_1(x) &= \frac{1-e^{x+3}}{x+3} + \int_0^1 e^{xt} A_0(t)dt, \\ &= \frac{1-e^{x+3}}{x+3} + \int_0^1 e^{xt} u_0^3(t)dt = 0, \\ u_{k+1}(x) &= \int_0^1 e^{xt} A_k(t)dt = 0, k \geqslant 1. \end{aligned} \tag{15.107}$$

This in turn gives the exact solution

$$u(x) = e^x, \tag{15.108}$$

that satisfies the integral equation. It is worth noting that we did not use the Adomian polynomials. This is due to the fact that the modified decomposition method accelerates the convergence of the solution.

Example 15.16

Use the modified Adomian decomposition method to solve the nonlinear Fredholm integral equation

$$u(x) = \sec x - 2x + \int_0^1 2x(u^2(t) - \tan^2(t))dt. \tag{15.109}$$

Substituting the series (15.85) and the Adomian polynomials (15.86) into the left side and the right side of (15.109) respectively we find

$$\sum_{n=0}^{\infty} u_n(x) = \sec x - 2x + \int_0^1 xt \left(\sum_{n=0}^{\infty} A_n(t) - \tan^2(t) \right) dt. \tag{15.110}$$

Using the modified decomposition method we set

$$\begin{aligned} u_0(x) &= \sec x, \\ u_1(x) &= -2x + \int_0^1 xt(A_0(t) - \tan^2(t))dt = 0. \end{aligned} \tag{15.111}$$

Consequently, the exact solution is given by

$$u(x) = \sec x. \tag{15.112}$$

Exercises 15.3.3

In Exercises 1–6, solve the following nonlinear Fredholm integral equations by using the Adomian decomposition method

1. $u(x) = 4 + \lambda \int_0^1 tu^2(t)dt$
2. $u(x) = 2 + \lambda \int_0^1 t^2u^2(t)dt$
3. $u(x) = 1 + \lambda \int_{-1}^1 t^2u^2(t)dt$
4. $u(x) = \frac{\sqrt{3}}{4} + \lambda \int_{-1}^1 xu^2(t)dt$
5. $u(x) = 3 - \lambda \int_0^1 tu^2(t)dt$
6. $u(x) = \frac{5}{4}x + \lambda \int_0^1 t^2u^2(t)dt$

In Exercises 7–12, solve the nonlinear Fredholm integral equations by using the modified Adomian decomposition method

7. $u(x) = \tan x - \frac{\pi}{2} + \int_0^{\pi} \frac{1}{1+u^2(t)}dt$
8. $u(x) = x - 1 + \int_0^{\sqrt{e-1}} \frac{2t}{1+u^2(t)}dt$
9. $u(x) = \sec x + \left(\frac{\pi}{4} - 1\right)x + \int_0^{\frac{\pi}{4}} x(u^2(t) - 1)dt$
10. $u(x) = \cosh x + \frac{1}{2} - x + \int_0^1 (x-t)(u^2 - \sinh^2 t)dt$
11. $u(x) = \ln x - 2x + \frac{1}{4} + \int_0^1 (x-t)u^2(t)dt$
12. $u(x) = \ln x - 4x + 1 + \int_0^1 (x-t)(1 - u(t) + u^2(t))dt$

15.3.4 The Successive Approximations Method

The successive approximations method or the Picard iteration method was introduced before in Chapter 3. The method provides a scheme that can be used for solving initial value problems or integral equations. This method solves any problem by finding successive approximations to the solution by starting with an initial guess as $u_0(x)$, called the zeroth approximation. As will be seen, the zeroth approximation is any selective real-valued function that will be used in a recurrence relation to determine the other approximations. The most commonly used values for the zeroth approximations are $0, 1$, or x. Of course, other real values can be selected as well.

Given Fredholm integral equation of the second kind

$$u(x) = f(x) + \lambda \int_a^b K(x,t)F(u(t))dt, \tag{15.113}$$

where $u(x)$ is the unknown function to be determined, $K(x,t)$ is the kernel, $F(u(t))$ is a nonlinear function of $u(t)$, and λ is a parameter. The successive approximations method introduces the recurrence relation

$$\begin{aligned} u_0(x) &= \text{ any selective real valued function,} \\ u_{n+1}(x) &= f(x) + \lambda \int_a^b K(x,t)u_n(t)dt, n \geqslant 0. \end{aligned} \tag{15.114}$$

The question of convergence of $u_n(x)$ for linear equation is justified by Theorem 1 presented in Chapter 3. However, for nonlinear Fredholm integral equation, it was proved in [1] that

$$\lambda < \frac{1}{k(b-1)}, \tag{15.115}$$

where k is the larger of the two numbers $K\left(1 + \frac{R}{|\lambda|K(b-a)}\right)$ and M as shown above. At the limit, the solution is determined by using the limit

$$u(x) = \lim_{n\to\infty} u_{n+1}(x). \tag{15.116}$$

The successive approximations method, or the Picard iteration method will be illustrated by studying the following examples.

Example 15.17

Use the successive approximations method to solve the nonlinear Fredholm integral equation

$$u(x) = \cos x - \frac{\pi^2}{48} + \frac{1}{12}\int_0^\pi tu^2(t)dt. \tag{15.117}$$

For the zeroth approximation $u_0(x)$, we can select

$$u_0(x) = 1. \tag{15.118}$$

The method of successive approximations admits the use of the iteration formula

$$u_{n+1}(x) = \cos x - \frac{\pi^2}{48} + \frac{1}{12}\int_0^\pi tu_n^2(t)dt, n \geqslant 0. \tag{15.119}$$

Substituting (15.118) into (15.119) we obtain

$$\begin{aligned} u_1(x) &= \cos x - \frac{\pi^2}{48} + \frac{1}{12}\int_0^\pi tu_0^2(t)dt = \cos(x) + 0.2056167584, \\ u_2(x) &= \cos x - \frac{\pi^2}{48} + \frac{1}{12}\int_0^\pi tu_1^2(t)dt = \cos(x) - 0.05115268549, \\ u_3(x) &= \cos x - \frac{\pi^2}{48} + \frac{1}{12}\int_0^\pi tu_2^2(t)dt = \cos(x) + 0.01812692764, \\ u_4(x) &= \cos x - \frac{\pi^2}{48} + \frac{1}{12}\int_0^\pi tu_3^2(t)dt = \cos(x) - 0.005907183842, \\ u_5(x) &= \cos x - \frac{\pi^2}{48} + \frac{1}{12}\int_0^\pi tu_4^2(t)dt = \cos(x) + 0.001983411200, \\ &\vdots \end{aligned} \tag{15.120}$$

and so on. Consequently, the solution $u(x)$ of (15.117)

$$u(x) = \lim_{n\to\infty} u_{n+1}(x) = \cos x. \tag{15.121}$$

Unlike the direct computation method where we may get more than a solution, the successive substitution method gives only one solution for the nonlinear Fredholm problem. However, using the direct computation method gives the two solutions

$$u(x) = \cos x, \cos x + \frac{32}{\pi^2}. \tag{15.122}$$

Example 15.18

Use the successive approximations method to solve the nonlinear Fredholm integral equation

$$u(x) = e^x - \frac{x}{192}(e^2+1) + \frac{1}{48}\int_0^1 xtu^2(t)dt. \tag{15.123}$$

For the zeroth approximation $u_0(x)$, we can select

$$u_0(x) = 1. \tag{15.124}$$

The method of successive approximations admits the use of the iteration formula

$$u_{n+1}(x) = e^x - \frac{x}{192}(e^2+1) + \frac{1}{48}\int_0^1 xtu_n^2(t)dt, n \geqslant 0. \tag{15.125}$$

Substituting (15.124) into (15.125) we obtain

$$\begin{aligned}
u_1(x) &= e^x - \frac{x}{192}(e^2+1) + \frac{1}{48}\int_0^1 xtu_0^2(t)dt = e^x - 0.03327633383x,\\
u_2(x) &= e^x - \frac{x}{192}(e^2+1) + \frac{1}{48}\int_0^1 xtu_1^2(t)dt = e^x - 0.0009901404827x,\\
u_3(x) &= e^x - \frac{x}{192}(e^2+1) + \frac{1}{48}\int_0^1 xtu_2^2(t)dt = e^x - 0.00002962822377x,\\
u_4(x) &= e^x - \frac{x}{192}(e^2+1) + \frac{1}{48}\int_0^1 xtu_3^2(t)dt = e^x - 0.000000886721005x,\\
&\vdots,
\end{aligned} \tag{15.126}$$

and so on. Consequently, the solution $u(x)$ of (15.123) is given by

$$u(x) = \lim_{n\to\infty} u_{n+1}(x) = e^x. \tag{15.127}$$

It is worth noting that the direct computation method gives an additional solution to this equation given by

$$u(x) = e^x + (208 - 8e)x. \tag{15.128}$$

Example 15.19

Use the successive approximations method to solve the nonlinear Fredholm integral equation

$$u(x) = \sin x + 1 - \frac{\pi}{12} - \frac{5\pi^2}{144} + \frac{1}{36}\int_0^{\pi} t(u(t) + u^2(t))dt. \qquad (15.129)$$

For the zeroth approximation $u_0(x)$, we can select

$$u_0(x) = 1. \qquad (15.130)$$

The method of successive approximations admits the use of the iteration formula

$$u_{n+1}(x) = \sin x + 1 - \frac{\pi}{12} - \frac{5\pi^2}{144} + \frac{1}{36}\int_0^{\pi} t(u_n(t) + u_n^2(t))dt, n \geqslant 0. \qquad (15.131)$$

Substituting (15.130) into (15.131), and proceeding as before we obtain the approximations

$$\begin{aligned} u_1(x) &= \sin x + 0.6696616927, \\ u_2(x) &= \sin x + 0.8214573046, \\ u_3(x) &= \sin x + 0.8997853785, \\ u_4(x) &= \sin x + 0.9426743063, \\ u_5(x) &= \sin x + 0.9668710030, \end{aligned} \qquad (15.132)$$

and so on. Consequently, the solution $u(x)$ of (15.129) is given by

$$u(x) = \lim_{n\to\infty} u_{n+1}(x) = 1 + \sin x. \qquad (15.133)$$

The direct computation method gives an additional solution to this equation given by

$$u(x) = \sin x - 2 - \frac{4}{\pi}\left(1 - \frac{18}{\pi}\right). \qquad (15.134)$$

Example 15.20

Use the successive approximations method to solve the nonlinear Fredholm integral equation

$$u(x) = \ln x + \frac{143}{144} + \frac{1}{36}\int_0^1 tu^2(t)dt. \qquad (15.135)$$

For the zeroth approximation $u_0(x)$, we can select

$$u_0(x) = 1. \qquad (15.136)$$

The method of successive approximations admits the use of the iteration formula

$$u_{n+1}(x) = \ln x + \frac{143}{144} + \frac{1}{36}\int_0^1 tu_n^2(t)dt, n \geqslant 0. \qquad (15.137)$$

Substituting (15.136) into (15.137), and proceeding as before we obtain the approximations

$$\begin{aligned}
u_1(x) &= \ln x + 1.006944444, \\
u_2(x) &= \ln x + 1.000097120, \\
u_3(x) &= \ln x + 1.000001349, \\
u_4(x) &= \ln x + 1.000000019, \\
u_5(x) &= \ln x + 1.000000000, \\
&\vdots
\end{aligned} \tag{15.138}$$

and so on. Consequently, the solution $u(x)$ of (15.135)

$$u(x) = \lim_{n\to\infty} u_{n+1}(x) = 1 + \ln x. \tag{15.139}$$

It is worth noting that the direct computation method gives an additional solution to this equation given by

$$u(x) = 72 + \ln x. \tag{15.140}$$

Exercises 15.3.4

Use the successive approximations method to solve the following nonlinear Fredholm integral equations:

1. $u(x) = \sin x - \frac{\pi^2}{64} + \frac{1}{48}\int_0^{\pi} t(1+u^2(t))dt$

2. $u(x) = \sin x + 1 - \frac{\pi}{16}\left(1+\frac{5\pi}{12}\right) + \frac{1}{48}\int_0^{\pi} t(u+u^2(t))dt$

3. $u(x) = \cos x + \frac{7}{6} - \frac{5\pi^2}{144} + \frac{1}{36}\int_0^{\pi} t(u+u^2(t))dt$

4. $u(x) = e^x + \frac{1}{144}(127-e^2) + \frac{1}{36}\int_0^{1} t(u+u^2(t))dt$

5. $u(x) = e^x + \frac{1}{144}(131-e^2) + \frac{1}{36}\int_0^{1} t(1+u^2(t))dt$

6. $u(x) = xe^x - \frac{1}{288}(3+e^2)x + \frac{1}{36}\int_0^{1} xtu^2(t)dt$

7. $u(x) = e^x + \frac{1}{384}(1-e^2)x + \frac{1}{96}\int_0^{1} xt^2u^2(t)dt$

8. $u(x) = e^x + \frac{1}{288}(1-e^3) + \frac{1}{96}\int_0^{1} u^3(t)dt$

9. $u(x) = \cos x - \frac{1}{144}x + \frac{1}{96}\int_0^{\frac{\pi}{2}} xu^3(t)dt$

10. $u(x) = \ln x - \frac{1}{18}x + \frac{1}{144} + \frac{1}{36}\int_0^{1} (x-t)u^2(t)dt$

11. $u(x) = x\ln x + \frac{10279}{10368}x + \frac{1}{36}\int_0^{1} tu^2(t)dt$

12. $u(x) = x + \ln x - \frac{5}{648} + \frac{1}{36}\int_0^1 tu^2(t)dt$

15.4 Homogeneous Nonlinear Fredholm Integral Equations

Substituting $f(x) = 0$ into the nonlinear Fredholm integral equation of the second kind

$$u(x) = f(x) + \lambda \int_a^b K(x,t)F(u(t))dt, \tag{15.141}$$

gives the homogeneous nonlinear Fredholm integral equation of the second kind given by

$$u(x) = \lambda \int_a^b K(x,t)F(u(t))dt. \tag{15.142}$$

In this section we will focus our study on the homogeneous nonlinear Fredholm integral equation (15.142) for the specific case where the kernel $K(x,t)$ is separable. The main goal for studying the homogeneous nonlinear Fredholm equation is to find nontrivial solution, because the trivial solution $u(x) = 0$ is a solution of this equation. Moreover, the Adomian decomposition method is not applicable here because it depends mainly on assigning a non-zero value for the zeroth component $u_0(x)$, and $f(x) = 0$ in this kind of equations. The direct computation method will be appropriate to be employed here to handle this kind of equations.

15.4.1 The Direct Computation Method

The direct computation method was used before in this chapter. This method replaces the homogeneous nonlinear Fredholm integral equations by a single algebraic equation or by a system of simultaneous algebraic equations depending on the number of terms of the separable kernel $K(x,t)$.

As stated before, the direct computation method handles Fredholm integral equations, homogeneous or nonhomogeneous, in a direct manner and gives the solution in an exact form and not in a series form.

The direct computation method will be applied in this section for the degenerate or separable kernels of the form

$$K(x,t) = \sum_{k=1}^{n} g_k(x)h_k(t). \tag{15.143}$$

The direct computation method can be applied as follows:

1. We first substitute (15.143) into the homogeneous nonlinear Fredholm integral equation the form

$$u(x) = \lambda \int_a^b K(x,t)\, F(u(t))dt. \tag{15.144}$$

2. This substitution leads to

$$u(x) = \lambda g_1(x) \int_a^b h_1(t)\, F(u(t))dt + \lambda g_2(x) \int_a^b h_2(t)\, F(u(t))dt + \cdots$$
$$+\lambda g_n(x) \int_a^b h_n(t)\, F(u(t))dt. \tag{15.145}$$

3. Each integral at the right side depends only on the variable t with constant limits of integration for t. This means that each integral is equivalent to a constant. Consequently, Equation (15.145) becomes

$$u(x) = \lambda\alpha_1 g_1(x) + \lambda\alpha_2 g_2(x) + \cdots + \lambda\alpha_n g_n(x), \tag{15.146}$$

where

$$\alpha_i = \int_a^b h_i(t)\, F(u(t))dt, 1 \leqslant i \leqslant n. \tag{15.147}$$

4. Substituting (15.146) into (15.147) gives a system of n simultaneous algebraic equations that can be solved to determine the constants $\alpha_i, 1 \leqslant i \leqslant n$. Using the obtained numerical values of α_i into (15.146), the solution $u(x)$ of the homogeneous nonlinear Fredholm integral equation (15.142) follows immediately.

Example 15.21

Solve the homogeneous nonlinear Fredholm integral equation by using the direct computation method

$$u(x) = \lambda \int_0^{\frac{\pi}{2}} \cos x \sin t u^2(t)dt. \tag{15.148}$$

This equation can be rewritten as

$$u(x) = \alpha\lambda \cos x, \tag{15.149}$$

where

$$\alpha = \int_0^{\frac{\pi}{2}} \sin t u^2(t)dt. \tag{15.150}$$

Substituting (15.149) into (15.150) gives

$$\alpha = \alpha^2\lambda^2 \int_0^{\frac{\pi}{2}} \sin t \cos^2 t dt, \tag{15.151}$$

that gives

$$\alpha = \frac{1}{3}\alpha^2\lambda^2. \tag{15.152}$$

Recall that $\alpha = 0$ gives the trivial solution. For $\alpha \neq 0$, we find that

$$\alpha = \frac{3}{\lambda^2}. \tag{15.153}$$

This in turn gives the eigenfunction $u(x)$ by

$$u(x) = \frac{3}{\lambda}\cos x. \tag{15.154}$$

Notice that $\lambda = 0$ is a singular point.

Example 15.22

Solve the homogeneous nonlinear Fredholm integral equation by using the direct computation method

$$u(x) = \lambda \int_0^1 e^{x-t} u^2(t)dt. \tag{15.155}$$

This equation can be rewritten as

$$u(x) = \alpha\lambda e^x, \tag{15.156}$$

where

$$\alpha = \int_0^1 e^{-t} u^2(t)dt. \tag{15.157}$$

Substituting (15.156) into (15.157) gives

$$\alpha = \alpha^2\lambda^2 \int_0^1 e^t dt, \tag{15.158}$$

that gives

$$\alpha = \alpha^2\lambda^2(e-1). \tag{15.159}$$

Recall that $\alpha = 0$ gives the trivial solution. For $\alpha \neq 0$, we find that

$$\alpha = \frac{1}{\lambda^2(e-1)}. \tag{15.160}$$

This in turn gives the eigenfunction $u(x)$ by

$$u(x) = \frac{1}{\lambda(e-1)}e^x. \tag{15.161}$$

Example 15.23

Solve the homogeneous nonlinear Fredholm integral equation by using the direct computation method

$$u(x) = \lambda \int_0^{\frac{\pi}{2}} \sin(x-2t)u^2(t)dt. \tag{15.162}$$

Notice that the kernel $\sin(x-2t) = \sin x\cos 2t - \cos x \sin 2t$ is separable. Equation (15.162) can be rewritten as

$$u(x) = \alpha\lambda\sin x - \beta\lambda\cos x, \tag{15.163}$$

where

$$\alpha = \int_0^{\frac{\pi}{2}} \cos 2t u^2(t)dt, \quad \beta = \int_0^{\frac{\pi}{2}} \sin 2t u^2(t)dt. \tag{15.164}$$

Substituting (15.163) into (15.164) and proceeding as before we obtain

$$\alpha = -\frac{\lambda^2\pi}{8}(\alpha^2 - \beta^2), \quad \beta = -\frac{\lambda^2}{4}(\alpha\beta\pi - 2\alpha^2 - 2\beta^2). \tag{15.165}$$

Solving this system for α and β gives

$$\alpha = -\frac{8\pi}{\lambda^2(\pi^2 - 16)}, \quad \beta = -\frac{32}{\lambda^2(\pi^2 - 16)}. \tag{15.166}$$

This in turn gives the eigenfunction $u(x)$ by

$$u(x) = -\frac{8}{\lambda(\pi^2 - 16)}(\pi \sin x - 4\cos x). \tag{15.167}$$

Example 15.24

Solve the homogeneous nonlinear Fredholm integral equation by using the direct computation method

$$u(x) = \lambda \int_{-1}^{1} (xt + t^2)(u(t) + u^2(t))dt. \tag{15.168}$$

Equation (15.168) can be rewritten as

$$u(x) = \alpha\lambda x + \beta\lambda, \tag{15.169}$$

where

$$\alpha = \int_{-1}^{1} t(u(t) + u^2(t))dt, \quad \beta = \int_{0}^{1} t^2(u(t) + u^2(t))dt. \tag{15.170}$$

Proceeding as before we find

$$\alpha = 0, \sqrt{\frac{5}{3}}\left(\frac{2\lambda - 3}{4\lambda^2}\right), \quad \beta = -\frac{2\lambda - 3}{2\lambda^2}, -\frac{2\lambda - 3}{4\lambda^2}. \tag{15.171}$$

This in turn gives the exact solutions

$$u(x) = -\frac{2\lambda - 3}{2\lambda}, \frac{(2\lambda - 3)(\sqrt{15}x - 3)}{12\lambda}. \tag{15.172}$$

Exercises 15.4.1

Use the direct computation method to solve the homogeneous nonlinear Fredholm integral equations

1. $u(x) = \lambda \int_0^{\frac{\pi}{2}} \sin x \cos t u^2(t)dt$
2. $u(x) = \lambda \int_0^{\frac{\pi}{2}} \sin x \cos t(u(t) + u^2(t))dt$
3. $u(x) = \lambda \int_0^{\pi} \cos x \cos t(t + u^2(t))dt$
4. $u(x) = \lambda \int_0^{1} e^{x-t}(u(t) + u^2(t))dt$
5. $u(x) = \lambda \int_0^{1} e^{x-2t}u^3(t)dt$
6. $u(x) = \lambda \int_0^{1} e^{x-2t}(3t^2 + e^{-2t}u^2(t))dt$
7. $u(x) = \lambda \int_0^{\frac{\pi}{2}} \cos(x - 2t)u^2(t)dt$
8. $u(x) = \lambda \int_0^{\frac{\pi}{2}} \sin(x + t)u^2(t)dt$
9. $u(x) = \lambda \int_0^{\pi} \cos(x + 2t)(1 + u^2(t))dt$
10. $u(x) = \lambda \int_{-1}^{1} (xt + t^2)u^2(t)dt$

11. $u(x) = \lambda \int_{-1}^{1} (xt + x^2t^2)u^2(t)dt$
12. $u(x) = \lambda \int_{0}^{1} (xt - xt^2)u^2(t)dt$

15.5 Nonlinear Fredholm Integral Equations of the First Kind

The standard form of the nonlinear Fredholm integral equations of the first kind is given by

$$f(x) = \int_a^b K(x,t)F(u(t))dt, \qquad (15.173)$$

where the kernel $K(x,t)$ and the function $f(x)$ are given real-valued functions, and $F(u(x))$ is a nonlinear function of $u(x)$. The linear Fredholm integral equation of the first kind is presented in Chapter 4 where the homotopy perturbation method was used for handling this type of equations.

To determine a solution for the nonlinear Fredholm integral equation of the first kind (15.173), we first convert it to a linear Fredholm integral equation of the first kind of the form

$$f(x) = \int_a^b K(x,t)v(t)dt, x \in D \qquad (15.174)$$

by using the transformation

$$v(x) = F(u(x)). \qquad (15.175)$$

We assume that $F(u(x))$ is invertible, then we can set

$$u(x) = F^{-1}(v(x)). \qquad (15.176)$$

The linear Fredholm integral equation of the first kind has been investigated in Chapter 4. An important remark has been reported in [3] and other references concerning the data function $f(x)$. The function $f(x)$ must lie in the range of the kernel $K(x,t)$ [3]. For example, if we set the kernel by

$$K(x,t) = e^x \sin t. \qquad (15.177)$$

Then if we substitute any integrable function $F(u(x))$ in (15.173), and we evaluate the integral, the resulting $f(x)$ must clearly be a multiple of e^x [3]. This means that if $f(x)$ is not a multiple of the x component of the kernel, then a solution for (15.173) does not exist. This necessary condition on $f(x)$ can be generalized. In other words, the data function $f(x)$ must contain components which are matched by the corresponding x components of the kernel $K(x,t)$

Nonlinear Fredholm integral equation of the first kind is considered ill-posed problem because it does not satisfy the following three properties:

1. Existence of a solution.
2. Uniqueness of a solution.

3. Continuous dependence of the solution $u(x)$ on the data $f(x)$. This property means that small errors in the data $f(x)$ should cause small errors [4] in the solution $u(x)$. The three properties were postulated by Hadamard [5]. Any problem that satisfies the three aforementioned properties is called well-posed problem. For any ill-posed problem, a very small change on the data $f(x)$ can give a large change in the solution $u(x)$. This means that nonlinear Fredholm integral equation of the first kind may lead to a lot of difficulties.

Several methods have been used to handle the linear and the nonlinear Fredholm integral equations of the first kind. The Legendre wavelets, the augmented Galerkin method, and the collocation method are examples of the methods used to handle this equation. The methods that we used so far in this text cannot handle this kind of equations independently if it is expressed in its standard form (15.174).

However, in this text, we will first apply the method of *regularization* that received a considerable amount of interest, especially in solving first order integral equations. We will second apply the homotopy perturbation method [6] to handle specific cases of the Fredholm integral equations where the kernel $K(x,t)$ is separable.

In what follows we will present a brief summary of the method of regularization and the homotopy perturbation method that will be used to handle the Fredholm integral equations of the first kind.

15.5.1 The Method of Regularization

The method of regularization was established independently by Phillips [7] and Tikhonov [8]. The method of regularization consists of replacing ill-posed problem by well-posed problem. The method of regularization transforms the linear Fredholm integral equation of the first kind

$$f(x) = \int_a^b K(x,t)v(t)dt, x \in D, \tag{15.178}$$

to the approximation Fredholm integral equation

$$\mu v_\mu(x) = f(x) - \int_a^b K(x,t)v_\mu(t)dt, x \in D, \tag{15.179}$$

where μ is a small positive parameter. It is clear that (15.179) is a Fredholm integral equation of the second kind that can be rewritten

$$v_\mu(x) = \frac{1}{\mu}f(x) - \frac{1}{\mu}\int_a^b K(x,t)v_\mu(t)\,dt, x \in D. \tag{15.180}$$

Moreover, it was proved in [1,3,9] that the solution v_μ of Equation (15.180) converges to the solution $v(x)$ of (15.178) as $\mu \to 0$ according to the following Lemma [10]:

Lemma 15.1

Suppose that the integral operator of (15.178) is continuous and coercive in the Hilbert space where $f(x), u(x)$, and $v_\mu(x)$ are defined, then:

1. $|v_\mu|$ *is bounded independently of* μ, *and*
2. $|v_\mu(x) - v(x)| \to 0$ *when* $\mu \to 0$.

The proof of this lemma can be found in [3, 9–10].

In summary, by combining the method of regulariztion with any of the methods used before for solving Fredholm integral equation of the second kind, we can solve Fredholm integral equation of the first kind (15.178). The method of regulariztion transforms the first kind equation to a second kind equation. The resulting integral equation (15.180) can be solved by any method that was presented before in this chapter. The exact solution $v(x)$ of (15.178) can thus be obtained by

$$v(x) = \lim_{\mu \to 0} v_\mu(x). \tag{15.181}$$

In what follows we will present four illustrative examples where we will use the method of regulariztion to transform the first kind integral equation to a second kind integral equation. The resulting equation will be solved by any appropriate method that we used before.

Example 15.25

Combine the method of regulariztion and the direct computation method to solve the nonlinear Fredholm integral equation of the first kind

$$e^x = \int_0^{\frac{1}{2}} 2e^{x-4t} u^2(t) dt. \tag{15.182}$$

We first set

$$v(x) = u^2(x), u(x) = \pm\sqrt{v(x)}, \tag{15.183}$$

to carry out (15.182) into

$$e^x = \int_0^{\frac{1}{2}} 2e^{x-4t} v(t) dt. \tag{15.184}$$

Using the method of regularization, Equation (15.184) can be transformed to

$$v_\mu(x) = \frac{1}{\mu} e^x - \frac{1}{\mu} \int_0^{\frac{1}{2}} 2e^{x-4t} v_\mu(t)\, dt. \tag{15.185}$$

To use the direct computation method, Equation (15.185) can be written as

$$v_\mu(x) = \left(\frac{1}{\mu} - \frac{\alpha}{\mu} \right) e^x, \tag{15.186}$$

where

$$\alpha = \int_0^{\frac{1}{2}} 2e^{-4t} v_\mu(t)\, dt. \tag{15.187}$$

To determine α, we substitute (15.186) into (15.187) to find

$$\alpha = \left(\frac{1}{\mu} - \frac{\alpha}{\mu}\right) \int_0^{\frac{1}{2}} 2e^{-3t}\, dt. \tag{15.188}$$

Integrating the right side and solve to find that

$$\alpha = \frac{2(1 - e^{\frac{1}{2}})}{2 - (2+\mu)e^{\frac{1}{2}}}. \tag{15.189}$$

This in turn gives

$$v_\mu(x) = \frac{1}{\mu}\left(1 - \frac{2(1 - e^{\frac{1}{2}})}{2 - (2+\mu)e^{\frac{1}{2}}}\right) e^x. \tag{15.190}$$

The exact solution $v(x)$ of (15.185) can be obtained by

$$v(x) = \lim_{\mu \to 0} v_\mu(x) = \frac{e^{x+\frac{1}{2}}}{2(e^{\frac{1}{2}} - 1)}. \tag{15.191}$$

Using (15.183) gives the exact solution of (15.182) by

$$u(x) = \pm\sqrt{\frac{e^{x+\frac{1}{2}}}{2(e^{\frac{1}{2}} - 1)}}. \tag{15.192}$$

Two more solutions to Equation (15.182) are given by

$$u(x) = \pm e^{2x}. \tag{15.193}$$

Example 15.26

Combine the method of regulariztion and the direct computation method to solve the nonlinear Fredholm integral equation of the first kind

$$\frac{\pi}{2}\sin x = \int_0^{\pi} \sin(x-t) u^2(t) dt. \tag{15.194}$$

We first set

$$v(x) = u^2(x), u(x) = \pm\sqrt{v(x)}, \tag{15.195}$$

to carry out (15.194) into

$$\frac{\pi}{2}\sin x = \int_0^{\pi} \sin(x-t) v(t) dt. \tag{15.196}$$

Using the method of regularization, Equation (15.196) can be transformed to

$$v_\mu(x) = \frac{\pi}{2\mu}\sin x - \frac{1}{\mu}\int_0^{\pi} \sin(x-t) v_\mu(t)\, dt. \tag{15.197}$$

The resulting Fredholm integral equation of the second kind will be solved by the direct computation method. Equation (15.197) can be written as

$$v_\mu(x) = \left(\frac{\pi}{2\mu} - \frac{\alpha}{\mu}\right)\sin x + \frac{\beta}{\mu}\cos x, \tag{15.198}$$

where

$$\alpha = \int_0^\pi \cos t\, v_\mu(t)\, dt, \quad \beta = \int_0^{\frac{1}{2}} \sin t\, v_\mu(t)\, dt. \tag{15.199}$$

To determine α and β, we substitute (15.198) into (15.199), integrate the resulting integral and solve to find that

$$\alpha = \frac{\pi^3}{2(\pi^2 + 4\mu^2)}, \quad \beta = \frac{\pi^2 \mu}{\pi^2 + 4\mu^2}. \tag{15.200}$$

Substituting this result into (15.198) gives the approximate solution

$$v_\mu(x) = \frac{2\pi\mu}{\pi^2 + 4\mu^2} \sin x + \frac{\pi^2}{\pi^2 + 4\mu^2} \cos x. \tag{15.201}$$

The exact solution $v(x)$ of (15.197) can be obtained by

$$v(x) = \lim_{\mu \to 0} v_\mu(x) = \cos x. \tag{15.202}$$

Using (15.195) gives the exact solution of (15.194) by

$$u(x) = \pm\sqrt{\cos x}. \tag{15.203}$$

Example 15.27

Combine the method of regulariztion and the Adomian decomposition method to solve the nonlinear Fredholm integral equation of the first kind

$$\frac{64}{15}x = \int_{-1}^{1} xtu^4(t)dt. \tag{15.204}$$

We first set

$$v(x) = u^4(x), u(x) = \pm\sqrt[4]{v(x)}, \tag{15.205}$$

to carry out (15.204) into

$$\frac{64}{15}x = \int_{-1}^{1} xtv(t)dt. \tag{15.206}$$

Using the method of regularization, Equation (15.206) can be transformed to

$$v_\mu(x) = \frac{64}{15\mu}x - \frac{1}{\mu}\int_{-1}^{1} xtv_\mu(t)\, dt. \tag{15.207}$$

The resulting Fredholm integral equation of the second kind will be solved by the Adomian decomposition method, where we first set

$$v_\mu(x) = \sum_{n=0}^{\infty} v_{\mu_n}(x), \tag{15.208}$$

and the recurrence relation

$$v_{\mu_0}(x) = \frac{64}{15\mu}x, \quad v_{\mu_{k+1}}(x) = -\frac{1}{\mu}\int_{-1}^{1} xtv_{\mu_k}(t)\, dt, k \geqslant 0. \tag{15.209}$$

This in turn gives the components

$$\begin{aligned} v_{\mu_0}(x) &= \frac{64}{15\mu}x, \qquad v_{\mu_1}(x) = -\frac{128}{45\mu^2}x, \\ v_{\mu_2}(x) &= \frac{256}{135\mu^3}x, \quad v_{\mu_3}(x) = -\frac{512}{405\mu^4}x, \end{aligned} \tag{15.210}$$

and so on. Substituting this result into (15.208) gives the approximate solution

$$v_\mu(x) = \frac{64}{5(2+3\mu)}x. \tag{15.211}$$

The exact solution $v(x)$ of (15.207) can be obtained by

$$v(x) = \lim_{\mu\to 0} v_\mu(x) = \frac{32}{5}x. \tag{15.212}$$

Using (15.205) gives the exact solution

$$u(x) = \pm\sqrt[4]{\frac{32}{5}x}. \tag{15.213}$$

It is interesting to point out that there are two more solutions to this equation given by

$$u(x) = \pm(1+x). \tag{15.214}$$

Example 15.28

Combine the method of regulariztion and the successive approximations method to solve the nonlinear Fredholm integral equation of the first kind

$$\frac{1}{5}x = \int_0^1 xtu^3(t)dt. \tag{15.215}$$

We first set

$$v(x) = u^3(x), u(x) = \pm\sqrt[3]{v(x)}, \tag{15.216}$$

to carry out (15.215) into

$$\frac{1}{5}x = \int_0^1 xtv(t)dt. \tag{15.217}$$

Using the method of regularization, Equation (15.215) can be transformed to

$$v_\mu(x) = \frac{1}{5\mu}x - \frac{1}{\mu}\int_0^1 xt\, v_\mu(t)\, dt. \tag{15.218}$$

To use the successive approximations method, we first select $u_{\mu_0}(x) = 0$. Consequently, we obtain the following approximations

$$\begin{aligned} v_{\mu_0}(x) &= 0, \quad v_{\mu_1}(x) = \frac{1}{5\mu}x, \\ v_{\mu_2}(x) &= \frac{1}{5\mu}x - \frac{1}{15\mu^2}x, \\ v_{\mu_3}(x) &= \frac{1}{5\mu}x - \frac{1}{15\mu^2}x + \frac{1}{45\mu^3}x, \\ v_{\mu_4}(x) &= \frac{1}{5\mu}x - \frac{1}{15\mu^2}x + \frac{1}{45\mu^3}x - \frac{1}{135\mu^4}x, \end{aligned} \tag{15.219}$$

and so on. Based on this we obtain the approximate solution

$$v_\mu(x) = \frac{3}{5(1+3\mu)}x. \tag{15.220}$$

The exact solution $v(x)$ of (15.218) can be obtained by

$$v(x) = \lim_{\mu \to 0} u_\mu(x) = \frac{3}{5}x. \tag{15.221}$$

Using (15.216) the exact solution is given by

$$u(x) = \sqrt[3]{\frac{3}{5}x}. \tag{15.222}$$

Another solution to this equation is given by

$$u(x) = x. \tag{15.223}$$

Exercises 15.5.1

Combine the regularization method with any method to solve the nonlinear Fredholm integral equations of the first kind

1. $e^{2x} = \int_0^1 e^{2x-3t}u^3(t)dt$
2. $\frac{1}{3}e^{-x} = \int_0^{\frac{1}{3}} e^{t-x}u^3(t)dt$
3. $-\frac{10}{27}x^2 = \int_0^1 x^2t^2u^3(t)dt$
4. $\frac{1}{32}x = \int_0^1 xtu^2(t)dt$
5. $\frac{29}{36}x = \int_0^1 xtu^2(t)dt$
6. $\frac{32}{63}x^2 = \int_{-1}^1 x^2t^2u^2(t)dt$
7. $\frac{466}{315}x^2 = \int_{-1}^1 x^2t^2u^2(t)dt$
8. $\frac{8}{35}x = \int_{-1}^1 xtu^2(t)dt$
9. $\frac{\pi}{2}\cos x = \int_0^\pi \cos(x-t)u^2(t)dt$
10. $\frac{\pi}{2}\sin x = \int_0^\pi \cos(x-t)u^2(t)dt$
11. $-\frac{\pi}{2}\cos x = \int_0^\pi \sin(x-t)u^2(t)dt$
12. $\frac{22}{3}x - \frac{709}{720} = \int_0^1 \cos(x-t)u^3(t)dt$

15.5.2 The Homotopy Perturbation Method

In what follows we present the homotopy perturbation method for handling the nonlinear Fredholm integral equations of the first kind of the form

$$f(x) = \int_a^b K(x,t)v(t)dt. \tag{15.224}$$

We first define the operator

$$L(u) = f(x) - \int_a^b K(x,t)u(t)dt = 0. \tag{15.225}$$

We next construct a convex homotopy of the form

$$H(u,p) = (1-p)u(x) + pL(u)(x) = 0. \tag{15.226}$$

The embedding parameter p monotonically increases from 0 to 1. The homotopy perturbation method admits the use of the expansion

$$u = \sum_{n=0}^{\infty} p^n u_n, \tag{15.227}$$

and consequently

$$v(x) = \lim_{p \to 1} \sum_{n=0}^{\infty} p^n u_n(x). \tag{15.228}$$

The series (15.228) converges to the exact solution if such a solution exists.

Substituting (15.227) into (15.226), and proceeding as in Chapter 4 we obtain the recurrence relation

$$\begin{aligned} u_0(x) &= 0, \quad u_1(x) = f(x), \\ u_{n+1}(x) &= u_n(x) - \int_a^b K(x,t) u_n(t) dt, n \geqslant 1. \end{aligned} \tag{15.229}$$

If the kernel is separable, i.e. $K(x,t) = g(x)h(t)$, then the following condition

$$\left| 1 - \int_a^b K(t,t) dt \right| < 1, \tag{15.230}$$

must be justified for convergence. We will concern ourselves only on the case where $K(x,t) = g(x)h(t)$. The HPM will be used to solve the following nonlinear Fredholm integral equations of the first kind.

Example 15.29

Use the homotopy perturbation method to solve the nonlinear Fredholm integral equation of the first kind

$$e^x = \int_0^1 e^{x-2t} u^2(t) dt. \tag{15.231}$$

We first set

$$v(x) = u^2(x), u(x) = \pm\sqrt{v(x)}, \tag{15.232}$$

to carry out (15.231) into

$$e^x = \int_0^1 e^{x-2t} v(t) dt. \tag{15.233}$$

Notice that

$$\left| 1 - \int_0^1 K(t,t) dt \right| = 0.3678 < 1. \tag{15.234}$$

Using the recurrence relation (15.229) we find

$$\begin{aligned} v_0(x) &= 0, \quad v_1(x) = e^x, \\ v_{n+1}(x) &= v_n(x) - \int_0^1 e^{x-2t} v_n(t) dt, n \geqslant 1. \end{aligned} \tag{15.235}$$

This in turn gives

$$\begin{aligned}
v_0(x) &= 0, \quad v_1(x) = e^x,\\
v_2(x) &= v_1(x) - \int_0^{\frac{1}{3}} e^{x-2t} v_1(t)dt = e^{x-1},\\
v_3(x) &= v_2(x) - \int_0^{\frac{1}{3}} e^{x-2t} v_2(t)dt = e^{x-2},\\
v_4(x) &= v_3(x) - \int_0^{\frac{1}{3}} e^{x-2t} v_3(t)dt = e^{x-3},\\
v_5(x) &= v_4(x) - \int_0^{\frac{1}{3}} e^{x-2t} v_4(t)dt = e^{x-4},
\end{aligned} \tag{15.236}$$

and so on. Consequently, the approximate solution is given by

$$v(x) = e^x(1 + e^{-1} + e^{-2} + e^{-3} + \cdots), \tag{15.237}$$

that converges to

$$v(x) = \frac{e^{x+1}}{e-1}. \tag{15.238}$$

Using (15.232) gives the exact solution

$$u(x) = \pm\sqrt{\frac{e^{x+1}}{e-1}}. \tag{15.239}$$

It is worthnoting that there are two more solutions to this equation given by

$$u(x) = \pm e^x. \tag{15.240}$$

The reason that the solution is not unique is due to the fact that the problem is nonlinear and ill-posed as well.

Example 15.30

Use the homotopy perturbation method to solve the nonlinear Fredholm integral equation of the first kind

$$\frac{1}{2}e^{-x} = \int_0^{\frac{1}{2}} e^{t-x} u^2(t)dt. \tag{15.241}$$

We first set

$$v(x) = u^2(x), u(x) = \pm\sqrt{v(x)}, \tag{15.242}$$

to carry out (15.241) into

$$\frac{1}{2}e^{-x} = \int_0^{\frac{1}{2}} e^{t-x} v(t)dt. \tag{15.243}$$

Notice that

$$\left|1 - \int_0^{\frac{1}{2}} K(t,t)dt\right| = 0.5 < 1. \tag{15.244}$$

Using the recurrence relation (15.229) we find

$$\begin{aligned} v_0(x) &= 0, \quad v_1(x) = \frac{1}{2}e^{-x}, \\ v_{n+1}(x) &= v_n(x) - \int_0^1 e^{t-x} v_n(t)dt, n \geqslant 1. \end{aligned} \tag{15.245}$$

This in turn gives

$$\begin{aligned} v_0(x) &= 0, \quad v_1(x) = \frac{1}{2}e^{-x}, \\ v_2(x) &= v_1(x) - \int_0^{\frac{1}{3}} e^{t-x} v_1(t)dt = \frac{1}{4}e^{-x}, \\ v_3(x) &= v_2(x) - \int_0^{\frac{1}{3}} e^{x-2t} v_2(t)dt = \frac{1}{8}e^{-x}, \end{aligned} \tag{15.246}$$

and so on. Consequently, the approximate solution is given by

$$v(x) = e^{-x}\left(\frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{16} + \cdots\right), \tag{15.247}$$

that converges to

$$v(x) = e^{-x}. \tag{15.248}$$

Using (15.242) gives the exact solution

$$u(x) = \pm e^{-\frac{1}{2}x}. \tag{15.249}$$

Example 15.31

Use the homotopy perturbation method to solve the nonlinear Fredholm integral equation of the first kind

$$-\frac{3}{8}x = \int_0^1 xtu^3(t)dt. \tag{15.250}$$

We first set

$$v(x) = u^3(x), u(x) = v^{\frac{1}{3}}(x), \tag{15.251}$$

to carry out (15.250) into

$$-\frac{3}{8}x = \int_0^1 xtv(t)dt. \tag{15.252}$$

Notice that

$$\left|1 - \int_0^1 K(t,t)dt\right| = \frac{2}{3} < 1. \tag{15.253}$$

Using the recurrence relation (15.229) we find

$$\begin{aligned} v_0(x) &= 0, \quad v_1(x) = -\frac{3}{8}x, \\ v_{n+1}(x) &= v_n(x) - \int_0^1 xtv_n(t)dt, n \geqslant 1. \end{aligned} \tag{15.254}$$

This in turn gives

$$\begin{aligned}
v_0(x) &= 0, \quad v_1(x) = -\frac{3}{8}x,\\
v_2(x) &= v_1(x) - \int_0^1 xt\, v_1(t)dt = -\frac{1}{4}x,\\
v_3(x) &= v_2(x) - \int_0^1 xt\, v_2(t)dt = -\frac{1}{6}x,\\
v_4(x) &= v_3(x) - \int_0^1 xt\, v_3(t)dt = -\frac{1}{9}x,
\end{aligned} \tag{15.255}$$

and so on. Consequently, the approximate solution is given by

$$v(x) = -\frac{3}{8}x\left(1 + \frac{2}{3} + \frac{4}{9} + \frac{8}{27} + \cdots\right), \tag{15.256}$$

that converges to

$$v(x) = -\frac{9}{8}x. \tag{15.257}$$

Using (15.251) gives the exact solution

$$u(x) = \sqrt[3]{-\frac{9}{8}x}. \tag{15.258}$$

It is interesting to point out that there is one more solution to this equation given by

$$u(x) = \ln x. \tag{15.259}$$

The reason that the solution is not unique is due to the fact that the problem is nonlinear and ill-posed as well.

Example 15.32

Use the homotopy perturbation method to solve the nonlinear Fredholm integral equation of the first kind

$$\frac{127}{252}x^2 = \int_0^1 x^2t^2u^2(t)dt. \tag{15.260}$$

We first set

$$v(x) = u^2(x), u(x) = \sqrt{v(x)}, \tag{15.261}$$

to carry out (15.260) into

$$\frac{127}{252}x^2 = \int_0^1 x^2t^2v(t)dt. \tag{15.262}$$

Notice that

$$\left|1 - \int_0^1 K(t,t)dt\right| = \frac{2}{3} < 1. \tag{15.263}$$

Using the recurrence relation (15.229) and proceeding as before we find

$$\begin{aligned} v_0(x) &= 0, \quad v_1(x) = \frac{127}{252}x^2, \\ v_2(x) &= v_1(x) - \int_0^1 xtv_1(t)dt = \frac{127}{315}x^2, \\ v_3(x) &= v_2(x) - \int_0^1 xtv_2(t)dt = \frac{508}{1575}x^2, \\ v_4(x) &= v_3(x) - \int_0^1 xtv_3(t)dt = \frac{2032}{7875}x^2, \end{aligned} \tag{15.264}$$

and so on. Consequently, the approximate solution is given by

$$v(x) = \frac{127}{252}x^2 \left(1 + \frac{4}{5} + \frac{16}{25} + \frac{64}{125} + \cdots\right), \tag{15.265}$$

that converges to

$$v(x) = \frac{635}{252}x^2. \tag{15.266}$$

Using (15.261) gives the exact solution

$$u(x) = \pm\sqrt{\frac{635}{252}}x. \tag{15.267}$$

It is interesting to point out that there are two more solutions to this equation given by

$$u(x) = \pm(x^2 + x^3). \tag{15.268}$$

The reason that the solution is not unique is due to the fact that the problem is nonlinear and ill-posed as well.

Exercises 15.5.2

Use the homotopy perturbation method to solve the nonlinear Fredholm integral equations of the first kind

1. $e^{2x} = \int_0^1 e^{2x-3t}\, u^3(t)\, dt$
2. $\frac{1}{3}e^{-x} = \int_0^{\frac{1}{3}} e^{t-x}\, u^3(t)\, dt$
3. $-\frac{10}{27}x^2 = \int_0^1 x^2t^2\, u^3(t)\, dt$
4. $\frac{1}{32}x = \int_0^1 xt\, u^2(t)\, dt$
5. $\frac{29}{36}x = \int_0^1 xt\, u^2(t)\, dt$
6. $\frac{32}{63}x^2 = \int_{-1}^1 x^2t^2\, u^2(t)\, dt$
7. $\frac{466}{315}x^2 = \int_{-1}^1 x^2t^2\, u^2(t)\, dt$
8. $\frac{8}{35}x = \int_{-1}^1 xt\, u^2(t)\, dt$

15.6 Systems of Nonlinear Fredholm Integral Equations

In this section, we will study systems of nonlinear Fredholm integral equations of the second kind given by

$$\begin{aligned} u(x) &= f_1(x) + \int_a^b \Big(K_1(x,t)F_1(u(t)) + \tilde{K}_1(x,t)\tilde{F}_1(v(t)) \Big)\, dt, \\ v(x) &= f_2(x) + \int_a^b \Big(K_2(x,t)F_2(u(t)) + \tilde{K}_2(x,t)\tilde{F}_2(v(t)) \Big)\, dt. \end{aligned} \tag{15.269}$$

The unknown functions $u(x)$ and $v(x)$, that will be determined, occur inside and outside the integral sign. The kernels $K_i(x,t)$ and $\tilde{K}_i(x,t)$, and the function $f_i(x)$ are given real-valued functions, for $i = 1, 2$. The functions F_i and $\tilde{F}_i$, for $i = 1, 2$ are nonlinear functions of $u(x)$ and $v(x)$.

Systems of linear Fredholm integral equations were presented in Chapter 11. In this chapter, the system of nonlinear Fredholm integral equations can be handled by four distinct methods, namely the direct computation method, the modified Adomian method, the successive approximations method, and the series solution method. Although the aforementioned methods work effectively for handling the systems of nonlinear Fredholm integral equations, but only the first two methods will be used in this section.

15.6.1 The Direct Computation Method

The direct computation method will be applied to solve the systems of nonlinear Fredholm integral equations of the second kind. The method approaches Fredholm integral equations in a direct manner and gives the solution in an exact form and not in a series form. In what follows we summarize the necessary steps needed to apply this method. The method will be applied for the degenerate or separable kernels of the form

$$\begin{aligned} K_1(x,t) &= \sum_{k=1}^{n} g_k(x)h_k(t), \quad \tilde{K}_1(x,t) = \sum_{k=1}^{n} \tilde{g}_k(x)\tilde{h}_k(t), \\ K_2(x,t) &= \sum_{k=1}^{n} r_k(x)s_k(t), \quad \tilde{K}_2(x,t) = \sum_{k=1}^{n} \tilde{r}_k(x)\tilde{s}_k(t). \end{aligned} \tag{15.270}$$

The direct computation method can be applied as follows:

1. We first substitute (15.270) into the system (15.269) to obtain

$$\begin{aligned} u(x) &= f_1(x) + \sum_{k=1}^{n} g_k(x) \int_a^b h_k(t)F_1(u(t))\, dt \\ &\quad + \sum_{k=1}^{n} \tilde{g}_k(x) \int_a^b \tilde{h}_k(t)\tilde{F}_1(v(t))\, dt, \\ v(x) &= f_2(x) + \sum_{k=1}^{n} r_k(x) \int_a^b s_k(t)F_2(u(t))\, dt \\ &\quad + \sum_{k=1}^{n} \tilde{r}_k(x) \int_a^b \tilde{s}_k(t)\tilde{F}_2(v(t))\, dt. \end{aligned} \tag{15.271}$$

2. Each integral at the right side depends only on the variable t with constant limits of integration for t. This means that each integral is equivalent to a constant. Based on this, Equation (15.271) becomes

$$\begin{aligned} u(x) &= f_1(x) + \alpha_1 g_1(x) + \cdots + \alpha_n g_n(x) + \beta_1 \tilde{g}_1(x) + \cdots + \beta_n \tilde{g}_n(x), \\ v(x) &= f_2(x) + \gamma_1 r_1(x) + \cdots + \gamma_n r_n(x) + \delta_1 \tilde{r}_1(x) + \cdots + \delta_n \tilde{r}_n(x), \end{aligned} \tag{15.272}$$

where

$$\begin{aligned} \alpha_i &= \int_a^b h_i(t)\, F_1(u(t))\, dt, 1 \leqslant i \leqslant n, \\ \beta_i &= \int_a^b \tilde{h}_i(t)\, \widetilde{F}_1(v(t)) dt, 1 \leqslant i \leqslant n, \\ \gamma_i &= \int_a^b s_i(t)\, F_2(u(t)) dt, 1 \leqslant i \leqslant n, \\ \delta_i &= \int_a^b \tilde{s}_i(t)\, \widetilde{F}_2(v(t)) dt, 1 \leqslant i \leqslant n. \end{aligned} \tag{15.273}$$

3. Substituting (15.272) into (15.273) gives a system of n algebraic equations that can be solved to determine the constants $\alpha_i, \beta_i, \gamma_i$, and δ_i. To facilitate the computational work, we can use the computer symbolic systems such as Maple and Mathematica. Using the obtained numerical values of these constants into (15.272), the solutions $u(x)$ and $v(x)$ of the system of nonlinear Fredholm integral equations (15.269) follow immediately. The analysis presented above can be explained by studying the following examples.

Example 15.33

Solve the following system of nonlinear Fredholm integral equations by using the direct computation method

$$\begin{aligned} u(x) &= \sin x + (1 - 2\pi)\cos x + \int_0^\pi \cos x (u^2(t) + v^2(t))\, dt, \\ v(x) &= \sin x - \cos x + \int_0^\pi (u^2(t) - v^2(t))\, dt. \end{aligned} \tag{15.274}$$

This system can be rewritten as

$$\begin{aligned} u(x) &= \sin x + (1 - 2\pi + \alpha + \beta)\cos x, \\ v(x) &= \sin x - \cos x + (\alpha - \beta), \end{aligned} \tag{15.275}$$

where

$$\alpha = \int_0^\pi u^2(t)\, dt, \quad \beta = \int_0^\pi v^2(t)\, dt. \tag{15.276}$$

To determine α, and β, we substitute (15.275) into (15.276), and solving the resulting system, we obtain

$$\alpha = \pi, \quad \beta = \pi. \tag{15.277}$$

Substituting this result into (15.275) leads to the exact solutions

$$(u(x), v(x)) = (\sin x + \cos x, \sin x - \cos x). \tag{15.278}$$

Example 15.34

Solve the following system of nonlinear Fredholm integral equations by using the direct computation method

$$\begin{aligned} u(x) &= \frac{\pi}{4}\sec x - \tan x + \int_0^{\frac{\pi}{4}} (\tan x\, u^2(t) + \sec x\, v^2(t))\, dt, \\ v(x) &= \left(1 - \frac{\pi}{4}\right)\sec x + \int_0^{\frac{\pi}{4}} (\tan x\, u^2(t) - \sec x\, v^2(t))\, dt. \end{aligned} \tag{15.279}$$

Following the analysis presented above this system can be rewritten as

$$\begin{aligned} u(x) &= \left(\frac{\pi}{4} + \beta\right)\sec x + (\alpha - 1)\tan x, \\ v(x) &= \left(1 - \frac{\pi}{4} - \beta\right)\sec x + \alpha\tan x, \end{aligned} \tag{15.280}$$

where

$$\alpha = \int_0^{\frac{\pi}{4}} u^2(t)\, dt, \quad \beta = \int_0^{\frac{\pi}{4}} v^2(t)\, dt. \tag{15.281}$$

To determine α and β, we substitute (15.280) into (15.281) and solve the resulting system, we find

$$\alpha = 1, \quad \beta = 1 - \frac{\pi}{4}. \tag{15.282}$$

Substituting (15.282) into (15.280) leads to the exact solutions

$$(u(x), v(x)) = (\sec x, \tan x). \tag{15.283}$$

Example 15.35

Solve the following system of nonlinear Fredholm integral equations by using the direct computation method

$$\begin{aligned} u(x) &= 1 - 6x + \ln x + \int_{0^+}^1 x(u^2(t) + v^2(t))\, dt, \\ v(x) &= 1 + 4x^2 - \ln x + \int_{0^+}^1 x^2(u^2(t) - v^2(t))\, dt. \end{aligned} \tag{15.284}$$

Proceeding as before, this system can be rewritten as

$$\begin{aligned} u(x) &= 1 + (\alpha + \beta - 6)x + \ln x, \\ v(x) &= 1 + (4 + \alpha - \beta)x^2 - \ln x, \end{aligned} \tag{15.285}$$

where

$$\alpha = \int_{0^+}^1 u^2(t)\, dt, \quad \beta = \int_{0^+}^1 v^2(t)\, dt. \tag{15.286}$$

To determine α, and β, we substitute (15.285) into (15.286), and proceeding as before to obtain

$$\alpha = 1, \beta = 5. \tag{15.287}$$

This in turn gives the exact solutions

$$(u(x), v(x)) = (1 + \ln x, 1 - \ln x). \tag{15.288}$$

Example 15.36

Solve the following system of nonlinear Fredholm integral equations by using the direct computation method

$$\begin{aligned} u(x) &= 27 + \sec^2 x + \int_0^{\frac{\pi}{4}} 35(v^2(t) - w^2(t))\, dt, \\ v(x) &= 115 - \sec^2 x + \int_0^{\frac{\pi}{4}} 35(w^2(t) - u^2(t))\, dt, \\ w(x) &= -3 - \sec^4 x + \int_0^{\frac{\pi}{4}} (u^2(t) - v^2(t))\, dt. \end{aligned} \tag{15.289}$$

Proceeding as before, this system can be rewritten as

$$\begin{aligned} u(x) &= 27 + \sec^2 x + 35(\beta - \gamma), \\ v(x) &= 115 - \sec^2 x + 35(\gamma - \alpha), \\ w(x) &= -3 - \sec^4 x + \alpha - \beta, \end{aligned} \tag{15.290}$$

where

$$\alpha = \int_0^{\frac{\pi}{4}} u^2(t)\, dt, \quad \beta = \int_0^{\frac{\pi}{4}} v^2(t)\, dt, \quad \gamma = \int_0^{\frac{\pi}{4}} w^2(t) dt. \tag{15.291}$$

To determine α, β, and γ, we substitute (15.290) into (15.291) and by solving the resulting system we find

$$\alpha = \frac{\pi}{4} + \frac{10}{3}, \quad \beta = \frac{\pi}{4} - \frac{2}{3}, \quad \gamma = \frac{\pi}{4} + \frac{8}{105}. \tag{15.292}$$

These results lead to the exact solutions

$$(u(x), v(x), w(x)) = (1 + \sec^2 x, 1 - \sec^2 x, 1 - \sec^4 x). \tag{15.293}$$

Exercises 15.6.1

Use the direct computation method to solve the following systems of nonlinear Fredholm integral equations

1. $\begin{cases} u(x) = x - \dfrac{4}{7} + \displaystyle\int_{-1}^{1} t\left(u^2(t) + v^2(t)\right)\, dt \\ v(x) = x^2 + x^3 + \dfrac{4}{7} + \displaystyle\int_{-1}^{1} t\left(u^2(t) - v^2(t)\right)\, dt \end{cases}$

2. $\begin{cases} u(x) = 22 + \ln x + \displaystyle\int_{0^+}^{1} \ln t\, \left(u^2(t) + v^2(t)\right)\, dt \\ v(x) = -14 - \ln x + \displaystyle\int_{0^+}^{1} \ln t\, \left(u^2(t) - v^2(t)\right)\, dt \end{cases}$

3. $$\begin{cases} u(x) = 22 - 11\ln x + \int_{0^+}^{1} \ln(xt)\, \left(u^2(t) + v^2(t)\right)\, dt \\ v(x) = -14 + 7\ln x + \int_{0^+}^{1} \ln(xt)\, \left(u^2(t) - v^2(t)\right)\, dt \end{cases}$$

4. $$\begin{cases} u(x) = \sin x + \cos x - \frac{\pi^2}{4}x + \int_{0}^{\frac{\pi}{2}} xt\left(u^2(t) + v^2(t)\right)\, dt \\ v(x) = \sin x - \cos x - \frac{\pi}{2}x + \int_{0}^{\frac{\pi}{2}} xt\left(u^2(t) - v^2(t)\right)\, dt \end{cases}$$

5. $$\begin{cases} u(x) = x + \sin^2 x - \frac{3\pi}{4} - \frac{2\pi^3}{3} + \int_{0}^{\pi} \left(u^2(t) + v^2(t)\right)\, dt \\ v(x) = x - \cos^2 x - \pi^2 x + \frac{2\pi^3}{3} + \int_{0}^{\pi} (x - t)\left(u^2(t) - v^2(t)\right)\, dt \end{cases}$$

6. $$\begin{cases} u(x) = \sec x - 2\sqrt{3} + \int_{0}^{\frac{\pi}{3}} \left(\tan^2 t\, u^2(t) + \sec^2 t\, v^2(t)\right)\, dt \\ v(x) = \tan x - 2\sqrt{3} + \frac{\pi}{3} + \int_{0}^{\frac{\pi}{3}} \left(\sec^2 t\, u^2(t) - \tan^2 t\, v^2(t)\right)\, dt \end{cases}$$

7. $$\begin{cases} u(x) = \frac{11}{35}x + \int_{-1}^{1} (x - t)\left(v^2(t) + w^2(t)\right)\, dt \\ v(x) = x^2 - \frac{20}{12}x + \int_{-1}^{1} (x - 2t)\left(w^2(t) + u^2(t)\right)\, dt \\ w(x) = x^3 - \frac{16}{15}x + \int_{-1}^{1} (x - 3t)\left(u^2(t) + v^2(t)\right)\, dt \end{cases}$$

8. $$\begin{cases} u(x) = \sec x \tan x - 2 + \frac{\pi}{4} + \int_{0}^{\frac{\pi}{4}} \left(v^2(t) - w^2(t)\right)\, dt \\ v(x) = \sec^2 x + 1 - \frac{\pi}{4} + \int_{0}^{\frac{\pi}{4}} \left(w^2(t) - u^2(t)\right)\, dt \\ w(x) = \sec^2 x + \int_{0}^{\frac{\pi}{4}} \left(u^2(t) - v^2(t)\right)\, dt \end{cases}$$

15.6.2 The modified Adomian Decomposition Method

The modified Adomian decomposition method [11–12] was frequently and thoroughly used in this text. The method decomposes the linear terms $u(x)$ and $v(x)$ by an infinite sum of components of the form

$$u(x) = \sum_{n=0}^{\infty} u_n(x), \quad v(x) = \sum_{n=0}^{\infty} v_n(x), \tag{15.294}$$

where the components $u_n(x)$ and $v_n(x)$ will be determined recurrently. The method can be used in its standard form, or combined with the noise terms phenomenon.

However, the nonlinear functions F_i and $\tilde{F}_i$, for $i = 1, 2$, in (15.269) should be replaced by the Adomian polynomials A_n defined by

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[F\left(\sum_{i=0}^{n}\lambda^i u_i\right)\right]_{\lambda=0}, \quad n = 0, 1, 2, \cdots, \tag{15.295}$$

Substituting the aforementioned assumptions for the linear and the nonlinear terms into the system (15.269) and using the recurrence relations we can determine the components $u_n(x)$ and $v_n(x)$. Having determined these components, the series solutions and the exact solutions are readily obtained.

Example 15.37

Use the modified Adomian decomposition method to solve the following system of nonlinear Fredholm integral equations

$$\begin{aligned} u(x) &= \sin x - \pi + \int_0^{\pi}\left((1+xt)u^2(t) + (1-xt)v^2(t)\right)dt, \\ v(x) &= \cos x + \frac{\pi^2}{2}x + \int_0^{\pi}\left((1-xt)u^2(t) - (1+xt)v^2(t)\right)dt. \end{aligned} \tag{15.296}$$

Substituting the linear terms $u(x)$ and $v(x)$ and the nonlinear terms $u^2(x)$ and $v^2(x)$ from (15.294) and (15.295) respectively into (15.296) gives

$$\begin{aligned} \sum_{n=0}^{\infty} u_n(x) &= \sin x - \pi \\ &\quad + \int_0^{\pi}\left((1+xt)\sum_{n=0}^{\infty} A_n(t) + (1-xt)\sum_{n=0}^{\infty} B_n(t)\right)dt, \\ \sum_{n=0}^{\infty} v_n(x) &= \cos x + \frac{\pi^2}{2}x \\ &\quad + \int_0^{\pi}\left((1-xt)\sum_{n=0}^{\infty} A_n(t) - (1+xt)\sum_{n=0}^{\infty} B_n(t)\right)dt. \end{aligned} \tag{15.297}$$

The modified decomposition method will be used here, hence we set the recursive relation

$$\begin{aligned} &u_0(x) = \sin x, \quad v_0(x) = \cos x, \\ &u_1(x) = -\pi + \int_0^{\pi}\left((1+xt)u_0^2(t) + (1-xt)v_0^2(t)\right)dt = 0, \\ &v_1(x) = \frac{\pi^2}{2}x + \int_0^{\pi}\left((1-xt)u_0^2(t) - (1+xt)v_0^2(t)\right)dt = 0. \end{aligned} \tag{15.298}$$

This in turn gives the exact solutions

$$(u(x), v(x)) = (\sin x, \cos x). \tag{15.299}$$

Example 15.38

Use the modified Adomian decomposition method to solve the following system of Fredholm integral equations

$$u(x) = x + \sec^2 x - \frac{8}{3} - \frac{\pi^3}{96} + \int_0^{\frac{\pi}{4}} \left(u^2(t) + v^2(t)\right) dt,$$
$$v(x) = x - \sec^2 x - \pi + 2\ln 2 + \int_0^{\frac{\pi}{4}} \left(u^2(t) - v^2(t)\right) dt. \tag{15.300}$$

Proceeding as before we obtain

$$\sum_{n=0}^{\infty} u_n(x) = x + \sec^2 x - \frac{8}{3} - \frac{\pi^3}{96} + \int_0^{\frac{\pi}{4}} \left(\sum_{n=0}^{\infty} A_n(t) + \sum_{n=0}^{\infty} B_n(t)\right) dt,$$
$$\sum_{n=0}^{\infty} v_n(x) = x - \sec^2 x - \pi + 2\ln 2 + \int_0^{\frac{\pi}{4}} \left(\sum_{n=0}^{\infty} A_n(t) = \sum_{n=0}^{\infty} B_n(t)\right) dt. \tag{15.301}$$

For simplicity, the modified decomposition method will be used again, therefore we set the recursive relation

$$u_0(x) = x + \sec^2 x, \quad v_0(x) = x - \sec^2 x,$$
$$u_1(x) = -\frac{8}{3} - \frac{\pi^3}{96} + \int_0^{\frac{\pi}{4}} \left(u_0^2(t) + v_0^2(t)\right) dt = 0, \tag{15.302}$$
$$v_1(x) = -\pi + 2\ln 2 + \int_0^{\frac{\pi}{4}} \left(u_0^2(t) + v_0^2(t)\right) dt = 0.$$

Consequently, the other components $(u_j, v_j) = (0, 0), j \geqslant 2$. As a result, the exact solutions are given by

$$(u(x), v(x)) = (x + \sec^2 x, x - \sec^2 x). \tag{15.303}$$

Example 15.39

Use the modified Adomian decomposition method to solve the following system of nonlinear Fredholm integral equations

$$u(x) = e^x - \frac{2}{x+2}\sinh(x+2) + \int_0^1 \left(e^{xt}u^2(t) + e^{-xt}v^2(t)\right) dt,$$
$$v(x) = e^{-x} - \frac{2}{x-2}\sinh(x-2) + \int_0^1 \left(e^{-xt}u^2(t) + e^{xt}v^2(t)\right) dt. \tag{15.304}$$

Proceeding as before and using the modified decomposition method, we find the recursive relation

$$u_0(x) = e^x, \quad v_0(x) = e^{-x},$$
$$u_1(x) = -\frac{2}{x+2}\sinh(x+2) + \int_0^1 \left(e^{xt}u_0^2(t) + e^{-xt}v_0^2(t)\right) dt = 0, \tag{15.305}$$
$$v_1(x) = -\frac{2}{x-2}\sinh(x-2) + \int_0^1 \left(e^{-xt}u_0^2(t) + e^{xt}v_0^2(t)\right) dt = 0.$$

The exact solutions are given by

$$(u(x), v(x)) = (e^x, e^{-x}). \tag{15.306}$$

Example 15.40

Use the modified Adomian decomposition method to solve the following system of nonlinear Fredholm integral equations

$$\begin{aligned} u(x) &= e^x - e^{x+1} + \int_0^1 \left(e^{x-3t}v^2(t) + e^{x-6t}w^2(t)\right) dt, \\ v(x) &= e^{2x} - 2e^x + \int_0^1 \left(e^{x-6t}w^2(t) + e^{x-2t}u^2(t)\right) dt, \\ w(x) &= e^{3x} - 2e^x + \int_0^1 \left(e^{x-2t}u^2(t) + e^{x-4t}v^2(t)\right) dt. \end{aligned} \tag{15.307}$$

Using the modified decomposition method, we set the recurrence relation

$$\begin{aligned} u_0(x) &= e^x, \quad v_0(x) = e^{2x}, \\ w_0(x) &= e^{3x}, \\ u_1(x) &= -e^{x+1} + \int_0^1 \left(e^{x-3t}v_0^2(t) + e^{x-6t}w_0^2(t)\right) dt = 0, \\ v_1(x) &= -2e^x + \int_0^1 \left(e^{x-6t}w_0^2(t) + e^{x-2t}u_0^2(t)\right) dt = 0, \\ w_1(x) &= -2e^x + \int_0^1 \left(e^{x-2t}u^2(t) + e^{x-4t}v^2(t)\right) dt = 0. \end{aligned} \tag{15.308}$$

Consequently, the exact solutions are given by

$$(u(x), v(x), w(x)) = (e^x, e^{2x}, e^{3x}). \tag{15.309}$$

Exercises 15.6.2

Use the modified Adomian decomposition method to solve the following systems of nonlinear Fredholm integral equations

1. $\begin{cases} u(x) = x - \dfrac{4}{7} + \displaystyle\int_{-1}^1 t\left(u^2(t) + v^2(t)\right) dt \\ v(x) = x^2 + x^3 + \dfrac{4}{7} + \displaystyle\int_{-1}^1 t\left(u^2(t) - v^2(t)\right) dt \end{cases}$

2. $\begin{cases} u(x) = \sin x + \cos x - 2\pi + \displaystyle\int_0^\pi \left(u^2(t) + v^2(t)\right) dt \\ v(x) = \sin x - \cos x - \pi x + \displaystyle\int_0^\pi xt\left(u^2(t) - v^2(t)\right) dt \end{cases}$

3. $\begin{cases} u(x) = x + \sin x + (4 - 2\pi) + \displaystyle\int_0^\pi \left(u^2(t) - v^2(t)\right) dt \\ v(x) = x - \cos x + (8 - 4\pi - 2\pi^2) + \displaystyle\int_0^\pi t\left(u^2(t) - v^2(t)\right) dt \end{cases}$

4. $$\begin{cases} u(x) = \sec x + \dfrac{\pi - 8}{4} + \displaystyle\int_0^{\frac{\pi}{4}} \left(u^2(t) + v^2(t)\right)\, dt \\ v(x) = \tan x - \dfrac{\pi}{4}x + \displaystyle\int_0^{\frac{\pi}{4}} x\left(u^2(t) - v^2(t)\right)\, dt \end{cases}$$

5. $$\begin{cases} u(x) = \sec x - \dfrac{\pi}{4} + \displaystyle\int_0^{\frac{\pi}{4}} \left(u^2(t) - v^2(t)\right)\, dt \\ v(x) = \tan x - \dfrac{\pi}{4}x + \displaystyle\int_0^{\frac{\pi}{4}} x\left(u^2(t) - v^2(t)\right)\, dt \end{cases}$$

6. $$\begin{cases} u(x) = \sec x \tan x + \dfrac{\sqrt{3}}{3} + \displaystyle\int_0^{\frac{\pi}{6}} \left(u^2(t) - v^2(t)\right)\, dt \\ v(x) = \sec^2 x - \dfrac{11\sqrt{3}}{27}x + \displaystyle\int_0^{\frac{\pi}{6}} x\left(u^2(t) + v^2(t)\right)\, dt \end{cases}$$

7. $$\begin{cases} u(x) = \sec x - \dfrac{\pi + 4}{4} + \displaystyle\int_0^{\frac{\pi}{4}} \left(u^2(t) + u(t)v(t)\right)\, dt \\ v(x) = \cos x + \dfrac{\pi - 4}{4} + \displaystyle\int_0^{\frac{\pi}{4}} \left(u^2(t) - u(t)v(t)\right)\, dt \end{cases}$$

8. $$\begin{cases} u(x) = \tan x + \dfrac{\pi - 5}{4} + \displaystyle\int_0^{\frac{\pi}{4}} \left(u^2(t) + u(t)v^2(t)\right)\, dt \\ v(x) = \cos x + \dfrac{\pi - 3}{4} + \displaystyle\int_0^{\frac{\pi}{4}} \left(u^2(t) - u(t)v^2(t)\right)\, dt \end{cases}$$

9. $$\begin{cases} u(x) = \sec x + \dfrac{\pi - 10}{8} + \displaystyle\int_0^{\frac{\pi}{4}} \left(v^2(t) + w^2(t)\right)\, dt \\ v(x) = \tan x - \dfrac{\pi + 10}{8} + \displaystyle\int_0^{\frac{\pi}{4}} \left(w^2(t) + u^2(t)\right)\, dt \\ w(x) = \cos x + \dfrac{\pi - 8}{4} + \displaystyle\int_0^{\frac{\pi}{4}} \left(u^2(t) + v^2(t)\right)\, dt \end{cases}$$

10. $$\begin{cases} u(x) = 1 + \pi \sec^2 x + \dfrac{3\pi - 10\pi^2}{6} + \displaystyle\int_0^{\frac{\pi}{4}} \left(v^2(t) + w^2(t)\right)\, dt \\ v(x) = 1 - \pi \sec^2 x - \dfrac{21\pi + 10\pi^2}{6} + \displaystyle\int_0^{\frac{\pi}{4}} \left(w^2(t) + u^2(t)\right)\, dt \\ w(x) = 1 + \dfrac{\pi}{2} \sec^2 x - \dfrac{3\pi + 16\pi^2}{6} + \displaystyle\int_0^{\frac{\pi}{4}} \left(u^2(t) + v^2(t)\right)\, dt \end{cases}$$

11. $$\begin{cases} u(x) = \sec x - \dfrac{1}{2} + \displaystyle\int_0^{\frac{\pi}{3}} \left(v(t)w(t)\right)\, dt \\ v(x) = \tan x + \dfrac{\pi}{3} + \displaystyle\int_0^{\frac{\pi}{3}} \left(w(t)u(t)\right)\, dt \\ w(x) = \cos x - 1 + \displaystyle\int_0^{\frac{\pi}{3}} \left(u(t)v(t)\right)\, dt \end{cases}$$

$$12.\quad \begin{cases} u(x) = \sec^2 x + \dfrac{2-\pi}{8} + \displaystyle\int_0^{\frac{\pi}{4}} (v(t)w(t))\, dt \\ v(x) = -\sec^2 x - \dfrac{1}{3} + \displaystyle\int_0^{\frac{\pi}{4}} (w(t)u(t))\, dt \\ w(x) = \tan^2 x - \dfrac{\pi}{4} + \displaystyle\int_0^{\frac{\pi}{4}} (u(t)v(t))\, dt \end{cases}$$

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Chapter 16
Nonlinear Fredholm Integro-Differential Equations

16.1 Introduction

The linear Fredholm integral equations and the linear Fredholm integro-differential equations were presented in Chapters 4 and 6 respectively. In Chapter 15, the nonlinear Fredholm integral equations were examined. It is our goal in this chapter to study the nonlinear Fredholm integro-differential equations [1–7] and the systems of nonlinear Fredholm integro-differential equations.

The nonlinear Fredholm integro-differential equations of the second kind is of the form

$$u^{(n)}(x) = f(x) + \lambda \int_0^1 K(x,t)F(u(t))\,dt, \tag{16.1}$$

where $u^{(n)}(x)$ is the nth derivative of $u(x)$. The kernel $K(x,t)$ and the function $f(x)$ are given real-valued functions, and $F(u(x))$ is a nonlinear function of $u(x)$.

In this chapter, we will mostly use degenerate or separable kernels. A degenerate or a separable kernel is a function that can be expressed as the sum of product of two functions each depends only on one variable. Such a kernel can be expressed in the form

$$K(x,t) = \sum_{i=1}^{n} g_i(x)\,h_i(t). \tag{16.2}$$

Several analytic and numerical methods have been used to handle the nonlinear Fredholm integro-differential equations. In this text we will apply three of the methods used in this text, namely, the direct computation method, the variational iteration method (VIM), and the Taylor series solution method to handle the nonlinear Fredholm integro-differential equations. The emphasis in this text will be on the use of these methods rather than proving theoretical concepts of convergence and existence. The theorems of uniqueness, existence, and convergence are important and can be found in the literature.

The concern will be on the determination of the solutions $u(x)$ of nonlinear Fredholm integro-differential equations of the second kind.

16.2 Nonlinear Fredholm Integro-Differential Equations

The linear Fredholm integro-differential equation, where both differential and integral operators appear together in the same equation, has been studied in Chapter 6. In this section, the nonlinear Fredholm integro-differential equation will be examined. The standard form of the nonlinear Fredholm integro-differential equations of reads

$$u^{(i)}(x) = f(x) + \lambda \int_0^1 K(x,t)F(u(t))dt, \tag{16.3}$$

where $u^{(i)}(x) = \frac{d^i u}{dx^i}$, and $F(u(x))$ is a nonlinear function of $u(x)$ such as $u^2(x), \sin(u(x))$, and $e^{u(x)}$. Because the equation in (16.3) combines the differential operator and the integral operator, then it is necessary to define initial conditions $u(0), u'(0), \ldots,\ u^{(i-1)}(0)$ for the determination of the particular solution $u(x)$ of this equation.

In Chapter 6, we applied four methods to handle the linear Fredholm integro-differential equations of the second kind. In this section we will use only three of the methods that we used in Chapter 6. However, the other methods presented in Chapter 6 can be used as well.

In what follows we will apply the direct computation method, the variational iteration method (VIM), and the series solution method to handle nonlinear Fredholm integro-differential equations of the second kind (16.3).

16.2.1 The Direct Computation Method

The direct computation method has been extensively introduced in this text. Without loss of generality, we may assume a standard form to the Fredholm integro-differential equation given by

$$u^{(n)}(x) = f(x) + \int_a^b K(x,t)\, F(u(t))dt,\ u^{(k)}(0) = b_k,\ 0 \leqslant k \leqslant (n-1), \tag{16.4}$$

where $u^{(n)}(x)$ indicates the n th derivative of $u(x)$ with respect to x, $F(u(t))$ is a nonlinear function of $u(x)$, and b_k are the initial conditions. It is important to point out that this method will be applied for equations where the kernels are degenerate or separable of the form

$$K(x,t) = \sum_{k=1}^{n} g_k(x)h_k(t). \tag{16.5}$$

Substituting (16.5) into the nonlinear Fredholm integro-differential equation (16.4) leads to

$$\begin{aligned} u^{(n)}(x) &= f(x) + g_1(x)\int_a^b h_1(t)\,F(u(t))dt \\ &\quad + g_2(x)\int_a^b h_2(t)\,F(u(t))dt + \cdots + g_n(x)\int_a^b h_n(t)\,F(u(t))dt. \end{aligned} \tag{16.6}$$

Each integral at the right side of (16.6) depends only on the variable t with constant limits of integration for t. This means that each integral is equivalent to a constant. Based on this, Equation (16.6) becomes

$$u^{(n)}(x) = f(x) + \alpha_1 g_1(x) + \alpha_2 g_2(x) + \cdots + \alpha_n g_n(x), \tag{16.7}$$

where

$$\alpha_i = \int_a^b h_i(t)\,F(u(t))dt, 1 \leqslant i \leqslant n. \tag{16.8}$$

Integrating both sides of (16.7) n times, using the initial conditions, we obtain

$$\begin{aligned} u(x) &= u(0) + xu'(0) + \frac{1}{2!}x^2u''(0) + \cdots + \frac{1}{(n-1)!}x^{n-1}u^{(n-1)} \\ &\quad + L^{-1}\left(f(x) + \alpha_1 g_1(x) + \alpha_2 g_2(x) + \cdots + \alpha_n g_n(x)\right), \end{aligned} \tag{16.9}$$

where L^{-1} is the n-fold integral operator. Substituting (16.9) into (16.8) gives a system of n algebraic equations that can be solved to determine the constants $\alpha_i, 1 \leqslant i \leqslant n$. Using the obtained numerical values of α_i into (16.7), the solution $u(x)$ of the nonlinear Fredholm integro-differential equation (16.4) is readily obtained.

It is interesting to point out that we may get more than one value for one or more of $\alpha_i, 1 \leqslant i \leqslant n$. This is normal because the equation is nonlinear and the solution $u(x)$ may not be unique for nonlinear problems. In what follows we present some examples to illustrate the use of the direct computation method.

Example 16.1

Solve the nonlinear Fredholm integro-differential equation by using the direct computation method

$$u'(x) = \cos x - \frac{\pi^2}{4}x + \int_0^\pi xtu^2(t)\,dt,\ u(0) = 0. \tag{16.10}$$

This equation may be written as

$$u'(x) = \cos x + \left(\alpha - \frac{\pi^2}{4}\right)x,\ u(0) = 0, \tag{16.11}$$

where

$$\alpha = \int_0^\pi tu^2(t)\,dt. \tag{16.12}$$

Integrating both sides of (16.11) from 0 to x, and using the initial condition, we find

$$u(x) = \sin x + \left(\frac{\alpha}{2} - \frac{\pi^2}{8}\right) x^2. \tag{16.13}$$

Substituting (16.13) into (16.12), evaluating the resulting integral, and solving the resulting equation for α we obtain

$$\alpha = \frac{\pi^2}{4}, \frac{\pi^8 - 96\pi^3 + 576\pi + 96}{4\pi^6}. \tag{16.14}$$

Consequently, the exact solutions are given by

$$u(x) = \sin x, \, \sin x - \left(\frac{12}{\pi^3} + \frac{72}{\pi^5} + \frac{12}{\pi^6}\right) x^2. \tag{16.15}$$

Example 16.2

Solve the nonlinear Fredholm integro-differential equation by using the direct computation method

$$u'(x) = e^x + \frac{1 - e^2}{2} x + \int_0^1 x u^2(t)\, dt, \; u(0) = 1. \tag{16.16}$$

This equation may be written as

$$u'(x) = e^x + \frac{1 - e^2 + 2\alpha}{2} x, \; u(0) = 1, \tag{16.17}$$

where

$$\alpha = \int_0^1 u^2(t)\, dt. \tag{16.18}$$

Integrating both sides of (16.17) from 0 to x, and by using the initial conditions we obtain

$$u(x) = e^x + \frac{1 - e^2 + 2\alpha}{4} x^2. \tag{16.19}$$

Substituting (16.19) into (16.18) and proceeding as before we get

$$\alpha = \frac{e^2 - 1}{2}, \frac{e^2 - 40e + 119}{2}. \tag{16.20}$$

The exact solutions are therefore given by

$$u(x) = e^x, e^x + (30 - 10e)x^2. \tag{16.21}$$

Example 16.3

Solve the nonlinear Fredholm integro-differential equation by using the direct computation method

$$u''(x) = 2 + \frac{11}{15}x + \frac{19}{35}x^2 + \frac{1}{2}\int_{-1}^1 (xt + x^2t^2)\left(u(t) - u^2(t)\right) dt, \; u(0) = 1, u'(0) = 1. \tag{16.22}$$

This equation may be written as

$$u''(x) = 2 + \left(\frac{1}{2}\alpha + \frac{11}{15}\right) x + \left(\frac{1}{2}\beta + \frac{19}{35}\right) x^2, \; u(0) = 1, u'(0) = 1, \tag{16.23}$$

obtained by setting

$$\alpha = \int_{-1}^{1} t(u(t) - u^2(t))\,dt, \quad \beta = \int_{-1}^{1} t^2(u(t) - u^2(t))\,dt. \tag{16.24}$$

Integrating both sides of (16.23) two times from 0 to x, and by using the given initial conditions we find

$$u(x) = 1 + x + x^2 + \left(\frac{1}{12}\alpha + \frac{11}{90}\right)x^3 + \left(\frac{1}{24}\beta + \frac{19}{420}\right)x^4. \tag{16.25}$$

Substituting (16.25) into (16.24), evaluating the integrals, and solving the resulting equations we find

$$\alpha = -\frac{22}{15}, \quad \beta = -\frac{38}{35}. \tag{16.26}$$

The exact solution is given by

$$u(x) = 1 + x + x^2. \tag{16.27}$$

Example 16.4

Solve the nonlinear Fredholm integro-differential equation by using the direct computation method

$$\begin{gathered} u'''(x) = \frac{2}{3}\sin x + \frac{1}{2}\int_0^{\pi} \cos(x-t)\,u^2(t)\,dt, \\ u(0) = 1, u'(0) = 0, u''(0) = -1. \end{gathered} \tag{16.28}$$

This equation may be written as

$$\begin{gathered} u'''(x) = \frac{2}{3}\sin x + \frac{1}{2}\alpha\cos x + \frac{1}{2}\beta\sin x, \\ u(0) = 1, u'(0) = 0, u''(0) = -1, \end{gathered} \tag{16.29}$$

obtained by setting

$$\alpha = \int_0^{\pi} \cos t u^2(t)\,dt, \quad \beta = \int_0^{\pi} \sin t u^2(t)\,dt. \tag{16.30}$$

Integrating both sides of (16.29) three times from 0 to x, and by using the given initial conditions we obtain

$$u(x) = \left(\frac{2}{3} + \frac{1}{2}\beta\right)\cos x + \frac{1}{2}\alpha(x - \sin x) + \left(\frac{1}{4}\beta - \frac{1}{6}\right)x^2 - \frac{1}{2}\beta + \frac{1}{3}. \tag{16.31}$$

Proceeding as in the previous examples gives

$$\alpha = 0, \quad \beta = \frac{2}{3}. \tag{16.32}$$

The exact solution is therefore given by

$$u(x) = \cos x. \tag{16.33}$$

Exercises 16.2.1

Solve the following nonlinear Fredholm integro-differential equations by using the direct computation method

1. $u'(x) = 1 - \frac{17}{24}x + \frac{1}{2}\int_0^1 xtu^2(t)\,dt,\ u(0) = 1$

2. $u'(x) = 1 + 2x - \frac{149}{240}x^2 + \frac{1}{4}\int_0^1 x^2tu^2(t)\,dt,\ u(0) = 1$

3. $u'(x) = \cos x - x + \frac{1}{4}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} xtu^2(t)\,dt,\ u(0) = 1$

4. $u'(x) = \sin x + \frac{1}{16}(\pi^2 + 8)x + \frac{1}{4}\int_0^{\pi} xt(u(t) - u^2(t))\,dt,\ u(0) = 0$

5. $u'(x) = 1 + e^x - \left(\frac{1}{3} + \frac{e}{4} + \frac{e^2}{8}\right)x + \frac{1}{4}\int_0^1 x(u(t) + u^2(t))\,dt,\ u(0) = 1$

6. $u'(x) = e^x + \frac{1}{16}(3 + e^2)x + \frac{1}{4}\int_0^1 xt(1 + u(t) - u^2(t))\,dt,\ u(0) = 2$

7. $u''(x) = -\sin x - \frac{\pi}{16}x + \frac{\pi^2}{32} + \frac{1}{8}\int_0^{\pi} (x - t)u^2(t)\,dt,\ u(0) = 0, u'(0) = 1$

8. $u''(x) = \frac{7}{3}\cos x + \frac{\pi}{2}\sin x + \frac{1}{2}\int_0^{\pi} \sin(x - t)u^2(t)\,dt,\ u(0) = 0, u'(0) = 0$

9. $u''(x) = \frac{1}{2}e^x + \frac{1}{2}\int_0^1 e^{x-2t}u^2(t)\,dt,\ u(0) = 1, u'(0) = 1$

10. $u'''(x) = \frac{3}{4}e^x + \frac{1}{4}\int_0^1 e^{x-3t}u^3(t)\,dt,\ u(0) = u'(0) = u''(0) = 1$

11. $u'''(x) = 8e^{2x} - \frac{1}{16}e^x + \frac{1}{16}\int_0^1 e^{x-4t}u^2(t)\,dt,\ u(0) = 1, u'(0) = 2, u''(0) = 4$

12. $u'''(x) = \sin x + \frac{\pi}{16} + \frac{1}{8}\int_0^{\pi} (u(t) - u^2(t))\,dt,\ u(0) = -u''(0) = 1, u'(0) = 0$

16.2.2 The Variational Iteration Method

The variational iteration method [8–10] was used before in previous chapters. The method handles linear and nonlinear problems in a straightforward manner. Unlike the Adomian decomposition method where we determine distinct components of the exact solution, the variational iteration method gives rapidly convergent successive approximations of the exact solution if such a closed form solution exists.

The standard ith order nonlinear Fredholm integro-differential equation is of the form

$$u^{(i)}(x) = f(x) + \int_0^1 K(x,t)F(u(t))dt, \tag{16.34}$$

where $u^{(i)}(x) = \frac{d^i u}{dx^i}$, and $F(u(x))$ is a nonlinear function of $u(x)$. The initial conditions should be prescribed for the complete determination of the exact solution.

The correction functional for the nonlinear integro-differential equation (16.34) is

$$u_{n+1}(x) = u_n(x) + \int_0^1 \lambda(t) \left(u_n^{(i)}(t) - f(t) - \int_0^t K(t,r)F(\tilde{u}_n(r))\, dr \right) dt. \tag{16.35}$$

To apply this method in an effective way, we should follow two essential steps:

(i) It is required first to determine the Lagrange multiplier λ that can be identified optimally via integration by parts and by using a restricted variation. The Lagrange multiplier λ may be a constant or a function.

(ii) Having λ determined, an iteration formula, without restricted variation, should be used for the determination of the successive approximations $u_{n+1}(x), n \geqslant 0$ of the solution $u(x)$. The zeroth approximation u_0 can be any selective function. However, the initial values are preferably used for the selective zeroth approximation u_0. Consequently, the solution is given by

$$u(x) = \lim_{n\to\infty} u_n(x). \tag{16.36}$$

The VIM will be illustrated by studying the following examples.

Example 16.5

Use the variational iteration method to solve the nonlinear Fredholm integro-differential equation

$$u'(x) = \cos x - \frac{\pi}{48}x + \frac{1}{24}\int_0^\pi xu^2(t)\, dt, u(0) = 0. \tag{16.37}$$

The correction functional for this equation is given by

$$u_{n+1}(x) = u_n(x) - \int_0^x \left(u_n'(t) - \cos t + \frac{\pi}{48}t - \frac{1}{24}\int_0^\pi tu_n^2(r)dr \right) dt, \tag{16.38}$$

where we used $\lambda = -1$ for first-order integro-differential equation.

We can use the initial condition to select $u_0(x) = u(0) = 0$. Using this selection into the correction functional gives the following successive approximations

$$\begin{aligned} u_0(x) &= 0, \\ u_1(x) &= \sin x - 0.03272492349x^2, \\ u_2(x) &= \sin x - 0.00663791983x^2, \\ u_3(x) &= \sin x - 0.00156723251x^2, \\ u_4(x) &= \sin x - 0.00038016125x^2, \end{aligned} \tag{16.39}$$

and so on. Using (16.36) gives the exact solution by

$$u(x) = \sin x. \tag{16.40}$$

Example 16.6

Use the variational iteration method to solve the nonlinear Fredholm integro-differential equation

$$u'(x) = \cos x - x\sin x - \frac{\pi(\pi^2+3)}{512}x + \frac{1}{64\pi}\int_0^{\pi} xtu^2(t)\,dt, u(0) = 0. \quad (16.41)$$

The correction functional for this equation is given by

$$u_{n+1}(x) = u_n(x) - \int_0^x \left(u_n'(t) - \cos t + t\sin t + \frac{\pi(\pi^2+3)}{512}t - \frac{1}{64\pi}\int_0^{\pi} tru_n^2(r)\,dr\right) dt. \quad (16.42)$$

We next select $u_0(x) = 0$ to find the successive approximations

$$\begin{aligned} u_0(x) &= 0,\\ u_1(x) &= x\cos x - 0.03948345181x^2,\\ u_2(x) &= x\cos x + 0.01017026496x^2,\\ u_3(x) &= x\cos x - 0.002418466659x^2,\\ u_4(x) &= x\cos x + 0.000587237401x^2, \end{aligned} \quad (16.43)$$

and so on. Proceeding as before, the exact solution is given by

$$u(x) = x\cos x. \quad (16.44)$$

Example 16.7

Use the variational iteration method to solve the nonlinear Fredholm integro-differential equation

$$u''(x) = -\cos x - \frac{3\pi}{128}x + \frac{1}{64}\int_0^{\pi} xu^2(t)dt, u(0) = 2, u'(0) = 0. \quad (16.45)$$

The correction functional for this equation is given by

$$u_{n+1}(x) = u_n(x) + \int_0^x (t-x)\left(u_n''(t) + \cos t + \frac{3\pi}{128}t - \frac{1}{64}\int_0^{\pi} tu_n^2(r)\,dr\right) dt. \quad (16.46)$$

We can use the initial condition to select $u_0(x) = u(0) + xu'(0) = 2$. Using this selection into the correction functional gives the following successive approximations

$$\begin{aligned} u_0(x) &= 2,\\ u_1(x) &= 1 + \cos x + 0.02045307718x^3,\\ u_2(x) &= 1 + \cos x + 0.00118839931x^3,\\ u_3(x) &= 1 + \cos x + 0.00004332607x^3,\\ u_4(x) &= 1 + \cos x + 0.00000152382x^3, \end{aligned} \quad (16.47)$$

and so on. The VIM gives the exact solution by

$$u(x) = 1 + \cos x. \quad (16.48)$$

Example 16.8

Use the variational iteration method to solve the nonlinear Fredholm integro-differential equation

$$u'''(x) = xe^x + \frac{1079}{360}e^x + \frac{1}{120}\int_0^1 e^{x-2t}u^2(t)\,dt, u(0) = 0, u'(0) = 1, u''(0) = 2. \tag{16.49}$$

The correction functional for this equation is given by

$$\begin{aligned} u_{n+1}(x) &= u_n(x) \\ &- \int_0^1 \frac{1}{2}(t-x)^2\left(u_n'''(t) - te^t - \frac{1079}{360}e^t - \frac{1}{120}\int_0^1 e^{t-2r}u_n^2(r)\,dr\right)dt. \end{aligned} \tag{16.50}$$

We can use the initial condition to select $u_0(x) = x + x^2$. Using this selection into the correction functional gives the following successive approximations

$$\begin{aligned} u_0(x) &= x + x^2, \\ u_1(x) &= xe^x + (0.88212x + 0.441063x^2 - 0.88212e^x + 0.88212) \times 10^{-3}, \\ u_2(x) &= xe^x + (0.267448x + 0.1337x^2 - 0.26744e^x + 0.26744) \times 10^{-6}, \\ u_3(x) &= xe^x + (0.8115x + 0.40602x^2 - 0.81286e^x + 0.81135) \times 10^{-10}, \end{aligned} \tag{16.51}$$

and so on. The VIM admits the use of

$$u(x) = \lim_{n\to\infty} u_n(x), \tag{16.52}$$

that gives the exact solution by

$$u(x) = xe^x. \tag{16.53}$$

Exercises 16.2.2

Use the variational iteration method to solve the nonlinear Fredholm integro-differential equations

1. $u'(x) = \sin x - \frac{\pi}{80} + \frac{1}{120}\int_0^\pi xu^2(t)\,dt, u(0) = 0$
2. $u'(x) = \cos x - \frac{\pi}{80}x - \frac{1}{30} + \frac{1}{120}\int_0^\pi xu^2(t)\,dt, u(0) = 1$
3. $u'(x) = x + x\cos x + \frac{\pi^2}{192}(3-\pi^2)x + \frac{1}{24}\int_0^\pi xtu^2(t)\,dt, u(0) = 0$
4. $u'(x) = 2e^{2x} - \frac{1}{24}e^x + \frac{1}{24}\int_0^1 e^{x-4t}u^2(t)\,dt, u(0) = 1$
5. $u'(x) = -2e^{-2x} - \frac{1}{24}xe^x + \frac{1}{24}\int_0^1 xe^{x+4t}u^2(t)\,dt, u(0) = 1$
6. $u'(x) = \cos x + \sin x - \frac{\pi}{96} + \frac{1}{96}\int_0^\pi xu^2(t)\,dt, u(0) = -1$
7. $u''(x) = -\sin x - \frac{\pi}{36}x + \frac{\pi^2}{72} + \frac{1}{18}\int_0^\pi (x-t)u^2(t)\,dt, u(0) = 0, u'(0) = 1$

8. $u''(x) = -\cos x - \frac{\pi^2}{288}x + \frac{1}{72}\int_0^{\pi} xtu^2(t)\,dt, u(0) = 1, u'(0) = 0$

9. $u''(x) = e^x + \frac{1}{4}(e^2-2)x + \frac{1}{2}\int_0^1 x(t-u^2(t))\,dt, u(0) = 1, u'(0) = 1$

10. $u'''(x) = \sin x + \frac{\pi}{144}x + \frac{1}{72}\int_0^{\pi} x(u(t)-u^2(t))\,dt, u(0) = 1, u'(0) = 0, u''(0) = -1$

11. $u'''(x) = \sin x - \cos x - \frac{\pi}{100}x + \frac{1}{100}\int_0^{\pi} xu^2(t)\,dt, u(0) = 1, u'(0) = 1, u''(0) = -1$

12. $u'''(x) = e^x + \frac{1}{200}(e^2-3)x + \frac{1}{100}\int_0^1 x(2u(t)-u^2(t))\,dt, u(0) = 2, u'(0) = 1, u''(0) = 1$

16.2.3 The Series Solution Method

The series solution method depends mainly on the Taylor series for analytic functions [1]. A real function $u(x)$ is called analytic if it has derivatives of all orders such that the generic form of Taylor series at $x = 0$ can be written as

$$u(x) = \sum_{n=0}^{\infty} a_n x^n. \tag{16.54}$$

The Taylor series method, or simply the series solution method will be used in this section for solving nonlinear Fredholm integro-differential equations of the second kind. We will assume that the solution $u(x)$ of the nonlinear Fredholm integro-differential equation

$$u^{(n)}(x) = f(x) + \lambda\int_0^1 K(x,t)F(u(t))dt, u^{(k)}(0) = k!a_k, 0 \leqslant k \leqslant (n-1), \tag{16.55}$$

is analytic, and therefore possesses a Taylor series of the form given in (16.54), where the coefficients a_n will be determined recurrently.

The first few coefficients a_k can be determined by using the initial conditions so that

$$a_0 = u(0),\ a_1 = u'(0),\ a_2 = \frac{1}{2!}u''(0), a_3 = \frac{1}{3!}u'''(0), \tag{16.56}$$

and so on. The remaining coefficients a_k of (16.54) will be determined by applying the series solution method to the nonlinear Fredholm integro-differential equation (16.55). Substituting (16.54) into both sides of (16.55) gives

$$\left(\sum_{k=0}^{\infty} a_k x^k\right)^{(n)} = T(f(x)) + \int_0^1 K(x,t)F\left(\sum_{k=0}^{\infty} a_k t^k\right) dt, \tag{16.57}$$

where $T(f(x))$ is the Taylor series for $f(x)$. The integro-differential equation (16.55) will be converted to a traditional integral in (16.57) where instead of integrating the unknown function $F(u(x))$, terms of the form t^n, $n \geqslant 0$ will be integrated. Notice that because we are seeking series solution, then if $f(x)$ includes elementary functions such as trigonometric functions, exponential functions, etc., then Taylor expansions for functions involved in $f(x)$ should be used.

We first integrate the right side of the integral in (16.57), and collect the coefficients of like powers of x. We next equate the coefficients of like powers of x into both sides of the resulting equation to determine a recurrence relation in $a_j, j \geqslant 0$. Solving the recurrence relation will lead to a complete determination of the coefficients $a_j, j \geqslant 0$, where some of these coefficients will be used from the initial conditions. Having determined the coefficients $a_j, j \geqslant 0$, the series solution follows immediately upon substituting the derived coefficients into (16.54). The exact solution may be obtained if such an exact solution exists. If an exact solution is not obtainable, then a truncated series can be used for numerical purposes. In this case, the more terms we evaluate, the higher accuracy level we achieve. The following examples will be used to illustrate the series solution method.

Example 16.9

Solve the nonlinear Fredholm integro-differential equation by using the series solution method

$$u'(x) = 1 - \frac{2}{15}x - \frac{226}{105}x^2 + \int_{-1}^{1}(xt + x^2t^2)u^2(t))dt, u(0) = 1. \qquad (16.58)$$

Substituting $u(x)$ by the series

$$u(x) = \sum_{n=0}^{\infty} a_n x^n, \qquad (16.59)$$

into both sides of the equation (16.58) leads to

$$\left(\sum_{n=0}^{\infty} a_n x^n\right)' = 1 - \frac{2}{15}x - \frac{226}{105}x^2 + \int_{-1}^{1}\left((xt + x^2t^2)(\sum_{n=0}^{\infty} a_n t^n)^2\right)dt. \quad (16.60)$$

Evaluating the integral at the right side, using $a_0 = 1$, we find

$$\begin{aligned} & a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + 5a_5x^4 + \cdots \\ & = 1 + \left(-\frac{2}{15} + \frac{4}{3}a_1 + \frac{4}{9}a_3a_4 + \frac{4}{5}a_1a_2 + \frac{4}{7}a_1a_4 + \frac{4}{7}a_2a_3 + \frac{4}{5}a_3\right)x \\ & \left(-\frac{52}{35} + \frac{2}{9}a_3^2 + \frac{4}{9}a_2a_4 + \frac{4}{7}a_1a_3 + \frac{2}{7}a_2^2 + \frac{4}{7}a_4 + \frac{4}{5}a_2 + \frac{2}{5}a_1^2 + \frac{2}{11}a_4^2\right)x^2. \end{aligned} \qquad (16.61)$$

Equating the coefficients of like powers of x in both sides, and solving the system of equations we obtain two sets of solutions

$$a_0 = 1, a_1 = 1, a_2 = 1, a_r = 0, r \geqslant 3, \qquad (16.62)$$

Consequently, the exact solutions is given by

$$u(x) = 1 + x + x^2, \qquad (16.63)$$

Example 16.10

Solve the nonlinear Fredholm integro-differential equation by using the series solution method

$$u'(x) = e^x - \frac{1}{4}(e^2 - 1)x + \frac{1}{2}\int_0^1 xu^2(t)dt, u(0) = 1. \tag{16.64}$$

Substituting $u(x)$ by the series

$$u(x) = \sum_{n=0}^{\infty} a_n x^n, \tag{16.65}$$

into both sides of the equation (16.64) leads to

$$\left(\sum_{n=0}^{\infty} a_n x^n\right)' = T_1(e^x - \frac{1}{4}(e^2 - 1)x) + \int_0^1 \left(x\sum_{n=0}^{\infty}(a_n t^n)^2\right) dt, \tag{16.66}$$

where T_1 is the Taylor series about $x = 0$. Evaluating the integral at the right side, using $a_0 = 1$, and proceeding as before we find

$$a_n = \frac{1}{n!}, n \geqslant 0. \tag{16.67}$$

Using (16.65) gives the exact solution

$$u(x) = e^x. \tag{16.68}$$

It is worth noting that we used the series assumption up to $O(x^{12})$ to get this result.

Example 16.11

Solve the nonlinear Fredholm integro-differential equation by using the series solution method

$$u'(x) = \cos x - \left(\frac{\pi}{8} + \frac{1}{3}\right)x + \left(\frac{\pi}{6} + \frac{\pi^2}{16}\right) + \frac{1}{12}\int_0^{\pi} (x - t)u^2(t)dt, u(0) = 1. \tag{16.69}$$

Substituting $u(x)$ by the series

$$u(x) = \sum_{n=0}^{\infty} a_n x^n, \tag{16.70}$$

into both sides of the equation (16.69) leads to

$$\left(\sum_{n=0}^{\infty} a_n x^n\right)' = T_1\left(\cos x - \left(\frac{\pi}{8} + \frac{1}{3}\right)x + \left(\frac{\pi}{6} + \frac{\pi^2}{16}\right)\right) + \frac{1}{12}\int_0^{\pi} \left((x - t)\left(\sum_{n=0}^{\infty} a_n t^n\right)^2\right) dt. \tag{16.71}$$

Evaluating the integral at the right side, using $a_0 = 1$, and proceeding as before we find

$$a_0 = 1,$$
$$a_{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}, n \geqslant 0, \tag{16.72}$$
$$a_{2n} = 0, n \geqslant 0.$$

Consequently, the series solution is given by

$$u(x) = 1 + \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}, \tag{16.73}$$

that converges to the exact solution

$$u(x) = 1 + \sin x. \tag{16.74}$$

Example 16.12

Solve the nonlinear Fredholm integro-differential equation by using the series solution method

$$u''(x) = 10 - \frac{146}{35}x + \frac{1}{2}\int_{-1}^{1} xtu^2(t)dt, u(0) = 1, u'(0) = 0. \tag{16.75}$$

Substituting $u(x)$ by the series

$$u(x) = \sum_{n=0}^{\infty} a_n x^n, \tag{16.76}$$

into both sides of the equation (16.75), evaluating the integral at the right side, using $a_0 = a_2 = 0, a_1 = 1$, and proceeding as before we find

$$a_0 = 1,\ a_1 = 0,\ a_2 = 5,\ a_3 = -1,\ a_k = 0, k \geqslant 4. \tag{16.77}$$

Consequently, the exact solution is given by

$$u(x) = 1 + 5x^2 - x^3. \tag{16.78}$$

Exercises 16.2.3

Solve the following nonlinear Fredholm integro-differential equations by using the series solution method

1. $u'(x) = -\frac{47}{45} - \frac{193}{90}x + \frac{1}{12}\int_{-1}^{1} (x-t)u^2(t)\, dt,\ u(0) = 1$

2. $u'(x) = \frac{53}{45} - \frac{197}{630}x + 3x^2 + \frac{1}{12}\int_{-1}^{1} (x-t)u^2(t)\, dt,\ u(0) = 1$

3. $u'(x) = 1 + \frac{82}{45}x + \frac{1}{12}\int_{-1}^{1} xtu^2(t)\, dt,\ u(0) = 1$

4. $u'(x) = e^x - \frac{1}{96}(e^2 - 3)x + \frac{1}{24}\int_{0}^{1} xt(u^2(t) - u(t))\, dt,\ u(0) = 1$

5. $u'(x) = e^x - \frac{1}{24}e^{x-1} + \frac{1}{24}\int_{0}^{1} e^{x-2t}(u^2(t) - u(t))\, dt,\ u(0) = 1$

6. $u''(x) = -2 - \frac{4}{15}x + \frac{1}{2}\int_{-1}^{1} xtu^2(t)\,dt,\ u(0) = u'(0) = 1$

7. $u''(x) = e^x - \frac{1}{48}(e^2 - 1)x + \frac{1}{12}\int_{0}^{1} xtu^2(t)\,dt,\ u(0) = u'(0) = 1$

8. $u''(x) = \frac{1796}{315}x + \frac{1}{2}\int_{0}^{1} xt^2u^2(t)\,dt,\ u(0) = 0, u'(0) = 1$

16.3 Homogeneous Nonlinear Fredholm Integro-Differential Equations

Substituting $f(x) = 0$ into the nonlinear Fredholm integral equation of the second kind

$$u^{(i)}(x) = f(x) + \lambda \int_a^b K(x,t)F(u(t))dt, \tag{16.79}$$

gives the homogeneous nonlinear Fredholm integral equation of the second kind given by

$$u^{(i)}(x) = \lambda \int_a^b K(x,t)F(u(t))dt, \tag{16.80}$$

where $F(u(t))$ is a nonlinear function of $u(t)$. The initial conditions should be prescribed to determine the exact solution.

In this section we will focus our study on the homogeneous nonlinear Fredholm integral equation (16.80) for the specific case where the kernel $K(x,t)$ is separable. The aim for studying the homogeneous nonlinear Fredholm equation is to find nontrivial solution. Moreover, the Adomian decomposition method is not applicable here because it depends mainly on assigning a nonzero value for the zeroth component $u_0(x)$, and $f(x) = 0$ in this kind of equations. The direct computation method works effectively to handle the homogeneous nonlinear Fredholm integro-differential equations.

16.3.1 The Direct Computation Method

The direct computation method was used before in this chapter. This method replaces the homogeneous nonlinear Fredholm integro-differential equations by a single algebraic equation or by a system of simultaneous algebraic equations depending on the number of terms of the separable kernel $K(x,t)$. The direct computation method handles Fredholm integro-differential equations, homogeneous or nonhomogeneous, in a direct manner and gives the solution in an exact form and not in a series form.

As stated before, the direct computation method will be applied to equations where the kernel $K(x,t_0)$ is degenerate or separable kernel of the form

$$K(x,t) = \sum_{k=1}^{n} g_k(x)h_k(t). \tag{16.81}$$

The direct computation method can be applied as follows:

1. We first substitute (16.81) into the homogeneous nonlinear Fredholm integro-differential equation

$$u^{(i)}(x) = \lambda \int_a^b K(x,t)\, F(u(t))dt. \tag{16.82}$$

2. This substitution leads to

$$u^{(i)}(x) = \lambda g_1(x) \int_a^b h_1(t)\, F(u(t))dt + \lambda g_2(x) \int_a^b h_2(t)\, F(u(t))dt + \cdots$$
$$+\lambda g_n(x) \int_a^b h_n(t)\, F(u(t))dt. \tag{16.83}$$

3. Each integral at the right side depends only on the variable t with constant limits of integration for t. This means that each integral is equivalent to a constant. Consequently, Equation (16.83) becomes

$$u^{(i)}(x) = \lambda\alpha_1 g_1(x) + \lambda\alpha_2 g_2(x) + \cdots + \lambda\alpha_n g_n(x), \tag{16.84}$$

where

$$\alpha_i = \int_a^b h_i(t)\, F(u(t))dt, 1 \leqslant i \leqslant n. \tag{16.85}$$

4. Integrating both sides of (16.84) i times from 0 to x, and using the given initial conditions we obtain an expression for $u(x)$ in terms of α_i and x.

5. Substituting the resulting expression for $u(x)$ into (16.85) gives a system of n simultaneous algebraic equations that can be solved to determine the constants $\alpha_i, 1 \leqslant i \leqslant n$. Using the obtained values of α_i into (16.84), the solution $u(x)$ of the integro-differential equation (16.80) follows immediately.

Example 16.13

Solve the homogeneous nonlinear Fredholm integro-differential equation by using the direct computation method

$$u'(x) = \frac{1}{12}\lambda \int_0^1 xu^2(t)\, dt,\ u(0) = 1. \tag{16.86}$$

This equation may be written as

$$u'(x) = \frac{1}{12}\lambda\alpha x,\ u(0) = 1, \tag{16.87}$$

where

$$\alpha = \int_0^1 u^2(t)\, dt. \tag{16.88}$$

Integrating both sides of (16.87) from 0 to x, and using the initial condition, we find

$$u(x) = 1 + \frac{1}{24}\lambda\alpha x^2. \tag{16.89}$$

Substituting (16.89) into (16.88), evaluating the integral, and solving the resulting equation for α we obtain

$$\alpha = \frac{2880 - 80\lambda \pm 32\sqrt{8100 - 450\lambda - 5\lambda^2}}{2\lambda^2}. \tag{16.90}$$

This in turn gives the exact solutions

$$u(x) = 1 + \frac{2880 - 80\lambda \pm 32\sqrt{8100 - 450\lambda - 5\lambda^2}}{48\lambda} x^2. \tag{16.91}$$

We now consider the following three cases:

1. Using $\lambda = 0$ into (16.89) gives the exact solution $u(x) = 1$. However, $u(x)$ is undefined by using $\lambda = 0$ into (16.91). Hence, $\lambda = 0$ is a singular point of homogeneous nonlinear Fredholm integro-differential equation (16.86).

2. For $\lambda = -45 \pm 27\sqrt{5}$, Equation (16.91) gives the exact solutions

$$u(x) = 1 \pm \sqrt{5}\, x^2. \tag{16.92}$$

Consequently, there are two *bifurcation points*, namely $-45 \pm 27\sqrt{5}$ for this equation. This shows that for $-45 - 27\sqrt{5} < \lambda < -45 + 27\sqrt{5}$, then equation (16.86) gives two real solutions, but has no real solutions for $\lambda > -45 + 27\sqrt{5}$ or $\lambda < -45 - 27\sqrt{5}$.

3. For $-45 - 27\sqrt{5} < \lambda < -45 + 27\sqrt{5}$, Equation (16.86) gives two exact real solutions. This is normal for nonlinear problems, where solution may not be unique.

Example 16.14

Solve the homogeneous nonlinear Fredholm integro-differential equation by using the direct computation method

$$u'(x) = \lambda \int_0^1 e^{x-t}\, u^2(t)\, dt, u(0) = 0. \tag{16.93}$$

This equation can be rewritten as

$$u'(x) = \alpha\lambda e^x, u(0) = 0, \tag{16.94}$$

where

$$\alpha = \int_0^1 e^{-t}\, u^2(t)\, dt. \tag{16.95}$$

Integrating both sides of (16.94) from 0 to x, and using the initial condition, we find

$$u(x) = \lambda\alpha(e^x - 1). \tag{16.96}$$

Substituting (16.96) into (16.95), evaluating the integral, and solving the resulting equation for α we obtain

$$\alpha = \frac{1}{\lambda^2(2\sinh 1 - 2)}. \tag{16.97}$$

This in turn gives the exact solution by

$$u(x) = \frac{e^x - 1}{\lambda(2\sinh 1 - 2)}. \tag{16.98}$$

Notice that $\lambda = 0$ is a singular point of this equation.

Example 16.15

Solve the homogeneous nonlinear Fredholm integro-differential equation by using the direct computation method

$$u'(x) = \lambda \int_{-1}^{1} (xt + x^2t^2)\, u^2(t)\, dt, u(0) = 0. \tag{16.99}$$

This equation can be rewritten as

$$u'(x) = \alpha\lambda x + \beta\lambda x^2, u(0) = 0, \tag{16.100}$$

where

$$\alpha = \int_{-1}^{1} t\, u^2(t)\, dt, \quad \beta = \int_{-1}^{1} t^2\, u^2(t)\, dt. \tag{16.101}$$

Integrating both sides of (16.100) from 0 to x, and using the initial condition, we find

$$u(x) = \frac{1}{2}\lambda\alpha x^2 + \frac{1}{3}\lambda\beta x^3. \tag{16.102}$$

Substituting (16.102) into (16.101), evaluating the integrals, and solving the resulting system of equations for α and β we obtain

$$\alpha = 0, \frac{14\sqrt{5}}{3\lambda^2}, \quad \beta = \frac{81}{2\lambda^2}, \frac{21}{2\lambda^2}. \tag{16.103}$$

This in turn gives the exact solutions

$$u(x) = \frac{27}{2\lambda}x^3, \frac{7\sqrt{5}}{3\lambda}x^2 + \frac{7}{2\lambda}x^3. \tag{16.104}$$

Notice that $\lambda = 0$ is a singular point of this equation.

Example 16.16

Solve the homogeneous nonlinear Fredholm integro-differential equation by using the direct computation method

$$u''(x) = \lambda \int_{-1}^{1} (1 + xt)\, (u(t) - u^2(t))\, dt, u(0) = u'(0) = 0. \tag{16.105}$$

This equation can be rewritten as

$$u''(x) = \alpha\lambda + \beta\lambda x, u(0) = u'(0) = 0, \tag{16.106}$$

where

$$\alpha = \int_{-1}^{1} (u(t) - u^2(t))\, dt, \quad \beta = \int_{-1}^{1} t\, (u(t) - u^2(t))\, dt. \tag{16.107}$$

Integrating both sides of (16.106) twice from 0 to x, and using the initial conditions, we find

$$u(x) = \frac{1}{2}\lambda\alpha x^2 + \frac{1}{6}\lambda\beta x^3. \tag{16.108}$$

Substituting (16.108) into (16.107), evaluating the integrals, and solving the resulting system of equations for α and β we obtain

$$\alpha = \frac{10(\lambda-3)}{3\lambda^2}, \frac{7(\lambda-15)}{5\lambda^2}, \quad \beta = 0, \pm\frac{7\sqrt{15(29\lambda+165)(\lambda-15)}}{25\lambda^2}. \quad (16.109)$$

(i) For $\alpha = \frac{10(\lambda-3)}{3\lambda^2}, \beta = 0$, the exact solution is given by

$$u(x) = \frac{5(\lambda-3)}{3\lambda}x^2. \quad (16.110)$$

(ii) For $\alpha = \frac{7(\lambda-15)}{5\lambda^2}, \beta = \pm\frac{7\sqrt{15(29\lambda+165)(\lambda-15)}}{25\lambda^2}$, the exact solutions are given by

$$u(x) = \frac{7}{150\lambda}\left(15(\lambda-15)x^2 \pm \sqrt{15(29\lambda+165)(\lambda-15)}\,x^3\right). \quad (16.111)$$

We now consider the following cases:

1. Using $\lambda = 0$ into (16.105) gives the trivial solution $u(x) = 0$. However, $u(x)$ is undefined by using $\lambda = 0$ into the last solution. Hence, $\lambda = 0$ is a singular point of equation (16.105).

2. There are two *bifurcation points*, namely $\lambda = -\frac{169}{29}$ and $\lambda = 15$ for this equation. This shows that for $\lambda < -\frac{169}{29}$, and for $\lambda > 15$, then equation (16.105) gives two real solutions, but has no real solutions for $-\frac{169}{29} < \lambda < 15$.

Exercises 16.3.1

Use the direct computation method to solve the following Fredholm integro-differential equations:

1. $u'(x) = \frac{1}{12}\lambda\int_{-1}^{1} x(u(t) - u^3(t))\,dt, u(0) = 0$

2. $u'(x) = \frac{1}{12}\lambda\int_{-1}^{1} x(u(t) - u^2(t))\,dt, u(0) = 0$

3. $u'(x) = \frac{1}{24}\lambda\int_{0}^{1} x(1 - u^2(t))\,dt, u(0) = 1$

4. $u'(x) = \frac{1}{24}\lambda\int_{0}^{\pi} \sin x \sin t(1 + u^2(t))\,dt, u(0) = 0$

5. $u'(x) = \frac{1}{12}\lambda\int_{0}^{\pi} (x + x^2 t)u^2(t)\,dt, u(0) = 0$

6. $u'(x) = \lambda\int_{-\pi}^{\pi} \sin(x+t)(1 - u^2(t))\,dt, u(0) = 0$

7. $u'(x) = \lambda\int_{-\pi}^{\pi} \cos(x+t)(1 - u^2(t))\,dt, u(0) = 0$

8. $u'(x) = \lambda\int_{-\pi}^{\pi} \cos(x+t)(1 - u^2(t))\,dt, u(0) = 0$

9. $u''(x) = \lambda\int_{-1}^{1} x(u(t) + u^2(t))\,dt, u(0) = 1, u'(0) = 0$

10. $u''(x) = \frac{1}{2}\lambda \int_{-1}^{1} x^2(1+u^2(t))\, dt, u(0) = 0, u'(0) = 1$

11. $u''(x) = \frac{1}{2}\lambda \int_{-1}^{1} (1-x^2t)(u(t)-u^2(t))\, dt, u(0) = u'(0) = 0$

12. $u''(x) = \lambda \int_{-\pi}^{\pi} \cos(x-t)(u(t)-u^2(t))\, dt, u(0) = u'(0) = 0$

16.4 Systems of Nonlinear Fredholm Integro-Differential Equations

In this section, systems of Fredholm integro-differential equations of the second kind given by

$$\begin{aligned} u^{(i)}(x) &= f_1(x) + \int_a^b \left(K_1(x,t)F_1(u(t)) + \tilde{K}_1(x,t)\tilde{F}_1(v(t))\right) dt, \\ v^{(i)}(x) &= f_2(x) + \int_a^b \left(K_2(x,t)F_2(u(t)) + \tilde{K}_2(x,t)\tilde{F}_2(v(t))\right) dt, \end{aligned} \tag{16.112}$$

will be studied. The unknown functions $u(x)$ and $v(x)$ occur inside the integral sign whereas the derivatives of $u(x), v(x)$ appear mostly outside the integral sign. The kernels $K_i(x,t)$ and $\tilde{K}_i(x,t)$, and the function $f_i(x)$ are given real-valued functions. The functions F_i and $\tilde{F}_i$ are nonlinear functions for $u(x)$ and $v(x)$ respectively.

In Chapter 11, two analytical methods were used for solving systems of linear Fredholm integro-differential equations. These methods are the direct computation method and the variational iteration method. The aforementioned methods can effectively handle the systems of nonlinear Fredholm integro-differential equations (16.112). The other methods presented in this text can also be used for handling such systems.

16.4.1 The Direct Computation Method

The direct computation method will be applied to solve the systems of nonlinear Fredholm integro-differential equations of the second kind. The method approaches any Fredholm equation in a direct manner and gives the solution in an exact form and not in a series form. The method will be applied for the degenerate or separable kernels of the form

$$\begin{aligned} K_1(x,t) &= \sum_{k=1}^{n} g_k(x)h_k(t), \quad \tilde{K}_1(x,t) = \sum_{k=1}^{n} \tilde{g}_k(x)\tilde{h}_k(t), \\ K_2(x,t) &= \sum_{k=1}^{n} r_k(x)s_k(t), \quad \tilde{K}_2(x,t) = \sum_{k=1}^{n} \tilde{r}_k(x)\tilde{s}_k(t). \end{aligned} \tag{16.113}$$

The direct computation method can be applied as follows:

1. We first substitute (16.113) into the system of Fredholm integro-differential equations (16.112) to obtain

$$u^{(i)}(x) = f_1(x) + \sum_{k=1}^{n} g_k(x) \int_a^b h_k(t)u(t)dt + \sum_{k=1}^{n} \tilde{g}_k(x) \int_a^b \tilde{h}_k(t)u(t)dt,$$
$$v^{(i)}(x) = f_2(x) + \sum_{k=1}^{n} r_k(x) \int_a^b s_k(t)v(t)\,dt + \sum_{k=1}^{n} \tilde{r}_k(x) \int_a^b \tilde{s}_k(t)v(t)\,dt. \quad (16.114)$$

2. Each integral at the right side depends only on the variable t with constant limits of integration for t. This means that Equation (16.114) becomes

$$u^{(i)}(x) = f_1(x) + \alpha_1 g_1(x) + \cdots + \alpha_n g_n(x) + \beta_1 \tilde{g}_1(x) + \cdots + \beta_n \tilde{g}_n(x),$$
$$v^{(i)}(x) = f_2(x) + \gamma_1 r_1(x) + \cdots + \gamma_n r_n(x) + \delta_1 \tilde{r}_1(x) + \cdots + \delta_n \tilde{r}_n(x), \quad (16.115)$$

where

$$\alpha_i = \int_a^b h_i(t)\,u(t)dt, 1 \leqslant i \leqslant n, \quad \beta_i = \int_a^b \tilde{h}_i(t)\,v(t)dt, 1 \leqslant i \leqslant n,$$
$$\gamma_i = \int_a^b s_i(t)\,u(t)dt, 1 \leqslant i \leqslant n, \quad \delta_i = \int_a^b \tilde{s}_i(t)\,v(t)dt, 1 \leqslant i \leqslant n. \quad (16.116)$$

3. Integrating both sides of (16.115) i times from 0 to x, and substituting the resulting equations for $u(x)$ and $v(x)$ into (16.116) gives a system of algebraic equations that can be solved to determine the constants $\alpha_i, \beta_i, \gamma_i$, and δ_i. Using the obtained numerical values of these constants, the solutions $u(x)$ and $v(x)$ of the system (16.112) follow immediately.

Example 16.17

Solve the system of nonlinear Fredholm integro-differential equations by using the direct computation method

$$u'(x) = \sin x + x\cos x - \frac{\pi^3}{3} + \int_0^{\pi} \left(u^2(t) + v^2(t)\right)\, dt, u(0) = 0$$
$$v'(x) = \cos x - x\sin x + \frac{\pi}{2} + \int_0^{\pi} \left(u^2(t) - v^2(t)\right)\, dt, v(0) = 0. \quad (16.117)$$

Following the analysis presented above, this system can be rewritten as

$$u'(x) = \sin x + x\cos x + \left(\alpha + \beta - \frac{\pi^3}{3}\right),$$
$$v'(x) = \cos x - x\sin x + \left(\alpha - \beta + \frac{\pi}{2}\right), \quad (16.118)$$

where

$$\alpha = \int_0^{\pi} u^2(t)\,dt, \quad \beta = \int_0^{\pi} v^2(t)\,dt. \quad (16.119)$$

Integrating both sides of (16.118) once from 0 to x gives

$$\begin{aligned} u(x) &= x\sin x + \left(\alpha + \beta - \frac{\pi^3}{3}\right)x, \\ v(x) &= x\cos x + \left(\alpha - \beta + \frac{\pi}{2}\right)x. \end{aligned} \tag{16.120}$$

Substituting (16.120) into (16.119), and solving the resulting system gives

$$\alpha = \frac{1}{6}\pi^3 - \frac{1}{4}\pi, \quad \beta = \frac{1}{6}\pi^3 + \frac{1}{4}\pi. \tag{16.121}$$

Substituting (16.121) into (16.120) leads to the exact solutions

$$(u(x), v(x)) = (x\sin x, x\cos x). \tag{16.122}$$

Example 16.18

Solve the system of nonlinear Fredholm integro-differential equations by using the direct computation method

$$\begin{aligned} u'(x) &= \sinh x - \frac{15}{16} + \int_0^{\ln 2} (u^2(t) + v^2(t))dt, u(0) = 1, \\ v'(x) &= \cosh x - \ln 2 + \int_0^{\ln 2} (u^2(t) - v^2(t))dt, v(0) = 0. \end{aligned} \tag{16.123}$$

This system can be rewritten as

$$\begin{aligned} u'(x) &= \sinh x + \left(\alpha + \beta - \frac{15}{16}\right), \\ v'(x) &= \cosh x + (\alpha - \beta - \ln 2), \end{aligned} \tag{16.124}$$

where

$$\alpha = \int_0^{\ln 2} u^2(t)\,dt, \quad \beta = \int_0^{\ln 2} v^2(t)\,dt. \tag{16.125}$$

Integrating both sides of (16.124) once from 0 to x, and using the initial conditions we find

$$\begin{aligned} u(x) &= \cosh x + \left(\alpha + \beta - \frac{15}{16}\right)x, \\ v(x) &= \sinh x + (\alpha - \beta - \ln 2)x. \end{aligned} \tag{16.126}$$

To determine α, and β, we substitute (16.126) into (16.125) and solving the resulting system we obtain

$$\alpha = \frac{15}{32} + \frac{1}{2}\ln 2, \quad \beta = \frac{15}{32} - \frac{1}{2}\ln 2. \tag{16.127}$$

Substituting (16.127) into (16.126) leads to the exact solutions

$$(u(x), v(x)) = (\cosh x, \sinh x). \tag{16.128}$$

Example 16.19

Solve the system of nonlinear Fredholm integro-differential equations by using the direct computation method

$$\begin{aligned} u'(x) &= e^x - 12 + \int_0^{\ln 2} (u^2(t) + v^2(t))dt, u(0) = 1, \\ v'(x) &= 3e^{3x} + 9 + \int_0^{\ln 2} (u^2(t) - v^2(t))dt, v(0) = 1. \end{aligned} \tag{16.129}$$

Proceeding as before we set

$$\begin{aligned} u'(x) &= e^x + (\alpha + \beta - 12), \\ v'(x) &= 3e^{3x} + (\alpha - \beta + 9), \end{aligned} \tag{16.130}$$

where

$$\alpha = \int_0^{\ln 2} u^2(t)\, dt, \quad \beta = \int_0^{\ln 2} v^2(t)\, dt. \tag{16.131}$$

Integrating both sides of (16.130) once from 0 to x, and using the initial conditions we find

$$\begin{aligned} u(x) &= e^x + (\alpha + \beta - 12)x, \\ v(x) &= e^{3x} + (\alpha - \beta + 9)x. \end{aligned} \tag{16.132}$$

To determine α, and β, we substitute (16.132) into (16.131) and solving the resulting system we obtain

$$\alpha = \frac{3}{2}, \quad \beta = \frac{21}{2}. \tag{16.133}$$

Substituting (16.133) into (16.132) leads to the exact solutions

$$(u(x), v(x)) = (e^x, e^{3x}). \tag{16.134}$$

Example 16.20

Solve the system of nonlinear Fredholm integro-differential equations by using the direct computation method

$$\begin{aligned} u''(x) &= -\sin x - \frac{2}{3} + \int_0^{\frac{\pi}{2}} (v(t)w(t))\, dt, \\ & u(0) = 0, u'(0) = 1 \\ v''(x) &= -\cos x - \frac{2}{3} + \int_0^{\frac{\pi}{2}} (w(t)u(t))\, dt, \\ & v(0) = 1, v'(0) = 0, \\ w''(x) &= -4\sin 2x - \frac{1}{2} + \int_0^{\frac{\pi}{2}} (u(t)v(t))\, dt, \\ & w(0) = 0, w'(0) = 2. \end{aligned} \tag{16.135}$$

This system can be rewritten as

$$\begin{aligned} u''(x) &= -\sin x + \left(\alpha - \frac{2}{3}\right), \\ v''(x) &= -\cos x + \left(\beta - \frac{2}{3}\right), \end{aligned} \tag{16.136}$$

$$w''(x) = -4\sin 2x + \left(\gamma - \frac{1}{2}\right),$$

where

$$\alpha = \int_0^{\frac{\pi}{2}} v(t)w(t)\,dt,$$
$$\beta = \int_0^{\frac{\pi}{2}} w(t)u(t)\,dt, \quad (16.137)$$
$$\gamma = \int_0^{\frac{\pi}{2}} u(t)v(t)\,dt.$$

Integrating both sides of (16.136) twice from 0 to x, and using the initial conditions we obtain

$$u(x) = \sin x + \left(\frac{1}{2}\alpha - \frac{1}{3}\right)x^2,$$
$$v(x) = \cos x + \left(\frac{1}{2}\beta - \frac{1}{3}\right)x^2, \quad (16.138)$$
$$w(x) = \sin 2x + \left(\frac{1}{2}\gamma - \frac{1}{4}\right)x^2.$$

Proceeding as before we obtain

$$\alpha = \frac{2}{3},\ \beta = \frac{2}{3},\ \gamma = \frac{1}{2}. \quad (16.139)$$

This in turn gives the exact solutions

$$(u(x), v(x), w(x)) = (\sin x, \cos x, \sin 2x). \quad (16.140)$$

Exercises 16.4.1

Use the direct computation method to solve the following systems of nonlinear Fredholm integro-differential equations

1. $$\begin{cases} u'(x) = \cos x - x\sin x - \pi^3 + \int_0^{\pi} 3\left(u^2(t) + v^2(t)\right)dt, u(0) = 0 \\ v'(x) = \sin x + x\cos x - \frac{\pi}{2} + \int_0^{\pi} \left(u^2(t) - v^2(t)\right)dt, v(0) = 0 \end{cases}$$

2. $$\begin{cases} u'(x) = \sinh 2x + \frac{1}{2}(\ln 2) + \frac{15}{32} + \int_0^{\ln 2} \left(u^2(t) - u(t)v(t)\right)dt \\ v'(x) = \sinh 2x - \frac{3}{2}\ln 2 - \frac{15}{32} + \int_0^{\ln 2} \left(v^2(t) - u(t)v(t)\right)dt \\ u(0) = 1, v(0) = 2 \end{cases}$$

3. $$\begin{cases} u'(x) = -\sin x - \frac{\pi}{4} + \int_0^{\frac{\pi}{4}} u^2(t)\,v^2(t)\,dt, u(0) = 1 \\ v'(x) = \sec x\tan x - \frac{\pi}{4} + \int_0^{\frac{\pi}{2}} u(t)\,v(t)\,dt, v(0) = 1 \end{cases}$$

4. $\begin{cases} u'(x) = \sinh(2x) - (\ln 2) - \dfrac{15}{16} + \displaystyle\int_0^{\ln 2} \left(u^2(t) + u(t)v(t)\right) dt \\ v'(x) = -\sinh(2x) - 3\ln 2 + \dfrac{15}{16} + \displaystyle\int_0^{\ln 2} \left(v^2(t) + u(t)v(t)\right) dt \\ u(0) = 1, v(0) = 1 \end{cases}$

5. $\begin{cases} u'(x) = (1+x)e^x - \dfrac{1}{3}(\ln 2)^3(e^x - 1) + \displaystyle\int_0^{\ln 2} \left(e^{x-2t}u^2(t) - u(t)v(t)\right) dt, u(0) = 0 \\ v'(x) = (1-x)e^{-x} - \dfrac{1}{3}(\ln 2)^3(e^x + 1) + \displaystyle\int_0^{\ln 2} \left(e^{x+2t}v^2(t) + u(t)v(t)\right) dt, v(0) = 0 \end{cases}$

6. $\begin{cases} u''(x) = -\cos x - \sin x - \dfrac{\pi}{2} - 1 + \displaystyle\int_0^{\frac{\pi}{2}} \left(u^2(t) + u(t)v(t)\right) dt \\ v''(x) = -\cos x + \sin x - \dfrac{\pi}{2} + 1 + \displaystyle\int_0^{\frac{\pi}{2}} \left(v^2(t) + u(t)v(t)\right) dt \\ u(0) = 1, u'(0) = 1, v(0) = 1, v'(0) = -1 \end{cases}$

7. $\begin{cases} u''(x) = e^x + \left(\dfrac{1}{3} - e^2\right) + \displaystyle\int_0^1 \left(u^2(t) + v^2(t)\right) dt, u(0) = 1, u'(0) = 2 \\ v''(x) = -e^x - 4 + \displaystyle\int_0^1 \left(u^2(t) - v^2(t)\right) dt, v(0) = -1, v'(0) = 0 \end{cases}$

8. $\begin{cases} u''(x) = e^x - \dfrac{255}{8} + \displaystyle\int_0^{\ln 2} v(t)w(t)dt, u(0) = 1, u'(0) = 1 \\ v''(x) = 9e^{3x} - \dfrac{21}{2} + \displaystyle\int_0^{\ln 2} w(t)u(t)dt, v(0) = 1, v'(0) = 3 \\ w''(x) = 25e^{5x} - \dfrac{15}{4} + \displaystyle\int_0^{\ln 2} u(t)v(t)dt, w(0) = 1, w'(0) = 5 \end{cases}$

16.4.2 The Variational Iteration Method

The variational iteration method was used to handle the Fredholm integral equations, the Fredholm integro-differential equations and the nonlinear Fredholm integro-differential equations. The method provides rapidly convergent successive approximations of the exact solution if such a closed form solution exists, and not components as in Adomian decomposition method. The variational iteration method handles linear and nonlinear problems in the same manner without any need to specific restrictions such as the so called Adomian polynomials that we need for nonlinear terms.

The correction functionals for the system of nonlinear Fredholm integro-differential equations

$$\begin{aligned} u^{(i)}(x) &= f_1(x) + \int_a^b \left(K_1(x,t)F_1(u(t)) + \widetilde{K}_1(x,t)\widetilde{F}_1(v(t))\right) dt, \\ v^{(i)}(x) &= f_2(x) + \int_a^b \left(K_2(x,t)F_2(u(t)) + \widetilde{K}_2(x,t)\widetilde{F}_2(v(t))\right) dt. \end{aligned} \quad (16.141)$$

are given by

$$\begin{aligned} u_{n+1}(x) &= u_n(x) + \int_0^x \lambda(t)\left(u_n^{(i)}(t) - f_1(t) - \Gamma_1(t)\right) dt, \\ v_{n+1}(x) &= v_n(x) + \int_0^x \lambda(t)\left(v_n^{(i)}(t) - f_2(t) - \Gamma_2(t)\right) dt. \end{aligned} \tag{16.142}$$

$$\begin{aligned} \Gamma_1 &= \int_a^b \left(K_1(t,r)F_1(\tilde{u}_n(r)) + \widetilde{K}_1(t,r)\widetilde{F}_1(\tilde{v}_n(r))\right) dr, \\ \Gamma_2 &= \int_a^b \left(K_2(t,r)F_2(\tilde{u}_n(r)) + \widetilde{K}_2(t,r)\widetilde{F}_2(\tilde{v}_n(r))\right) dr. \end{aligned} \tag{16.143}$$

As stated before it is necessary to determine the Lagrange multiplier λ that can be identified optimally. An iteration formula, without restricted variation, should be used for the determination of the successive approximations $u_n(x), v_n(x), n \geqslant 0$ of the solutions $u(x)$ and $v(x)$. The zeroth approximations $u_0(x)$ and $v_0(x)$ can be selected by using the initial conditions. Consequently, the solutions are given by

$$u(x) = \lim_{n\to\infty} u_n(x), v(x) = \lim_{n\to\infty} v_n(x). \tag{16.144}$$

The VIM will be illustrated by studying the following examples.

Example 16.21

Use the variational iteration method to solve the system of nonlinear Fredholm integro-differential equations

$$\begin{aligned} u'(x) &= \cos x - 5\sin x + \int_0^\pi \cos(x-t)(u^2(t)+v^2(t))dt, u(0) = 1, \\ v'(x) &= 3\cos x - \sin x + \int_0^\pi \sin(x-t)(u^2(t)+v^2(t))dt, v(0) = 1. \end{aligned} \tag{16.145}$$

The correction functionals for this system are given by

$$\begin{aligned} u_{n+1}(x) &= u_n(x) - \int_0^x \left(u_n'(t) - \cos t + 5\sin t - \rho_1\right) dt, \\ v_{n+1}(x) &= v_n(x) - \int_0^x \left(v_n'(t) - 3\cos t + \sin t - \rho_2\right) dt, \end{aligned} \tag{16.146}$$

where

$$\begin{aligned} \rho_1 &= \int_0^\pi \cos(t-r)(u_n^2(r) + v_n^2(r))\, dr, \\ \rho_2 &= \int_0^\pi \sin(t-r)(u_n^2(r) + v_n^2(r))\, dr. \end{aligned} \tag{16.147}$$

We can select the zeroth approximations $u_0(x) = u(0) = 1$ and $v_0(x) = v(0) = 1$. Using this selection, the correction functionals give the following successive approximations

$$\begin{aligned}
u_0(x) &= 1, \quad v_0(x) = 1,\\
u_1(x) &= u_0(x) - \int_0^x \left(u_0'(t) - \cos t + 5\sin t - \int_0^{\pi} \cos(t-r)(u_0^2(r) + v_0^2(r))dr \right) dt,\\
&= \cos x + \sin x, \qquad (16.148)\\
v_1(x) &= v_0(x) - \int_0^x \left(v_0'(t) - 3\cos t + \sin t - \int_0^{\pi} \sin(t-r)(u_0^2(r) + v_0^2(r))dr \right) dt,\\
&= \cos x - \sin x,
\end{aligned}$$

where we obtained the same approximations for $u_j(x)$ and $v_j(x), j \geqslant 2$. The exact solutions are therefore given by

$$(u(x), v(x)) = (\cos x + \sin x, \cos x - \sin x). \qquad (16.149)$$

Example 16.22

Use the variational iteration method to solve the system of nonlinear Fredholm integro-differential equations

$$\begin{aligned}
u'(x) &= e^x - \frac{7}{64} + \frac{1}{48}\int_0^{\ln 2} (u^2(t) + v^2(t))dt, u(0) = 1,\\
v'(x) &= 2e^{2x} + \frac{3}{64} + \frac{1}{48}\int_0^{\ln 2} (u^2(t) - v^2(t))dt, v(0) = 1.
\end{aligned} \qquad (16.150)$$

The correction functionals for this system are given by

$$\begin{aligned}
u_{n+1}(x) &= u_n(x)\\
&\quad - \int_0^x \left(u_n'(t) - e^t + \frac{7}{64} - \frac{1}{48}\int_0^{\ln 2} (u_n^2(r) + v_n^2(r))\, dr \right) dt,\\
v_{n+1}(x) &= v_n(x)\\
&\quad - \int_0^x \left(v_n'(t) - 2e^{2t} - \frac{3}{64} - \frac{1}{48}\int_0^{\ln 2} (u_n^2(r) - v_n^2(r))\, dr \right) dt.
\end{aligned} \qquad (16.151)$$

Using the initial conditions to select $u_0(x) = 1$ and $v_0(x) = 1$. Consequently, the correction functionals will give the following successive approximations

$$\begin{aligned}
u_0(x) &= 1, \quad v_0(x) = 1,\\
u_1(x) &= e^x - 8.049386747 \times 10^{-3}x,\\
v_1(x) &= e^{2x} + 4.6875 \times 10^{-3}x,\\
u_2(x) &= e^x - 3.2768581 \times 10^{-5}x,\\
v_2(x) &= e^{2x} - 2.528456529 \times 10^{-3}x, \qquad (16.152)\\
u_3(x) &= e^x - 6.754775182 \times 10^{-5}x,\\
v_3(x) &= e^{2x} + 6.649289722 \times 10^{-5}x,\\
u_4(x) &= e^x + 6.7567661461 \times 10^{-7}x,\\
v_4(x) &= e^{2x} - 2.850098438 \times 10^{-6}x,
\end{aligned}$$

and so on. The exact solutions are therefore given by

$$(u(x), v(x)) = (e^x, e^{2x}). \tag{16.153}$$

Example 16.23

Use the variational iteration method to solve the system of nonlinear Fredholm integro-differential equations

$$\begin{aligned} u'(x) &= 2x + \frac{149}{64} + \frac{1}{64}\int_0^1 (u^2(t) + v^2(t))dt, u(0) = 1, \\ v'(x) &= 2x - \frac{67}{64} + \frac{1}{64}\int_0^1 (u^2(t) - v^2(t))dt, v(0) = 1. \end{aligned} \tag{16.154}$$

The correction functionals for this system are given by

$$\begin{aligned} u_{n+1}(x) &= u_n(x) \\ &\quad - \int_0^x \left(u'_n(t) - 2t - \frac{149}{64} - \frac{1}{64}\int_0^1 (u_n^2(r) + v_n^2(r))\,dr\right) dt, \\ v_{n+1}(x) &= v_n(x) \\ &\quad - \int_0^x \left(v'_n(t) - 2t + \frac{67}{64} - \frac{1}{64}\int_0^1 (u_n^2(r) - v_n^2(r))\,dr\right) dt. \end{aligned} \tag{16.155}$$

We can use the initial conditions to select $u_0(x) = 1$ and $v_0(x) = 1$. Using this selection into the correction functionals gives the following successive approximations

$$\begin{aligned} u_0(x) &= 0, \quad v_0(x) = 1, \\ u_1(x) &= 1 + 0.9625x + x^2, \\ v_1(x) &= 1 - 1.046875x + x^2, \\ u_2(x) &= 1 + 0.9981388855x + x^2, \\ v_2(x) &= 1 - 1.0006633x + x^2, \\ u_3(x) &= 1 + 0.9999283771x + x^2, \\ v_3(x) &= 1 - 1.000054354x + x^2, \\ u_4(x) &= 1 + 0.9999968676x + x^2, \\ v_4(x) &= 1 - 1.000001717x + x^2, \end{aligned} \tag{16.156}$$

and so on. It is obvious that the constant and the coefficient of x^2 are fixed for both $u(x)$ and $v(x)$. However, the coefficient of x increases to 1 for $u(x)$ and decreases to 1 for $v(x)$. Consequently, the approximations converge and give the exact solutions by

$$(u(x), v(x)) = (1 + x + x^2, 1 - x + x^2). \tag{16.157}$$

Example 16.24

Use the variational iteration method to solve the system of nonlinear Fredholm integro-differential equations

$$u''(x) = -\cos x - \frac{3\pi}{128} + \frac{1}{64}\int_0^{\frac{\pi}{2}} (u^2(t) + v^2(t))dt, u(0) = 2, u'(0) = 0,$$
$$v''(x) = \sin x - \frac{1}{16} + \frac{1}{64}\int_0^{\frac{\pi}{2}} (u^2(t) - v^2(t))dt, v(0) = 1, v'(0) = -1. \tag{16.158}$$

The correction functionals for this system are given by

$$u_{n+1}(x) = u_n(x) + \int_0^x \left((t-x)(u_n''(t) + \cos t + \frac{3\pi}{128} - \frac{1}{64}\int_0^{\frac{\pi}{2}} \left(u_n^2(r) + v_n^2(r)\right) dr \right) dt,$$
$$v_{n+1}(x) = v_n(x) + \int_0^x \left((t-x)(v_n''(t) - \sin t + \frac{1}{16} - \frac{1}{64}\int_0^{\frac{\pi}{2}} \left(u_n^2(r) - v_n^2(r)\right) dr \right) dt. \tag{16.159}$$

Notice that the Lagrange multiplier $\lambda = (t - x)$ because each equation is of second order. We can use the initial conditions to select $u_0(x) = 2$ and $v_0(x) = 1 - x$. Using this selection into the correction functionals and proceeding as before we obtain the following successive approximations

$$\begin{aligned}
u_0(x) &= 2, \quad v_0(x) = 1 - x,\\
u_1(x) &= 1 + \cos x + 0.01536031054x^2,\\
v_1(x) &= 1 - \sin x - 0.01474892098x^2,\\
u_2(x) &= 1 + \cos x + 0.00046366859x^2,\\
v_2(x) &= 1 - \sin x + 0.0003878775265x^2,\\
u_3(x) &= 1 + \cos x + 0.00001366260916x^2,\\
v_3(x) &= 1 - \sin x + 0.0007639191808x^2,\\
u_4(x) &= 1 + \cos x + 0.000002178741859x^2,\\
v_4(x) &= 1 - \sin x - 0.00000142757975x^2,
\end{aligned} \tag{16.160}$$

and so on. Consequently, the exact solutions are given by

$$(u(x), v(x)) = (1 + \cos x, 1 - \sin x). \tag{16.161}$$

Exercises 16.4.2

Use the variational iteration method to solve the following systems of Fredholm integro-differential equations

1. $$\begin{cases} u'(x) = \cos x - \dfrac{\pi}{128} + \dfrac{1}{64}\displaystyle\int_0^{\frac{\pi}{2}} \left(u^2(t) + v^2(t)\right) dt, u(0) = 0 \\ v'(x) = -\sin x + \dfrac{1}{64}\displaystyle\int_0^{\frac{\pi}{2}} \left(u^2(t) - v^2(t)\right) dt, v(0) = 1 \end{cases}$$

2. $$\begin{cases} u'(x) = \sin x + x\cos x - \dfrac{\pi^3}{576} + \dfrac{1}{24}\displaystyle\int_0^{\frac{\pi}{2}} \left(u^2(t) + v^2(t)\right)\,dt, u(0) = 0 \\ v'(x) = \cos x - x\sin x - \dfrac{\pi}{96} + \dfrac{1}{24}\displaystyle\int_0^{\frac{\pi}{2}} \left(u^2(t) - v^2(t)\right)\,dt, v(0) = 0 \end{cases}$$

3. $$\begin{cases} u'(x) = e^x - \dfrac{5}{64} + \dfrac{1}{24}\displaystyle\int_0^{\ln 2} \left(u^2(t) + v^2(t)\right)\,dt, u(0) = 1 \\ v'(x) = -e^{-x} - \dfrac{3}{64} + \dfrac{1}{24}\displaystyle\int_0^{\ln 2} \left(u^2(t) - v^2(t)\right)\,dt, v(0) = 1 \end{cases}$$

4. $$\begin{cases} u'(x) = e^x - \dfrac{1}{2} + \dfrac{1}{24}\displaystyle\int_0^{\ln 2} \left(u^2(t) + v^2(t)\right)\,dt, u(0) = 1 \\ v'(x) = 3e^{3x} + \dfrac{3}{8} + \dfrac{1}{24}\displaystyle\int_0^{\ln 2} \left(u^2(t) - v^2(t)\right)\,dt, v(0) = 1 \end{cases}$$

5. $$\begin{cases} u'(x) = e^x - \dfrac{7}{32} + \dfrac{1}{24}\displaystyle\int_0^{\ln 2} \left(u^2(t) + u(t)v(t)\right)\,dt, u(0) = 1 \\ v'(x) = 3e^{3x} + \dfrac{3}{32} + \dfrac{1}{24}\displaystyle\int_0^{\ln 2} \left(u^2(t) - u(t)v(t)\right)\,dt, v(0) = 1 \end{cases}$$

6. $$\begin{cases} u'(x) = 3x^2 + \dfrac{15}{16} + \dfrac{1}{64}\displaystyle\int_{-1}^{1} \left(u^2(t) + u(t)v(t)\right)\,dt, u(0) = 1 \\ v'(x) = -3x^2 - \dfrac{17}{16} + \dfrac{1}{64}\displaystyle\int_{-1}^{1} \left(v^2(t) + u(t)v(t)\right)\,dt, v(0) = 1 \end{cases}$$

7. $$\begin{cases} u''(x) = e^x + 4e^{2x} - \dfrac{23}{192} + \dfrac{1}{64}\displaystyle\int_0^{\ln 2} \left(u^2(t) + u(t)v(t)\right)\,dt, u(0)2, u'(0) = 3 \\ v''(x) = e^x - 4e^{2x} + \dfrac{5}{192} + \dfrac{1}{64}\displaystyle\int_0^{\ln 2} \left(v^2(t) + u(t)v(t)\right)\,dt, v(0) = 0, v'(0) = -1 \end{cases}$$

8. $$\begin{cases} u''(x) = 12x^2 + \dfrac{457}{240} + \dfrac{1}{64}\displaystyle\int_{-1}^{1} \left(u^2(t) + u(t)v(t)\right)\,dt, u(0) = 1, u'(0) = 0 \\ v''(x) = -12x^2 - \dfrac{487}{240} + \dfrac{1}{64}\displaystyle\int_{-1}^{1} \left(v^2(t) + u(t)v(t)\right)\,dt, v(0) = 1, v'(0) = 0 \end{cases}$$

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Chapter 17
Nonlinear Singular Integral Equations

17.1 Introduction

Abel's integral equation, linear or nonlinear, occurs in many branches of scientific fields [1], such as microscopy, seismology, radio astronomy, electron emission, atomic scattering, radar ranging, plasma diagnostics, X-ray radiography, and optical fiber evaluation. Linear Abel's integral equation is the earliest example of an integral equation. In Chapter 2, Abel's integral equation was defined as a singular integral equation. Volterra integral equations of the first kind

$$f(x) = \lambda \int_{g(x)}^{h(x)} K(x,t)u(t)\,dt, \tag{17.1}$$

or of the second kind

$$u(x) = f(x) + \int_{g(x)}^{h(x)} K(x,t)u(t)\,dt, \tag{17.2}$$

are called *singular* [2–8] if:

1. one of the limits of integration $g(x)$, $h(x)$ or both are *infinite*, or
2. if the kernel $K(x,t)$ becomes *infinite* at one or more points at the range of integration.

In a similar manner, nonlinear Volterra integral equations of the first kind

$$f(x) = \lambda \int_{g(x)}^{h(x)} K(x,t)F(u(t))\,dt, \tag{17.3}$$

or of the second kind

$$u(x) = f(x) + \int_{g(x)}^{h(x)} K(x,t)F(u(t))\,dt, \tag{17.4}$$

where $F(u(t))$ is a nonlinear function of $u(t)$, are called *singular* if:

1. one of the limits of integration $g(x)$, $h(x)$ or both are *infinite*, or

2. if the kernel $K(x,t)$ becomes *infinite* at one or more points at the range of integration.

Examples of the second style are the nonlinear Abel's integral equation, generalized nonlinear Abel's integral equation, and the nonlinear weakly-singular integral equations given by

$$f(x) = \int_0^x \frac{1}{\sqrt{(x-t)}} u^2(t)\, dt, \tag{17.5}$$

$$f(x) = \int_0^x \frac{1}{(x-t)^\alpha} u^3(t)\, dt, 0 < \alpha < 1, \tag{17.6}$$

and

$$u(x) = f(x) + \int_0^x \frac{1}{(x-t)^\alpha} u^2(t)\, dt, 0 < \alpha < 1, \tag{17.7}$$

respectively. It is clear that the kernel in each equation becomes infinite at the upper limit $t = x$.

In this chapter we will focus our study on the second style of nonlinear singular integral equations, namely the equations where the kernel $K(x,t)$ becomes unbounded at one or more points of singularities in its domain of validity. The equations that will be investigated are nonlinear Abel's problem, generalized nonlinear Abel integral equations and the nonlinear weakly-singular Volterra integral equations.

In the previous chapters, we focused our study on the techniques that determine the solution to any integral equation. In this chapter, we will run our study in a manner parallel to the approaches used before and focus our concern on solving the nonlinear singular integral equation.

It is interesting to point out that although the nonlinear singular integral equations (17.5) and (17.6) are Volterra integral equations of the first kind, and the singular equation (17.7) is a Volterra equation of the second kind, but two of the methods used before in Section 3.3, namely the series solution method and the conversion to a second kind Volterra equation, are not applicable for these singular cases. This is due to the fact that the series solution cannot be used to handle singular integral equations especially if $u(x)$ is not analytic. Moreover, converting singular integral equation to a second kind Volterra equation is not obtainable because we cannot use Leibnitz rule due to the singularity behavior of the kernel of this equation.

17.2 Nonlinear Abel's Integral Equation

The standard form of the nonlinear Abel's integral equation [9] is given by

$$f(x) = \int_0^x \frac{1}{\sqrt{(x-t)}} F(u(t))\, dt, \tag{17.8}$$

where the function $f(x)$ is a given real-valued function, and $F(u(x))$ is a nonlinear function of $u(x)$. Recall that the unknown function $u(x)$ occurs only inside the integral sign for the Abel's integral equation (17.8).

To determine a solution for the nonlinear Abel's integral equation (17.8), we first convert it to a linear Abel's integral equation of the form

$$f(x) = \int_0^x \frac{1}{\sqrt{(x-t)}} v(t)\, dt, \tag{17.9}$$

by using the transformation

$$v(x) = F(u(x)), \tag{17.10}$$

where $F(u(x))$ is invertible, i.e $F^{-1}(u(x))$ exists. This in turn means that

$$u(x) = F^{-1}(v(x)). \tag{17.11}$$

In this section we will handle Equation (17.9) by using the Laplace transform method.

17.2.1 The Laplace Transform Method

Although the Laplace transform method was presented before, but a brief summary will be helpful. The Abel's integral equation can be expressed as

$$f(x) = \int_0^x K(x-t)u(t)\, dt. \tag{17.12}$$

Consider two functions $f_1(x)$ and $f_2(x)$ that possess the conditions needed for the existence of Laplace transform for each. Let the Laplace transforms for the functions $f_1(x)$ and $f_2(x)$ be given by

$$\begin{aligned} \mathcal{L}\{f_1(x)\} &= F_1(s), \\ \mathcal{L}\{f_2(x)\} &= F_2(s). \end{aligned} \tag{17.13}$$

The Laplace convolution product of these two functions is defined by

$$(f_1 * f_2)(x) = \int_0^x f_1(x-t)f_2(t)\, dt, \tag{17.14}$$

or

$$(f_2 * f_1)(x) = \int_0^x f_2(x-t)f_1(t)\, dt. \tag{17.15}$$

Recall that

$$(f_1 * f_2)(x) = (f_2 * f_1)(x). \tag{17.16}$$

We can easily show that the Laplace transform of the convolution product $(f_1 * f_2)(x)$ is given by

$$\mathcal{L}\{(f_1 * f_2)(x)\} = F_1(s)F_2(s). \tag{17.17}$$

Based on this summary, we will examine Abel's integral equation where the kernel is a difference kernel. Recall that we will apply the Laplace transform method and the inverse of the Laplace transform using Table 2 in section 1.5.

Taking Laplace transforms of both sides of (17.9) leads to

$$\mathcal{L}\{f(x)\} = \mathcal{L}\{v(x)\}\,\mathcal{L}\{x^{-\frac{1}{2}}\}, \tag{17.18}$$

or equivalently

$$F(s) = V(s)\frac{\Gamma(1/2)}{s^{1/2}} = V(s)\frac{\sqrt{\pi}}{s^{1/2}}, \tag{17.19}$$

that gives

$$V(s) = \frac{s^{1/2}}{\sqrt{\pi}}F(s), \tag{17.20}$$

where Γ is the gamma function, and $\Gamma(\frac{1}{2}) = \sqrt{\pi}$. The last equation (17.20) can be rewritten as

$$V(s) = \frac{s}{\pi}(\sqrt{\pi}s^{-\frac{1}{2}}F(s)), \tag{17.21}$$

which can be rewritten by

$$\mathcal{L}\{v(x)\} = \frac{s}{\pi}\mathcal{L}\{y(x)\}, \tag{17.22}$$

where

$$y(x) = \int_0^x (x-t)^{-\frac{1}{2}} f(t)dt. \tag{17.23}$$

Using the fact

$$\mathcal{L}\{y'(x)\} = s\,\mathcal{L}\{y(x)\} - y(0), \tag{17.24}$$

into (17.22) we obtain

$$\mathcal{L}\{v(x)\} = \frac{1}{\pi}\,\mathcal{L}\{y'(x)\}. \tag{17.25}$$

Applying L^{-1} to both sides of (17.25) gives the formula

$$v(x) = \frac{1}{\pi}\frac{d}{dx}\int_0^x \frac{f(t)}{\sqrt{x-t}}dt, \tag{17.26}$$

that will be used for the determination of the solution $v(x)$. Having determined $v(x)$, then the solution $u(x)$ of (17.8) follows immediately by using

$$u(x) = F^{-1}(v(x)). \tag{17.27}$$

Notice that the formula (17.26) will be used for solving nonlinear Abel's integral equation, and it is not necessary to use Laplace transform method for each problem. The nonlinear Abel's problem (17.8) can be solved directly by using the formulas (17.26) and (17.27). Appendix B, calculators, and computer programs can be used to evaluate the resulting integrals. The presented analysis will be used to determine the solution of Abel's problem (17.8) in the following examples.

Example 17.1

Solve the nonlinear Abel integral equation

$$2\pi\sqrt{x} = \int_0^x \frac{1}{\sqrt{x-t}}u^2(t)\,dt. \tag{17.28}$$

Assume $u^2(x)$ is invertible. The transformation

$$v(x) = u^2(x),\ u(x) = \pm\sqrt{v(x)}, \tag{17.29}$$

carries (17.28) into

$$2\pi\sqrt{x} = \int_0^x \frac{1}{\sqrt{x-t}} v(t)\,dt. \tag{17.30}$$

Substituting $f(x) = 2\pi\sqrt{x}$ in (17.26) gives

$$v(x) = \frac{1}{\pi}\frac{d}{dx}\int_0^x \frac{2\pi\sqrt{t}}{\sqrt{x-t}} dt = \pi. \tag{17.31}$$

This in turn gives the solutions

$$u(x) = \pm\sqrt{\pi}, \tag{17.32}$$

obtained upon using (17.29).

Example 17.2

Solve the nonlinear Abel integral equation

$$\frac{32}{35} x^{\frac{7}{2}} = \int_0^x \frac{1}{\sqrt{x-t}} u^3(t)\,dt. \tag{17.33}$$

The transformation

$$v(x) = u^3(x),\ u(x) = v^{\frac{1}{3}}(x), \tag{17.34}$$

carries (17.33) into

$$\frac{32}{35} x^{\frac{7}{2}} = \int_0^x \frac{1}{\sqrt{x-t}} v(t)\,dt. \tag{17.35}$$

Substituting $f(x) = \frac{32}{35} x^{\frac{7}{2}}$ in (17.26) gives

$$v(x) = \frac{1}{\pi}\frac{d}{dx}\int_0^x \frac{\frac{32}{35} t^{\frac{7}{2}}}{\sqrt{x-t}} dt = x^3. \tag{17.36}$$

The exact solution is therefore given by

$$u(x) = x, \tag{17.37}$$

obtained upon using (17.34).

Example 17.3

Solve the nonlinear Abel integral equation

$$\frac{4}{3} x^{\frac{3}{2}} = \int_0^x \frac{1}{\sqrt{x-t}} \ln(u(t))\,dt. \tag{17.38}$$

The transformation

$$v(x) = \ln(u(x)),\ u(x) = e^{v(x)}, \tag{17.39}$$

carries the integral equation into

$$\frac{4}{3} x^{\frac{3}{2}} = \int_0^x \frac{1}{\sqrt{x-t}} v(t)\,dt. \tag{17.40}$$

Proceeding as before we obtain

$$v(x) = \frac{1}{\pi}\frac{d}{dx}\int_0^x \frac{\frac{4}{3} t^{\frac{3}{2}}}{\sqrt{x-t}} dt = x. \tag{17.41}$$

The exact solution is therefore given by

$$u(x) = e^x. \tag{17.42}$$

Example 17.4

Solve the nonlinear Abel integral equation

$$6x^{\frac{1}{2}}(1+\frac{2}{9}x) = \int_0^x \frac{1}{\sqrt{x-t}}\sin^{-1}(u(t))\,dt. \tag{17.43}$$

The transformation

$$v(x) = \sin^{-1}(u(x)),\; u(x) = \sin(v(x)), \tag{17.44}$$

carries the integral equation into

$$6x^{\frac{1}{2}}(1+\frac{2}{9}x) = \int_0^x \frac{1}{\sqrt{x-t}}v(t)\,dt. \tag{17.45}$$

Proceeding as before we obtain

$$v(x) = \frac{1}{\pi}\frac{d}{dx}\int_0^x \frac{6t^{\frac{1}{2}}(1+\frac{2}{9}t)}{\sqrt{x-t}}dt = x+3. \tag{17.46}$$

This gives the exact solution by

$$u(x) = \sin(x+3), \tag{17.47}$$

obtained upon using (17.44).

Exercises 17.2.1

Solve the following nonlinear Abel integral equations

1. $\frac{4}{3}x^{\frac{3}{2}} = \int_0^x \frac{1}{\sqrt{x-t}}u^2(t)\,dt$
2. $\frac{1}{15}x^{\frac{1}{2}}(30+40x+16x^2) = \int_0^x \frac{1}{\sqrt{x-t}}u^2(t)\,dt$
3. $\frac{16}{15}x^{\frac{5}{2}} = \int_0^x \frac{1}{\sqrt{x-t}}u^3(t)\,dt$
4. $2x^{\frac{1}{2}}(1+2x)+\frac{3\pi}{8}(4x+x^2) = \int_0^x \frac{1}{\sqrt{x-t}}u^3(t)\,dt$
5. $\frac{2}{3}x^{\frac{1}{2}}(3+2x) = \int_0^x \frac{1}{\sqrt{x-t}}\cos^{-1}(u(t))\,dt$
6. $\frac{2}{3}x^{\frac{1}{2}}(3+2x) = \int_0^x \frac{1}{\sqrt{x-t}}\ln(u(t))\,dt$
7. $\frac{2}{3}ex^{\frac{1}{2}}(3+2x) = \int_0^x \frac{1}{\sqrt{x-t}}e^{u(t)}\,dt$
8. $\frac{\pi^{\frac{3}{2}}}{2}x = \int_0^x \frac{1}{\sqrt{x-t}}e^{u(t)}\,dt$

17.3 The Generalized Nonlinear Abel Equation

The generalized nonlinear Abel integral equation [3,5] is of the form

$$f(x) = \int_0^x \frac{1}{(x-t)^\alpha}F(u(t))dt,\; 0<\alpha<1, \tag{17.48}$$

where α is a known constant such that $0 < \alpha < 1$, $f(x)$ is a given function, and $F(u(x))$ is a nonlinear function of $u(x)$. The nonlinear Abel integral equation is a special case of the generalized equation where $\alpha = \frac{1}{2}$. The expression $(x-t)^{-\alpha}$ is called the kernel of the integral equation. The Laplace transform method will be used to handle the generalized nonlinear Abel integral equation (17.48)

17.3.1 The Laplace Transform Method

To determine a solution for the generalized nonlinear Abel integral equation (17.48), we first convert it to a linear Abel integral equation of the form

$$f(x) = \int_0^x \frac{1}{(x-t)^\alpha} v(t)\, dt,\ 0 < \alpha < 1, \tag{17.49}$$

by using the transformation

$$v(x) = F(u(x)). \tag{17.50}$$

This in turn means that

$$u(x) = F^{-1}(v(x)). \tag{17.51}$$

Taking Laplace transforms of both sides of (17.49) leads to

$$\mathcal{L}\{f(x)\} = \mathcal{L}\{v(x)\}\, \mathcal{L}\{x^{-\alpha}\}, \tag{17.52}$$

or equivalently

$$F(s) = V(s)\frac{\Gamma(1-\alpha)}{s^{1-\alpha}}, \tag{17.53}$$

that gives

$$V(s) = \frac{s^{1-\alpha}}{\Gamma(1-\alpha)} F(s), \tag{17.54}$$

where Γ is the gamma function. The last equation (17.54) can be rewritten as

$$\mathcal{L}\{v(x)\} = \frac{s}{\Gamma(\alpha)\Gamma(1-\alpha)} \mathcal{L}\{y(x)\}, \tag{17.55}$$

where

$$y(x) = \int_0^x \frac{1}{(x-t)^{\alpha-1}} f(t) dt. \tag{17.56}$$

Using the facts

$$\mathcal{L}\{y'(x)\} = s\, \mathcal{L}\{y(x)\} - y(0), \tag{17.57}$$

and

$$\Gamma(\alpha)\Gamma(1-\alpha) = \frac{\pi}{\sin \alpha\pi}, \tag{17.58}$$

into (17.55) we obtain

$$\mathcal{L}\{v(x)\} = \frac{\sin \alpha\pi}{\pi}\, \mathcal{L}\{y'(x)\}. \tag{17.59}$$

Applying L^{-1} to both sides of (17.59) gives the formula

$$v(x) = \frac{\sin \alpha\pi}{\pi} \frac{d}{dx} \int_0^x \frac{f(t)}{(x-t)^{1-\alpha}} dt. \tag{17.60}$$

Integrating the integral at the right side of (17.60) by parts, and differentiating the result we obtain the more suitable formula

$$v(x) = \frac{\sin \alpha\pi}{\pi} \left(\frac{f(0)}{x^{1-\alpha}} + \int_0^x \frac{f'(t)}{(x-t)^{1-\alpha}} dt \right), \quad 0 < \alpha < 1. \tag{17.61}$$

The derivation of (17.61) is left to the reader. Having determined $v(x)$, the solution is given by

$$u(x) = F^{-1}(v(x)). \tag{17.62}$$

Three remarks can be made here:

1. The exponent of the kernel of the generalized nonlinear Abel integral equation is $-\alpha$, but the exponent of the kernel of the two formulae (17.60) and (17.61) is $\alpha - 1$.

2. The unknown function in (17.49) has been replaced by $f(t)$ and $f'(t)$ in (17.60) and (17.61) respectively.

3. In (17.60), differentiation is used after integrating the integral at the right side, whereas in (17.61), integration is only required.

Example 17.5

Solve the generalized nonlinear Abel integral equation

$$\frac{243}{440} x^{\frac{11}{3}} = \int_0^x \frac{1}{(x-t)^{\frac{1}{3}}} u^3(t)\, dt. \tag{17.63}$$

Notice that $\alpha = \frac{1}{3}, f(x) = \frac{243}{440} x^{\frac{11}{3}}$. The transformation

$$v(x) = u^3(x),\ u(x) = v^{\frac{1}{3}}(x), \tag{17.64}$$

carries the integral equation into

$$\frac{243}{440} x^{\frac{11}{3}} = \int_0^x \frac{1}{(x-t)^{\frac{1}{3}}} v(t)\, dt. \tag{17.65}$$

Using (17.60) gives

$$v(x) = \frac{\sqrt{3}}{2\pi} \frac{d}{dx} \int_0^x \frac{\frac{243}{440} t^{\frac{11}{3}}}{(x-t)^{\frac{2}{3}}} dt = x^3. \tag{17.66}$$

This gives the exact solution by

$$u(x) = x. \tag{17.67}$$

Example 17.6

Solve the generalized nonlinear Abel integral equation

$$\frac{3}{40} x^{\frac{2}{3}} (20 - 24x + 9x^2) = \int_0^x \frac{1}{(x-t)^{\frac{1}{3}}} u^2(t)\, dt. \tag{17.68}$$

Notice that $\alpha = \frac{1}{3}, f(x) = \frac{3}{40} x^{\frac{2}{3}} (20 - 24x + 9x^2)$. The transformation

$$v(x) = u^2(x),\ u(x) = \pm v^{\frac{1}{2}}(x), \tag{17.69}$$

carries the integral equation into

$$\frac{3}{40}x^{\frac{2}{3}}(20 - 24x + 9x^2) = \int_0^x \frac{1}{(x-t)^{\frac{1}{3}}} v(t)\,dt. \tag{17.70}$$

Using (17.60) gives

$$v(x) = \frac{\sqrt{3}}{2\pi}\frac{d}{dx}\int_0^x \frac{\frac{3}{40}t^{\frac{2}{3}}(20 - 24t + 9t^2)}{(x-t)^{\frac{2}{3}}}dt = (1-x)^2. \tag{17.71}$$

The exact solution is therefore given by

$$u(x) = \pm(1-x). \tag{17.72}$$

Example 17.7

Solve the generalized nonlinear Abel integral equation

$$\frac{6}{55}x^{\frac{5}{6}}(11 + 6x) = \int_0^x \frac{1}{(x-t)^{\frac{1}{6}}} \sin^{-1}(u(t))\,dt. \tag{17.73}$$

The transformation

$$v(x) = \sin^{-1}(u(t)),\ u(x) = \sin(v(x)), \tag{17.74}$$

carries the integral equation into

$$\frac{6}{55}x^{\frac{5}{6}}(11 + 6x) = \int_0^x \frac{1}{(x-t)^{\frac{1}{6}}} v(t)\,dt. \tag{17.75}$$

Using (17.60) gives

$$v(x) = \frac{1}{2\pi}\frac{d}{dx}\int_0^x \frac{\frac{6}{55}t^{\frac{5}{6}}(11 + 6t)}{(x-t)^{\frac{5}{6}}}dt = 1 + x. \tag{17.76}$$

The exact solution is given by

$$u(x) = \sin(1 + x). \tag{17.77}$$

Example 17.8

Solve the following generalized Abel integral equation

$$\frac{9\Gamma(\frac{2}{3})\Gamma(\frac{5}{6})}{7\sqrt{\pi}}x^{\frac{7}{6}} = \int_0^x \frac{1}{(x-t)^{\frac{1}{3}}} \ln(u(t))\,dt. \tag{17.78}$$

The transformation

$$v(x) = \ln(u(t)),\ u(x) = e^{v(x)}, \tag{17.79}$$

carries the integral equation into

$$\frac{9\Gamma(\frac{2}{3})\Gamma(\frac{5}{6})}{7\sqrt{\pi}}x^{\frac{7}{6}} = \int_0^x \frac{1}{(x-t)^{\frac{1}{3}}} v(t)\,dt. \tag{17.80}$$

Using (17.60) gives

$$v(x) = \frac{1}{2\pi}\frac{d}{dx}\int_0^x \frac{\frac{9\Gamma(\frac{2}{3})\Gamma(\frac{5}{6})}{7\sqrt{\pi}}t^{\frac{7}{6}}}{(x-t)^{\frac{2}{3}}}dt = \sqrt{x}. \tag{17.81}$$

Consequently, the exact solution is given by

$$u(x) = e^{\sqrt{x}}. \tag{17.82}$$

17.3.2 The Main Generalized Nonlinear Abel Equation

A further generalization for linear Abel integral equation was presented in Chapter 7. In this section, the further generalized nonlinear Abel integral equation [3,5] is of the form

$$f(x) = \int_0^x \frac{1}{[g(x) - g(t)]^{\alpha}} F(u(t))\, dt, 0 < \alpha < 1, \tag{17.83}$$

where $g(t)$ is strictly monotonically increasing and differentiable function in some interval $0 < t < b$, $g'(t) \neq 0$ for every t in the interval, and $F(u(t))$ is a nonlinear function of $u(t)$). As presented before, we first convert this equation to a linear Abel integral equation of the form

$$f(x) = \int_0^x \frac{1}{[g(x) - g(t)]^{\alpha}} v(t)\, dt,\ 0 < \alpha < 1, \tag{17.84}$$

by using the transformation

$$v(x) = F(u(x)). \tag{17.85}$$

This in turn means that

$$u(x) = F^{-1}(v(x)). \tag{17.86}$$

The solution $u(x)$ of (17.84) is given by

$$v(x) = \frac{\sin \alpha\pi}{\pi} \frac{d}{dx} \int_0^x \frac{g'(t) f(t)}{[g(x) - g(t)]^{1-\alpha}} dt, 0 < \alpha < 1. \tag{17.87}$$

The formula (17.87) was formally derived in Chapter 7. Having determined $v(x)$ by using (17.87), the solution $u(x)$ is obtained by using (17.86).

The proposed approach can be explained by studying the following illustrative examples.

Example 17.9

Solve the generalized nonlinear Abel integral equation

$$\frac{6}{5}(\sin x)^{\frac{5}{6}} = \int_0^x \frac{u^2(t)}{(\sin x - \sin t)^{\frac{1}{6}}}\, dt, \tag{17.88}$$

where $0 < x < \frac{\pi}{2}$. The transformation

$$v(x) = u^2(x),\ u(x) = \pm v^{\frac{1}{2}}(x), \tag{17.89}$$

carries the integral equation into

$$\frac{6}{5}(\sin x)^{\frac{5}{6}} = \int_0^x \frac{v(t)}{(\sin x - \sin t)^{\frac{1}{6}}}\, dt. \tag{17.90}$$

Notice that $\alpha = \frac{1}{6}$, $f(x) = \frac{6}{5}(\sin x)^{\frac{5}{6}}$. Besides, $g(x) = \sin x$ is strictly monotonically increasing in $0 < x < \frac{\pi}{2}$, and $g'(x) = \cos x \neq 0$ for every x in $0 < x < \frac{\pi}{2}$. Using (17.87) gives

$$v(x) = \frac{3}{5\pi}\frac{d}{dx}\int_0^x \frac{\cos t\,(\sin t)^{\frac{5}{6}}}{(\sin x - \sin t)^{\frac{5}{6}}}dt = \cos x. \tag{17.91}$$

The exact solution is given by

$$u(x) = \pm\sqrt{\cos x}. \tag{17.92}$$

Example 17.10

Solve the generalized nonlinear Abel integral equation

$$\frac{3}{10}x^{\frac{10}{3}} = \int_0^x \frac{u^3(t)}{(x^4 - t^4)^{\frac{1}{6}}}dt, \tag{17.93}$$

where $0 < x < 2$. The transformation

$$v(x) = u^3(x),\ u(x) = v^{\frac{1}{3}}(x), \tag{17.94}$$

carries the integral equation into

$$\frac{3}{10}x^{\frac{10}{3}} = \int_0^x \frac{v(t)}{(x^4 - t^4)^{\frac{1}{6}}}dt. \tag{17.95}$$

Notice that $\alpha = \frac{1}{6}$, $f(x) = \frac{3}{10}x^{\frac{10}{3}}$. Besides, $g(x) = x^4$ is strictly monotonically increasing in $0 < x < 2$, and $g'(x) = 4x^3 \neq 0$ for every x in $0 < x < 2$. Using (17.87) gives

$$v(x) = \frac{3}{5\pi}\frac{d}{dx}\int_0^x \frac{t^{\frac{19}{3}}}{(x^4 - t^4)^{\frac{5}{6}}}dt = x^3. \tag{17.96}$$

$$u(x) = x. \tag{17.97}$$

Example 17.11

Solve the generalized nonlinear Abel integral equation

$$\frac{4}{3}\sin^{\frac{3}{4}} x = \int_0^x \frac{\ln(u(t))}{(\sin x - \sin t)^{\frac{1}{4}}}dt, \tag{17.98}$$

where $0 < x < \frac{\pi}{2}$. The transformation

$$v(x) = \ln(u(x)),\ u(x) = e^{v(x)}, \tag{17.99}$$

carries the integral equation into

$$\frac{4}{3}\sin^{\frac{3}{4}} x = \int_0^x \frac{v(t)}{(\sin x - \sin t)^{\frac{1}{4}}}dt. \tag{17.100}$$

Proceeding as before, and using (17.87) we obtain

$$v(x) = \frac{4}{3\sqrt{2}\pi}\frac{d}{dx}\int_0^x \frac{\cos t \sin^{\frac{3}{4}} t}{(\sin x - \sin t)^{\frac{3}{4}}}dt = \cos x. \tag{17.101}$$

Consequently, the exact solution is given by

$$u(x) = e^{\cos x}. \tag{17.102}$$

Example 17.12

Solve the generalized nonlinear Abel integral equation

$$\pi + x = \int_0^x \frac{\sin^{-1}(u(t))}{(x^2-t^2)^{\frac{1}{2}}}dt, \tag{17.103}$$

where $0 < x < 2$. The transformation

$$v(x) = \sin^{-1}(u(t)),\; u(x) = \sin(v(x)), \tag{17.104}$$

carries the integral equation into

$$\pi + x = \int_0^x \frac{v(t)}{(x^2-t^2)^{\frac{1}{2}}}dt. \tag{17.105}$$

Proceeding as before we find

$$v(x) = x + 2. \tag{17.106}$$

The exact solution is given by

$$u(x) = \sin(x+2). \tag{17.107}$$

Exercises 17.3.1

In Exercises 1–8, solve the generalized nonlinear Abel integral equations

1. $\frac{9}{10}x^{\frac{5}{3}} = \int_0^x \frac{1}{(x-t)^{\frac{1}{3}}}u^2(t)dt$
2. $\frac{3}{40}x^{\frac{2}{3}}(20+24x+9x^2) = \int_0^x \frac{1}{(x-t)^{\frac{1}{3}}}u^2(t)dt$
3. $\frac{16}{21}x^{\frac{7}{4}} = \int_0^x \frac{1}{(x-t)^{\frac{1}{4}}}u^3(t)dt$
4. $\frac{4}{231}x^{\frac{3}{4}}(77-88x+32x^2) = \int_0^x \frac{1}{(x-t)^{\frac{1}{4}}}u^2(t)dt$
5. $\frac{36}{55}x^{\frac{11}{6}} = \int_0^x \frac{1}{(x-t)^{\frac{1}{6}}}\cos^{-1}(u(t))dt$
6. $\frac{6}{55}x^{\frac{5}{6}}(11+6x) = \int_0^x \frac{1}{(x-t)^{\frac{1}{6}}}e^{u(t)}dt$
7. $\frac{5}{36}x^{\frac{4}{5}}(9-5x) = \int_0^x \frac{1}{(x-t)^{\frac{1}{5}}}\ln(u(t))dt$
8. $\frac{3}{10}x^{\frac{2}{3}}(5\pi+3x) = \int_0^x \frac{1}{(x-t)^{\frac{1}{3}}}\sinh^{-1}(u(t))dt$

In Exercises 9–16, use the formula (17.87) to solve the following generalized Abel integral equations

9. $\frac{6}{5}\sin^{\frac{5}{6}}x = \int_0^x \frac{1}{(\sin x - \sin t)^{\frac{1}{6}}}u^3(t)dt$
10. $x = \int_0^x \frac{1}{(x^2-t^2)^{\frac{1}{2}}}\sqrt{u(t)}dt$
11. $\frac{\pi}{2} + 2x + \frac{\pi}{4}x^2 = \int_0^x \frac{1}{(x^2-t^2)^{\frac{1}{2}}}u^2(t)dt$
12. $\frac{\pi}{2}(1 + x^2 + \frac{3}{16}x^4) = \int_0^x \frac{1}{(x^2-t^2)^{\frac{1}{2}}}u^2(t)dt$
13. $-\frac{4}{3}\sin^{\frac{3}{4}}x = \int_0^x \frac{1}{(\sin x - \sin t)^{\frac{1}{4}}}\ln(u(t))dt$

14. $\frac{4}{3}\sin^{\frac{3}{4}} x = \int_0^x \frac{1}{(\sin x - \sin t)^{\frac{1}{4}}} e^{u(t)} dt$

15. $\frac{\pi^2}{2} + x = \int_0^x \frac{1}{(x^2 - t^2)^{\frac{1}{2}}} \sinh^{-1}(u(t))dt$ 16. $\frac{9}{20} x^{\frac{10}{3}} = \int_0^x \frac{1}{(x^2 - t^2)^{\frac{1}{3}}} e^{u(t)} dt$

17.4 The Nonlinear Weakly-Singular Volterra Equations

The nonlinear weakly-singular Volterra integral equations of the second kind are given by

$$u(x) = f(x) + \int_0^x \frac{\beta}{\sqrt{x-t}} F(u(t))dt,\ x \in [0, T], \tag{17.108}$$

and

$$u(x) = f(x) + \int_0^x \frac{\beta}{[g(x) - g(t)]^\alpha} F(u(t))dt, 0 < \alpha < 1,\ x \in [0, T], \tag{17.109}$$

where β is a constant, and $F(u(t))$ is a nonlinear function of $u(t)$). Equation (17.109) is known as the generalized nonlinear weakly-singular Volterra equation. These equations arise in many mathematical physics and chemistry applications such as stereology, heat conduction, crystal growth and the radiation of heat from a semi-infinite solid. It is also assumed that the function $f(x)$ is a given real valued function. The nonlinear weakly-singular and the generalized nonlinear weakly-singular equations (17.108) and (17.109) fall under the category of singular equations with singular kernels

$$\begin{aligned} K(x,t) &= \frac{1}{\sqrt{x-t}}, \\ K(x,t) &= \frac{1}{[g(x) - g(t)]^\alpha}, 0 < \alpha < 1, \end{aligned} \tag{17.110}$$

respectively.

In this section we will use the Adomian decomposition method to handle the nonlinear weakly-singular Volterra integral equations. The modified decomposition method and the noise terms phenomenon will be used wherever it is appropriate. We will only present a summary of the necessary steps for the Adomian method.

17.4.1 The Adomian Decomposition Method

The Adomian decomposition method will be applied on the generalized nonlinear weakly-singular Volterra equation (17.109), because Eq. (17.108) is a special case of the generalized equation with $\alpha = \frac{1}{2}, g(x) = x$. As stated before, we will outline a brief framework of the method. To determine the solution $u(x)$ of (17.109) we substitute the decomposition series for the linear term $u(x)$

$$u(x) = \sum_{n=0}^{\infty} u_n(x), \tag{17.111}$$

and

$$F(u(x)) = \sum_{n=0}^{\infty} A_n(x),\ A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[F\left(\sum_{i=0}^{n} \lambda^i\, u_i\right)\right]_{\lambda=0}, n = 0, 1, 2, \cdots \tag{17.112}$$

where A_n are the Adomian polynomials, into both sides of (17.109) to obtain

$$\sum_{n=0}^{\infty} u_n(x) = f(x) + \int_0^x \frac{\beta}{[g(x)-g(t)]^{\alpha}}\left(\sum_{n=0}^{\infty} A_n(t)\right) dt, 0 < \alpha < 1, \tag{17.113}$$

The components $u_0(x), u_1(x), u_2(x), \ldots$ are usually determined by using the recurrence relation

$$\begin{aligned} u_0(x) &= f(x) \\ u_1(x) &= \int_0^x \frac{\beta}{[g(x)-g(t)]^{\alpha}} A_0(t)dt, \\ u_2(x) &= \int_0^x \frac{\beta}{[g(x)-g(t)]^{\alpha}} A_1(t)dt, \\ u_3(x) &= \int_0^x \frac{\beta}{[g(x)-g(t)]^{\alpha}} A_2(t)dt, \end{aligned} \tag{17.114}$$

and so on. Having determined the components $u_0(x), u_1(x), u_2(x), \cdots$, the solution $u(x)$ of (17.109) will be determined in the form of a series by substituting the derived components in (17.111). The determination of the previous components can be obtained by using Appendix B, calculator, or any computer symbolic systems such as Maple or Mathematica. The obtained series converges to the exact solution if such a solution exists. For concrete problems, we use as many terms as we need for numerical purposes. The method has been proved to be effective in handling this kind of integral equations.

It is normal that we use the modified decomposition method and the noise terms phenomenon wherever it is appropriate. The Adomian decomposition method and the related forms will be studied in the following examples.

Example 17.13

Solve the nonlinear weakly-singular Volterra integral equation

$$u(x) = x - \frac{16}{15}x^{\frac{5}{2}} + \int_0^x \frac{u^2(t)}{\sqrt{x-t}}\, dt. \tag{17.115}$$

Using the recurrence relation we set

$$\begin{aligned} u_0(x) &= x - \frac{16}{15}x^{\frac{5}{2}}, \\ u_1(x) &= -\int_0^x \frac{\left(t - \frac{16}{15}t^{\frac{5}{2}}\right)^2}{\sqrt{x-t}}\, dt = \frac{16}{15}x^{\frac{5}{2}} - \frac{7\pi}{12}x^4 + \frac{131072}{155925}x^{\frac{11}{2}}. \end{aligned} \tag{17.116}$$

The noise terms $\mp\frac{16}{15}x^{\frac{5}{2}}$ appear between $u_0(x)$ and $u_1(x)$. By canceling the noise term $-\frac{16}{15}x^{\frac{5}{2}}$ from $u_0(x)$ and verifying that the non-canceled term in $u_0(x)$ justifies the equation (17.115), the exact solution is therefore given by

$$u(x) = x. \tag{17.117}$$

We can easily obtain the exact solution by using the modified decomposition method. This can be done by splitting the nonhomogeneous part $f(x)$ into two parts $f_1(x) = x$ and $f_2(x) = -\frac{16}{15}x^{\frac{5}{2}}$. Accordingly, we set the modified recurrence relation

$$u_0(x) = x, \quad u_1(x) = -\frac{16}{15}x^{\frac{5}{2}} - \int_0^x \frac{t^2}{\sqrt{x-t}}\,dt = 0. \tag{17.118}$$

Example 17.14

Solve the nonlinear-weakly singular Volterra integral equation

$$u(x) = \cos^{\frac{1}{2}} x - \frac{3}{2}\sin^{\frac{2}{3}} x + \int_0^x \frac{u^2(t)}{(\sin x - \sin t)^{\frac{1}{3}}}\,dt. \tag{17.119}$$

In this example, the modified decomposition method will be used. We first set

$$f_1(x) = \cos^{\frac{1}{2}} x, \quad f_2(x) = -\frac{3}{2}\sin^{\frac{2}{3}} x. \tag{17.120}$$

Using the modified decomposition method we set

$$\begin{aligned} u_0(x) &= \cos^{\frac{1}{2}} x, \\ u_1(x) &= -\frac{3}{2}\sin^{\frac{2}{3}} x + \int_0^x \frac{1}{(\sin x - \sin t)^{\frac{1}{3}}}u_0^2(t)\,dt = 0. \end{aligned} \tag{17.121}$$

The exact solution is therefore given by

$$u(x) = \cos^{\frac{1}{2}} x. \tag{17.122}$$

Example 17.15

Solve the nonlinear weakly-singular Volterra integral equation

$$u(x) = e^x - \frac{9}{40}(e^x - 1)^{\frac{2}{3}}(9e^{2x} + 6e^x + 5) + \int_0^x \frac{u^3(t)}{(e^x - e^t)^{\frac{1}{3}}}\,dt. \tag{17.123}$$

In this example, the modified decomposition method will be used. We first set

$$f_1(x) = e^x, \quad f_2(x) = -\frac{9}{40}(9e^{2x} + 6e^x + 5)(e^x - 1)^{\frac{2}{3}}. \tag{17.124}$$

The modified recurrence relation is given by

$$\begin{aligned} u_0(x) &= e^x, \\ u_1(x) &= -\frac{9}{40}(9e^{2x} + 6e^x + 5)(e^x - 1)^{\frac{2}{3}} + \int_0^x \frac{e^{3t}}{(e^x - e^t)^{\frac{1}{3}}}\,dt = 0. \end{aligned} \tag{17.125}$$

The exact solution is therefore given by

$$u(x) = e^x. \tag{17.126}$$

Example 17.16

Solve the nonlinear weakly-singular Volterra integral equation

$$u(x) = (1+x)^3 - x - \frac{\pi}{2} + \int_0^x \frac{u^{\frac{1}{3}}(t)}{\sqrt{x^2-t^2}}\,dt. \tag{17.127}$$

To use the modified decomposition method, we select

$$f_1(x) = (1+x)^3, \quad f_2(x) = -x - \frac{\pi}{2}. \tag{17.128}$$

We then set the modified recurrence relation

$$u_0(x) = (1+x)^3, \quad u_1(x) = -x - \frac{\pi}{2} + \int_0^x \frac{1+t}{\sqrt{x^2-t^2}}\,dt = 0. \tag{17.129}$$

The exact solution is therefore given by

$$u(x) = (1+x)^3. \tag{17.130}$$

Exercises 17.4.1

Solve the following nonlinear weakly-singular Volterra equations

1. $u(x) = 1 - x - \frac{2}{15}x^{\frac{1}{2}}(8x^2 - 20x + 15) + \int_0^x \frac{u^2(t)}{\sqrt{x-t}}\,dt$
2. $u(x) = x^{\frac{1}{4}} - x + \int_0^x \frac{u^4(t)}{\sqrt{x^2-t^2}}\,dt$
3. $u(x) = \sin^{\frac{1}{3}} x + \frac{3}{2}(\cos x - 1)^{\frac{2}{3}} + \int_0^x \frac{u^3(t)}{(\cos x - \cos t)^{\frac{1}{3}}}\,dt$
4. $u(x) = \cos^{\frac{1}{4}} x - 2\sin^{\frac{1}{2}} x + \int_0^x \frac{u^4(t)}{\sqrt{\sin x - \sin t}}\,dt$
5. $u(x) = e^{\frac{1}{2}x} - \frac{2}{3}(2e^x + 1)\sqrt{e^x - 1} + \int_0^x \frac{u^4(t)}{\sqrt{e^x - e^t}}\,dt$
6. $u(x) = e^{2x} - 2\sqrt{e^x - 1} + \int_0^x \frac{u^{\frac{1}{2}}(t)}{\sqrt{e^x - e^t}}\,dt$
7. $u(x) = (\ln x)^2 + 2\sqrt{x}(2 - 2\ln 2 - \ln x) + \int_{0^+}^x \frac{u^{\frac{1}{2}}(t)}{\sqrt{x-t}}\,dt$
8. $u(x) = (x + x^2)^4 - \frac{9}{40}x^{\frac{5}{3}}(3x + 4) + \int_0^x \frac{u^{\frac{1}{2}}(t)}{(x-t)^{\frac{1}{3}}}\,dt$

17.5 Systems of Nonlinear Weakly-Singular Volterra Integral Equations

The weakly-singular Volterra integral equations were examined in Chapter 7. Three powerful methods, namely, the Adomian decomposition method, the successive approximations method, and the Laplace transform method, were used to handle this type of equations.

In this section, we will extend our previous work in Chapter 7 to study the systems of nonlinear weakly-singular Volterra integral equations in two unknowns $u(x)$ and $v(x)$. Generalization to any number of unknowns can be followed in a parallel manner. To achieve our goal, we will use only the modified Adomian decomposition method in studying this type of systems of equations. The other methods presented in this text may be used, but it will be left to the reader to select the appropriate method.

The system of nonlinear weakly-singular Volterra integral equations of the convolution type in two unknowns is of the form

$$\begin{aligned} u(x) &= f_1(x) + \int_0^x \left(K_{11}(x,t)F_{11}(u(t)) + K_{12}(x,t)F_{12}(v(t))\right) dt, \\ v(x) &= f_2(x) + \int_0^x \left(K_{21}(x,t)F_{21}(u(t)) + K_{22}(x,t)F_{22}(v(t))\right) dt. \end{aligned} \tag{17.131}$$

The kernels $K_{ij}(x,t), 1 \leqslant i,j \leqslant 2$ and the functions $f_i(x), i = 1,2$ are given real-valued functions, and the functions $F_{ij}(x,t), 1 \leqslant i,j \leqslant 2$ are nonlinear functions of $u(x)$ and $v(x)$. The kernels K_{ij} are singular kernels of the generalized form given by

$$K_{ij} = \frac{1}{[g(x) - g(t)]^{\alpha_{ij}}}, 1 \leqslant i,j \leqslant 2. \tag{17.132}$$

Notice that the kernel is called weakly singular as the singularity may be transformed away by a change of variable [3].

17.5.1 The Modified Adomian Decomposition Method

The Adomian decomposition method [10], as presented before, decomposes each solution as an infinite sum of components, where these components are determined recurrently. This method can be used in its standard form, or combined with the noise terms phenomenon. It will be shown that the modified decomposition method is effective and reliable in handling the systems of nonlinear weakly-singular Volterra integral equations. In view of this, the modified decomposition method will be used extensively in this section.

We will focus our work on the systems of the generalized nonlinear weakly-singular Volterra integral equations in two unknowns of the form

$$\begin{aligned} u(x) &= f_1(x) + \int_0^x \left(K_{11}(x,t)F_{11}(u(t)) + K_{12}(x,t)F_{12}(v(t))\right) dt, \\ v(x) &= f_2(x) + \int_0^x \left(K_{21}(x,t)F_{21}(u(t)) + K_{22}(x,t)F_{22}(v(t))\right) dt. \end{aligned} \tag{17.133}$$

The kernels $K_{ij}(x,t), 1 \leqslant i,j \leqslant 2$ and the functions $f_i(x), i = 1,2$ are given real-valued functions. The kernels K_{ij} are singular kernels of the generalized form given by

$$K_{ij} = \frac{1}{[g(x)-g(t)]^{\alpha_{ij}}}, 1 \leqslant i,j \leqslant 2. \tag{17.134}$$

For revision purposes, we give a brief review of the modified decomposition method. In this method we usually split each of the source terms $f_i(x), i = 1,2$ into two parts $f_{i1}(x)$ and $f_{i2}(x)$, where the first parts $f_{i1}(x), i = 1,2$ are assigned to the zeroth components $u_0(x)$ and $v_0(x)$. However, the other parts $f_{i2}(x)$ are assigned to the components $u_1(x)$ and $v_1(x)$. Based on this, we use the modified recurrence relation as follows:

$$\begin{aligned} u_0(x) &= f_{11}(x), \quad v_0(x) = f_{21}(x),\\ u_1(x) &= f_{12}(x) + \int_0^x (K_{11}(x,t)A_{11}(t) + K_{12}(x,t)B_{12}(t))\,dt,\\ v_1(x) &= f_{22}(x) + \int_0^x (K_{21}(x,t)A_{21}(t) + K_{22}(x,t)B_{22}(t))\,dt, \end{aligned} \tag{17.135}$$

where A_{i1} and B_{i2} are the Adomian polynomials for the nonlinear functions F_{i1} and F_{i2} respectively.

Example 17.17

Solve the system of the nonlinear weakly-singular Volterra integral equations by using the modified decomposition method

$$\begin{aligned} u(x) &= e^x - \frac{2}{3}(e^{2x}+2)(e^{2x}-1)^{\frac{1}{2}} + \int_0^x \left(\frac{1}{(e^{2x}-e^{2t})^{\frac{1}{2}}}(u^2(t)+v^2(t))\right) dt,\\ v(x) &= e^{2x} + \frac{2}{3}(e^{2x}-1)^{\frac{3}{2}} + \int_0^x \left(\frac{1}{(e^{2x}-e^{2t})^{\frac{1}{2}}}(u^2(t)-v^2(t))\right) dt. \end{aligned} \tag{17.136}$$

Following the discussion presented above, we set the modified recurrence relation

$$\begin{aligned} u_0(x) &= e^x, \quad v_0(x) = e^{2x},\\ u_1(x) &= -\frac{2}{3}(e^{2x}+2)(e^{2x}-1)^{\frac{1}{2}} + \int_0^x \left(\frac{1}{(e^{2x}-e^{2t})^{\frac{1}{2}}}(u_0^2(t)+v_0^2(t))\right) dt\\ &= 0, \qquad\qquad (17.137)\\ v_1(x) &= \frac{2}{3}(e^{2x}-1)^{\frac{3}{2}} + \int_0^x \left(\frac{1}{(e^{2x}-e^{2t})^{\frac{1}{2}}}(u_0^2(t)-v_0^2(t))\right) dt = 0,\\ u_{k+1}(x) &= 0, k \geqslant 1, \quad v_{k+1}(x) = 0, k \geqslant 1. \end{aligned}$$

The exact solutions are therefore given by

$$(u(x), v(x)) = (e^x, e^{2x}). \tag{17.138}$$

Example 17.18

Solve the system of the nonlinear weakly-singular Volterra integral equations by using the modified decomposition method

$$\begin{aligned}
u(x) &= \sin^{\frac{1}{2}} x + \frac{3}{2}(\cos x - 1)^{\frac{2}{3}} - \frac{3}{2}\sin^{\frac{2}{3}} x \\
&\quad + \int_0^x \left(\frac{1}{(\cos x - \cos t)^{\frac{1}{3}}} u^2(t) + \frac{1}{(\sin x - \sin t)^{\frac{1}{3}}} v^2(t) \right) dt, \\
v(x) &= \cos^{\frac{1}{2}} x + 3(\cos x - 1)^{\frac{1}{3}} + 3\sin^{\frac{1}{3}} x \\
&\quad + \int_0^x \left(\frac{1}{(\cos x - \cos t)^{\frac{2}{3}}} u^2(t) - \frac{1}{(\sin x - \sin t)^{\frac{2}{3}}} v^2(t) \right) dt.
\end{aligned} \tag{17.139}$$

Proceeding as before, we set the modified recurrence relation

$$\begin{aligned}
u_0(x) &= \sin^{\frac{1}{2}} x, \quad v_0(x) = \cos^{\frac{1}{2}} x, \\
u_1(x) &= \frac{3}{2}(\cos x - 1)^{\frac{2}{3}} - \frac{3}{2}\sin^{\frac{2}{3}} x \\
&\quad + \int_0^x \left(\frac{1}{(\cos x - \cos t)^{\frac{1}{3}}} u_0^2(t) + \frac{1}{(\sin x - \sin t)^{\frac{1}{3}}} v_0^2(t) \right) dt = 0, \\
v_1(x) &= \cos^{\frac{1}{2}} x + 3(\cos x - 1)^{\frac{1}{3}} + 3\sin^{\frac{1}{3}} x \\
&\quad + \int_0^x \left(\frac{1}{(\cos x - \cos t)^{\frac{2}{3}}} u_0^2(t) - \frac{1}{(\sin x - \sin t)^{\frac{2}{3}}} v_0^2(t) \right) dt = 0, \\
u_{k+1}(x) &= 0, k \geqslant 1, \quad v_{k+1}(x) = 0, k \geqslant 1.
\end{aligned} \tag{17.140}$$

The exact solutions are therefore given by

$$(u(x), v(x)) = (\sin^{\frac{1}{2}} x, \cos^{\frac{1}{2}} x). \tag{17.141}$$

Example 17.19

Solve the system of the nonlinear weakly-singular Volterra integral equations by using the modified decomposition method

$$\begin{aligned}
u(x) &= x^2 - \frac{2}{5}x^{\frac{5}{2}} - \frac{3}{14}x^{\frac{14}{3}} + \int_0^x \left(\frac{1}{(x^5 - t^5)^{\frac{1}{2}}} u^2(t) + \frac{1}{(x^7 - t^7)^{\frac{1}{3}}} v^2(t) \right) dt, \\
v(x) &= x^3 - \frac{4}{15}x^{\frac{15}{4}} - \frac{5}{28}x^{\frac{28}{5}} + \int_0^x \left(\frac{1}{(x^5 - t^5)^{\frac{1}{4}}} u^2(t) + \frac{1}{(x^7 - t^7)^{\frac{1}{5}}} v^2(t) \right) dt.
\end{aligned} \tag{17.142}$$

We next set the modified recurrence relation and proceeding as before we obtain

$$\begin{aligned}
u_0(x) &= x^2, & v_0(x) &= x^3, \\
u_1(x) &= 0, & v_1(x) &= 0, \\
u_{k+1}(x) &= 0, k \geqslant 1, & v_{k+1}(x) &= 0, k \geqslant 1.
\end{aligned} \tag{17.143}$$

The exact solutions are therefore given by

$$(u(x), v(x)) = (x^2, x^3). \tag{17.144}$$

Example 17.20

Solve the system of the nonlinear weakly-singular Volterra integral equations by using the modified decomposition method

$$
\begin{aligned}
u(x) &= e^{x} - \frac{1}{2}(e^{4x}-1)^{\frac{1}{2}} - \frac{1}{3}(e^{6x}-1)^{\frac{1}{2}} \\
&\quad + \int_0^x \left(\frac{1}{(e^{4x}-e^{4t})^{\frac{1}{2}}} v^2(t) + \frac{1}{(e^{6x}-e^{6t})^{\frac{1}{2}}} w^2(t) \right) dt, \\
v(x) &= e^{2x} - \frac{1}{3}(e^{6x}-1)^{\frac{1}{2}} - (e^{2x}-1)^{\frac{1}{2}} \\
&\quad + \int_0^x \left(\frac{1}{(e^{6x}-e^{6t})^{\frac{1}{2}}} w^2(t) + \frac{1}{(e^{2x}-e^{2t})^{\frac{1}{2}}} u^2(t) \right) dt, \\
w(x) &= e^{3x} - (e^{2x}-1)^{\frac{1}{2}} - \tfrac{1}{2}(e^{4x}-1)^{\frac{1}{2}} \\
&\quad + \int_0^x \left(\frac{1}{(e^{2x}-e^{2t})^{\frac{1}{2}}} u^2(t) + \frac{1}{(e^{4x}-e^{4t})^{\frac{1}{2}}} v^2(t) \right) dt.
\end{aligned}
\tag{17.145}
$$

Now we use the modified recurrence relation

$$
\begin{aligned}
&u_0(x) = e^{x}, \quad v_0(x) = e^{2x}, \ w_0(x) = e^{3x}, \\
&u_1(x) = 0, \quad\ v_1(x) = 0, \quad\ w_1(x) = 0, \\
&u_{k+1}(x) = 0,\ v_{k+1}(x) = 0,\ w_{k+1}(x) = 0, k \geqslant 1.
\end{aligned}
\tag{17.146}
$$

The exact solutions are therefore given by

$$
(u(x), v(x), w(x)) = (e^{x}, e^{2x}, e^{3}). \tag{17.147}
$$

Exercises 17.5.1

Solve the following systems of generalized weakly singular Volterra integral equations by using the modified decomposition method

1. $$\begin{cases} u(x) = x - \dfrac{2}{5}x^{\frac{5}{2}} + \displaystyle\int_0^x \left(\frac{1}{(x^5-t^5)^{\frac{1}{2}}} u^2(t)v(t) \right) dt \\ v(x) = x^2 - \dfrac{1}{4}x^4 + \displaystyle\int_0^x \left(\frac{1}{(x^6-t^6)^{\frac{1}{2}}} u(t)v^2(t) \right) dt \end{cases}$$

2. $$\begin{cases} u(x) = x - \dfrac{2}{3}x^{\frac{3}{2}} - \dfrac{1}{2}x^2 + \displaystyle\int_0^x \left(\frac{1}{(x^3-t^3)^{\frac{1}{2}}} u^2(t) + \frac{1}{(x^3-t^3)^{\frac{1}{3}}} v(t) \right) dt \\ v(x) = x^2 - \dfrac{2}{3}x^{\frac{3}{2}} - \dfrac{1}{4}x^4 + \displaystyle\int_0^x \left(\frac{1}{(x^2-t^2)^{\frac{1}{4}}} u(t) + \frac{1}{(x^5-t^5)^{\frac{1}{5}}} v^2(t) \right) dt \end{cases}$$

3. $$\begin{cases} u(x) = e^{x} - \dfrac{4}{3}(e^{x}-1)^{\frac{1}{2}}(e^{x}+2) + \displaystyle\int_0^x \frac{1}{(e^{x}-e^{t})^{\frac{1}{2}}} (u^2(t)+v^2(t))\, dt \\ v(x) = e^{\frac{1}{2}x} - \dfrac{4}{3}(e^{x}-1)^{\frac{1}{2}}(e^{x}-1) + \displaystyle\int_0^x \frac{1}{(e^{x}-e^{t})^{\frac{1}{2}}} (u^2(t)-v^2(t))\, dt \end{cases}$$

4. $$\begin{cases} u(x) = e^{x} - 2(e^{x}-1)^{\frac{1}{2}} + \displaystyle\int_0^x \frac{1}{(e^{x}-e^{t})^{\frac{1}{2}}} u^2(t)v(t)\, dt \\ v(x) = e^{-x} + 2(e^{-x}-1)^{\frac{1}{2}} + \displaystyle\int_0^x \frac{1}{(e^{-x}-e^{-t})^{\frac{1}{2}}} u(t)v^2(t)\, dt \end{cases}$$

5. $$\begin{cases} u(x) = \cos^{\frac{1}{2}} x + 2(\cos x - 1)^{\frac{1}{2}} - 2\sin^{\frac{1}{2}} x \\ \qquad + \int_0^x \left(\frac{1}{(\sin x - \sin t)^{\frac{1}{2}}} u^2(t) + \frac{1}{(\cos x - \cos t)^{\frac{1}{2}}} v^2(t) \right) dt \\ v(x) = \sin^{\frac{1}{2}} x - \frac{3}{2}(\cos x - 1)^{\frac{2}{3}} - \frac{3}{2}\sin^{\frac{2}{3}} x \\ \qquad + \int_0^x \left(\frac{1}{(\sin x - \sin t)^{\frac{1}{3}}} u^2(t) - \frac{1}{(\cos x - \cos t)^{\frac{1}{3}}} v^2(t) \right) dt \end{cases}$$

6. $$\begin{cases} u(x) = \cos x + \frac{2}{15}(15 - 8\sin^2 x)\sin^{\frac{1}{2}} x + \int_0^x \left(\frac{1}{(\sin x - \sin t)^{\frac{1}{2}}} u^2(t)v(t) \right) dt \\ v(x) = -\cos x - \frac{3}{40}(20 - 9\sin^2 x)\sin^{\frac{2}{3}} x + \int_0^x \left(\frac{1}{(\sin x - \sin t)^{\frac{1}{3}}} u(t)v^2(t) \right) dt \end{cases}$$

7. $$\begin{cases} u(x) = x - \frac{1}{3}x^2 + \int_0^x \frac{1}{(x^6 - t^6)^{\frac{1}{2}}} v(t)w(t)\,dt \\ v(x) = x^2 - \frac{3}{10}x^{\frac{10}{3}} + \int_0^x \frac{1}{(x^5 - t^5)^{\frac{1}{3}}} w(t)u(t)\,dt \\ w(x) = x^3 - \frac{1}{3}x^3 + \int_0^x \frac{1}{(x^4 - t^4)^{\frac{1}{4}}} u(t)v(t)\,dt \end{cases}$$

8. $$\begin{cases} u(x) = x - \frac{2}{35}(7 + 5x)x^{\frac{5}{2}} + \int_0^x \left(\frac{1}{(x^5 - t^5)^{\frac{1}{2}}} v^2(t) + \frac{1}{(x^7 - t^7)^{\frac{1}{2}}} w^2(t) \right) dt \\ v(x) = x^2 - \frac{2}{21}(7 + 3x^2)x^{\frac{3}{2}} + \int_0^x \left(\frac{1}{(x^7 - t^7)^{\frac{1}{2}}} w^2(t) + \frac{1}{(x^3 - t^3)^{\frac{1}{2}}} u^2(t) \right) dt \\ w(x) = x^3 - \frac{2}{15}(5 + 3x)x^{\frac{3}{2}} + \int_0^x \left(\frac{1}{(x^3 - t^3)^{\frac{1}{2}}} u^2(t) + \frac{1}{(x^5 - t^5)^{\frac{1}{2}}} v^2(t) \right) dt \end{cases}$$

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Chapter 18
Applications of Integral Equations

18.1 Introduction

Integral equations arise in many scientific and engineering problems. A large class of initial and boundary value problems can be converted to Volterra or Fredholm integral equations. The potential theory contributed more than any field to give rise to integral equations. Mathematical physics models, such as diffraction problems, scattering in quantum mechanics, conformal mapping, and water waves also contributed to the creation of integral equations.

Many other applications in science and engineering are described by integral equations or integro-differential equations. The Volterra's population growth model, biological species living together, propagation of stocked fish in a new lake, the heat transfer and the heat radiation are among may areas that are described by integral equations. Many scientific problems give rise to integral equations with logarithmic kernels. Integral equations often arise in electrostatic, low frequency electro magnetic problems, electro magnetic scattering problems and propagation of acoustical and elastical waves.

In this text we presented a variety of methods to handle all types of integral equations. The problems we handled led in most cases to the determination of exact solutions. Because it is not always possible to find exact solutions to problems of physical sciences, much work is devoted to obtaining qualitative approximations that highlight the structure of the solution.

It is the aim of this chapter to handle some integral applications taken from a variety of fields. The obtained series will be handled to give insight into the behavior of the solution and some of the properties of the examined phenomenon. For each application, we will select one or two proper approaches from the methods that were presented in this text. The reader can try different methods for each application.

Polynomials are frequently used to approximate power series. However, polynomials tend to exhibit oscillations that may produce an approximation error bounds. In addition, polynomials can never blow up in a finite plane;

and this makes the singularities not apparent. To overcome these difficulties, the Taylor series is best manipulated by Padé approximants for numerical approximations.

18.2 Volterra's Population Model

In this section we will study the Volterra model for population growth of a species within a closed system. The Volterra's population model is characterized by the nonlinear Volterra integro-differential equation

$$\frac{dP}{dT} = aP - bP^2 - cP\int_0^T P(x)dx,\ \ P(0) = P_0, \tag{18.1}$$

where $P = P(T)$ denotes the population at time T, a, b, and c are constants and positive parameters, $a > 0$ is the birth rate coefficient, $b > 0$ is the crowding coefficient, $c > 0$ is the toxicity coefficient, and P_0 is the initial population. The coefficient c indicates the essential behavior of the population evolution before its level falls to zero in the long run.

When $b = 0$ and $c = 0$, Equation (18.1) becomes the Malthus differential equation

$$\frac{dP}{dT} = aP,\ P(0) = P_0. \tag{18.2}$$

The Malthus equation (18.2) assumes that population growth is proportional to the current population. Equation (18.2) is separable with a solution given by

$$P(T) = P_0 e^{aT}. \tag{18.3}$$

It is obvious that Equation (18.3) represents a population growth for $a > 0$, and a population decay for $a < 0$.

When $c = 0$, Equation (18.1) becomes the logistic growth model that reads

$$\frac{dP}{dT} = aP - bP^2,\ \ P(0) = P_0. \tag{18.4}$$

Verhulst instituted the logistic growth model (18.4) that eliminates the undesirable effect of unlimited growth by introducing the growth-limiting term $-bP^2$. The solution to logistic growth model (18.4) is

$$P(T) = \frac{aP_0 e^{aT}}{a + bP_0(e^{aT} - 1)}, \tag{18.5}$$

where

$$\lim_{T\to\infty} = \frac{a}{b}. \tag{18.6}$$

Volterra introduced an integral term $-cP\int_0^T P(x)dx$ to the logistic growth model (18.4) to get the Volterra's population growth model (18.1). The additional integral term characterizes the accumulated toxicity produced since time zero.

Many time scales and population scales may be applied. However, we apply the scale time and population by introducing the non-dimensional variables

$$t = \frac{cT}{b},\ u = \frac{bP}{a}, \tag{18.7}$$

to obtain the non-dimensional Volterra's population growth model

$$\kappa\frac{du}{dt} = u - u^2 - u\int_0^t u(x)dx,\ u(0) = u_0, \tag{18.8}$$

where $u \equiv u(t)$ is the scaled population of identical individuals at a time t, and the non-dimensional parameter $\kappa = c/(ab)$ is a prescribed parameter. Volterra introduced this model for a population $u(t)$ of identical individuals which exhibits crowding and sensitivity to the amount of toxins produced.

A considerable amount of research work has been invested to determine numerical and analytic solutions [1–5] of the population growth model (18.8). The analytical solution

$$u(t) = u_0 e^{\left(\frac{1}{\kappa}\int_0^t \left[1-u(\tau)-\int_0^\tau u(x)dx\right] d\tau\right)}, \tag{18.9}$$

shows that $u(t) > 0$ for all t if the initial population $u_0 > 0$. However, this closed form solution cannot lead to an insight into the behavior of the population evolution. As a result, research was directed towards the analysis of the population rapid rise along the logistic curve followed by its decay to zero in the long run. The non-dimensional parameter κ plays a great role in the behavior of $u(t)$ concerning the rapid rise to a certain amplitude followed by an exponential decay to extinction. For κ small, the population is not sensitive to toxins, whereas the population is strongly sensitive to toxins for large κ [5]. Furthermore, for κ large, it was shown by [2] that the solution is proportional to $\text{sech}^2(t)$.

As stated before, many analytical and numerical methods have been used to determine closed form solution and numerical approximations to the Volterra's population model (18.8). We have presented six different methods in Chapter 5 to handle Volterra integro-differential equations. In this section, we will apply only two of these six methods to determine an approximation of a reasonable accuracy level to the solution of the Volterra's population model (18.8). The two methods are the variational iteration method and the series solution method. The efficiency of these two methods can be dramatically improved by determining further terms of the power series. Further, the behavior of the solution structure in that it increases rapidly in the logistic curve and it decreases exponentially to extinction in the long run can be formally determined by using the Padé approximants of the obtained series.

18.2.1 The Variational Iteration Method

To avoid the cumbersome work of Adomian polynomials we will apply the variational iteration method to formally derive an approximation to the

Volterra's population growth model

$$\frac{du}{dt} = 10u(t) - 10u^2(t)\ - 10u(t)\int_0^t u(x)dx,\ u(0) = 0.1, \tag{18.10}$$

where, for simplicity reasons, we selected $\kappa = 0.1$ and $u(0) = 0.1$. The variational iteration method provides rapidly convergent successive approximations of the solution. The correction functional for the integro-differential equation (18.10) is

$$\begin{aligned} u_{n+1}(t) &= u_n(t) \\ &- \int_0^t \left(u_n'(r) - 10u_n(r) + 10u_n^2(r) + 10u_n(r)\int_0^r u_n(x)dx\right) dr, n \geqslant 0, \end{aligned} \tag{18.11}$$

where we used $\lambda = -1$ for first-order integro-differential equations. We can select $u_0(x) = u(0) = 0.1$. Using this selection into the correction functional (18.11) gives the following successive approximations

$$\begin{aligned} u_0(x) &= 0.1, \\ u_1(x) &= 0.1 + 0.9t - 0.05t^2, \\ u_2(x) &= 0.1 + 0.9t + 3.55t^2 - 3.283333333t^3 - 0.7708333333t^4 \\ &\quad + 0.07t^5 - 0.001388888889t^6, \\ u_3(x) &= 0.1 + 0.9t + 3.55t^2 + 6.316666667t^3 - 24.7375t^4 - 19.1225t^5 \\ &\quad + 38.10513889t^6 + 2.631746032t^7 - 8.542214780t^8 + O(t^9), \\ &\ \ \vdots\ , \\ u_8(t) &= 0.1 + 0.9\,t + 3.55\,t^2 + 6.316666667\,t^3 - 5.5375\,t^4 \\ &\quad - 63.70916667\,t^5 - 156.0804167\,t^6 - 18.47323411\,t^7 \\ &\quad + 1056.288569\,t^8 + O(t^9), \end{aligned} \tag{18.12}$$

and so on. Based on this, the approximation of $u(t)$ is given by

$$\begin{aligned} u(t) &= 0.1 + 0.9\,t + 3.55\,t^2 + 6.316666667\,t^3 - 5.5375\,t^4 - 63.70916667\,t^5 \\ &\quad -156.0804167\,t^6 - 18.47323411\,t^7 + 1056.288569\,t^8 + O(t^9), \end{aligned} \tag{18.13}$$

To increase the degree of accuracy, more approximations should be determined and more terms should be included.

18.2.2 The Series Solution Method

We will apply the series solution method to the nonlinear Volterra integro-differential equation

$$\frac{du}{dt} = 10\,u(t) - 10u^2(t) - 10u(t)\int_0^t u(x)\,dx,\ u(0) = 0.1, \tag{18.14}$$

where we selected the initial condition $u(0) = 0.1$ and the non-dimensional parameter $\kappa = 0.1$.

Assuming that $u(t)$ is analytic, then it can be represented by a power series of the form

$$u(t) = \sum_{n=0}^{\infty} a_n\, t^n. \tag{18.15}$$

Substituting (18.15) into both sides of (18.14) gives

$$\begin{aligned}\sum_{n=1}^{\infty} n a_n\, t^{n-1} &= 10\left(\sum_{n=0}^{\infty} a_n t^n\right) - 10\left(\sum_{n=0}^{\infty} a_n t^n\right)^2 \\ &\quad -10\left(\sum_{n=0}^{\infty} a_n\, t^n\right)\int_0^t\left(\sum_{n=0}^{\infty} a_n x^n\right)\, dx.\end{aligned} \tag{18.16}$$

Evaluating the integral at the right side, equating the coefficients of identical powers of t from both sides, and using the initial condition lead to

$$\begin{aligned} &a_0 = 0.1, && a_1 = 0.9, && a_2 = 3.55, \\ &a_3 = 6.316666667, && a_4 = -5.537500000, && a_5 = -63.70916667, \\ &a_6 = -156.0804167, && a_7 = -18.47323411, && a_8 = 1056.288569, \end{aligned} \tag{18.17}$$

and so on.

The approximation of $u(t)$ given by proceeding as before, the series solution

$$\begin{aligned} u(t) &= 0.1 + 0.9\,t + 3.55\,t^2 + 6.316666667\,t^3 - 5.5375\,t^4 - 63.70916667\,t^5 \\ &\quad -156.0804167\,t^6 - 18.47323411\,t^7 + 1056.288569\,t^8 + O(t^9), \end{aligned} \tag{18.18}$$

obtained by using (18.17) into (18.15). The obtained approximation (18.18) is consistent with the approximation (18.13) obtained before by using the variational iteration method.

18.2.3 The Padé Approximants

To examine more closely the mathematical structure of $u(t)$ as shown above, we seek to study the rapid growth along the logistic curve that will reach a peak, then followed by the slow exponential decay where $u(t) \to 0$ as $t \to \infty$.

Polynomials are frequently used to approximate power series. However, polynomials tend to exhibit oscillations that may produce an approximation error bounds. In addition, polynomials can never blow up in a finite plane; and this makes the singularities not apparent. To overcome these difficulties, the Taylor series is best manipulated by Padé approximants for numerical approximations.

Padé approximants [4] have the advantage of manipulating the polynomial approximation into a rational function to gain more information about $u(t)$. Padé approximant represents a function by the ratio of two polynomials [4].

The coefficients of the polynomials in the numerator and in the denominator are determined by using the coefficients in the Taylor expansion of the function. To determine the Padé approximants, the reader is advised to read [4] and to use computer programs such as Maple or Mathematica.

Consequently, the series (18.18) should be manipulated to construct several Padé approximants where the performance of the approximants show superiority over series solutions. Using computer tools we obtain the following approximant

$$[4/4] = \frac{0.1 + 0.4687931695t + 0.9249573236t^2 + 0.9231293234t^3 + 0.400423311t^4}{1 - 4.312068305t + 12.55818798t^2 - 13.88064046t^3 + 10.8683052t^4}. \tag{18.19}$$

Figure 18.1 shows the behavior of $u(t)$ and explores the rapid growth that will reach a peak followed by a slow exponential decay. This behavior cannot be obtained if we graph the truncated polynomial of the series solution.

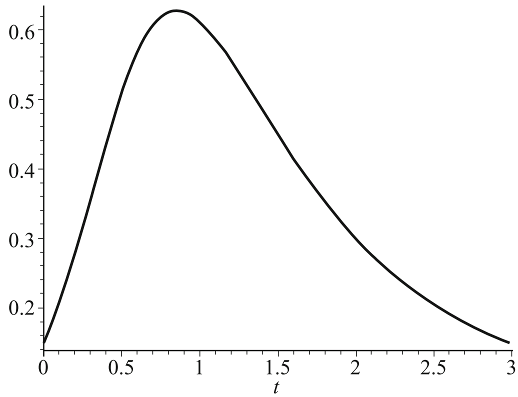

Fig. 18.1 Padé approximant [4/4] shows a rapid growth followed by a slow exponential decay.

Figure 18.2 shows the relation between Padé approximant [4/4] of $u(t)$ and t for $\kappa = 0.05, 0.1, 0.5$, where the graph of larger amplitude is related to the small $\kappa = 0.05$, whereas the graph with smaller amplitude is related to the large $\kappa = 0.5$.

18.3 Integral Equations with Logarithmic Kernels

In this section, we will study first and second kind Fredholm integral equations with logarithmic kernels. An example of such equations is the exterior boundary value problem for the two-dimensional Helmholtz equation char-

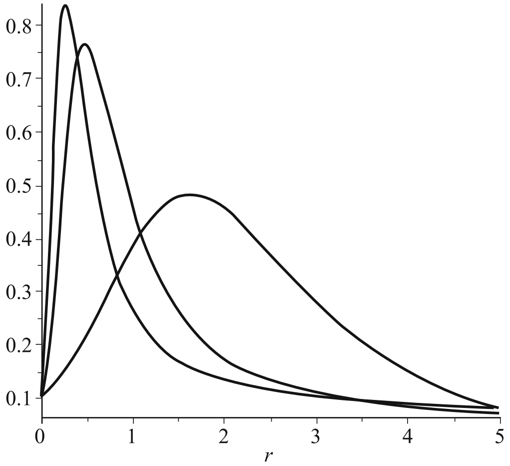

Fig. 18.2 Relation between Padé approximant [4/4] of $u(t)$ and t for $\kappa = 0.1, 0.5, 0.05$.

acterized by a second kind Fredholm integral equation with a logarithmic kernel.

It is useful first to introduce and prove the following identities:

Identity 1

The first identity reads

$$\int_{-\pi}^{\pi} \log\left|2\sin\frac{x-t}{2}\right| \cos t\, dt = -\pi\cos x. \tag{18.20}$$

To prove this identity, we integrate the left side by parts to find

$$\begin{aligned}
&\int_{-\pi}^{\pi} \log\left|2\sin\frac{x-t}{2}\right| \cos t dt \\
&= \left[\log\left|2\sin\frac{x-t}{2}\right| \sin t\right]_{-\pi}^{\pi} + \frac{1}{2}\int_{-\pi}^{\pi} \sin t\cot\frac{x-t}{2} dt, \\
&= \frac{1}{2}\int_{-\pi}^{\pi} \sin t\cot\frac{x-t}{2} dt.
\end{aligned} \tag{18.21}$$

Substituting

$$w(x,t) = \frac{x-t}{2}, \tag{18.22}$$

into the right hand side of (18.21) gives

$$\begin{aligned}
\int_{-\pi}^{\pi} \log\left|2\sin\frac{x-t}{2}\right| \cos t\, dt = -\sin x\int_{\frac{x+\pi}{2}}^{\frac{x-\pi}{2}} (\cot w - \sin 2w)\ dw \\
+2\cos x\int_{\frac{x+\pi}{2}}^{\frac{x-\pi}{2}} \cos^2 w\ dw.
\end{aligned} \tag{18.23}$$

Integrating the right side of (18.23) proves the identity (18.20).

Identity 2
The second identity reads

$$\int_{-\pi}^{\pi} \log\left|2\sin\frac{x-t}{2}\right| \sin t\,dt = -\pi\sin x. \tag{18.24}$$

Integrating the left side of (18.24) by parts gives

$$\begin{aligned}
&\int_{-\pi}^{\pi} \log\left|2\sin\frac{x-t}{2}\right| \sin t dt\\
&= \left[-\log\left|2\sin\frac{x-t}{2}\right|\cos t\right]_{-\pi}^{\pi} - \frac{1}{2}\int_{-\pi}^{\pi}\cos t\cot\frac{x-t}{2}dt,\\
&= -\frac{1}{2}\int_{-\pi}^{\pi}\cos t\cot\frac{x-t}{2}dt.
\end{aligned} \tag{18.25}$$

Substituting

$$w(x,t) = \frac{x-t}{2}, \tag{18.26}$$

into (18.25), and proceeding as before, we obtain the second identity (18.24).

Identity 3
The third identity reads

$$\int_{-\pi}^{\pi} \log\left|2\sin\frac{x-t}{2}\right| \cos 2t\,dt = -\frac{\pi}{2}\cos 2x, \tag{18.27}$$

Identity 4
The third identity reads

$$\int_{-\pi}^{\pi} \log\left|2\sin\frac{x-t}{2}\right| \sin 2t\,dt = -\frac{\pi}{2}\sin 2x. \tag{18.28}$$

The last two identities can be proved in a similar manner.

Many scientific problems give rise to integral equations with logarithmic kernels. The exterior boundary value problem for the two-dimensional Helmholtz equation usually lead to a Fredholm integral equation of the second kind with logarithmic kernel. Many inverse problems, such as tomography, geophysics and non-destructive detection give rise to Fredholm integral equation of the first kind with logarithmic kernel that is severely ill-posed in Hadamard's sense.

The few topics where a formulation of a problem by means of integral equations with logarithmic kernels has been used are reported in [6–7] as follows:

1. Investigation of electrostatic, and low frequency electromagnetic problems.
2. Methods for computing the conformal mapping of a given domain.
3. Solution of electromagnetic scattering problems.

4. Determination of propagation of acoustical and elastical waves.
For more information about these examples from mathematical physics, the reader is advised to read [6–7] and the references therein.

In what follows we will study the two kinds of Fredholm integral equations with logarithmic kernels.

18.3.1 Second Kind Fredholm Integral Equation with a Logarithmic Kernel

In this section we selected the exterior boundary value problem for the two-dimensional Helmholtz equation characterized by a second kind Fredholm integral equation

$$u(x) = f(x) - \int_{-\pi}^{\pi} K(x,t)u(t)\,dt, x \in [-\pi, \pi], \tag{18.29}$$

where the kernel $K(x,t)$ is logarithmic given by

$$K(x,t) = -\frac{a(x,t)}{\pi}\log\left|2\sin\frac{x-t}{2}\right| + b(x,t), \tag{18.30}$$

with

$$a(x,t) = a_0 + a_1(x,t)\sin^2\frac{x-t}{2}, \tag{18.31}$$

where a_0 is a constant, $a_1(x,t)$ and $b(x,t)$ are continuous functions of (x,t) and are 2π periodic in each variable.

Many numerical methods have been used to investigate this equation. Most of these methods suffer from the computational difficulties. In this text, we presented many methods to handle Fredholm integral equations of the second kind. Some of the methods that we applied before require that the kernel being separable. In (18.29), the kernel is not separable, therefore we select the Adomian decomposition method to handle the Fredholm integral equation of the second kind (18.29).

The Adomian Decomposition Method

The Adomian decomposition method will be used to handle the Fredholm integral equation of the second kind given by

$$u(x) = f(x) - \int_{-\pi}^{\pi} K(x,t)u(t)\,dt, x \in [-\pi, \pi], \tag{18.32}$$

where the kernel $K(x,t)$ is logarithmic given by

$$K(x,t) = -\frac{a(x,t)}{\pi}\log\left|2\sin\frac{x-t}{2}\right| + b(x,t), \tag{18.33}$$

with

$$a(x,t) = a_0 + a_1(x,t)\sin^2\frac{x-t}{2}, \tag{18.34}$$

where a_0 is a constant, $a_1(x,t)$ and $b(x,t)$ are continuous functions of (x,t) and are 2π periodic in each variable.

The Adomian decomposition method decomposes the linear term $u(x)$ by an infinite series

$$u(x) = \sum_{n=0}^{\infty} u_n(x), \tag{18.35}$$

where the components $u_0(x), u_1(x), u_2(x), \ldots$ will be determined recurrently. Substituting (18.35) into both sides of (18.32) gives

$$\sum_{n=0}^{\infty} u_n(x) \;=\; f(x) - \int_{-\pi}^{\pi} \left(\sum_{n=0}^{\infty} u_n(t)\right) K(x,t)dt,\ x \in [-\pi, \pi]. \tag{18.36}$$

The components can be obtained by using the recurrence relation

$$\begin{aligned} u_0(x) &= f(x), \\ u_{k+1}(x) &= -\int_{-\pi}^{\pi} u_k(t)K(x,t)dt, k \geqslant 0. \end{aligned} \tag{18.37}$$

Having determined the components, the solution in a series form is obtained upon using (18.35). This will be illustrated by the following applications.

Application 1

As a first application, we consider the case examined in [8], where the domain is the circular region

$$x_1^2 + x_2^2 \leqslant R^2, x_1 = R\cos x, x_2 = R\sin x. \tag{18.38}$$

For the first application, it was assumed that

$$\begin{aligned} f(x) &= 2f(x_1, x_2) = 2x_1 = 2R\cos x, \\ a(x,t) &= -i, \quad i = \sqrt{-1}, \\ b(x,t) &= -\frac{1}{2\pi}(1+2i), \end{aligned} \tag{18.39}$$

where R is the radius of the circular region. Using (18.37) gives

$$\begin{aligned} u_0(x) &= 2R\cos x, & u_1(x) &= 2iR\cos x, \\ u_2(x) &= 2i^2R\cos x, & u_3(x) &= 2i^3R\cos x, \end{aligned} \tag{18.40}$$

obtained upon using the first identity. As a result, the series solution is given by

$$u(x) = 2R\cos x(1 + i + i^2 + i^3 + \cdots), \tag{18.41}$$

that converges to

$$u(x) = R(1+i)\cos x, \tag{18.42}$$

obtained after evaluating the sum of the geometric series.

Application 2

As a second application, we consider the case examined in [8], where the domain is the circular region

$$x_1^2 + x_2^2 \leqslant R^2, x_1 = R\cos x, x_2 = R\sin x. \tag{18.43}$$

For the second application, it was assumed that

$$\begin{aligned} f(x) &= 2f(x_1, x_2) = 2x_2 = 2R\sin x, \\ a(x,t) &= -i, \quad i = \sqrt{-1}, \\ b(x,t) &= -\frac{1}{2\pi}(1+2i), \end{aligned} \tag{18.44}$$

where R is the radius of the circular region. Using (18.37) gives

$$\begin{aligned} u_0(x) &= 2R\sin x, & u_1(x) &= 2iR\sin x, \\ u_2(x) &= 2i^2R\sin x, & u_3(x) &= 2i^3R\sin x, \end{aligned} \tag{18.45}$$

obtained upon using the second identity. As a result, the series solution is given by

$$u(x) = 2R\sin x(1 + i + i^2 + i^3 + \cdots), \tag{18.46}$$

that converges to

$$u(x) = R(1+i)\sin x. \tag{18.47}$$

Application 3

As a third application, we consider the case examined in [8], where the domain is the circular region

$$x_1^2 + x_2^2 \leqslant R^2, x_1 = R\cos x, x_2 = R\sin x. \tag{18.48}$$

For the third application, it was assumed that

$$\begin{aligned} & f(x) = 2(x_1^2 - x_2^2) = 2R^2\cos 2x, \\ & a(x,t) = -i, i = \sqrt{-1}, \quad b(x,t) = -\frac{1}{2\pi}(1+2i), \end{aligned} \tag{18.49}$$

where R is the radius of the circular region. Using (18.37) gives

$$\begin{aligned} u_0(x) &= 2R^2\cos 2x, & u_1(x) &= R^2 i\cos 2x, \\ u_2(x) &= \frac{R^2}{2}i^2\cos 2x, & u_3(x) &= \frac{R^2}{4}i^3\cos 2x, \end{aligned} \tag{18.50}$$

obtained upon using the third identity. As a result, the series solution is given by

$$u(x) = 2R^2\cos 2x\left(1 + \frac{i}{2} + \frac{i^2}{4} + \frac{i^3}{8} + \cdots\right), \tag{18.51}$$

and in closed form

$$u(x) = \frac{4}{5}R^2(2+i)\,\cos 2x. \tag{18.52}$$

Application 4

As a last application, we consider the case where the domain is the circular region

$$x_1^2 + x_2^2 \leqslant R^2, x_1 = R\cos x, x_2 = R\sin x. \tag{18.53}$$

For the last application, it was assumed that

$$f(x) = 4x_1x_2 = 2R^2 \sin 2x,$$
$$a(x,t) = -i, i = \sqrt{-1}, \quad b(x,t) = -\frac{1}{2\pi}(1+2i), \tag{18.54}$$

where R is the radius of the circular region. Using (18.37) gives

$$u_0(x) = 2R^2 \sin 2x, \qquad u_1(x) = R^2 i \sin 2x,$$
$$u_2(x) = \frac{R^2}{2} i^2 \sin 2x, \quad u_3(x) = \frac{R^2}{4} i^3 \sin 2x, \tag{18.55}$$

obtained upon using the third identity. As a result, the series solution is given by

$$u(x) = 2R^2 \sin 2x \left(1 + \frac{i}{2} + \frac{i^2}{4} + \frac{i^3}{8} + \cdots\right), \tag{18.56}$$

and in closed form

$$u(x) = \frac{4}{5} R^2 (2+i) \sin 2x. \tag{18.57}$$

18.3.2 First Kind Fredholm Integral Equation with a Logarithmic Kernel

In [6], an integral equation whose kernel is equal to the logarithm of the distance between two points on a plane, closed, smooth, and simple curve is investigated. The problem is characterized by a Fredholm integral equation with logarithmic kernel given by

$$f(x) = \int_{-\pi}^{\pi} K(x,t)u(t)\,dt, x \in [-\pi, \pi], \tag{18.58}$$

where the kernel $K(x,t)$ is logarithmic given by

$$K(x,t) = \log \left| 2 \sin \frac{x-t}{2} \right|. \tag{18.59}$$

The unknown function $u(x)$ and the known function $f(x)$ are 2π periodic functions.

Many numerical methods have been used to investigate this equation, but most of these methods suffer from the computational difficulties. In this text, we presented two methods to handle Fredholm integral equations of the first kind. In (18.58), the kernel is not separable, therefore we select the regularization method combined with the Adomian decomposition method to handle the Fredholm integral equation of the first kind (18.58).

The Regularization method combined with the Adomian Decomposition Method

We will use the regularization method combined with the Adomian decomposition method to handle the Fredholm integral equation of the first kind given by

$$f(x) = \int_{-\pi}^{\pi} K(x,t)u(t)\,dt, x \in [-\pi, \pi], \tag{18.60}$$

where the kernel $K(x,t)$ is logarithmic given by

$$K(x,t) = \log\left|2\sin\frac{x-t}{2}\right|. \tag{18.61}$$

The method of regularization consists of replacing ill-posed problem by well-posed problem. The method of regularization transforms the linear Fredholm integral equation of the first kind (18.60) to the approximation Fredholm integral equation

$$\mu u_\mu(x) = f(x) - \int_{-\pi}^{\pi} K(x,t)u_\mu(t)\,dt, x \in [-\pi, \pi], \tag{18.62}$$

where μ is a small positive parameter. It is clear that (18.62) is a Fredholm integral equation of the second kind that can be rewritten as

$$u_\mu(x) = \frac{1}{\mu}f(x) - \frac{1}{\mu}\int_{-\pi}^{\pi} K(x,t)u_\mu(t)\,dt, x \in [-\pi, \pi]. \tag{18.63}$$

Notice that Equation (18.63) will be solved by using the Adomian decomposition method. As proved in Chapter 4, the solution u_μ of equation (18.63) converges to the solution $u(x)$ of (18.60) as $\mu \to 0$. The exact solution $u(x)$ can thus be obtained by

$$u(x) = \lim_{\mu\to 0} u_\mu(x). \tag{18.64}$$

The Adomian decomposition method decomposes the linear term $u(x)$ by an infinite series

$$u_\mu(x) = \sum_{n=0}^{\infty} u_{\mu_n}(x), \tag{18.65}$$

where the components $u_{\mu_0}(x), u_{\mu_1}, u_{\mu_2}, \ldots$ will be determined recurrently. Substituting (18.65) into both sides of (18.63) gives

$$\sum_{n=0}^{\infty} u_{\mu_n}(x) = \frac{1}{\mu}f(x) - \frac{1}{\mu}\int_{-\pi}^{\pi}\left(\sum_{n=0}^{\infty} u_{\mu_n}(t)\right)K(x,t)dt,\ x \in [-\pi, \pi]. \tag{18.66}$$

The components can be obtained by using recurrence relation

$$\begin{aligned} u_{\mu_0}(x) &= \frac{1}{\mu}f(x), \\ u_{\mu_{k+1}}(x) &= -\frac{1}{\mu}\int_{-\pi}^{\pi} u_{\mu_k}(t)K(x,t)dt, k \geqslant 0. \end{aligned} \tag{18.67}$$

Having determined the components $u_{\mu_0}(x), u_{\mu_1}, u_{\mu_2}, \ldots$, the solution in a series form is obtained upon using (18.65). The obtained series solution may converge to a closed form solution if such a solution exists. Otherwise, the series can be used for numerical purposes.

Application 1

As a first application, we consider the case examined in [6] and given by

$$-\frac{\pi}{2}\cos 2x = \int_{-\pi}^{\pi} \log\left|2\sin\frac{x-t}{2}\right| u(t)\,dt, x \in [-\pi, \pi]. \quad (18.68)$$

Following the regularization technique, Equation (18.68) is transformed to the second kind Fredholm equation

$$u_\mu(x) = -\frac{\pi}{2\mu}\cos 2x - \frac{1}{\mu}\int_{-\pi}^{\pi} \log\left|2\sin\frac{x-t}{2}\right| u_\mu(t)\,dt, x \in [-\pi, \pi], \quad (18.69)$$

The Adomian decomposition method admits the use of

$$u_\mu(x) = \sum_{n=0}^{\infty} u_{\mu_n}(x), \quad (18.70)$$

and the recurrence relation

$$\begin{aligned} u_{\mu_0}(x) &= -\frac{\pi}{2\mu}\cos 2x, \\ u_{\mu_{k+1}}(x) &= -\frac{1}{\mu}\int_{-\pi}^{\pi} \log\left|2\sin\frac{x-t}{2}\right| u_{\mu_k}(t)\,dt, k \geqslant 0. \end{aligned} \quad (18.71)$$

Using the third identity given above gives the components

$$\begin{aligned} u_{\mu_0}(x) &= -\frac{\pi}{2\mu}\cos 2x, \quad u_{\mu_1}(x) = -\frac{\pi^2}{4\mu^2}\cos 2x, \\ u_{\mu_2}(x) &= -\frac{\pi^3}{8\mu^3}\cos 2x, \quad u_{\mu_3}(x) = -\frac{\pi^4}{16\mu^4}\cos 2x, \end{aligned} \quad (18.72)$$

and so on. Substituting this result into (18.70) gives the approximate solution

$$u_\mu(x) = -\frac{\pi}{2\mu}\cos 2x\left(1 + \frac{\pi}{2\mu} + \frac{\pi^2}{4\mu^2} + \frac{\pi^3}{8\mu^3} + \cdots\right), \quad (18.73)$$

that gives

$$u_\mu(x) = \frac{-\pi}{2\mu - \pi}\cos 2x. \quad (18.74)$$

The exact solution $u(x)$ of (18.68) can be obtained by

$$u(x) = \lim_{\mu \to 0} u_\mu(x) = \cos 2x. \quad (18.75)$$

Application 2

As a last application, we consider the problem

$$-\frac{\pi}{2}\sin 2x = \int_{-\pi}^{\pi} \log\left|2\sin\frac{x-t}{2}\right| u(t)\,dt, x \in [-\pi, \pi]. \quad (18.76)$$

Following the regularization technique, Equation (18.76) is transformed to the second kind Fredholm equation

$$u_\mu(x) = -\frac{\pi}{2\mu}\sin 2x - \frac{1}{\mu}\int_{-\pi}^{\pi} \log\left|2\sin\frac{x-t}{2}\right| u_\mu(t)\,dt, x \in [-\pi, \pi]. \quad (18.77)$$

The Adomian decomposition method admits the use of

$$u_\mu(x) = \sum_{n=0}^{\infty} u_{\mu_n}(x), \tag{18.78}$$

and the recurrence relation

$$\begin{aligned} u_{\mu_0}(x) &= -\frac{\pi}{2\mu}\sin 2x, \\ u_{\mu_{k+1}}(x) &= -\frac{1}{\mu}\int_{-\pi}^{\pi}\log\left|2\sin\frac{x-t}{2}\right| u_{\mu_k}(t)\,dt, k \geqslant 0. \end{aligned} \tag{18.79}$$

Using the fourth identity given above gives the components

$$\begin{aligned} u_{\mu_0}(x) &= -\frac{\pi}{2\mu}\sin 2x, \quad u_{\mu_1}(x) = -\frac{\pi^2}{4\mu^2}\sin 2x, \\ u_{\mu_2}(x) &= -\frac{\pi^3}{8\mu^3}\sin 2x, \quad u_{\mu_3}(x) = -\frac{\pi^4}{16\mu^4}\sin 2x, \end{aligned} \tag{18.80}$$

and so on. Substituting this result into (18.78) gives the approximate solution

$$u_\mu(x) = \frac{-\pi}{2\mu - \pi}\sin 2x. \tag{18.81}$$

The exact solution $u(x)$ of (18.76) can be obtained by

$$u(x) = \lim_{\mu \to 0} u_\mu(x) = \sin 2x. \tag{18.82}$$

18.3.3 Another First Kind Fredholm Integral Equation with a Logarithmic Kernel

We now consider another first kind Fredholm integral equation [9–10] with a logarithmic kernel given by

$$\int_{-1}^{1} \ln|x-t|\, u(t)dt = \alpha, -1 < x < 1, \tag{18.83}$$

where α is a constant.

Using the substitutions

$$x = \cos A,\; t = \cos B, \tag{18.84}$$

carries (18.83) into

$$\int_0^{\pi} \ln|\cos A - \cos B|\, g(B)\,dB = \alpha, 0 < A < \pi, \tag{18.85}$$

where

$$g(B) = -\sin B u(\cos B). \tag{18.86}$$

Substituting the expansion

$$g(B) = \sum_{n=0}^{\infty} a_n \cos(nB), \tag{18.87}$$

and using the summation formula, from Appendix F,

$$\ln|\cos x - \cos y| = -\ln 2 - 2\sum_{k=1}^{\infty} \frac{\cos(kx)\cos(ky)}{k}, \tag{18.88}$$

into (18.85) we obtain

$$\int_0^{\pi} \left(-\ln 2 - 2\sum_{k=1}^{\infty} \frac{\cos kA \cos kB}{k}\right)\left(\sum_{n=0}^{\infty} a_n \cos nB\right) dB = \alpha. \tag{18.89}$$

Using the orthogonality conditions of cosine functions we obtain

$$-\pi a_0 \ln 2 - \sum_{n=1}^{\infty} \pi a_n \frac{\cos nA}{n} = \alpha. \tag{18.90}$$

Equating the coefficients of like terms from both sides gives

$$a_0 = -\frac{\alpha}{\pi \ln 2}, \quad a_n = 0, n \geqslant 1. \tag{18.91}$$

Substituting this result into (18.87) gives the exact solution

$$u(x) = -\frac{\alpha}{\pi \ln 2}\left(\frac{1}{\sqrt{1-x^2}}\right). \tag{18.92}$$

Substituting (18.92) into (18.83) gives the following identity

$$\int_{-1}^{1} \frac{\ln|x-t|}{\sqrt{1-t^2}}\, dt = -\pi \ln 2, -1 < x < 1. \tag{18.93}$$

18.4 The Fresnel Integrals

The Fresnel integrals Fresnel$S(x)$ and Fresnel$C(x)$, denoted by $S(x)$ and $C(x)$ respectively, are two transcendental functions used in optics. They arise in the diffraction phenomena, and are defined by

$$S(x) = \int_0^x \sin\frac{\pi}{2}t^2\, dt, \tag{18.94}$$

and

$$C(x) = \int_0^x \cos\frac{\pi}{2}t^2\, dt. \tag{18.95}$$

The coefficient $\frac{\pi}{2}$ is called the normalization factor. The Fresnel integrals are entire functions, i.e analytic, with the following series representation

$$S(x) = \sum_{n=0}^{\infty} (-1)^n \frac{\left(\frac{\pi}{2}\right)^{2n+1}}{(2n+1)!(4n+3)} x^{4n+3}, \tag{18.96}$$

and

$$C(x) = \sum_{n=0}^{\infty} (-1)^n \frac{\left(\frac{\pi}{2}\right)^{2n}}{(2n)!(4n+1)} x^{4n+1}. \tag{18.97}$$

The Fresnel integrals possess the special values

$$S(0)=0,\ \lim_{x\to\infty} S(x)=\frac{1}{2},\ \lim_{x\to-\infty} S(x)=-\frac{1}{2},$$
$$C(0)=0,\ \lim_{x\to\infty} C(x)=\frac{1}{2},\ \lim_{x\to-\infty} C(x)=-\frac{1}{2}. \tag{18.98}$$

In this section we will study four applications that include the Fredholm integral equation and the Volterra integral equation where the Fresnel integrals will be involved.

Application 1

As a first application, we consider the Fredholm integral equation of the second kind

$$u(x)=\cos x+\int_0^{\frac{\pi}{2}} \cos\frac{\pi}{2}x^2 u(t)dt. \tag{18.99}$$

The direct computation method will be used to handle this problem, hence it may be rewritten as

$$u(x)=\cos x+\alpha\cos\frac{\pi}{2}x^2, \tag{18.100}$$

where

$$\alpha=\int_0^{\frac{\pi}{2}} u(t)dt. \tag{18.101}$$

To determine α, we substitute (18.100) into (18.101), evaluate the resulting integral to find that

$$\alpha=\frac{1}{1-C\left(\frac{\pi}{2}\right)}. \tag{18.102}$$

Substituting (18.102) into (18.100) gives the exact solution

$$u(x)=\cos x+\left(\frac{1}{1-C(\frac{\pi}{2})}\right)\cos\frac{\pi}{2}x^2, \tag{18.103}$$

Application 2

As a second application, we consider the Fredholm integral equation of the first kind

$$\frac{\pi^2}{72}\sin\left(\frac{\pi}{2}x^2\right)=\int_0^{\frac{\pi}{6}} \sin\left(\frac{\pi}{2}x^2\right)u(t)dt. \tag{18.104}$$

The regularization method will be combined with the direct computation method to handle this problem. The regularization method transforms (18.104) to the Fredholm integral equation of the second kind

$$u_\mu(x)=\frac{\pi^2}{72\mu}\sin\frac{\pi}{2}x^2-\frac{1}{\mu}\int_0^{\frac{\pi}{6}} \sin\frac{\pi}{2}x^2 u_\mu(t)dt. \tag{18.105}$$

The direct computation method carries (18.105) to

$$u_\mu(x)=\left(\frac{\pi^2}{72\mu}-\frac{\alpha}{\mu}\right)\sin\frac{\pi}{2}x^2, \tag{18.106}$$

where

$$\alpha = \int_0^{\frac{\pi}{6}} u_\mu(t)dt. \tag{18.107}$$

To determine α, we substitute (18.106) into (18.107), evaluate the resulting integral to find that

$$\alpha = \frac{\pi^2}{72}\left(\frac{S\left(\frac{\pi}{6}\right)}{S\left(\frac{\pi}{6}\right)+\mu}\right). \tag{18.108}$$

Substituting (18.108) into (18.106) gives the approximate solution

$$u_\mu(x) = \frac{\pi^2}{72}\left(\frac{1}{S\left(\frac{\pi}{6}\right)+\mu}\right)\sin\left(\frac{\pi}{2}x^2\right). \tag{18.109}$$

Consequently, the exact solution of (18.104) is obtained by

$$u(x) = \lim_{\mu\to 0} u_\mu(x) = \left(\frac{\pi^2}{72}\right)\left(\frac{1}{S\left(\frac{\pi}{6}\right)}\right)\sin\frac{\pi}{2}x^2. \tag{18.110}$$

Notice that $u(x) = x$ is another solution to this equation. This is possible because this problem is an ill-posed equation. It is normal for ill-posed problems to get more than one solution.

Application 3

As a third application, we consider the Volterra integral equation of the second kind

$$u(x) = 1 - x + \cos x^2 - \sqrt{\frac{\pi}{2}}C\left(\sqrt{\frac{2}{\pi}}\,x\right) + \int_0^x u(t)dt. \tag{18.111}$$

We select the Adomian decomposition method to handle this problem. The Adomian decomposition method decomposes the linear term $u(x)$ by an infinite series

$$u(x) = \sum_{n=0}^{\infty} u_n(x), \tag{18.112}$$

where the components $u_0(x), u_1(x), u_2(x), \ldots$ will be determined recurrently. Substituting (18.112) into both sides of (18.111) gives

$$\sum_{n=0}^{\infty} u_n(x) = 1 - x + \cos x^2 - \sqrt{\frac{\pi}{2}}C\left(\sqrt{\frac{2}{\pi}}\,x\right) + \int_0^x \left(\sum_{n=0}^{\infty} u_n(t)\right)dt. \tag{18.113}$$

The components can be obtained by using the recurrence relation

$$\begin{aligned} u_0(x) &= 1 - x + \cos x^2 - \sqrt{\frac{\pi}{2}}C\left(\sqrt{\frac{2}{\pi}}\,x\right), \\ u_{k+1}(x) &= \int_0^x u_k(t)dt, k \geqslant 0. \end{aligned} \tag{18.114}$$

This in turn gives

$$u_0(x) = 1 - x + \cos x^2 - \sqrt{\frac{\pi}{2}}C\left(\sqrt{\frac{2}{\pi}}\,x\right),$$
$$u_1(x) = x - \frac{1}{2}x^2 + \sqrt{\frac{\pi}{2}}C\left(\sqrt{\frac{2}{\pi}}\,x\right) + \frac{1}{2}\sin x^2 - \sqrt{\frac{\pi}{2}}xC\left(\sqrt{\frac{2}{\pi}}\,x\right). \tag{18.115}$$

The noise terms appear between $u_0(x)$ and $u_1(x)$. By canceling the noise terms from $u_0(x)$, the exact solution is given by

$$u(x) = 1 + \cos x^2. \tag{18.116}$$

Application 4

As a last application, we consider the nonlinear Volterra integral equation of the second kind

$$u(x) = \cos\left(\frac{\pi}{2}x^2\right) - \frac{1}{2}\sin\left(\frac{\pi}{2}x^2\right)\left(\frac{1}{\sqrt{2}}C(\sqrt{2}\,x) + x\right) + \int_0^x \sin\left(\frac{\pi}{2}x^2\right)u^2(t)dt. \tag{18.117}$$

We select the modified Adomian decomposition method to handle this problem. Proceeding as before, we substitute $u(x)$ by the decomposition series. The modified Adomian decomposition method admits the use of the recurrence relation

$$u_0(x) = \cos\frac{\pi}{2}x^2, \tag{18.118}$$
$$u_1(x) = -\frac{1}{2}\sin\frac{\pi}{2}x^2\left(\frac{1}{\sqrt{2}}C(\sqrt{2}\,x) + x\right) + \int_0^x \sin\frac{\pi}{2}x^2 u_0^2(t)dt = 0.$$

As a result, the exact solution is given by

$$u(x) = \cos\left(\frac{\pi}{2}x^2\right). \tag{18.119}$$

18.5 The Thomas-Fermi Equation

The Thomas-Fermi equation was derived independently by Thomas (1927) and Fermi (1928) to study the potentials and the electron distribution in an atom. This equation plays an important role in mathematical physics. It was introduced first to study the multi-electron atoms. It was used in the description of the charge density in atoms of high atomic number. The Thomas-Fermi equation was also used to address the molecular theory, solid state theory, and hydrodynamic codes. The dimensionless Thomas-Fermi equation is a nonlinear Volterra integro-differential equation of the second kind given by

$$u' = B + \int_0^x u^{\frac{3}{2}}(t)t^{-\frac{1}{2}}\,dt. \tag{18.120}$$

For an isolated atom, the boundary conditions are given by

$$u(0) = 1,\ \lim_{x\to\infty} u(x) = 0. \tag{18.121}$$

Notice that the potential $u'(0) = B$ is important that will be the focus of this study.

Many numerical and analytic approaches were used to handle the Thomas-Fermi equation (18.120). In this text we presented three distinct methods to handle nonlinear Volterra integro-differential equations of the second kind. To carry this study, we select the variational iteration method to conduct a proper analysis on this equation. The VIM introduces the correction functional

$$u_{n+1}(x) = u_n(x) - \int_0^x \left(u_n'(t) - B - \int_0^t (u_n^{\frac{1}{2}}(r) r^{-\frac{1}{2}}\, dr) \right) dt. \qquad (18.122)$$

We first select the zeroth approximation $u_0(x) = 1$. Using this selection of $u_0(x)$ in the correction functional gives successive approximations, and by setting $x^{\frac{1}{2}} = t$, the approximation of $u_8(t)$ is readily found to be

$$\begin{aligned}
u_8(t) &= 1 + Bt^2 + \frac{4}{3}t^3 + \frac{2}{5}Bt^5 + \frac{1}{3}t^6 + \frac{3}{70}B^2t^7 + \frac{2}{15}Bt^8 \\
&+ \left(-\frac{1}{252}B^3 + \frac{2}{27}\right)t^9 + \frac{1}{175}B^2t^{10} + \left(\frac{1}{1056}B^4 + \frac{31}{1485}B\right)t^{11} \\
&+ \left(\frac{4}{405} + \frac{4}{1575}B^3\right)t^{12} + \left(-\frac{3}{9152}B^5 + \frac{557}{100100}B^2\right)t^{13} \\
&+ \left(-\frac{29}{24255}B^4 + \frac{4}{693}B\right)t^{14} \\
&+ \left(\frac{101}{52650} + \frac{7}{49920}B^6 - \frac{623}{351000}B^3\right)t^{15} - \left(\frac{46}{45045}B^2 - \frac{68}{105105}B^5\right)t^{16} \\
&+ \left(-\frac{3}{43520}B^7 - \frac{113}{1178100}B + \frac{153173}{116424000}B^4\right)t^{17} \\
&+ \left(-\frac{4}{10395}B^6 + \frac{1046}{675675}B^3 + \frac{23}{473850}\right)t^{18} \\
&+ \left(\frac{799399}{698377680}B^2 - \frac{1232941}{1278076800}B^5 + \frac{99}{2646016}B^8\right)t^{19} \\
&+ \left(-\frac{99856}{70945875}B^4 + \frac{51356}{103378275}B + \frac{256}{1044225}B^7\right)t^{20} \\
&+ \left(\frac{705965027}{966226060800}B^6 - \frac{33232663}{25881055200}B^3 + \frac{35953}{378132300} - \frac{143}{6537216}B^9\right)t^{21} \\
&+ \left(\frac{43468}{33622875}B^5 - \frac{6272}{38105925}B^8 - \frac{250054}{342953325}B^2\right)t^{22} + O(t^{23}).
\end{aligned}$$

It was stated before, that we aim to study the mathematical behavior of the potential $u(x)$ to enable us to determine the initial slope of the potential $B = u'(0)$. This can be achieved by forming Padé approximants which have the advantage of manipulating the polynomial approximation into a rational function to gain more information about $u(x)$. It is to be noted that Padé finding algorithms are built-in utilities in most manipulation languages such as Maple and Mathematica.

It was found that the diagonal approximant is the most effective approximant, therefore we constructed only the diagonal approximants [2/2], [4/4], [5/5], and [10/10]. Using the boundary condition $u(\infty) = 0$, the diagonal approximant [M/M] vanish if the coefficient of t with the highest power in the numerator vanishes. Using the computer built-in utilities to solving the resulting polynomials, we obtain the values of the initial slope $B = u'(0)$ listed above in Table 18.1.

Table 18.1 Padé approximants and initial slopes $B = u'(0)$

Padé approximants	Initial slope $B = u'(0)$
[2/2]	−1.211413729
[4/4]	−1.550525919
[5/5]	−1.586834763
[10/10]	−1.587652774

It is important to note that in applying the boundary condition $u(\infty) = 0$ to the diagonal approximants $[M/M]$, a polynomial equation for B results that gives many roots although the Thomas-Fermi equation has a unique solution. Recall that $u(x)$ is a decreasing function, hence $u'(0) < 0$. Using this fact, complex roots and unphysical positive roots were excluded. It is also obvious that the error decreases dramatically with the increase of the degree of the Padé approximants. Moreover, by graphing a variety of Padé approximants of $u(t)$, then for $|u'(0) < 1.587652774|$, we can easily observe

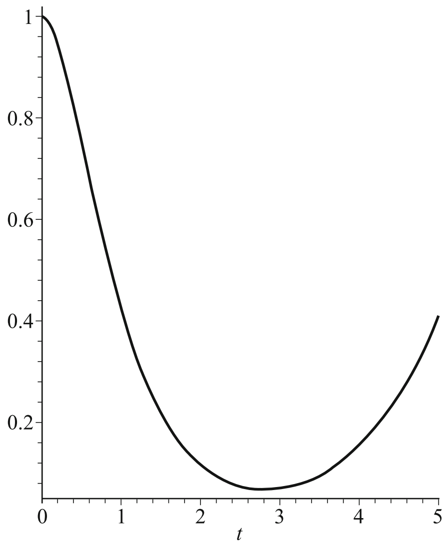

Fig. 18.3 The Padé approximant [8/8] of $u(t)$.

the decrease of $u(t)$ that will reach a minimum followed by an increase after that. This is shown by Figure 18.3.

Figure 18.3 above shows the behavior of the Padé approximants [8/8] of $u(t)$ and explore the decrease of $u(t)$ that will reach a minimum followed by an increase after that.

18.6 Heat Transfer and Heat Radiation

In this section we will study two mathematical physics models. The first model is an Abel-type Volterra integral equation that describes the temperature distribution along the surface when the heat transfer to it is balanced by radiation from it. The second model is also Abel-type Volterra integral equation that determines the temperature in a semi-infinite solid, whose surface can dissipate heat by nonlinear radiation [11].

18.6.1 Heat Transfer: Lighthill Singular Integral Equation

Lighthill presented a nonlinear singular integral equation [12–13] which describes the temperature distribution of the surface of a projectile moving through a laminar layer. Lighthill considered a uniform stream at arbitrary Mach number [12] and deduced the temperature distribution along the surface when the heat transfer to it is balanced by radiation from it. This led to the Lighthill singular integral equation

$$\{F(z)\}^4 = -\frac{1}{2\sqrt{z}} \int_0^z \frac{F'(s)}{(z^{\frac{3}{2}} - s^{\frac{3}{2}})^{\frac{1}{3}}}\, ds, \tag{18.123}$$

with boundary conditions

$$F(0) = 1, \lim_{t\to\infty} F(t) = 0. \tag{18.124}$$

In [13], suitable variable transformations were applied to convert (18.123) to the nonlinear singular Volterra equation of the second kind given by

$$u(x) = 1 - \frac{\sqrt{3}}{\pi} \int_0^x \frac{t^{\frac{1}{3}} u^4(t)}{(x-t)^{\frac{2}{3}}} dt, x \in [0, 1], \tag{18.125}$$

where

$$u(x) = F(x^{\frac{2}{3}}), \tag{18.126}$$

and

$$u(0) = 1, \lim_{t\to\infty} u(t) = 0. \tag{18.127}$$

The product integration method and collocation method based on piecewise polynomials were applied to obtain a series solution. However, we select the

Adomian decomposition method to handle the nonlinear singular equation (18.125). The Adomian method assumes that the linear term $u(x)$ be decomposed by the series

$$u(x) = \sum_{n=0}^{\infty} u_n(x), \tag{18.128}$$

and the nonlinear term $u^4(x)$ by the series

$$u^4(x) = \sum_{n=0}^{\infty} A_n(x), \tag{18.129}$$

where $A_n(x)$ are the Adomian polynomials. In what follows we list some of the Adomian polynomials for $u^4(x)$:

$$\begin{aligned}
A_0 &= u_0^4,\\
A_1 &= 4u_0^3u_1,\\
A_2 &= 4u_0^3u_2 + 6u_0^2u_1^2,\\
A_3 &= 4u_0^3u_2 + 12u_0^2u_1u_2 + 4u_0u_1^3,\\
A_4 &= 4u_0^3u_4 + 12u_0^2\left(\frac{1}{2}u_2^2 + u_1u_3\right) + 12u_0u_1^2u_2 + u_1^4,\\
A_5 &= 4u_0^3u_5 + 12u_0^2(u_1u_4 + u_2u_3) + 12u_0(u_1^2u_3 + u_1u_2^2) + 4u_1^3u_2,\\
A_6 &= 4u_0^3u_6 + 12u_0^2\left(u_1u_5 + u_2u_4 + \frac{1}{2}u_3^2\right) + 4u_0(3u_1^2u_4 + 6u_1u_2u_3 + u_2^3)\\
&\quad +(4u_1^3u_3 + 6u_1^2u_2^2),
\end{aligned} \tag{18.130}$$

and so on for other Adomian polynomials. We can use the recurrence relation

$$\begin{aligned}
u_0(x) &= 1,\\
u_{k+1}(x) &= -\frac{\sqrt{3}}{\pi}\int_0^x \frac{t^{\frac{1}{3}}A_k(t)}{(x-t)^{\frac{2}{3}}}dt, k \geqslant 0.
\end{aligned} \tag{18.131}$$

Accordingly, we obtain the following components

$$\begin{aligned}
u_0(x) &= 1,\\
u_1(x) &= -\frac{\sqrt{\pi}\sqrt[3]{2}}{\Gamma(\frac{2}{3})\Gamma(\frac{5}{6})}x^{\frac{2}{3}} = -1.460998487x^{\frac{2}{3}},\\
u_2(x) &= \frac{18}{\left(\Gamma(\frac{2}{3})\right)^3}x^{\frac{4}{3}} = 7.249416142x^{\frac{4}{3}},\\
u_3(x) &= -\frac{80(9\left(\Gamma(\frac{2}{3})\right)^3 + \pi^2)}{9\left(\Gamma(\frac{2}{3})\right)^6}x^2 = -46.44973783x^2,\\
u_4(x) &= \frac{112\sqrt{3}\pi\left(180\left(\Gamma(\frac{2}{3})\right)^6 + \pi\left(\Gamma(\frac{2}{3})\right)^3(81\sqrt{3}+20\pi) + 2\sqrt{3}\pi^3\right)}{135\left(\Gamma(\frac{2}{3})\right)^{12}}\\
&= 332.7552332x^{\frac{8}{3}},\\
u_5(x) &= -2536.820572x^{\frac{10}{3}},\\
u_6(x) &= 20120.06098x^4,\\
u_7(x) &= -163991.8463x^{\frac{14}{3}},
\end{aligned} \tag{18.132}$$

$$u_8(x) = 1363564.301x^{\frac{16}{3}},$$
$$u_9(x) = -11511356x^6.$$

Combining (18.126) and (18.133) gives the series solution

$$\begin{aligned} F(x) = 1 - 1.460998487x + 7.249416142x^2 - 46.44973783x^3 \\ +332.7552332x^4 - 2536.820572x^5 + 20120.06098x^6 \\ -163991.8463x^7 + 1363564.301x^8 - 11511356x^9 + \cdots, \end{aligned} \tag{18.133}$$

which is consistent with the results obtained in [12,14].

To show that the obtained series satisfies the boundary condition at $x = \infty$, we construct the Padé approximants $[1/1], [2/2], [3/3]$ and $[4/4]$, and by evaluating the limits of these approximants at $x = \infty$ we obtain a sequence that converges to 0. This confirms the convergence to 0 as deducted by [14] and other works..

Figure 18.4 above shows the behavior of the Padé approximant [4/4] of $u(x)$ and explores the rapid decay of that curve.

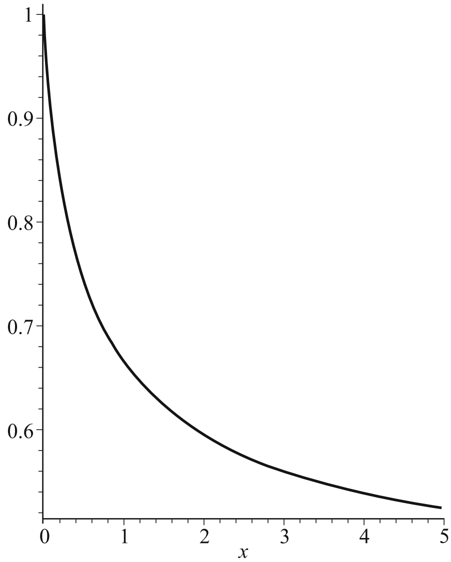

Fig. 18.4 The Padé approximant [4/4] of $u(x)$.

18.6.2 Heat Radiation in a Semi-Infinite Solid

In this part we will study an Abel-type nonlinear Volterra integral equation given by

$$u(x) = \frac{1}{\sqrt{\pi}} \int_0^x \frac{f(t) - u^n(t)}{\sqrt{x-t}}\, dt, \qquad (18.134)$$

where $u(x)$ gives the temperature at the surface for all time. The physical problem which motivated consideration of (18.134) is that of determining [11] the temperature in a semi-infinite solid, whose surface can dissipate heat by nonlinear radiation. At the surface, energy is supplied according to the given function $f(t)$, while radiated energy [11] escapes in proportion to $u^n(t)$.

Equation (18.134) may be rewritten as

$$u(x) = \frac{1}{\sqrt{\pi}} \int_0^x \frac{f(t)}{\sqrt{x-t}} dt - \frac{1}{\sqrt{\pi}} \int_0^x \frac{1}{\sqrt{x-t}} u^n(t) dt. \qquad (18.135)$$

The Adomian decomposition method handles such problems effectively. In what follows, we will select only two cases for $f(x)$ and n.

Application 1

As a first application, we select $f(x) = \frac{1}{2}$ and $n = 4$. Based on this selection, Equation (18.135) becomes

$$u(x) = \sqrt{\frac{x}{\pi}} - \frac{1}{\sqrt{\pi}} \int_0^x \frac{1}{\sqrt{x-t}} u^4(t) dt, \qquad (18.136)$$

that governs the radiation of heat from a semi-infinite solid having a constant heat source.

Using the Adomian assumptions for $u(x)$ and $u^4(x)$ as shown above by (18.128) and (18.131), and by setting the recurrence relation

$$u_0(x) = \sqrt{\frac{x}{\pi}}, \quad u_{k+1}(x) = -\frac{1}{\sqrt{\pi}} \int_0^x \frac{A_k(t)}{(x-t)^{\frac{2}{3}}} dt, k \geqslant 0. \qquad (18.137)$$

we obtain the following components

$$\begin{aligned}
u_0(x) &= \sqrt{\frac{x}{\pi}} = 0.5641895835 x^{\frac{1}{2}}, \\
u_1(x) &= -\frac{16}{15}\sqrt{\left(\frac{x}{\pi}\right)^5} = -0.06097531350 x^{\frac{5}{2}}, \\
u_2(x) &= \frac{16384}{4725}\sqrt{\left(\frac{x}{\pi}\right)^9} = 0.02008369878 x^{\frac{9}{2}}, \\
u_3(x) &= -\frac{200278016}{14189175}\sqrt{\left(\frac{x}{\pi}\right)^{13}} = -0.008283272764 x^{\frac{13}{2}}, \\
u_4(x) &= \frac{491444116652032}{7761123995625}\sqrt{\left(\frac{x}{\pi}\right)^{17}} = 0.003765092063 x^{\frac{17}{2}}, \\
u_5(x) &= -0.001806440429 x^{\frac{21}{2}}, \\
u_6(x) &= 0.0008970031915 x^{\frac{25}{2}}, \\
u_7(x) &= -0.0004561205497 x^{\frac{29}{2}},
\end{aligned} \qquad (18.138)$$

and so on. The series solution is given by

$$u(x) = x^{\frac{1}{2}} \times (0.5641895835 - 0.06097531350x^2 + 0.02008369878x^4 - 0.008283272764x^6 + 0.003765092063x^8 - 0.001806440429x^{10} + 0.0008970031915x^{12} - 0.0004561205497x^{14} + \cdots) \quad (18.139)$$

Application 2

As a first application, we select $f(x) = 2\sqrt{\frac{x}{\pi}}$ and $n = 3$. Based on this selection, Equation (18.135) becomes

$$u(x) = x - \frac{1}{\sqrt{\pi}} \int_0^x \frac{1}{\sqrt{x-t}} u^3(t)dt. \quad (18.140)$$

Using the Adomian assumptions for $u(x)$ and $u^3(x)$, and by setting the recurrence relation

$$\begin{aligned} u_0(x) &= x, \\ u_{k+1}(x) &= -\frac{1}{\sqrt{\pi}} \int_0^x \frac{A_k(t)}{\sqrt{x-t}} dt, k \geqslant 0. \end{aligned} \quad (18.141)$$

we obtain the following components

$$\begin{aligned} u_0(x) &= x, & u_1(x) &= -0.5158304764x^{\frac{7}{2}}, \\ u_2(x) &= 0.6187500000x^6, & u_3(x) &= -0.8971997530x^{\frac{17}{2}}, \\ u_4(x) &= 1.414175172x^{11}, & u_5(x) &= -2.335723848x^{\frac{27}{2}}, \\ u_6(x) &= 3.974710222x^{16}, \end{aligned} \quad (18.142)$$

and so on. The series solution is given by

$$\begin{aligned} u(x) = x &- 0.5158304764x^{\frac{7}{2}} + 0.6187500000x^6 - 0.8971997530x^{\frac{17}{2}} \\ &+ 1.414175172x^{11} - 2.335723848x^{\frac{27}{2}} + 3.974710222x^{16} + \cdots. \end{aligned} \quad (18.143)$$

To study the structure of the obtained solution, we first substitute $t = x^{\frac{1}{2}}$, construct proper Padé approximants and follow the discussion used before.

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Appendix A
Table of Indefinite Integrals

A.1 Basic Forms

1. $\int x^n \, dx = \frac{1}{n+1} x^{n+1} + C, n \neq -1$

2. $\int \frac{1}{x} \, dx = \ln |x| + C$

3. $\int e^{ax} \, dx = \frac{1}{a} e^{ax} + C$

4. $\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a} + C$

5. $\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a} + C$

6. $\int \frac{1}{x\sqrt{x^2 - 1}} \, dx = \sec^{-1} x + C$

7. $\int \cos ax \, dx = \frac{1}{a} \sin ax + C$

8. $\int \sin ax \, dx = -\frac{1}{a} \cos ax + C$

9. $\int \tan ax \, dx = -\frac{1}{a} \ln |\cos ax| + C$

10. $\int \cot ax \, dx = \frac{1}{a} \ln \sin ax + C$

11. $\int \tan ax \, \sec ax \, dx = \frac{1}{a} \sec ax + C$

12. $\int \sec x \, dx = -\ln(\sec x - \tan x) + C$

13. $\int \csc x \, dx = -\ln(\csc x + \cot x) + C$

14. $\int \sec^2 ax \, dx = \frac{1}{a} \tan ax + C$

15. $\int \csc^2 ax \, dx = -\frac{1}{a} \cot ax + C$

A.2 Trigonometric Forms

1. $\int \sin^2 x \, dx = \frac{1}{2} x - \frac{1}{4} \sin 2x + C$

2. $\int \cos^2 x \, dx = \frac{1}{2} x + \frac{1}{4} \sin 2x + C$

3. $\int \sin^3 x \, dx = -\frac{1}{3} \cos x \left(2 + \sin^2 x\right) + C$

4. $\int \cos^3 x \, dx = \frac{1}{3} \sin x \left(2 + \cos^2 x\right) + C$

5. $\int \tan^2 x \, dx = \tan x - x + C$

6. $\int \cot^2 x \, dx = -\cot x - x + C$

7. $\int x \sin x \, dx = \sin x - x \cos x + C$

8. $\int x \cos x \, dx = \cos x + x \sin x + C$

9. $\int x^2 \sin x \, dx = 2x \sin x - \left(x^2 - 2\right) \cos x + C$

10. $\int x^2 \cos x \, dx = 2x \cos x + \left(x^2 - 2\right) \sin x + C$

11. $\int \sin x \cos x \, dx = \frac{1}{2}\sin^2 x + C$

12. $\int \frac{1}{1 + \sin x} \, dx = -\tan(\frac{1}{4}\pi - \frac{1}{2}x) + C$

13. $\int \frac{1}{1 - \sin x} \, dx = \tan(\frac{1}{4}\pi + \frac{1}{2}x) + C$

14. $\int \frac{1}{1 + \cos x} \, dx = \tan(\frac{1}{2}x) + C$

15. $\int \frac{1}{1 - \cos x} \, dx = -\cot(\frac{1}{2}x) + C$

A.3 Inverse Trigonometric Forms

1. $\int \sin^{-1} x \, dx = x\sin^{-1} x + \sqrt{1 - x^2} + C$

2. $\int \cos^{-1} x \, dx = x\cos^{-1} x - \sqrt{1 - x^2} + C$

3. $\int \tan^{-1} x \, dx = x\tan^{-1} x - \frac{1}{2} \ln(1 + x^2) + C$

4. $\int x\sin^{-1} x \, dx = \frac{1}{4}[(2x^2 - 1)\sin^{-1} x + x\sqrt{1 - x^2}] + C$

5. $\int x\cos^{-1} x \, dx = \frac{1}{4}[(2x^2 - 1)\cos^{-1} x - x\sqrt{1 - x^2}] + C$

6. $\int x\tan^{-1} x \, dx = \frac{1}{2}[(x^2 + 1)\tan^{-1} x - x] + C$

7. $\int x\cot^{-1} x \, dx = \frac{1}{2}[(x^2 + 1)\cot^{-1} x + x] + C$

8. $\int \sec^{-1} x \, dx = x\sec^{-1} x - \ln(x + \sqrt{x^2 - 1}) + C$

9. $\int x\sec^{-1} x \, dx = \frac{1}{2}[x^2\sec^{-1} x - \sqrt{x^2 - 1}] + C$

A.4 Exponential and Logarithmic Forms

1. $\int e^{ax} \, dx = \frac{1}{a}e^{ax} + C$

2. $\int xe^{ax} \, dx = \frac{1}{a^2}(ax - 1)e^{ax} + C$

3. $\int x^2 e^{ax} \, dx = \frac{1}{a^3}(a^2 x^2 - 2ax + 2)e^{ax} + C$

4. $\int x^3 e^{ax} \, dx = \frac{1}{a^4}(a^3 x^3 - 3a^2 x^2 + 6ax - 6)e^{ax} + C$

5. $\int e^x \sin x \, dx = \frac{1}{2}(\sin x - \cos x)e^x + C$

6. $\int e^x \cos x \, dx = \frac{1}{2}(\sin x + \cos x)e^x + C$

7. $\int \ln x \, dx = x \ln x - x + C$

8. $\int x \ln x \, dx = \frac{1}{2}x^2(\ln x - \frac{1}{2}) + C$

A.5 Hyperbolic Forms

1. $\int \sinh ax \, dx = \frac{1}{a} \cosh ax + C$

2. $\int \cosh ax \, dx = \frac{1}{a} \sinh ax + C$

3. $\int x \sinh x \, dx = x \cosh x - \sinh x + C$

4. $\int x \cosh x \, dx = x \sinh x - \cosh x + C$

5. $\int \sinh^2 x \, dx = \frac{1}{2}(\sinh x \cosh x - x) + C$

6. $\int \cosh^2 x \, dx = \frac{1}{2}(\sinh x \cosh x + x) + C$

7. $\int \tanh ax \, dx = \frac{1}{a} \ln \cosh ax + C$

8. $\int \coth ax \, dx = \frac{1}{a} \ln \sinh ax + C$

9. $\int \operatorname{sech}^2 x \, dx = \tanh x + C$

10. $\int \operatorname{csch}^2 x \, dx = -\coth x + C$

A.6 Other Forms

1. $\int \frac{1}{\sqrt{a^2 - x^2}} dx = \arcsin \frac{x}{a} + C$

2. $\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \arctan \frac{x}{a} + C$

3. $\int \frac{1}{\sqrt{2ax - x^2}} dx = \arccos \frac{a - x}{a} + C$

4. $\int \frac{1}{a^2 - x^2} dx = \frac{1}{2a} \ln \frac{x + a}{x - a} + C$

Appendix B

Integrals Involving Irrational Algebraic Functions

B.1 Integrals Involving $\frac{t^n}{\sqrt{x-t}}$, n is an integer, $n \geqslant 0$

1. $\displaystyle\int_0^x \frac{1}{\sqrt{x-t}}dt = 2\sqrt{x}$

2. $\displaystyle\int_0^x \frac{t}{\sqrt{x-t}}dt = \frac{4}{3}x^{\frac{3}{2}}$

3. $\displaystyle\int_0^x \frac{t^2}{\sqrt{x-t}}dt = \frac{16}{15}x^{\frac{5}{2}}$

4. $\displaystyle\int_0^x \frac{t^3}{\sqrt{x-t}}dt = \frac{32}{35}x^{\frac{7}{2}}$

5. $\displaystyle\int_0^x \frac{t^4}{\sqrt{x-t}}dt = \frac{256}{315}x^{\frac{9}{2}}$

6. $\displaystyle\int_0^x \frac{t^5}{\sqrt{x-t}}dt = \frac{512}{693}x^{\frac{11}{2}}$

7. $\displaystyle\int_0^x \frac{t^6}{\sqrt{x-t}}dt = \frac{2048}{3003}x^{\frac{13}{2}}$

8. $\displaystyle\int_0^x \frac{t^7}{\sqrt{x-t}}dt = \frac{4096}{6435}x^{\frac{15}{2}}$

9. $\displaystyle\int_0^x \frac{t^8}{\sqrt{x-t}}dt = \frac{65536}{109395}x^{\frac{17}{2}}$

10. $\displaystyle\int_0^x \frac{t^n}{\sqrt{x-t}}dt = \frac{2^{n+1}\Gamma(n+1)}{1\cdot 3\cdot 5\cdots(2n+1)}x^{n+\frac{1}{2}}$

B.2 Integrals Involving $\frac{t^{\frac{n}{2}}}{\sqrt{x-t}}$, n is an odd integer, $n \geqslant 1$

1. $\displaystyle\int_0^x \frac{t^{\frac{1}{2}}}{\sqrt{x-t}}\,dt = \frac{1}{2}\pi x$

2. $\displaystyle\int_0^x \frac{t^{\frac{3}{2}}}{\sqrt{x-t}}\,dt = \frac{3}{8}\pi x^2$

3. $\displaystyle\int_0^x \frac{t^{\frac{5}{2}}}{\sqrt{x-t}}\,dt = \frac{5}{16}\pi x^3$

4. $\displaystyle\int_0^x \frac{t^{\frac{7}{2}}}{\sqrt{x-t}}\,dt = \frac{35}{128}\pi x^4$

5. $\displaystyle\int_0^x \frac{t^{\frac{9}{2}}}{\sqrt{x-t}}\,dt = \frac{63}{256}\pi x^5$

6. $\displaystyle\int_0^x \frac{t^{\frac{11}{2}}}{\sqrt{x-t}}\,dt = \frac{231}{1024}\pi x^6$

7. $\displaystyle\int_0^x \frac{t^{\frac{13}{2}}}{\sqrt{x-t}}\,dt = \frac{429}{2048}\pi x^7$

8. $\displaystyle\int_0^x \frac{t^{\frac{15}{2}}}{\sqrt{x-t}}\,dt = \frac{6435}{32768}\pi x^8$

9. $\displaystyle\int_0^x \frac{t^{\frac{17}{2}}}{\sqrt{x-t}}\,dt = \frac{12155}{65536}\pi x^9$

10. $\displaystyle\int_0^x \frac{t^{\frac{n}{2}}}{\sqrt{x-t}}\,dt = \frac{\Gamma(\frac{n+2}{2})}{\Gamma(\frac{n+3}{2})}\sqrt{\pi}x^{\frac{n+1}{2}}$

Appendix C
Series Representations

C.1 Exponential Functions Series

1. $e^{ax} = 1 + (ax) + \frac{(ax)^2}{2!} + \frac{(ax)^3}{3!} + \frac{(ax)^4}{4!} + \cdots$

2. $e^{-ax} = 1 - (ax) + \frac{(ax)^2}{2!} - \frac{(ax)^3}{3!} + \frac{(ax)^4}{4!} + \cdots$

3. $e^{-x^2} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots$

4. $a^x = 1 + x \ln a + \frac{1}{2!}(x \ln a)^2 + \frac{1}{3!}(x \ln a)^3 + \cdots, a > 0$

5. $e^{\sin x} = 1 + x + \frac{x^2}{2!} - \frac{3x^4}{4!} - \frac{8x^5}{5!} - \frac{3x^6}{6!} + \cdots$

6. $e^{\cos x} = e(1 - \frac{x^2}{2!} - \frac{4x^4}{4!} - \frac{31x^6}{6!} + \cdots)$

7. $e^{\tan x} = 1 + x + \frac{x^2}{2!} + \frac{3x^3}{3!} + \frac{9x^4}{4!} + \frac{57x^5}{5!} + \cdots$

8. $e^{\sin^{-1} x} = 1 + x + \frac{x^2}{2!} + \frac{2x^3}{3!} + \frac{5x^4}{4!} + \cdots$

C.2 Trigonometric Functions

1. $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$

2. $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$

3. $\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \cdots$

4. $\sec x = 1 + \frac{x^2}{2!} + \frac{5x^4}{4!} + \frac{61x^6}{6!} + \frac{1385x^8}{8!} + \cdots$

C.3 Inverse Trigonometric Functions

1. $\sin^{-1}x = x + \frac{1}{2}\frac{x^3}{3} + \frac{1\cdot 3}{2\cdot 4}\frac{x^5}{5} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{x^7}{7} + \cdots,\ x^2 < 1$

2. $\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$

C.4 Hyperbolic Functions

1. $\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots$
2. $\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots$

3. $\tanh x = x - \frac{x^3}{3} + \frac{2x^5}{15} - \frac{17x^7}{315} + \cdots$

C.5 Inverse Hyperbolic Functions

1. $\sinh^{-1}x = x - \frac{1}{2}\frac{x^3}{3} + \frac{1\cdot 3}{2\cdot 4}\frac{x^5}{5} - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{x^7}{7} + \cdots$

2. $\tanh^{-1}x = x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \cdots$

C.6 Logarithmic Functions

1. $\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots, -1 < x \leqslant 1$

2. $\ln(1-x) = -(x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4 + \cdots), -1 \leqslant x < 1$

3. $\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$

Appendix D
The Error and the Complementary Error Functions

D.1 The Error Function

The error function $\text{erf}(x)$ is defined by:

1. $\text{erf}(x) = \dfrac{2}{\sqrt{\pi}} \displaystyle\int_0^x e^{-u^2} du$

2. $\text{erf}(x) = \dfrac{2}{\sqrt{\pi}} \left(x - \dfrac{x^3}{3} + \dfrac{x^5}{5 \cdot 2!} - \dfrac{x^7}{7 \cdot 3!} + \cdots \right)$

3. $\text{erf}(-x) = -\text{erf}(x)$

4. $\dfrac{d}{dx}[\text{erf}(x)] = \dfrac{2}{\sqrt{\pi}} e^{-x^2}$

5. $\dfrac{d^2}{dx^2}[\text{erf}(x)] = -\dfrac{4}{\sqrt{\pi}} e^{-x^2}$

6. $\dfrac{d^3}{dx^3}[\text{erf}(x)] = -\dfrac{1}{\sqrt{\pi}} (8x^2 - 4) e^{-x^2}$

7. $\dfrac{d^4}{dx^4}[\text{erf}(x)] = -\dfrac{8}{\sqrt{\pi}} (3x - 2x^3) e^{-x^2}$

8. $\text{erf}(0) = 0$

9. $\lim\limits_{x \to \infty} \text{erf}(x) = 1$

D.2 The Complementary Error Function

The complementary error function $\text{erfc}(x)$ is defined by

1. $\text{erfc}(x) = \dfrac{2}{\sqrt{\pi}} \displaystyle\int_x^\infty e^{-u^2} du$

2. $\text{erf}(x) + \text{erfc}(x) = 1$

3. $\text{erfc}(x) = 1 - \dfrac{2}{\sqrt{\pi}} \left(x - \dfrac{x^3}{3} + \dfrac{x^5}{5 \cdot 2!} - \dfrac{x^7}{7 \cdot 3!} + \cdots \right)$

Appendix E
Gamma Function

1. $\Gamma(x) = \int_0^\infty t^{x-1}\, e^{-t} dt$

2. $\Gamma(x+1) = x\Gamma(x)$

3. $\Gamma(n+1) = n!, n$ is an integer

   $\Gamma(1) = \Gamma(2) = 1, \Gamma(3) = 2$

4. $\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x}$

5. $\Gamma(\frac{1}{2}) = \sqrt{\pi}$

6. $\Gamma(\frac{3}{2}) = \frac{1}{2}\sqrt{\pi}$

7. $\Gamma(\frac{1}{2})\Gamma(-\frac{1}{2}) = -2\pi$

Appendix F
Infinite Series

F.1 Numerical Series

1. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} = \ln 2$

2. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = \frac{\pi}{4}$

3. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$

4. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}$

5. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$

6. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{(n+1)!} = 1$

7. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$

8. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n(n+1)} = 1 - 2\ln 2$

9. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+2)} = \frac{3}{4}$

10. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n(n+2)} = -\frac{1}{4}$

11. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n+1)} = \frac{1}{2}$

12. $\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k}{n^{2k}} = \frac{n^2}{n^2+1}$

13. $\displaystyle\sum_{k=0}^{\infty} \frac{1}{k!} = e = 2.718281828\cdots$

14. $\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k}{k!} = \frac{1}{e} = 0.3678794412\cdots$

F.2 Trigonometric Series

1. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k}\sin(kx) = \frac{1}{2}(\pi - x), 0 < x < 2\pi$

2. $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k}\sin(kx) = \frac{1}{2}x, -\pi < x < \pi$

3. $\displaystyle\sum_{k=1}^{\infty} \frac{\sin[(2k-1)x]}{2k-1} = \frac{\pi}{4}, 0 < x < \pi$

4. $\displaystyle\sum_{k=1}^{\infty} (-1)^{k-1} \frac{\sin[(2k-1)x]}{2k-1} = \frac{1}{2} \ln \tan\left(\frac{x}{2} + \frac{\pi}{4}\right), -\frac{1}{2}\pi < x < \frac{1}{2}\pi$

5. $\displaystyle\sum_{k=1}^{\infty} (-1)^{k} \frac{\sin[(2k-1)x]}{(2k+1)^2} = \begin{cases} \frac{1}{4}\pi x & \text{if } -\frac{1}{2}\pi < x < \frac{1}{2}\pi, \\ \frac{1}{4}\pi(\pi - x) & \text{if } \frac{1}{2}\pi < x < \frac{3}{2}\pi. \end{cases}$

6. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k} \cos(kx) = -\ln\left(2 \sin\frac{1}{2}x\right), 0 < x < 2\pi$

7. $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \cos(kx) = \ln\left(2 \cos\frac{1}{2}x\right), -\pi < x < \pi$

8. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2} \cos(kx) = \frac{1}{12}\left(3x^2 - 6\pi x + 2\pi^2\right), 0 < x < 2\pi$

9. $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k}}{k^2} \cos(kx) = \frac{1}{12}\left(3x^2 - \pi^2\right), -\pi < x < \pi$

10. $\displaystyle\sum_{k=1}^{\infty} (-1)^{k-1} \frac{\cos[(2k-1)x]}{2k-1} = \frac{\pi}{4}, 0 < x < \pi$

11. $\displaystyle\sum_{k=1}^{\infty} \frac{\cos[(2k-1)x]}{(2k-1)^2} = \frac{\pi}{4}\left(\frac{\pi}{2} - |x|\right), -\pi < x < \pi$

12. $\displaystyle\ln|\sin x| = -\ln 2 - \sum_{k=1}^{\infty} \frac{\cos(2kx)}{k},\ x \neq 0, \pm\pi, \pm 2\pi, \cdots$

13. $\displaystyle\ln|\cos x| = -\ln 2 - \sum_{k=1}^{\infty} (-1)^{k} \frac{\cos(2kx)}{k},\ x \neq \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm 2\pi, \cdots$

14. $\displaystyle\ln|\cos x - \cos y| = -\ln 2 - 2\sum_{k=1}^{\infty} \frac{\cos(kx)\cos(ky)}{k}$

Appendix G
The Fresnel Integrals

G.1 The Fresnel Cosine Integral

The Fresnel cosine integral $C(x)$ is defined by:

1. $C(x) = \int_0^x \cos\left(\frac{\pi}{2}t^2\right)\, dt$

2. $C(x) = \sum_{n=0}^{\infty} (-1)^n \frac{(\frac{\pi}{2})^{2n}}{(2n)!(4n+1)} x^{4n+1}$

3. $C(0) = 0,\ \lim_{x\to\infty} C(x) = \frac{1}{2},\ \lim_{x\to-\infty} C(x) = -\frac{1}{2}$

G.2 The Fresnel Sine Integral

The Fresnel sine integral $S(x)$ is defined by:

1. $S(x) = \int_0^x \sin\left(\frac{\pi}{2}t^2\right)\, dt$

2. $S(x) = \sum_{n=0}^{\infty} (-1)^n \frac{(\frac{\pi}{2})^{2n+1}}{(2n+1)!(4n+3)} x^{4n+3}$

3. $S(0) = 0,\ \lim_{x\to\infty} S(x) = \frac{1}{2},\ \lim_{x\to-\infty} S(x) = -\frac{1}{2}$

Answers

Exercises 1.1

1. $f(x) = e^{2x} - 1$
2. $f(x) = e^{-3x}$
3. $f(x) = e^x - 1$
4. $f(x) = \cos x - \sin x$
5. $f(x) = \sin 3x$
6. $f(x) = \sinh 2x$
7. $f(x) = \cosh 2x$
8. $f(x) = \cosh 3x - 1$
9. $f(x) = 1 + \cos 2x$
10. $f(x) = 1 + \sin x$
11. $f(x) = \tan x$
12. $f(x) = \tanh x$
13. $f(x) = 1 + x + e^x$
14. $f(x) = 1 + x + \cos x$

Exercises 1.2.1

1. $u(x) = (x + c)e^{-x}$
2. $u(x) = x^4(c + e^x)$
3. $u(x) = \dfrac{c}{x^2 + 9}$
4. $u(x) = x^4(c + x + x^2)$
5. $u(x) = x + \dfrac{c}{x}$
6. $u(x) = x(c - \cos x)$
7. $u(x) = x^2 e^x$
8. $u(x) = x$
9. $u(x) = \cot x e^{2x}$
10. $u(x) = (1 + x^4)e^{3x}$
11. $u(x) = \dfrac{x}{1 + x^3}$
12. $u(x) = (1 + x)\cos x$

Exercises 1.2.2

1. $u(x) = e^{2x}(A + Bx)$
2. $u(x) = Ae^{-x} + Be^{3x}$
3. $u(x) = Ae^{-x} + Be^{2x}$
4. $u(x) = A + Be^{2x}$
5. $u(x) = e^{3x}(A + Bx)$
6. $u(x) = (A\sin 2x + B\cos 2x)$
7. $u(x) = e^x \cos x$
8. $u(x) = e^{3x}(1 + x)$
9. $u(x) = e^{-2x} + e^{5x}$
10. $u(x) = \cos 3x$
11. $u(x) = 2 + e^{9x}$
12. $u(x) = \cosh 3x$
13. $u(x) = A + Be^x - x$
14. $u(x) = 3 + A\sin x + B\cos x$
15. $u(x) = A\cosh x - 3x$
16. $u(x) = A\cosh x - \cos x$
17. $u(x) = 8e^x - 6x - 5$
18. $u(x) = -\sin x + 3e^x$
19. $u(x) = e^x + \sin x$
20. $u(x) = 1 + x + 2e^{4x}$

Exercises 1.2.3

1. $u(x) = a_0(1 - \frac{1}{2}x^2 + \frac{1}{8}x^4 + \cdots) + a_1(x - \frac{1}{3}x^3 + \frac{1}{15}x^5 + \cdots)$

2. $u(x) = a_0(1 - \frac{1}{6}x^3 - \frac{1}{40}x^5 + \cdots) + a_1(x + \frac{1}{6}x^3 - \frac{1}{12}x^4 + \cdots)$

3. $u(x) = a_0(1 - \frac{1}{2}x^2 - \frac{1}{6}x^3 + \cdots) + a_1(x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \cdots)$

4. $u(x) = a_0(1 - \frac{1}{6}x^3 - \frac{1}{24}x^4 + \cdots) + a_1(x + \frac{1}{2}x^2 + \frac{1}{6}x^3 - \frac{1}{24}x^4 + \cdots)$

5. $u(x) = a_0(1 - \frac{1}{6}x^3 - \frac{1}{24}x^4 + \cdots) + a_1(x + \frac{1}{2}x^2 + \frac{1}{6}x^3 - \frac{1}{24}x^4 + \cdots)$
$+(\frac{1}{6}x^3 + \frac{1}{24}x^4 - \frac{1}{180}x^6 + \cdots)$

6. $u(x) = a_0(1 - \frac{1}{6}x^3 - \frac{1}{40}x^5 + \cdots) + a_1(x + \frac{1}{6}x^3 - \frac{1}{12}x^4 + \cdots)$
$+(\frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{8}x^4 + \cdots)$

7. $u(x) = a_0(1 + \frac{1}{6}x^3 + \frac{1}{180}x^6 + \cdots) + a_1(x + \frac{1}{12}x^4 + \frac{1}{504}x^7 + \cdots)$
$+(\frac{1}{2}x^2 - \frac{1}{24}x^4 + \frac{1}{40}x^5 + \frac{1}{40}x^5 + \cdots)$

8. $u(x) = a_0(1 + \frac{1}{12}x^4 + \frac{1}{672}x^8 + \cdots) + a_1(x + \frac{1}{20}x^5 + \frac{1}{1440}x^9 + \cdots)$
$-(\frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{60}x^5 + \cdots)$

Exercises 1.3

1. $F'(x) = e^{-x^4} - \int_0^x 2xe^{-x^2t^2}\,dt$

2. $F'(x) = 2x\ln(1+x^3) - \ln(1+x^2) + \int_x^{x^2} \frac{t}{1+xt}\,dt$

3. $F'(x) = \sin(2x^2) + \int_0^x 2x\cos(x^2+t^2)\,dt$

4. $F'(x) = \cosh(2x^3) + \int_0^x 3x^2\sinh(x^3+t^3)\,dt$

5. $F'(x) = \int_0^x u(t)\,dt$

6. $F'(x) = \int_0^x 2(x-t)u(t)\,dt$

7. $F'(x) = \int_0^x 3(x-t)^2u(t)\,dt$

8. $F'(x) = \int_0^x 4(x-t)^3u(t)\,dt$

9. $3x^2 + x^5 = 4u(x) + \int_0^x u(t)\,dt$

10. $(1+x)e^x = u(x) + \int_0^x e^{x-t}u(t)\,dt$

11. $4x + 9x^2 = 6u(x) + 5\int_0^x u(t)\,dt$

12. $\cosh x + \cot x = 3u(x) + \int_0^x u(t)\,dt$

13. 2 14. 3 15. 4 16. 5

Exercises 1.5

1. $y = \frac{1}{2}\sin 2x$ 2. $y = x$ 3. $y = 2x + e^{-x}$ 4. $y = e^{x} + e^{2x}$

5. $\frac{1}{s^2} + \frac{1}{s^2+1}$ 6. $\frac{1}{s-1} - \frac{s}{s^2+1}$ 7. $\frac{1}{s} + \frac{1}{(s-1)^2}$ 8. $\frac{2s^2}{s^4-1}$

9. $\frac{1}{s^2-1}Y(s)$ 10. $\frac{2}{s^3} + \frac{1}{s-1}Y(s)$ 11. $\frac{1}{(s-1)^2}Y(s)$ 12. $\frac{1}{s} + \frac{1}{s^2} - \frac{1}{s^2}Y(s)$

13. $y(x) = 2x^2 + \sin x$ 14. $y(x) = \sin 3x + \sinh x$

15. $y(x) = \sin x + \cosh 2x$ 16. $y(x) = 2 - \cos x - \cosh x$

Exercises 1.6

1. 1 2. $\frac{1}{3}$ 3. x 4. $\sin x$ 5. $\frac{1}{e^2-1}$ 6. $\frac{9}{9-n}x$ 7. $2\pi x$ 8. $\frac{e}{\pi - e}$

Exercises 2.1

1. Volterra, second kind
2. Volterra, first kind
3. Fredholm, second kind
4. Fredholm, first kind
5. Fredholm, second kind
6. Volterra, second kind
7. Volterra, first kind
8. Fredholm, first kind
9. Volterra-Fredholm, second kind
10. Volterra-Fredholm, second kind
11. Generalized Abel's singular
12. Weakly singular equation

Exercises 2.2

1. Volterra I-DE
2. Fredholm I-DE
3. Volterra-Fredholm I-DE
4. Volterra-Fredholm I-DE
5. Fredholm I-DE
6. Volterra I-DE

Exercises 2.3

1. Volterra, Linear, Inhomogeneous
2. Fredholm, Linear, Inhomogeneous
3. Volterra, Linear, Homogeneous
4. Fredholm, Linear, Homogeneous
5. Volterra, Nonlinear, Inhomogeneous
6. Fredholm, Nonlinear, Inhomogeneous
7. Fredholm(Integro-diff), Linear, Inhomogeneous
8. Volterra(Integro-diff), Linear, Homogeneous

Exercises 2.5

1. $u(x) = 4 + 4\int_0^x u(t)\,dt$ 2. $u(x) = e^{-2x^2} - 4x\int_0^x u(t)\,dt$

3. $u(x) = -4x - 4\int_0^x (x-t)u(t)\,dt$

4. $u(x) = -1 - 8x + \int_0^x [6 - 8(x-t)]\,u(t)\,dt$

5. $u(x) = 1 + x + \int_0^x (x-t)u(t)dt$

6. $u(x) = 1 + x - \frac{1}{2}x^2 + 2\int_0^x \left[1 - \frac{1}{2}(x-t)^2\right] u(t)\,dt$

7. $u(x) = 2 + x + \int_0^x (x-t)u(t)dt$

8. $u(x) = -1 - \int_0^x (x-t)\left[1 + \frac{1}{6}(x-t)^2\right] u(t)\,dt$

9. $u'(x) - 2u(x) = 1, u(0) = 0$ 10. $u'(x) + u(x) = e^x, u(0) = 2$

11. $u''(x) - u(x) = 2, u(0) = 1, u'(0) = 0$

12. $u''(x) + u(x) = -\sin x, u(0) = 0, u'(0) = 1$

13. $u'''(x) - 4u(x) = -\sin x, u(0) = u'(0) = 0, u''(0) = 1$

14. $u'''(x) - 2u(x) = \cosh x, u(0) = 2, u'(0) = 1, u''(0) = 0$

15. $u^{\text{iv}}(x) - 12u(x) = 0, u(0) = 1, u'(0) = u''(0) = u''' = 0$

16. $u^{\text{iv}}(x) - u'''(x) - 3u''(x) - 6u(x) = e^x,$
$u(0) = 2, u'(0) = 3, u''(0) = 10, u''' = 32$

Exercises 2.6

1. $u(x) = \int_0^1 K(x,t)u(t)\,dt,\ K(x,t) = \begin{cases} 4t(1-x) & \text{for } 0 \leqslant t \leqslant x \\ 4x(1-t) & \text{for } x \leqslant t \leqslant 1 \end{cases}$

2. $u(x) = \int_0^1 K(x,t)u(t)\,dt,\ K(x,t) = \begin{cases} xt(1-x) & \text{for } 0 \leqslant t \leqslant x \\ x^2(1-t) & \text{for } x \leqslant t \leqslant 1 \end{cases}$

3. $u(x) = 3x - 2 + \int_0^1 K(x,t)u(t)\,dt,\ K(x,t) = \begin{cases} 2t(1-x) & \text{for } 0 \leqslant t \leqslant x \\ 2x(1-t) & \text{for } x \leqslant t \leqslant 1 \end{cases}$

4. $u(x) = 4 - 3x^2 + \int_0^1 K(x,t)u(t)\,dt,\ K(x,t) = \begin{cases} 3xt(1-x) & \text{for } 0 \leqslant t \leqslant x \\ 3x^2(1-t) & \text{for } x \leqslant t \leqslant 1 \end{cases}$

5. $u(x) = \int_0^1 K(x,t)u(t)\,dt,\ K(x,t) = \begin{cases} 4t & \text{for } 0 \leqslant t \leqslant x \\ 4x & \text{for } x \leqslant t \leqslant 1 \end{cases}$

6. $u(x) = \int_0^1 K(x,t)u(t)\,dt,\ K(x,t) = \begin{cases} xt & \text{for } 0 \leqslant t \leqslant x \\ x^2 & \text{for } x \leqslant t \leqslant 1 \end{cases}$

7. $u(x) = x - 4 + \int_0^1 K(x,t)u(t)\,dt,\ K(x,t) = \begin{cases} 4t & \text{for } 0 \leqslant t \leqslant x \\ 4x & \text{for } x \leqslant t \leqslant 1 \end{cases}$

8. $u(x) = 2 - 4x + \int_0^1 K(x,t)u(t)\,dt,\ K(x,t) = \begin{cases} 4xt & \text{for } 0 \leqslant t \leqslant x \\ 4x^2 & \text{for } x \leqslant t \leqslant 1 \end{cases}$

9. $u'' + 3u = 4e^x,\ 0 < x < 1,\ u(0) = 1, u(1) = e^2$

10. $u'' + u = 6,\ 0 < x < 1,\ u(0) = 0, u(1) = 3$

11. $u'' + 6u = -\cos x,\ 0 < x < 1,\ u(0) = 1, u(1) = \cos 1$

12. $u'' + 4u = \sinh x,\ 0 < x < 1,\ u(0) = 0, u(1) = \sinh 1$

13. $u'' + u = 9e^{3x},\ 0 < x < 1,\ u(0) = 1, u'(1) = 3e^3$

14. $u'' + 6u = 12x^2,\ 0 < x < 1,\ u(0) = 0, u'(1) = 4$

15. $u'' + 4u = 4,\ 0 < x < 1,\ u(0) = 3, u'(1) = 4$

16. $u'' + 2u = e^x,\ 0 < x < 1,\ u(0) = 2, u'(1) = e$

Exercises 2.7

17. $f(x) = 1 + 4x,$
18. $\alpha = \frac{1}{4},$
19. $f(x) = x,$
20. $\alpha = 2$
21. $f(x) = 2 - e^{2x},$
22. $f(x) = \sin x - x$
23. $f(x) = 3x^2,$
24. $f(x) = 4x + 2x^3$

Exercises 3.2.1

1. $u(x) = 6x$
2. $u(x) = 6x$
3. $u(x) = 1 + x$
4. $u(x) = x - x^2$
5. $u(x) = e^x$
6. $u(x) = e^{-x}$
7. $u(x) = \sin x + \cos x$
8. $u(x) = \cos x - \sin x$
9. $u(x) = \cos x$
10. $u(x) = \cosh x$
11. $u(x) = \sin x$
12. $u(x) = \sinh x$
13. $u(x) = e^x$
14. $u(x) = e^{-x}$
15. $u(x) = e^{x^2}$
16. $u(x) = e^{-x^2}$
17. $u(x) = 3\cos x - 2$
18. $u(x) = 3\sin x - 2$
19. $u(x) = 2\cosh x - 2$
20. $u(x) = 2\sinh x - 2$
21. $u(x) = e^{2x}$
22. $u(x) = 4 + \cos x$
23. $u(x) = e^x$
24. $u(x) = \cos x$
25. $u(x) = e^x$
26. $u(x) = e^{-x}$
27. $u(x) = 3 + \frac{1}{4}x^4 + \frac{1}{112}x^8 + \frac{1}{4928}x^{12} + \cdots$
28. $u(x) = 3 + \frac{3}{4}x^2 + \frac{1}{4}x^3 + \frac{1}{16}x^4 + \cdots$
29. $u(x) = 1 + \frac{1}{3}x^3 + \frac{1}{72}x^6 + \frac{1}{4536}x^9 + \cdots$
30. $u(x) = 1 + \frac{1}{2}x^3 + \frac{1}{16}x^6 + \frac{1}{224}x^9 + \cdots$

Exercises 3.2.2

1. $u(x) = \cos x$
2. $u(x) = \sinh x$
3. $u(x) = 2x + 3x^2$
4. $u(x) = 3x^2$
5. $u(x) = 2x$
6. $u(x) = e^{-x^2}$
7. $u(x) = \cosh x$
8. $u(x) = e^x$
9. $u(x) = 1 + \sin x$
10. $u(x) = e^x + \sin x$
11. $u(x) = 1 + x + \cosh x$
12. $u(x) = \cos x$
13. $u(x) = \sec^2 x$
14. $u(x) = \cosh x$
15. $u(x) = \sinh x$
16. $u(x) = x^3$

Exercises 3.2.3

1. $u(x) = 6x$
2. $u(x) = 6x$
3. $u(x) = 6x$
4. $u(x) = x + x^2$
5. $u(x) = \sin x$
6. $u(x) = \cos x$
7. $u(x) = \sinh x$
8. $u(x) = \cosh x$
9. $u(x) = \sec^2 x$
10. $u(x) = \cos^2 x$
11. $u(x) = \sin^2 x$
12. $u(x) = \tan^2 x$

Exercises 3.2.4

1. $u(x) = e^{-x}$
2. $u(x) = x + x^4$
3. $u(x) = 1 + x$
4. $u(x) = 1 - \sinh x$
5. $u(x) = \cos x$
6. $u(x) = \sinh x$
7. $u(x) = e^{2x}$
8. $u(x) = 4 + \cos x$
9. $u(x) = e^x$
10. $u(x) = e^x$
11. $u(x) = e^{-x}$
12. $u(x) = \cos x$
13. $u(x) = \sin x$
14. $u(x) = \cosh x$
15. $u(x) = e^x$
16. $u(x) = 1 + e^{-x}$
17. $u(x) = \cos x$
18. $u(x) = \sinh x$
19. $u(x) = xe^x$
20. $u(x) = \sin x + \cos x$

Exercises 3.2.5

1. $u(x) = e^x - 1$
2. $u(x) = \sinh x$
3. $u(x) = x - \sin x$
4. $u(x) = e^{2x}$
5. $u(x) = -x + \sinh x$
6. $u(x) = e^{-x}$
7. $u(x) = -1 + \cos x$
8. $u(x) = \cos x$
9. $u(x) = e^{3x}$
10. $u(x) = \cosh x$
11. $u(x) = 2 + \cos x$
12. $u(x) = \cos x$
13. $u(x) = \sinh x$
14. $u(x) = \cos x - \sin x$
15. $u(x) = e^{-x^3}$
16. $u(x) = \sinh 2x$
17. $u(x) = \sinh x + \cos x$
18. $u(x) = \sin x + \cosh x$
19. $u(x) = x \sin x$
20. $u(x) = x \cosh x$

Exercises 3.2.6

1. $u(x) = \sinh x$
2. $u(x) = \cos x - \sin x$
3. $u(x) = \cos x$
4. $u(x) = e^{3x}$
5. $u(x) = \sinh x - \cosh x$
6. $u(x) = e^{x}$
7. $u(x) = \sin x$
8. $u(x) = \frac{2}{5}e^{x}(2\cos x + \sin x) + \frac{1}{5}e^{-x}$
9. $u(x) = \cosh x$
10. $u(x) = \sinh x$
11. $u(x) = \sin x$
12. $u(x) = \cos x$
13. $u(x) = xe^{x}$
14. $u(x) = x\sinh x$
15. $u(x) = x\sin x$
16. $u(x) = 1 + \frac{1}{2}x^2$

Exercises 3.2.7

1. $u(x) = e^{-x}$
2. $u(x) = \cos x$
3. $u(x) = \sinh x$
4. $u(x) = e^{x}$
5. $u(x) = e^{x}$
6. $u(x) = e^{2x}$
7. $u(x) = 2 + \cos x$
8. $u(x) = \sin x + \cos x$
9. $u(x) = \sin x$
10. $u(x) = \sinh x$
11. $u(x) = x\sinh x$
12. $u(x) = \cos x - \sin x$
13. $u(x) = \ln(1 + x)$
14. $u(x) = x\ln(1 + x)$
15. $u(x) = \sec x$
16. $u(x) = \tan x$

Exercises 3.3.1

1. $u(x) = \sinh x$
2. $u(x) = x\sinh x$
3. $u(x) = e^{x}$
4. $u(x) = x + e^{x}$
5. $u(x) = x + e^{x}$
6. $u(x) = x\cos x$
7. $u(x) = \cosh x$
8. $u(x) = x\cosh x$
9. $u(x) = \sin x + \cos x$
10. $u(x) = -e^{-x}$
11. $u(x) = x + \ln(1 + x)$
12. $u(x) = xe^{x}$

Exercises 3.3.2

1. $u(x) = \sin x$
2. $u(x) = \cos x$
3. $u(x) = x + \cos x$
4. $u(x) = \sin x + \cos x$
5. $u(x) = e^{-2x}$
6. $u(x) = e^{-x}$
7. $u(x) = e^{-x}$
8. $u(x) = e^{-x}$
9. $u(x) = x + e^{x}$
10. $u(x) = x + \sin x$
11. $u(x) = 2 + \sinh x$
12. $u(x) = x$

Exercises 3.3.3

1. $u(x) = \cos x$
2. $u(x) = \sin x + \cos x$
3. $u(x) = e^{-x}$

4. $u(x) = e^x$ 5. $u(x) = \sinh x$ 6. $u(x) = xe^x$
7. $u(x) = \cosh x$ 8. $u(x) = \cos x$ 9. $u(x) = 20x^3$
10. $u(x) = xe^x$ 11. $u(x) = x + e^x$ 12. $u(x) = \sec^2 x$
13. $u(x) = \cos x$ 14. $u(x) = e^x$ 15. $u(x) = e^{-x}$
16. $u(x) = x\cos x$

Exercises 4.2.1

1. $u(x) = e^x$ 2. $u(x) = e^x + 1$ 3. $u(x) = \cos x$
4. $u(x) = \sin x$ 5. $u(x) = e^x$ 6. $u(x) = e^x$
7. $u(x) = e^x$ 8. $u(x) = 1 + \dfrac{2\pi}{8-\pi}\sin^2 x$ 9. $u(x) = xe^x$
10. $u(x) = x\sin x$ 11. $u(x) = x\cos x$ 12. $u(x) = 2\sin x$
13. $u(x) = \sin x$ 14. $u(x) = 1 + x^2$ 15. $u(x) = 1 - x^2$
16. $u(x) = \sin x$ 17. $u(x) = e^{-x}$ 18. $u(x) = e^x$
19. $u(x) = \cos x$ 20. $u(x) = -\tan^2 x$

Exercises 4.2.2

1. $u(x) = \sin x$ 2. $u(x) = \sin x - \cos x$ 3. $u(x) = e^x + 12x^2$
4. $u(x) = \sin^{-1}\left(\dfrac{x+1}{2}\right) - \sin^{-1}\left(\dfrac{x-1}{2}\right)$ 5. $u(x) = x + 21x^2$
6. $u(x) = x + x^4$ 7. $u(x) = x + e^x$ 8. $u(x) = xe^x$
9. $u(x) = e^{x+1} + e^{x-1}$ 10. $u(x) = x^2 + x^3$ 11. $u(x) = e^{2x}$
12. $u(x) = e^{2x}$ 13. $u(x) = e^x$ 14. $u(x) = x(\sin x - \cos x)$
15. $u(x) = x\tan^{-1} x$ 16. $u(x) = \dfrac{e^x}{1+e^x}$

Exercises 4.2.3

1. $u(x) = \dfrac{1+\sin x+\cos x}{1+\sin x}$ 2. $u(x) = x + x\sin x$ 3. $u(x) = x^2 + \sec^2 x$
4. $u(x) = x(\sin x + \cos x)$ 5. $u(x) = \dfrac{x + x\cos x + \sin x}{1+\cos x}$
6. $u(x) = \dfrac{\sin x + \sin^2 x + \cos x}{1+\sin x}$ 7. $u(x) = \dfrac{2\sin x + \sin^2 x}{1+\sin x}$
8. $u(x) = \dfrac{\sin x}{1+\sin x}$ 9. $u(x) = \dfrac{\cos x}{1+\cos x}$ 10. $u(x) = x^2(\sin x + \cos x)$

11. $u(x) = x \sin 2x$ 12. $u(x) = x \cos 2x$ 13. $u(x) = \dfrac{1}{1+x^2}$
14. $u(x) = \cos^{-1} x$ 15. $u(x) = x \cos^{-1} x$ 16. $u(x) = x \tan^{-1} x$

Exercises 4.2.4

1. $u(x) = \cos x$ 2. $u(x) = \sin x$ 3. $u(x) = 1 + x^2$
4. $u(x) = 1 - x^2$ 5. $u(x) = \sin x - \cos x$ 6. $u(x) = e^{2x}$
7. $u(x) = 1 - x^2 + x^3$ 8. $u(x) = \sin x - \cos x$ 9. $u(x) = 1 + e^x$
10. $u(x) = x + e^x$ 11. $u(x) = x \sin 2x$ 12. $u(x) = e^x$
13. $u(x) = x - \cos x$ 14. $u(x) = x - \sin x$ 15. $u(x) = \sec x + \tan x$
16. $u(x) = \sin x + \cos x$

Exercises 4.2.5

1. $u(x) = 1 - x^2 + x^3$ 2. $u(x) = 1 - x^2 + x^3$ 3. $u(x) = x^3 + x^4$
4. $u(x) = x^3 + x^4$ 5. $u(x) = 1 + x^2 + x^3$ 6. $u(x) = \sec x \tan x$
7. $u(x) = \sec x \tan x$ 8. $u(x) = 1 + \dfrac{1}{2} \ln x$ 9. $u(x) = 1 + \dfrac{1}{2} \ln x$
10. $u(x) = 1 + \dfrac{1}{2} \ln x$ 11. $u(x) = \sin x + \cos x$ 12. $u(x) = \sin x - \cos x$
13. $u(x) = 1 + \dfrac{\pi}{8} \sec^2 x$ 14. $u(x) = 1 - \dfrac{\pi}{8} \sec^2 x$ 15. $u(x) = 1 + e^x$
16. $u(x) = x + e^x$

Exercises 4.2.6

1. $u(x) = \dfrac{3x}{3 - 2\lambda}, 0 < \lambda < \dfrac{3}{2}$ 2. $u(x) = 1 + x^3 + \dfrac{6\lambda x}{5(3 - 2\lambda)}, 0 < \lambda < \dfrac{3}{2}$
3. $u(x) = e^x$ 4. $u(x) = 2 + e^x$ 5. $u(x) = \sec^2 x$
6. $u(x) = 1 + \sec^2 x$ 7. $u(x) = \cos x$ 8. $u(x) = x - \cos x$
9. $u(x) = x - \sin x$ 10. $u(x) = \sin x$ 11. $u(x) = \sec x \tan x$
12. $u(x) = \sec x \tan x$ 13. $u(x) = \sec x + \tan x$ 14. $u(x) = \sin x + \cos x$
15. $u(x) = \sin x - \cos x$ 16. $u(x) = \ln(xt)$

Exercises 4.2.7

1. $u(x) = \dfrac{8}{3} - 2x,$ 2. $u(x) = 6x$ 3. $u(x) = 5x$

4. $u(x) = 3x$
5. $u(x) = 3x - 3x^2$
6. $u(x) = 3x - 5x^3$
7. $u(x) = 5x^4 + 7x^5$
8. $u(x) = 3x^2 - 5x^3 - 2x^4$
9. $u(x) = \sin x$
10. $u(x) = x^2 + \sin x$
11. $u(x) = \sec^2 x$
12. $u(x) = \ln(1 + x)$

Exercises 4.3

1. $u(x) = \frac{2}{\pi}\alpha \sin^2 x$
2. $u(x) = \alpha \tan x$
3. $u(x) = \alpha \sec^2 x$
4. $u(x) = \frac{1}{2}\alpha \sin x$
5. $u(x) = \frac{3}{8}\alpha x$
6. $u(x) = \alpha x$
7. $u(x) = \frac{8}{\pi}\alpha \sin^{-1} x$
8. $u(x) = \frac{8}{\pi}\alpha \cos^{-1} x$
9. $u(x) = \pm\frac{\sqrt{3}}{2}(1 \pm \sqrt{3}x)$
10. $u(x) = -\frac{3}{20}\beta(3x - 10)$
11. $u(x) = \frac{1}{\pi}(\alpha \sin x + \beta \cos x)$
12. $u(x) = 6\beta(1 - x), 6\beta(3 - 4x)$
13. $u(x) = \frac{1}{2}\alpha(1 - x)$
14. $u(x) = \gamma(-50x^2 + 32x + 9)$

Exercises 4.4.1

1. $u(x) = (1 + e^{-1})\frac{e^{3x}}{2}, e^{2x}$
2. $u(x) = e^{3x}$
3. $u(x) = 3x$
4. $u(x) = 6x^2, 3 + x^2$
5. $u(x) = x^2, x^2 + x^n, n \geqslant 0,$
6. $u(x) = \frac{3}{5}x, x^3$
7. $u(x) = \frac{5}{6}x^2, x^3$
8. $u(x) = \frac{5}{3}x^2, 1 + x$
9. $u(x) = -\frac{3}{4}x, \ln x$
10. $u(x) = \frac{3}{4}x, \ln(1 + x)$
11. $u(x) = \frac{1}{4}x, x + \ln x$
12. $u(x) = \frac{7}{4}x, x - \ln x$
13. $u(x) = \sin x$
14. $u(x) = \cos x$
15. $u(x) = \cos x + \sin x$
16. $u(x) = \cos x - \sin x$

Exercises 4.4.2

1. $u(x) = (1 + e^{-1})\frac{e^{3x}}{2}, e^{2x}$
2. $u(x) = e^{3x}$
3. $u(x) = 3x$
4. $u(x) = 6x^2, 3 + x^2$
5. $u(x) = x^2, x^2 + x^n,\ n \geqslant 0$
6. $u(x) = \frac{3}{5}x, x^3$
7. $u(x) = \frac{5}{6}x^2, x^3$
8. $u(x) = \frac{5}{3}x^2, 1 + x$
9. $u(x) = -\frac{3}{4}x, \ln x$
10. $u(x) = \frac{3}{4}x, \ln(1 + x)$
11. $u(x) = \frac{1}{4}x, x + \ln x$
12. $u(x) = \frac{7}{4}x, x - \ln x$

Exercises 5.2.1

1. $u(x) = \sinh x$
2. $u(x) = e^x$
3. $u(x) = e^{2x}$
4. $u(x) = \cosh x$
5. $u(x) = \cos x$
6. $u(x) = e^x$
7. $u(x) = \sin x + \cos x$
8. $u(x) = x \sin x$
9. $u(x) = x + \sin x$
10. $u(x) = x + e^x$
11. $u(x) = \cos x - \sin x$
12. $u(x) = e^x - x$
13. $u(x) = x^2 + e^x$
14. $u(x) = \sin x + \cos x$
15. $u(x) = x^3 + e^x$
16. $u(x) = 1 + x + e^x$

Exercises 5.2.2

1. $u(x) = e^{-x}$
2. $u(x) = e^x$
3. $u(x) = xe^x$
4. $u(x) = \sinh x$
5. $u(x) = x + \sinh x$
6. $u(x) = x + \cosh x$
7. $u(x) = e^x$
8. $u(x) = \sin x + \cos x$
9. $u(x) = x + \cos x$
10. $u(x) = x^2 + \sin x$
11. $u(x) = \cos x - \sin x$
12. $u(x) = x + e^x$
13. $u(x) = xe^x$
14. $u(x) = \sin x + \cos x$
15. $u(x) = x^2 + e^x$
16. $u(x) = 2 + e^x$

Exercises 5.2.3

1. $u(x) = 2 + e^x$
2. $u(x) = x + e^x$
3. $u(x) = ae^{-ax} - be^{-bx}$
4. $u(x) = \sin x$
5. $u(x) = x + \sin x$
6. $u(x) = e^x$
7. $u(x) = e^{-x}$
8. $u(x) = e^{2x}$
9. $u(x) = \sin x$
10. $u(x) = \cosh x$
11. $u(x) = 4 + e^x$
12. $u(x) = \sinh x$
13. $u(x) = x + \cosh x$
14. $u(x) = e^x$
15. $u(x) = xe^x$
16. $u(x) = x + e^x$

Exercises 5.2.4

1. $u(x) = \sinh x$
2. $u(x) = e^x$
3. $u(x) = x - e^{-x}$
4. $u(x) = \cosh x$
5. $u(x) = x + \sin x$
6. $u(x) = e^x$
7. $u(x) = \cos x + \sin x$
8. $u(x) = x^2 + \sin x$
9. $u(x) = \sin x$
10. $u(x) = x + e^x$
11. $u(x) = e^x - x$
12. $u(x) = \sin x$
13. $u(x) = x^2 + e^x$
14. $u(x) = x^3 + e^x$
15. $u(x) = xe^x$
16. $u(x) = 1 + x + e^x$

Exercises 5.2.5

1. $u(x) = \sinh x$ 2. $u(x) = e^{-x}$ 3. $u(x) = \cosh x$
4. $u(x) = \sinh x$ 5. $u(x) = x + \cos x$ 6. $u(x) = \sin x$
7. $u(x) = x + e^{x}$ 8. $u(x) = 1 + 4x$

Exercises 5.2.6

1. $u(x) = \sin x$ 2. $u(x) = \cos x + \sin x$ 3. $u(x) = \sinh x$
4. $u(x) = e^{x}$ 5. $u(x) = x + \sin x$ 6. $u(x) = x + e^{x}$
7. $u(x) = e^{x} - x$ 8. $u(x) = \cosh x$

Exercises 5.3.1

1. $u(x) = x + \sin x$ 2. $u(x) = x \sin x$ 3. $u(x) = 4 + e^{x}$
4. $u(x) = x + \cosh x$ 5. $u(x) = \sin x - \cos x$ 6. $u(x) = x - \cos x$
7. $u(x) = \sin x$ 8. $u(x) = \sinh x$ 9. $u(x) = e^{x}$
10. $u(x) = \sin x + \cos x$ 11. $u(x) = \sin 2x$ 12. $u(x) = x + \sinh 2x$

Exercises 5.3.2

1. $u(x) = xe^{x}$ 2. $u(x) = \sin x$ 3. $u(x) = \sin x$
4. $u(x) = x + \cos x$ 5. $u(x) = \sin x - \cos x$ 6. $u(x) = x + \cosh x$
7. $u(x) = \cosh x$ 8. $u(x) = e^{-x}$ 9. $u(x) = \sin x + \cos x$
10. $u(x) = \sin x - \cos x$ 11. $u(x) = x + \cosh x$ 12. $u(x) = x + e^{x}$

Exercises 6.2.1

1. $u(x) = 4x + 6x^2$ 2. $u(x) = 1 + 8x + 12x^3$ 3. $u(x) = \tan x$
4. $u(x) = 1 - 6x^2$ 5. $u(x) = \sec^2 x$ 6. $u(x) = \sin x$
7. $u(x) = \sin x$ 8. $u(x) = \cos x$ 9. $u(x) = \sin x - \cos x$
10. $u(x) = 4 \cosh x$ 11. $u(x) = x \sin x$ 12. $u(x) = \sin x$
13. $u(x) = 4 \sinh x$ 14. $u(x) = xe^{x}$ 15. $u(x) = \sin x$
16. $u(x) = \sin x$

Exercises 6.2.2

1. $u(x) = 4x + 6x^2$
2. $u(x) = 2 - 2x + 3x^2$
3. $u(x) = \tan x$
4. $u(x) = 2 - 3x^2$
5. $u(x) = \sec^2 x$
6. $u(x) = \sin x$
7. $u(x) = \sin x$
8. $u(x) = \cos x$
9. $u(x) = \sin x - \cos x$
10. $u(x) = 4\cosh x$
11. $u(x) = x \sin x$
12. $u(x) = \cos x$
13. $u(x) = 4\sinh x$
14. $u(x) = e^x$
15. $u(x) = \sin x$
16. $u(x) = \cos x$

Exercises 6.2.3

1. $u(x) = \tan x$
2. $u(x) = 6x + 12x^2$
3. $u(x) = 6x + 12x^2$
4. $u(x) = \sec^2 x$
5. $u(x) = \sin^2 x$
6. $u(x) = \sin x$
7. $u(x) = \cos^2 x$
8. $u(x) = \cos x$
9. $u(x) = \sin x - \cos x$
10. $u(x) = \cos x$
11. $u(x) = 4\cosh x$
12. $u(x) = \ln(1 + x)$
13. $u(x) = e^{-x}$
14. $u(x) = xe^x$
15. $u(x) = \sin x + \cos x$
16. $u(x) = \ln(1 + x)$

Exercises 6.2.4

1. $u(x) = 2 + 6x^2$
2. $u(x) = -2 + \frac{4}{9}x^3$
3. $u(x) = 3x + 3x^2$
4. $u(x) = -2x + 5x^2$
5. $u(x) = 9x^2 + x^3$
6. $u(x) = 3 - 9x^2$
7. $u(x) = 3 - 9x^2$
8. $u(x) = 1 + \frac{3}{2}x^2 + 5x^3$
9. $u(x) = \sin x + \cos x$
10. $u(x) = \cos x$
11. $u(x) = \cos x$
12. $u(x) = e^x$

Exercises 7.2.1

1. $u(x) = \frac{1}{\sqrt{x}}$
2. $u(x) = \sqrt{x}$
3. $u(x) = 1$
4. $u(x) = 1 + 2x$
5. $u(x) = \sqrt{x}(\frac{4}{3}x - 1)$
6. $u(x) = x^{\frac{3}{2}}$
7. $u(x) = x$
8. $u(x) = \sqrt{x} + x$
9. $u(x) = 1 - \sqrt{x}$
10. $u(x) = \frac{16}{5\pi}x^{\frac{5}{2}}$
11. $u(x) = 3x - \frac{2}{\pi}\sqrt{x}$
12. $u(x) = \frac{8}{\pi}x^{\frac{3}{2}}$
13. $u(x) = 2\sqrt{x} + \frac{8}{15}x^{\frac{5}{2}}$
14. $u(x) = \frac{8}{3}x^{\frac{3}{2}} - \frac{8}{5}x^{\frac{5}{2}}$
15. $u(x) = 2\sqrt{x} - \frac{4}{3}x^{\frac{3}{2}}$
16. $u(x) = \frac{8}{3}x^{\frac{3}{2}} + \frac{64}{105}x^{\frac{7}{2}}$

Exercises 7.3

1. $u(x) = x$ 2. $u(x) = x^2$ 3. $u(x) = 4$ 4. $u(x) = 1 + x$
5. $u(x) = x^3$ 6. $u(x) = \pi - x^2$ 7. $u(x) = x + x^2$ 8. $u(x) = x$
9. $u(x) = 3x^2$ 10. $u(x) = 3x^2$ 11. $u(x) = x$ 12. $u(x) = \cos x$
13. $u(x) = e^x$ 14. $u(x) = e^{2x}$ 15. $u(x) = \cos x$ 16. $u(x) = x^5$

Exercises 7.4.1

1. $u(x) = x$ 2. $u(x) = x$ 3. $u(x) = \sin x$ 4. $u(x) = \cos x$
5. $u(x) = e^{-x}$ 6. $u(x) = x$ 7. $u(x) = x^3$ 8. $u(x) = x^3$
9. $u(x) = x^5$ 10. $u(x) = x^5$ 11. $u(x) = e^x$ 12. $u(x) = e^x + e^{2x}$
13. $u(x) = \sin 2x$ 14. $u(x) = \sinh x$ 15. $u(x) = 1 + x$ 16. $u(x) = \pi + x^2$

Exercises 7.4.2

1. $u(x) = \sqrt{x}$ 2. $u(x) = x^{\frac{5}{2}}$ 3. $u(x) = 3$ 4. $u(x) = 1 + \sqrt{x}$
5. $u(x) = x^{\frac{3}{2}}$ 6. $u(x) = x^3$ 7. $u(x) = 1 + x^2$ 8. $u(x) = 1 - x$
9. $u(x) = x^2$ 10. $u(x) = x + \sqrt{x}$ 11. $u(x) = \pi$ 12. $u(x) = x + x^2$

Exercises 7.4.3

1. $u(x) = \sqrt{x}$ 2. $u(x) = x^2$ 3. $u(x) = 4$ 4. $u(x) = x^3$
5. $u(x) = 1 + \sqrt{x}$ 6. $u(x) = 1 + x$ 7. $u(x) = x^{\frac{3}{2}}$ 8. $u(x) = x + \sqrt{x}$
9. $u(x) = x$ 10. $u(x) = x^2$ 11. $u(x) = x$ 12. $u(x) = x^3$
13. $u(x) = 1 + x$ 14. $u(x) = x$ 15. $u(x) = x$ 16. $u(x) = x^2$

Exercises 8.2.1

1. $u(x) = 2 + 6x + 12x^2$ 2. $u(x) = 6 + 12x$ 3. $u(x) = 2 + 6x$
4. $u(x) = 2 + 12x^2$ 5. $u(x) = 6x + 20x^3$ 6. $u(x) = x$
7. $u(x) = e^x$ 8. $u(x) = \sin x$ 9. $u(x) = \sin x$
10. $u(x) = x \sin x$ 11. $u(x) = \ln(1 + x)$ 12. $u(x) = xe^x$

Exercises 8.2.2

1. $u(x) = x$
2. $u(x) = \sec^2 x$
3. $u(x) = \sin x$
4. $u(x) = x^3$
5. $u(x) = x^3$
6. $u(x) = \sin x$
7. $u(x) = \cos x$
8. $u(x) = \cos x$
9. $u(x) = \tan x$
10. $u(x) = \cot x$
11. $u(x) = x + \sin x$
12. $u(x) = x + \cos x$

Exercises 8.3.1

1. $u(x) = 2 + 6x + 12x^2$
2. $u(x) = 6 + 12x$
3. $u(x) = 2 + 6x$
4. $u(x) = 2 + 12x^2$
5. $u(x) = 4x + 12x^3$
6. $u(x) = 4 + 10x$
7. $u(x) = 6 + 9x + 5x^2$
8. $u(x) = \sin x$
9. $u(x) = \ln(1 + x)$
10. $u(x) = \ln(1 + x)$
11. $u(x) = x + \cos x$
12. $u(x) = \tan x$

Exercises 8.3.2

1. $u(x) = x$
2. $u(x) = x$
3. $u(x) = x^2$
4. $u(x) = x^3$
5. $u(x) = x^3$
6. $u(x) = x^2$
7. $u(x) = x \sin x$
8. $u(x) = x + \cos x$
9. $u(x) = \ln(1 + x)$
10. $u(x) = \tan x$
11. $u(x) = \sec^2 x$
12. $u(x) = 2xe^x$

Exercises 8.4.1

1. $u(x) = xt$
2. $u(x) = x + t$
3. $u(x) = x^2 - t^2$
4. $u(x) = x^2 - t^2$
5. $u(x) = 1 + x^3 + t^3$
6. $u(x) = x^2 t - xt^2$
7. $u(x) = e^{-t} \cos x$
8. $u(x) = \cos(x - t)$
9. $u(x) = xe^t$
10. $u(x) = \cos(x + t)$
11. $u(x) = \cos x \cos t$
12. $u(x) = e^{-t} \sin x$

Exercises 9.2.1

1. $u(x) = 6x$
2. $u(x) = 1 + 6x$
3. $u(x) = 1 + 3x^2$
4. $u(x) = 1 + 3x^2$
5. $u(x) = 1 - 3x - 3x^2$
6. $u(x) = 69 - 168x - 250x^2$
7. $u(x) = \sin x - \cos x$
8. $u(x) = e^x$
9. $u(x) = 2 + 6x + 12x^2$
10. $u(x) = -6x - 12x^2$
11. $u(x) = x \cos x$
12. $u(x) = x \sin x$

Exercises 9.2.2

1. $u(x) = 6x$
2. $u(x) = 1 + 6x$
3. $u(x) = 1 + 3x^2$

4. $u(x) = 1 + 3x^2$ 5. $u(x) = 1 - 3x - 3x^2$ 6. $u(x) = 3 + x^3$
7. $u(x) = x + \sin x$ 8. $u(x) = \sin x + \cos x$ 9. $u(x) = x + e^x$
10. $u(x) = x^3 + \sin x$ 11. $u(x) = x \cos x$ 12. $u(x) = x \sin x$

Exercises 9.3.1

1. $u(x) = 2 + 6x^2$ 2. $u(x) = 1 + x + x^2$ 3. $u(x) = 1 + x - x^3$
4. $u(x) = 1 + x - x^3$ 5. $u(x) = x + \cos x$ 6. $u(x) = x + e^x$
7. $u(x) = 1 - e^x$ 8. $u(x) = \sin x - \cos x$ 9. $u(x) = 1 + 9x$
10. $u(x) = x \cos x$ 11. $u(x) = 1 + \frac{9}{2}x + e^x$ 12. $u(x) = \sin(2x)$

Exercises 9.3.2

1. $u(x) = 2 + 6x^2$ 2. $u(x) = 1 + x - \frac{10}{3}x^2$ 3. $u(x) = 1 + x - x^3$
4. $u(x) = 1 + x - x^2 - \frac{5}{3}x^3$ 5. $u(x) = x + \cos x$ 6. $u(x) = 1 + e^x$
7. $u(x) = 1 - e^x$ 8. $u(x) = e^{-x}$ 9. $u(x) = 1 + 9x$
10. $u(x) = 1 + x - \frac{5}{3}x^2$ 11. $u(x) = 1 + x + x^2 + x^3$ 12. $u(x) = 1 - \frac{5}{2}x^3$

Exercises 9.4.1

1. $u(x) = x + t$ 2. $u(x) = x^2 + t^2$ 3. $u(x) = 1 + 2xt$ 4. $u(x) = x^2 + t$
5. $u(x) = x^2t^2$ 6. $u(x) = x^3 - t^3$ 7. $u(x) = t \cos x$ 8. $u(x) = t + \sin x$
9. $u(x) = \sin t + \cos x$ 10. $u(x) = t + e^x$ 11. $u(x) = xe^t$ 12. $u(x) = te^x$

Exercises 10.2.1

1. $(u(x), v(x)) = (x^2, x^3)$ 2. $(u(x), v(x)) = (1 + x + x^2, 1 - x - x^2)$
3. $(u(x), v(x)) = (1 + x^3, 1 - x^3)$ 4. $(u(x), v(x)) = (x^2 + x^3, x^2 - x^3)$
5. $(u(x), v(x)) = (\cos x, \sin x)$ 6. $(u(x), v(x)) = (\sin x, \cos x)$
7. $(u(x), v(x)) = (\sec^2 x, \tan^2 x)$ 8. $(u(x), v(x)) = (\sin^2 x, \cos^2 x)$
9. $(u(x), v(x)) = (e^{-x}, e^x)$ 10. $(u(x), v(x)) = (1 + e^x, 1 - e^x)$
11. $(u(x), v(x)) = (1 + \sin x, 1 - \sin x)$ 12. $(u(x), v(x)) = (x + \cos x, x - \cos x)$

Exercises 10.2.2

1. $(u, v) = (x^2, x^3)$
2. $(u, v) = (1 + x + x^2, 1 - x - x^2)$
3. $(u, v) = (1 + x^3, 1 - x^3)$
4. $(u, v) = (x^2 + x^3, x^2 - x^3)$
5. $(u, v) = (x + \sin x, x - \cos x)$
6. $(u, v) = (\sin x, \cos x)$
7. $(u, v) = (1 + \sinh x, 1 - \cosh x)$
8. $(u, v) = (e^x, e^{-x})$
9. $(u, v) = (\sin x + \cos x, \sin x - \cos x)$
10. $(u, v) = (e^x \sin x, e^x \cos x)$
11. $(u, v, w) = (\sin x, \cos x, \sin x + \cos x)$
12. $(u, v, w) = (1 + x, x + x^2, x^2 + x^3)$

Exercises 10.3.1

1. $(u, v) = (x^2, x^3)$
2. $(u, v) = (1 + x + x^2, 1 - x - x^2)$
3. $(u, v) = (1 + x^3, 1 - x^3)$
4. $(u, v) = (x^2 + x^3, x^2 - x^3)$
5. $(u, v) = (x + \sin x, x - \cos x)$
6. $(u, v) = (\sin x, \cos x)$
7. $(u, v) = (1 + e^x, 1 - e^x)$
8. $(u, v) = (1 + e^x, 1 - xe^x)$
9. $(u, v) = (1 + x + e^x, 1 - x + e^x)$
10. $(u, v, w) = (1 + x, 1 + x^2, 1 + x^3)$
11. $(u, v, w) = (1, \sin x, \cos x)$
12. $(u, v, w) = (1 + \cos x, 1 + \sin x, \sin x - \cos x)$

Exercises 10.4.1

1. $(u, v) = (1 + x^2, 1 - x^2)$
2. $(u, v) = (1 + 3x, 2 - 3x)$
3. $(u, v) = (1 + x - x^2, 1 - x + x^2)$
4. $(u, v) = (1 + \sin x, 1 - \sin x)$
5. $(u, v) = (1 + \sin x, 1 + \cos x)$
6. $(u, v) = (x + \cos x, x - \cos x)$
7. $(u, v) = (e^x, 2e^{2x})$
8. $(u, v) = (1 + e^x, 2 - e^x)$
9. $(u, v) = (x + e^x, x - e^x)$
10. $(u, v, w) = (1 + e^x, 1 - e^x, x + e^x)$
11. $(u, v, w) = (1 + \cos x, 1 - \cos x, x + \cos x)$
12. $(u, v, w) = (1 + \cos x, 1 - \sin x, e^x)$

Exercises 10.4.2

1. $(u, v) = (1 + \frac{1}{2}x^2, 1 - \frac{1}{2}x^2)$
2. $(u, v) = (x + \frac{1}{2}x^2, x - \frac{1}{2}x^2)$
3. $(u, v) = (1 + 2x, 1 - 2x)$
4. $(u, v) = (1 + 2x, 1 - 2x)$
5. $(u, v) = (\sin x, \cos x)$
6. $(u, v) = (\sin x, \cos x)$
7. $(u, v) = (2 + \sin x, 3 - \cos x)$
8. $(u, v) = (2 + e^x, 3 - e^x)$
9. $(u, v) = (x - e^x, x + e^x)$
10. $(u, v, w) = (1 + \cos x, 1 - \cos x, x + \cos x)$

11. $(u, v, w) = (1 + \cos x, 1 - \sin x, e^x)$

12. $(u, v, w) = (x, x^2, x^3)$

Exercises 11.2.1

1. $(u, v) = (x, x^2 + x^3)$
2. $(u, v) = (x, x^2 + x^3)$
3. $(u, v) = (x + x^2, x^3 + x^4)$
4. $(u, v) = (\sin x + \cos x, \sin x - \cos x)$
5. $(u, v) = (x + \sin^2 x, x - \cos^2 x)$
6. $(u, v) = (x^2 + \sin x, x^2 + \cos x)$
7. $(u, v) = (x \tan^{-1} x, x + \tan^{-1} x)$
8. $(u, v) = (\dfrac{e^x}{1 + e^x}, \dfrac{1}{1 + e^x})$
9. $(u, v) = (\dfrac{\sin x}{1 + \sin x}, \dfrac{\cos x}{1 + \cos x})$
10. $(u, v) = (\sec x \tan x, \sec^2 x)$
11. $(u, v, w) = (x, x^2, x^3)$
12. $(u, v, w) = (1 + \dfrac{\pi}{8} \sec^2 x, 1 - \dfrac{\pi}{8} \sec^2 x, 1 + \dfrac{\pi}{2} \sec^2 x)$

Exercises 11.2.2

1. $(u, v) = (x, x^2 + x^3)$
2. $(u, v) = (2 + \ln x, 2 - \ln x)$
3. $(u, v) = (x + x^2, x^3 + x^4)$
4. $(u, v) = (\sin x + \cos x, \sin x - \cos x)$
5. $(u, v) = (x + \sin^2 x, x - \cos^2 x)$
6. $(u, v) = (\tan x, \sec x)$
7. $(u, v) = (x^2 + \sin x, x^2 + \cos x)$
8. $(u, v) = (\dfrac{e^x}{1 + e^x}, \dfrac{1}{1 + e^x})$
9. $(u, v) = (\dfrac{\sin x}{1 + \sin x}, \dfrac{\cos x}{1 + \cos x})$
10. $(u, v) = (\sec x \tan x, \sec^2 x)$
11. $(u, v, w) = (x, x^2, x^3)$
12. $(u, v, w) = (1 + \sec^2 x, 1 - \sec^2 x, \sec x \tan x)$

Exercises 11.3.1

1. $(u, v) = (\sin x, \cos x)$
2. $(u, v) = (1 + \cos x, 1 - \sin x)$
3. $(u, v) = (\cos(2x), \sin(2x))$
4. $(u, v) = (1 + \sinh^2 x, 1 + \cosh^2 x)$
5. $(u, v) = (1 + \cosh^2 x, 1 - \cosh^2 x)$
6. $(u, v) = (x + \sinh x, x + \cosh x)$
7. $(u, v) = (x + e^x, x - e^x)$
8. $(u, v) = (xe^x, xe^{-x})$
9. $(u, v) = (e^x, e^{2x})$
10. $(u, v) = (\sin^2 x, \cos^2 x)$
11. $(u, v) = (\sin x, \cos x)$
12. $(u, v) = (\sin x + \cos x, \sin x - \cos x)$

Exercises 11.3.2

1. $(u,v)=(\sin x,\cos x)$
2. $(u,v)=(\sin x,\cos x)$
3. $(u,v)=(x\sin x,x\cos x)$
4. $(u,v)=(1+\sinh^2 x,1+\cosh^2 x)$
5. $(u,v)=(1+\sinh^2 x,1-\sinh^2 x)$
6. $(u,v)=(x+\sinh x,x+\cosh x)$
7. $(u,v)=(e^x,e^{-x})$
8. $(u,v)=(xe^x,xe^{-x})$
9. $(u,v)=(e^x,e^{3x})$
10. $(u,v)=(e^x,e^{3x})$
11. $(u,v)=(\sin x,\cos x)$
12. $(u,v)=(\sin x+\cos x,\sin x-\cos x)$

Exercises 12.2.1

1. $(u,v)=(2+3x,3+4x)$
2. $(u,v)=(\pi+x,\pi-x)$
3. $(u,v)=(x+6,x-6)$
4. $(u,v)=(x^4,4)$
5. $(u,v)=(1+x^2,1-x^2)$
6. $(u,v)=(x^3+1,x^3-1)$
7. $(u,v)=(1+x-x^2,1-x+x^2)$
8. $(u,v)=(1+x+x^3,1-x-x^3)$

Exercises 12.2.2

1. $(u,v,w)=(x,x^2,x^3)$
2. $(u,v,w)=(2+3x,3+4x,4+5x)$
3. $(u,v,w)=(1+x,x+x^2,x^2+x^3)$
4. $(u,v,w)=(1+x,x+x^2,x^2+x^3)$
5. $(u,v,w)=(6x,6+x^2,6-x^3)$
6. $(u,v,w)=(2+x+x^2,1-2x+x^2,1+x-2x^2)$
7. $(u,v,w)=(x+x^2+3x^3,x+3x^2-x^3,x-x^2+3x^3)$
8. $(u,v,w)=(x+x^2+3x^3,x+3x^2-x^3,x-x^2+3x^3)$

Exercises 12.3.1

1. $(u,v)=(x+x^2,x-x^2)$
2. $(u,v)=(x^2,x)$
3. $(u,v)=(1+x^2,1-x^2)$
4. $(u,v)=(1+3x,1+3x^2)$
5. $(u,v)=(1+x+x^2,1-x-x^2)$
6. $(u,v)=(1+x+2x^2,1-2x-x^2)$
7. $(u,v)=(1+x-x^2,1-x+x^2)$
8. $(u,v)=(6+x^2,6-x^2)$

Exercises 12.3.2

1. $(u,v)=(1+x^2,1-x^2)$
2. $(u,v)=(x,1+x)$
3. $(u,v)=(x,x^2)$
4. $(u,v)=(\sin x,\cos x)$
5. $(u,v)=(\cos x,\sin x)$
6. $(u,v)=(\sinh x,\cosh x)$

7. $(u,v) = (\cosh x, \sinh x)$ 8. $(u,v) = (e^x, e^{-x})$

Exercises 13.3.1

1. $u(x) = \dfrac{1}{1-x}$ 2. $u(x) = 1 + 3x$ 3. $u(x) = 1 + 3x$
4. $u(x) = 1 + x$ 5. $u(x) = \sin x$ 6. $u(x) = \sin x + \cos x$
7. $u(x) = \cos x - \sin x$ 8. $u(x) = 1 + \cos x$ 9. $u(x) = 1 - \sinh x$
10. $u(x) = e^x$ 11. $u(x) = e^x$ 12. $u(x) = 1 + e^x$

Exercises 13.3.2

1. $u(x) = 1 + x$ 2. $u(x) = 1 + x^2$ 3. $u(x) = 1 + 3x$
4. $u(x) = 1 + 2x$ 5. $u(x) = \sin x$ 6. $u(x) = \sin x + \cos x$
7. $u(x) = \cosh x$ 8. $u(x) = \cosh x$ 9. $u(x) = 1 - \sinh x$
10. $u(x) = e^x$ 11. $u(x) = e^x$ 12. $u(x) = e^{-x}$

Exercises 13.3.3

1. $u(x) = \tanh x$ 2. $u(x) = 1 + x^2$ 3. $u(x) = 1 + x^2$
4. $u(x) = 1 + 2x$ 5. $u(x) = x$ 6. $u(x) = \sin x + \cos x$
7. $u(x) = 1 + x^2$ 8. $u(x) = e^x$ 9. $u(x) = 1 - \sinh x$
10. $u(x) = \sin x$ 11. $u(x) = \sin x$ 12. $u(x) = \cos x$

Exercises 13.4.1

1. $u(x) = \pm(1 + x)$ 2. $u(x) = \pm(\sin x + \cos x)$ 3. $u(x) = \pm e^{2x}$
4. $u(x) = \pm \sin x$ 5. $u(x) = \pm \sin x$ 6. $u(x) = 1 + x$
7. $u(x) = 3x$ 8. $u(x) = 3x$

Exercises 13.4.2

1. $u(x) = \pm(\sin x - \cos x)$ 2. $u(x) = \pm(\sin x + \cos x)$ 3. $u(x) = \sin x$
4. $u(x) = \pm \sin x$ 5. $u(x) = \pm(1 - 2x)$ 6. $u(x) = \pm \sinh x$
7. $u(x) = 3x$ 8. $u(x) = 3x$

Exercises 13.5.1

1. $(u(x), v(x)) = (x, x^2)$
2. $(u(x), v(x)) = (1 + x^2, 1 - x^2)$
3. $(u(x), v(x)) = (1 + e^x, 1 - e^x)$
4. $(u(x), v(x)) = (e^x, e^{-x})$
5. $(u(x), v(x)) = (\cos x, \sin x)$
6. $(u(x), v(x)) = (1 + \sin x, 1 - \sin x)$
7. $(u(x), v(x)) = (\cosh x, \sinh x)$
8. $(u(x), v(x)) = (1 + \cosh x, 1 - \cosh x)$

Exercises 13.5.2

1. $(u(x), v(x)) = (x, x^2)$
2. $(u(x), v(x)) = (x^3, x^5)$
3. $(u(x), v(x)) = (\sin x, \cos x)$
4. $(u(x), v(x)) = (\cosh x, \sinh x)$
5. $(u(x), v(x)) = (\sin x + \cos x, \sin x - \cos x)$
6. $(u(x), v(x)) = (e^{2x}, e^{-2x})$
7. $(u(x), v(x)) = (1 + e^x, 1 - e^x)$
8. $(u(x), v(x)) = (1 + \sin x, 1 - \sin x)$

Exercises 14.2.1

1. $u(x) = 2 + e^x$
2. $u(x) = x + e^x$
3. $u(x) = 1 - e^{-x}$
4. $u(x) = \sin x$
5. $u(x) = \sin x$
6. $u(x) = \cosh x$
7. $u(x) = e^x$
8. $u(x) = \sin x + \cos x$

Exercises 14.2.2

1. $u(x) = \sin x$
2. $u(x) = e^{-x}$
3. $u(x) = x$
4. $u(x) = \cos x$
5. $u(x) = \cos x$
6. $u(x) = e^x$
7. $u(x) = e^x$
8. $u(x) = \sec x$

Exercises 14.2.3

1. $u(x) = \operatorname{sech} x$
2. $u(x) = \sinh x$
3. $u(x) = \sin x - \cos x$
4. $u(x) = e^{-x}$
5. $u(x) = e^{-x}$
6. $u(x) = 1 + e^x$
7. $u(x) = 1 + e^x$
8. $u(x) = \sin(2x)$

Exercises 14.3.1

1. $u(x) = \cos x$
2. $u(x) = \cosh x$
3. $u(x) = \sin x + \cos x$
4. $u(x) = 1 + e^x$
5. $u(x) = 3 + e^x$
6. $u(x) = \sin(2x)$
7. $u(x) = x^2$
8. $u(x) = e^x$

Exercises 14.3.2

1. $u(x) = 1 + \cosh x$ 2. $u(x) = 1 + \cos x$ 3. $u(x) = x + e^x$
4. $u(x) = \sin x - \cos x$ 5. $u(x) = \sin x + \cos x$ 6. $u(x) = x + \sin x$
7. $u(x) = e^{2x}$ 8. $u(x) = \sinh x$

Exercises 14.4.1

1. $(u(x), v(x)) = (1 + x^2, 1 - x^2)$ 2. $(u(x), v(x)) = (x + x^3, x - x^3)$
3. $(u(x), v(x)) = (1 + e^x, 1 - e^x)$ 4. $(u(x), v(x)) = (x + \cos x, x - \cos x)$
5. $(u(x), v(x)) = (1 + x + x^2, 1 - x - x^2)$ 6. $(u(x), v(x)) = (x + \sin x, x - \sin x)$
7. $(u(x), v(x), w(x)) = (e^x, e^{2x}, e^{3x})$ 8. $(u(x), v(x), w(x)) = (e^x, 2e^{2x}, 3e^{3x})$

Exercises 14.4.2

1. $(u(x), v(x)) = (1 + x^2, 1 - x^2)$ 2. $(u(x), v(x)) = (1 + x^3, 1 - x^3)$
3. $(u(x), v(x)) = (1 + \sin x, 1 + \cos x)$
4. $(u(x), v(x)) = (\sin x + \cos x, \sin x - \cos x)$
5. $(u(x), v(x)) = (e^{2x}, e^{3x})$ 6. $(u(x), v(x)) = (\cos x, \sin x)$
7. $(u(x), v(x)) = (e^x, e^{-x})$ 8. $(u(x), v(x)) = (1 + x + x^2, 1 - x + x^2)$

Exercises 15.3.1

1. $u(x) = \dfrac{1 \pm \sqrt{1 - 8\lambda}}{\lambda}, \lambda < \dfrac{1}{8}$
   $\lambda = 0$ is a singular point, $\lambda = 1/8$ is a bifurcation point
2. $u(x) = \dfrac{3 \pm \sqrt{9 - 24\lambda}}{2\lambda}, \lambda < \dfrac{3}{8}$
   $\lambda = 0$ is a singular point, $\lambda = 3/8$ is a bifurcation point
3. $u(x) = \dfrac{3 \pm \sqrt{9 - 24\lambda}}{4\lambda}, \lambda < \dfrac{3}{8}$
   $\lambda = 0$ is a singular point, $\lambda = 3/8$ is a bifurcation point
4. $u(x) = \dfrac{\sqrt{3}}{4} + \dfrac{3(1 \pm \sqrt{1 - \lambda^2})}{4\lambda} x, -1 < \lambda < 1$
   $\lambda = 0$ is a singular point, $\lambda = \pm 1$ are bifurcation points
5. $u(x) = \dfrac{3}{2\lambda} - \dfrac{(2 + 4\lambda) \pm \sqrt{4 + 16\lambda - 2\lambda^2}}{\lambda} x, 4 - 3\sqrt{2} < \lambda < 4 + 3\sqrt{2}$
   $\lambda = 0$ is a singular point, $\lambda = 4 \pm 3\sqrt{2}$ are bifurcation points

6. $u(x) = \dfrac{5(1 \pm \sqrt{1-\lambda})}{2\lambda}, \lambda < 1$

$\lambda = 0$ is a singular point, $\lambda = 1$ is a bifurcation point

7. $u(x) = 1 - x^2, 1 - 4x^2, 1 + \dfrac{4}{5}x^2$ 8. $u(x) = 1 + x, 1 - \dfrac{5}{8}x$

9. $u(x) = x - x^2$ 10. $u(x) = \sin x, -\dfrac{7}{4}\sin x, \dfrac{3}{4}\sin x + \dfrac{\sqrt{5}}{2}\cos x$

11. $u(x) = 1 + x - x^2$ 12. $u(x) = \cos x$

Exercises 15.3.2

1. $u(x) = 1 - x - x^2$ 2. $u(x) = 1 - x - x^2$ 3. $u(x) = 1 + x^2$

4. $u(x) = 1 + x^2$ 5. $u(x) = 1 + x - x^2 - x^3$ 6. $u(x) = e^x$

7. $u(x) = e^x$ 8. $u(x) = \cos x$

Exercises 15.3.3

1. $u(x) = \dfrac{1 \pm \sqrt{1-8\lambda}}{\lambda}, \lambda < \dfrac{1}{8}$ 2. $u(x) = \dfrac{3 \pm \sqrt{9-24\lambda}}{2\lambda}, \lambda < \dfrac{3}{8}$

3. $u(x) = \dfrac{3 \pm \sqrt{9-24\lambda}}{4\lambda}, \lambda < \dfrac{3}{8}$ 4. $u(x) = \dfrac{\sqrt{3}}{4} + \dfrac{3(1 \pm \sqrt{1-\lambda^2})}{4\lambda}x, -1 < \lambda < 1$

5. $u(x) = \dfrac{3}{2\lambda} - \dfrac{(2+4\lambda) \pm \sqrt{4+16\lambda-2\lambda^2}}{\lambda}x, 4 - 3\sqrt{2} < \lambda < 4 + 3\sqrt{2}$

6. $u(x) = \dfrac{5(1 \pm \sqrt{1-\lambda})}{2\lambda}, \lambda < 1$

7. $u(x) = \tan x$ 8. $u(x) = x$ 9. $u(x) = \sec x$

10. $u(x) = \cosh x$ 11. $u(x) = \ln x$ 12. $u(x) = \ln x$

Exercises 15.3.4

1. $u(x) = \sin x$ 2. $u(x) = 1 + \sin x$ 3. $u(x) = 1 + \cos x$ 4. $u(x) = 1 + e^x$

5. $u(x) = 1 + e^x$ 6. $u(x) = xe^x$ 7. $u(x) = e^x$ 8. $u(x) = e^x$

9. $u(x) = \cos x$ 10. $u(x) = \ln x$ 11. $u(x) = x\ln x$ 12. $u(x) = x + \ln x$

Exercises 15.4.1

1. $u(x) = \dfrac{3}{\lambda}\sin x$ 2. $u(x) = \dfrac{3(\lambda-2)}{2\lambda}\sin x$ 3. $u(x) = -2\lambda\cos x$

4. $u(x) = \dfrac{\lambda-1}{\lambda(e-1)}e^x$ 5. $u(x) = \dfrac{1}{\sqrt{\lambda(e-1)}}e^x$ 6. $u(x) = \dfrac{1 \pm \sqrt{1-4\lambda^2}}{2\lambda}e^x$

7. $u(x) = \dfrac{8}{\lambda(\pi^2 - 16)}(\pi \cos x - 4 \sin x)$

8. $u(x) = \dfrac{3}{5\lambda}(\cos x + \sin x)$

9. $u(x) = \dfrac{4}{\lambda\pi} \cos x, -\dfrac{2}{\lambda\pi}(\cos x + \sqrt{3} \sin x)$

10. $u(x) = -\dfrac{3}{2\lambda}, \dfrac{3 + \sqrt{15}x}{4\lambda}$

11. $u(x) = \dfrac{7}{2\lambda}x^2, \dfrac{1}{28\lambda}(15\sqrt{7}x + 35x^2)$

12. $u(x) = \dfrac{20}{\lambda}x$

Exercises 15.5.1

1. $u(x) = \left(\dfrac{e^{2x+1}}{e-1}\right)^{\frac{1}{3}}, \ e^x$

2. $u(x) = e^{-\frac{1}{3}x}$

3. $u(x) = \left(\dfrac{-10x^2}{27}\right)^{\frac{1}{3}}, \ \ln x$

4. $u(x) = \sqrt{\dfrac{3x}{32}}, \ x \ln x$

5. $u(x) = \sqrt{\dfrac{29x}{12}}, \ 2x + \ln x$

6. $u(x) = \sqrt{\dfrac{80}{63}x}, \ x^2 + x^3$

7. $u(x) = \sqrt{\dfrac{233}{63}x}, \ x + x^2 + x^3$

8. $u(x) = \sqrt{\dfrac{12x}{35}}, \ x + x^2 - x^3$

9. $u(x) = \sqrt{\cos x}$

10. $u(x) = \sqrt{\sin x}$

11. $u(x) = \sqrt{\sin x}$

12. $u(x) = \sqrt[3]{\dfrac{937}{40} - \dfrac{1931}{60}x}, \ x - \ln x$

Exercises 15.5.2

1. $u(x) = \left(\dfrac{e^{2x+1}}{e-1}\right)^{\frac{1}{3}}, \ e^x$

2. $u(x) = e^{-\frac{1}{3}x}$

3. $u(x) = \left(\dfrac{-10x^2}{27}\right)^{\frac{1}{3}}, \ \ln x$

4. $u(x) = \sqrt{\dfrac{3x}{32}}, \ x \ln x$

5. $u(x) = \sqrt{\dfrac{29x}{12}}, \ 2x + \ln x$

6. $u(x) = \sqrt{\dfrac{80}{63}x}, \ x^2 + x^3$

7. $u(x) = \sqrt{\dfrac{233}{63}x}, \ x + x^2 + x^3$

8. $u(x) = \sqrt{\dfrac{12x}{35}}, \ x + x^2 - x^3$

Exercises 15.6.1

1. $(u, v) = (x, x^2 + x^3)$

2. $(u, v) = (2 + \ln x, 2 - \ln x)$

3. $(u, v) = (2 + \ln x, 2 - \ln x)$

4. $(u, v) = (\sin x + \cos x, \sin x - \cos x)$

5. $(u, v) = (x + \sin^2 x, x - \cos^2 x)$

6. $(u, v) = (\sec x, \tan x)$

7. $(u, v, w) = (x, x^2, x^3)$

8. $(u, v, w) = (\sec x \tan x, \sec^2 x, \tan^2 x)$

Exercises 15.6.2

1. $(u,v)=(x,x^2+x^3)$
2. $(u,v)=(\sin x+\cos x,\sin x-\cos x)$
3. $(u,v)=(x+\sin x,x-\cos x)$
4. $(u,v)=(\sec x,\tan x)$
5. $(u,v)=(\sec x,\tan x)$
6. $(u,v)=(\sec x\tan x,\sec^2 x)$
7. $(u,v)=(\sec x,\cos x)$
8. $(u,v)(\tan x,\cos x)$
9. $(u,v,w)=(\sec x,\tan x,\cos x)$
10. $(u,v,w)=(1+\pi\sec^2 x,1-\pi\sec^2 x,1+\frac{\pi}{2}\sec^2 x)$
11. $(u,v,w)=(\sec x,\tan x,\cos x)$
12. $(u,v,w)=(\sec^2 x,\cos^2 x,\tan^2 x)$

Exercises 16.2.1

1. $u(x)=1+x,1+x+\frac{98}{5}x^2$
2. $u(x)=1+x+x^2,1+x+x^2+\frac{9224}{105}x^3$
3. $u(x)=1+\sin x$
4. $u(x)=1-\cos x$
5. $u(x)=x+e^x$
6. $u(x)=1+e^x,1+e^x+(24e-\frac{243}{2})x^2$
7. $u(x)=\sin x$
8. $u(x)=1-\cos x$
9. $u(x)=e^x$
10. $u(x)=e^x$
11. $u(x)=e^{2x}$
12. $u(x)=\cos x$

Exercises 16.2.2

1. $u(x)=1-\cos x$
2. $u(x)=1+\sin x$
3. $u(x)=x\sin x$
4. $u(x)=e^{2x}$
5. $u(x)=e^{-2x}$
6. $u(x)=\sin x-\cos x$
7. $u(x)=\sin x$
8. $u(x)=\cos x$
9. $u(x)=e^x$
10. $u(x)=\cos x$
11. $u(x)=\sin x+\cos x$
12. $u(x)=1+e^x$

Exercises 16.2.3

1. $u(x)=1-x-x^2$
2. $u(x)=1+x+x^3$
3. $u(x)=1+x+x^2$
4. $u(x)=e^x$
5. $u(x)=e^x$
6. $u(x)=1+x-x^2$
7. $u(x)=e^x$
8. $u(x)=x+x^3$

Exercises 16.3.1

1. $u(x)=\frac{\sqrt{21\lambda(\lambda-36)}}{3\lambda}x^2$
2. $u(x)=\frac{5(\lambda-36)}{3\lambda}x^2$

3. $u(x) = 1 - \dfrac{10(\lambda+72)}{3\lambda}\, x^2$

4. $u(x) = \dfrac{36 \pm 2\sqrt{324 - 6\lambda^2}}{24\lambda} \sin x$

5. $u(x) = \dfrac{60}{\lambda}\, x^2$

6. $u(x) = \dfrac{1 - \cos x}{2\pi\lambda}$

7. $u(x) = \dfrac{\pm \sin x + \cos x - 1}{2\pi\lambda}$

8. $u(x) = \dfrac{(1 \pm \pi\lambda)(\sin x \mp \cos x \pm 1)}{2\pi\lambda}$

9. $u(x) = 1 + \dfrac{21 \pm \sqrt{441 - 56\lambda^2}}{2\lambda}\, x^3$

10. $u(x) = x + \dfrac{54 \pm 2\sqrt{729 - 3\lambda^2}}{\lambda}\, x^4$

11. $u(x) = \dfrac{5(\lambda - 6)}{3\lambda}\, x^2$

12. $u(x) = \dfrac{\lambda\pi + 1}{2\lambda}(1 - \cos x), \; \dfrac{\lambda\pi - 1}{3\pi\lambda}(1 - \cos x)$
$+ \dfrac{\sqrt{3(\pi\lambda + 5)(\pi\lambda - 1)}}{9\pi\lambda}\,(x - \sin x)$

Exercises 16.4.1

1. $(u, v) = (x \cos x, x \sin x)$
2. $(u, v) = (1 + \sinh^2 x, 1 + \cosh^2 x)$
3. $(u, v) = (\cos x, \sec x)$
4. $(u, v) = (1 + \sinh^2 x, 1 - \sinh^2 x)$
5. $(u, v) = (xe^x, xe^{-x})$
6. $(u, v) = (\cos x + \sin x, \cos x - \sin x)$
7. $(u, v) = (x + e^x, x - e^x)$
8. $(u, v, w) = (e^x, e^{3x}, e^{5x})$

Exercises 16.4.2

1. $(u, v) = (\sin x, \cos x)$
2. $(u, v) = (x \sin x, x \cos x)$
3. $(u, v) = (e^x, e^{-x})$
4. $(u, v) = (e^x, e^{3x})$
5. $(u, v) = (e^x, e^{3x})$
6. $(u, v) = (1 + x + x^3, 1 - x - x^3)$
7. $(u, v) = (e^x + e^{2x}, e^x - e^{2x})$
8. $(u, v) = (1 + x^2 + x^4, 1 - x^2 - x^4)$

Exercises 17.2.1

1. $u(x) = \pm x$
2. $u(x) = \pm(1 + x)$
3. $u(x) = x^{\frac{2}{3}}$
4. $u(x) = 1 + \sqrt{x}$
5. $u(x) = \cos(x + 1)$
6. $u(x) = e^{x+1}$
7. $u(x) = 1 + \ln(x + 1)$
8. $u(x) = \dfrac{1}{2}\ln(\pi x)$

Exercises 17.3.1

1. $u(x) = \pm\sqrt{x}$
2. $u(x) = \pm(1 + x)$
3. $u(x) = x^{\frac{1}{3}}$
4. $u(x) = 1 - x$
5. $u(x) = \cos x$
6. $u(x) = \ln(1 + x)$

7. $u(x) = e^{1-x}$
8. $u(x) = \sinh(\pi + x)$
9. $u(x) = \cos^{\frac{1}{3}} x$
10. $u(x) = x^2$
11. $u(x) = 1 + x$
12. $u(x) = 1 + x^2$
13. $u(x) = e^{-\cos x}$
14. $u(x) = \ln(\cos x)$
15. $u(x) = \sinh(\pi + x)$
16. $u(x) = 3\ln x$

Exercises 17.4.1

1. $u(x) = 1 - x$
2. $u(x) = x^{\frac{1}{4}}$
3. $u(x) = \sin^{\frac{1}{3}} x$
4. $u(x) = \cos^{\frac{1}{4}} x$
5. $u(x) = e^{\frac{1}{2}x}$
6. $u(x) = e^{2x}$
7. $u(x) = (\ln x)^2$
8. $u(x) = (x + x^2)^4$

Exercises 17.5.1

1. $(u, v) = (x, x^2)$
2. $(u, v) = (x, x^2)$
3. $(u, v) = (e^x, e^{\frac{1}{2}x})$
4. $(u, v) = (e^x, e^{-x})$
5. $(u, v) = (\cos^{\frac{1}{2}} x, \sin^{\frac{1}{2}} x)$
6. $(u, v) = (\cos x, -\cos x)$
7. $(u, v, w) = (x, x^2, x^3)$
8. $(u, v, w) = (x, x^2, x^3)$

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