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Miroslav Bulíček, Eduard Feireisl
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New Trends and Results in Mathematical Description of Fluid Flows

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Miroslav Bulíček • Eduard Feireisl • Milan Pokorný
Editors

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Editors
Miroslav Bulíček
Faculty of Mathematics and Physics
Charles University
Prague, Czech Republic

Eduard Feireisl
Institute of Mathematics
Czech Academy of Sciences
Prague, Czech Republic

Milan Pokorný
Faculty of Mathematics and Physics
Charles University
Prague, Czech Republic

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Preface

This special volume consists of four surveys that are focussed on several aspects in fluid dynamics. The basis for these surveys were series of lectures delivered by Dominic Breit (Heriot-Watt University Edinburgh, United Kingdom), Yann Brenier (Ecole Polytechnique, Palaiseau, France), Pierre-Emmanuel Jabin (University of Maryland, USA) and Christian Rohde (Universität Stuttgart, Germany) at the EMS School in Applied Mathematics *Mathematical Aspects of Fluid Flows* held at Kácov, Czech Republic, May 28–June 2, 2017.

The four surveys cover the following subjects: stochastic partial differential equations in fluid mechanics, different concepts of solutions in ideal fluid mechanics connected with the least action principle, new views on the theory of the transport and continuity equations in connection with the compressible Navier–Stokes equations and related models, and modelling, mathematical and numerical analysis of models as well as the numerical solution of equations describing multi-phase fluid flows.

The main objective of the series of the Kácov Schools is to present new, modern methods, tools and results in the mathematical theory of compressible and incompressible fluids, and more complex systems of partial differential equations. Such a goal must be, however, motivated by studying physically relevant problems. In particular, the rigorous analytical results should lead to good understanding of the behaviour of the model, should predict its limitations and should indicate the kind of numerical method that could be used in order to solve the problem in a stable, accurate and efficient manner. Hand-in-hand with these mathematical properties, the analysis of the models should also lead to the relevant qualitative predictions of the model that are compatible with the physical expectations/experiments. Furthermore the model must obey basic physical principles, such as energy conservation or an entropy inequality. We believe that each of the four surveys fulfills these requirements and presents the most recent results and points of view, and addresses important problems in the mathematical theory of fluids.

One of the popular recent directions is to include suitable small perturbations into the models, which may be of numerical or empirical nature, or may originate in physical uncertainties (such as thermodynamic fluctuations occurring in fluid flows). This issue is discussed in the first part of the volume authored by Dominic Breit. He explains how the classical (deterministic) equations describing fluid flow can be modified to include stochastic effects, leading to stochastic partial

differential equations, where all of the quantities involved (such as the density, the velocity and the temperature) are defined on a probability space, and he shows how to include an appropriate Wiener process into the equations. In the introduction the author presents a nice overview of the methods used in the theory of stochastic PDEs. In the rest of the survey, the author discusses two notions of solution: a deterministic equation with random coefficients (semi-deterministic approach) and a fully stochastic problem (finite energy weak martingale solutions), and he highlights the strong and the weak points of both approaches. The available theory for both incompressible and compressible fluids is described in detail. Finally, the main steps of the existence proofs are presented.

Yann Brenier discusses various concepts of generalized and approximate solutions in the mathematical theory of ideal (inviscid) incompressible fluids. The central concept is the relation between solutions of the fluid system in the Eulerian description given by the standard Euler system of partial differential equations and a variational approach based on the least action principle. The least action principle characterizes the admissible trajectories as those minimizing the action functional. It is shown that smooth solutions of the Euler system when restricted to a sufficiently short time interval comply with the least action principle, while they represent critical points of the latter when the time lap is large. Furthermore, various concepts of generalized solutions to the least action principle are introduced and solvability of the latter in this new framework is discussed. Finally, the approximation of the Euler equations, based on a modified least action principle taking into account the energy dissipation, is considered. The intimate relationship of this concept to stochastic and quantum phenomena is discussed in the final part of the chapter.

The next part of the volume is the survey written by Didier Bresch and Pierre-Emmanuel Jabin and is focussed on the regularity and qualitative theory and estimates for the advection equation driven by a nonsmooth velocity field. The authors present a nice overview of the available results, but also show the essential tools and methods used in the most recent results. These are then used to obtain new results for complex systems where the transport equation is coupled to other PDE's, e.g., to the compressible Navier–Stokes system. The aim is also to impose relevant and available estimates for the velocity field or for the unknown, i.e., one cannot impose a bound on the divergence of the velocity or upper or lower bounds on the unknown itself. The whole theory is therefore built for situations when the velocity belongs to a certain Bochner–Sobolev space, which is the natural candidate appearing in the theory of compressible fluids. The survey is split into three main parts: the first one deals with the Lagrangian approach and the logarithmic scale for advective equations; the second part concerns the Eulerian description of the problem and takes into account the power of the technique of renormalized solutions to deduce compactness results for the compressible Navier–Stokes system; the last part provides a beautiful example of how the method introduced by the authors can be applied to the Stokes system coupled with a nonmonotone pressure law.

The last survey, authored by Christian Rohde, ranges from mathematical modelling to sophisticated numerical methods for multi-phase compressible flows. He considers the compressible free flow of homogeneous fluids that occur in liquid and vapour phases and he aims to describe also phase-change phenomena. Two methods in particular are discussed in the survey. First, models which display the phase boundary as a sharp interface are discussed, and second, diffuse interphase models are considered. For both classes of models the associated thermodynamic framework is set up, the applicability of the model is discussed, and finally a suitable numerical scheme is introduced. The most developed among these is the isothermal sharp interphase model, which is understood as a free boundary value problem with appropriate coupling conditions across the interface. These conditions are of the form of an algebraic constitutive relation and the choices of kinetic relations, leading to consistency, well-posedness and a thermodynamic theory, are discussed. Ultimately, these results appear then to be only a weak basis for (multi-dimensional) numerics but it turns out that an appropriate Riemann solver is a key for the construction of moving-mesh finite volume methods for the thermodynamically consistent tracking of interfaces with mass transfer. The sharp interphase model is not able to describe topological changes of the interface. This issue is then solved by using the Navier–Stokes–Korteweg model, where the free energy functional is extended by higher-order terms. The stress tensor is then changed accordingly in order to retain thermodynamical consistency. Finally, a numerical discretization that is able to deal with higher-order and nonlocal terms in the stress tensor is proposed, with a focus on the validity of the entropy inequality also on the level of the numerical approximation.

In conclusion, we are grateful to all the lecturers, the authors and also the participants for their effort and enthusiasm that led to top quality surveys in the field of mathematical theory for fluid flows on the one hand, and also to the beautiful, friendly and inspiring atmosphere during the school at Kácov. We would also like to use this opportunity to invite all interested researchers to the next, already sixteenth, school that will be held again at Kácov in 2019. Further details will be available from the webpage http://essam-maff.cuni.cz, where all links to lecture notes as well as to the history of the School can be found.

Miroslav Bulíček
Eduard Feireisl
Milan Pokorný

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An Introduction to Stochastic Navier–Stokes Equations

Dominic Breit

Abstract. The dynamics of liquids and gases can be modeled by the Navier–Stokes system of partial differential equations describing the balance of mass and momentum in the fluid flow. In recent years their has been an increasing interest in random influences on the fluid motion modeled via stochastic partial differential equations.

In this lecture notes we study the existence of weak martingale solutions to the stochastic Navier-Stokes equations (both incompressible and compressible). These solutions are weak in the analytical sense (derivatives exists only in the sense of distributions) and weak in the stochastic sense (the underlying probability space is not a priori given but part of the problem). In particular, we give a detailed introduction to the stochastic compactness method based on Skorokhod's representation theorem.

Mathematics Subject Classification (2010). 60H15, 35R60, 76D05, 76N10, 35Q30.

Keywords. Stochastic Navier–Stokes equations, stochastic PDEs, martingale solutions, weak solutions, compressible fluids.

1. Introduction

In continuum mechanics the motion of an isentropic fluid in a physical domain $\mathcal{O} \subset \mathbb{R}^d$, $d = 2, 3$, is described by the conservation of mass and momentum respectively. Stochastic components in the equations of motions are commonly used to model small perturbations (numerical, empirical, and physical uncertainties) or thermodynamic fluctuations present in fluid flows. Moreover, it is used for a better understanding of turbulence. As a consequence stochastic partial differential equations (SPDEs) such as the stochastic Navier–Stokes equations are gaining more and more interest in fluid mechanical research. All involved quantities are defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$. The basic model for homogeneous

incompressible fluids reads as:

$$\begin{cases} \mathrm{d}(\varrho \mathbf{v}) = \operatorname{div} \mathbf{S}\,\mathrm{d}t - \operatorname{div}(\varrho \mathbf{v} \otimes \mathbf{v})\,\mathrm{d}t - \nabla p\,\mathrm{d}t + \mathbb{G}(\mathbf{v})\mathrm{d}W_t & \text{in } Q, \\ \operatorname{div} \mathbf{v} = 0 & \text{in } Q, \end{cases} \tag{1.1}$$

$\mathbb{P}$-a.s. subject to initial and boundary conditions. Here the unknown quantities are the velocity field $\mathbf{v} : \Omega \times Q \to \mathbb{R}^d$, $Q := (0,T) \times \mathcal{O}$, and the pressure $p : \Omega \times Q \to \mathbb{R}$. The function $\mathbf{S} : \Omega \times Q \to \mathbb{R}^{d \times d}_{\mathrm{sym}}$ is the stress deviator and $\varrho > 0$ is the constant density of the fluid. The quantity $\mathbf{W}$ in $(1.1)_1$ is a (possibly infinite-dimensional) $(\mathcal{F}_t)$-Wiener process and the coefficient $\mathbb{G}(\mathbf{v})$ grows linearly in $\mathbf{v}$. The precise assumptions are given in Section 3. For the stress deviator we assume Newton's rheological law

$$\mathbf{S} = \mathbf{S}(\nabla \mathbf{v}) = \nu \left(\nabla \mathbf{v} + \nabla^t \mathbf{v} - \frac{2}{3} \operatorname{div} \mathbf{v}\, \mathbf{I}_d \right) + \lambda \operatorname{div} \mathbf{v}\, \mathbf{I}_d, \tag{1.2}$$

where $\nu > 0$, $\lambda \geq 0$ are constant viscosity coefficients. Under the incompressibility assumption $\operatorname{div} \mathbf{v} = 0$ relation (4.3) reduces to

$$\mathbf{S} = \mathbf{S}(\nabla \mathbf{v}) = \nu \left(\nabla \mathbf{v} + \nabla^t \mathbf{v} \right) \tag{1.3}$$

and we have $\operatorname{div} \mathbf{S} = \nu \Delta \mathbf{v}$. Let us finally remark that $(1.1)_1$ has to be understood as an abbreviation for a stochastic integral equation, where we can give a rigorous meaning to the stochastic integral $\int_0^t \mathbb{G}(\mathbf{v})\mathrm{d}W$, details are given in Sections 2.2 and 2.5.

Equation (1.1) has a long history starting with the pioneering work of Bensoussan and Temam [3]. In [3] equation $(1.1)_1$ is transformed by introducing the new variable $e^{-W}\mathbf{u}$. This leads to a deterministic equation with random coefficients. It can be solved pathwise for fixed $\omega \in \Omega$. A theorem on measurable selection allows to construct a random variable, i.e., a measurable mapping from $(\Omega, \mathcal{F})$ to an appropriate function space. This semi-deterministic approach has two drawbacks. First of all, the solution is not progressively measurable and hence the stochastic integral $\int_0^t \mathbb{G}(\mathbf{v})\mathrm{d}W$ cannot be defined. Moreover, it is not possible to control the energy and study the asymptotic behavior of (1.1).

A fully stochastic theory for equation (1.1) has been developed by Flandoli and Gątarek in [17]. After finding a suitable approximation one has to perform the limit procedure in the convective term and in the stochastic integral. There is a significant difference in comparison to the deterministic situation leading to the concept of martingale solutions: In general it is not possible to get any compactness in $\omega \in \Omega$ as no topological structure on the sample space Ω is assumed. To overcome this difficulty, it is classical to rather concentrate on compactness of the set of laws of the approximations and apply the Skorokhod representation theorem (see Lemma 2.6.2). It yields existence of a new probability space with a sequence of random variables that have the same laws as the original ones and that in addition converges almost surely. The result of [17] is the existence of a weak martingale solution to (1.1). These solutions are weak in the analytical sense (derivatives have to be understood in the sense of distributions) and weak in the stochastic

sense (the underlying probability space is an integral part of the solution and not a priori known). Strong solutions in the stochastic results can only be expected when uniqueness holds (even in the case of stochastic ODEs, see Section 2.4 for further details). Unfortunately, this seems to be out of reach for the three-dimensional Navier–Stokes equations.

Today there exists an abundant amount of literature concerning the dynamics of incompressible fluids driven by stochastic forcing. We refer to the lecture notes by Flandoli [16], the monograph of Kuksin and Shyrikian [25], the recent survey by Romito [32] as well as the references cited therein for a recent overview. Definitely much less is known if compressibility of the fluid is taken into account. In the compressible case, when the assumption of a constant density is not appropriate anymore (this assumption is always an idealization), the balance of momentum and mass read as

$$\begin{cases} \mathrm{d}(\varrho\mathbf{v}) = \operatorname{div}\mathbf{S}\,\mathrm{d}t - \operatorname{div}(\varrho\mathbf{v}\otimes\mathbf{v})\,\mathrm{d}t - \nabla p(\varrho)\,\mathrm{d}t + \mathbb{G}(\varrho,\varrho\mathbf{v})\mathrm{d}W_t & \text{in } Q, \\ \mathrm{d}\varrho + \operatorname{div}(\varrho\mathbf{v})\,\mathrm{d}t = 0 & \text{in } Q, \end{cases} \tag{1.4}$$

where $\mathbf{S}(\nabla\mathbf{v})$ is given by (1.2). The unknowns are the velocity field $\mathbf{v} : \Omega\times Q \to \mathbb{R}^d$ and the density $\varrho : \Omega \times Q \to \mathbb{R}$. The noise coefficient $\mathbb{G}(\varrho, \varrho\mathbf{v})$ grows linearly in ϱ and $\varrho\mathbf{v}$. Equation $(1.4)_1$ must be supplemented with a constitutive law which relates the pressure to the density. The common assumption is the γ-law $p(\varrho) = a\varrho^\gamma$ with $a > 0$ and $\gamma > 1$.

First existence results for (1.4) were based on a suitable transformation formula that allows to reduce the problem to a random system of PDEs. Similar to the results for incompressible fluids from [3] Feireisl, Maslowski & Novotný [14] propose a solution to (1.4) by means of a semi-deterministic approach. The first "truly" stochastic existence result for the compressible Navier–Stokes system perturbed by a general nonlinear multiplicative noise was obtained by Breit and Hofmanová [8]. The existence of the so-called finite energy weak martingale solutions in three space dimensions with periodic boundary conditions was established. These solutions are weak in both senses and the time evolution of the system can be controlled in terms of the initial energy. The proof is based on a four layer approximation scheme together with a refined stochastic compactness method and a careful identification of the limit procedure. In order to deal with weak topologies of Banach spaces – typical for the compressible Navier–Stokes system – Jakubowski's extension of the classical Skorokhod representation theorem (see Lemma 2.6.3) comes into play. Extension of the results from [8] to the zero Dirichlet boundary conditions then appeared in [34].

The lecture notes are organized as follows. In the next section we present basic material on stochastic analysis. In particular, we explain the construction of the stochastic integral and present the stochastic compactness method for SDEs. In Section 3 we consider stochastic Navier–Stokes equations and show the existence of martingale solutions. Section 4 is concerned with stochastically forced compressible fluid flows.

2. Preliminaries

2.1. Stochastic processes

We consider random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with associated filtration $(\mathcal{F}_t)_{t\geq 0}$. A real-valued *stochastic process* is a set of random variables $X = (X_t)_{t\geq 0}$ on $(\Omega, \mathcal{F})$ with values in $(\mathbb{R}, \mathfrak{B}(\mathbb{R}))$, where $\mathfrak{B}$ denotes the Borelian σ-algebra. A stochastic process can be interpreted as a function of t and ω, where t can be interpreted as time. For fixed $\omega \in \Omega$ the mapping $t \mapsto X_t(\omega)$ is called *path* or *trajectory* of X. We follow the presentation from [24] where the interested reader may also find details of the proofs.

Definition 2.1.1. A stochastic process X is called *measurable*, if the mapping

$$(t, \omega) \mapsto X_t(\omega) : \ ([0, \infty) \times \Omega, \mathfrak{B}([0, \infty)) \otimes \mathcal{F}) \to (\mathbb{R}, \mathfrak{B}(\mathbb{R}))$$

is measurable.

Definition 2.1.2. A stochastic process X is called *adapted* to the filtration $(\mathcal{F}_t)_{t\geq 0}$, if the mapping

$$\omega \mapsto X_t(\omega) : \ (\Omega, \mathcal{F}_t) \to (\mathbb{R}, \mathfrak{B}(\mathbb{R}))$$

is measurable for all $t \geq 0$.

Definition 2.1.3. A stochastic process X is called *progressively measurable*, if the mapping

$$(s, \omega) \mapsto X_s(\omega) : \ ([0, t] \times \Omega, \mathfrak{B}([0, t]) \otimes \mathcal{F}_t) \to (\mathbb{R}, \mathfrak{B}(\mathbb{R}))$$

is measurable for all $t \geq 0$.

Remark 2.1.1. Progressive measurability implies adaptedness.

Theorem 2.1.2. *If a stochastic process X is adapted to the filtration $(\mathcal{F}_t)_{t\geq 0}$ and a.e. path is left-continuous or right-continuous, then X is progressively measurable.*

The most important process is the Wiener process.

Definition 2.1.4 (Wiener process). A Wiener process is a real-valued stochastic process $W = (W_t)_{t\geq 0}$ with the following properties.

i) The increments of W are independent, i.e., for arbitrary $0 \leq t_0 < t_1 < \cdots < t_n$ the random variables $W_{t_1} - W_{t_0}, W_{t_2} - W_{t_1}, \ldots, W_{t_n} - W_{t_{n-1}}$ are independent.
ii) For all $t > s \geq 0$ we have $W_t - W_s \sim \mathcal{N}(0, t - s)$.
iii) There holds $W_0 = 0$ almost surely.
iv) The mapping $t \mapsto W_t(\omega)$ is continuous for a.e. $\omega \in \Omega$.

Definition 2.1.5. A filtration $(\mathcal{F}_t)_{t\geq 0}$ is called *right-continuous*, if

$$\mathcal{F}_t = \bigcap_{\varepsilon > 0} \mathcal{F}_{t+\varepsilon} \quad \forall t \geq 0$$

and *left-continuous*, if

$$\mathcal{F}_t = \bigcup_{s<t} \mathcal{F}_s \quad \forall t > 0.$$

Definition 2.1.6. A filtration $(\mathcal{F}_t)_{t\geq 0}$ satisfies the *usual conditions*, if it is right-continuous and $\mathcal{F}_0$ contains all the $\mathbb{P}$-nullsets of $\mathcal{F}$.

Definition 2.1.7. Let X be a stochastic process and let $\mathcal{T}$ be an $\mathcal{F}$-measurable random variable with values in $[0,\infty]$. We define $X_{\mathcal{T}}$ in $\{\mathcal{T}<\infty\}$ by

$$X_{\mathcal{T}}(\omega) = X_{\mathcal{T}(\omega)}(\omega).$$

If $X_\infty(\omega) = \lim\limits_{t\to\infty} X_t(\omega)$ exists for a.e. $\omega\in\Omega$ we set $X_{\mathcal{T}}(\omega) = X_\infty(\omega)$ in $\{\mathcal{T}=\infty\}$.

Definition 2.1.8. Let X be a stochastic process on a probability space $(\Omega,\mathcal{F},\mathbb{P})$ with filtration $(\mathcal{F}_t)_{t\geq 0}$. A random variable $\mathcal{T}$ is called a *stopping time* if the set $\{\mathcal{T}\leq t\}$ belongs to $\mathcal{F}_t$ for all $t\geq 0$.
The by T induced σ-algebra $\mathcal{F}_{\mathcal{T}}$ is given by

$$\mathcal{F}_{\mathcal{T}} := \{A\in\mathcal{F} : A\cap\{\mathcal{T}\leq t\}\in\mathcal{F}_t\ \forall t\geq 0\}.$$

Definition 2.1.9. Let $M=(M_t)_{t\geq 0}$ be an $(\mathcal{F}_t)_{t\geq 0}$-adapted stochastic process with $\mathbb{E}[|M_t|]<\infty$ for all $t\geq 0$. M is called a sub-martingale (super-martingale, respectively) if we have for all $0\leq s\leq t<\infty$ that $\mathbb{P}$-a.s. $\mathbb{E}[M_t|\mathcal{F}_s]\geq M_s$ ($\mathbb{E}[M_t|\mathcal{F}_s]\leq M_s$, respectively). M is called a martingale if it is a sub-martingale and a super-martingale.[1]

Definition 2.1.10. Let A be an adapted stochastic process. A is called *increasing*, if we have for $\mathbb{P}$-a.e. $\omega\in\Omega$

i) $A_0=0$;
ii) $t\mapsto A_t(\omega)$ is increasing and right-continuous;
iii) $\mathbb{E}[A_t]<\infty$ for all $t\in[0,\infty)$.

An increasing process is called integrable if

$$\mathbb{E}[A_\infty]<\infty,\ \text{where } A_\infty(\omega)=\lim_{t\to\infty}A_t(\omega) \text{ for } \omega\in\Omega.$$

Theorem 2.1.3 (Doob–Meyer decomposition, [24] (Thm. 4.10, p. 24)). *Let Y be a nonnegative sub-martingale with a.s. continuous trajectories. There is a continuous martingale M and a $\mathbb{P}$-a.s. increasing continuous and adapted process A, such that*

$$Y_t = M_t + A_t.$$

The decomposition is unique.

Definition 2.1.11. Let X be a right-continuous martingale. We call X *quadratically integrable*, if $\mathbb{E}\left[X_t^2\right]<\infty$, for all $t\geq 0$. If we have in addition that $X_0=0$ $\mathbb{P}$-a.s., we write $X\in\mathscr{M}_2$ or $X\in\mathscr{M}_2^c$, if X is continuous.

Definition 2.1.12. For $X\in\mathscr{M}_2$ and $0\leq t<\infty$ we define

$$\|X\|_{\mathscr{M}_2,t} := \sqrt{\mathbb{E}[X_t^2]}.$$

[1] $\mathbb{E}$ denotes the expectation with respect to the probability measure $\mathbb{P}$.

We set

$$\|X\|_{\mathscr{M}_2} := \sum_{n=1}^{\infty} \frac{\|X\|_{\mathscr{M}_2,n} \wedge 1}{2^n}$$

which induces a metric $d_{\mathscr{M}_2}$ on $\mathscr{M}_2$ given by

$$d_{\mathscr{M}_2}(X,Y) = \|X-Y\|_{\mathscr{M}_2}.$$

Definition 2.1.13 (Total variation). A function $f : [0,t] \to \mathbb{R}$ is called of bounded variation if there is $M > 0$ such that $\sum_{i=1}^{n} |f(t_i) - f(t_{i-1})| \leq M$ for all finite partitions $\Pi = \{t_0, t_1, \dots, t_n\} \subset [0,t]$ $(n \in \mathbb{N})$ with $0 = t_0 < t_1 < \cdots < t_n = t$. The quantity

$$V(f) := \sup\left\{ \sum_{i=1}^{n} |f(t_i) - f(t_{i-1})| : 0 = t_0 < t_1 < \cdots < t_n = t, n \in \mathbb{N} \right\}$$

is called total variation of f over $[0,t]$.

It is easy to show that, for $X \in \mathscr{M}_2$, the process $X^2 = (X_t^2)_{t\geq 0}$ is a non-negative submartingale. Hence we can apply Theorem 2.1.3 to $Y = X^2$ and the following definition makes sense.

Definition 2.1.14 (Quadratic variation). For $X \in \mathscr{M}_2$ we define the *quadratic variation* of X, as the process $\langle\langle X \rangle\rangle := A$, where A is the increasing process from the Doob–Meyer decomposition of X^2.

Definition 2.1.15 (Covariation). For $X, Y \in \mathscr{M}_2$ we define the *covariation* $\langle\langle X,Y \rangle\rangle$ by

$$\langle\langle X,Y \rangle\rangle_t := \frac{1}{4}\left[\langle\langle X+Y \rangle\rangle_t - \langle\langle X-Y \rangle\rangle_t\right],$$

for $t \geq 0$. The process $XY - \langle\langle X,Y \rangle\rangle$ is a martingale. In particular, we have $\langle\langle X,X \rangle\rangle = \langle\langle X \rangle\rangle$.

Definition 2.1.16 (*p*th Variation). Let X be a stochastic process, $p \geq 1$, $t > 0$ fixed and $\Pi = \{t_0, t_1, \dots, t_n\}$, with $0 = t_0 < t_1 < \cdots < t_n = t$, $n \in \mathbb{N}$, a partition of $[0,t]$. The *pth variation* of X in Π is defined by

$$V_t^{(p)}(\Pi) = \sum_{k=0}^{n} |X_{t_k} - X_{t_{k-1}}|^p.$$

Theorem 2.1.4 ([24] (Thm. 5.8, p. 32)).
Let $X \in \mathscr{M}_2^c$, Π be a partition of $[0,t]$ and $\|\Pi\| := \max_{1\leq k\leq n} |t_k - t_{k-1}|$ the size of Π. Then we have $\lim_{\|\Pi\|\to 0} V_t^{(2)}(\Pi) = \langle\langle X \rangle\rangle_t$ in probability, i.e., for all $\varepsilon > 0$, $\eta > 0$ there is $\delta > 0$, such that we have that

$$\mathbb{P}\left(|V_t^{(2)}(\Pi) - \langle\langle X \rangle\rangle_t| > \varepsilon\right) < \eta,$$

for $\|\Pi\| < \delta$.

Very useful is also the Burkholder–Davis–Gundy inequality.

Lemma 2.1.5 ([24] (Thm. 3.28, p. 166)). *Let $X \in \mathscr{M}_2^c$ and $T > 0$. Then we have for all $p > 0$*

$$c_p\,\mathbb{E}\Big[\sup_{t\in(0,T)}|X_t|\Big]^p \le \mathbb{E}\Big[\langle\langle X,X\rangle\rangle_T\Big]^{p/2} \le C_p\,\mathbb{E}\Big[\sup_{t\in(0,T)}|X_t|\Big]^p,$$

where c_p, C_p are positive constants.

2.2. Stochastic integration

The aim of this section is to define stochastic integrals of the form

$$I_T(X) = \int_0^T X_t(\omega)\,\mathrm{d}M_t(\omega). \tag{2.1}$$

Here M is a square integrable martingale, X a stochastic process and $T > 0$. Throughout the section we assume that $M_0 = 0$ $\mathbb{P}$-a.s. Moreover, we suppose that M is $(\mathcal{F}_t)_{t\ge0}$-adapted, where $(\mathcal{F}_t)_{t\ge0}$ is a filtration which satisfies the usual conditions (see Definition 2.1.6). A process $M \in \mathscr{M}_2^c$ could be of unbounded variation in any finite subinterval of $[0,T]$. Hence integrals of the form (2.1) cannot be defined pointwise in $\omega \in \Omega$. However, M has finite quadratic variation given by the continuous and increasing process $\langle\langle M\rangle\rangle$ (see Theorem 2.1.4). Due to this fact, the stochastic integral can be defined with respect to continuous integrable martingales M for an appropriate class of integrands X.

The definition of the stochastic integral goes back to Itô. He studied the case where M is a Wiener process. His students Kunita and Watanabe considered the general case $M \in \mathscr{M}_2^c$. In the following we have a look at the class of integrands which are allowed in (2.1). We define a measure μ_M on $([0,\infty)\times\Omega, \mathfrak{B}([0,\infty))\otimes\mathcal{F})$ by

$$\mu_M(A) = \mathbb{E}\left[\int_0^\infty \mathbb{I}_A(t,\omega)\,\mathrm{d}\langle\langle M\rangle\rangle_t(\omega)\right] \quad \text{for} \quad A \in \mathfrak{B}([0,\infty))\otimes\mathcal{F}. \tag{2.2}$$

We call two $(\mathcal{F}_t)_{t\ge0}$-adapted stochastic processes $X = (X_t)_{t\ge0}$ and $Y = (Y_t)_{t\ge0}$ *equivalent with respect to* M, if $X_t(\omega) = Y_t(\omega)$ μ_M-a.e. This leads to the following equivalence relation: for an $(\mathcal{F}_t)_{t\ge0}$-adapted process X we define

$$[X]_T^2 := \mathbb{E}\left[\int_0^T X_t^2(\omega)\,\mathrm{d}\langle\langle M\rangle\rangle_t(\omega)\right], \tag{2.3}$$

provided the right-hand side exists. So $[X]_T$ is the L^2-norm of X as a function of (t,ω) with respect to the measure μ_M. We define the equivalence relation

$$X \sim Y \Leftrightarrow [X-Y]_T = 0 \quad \forall T > 0. \tag{2.4}$$

Our definition of the stochastic integral will imply that $I(X)$ and $I(Y)$ coincide provided X and Y are equivalent.

Definition 2.2.1. We define $\mathcal{L}^*$ as the space of equivalence classes of progressively $(\mathcal{F}_t)_{t\ge0}$-measurable processes X with $[X]_T < \infty$ for all $T > 0$.

Remark 2.2.1. By setting $[X-Y] := \sum_{n=0}^\infty 2^{-n}(1 \wedge [X-Y]_n)$ we can define a metric on $\mathcal{L}^*$.

Remark 2.2.2. In the following we do not distinguish between X and the equivalence class X^* of X.

For $0 < T < \infty$ we define $\mathcal{L}_T^*$ as the space of processes $X \in \mathcal{L}^*$ with $X_t(\omega) = 0$ for all $t \geq T$ and a.e. $\omega \in \Omega$ and set

$$\mathcal{L}_\infty := \bigg\{ X \in \mathcal{L}^* : \mathbb{E}\bigg[\int_0^\infty X_t^2 \, \mathrm{d}\langle\langle M \rangle\rangle_t \bigg] < \infty \bigg\}.$$

A process $X \in \mathcal{L}_T^*$ can be identified with a process only defined on $\Omega \times [0,T]$. In particular we have that $\mathcal{L}_T^*$ is a closed subspace of the Hilbert space

$$\mathscr{H}_T := L^2(\Omega \times (0,T), \mathcal{F}_T \otimes \mathfrak{B}([0,T]), \mu_M). \tag{2.5}$$

Definition 2.2.2. A process X is called *step process* if there is a strictly increasing sequence $(t_n)_{n\in\mathbb{N}} \subset \mathbb{R}$ with $t_0 = 0$ and $\lim_{n\to\infty} t_n = \infty$, a sequence of random variables $(\xi_n)_{n\in\mathbb{N}_0}$ and $C < \infty$ with $\sup_{n\in\mathbb{N}_0} |\xi_n(\omega)| \leq C$ such that the following holds: we have that ξ_n is $\mathcal{F}_{t_n}$-measurable for every $n \in \mathbb{N}_0$ and we have the representation

$$X_t(\omega) = \xi_0(\omega)\mathbb{I}_{\{0\}}(t) + \sum_{i=0}^{\infty} \xi_i(\omega)\mathbb{I}_{(t_i, t_i+1]}(t), \tag{2.6}$$

for all $0 \leq t < \infty$. The space of step processes is denoted by $\mathcal{L}_0$.

Remark 2.2.3. (i) Step processes are progressively measurable and bounded.
(ii) There holds $\mathcal{L}_0 \subseteq \mathcal{L}^*$.

Definition 2.2.3. Let $X \in \mathcal{L}_0$ and $M \in \mathscr{M}_2^c$. The stochastic integral of X with respect to M is the *martingale transformation*

$$I_t(X) := \sum_{i=0}^{n-1} \xi_i(M_{t_{i+1}} - M_{t_i}) + \xi_n(M_t - M_{t_n}) = \sum_{i=0}^{\infty} \xi_i(M_{t\wedge t_{i+1}} - M_{t\wedge t_i}), \tag{2.7}$$

for $0 \leq t < \infty$. Here $n = n(t) \in \mathbb{N}$ is the unique natural number such that $t_n \leq t < t_{n+1}$.

In order to define the stochastic integral for $X \in \mathcal{L}^*$ we have to approximate the elements of $\mathcal{L}^*$ in an appropriate way by step processes, i.e., by processes from $\mathcal{L}_0$. This can be done thanks to the following theorem.

Theorem 2.2.4 ([24] (Prop. 2.8, p. 137)). *The space of step processes $\mathcal{L}_0$ is dense in $\mathcal{L}^*$ with respect to the metric defined in Remark* 2.2.1.

Definition 2.2.4. Let $X \in \mathcal{L}^*$ and $M \in \mathscr{M}_2^c$. The stochastic integral of X with respect to M is the unique quadratically integrable martingale

$$I(X) = \{I_t(X), (\mathcal{F}_t)_{t\geq 0}, 0 \leq t < \infty\},$$

which satisfies $\lim_{n\to\infty} \|I(X^{(n)}) - I(X)\|_{\mathscr{M}_2} = 0$, for every sequence $(X^{(n)})_{n\in\mathbb{N}} \subseteq \mathcal{L}_0$ with $\lim_{n\to\infty} [X^{(n)} - X] = 0$. We write

$$I_t(X) = \int_0^t X_s \, \mathrm{d}M_s; \ 0 \le t < \infty.$$

The stochastic integral, as defined in Definition 2.2.4, has the following elementary properties.

Theorem 2.2.5 ([24] (pp. 137–141)). *Let $X, Y \in \mathcal{L}^*$ and $0 \le s < t < \infty$. For the stochastic integrals $I(X), I(Y)$ we have*

a) $I_0(X) = 0$ $\mathbb{P}$-*a.s.*
b) $\mathbb{E}[I_t(X)|\mathcal{F}_s] = I_s(X)$ $\mathbb{P}$-*a.s.* (*martingale property*)
c) $\mathbb{E}\left[(I_t(X))^2\right] = \mathbb{E}\left[\int_0^t X_u^2 \, \mathrm{d}\langle\langle M\rangle\rangle_u\right]$ (*Itô-isometry*)
d) $\|I(X)\|_{\mathscr{M}_2} = [X]$
e) $\mathbb{E}\left[(I_t(X) - I_s(X))^2|\mathcal{F}_s\right] = \mathbb{E}\left[\int_s^t X_u^2 \ \mathrm{d}\langle\langle M\rangle\rangle_u|\mathcal{F}_s\right]$ $\mathbb{P}$-*a.s.*
f) $I(\alpha X + \beta Y) = \alpha I(X) + \beta I(Y)$, *for* $\alpha, \beta \in \mathbb{R}$.

2.3. Itô's Lemma

One of the most important tools in stochastic analysis is Itô's Lemma. It is a chain-rule for paths of stochastic processes. In contrast to the deterministic case it can only be interpreted as an integral equation because the stochastic processes we are interested in (for instance the Wiener process) are in general not differentiable.

Definition 2.3.1 (Continuous local martingale). Let X be a continuous process adapted to $(\mathcal{F}_t)_{t\ge0}$. Assume that there is a sequence of stopping times $(\mathcal{T}_n)_{n\in\mathbb{N}}$ of the filtration $(\mathcal{F}_t)_{t\ge0}$, such that $(X_t^{(n)} := X_{t\wedge\mathcal{T}_n})_{t\ge0}$ is an $(\mathcal{F}_t)_{t\ge0}$-martingale for all $n \in \mathbb{N}$ and $\mathbb{P}(\lim_{n\to\infty} \mathcal{T}_n = \infty) = 1$. In this case we call X a *continuous local martingale*. If, in addition, $X_0 = 0$ $\mathbb{P}$-a.s., we write $X \in \mathscr{M}_2^{c,\mathrm{loc}}$.

Definition 2.3.2 (Continuous semi-martingale). A *continuous semi-martingale* X is a $(\mathcal{F}_t)_{t\ge0}$-adapted process such that the following (unique) decomposition holds:

$$X_t = X_0 + M_t + B_t, \quad 0 \le t < \infty. \tag{2.8}$$

In the above X_0 is an $\mathcal{F}_0$-measurable random variable, we have $M = (M_t)_{t\ge0} \in \mathscr{M}_2^{c,\mathrm{loc}}$ and $B = (B_t)_{t\ge0}$ is a continuous $(\mathcal{F}_t)_{t\ge0}$-adapted process such that a.s. $B(0) = 0$ and its paths are of bounded variation, i.e., we have $\mathbb{P}$-a.s.

$$\lim_{|\Pi(t)|\to0} \sum_{k=1}^{m} |B_{t_k} - B_{t_{k-1}}| < \infty,$$

for all $t < \infty$, where $\Pi(t) = \{0 = t_0 < t_1 < \cdots < t_m = t\}$ is a partition of $[0, t]$.

Theorem 2.3.1 (Itô's Lemma, [24] (Thm. 3.3, p. 149)). *Let $f : \mathbb{R} \to \mathbb{R}$ be a C^2-function and $(X_t)_{t\geq 0}$ be a continuous $(\mathcal{F}_t)_{t\geq 0}$ semi-martingale with the decomposition* (2.8)*. The following holds $\mathbb{P}$-a.s. for $0 \leq t < \infty$*

$$\begin{aligned} f(X_t) &= f(X_0) + \int_0^t f'(X_s)\,\mathrm{d}X_s + \frac{1}{2}\int_0^t f''(X_s)\,\mathrm{d}\langle\langle X\rangle\rangle_s \\ &= f(X_0) + \int_0^t f'(X_s)\,\mathrm{d}M_s + \int_0^t f'(X_s)\,\mathrm{d}B_s + \frac{1}{2}\int_0^t f''(X_s)\,\mathrm{d}\langle\langle M\rangle\rangle_s. \end{aligned} \tag{2.9}$$

Remark 2.3.2. The stochastic integral $\int_0^t f'(X_s)\,\mathrm{d}M_s$ in (2.9) is a continuous local martingale. The other two integrals in (2.9) are Lebesgue–Stieltjes integrals. They are of bounded variation as a function of t. Due to this $(f(X_t))_{t\geq 0}$ is a continuous $(\mathcal{F}_t)_{t\geq 0}$ semi-martingale.

Remark 2.3.3. Equation (2.9) is often written in differential form

$$\begin{aligned} \mathrm{d}f(X_t) &= f'(X_t)\,\mathrm{d}X_t + \frac{1}{2}f''(X_t)\,\mathrm{d}\langle\langle X\rangle\rangle_t \\ &= f'(X_t)\,\mathrm{d}M_t + f'(X_t)\,\mathrm{d}B_t + \frac{1}{2}f''(X_t)\,\mathrm{d}\langle\langle M\rangle\rangle_t. \end{aligned}$$

Note that this does not have a rigorous meaning. It only serves as an abbreviation for (2.9).

2.4. Stochastic ODEs

In this section we are concerned with stochastic differential equations. We seek a real-valued process $(X_t)_{t\in[0,T]}$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with filtration $(\mathcal{F}_t)_{t\geq 0}$ such that

$$\begin{cases} \mathrm{d}X_t &= \mu(t, X)\,\mathrm{d}t + \Sigma(t, X)\,\mathrm{d}W_t, \\ X(0) &= X_0, \end{cases} \tag{2.10}$$

which holds true $\mathbb{P}$-a.s. and for all $t \in [0, T]$. Here W is a Wiener process with respect to $(\mathcal{F}_t)_{t\geq 0}$. The functions $\mu, \Sigma : [0, T] \times \mathbb{R} \to \mathbb{R}$ are assumed to be continuous. As in Remark 2.3.3, equation $(2.10)_1$ is only an abbreviation for the integral equation

$$X(t) = X(0) + \int_0^t \mu(s, X(s))\,\mathrm{d}s + \int_0^t \Sigma(s, X(s))\,\mathrm{d}W_s. \tag{2.11}$$

There are two different concepts of solutions to (2.11).

i) We talk about a strong solution (in the probabilistic sense) if the solution exists on a given filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ with a given Wiener process W. A strong solution exists for a given initial datum X_0 (an $\mathcal{F}_0$-measurable random variable) and we have $X(0) = X_0$ a.s.

ii) We talk about a weak solution (in the probabilistic sense) or martingale solution if there is a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ and a Wiener

process such that (2.11) holds true. The solution is usually written as

$$((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}), W, X).$$

This means that when seeking a weak solution, constructing the probability space (and the Wiener process on it) is part of the problem. A solution typically exists for a given initial law Λ_0 (a Radon measure on $\mathbb{R}$) and we have $\mathbb{P} \circ X^{-1}(0) = \Lambda_0$. Even if an initial datum X_0 is given it might live on a different probability space. Hence $X(0)$ and X_0 can only coincide in law.

In the following we will have a look at existence results for (2.10).

Theorem 2.4.1. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with filtration $(\mathcal{F}_t)_{t\geq 0}$ and $X_0 \in L^2(\Omega, \mathcal{F}_0, \mathbb{P})$. Assume that μ and Σ are continuous on $[0,T]\times\mathbb{R}$ and globally Lipschitz continuous with respect to the second variable. Then there is a unique $(\mathcal{F}_t)_{t\geq 0}$-adapted process X such that (2.11) holds $\mathbb{P}$-a.s. for every $0 \leq t \leq T$ and we have $X(0) = X_0$ a.s. The trajectories of X are a.s. continuous and we have*

$$\mathbb{E}\left[\sup_{t\in(0,T)} |X_t|^2\right] < \infty.$$

The proof of Theorem 2.4.1 is classical and consists in finding a fixed point of the mapping

$$\begin{aligned}
&\mathfrak{T} : L^2(\Omega, C^0([0,T])) \to L^2(\Omega, C^0([0,T])),\\
&X \mapsto X_0 + \int_0^{\cdot} \mu(s, X_s)\,\mathrm{d}s + \int_0^{\cdot} \Sigma(s, X_s)\,\mathrm{d}W_s,
\end{aligned}$$

similar to the proof by Picard–Lindelöff for deterministic ODEs, see, e.g., [1] and [18, 19] for details. In the following we will derive a priori estimates for solutions to (2.11).

Corollary 2.4.2. *Let the assumptions of Theorem* 2.4.1 *be satisfied. Assume further that there is $K \geq 0$ such that*

$$|\mu(t,X)| + |\Sigma(t,X)| \leq K(1+|X|) \quad \forall (t,X) \in [0,T]\times\mathbb{R}. \tag{2.12}$$

Then we have

$$\mathbb{E}\left[\sup_{t\in(0,T)} |X_t|^2\right] \leq C\,\mathbb{E}\Big[|X_0|^2 + 1\Big],$$

where C only depends on T and K.

Remark 2.4.3. The linear growth assumed in Corollary 2.4.2 certainly follows from the Lipschitz continuity supposed in Theorem 2.4.1. However, the estimate in Corollary 2.4.2 only depends on the constant K in (2.12) but is independent of the Lipschitz constant.

Proof. We apply Itô's formula (Theorem 2.3.1) to the function $f(X) = |X|^2$ and obtain

$$\begin{aligned}|X_t|^2 &= |X_0|^2 + 2\int_0^t X_s \,\mathrm{d}X_s + \int_0^t \mathrm{d}\langle\langle X\rangle\rangle_s \\ &= |X_0|^2 + 2\int_0^t X_s \mu(s,X_s)\,\mathrm{d}s + 2\int_0^t X_s\,\Sigma(s,X_s)\,\mathrm{d}W + \int_0^t \Sigma^2(s,X_s)\,\mathrm{d}s \\ &\leq |X_0|^2 + C\bigg(\int_0^t |X_s|^2\,\mathrm{d}s + 1\bigg) + 2\int_0^t X_s\,\Sigma(s,X_s)\,\mathrm{d}W.\end{aligned}$$

By the Burgholder–Davis–Gundy inequality (Lemma 2.1.5 with $p = 1$) we have

$$\begin{aligned}\mathbb{E}\bigg[\sup_{t\in(0,T)}\Big|\int_0^t X_s\Sigma(s,X_s)\,\mathrm{d}W\Big|\bigg] &\leq C\,\mathbb{E}\bigg[\int_0^T |X_t|^2|\Sigma(t,X_t)|^2\,\mathrm{d}t\bigg]^{1/2} \\ &\leq C\,\mathbb{E}\bigg[\int_0^T |X_t|^4\,\mathrm{d}t + 1\bigg]^{1/2} \leq C\,\mathbb{E}\bigg[\sup_{t\in(0,T)}|X_t|^2\int_0^T |X_t|^2\,\mathrm{d}t + 1\bigg]^{1/2}.\end{aligned}$$

Finally, by Young's inequality we gain for every $\delta > 0$

$$\mathbb{E}\bigg[\sup_{t\in(0,T)}\Big|\int_0^t X_s\Sigma(s,X_s)\,\mathrm{d}W\Big|\bigg] \leq \delta\,\mathbb{E}\bigg[\sup_{t\in(0,T)}|X_t|^2\bigg] + C(\delta)\mathbb{E}\bigg(\int_0^T |X_t|^2\,\mathrm{d}t + 1\bigg).$$

Choosing δ small enough and applying Gronwall's lemma yields the claim. □

If the assumptions on the coefficients in Theorem 2.4.1 are weakened from Lipschitz continuity to just continuity, strong solutions might not exist, see [2]. In this case we can only hope for a weak solution. We refer to [21] for a nice proof and further references.

Theorem 2.4.4. *Let Λ_0 be a Borel probability measure on $\mathbb{R}$. Assume that μ and Σ are continuous in $[0,T]\times\mathbb{R}$ and have linear growth, i.e., there is $K \geq 0$ such that*

$$|\mu(t,X)| + |\Sigma(t,X)| \leq K(1+|X|) \quad \forall (t,X) \in [0,T]\times\mathbb{R}.$$

There is a quantity $((\Omega,\mathcal{F},(\mathcal{F}_t)_{t\geq 0},\mathbb{P}),W,X)$ (called martingale solution to (2.10)) with the following properties.

i) *$(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\geq 0},\mathbb{P})$ is a stochastic basis with a complete right-continuous filtration.*
ii) *W is an $(\mathcal{F}_t)_{t\geq 0}$-Wiener process on $(\Omega,\mathcal{F},\mathbb{P})$.*
iii) *X is an $(\mathcal{F}_t)_{t\geq 0}$-adapted stochastic process with a.s. continuous trajectories such that*

$$\mathbb{E}\bigg[\sup_{t\in(0,T)}|X_t|^2\bigg] < \infty.$$

iv) *Equation (2.11) holds $\mathbb{P}$-a.s. for every $0 \leq t \leq T$.*
v) *We have $\mathbb{P}\circ X(0)^{-1} = \Lambda_0$ (that is $\mathbb{P}(X(0)\in B) = \Lambda_0(B)$ for all $B \in \mathfrak{B}(\mathbb{R})$).*

Instead of giving a full proof of Theorem 2.4.4 let us show how to gain compactness of approximate solutions. Let (X_t^m) be a sequence of solutions to

$$\mathrm{d}X_t^m = \mu(t, X_t^m)\,\mathrm{d}t + \Sigma(t, X_t^m)\,\mathrm{d}W_t, \quad X^m(0) = X_0,$$

here X_0 is an $\mathcal{F}_0$-measurable random variable with law Λ_0. Assume further there is $C > 0$ such that we have

$$\mathbb{E}\Big[\sup_{t\in(0,T)} |X_t^m|^p\Big] \le C \tag{2.13}$$

uniformly in m for some $p > 2$. For a rigorous proof we have to approximate μ and Σ by Lipschitz continuous functions μ_m and Σ_m such that uniformly

$$|\mu_m(t, X)| + |\Sigma_m(t, X)| \le K(1 + |X|) \quad \forall (t, X) \in [0, T] \times \mathbb{R}.$$

In order to get compactness estimate (2.13) is not sufficient. We have to show time regularity. We decompose X^m into a deterministic and a stochastic part $X_t^m = Y_t^m + Z_t^m$. Concerning the stochastic integral, we apply the Burgholder–Davis–Gundy inequality (Lemma 2.1.5) for $\theta > 2$ and get due to (2.13)

$$\begin{aligned}
\mathbb{E}\,|Z^m(t) - Z^m(s)|^\theta &= \mathbb{E}\bigg|\int_s^t \Sigma(r, X_r^m)\,\mathrm{d}W\bigg|^\theta \\
&\le C\,\mathbb{E}\bigg(\int_s^t |\Sigma(r, X_r^m)|^2\,\mathrm{d}r\bigg)^{\frac{\theta}{2}} \le C\,\mathbb{E}\bigg(\int_s^t (1 + |X_r^m|^2)\,\mathrm{d}r\bigg)^{\frac{\theta}{2}} \\
&\le C|t - s|^{\theta/2}\Big(1 + \mathbb{E}\sup_{0\le t\le T} |X_t^m|^2\Big) \le C|t - s|^{\theta/2}
\end{aligned}$$

and the Kolmogorov continuity criterion applies (see Theorem 2.5.5). We obtain

$$\mathbb{E}\big\|Z^m\big\|_{C^\alpha([0,T])} \le C$$

for any $\alpha \in (0, \frac{1}{2})$. Y^m can be estimated similarly and we obtain

$$\mathbb{E}\big\|X^m\big\|_{C^\alpha([0,T])} \le C. \tag{2.14}$$

Now, we consider the joint law ν^m of (X^m, W) on $\mathcal{X}$, where

$$\mathcal{X} = C^0([0, T]; \mathbb{R}) \times C^0([0, T]; \mathbb{R}).$$

We have to show tightness of the probability law ν^m. The law of (X^m) is tight by $C^\alpha \hookrightarrow^c C^0$ and

$$\mathbb{P}\big(\|X^m\|_{C^\alpha} \le R\big) \ge 1 - \tfrac{C}{R},$$

which is a consequence of (2.14). The law of W is tight as being a Radon measure on the Polish space $C^0([0, T]; \mathbb{R})$. Hence, ν^m is tight. By Prokhorov's theorem (Lemma 2.6.1) there is a subsequence such that the probability laws convergence weakly in sense of measures. Now we apply Skorokhod's theorem (Lemma 2.6.2). It yields the existence of a subsequence ν^m (not relabeled), a probability space $(\tilde\Omega, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ with $\mathcal{X}$-valued Borel measurable random variables $(\tilde X^m, \tilde W^m)$, $m \in \mathbb{N}$, and $(\tilde X, \tilde W)$ such that

- the law of $(\tilde{X}^m, \tilde{W}^m)$ is given by ν^m, $m \in \mathbb{N}$,
- the law of $(\tilde{X}, \tilde{W})$, denoted by ν, is a Radon measure,
- $(\tilde{X}^m, \tilde{W}^m)$ converges $\tilde{\mathbb{P}}$-a.s. to $(\tilde{X}, \tilde{W})$ in the topology of $\mathcal{X}$, i.e., we have $\tilde{\mathbb{P}}$-a.s.

$$\tilde{X}^m \to \tilde{X} \quad \text{in} \quad C^0([0,T];\mathbb{R}), \tag{2.15}$$

$$\tilde{W}^m \to \tilde{W} \quad \text{in} \quad C^0([0,T];\mathbb{R}). \tag{2.16}$$

Now we introduce the filtration on the new probability space which ensures the correct measurabilities of the new variables. We denote by $\mathbf{r}_t$ the operator of restriction to the interval $[0,t]$ acting on $C^0([0,T];\mathbb{R})$, that is

$$\mathbf{r}_t : X \to X|_{[0,t]}, \quad f \mapsto f|_{[0,t]}. \tag{2.17}$$

Clearly, $\mathbf{r}_t$ is a continuous mapping. Let $(\tilde{\mathcal{F}}_t)_{t\geq 0}$ and $(\tilde{\mathcal{F}}_t^m)_{t\geq 0}$ be the $\tilde{\mathbb{P}}$-augmented canonical filtration of the processes $\big(\tilde{X}, \tilde{W}\big)$ and $\big(\tilde{X}^m, \tilde{W}^m\big)$, respectively, that is

$$\tilde{\mathcal{F}}_t = \sigma\Big(\sigma\big(\mathbf{r}_t\tilde{X}, \mathbf{r}_t\tilde{W}\big) \cup \big\{\mathcal{N} \in \tilde{\mathcal{F}};\ \tilde{\mathbb{P}}(\mathcal{N}) = 0\big\}\Big), \quad t \in [0,T].$$

$$\tilde{\mathcal{F}}_t^m = \sigma\Big(\sigma\big(\mathbf{r}_t\tilde{X}^m, \mathbf{r}_t\tilde{W}^m\big) \cup \big\{\mathcal{N} \in \tilde{\mathcal{F}};\ \tilde{\mathbb{P}}(\mathcal{N}) = 0\big\}\Big), \quad t \in [0,T].$$

This definition guarantees that the processes are adapted and we can define stochastic integrals. We have to show that $\tilde{\mathbb{P}}$-a.s.

$$\tilde{X}_t^m = \tilde{X}^m(0) + \int_0^t \mu(s, \tilde{X}_s^m)\,\mathrm{d}s + \int_0^t \Sigma(s, \tilde{X}_s^m)\,\mathrm{d}\tilde{W}^m, \tag{2.18}$$

i.e., the equation continues to hold on the new probability space.

The mapping $X \mapsto \int_0^\cdot \mu(t, X_t)\,\mathrm{d}t$ is continuous on the pathspace. However, the mapping $(X, W) \mapsto \int_s^t \Sigma(r, X_r)\,\mathrm{d}W$ is not. So, we cannot identify it immediately. We will make use of the fact that a martingale is uniquely determined by its quadratic variations. This can be done with the help of an elementary method by Brzezniak and Ondreját [9]. It replaces the use of the martingale representation theorem. We consider the functionals

$$\mathfrak{M}(Y)_t = Y_t - Y(0) - \int_0^t \mu(r, Y_r)\mathrm{d}r,$$

$$\mathfrak{N}(Y)_t = \int_0^t |\Sigma(r, Y_r)|^2\mathrm{d}r, \quad \mathfrak{L}(Y)_t = \int_0^t \Sigma(r, Y_r)\mathrm{d}r.$$

Obviously $\mathfrak{M}$, $\mathfrak{N}$ and $\mathfrak{L}$ are continuous on the pathspace. Consequently, by equality of laws, we have

$$\mathfrak{M}(X^m) \sim^d \mathfrak{M}(\tilde{X}^m), \quad \mathfrak{N}(X^m) \sim^d \mathfrak{N}(\tilde{X}^m), \quad \mathfrak{L}(X^m) \sim^d \mathfrak{L}(\tilde{X}^m).$$

Let $\mathfrak{M}(X^m)_{s,t}$ denote the increment $\mathfrak{M}(X^m)_t - \mathfrak{M}(X^m)_s$ and similarly for $\mathfrak{N}(X^m)_{s,t}$ and $\mathfrak{L}(X^m)_{s,t}$. Note that the proof will be complete once we show that

the process $\mathfrak{M}(\tilde{X}^m)$ is an $(\tilde{\mathcal{F}}_t^m)_{t\geq 0}$-martingale and its quadratic and cross variations satisfy, respectively,

$$\langle\langle \mathfrak{M}(\tilde{X}^m)\rangle\rangle = \mathfrak{N}(\tilde{X}^m), \quad \langle\langle \mathfrak{M}(\tilde{X}^m), \tilde{W}\rangle\rangle = \mathfrak{L}(\tilde{X}^m). \tag{2.19}$$

Indeed, in that case we have

$$\left\langle\left\langle \mathfrak{M}(\tilde{X}^m) - \int_0^{\cdot} \Sigma(s, \tilde{X}_s^m)\,\mathrm{d}\tilde{W}\right\rangle\right\rangle = 0 \tag{2.20}$$

which implies the desired equation (2.18) on the new probability space. Let us verify (2.19). To this end, we fix times $s, t \in [0, T]$ such that $s < t$ and let

$$h : \mathcal{X}\big|_{[0,s]} \to [0, 1]$$

be a continuous function. Since $\mathfrak{M}(X^m)$ is a square integrable $(\mathcal{F}_t)_{t\geq 0}$-martingale, we infer that

$$\big[\mathfrak{M}(X^m)\big]^2 - \mathfrak{N}(X^m), \quad \mathfrak{M}(X^m)W - \mathfrak{L}(X^m),$$

are $(\mathcal{F}_t)_{t\geq 0}$-martingales (recall Definition 2.1.14). Then it follows from the equality of laws that

$$\tilde{\mathbb{E}}\big[\, h\big(\mathbf{r}_s\tilde{X}^m, \mathbf{r}_s\tilde{W}^m\big)\mathfrak{M}(\tilde{X}^m)_{s,t}\big] = \mathbb{E}\big[\, h\big(\mathbf{r}_s X^m, \mathbf{r}_s W\big)\mathfrak{M}(X^m)_{s,t}\big] = 0, \tag{2.21}$$

$$\begin{aligned} &\tilde{\mathbb{E}}\Big[\, h\big(\mathbf{r}_s\tilde{X}^m, \mathbf{r}_s\tilde{W}^m\big)\Big([\mathfrak{M}(\tilde{X}^m)^2]_{s,t} - \mathfrak{N}(\tilde{X}^m)_{s,t}\Big)\Big] \\ &= \mathbb{E}\Big[\, h\big(\mathbf{r}_s X^m, \mathbf{r}_s W^m\big)\Big([\mathfrak{M}(X^m)^2]_{s,t} - \mathfrak{N}(X^m)_{s,t}\Big)\Big] = 0, \end{aligned} \tag{2.22}$$

$$\begin{aligned} &\tilde{\mathbb{E}}\Big[\, h\big(\mathbf{r}_s\tilde{X}^m, \mathbf{r}_s\tilde{W}^m\big)\Big([\mathfrak{M}(\tilde{X}^m)\tilde{W}^m]_{s,t} - \mathfrak{L}(\tilde{X}^m)_{s,t}\Big)\Big] \\ &= \mathbb{E}\Big[\, h\big(\mathbf{r}_s X^m, \mathbf{r}_s W\big)\Big([\mathfrak{M}(X^m)W]_{s,t} - \mathfrak{L}(X^m)_{s,t}\Big)\Big] = 0. \end{aligned} \tag{2.23}$$

So we have shown (2.19) and hence (2.20). This means on the new probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ we have (2.18). Finally, on account of (2.15) and (2.16), we can pass to the limit in (2.21)–(2.23). This yields

$$\tilde{X}_t = \tilde{X}(0) + \int_0^t \mu(s, \tilde{X}_s)\,\mathrm{d}s + \int_0^t \Sigma(s, \tilde{X}_s)\,\mathrm{d}\tilde{W}, \tag{2.24}$$

i.e., $\tilde{X}$ is the solution we are looking for.

The stochastic ODEs considered in Theorem 2.4.1 have two drawbacks. First, we need vector-valued processes and, secondly, we have to weaken the assumptions on the drift μ (linear growth in X is too strong). Everything in this section can be obviously extended to the multi-dimensional setting. Here, a standard Wiener process in $\mathbb{R}^M$ is a vector-valued stochastic process and each of its components is a real-valued Wiener process (recall Definition 2.1.4). Moreover, the components are stochastically independent. Getting rid of the linear growth assumption for μ

is more difficult. Now we seek an $\mathbb{R}^N$-valued process $(\mathbf{X}_t)_{t\in[0,T]}$ on a probability space $(\Omega,\mathcal{F},\mathbb{P})$ with filtration $(\mathcal{F}_t)_{t\geq 0}$ such that

$$\begin{cases} \mathrm{d}\mathbf{X}_t &= \boldsymbol{\mu}(t,\mathbf{X})\,\mathrm{d}t + \boldsymbol{\Sigma}(t,\mathbf{X})\,\mathrm{d}\mathbf{W}_t, \\ \mathbf{X}(0) &= \mathbf{X}_0. \end{cases} \tag{2.25}$$

Here $\mathbf{W}$ is a standard $\mathbb{R}^M$-valued Wiener process with respect to $(\mathcal{F}_t)_{t\geq 0}$ and $\mathbf{X}_0 \in L^2(\Omega,\mathcal{F}_0,\mathbb{P})$ is some initial datum. The functions

$$\begin{aligned} \boldsymbol{\mu} &: [0,T]\times\mathbb{R}^N \to \mathbb{R}^N, \\ \boldsymbol{\Sigma} &: [0,T]\times\mathbb{R}^N \to \mathbb{R}^{N\times M}, \end{aligned}$$

are continuous in $\mathbf{X}\in\mathbb{R}^N$ for each fixed $t\in[0,T]$, $\omega\in\Omega$. The application in Section 3 requires weaker assumptions as in the classical existence theorems mentioned above. Fortunately, some more recent results apply. In the following we state the assumptions which are in fact a special case of the assumptions in [31, Thm. 3.1.1].

(A1) We assume that the following integrability condition on $\boldsymbol{\mu}$ holds for all $R<\infty$

$$\int_0^T \sup_{|\mathbf{X}|\leq R} |\boldsymbol{\mu}(t,\mathbf{X})|^2\,\mathrm{d}t < \infty.$$

(A2) $\boldsymbol{\mu}$ is weakly coercive, i.e., for all $(t,\mathbf{X})\in[0,T]\times\mathbb{R}^N$ we have that

$$\boldsymbol{\mu}(t,\mathbf{X})\cdot\mathbf{X} \leq c$$

for some $c\geq 0$.

(A3) $\boldsymbol{\mu}$ is locally weakly monotone, i.e., for all $t\in[0,T]$ and all $\mathbf{X},\mathbf{Y}\in\mathbb{R}^N$ with $|\mathbf{X}|,|\mathbf{Y}|\leq R$ the following holds

$$\big(\boldsymbol{\mu}(t,\mathbf{X})-\boldsymbol{\mu}(t,\mathbf{Y})\big) : (\mathbf{X}-\mathbf{Y}) \leq c(R)|\mathbf{X}-\mathbf{Y}|^2.$$

(A4) $\boldsymbol{\Sigma}$ is Lipschitz continuous, i.e., for all $t\in[0,T]$ and all $\mathbf{X},\mathbf{Y}\in\mathbb{R}^N$ the following holds

$$|\boldsymbol{\Sigma}(t,\mathbf{X})-\boldsymbol{\Sigma}(t,\mathbf{Y})| \leq c\,|\mathbf{X}-\mathbf{Y}|.$$

Theorem 2.4.5. *Let $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ satisfy* (A1)–(A4)*. Assume we have a given probability space $(\Omega,\mathcal{F},\mathbb{P})$ with filtration $(\mathcal{F}_t)_{t\geq 0}$, an initial datum $\mathbf{X}_0\in L^2(\Omega,\mathcal{F}_0,\mathbb{P})$ and an $(\mathcal{F}_t)_{t\geq 0}$-Wiener process $\mathbf{W}$. Then there is a unique $(\mathcal{F}_t)_{t\geq 0}$-adapted process $\mathbf{X}$ satisfying*

$$\mathbf{X}(t) = \mathbf{X}_0 + \int_0^t \boldsymbol{\mu}(\sigma,\mathbf{X}(\sigma))\,\mathrm{d}\sigma + \int_0^t \boldsymbol{\Sigma}(\sigma,\mathbf{X}(\sigma))\,\mathrm{d}\mathbf{W}_\sigma, \quad \mathbb{P}\text{-a.s.},$$

for every $t\in[0,T]$. The trajectories of $\mathbf{X}$ are $\mathbb{P}$-a.s. continuous and we have

$$\mathbb{E}\bigg[\sup_{t\in(0,T)} |\mathbf{X}_t|^2\bigg] < \infty.$$

Theorem 2.4.6. *Let the assumptions of Theorem* 2.4.5 *hold. Assume that* $\mathbf{X}_0 \in L^\beta(\Omega, \mathcal{F}_0, \mathbb{P})$ *for some* $\beta > 2$. *Then we have*

$$\mathbb{E}\left[\sup_{t\in(0,T)} |\mathbf{X}_t|^\beta\right] < \infty.$$

2.5. Stochastic analysis in infinite dimensions

In the following we extend the setup from the previous sections to the case of Banach or Hilbert space-valued stochastic processes (see [10]).
Let $(\mathscr{V}, \|\cdot\|_{\mathscr{V}})$ be a Banach space and $1 \le p < \infty$. We denote by $L^p(\Omega, \mathcal{F}, \mathbb{P}; \mathscr{V})$ the Banach space of all measurable mappings $v : (\Omega, \mathcal{F}) \to (\mathscr{V}, \mathfrak{B}(\mathscr{V}))$ such that

$$\mathbb{E}\big[\|v\|_{\mathscr{V}}^p\big] < \infty,$$

where the expectation is taken with respect to $(\Omega, \mathcal{F}, \mathbb{P})$. The measurability has to be understood via the approximation by step functions as usual for Bochner spaces. The definitions of adaptedness and progressive measurability extend in a straightforward manner to Banach space-valued processes. The same is true for the definitions of martingale and semi-martingale. The definition of the stochastic integral can be extended to Hilbert spaces, where the process X as well as the stochastic integral take values in some separable Hilbert spaces $(\mathscr{H}, \|\cdot\|_{\mathscr{H}})$. Let $\mathfrak{U}$ be a separable Hilbert space with orthonormal basis $(e_k)_{k\in\mathbb{N}}$ and let $L_2(\mathfrak{U}, L^2(\mathcal{O}))$ be the set of Hilbert–Schmidt operators from $\mathfrak{U}$ to $L^2(\mathcal{O})$ with $\mathcal{O} \subset \mathbb{R}^d$ and $d \in \mathbb{N}$. Recall that a bounded linear operator $\mathbb{G} : \mathfrak{U} \to L^2(\mathcal{O})$ is called Hilbert–Schmidt operator iff

$$\sum_{k\in\mathbb{N}} \|\mathbb{G}e_k\|_{L^2(\mathcal{O})}^2 < \infty.$$

We consider a cylindrical Wiener process $W = (W_t)_{t\in[0,T]}$ which has the form

$$W(\sigma) = \sum_{k\in\mathbb{N}} e_k \beta_k(\sigma) \tag{2.26}$$

with a sequence $(\beta_k)_{k\in\mathbb{N}}$ of independent real-valued Brownian motions on $(\Omega, \mathcal{F}, \mathbb{P})$. Define further the auxiliary space $\mathfrak{U}_0 \supset \mathfrak{U}$ as

$$\begin{aligned} \mathfrak{U}_0 &:= \left\{ e = \sum_k \alpha_k e_k : \ \sum_k \frac{\alpha_k^2}{k^2} < \infty \right\}, \\ \|e\|_{\mathfrak{U}_0}^2 &:= \sum_{k=1}^\infty \frac{\alpha_k^2}{k^2}, \quad e = \sum_k \alpha_k e_k, \end{aligned} \tag{2.27}$$

thus the embedding $\mathfrak{U} \hookrightarrow \mathfrak{U}_0$ is Hilbert–Schmidt and trajectories of W belong $\mathbb{P}$-a.s. to the class $C([0,T]; \mathfrak{U}_0)$ (see [10]).

For $\mathbb{G} \in L^2(\Omega, \mathcal{F}, \mathbb{P}; L^2(0,T; L_2(\mathfrak{U}, L^2(\mathcal{O}))))$ progressively $(\mathcal{F}_t)_{t\ge0}$-measurable we see that the equality

$$\int_0^t \mathbb{G}(\sigma)\, \mathrm{d}W_\sigma = \sum_{k=1}^\infty \int_0^t \mathbb{G}(\sigma)(e_k)\, \mathrm{d}\beta_k(\sigma) \tag{2.28}$$

defines a $\mathbb{P}$-a.s. continuous $L^2(\mathcal{O})$-valued $(\mathcal{F}_t)_{t\geq 0}$-martingale. Moreover, we can multiply the above with test functions since

$$\int_{\mathcal{O}}\int_0^t \mathbb{G}(\sigma)\,\mathrm{d}W_\sigma\cdot\boldsymbol{\varphi}\,\mathrm{d}x=\sum_{k=1}^{\infty}\int_0^t\int_{\mathcal{O}}\mathbb{G}(\sigma)(e_k)\cdot\boldsymbol{\varphi}\,\mathrm{d}x\,\mathrm{d}\beta_k(\sigma),\quad \boldsymbol{\varphi}\in L^2(\mathcal{O}),$$

is well defined and $\mathbb{P}$-a.s. continuous.

In the following we define the quadratic variation of a stochastic process with values in a Hilbert space (see [10]).

Definition 2.5.1. Let $(X_t)_{t\geq 0}$ be a continuous semi-martingale on a probability space $(\Omega,\mathcal{F},\mathbb{P})$ with values in a separable Hilbert space $(\mathscr{H},\langle\cdot,\cdot\rangle_{\mathscr{H}})$ with basis $(h_i)_{i\in\mathbb{N}}$. Then its quadratic variation process is defined as

$$\langle\langle X,X\rangle\rangle_t^{\mathscr{H}}:=\sum_{i,j\in\mathbb{N}}\Big\langle\Big\langle\langle X,h_i\rangle_{\mathscr{H}},\langle X,h_j\rangle_{\mathscr{H}}\Big\rangle\Big\rangle_t\langle h_j,\cdot\rangle_{\mathscr{H}}\,h_i$$

and has values in $\mathcal{N}(\mathscr{H})$ (the set of nuclear operators on $\mathscr{H}$). Moreover, we define the trace of $\langle\langle X,X\rangle\rangle_t^{\mathscr{H}}$ by

$$\operatorname{tr}\langle\langle X,X\rangle\rangle_t^{\mathscr{H}}:=\sum_{i\in\mathbb{N}}\Big\langle\Big\langle\langle X,h_i\rangle_{\mathscr{H}},\langle X,h_i\rangle_{\mathscr{H}}\Big\rangle\Big\rangle_t\langle h_i,\cdot\rangle_{\mathscr{H}}\,h_i.$$

The Burkholder–Davis–Gundy inequality from Lemma 2.5.1 extends to the infinite-dimensional situations as follows.

Lemma 2.5.1. *Let $(\mathscr{H},\langle\cdot,\cdot\rangle_{\mathscr{H}})$ be a separable Hilbert space, $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space and $(X_t)_{t\geq 0}$ be a continuous martingale on $(\Omega,\mathcal{F},\mathbb{P})$ with values in $\mathscr{H}$. Then we have for all $p>0$ and all $T>0$*

$$c_p\,\mathbb{E}\bigg[\sup_{t\in(0,T)}\|X_t\|_{\mathscr{H}}\bigg]^p\leq\mathbb{E}\Big[\|\operatorname{tr}\langle\langle X,X\rangle\rangle_T^{\mathscr{H}}\|_{\mathcal{N}(\mathscr{H})}\Big]^{p/2}\leq C_p\,\mathbb{E}\bigg[\sup_{t\in(0,T)}\|X_t\|_{\mathscr{H}}\bigg]^p,$$

where c_p,C_p are positive constants. Here, we have

$$\|\operatorname{tr}\langle\langle X,X\rangle\rangle_T^{\mathscr{H}}\|_{\mathcal{N}(\mathscr{H})}=\sum_{i\in\mathbb{N}}\Big\langle\Big\langle\langle X,e_i\rangle_{\mathscr{H}}\Big\rangle\Big\rangle_t.$$

Now we present an infinite-dimensional version of Itô's formula which is appropriate at least to obtain energy estimates for linear SPDEs, see [26, Theorem 3.1] or [33, Chapter 4.2, Theorem 2].

Theorem 2.5.2. *Let $(\mathscr{V},\|\cdot\|_{\mathscr{V}})$ be a Banach space which is continuously embedded into a separable Hilbert space $(\mathscr{H},\langle\cdot,\cdot\rangle_{\mathscr{H}})$. Let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space with filtration $(\mathcal{F}_t)_{t\geq 0}$. Assume that the processes $(X_t)_{t\in[0,T]}$ and $(Y_t)_{t\in[0,T]}$, taking values in $\mathscr{V}$ and $\mathscr{V}'$, respectively, are progressively $(\mathcal{F}_t)_{t\geq 0}$-measurable and*

$$\mathbb{P}\bigg\{\int_0^T\big(\|X\|_{\mathscr{V}}^2+\|Y\|_{\mathscr{V}'}^2\big)\,\mathrm{d}t<\infty\bigg\}=1.$$

Assume further that there is a continuous martingale $(M_t)_{t\in[0,T]}$*, taking values in* $\mathscr{H}$*, such that, for* $\mathbb{P}\times\mathcal{L}^1$*-a.e.* (ω,t)*, the following equality holds:*

$$\langle X(t),\varphi\rangle_{\mathscr{H}} = \langle X(0),\varphi\rangle_{\mathscr{H}} + \int_0^t {}_{\mathscr{V}'}\langle Y(\sigma),\varphi\rangle_{\mathscr{V}}\,\mathrm{d}\sigma + \langle M_t,\varphi\rangle_{\mathscr{H}} \quad \forall\varphi\in\mathscr{V}.$$

Then we have

$$\|X(t)\|_{\mathscr{H}}^2 = \|X(0)\|_{\mathscr{H}}^2 + \int_0^t {}_{\mathscr{V}'}\langle Y(\sigma),X(\sigma)\rangle_{\mathscr{V}}\,\mathrm{d}\sigma + 2\int_0^t \langle X(\sigma),\mathrm{d}M_\sigma\rangle_{\mathscr{H}} + \|\operatorname{tr}\langle\langle M,M\rangle\rangle_t^{\mathscr{H}}\|_{\mathcal{N}(\mathscr{H})}, \quad \mathbb{P}\times\mathcal{L}^1\text{-a.e.}$$

In some applications we need fractional time derivatives of stochastic integrals. The following lemma is concerned with fractional derivatives of stochastic integrals in Hilbert spaces (see [17] [Lemma 2.1] for a proof).

Lemma 2.5.3. *Let* $\mathbb{G}\in L^p(\Omega,\mathcal{F},\mathbb{P};L^p(0,T;L_2(\mathfrak{U},L^2(\mathcal{O}))))$ $(p\geq 2)$ *be progressively* $(\mathcal{F}_t)_{t\geq 0}$*-measurable and* W *a cylindrical* $(\mathcal{F}_t)_{t\geq 0}$*-Wiener process as in* (2.26)*. Then the following holds for any* $\alpha\in(0,1/2)$

$$\mathbb{E}\left[\left\|\int_0^{\cdot}\mathbb{G}\,\mathrm{d}W_\sigma\right\|^p_{W^{\alpha,p}(0,T;L^2(\mathcal{O}))}\right] \leq c(\alpha,p)\,\mathbb{E}\left[\int_0^T \|\mathcal{O}\|^p_{L_2(\mathfrak{U},L^2(\mathcal{O}))}\,\mathrm{d}t\right].$$

The following lemma is very useful in order to pass to the limit in stochastic integrals (see [11, Lemma 2.1])

Lemma 2.5.4. *Consider a sequence of cylindrical Wiener processes* (W^n) *over* $\mathfrak{U}$ *(see* (2.26)*) with respect to the filtration* $(\mathcal{F}_t^n)_{t\geq 0}$*. Assume that* (Ψ^n) *is a sequence of progressively* $(\mathcal{F}_t^n)_{t\geq 0}$*-measurable processes such that*

$$\Psi^n\in L^2(0,T;L_2(\mathfrak{U},L^2(\mathcal{O})))\ \mathbb{P}\text{-a.s.}$$

Suppose there is a cylindrical $(\mathcal{F}_t)_{t\geq 0}$*-Wiener process* W *and*

$$\Psi\in L^2(0,T;L_2(\mathfrak{U},L^2(\mathcal{O}))),$$

progressively $(\mathcal{F}_t)_{t\geq 0}$*-measurable, such that*

$$W^n\to W \quad \text{in} \quad C^0([0,T];\mathfrak{U}_0),$$
$$\Psi^n\to\Psi \quad \text{in} \quad L^2(0,T;L_2(\mathfrak{U},L^2(\mathcal{O}))),$$

in probability. Then we have

$$\int_0^{\cdot}\Psi^n\,\mathrm{d}\mathbf{W}^n\to\int_0^{\cdot}\Psi\,\mathrm{d}W \quad \text{in} \quad L^2(0,T;L^2(\mathcal{O})),$$

in probability.

The approach to establish Hölder continuity of a stochastic integral relies on the Kolmogorov continuity theorem. This is a classical result that allows to show existence of a Hölder continuous modification for stochastic processes.[2]

[2] A modification of a stochastic process $\mathbf{U}$ is a stochastic process $\mathbf{V}$ such that $\mathbb{P}\big(\mathbf{U}(t)=\mathbf{V}(t)\big)=1$ for all $t\in(0,\infty)$.

Theorem 2.5.5 (Kolmogorov continuity theorem, [10] (Theorem 3.3)). *Let* $\mathbf{U}$ *be a stochastic process taking values in a separable Banach space* X*. Assume that there exist constants* $K > 0$*,* $a \geq 1$*,* $b > 0$ *such that for all* $s, t \in [0, T]$

$$\mathbb{E}\|\mathbf{U}(t) - \mathbf{U}(s)\|_X^a \leq K|t - s|^{1+b}.$$

Then there exists $\mathbf{V}$*, a modification of* $\mathbf{U}$*, which has* $\mathbb{P}$*-a.s. Hölder continuous trajectories with exponent* γ *for every* $\gamma \in (0, \frac{b}{a})$*. In addition, it holds true that*

$$\mathbb{E}\|\mathbf{V}\|_{C_t^\gamma X}^a \lesssim K,$$

where the proportional constant does not depend on $\mathbf{V}$*.*

2.6. Tools for compactness

In this section we present some (mainly basic) tools from probability theory which are quite crucial to obtain compactness for SPDEs. Let $(\mathscr{V}, \tau)$ be a topological space. The smallest σ-field $\mathfrak{B}(\mathscr{V})$ on $(\mathscr{V}, \tau)$ which contains all open sets is called topological σ-field. A random variable with values in the topological space $(\mathscr{V}, \tau)$ is a measurable map $X : (\Omega, \mathcal{F}) \to (\mathscr{V}, \mathfrak{B}(\mathscr{V}))$. The probability law μ of X on $(\mathscr{V}, \tau)$ will be given by $\mu = \mathbb{P} \circ X^{-1}$. An important concept for applications is the pre-compactness of families of random variables. We will need the following definition.

Definition 2.6.1 (Tightness). A family $(\mu_\alpha)_{\alpha \in \mathcal{I}}$ of probability laws on a topological space $(\mathscr{V}, \mathfrak{B}(\mathscr{V}))$ is called tight if for every $\varepsilon > 0$ there is a compact subset $K \subset \mathscr{V}$ such that $\mu_\alpha(K) \geq 1 - \varepsilon$ for every $\alpha \in \mathcal{I}$.

Lemma 2.6.1 (Prokhorov; [22], Thm. 2.6). *Let* $(\mu_\alpha)_{\alpha \in \mathcal{I}}$ *be a family of probability laws on a metric space* $(\mathscr{V}, \rho)$*. The family* $(\mu_\alpha)_{\alpha \in \mathcal{I}}$ *is tight if and only if it is relatively weakly compact.*

Lemma 2.6.2 (Skorokhod; [22], Thm. 2.7). *Let* $(\mu_n)_{n \in \mathbb{N}}$ *be a sequence of probability laws on a complete separable metric space* $(\mathscr{V}, \rho)$ *such that* $\mu_n \to \mu$ *weakly in the sense of measures as* $n \to \infty$*. Then there is a probability space* $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ *and random variables* $(\tilde{X}_n)_{n \in \mathbb{N}}, \tilde{X} : (\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}) \to (\mathscr{V}, \mathfrak{B}(\mathscr{V}))$ *such that*

- *The laws of* $\tilde{X}_n$ *and* $\tilde{X}$ *under* $\tilde{\mathbb{P}}$ *coincide with* μ_n *and* μ *respectively,* $n \in \mathbb{N}$*.*
- *we have* $\tilde{\mathbb{P}}$ *a.s. that* $\tilde{X}_n \to^\rho \tilde{X}$ *for* $n \to \infty$*.*

The proof of Lemma 2.6.2 in the general case is not very long but quite technical and it is hard to grasp the main ideas. We will therefore briefly outline the case of real-valued random variables, i.e., $\mathscr{V} = \mathbb{R}$ and $\rho(x, y) = |x - y|$. Let μ_n be a probability law on $\mathbb{R}$ such that $\mu_n \to \mu$ weakly in the sense of measures as $n \to \infty$. We denote by F_n and F the distribution functions of μ_n and μ respectively, that is

$$F_n(x) = \mu_n((-\infty, x]), \quad F(x) = \mu((-\infty, x]), \quad x \in \mathbb{R}.$$

Let us assume for simplicity that they are injective (otherwise one can argue via their generalized inverse functions). In this case we have $F_n \to F$ pointwise. Now

we set $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}) = ([0,1], \overline{\mathfrak{B}([0,1])}, \mathcal{L}^1|_{[0,1]})$. Let us assume for simplicity that the distribution functions F_n $(n \in \mathbb{N})$ and F are continuous. We define random variables (for $\omega \in (0,1)$)

$$\tilde{X}_n(\omega) = F_n^{-1}(\omega), \quad \tilde{X}(\omega) = F^{-1}(\omega).$$

Now one can easily see that for $n \in \mathbb{N}$

$$\mu_{\tilde{X}_n} = \tilde{\mathbb{P}} \circ \tilde{X}_n^{-1} = \mathcal{L}^1 \circ F_n = \mu_n$$

and similarly $\mu_{\tilde{X}} = \mu$. Moreover, we have

$$\tilde{X}_n(\omega) = F_n^{-1}(\omega) \to F^{-1}(\omega) = \tilde{X}(\omega)$$

for every $\omega \in (0,1)$. In the general case this convergence only holds true in points where F is continuous (which holds for $\mathcal{L}^1$-a.e. ω). This means we have $\tilde{X}_n \to \tilde{X}$ a.s.

Lemma 2.6.2 only applies to metric spaces. Unfortunately, this does not cover Banach spaces with the weak topology which will be crucial for compressible fluids in Section 4. Therefore we need the following generalization.

Definition 2.6.2 (Quasi-Polish space). Let $(\mathscr{V}, \tau)$ be a topological space such that there exists a countable family

$$\{f_n : \mathscr{V} \to [-1,1];\ n \in \mathbb{N}\}$$

of continuous functions that separates points of $\mathscr{V}$. Then $(\mathscr{V}, \tau, (f_n)_{n\in\mathbb{N}})$ is called a quasi-Polish space.

Lemma 2.6.3 (Jakubowski–Skorokhod, [23]). *Let $(\mu_n)_{n\in\mathbb{N}}$ be a family of probability laws on a quasi-Polish space $(\mathscr{V}, \tau, (f_n)_{n\in\mathbb{N}})$ and let $\mathscr{S}$ be the σ-algebra generated by the maps $(f_n)_{n\in\mathbb{N}}$. Let $(\mu_n)_{n\in\mathbb{N}}$ be a tight sequence of probability laws on $(\mathscr{V}, \mathscr{S})$. Then there is a subsequence $(\mu_{n_k})_{k\in\mathbb{N}}$ such that the following holds. There is a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ and $\mathscr{V}$-valued Borel measurable random variables $(\tilde{X}_k)_{k\in\mathbb{N}}, \tilde{X} : (\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}) \to (\mathscr{V}, \mathscr{S})$ such that*

- *The laws of $\tilde{X}_k$ under $\tilde{\mathbb{P}}$ coincide with μ_{n_k}, $k \in \mathbb{N}$.*
- *we have $\tilde{\mathbb{P}}$-a.s. that $\tilde{X}_k \to^\tau \tilde{X}$ for $k \to \infty$.*
- *The law of $\tilde{X}$ under $\tilde{\mathbb{P}}$ is a Radon measure.*

3. Incompressible fluids

In this section we discuss the existence of martingale solutions to the stochastic incompressible Navier–Stokes equations

$$\begin{cases} \mathrm{d}(\varrho \mathbf{v}) = \nu \boldsymbol{\Delta} \mathbf{v}\, \mathrm{d}t - \operatorname{div}(\varrho \mathbf{v} \otimes \mathbf{v})\, \mathrm{d}t - \nabla p\, \mathrm{d}t + \mathbb{G}(\mathbf{v}) \mathrm{d}W_t & \text{in } Q, \\ \operatorname{div} \mathbf{v} = 0 & \text{in } Q, \end{cases} \tag{3.1}$$

subject to zero boundary conditions for the velocity and some given initial law. The physical body is prescribed by a bounded domain $\mathcal{O} \in \mathbb{R}^d$ $(d = 2, 3)$ with smooth boundary and $Q = (0,T) \times \mathcal{O}$. All quantities are defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$. The filtration $(\mathcal{F}_t)_{t\geq 0}$ satisfies the *usual conditions*, see

Definition 2.1.6. The unknowns are the velocity field $\mathbf{v}$ and the pressure p. The latter one does not appear in the weak formulation due to the use of divergence-free test functions. In order to rigorously define the noise let $\mathfrak{U}$ be a separable Hilbert space with orthonormal basis $(e_k)_{k\in\mathbb{N}}$ and let $L_2(\mathfrak{U}, L^2(\mathcal{O}))$ be the set of Hilbert–Schmidt operators from $\mathfrak{U}$ to $L^2(\mathcal{O})$. The most natural choice is $\mathfrak{U} = L^2(\mathcal{O})$. We consider a cylindrical Wiener process $W = (W_t)_{t\geq 0}$ which has the form

$$W_\sigma = \sum_{k\in\mathbb{N}} e_k \beta_k(\sigma) \tag{3.2}$$

with a sequence (β_k) of independent real-valued Wiener processes. We suppose the following linear growth assumptions on $\mathbb{G}$: For each $\mathbf{z} \in L^2(\mathcal{O})$ there is a mapping $\mathbb{G}(\mathbf{z}) : \mathfrak{U} \to L^2(\mathcal{O})$ defined by $\mathbb{G}(\mathbf{z})e_k = \mathbf{g}_k(\mathbf{z}(\cdot))$. In particular, we suppose that $\mathbf{g}_k \in C^1(\mathbb{R}^d)$ and the following conditions for some $L \geq 0$

$$\sum_{k\in\mathbb{N}} |\mathbf{g}_k(\boldsymbol{\xi})|^2 \leq L(1+|\boldsymbol{\xi}|^2), \quad \sum_{k\in\mathbb{N}} |\nabla \mathbf{g}_k(\boldsymbol{\xi})|^2 \leq L, \quad \boldsymbol{\xi} \in \mathbb{R}^d. \tag{3.3}$$

As already indicated in Section 2.4 the lack of uniqueness of solutions requires the consideration of martingale solutions. The underlying probability space is not a priori known but becomes an integral part of the solution. In the following we define martingale solutions for (3.1) starting with an initial law defined on

$$L^2_{\mathrm{div}}(\mathcal{O}) := \overline{C^\infty_{c,\mathrm{div}}(\mathcal{O})}^{L^2(\mathcal{O})}.$$

Definition 3.0.1 (Solution). Let Λ_0 be a Borel probability measure on $L^2_{\mathrm{div}}(\mathcal{O})$. Then

$$\big((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}), \mathbf{v}, W\big)$$

is called a weak martingale solution to (3.1) with the initial datum Λ_0 provided the following holds.

(a) $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ is a stochastic basis with a complete right-continuous filtration,
(b) W is a $(\mathcal{F}_t)_{t\geq 0}$-cylindrical Wiener process,
(c) $\mathbf{v} \in L^2(\Omega, \mathcal{F}, \mathbb{P}; L^2(0,T; W^{1,2}_{0,\mathrm{div}}(\mathcal{O})))$ is progressively $(\mathcal{F}_t)_{t\geq 0}$-measurable with $\mathbf{v} \in C_w([0,T]; L^2(\mathcal{O}))$ a.s. and

$$\mathbb{E}\bigg[\sup_{t\in(0,T)} \int_{\mathcal{O}} |\mathbf{v}|^2 \,\mathrm{d}x\bigg] < \infty,$$

(d) $\Lambda_0 = \mathbb{P} \circ \mathbf{v}(0)^{-1}$ (that is $\mathbb{P}(\mathbf{v}(0) \in B) = \Lambda_0(B)$ for all $B \in \mathfrak{B}(L^2_{\mathrm{div}}(\mathcal{O}))$),
(e) for all $\boldsymbol{\varphi} \in C^\infty_{c,\mathrm{div}}(\mathcal{O})$ and all $t \in [0,T]$ we have

$$\begin{aligned}\int_{\mathcal{O}} \mathbf{v}(t)\cdot\boldsymbol{\varphi}\,\mathrm{d}x + \int_0^t\int_{\mathcal{O}} \nu\nabla\mathbf{v} : \nabla\boldsymbol{\varphi}\,\mathrm{d}x\,\mathrm{d}\sigma - \int_0^t\int_{\mathcal{O}} \mathbf{v}\otimes\mathbf{v} : \nabla\boldsymbol{\varphi}\,\mathrm{d}x\,\mathrm{d}\sigma \\ = \int_{\mathcal{O}} \mathbf{v}(0)\cdot\boldsymbol{\varphi}\,\mathrm{d}x + \int_{\mathcal{O}}\int_0^t \mathbb{G}(\mathbf{v})\,\mathrm{d}W_\sigma\cdot\boldsymbol{\varphi}\,\mathrm{d}x\end{aligned}$$

$\mathbb{P}$-a.s.

The main result of this section is the following theorem.[3]

Theorem 3.0.1 (Existence). *Assume that* (3.3) *holds and we have*

$$\int_{L^2_{\mathrm{div}}(\mathcal{O})} \|\mathbf{u}\|^{\beta}_{L^2(\mathcal{O})}\, \mathrm{d}\Lambda_0(\mathbf{u}) < \infty \tag{3.4}$$

for some $\beta > 2$. *Then there is a weak martingale solution to* (3.1) *in the sense of Definition* 3.0.1.

The key ideas for the proof of Theorem 3.0.1 go back to [17]. A main difference is that we do not use a martingale representation theorem to identify the stochastic integral after the limit procedure. Instead we use the elementary approach from [9] (see also [30]). We follow the presentation from [5] which originally aims to study non-Newtonian fluid flows.

3.1. The approximated system

We will try to find a solution by separating space and time via a Galerkin ansatz. Then we seek for an approximated solution by solving an ordinary stochastic differential equation.

There is a sequence $(\lambda_k) \subset \mathbb{R}$ and a sequence of functions $(\mathbf{w}_k) \subset W^{l,2}_{0,\mathrm{div}}(\mathcal{O})$, $l \in \mathbb{N}$, such that[4]

i) $\mathbf{w}_k$ is an eigenvector to the eigenvalue λ_k of the Stokes operator in the sense that:

$$\langle \mathbf{w}_k, \boldsymbol{\varphi} \rangle_{W^{l,2}_0} = \lambda_k \int_{\mathcal{O}} \mathbf{w}_k \cdot \boldsymbol{\varphi}\, dx \quad \text{for all} \quad \boldsymbol{\varphi} \in W^{l,2}_{0,\mathrm{div}}(\mathcal{O}),$$

ii) $\int_{\mathcal{O}} \mathbf{w}_k \mathbf{w}_m\, dx = \delta_{km}$ for all $k, m \in \mathbb{N}$,
iii) $1 \le \lambda_1 \le \lambda_2 \le \cdots$ and $\lambda_k \to \infty$,
iv) $\langle \frac{\mathbf{w}_k}{\sqrt{\lambda_k}}, \frac{\mathbf{w}_m}{\sqrt{\lambda_m}} \rangle_{W^{l,2}_0} = \delta_{km}$ for all $k, m \in \mathbb{N}$,
v) $(\mathbf{w}_k)$ is a basis of $W^{l,2}_{0,\mathrm{div}}(\mathcal{O})$.

We choose $l > 1 + \frac{d}{2}$ such that $W^{l,2}_0(\mathcal{O}) \hookrightarrow W^{1,\infty}(\mathcal{O})$. Let $\mathbf{v}_0$ be an $\mathcal{F}_0$-measurable random variable with values in $L^2_{\mathrm{div}}(\mathcal{O})$ with law Λ_0. We are looking for an approximated solution $\mathbf{v}_N$ of the form

$$\mathbf{v}_N = \sum_{k=1}^{N} c_i^N \mathbf{w}_k = \mathbf{C}_N \cdot \boldsymbol{\omega}_N, \quad \boldsymbol{\omega}_N = (\mathbf{w}_1, \ldots, \mathbf{w}_N),$$

[3]The theorem holds in any dimension $d \ge 2$ even so the physical applications are restricted to the cases $d = 2, 3$.
[4]see [29], Appendix

where $\mathbf{C}_N = (c_N^i) : \Omega \times (0,T) \to \mathbb{R}^N$. Therefore, we would like to solve the system $(k = 1, \dots, N)$

$$\begin{aligned}\int_{\mathcal{O}} \mathrm{d}\mathbf{v}_N \cdot \mathbf{w}_k \,\mathrm{d}x + \int_{\mathcal{O}} \nu \nabla \mathbf{v}_N : \nabla \mathbf{w}_k \,\mathrm{d}x \,\mathrm{d}t \\ = \int_{\mathcal{O}} \mathbf{v}_N \otimes \mathbf{v}_N : \nabla \mathbf{w}_k \,\mathrm{d}x \,\mathrm{d}t + \int_{\mathcal{O}} \mathbb{G}(\mathbf{v}_N) \,\mathrm{d}W_\sigma^N \cdot \mathbf{w}_k \,\mathrm{d}x, \\ \mathbf{v}_N(0) = \mathcal{P}_N \mathbf{v}_0. \end{aligned} \tag{3.5}$$

Here $\mathcal{P}_N : L^2_{\mathrm{div}}(\mathcal{O}) \to \mathcal{X}_N := \mathrm{span}\,\{\mathbf{w}_1, \dots, \mathbf{w}_N\}$ is the orthogonal projection, i.e.,

$$\mathcal{P}_N \mathbf{u} = \sum_{k=1}^N \langle \mathbf{u}, \mathbf{w}_k \rangle_{L^2} \mathbf{w}_k.$$

The equation above is to be understood $\mathbb{P}$-a.s. and for all $t \in [0,T]$. Moreover, we have set

$$W^N(\sigma) = \sum_{k=1}^N e_k \beta_k(\sigma) = \mathbf{E}_N \cdot \boldsymbol{\beta}^N(\sigma), \quad \mathbf{E}_N = (e_1, \dots, e_N).$$

The system (3.5) is equivalent to solving

$$\begin{cases} \mathrm{d}\mathbf{C}_N &= \big[\boldsymbol{\mu}(\mathbf{C}_N)\big] \,\mathrm{d}t + \boldsymbol{\Sigma}(\mathbf{C}_N) \,\mathrm{d}\boldsymbol{\beta}_t^N, \\ \mathbf{C}_N(0) &= \mathbf{C}_0, \end{cases} \tag{3.6}$$

with the abbreviations

$$\begin{aligned} \boldsymbol{\mu}(\mathbf{C}^N) &= \bigg(-\int_{\mathcal{O}} \mathbf{C}_N \cdot \nu \nabla \boldsymbol{\omega}_N : \nabla \mathbf{w}_k \,\mathrm{d}x + \int_{\mathcal{O}} (\mathbf{C}_N \cdot \boldsymbol{\omega}_N) \otimes (\mathbf{C}_N \cdot \boldsymbol{\omega}_N) : \nabla \mathbf{w}_k \,\mathrm{d}x \bigg)_{k=1}^N, \\ \boldsymbol{\Sigma}(\mathbf{C}_N) &= \bigg(\int_{\mathcal{O}} \mathbb{G}(\mathbf{C}_N \cdot \boldsymbol{\omega}_N) e_l \cdot \mathbf{w}_k \,\mathrm{d}x \bigg)_{k,l=1}^N, \\ \mathbf{C}_0 &= \Big(\langle \mathbf{v}_0, \mathbf{w}_k \rangle_{L^2(G)} \Big)_{k=1}^N. \end{aligned}$$

In the following we will check the assumptions of Theorem 2.4.5. We have

$$\begin{aligned} &\big(\boldsymbol{\mu}(\mathbf{C}_N) - \boldsymbol{\mu}(\tilde{\mathbf{C}}_N)\big) \cdot \big(\mathbf{C}_N - \tilde{\mathbf{C}}_N\big) \\ &\quad = -\nu \int_{\mathcal{O}} \big| \nabla \mathbf{v}_N - \nabla \tilde{\mathbf{v}}_N \big|^2 \,\mathrm{d}x + \int_{\mathcal{O}} \big(\mathbf{v}_N \otimes \mathbf{v}_N - \tilde{\mathbf{v}}_N \otimes \tilde{\mathbf{v}}_N \big) : \big(\nabla \mathbf{v}_N - \nabla \tilde{\mathbf{v}}_N \big) \,\mathrm{d}x \\ &\quad \le \int_{\mathcal{O}} \big(\mathbf{v}_N \otimes \mathbf{v}_N - \tilde{\mathbf{v}}_N \otimes \tilde{\mathbf{v}}_N \big) : \big(\nabla \mathbf{v}_N - \nabla \tilde{\mathbf{v}}_N \big) \,\mathrm{d}x. \end{aligned}$$

If $|\mathbf{C}_N| \le R$ and $|\tilde{\mathbf{C}}_N| \le R$ the following holds

$$\big(\boldsymbol{\mu}(t, \mathbf{C}_N) - \boldsymbol{\mu}(t, \tilde{\mathbf{C}}_N)\big) \cdot \big(\mathbf{C}_N - \tilde{\mathbf{C}}_N\big) \le c(R,N) |\mathbf{C}_N - \tilde{\mathbf{C}}_N|^2.$$

Here, we took into account boundedness of $\mathbf{w}_k$ and $\nabla \mathbf{w}_k$. The above implies weak monotonicity in the sense of (A3) by the Lipschitz continuity $\boldsymbol{\Sigma}$ in $\mathbf{C}_N$, cf. (3.3).

On account of $\int_G \mathbf{v}_N \otimes \mathbf{v}_N : \nabla \mathbf{v}_N \,\mathrm{d}x = 0$ the following holds

$$\boldsymbol{\mu}(t, \mathbf{C}_N) \cdot \mathbf{C}_N = -\nu \int_{\mathcal{O}} |\nabla \mathbf{v}_N|^2 \,\mathrm{d}x \,\mathrm{d}x \leq 0.$$

This implies weak coercivity in the sense of (A2). Finally, (A4) follows from (3.3). We obtain a unique strong solution $\mathbf{C}_N$ to the SDE (3.6) with $\mathbb{P}$-a.s. continuous trajectories.

In the following we will derive some uniform estimates.

Theorem 3.1.1. *Assume* (3.3) *and*

$$\int_{L^2_{\mathrm{div}}(\mathcal{O})} \|\mathbf{u}\|^2_{L^2(\mathcal{O})} \,\mathrm{d}\Lambda_0(\mathbf{u}) < \infty. \tag{3.7}$$

Then the following holds uniformly in N

$$\begin{aligned} &\mathbb{E}\bigg[\sup_{t\in(0,T)} \int_{\mathcal{O}} |\mathbf{v}_N(t)|^2 \,\mathrm{d}x + \int_Q |\nabla \mathbf{v}_N|^2 \,\mathrm{d}x \,\mathrm{d}t\bigg] \\ &\quad \leq c\left(1 + \int_{L^2_{\mathrm{div}}(\mathcal{O})} \|\mathbf{u}\|^2_{L^2(\mathcal{O})} \,\mathrm{d}\Lambda_0(\mathbf{u})\right). \end{aligned}$$

Proof. We apply Itô's formula to the function $f(\mathbf{C}) = \frac{1}{2}|\mathbf{C}|^2$ which shows

$$\begin{aligned} \frac{1}{2}\|\mathbf{v}_N(t)\|^2_{L^2(G)} &= \frac{1}{2}\|\mathbf{C}_N(0)\|^2_{L^2(G)} + \sum_{k=1}^N \int_0^t c_N^k \,\mathrm{d}(c_N^k)_\sigma + \frac{1}{2}\sum_{k=1}^N \int_0^t \mathrm{d}\langle\langle c_N^k \rangle\rangle_\sigma \\ &= \frac{1}{2}\|\mathcal{P}_N \mathbf{v}_0\|^2_{L^2(\mathcal{O})} - \nu \int_0^t \int_{\mathcal{O}} |\nabla \mathbf{v}_N|^2 \,\mathrm{d}x \,\mathrm{d}\sigma \\ &\quad + \int_{\mathcal{O}} \int_0^t \mathbf{v}_N \cdot \mathbb{G}(\mathbf{v}_N) \,\mathrm{d}W_\sigma^N \,\mathrm{d}x \\ &\quad + \frac{1}{2}\int_{\mathcal{O}} \int_0^t \mathrm{d}\Big\langle\Big\langle \int_0^{\cdot} \mathcal{P}_N \mathbb{G}(\mathbf{v}_N) \,\mathrm{d}W^N \Big\rangle\Big\rangle_\sigma \,\mathrm{d}x. \end{aligned} \tag{3.8}$$

Here, we used $\mathrm{d}\mathbf{v}_N = \sum_{k=1}^N \mathrm{d}c_N^k \mathbf{w}_k$, $\int_{\mathcal{O}} \mathbf{v}_N \otimes \mathbf{v}_N : \nabla \mathbf{v}_N \,\mathrm{d}x = 0$, property (ii) of the base $(\mathbf{w}_k)$ as well as

$$\mathrm{d}c_N^k = -\nu \int_{\mathcal{O}} |\nabla \mathbf{w}_k|^2 \,\mathrm{d}x \,\mathrm{d}t + \int_{\mathcal{O}} \mathbf{v}_N \otimes \mathbf{v}_N : \nabla \mathbf{w}_k \,\mathrm{d}x \,\mathrm{d}t + \int_{\mathcal{O}} \mathbb{G}(\mathbf{v}_N) \,\mathrm{d}W_t^N \cdot \mathbf{w}_k \,\mathrm{d}x.$$

Now we can follow, by taking the supremum in time and building expectations, that

$$\begin{aligned} &\mathbb{E}\bigg[\sup_{t\in(0,T)} \int_{\mathcal{O}} |\mathbf{v}_N(t)|^2 \,\mathrm{d}x + \int_0^T \int_{\mathcal{O}} |\nabla \mathbf{v}_N|^2 \,\mathrm{d}x \,\mathrm{d}\sigma\bigg] \\ &\quad \leq c\bigg(\mathbb{E}\Big[\|\mathbf{v}_0\|^2_{L^2(\mathcal{O})}\Big] + \mathbb{E}\bigg[\sup_{t\in(0,T)} J_2(t)\bigg] + \mathbb{E}\Big[J_3(T)\Big]\bigg). \end{aligned}$$

Here, we abbreviated

$$J_2(t) = \int_{\mathcal{O}} \int_0^t \mathbf{v}_N \cdot \mathbb{G}(\mathbf{v}_N)\,\mathrm{d}W_\sigma^N\,\mathrm{d}x,$$
$$J_3(t) = \int_{\mathcal{O}} \int_0^t \mathrm{d}\Big\langle\Big\langle \int_0^{\cdot} \mathcal{P}_N \mathbb{G}(\mathbf{v}_N)\,\mathrm{d}W^N\Big\rangle\Big\rangle_\sigma\,\mathrm{d}x.$$

Straightforward calculations show on account of (3.2) and (3.3)

$$\begin{aligned}
\mathbb{E}[J_3(t)] &= \mathbb{E}\bigg[\sum_{i=1}^N \int_0^t \int_{\mathcal{O}} |\mathcal{P}_N \mathbb{G}(\mathbf{v}_N) e_i|^2\,\mathrm{d}x\,\mathrm{d}\sigma\bigg] \\
&\leq \mathbb{E}\bigg[\sum_{i=1}^\infty \int_0^t \int_{\mathcal{O}} |\mathcal{P}_N \mathbf{g}_i(\mathbf{v}_N)|^2\,\mathrm{d}x\,\mathrm{d}\sigma\bigg] \\
&\leq c\,\mathbb{E}\bigg[\sum_{i=1}^\infty \int_0^t \int_{\mathcal{O}} |\mathbf{g}_i(\mathbf{v}_N)|^2\,\mathrm{d}x\,\mathrm{d}\sigma\bigg] \\
&\leq c\,\mathbb{E}\bigg[1 + \int_0^t \int_{\mathcal{O}} |\mathbf{v}_N|^2\,\mathrm{d}x\,\mathrm{d}\sigma\bigg].
\end{aligned}$$

So, we have

$$\begin{aligned}
&\mathbb{E}\bigg[\sup_{t\in(0,T)} \int_{\mathcal{O}} |\mathbf{v}_N(t)|^2\,\mathrm{d}x\bigg] + \mathbb{E}\bigg[\int_Q |\nabla \mathbf{v}_N|^2\,\mathrm{d}x\,\mathrm{d}t\bigg] \\
&\quad \leq c\,\mathbb{E}\bigg[\int_{\mathcal{O}} |\mathbf{v}_0|^2\,\mathrm{d}x + \int_0^T \int_{\mathcal{O}} |\mathbf{v}_N|^2\,\mathrm{d}x\,\mathrm{d}t\bigg] + \mathbb{E}\bigg[\sup_{t\in(0,T)} |J_2(t)|\bigg].
\end{aligned} \tag{3.9}$$

On account of the Burkholder–Davis–Gundy inequality (Lemma 2.1.5), Young's inequality and (3.3) we obtain

$$\begin{aligned}
\mathbb{E}\bigg[\sup_{t\in(0,T)} |J_2(t)|\bigg] &= \mathbb{E}\bigg[\sup_{t\in(0,T)} \bigg|\int_0^t \int_{\mathcal{O}} \mathbf{v}_N \cdot \mathbb{G}(\mathbf{v}_N)\,\mathrm{d}x\,\mathrm{d}W_\sigma^N\bigg|\bigg] \\
&= \mathbb{E}\bigg[\sup_{t\in(0,T)} \bigg|\sum_{i=1}^N \int_0^t \int_{\mathcal{O}} \mathbf{v}_N \cdot \mathbb{G}(\mathbf{v}_N) e_i\,\mathrm{d}x\,\mathrm{d}\beta_i(\sigma)\bigg|\bigg] \\
&= \mathbb{E}\bigg[\sup_{t\in(0,T)} \bigg|\sum_{i=1}^N \int_0^t \int_{\mathcal{O}} \mathbf{v}_N \cdot \mathbf{g}_i(\mathbf{v}_N)\,\mathrm{d}x\,\mathrm{d}\beta_i(\sigma)\bigg|\bigg] \\
&\leq c\,\mathbb{E}\bigg[\int_0^T \sum_{i=1}^N \bigg(\int_{\mathcal{O}} \mathbf{v}_N \cdot \mathbf{g}_i(\mathbf{v}_N)\,\mathrm{d}x\bigg)^2 \mathrm{d}t\bigg]^{1/2} \\
&\leq c\,\mathbb{E}\bigg[\bigg(\int_0^T \bigg(\sum_{i=1}^N \int_{\mathcal{O}} |\mathbf{v}_N|^2\,\mathrm{d}x \int_{\mathcal{O}} |\mathbf{g}_i(\mathbf{v}_N)|^2\,\mathrm{d}x\bigg)\,\mathrm{d}t\bigg]^{1/2} \\
&\leq c\,\mathbb{E}\bigg[1 + \int_0^T \bigg(\int_{\mathcal{O}} |\mathbf{v}_N|^2\,\mathrm{d}x\bigg)^2 \mathrm{d}t\bigg]^{1/2}
\end{aligned}$$

$$\leq \delta\,\mathbb{E}\bigg[\sup_{t\in(0,T)}\int_{\mathcal{O}}|\mathbf{v}_N|^2\,\mathrm{d}x\bigg]+c(\delta)\,\mathbb{E}\bigg[1+\int_0^T\int_{\mathcal{O}}|\mathbf{v}_N|^2\,\mathrm{d}x\,\mathrm{d}t\bigg].$$

This finally proves the claim for δ sufficiently small using Gronwall's lemma as well as $\Lambda_0=\mathbb{P}\circ\mathbf{v}_0^{-1}$. □

A similar proof yields to following estimate (one has to apply the $\frac{\beta}{2}$th power to (3.8) before taking expectations).

Corollary 3.1.2. *Let the assumptions of Theorem* 3.1.1 *be satisfied and in addition*

$$\int_{L^2_{\mathrm{div}}(\mathcal{O})}\|\mathbf{u}\|^{\beta}_{L^2(\mathcal{O})}\,\mathrm{d}\Lambda_0(\mathbf{u})<\infty,$$

for some $\beta\geq 2$. *Then we have*

$$\mathbb{E}\bigg[\sup_{t\in(0,T)}\int_{\mathcal{O}}|\mathbf{v}_N(t)|^2\,\mathrm{d}x+\int_Q|\nabla\mathbf{v}_N|^2\,\mathrm{d}x\,\mathrm{d}t\bigg]^{\beta/2}$$
$$\leq c_\beta\left(1+\mathbb{E}\bigg[\int_{L^2_{\mathrm{div}}(\mathcal{O})}\|\mathbf{u}\|^2_{L^2(\mathcal{O})}\,\mathrm{d}\Lambda_0(\mathbf{u})\bigg]^{\beta/2}\right).$$

3.2. Compactness

We have to pass to the limit in the convective term as well as the nonlinear noise coefficient. This will be a consequence of some compactness arguments. We consider $\boldsymbol{\varphi}\in C^\infty_{c,\mathrm{div}}(\mathcal{O})$ and obtain by (3.5)

$$\begin{aligned}\int_{\mathcal{O}}\mathbf{v}_N(t)\cdot\boldsymbol{\varphi}\,\mathrm{d}x&=\int_{\mathcal{O}}\mathbf{v}_N(t)\cdot\mathcal{P}^l_N\boldsymbol{\varphi}\,\mathrm{d}x\\&=\int_G\mathbf{v}_0\cdot\mathcal{P}^l_N\boldsymbol{\varphi}\,\mathrm{d}x+\int_0^t\int_{\mathcal{O}}\mathbf{H}_N:\nabla\mathcal{P}^l_N\boldsymbol{\varphi}\,\mathrm{d}x\,\mathrm{d}\sigma\\&\quad+\int_{\mathcal{O}}\int_0^t\mathbb{G}(\mathbf{v}_N)\mathrm{d}W^N_\sigma\cdot\mathcal{P}^l_N\boldsymbol{\varphi}\,\mathrm{d}x,\\\mathbf{H}_N&:=-\nu\nabla\mathbf{v}_N+\mathbf{v}_N\otimes\mathbf{v}_N.\end{aligned}$$

Here $\mathcal{P}^l_N$ denotes the projection into $\mathcal{X}_N$ with respect to the $W^{l,2}_{0,\mathrm{div}}(\mathcal{O})$ inner product. From the a priori estimates in Theorem 3.1.1 and Corollary 3.1.2 we obtain

$$\mathbf{H}_N\in L^{p_0}(\Omega\times Q;\mathbb{P}\otimes\mathcal{L}^{d+1}) \tag{3.10}$$

for some $p_0>1$ uniformly in N (provided $\beta>2$). Let us consider the functional

$$\mathscr{H}_N(t,\boldsymbol{\varphi}):=\int_0^t\int_{\mathcal{O}}\mathbf{H}_N:\nabla\mathcal{P}^l_N\boldsymbol{\varphi}\,\mathrm{d}x\,\mathrm{d}\sigma,\quad\boldsymbol{\varphi}\in C^\infty_{c,\mathrm{div}}(\mathcal{O}).$$

Then we deduce from (3.10) and the embedding $W_0^{\tilde{l},p_0}(\mathcal{O}) \hookrightarrow W_0^{l,2}(\mathcal{O})$ for $\tilde{l} \geq l + d\big(\frac{1}{p_0} - \frac{1}{2}\big)$ the estimate

$$\mathbb{E}\bigg[\big\|\mathscr{H}_N\big\|_{W^{1,p_0}([0,T];W^{-\tilde{l},p_0}_{\mathrm{div}}(\mathcal{O}))}\bigg] \leq c.$$

For the stochastic term we use Lemma 2.5.3 and (3.3) to estimate for all $\alpha < 1/2$

$$\begin{aligned}\mathbb{E}\bigg[\Big\|\int_0^\cdot \mathbb{G}(\mathbf{v}_N)\,\mathrm{d}W_\sigma^N\Big\|_{W^{\alpha,2}(0,T;L^2(\mathcal{O}))}\bigg] &\leq c\,\mathbb{E}\bigg[\int_0^T \|\mathbb{G}(\mathbf{v}_N)\|^2_{L_2(\mathfrak{U},L^2(\mathcal{O}))}\,\mathrm{d}t\bigg]\\ &\leq c\,\mathbb{E}\bigg[\sum_k \int_0^T\int_{\mathcal{O}} |\mathbf{g}_k(\cdot,\mathbf{v}_N)|^2\,\mathrm{d}x\,\mathrm{d}t\bigg] \leq c\,\mathbb{E}\bigg[1+\int_Q |\mathbf{v}_N|^2\,\mathrm{d}x\,\mathrm{d}t\bigg].\end{aligned}$$

So we have due to Theorem 3.1.1 and $p_0 \leq 2$ that

$$\mathbb{E}\bigg[\Big\|\int_0^\cdot \mathbb{G}(\mathbf{v}_N)\,\mathrm{d}W_\sigma^N\Big\|_{W^{\alpha,p_0}((0,T);L^2(\mathcal{O}))}\bigg] \leq c.$$

Combining the both informations above shows

$$\mathbb{E}\Big[\|\mathbf{v}_N\|_{W^{\alpha,p_0}(0,T;W^{-\tilde{l},p_0}_{\mathrm{div}}(\mathcal{O}))}\Big] \leq c. \tag{3.11}$$

An interpolation with $L^{p_0}(0,T;W^{1,p_0}_{0,\mathrm{div}}(\mathcal{O}))$ implies on account of Theorem 3.1.1

$$\mathbb{E}\Big[\|\mathbf{v}_N\|_{W^{\kappa,p_0}(0,T;L^{p_0}_{\mathrm{div}}(\mathcal{O}))}\Big] \leq c \tag{3.12}$$

for some $\kappa > 0$. Note that we have

$$\begin{aligned}W^{\kappa,p_0}(0,T;L^{p_0}_{\mathrm{div}}(\mathcal{O})) \cap L^2(0,T;W^{1,2}_{0,\mathrm{div}}(\mathcal{O})) \cap L^\infty(0,T;L^2_{\mathrm{div}}(\mathcal{O}))\\ \hookrightarrow\hookrightarrow L^r(0,T;L^r_{\mathrm{div}}(\mathcal{O}))\end{aligned} \tag{3.13}$$

compactly for all $r < \frac{10}{3}$. We will use this embedding in order to show compactness of $\mathbf{v}_N$. We consider the path space

$$\mathscr{Y} := L^r(0,T;L^r(\mathcal{O})) \otimes C([0,T],\mathfrak{U}_0) \otimes L^2_{\mathrm{div}}(\mathcal{O}).$$

In the following we introduce some notations.

- $\nu_{\mathbf{v}_N}$ is the law of $\mathbf{v}_N$ on $L^r(0,T;L^r(\mathcal{O}))$;
- ν_W is the law of W on $C([0,T],\mathfrak{U}_0)$, where $\mathfrak{U}_0$ is defined in (2.27);
- $\boldsymbol{\nu}_N$ is the joint law of $\mathbf{v}_N$, W, $\mathbf{v}_0$ on $\mathscr{Y}$.

We consider the ball $\mathcal{B}_R$ in the space

$$W^{\kappa,p_0}(0,T;L^{p_0}_{\mathrm{div}}(\mathcal{O})) \cap L^2(0,T;W^{1,2}_{0,\mathrm{div}}(\mathcal{O})) \cap L^\infty(0,T;L^2_{\mathrm{div}}(\mathcal{O}))$$

and obtain for its complement $\mathcal{B}_R^C$ by Theorem 3.1.1 and (3.12)

$$\begin{aligned}\nu_{\mathbf{v}_N}(\mathcal{B}_R^C) &= \mathbb{P}\Big(\|\mathbf{v}_N\|_{W^{\kappa,p_0}(L^{p_0})} + \|\mathbf{v}_N\|_{L^2(W^{1,2})} + \|\mathbf{v}_N\|_{L^\infty(L^2)} \geq R\Big)\\ &\leq \frac{1}{R}\,\mathbb{E}\Big[\|\mathbf{v}_N\|_{W^{\kappa,p_0}(L^{p_0})} + \|\mathbf{v}_N\|_{L^2(W^{1,2})} + \|\mathbf{v}_N\|_{L^\infty(L^2)}\Big] \leq \frac{c}{R}.\end{aligned}$$

So, for a fixed $\eta > 0$, we find $R(\eta)$ with

$$\nu_{\mathbf{v}_N}(\mathcal{B}_{R(\eta)}) \geq 1 - \frac{\eta}{3},$$

i.e., $\nu_{\mathbf{v}_N}$ is tight by (3.13). Since also the law ν_W is tight, as being a Radon measure on the Polish space $C([0,T],\mathfrak{U}_0)$, there exists a compact set $C_\eta \subset C([0,T],\mathfrak{U}_0)$ such that $\nu_W(C_\eta) \geq 1 - \frac{\eta}{3}$. For the same reason we find compact subsets of $L^2_{\mathrm{div}}(\mathcal{O})$ such that its measure Λ_0 is smaller than $1 - \frac{\eta}{3}$. Hence, we can find a compact subset $\mathscr{V}_\eta \subset \mathscr{V}$ such that $\boldsymbol{\nu}_N(\mathscr{V}_\eta) \geq 1 - \eta$. Thus, $\{\boldsymbol{\nu}_N,\ N \in \mathbb{N}\}$ is tight in the same space. Prokhorov's Theorem (see Lemma 2.6.1) therefore implies that $\boldsymbol{\nu}_N$ is also relatively weakly compact. This means that we have a weakly convergent subsequence with limit $\boldsymbol{\nu}$. Now we use Skorohod's representation theorem (see Lemma 2.6.2) to infer the existence of a probability space $(\tilde{\Omega},\tilde{\mathcal{F}},\tilde{\mathbb{P}})$, a sequence $(\tilde{\mathbf{v}}_N,\tilde{W}^N,\tilde{\mathbf{v}}_{0,N})$ and $(\tilde{\mathbf{v}},\tilde{W},\tilde{\mathbf{v}}_0)$ on $(\tilde{\Omega},\tilde{\mathcal{F}},\tilde{\mathbb{P}})$, both with values in $\mathscr{V}$, such that the following holds.

- The laws of $(\tilde{\mathbf{v}}_N,\tilde{W}^N,\tilde{\mathbf{v}}_{0,N})$ and $(\tilde{\mathbf{v}},\tilde{W},\tilde{\mathbf{v}}_0)$ under $\tilde{\mathbb{P}}$ coincide with $\boldsymbol{\nu}_N$ and $\boldsymbol{\nu}$.
- We have the convergences

$$\begin{aligned}
\tilde{\mathbf{v}}_N &\longrightarrow \tilde{\mathbf{v}} \quad \text{in} \quad L^r(0,T;L^r(\mathcal{O})),\\
\tilde{W}^N &\longrightarrow \tilde{W} \quad \text{in} \quad C([0,T],\mathfrak{U}_0),\\
\tilde{\mathbf{v}}_{0,N} &\longrightarrow \tilde{\mathbf{v}}_0 \quad \text{in} \quad L^2(\mathcal{O}),
\end{aligned}$$

  $\tilde{\mathbb{P}}$-a.s.

After choosing a subsequence we obtain by Vitali's convergence theorem

$$\tilde{W}^N \longrightarrow \tilde{W} \quad \text{in} \quad L^2(\tilde{\Omega},\tilde{\mathcal{F}},\tilde{\mathbb{P}};C([0,T],\mathfrak{U}_0)), \tag{3.14}$$

$$\tilde{\mathbf{v}}_N \longrightarrow \tilde{\mathbf{v}} \quad \text{in} \quad L^q(\tilde{\Omega}\times Q;\tilde{\mathbb{P}}\otimes\mathcal{L}^{d+1}), \tag{3.15}$$

$$\tilde{\mathbf{v}}_{0,N} \longrightarrow \tilde{\mathbf{v}}_0 \quad \text{in} \quad L^2(\tilde{\Omega}\times\mathcal{O},\tilde{\mathbb{P}}\otimes\mathcal{L}^{d+1}), \tag{3.16}$$

for all $q < \min\{\beta, r\}$. Now we introduce the filtration on the new probability space which ensure the correct measurabilities of the new variables. We denote by $\mathbf{r}_t$ the operator of restriction to the interval $[0,t]$ acting on various path spaces. In particular, if X stands for one of the path spaces $L^r(0,T;L^r(\mathcal{O}))$ or $C([0,T],\mathfrak{U}_0)$ and $t \in [0,T]$, we define

$$\mathbf{r}_t : X \to X|_{[0,t]}, \quad f \mapsto f|_{[0,t]}. \tag{3.17}$$

Clearly, $\mathbf{r}_t$ is a continuous mapping. Let $(\tilde{\mathcal{F}}_t)_{t\geq 0}$ and $(\tilde{\mathcal{F}}^N_t)_{t\geq 0}$ be the $\tilde{\mathbb{P}}$-augmented canonical filtration of the processes $\big(\tilde{\mathbf{v}},\tilde{W}\big)$ and $\big(\tilde{\mathbf{v}}_N,\tilde{W}^N\big)$, respectively, that is

$$\begin{aligned}
\tilde{\mathcal{F}}_t &= \sigma\Big(\sigma\big(\mathbf{r}_t\tilde{\mathbf{v}},\mathbf{r}_t\tilde{W}\big)\cup\big\{\mathcal{N}\in\tilde{\mathcal{F}};\ \tilde{\mathbb{P}}(\mathcal{N})=0\big\}\Big), \qquad t\in[0,T].\\
\tilde{\mathcal{F}}^N_t &= \sigma\Big(\sigma\big(\mathbf{r}_t\tilde{\mathbf{v}}_N,\mathbf{r}_t\tilde{W}^N\big)\cup\big\{\mathcal{N}\in\tilde{\mathcal{F}};\ \tilde{\mathbb{P}}(\mathcal{N})=0\big\}\Big), \quad t\in[0,T].
\end{aligned}$$

This definition guarantees that the processes are adapted and we can define stochastic integrals.

3.3. The system on the new probability space

Now we are going to show that the approximated equations also hold on the new probability space. Similar to the proof of Theorem 2.4.4 we use the elementary method from [9] which has already been generalized to different settings (see, for instance, [8, 20]). The keystone is to identify not only the quadratic variation of the corresponding martingale but also its cross variation with the limit Wiener process obtained through compactness. First we notice that $\tilde{W}^N$ has the same law as W. As a consequence, there exists a collection of mutually independent real-valued $(\tilde{\mathcal{F}}_t^N)_{t\geq 0}$-Wiener processes $(\tilde{\beta}_k^N)_k$ such that $\tilde{W}^N = \sum_k \tilde{\beta}_k^N e_k$. In particular, there exists a collection of mutually independent real-valued $(\tilde{\mathcal{F}}_t)_{t\geq 0}$-Wiener processes $(\tilde{\beta}_k)_{k\geq 1}$ such that $\tilde{W} = \sum_k \tilde{\beta}_k e_k$. We abbreviate $\tilde{W}^{N,N} := \sum_{k=1}^N \tilde{\beta}_k^N e_k$. Let us now define for all $t \in [0,T]$ and $\boldsymbol{\varphi} \in C^\infty_{c,\mathrm{div}}(\mathcal{O})$ the functionals

$$\begin{aligned}
\mathfrak{M}(\mathbf{u},\mathbf{u}_0)_t &= \int_{\mathcal{O}} \mathbf{u}(t)\cdot\boldsymbol{\varphi}\,\mathrm{d}x - \int_{\mathcal{O}} \mathbf{u}_0\cdot\boldsymbol{\varphi}\,\mathrm{d}x + \int_0^t\int_{\mathcal{O}} \mathbf{u}\otimes\mathbf{u} : \nabla\mathcal{P}_N^l\boldsymbol{\varphi}\,\mathrm{d}x\,\mathrm{d}\sigma \\
&\quad + \int_0^t\int_{\mathcal{O}} \nu\nabla\mathbf{u} : \nabla\mathcal{P}_N^l\boldsymbol{\varphi}\,\mathrm{d}x\,\mathrm{d}\sigma, \\
\mathfrak{N}(\mathbf{u})_t &= \sum_{k=1}^N \int_0^t \left(\int_{\mathcal{O}} \mathbf{g}_k(\mathbf{u})\cdot\mathcal{P}_N^l\boldsymbol{\varphi}\,\mathrm{d}x\right)^2 \mathrm{d}\sigma, \\
\mathfrak{N}_k(\mathbf{u})_t &= \int_0^t\int_{\mathcal{O}} \mathbf{g}_k(\mathbf{u})\cdot\mathcal{P}_N^l\boldsymbol{\varphi}\,\mathrm{d}x\,\mathrm{d}\sigma,
\end{aligned}$$

let $\mathfrak{M}(\mathbf{v}_N,\mathbf{v}_0)_{s,t}$ denote the increment $\mathfrak{M}(\mathbf{v}_N,\mathbf{v}_0)_t - \mathfrak{M}(\mathbf{v}_N,\mathbf{v}_0)_s$ and similarly for $\mathfrak{N}(\mathbf{v}_N)_{s,t}$ and $\mathfrak{N}_k(\mathbf{v}_N)_{s,t}$. Note that the proof will be complete once we show that the process $\mathfrak{M}(\tilde{\mathbf{v}}_N)$ is an $(\tilde{\mathcal{F}}_t^N)_{t\geq 0}$-martingale and its quadratic and cross variations satisfy, respectively,

$$\langle\langle \mathfrak{M}(\tilde{\mathbf{v}}_N,\tilde{\mathbf{v}}_0)\rangle\rangle = \mathfrak{N}(\tilde{\mathbf{v}}^N), \qquad \langle\langle \mathfrak{M}(\tilde{\mathbf{v}}_N,\tilde{\mathbf{v}}_0),\tilde{\beta}_k\rangle\rangle = \mathfrak{N}_k(\tilde{\mathbf{v}}_N). \tag{3.18}$$

Indeed, in that case we have

$$\left\langle\left\langle \mathfrak{M}(\tilde{\mathbf{v}}_N,\tilde{\mathbf{v}}_0) - \int_0^\cdot\int_G \mathbb{G}(\tilde{\mathbf{v}}_N)\,\mathrm{d}\tilde{W}^{N,N}\cdot\mathcal{P}_N^l\boldsymbol{\varphi}\,\mathrm{d}x\right\rangle\right\rangle = 0 \tag{3.19}$$

which implies the desired equation on the new probability space. Let us verify (3.18). To this end, we claim that with the above uniform estimates in hand, the mappings

$$(\mathbf{v}_N,\mathbf{v}_0)\mapsto \mathfrak{M}(\mathbf{v}_N,\mathbf{v}_0)_t, \quad \mathbf{v}_N\mapsto\mathfrak{N}(\mathbf{v}_N)_t, \quad \mathbf{v}_N\mapsto\mathfrak{N}_k(\mathbf{v}_N)_t$$

are well defined and measurable on a subspace of the path space where the joint law of $(\tilde{\mathbf{v}}_N,\tilde{\mathbf{v}}_0)$ is supported, i.e., the uniform estimates from Theorem 3.1.1 hold true. Indeed, in the case of $\mathfrak{N}(\mathbf{u})_t$ we have by (3.3) and the continuity of $\mathcal{P}_N^l$

in $L^2(\mathcal{O})$

$$\sum_{k=1}^{N}\int_0^t\left(\int_{\mathcal{O}}\mathbf{g}_k(\mathbf{u})\cdot\mathcal{P}_N^l\varphi\,\mathrm{d}x\right)^2\mathrm{d}\sigma\le c(\boldsymbol{\varphi})\sum_{k=1}^{\infty}\int_0^t\int_{\mathcal{O}}|\mathbf{g}_k(\mathbf{u})|^2\,\mathrm{d}x\,\mathrm{d}\sigma$$
$$\le c(\boldsymbol{\varphi})\left(1+\int_Q|\mathbf{u}|^2\,\mathrm{d}x\,\mathrm{d}t\right)$$

which is finite for $\mathbf{u}\in L^2(Q)$. $\mathfrak{M}(\mathbf{v}_N,\mathbf{v}_0)$ and $\mathfrak{N}_k(\mathbf{v}_N)_t$ can be handled similarly and therefore, the following random variables have the same laws

$$\mathfrak{M}(\mathbf{v}_N,\mathbf{v}_0)\sim^d\mathfrak{M}(\tilde{\mathbf{v}}_N,\tilde{\mathbf{v}}_0),\quad\mathfrak{N}(\mathbf{v}_N)\sim^d\mathfrak{N}(\tilde{\mathbf{v}}_N),\quad\mathfrak{N}_k(\mathbf{v}_N)\sim^d\mathfrak{N}_k(\tilde{\mathbf{v}}_N).$$

Let us now fix times $s,t\in[0,T]$ such that $s<t$ and let

$$h:\mathscr{V}\big|_{[0,s]}\to[0,1]$$

be a continuous function. Since

$$\mathfrak{M}(\mathbf{v}_N,\mathbf{v}_0)_t=\int_0^t\int_{\mathcal{O}}\mathbb{G}(\mathbf{v}_N)\,\mathrm{d}W_\sigma^N\cdot\mathcal{P}_N\boldsymbol{\varphi}\,\mathrm{d}x=\sum_{k=1}^{N}\int_0^t\int_{\mathcal{O}}\mathbf{g}_k(\mathbf{v}_N)\cdot\mathcal{P}_N\boldsymbol{\varphi}\,\mathrm{d}x\,\mathrm{d}\beta_k$$

is a square integrable $(\mathcal{F}_t)_{t\ge0}$-martingale, we infer that

$$\big[\mathfrak{M}(\mathbf{v}_N,\mathbf{v}_0)\big]^2-\mathfrak{N}(\mathbf{v}^N),\quad\mathfrak{M}(\mathbf{v}_N)\beta_k-\mathfrak{N}_k(\mathbf{v}_N),$$

are $(\mathcal{F}_t)_{t\ge0}$-martingales. Let $\mathbf{r}_s$ be the restriction of a function to the interval $[0,s]$. Then it follows from the equality of laws that

$$\tilde{\mathbb{E}}\big[h\big(\mathbf{r}_s\tilde{\mathbf{v}}_N,\mathbf{r}_s\tilde{W}^N\big)\mathfrak{M}(\tilde{\mathbf{v}}_N,\tilde{\mathbf{v}}_0)_{s,t}\big]=\mathbb{E}\big[h\big(\mathbf{r}_s\mathbf{v}_N,\mathbf{r}_sW\big)\mathfrak{M}(\mathbf{v}_N,\mathbf{v}_0)_{s,t}\big]=0,$$
$$\tilde{\mathbb{E}}\left[h\big(\mathbf{r}_s\tilde{\mathbf{v}}_N,\mathbf{r}_s\tilde{W}^N\big)\Big([\mathfrak{M}(\tilde{\mathbf{v}}_N,\tilde{\mathbf{v}}_0)^2]_{s,t}-\mathfrak{N}(\tilde{\mathbf{v}}_N)_{s,t}\Big)\right]$$
$$=\mathbb{E}\left[h\big(\mathbf{r}_s\mathbf{v}_N,\mathbf{r}_sW\big)\Big([\mathfrak{M}(\mathbf{v}_N,\mathbf{v}_0)^2]_{s,t}-\mathfrak{N}(\mathbf{v}_N)_{s,t}\Big)\right]=0,$$
$$\tilde{\mathbb{E}}\left[h\big(\mathbf{r}_s\tilde{\mathbf{v}}_N,\mathbf{r}_s\tilde{W}^N\big)\Big([\mathfrak{M}(\tilde{\mathbf{v}}_N,\tilde{\mathbf{v}}_0)\tilde{\beta}_k^N]_{s,t}-\mathfrak{N}_k(\tilde{\mathbf{v}}_N)_{s,t}\Big)\right]$$
$$=\mathbb{E}\left[h\big(\mathbf{r}_s\mathbf{v}_N,\mathbf{r}_sW\big)\Big([\mathfrak{M}(\mathbf{v}_N,\mathbf{v}_0)\beta_k]_{s,t}-\mathfrak{N}_k(\mathbf{v}_N)_{s,t}\Big)\right]=0.$$

So we have shown (3.18) and hence (3.19). This means on the new probability space $(\tilde{\Omega},\tilde{\mathcal{F}},\tilde{\mathbb{P}})$ we have the equations $(k=1,\dots,N)$

$$\int_{\mathcal{O}}\mathrm{d}\tilde{\mathbf{v}}_N\cdot\mathbf{w}_k\,\mathrm{d}x+\int_{\mathcal{O}}\nu\nabla\tilde{\mathbf{v}}_N:\nabla\mathbf{w}_k\,\mathrm{d}x\,\mathrm{d}t$$
$$=\int_{\mathcal{O}}\tilde{\mathbf{v}}_N\otimes\tilde{\mathbf{v}}_N:\nabla\mathbf{w}_k\,\mathrm{d}x\,\mathrm{d}t+\int_G\mathbb{G}(\tilde{\mathbf{v}}_N)\,\mathrm{d}\tilde{W}_\sigma^{N,N}\cdot\mathbf{w}_k\,\mathrm{d}x,$$
$$\tilde{\mathbf{v}}_N(0)=\mathcal{P}_N\tilde{\mathbf{v}}_0,\tag{3.20}$$

and the convergences

$$\begin{aligned} \tilde{\mathbf{v}}_N &\rightharpoonup \tilde{\mathbf{v}} && \text{in} \quad L^2(\tilde{\Omega},\tilde{\mathcal{F}},\tilde{\mathbb{P}};L^2(0,T;W^{1,2}_{0,\mathrm{div}}(\mathcal{O}))),\\ \tilde{\mathbf{v}}_N\otimes\tilde{\mathbf{v}}_N &\rightharpoonup \tilde{\mathbf{v}}\otimes\tilde{\mathbf{v}} && \text{in} \quad L^{q/2}(\tilde{\Omega},\tilde{\mathcal{F}},\tilde{\mathbb{P}};L^{q/2}(Q)),\\ \mathbb{G}(\tilde{\mathbf{v}}_N) &\rightharpoonup \mathbb{G}(\tilde{\mathbf{v}}) && \text{in} \quad L^2(\tilde{\Omega},\tilde{\mathcal{F}},\tilde{\mathbb{P}};L^2(0,T;L_2(\mathfrak{U},L^2(G)))). \end{aligned} \tag{3.21}$$

We obtain from (3.14), (3.20) and (3.21) the limit equation

$$\begin{aligned} &\int_{\mathcal{O}}\tilde{\mathbf{v}}(t)\cdot\boldsymbol{\varphi}\,\mathrm{d}x+\int_0^t\int_{\mathcal{O}}\nu\nabla\tilde{\mathbf{v}}:\nabla\boldsymbol{\varphi}\,\mathrm{d}x+\int_0^t\int_{\mathcal{O}}\tilde{\mathbf{v}}\otimes\tilde{\mathbf{v}}:\nabla\boldsymbol{\varphi}\,\mathrm{d}x\,\mathrm{d}\sigma\\ &\quad=\int_{\mathcal{O}}\int_0^t\mathbb{G}(\tilde{\mathbf{v}})\,\mathrm{d}\tilde{W}_\sigma\cdot\boldsymbol{\varphi}\,\mathrm{d}x \end{aligned} \tag{3.22}$$

for all $\boldsymbol{\varphi}\in C^\infty_{c,\mathrm{div}}(\mathcal{O})$. The limit in the stochastic term needs some explanations. We have the convergences

$$\begin{aligned} \tilde{W}^N &\longrightarrow \tilde{W} && \text{in} \quad C([0,T],\mathfrak{U}_0),\\ \mathbb{G}(\tilde{\mathbf{v}}_N) &\longrightarrow \mathbb{G}(\tilde{\mathbf{v}}) && \text{in} \quad L^2(0,T;L_2(U,L^2(\mathcal{O}))), \end{aligned}$$

in probability. For the second one we use (3.3) and (3.15). These convergences imply

$$\int_0^t\mathbb{G}(\tilde{\mathbf{v}}_N)\,\mathrm{d}\tilde{W}^N_\sigma\longrightarrow\int_0^t\mathbb{G}(\tilde{\mathbf{v}})\,\mathrm{d}\tilde{W}_\sigma\quad\text{in}\quad L^2(0,T;L^2(\mathcal{O}))$$

in probability by Lemma 2.5.4. So we can pass to the limit in the stochastic integral.

Remark 3.3.1. According to the remarks in [22] (beginning of the proof of Thm. 2.7. on p. 9) it is possible to choose the new probability space obtained by Skorokhod's representation theorem as

$$(\tilde{\Omega},\tilde{\mathcal{F}},\tilde{\mathbb{P}})=([0,1];\overline{\mathcal{B}([0,1])};\mathcal{L}^1|_{[0,1]}).$$

4. Compressible fluids

In this section we discuss the questions of how the Navier–Stokes equations for *compressible* fluids are affected by random perturbations. If compressibility is taken into account, the basic *field equations* in $Q=(0,T)\times\mathcal{O}$ read as

$$\mathrm{d}(\varrho\mathbf{u})+\mathrm{div}(\varrho\mathbf{u}\otimes\mathbf{u})\,\mathrm{d}t=\mathrm{div}\,\mathbf{S}\,\mathrm{d}t-\nabla p\,\mathrm{d}t+\varrho\mathbb{F}\,\mathrm{d}W, \tag{4.1}$$

$$\mathrm{d}\varrho+\mathrm{div}(\varrho\mathbf{u})\,\mathrm{d}t=0. \tag{4.2}$$

The unknowns are velocity $\mathbf{u}$ (for which we suppose the no-slip boundary condition $\mathbf{u}|_{\partial\mathcal{O}}=0$) and density ϱ. All quantities are defined over a probability space $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\geq0},\mathbb{P})$ with a complete, right-continuous filtration. Equation (4.1) – *the equation of continuity* – represents a mathematical formulation of the physical principle of mass conservation. Equation (4.2) – *the momentum equation* –

reflects Newton's second law of momentum conservation. Here $\mathbf{S}$ is the viscous stress tensor for which we assume Newton's rheological law, i.e.,

$$\mathbf{S} = \mathbf{S}(\nabla \mathbf{u}) = \nu \left(\nabla \mathbf{u} + \nabla^t \mathbf{u} - \frac{2}{3} \operatorname{div} \mathbf{u} \mathbf{I}_d \right) + \lambda \operatorname{div} \mathbf{u} \mathbf{I}_d, \tag{4.3}$$

where $\nu > 0$, $\lambda \geq -\frac{1}{3}\nu$ are constant viscosity coefficients. The symbol $p = p(\varrho)$ denotes the pressure, typically given by the isentropic state equation

$$p(\varrho) = a\varrho^\gamma, \ a > 0. \tag{4.4}$$

The parameter a is the squared reciprocal of the Mach number, that is, the ratio of flow velocity and speed of sound. For the adiabatic exponent γ, also called the isentropic expansion factor, we suppose $\gamma > \frac{3}{2}$ ($\gamma > 1$ if $d = 2$). Random effects are incorporated in the forcing term $\varrho \mathbb{F} \, \mathrm{d}W$. The process W is a cylindrical Wiener process, that is, $W(t) = \sum_{k \geq 1} \beta_k(t) e_k$ with $(\beta_k)_{k \geq 1}$ being mutually independent real-valued standard Wiener processes relative to $(\mathcal{F}_t)_{t \geq 0}$. Here $(e_k)_{k \geq 1}$ denotes a complete orthonormal system in a separable Hilbert space $\mathfrak{U}$ (e.g., $\mathfrak{U} = L^2(\mathcal{O})$ would be a natural choice). The diffusion coefficient $\mathbb{F}$ belongs to the class of Hilbert–Schmidt operators $L_2(\mathfrak{U}; L^2(\mathcal{O}))$ and satisfies uniformly in $x \in \mathcal{O}$

$$\sum_{k \geq 1} |\mathbf{F}_k(x)|^2 \leq C, \tag{4.5}$$

where $\mathbf{F}_k = \mathbb{F} e_k$. This assumption can be generalized in several ways. In particular, $\mathbb{F}$ can depend in a nonlinear way on ϱ and $\mathbf{u}$ as in [8] and [34].

The approach to (4.1)–(4.4) we present here is based on the concept of the *finite energy weak martingale solution* introduced by Breit and Hofmanová [8]. From the probabilistic point of view, finite energy weak martingale solutions to (4.1)–(4.4) are weak solutions in the sense that neither the underlying probability space nor the driving Wiener process can be specified in advance and these stochastic elements become part of the solution. As discussed in Section 2.4, this is intimately related to the lack of uniqueness (as in the incompressible case, uniqueness is a big open problem for the compressible Navier–Stokes equations). From the PDE point of view, finite energy weak martingale solutions are also weak, that is, (4.1) and (4.2) are satisfied in the sense of distributions. In addition, the continuity equation (4.2) is satisfied in the renormalized sense and an energy inequality holds true. The concept of renormalized solution was introduced in [12] in the context of linear transport equations. In compressible fluid mechanics, it is an essential tool to pass to the limit in the nonlinear pressure. The energy inequality has to be understood as an integral part of the definition of a solution. It encodes certain pieces of information concerning stability which would be otherwise lost in the construction process of conventional weak martingale solutions. Its important role is demonstrated by the fact that it allows to prove a weak-strong uniqueness principle, cf. [7].

The previous discussion is summarized in the following definition. We remark that the initial law describes the initial state of $(\varrho, \varrho\mathbf{u})$ rather than $(\varrho, \mathbf{u})$ (the natural integrability of $\varrho\mathbf{u}$ is $\frac{2\gamma}{\gamma+1}$).

Definition 4.0.1 (Finite energy weak martingale solution). Let Λ be a Borel probability measure on $L^\gamma(\mathcal{O}) \times L^{\frac{2\gamma}{\gamma+1}}(\mathcal{O})$. Then

$$\big((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}), \varrho, \mathbf{u}, W\big)$$

is called a *finite energy weak martingale solution* to (4.1)–(4.4) with the initial law Λ provided

(a) $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ is a stochastic basis with a complete right-continuous filtration,
(b) W is a cylindrical $(\mathcal{F}_t)_{t\geq 0}$-Wiener process,
(c) the density ϱ satisfies $\varrho \geq 0$, $t \mapsto \langle \varrho(t), \psi\rangle \in C([0,T])$ for any $\psi \in C^\infty(\mathcal{O})$ $\mathbb{P}$-a.s., the stochastic process $t \mapsto \langle \varrho(t), \psi\rangle$ is $(\mathcal{F}_t)_{t\geq 0}$-adapted, and

$$\mathbb{E}\bigg[\sup_{t\in(0,T)} \|\varrho(t)\|_{L^\gamma_x}^p\bigg] < \infty \text{ for all } p \in (1,\infty), \tag{4.6}$$

(d) the velocity field $\mathbf{u} \in L^2(\Omega \times (0,T); W_0^{1,2}(\mathcal{O}))$ satisfies

$$\mathbb{E}\bigg(\int_0^T \|\mathbf{u}\|^2_{W^{1,2}_x}\ \mathrm{d}t\bigg)^p < \infty \text{ for all } p \in (1,\infty), \tag{4.7}$$

(e) the momentum $\varrho\mathbf{u}$ satisfies $t \mapsto \langle \varrho\mathbf{u}(t), \boldsymbol{\varphi}\rangle \in C([0,T])$ for any $\boldsymbol{\varphi} \in C_c^\infty(\mathcal{O})$ $\mathbb{P}$-a.s., the stochastic process $t \mapsto \langle \varrho\mathbf{u}(t), \boldsymbol{\varphi}\rangle$ is $(\mathcal{F}_t)$-adapted,

$$\mathbb{E}\bigg[\sup_{t\in(0,T)} \|\varrho\mathbf{u}(t)\|^p_{L^{\frac{2\gamma}{\gamma+1}}_x}\bigg] < \infty \text{ for all } p \in (1,\infty), \tag{4.8}$$

(f) $\Lambda = \mathbb{P} \circ \big(\varrho(0), \varrho\mathbf{u}(0)\big)^{-1}$,
(g) for all $\psi \in C^\infty(\mathcal{O})$ and $\boldsymbol{\varphi} \in C_c^\infty(\mathcal{O})$ and all $t \in [0,T]$ there holds $\mathbb{P}$-a.s.

$$\begin{aligned}
\big\langle \varrho(t), \psi\big\rangle &= \big\langle \varrho(0), \psi\big\rangle + \int_0^t \big\langle \varrho\mathbf{u}, \nabla\psi\big\rangle\,\mathrm{d}s,\\
\big\langle \varrho\mathbf{u}(t), \boldsymbol{\varphi}\big\rangle &= \big\langle \varrho\mathbf{u}(0), \boldsymbol{\varphi}\big\rangle + \int_0^t \big\langle \varrho\mathbf{u}\otimes\mathbf{u}, \nabla\boldsymbol{\varphi}\big\rangle\,\mathrm{d}s - \int_0^t \big\langle \mathbf{S}(\nabla\mathbf{u}), \nabla\boldsymbol{\varphi}\big\rangle\,\mathrm{d}s\\
&\quad + \int_0^t \big\langle p(\varrho), \operatorname{div}\boldsymbol{\varphi}\big\rangle\,\mathrm{d}s + \int_0^t \big\langle \varrho\mathbb{F}\,\mathrm{d}W, \boldsymbol{\varphi}\big\rangle,
\end{aligned}$$

(h) for all $\varphi \in C_c^\infty([0,T))$, $\varphi \geq 0$, the following energy inequality holds true $\mathbb{P}$-a.s.

$$\begin{aligned} &-\int_0^T \partial_t \varphi \int_{\mathcal{O}} \Big[\frac{1}{2}\varrho|\mathbf{u}|^2 + P(\varrho)\Big]\,\mathrm{d}x\,\mathrm{d}t + \int_0^T \varphi \int_{\mathcal{O}} \mathbf{S}(\nabla \mathbf{u}) : \nabla \mathbf{u}\,\mathrm{d}x\,\mathrm{d}t \\ &\quad \leq \varphi(0)\int_{\mathcal{O}} \Big[\frac{1}{2}\frac{|(\varrho\mathbf{u})(0)|^2}{\varrho(0)} + P(\varrho(0))\Big]\,\mathrm{d}x + \frac{1}{2}\int_0^T \varphi \int_{\mathcal{O}} \sum_{k=1}^\infty \varrho|\mathbf{F}_k|^2\,\mathrm{d}x\,\mathrm{d}t \\ &\quad\quad + \sum_{k=1}^\infty \int_0^T \varphi \int_{\mathcal{O}} \varrho \mathbf{F}_k \cdot \mathbf{u}\,\mathrm{d}x\,\mathrm{d}\beta_k, \end{aligned} \tag{4.9}$$

where

$$P(\rho) = \rho \int_0^\rho \frac{p(z)}{z^2}\,\mathrm{d}z = \frac{a}{\gamma - 1}\rho^\gamma$$

is the pressure potential,

(i) if $b \in C^1(\mathbb{R})$ such that $b'(z) = 0$ for all $z \geq M_b$, then for all $\psi \in C^\infty(\mathcal{O})$ and all $t \in [0,T]$ there holds $\mathbb{P}$-a.s.

$$\begin{aligned} \big\langle b(\varrho(t)), \psi\big\rangle = \big\langle b(\varrho(0)), \psi\big\rangle &+ \int_0^t \big\langle b(\varrho)\mathbf{u}, \nabla\psi\big\rangle\,\mathrm{d}s \\ &- \int_0^t \big\langle \big(b'(\varrho)\varrho - b(\varrho)\mathbf{u}\big)\,\mathrm{div}\,\mathbf{u}, \psi\big\rangle\,\mathrm{d}s. \end{aligned}$$

The existence of a solution to (4.1)–(4.4) in the sense of Definition 4.0.1 has been shown in [8] under periodic boundary conditions. The Dirichlet case appeared later in [34]. The proof relies on a multi-layer approximation scheme whose core follows the technique developed by Feireisl, Novotný and Petzeltová [15] in order to deal with the deterministic counterpart. It makes an essential use of the stochastic compactness method and, in particular, of the Jakubowski–Skorokhod representation Theorem 2.6.3. In comparison to the existence result from [8], the energy inequality (4.9) originally introduced in [6] is included in the definition of a solution. As a simplification we assume that $\gamma > 3$. This is very restrictive from a physical point of view. However, it allows us to focus more on the bulk ideas and to avoid several technical difficulties. First of all the theory by DiPerna–Lions [12] on renormalized solutions becomes applicable (provided $\gamma \geq \frac{9}{5}$). The case $\frac{3}{2} < \gamma < \frac{9}{5}$ can be included by using the theory by Feireisl et al. [15] which has been extended to the stochastic setting in [8, Section 6]. A second point is the continuity of the effective viscous flux, cf. (4.39). If $\gamma < 3$, the integrability of the density from Proposition 4.1.1 cannot be shown in the present form and one cannot guarantee well-definedness of $\varrho^{\gamma+1}$. In order to overcome this an L^∞-type truncation has to be applied (see [8, Section 6] for the stochastic case).

Instead of giving a full proof of the existence of a solution to (4.1)–(4.4) we will focus on sequential compactness. In Section 4.1 we show how to derive formally the a priori estimates. Based on this, we prove in Section 4.2 how to pass to the limit in a sequence of solutions which enjoy appropriate regularity properties.

4.1. A priori estimates

We are going to show that smooth solutions to (4.1)–(4.4) satisfy the following energy estimate

$$\begin{aligned}\mathbb{E}\bigg[\sup_{t\in[0,T]}\int_{\mathcal{O}}\Big[\frac{|\varrho\mathbf{u}|^2}{2\varrho}+P(\varrho)\Big]\,\mathrm{d}x\bigg]^p&\\ +\mathbb{E}\bigg[\int_0^T\int_{\mathcal{O}}\nu|\nabla\mathbf{u}|^2+\eta|\operatorname{div}\mathbf{u}|^2\,\mathrm{d}x\,\mathrm{d}t\bigg]^p&\\ \leq c(p,T)\,\mathbb{E}\bigg[\bigg(\int_{\mathcal{O}}\Big[\frac{|(\varrho\mathbf{u})(0)|^2}{2\varrho(0)}+P(\varrho(0))\Big]\,\mathrm{d}x\bigg)^p+1\bigg]&\end{aligned}\tag{4.10}$$

for all $1\leq p<\infty$, where $\eta=\lambda+\frac{\nu}{3}\geq 0$. In order to prove (4.10) we apply Itô's formula to the functional $f(\mathbf{q},\varrho)=\frac{1}{2}\int_{\mathcal{O}}\frac{|\mathbf{q}|^2}{\varrho}\,\mathrm{d}x$. This corresponds exactly to the test with $\mathbf{u}$ in the momentum equation and $\frac{1}{2}|\mathbf{u}|^2$ in the continuity equation from the deterministic case. We gain

$$\begin{aligned}\frac{1}{2}\int_{\mathcal{O}}\varrho|\mathbf{u}|^2\,\mathrm{d}x=\frac{1}{2}\int_{\mathcal{O}}\frac{|(\varrho\mathbf{u})(0)|^2}{\varrho(0)}\,\mathrm{d}x-\nu\int_0^t\int_{\mathcal{O}}|\nabla\mathbf{u}|^2\,\mathrm{d}x\,\mathrm{d}\sigma-\eta\int_0^t\int_{\mathcal{O}}|\operatorname{div}\mathbf{u}|^2\,\mathrm{d}x\,\mathrm{d}\sigma&\\ +\int_0^t\int_{\mathcal{O}}\varrho\mathbf{u}\otimes\mathbf{u}:\nabla\mathbf{u}\,\mathrm{d}x\,\mathrm{d}\sigma+\int_0^t\int_{\mathcal{O}}\varrho^\gamma\operatorname{div}\mathbf{u}\,\mathrm{d}x\,\mathrm{d}\sigma-\frac{1}{2}\int_0^t\int_{\mathcal{O}}|\mathbf{u}|^2\,\mathrm{d}\varrho&\\ +\int_0^t\int_{\mathcal{O}}\mathbf{u}\cdot\varrho\mathbb{F}\,\mathrm{d}x\,\mathrm{d}W+\frac{1}{2}\int_0^t\varrho^{-1}\,\mathrm{d}\bigg\langle\bigg\langle\int_0^{\cdot}\varrho\mathbb{F}\,\mathrm{d}W\bigg\rangle\bigg\rangle.&\end{aligned}$$

In the following we use the renormalized equation of continuity to get

$$\int_0^t\int_{\mathcal{O}}\varrho^\gamma\operatorname{div}\mathbf{u}\,\mathrm{d}x\,\mathrm{d}\sigma=-\int_{\mathcal{O}}P(\varrho)\,\mathrm{d}x+\int_{\mathcal{O}}P(\varrho(0))\,\mathrm{d}x\tag{4.11}$$

Using (4.11) we gain

$$\begin{aligned}&\frac{1}{2}\int_{\mathcal{O}}\varrho|\mathbf{u}|^2\,\mathrm{d}x+\nu\int_0^t\int_{\mathcal{O}}|\nabla\mathbf{u}|^2\,\mathrm{d}x\,\mathrm{d}\sigma+\eta\int_0^t\int_{\mathcal{O}}|\operatorname{div}\mathbf{u}|^2\,\mathrm{d}x\,\mathrm{d}\sigma+\int_{\mathcal{O}}P(\varrho)\,\mathrm{d}x\\ &\quad\leq\frac{1}{2}\int_{\mathcal{O}}\frac{|(\varrho\mathbf{u})(0)|^2}{\varrho(0)}\,\mathrm{d}x+\int_{\mathcal{O}}P(\varrho(0))\,\mathrm{d}x\\ &\qquad+\int_0^t\int_{\mathcal{O}}\mathbf{u}\cdot\varrho\mathbb{F}\,\mathrm{d}x\,\mathrm{d}W+\frac{1}{2}\int_0^t\varrho^{-1}\,\mathrm{d}\bigg\langle\bigg\langle\int_0^{\cdot}\varrho\mathbb{F}\,\mathrm{d}W\bigg\rangle\bigg\rangle\\ &\quad=:\frac{1}{2}\int_{\mathcal{O}}\frac{|(\varrho\mathbf{u})(0)|^2}{\varrho(0)}\,\mathrm{d}x+\int_{\mathcal{O}}P(\varrho(0))\,\mathrm{d}x+T_1(t)+T_2(t).\end{aligned}$$

We apply the pth power on both sides and then take the expectation. Due to (4.5) we have

$$T_2(t)\leq\frac{1}{2}\sum_k\int_0^t\int_{\mathcal{O}}\varrho|\mathbf{F}_k|^2\,\mathrm{d}x\,\mathrm{d}t\leq c\int_0^T\int_{\mathcal{O}}\varrho\,\mathrm{d}x\,\mathrm{d}t\leq cT.\tag{4.12}$$

In fact, (4.12) is a consequence of the conservation of mass

$$\int_{\mathcal{O}} \varrho(t)\,\mathrm{d}x = \int_{\mathcal{O}} \varrho(0)\,\mathrm{d}x \quad \forall t \in [0,T] \tag{4.13}$$

which holds due to (4.2). As a consequence of the Burgholder–Davis–Gundy inequality (Lemma 2.1.5), (4.5) and (4.13) we gain for $p \geq 1$

$$\begin{aligned}
\mathbb{E}\bigg[\sup_{t\in(0,T)} |T_1(t)|\bigg]^p &= \mathbb{E}\bigg[\sup_{t\in(0,T)} \Big| \int_0^t \int_{\mathcal{O}} \mathbf{u}\cdot\varrho\mathbb{F}\,\mathrm{d}x\,\mathrm{d}W_\sigma\Big|\bigg]^p \\
&= \mathbb{E}\bigg[\sup_{t\in(0,T)} \Big| \int_0^t \sum_k \int_{\mathcal{O}} \mathbf{u}\cdot\varrho\mathbf{F}_k\,\mathrm{d}x\,\mathrm{d}\beta_k(\sigma)\Big|\bigg]^p \\
&\leq c\,\mathbb{E}\bigg[\int_0^T \sum_k \bigg(\int_{\mathcal{O}} \mathbf{u}\cdot\varrho\mathbf{F}_k\,\mathrm{d}x\bigg)^2 \mathrm{d}t\bigg]^{p/2} \\
&\leq c\,\mathbb{E}\bigg[\int_0^T \sum_k \bigg(\int_{\mathcal{O}} \varrho|\mathbf{u}|^2\,\mathrm{d}x\bigg)\bigg(\int_{\mathcal{O}} \varrho|\mathbf{F}_k|^2\,\mathrm{d}x\bigg)\,\mathrm{d}t\bigg]^{p/2} \\
&\leq c\,\mathbb{E}\bigg[\int_0^T \int_{\mathcal{O}} \varrho|\mathbf{u}|^2\,\mathrm{d}x\,\mathrm{d}t\bigg]^{p/2}.
\end{aligned}$$

This implies by Young's inequality for every $\delta > 0$ that

$$\mathbb{E}\bigg[\sup_{t\in(0,T)} |T_1(t)|\bigg]^p \leq \delta\,\mathbb{E}\bigg[\sup_{t\in(0,T)} \int_{\mathcal{O}} \varrho|\mathbf{u}|^2\,\mathrm{d}x\bigg]^p + c(\delta,T).$$

Finally, taking δ small enough, inequality (4.10) follows.

The aim in the following is to show sequential compactness of solutions to (4.1)–(4.4). To this end, we assume that for every $\varepsilon \in (0,1)$ there exists

$$\big((\Omega^\varepsilon, \mathcal{F}^\varepsilon, (\mathcal{F}_t^\varepsilon)_{t\geq 0}, \mathbb{P}^\varepsilon), \varrho_\varepsilon, \mathbf{u}_\varepsilon, W_\varepsilon\big)$$

which is a weak martingale solution to (4.1)–(4.4) that satisfies (4.10). We further assume that the initial data $(\varrho_\varepsilon(0), \varrho_\varepsilon\mathbf{u}_\varepsilon(0))$ belong $\mathbb{P}$-a.s. to the set

$$\Big\{(\rho,\mathbf{q}) \in L^\gamma(\mathcal{O}) \times L^{\frac{2\gamma}{\gamma+1}}(\mathcal{O});\ \rho \geq 0,\ (\rho)_{\mathcal{O}} \leq M,\ \mathbf{q}(x) = 0 \text{ whenever } \rho(x) = 0\Big\}, \tag{4.14}$$

with some $M > 0$ and satisfy

$$\mathbb{E}\bigg[\int_{\mathcal{O}} \Big[\frac{|(\varrho_\varepsilon\mathbf{u}_\varepsilon(0)|^2}{2\varrho_\varepsilon(0)} + P(\varrho_\varepsilon(0))\Big]\,\mathrm{d}x\bigg]^p \leq C(p), \tag{4.15}$$

for all $1 \leq p < \infty$ uniformly in ε. Hence the right-hand side of (4.10) is bounded uniformly in ε. We assume that ϱ_ε and $\mathbf{u}_\varepsilon$ are smooth enough such that all the

following computations are well defined. It was shown in [23] that it is enough to consider only one probability space, namely,

$$(\Omega^\varepsilon, \mathcal{F}^\varepsilon, \mathbb{P}^\varepsilon) = \big([0,1], \overline{\mathfrak{B}([0,1])}, \mathcal{L}|_{[0,1]}\big) \qquad \forall \varepsilon \in (0,1)$$

where $\mathcal{L}$ denotes the Lebesgue measure on $[0,1]$. Moreover, we can assume without loss of generality that there exists one common Wiener process W for all ε. Indeed, this can be achieved by performing the compactness argument from any chosen subsequence $(\varepsilon_n)_{n\in\mathbb{N}}$ at once.

As the functions $\mathbf{u}_\varepsilon$ and ϱ_ε satisfy the energy inequality

$$\begin{aligned}\mathbb{E}\bigg[\sup_{0\leq t\leq T}\int_{\mathcal{O}}\Big(\frac{1}{2}\varrho_\varepsilon|\mathbf{u}_\varepsilon|^2+\frac{a}{\gamma-1}\varrho_\varepsilon^\gamma\Big)\,\mathrm{d}x\\ +\int_0^T\int_{\mathcal{O}}\nu|\mathbf{u}_\varepsilon|^2+\eta|\operatorname{div}\mathbf{u}_\varepsilon|^2\,\mathrm{d}x\,\mathrm{d}s\bigg]^p\leq C(p)\end{aligned} \tag{4.16}$$

for all $1\leq p<\infty$ we have the following uniform bounds

$$\mathbf{u}_\varepsilon \in L^p(\Omega; L^2(0,T; W_0^{1,2}(\mathcal{O}))), \tag{4.17}$$

$$\sqrt{\varrho_\varepsilon}\mathbf{u}_\varepsilon \in L^p(\Omega; L^\infty(0,T; L^2(\mathcal{O}))), \tag{4.18}$$

$$\varrho_\varepsilon \in L^p(\Omega; L^\infty(0,T; L^\gamma(\mathcal{O}))), \tag{4.19}$$

$$\varrho_\varepsilon\mathbf{u}_\varepsilon \in L^p(\Omega; L^\infty(0,T; L^{\frac{2\gamma}{\gamma+1}}(\mathcal{O}))), \tag{4.20}$$

$$\varrho_\varepsilon\mathbf{u}_\varepsilon\otimes\mathbf{u}_\varepsilon \in L^p(\Omega; L^2(0,T; L^{\frac{6\gamma}{4\gamma+3}}(\mathcal{O}))). \tag{4.21}$$

As the next step, we improve the space integrability of the density.

Proposition 4.1.1. *Let $\gamma>3$. Then the following holds*

$$\mathbb{E}\int_0^T\int_{\mathcal{O}}\varrho_\varepsilon^{\gamma+1}\,\mathrm{d}x\,\mathrm{d}t\leq C, \tag{4.22}$$

uniformly in ε.

Proof. In the deterministic case, this is achieved by testing (4.1) with

$$\mathcal{B}\varrho_\varepsilon = \mathrm{Bog}_{\mathcal{O}}(\varrho_\varepsilon - (\varrho_\varepsilon)_{\mathcal{O}})$$

(that is a right-inverse to the divergence operator). Here $\mathrm{Bog}_{\mathcal{O}}$ is the Bogovskiĭ operator on $\mathcal{O}$. Note that $\mathcal{B}$ is continuous from $L^p(\mathcal{O})$ to $W_0^{1,p}(\mathcal{O})$ for all $1<p<\infty$, cf. [4]. In the stochastic setting we apply Itô's formula (Theorem 2.5.2) to the functional $f(\mathbf{q},\rho)=\int_{\mathcal{O}}\mathbf{q}\cdot\mathcal{B}\rho\,\mathrm{d}x$. Note that f is linear in $\mathbf{q}=\varrho\mathbf{u}$ and the quadratic

variation of ϱ is zero. Hence we do not need a correction term. We gain

$$\begin{aligned}\mathbb{E}J_0 &= \mathbb{E}\int_{\mathcal{O}} \varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \mathcal{B}\varrho_\varepsilon \,\mathrm{d}x \\ &= \mathbb{E}\int_{\mathcal{O}} \varrho_\varepsilon \mathbf{u}_\varepsilon(0)\cdot \mathcal{B}\varrho_\varepsilon(0)\,\mathrm{d}x\,\mathrm{d}\sigma - \nu\mathbb{E}\int_0^t\int_{\mathcal{O}} \nabla \mathbf{u}_\varepsilon : \nabla\mathcal{B}\varrho_\varepsilon \,\mathrm{d}x\,\mathrm{d}\sigma \\ &\quad - \eta\mathbb{E}\int_0^t\int_{\mathcal{O}} \operatorname{div}\mathbf{u}_\varepsilon\, \varrho_\varepsilon\,\mathrm{d}x + \mathbb{E}\int_0^t\int_{\mathcal{O}} \varrho\mathbf{u}_\varepsilon\otimes\mathbf{u}_\varepsilon : \nabla\mathcal{B}\varrho_\varepsilon\,\mathrm{d}x\,\mathrm{d}\sigma \\ &\quad + \mathbb{E}\int_0^t\int_{\mathcal{O}} a\varrho_\varepsilon^{\gamma+1}\,\mathrm{d}x\,\mathrm{d}\sigma - \mathbb{E}\int_0^t (\varrho_\varepsilon)_{\mathcal{O}}\int_{\mathcal{O}} a\varrho_\varepsilon^\gamma\,\mathrm{d}x\,\mathrm{d}\sigma \\ &\quad - \mathbb{E}\int_0^t\int_{\mathcal{O}} \varrho_\varepsilon\mathbf{u}\cdot\mathcal{B}\operatorname{div}(\varrho_\varepsilon\mathbf{u}_\varepsilon)\,\mathrm{d}x\,\mathrm{d}\sigma \\ &= \mathbb{E}J_1 + \cdots + \mathbb{E}J_7\end{aligned}$$

using $\mathrm{d}\varrho_{\boldsymbol{\varepsilon}} = -\operatorname{div}(\varrho_{\boldsymbol{\varepsilon}}\mathbf{u}_{\boldsymbol{\varepsilon}})\,\mathrm{d}t$. Note that the expectation of the stochastic integral vanishes. We want to bound the term J_5, so we have to estimate all the others. By (4.14) we have for all $t\in[0,T]$

$$(\varrho_\varepsilon(t))_{\mathcal{O}} = \varrho_\varepsilon(0))_{\mathcal{O}} \le C$$

So (4.19) yields $\mathbb{E}J_6 \le C$. The most critical term is J_4 which we estimate by

$$\mathbb{E}J_4 \le \mathbb{E}\int_0^t \|\varrho_\varepsilon\|_{L_x^\gamma}\|\mathbf{u}_\varepsilon\|_{L_x^6}^2\|\varrho_\varepsilon\|_{L_x^r}\,\mathrm{d}t,$$

where $r := \frac{3\gamma}{2\gamma-3}$. We proceed, using continuity of $\nabla\mathcal{B}$ and Sobolev's embedding, by

$$\begin{aligned}\mathbb{E}J_4 &\le C\,\mathbb{E}\Big(\sup_{0\le s\le t}\|\varrho_\varepsilon\|_{L_x^\gamma}\Big)\Big(\sup_{0\le s\le t}\|\varrho_\varepsilon\|_{L_x^r}\Big)\int_0^t \|\nabla\mathbf{u}_\varepsilon\|_{L_x^2}^2\,\mathrm{d}\sigma \\ &\le C\left(\mathbb{E}\sup_{0\le s\le t}\|\varrho_\varepsilon\|_{L_x^\gamma}^{q_1}\right)^{1/q_1}\left(\mathbb{E}\sup_{0\le s\le t}\|\varrho_\varepsilon\|_{L_x^r}^{q_2}\right)^{1/q_2}\left(\mathbb{E}\left[\int_0^t\|\nabla\mathbf{u}_\varepsilon\|_{L_x^2}^2\,\mathrm{d}\sigma\right]^{q_3}\right)^{1/q_3}\end{aligned}$$

as a consequence of Hölder's inequality ($\frac{1}{q_1}+\frac{1}{q_2}+\frac{1}{q_2}=1$, for instance $q_1 = q_2 = q_3 = 3$). If $r\le\gamma$ ($\Leftrightarrow \gamma\ge 3$) we can conclude from (4.17) and (4.19) that $\mathbb{E}J_4\le C$. In order to estimate J_0 we use the following estimate which follows from the continuity of $\nabla\mathcal{B}$ and Sobolev's theorem for $q=\frac{6\gamma}{5\gamma-3}\in(1,3)$

$$\|\mathcal{B}\varrho_\varepsilon\|_{L_x^{\frac{3q}{3-q}}} \le C\,\|\nabla\mathcal{B}\varrho_\varepsilon\|_{L_x^q} \le C\,\|\varrho_\varepsilon\|_{L_x^q}.$$

We gain $|\mathbb{E}J_0|\le C$ as a consequence of (4.19). We have due to the continuity of $\nabla\mathcal{B}$

$$\mathbb{E}J_2 + \mathbb{E}J_4 \le \mathbb{E}\left[\int_0^t\int_{\mathcal{O}}|\nabla\mathbf{u}_\varepsilon|^2\,\mathrm{d}x\,\mathrm{d}\sigma\right] + \mathbb{E}\left[\int_0^t\int_{\mathcal{O}}|\varrho_\varepsilon|^2\,\mathrm{d}x\,\mathrm{d}\sigma\right] \le C.$$

Moreover, we obtain

$$\begin{aligned}\mathbb{E}[|J_7|] &\leq C\,\mathbb{E}\bigg[\int_0^t \|\varrho_\varepsilon\|_{L_x^\gamma}\|\mathbf{u}_\varepsilon\|_{L_x^6}\|\mathcal{B}(\mathrm{div}(\varrho_\varepsilon\mathbf{u}_\varepsilon))\|_{L_x^p}\,\mathrm{d}t\bigg]\\ &\leq C\,\mathbb{E}\bigg[\int_0^t \|\varrho_\varepsilon\|_{L_x^\gamma}\|\mathbf{u}_\varepsilon\|_{L_x^6}\|\varrho_\varepsilon\mathbf{u}_\varepsilon\|_{L_x^p}\,\mathrm{d}t\bigg]\\ &\leq C\,\mathbb{E}\bigg[\int_0^t \|\varrho_\varepsilon\|_{L_x^\gamma}\|\mathbf{u}_\varepsilon\|_{6}^2\|\varrho_\varepsilon\|_{L_x^r}\,\mathrm{d}t\bigg],\end{aligned}$$

using continuity properties of $\mathcal{B}$ on negative Sobolev spaces, where $\frac{1}{p}=\frac{1}{r}+\frac{1}{6}$. We proceed by

$$\begin{aligned}\mathbb{E}[|J_7|] &\leq C\,\mathbb{E}\bigg[\Big(\sup_{0\leq s\leq t}\|\varrho_\varepsilon\|_{L_x^\gamma}\Big)\Big(\sup_{0\leq s\leq t}\|\varrho_\varepsilon\|_{L_x^r}\Big)\int_0^t \|\nabla\mathbf{u}_\varepsilon\|_{L_x^2}^2\,\mathrm{d}\sigma\bigg]\\ &\leq C\left(\mathbb{E}\sup_{0\leq s\leq t}\|\varrho_\varepsilon\|_{L_x^\gamma}^{q_1}\right)^{1/q_1}\left(\mathbb{E}\sup_{0\leq s\leq t}\|\varrho_\varepsilon\|_{L_x^r}^{q_2}\right)^{1/q_2}\left(\mathbb{E}\bigg[\int_0^t \|\nabla\mathbf{u}_\varepsilon\|_{L_x^2}^2\,\mathrm{d}\sigma\bigg]^{q_3}\right)^{1/q_3}\\ &\leq C,\end{aligned}$$

using again (4.17) and (4.19). Plugging all together we conclude the claimed estimate. □

4.2. Compactness

Let us define the path space $\mathcal{X}=\mathcal{X}_\varrho\times\mathcal{X}_\mathbf{u}\times\mathcal{X}_{\varrho\mathbf{u}}\times\mathcal{X}_W$ where[5]

$$\begin{aligned}\mathcal{X}_\varrho &= C_w([0,T];L^\gamma(\mathcal{O}))\cap\big(L^{\gamma+1}(Q),w\big), &\qquad \mathcal{X}_\mathbf{u} &= \big(L^2(0,T;W^{1,2}(\mathcal{O})),w\big),\\ \mathcal{X}_{\varrho\mathbf{u}} &= C_w([0,T];L^{\frac{2\gamma}{\gamma+1}}(\mathcal{O})), &\qquad \mathcal{X}_W &= C([0,T];\mathfrak{U}_0).\end{aligned}$$

Let us denote by $\mu_{\varrho_\varepsilon}$, $\mu_{\mathbf{u}_\varepsilon}$ and $\mu_{\varrho_\varepsilon\mathbf{u}_\varepsilon}$, respectively, the law of ϱ_ε, $\mathbf{u}_\varepsilon$ and $\varrho_\varepsilon\mathbf{u}_\varepsilon$ on the corresponding path space. By μ_W we denote the law of W on $\mathcal{X}_W$ and their joint law on $\mathcal{X}$ is denoted by μ^ε.

Proposition 4.2.1. *The set $\{\mu_{\mathbf{u}_\varepsilon};\,\varepsilon\in(0,1)\}$ is tight on $\mathcal{X}_\mathbf{u}$.*

Proof. The proof follows directly from (4.17). Indeed, for any $R>0$ the set

$$B_R=\big\{\mathbf{u}\in L^2(0,T;W_0^{1,2}(\mathcal{O}));\ \|\mathbf{u}\|_{L^2(0,T;W^{1,2}(\mathcal{O}))}\leq R\big\}$$

is relatively compact in $\mathcal{X}_\mathbf{u}$ and

$$\mu_{\mathbf{u}_\varepsilon}(B_R^c)=\mathbb{P}\big(\|\mathbf{u}_\varepsilon\|_{L^2(0,T;W^{1,2}(\mathcal{O}))}\geq R\big)\leq\frac{1}{R}\mathbb{E}\|\mathbf{u}_\varepsilon\|_{L^2(0,T;W^{1,2}(\mathcal{O}))}\leq\frac{C}{R}$$

which yields the claim. □

Proposition 4.2.2. *The set $\{\mu_{\varrho_\varepsilon};\,\varepsilon\in(0,1)\}$ is tight on $\mathcal{X}_\varrho$.*

[5] C_w denotes the space of functions being continuous with respect to the weak topology.

Proof. Due to (4.20) we obtain that

$$\{\operatorname{div}(\varrho_\varepsilon \mathbf{u}_\varepsilon)\} \quad \text{is bounded in} \quad L^p(\Omega; L^\infty(0,T; W^{-1,\frac{2\gamma}{\gamma+1}}(\mathcal{O}))). \tag{4.23}$$

As a consequence,

$$\mathbb{E}\|\varrho_\varepsilon\|^p_{C^{0,1}([0,T];W^{-1,\frac{2\gamma}{\gamma+1}}(\mathcal{O}))} \leq C$$

due the continuity equation (4.2). Now, the required tightness on $C_w([0,T]; L^\gamma(\mathcal{O}))$ follows by a similar reasoning as in Proposition 4.2.1 together with the compact embedding

$$L^\infty(0,T; L^\gamma(\mathcal{O})) \cap C^{0,1}([0,T]; W^{-1,\frac{2\gamma}{\gamma+1}}(\mathcal{O})) \overset{c}{\hookrightarrow} C_w([0,T]; L^\gamma(\mathcal{O})). \qquad \square$$

Proposition 4.2.3. *The set* $\{\mu_{\varrho_\varepsilon \mathbf{u}_\varepsilon};\, \varepsilon \in (0,1)\}$ *is tight on* $\mathcal{X}_{\varrho\mathbf{u}}$.

Proof. We decompose $\varrho_\varepsilon \mathbf{u}_\varepsilon$ into two parts, namely, $\varrho_\varepsilon \mathbf{u}_\varepsilon(t) = Y^\varepsilon(t) + Z^\varepsilon(t)$, where

$$Y^\varepsilon(t) = \mathbf{q}(0) - \int_0^t \big[\operatorname{div}(\varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon) - \nu\Delta \mathbf{u}_\varepsilon + \eta \nabla \operatorname{div} \mathbf{u}_\varepsilon + a\nabla \varrho_\varepsilon^\gamma\big]\,\mathrm{d}s,$$

$$Z^\varepsilon(t) = \int_0^t \varrho_\varepsilon \mathbb{F}\,\mathrm{d}W.$$

Due to the uniform bounds (4.17)–(4.21) we obtain the Hölder continuity of Y^ε, namely, there exist $\vartheta > 0$ and $b > 3/2$ such that

$$\mathbb{E}\big\|Y^\varepsilon\|_{C^\vartheta([0,T];W^{-b,2}(\mathcal{O}))} \leq C.$$

Concerning the stochastic integral, we apply the Burkholder–Davis–Gundy inequality (Lemma 2.1.5). We obtain due to (4.5) and (4.19) that for any $b > \frac{3}{2}$ and $\theta \geq 2$

$$\begin{aligned}
\mathbb{E}\left\|Z^\varepsilon(t) - Z^\varepsilon(s)\right\|^\theta_{W_x^{-b,2}} &\leq C\,\mathbb{E}\bigg(\int_s^t \sum_{k\geq 1} \big\|\varrho_\varepsilon \mathbf{F}_k\big\|^2_{W_x^{-b,2}}\,\mathrm{d}r\bigg)^{\theta/2} \\
&\leq C\,\mathbb{E}\bigg(\int_s^t \sum_{k\geq 1} \big\|\varrho_\varepsilon \mathbf{F}_k\big\|^2_{L^1_x}\,\mathrm{d}r\bigg)^{\theta/2} \leq C\,\mathbb{E}\bigg(\int_s^t \|\varrho_\varepsilon\|^2_{L^1_x}\,\mathrm{d}r\bigg)^{\theta/2} \\
&\leq C|t-s|^{\theta/2}\Big(1 + \mathbb{E}\sup_{0\leq t\leq T}\|\varrho_\varepsilon\|^{\theta\gamma/2}_{L^\gamma_x}\Big) \leq C|t-s|^{\theta/2}
\end{aligned}$$

and the Kolmogorov continuity criterion (Theorem 2.5.5) applies. We obtain

$$\mathbb{E}\big\|Z^\varepsilon\|_{C^\vartheta([0,T];W^{-b,2}(\mathcal{O}))} \leq C$$

for any $\vartheta \in (0, \frac{1}{2})$. Combining the estimates for Y^ε and Z^ε we have that

$$\mathbb{E}\big\|\varrho_\varepsilon \mathbf{u}_\varepsilon\|_{C^\vartheta([0,T];W^{-b,2}(\mathcal{O}))} \leq C. \tag{4.24}$$

Let us define the sets

$$B_R = \big\{h \in L^\infty(0,T; L^{\frac{2\beta}{\beta+1}}(\mathcal{O}));\ \|h\|_{L^\infty(0,T;L^{\frac{2\gamma}{\gamma+1}}(\mathcal{O}))} \leq R\big\}$$

$$C_R = \big\{h \in C^\vartheta([0,T]; W^{-b,2}(\mathcal{O}));\ \|h\|_{C^\vartheta([0,T];W^{-b,2}(\mathcal{O}))} \leq R\big\}$$

and

$$K_R = B_R \cap C_R.$$

Then it can be shown that K_R is relatively compact in $\mathcal{X}_{\varrho\mathbf{u}}$. The proof is based on the Arzelà–Ascoli theorem and follows closely the lines of the proof of [30, Corollary B.2]. Moreover, we obtain by (4.20) and (4.24)

$$\begin{aligned}\mu_{\varrho_\varepsilon \mathbf{u}_\varepsilon}\big(K_R^c\big) &= \mathbb{P}\big([\varrho_\varepsilon \mathbf{u}_\varepsilon \notin B_R] \cup [\varrho_\varepsilon \mathbf{u}_\varepsilon \notin C_R]\big) \\ &\leq \mathbb{P}\Big(\|\varrho_\varepsilon \mathbf{u}_\varepsilon\|_{L^\infty(0,T;L^{\frac{2\gamma}{\gamma+1}}(\mathcal{O}))} > R\Big) + \mathbb{P}\Big(\|\varrho_\varepsilon \mathbf{u}_\varepsilon\|_{C^\vartheta([0,T];W^{-b,2}(\mathcal{O}))} > R\Big) \\ &\leq \frac{C}{R}.\end{aligned}$$

A suitable choice of R completes the proof. □

Since also the law of μ_W is tight as being Radon measures on the Polish spaces $\mathcal{X}_W$ we can deduce tightness of the joint laws μ^ε.

Corollary 4.2.4. *The set $\{\mu^\varepsilon;\, \varepsilon \in (0,1)\}$ is tight on $\mathcal{X}$.*

Now we have all in hand to apply the Jakubowski–Skorokhod representation theorem (Lemma 2.6.3). It yields the following.

Proposition 4.2.5. *There exists a subsequence μ^ε (not relabeled), a probability space $(\tilde\Omega, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ with $\mathcal{X}$-valued Borel measurable random variables $(\tilde\varrho_{\boldsymbol{\varepsilon}}, \tilde{\mathbf{u}}_\varepsilon, \tilde{\mathbf{q}}_\varepsilon, \tilde W_\varepsilon)$, $\boldsymbol{\varepsilon} > 0$, and $(\tilde\varrho, \tilde{\mathbf{u}}, \tilde{\mathbf{q}}, \tilde W)$ such that*

(a) *the law of $(\tilde\varrho_\varepsilon, \tilde{\mathbf{u}}_\varepsilon, \tilde{\mathbf{q}}_\varepsilon, \tilde W_\varepsilon)$ is given by μ^ε, $\varepsilon \in (0,1)$,*
(b) *the law of $(\tilde\varrho, \tilde{\mathbf{u}}, \tilde{\mathbf{q}}, \tilde W)$, denoted by μ, is a Radon measure,*
(c) *$(\tilde\varrho_\varepsilon, \tilde{\mathbf{u}}_\varepsilon, \tilde{\mathbf{q}}_\varepsilon, \tilde W_\varepsilon)$ converges $\tilde{\mathbb{P}}$-a.s. to $(\tilde\varrho, \tilde{\mathbf{u}}, \tilde{\mathbf{q}}, \tilde W)$ in the topology of $\mathcal{X}$, i.e., we have $\tilde{\mathbb{P}}$-a.s.*

$$\begin{aligned}\tilde\varrho_\varepsilon &\to \tilde\varrho \quad &&in \quad C_w([0,T];L^\gamma(\mathcal{O})),\\ \tilde{\mathbf{u}}_\varepsilon &\rightharpoonup \tilde{\mathbf{u}} \quad &&in \quad L^2(0,T;W_0^{1,2}(\mathcal{O})),\\ \tilde{\mathbf{q}}_\varepsilon &\to \tilde{\mathbf{q}} \quad &&in \quad C_w([0,T];L^{\frac{2\gamma}{\gamma+1}}(\mathcal{O})),\\ \tilde W_\varepsilon &\to \tilde W \quad &&in \quad C([0,T],\mathfrak{U}_0),\end{aligned}$$

The main difficulty is to pass to the limit in the nonlinear pressure. In the next subsection, we introduce a stochastic generalization of the technique based on the regularity of the effective viscous flux, which is originally due to Lions [28]. By this we establish strong convergence of the approximate densities and identify the pressure terms as well as the stochastic integral.

Lemma 4.2.6. *The following convergences hold true $\tilde{\mathbb{P}}$-a.s.*

$$\tilde\varrho_\varepsilon \tilde{\mathbf{u}}_\varepsilon \to \tilde\varrho \tilde{\mathbf{u}} \qquad in \quad L^2(0,T;W^{-1,2}(\mathcal{O})), \tag{4.25}$$

$$\tilde\varrho_\varepsilon \tilde{\mathbf{u}}_\varepsilon \otimes \tilde{\mathbf{u}}_\varepsilon \rightharpoonup \tilde\varrho \tilde{\mathbf{u}} \otimes \tilde{\mathbf{u}} \quad in \quad L^1(0,T;L^1(\mathcal{O})). \tag{4.26}$$

Proof. In order to identify the limit $\tilde{\mathbf{q}}$, note that we have

$$\tilde{\varrho}_\varepsilon \tilde{\mathbf{u}}_\varepsilon \rightharpoonup \tilde{\varrho}\tilde{\mathbf{u}} \quad \text{in} \quad L^1(0,T;L^1(\mathcal{O})) \quad \tilde{\mathbb{P}}\text{-a.s.}$$

as a consequence of the convergence of $\tilde{\varrho}_\varepsilon$ and $\tilde{\mathbf{u}}_\varepsilon$ in $\mathcal{X}_\varrho$ and $\mathcal{X}_{\mathbf{u}}$, respectively. This yields together with Proposition 4.2.5 and the compactness of the embedding $L^{\frac{2\gamma}{\gamma+1}}(\mathcal{O}) \hookrightarrow W^{-1,2}(\mathcal{O})$ that

$$\tilde{\varrho}_\varepsilon \tilde{\mathbf{u}}_\varepsilon \rightharpoonup \tilde{\varrho}\tilde{\mathbf{u}} \quad \text{in} \quad L^2(0,T;W^{-1,2}(\mathcal{O})) \quad \tilde{\mathbb{P}}\text{-a.s.}$$

Combining this with the convergence of $\tilde{\mathbf{u}}_\varepsilon$ implies the second claim. □

Let $(\tilde{\mathcal{F}}_t^\varepsilon)$ and $(\tilde{\mathcal{F}}_t)$, respectively, be the $\tilde{\mathbb{P}}$-augmented canonical filtration of the process $(\tilde{\varrho}_\varepsilon, \tilde{\mathbf{u}}_\varepsilon, \tilde{W}_\varepsilon)$ and $(\tilde{\varrho}, \tilde{\mathbf{u}}, \tilde{W})$, respectively, that is

$$\begin{aligned}
\tilde{\mathcal{F}}_t^\varepsilon &= \sigma\big(\sigma\big(\mathbf{r}_t\tilde{\varrho}_\varepsilon, \mathbf{r}_t(\tilde{\varrho}_\varepsilon\tilde{\mathbf{u}}_\varepsilon), \mathbf{r}_t\tilde{W}_\varepsilon\big) \cup \big\{N \in \tilde{\mathcal{F}};\ \tilde{\mathbb{P}}(N) = 0\big\}\big), \quad t \in [0,T],\\
\tilde{\mathcal{F}}_t &= \sigma\big(\sigma\big(\mathbf{r}_t\tilde{\varrho}, \mathbf{r}_t(\tilde{\varrho}\tilde{\mathbf{u}}), \mathbf{r}_t\tilde{W}\big) \cup \big\{N \in \tilde{\mathcal{F}};\ \tilde{\mathbb{P}}(N) = 0\big\}\big), \qquad t \in [0,T].
\end{aligned}$$

We obtain the following result.

Proposition 4.2.7. *For every $\varepsilon \in (0,1)$, $\big((\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t^\varepsilon)_{t\geq 0}, \tilde{\mathbb{P}}), \tilde{\varrho}_\varepsilon, \tilde{\mathbf{u}}_\varepsilon, \tilde{W}_\varepsilon\big)$ is a weak martingale solution to* (4.1)–(4.4)*. Furthermore, there exists $b > \frac{3}{2}$ together with a $W^{-b,2}(\mathcal{O})$-valued continuous square integrable $(\tilde{\mathcal{F}}_t)_{t\geq 0}$-martingale $\tilde{M}$ and*

$$\tilde{p} \in L^{\gamma+1}(\tilde{\Omega} \times Q)$$

such that $\big((\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t)_{t\geq 0}, \tilde{\mathbb{P}}), \tilde{\varrho}, \tilde{\mathbf{u}}, \tilde{p}, \tilde{M}\big)$ is a weak martingale solution to

$$\mathrm{d}\tilde{\varrho} + \operatorname{div}(\tilde{\varrho}\tilde{\mathbf{u}})\mathrm{d}t = 0, \tag{4.27a}$$

$$\mathrm{d}(\tilde{\varrho}\tilde{\mathbf{u}}) + \big[\operatorname{div}(\tilde{\varrho}\tilde{\mathbf{u}} \otimes \tilde{\mathbf{u}}) - \nu\Delta\tilde{\mathbf{u}} - \eta\nabla\operatorname{div}\tilde{\mathbf{u}} + \nabla\tilde{p}\big]\mathrm{d}t = \mathrm{d}\tilde{M}. \tag{4.27b}$$

Besides, (4.27a) *holds true in the renormalized sense.*

Proof. The passage to the limit in (4.2) follows from (4.25) and Proposition 4.2.5. Concerning the passage to the limit in (4.1), we follow the approach of Section 3.3 and define for all $t \in [0,T]$ and $\boldsymbol{\varphi} \in C_c^\infty(\mathcal{O})$ the functionals

$$\begin{aligned}
\mathfrak{M}_\varepsilon(\rho,\mathbf{v},\mathbf{q})_t &= \big\langle \mathbf{q}(t), \boldsymbol{\varphi}\big\rangle - \big\langle \mathbf{q}(0), \boldsymbol{\varphi}\big\rangle - \int_0^t \big\langle \mathbf{q}\otimes\mathbf{v}, \nabla\boldsymbol{\varphi}\big\rangle\,\mathrm{d}r - \nu\int_0^t \big\langle \nabla\mathbf{v}, \nabla\boldsymbol{\varphi}\big\rangle\,\mathrm{d}r\\
&\quad - \eta\int_0^t \big\langle \operatorname{div}\mathbf{v}, \operatorname{div}\boldsymbol{\varphi}\big\rangle\,\mathrm{d}r + a\int_0^t \big\langle \rho^\gamma, \operatorname{div}\boldsymbol{\varphi}\big\rangle\,\mathrm{d}r\\
\mathfrak{N}(\rho)_t &= \sum_{k\geq 1}\int_0^t \big\langle \rho\mathbf{F}_k, \boldsymbol{\varphi}\big\rangle^2\,\mathrm{d}r,\\
\mathfrak{N}_k(\rho,\mathbf{q})_t &= \int_0^t \big\langle \rho\mathbf{F}_k, \boldsymbol{\varphi}\big\rangle\,\mathrm{d}r,
\end{aligned}$$

and deduce that for any continuous function $h : \mathcal{X}|_{[0,s]} \to [0,1]$ we have

$$\tilde{\mathbb{E}}\, h\big(\mathbf{r}_s\tilde{\varrho}_\varepsilon, \mathbf{r}_s\tilde{\mathbf{u}}_\varepsilon, \mathbf{r}_s\tilde{W}_\varepsilon\big)\big[\mathfrak{M}_\varepsilon(\tilde{\varrho}_\varepsilon, \tilde{\mathbf{u}}_\varepsilon, \tilde{\varrho}_\varepsilon\tilde{\mathbf{u}}_\varepsilon)_{s,t}\big] = 0, \tag{4.28}$$

$$\tilde{\mathbb{E}}\, h\big(\mathbf{r}_s\tilde{\varrho}_\varepsilon, \mathbf{r}_s\tilde{\mathbf{u}}_\varepsilon, \mathbf{r}_s\tilde{W}_\varepsilon\big)\Big[[\mathfrak{M}_\varepsilon(\tilde{\varrho}_\varepsilon, \tilde{\mathbf{u}}_\varepsilon, \tilde{\varrho}_\varepsilon\tilde{\mathbf{u}}_\varepsilon)^2]_{s,t} - \mathfrak{N}(\tilde{\varrho}_\varepsilon, \tilde{\varrho}_\varepsilon\tilde{\mathbf{u}}_\varepsilon)_{s,t}\Big] = 0, \tag{4.29}$$

$$\tilde{\mathbb{E}}\, h\big(\mathbf{r}_s\tilde{\varrho}_\varepsilon, \mathbf{r}_s\tilde{\mathbf{u}}_\varepsilon, \mathbf{r}_s\tilde{W}_\varepsilon\big)\Big[[\mathfrak{M}_\varepsilon(\tilde{\varrho}_\varepsilon, \tilde{\mathbf{u}}_\varepsilon, \tilde{\varrho}_\varepsilon\tilde{\mathbf{u}}_\varepsilon)\tilde{\beta}_k^\varepsilon]_{s,t} - \mathfrak{N}_k(\tilde{\varrho}_\varepsilon, \tilde{\varrho}_\varepsilon\tilde{\mathbf{u}}_\varepsilon)_{s,t}\Big] = 0, \tag{4.30}$$

which implies the first part of the statement.

As the next step, we will pass to the limit in (4.28). We apply (4.21) and (4.26) for the convective term. As far as the pressure is concerned, we see that according to (4.22) there exists $\tilde{p} \in L^{\gamma+1}(\tilde{\Omega} \times Q)$ such that

$$a\tilde{\varrho}_\varepsilon^\gamma \rightharpoonup \tilde{p} \qquad \text{in} \qquad L^{\gamma+1}(\tilde{\Omega} \times Q).$$

Hence, in view of Proposition 4.2.5, we deduce

$$\begin{aligned}&\tilde{\mathbb{E}}\, h\big(\mathbf{r}_s\tilde{\varrho}_\varepsilon, \mathbf{r}_s\tilde{\mathbf{u}}_\varepsilon, \mathbf{r}_s\tilde{W}_\varepsilon\big)\bigg[a\int_0^t \big\langle \tilde{\varrho}_\varepsilon^\gamma, \operatorname{div}\boldsymbol{\varphi}\big\rangle\,\mathrm{d}r\bigg]\\ &\to \tilde{\mathbb{E}}\, h\big(\mathbf{r}_s\tilde{\varrho}, \mathbf{r}_s\tilde{\mathbf{u}}, \mathbf{r}_s\tilde{W}\big)\bigg[\int_0^t \big\langle \tilde{p}, \operatorname{div}\boldsymbol{\varphi}\big\rangle\,\mathrm{d}r\bigg].\end{aligned}$$

Convergence of the remaining terms is obvious and therefore we have proved that

$$\tilde{\mathbb{E}}\, h\big(\mathbf{r}_s\tilde{\varrho}, \mathbf{r}_s\tilde{\mathbf{u}}, \mathbf{r}_s\tilde{W}\big)\big[\big\langle \tilde{M}, \boldsymbol{\varphi}\big\rangle_{s,t}\big] = 0, \tag{4.31}$$

where

$$\begin{aligned}\tilde{M}_t = \tilde{\varrho}\tilde{\mathbf{u}}(t) - \tilde{\varrho}\tilde{\mathbf{u}}(0) &+ \int_0^t \operatorname{div}(\tilde{\varrho}\tilde{\mathbf{u}} \otimes \tilde{\mathbf{u}})\,\mathrm{d}r - \nu\int_0^t \Delta\tilde{\mathbf{u}}\,\mathrm{d}r\\ &- \eta\int_0^t \nabla\operatorname{div}\tilde{\mathbf{u}}\,\mathrm{d}r + \int_0^t \nabla\tilde{p}\,\mathrm{d}r.\end{aligned}$$

Hence $\tilde{M}$ is a continuous $(\tilde{\mathcal{F}}_t)_{t\geq 0}$-martingale and possesses moments of any order due to our uniform estimates.

To conclude the proof, we will show that $(\tilde{\varrho}, \tilde{\mathbf{u}})$ solves the continuity equation in the renormalized sense. Both density and velocity are extended by zero to the whole space. We apply to (4.27) a standard smoothing operator S^m (which is the convolution with an approximation to the identity in space) such that $\tilde{\mathbb{P}} \otimes \mathcal{L}^4$-a.e. in $\tilde{\Omega} \times Q$

$$\partial_t S^m[\tilde{\varrho}] + \operatorname{div}\big(S^m[\tilde{\varrho}]\tilde{\mathbf{u}}\big) = \operatorname{div}\big(S^m[\tilde{\varrho}]\tilde{\mathbf{u}} - S^m[\tilde{\varrho}\tilde{\mathbf{u}}]\big). \tag{4.32}$$

Setting $\tilde{r}_m := \operatorname{div}\big(S^m[\tilde{\varrho}]\tilde{\mathbf{u}} - S^m[\tilde{\varrho}\tilde{\mathbf{u}}]\big)$ we infer from the commutation lemma (see, e.g., [27, Lemma 2.3]) that $\tilde{\mathbb{P}} \otimes \mathcal{L}^1$-a.e.

$$\|\tilde{r}_m\|_{L_x^q} \leq \|\tilde{\mathbf{u}}\|_{W_x^{1,2}}\|\tilde{\varrho}\|_{L_x^{\gamma+1}}, \qquad \tfrac{1}{q} = \tfrac{1}{2} + \tfrac{1}{\gamma+1},$$

as well as $\tilde{r}_m \to 0$ in $L^1(\mathbb{R}^3)$. Both together imply $\tilde{r}_m \to 0$ in $L^1(\tilde{\Omega} \times Q)$. Let $b : \mathbb{R} \to \mathbb{R}$ be a C^1-function with compact support. We multiply (4.32) by $b'(S^m[\tilde{\varrho}])$ to obtain

$$\partial_t b(S^m[\tilde{\varrho}]) + \operatorname{div}\big(b(S^m[\tilde{\varrho}])\tilde{\mathbf{u}}\big) + \big(b'(S^m[\tilde{\varrho}])S^m[\tilde{\varrho}] - b(S^m[\tilde{\varrho}])\big)\operatorname{div}\tilde{\mathbf{u}} = \tilde{r}_m b'(S^m[\tilde{\varrho}]).$$

As b' is bounded the right-hand side vanishes for $m \to \infty$ (in the $L^1(\tilde{\Omega} \times Q)$-sense) and we gain

$$\partial_t b(\tilde{\varrho}) + \operatorname{div}\big(b(\tilde{\varrho})\tilde{\mathbf{u}}\big) + \big(b'(\tilde{\varrho})\tilde{\varrho} - b(\tilde{\varrho})\big)\operatorname{div}\tilde{\mathbf{u}} = 0 \tag{4.33}$$

in the sense of distributions, i.e.,

$$\begin{aligned}\int_Q b(\tilde{\varrho})\,\partial_t\varphi\,\mathrm{d}x\,\mathrm{d}t = &-\int_Q \big(b(\tilde{\varrho})\tilde{\mathbf{u}}\big)\cdot\nabla\varphi\,\mathrm{d}x\,\mathrm{d}t + \int_Q \big(b'(\tilde{\varrho})\tilde{\varrho} - b(\tilde{\varrho})\big)\operatorname{div}\tilde{\mathbf{u}}\,\varphi\,\mathrm{d}x\,\mathrm{d}t\\ &-\int_{\mathcal{O}} b(\tilde{\varrho}(0))\varphi(0)\,\mathrm{d}x\end{aligned}$$

for all $\varphi \in C_c^\infty([0,T)\times\mathcal{O})$ which is equivalent to

$$\begin{aligned}\int_{\mathcal{O}} b(\tilde{\varrho})\,\psi\,\mathrm{d}x = &\int_{\mathcal{O}} b(\tilde{\varrho}(0))\psi(0)\,\mathrm{d}x + \int_0^t\int_{\mathcal{O}} \big(b(\tilde{\varrho})\tilde{\mathbf{u}}\big)\cdot\nabla\psi\,\mathrm{d}x\,\mathrm{d}\sigma\\ &-\int_0^t\int_{\mathcal{O}} \big(b'(\tilde{\varrho})\tilde{\varrho} - b(\tilde{\varrho})\big)\operatorname{div}\tilde{\mathbf{u}}\,\psi\,\mathrm{d}x\,\mathrm{d}\sigma\end{aligned}$$

for all $\psi \in C_c^\infty(\mathcal{O})$. □

4.3. Strong convergence of the density

In the first step, we proceed similar to Proposition 4.1.1 and test (4.2) by $\psi\Delta^{-1}\nabla\tilde{\varrho}_\varepsilon$ ($\psi \in C_c^\infty(Q)$), that is, we apply Itô's formula (Theorem 2.5.2) to the function $f(\rho,\mathbf{q}) = \int_{\mathcal{O}} \psi\mathbf{q}\cdot\Delta^{-1}\nabla\rho\,\mathrm{d}x$ (ϱ_ε is extended by zero to the whole space). This yields

$$\begin{aligned}&\tilde{\mathbb{E}}\int_{\mathcal{O}} \psi\tilde{\varrho}_\varepsilon\tilde{\mathbf{u}}_\varepsilon\cdot\Delta^{-1}\nabla\tilde{\varrho}_\varepsilon\,\mathrm{d}x = -\nu\,\tilde{\mathbb{E}}\int_0^t\int_{\mathcal{O}} \psi\nabla\tilde{\mathbf{u}}_\varepsilon : \nabla\Delta^{-1}\nabla\tilde{\varrho}_\varepsilon\,\mathrm{d}x\,\mathrm{d}\sigma\\
&-\nu\,\tilde{\mathbb{E}}\int_0^t\int_{\mathcal{O}} \nabla\tilde{\mathbf{u}}_\varepsilon : \nabla\psi\otimes\Delta^{-1}\nabla\tilde{\varrho}_\varepsilon\,\mathrm{d}x\,\mathrm{d}\sigma - \eta\,\tilde{\mathbb{E}}\int_0^t\int_{\mathcal{O}} \psi\operatorname{div}\tilde{\mathbf{u}}_\varepsilon\,\tilde{\varrho}_\varepsilon\,\mathrm{d}x\,\mathrm{d}\sigma\\
&-\eta\,\tilde{\mathbb{E}}\int_0^t\int_{\mathcal{O}} \operatorname{div}\tilde{\mathbf{u}}_\varepsilon\,\nabla\psi\cdot\Delta^{-1}\nabla\tilde{\varrho}_\varepsilon\,\mathrm{d}x\,\mathrm{d}\sigma + \tilde{\mathbb{E}}\int_0^t\int_{\mathcal{O}} \psi\tilde{\varrho}\tilde{\mathbf{u}}\otimes\tilde{\mathbf{u}}_\varepsilon : \nabla\Delta^{-1}\nabla\tilde{\varrho}_\varepsilon\,\mathrm{d}x\,\mathrm{d}\sigma\\
&+\tilde{\mathbb{E}}\int_0^t\int_{\mathcal{O}} \tilde{\varrho}_\varepsilon\tilde{\mathbf{u}}_\varepsilon\otimes\tilde{\mathbf{u}}_\varepsilon : \nabla\psi\otimes\Delta^{-1}\nabla\tilde{\varrho}_\varepsilon\,\mathrm{d}x\,\mathrm{d}\sigma + \tilde{\mathbb{E}}\int_0^t\int_{\mathcal{O}} a\psi\tilde{\varrho}_\varepsilon^{\gamma+1}\,\mathrm{d}x\,\mathrm{d}\sigma\\
&+\tilde{\mathbb{E}}\int_0^t\int_{\mathcal{O}} a\tilde{\varrho}_\varepsilon^{\gamma}\,\nabla\psi\cdot\Delta^{-1}\nabla\tilde{\varrho}_\varepsilon\,\mathrm{d}x\,\mathrm{d}\sigma + \tilde{\mathbb{E}}\int_0^t\int_{\mathcal{O}} \partial_t\psi\tilde{\varrho}_\varepsilon\tilde{\mathbf{u}}_\varepsilon\nabla\Delta^{-1}\tilde{\varrho}_\varepsilon\,\mathrm{d}x\,\mathrm{d}\sigma\\
&-\tilde{\mathbb{E}}\int_0^t\int_{\mathcal{O}} \psi\tilde{\varrho}_\varepsilon\tilde{\mathbf{u}}_\varepsilon\nabla\Delta^{-1}\operatorname{div}(\tilde{\varrho}_\varepsilon\tilde{\mathbf{u}}_\varepsilon)\,\mathrm{d}x\,\mathrm{d}\sigma\\
&= \tilde{\mathbb{E}}J_1 + \cdots + \tilde{\mathbb{E}}J_{10}.\end{aligned} \tag{4.34}$$

Note that the expectation of the stochastic integral vanishes. This can be written as

$$\begin{aligned}\tilde{\mathbb{E}}\int_0^t\int_{\mathcal{O}}\psi\big(a\tilde{\varrho}_\varepsilon^\gamma-(\nu+\eta)\operatorname{div}\tilde{\mathbf{u}}_\varepsilon\big)\tilde{\varrho}_\varepsilon\,\mathrm{d}x\,\mathrm{d}t\\ =\tilde{\mathbb{E}}\big[J_0-J_2-J_4-J_6-J_8-J_9\big]\\ +\tilde{\mathbb{E}}\int_0^t\int_{\mathcal{O}}\psi\tilde{u}_\varepsilon^i\big(\tilde{\varrho}_\varepsilon\mathcal{R}_{ij}[\tilde{\varrho}_\varepsilon\tilde{u}_\varepsilon^j]-\tilde{\varrho}_\varepsilon\tilde{u}_\varepsilon^j\mathcal{R}_{ij}[\tilde{\varrho}_\varepsilon]\big)\,\mathrm{d}x\,\mathrm{d}\sigma,\end{aligned}\tag{4.35}$$

where the operator $\mathcal{R}$ is defined by $\mathcal{R}_{ij}=\partial_i\Delta^{-1}\partial_j$. We proceed similarly for the limit equation (4.27) and obtain

$$\begin{aligned}\tilde{\mathbb{E}}\int_{\mathcal{O}}\psi\tilde{\varrho}\tilde{\mathbf{u}}\cdot\Delta^{-1}\nabla\tilde{\varrho}\,\mathrm{d}x&=-\nu\tilde{\mathbb{E}}\int_0^t\int_{\mathcal{O}}\psi\nabla\tilde{\mathbf{u}}:\nabla\Delta^{-1}\nabla\tilde{\varrho}\,\mathrm{d}x\,\mathrm{d}\sigma\\
&-\nu\tilde{\mathbb{E}}\int_0^t\int_{\mathcal{O}}\nabla\tilde{\mathbf{u}}:\nabla\psi\otimes\Delta^{-1}\nabla\tilde{\varrho}\,\mathrm{d}x\,\mathrm{d}\sigma-\eta\tilde{\mathbb{E}}\int_0^t\int_{\mathcal{O}}\psi\operatorname{div}\tilde{\mathbf{u}}\,\tilde{\varrho}\,\mathrm{d}x\,\mathrm{d}\sigma\\
&-\eta\tilde{\mathbb{E}}\int_0^t\int_{\mathcal{O}}\operatorname{div}\tilde{\mathbf{u}}\,\nabla\psi\cdot\Delta^{-1}\nabla\tilde{\varrho}\,\mathrm{d}x\,\mathrm{d}\sigma+\tilde{\mathbb{E}}\int_0^t\int_{\mathcal{O}}\psi\tilde{\varrho}\tilde{\mathbf{u}}\otimes\tilde{\mathbf{u}}:\nabla\Delta^{-1}\nabla\tilde{\varrho}\,\mathrm{d}x\,\mathrm{d}\sigma\\
&+\tilde{\mathbb{E}}\int_0^t\int_{\mathcal{O}}\tilde{\varrho}\tilde{\mathbf{u}}\otimes\tilde{\mathbf{u}}:\nabla\psi\otimes\Delta^{-1}\nabla\tilde{\varrho}\,\mathrm{d}x\,\mathrm{d}\sigma+\tilde{\mathbb{E}}\int_0^t\int_{\mathcal{O}}\psi\tilde{\varrho}\,\tilde{p}\,\mathrm{d}x\,\mathrm{d}\sigma\\
&+\tilde{\mathbb{E}}\int_0^t\int_{\mathcal{O}}\tilde{p}\,\nabla\psi\cdot\Delta^{-1}\nabla\tilde{\varrho}\,\mathrm{d}x\,\mathrm{d}\sigma+\tilde{\mathbb{E}}\int_0^t\int_{\mathcal{O}}\partial_t\psi\tilde{\varrho}\tilde{\mathbf{u}}\nabla\Delta^{-1}\tilde{\varrho}\,\mathrm{d}x\,\mathrm{d}\sigma\\
&-\tilde{\mathbb{E}}\int_0^t\int_{\mathcal{O}}\psi\tilde{\varrho}\tilde{\mathbf{u}}\nabla\Delta^{-1}\operatorname{div}(\tilde{\varrho}\tilde{\mathbf{u}})\,\mathrm{d}x\,\mathrm{d}\sigma\\
&=\tilde{\mathbb{E}}K_1+\cdots+\tilde{\mathbb{E}}K_{10}\end{aligned}\tag{4.36}$$

and

$$\begin{aligned}\tilde{\mathbb{E}}\int_0^t\int_{\mathcal{O}}\psi\big(\tilde{p}-(\lambda+\eta)\operatorname{div}\tilde{\mathbf{u}}\big)\tilde{\varrho}\,\mathrm{d}x\,\mathrm{d}t=\tilde{\mathbb{E}}\big[K_0-K_2-K_4-K_6-K_8-K_9-K_{10}\big]\\
+\tilde{\mathbb{E}}\int_0^t\int_{\mathcal{O}}\psi\tilde{u}^i\big(\tilde{\varrho}\mathcal{R}_{ij}[\tilde{\varrho}\tilde{u}^j]-\tilde{\varrho}\tilde{u}^j\mathcal{R}_{ij}[\tilde{\varrho}]\big)\,\mathrm{d}x\,\mathrm{d}\sigma,\end{aligned}\tag{4.37}$$

where we used the Einstein summation convention. Now we prove that $\tilde{\mathbb{E}}J_0\to\tilde{\mathbb{E}}K_0$. Due to Proposition 4.2.5, (4.25) and the compactness of the operator $\Delta^{-1}\nabla$ on $L^\gamma(\mathcal{O})$ we have for any fixed $t\in[0,T]$,

$$\begin{aligned}\Delta^{-1}\nabla\tilde{\varrho}_\varepsilon(t)\to\Delta^{-1}\nabla\tilde{\varrho}(t)\quad&\text{in}\quad L^\gamma(\mathcal{O})\quad\tilde{\mathbb{P}}\text{-a.s.},\\
\tilde{\varrho}_\varepsilon\tilde{\mathbf{u}}_\varepsilon(t)\rightharpoonup\tilde{\varrho}\tilde{\mathbf{u}}(t)\quad&\text{in}\quad L^{\frac{2\gamma}{\gamma+1}}(\mathcal{O})\quad\tilde{\mathbb{P}}\text{-a.s.}\end{aligned}$$

Hence due to the assumption $\gamma>3$

$$\int_{\mathcal{O}}\psi\tilde{\varrho}_\varepsilon\tilde{\mathbf{u}}_\varepsilon(t)\cdot\Delta^{-1}\nabla\tilde{\varrho}_\varepsilon(t)\,\mathrm{d}x\to\int_{\mathcal{O}}\psi\tilde{\varrho}\tilde{\mathbf{u}}(t)\cdot\Delta^{-1}\nabla\tilde{\varrho}(t)\,\mathrm{d}x\quad\tilde{\mathbb{P}}\text{-a.s.}$$

This, together with the following bound, for all $p \geq 1$,

$$\tilde{\mathbb{E}} \left| \int_{\mathcal{O}} \psi \tilde{\varrho}_\varepsilon \tilde{\mathbf{u}}_\varepsilon(t) \cdot \Delta^{-1} \nabla \tilde{\varrho}_\varepsilon(t) \, \mathrm{d}x \right|^p$$
$$\leq C \, \tilde{\mathbb{E}} \|\Delta^{-1} \nabla \tilde{\varrho}_\varepsilon\|^{2p}_{L^\infty(\mathcal{O})} + C \, \tilde{\mathbb{E}} \left[\int_{\mathcal{O}} |\tilde{\varrho}_\varepsilon \tilde{\mathbf{u}}_\varepsilon| \mathrm{d}x \right]^p \leq C$$

yields the claim (note that $\Delta^{-1}\nabla : L^\gamma(\mathcal{O}) \to L^\infty(\mathcal{O})$ provided $\gamma > 3$). The remaining terms (those containing derivatives of ψ) are of lower order and hence even easier to handle.

Now we come to the crucial point. In order to establish convergence of the left-hand side of (4.35) to the left-hand side of (4.37), we need to verify convergence of the remaining term on the right-hand side of (4.35) to the corresponding one in (4.37). Since $\tilde{\mathbf{u}}_\varepsilon$ is weakly convergent in $L^2(\Omega; L^2(0,T; W_0^{1,2}(\mathcal{O})))$, we have to show that $\tilde{\varrho}_\varepsilon \mathcal{R}[\tilde{\varrho}_\varepsilon \tilde{\mathbf{u}}_\varepsilon] - \tilde{\varrho}_\varepsilon \tilde{\mathbf{u}}_\varepsilon \mathcal{R}[\tilde{\varrho}_\varepsilon]$ converges strongly in $L^2(\Omega; L^2(0,T; W^{-1,2}(\mathcal{O})))$. For the identification of the limit we make use of the div-curl lemma.

From Proposition 4.2.5 we obtain that

$$\tilde{\varrho}_\varepsilon \rightharpoonup \tilde{\varrho} \quad \text{in} \quad L^\gamma(\mathcal{O}) \quad \tilde{\mathbb{P}} \otimes \mathcal{L}\text{-a.e.},$$
$$\tilde{\varrho}_\varepsilon \tilde{\mathbf{u}}_\varepsilon \rightharpoonup \tilde{\varrho} \tilde{\mathbf{u}} \quad \text{in} \quad L^{\frac{2\gamma}{\gamma+1}}(\mathcal{O}) \quad \tilde{\mathbb{P}} \otimes \mathcal{L}\text{-a.e.}$$

Hence we can apply [15, Lemma 3.4] to conclude that

$$\tilde{\varrho}_\varepsilon \mathcal{R}_{ij}[\tilde{\varrho}_\varepsilon \tilde{\mathbf{u}}_\varepsilon] - \tilde{\varrho}_\varepsilon \tilde{\mathbf{u}}_\varepsilon \mathcal{R}_{ij}[\tilde{\varrho}_\varepsilon] \rightharpoonup \tilde{\varrho} \mathcal{R}_{ij}[\tilde{\varrho} \tilde{\mathbf{u}}] - \tilde{\varrho} \tilde{\mathbf{u}} \mathcal{R}_{ij}[\tilde{\varrho}] \quad \text{in} \quad L^r(\mathcal{O}) \quad \tilde{\mathbb{P}} \otimes \mathcal{L}\text{-a.e.},$$

where

$$\frac{1}{r} = \frac{1}{\gamma} + \frac{\gamma+1}{2\gamma} < 1$$

due to $\gamma > 3$. Therefore $L^r(\mathcal{O})$ is compactly embedded into $W^{-1,\frac{3}{2}}(\mathcal{O})$ and as a consequence,

$$\tilde{\varrho}_\varepsilon \mathcal{R}_{ij}[\tilde{\varrho}_\varepsilon \tilde{\mathbf{u}}_\varepsilon] - \tilde{\varrho}_\varepsilon \tilde{\mathbf{u}}_\varepsilon \mathcal{R}_{ij}[\tilde{\varrho}_\varepsilon] \to \tilde{\varrho} \mathcal{R}_{ij}[\tilde{\varrho} \tilde{\mathbf{u}}] - \tilde{\varrho} \tilde{\mathbf{u}} \mathcal{R}_{ij}[\tilde{\varrho}] \quad \text{in} \quad W^{-1,\frac{3}{2}}(\mathcal{O}) \quad \tilde{\mathbb{P}} \otimes \mathcal{L}\text{-a.e.}$$

Moreover, it is possible to show that for any $p \in \left(2, \frac{\gamma}{2}\right)$ (using continuity of $\mathcal{R}_{ij}$, Hölder's inequality as well as Proposition 4.2.5)

$$\tilde{\mathbb{E}} \int_0^T \left\| \tilde{\varrho}_\varepsilon \mathcal{R}_{ij}[\tilde{\varrho}_\varepsilon \tilde{\mathbf{u}}_\varepsilon] - \tilde{\varrho}_\varepsilon \tilde{\mathbf{u}}_\varepsilon \mathcal{R}_{ij}[\tilde{\varrho}_\varepsilon] \right\|^p_{W_x^{-1,\frac{3}{2}}}$$
$$\leq C \, \tilde{\mathbb{E}} \int_0^T \|\tilde{\varrho}_\varepsilon\|^{2p}_{L^\gamma_x} \mathrm{d}t + C \, \tilde{\mathbb{E}} \sup_{t \in (0,T)} \|\tilde{\varrho}_\varepsilon \tilde{\mathbf{u}}_\varepsilon\|^{2p}_{L_x^{\frac{2\gamma}{\gamma+1}}} \leq C$$

which gives the desired convergence

$$\tilde{\varrho}_\varepsilon \mathcal{R}_{ij}[\tilde{\varrho}_\varepsilon \tilde{\mathbf{u}}_\varepsilon] - \tilde{\varrho}_\varepsilon \tilde{\mathbf{u}}_\varepsilon \mathcal{R}_{ij}[\tilde{\varrho}_\varepsilon] \to \tilde{\varrho} \mathcal{R}_{ij}[\tilde{\varrho} \tilde{\mathbf{u}}] - \tilde{\varrho} \tilde{\mathbf{u}} \mathcal{R}_{ij}[\tilde{\varrho}] \quad \text{in} \quad L^2(\Omega; L^2(0,T; W^{-1,\frac{3}{2}}(\mathcal{O}))).$$

Unfortunately, the space integrability of $\nabla \mathbf{u}$ is not sufficient to use this immediately. Hence we apply a spatial regularization $(\cdot)_\kappa$ with parameter $\kappa > 0$ and gain

for any fixed κ

$$\tilde{\mathbb{E}}\int_Q \psi(\tilde{u}_\varepsilon^i)_\kappa\big(\tilde{\varrho}_\varepsilon\mathcal{R}_{ij}[\tilde{\varrho}_\varepsilon\tilde{u}_\varepsilon^j]-\tilde{\varrho}_\varepsilon\tilde{u}_\varepsilon^j\mathcal{R}_{ij}[\tilde{\varrho}_\varepsilon]\big)\,\mathrm{d}x\,\mathrm{d}t \\ \to \tilde{\mathbb{E}}\int_Q \psi(\tilde{u}^i)_\kappa\big(\tilde{\varrho}\mathcal{R}_{ij}[\tilde{\varrho}\tilde{u}^j]-\tilde{\varrho}\tilde{u}^j\mathcal{R}_{ij}[\tilde{\varrho}]\big)\,\mathrm{d}x\,\mathrm{d}t.$$

Passing with κ to zero on both sides yields

$$\tilde{\mathbb{E}}\int_Q \psi\tilde{u}_\varepsilon^i\big(\tilde{\varrho}_\varepsilon\mathcal{R}_{ij}[\tilde{\varrho}_\varepsilon\tilde{u}_\varepsilon^j]-\tilde{\varrho}_\varepsilon\tilde{u}_\varepsilon^j\mathcal{R}_{ij}[\tilde{\varrho}_\varepsilon]\big)\,\mathrm{d}x\,\mathrm{d}t \\ \to \tilde{\mathbb{E}}\int_Q \psi\tilde{u}^i\big(\tilde{\varrho}\mathcal{R}_{ij}[\tilde{\varrho}\tilde{u}^j]-\tilde{\varrho}\tilde{u}^j\mathcal{R}_{ij}[\tilde{\varrho}]\big)\,\mathrm{d}x\,\mathrm{d}t. \tag{4.38}$$

and accordingly

$$\tilde{\mathbb{E}}\int_Q \psi\big(a\tilde{\varrho}_\varepsilon^\gamma-(\nu+\eta)\operatorname{div}\tilde{\mathbf{u}}_\varepsilon\big)\,\tilde{\varrho}_\varepsilon\,\mathrm{d}x\,\mathrm{d}t\to\tilde{\mathbb{E}}\int_Q\psi\big(\tilde{p}-(\nu+\eta)\operatorname{div}\tilde{\mathbf{u}}\big)\,\tilde{\varrho}\,\mathrm{d}x\,\mathrm{d}t. \tag{4.39}$$

As a consequence of the integrabilities from Proposition 4.2.5 we can exclude concentrations at the boundary. In order to deal with the local nature of (4.39) we use ideas from [13]. First of all, by the monotonicity of the mapping $z\mapsto az^\gamma$, we find for arbitrary nonnegative $\psi\in C_0^\infty(Q)$

$$\begin{aligned}
&(\nu+\eta)\liminf_{\varepsilon\to0}\tilde{\mathbb{E}}\int_Q\psi\big(\operatorname{div}\tilde{\mathbf{u}}_\varepsilon\,\tilde{\varrho}_\varepsilon-\operatorname{div}\tilde{\mathbf{u}}\,\tilde{\varrho}\big)\,\mathrm{d}x\,\mathrm{d}t\\
&\quad=\liminf_{\varepsilon\to0}\tilde{\mathbb{E}}\int_Q\Big(\psi\big(\tilde{p}-(\nu+\eta)\operatorname{div}\tilde{\mathbf{u}}\big)\tilde{\varrho}-\psi\big(a\tilde{\varrho}_\varepsilon^\gamma-(\nu+\eta)\operatorname{div}\tilde{\mathbf{u}}_\varepsilon\big)\,\tilde{\varrho}_\varepsilon\Big)\,\mathrm{d}x\,\mathrm{d}t\\
&\qquad+\liminf_{\varepsilon\to0}\tilde{\mathbb{E}}\int_Q\psi\big(a\tilde{\varrho}_\varepsilon^{\gamma+1}-\tilde{p}\tilde{\varrho}\big)\,\mathrm{d}x\,\mathrm{d}t\\
&\quad=\liminf_{\varepsilon\to0}\tilde{\mathbb{E}}\int_Q\psi\big(a\tilde{\varrho}_\varepsilon^\gamma-\tilde{p}\big)\big(\tilde{\varrho}_\varepsilon-\tilde{\varrho}\big)\,\mathrm{d}x\,\mathrm{d}t\geq0
\end{aligned}$$

using (4.39). As ψ is arbitrary we conclude

$$\overline{\operatorname{div}\tilde{\mathbf{u}}\,\tilde{\varrho}}\geq\operatorname{div}\tilde{\mathbf{u}}\,\tilde{\varrho}\quad\text{a.e. in}\quad\Omega\times Q, \tag{4.40}$$

where

$$\operatorname{div}\tilde{\mathbf{u}}_\varepsilon\,\tilde{\varrho}_\varepsilon\rightharpoonup\overline{\operatorname{div}\tilde{\mathbf{u}}\,\tilde{\varrho}}\quad\text{in}\quad L^1(\Omega\times Q),$$

recall Proposition 4.2.5. Now, we compute both sides of (4.40) by means of the corresponding continuity equations. As $\tilde{\varrho}_\varepsilon$ solves (4.2) a.e. we gain

$$\partial_t b(\tilde{\varrho}_\varepsilon)+\operatorname{div}(b(\tilde{\varrho}_\varepsilon)\tilde{\mathbf{u}}_\varepsilon)+(b'(\tilde{\varrho}_\varepsilon)\tilde{\varrho}_\varepsilon-b(\tilde{\varrho}_\varepsilon))\operatorname{div}\tilde{\mathbf{u}}_\varepsilon=0$$

$\mathbb{P}\otimes\mathcal{L}^4$-a.e. and hence

$$\int_0^t\int_{\mathcal{O}}\big(b'(\tilde{\varrho}_\varepsilon)\tilde{\varrho}_\varepsilon-b(\tilde{\varrho}_\varepsilon)\big)\operatorname{div}\tilde{\mathbf{u}}_\varepsilon\,\mathrm{d}x\,\mathrm{d}t=\int_{\mathcal{O}}b\big(\tilde{\varrho}_\varepsilon(0)\big)\,\mathrm{d}x-\int_{\mathcal{O}}b\big(\tilde{\varrho}_\varepsilon(t)\big)\,\mathrm{d}x.$$

For $b(z) = z \ln z$ we have

$$\int_0^t \int_{\mathcal{O}} \tilde{\varrho}_\varepsilon \operatorname{div} \tilde{\mathbf{u}}_\varepsilon \,\mathrm{d}x \,\mathrm{d}t = \int_{\mathcal{O}} \tilde{\varrho}_\varepsilon(0) \ln \tilde{\varrho}_\varepsilon(0) \,\mathrm{d}x - \int_{\mathcal{O}} \tilde{\varrho}_\varepsilon(t) \ln \tilde{\varrho}_\varepsilon(t) \,\mathrm{d}x. \tag{4.41}$$

Since the limit functions $(\tilde{\varrho}, \tilde{\mathbf{u}})$ solve (4.27a) in the renormalized sense as shown in Proposition 4.2.7, it follows that

$$\int_0^t \int_{\mathcal{O}} \tilde{\varrho} \operatorname{div} \tilde{\mathbf{u}} \,\mathrm{d}x \,\mathrm{d}t = \int_{\mathcal{O}} \tilde{\varrho}(0) \ln \tilde{\varrho}(0) \,\mathrm{d}x - \int_{\mathcal{O}} \tilde{\varrho}(t) \ln \tilde{\varrho}(t) \,\mathrm{d}x. \tag{4.42}$$

Combining (4.40)–(4.42) shows

$$\limsup_{\varepsilon \to 0} \tilde{\mathbb{E}} \int_{\mathcal{O}} \tilde{\varrho}_\varepsilon(t) \ln(\varrho_\varepsilon(t)) \,\mathrm{d}x \le \tilde{\mathbb{E}} \int_{\mathcal{O}} \tilde{\varrho}(t) \ln(\varrho(t)) \,\mathrm{d}x$$

for any $t \in I$. This gives the claimed convergence $\tilde{\varrho}_\varepsilon \to \tilde{\varrho}$ in $L^1(\Omega \times Q)$ by convexity of $z \mapsto z \ln z$. Consequently, we have $\tilde{p} = a\tilde{\varrho}^\gamma$ and the following strong convergence holds true

$$\tilde{\varrho}_\varepsilon \to \tilde{\varrho} \qquad \tilde{\mathbb{P}} \otimes \mathcal{L}^4\text{-a.e.} \tag{4.43}$$

With this in hand, we can finally identify the limit in the stochastic term.

Proposition 4.3.1. $\big((\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t)_{t \ge 0}, \tilde{\mathbb{P}}), \tilde{\varrho}, \tilde{\mathbf{u}}, \tilde{W}\big)$ *is a finite energy weak martingale solution to* (4.1)–(4.4).

Proof. According to Proposition 4.2.7, it remains to show that

$$\tilde{M} = \int_0^{\cdot} \tilde{\varrho} \mathbb{F} \,\mathrm{d}\tilde{W}.$$

Towards this end, it is enough to pass to the limit in (4.29), (4.30) and establish

$$\tilde{\mathbb{E}}\, h\big(\mathbf{r}_s \tilde{\varrho}, \mathbf{r}_s \tilde{\mathbf{u}}, \mathbf{r}_s \tilde{W}\big) \left[\big[\langle \tilde{M}, \boldsymbol{\varphi} \rangle^2\big]_{s,t} - \sum_{k \ge 1} \int_s^t \big\langle \varrho \mathbf{F}_k, \boldsymbol{\varphi} \big\rangle^2 \,\mathrm{d}r \right] = 0, \tag{4.44}$$

$$\tilde{\mathbb{E}}\, h\big(\mathbf{r}_s \tilde{\varrho}, \mathbf{r}_s \tilde{\mathbf{u}}, \mathbf{r}_s \tilde{W}\big) \left[\big[\langle \tilde{M}, \boldsymbol{\varphi} \rangle \tilde{\beta}_k\big]_{s,t} - \int_s^t \big\langle \tilde{\varrho} \mathbf{F}_k, \boldsymbol{\varphi} \big\rangle \,\mathrm{d}r \right] = 0. \tag{4.45}$$

The convergence in the terms that involve $M_\varepsilon(\tilde{\varrho}_\varepsilon, \tilde{\mathbf{u}}_\varepsilon, \tilde{\varrho}_\varepsilon \tilde{\mathbf{u}}_\varepsilon)$ follows from the convergences from Proposition 4.2.5 together with the fact that, due to our estimates, $M_\varepsilon(\tilde{\varrho}_\varepsilon, \tilde{\mathbf{u}}_\varepsilon, \tilde{\varrho}_\varepsilon \tilde{\mathbf{u}}_\varepsilon)$ possesses moments of any order (uniformly in ε). The convergence in terms coming from the stochastic integral can be justified similarly to (3.22) using Lemma 2.5.3 (recall strong convergence of $\tilde{\varrho}_\varepsilon$ which follows from (4.43)). □

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Dominic Breit
School of Mathematical & Computer Science
Colin McLaurin Building G.09
Heriot-Watt University
Riccarton Edinburgh
EH14 4AS, UK
e-mail: d.breit@hw.ac.uk

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Some Concepts of Generalized and Approximate Solutions in Ideal Incompressible Fluid Mechanics Related to the Least Action Principle

Yann Brenier

Abstract. Various concepts of generalized and approximate solutions related to the mathematical theory of ideal incompressible fluids are discussed in relation with variational and stochastic approaches, in close connection with the least action principle.

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1. The Least Action Principle for an ideal incompressible fluid

1.1. The configuration space of an incompressible fluid

In Classical Continuum Mechanics [8, 41, 42], the motion of an incompressible fluid moving in a compact domain D of the Euclidean space $\mathbb{R}^d$ can be seen as a trajectory $t \to g(t)$ on the configuration space $S\,\mathrm{Diff}(D)$ of all diffeomorphisms of D with Jacobian determinant equal to one. This configuration space can be embedded in a larger one, namely the set $S(D)$ of all maps h from D into itself, not necessarily one-to-one, such that, for all Borel subset B of D, $h^{-1}(B)$ is a Borel subset of D having the same Lebesgue measure as B. Such mappings $h \in S(D)$ satisfy the change of variable formula

$$\int_D \phi(h(x))dx = \int_D \phi(x)dx$$

for all $\phi \in C(D)$, where dx denotes the Lebesgue measure, normalized so that the measure of D is 1. For the composition rule, $S\,\mathrm{Diff}(D)$ is a group (the identity map I being the unity of the group), meanwhile $S(D)$ is a semi-group. Both

$S\,\mathrm{Diff}(D)$ and $S(D)$ are naturally embedded in the Hilbert space $L^2(D,\mathbb{R}^d)$ of all square integrable mapping from D into $\mathbb{R}^d$. In this space, $S(D)$ is a closed subset, meanwhile $S\,\mathrm{Diff}(D)$ is never closed, except in the degenerate case $d=1$. (If D is an interval on the real line, then $S\,\mathrm{Diff}(D)=\{I\}$.) $S(D)$ is not a convex set and is contained in a sphere of L^2. Indeed, each $h\in S(D)$ satisfies

$$\|h\|_{L^2}^2=\int_D |h(x)|^2dx=\int_D |x|^2dx=cst,$$

where $|.|$ is the Euclidean norm and $\|.\|_{L^2}$ the corresponding L^2 norm.

1.2. The Euler equations

An ideal incompressible fluid moving inside D is usually described by a velocity field $v(t,x)$ and a pressure field $p(t,x)$, subject to the classical Euler equations [36]

$$\partial_t v+(v.\nabla)v=-\nabla p,$$

$$\mathrm{div}\ v=0,$$

with the boundary condition that v is parallel to ∂D. The flowmap $(t,x)\to g(t,x)$ describing the motion of fluid particles is defined by

$$\partial_t g(t,x)=v(t,g(t,x)),\quad g(0,x)=x,$$

and an equivalent set of equations is given by

$$\partial_{tt}^2 g(t,x)=-(\nabla p)(t,g(t,x)),$$

$$det(\partial_x g(t,x))=1,$$

which insures that $t\to g(t)$ is valued in the configuration space $S\,\mathrm{Diff}(D)$, provided that v is smooth enough. For a review on the Euler equations and their importance in the field of nonlinear PDEs, see [8, 40, 41].

1.3. Geometric interpretation of the Euler equations

A formal Riemannian metric can be induced on the configuration space $S\,\mathrm{Diff}(D)$ from the L^2 norm by defining, for any g_0, g_1 in $S\,\mathrm{Diff}(D)$, the geodesic distance

$$\delta_D(g_0,g_1)=\inf\int_0^1 \|\partial_t g(t,.)\|_{L^2}dt$$

where the infimum is performed over all smooth trajectories $t\to g(t)\in G\,\mathrm{Diff}(D)$ satisfying

$$g(0)=g_0,\quad g(1)=g_1.$$

A geodesic curve can be defined as a curve $t\to g(t)\in S\,\mathrm{Diff}(D)$ such that for all $t_0\in R$, there is $\delta>0$ such that if $t_0<t_1<t_0+\delta$, then

$$\delta_D(g(t_0),g(t_1))=\int_{t_0}^{t_1}\|\partial_t g(t,.)\|_{L^2}dt. \tag{1.1}$$

If, in addition, the t parametrization of g is chosen so that $\|\partial_t g(t,.)\|_{L^2}$ is t-independent, then (1.1) means that g minimizes the Action

$$A(g) = A_{D,t_0,t_1}(g) = \frac{1}{2}\int_{t_0}^{t_1}\int_D |\partial_t g(t,x)|^2 dxdt$$

among all smooth trajectories $t \in [t_0,t_1] \to \gamma$ on $S\,\mathrm{Diff}(D)$ satisfying

$$\gamma(t_0) = g(t_0), \quad \gamma(t_1) = g(t_1). \tag{1.2}$$

This defines a Least Action Principle for the configuration space $S\,\mathrm{Diff}(D)$, which exactly corresponds to the motion of an ideal incompressible in D, as is well known since Arnold (see [7, 8]). Let us prove, for example, that any smooth solution of the Euler equations satisfies the Least Action Principle.

Proof. Let us compare g and γ subject to (1.2), fix $x \in D$ and denote $z(t) = g(t,x)$, $\zeta(t) = \gamma(t,x)$. Since p is smooth, there is a constant $K = K(p) \geq 0$ such that

$$p(t,\zeta(t)) \leq p(t,z(t)) + \nabla p(t,z(t)).(\zeta(t) - z(t)) + \frac{1}{2}K(p)|\zeta(t) - z(t)|^2.$$

By using the one-dimensional Poincaré inequality, we get

$$\int_{t_0}^{t_1} |\zeta(t) - z(t)|^2 dt \leq \frac{(t_1 - t_0)^2}{\pi^2}\int_{t_0}^{t_1} |\zeta'(t) - z'(t)|^2 dt,$$

since $\zeta(t_j) = z(t_j)$ for $j = 0, 1$. Thus

$$\int_{t_0}^{t_1} [p(t,\zeta(t)) - p(t,z(t)) - \nabla p(t,z(t)).(\zeta(t) - z(t))]dt \leq \int_{t_0}^{t_1} \frac{1}{2}|\zeta'(t) - z'(t)|^2 dt,$$

provided that $t_1 - t_0$ is small enough so that

$$\frac{(t_1 - t_0)^2}{\pi^2}K(p) \leq 1. \tag{1.3}$$

Since g is a solution to the Euler equations, we have

$$z''(t) = \partial_{tt}^2 g(t,x) = -\nabla p(t,z(t)).$$

It follows, after integrating by parts, that

$$\int_{t_0}^{t_1} [p(t,\zeta(t)) - p(t,z(t)) - z'(t).(\zeta'(t) - z'(t))]dt \leq \int_{t_0}^{t_1} \frac{1}{2}|\zeta'(t) - z'(t)|^2 dt,$$

which leads to

$$\int_{t_0}^{t_1} \left[-p(t,z(t)) + \frac{1}{2}|z'(t)|^2\right] dt \leq \int_{t_0}^{t_1} \left[-p(t,\zeta(t)) + \frac{1}{2}|\zeta'(t)|^2\right] dt.$$

After integrating over $x \in D$, we get

$$\int_{t_0}^{t_1}\int_D [-p(t,g(t,x)) + \frac{1}{2}|\partial_t g(t,x)|^2]dxdt$$
$$\leq \int_{t_0}^{t_1}\int_D \left[-p(t,\gamma(t,x)) + \frac{1}{2}|\partial_t \gamma(t,x)|^2\right] dxdt.$$

Since both g and γ are volume preserving

$$\int_{t_0}^{t_1}\int_D p(t,g(t,x))dxdt = \int_{t_0}^{t_1}\int_D p(t,\gamma(t,x))dxdt = \int_{t_0}^{t_1}\int_D p(t,x)dxdt,$$

which shows that

$$\int_{t_0}^{t_1}\int_D \frac{1}{2}|\partial_t g(t,x)|^2 dxdt \le \int_{t_0}^{t_1}\int_D \frac{1}{2}|\partial_t \gamma(t,x)|^2 dxdt$$

and achieves the proof. □

Remark 1.1. The Least Action Principle is satisfied only on sufficiently short time intervals. On larger time intervals, g is no longer a minimizer but rather a critical point of the Action. When D is convex, the constant $K(p)$ can be taken as the largest eigenvalue of the Hessian matrix of p. Condition (1.3) is sharp in the following case: D is the unique disk in R^2, $t_0 = 0$, $t_1 = \pi$, $v(x) = (-x_2, x_1)$, $p(x) = \frac{1}{2}(x_1^2 + x_2^2)$ and $g(t,x) = xe^{it}$ (where the complex notation $x = x_1 + ix_2$ is used). This fairly trivial solution to the Euler equations fails in minimizing the Action as soon as $t_1 > \pi$ (that is, after half a rotation of the disk).

1.4. The Least Action Problem (LAP)

Let us now define the Least Action Problem:

Minimize the Action among all smooth trajectories on $S\,\mathrm{Diff}(D)$ connecting two given elements g_0, g_1 of $S\,\mathrm{Diff}(D)$. This is the same as finding a shortest path between g_0 and g_1 along $S\,\mathrm{Diff}(D)$. Because of the group property, we can assume g_0 to be the identity map I and denote g_1 by h. Thus, we are looking for a curve $t \in [0,1] \to g(t) \in S\,\mathrm{Diff}(D)$, such that $g(0) = I$, $g(1) = h$, minimizing the Action

$$A(g) = A_D(g) = \frac{1}{2}\int_0^1 \int_D |\partial_t g(t,x)|^2 dxdt.$$

As mentioned before, the corresponding system of PDEs are the Euler equations, written in Lagrangian form

$$\partial_{tt}^2 g(t,x) = -\nabla p(t, g(t,x)),$$

with two point boundary conditions in time, which is different from solving the Cauchy problem, where only initial conditions are prescribed, namely $g(t=0,x)$ and $\partial_t g(t=0,x)$ for all $x \in D$. A local existence and uniqueness theorem for the LAP can be found in Ebin and Marsden paper [33]: if h and I are sufficiently close in a sufficiently high-order Sobolev norm, then there is a unique shortest path. In the large, uniqueness can fail for the LAP. For example, in the case when D is the unit disk, $g_0(x) = x = -g_1(x)$, the LAP has two solutions $g(t,x) = xe^{+i\pi t}$ and $g(t,x) = xe^{-i\pi t}$, where complex notations are used. In 1985, A. Shnirelman [47, 48] obtained a remarkable estimate on the geodesic distance, in the special case $D = [0,1]^3$, showing that there are two positive constants C and $\alpha \in]0,1[$ such that

$$\delta_{[0,1]^3}(g_0, g_1) \le C\|g_0 - g_1\|_{L^2}^{\alpha}, \tag{1.4}$$

for all g_0, g_1 in $S\,\mathrm{Diff}(D)$. (Strictly speaking, the result proved in [47] involves a mild restriction on g_0, g_1, that Shnirelman removed in 1992 [48].) The proof uses a combinatorial construction based on the concept of discrete flows (somewhat related to the numerical scheme described in [16] to solve the Euler equations). In particular, the geodesic diameter of $S\,\mathrm{Diff}([0,1]^3)$ is finite. This result is not trivial at all, since in the case $D = [0,1]^2$ the geodesic diameter is known to be infinite (due to the symplectic nature of $S\,\mathrm{Diff}(D)$ when $d = 2$, see [8, 35]). Thanks to estimate (1.4), Shnirelman was able to find data for which the global LAP has no (classical) solution. More precisely, in the case $D = [0,1]^3$, there are data h of the form

$$h(x_1, x_2, x_3) = (H(x_1, x_2), x_3),$$

where H is an area preserving mapping of the unit square, i.e., an element of $S\,\mathrm{Diff}([0,1]^2)$, for which there is no shortest path. Indeed, if H satisfies

$$\delta_{[0,1]^3}(I, h) < \delta_{[0,1]^2}(I, H) < +\infty$$

(which is possible and means that the Action can be reduced if the third dimension motion is used, even when the data are two-dimensional), then, for each trajectory γ connecting I and h on $S\,\mathrm{Diff}([0,1]^3)$, Shnirelman shows that there is such a trajectory γ' satisfying

$$A_{[0,1]^3}(\gamma') < A_{[0,1]^3}(\gamma).$$

What happens can be seen as a homogenization phenomenon. Minimizing sequences are genuinely three-dimensional flows that try to be as two-dimensional as possible, with a vanishing third component of the velocity field, but cannot converge in any strong sense to a two-dimensional flow. (Otherwise the strict inequality between two- and three-dimensional geodesic distance would be contradicted.) They rather weakly converge toward some "generalized flow", where fluid trajectories can cross each other. This picture is fully consistent with the description given in [13] where the concept of generalized flow was used, independently of Shnirelman's work, to provide generalized solutions to the LAP. Before reviewing this concept, it is worth considering several model problems for the LAP, including the L^2 projection problem onto $S\,\mathrm{Diff}(D)$, which is linked with another important nonlinear PDE, the Monge–Ampère equation.

2. From the Least Action Problem to the polar decomposition of maps

2.1. The semi-discrete Least Action Problem

A semi-discrete version of the LAP can be defined as follows. Let $N > 2$ be a given integer. We call a semi-discrete shortest path a sequence $g_1, \ldots, g_N$ in $S\,\mathrm{Diff}(D)$ that minimizes

$$A_N(g_1, \ldots, g_N) = \frac{1}{2} \sum_{i=2}^{N} \|g_i - g_{i-1}\|_{L^2}^2,$$

subject to the constraint

$$g_1 = I, \quad g_N = h,$$

where I denotes the identity map and h is the final configuration to be reached. Since $S\,\mathrm{Diff}(D)$ is included in a sphere of $L^2(D, \mathbb{R}^d)$, this amounts to maximize

$$A'_N(g_1, \ldots, g_N) = \sum_{i=2}^{N} ((g_i, g_{i-1})),$$

where $((.,.))$ denotes the L^2 inner product.

2.2. The mid-point problem and the polar decomposition of maps

In the particular case $N = 3$, there is a single unknown g_2 (since g_1 and g_3 are prescribed) supposed to maximize $((g_2, I + h))$, or, equivalently, minimize

$$\left\| g_2 - \frac{1}{2}(I + h) \right\|_{L^2}^2 .$$

In other words, g_2 is the L^2 projection of the mid-point $\frac{1}{2}(I + h)$ onto $S\,\mathrm{Diff}(D)$. Since $S\,\mathrm{Diff}(X)$ is neither convex nor closed in L^2, this problem is not trivial. If we substitute the set of all measure preserving maps $S(D)$ for $S\,\mathrm{Diff}(D)$, we get a closed bounded subset of the Hilbert space L^2. Then, it follows from Edelstein's theorem [9] that almost every element $u \in L^2(D, \mathbb{R}^d)$, in the sense of Baire, has a unique L^2 projection onto $S(D)$. As a matter of fact, the L^2 projection onto $S(D)$ (rather than $S\,\mathrm{Diff}(D)$) induces a "polar decomposition" of the space $L^2(D, \mathbb{R}^d)$ by $S(D)$ and the dual convex cone

$$K(D) = \{u \in L^2(D, \mathbb{R}^d)\,; \quad ((u, I - h)) \geq 0, \quad \forall h \in S(D)\}.$$

$K(D)$ can be characterized as the set of all square integrable mappings from D into $\mathbb{R}^d$ that coincide almost everywhere on D with the gradient of some lower semi-continuous convex function defined on $\mathbb{R}^d$. More precisely, the following "polar decomposition" theorem is stated in [15] (preceded by [13]).

Theorem 2.1. *Assume that $u \in L^2(D, \mathbb{R}^d)$ satisfies the following non-degeneracy condition: if N is a Lebesgue negligible subset of D, then $u^{-1}(N)$ is also Lebesgue negligible. Then there is a unique decomposition*

$$u(x) = \nabla\Phi(h(x)), \quad a.e.\ x \in D,$$

where h belongs to $S(D)$ and Φ (defined up to an additive constant) is the restriction to D of a lower semi-continuous convex function on $\mathbb{R}^d$. Moreover, h is the unique L^2 projection of u onto D, $\nabla\Phi$ is the unique rearrangement of u in the class $K(D)$.

By rearrangement of u, we mean any map v from D into $\mathbb{R}^d$ such that

$$\int_D \phi(v(x))dx = \int_D \phi(u(x))dx$$

holds for all $\phi \in C_c(\mathbb{R}^d)$. Theorem 2.1 shows that a vector-valued mapping u has a unique rearrangement as a gradient of some convex potential, which generalizes the classical theory on non-decreasing rearrangements of real-valued functions. Theorem 2.1 can also be seen as a nonlinear Hodge decomposition theorem. Indeed, when linearized about the identity map, the polar decomposition yields the classical unique decomposition of vector fields

$$z = w + \nabla p,$$

where z is a given vector field, w a divergence free vector field, parallel to the boundary of D, and p is a real-valued function.

A corresponding regularity result follows from Caffarelli's work [29]:

Theorem 2.2. *Let u be a smooth function from D to $\mathbb{R}^d$ where both D and $u(D)$ are supposed to be smooth and uniformly strictly convex. Assume the Jacobian determinant of u to be positive and bounded away from zero. Then u admits a unique polar factorization $u = D\Phi \circ h$, where Φ is smooth and strictly uniformly convex on D and h belongs to $S\,\mathrm{Diff}(D)$. In addition Φ can be recovered by solving a Monge–Ampère equation.*

This regularity result shows that, under strong assumptions on D and u, u has a unique L^2 projection on $S\,\mathrm{Diff}(D)$. The proof is based on the fact that the Legendre–Fenchel transform of Φ, namely

$$\Psi(y) = \sup_{x \in \mathbb{R}^d} (x.y - \Phi(x)),$$

is a weak solution to the Monge–Ampère equation

$$det D^2 \Psi = \rho,$$

where $\rho(x)dx$ is the image measure of dx by u. Caffarelli shows that Ψ is a solution in the sense of Alexandrov and is strictly convex. Then, he obtains regularity results [29]. In the meantime this theorem has been considerably extended, in particular to the case of Riemannian manifolds (see [51] for a detailed description of these results and their many contributors).

The proof of Theorem 2.1 relies on a the relaxation technique introduced by Kantorovich to solve Monge's optimal transport problem (cf. [50, 51]). Let us quote a typical result (which does not differ essentially from Theorem 2.1).

Theorem 2.3. *Assume ρ_0 and ρ_1 to be two nonnegative Lebesgue integrable compactly supported functions on $\mathbb{R}^d$, such that*

$$\int_{\mathbb{R}^d} \rho_0(x)dx = \int_{\mathbb{R}^d} \rho_1(x)dx = 1.$$

Then there is a Lipschitz continuous convex function Φ on $\mathbb{R}^d$ such that

$$\int_{\mathbb{R}^d} f(\nabla\Phi(x))\rho_0(x)dx = \int_{\mathbb{R}^d} f(x)\rho_1(x)dx$$

holds for any continuous function f on $\mathbb{R}^d$.

A proof can be sketched as follows (see [15, 50]).

Sketch of the proof. Let us consider a ball B in $\mathbb{R}^d$ containing the supports of both ρ_0 and ρ_1 and introduce the set M of all Borel regular probability measures ν on $B \times B$ having $\rho_0(x)dx$ and $\rho_1(x)dx$ as margins, which means

$$\int_{B\times B} f(x)\nu(dx,dy) = \int_B f(x)\rho_0(x)dx,$$
$$\int_{B\times B} f(y)\nu(dx,dy) = \int_B f(y)\rho_1(y)dy,$$

for all continuous functions f on $\mathbb{R}^d$. By using the Riesz representation theorem on Borel measures and elementary convex analysis (as the Rockafellar theorem stated in [28]), we obtain the duality equality

$$\max_{\nu\in M} \int_{B\times B} x.y\ \nu(dx,dy) = \inf \int_B [\Phi(x)\rho_0(x) + \Psi(x)\rho_1(x)]dx,$$

where the infimum is taken over all pairs (Φ, Ψ) of continuous functions on B satisfying

$$\Phi(x) + \Psi(y) \geq x.y, \quad \forall x \in B, \quad \forall y \in B.$$

Then, it can be established that the infimum is attained by a pair (Φ, Ψ) such that Φ is the restriction of a Lipschitz continuous convex function defined on $\mathbb{R}^d$, and for $\rho_0(x)dx$ almost every point of $\mathbb{R}^d$, Ψ coincide with the Legendre–Fenchel transform of Φ,

$$LF(\Phi)(y) = \sup_{x\in\mathbb{R}^d} (x.y - \Phi(x)).$$

Moreover, if $\nu = \nu_{\rm opt} \in M$ maximizes $\int_{B\times B} x.y\ \nu(dx,dy)$, then

$$\Phi(x) + \Psi(y) = x.y$$

holds for $\nu_{\rm opt}$-almost every $(x, y) \in \mathbb{R}^d \times \mathbb{R}^d$. Using well-known properties of the Legendre–Fenchel transform, one deduces that $\nu_{\rm opt}$ is necessarily of the form

$$\nu_{opt}(dx,dy) = \delta(y - \nabla\Phi(x))\rho_0(x)dx$$

which implies

$$\int_{\mathbb{R}^d\times\mathbb{R}^d} f(y)\nu_{\rm opt}(dx,dy) = \int_{\mathbb{R}^d} f(\nabla\Phi(x))\rho_0(x)dx,$$

for all continuous functions f on $\mathbb{R}^d$ and achieves the proof since the second margin of $\nu_{\rm opt}$ is $\rho_1(x)dx$. □

Remark 2.1. We can define the Monge–Kantorovich (often called Wasserstein) distance (see [50] for example) between ρ_0 and ρ_1 by setting

$$\Delta(\rho_0,\rho_1) = \inf_{\nu\in M} \left(\int_{D\times D} |x-y|^2 \nu(dx,dy) \right)^{1/2}. \tag{2.1}$$

Then we get

$$\int_D |\nabla\Phi(x) - x|^2 \rho_0(x)dx = \Delta(\rho_0, \rho_1)^2.$$

Indeed,

$$\begin{aligned}\int_D |\nabla\Phi(x) - x|^2 \rho_0(x)dx &= \int_{D\times D} |y - x|^2 \nu_{\text{opt}}(dx, dy)\\ &= \int_D |x|^2(\rho_0(x) + \rho_1(x))dx - \int_{D\times D} 2y.x\ \nu_{\text{opt}}(dx, dy)\end{aligned}$$

(since ρ_0 and ρ_1 are the margins of ν_{opt})

$$\leq \int_D |x|^2(\rho_0(x) + \rho_1(x))dx - \int_{D\times D} 2y.x\ \nu(dx, dy)$$

for every $\nu \in M$ (since ν_{opt} maximizes $\int y.x\ \nu(dx, dy)$),

$$= \int_{D\times D} |y - x|^2 \nu(dx, dy)$$

(since ρ_0 and ρ_1 are also the margins of ν).

Remark 2.2. The proof of Theorem 2.1 uses similar arguments and corresponds to the special case where $\rho_0(x) = 1$ and $\rho_1(x)dx$ is the image measure of dx by the mapping u. However the proof is more complicated, partly due to the assumption that u belongs to L^2, which rules out the assumption that ρ_1 is compactly supported.

3. Generalized solutions to the Least Action Problem

3.1. The concept of generalized flows

Following the idea of the proof of Theorem 2.3, the Least Action Problem has been considered in [13, 15] in a generalized, convexified, framework. Let us briefly review the concept of "generalized flows", closely related to the theory of Young's measures [53, 49]. Let us introduce the space $\Omega = D^{[0,1]}$ of all path $t \in [0, 1] \to \omega(t) \in D$. By Tychonov's theorem, Ω is a compact Hausdorff space for the product topology and the space $C(\Omega)$ of all continuous functions on Ω which can be seen, by the Stone–Weierstrass theorem, as the completion of the space $C_{\text{fin}}(\Omega)$ of all path functionals of finite type defined as follows. F is of finite type if there is a sequence $0 \leq t_1 < \cdots < t_n \leq 1$ and a continuous function f on D^n such that

$$F(\omega) = f(\omega(t_1), \ldots, \omega(t_n)), \quad \forall\omega \in \Omega.$$

The dual space $C'(\Omega)$ is also, by Riesz' theorem, the set of all Borel signed measures on Ω. Then, we define a generalized flow to be a probability Borel measure on Ω. If g is a classical flow, that is a smooth one parameter family of diffeomorphisms of D, then one can associate a unique probability measure $\mu = \mu_g$ on Ω defined by

$$\int_\Omega F(\omega)\mu_g(d\omega) = \int_D f(g(t_0, x), \ldots, g(t_n, x))dx,$$

for all $F \in C_{\rm fin}(\Omega)$. Thus there is a natural embedding of classical flows in the set of generalized flows. If $t \to g(t)$ is valued in $S\,{\rm Diff}(D)$ and satisfies the initial and final conditions $g(0) = I$, $g(1) = h$, then the corresponding probability measure μ_g satisfies the following properties. For all $\phi \in C(D)$ and $\tau \in [0,1]$,

$$\int_\Omega \phi(\omega(\tau))\mu_g(d\omega) = \int_D \phi(g(\tau,x))dx = \int_D \phi(x)dx.$$

For all $f \in C(D^2)$,

$$\int_\Omega f(\omega(0),\omega(1))\mu_g(d\omega) = \int_D f(g(0,x),g(1,x))dx = \int_D f(x,h(x))dx.$$

Moreover, the Action of g, $A(g)$ can be expressed as follows:

$$A(g) = \int_\Omega a(\omega)\mu_g(d\omega),$$

where a is the lsc function on Ω, valued in $[0,+\infty]$, defined by

$$a(\omega) = \frac{1}{2}\int_0^1 \left|\frac{d}{dt}\omega(t)\right|^2 dt,$$

whenever ω is in the Sobolev space $W^{1,2}([0,1],\mathbb{R}^d)$ and $a(\omega) = +\infty$ otherwise. See [13] for more details.

3.2. The weak formulation of the LAP

Let μ be a generalized flow. For any finite sequence $0 \le t_1 < \cdots < t_n \le 1$, we denote by $\mu_{|t_1,\ldots,t_n}$ the corresponding projection (often called "margin") of μ onto $D^{(t_1,\ldots,t_n)}$ defined by

$$\int_{D^{(t_1,\ldots,t_n)}} f(x_1,\ldots,x_n)\mu_{|t_1,\ldots,t_n}(dx_1,\ldots,dx_n) = \int_\Omega f(\omega(t_1),\ldots,\omega(t_n))\mu(d\omega),$$

for all $f \in C(D^n)$. We say that μ is incompressible if $\mu_{|t} = dx$ for all $t \in [0,1]$. By definition, a generalized solution to the LAP is any generalized incompressible flow that minimizes the Action, now defined by

$$A(\mu) = \int_\Omega a(\omega)\mu(d\omega),$$

and is compatible with h in the sense that

$$\mu_{|0,1} = \eta$$

where $\eta = \eta(h)$ is defined by

$$\eta(dx_0,dx_1) = \delta(x_1 - h(x_0))dx_0,$$

which means

$$\int_{D^2} f(x_0,x_1)\eta(dx_0,dx_1) = \int_D f(x,h(x))dx,$$

for all $f \in C(D^2)$.

3.3. The semi-discrete version of the weak LAP

A semi-discrete version of the weak LAP can be defined as follows. Let $N > 0$ be a given integer. We define a semi-discrete generalized flow as a Borel probability measure ν on D^N. We say that ν is incompressible if its margins are equal to dx

$$\nu_{|1} = \cdots = \nu_{|N} = dx \tag{3.1}$$

with the same notations as in the previous subsection. We denote by M_N the set of all semi-discrete generalized incompressible flows. If

$$\nu_{|1,N} = \eta, \qquad \text{where} \qquad \eta(dx_0, dx_1) = \delta(x_1 - h(x_0))dx_0,$$

we say that ν is compatible with h. The semi-discrete weak LAP consists in finding $\nu \in M_N$, compatible with h, that minimizes

$$A_N(\nu) = \int_{D^N} a_N(x_1, \ldots, x_N)\nu(dx_1, \ldots, dx_N)$$

where

$$a_N(x_1, \ldots, x_N) = \frac{1}{2}\sum_{i=2}^{N} |x_i - x_{i-1}|^2.$$

This problem is similar to the "multi-dimensional Monge–Kantorovich" problems considered in Rachev's paper [44]. Since A_N is continuous and M_N is compact for the vague topology, it is obvious that the semi-discrete weak LAP always has a solution (not necessarily unique). More information can be obtained by using duality arguments, as for the mid-point problem mentioned earlier, that may be summarized as follows (see [15]).

Theorem 3.1. *There is a unique sequence $p_2, \ldots, p_{N-1}$ of Lipschitz continuous functions on D, with zero mean, and a unique function $\pi \in C(D^2)$, such that inequality*

$$\pi(x_1, x_N) + p_2(x_2) + \cdots + p_{N-1}(x_{N-1}) \leq a_N(x_1, \ldots, x_N)$$

holds for all $(x_1, \ldots, x_N) \in D^N$, the corresponding equality being satisfied ν almost everywhere, for any solution ν to the semi-discrete weak LAP. In addition,

$$A_N(\nu) = \int_{D^2} \pi(x, y)\eta(dx, dy).$$

4. Results on the generalized Least Action Problem

In [13], it is shown (in the case $D = \mathbb{R}^d/\mathbb{Z}^d$), that the weak LAP always has a solution. It is also shown in [13] that a smooth classical solution to the Euler equations satisfies the Least Action Principle in the generalized framework in the same conditions as in the classical framework, namely under condition (1.3), which can be seen as a consistency result for the generalized framework. In the case $D = [0, 1]^3$, the minimizing sequences of the classical LAP converge to the generalized solutions of the weak LAP, as can be deduced from the density result by Shnirelman [48].

To get more information on the generalized solutions, duality arguments must be used. In particular, the duality result obtained for the semi-discrete weak LAP (Theorem 3.1) has been, in a suitable sense, extended to the continuous weak LAP in a series of papers by the author [14, 16, 17], followed by Ambrosio and Figalli [3, 2], from which we select the following statement [2]:

Theorem 4.1. *There is a unique function $p(t,x)$ defined on $[0,1]\times D$, with*

$$\int_D p(t,x)dx = 0, \quad \forall t \in [0,1],$$

which is locally square integrable in time and valued in the space of locally bounded variation in space such that inequality

$$\pi(\omega(0),\omega(1)) + \int_0^1 p(t,\omega(t))dt \le \frac{1}{2}\int_0^1 \left|\frac{d}{dt}\omega(t)\right|^2 dt \tag{4.1}$$

holds for each path ω in $W^{1,2}([0,1],\mathbb{R}^d)$ valued in D, the corresponding equality being satisfied μ almost everywhere, for any solution μ to the weak LAP. In addition,

$$A(\mu) = \int_{D^2} \pi(x,y)\eta(dx,dy).$$

Let us point out that, in this result, p must be considered as the pressure field governing the motion of fluid particles. Indeed, the result implies that, μ almost surely, every path ω minimizes

$$\frac{1}{2}\int_0^1 \left|\frac{d}{dt}\omega(t)\right|^2 dt - \int_0^1 p(t,\omega(t))dt$$

among all paths ζ such that $\zeta(0)=\omega(0)$, $\zeta(1)=\omega(1)$. In particular, if p is smooth enough, μ almost surely, ω is solution to the dynamical equation

$$\frac{d^2}{dt^2}\omega(t) = -(\nabla p)(t,\omega(t)),$$

which means that the probability measure μ concentrates on trajectories that are driven by the acceleration field $-\nabla p$ and, therefore, p can be considered as the pressure field of the fluid.

4.1. Continuity of generalized solutions with respect to data

A fairly straightforward consequence of the duality arguments used to solve the semi-discrete weak LAP is the continuous dependence of solutions with respect to data. More precisely, if $h_\epsilon \in S(D)$ converges to $h \in S(D)$ for the strong L^2 topology, then the corresponding (not necessarily unique) generalized solutions ν_ϵ have a subsequence that vaguely converges to a solution ν of the limit problem (the number of step N being kept fixed). As a matter of fact, the same result is true for the continuous weak LAP, but the proof involves more delicate arguments, involving Shnirelman's estimate (1.4).

4.2. Example of generalized solutions

Explicit examples of nontrivial generalized solutions to the weak LAP are described in [13]. Let us quote a first example, when D is the unique disk and $h(x) = -x$. Then, the classical LAP has two distinct solutions $g_+(t,x) = e^{i\pi t}x$ and $g_-(t,x) = e^{-i\pi t}x$, with the same pressure field $p = \pi^2|x|^2/2$, where complex notations are used. Simple minded generalized solutions can be obtained by mixing g_+ and g_- in the following way. Let θ be any measurable function from $[0,1]$ into itself. Then we define a generalized solution μ_θ by setting for each $f \in C(\Omega)$,

$$\int_\Omega f(t \to \omega(t))\mu_\theta(d\omega) = \int_{[0,1]^2} [\theta(r) f(t \to g_+(t, re^{2i\pi\sigma})) + (1-\theta(r)) f(t \to g_-(t, re^{2i\pi\sigma}))] dr d\sigma.$$

Such a generalized solution can be seen as a two phase flow, with two phases moving through each other and driven by the same pressure field. A much more interesting solution μ is given by

$$\int_\Omega f(t \to \omega(t))\mu(d\omega) = \int_D \int_0^1 f(t \to x\cos(\pi t) + (1-|x|^2)^{1/2} e^{2i\pi\sigma} \sin(\pi t)) d\sigma dx. \tag{4.2}$$

The corresponding generalized fluid motion is very peculiar and looks like a classical wave propagation on the two-dimensional sphere. For $0 < t < 1$, the fluid particle initially located at x splits up along a circle of radius $(1-|x|^2)^{1/2}\sin(\pi t)$, with center $x\cos(\pi t)$, that moves across the unit disk and shrinks down to the point $-x$ when $t = 1$. Note that all these generalized solutions are concentrated along path that are driven by the same pressure field $p = \pi^2|x|^2/2$, which is consistent with Theorem 4.1.

In the one-dimensional case $D = [0,1]$ the classical LAP is void, since $S\,\mathrm{Diff}(D) = \{I\}$. However, the weak LAP, when h is given in $S(D)$ is not trivial. It is shown in [13], for example, that, for $h(x) = 1 - x$, the unique generalized solution μ is given by

$$\int_\Omega f(t \to \omega(t))\mu(d\omega) = \int_D \int_0^1 f(t \to x\cos(\pi t) + (1-x^2)^{1/2} \cos(2\pi\sigma)\sin(\pi t)) d\sigma dx.$$

This solution can be seen as the one-dimensional projection of the two-dimensional solution defined by (4.2).

Another very interesting case is when $D = [0,1]$ and h is the map that exchanges subintervals $[0,1/2]$ and $[1/2,0]$, $h(x) = x + 1/2$ if $0 \le x < 1/2$, $h(x) = x - 1/2$, otherwise. This example is related to Shnirelman's negative result for the classical LAP (see [47]). The one-dimensional case has been thoroughly studied in [11].

Finally, let us mention that reliable numerical solutions of the generalized LAP can be obtained in different ways: combinatorial scheme as in [19] (but only as $D = [0,1]$), computational geometric methods as in [43], entropic regularization as in [31].

4.3. Two phase flows in one space dimension

Let us go back to the weak LAP when $D = [0,1]$ and h exchanges $[0,1/2]$ and $[1/2,1]$. Then, the solution looks like a two phase flow, each subinterval corresponding to a distinct phase. More generally, a one-dimensional two phase flow solution μ to the weak LAP can be defined by a pressure field p, two density fields, ρ, ρ', and two velocity fields v, v', that satisfy the following set of equations:

$$\rho + \rho' = 1,$$

$$\partial_t \rho + \partial_x(\rho v) = 0, \quad \partial_t v + \partial_x \left(\frac{v^2}{2} + p \right) = 0,$$

$$\partial_t \rho' + \partial_x(\rho' v') = 0, \quad \partial_t v' + \partial_x \left(\frac{v'^2}{2} + p \right) = 0.$$

By elimination, we get

$$\rho v + \rho' v' = 0,$$

which implies $vv' \leq 0$, and

$$-p = \rho v^2 + \rho' v'^2 = -vv'.$$

Then (v, v') solve the nonlinear first-order system

$$\partial_t v + \partial_x(\frac{v^2}{2} + vv') = 0, \quad \partial_t v' + \partial_x(\frac{v'^2}{2} + vv') = 0.$$

Such a solution μ can also be obtained directly from a variational approach. Let us consider the first phase and introduce the flowmap g associated with v. Then g must satisfy

$$\partial_x g \geq 1.$$

Indeed, the first phase must expand when moving through the second phase so that the two phase mixture keeps a constant density. $A(\mu)$ is the sum of the Action of each phase and can be computed in terms of g only, or, even simpler, in terms of

$$w(t,x) = g(t,x) - x,$$

subject to

$$\partial_x w(t,x) \geq 0.$$

We get, after simple calculations,

$$A(\mu) = \int \alpha(\partial_t w, \partial_x w) dx dt,$$

where

$$\alpha(a,b) = a^2 \left(1 + \frac{1}{b} \right)$$

is a convex function for $b > 0$, and strictly convex for $a \neq 0$. Thus, w must be solution to the degenerate quasilinear second-order elliptic equation

$$\partial_t \left(\left(1 + \frac{1}{\partial_x w} \right) \partial_t w \right) - \frac{1}{2} \partial_x \left(\frac{\partial_t w}{\partial_x w} \right)^2 = 0.$$

5. A dissipative least action principle for approximations of the Euler equations

Our purpose is to exhibit relevant approximations of the Euler equations for which a *modified* least action principle can be designed that can include *energy dissipation*. There are examples, typically in infinite dimension (but not necessarily), of *formally* Hamiltonian systems which *do not* necessarily *preserve* the energy because of some hidden dissipative mechanism:

i) the (inviscid) Burger equation

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(\frac{u^2}{2}\right) = 0, \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R} \to u(t,x) \in \mathbb{R};$$

ii) the Euler equations of incompressible fluids: at least at the physical level, it is often believed that the energy could dissipate according to Kolmogorov's "K41" theory of turbulence [37].

Let us start the discussion with special examples of finite-dimensional dynamical systems for which a dissipative version of the least action principle can be designed.

5.1. Finite-dimensional examples

Given a Euclidean space H (or more generally a Hilbert space) with norm $\|\cdot\|$ and a potential $Q : H \to \mathbb{R}$,

$$\frac{1}{2}\|V_t\|^2 + Q[X_t]$$

is the conserved energy (or Hamiltonian) for the dynamical system

$$\frac{dV_t}{dt} = -\nabla Q[X_t], \quad \frac{dX_t}{dt} = V_t, \quad (X_t, V_t) \in H \times H.$$

As is well known, its solutions can be obtained from the "least action principle" by looking for critical points of the "action"

$$\int_{t_0}^{t_1} \frac{1}{2}\left\|\frac{dX_t}{dt}\right\|^2 - Q[X_t] \quad dt,$$

among all curves $t \in [t_0, t_1] \to X_t$ with fixed values at t_0 and t_1.

We are going to define a special class of Hamiltonian systems (in finite dimension), for which a *modified* least action principle can be designed that can include *energy dissipation*. This issue has been already discussed by various authors, Shnirelman and Wolansky, for instance [46, 52]. The systems we are going to discuss are very special but, among them, we will get discrete or approximate versions of the Euler model of incompressible fluids.

Let H be a Euclidean space and S a bounded closed subset. Set

$$Q[X] = -\frac{1}{2}\mathrm{dist}^2(X,S) = -\inf_{s\in S}\frac{\|X-s\|^2}{2}$$

and consider the corresponding dynamical system

$$\frac{d^2X_t}{dt^2} = -\nabla Q[X_t]$$

N.B.: Q is semi-convex, but not smooth (unless S is convex).

Indeed: $Q[X] = -\frac{1}{2}\|X\|^2 + R[X]$, where $R[X] = \sup_{s\in S}((X,s)) - \frac{1}{2}\|s\|^2$ is convex.

5.2. The main example and the Vlasov–Monge–Ampère system

Let us now describe our main example. Let $\{A(1),\dots,A(N)\}$ be a cubic lattice of N points approximating $D = [-1/2,1/2]^d \subset \mathbb{R}^d$ as N tends to infinity. Define

$$H = (\mathbb{R}^d)^N, \quad S = \{(A(\sigma_1),\dots,A(\sigma_N)) \in H, \quad \sigma \in \mathcal{S}_n\}$$

(where $\mathcal{S}_N$ denotes the group of all permutations of the first N integers).

Then, the dynamical system introduced in the previous subsection reads, after elementary calculations,

$$\beta\frac{d^2X_t(\alpha)}{dt^2} = X_t(\alpha) - A(\sigma_{\mathrm{opt}}(\alpha))\ , \quad X_t(\alpha)\in\mathbb{R}^d, \ \ \alpha = 1,\dots,N \tag{5.1}$$

$$\sigma_{\mathrm{opt}} = \mathrm{Arginf}_{\ \sigma\in\mathcal{S}_N}\ \sum_{\alpha=1}^{N}|X_t(\alpha) - A(\sigma(\alpha))|^2 \tag{5.2}$$

with $\beta = 1$, involving, at each time t, a discrete optimal transport problem.

This system was introduced, in the case $\beta = -1$, in [18], where its hydrodynamic limit to the Euler equations has been established.

Notice that, as $d = 1$, this system reduces to

$$\beta\frac{d^2X_t(\alpha)}{dt^2} = X_t(\alpha) - \frac{1}{2N}\sum_{\alpha'\neq\alpha}\mathrm{sgn}(X_t(\alpha) - X_t(\alpha')).$$

This describes the Newtonian gravitational interaction of N parallel planes as $\beta = 1$ (with a global neutralization of the total mass, expressed by the linear term X_t).

The continuous version, involving the Monge–Ampère equation, closely related to optimal transport theory, was introduced by Brenier and Loeper [27], and studied by Cullen, Gangbo, Pisante [30], Ambrosio–Gangbo [4]. We find

$$\partial_t f(t,x,\xi) + \nabla_x\cdot(\xi\ f(t,x,\xi)) - \nabla_\xi\cdot(\nabla_x\varphi(t,x)f(t,x,\xi)) = 0 \tag{5.3}$$

$$\det(\mathbb{I} - \beta D_x^2\varphi(t,x)) = \int_{\mathbb{R}^d} f(t,x,\xi)d\xi, \quad (t,x,\xi)\in\mathbb{R}\times D\times\mathbb{R}^d. \tag{5.4}$$

This fully nonlinear version of the Vlasov–Poisson system is related to Electrodynamics ($\beta = -1$) and Gravitation ($\beta = 1$). The formal limit $\beta = 0$ reads

$$\partial_t f + \nabla_x \cdot (\xi\, f) - \nabla_\xi \cdot (\nabla_x p\, f) = 0, \int_{\mathbb{R}^d} f(t,x,\xi) d\xi = 1,$$

where $p = p(t,x)$ substitutes for φ as a Lagrange multiplier of constraint $\int f d\xi = 1$. It can be understood as a "kinetic formulation" of the Euler equations of homogeneous incompressible fluids (see [14, 17], for this concept). Classical solutions (v,p) to the Euler equations correspond to very special and singular solutions of the kinetic version of form

$$f(t,x,\xi) = \delta(\xi - v(t,x)).$$

5.3. Conservative solutions à la Bouchut–Ambrosio

Let us go back to the general case, where H and S can be chosen freely, respectively as a Euclidean space and a bounded closed subset. The dynamical system

$$\frac{d^2 X_t}{dt^2} = -\nabla Q[X_t]$$

with $Q[X] = -\frac{1}{2}\|X\|^2 + R[X]$, where $R[X] = \sup_{s\in S}((X,s)) - \frac{1}{2}\|s\|^2$ is convex, Lipschitz continuous, but not smooth (unless S is convex), cannot be treated by the usual Cauchy–Lipschitz theory. However the second derivatives of R are nonnegative bounded measures and we may apply the DiPerna–Lions theory [32], as generalized by Bouchut and Ambrosio to second-order ODEs with "coefficients of bounded variation" [1, 12]: for "almost every initial condition"

$$\left(X_0,\ \frac{dX_0}{dt}\right) \in H \times H,$$

$$\frac{d^2 X_t}{dt^2} = -\nabla Q[X_t] = X_t - \nabla R[X_t]$$

admits a global $C^{1,1}$ solution, unique in a sense precised by Ambrosio.

Such a solution is "conservative" and time-reversible. For the system of particles discussed in the previous subsection, in particular in the framework of 1D-Newtonian gravitation, this corresponds to elastic, non-dissipative collisions.

5.4. Rewriting of the action for "good" curves

There is a subset $N \subset H$, which is small in both the Baire category sense and the Lebesgue measure sense (but not empty unless S is convex), outside of which every point $X \in H \setminus N$ admits a unique closest point $\pi[X]$ on S and

$$Q = -\frac{1}{2}\mathrm{dist}^2(\cdot, S)$$

is differentiable at X with:

$$-\nabla Q[X] = X - \pi[X], \quad Q[X] = -\frac{1}{2}\|X - \pi[X]\|^2 = -\frac{1}{2}\|\nabla Q[X]\|^2.$$

So, the potential can be rewritten as a negative squared gradient.

Thus, for any "good" curve which almost never hits the bad set N, the action can be written

$$\frac{1}{2}\int_{t_0}^{t_1} \left\| \frac{dX_t}{dt} \right\|^2 + \|\nabla Q[X_t]\|^2 \quad dt$$

which can be rearranged as a perfect square up to a boundary term that does not play any role in the least action principle

$$\frac{1}{2}\int_{t_0}^{t_1} \left\| \frac{dX_t}{dt} + \nabla Q[X_t] \right\|^2 \quad dt \quad - Q[X_{t_1}] + Q[X_{t_0}].$$

5.5. Gradient-flow solutions as special least-action solutions

Due to the very special structure of the action, we find as particular least action solutions any solution to the first-order "gradient-flow equation"

$$\frac{dX_t}{dt} = -\nabla Q[X_t]$$

(somewhat like "instantons" in Yang–Mills theory). However, this is correct only when $t \to X_t \in H$ is a "good" curve (i.e., almost never hits the "bad set" where Q is not differentiable).

5.6. Global dissipative solutions of the gradient-flow

Since Q is semi-convex, we may use the classical theory of maximal monotone operators (going back to the 70s, as in the book by H. Brezis [28]) to solve the initial value problem for the gradient-flow equation.

For each initial condition, there is a unique global solution s.t

$$\frac{d_+X_t}{dt} = -\overline{\nabla} Q[X_t]\ , \quad \forall t \geq 0., \quad X \in C^0([0, +\infty[, H). \tag{5.5}$$

Here, $\frac{d_+}{dt}$ denotes the right-derivative at t, and, for each X,

$$\overline{\nabla} Q[X] = -X + \overline{\nabla} R[X]$$

where $\overline{\nabla} R[X]$ is the "relaxed" gradient of the convex function R at point X, i.e., the unique $w \in H$ with lowest norm, $\|w\|$, such that

$$R[Z] \geq R[X] + ((w, Z - X)), \quad \forall Z \in H.$$

The relaxed gradient is well defined for every X and extends the usual gradient to the "bad set" N. These solutions in the sense of maximal monotone operator theory are in general not conservative solutions (in the sense of Bouchut–Ambrosio) to the original dynamical system. Indeed, they allow velocity jumps and are generally only Lipshitz continuous and not C^1.

However, they have interesting dissipative features. Indeed, the velocity may jump with an instantaneous loss of kinetic energy.

In the case of one-dimensional gravitating particles, these jumps precisely correspond to sticky collisions [25, 24]. The bad set N is just the collision set and the relaxed gradient precisely encodes sticky collisions instead of elastic collisions.

5.7. A proposal for a modified action

The conservative solutions, that are only defined for almost every initial condition, manage to hit the bad set only for a negligible amount of time, while the gradient flow solutions enjoy very much staying in it as soon as they enter it.

Our proposal is to pick up the nice dissipative property of the gradient flow solutions and to lift them to the full dynamical system. For that purpose, we introduce the "modified action"

$$\int_{t_0}^{t_1} \left\| \frac{dX_t}{dt} \right\|^2 + \left\| \overline{\nabla} Q[X_t] \right\|^2 \quad dt \tag{5.6}$$

which favors "bad" curves that stay on the "bad set" for a while. Let us recall that $\overline{\nabla} Q$ denotes the "relaxed" gradient of the semi-convex function

$$Q[X] = -\frac{1}{2}\mathrm{dist}^2(X, S) = -\frac{1}{2}\|X\|^2 + \sup_{s \in S} \left\{ ((X, s)) - \frac{1}{2}\|s\|^2 \right\}. \tag{5.7}$$

6. Stochastic and quantum origin of the dissipative least action principle

Using large deviation principles (or alternatively the concept of guiding wave coming from quantum mechanics), we will derive, following [22] and from essentially nothing (namely N independent Brownian particles without any interaction nor external potential), the dissipative least action principle (5.6, 5.7), for the special system (5.1, 5.2), in the "gravitational" case $\beta = 1$. Let us recall that this system is a discretization of the Vlasov–Monge–Ampère system (5.3, 5.4) as well as an approximation of the Euler equations.

The first step of our analysis is very much related to the Schrödinger problem, as analyzed by Christian Léonard [39] and also to recent results by Robert Berman on permanental processes related to Kählerian Geometry [10].

6.1. Localization of a Brownian point cloud

Given a point cloud

$$\{A(\alpha) \in \mathbb{R}^d,\ \alpha = 1, \dots, N\},$$

we consider N independent Brownian curves issued from this cloud

$$Y_t(\alpha) = A(\alpha) + \sqrt{\epsilon} B_t(\alpha), \quad \alpha = 1, \dots, N.$$

At a fixed time $T > 0$, the probability for the moving cloud to reach position $X = (X(\alpha),\ \alpha = 1, \dots, N) \in \mathbb{R}^{dN}$ has density

$$\frac{1}{Z} \sum_{\sigma \in \mathcal{S}_N} \prod_{\alpha=1}^{N} \exp\left(-\frac{|X(\alpha) - A(\sigma(\alpha))|^2}{2\epsilon T} \right) = \frac{1}{Z} \sum_{\sigma \in \mathcal{S}_N} \exp\left(-\frac{\|X - A_\sigma\|^2}{2\epsilon T} \right)$$

(here $\mathcal{S}_N$ denotes the group of all permutations of the first N integers, while $|\cdot|$ and $\|\cdot\| =$ are the euclidean norms respectively on $\mathbb{R}^d$ and $\mathbb{R}^{Nd}$.)

Since

$$-\epsilon \log \frac{1}{Z} \sum_{\sigma \in \mathcal{S}_N} \exp\left(-\frac{\|X - A_\sigma\|^2}{2\epsilon T}\right) \sim \frac{1}{2T} \inf_{\sigma \in \mathcal{S}_N} \|X - A_\sigma\|^2$$

as $\epsilon \to 0$, an observer at time T feels that the particles arrived at $X_T \in \mathbb{R}^{dN}$, have travelled along straight lines by "optimal transport"

$$X_t = \left(1 - \frac{t}{T}\right) A_{\sigma_{\rm opt}} + \frac{t}{T} X_T \ , \quad \sigma_{\rm opt} = {\rm Arginf}_{\sigma \in \mathcal{S}_N} \ \|X_T - A_\sigma\|^2.$$

This formula implies (through a simple argument)

$$\frac{dX_t}{dt} = \frac{X_t - A_{\sigma_{\rm opt}}}{t} \ , \quad \sigma_{\rm opt} = {\rm Arginf}_{\sigma \in \mathcal{S}_N} \ \|X_t - A_\sigma\|^2.$$

The resulting "deterministic" process is, as a matter fact, just the output of the pure observation of a random process as the level of noise vanishes. From a physical viewpoint, it is equivalent to the Zeldovich model in Cosmology [54, 45, 38, 26].

6.2. An alternative viewpoint: the pilot wave

Introducing the heat equation in the space of "clouds" $X \in \mathbb{R}^{Nd}$

$$\frac{\partial \rho}{\partial t}(t, X) = \frac{\epsilon}{2} \triangle \rho(t, X), \quad \rho(t = 0, X) = \frac{1}{N!} \sum_{\sigma \in \mathcal{S}_N} \delta(X - A_\sigma),$$

we follow the "pilot wave" à la de Broglie, solving the ODE $\frac{dX_t}{dt} = v(t, X_t)$ with "velocity" $v(t, X) = -\frac{\epsilon}{2} \nabla_X \log \rho(t, X)$ and find

$$\frac{dX_t}{dt} = \frac{X_t \ - \langle A \rangle}{2t} \ , \ \ \langle A \rangle = \frac{\sum_{\sigma \in \mathcal{S}_N} A_\sigma \exp\left(\frac{-\|X_t - A_\sigma\|^2}{2\epsilon t}\right)}{\sum_{\sigma \in \mathcal{S}_N} \exp\left(\frac{-\|X_t - A_\sigma\|^2}{2\epsilon t}\right)}.$$

[As a matter of fact, a similar calculation also works for the free bosonic Schrödinger equation: $(i\partial_t + 1/2\triangle)\psi = 0, \quad \psi(0, X) = \sum_\sigma \exp(-\|X - A_\sigma\|^2/a^2), \quad v = \mathcal{I}m \nabla \log \psi$.]

Using exponential time $t = \exp(2\theta)$, we may also write:

$$\frac{dX_\theta}{d\theta} = X_\theta \ - \langle A \rangle \ , \quad \langle A \rangle \ = \frac{\sum_{\sigma \in \mathcal{S}_N} A_\sigma \exp\left(\frac{-\|X_\theta - A_\sigma\|^2}{2\epsilon \exp(2\theta)}\right)}{\sum_{\sigma \in \mathcal{S}_N} \exp\left(\frac{-\|X_t - A_\sigma\|^2}{2\epsilon \exp(2\theta)}\right)}.$$

As $\epsilon \to 0$, we obtain (5.5) in the sense of maximal monotone operator theory:

$$\frac{d_+ X_\theta}{d\theta} = -\overline{\nabla} Q[X_\theta] \ , \quad Q[X] = \inf_{\sigma \in \mathcal{S}_N} \ \|X - A_\sigma\|^2/2.$$

6.3. Large deviations of the pilot system

Let us add some noise η to the "guided" trajectories (with fixed ϵ)

$$\frac{dX_\theta^\epsilon}{d\theta} = X_\theta^\epsilon - \langle A\rangle + \eta\frac{dB_\theta}{d\theta}\ , \quad \langle A\rangle = \frac{\sum_{\sigma\in\mathcal{S}_N} A_\sigma \exp\left(\frac{-\|X_\theta^\epsilon - A_\sigma\|^2}{2\epsilon\exp(2\theta)}\right)}{\sum_{\sigma\in\mathcal{S}_N}\exp\left(\frac{-\|X_\theta^\epsilon - A_\sigma\|^2}{2\epsilon\exp(2\theta)}\right)}\ .$$

For ϵ fixed and $\eta\to 0$, we first get the corresponding large deviation rate function. Then, as $\epsilon\to 0$ we can pass to the Γ-limit[1] and obtain the dissipative action (5.6, 5.7), namely

$$\int\left(\left\|\frac{dX_\theta}{d\theta}\right\|^2 + \left\|\overline{\nabla}Q[X_\theta]\right\|^2\right)d\theta\ , \quad Q[X] = \inf_{\ \sigma\in\mathcal{S}_N}\ \|X - A_\sigma\|^2/2,$$

from which we may recover, through the least action principle, a dissipative version of the discrete VMA system (5.1, 5.2)

$$\frac{d^2X_\theta}{d\theta^2} = X_\theta - A_{\sigma_{\rm opt}}\ , \quad \sigma_{\rm opt} = \mathrm{Arginf}_{\ \sigma\in\mathcal{S}_N}\ \|X_\theta - A_\sigma\|^2$$

which, in particular, includes sticky collisions in the case $d = 1$.

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Yann Brenier
CNRS, Centre de mathématiques Laurent Schwartz
Ecole Polytechnique
F-91128 Palaiseau, France
e-mail: brenier@math.polytechnique.fr

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Quantitative Regularity Estimates for Compressible Transport Equations

Didier Bresch and Pierre-Emmanuel Jabin

Abstract. These notes aim at presenting some recent estimates for transport equations with rough, i.e., non-smooth, velocity fields. Our final goal is to use those estimates to obtain new results on complex systems where the transport equation is coupled to other PDE's: A driving example being the compressible Navier–Stokes system. But for simplicity, we work in the linear setting where the velocity field is given and only briefly sketch at the end of the notes how to use the new theory for nonlinear estimates.

After reviewing some of the classical results, we focus on /quantitative/estimates, in the absence of any bounds on the divergence of the velocity fields (or any corresponding bound on the Jacobian of the Lagrangian flow) for which a new approach is needed.

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Preface

We will investigate in these notes the regularity of weak solutions ρ to the advection equation in the conservative form

$$\partial_t \rho + \mathrm{div}(\rho\, u) = 0 \text{ in } (0,T)\times\Omega,$$

for a velocity field u that is not smooth with $u \in L^2(0,T;\ H^1(\Omega))$ as the typical example from Fluid Mechanics and where Ω is some smooth domain. There already exists a large body of literature around the well-posedness of such an equation with

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fields $u \in L^p(0,T;\ W^{1,p}(\Omega))$ where $p > 1$; see, for example, the surveys [5, 23]. What makes our investigation in these notes specific is that

- We require quantitative estimates of regularity. While linear advection equations are interesting in themselves, we ultimately want to consider general coupled systems for which quantitative bounds are easier to use.
- We want estimates that are compatible with strong compression effects, leading to large values of ρ, or rarefactions, leading to small values of ρ (or even vacuum in extreme cases). This means that we cannot impose a bound on $\operatorname{div} u$, upper or lower bounds on ρ. Instead, we will only assume that $\rho \in L^\infty(0,T;\ L^p(\Omega))$ for some $p > 1$.

As we will see, there now exist several types of quantitative estimates that satisfy our first constraint. But many of those are not obviously compatible with our second constraint, so that we will only introduce our main such estimate in Section 2.

Because compression or rarefaction plays a strong role here, the conservative advection equation may have a different behavior from the advective form

$$\partial_t \phi + u \cdot \nabla \phi = 0.$$

The duality between those two equations and their appropriate combination will play an important role in our calculations. Finally we should mention here that most of the material and ideas presented here are valid for many types of spatial domains Ω: Either $\Omega = \mathbb{R}^d$ with appropriate decay at infinity or Ω a smooth bounded domain with appropriate boundary conditions. Because we want to focus on the main ideas behind the method, we will however work in the torus Π^d for simplicity.

The notes are divided in three parts: the first part focuses on Lagrangian approaches and the log scale for advective equations (Corollary 1.7 deduced from Theorem 1.6 due to Crippa–De Lellis [22]); the second part concerns the Eulerian approaches (with renormalized technics due to Diperna–Lions in Theorem15 and Theorem 2.3 and deduced compactness for compressible Navier–Stokes through appropriate defect measure quantities due respectively to Lions in Theorem 2.5) and the log-log scale for compressible transport equations (Theorem 2.14 in the spirit of [17]); the last part provides an example of applications with a coupled Stokes system with a non-monotone pressure law of the method introduced by the two authors (see [18]).

1. Lagrangian approaches

This section is devoted to the study of the trajectories of ODEs flows. We, of course, hope to derive the regularity of the solution to advection equations from the regularity of the trajectories, through the method of characteristics. For this reason, we consider all possible trajectories and consider the flow

$$\frac{d}{dt}X(t,x,s) = u(t,X(t,x,s)), \quad X(t=s,x,s) = x \in \Pi^d. \tag{1}$$

Our perspective here favors Eulerian approaches as they are easier to use when transport equations are coupled to other PDE's. But in other settings there can be many advantages to direct Lagrangian methods, not least of all the simplicity of the formulation. For reader's convenience, we recall some researchers (and corresponding dates) who have obtained important results on the subject: Lipschitz (1868), Peano (1886), Lindenhöf (1894), Osgood (1900), Nagumo (1926), Filippov (1960), Di Perna–Lions (1989), Ambrosio (2004), Crippa–De Lellis (2008) and others.

First we recall the Cauchy–Lipschitlz theory for Lipschitz velocity field, then we focus on explicit regularity estimates for the velocity field belonging to Sobolev spaces in the spirit of Crippa–Delellis. Finally we explain how to find an Eulerian formulation of those Lagrangian estimates introducing appropriate weights that identify "good" trajectories as introduced recently in [17].

1.1. The Cauchy–Lipschitz theory

We start with the best-known approach to well-posedness of ODE's and advection equations whose main result can be summarized by

Theorem 1.1. *Assume that $u \in L^\infty(0,T;\ W^{1,\infty}(\Pi^d))$, then there exists a unique solution $X \in W^{1,\infty}([0,\ T] \times \Pi^d \times [0,\ T])$ to (1) which satisfies*

$$|X(t,x,s) - X(t,y,s)| \leq |x-y| \exp \int_{[s,\ t]} \|\nabla_x u(r,.)\|_{L^\infty(\Pi^d)}\, dr. \tag{2}$$

Moreover the map $x \longrightarrow X(t,x,s)$ is a homeomorphism of Π^d for any fixed t and s, with

$$X(t,X(s,x,r),s) = X(t,x,r), \quad \textit{in particular} \quad X(t,X(s,x,t),s) = x. \tag{3}$$

We do not give the proof of this theorem which is already well known (even for $u \in L^1(0,T;\ W^{1,\infty}(\Pi^d))$, but we emphasize that the main point is to derive estimate (2) through the use of the Gronwall lemma and the well-known inequality

$$|u(t,x) - u(t,y)| \leq \|\nabla u\|_{L^\infty([0,\ T]\times\Pi^d)}\, |x-y|. \tag{4}$$

We also observe that since we work on the torus, we are able to bypass all the discussion about trajectories going to infinity and the need for maximal solutions.

The method of characteristics allows to translate such regularity on the solution to advection equations as per

Theorem 1.2. *Assume that* $u \in L^\infty(0,T;\ W^{1,\infty}(\Pi^d))$ *and that* $\phi^0 \in L^1(\Pi^d)$*, then there exists a unique solution in the sense of distribution to*

$$\partial_t \phi + u \cdot \nabla \phi = 0, \quad \phi|_{t=0} = \phi^0, \tag{5}$$

which is given by

$$\phi(t,x) = \phi^0(X(0,x,t)).$$

In addition if $\phi^0 \in W^{s,p}(\Pi^d)$ *(with* $0 \leq s \leq 1$*) then*

$$\|\phi(t,.)\|_{W^{s,p}(\Pi^d)} \leq \|\phi(t,.)\|_{W^{s,p}(\Pi^d)} \exp \int_{[0,\ t]} \|\nabla_x u(r,.)\|_{L^\infty(\Pi^d)}\, dr.$$

We again skip the proof which is straightforward using Theorem 1.1, and in particular Equation (3). If one instead wishes to solve the conservative form of the transport equation, it is necessary to look at the Jacobian of the map X. Define hence

$$J(t,x,s) = \det \nabla_x X(t,x,s).$$

Then this Jacobian solves the ODE

$$\frac{d}{dt} J(t,x,s) = J(t,x,s)\ \operatorname{div} u(t, X(t,x,s)), \tag{6}$$

which is well posed in this theory as $\operatorname{div} u$ is bounded. One then has

Corollary 1.3. *Assume that* $u \in L^\infty(0,T;\ W^{1,\infty}(\Pi^d))$ *and that* $\phi^0 \in L^1(\Pi^d)$*, then there exists a unique solution in the sense of distribution to*

$$\partial_t \rho + \operatorname{div}(\rho\, u) = 0, \quad \rho|_{t=0} = \rho^0, \tag{7}$$

which is given by

$$\rho(t,x) = \frac{\rho^0(X(0,x,t))}{J(t, X(0,x,t))}.$$

While we have well-posedness under the same condition, one immediately sees that the regularity of the solution ρ also requires the corresponding regularity of $\operatorname{div} u$. This will be a recurring theme as the regularity for the convective or conservative form (7) will be consistently more difficult to obtain.

1.2. Lagrangian estimates for $u \in L^1(0,T;\ W^{1,p}(\Pi^d))$

It is possible to slightly extend the Gronwall like estimates in the previous section, for example to log-Lipschitz velocity field which is critical for the uniqueness theory of 2d-Euler (see for instance Youdovitch, J.-Y. Chemin and N. Lerner, E. Zuazua). But the first results on global well-posedness for velocity fields in Sobolev spaces were obtained in [25] in an Eulerian framework that we will present in the next section.

Instead, a corresponding Lagrangian approach was only introduced much later in [22]. In addition, [22] also provided the first explicit regularity estimates when $u \in W^{1,p}$, which makes it especially relevant for our purpose. The approach introduced in [22] has proved to be very fruitful with now many extensions. We only quote a few examples: [9] and [11] are concerned with velocity fields u that

are obtained from a singular integral or the Riesz transform of a measure; [19] and [34] apply to special Hamiltonian dynamics similar to Newton's second law and use this specific structure to require less than one derivative on u.

Let us from now on assume that $u \in L^1(0,T,\ W^{1,p}(\Pi^d))$ with $p > 1$. The first question is in what sense we can solve Equation (1) as $u(t,x)$ may not be defined at every point x. We hence rely on some a priori estimates on the flow,

Definition 1.4. For some exponent q with $1/p + 1/q \leq 1$, the Jacobian of the transform $x \longrightarrow X(t,x,s)$ is bounded in L^q in the sense iff for any $\psi \in \mathcal{C}(\Pi^d)$

$$\begin{aligned} &\int_{\Pi^d} \psi(X(t,x,s))\,dx = \int_{\Pi^d} \psi(x)\,w(t,x,s)\,dx, \\ &\text{with} \sup_{s\in[0,\ T]} \sup_{t\in[0,\ s]} \|w(t,.,s)\|_{L^q(\Pi^d)} = L < \infty. \end{aligned} \tag{8}$$

The function w can be interpreted as the law of the random variable $X(t,x,s)$ and would correspond to the previous $1/J(t,X(s,x,t),s)$. But it may not always be calculated directly like that as our flow may not be differentiable. In particular and contrary to the Lipschitz case, w may in fact vanish over large sets.

The reason for (8) will become apparent in the next section as it corresponds to natural L^q estimates on a solution ρ to (7). The original result in [22] instead assumed that w is bounded from below and from above. This allows to obtain additional properties from the regularity we present here, such as the full reversibility of the flow.

With (8), we can now define our notion of solution:

Definition 1.5. For $u \in L^1([0,\ s] \times \Pi^d)$, we say that a measurable map $X(t,x,s)$ solves (1) iff for all test function $\psi \in C^\infty([0,s] \times \Pi^{2d})$,

$$\begin{aligned} &\int_{\Pi^d} \psi(t,x,X(t,x,s))\,dx = \int_{\Pi^d} \psi(s,x,x)\,dx \\ &- \int_t^s \int_{\Pi^d} \left(\partial_t\psi(r,x,X(t,x,s)) + u(r,X(r,x,s)) \cdot \nabla_X \psi(r,X(r,x,s))\right)\,dx\,dr. \end{aligned} \tag{9}$$

The regularity estimate obtained in [22] reads

Theorem 1.6 ([22]). *Assume that $u \in L^1(0,T,\ W^{1,p}(\Pi^d))$ for some $1 < p < \infty$. Consider any solution $X(t,x,s)$ to (1) in the sense of (9) which also satisfies (8) with $1/p + 1/q \leq 1$. Then there exists a constant C depending only on the dimension s.t. for any $\omega \in \mathbb{S}^{d-1}$ and any h*

$$\int_{\Pi^d} \log\left(1 + \frac{|X(t,x,s) - X(t,x+h\,\omega,s)|}{h}\right)\,dx \leq C + C\,L \int_{[t,s]} \|\nabla u(r,.)\|_{L^p(\Pi^d)}\,dr.$$

Sketch of proof. Denote for simplicity

$$Q_h(t) = \int_{\Pi^d} \log\left(1 + \frac{|X(t,x,s) - X(t,x+h\,\omega,s)|}{h}\right)\,dx.$$

We use a doubling of variables argument by defining

$$Q_h^K(t) = \int_{\Pi^d} \int_{[t,\ s]} \log\left(1 + \frac{|X(t,x,s) - X(t',x+h\,\omega,s)|}{h}\right) dx\, K(t,t')\, dt',$$

$$\tilde{Q}_h^K(t) = \int_{\Pi^d} \int_{[t,\ s]} \log\left(1 + \frac{|X(t',x,s) - X(t,x+h\,\omega,s)|}{h}\right) dx\, K(t,t')\, dt',$$

for some kernel K.

From the definition of a solution X to (1) in the sense of (9) which also satisfies (8) with $1/p + 1/q \le 1$, one directly obtains that

$$Q_h^K(t) = Q_h^K(s) - \int_{[t,\ s]^2} \int_{\Pi^d} \frac{u(r, X(r,x,s))}{h + |X(r,x,s) - X(t',x+h\,\omega,s)|}\, dx\, K(r,t')\, dr\, dt'$$
$$- \int_{[t,\ s]^2} \int_{\Pi^d} \log\left(1 + \frac{|X(t,x,s) - X(t',x+h\,\omega,s)|}{h}\right) dx\, \partial_t K(r,t')\, dr\, dt',$$

and similarly

$$\tilde{Q}_h^K(t) = \tilde{Q}_h^K(s) - \int_{[t,\ s]^2} \int_{\Pi^d} \frac{u(r, X(r,x+h\,\omega,s))}{h + |X(t',x,s) - X(r,x+h\,\omega,s)|}\, dx\, K(r,t')\, dr\, dt'$$
$$- \int_{[t,\ s]^2} \int_{\Pi^d} \log\left(1 + \frac{|X(t,x,s) - X(t',x+h\,\omega,s)|}{h}\right) dx\, \partial_t K(r,t')\, dr\, dt'.$$

Now let K converge to $\delta(t-t')$ so that

$$Q_h(t) \le Q_h(s) + \int_{[t,\ s]} \int_{\Pi^d} \frac{|u(r, X(r,x,s)) - u(r, X(r,x+h\,\omega,s))|}{h + |X(t,x,s) - X(t,x+h\,\omega,s)|}\, dx\, dr.$$

Remark that Hypothesis (8) with (9) satisfied provides regular Lagrangian flow associated to u in the sense of Crippa–De Lellis. The key is now instead of using Lipschitz estimate to use the more precise inequality: There exists a constant C depending only on the dimension s.t. for any $u \in BV(\Pi^d)$

$$|u(x) - u(y)| \le C\,|x-y|\,(M\,|\nabla u|(x) + M\,|\nabla u|(y)), \tag{10}$$

where $M\,f$ is the maximal function of f

$$M\,f(x) = \sup_r \frac{1}{|B(x,r)|} \int_{B(x,r)} f(y)\, dy.$$

We will later give a proof of (10) in Lemma 2.9 as we will require more precise estimates. Assuming it for the time being (one can also see [46]), this leads to

$$Q_h(t) \le Q_h(s) + C \int_t^s \int_{\Pi^d} (M\,|\nabla u(r, X(r,x,s))| + M\,|\nabla u(r, X(r,x+h\,\omega,s))|)\, dx\, dr$$
$$= Q_h(s) + 2\,C \int_t^s \int_{\Pi^d} M\,|\nabla u(r, X(r,x,s))|\, dx\, dr.$$

Using now (8), we obtain the precise intermediary estimate

$$Q_h(t) \le Q_h(s) + 2\,C \int_t^s \int_{\Pi^d} M\,|\nabla u(r,x)|\, w(r,x,s)\, dx\, dr. \tag{11}$$

By Hölder's inequality, this implies

$$Q_h(t) \leq Q_h(s) + 2\,C \sup_{s\in[0,\ T]} \sup_{t\leq s} \|w(t,.,s)\|_{L^q} \int_t^s \int_{\Pi^d} \|M\,|\nabla u(r,.)\|_{L^p}\,dr,$$

which allows us to conclude the proof by recalling that the maximal function is bounded on L^p for $p>1$, that is for some constant C depending only on d

$$\|M\,f\|_{L^p(\Pi^d)} \leq C\,\|f\|_{L^p(\Pi^d)}.$$

Interested readers are referred to [22] for details.

Compactness. While its proof is relatively straightforward, the regularity that is provided by Theorem 1.6 may not be very clear at first. For example does it even imply compactness in L^1? We recall the Riesz–Fréchet–Kolmogorov criterion which is an easy way to check for compactness in a space and to measure regularity by bounding for a given smooth convolution kernel K,

$$\int_{\Pi^d} |X(t,x,s) - K_h \star X(t,.,s)|\,dx,$$

where as usual $K_h(x) = h^{-d}\,K(x/h)$. With a simple bound and the use of spherical coordinates, one may bound

$$\begin{aligned}\int_{\Pi^d} &|X(t,x,s) - K_h \star X(t,.,s)|\,dx \\ &\leq \int \int_{\mathbb{S}^{d-1}} \int_{\Pi^d} |X(t,x,s) - X(t,x+h\,r\,\omega,s)|\,dx\,K(r\,\omega)\,d\omega\,r^{d-1}\,dr.\end{aligned} \tag{12}$$

For any $t,\ s,\ h,\ r,\ \omega, R$, denote $I = \{x\in\Pi^d\,,\ |X(t,x,s) - X(t,x+h\,r\,\omega,s)\ \geq Rhr\}$, then

$$\begin{aligned}\int_{\Pi^d} &\log\left(1+\frac{|X(t,x,s)-X(t,x+h\,r\,\omega,s)|}{h\,r}\right)\,dx \\ &\geq \int_I \log\left(1+\frac{|X(t,x,s)-X(t,x+h\,r\,\omega,s)|}{h\,r}\right)\,dx \\ &\geq |I|\log(1+R)\end{aligned}$$

and therefore using Theorem 1.6

$$|I| = |\{x\,,\ |X(t,x,s)-X(t,x+h\,r\,\omega,s)| \geq R\,h\,r\}| \leq \frac{C\,L\,\|\nabla u\|_{L^1(0,T;\,L^p(\Pi^d))}+C}{\log(1+R)}.$$

Therefore writing $\Pi^d = I\cup(\Pi^d\setminus\overline{I})$ and using (12):

$$\int_{\Pi^d} |X(t,x,s) - K_h\star X(t,.,s)|\,dx \leq \left[\frac{CL\|\nabla u\|_{L^1(0,T;\,L^p(\Pi^d))}+C}{\log(1+R)}\right]^{1/p} + C_K\,R\,h,$$

where $C_K = \int |z|\,K(z)\,dz$ and where we recall that by definition $X(t,x,s)\in\Pi^d$. By using the Young inequality and choosing $R = \frac{1}{h\,\log(1/h)}$ to get convergence to

0 with respect to h, we finally obtain

$$\begin{aligned}\int_{\Pi^d} |X(t,x,s) - K_h \star X(t,.,s)|\, dx \\ \leq \frac{(C_K + C)\, L\, \|\nabla u\|_{L^1(0,\ T;\ L^p(\Pi^d))} + (C_K + C)}{\log 1/h},\end{aligned} \tag{13}$$

which proves that we control compactness through what is essentially a log of a derivative on X. This regularity may now be translated as a regularity on the transport equation in advective form

Corollary 1.7. *Assume that $u \in L^1(0,T,\ W^{1,p}(\Pi^d))$ for some $1 < p < \infty$. Assume that there exists a solution $X(t,x,s)$ to (1) in the sense of (9) which also satisfies (8) with $1/p + 1/q \leq 1$. Then for any $\phi^0 \in L^\infty(\Pi^d)$, there exists a weak solution ϕ to Equation (5) given by*

$$\phi(t,x) = \phi^0(X(0,x,t)).$$

Moreover if $\phi^0 \in W^{s,q}$ for some $s > 0$ and $q \geq 1$ then

$$\|\phi(t,.) - K_h \star \phi(t,.)\|_{L^1} \leq \frac{C\, L\, \|\nabla u\|_{L^1(0,T;\ L^p(\Pi^d))} + C}{\log 1/h},$$

for some C depending only on moments of K, s and q.

As before, we only obtain directly the advective equation. Obtaining the conservative form would require to also solve the differential equation on the Jacobian and derive regularity from it which is not obvious in this framework.

In the spirit of what we have proved on X previously, the following compactness lemma will be important especially in the third part of these notes.

Proposition 1.8. *Let a_k be a sequence uniformly bounded in some $L^p((0,T) \times \Pi^d)$ with $1 \leq p < \infty$. Assume that $\mathcal{K}_h$ is a sequence of positive, bounded functions s.t.*

i) $\forall \eta > 0, \quad \sup_h \int_{\Pi^d} \mathcal{K}_h(x)\, 1_{\{x\,:\,|x| \geq \eta\}} dx < \infty,$

ii) $\|\mathcal{K}_h\|_{L^1(\Pi^d)} \longrightarrow +\infty$ *as* $h \to +\infty$

If $\partial_t a_k \in L^q([0,\ T] \times W^{-1,q}(\Pi^d))$ (with $q \geq 1$) uniformly in k and

$$\limsup_k \Big[\frac{1}{\|\mathcal{K}_h\|_{L^1}} \int_0^T \int_{\Pi^{2d}} \mathcal{K}_h(x-y)\, |a_k(t,x) - a_k(t,y)|^p\, dx\, dy\, dt \Big] \longrightarrow 0, \quad \text{as } h \to 0$$

then a_k is compact in $L^p([0,\ T] \times \Pi^d)$. Conversely if a_k is compact in $L^p([0,\ T] \times \Pi^d)$ then the above quantity converges to 0 with h.

Remark. If we denote $\bar{\mathcal{K}}_h$ the normalized kernel

$$\bar{\mathcal{K}}_h = \frac{\mathcal{K}_h}{\|\mathcal{K}_h\|_{L^1}}.$$

Write

$$\|a_k - \bar{\mathcal{K}}_h \star_x a_k\|_{L^p}^p \leq \frac{1}{\|\mathcal{K}_h\|_{L^1}^p} \int_{\Pi^d} \Big(\int_{\Pi^d} \mathcal{K}_h(x-y)|a_k(t,x) - a_k(t,y)|dx \Big)^p dy$$
$$\leq \frac{1}{\|\mathcal{K}_h\|_{L^1}} \int_{\Pi^{2d}} \mathcal{K}_h(x-y)|a_k(t,x) - a_k(t,y)|^p dx\, dy$$

which converges to zero as $h \to 0$ uniformly in k by assumption. On the other hand for a fixed h, the sequence $\overline{K}_h \star_x u_k$ in k is compact in x. This completes the compactness in space. Concerning the compactness in time, we just have to couple everything and use the uniform bound on $\partial_t a_k$ as per the usual Aubin–Lions–Simon Lemma.

1.3. An Eulerian formulation

A very natural question following from the previous analysis is whether one can find an Eulerian formulation of those Lagrangian estimates. One could think for example of trying Wasserstein distances; we will not pursue this idea here but refer, for example, to [45]. Instead, here we will interpret the proof of Theorem 1.6 as identifying the "good" trajectories where the flow has some regularity and then proving through a sort of equivalent of (8) that those good trajectories have large probability, that means w does not vanish too much.

The tracking of good trajectories may be done through an auxiliary equation on an appropriate weight $w(t,x)$ through

$$\partial_t w + u \cdot \nabla w = -\lambda M |\nabla u| \, w, \quad w|_{t=0} = 1. \tag{14}$$

Then one may prove as a first step

Proposition 1.9. *Assume that ϕ is a renormalized solution to the transport equation in advective form, Equation* (5)*. Then if w solves Equation* (14) *with λ large enough, one has that for any $k > 0$, any h*

$$\int_{\Pi^{2\,d}} \frac{|\phi(t,x) - \phi(t,y)|}{(h + |x-y|)^k} w(t,x)\, w(t,y)\, dx\, dy \leq \int_{\Pi^{2\,d}} \frac{|\phi^0(x) - \phi^0(y)|}{(h + |x-y|)^k} dx\, dy.$$

Remark 1.10. We will define precisely what is meant by renormalized solutions in the next section. At this time, it should be interpreted as allowing to perform similar calculations as if u was smooth. Even in such a case, the proposition could for example be used to consider a sequence of solutions ϕ_n for a sequence of regularized velocity fields u_n. Quantitative regularity estimates would then be used to derive compactness.

Remark 1.11. Without weights, a quantity like

$$\sup_{h \leq 1} \int_{\Pi^{2\,d}} \frac{|\phi(t,x) - \phi(t,y)|}{(h + |x-y|)^k} dx\, dy,$$

would actually control a Besov regularity of ϕ at order $k - d$ (and hence Sobolev regularity at any lower order). Unfortunately, such regularity cannot hold for solutions to (5) with rough velocity fields, as the examples in [2] and [33].

Sketch of proof. Formal calculations show that

$$\partial_t|\phi(t,x)-\phi(t,y)|+u(t,x)\cdot\nabla_x|\phi(t,x)-\phi(t,y)|+u(t,y)\cdot\nabla_y|\phi(t,x)-\phi(t,y)|=0.$$

Such an equality will again be justified in the next section. Hence still formally

$$\begin{aligned}\partial_t(|\phi(t,x)-\phi(t,y)|\,w(t,x)\,w(t,y))+u(t,x)\cdot\nabla_x(|\phi(t,x)-\phi(t,y)|\,w(t,x)\,w(t,y))\\ +u(t,y)\cdot\nabla_y(|\phi(t,x)-\phi(t,y)|\,w(t,x)\,w(t,y))\\ =-\lambda\,(M\,|\nabla u|(t,x)+M\,|\nabla u|(t,y))\,|\phi(t,x)-\phi(t,y)|\,w(t,x)\,w(t,y).\end{aligned}$$

Multiplying by $(h+|x-y|)^{-k}$ and integrating by parts yields

$$\begin{aligned}&\frac{d}{dt}\int_{\Pi^{2\,d}}\frac{|\phi(t,x)-\phi(t,y)|}{(h+|x-y|)^k}\,w(t,x)\,w(t,y)\,dx\,dy\\ &\quad=k\int_{\Pi^{2\,d}}\frac{|\phi(t,x)-\phi(t,y)|}{(h+|x-y|)^{k+1}}\,w(t,x)\,w(t,y)\,(u(t,x)-u(t,y))\cdot\frac{x-y}{|x-y|}\,dx\,dy\\ &\qquad+\int_{\Pi^{2\,d}}\frac{|\phi(t,x)-\phi(t,y)|}{(h+|x-y|)^k}\,w(t,x)\,w(t,y)\,(\operatorname{div}u(t,x)+\operatorname{div}u(t,y))\,dx\,dy\\ &\qquad-\lambda\int_{\Pi^{2\,d}}\frac{|\phi(t,x)-\phi(t,y)|}{(h+|x-y|)^k}\,w(t,x)\,w(t,y)\,(M\,|\nabla u|(t,x)+M\,|\nabla u|(t,y))\,dx\,dy.\end{aligned}$$

Using inequality (10) and the symmetry in x and y, one finally deduces

$$\begin{aligned}&\frac{d}{dt}\int_{\Pi^{2\,d}}\frac{|\phi(t,x)-\phi(t,y)|}{(h+|x-y|)^k}\,w(t,x)\,w(t,y)\,dx\,dy\\ &\le 2\int_{\Pi^{2\,d}}\frac{|\phi(t,x)-\phi(t,y)|}{(h+|x-y|)^k}\,w(t,x)\,w(t,y)\,(\operatorname{div}u(t,x)-(\lambda-k)\,M\,|\nabla u|(t,x))\,dx\,dy.\end{aligned}$$

Since $M\,|\nabla u|(t,x)\ge|\nabla u(t,x)|\ge\operatorname{div}u(t,x)$, taking $\lambda\ge k+1$ gives the result.

Proposition 1.9 is in itself insufficient as obviously if w vanishes everywhere for instance then it contains no information. It is hence necessary to control the set where w is small, as by

Lemma 1.12. *Assume that $u\in L^1(0,T;\ W^{1,p}(\Pi^{2d}))$ with $p>1$ and that there exists a renormalized solution $\rho\in L^\infty(0,T;\ L^q(\Pi^{2d}))$ to Equation (7) with $1/p+1/q\le1$, then*

$$\int_{\Pi^d}|\log w(t,x)|\,\rho(t,x)\,dx\le C_d\,\lambda\,\|u\|_{L^1(0,T;\,W^{1,p}(\Pi^{2d}))}\,\|\rho\|_{L^\infty(0,T;\,L^q(\Pi^{2d}))}.$$

Sketch of proof. Note that since w solves (14) then using maximum principle $w\le1$ *a.e.* Hence $|\log w|=-\log w$ and using renormalization

$$\partial_t|\log w(t,x)|+u\cdot\nabla_x|\log w(t,x)|=\lambda\,M\,|\nabla u|(t,x).$$

Multiplying by ρ and integrating yields

$$\int_{\Pi^d}|\log w(t,x)|\,\rho(t,x)\,dx\le\lambda\int_0^t\int_{\Pi^d}M\,|\nabla u|(s,x)\,\rho(s,x)\,dx\,ds,$$

or by the Hölder estimate

$$\int_{\Pi^d} |\log w(t,x)|\,\rho(t,x)\,dx \le \lambda\,\|M\,|\nabla u|\|_{L^1(0,\ T;\ L^p(\Pi^{2d}))}\,\|\rho\|_{L^\infty(0,\ T;\ L^q(\Pi^{2d}))}.$$

Using the fact that the maximal operator is continuous on L^p for $p > 1$ concludes the proof.

It is now relatively simple to combine Lemma 1.12 to Proposition 1.9 to obtain

Theorem 1.13. *Assume that $u \in L^1(0,T;\ W^{1,p}(\Pi^{2d}))$ with $p > 1$ and that there exists a renormalized solution $\rho \in L^\infty(0,T;\ L^q(\Pi^{2d}))$ to Equation (7) with $1/p + 1/q \le 1$. Consider any renormalized solution $\phi \in L^\infty(0,T;\ L^r(\Pi^d))$ to Equation (5) then for any $\alpha > 0$*

$$\begin{aligned}
&\int_{\Pi^{2\,d}} \frac{|\phi(t,x)-\phi(t,y)|}{(h+|x-y|)^k}\,1\wedge\rho(t,x)\,1\wedge\rho(t,y)\,dx\,dy\\
&\quad\le h^{-\alpha}\int_{\Pi^{2\,d}} \frac{|\phi^0(x)-\phi^0(y)|}{(h+|x-y|)^k}\,dx\,dy\\
&+C\,\frac{\lambda\,h^{d-k}}{|\log h|^{1-1/r}}\,\|\phi\|_{L^\infty(0,T;\ L^r(\Pi^d))}\,\|u\|^{1-1/r}_{L^1(0,\ T;\ W^{1,p}(\Pi^{2d}))}\,\|\rho\|^{1-1/r}_{L^\infty(0,T;\ L^q(\Pi^{2d}))},
\end{aligned}$$

for some constant C depending only on the dimension d and α.

Remark 1.14. The theorem provides compactness on ϕ where ρ does not vanish. For instance if $\rho(t,x) \ge \bar\rho > 0$ and $\phi^0 \in W^{\alpha,1}$, then we may deduce that

$$\int_{\Pi^d} |\phi(t,x) - K_h \star \phi(t,x)|\,dx \le L\,|\log h|^{-1},$$

where L depends on the various norm and $K_h = C^{-1}\,(h+|x|)^{d-k}$ can be interpreted as a convolution kernel. We hence have in that case an equivalent of Corollary 1.7.

Remark 1.15. In general however, and contrary to Corollary 1.7, we only control the regularity of ϕ where $\rho > 0$. This is because the assumption that there exists a ρ in L^q is much weaker than the assumption on the Jacobian (8). In fact, translated in Eulerian framework, (8) is equivalent to asking that for any $t_0 \in [0,\ T]$, there exists $\rho_{t_0} \in L^\infty_t L^q_x$ solving (see [23] for example)

$$\partial_t \rho_{t_0} + \operatorname{div}(\rho_{t_0}\,u) = 0, \quad \rho_{t_0}|_{t=t_0} = 1.$$

Unfortunately, in our applications, we will not have such a family of solutions but only one...

Proof. The first part of the proof would be to prove that there exists an appropriate weight $w(t,x)$ solving (14) and such that we may apply Proposition 1.9 and Lemma 1.12. We will however skip this argument at the time being before we go back to the existence question for transport equations in the next section.

The rest of the proof is a simple interpolation, by decomposing Π^{2d} into the set $\{x,\, y \mid w(t,x) > \eta,\ w(t,y) > \eta\}$ and the complementary set

$$\begin{aligned}
&\int_{\Pi^{2\,d}} \frac{|\phi(t,x)-\phi(t,y)|}{(h+|x-y|)^k}\, 1\wedge\rho(t,x)\, 1\wedge\rho(t,y)\, dx\, dy \\
&\quad\le \frac{1}{\eta^2}\int_{\Pi^{2\,d}} \frac{|\phi(t,x)-\phi(t,y)|}{(h+|x-y|)^k}\, w(t,x)\, w(t,y)\, dx\, dy \\
&\qquad + \int_{x,\,y \mid \substack{w(t,x)\le\eta, \\ \text{or} \\ w(t,y)\le\eta}} \frac{|\phi(t,x)|+|\phi(t,y)|}{(h+|x-y|)^k}\, 1\wedge\rho(t,x)\, 1\wedge\rho(t,y)\, dx\, dy.
\end{aligned}$$

By symmetry the last term is bounded by

$$\begin{aligned}
&\int_{x,\,y \mid \substack{w(t,x)\le\eta, \\ \text{or} \\ w(t,y)\le\eta}} \frac{|\phi(t,x)|+|\phi(t,y)|}{(h+|x-y|)^k}\, 1\wedge\rho(t,x)\, 1\wedge\rho(t,y)\, dx\, dy \\
&\quad\le C\, h^{d-k} \int_{x,\ w(t,x)\le\eta} (|\phi(t,x)| + K_h \star |\phi|)\, 1\wedge\rho(t,x)\, dx,
\end{aligned}$$

where $K_h(x) = C^{-1}\, h^{k-d}\, (h+|x|)^k$ with C s.t. $\|K_h\|_{L^1} = 1$. By the Hölder estimate

$$\begin{aligned}
&\int_{x,\ w(t,x)\le\eta} (|\phi(t,x)| + K_h \star |\phi|)\, 1\wedge\rho(t,x)\, dx \\
&\quad\le 2\, \|\phi\|_{L^\infty(0,T;\, L^r(\Pi^d))} \left(\int_{x,\ w(t,x)\le\eta} 1\wedge\rho(t,x)\, dx\right)^{1-1/r}.
\end{aligned}$$

Now

$$\begin{aligned}
&\int_{x,\ w(t,x)\le\eta} 1\wedge\rho(t,x)\, dx \\
&\quad\le \frac{1}{|\log\eta|}\int_{x,\ w(t,x)\le\eta} |\log w(t,x)|\, \rho(t,x)\, dx \\
&\quad\le C_d\, \frac{\lambda}{|\log\eta|}\, \|u\|_{L^1(0,T;\, W^{1,p}(\Pi^{2d}))}\, \|\rho\|_{L^\infty(0,T;\, L^q(\Pi^{2d}))},
\end{aligned}$$

by using Lemma 1.12. Using now Proposition 1.9 and combining our estimates, we find

$$\begin{aligned}
&\int_{\Pi^{2\,d}} \frac{|\phi(t,x)-\phi(t,y)|}{(h+|x-y|)^k}\, 1\wedge\rho(t,x)\, 1\wedge\rho(t,y)\, dx\, dy \\
&\quad\le \frac{1}{\eta^2}\int_{\Pi^{2\,d}} \frac{|\phi(t,x)-\phi(t,y)|}{(h+|x-y|)^k}\, w(t,x)\, w(t,y)\, dx\, dy \\
&\qquad + C\, \frac{\lambda^{1-1/r}\, h^{d-k}}{|\log\eta|^{1-1/r}}\, \|\phi\|_{L^\infty(0,T;\, L^r(\Pi^d))}\, \|u\|^{1-1/r}_{L^1(0,T;\, W^{1,p}(\Pi^{2d}))}\, \|\rho\|^{1-1/r}_{L^\infty(0,T;\, L^q(\Pi^{2d}))},
\end{aligned}$$

which gives the desired result after taking $\eta = h^{\alpha/2}$. □

We can develop almost the same estimates and theory for the conservative form, starting with

Proposition 1.16. *Assume that ρ is a renormalized solution to the transport equation in conservative form, Equation* (7)*. Then if w solves Equation* (14) *with λ large enough, one has that for any $k > 0$, any h*

$$\begin{aligned}
&\int_{\Pi^{2\,d}} \frac{|\rho(t,x)-\rho(t,y)|}{(h+|x-y|)^k}\, w(t,x)\, w(t,y)\, dx\, dy \\
&\quad \le \int_{\Pi^{2\,d}} \frac{|\rho^0(x)-\rho^0(y)|}{(h+|x-y|)^k}\, dx\, dy \\
&\qquad + \frac{1}{2}\int_0^t \int_{\Pi^{2\,d}} \frac{|\operatorname{div} u(s,x) - \operatorname{div} u(s,y)|}{(h+|x-y|)^k}\, (\rho(s,x)+\rho(s,y))\, dx\, dy\, ds.
\end{aligned}$$

Sketch of proof. The argument follows exactly the same steps as before, starting with the modified equation

$$\begin{aligned}
&\partial_t |\rho(t,x)-\rho(t,y)| + u(x)\cdot\nabla_x |\rho(t,x)-\rho(t,y)| + u(y)\cdot\nabla_y |\rho(t,x)-\rho(t,y)| \\
&\quad \le \frac{|\operatorname{div} u(t,x) - \operatorname{div} u(t,y)|}{2}\, (\rho(t,x)+\rho(t,y)) \\
&\qquad + \frac{|\rho(t,x)-\rho(t,y)|}{2}\, (|\operatorname{div} u(t,x)| + |\operatorname{div} u(t,y)|.
\end{aligned}$$

Hence, we obtain

$$\begin{aligned}
&\frac{d}{dt}\int_{\Pi^{2\,d}} \frac{|\rho(t,x)-\rho(t,y)|}{(h+|x-y|)^k}\, w(t,x)\, w(t,y)\, dx\, dy \\
&\quad \le k\int_{\Pi^{2\,d}} \frac{|\rho(t,x)-\rho(t,y)|}{(h+|x-y|)^{k+1}}\, w(t,x)\, w(t,y)\, (u(t,x)-u(t,y))\cdot\frac{x-y}{|x-y|}\, dx\, dy \\
&\qquad + 2\int_{\Pi^{2\,d}} \frac{|\rho(t,x)-\rho(t,y)|}{(h+|x-y|)^k}\, w(t,x)\, w(t,y)\, (|\operatorname{div} u(t,x)| + |\operatorname{div} u(t,y)|)\, dx\, dy \\
&\qquad - \lambda\int_{\Pi^{2\,d}} \frac{|\rho(t,x)-\rho(t,y)|}{(h+|x-y|)^k}\, w(t,x)\, w(t,y)\, (M\,|\nabla u|(t,x) + M\,|\nabla u|(t,y))\, dx\, dy \\
&\qquad + \frac{1}{2}\int_{\Pi^{2\,d}} \frac{|\operatorname{div} u(t,x) - \operatorname{div} u(t,y)|}{(h+|x-y|)^k}\, (\rho(t,x)+\rho(t,y))\, w(t,x)\, w(t,y) dx\, dy.
\end{aligned}$$

The rest follows as before with only the last term remaining which yields the proposition since $w \le 1$ by the maximum principle.

Lemma 1.12 does not need to be modified and thus from Proposition 1.16, we may deduce

Theorem 1.17. *Assume that $u \in L^1(0,T;\ W^{1,p}(\Pi^{2d}))$ with $p > 1$ and that there exists a renormalized solution $\rho \in L^\infty(0,T;\ L^q(\Pi^{2d}))$ to Equation* (7) *with $1/p +$*

$1/q \leq 1$. *Then for any* $\alpha > 0$

$$\int_{\Pi^{2\,d}} \frac{|\rho(t,x)-\rho(t,y)|}{(h+|x-y|)^k}\, 1\wedge\rho(t,x)\, 1\wedge\rho(t,y)\, dx\, dy$$
$$\leq h^{-\alpha} \int_{\Pi^{2\,d}} \frac{|\rho^0(x)-\rho^0(y)|}{(h+|x-y|)^k}\, dx\, dy$$
$$+ \frac{1}{2\,h^\alpha} \int_0^t \int_{\Pi^{2\,d}} \frac{|\operatorname{div} u(s,x) - \operatorname{div} u(s,y)|}{(h+|x-y|)^k} \left(\rho(t,x)+\rho(t,y)\right) dx\, dy\, ds$$
$$+ C\, \frac{\lambda\, h^{d-k}}{|\log h|^{1-1/r}}\, \|u\|^{1-1/q}_{L^1(0,T;\, W^{1,p}(\Pi^{2d}))}\, \|\rho\|^{2-1/q}_{L^\infty(0,T;\, L^q(\Pi^{2d}))},$$

for some constant C *depending only on the dimension* d *and* α.

Compared to the result for the advective equation (5), this new estimate includes a term with $\operatorname{div} u(t,x) - \operatorname{div} u(t,y)$. As we have seen early on, it is natural that the regularity of ρ involves the corresponding regularity of $\operatorname{div} u$.

As before the regularity is obtained only where ρ does not vanish. However, there is an added twist that shows grounds for some optimism here, as now ρ is the same function for weight and for the regularity.

So, for instance, if $\rho(t,x) = \rho(t,y) = 0$, then obviously one has as well that $\rho(t,x) - \rho(t,y) = 0$ and there is nothing to control.

Unfortunately, this does not quite work: The problem occurs when only one of $\rho(t,x)$ or $\rho(t,y)$ vanishes (or is small). If $\rho(t,y) = 0$, then Theorem 1.17 does not provide any bound and therefore $\rho(t,x) - \rho(t,y)$ could well be large.

The problem is that we are using the products, $1\wedge\rho(t,x)\, 1\wedge\rho(t,y)$ and earlier $w(t,x)\, w(t,y)$, as weights. Instead, one would like to work with weights which only vanish if both $\rho(t,x)$ and $\rho(t,y)$ vanish; a good example is the sum

$$1\wedge\rho(t,x) + 1\wedge\rho(t,y), \quad w(t,x) + w(t,y).$$

Contrary to what it may first seem, this will impose major changes in our approach. Theorem 1.17 compares "good" trajectories and we would now have to compare a "good" to a "bad" trajectory. This will require proving that there are not too many bad trajectories around a good one and forces to move away from Lagrangian approaches.

2. Examples of Eulerian approaches: Renormalized solutions

We now start by reviewing the classical notion of renormalized solutions. Those provide the basic tools to obtain well-posedness for the various equations (or auxiliary equations) and are hence useful to justify our formal calculations. By emphasizing the notion of commutator estimates central to Eulerian approaches, they also lead to the method presented at the end of this section which is finally able to answer our main question. Then we present a log-log scale for compressible transport equations propagating regularity with weights and describing their properties.

2.1. Basic notions of renormalized solutions

Renormalized solutions were introduced in the seminal contribution [25]. This was the first result to obtain well-posedness for transport equations with velocity fields in $W^{1,p}$. And whereas almost all previous contributions were based on the study of the characteristics, [25] introduced a purely Eulerian method from which one could deduce the properties of the ODE and the flow if so desired.

We recall here our conservative or continuity equation

$$\partial_t \rho + \operatorname{div}(\rho\, u) = 0, \quad \rho|_{t=0} = \rho^0. \tag{15}$$

A weak solution for (15) is any $\rho \in L^1_{\rm loc}(\mathbb{R}_+ \times \Pi^d)$ s.t. $\rho\, u \in L^1_{\rm loc}(\mathbb{R}_+ \times \Pi^d)$ and for any test function $\psi \in C^\infty_c(\mathbb{R}_+ \times \Pi^d)$

$$\int_{\mathbb{R}_+} \int_{\Pi^d} (\partial_t \psi(t,x) + u \cdot \nabla_x \psi(t,x))\, \rho(t,x)\, dx\, dt = -\int_{\Pi^d} \rho^0(x)\, \psi(0,x)\, dx.$$

The dual advective form is

$$\partial_t \phi + u \cdot \nabla \phi = 0, \quad \phi|_{t=0} = \phi^0, \tag{16}$$

and a weak solution for (16) is any $\phi \in L^1_{\rm loc}(\mathbb{R}_+ \times \Pi^d)$ s.t. $\phi\, u \in L^1_{\rm loc}(\mathbb{R}_+ \times \Pi^d)$, $\phi \operatorname{div} u \in L^1_{\rm loc}(\mathbb{R}_+ \times \Pi^d)$ and for any test function $\psi \in C^\infty_c(\mathbb{R}_+ \times \Pi^d)$

$$\begin{aligned}\int_{\mathbb{R}_+} \int_{\Pi^d} &(\partial_t \psi(t,x) + u \cdot \nabla_x \psi(t,x) + \operatorname{div}\, \mathrm{u}\, \psi(t,x))\, \rho(t,x)\, dx\, dt \\ &= -\int_{\Pi^d} \rho^0(x)\, \psi(0,x)\, dx.\end{aligned}$$

Following the presentation of the theory given in [25], one defines the key notion of renormalized solutions

Definition 2.1. A function $\rho \in L^\infty(0,T;\ L^q(\Pi^d))$ is a renormalized solution to Equation (15), where $u \in L^1(0,T;\ L^p(\Pi^d))$ and $\operatorname{div} u \in L^1(0,T;\ L^p(\Pi^d))$ with $1/p + 1/q \le 1$, if ρ is a weak solution and for any $\chi \in C^1 \cap W^{1,\infty}(\mathbb{R})$, one has in the sense of distributions

$$\partial_t \chi(\rho) + \operatorname{div}(\chi(\rho)\, u) = \operatorname{div}\, \mathrm{u}\, (\chi(\rho) - \rho\, \chi'(\rho)). \tag{17}$$

Remark that the various products ρu, $\chi(\rho)\, u$ from the assumed bounds on ρ, u and $\operatorname{div} u$ since $|\chi(\rho)| \le C + C\,|\rho|$. Of course, a similar definition could be introduced for the advective form (16).

Ideally for a given velocity field u, all weak solutions would automatically be renormalized, leading to the definition

Definition 2.2. Assume that $u \in L^1(0,T;\ L^p(\Pi^d))$, $\operatorname{div} u \in L^1(0,T;\ L^p(\Pi^d))$. Equation 15 is said to have the renormalization property for this particular u if any weak solution $\rho \in L^\infty(0,T;\ L^q(\Pi^d))$ with $1/p + 1/q \le 1$ is renormalized.

Readers will immediately perceive the convenience of having a renormalized solution as it easily allows to manipulate various nonlinear quantities. However, the key point is that renormalized solutions to (16) are well behaved.

We start with uniqueness:

Theorem 2.3 ([25]). *Assume* $u \in L^1(0,T;\ L^p(\Pi^d))$, $\operatorname{div} u \in L^1(0,T;\ L^p(\Pi^d))$. *Assume moreover that Equation* (15) *has the renormalization property for* u. *Then there exists at most one weak solution* $\rho \in L^\infty(0,T;\ L^q(\Pi^d))$ *with* $1/p + 1/q \leq 1$ *for a given* ρ^0.

Proof. Given two solutions ρ_1 and ρ_2 in $L^\infty(0,T;\ L^q(\Pi^d))$, we define $\rho = \rho_1 - \rho_2$. ρ is also a weak solution and hence a renormalized one.

Choose a sequence $\chi_n \in C^1 \cap W^{1,\infty}$ s.t. $\chi_n(\xi) \to |\xi|$ in L^∞. By applying the definition of a renormalized solution to $\chi_n(\rho)$ and passing to the limit in n, one finds that in the sense of distributions

$$\partial_t |\rho| + \operatorname{div}(|\rho|\, u) = 0.$$

Let us now use the function constant and equal to 1 as test function; one has that

$$\frac{d}{dt} \int_{\Pi^d} |\rho(t,x)|\, dx = 0.$$

Since $\rho^0 = 0$, we conclude that $\rho(t,x) = 0$ for *a.e.* $t,\ x$. □

Existence can be obtained trivially but is a priori more demanding as it requires an L^∞ bound on the divergence

Theorem 2.4. *Assume* $u \in L^1(0,T;\ L^p(\Pi^d))$ *for* $p < \infty$, *and assume now that* $\operatorname{div} u \in L^1(0,T;\ L^\infty(\Pi^d))$. *Then for a given* $\rho^0 \in L^q(\Pi^d)$ *with* $q > 1$ *and* $1/q + 1/p \leq 1$, *there exists at least one weak solution* $\rho \in L^\infty(0,T;\ L^q(\Pi^d))$ *to Equation* (15).

Proof. We consider a sequence of smooth (for example Lipschitz) u_n which converges to u in $L^1(0,T;\ L^p(\Pi^d))$. Since u_n is smooth, the Cauchy–Lipschitz theory provides a sequence ρ_n of solutions to

$$\partial_t \rho_n + \operatorname{div}(\rho_n\, u_n) = 0, \quad \rho_n|_{t=0} = \rho^0.$$

Since $\operatorname{div} u \in L^1(0,\ T;\ L^\infty(\Pi^d))$, it is possible to choose the sequence u_n s.t.

$$\sup_n \| \operatorname{div} u_n \|_{L^1(0,T;\ L^\infty(\Pi^d))} < \infty.$$

On the other hand, a direct calculation shows that

$$\|\rho_n(t,.)\|^q_{L^q(\Pi^d)} \leq (q-1)\, \|\rho_n(t,.)\|^q_{L^q(\Pi^d)}\, \exp \| \operatorname{div} u \|_{L^1(0,T;\ L^\infty(\Pi^d))}.$$

Therefore ρ_n is uniformly bounded in $L^\infty(0,T;\ L^q(\Pi^d))$. We may hence extract a weak-* subsequence converging to $\rho \in L^\infty(0,T;\ L^q(\Pi^d))$.

Passing to the limit in every term, one obtains a weak solution to (15). □

This existence result does not use the renormalization property and it is natural to ask if it can be improved so that we may obtain strong convergence of the sequence of approximation. We give such an argument below based on using $\chi(\xi) = \xi \log \xi$ which forms the basis of the compactness method introduced in [38] for the compressible Navier–Stokes equation.

Theorem 2.5 ([25, 38]). *Consider a sequence u_n converging strongly, in the space $L^1(0,T;L^p(\Pi^d))$, to u for $p<\infty$, and assume moreover that $\operatorname{div} u_n$ converges to $\operatorname{div} u \in L^1(0,T;\ L^\infty(\Pi^d))$. Consider further any sequence ρ_n of renormalized solutions to*

$$\partial_t \rho_n + \operatorname{div}(\rho_n\, u_n) = 0.$$

Assume that ρ_n is uniformly bounded in $\rho \in L^\infty(0,T;\ L^q(\Pi^d))$ with $q>1$ and $1/q+1/p\le 1$, that ρ_n converges weak- to ρ, that ρ_n^0 converges strongly to $\rho^0 \in L^q(\Pi^d)$ and that ρ is a renormalized solution to (15). Then ρ_n converges strongly to ρ in $L^1([0,\ T]\times\Pi^d)$.*

Proof. First of all, we remark that we may use $\chi(\xi)=\xi\log\xi$ in the definition of a renormalized solution even though $\chi\notin W^{1,\infty}$. Consider any velocity field $u\in L^1([0,\ T],\ L^p(\Pi^d))$ with $\operatorname{div} u \in L^1([0,\ T],\ L^p(\Pi^d))$, and a renormalized solution to (15), $\rho\in L^\infty([0,\ T],\ L^q(\Pi^d))$ with $1/p+1/q<1$.

Choose any sequence $\chi_k\in C^1\cap W^{1,\infty}$ that converges pointwise to χ and is bounded by $\chi(\xi)$ for ξ large. Since $1/p+1/q<1$, it is straightforward to check that $\chi_k(\rho)\,u$ converges to $\chi(\rho)\,u$ and that Equation (17) holds for $\chi(\xi)=\xi\log\xi$.

We first apply this to ρ_n and u_n to find that

$$\partial_t \rho_n\, \log\rho_n + \operatorname{div}\left(\rho_n\, \log\rho_n\, u_n\right) = -\operatorname{div} u_n\, \rho_n.$$

We may pass to the limit in this equation. But of course, since we have not proved compactness of ρ_n yet, we cannot identify the weak-* limit of ρ_n. Let us hence denote

$$\overline{\rho\log\rho} = \text{weak-}^*\lim \rho_n\, \log\rho_n.$$

We obtain

$$\partial_t \overline{\rho\, \log\rho} + \operatorname{div}\left(\overline{\rho\, \log\rho}\, u\right) = -\operatorname{div} u\, \rho.$$

From the proof of Theorem 2.4, we know that ρ and u solve (15) which by assumption has the renormalization property. Therefore we also have

$$\partial_t \rho\, \log\rho + \operatorname{div}(\rho\, \log\rho\, u) = -\operatorname{div} u\, \rho.$$

By taking the difference and integrating over Π^d, this leads to

$$\frac{d}{dt}\int_{\Pi^d} (\overline{\rho\, \log\rho} - \rho\, \log\rho)\, dx = 0.$$

By the compactness of ρ_n^0, we finally deduce that

$$\int_{\Pi^d} (\overline{\rho\, \log\rho} - \rho\, \log\rho)\, dx = 0.$$

But by the convexity of $\chi=\xi\, \log\xi$, one has that

$$\overline{\rho\, \log\rho} - \rho\, \log\rho \ge 0,$$

concluding that

$$\overline{\rho\, \log\rho} = \rho\, \log\rho,$$

and proving the compactness of ρ_n. □

Theorem 2.5 is our first result proving compactness of a sequence ρ_n without any assumption on the essential boundedness on the divergence or any comparable assumption on the vacuum. Of course, it does not provide a quantitative regularity estimate and it relies explicitly on the structure of the limit equation, which can be a clear drawback to study nonlinear coupled models. It also remains an if-theorem at this stage as we have not yet found any sufficient condition on u to guarantee that Equation (15) has the renormalization property. This will be the object of the next section around the so-called commutator estimates.

One may make a last remark on our approach so far, which is the requirement that $1/p + 1/q \leq 1$. We, of course, need to make sense of the product $\rho\, u$ but also of the product $\rho \operatorname{div} u$. However, this last requirement does not seem optimal as a more clever use of the renormalization χ should make it unnecessary. This is in fact the basis for the improvement on the Lions theory developed in particular in [29, 30].

2.2. Proving the renormalization property: Commutator estimates

The main breakthrough of [25] was to present a very straightforward proof of

Theorem 2.6 ([25]). *Assume that $u \in L^1(0,T;W^{1,p}(\Pi^d))$, then Equation* (15) *has the renormalization property.*

Obviously, if $u \in L^1(0,T;W^{1,p}(\Pi^d))$ then $\operatorname{div} u \in L^1(0,T;L^p(\Pi^d))$ and all the results of the previous section automatically apply. There are even more consequences to having the renormalization property and we refer again to [5, 23] for a more thorough treatment.

The ideas introduced in [25] started a now very active field of research about the minimal conditions on u guaranteeing that (15) has the renormalization property. A crucial initial effort culminated in [3, 4] (after corresponding results in the kinetic case in [10]) to lower the requirement to $u \in L^1_t BV_x$ with $\operatorname{div} u \in L^1_{t,x}$, which is critical to many applications to hyperbolic systems. In view of the counterexample developed in [24], the BV regularity seems to be optimal in such a general setting.

The commutator estimates can also be partially translated on the characteristics as in [32] and renormalized solutions applied to various settings such as degenerate diffusion in [35].

It is possible to study further the regularity of renormalized solutions to (15); almost everywhere differentiability in [6] for instance. But as we mentioned before the first quantitative regularity estimate had been obtained in [22].

Proof. Consider any $\rho \in L^\infty(0,T;L^q(\Pi^d))$, weak solution to (15). For any $\chi \in C^1 \cap W^{1,\infty}$, we have to prove that (17) holds. If ρ was smooth then showing (17) would be a straightforward consequence of the chain rule. The main idea in the proof of Theorem 2.6 is hence simply to regularize ρ by convolution.

Hence choose any smooth $K \in C^1(\Pi^d)$ with $\operatorname{supp} K$ concentrated near 0 so that $K_\varepsilon(x) = \varepsilon^{-d} K(x/\varepsilon)$ is an approximation of the identity as $\varepsilon \to 0$.

Denote $\rho_\varepsilon = K_\varepsilon \star \rho$. ρ_ε cannot solve (15) exactly (unless u is constant) but one may write

$$\partial_t \rho_\varepsilon + \operatorname{div}(\rho_\varepsilon\, u) = R_\varepsilon, \tag{18}$$

where the commutator reads

$$R_\varepsilon(x) = \int_{\Pi^d} \nabla K_\varepsilon(x-y)\cdot(u(t,x)-u(t,y))\,\rho(t,y)\,dy + \rho_\varepsilon(t,x)\,\operatorname{div} u(t,x). \tag{19}$$

The heart of the method is hence to prove through a commutator estimate that R_ε converges strongly to 0. For a fixed ρ and u, this is straightforward through

Proposition 2.7 (Commutator estimate from [25]). *Let $u \in L^1(0,T;W^{1,p}(\Pi^d))$ and that $\rho \in L^\infty([0,\ T],\ L^q(\Pi^d))$ then $R_\varepsilon \longrightarrow 0$ in $L^1([0,\ T]\times\Pi^d)$ as $\varepsilon \to 0$ where R_ε is defined by* (19).

Assuming for the time being that Proposition 2.7 holds, we can easily conclude. Now ρ_ε is smooth and we may apply the chain rule on Equation (18) to find

$$\partial_t \chi(\rho_\varepsilon) + \operatorname{div}(\chi(\rho_\varepsilon)\,u) = \operatorname{div} u\,(\chi(\rho_\varepsilon) - \rho_\varepsilon\,\chi'(\rho_\varepsilon)) + \chi'(\rho_\varepsilon)\,R_\varepsilon. \tag{20}$$

Since χ' is bounded we know from the proposition that $\chi'(\rho_\varepsilon)\,R_\varepsilon \to 0$.

Since K_ε is an approximation of the identity as $\varepsilon \to 0$, then ρ_ε converges strongly to ρ in $L^r(0,T;L^q(\Pi^d))$ for any $r < \infty$. Therefore $\chi(\rho_\varepsilon)$ converges strongly to $\chi(\rho_\varepsilon)$ in the same space. Therefore $\chi(\rho_\varepsilon)$ converges weak-* to $\chi(\rho)$ in $L^\infty(0,T;L^q(\Pi^d))$. Passing to the limit in each term in Equation (20), we deduce Equation (17).

Proof of Proposition 2.7. There only remains to prove Proposition 2.7. Note that for *a.e.* $x,\ y$

$$u(t,x) - u(t,y) = \int_0^1 (x-y)\cdot\nabla u(t,\theta\,x + (1-\theta)\,y)\,d\theta,$$

which lets us write

$$\begin{aligned}&\int_{\Pi^d} \nabla K_\varepsilon(x-y)\cdot(u(t,x)-u(t,y))\,\rho(t,y)\,dy\\ &\quad= \int_0^1 \int_{\Pi^d} (x-y)\otimes\nabla K_\varepsilon(x-y)\ :\ \nabla u(t,\theta\,x+(1-\theta)\,y)\,\rho(t,y)\,dy\,d\theta.\end{aligned}$$

Remark that

$$(x-y)\otimes\nabla K_\varepsilon(x-y) = \varepsilon^{-d}\,\frac{x-y}{\varepsilon}\otimes\nabla K((x-y)/\varepsilon) = \varepsilon^{-d}\,L((x-y)/\varepsilon) = L_\varepsilon(x-y),$$

with $L(x) = x\otimes\nabla K(x)$. Observe that by integration by parts

$$\int_{\Pi^d} L_{ij}(x)\,dx = \int_{\Pi^d} x_i\,\partial_j K(x)\,dx = -\delta_{ij}\int_{\Pi^d} K(x)\,dx = -\delta_{ij}.$$

Hence as a convolution operator, L_ε is an approximation of $-\delta(x)\, I$ with $\delta(x)$ the Dirac mass and I the identity matrix. Therefore strongly in L^1

$$\int_0^1 \int_{\Pi^d} (x-y) \otimes \nabla K_\varepsilon(x-y) \, : \, \nabla u(t, \theta\, x + (1-\theta)\, y)\, \rho(t,y)\, dy\, d\theta \to - \operatorname{div} u(t,x)\, \rho(t,x),$$

proving that $R_\varepsilon \to 0$ in L^1. □

As simple as the previous proof is, since we are looking for quantitative estimates, a natural question is whether it would be possible to quantify the previous argument and in particular Proposition 2.7. This does not seem to be easy as it would imply giving an explicit rate of convergence on the commutator R_ε without using any additional regularity on ρ or ∇u.

Such an approach was nevertheless initiated in [7] and simplified in [8] from which we quote

Proposition 2.8. *Let $1 < p < \infty$, $\exists C < \infty$ depending only on p and the dimension s.t. for all $u \in W^{1,p}(\Pi^d)$ with $1 \le p \le 2$ and for all $g \in L^{2p^*}$ with $1/p^* = 1 - 1/p$,*

$$\begin{aligned} &\int_{\Pi^{2d}} \nabla K_h\,(x-y)\,(u\,(x) - u\,(y))\, \left|g\,(x) - g\,(y)\right|^2\, dx\, dy \\ &\quad \le C\, \|\nabla u\|_{B^0_{p,q}}\, |\log h|^{1-1/q}\, \|g\|^2_{L^{2\,p^*}} \\ &\qquad + C\, \|\operatorname{div} u\|_{L^\infty} \int_{\mathbb{R}^{2d}} K_h\,(x-y)\, \left|g\,(x) - g\,(y)\right|^2\, dx\, dy, \end{aligned}$$

where $K_h(x) = (h + |x|)^d$ for x small enough. In particular using $q = 2$,

$$\begin{aligned} &\int_{\Pi^{2d}} \nabla K_h\,(x-y)\,(u\,(x) - u\,(y))\, \left|g\,(x) - g\,(y)\right|^2\, dx\, dy \\ &\quad \le C\, \|\nabla u\|_{L^p}\, |\log h|^{1/2}\, \|g\|_{L^{2\,p^*}} \\ &\qquad + C\, \|\operatorname{div} u\|_{L^\infty} \int_{\mathbb{R}^{2d}} K_h\,(x-y)\, \left|g\,(x) - g\,(y)\right|^2\, dx\, dy. \end{aligned}$$

The proof of this proposition will not be given here, as it is rather complex and requires a careful analysis of the cancellations in the expression. We emphasize that this commutator estimate only works for kernels with the critical singularity in $|x|^{-d}$ at $x = 0$; we will understand better the reason for that in the next section.

The straightforward estimate would give

$$\int_{\Pi^{2d}} \nabla K_h\,(x-y)\,(a\,(x) - a\,(y))\, \left|g\,(x) - g\,(y)\right|^2\, dx\, dy \sim |\log h|,$$

and therefore Proposition 2.8 gains a factor $|\log h|^{1/2}$ as a rate of convergence and would later yield a corresponding gain of derivative. We are hence again, as in the first section, at a log scale for the gain of regularity.

The underlying result behind Proposition 2.8 has recently been improved in [44] to a gain of a full $|\log h|$ (at the cost of a much more complicated analysis),

see also [36]. This kind of critical semi-norm has also been used in other contexts, see, for example, [12].

However, from our point of view in these notes, the major drawback of Proposition 2.8 is that it requires $\operatorname{div} u$ to be bounded. There are major benefits to having a self-contained and quantitative commutator estimate, which would be more apparent if we were to consider vanishing viscosity or other approximations of (15). But our goal of obtaining estimates that do not require a bounded divergence will instead lead us to combine some of the ideas in the proof of Proposition 2.8 with the Lagrangian approach (or the Eulerian formulation of the Lagrangian approach) explained in the previous section.

2.3. The log log scale for compressible transport equations

We here present the main estimate of these notes for the linear convective equation (15). This estimate will also form the basis for the analysis of some simple nonlinear models in the next section. We follow closely here [17, 18] where the method has been introduced.

2.3.1. Technical preliminaries. As we had seen in the previous section, there is a technical difficulty if we try to use weights like $w(t,x) + w(t,y)$. To be more precise here, we would have to try (and fail) to control $M|\nabla u_k|(y)$ by $M|\nabla u_k|(x)$. Instead, we have to be more precise than (10) in order to avoid this and use more sophisticated tools. First one replaces (10) by

Lemma 2.9. *There exists $C > 0$ s.t. for any $u \in W^{1,1}(\Pi^d)$, one has*

$$|u(x) - u(y)| \leq C\,|x-y|\,(D_{|x-y|}u(x) + D_{|x-y|}u(y)),$$

where we denote

$$D_h u(x) = \frac{1}{h}\int_{|z|\leq h} \frac{|\nabla u(x+z)|}{|z|^{d-1}}\,dz.$$

Proof. A full proof of such a well-known result can, for instance, be found in [19] in a more general setting namely $u \in BV$. One possibility is simply to consider trajectories $\gamma(t)$ from x to y which stays within the ball of diameter $|x-y|$ to control

$$|u(x) - u(y)| \leq \int_0^1 \gamma'(t)\cdot\nabla u(\gamma(t))\,dt.$$

And then to average over all such trajectories with length of order $|x-y|$. □

Note that this result actually implies the estimate (10) as one can check, through a simple dyadic decomposition, that there exists $C > 0$, for any $u \in W^{1,p}(\Pi^d)$ with $p \geq 1$

$$D_h\, u(x) \leq C\, M|\nabla u|(x). \tag{21}$$

We leave such a proof to the reader and instead emphasize that the key improvement in using D_h is that small translations of the operator D_h are actually easy to control.

Let us first specify precisely the kernel K_h that we will use from now on. Choosing $[-1,\ 1]^d$ as a representative of the torus Π^d, we choose $K_h \in W^{1,\infty}$ with

$$K_h(x) = \frac{1}{(h+|x|)^d}, \quad \text{for } |x| \leq \frac{1}{2}, \quad 0 \leq K_h(x) \leq 1, \quad \text{supp}\, K_h \subset [-3/4,\ 3/4]^d. \tag{22}$$

We insist here on the precise exponent d in K_h which is critical for integrability and that we have seen in Proposition 2.8. In particular

$$\frac{|\log h|}{C} \leq \int_{\Pi^d} K_h(x)\, dx \leq C\, |\log h|.$$

We hence also define the normalized kernel

$$\overline{K}_h(x) = \frac{K_h(x)}{\int_{\Pi^d} K_h(y)\, dy}, \tag{23}$$

which is now a standard convolution kernel or approximation of identity.

The main point here is the estimate

Lemma 2.10. *For any $1 < p < \infty$, there exists $C > 0$ s.t. for any $u \in H^1(\Pi^d)$*

$$\int_{\Pi^d} K_h(z)\, \|D_{|z|}\, u(.) - D_{|z|}\, u(.+z)\|_{L^p}\, dz \leq C\ \|u\|_{B^1_{p,1}}, \tag{24}$$

where $B^1_{p,1}$ is the classical Besov space. As a consequence for any $1 < p < 2$

$$\int_{\Pi^d} K_h(z)\, \|D_{|z|}\, u(.) - D_{|z|}\, u(.+z)\|_{L^p}\, dz \leq C\, |\log h|^{1/2}\, \|u\|_{W^{1,p}}. \tag{25}$$

This lemma is in fact a corollary of a classical result

Lemma 2.11. *Let any family L_r of kernels satisfy for some $s > 0$*

$$\int L_r = 0, \quad \sup_r \left(\|L_r\|_{L^1} + r^s\, \|L_r\|_{W^{s,1}}\right) \leq C_L, \quad \sup_r r^{-s} \int |z|^s\, |L_r(z)|\, dz \leq C_L. \tag{26}$$

For any $1 < p < \infty$, there exists $C > 0$ depending only on C_L above s.t. for any $u \in L^p(\Pi^d)$

$$\int_0^1 \|L_r \star u\|_{L^p}\, \frac{dr}{h+r} \leq C\ \|u\|_{B^0_{p,1}}. \tag{27}$$

As a consequence, for $p \leq 2$

$$\int_0^1 \|L_r \star\ u\|_{L^p}\, \frac{dr}{h+r} \leq C\, |\log h|^{1/2}\, \|u\|_{L^p}. \tag{28}$$

Remark. We skip the proof of Lemma 2.11 which is rather classical. The bounds (25) and (28) can, for example, be obtained by a straightforward application of the so-called square function, see the book by E.M. Stein [46, p. 159] or Lemma 2.3.3 in [17].

Proof of Lemma 2.10 *assuming Lemma* 2.11. Using spherical coordinates

$$\int_{\Pi^d} K_h(z)\,\|D_{|z|}\,u(.) - D_{|z|}\,u(.+z)\|_{L^p}\,dz$$
$$\leq C\int_{S^{d-1}}\int_0^{1/2} \|D_r\,u(.) - D_r\,u(.+r\,\omega)\|_{L^p}\,\frac{dr}{r+h}\,d\omega.$$

Denote

$$L_\omega(x) = \frac{\mathbb{I}_{|x|\leq 1/2}}{|x|^{d-1}} - \frac{\mathbb{I}_{|x-\omega|\leq 1/2}}{|x-\omega|^{d-1}}, \quad L_{\omega,r}(x) = r^{-d}\,L_\omega(x/r),$$

and remark that $L_\omega \in W^{s,1}$ with a norm uniform in ω and with support in the unit ball. Moreover,

$$D_r u(x) - D_r u(x+r\omega) = \int |\nabla u|(x - r\,z)\,L_\omega(z)\,dz = L_{\omega,r} \star |\nabla u|.$$

We hence apply Lemma 2.11 since the family $L_{\omega,r}$ satisfies the required hypothesis and we get

$$\int_{h_0}^1 \|L_{\omega,r} \star \nabla\, u\|_{L^p}\,\frac{dr}{r} \leq C\,\|u\|_{B^1_{1,p}},$$

with a constant C independent of ω and so

$$\int_{h_0}^1\int_{\Pi^d} K_h(z)\,\|D_{|z|}\,u(.) - D_{|z|}\,u(.+z)\|_{L^p}\,dz\,dh$$
$$\leq C\int_{S^{d-1}}\int_{h_0}^1 \|L_{\omega,r} \star \nabla u\|_{L^p}\,\frac{dr}{r}\,d\omega \leq C\int_{S^{d-1}} \|u\|_{B^1_{1,p}}\,d\omega,$$

yielding (24). The bound (25) is deduced in the same manner. □

2.3.2. Propagating regularity with weights. We now come back to the basic strategy outlined at the end of the previous section and consider again the auxiliary equation on the weights

$$\partial_t w + u\cdot\nabla w = -\lambda\,M\,|\nabla u|\,w - D\,w \text{ in } (0,T)\times\Pi^d, \qquad w|_{t=0} = 1 \text{ in } \Pi^d \quad (29)$$

where we allow for an abstract additional penalization $D(t,x)$ which we will need in the next section. By using the tools for renormalized solutions that we briefly explained at the beginning of the section and the maximum principle, one can ensure

Lemma 2.12. *Assume that* $u \in L^1(0,T;W^{1,p}(\Pi^d))$ *and that* $D \geq 0$ *with* $D \in L^1((0,T)\times\Pi^d)$. *Then there exists a renormalized solution* w *to Equation* (29) *with* $w \in L^\infty((0,T)\times\Pi^d)$.

We skip the proof of Lemma 2.12 which essentially follows the existence strategy of Theorem 2.4 while using Theorem 2.6 for the renormalization property and the maximum principle using the sign of the right-hand side.

We are now ready to prove an equivalent of Proposition 1.9 or Proposition 1.16 but for the weight $w(t,x)+w(t,y)$.

Proposition 2.13. *Assume that $u \in L^1(0,T;W^{1,p}(\Pi^d))$, $\rho \in L^\infty(0,T;\ L^q(\Pi^d))$ is a renormalized solution to Equation (15) with $1/p+1/q \le 1$. Then for w a renormalized solution to Equation (14) with λ large enough, one has that for any h*

$$\begin{aligned}
&\int_{\Pi^{2\,d}} |\rho(t,x)-\rho(t,y)|\, K_h(x-y)\,(w(t,x)+w(t,y))\,dx\,dy \\
&\quad\le \int_{\Pi^{2\,d}} |\rho^0(x)-\rho^0(y)|\, K_h(x-y)\,dx\,dy \\
&\qquad + C\,|\log h|^{1/2}\,\|u\|_{L^1(0,T;W^{1,p}(\Pi^d))}\,\|\rho\|_{L^\infty(0,T;L^q(\Pi^d))} \\
&\qquad - 2\int_0^t \int_{\Pi^{2\,d}} (\operatorname{div} u(t,x) - \operatorname{div}\ u(t,y)) \\
&\qquad\qquad \times K_h(x-y)\,w(t,x)\,(\rho(t,x)+\rho(t,y))\,s(x,y)\,dx\,dy\,dt,
\end{aligned}$$

where $s(x,y)=\operatorname{sign}(\rho(t,x)-\rho(t,y))$.

Proof. The argument initially follows the same steps as Proposition 1.9 or Proposition 1.16. We first specify more (for further use in the next section) the equation

$$\begin{aligned}
&\partial_t|\rho(t,x)-\rho(t,y)| + u(x)\cdot\nabla_x|\rho(t,x)-\rho(t,y)| + u(y)\cdot\nabla_y|\rho(t,x)-\rho(t,y)| \\
&\quad = \frac{\operatorname{div} u(t,y)-\operatorname{div} u(t,x)}{2}\,(\rho(t,x)+\rho(t,y))\,s(x,y) \\
&\qquad - \frac{|\rho(t,x)-\rho(t,y)|}{2}\,(\operatorname{div} u(t,x)+\operatorname{div} u(t,y)),
\end{aligned}$$

where again $s(x,y)=\operatorname{sign}(\rho(t,x)-\rho(t,y))$ and where we can now fully justify the calculations as ρ is a renormalized solution. Multiplying by $w(t,x)+w(t,y)$ and using Equation (29) and the symmetry between x and y, we find the modified

$$\begin{aligned}
&\frac{d}{dt}\int_{\Pi^{2\,d}} |\rho(t,x)-\rho(t,y)|\, K_h(x-y)\,(w(t,x)+w(t,y))\,dx\,dy \\
&\quad\le 2\int_{\Pi^{2\,d}} |\rho(t,x)-\rho(t,y)|\, w(t,x)\,\nabla K_h(x-y)\cdot(u(t,x)-u(t,y))\,dx\,dy \\
&\qquad + \int_{\Pi^{2\,d}} |\rho(t,x)-\rho(t,y)|\, K_h(x-y)\,w(t,x)\,(\operatorname{div} u(t,x)+\operatorname{div} u(t,y))\,dx\,dy \\
&\qquad - 2\int_{\Pi^{2\,d}} |\rho(t,x)-\rho(t,y)|\, K_h(x-y)\,w(t,x)\,(D+\lambda\,M\,|\nabla u|(t,x))\,dx\,dy \\
&\qquad - \int_{\Pi^{2\,d}} (\operatorname{div} u(t,x)-\operatorname{div} u(t,y)) \\
&\qquad\qquad \times K_h(x-y)\,w(t,x)\,(\rho(t,x)+\rho(t,y))\,s(x,y)\,dx\,dy.
\end{aligned}$$

Remark that

$$\begin{aligned}
&\int_{\Pi^{2\,d}} |\rho(t,x)-\rho(t,y)|\, K_h(x-y)\, w(t,x)\ \mathrm{div}\, u(t,y))\, dx\, dy \\
&= \int_{\Pi^{2\,d}} |\rho(t,x)-\rho(t,y)|\, K_h(x-y)\, w(t,x)\ \mathrm{div}\ u(t,x))\, dx\, dy \\
&\quad - \int_{\Pi^{2\,d}} (\mathrm{div}\, u(t,x) - \mathrm{div}\ u(t,y)) \\
&\qquad\qquad \times K_h(x-y)\, w(t,x)\, (\rho(t,x)+\rho(t,y))\, s(x,y)\, dx\, dy.
\end{aligned}$$

Recalling that div $\mathrm{u}(t,x) \le M\, |\nabla u|(t,x)$, we may thus simplify for λ large enough

$$\begin{aligned}
&\frac{d}{dt}\int_{\Pi^{2\,d}} |\rho(t,x)-\rho(t,y)|\, K_h(x-y)\, (w(t,x)+w(t,y))\, dx\, dy \\
&\quad \le 2\int_{\Pi^{2\,d}} |\rho(t,x)-\rho(t,y)|\, w(t,x)\, \nabla K_h(x-y)\cdot (u(t,x)-u(t,y))\, dx\, dy \\
&\qquad - \lambda \int_{\Pi^{2\,d}} |\rho(t,x)-\rho(t,y)|\, K_h(x-y)\, w(t,x)\, M\, |\nabla u|(t,x)\, dx\, dy \\
&\qquad - 2\int_{\Pi^{2\,d}} (\mathrm{div}\, u(t,x) - \mathrm{div}\, u(t,y)) \\
&\qquad\qquad \times K_h(x-y)\, w(t,x)\, (\rho(t,x)+\rho(t,y))\, s(x,y)\, dx\, dy,
\end{aligned}$$

and we are back to our commutator estimate.

However, now we cannot use estimate (10) as we would then have to bound $w(t,x)\, M|\nabla u|(t,y)$ by $w(t,x)\, M|\nabla u|(t,x)$ which is simply not possible absent some more regularity on ∇u.

Instead, we use Lemma 2.9 to bound

$$\begin{aligned}
&\int_{\Pi^{2\,d}} |\rho(t,x)-\rho(t,y)|\, w(t,x)\, \nabla K_h(x-y)\cdot (u(t,x)-u(t,y))\, dx\, dy \\
&\quad \le C\int_{\Pi^{2\,d}} |\rho(t,x)-\rho(t,y)|\, w(t,x)\, (1+K_h(x-y)) \\
&\qquad\qquad \times (D_{|x-y|}u(t,x) + D_{|x-y|}u(t,y))\, dx\, dy,
\end{aligned}$$

since we recall that for small x, $|\nabla K_h(x)| \le C\, |x|^{-1}\, K_h(x)$ and that K_h is smooth for x of order 1.

By (21), we may bound directly the term without K_h

$$\begin{aligned}
&\int_{\Pi^{2\,d}} |\rho(t,x)-\rho(t,y)|\, w(t,x)\, (D_{|x-y|}u(t,x) + D_{|x-y|}u(t,y))\, dx\, dy \\
&\qquad \le \|\rho(t,.)\|_{L^q(\Pi^d)}\, \|M\, |\nabla u(t,.)|\|_{L^p(\Pi^d)}.
\end{aligned}$$

As for the other term, we may now use Lemma 2.10 to move $D_{|x-y|}u(t,y)$ to $D_{|x-y|}u(t,x)$. By a change of variable

$$\begin{aligned}
&\int_{\Pi^{2\,d}} |\rho(t,x)-\rho(t,y)|\, w(t,x)\, K_h(x-y)\, D_{|x-y|}u(t,y)\, dx\, dy \\
&= \int_{\Pi^{2\,d}} |\rho(t,x)-\rho(t,y)|\, w(t,x)\, K_h(x-y)\, D_{|x-y|}u(t,x)\, dx\, dy \\
&\quad + \int_{\Pi^{2\,d}} |\rho(t,x)-\rho(t,x+z)|\, w(t,x)\, K_h(z)\, (D_{|z|}u(t,x+z)-D_{|z|}u(t,x))\, dx\, dz.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\int_{\Pi^{2\,d}} |\rho(t,x)-\rho(t,x+z)|\, w(t,x)\, K_h(z)\, (D_{|z|}u(t,x+z)-D_{|z|}u(t,x))\, dx\, dz \\
&\quad \le \int_{\Pi^d} \|\rho(t,.)-\rho(t,.+z)\|_{L^q}\, K_h(z)\, \|D_{|z|}u(t,.+z)-D_{|z|}u(t,.)\|_{L^p}\, dz \\
&\quad \le C\, |\log h|^{1/2}\, \|\rho(t,.)\|_{L^q(\Pi^d)}\, \|u\|_{W^{1,p}},
\end{aligned}$$

by bounding $\|\rho(t,.)-\rho(t,.+z)\|_{L^q} \le 2\|\rho(t,.)\|_{L^q(\Pi^d)}$ and a direct application of Lemma 2.10. We want to emphasize here that this is the key part of the proof. Even though it remains relatively straightforward technically (also thanks to the preliminaries), this is what forces us to use this specific K_h.

Combining those estimates, we get that

$$\begin{aligned}
&\frac{d}{dt}\int_{\Pi^{2\,d}} |\rho(t,x)-\rho(t,y)|\, K_h(x-y)\, (w(t,x)+w(t,y))\, dx\, dy \\
&\le 2\int_{\Pi^{2\,d}} |\rho(t,x)-\rho(t,y)|\, w(t,x)\, K_h(x-y)\, D_{|x-y|}u(t,x)\, dx\, dy \\
&\quad - \lambda \int_{\Pi^{2\,d}} |\rho(t,x)-\rho(t,y)|\, K_h(x-y)\, w(t,x)\, M\, |\nabla u|(t,x)\, dx\, dy \\
&\quad + C\, |\log h|^{1/2}\, \|\rho(t,.)\|_{L^q(\Pi^d)}\, \|u\|_{W^{1,p}} \\
&\quad - 2\int_{\Pi^{2\,d}} (\operatorname{div}, u(t,x) - \operatorname{div} u(t,y)) \\
&\qquad\qquad \times K_h(x-y)\, w(t,x)\, (\rho(t,x)+\rho(t,y))\, s(x,y)\, dx\, dy,
\end{aligned}$$

which lets us conclude the proof by applying (21) and integrating in time. □

2.3.3. The final estimate. We are now ready to state the concluding result of our linear analysis,

Theorem 2.14. *Assume that* $u \in L^1(0,T;W^{1,p}(\Pi^d))$. $\rho \in L^\infty(0,T;\ L^q(\Pi^d))$ *is a renormalized solution to Equation* (15) *with* $1/p+1/q \le 1$. *Assume further that*

$$\int_{\Pi^d}\int_0^T \|\operatorname{div} u(s,.) - \operatorname{div} u(s,.+z)\|_{L^p(\Pi^d))}\, K_h(z)\, ds\, dz \le L,$$

and that

$$\int_{\Pi^{2\,d}} |\rho^0(x) - \rho^0(y)|\, K_h(x-y)\, dx\, dy \le L.$$

Then there exists a constant C depending only on the dimension and L such that one has for any h

$$\int_{\Pi^{2\,d}} |\rho(t,x) - \rho(t,y)|\, K_h(x-y)\, dx\, dy \le C\, N\, \frac{|\log h|}{\log|\log h|} + N\, L\, |\log h|^{3/4},$$

with

$$N = (1 + \|\rho\|_{L^\infty(0,T;L^q(\Pi^d))})\,(1 + \|u\|_{L^1(0,T;W^{1,p}(\Pi^d))}).$$

This is the result that we had been looking for:

- It does not require any bound on $\operatorname{div} u$ or on the Jacobian of the flow in general. It only requires *one* solution ρ bounded in some L^q.
- It provides an explicit regularity estimate on the solution ρ. And it only requires minimal regularity on $\operatorname{div} u$ (in fact any compactness on $\operatorname{div} u$ would give compactness on ρ by an easy modification of the proof).

Observe that in general the regularity provided by Theorem 2.14 is getting worse when T increases: In particular $\|u\|_{L^1(0,T;W^{1,p}(\Pi^d))}$ and thus N increases if T increases.

Theorem 2.14 essentially provides a $\log\log$ derivative on ρ. This appears to be a new scale in the problem (recall that we had a log scale previously), one that is due to possible concentration or vacuum.

Proof. Remark that by a change of variable

$$\begin{aligned}
&\int_0^t \int_{\Pi^{2\,d}} (\operatorname{div} u(t,x) - \operatorname{div} u(t,y))\, K_h\, w(t,x)\, (\rho(t,x) + \rho(t,y))\, s(x,y)\, dx\, dy\, dt \\
&\quad \le 2 \int_{\Pi^d} \|\operatorname{div} u(t,.) - \operatorname{div} u(t,.+z)\|_{L^1(0,T;L^p(\Pi^d))}\, K_h(z)\, \|\rho\|_{L^\infty(0,T;L^q(\Pi^d))}\, dz \\
&\quad \le L\, \|\rho\|_{L^\infty(0,T;L^q(\Pi^d))}.
\end{aligned}$$

Then we choose $D = 0$ and since there exists a weight by Lemma 2.12, we may directly apply Proposition 2.13 to find

$$\begin{aligned}
&\int_{\Pi^{2\,d}} |\rho(t,x) - \rho(t,y)|\, K_h(x-y)\, (w(t,x) + w(t,y))\, dx\, dy \\
&\quad \le L\, (1 + \|\rho\|_{L^\infty(0,T;L^q(\Pi^d))}) \\
&\qquad + C\, |\log h|^{1/2}\, \|u\|_{L^1(0,T;W^{1,p}(\Pi^d))}\, \|\rho\|_{L^\infty(0,T;L^q(\Pi^d))} \\
&\quad \le N\, (L + C\, |\log h|^{1/2}).
\end{aligned}$$

where we recall the definition of N

$$N = (1 + \|\rho\|_{L^\infty(0,T;L^q(\Pi^d))})\,(1 + \|u\|_{L^1(0,T;W^{1,p}(\Pi^d))}).$$

Now it only remains to remove the weight $w(t,x)+w(t,y)$. But those only vanish if both $w(t,x)$ and $w(t,y)$ vanish. Defining

$$\Omega = \{x,\ w(t,x) \leq \eta\},$$

we may simply write

$$\int_{\Pi^{2\,d}} |\rho(t,x)-\rho(t,y)|\, K_h(x-y)\, dx\, dy \leq \int_{\Omega^2} \cdots + \int_{x\notin\Omega \text{ or } y\notin\Omega} \cdots$$

If $x \notin \Omega$ or $y \notin \Omega$ then $w(t,x)+w(t,y) \geq \eta$, thus

$$\begin{aligned}
&\int_{x\notin\Omega \text{ or } y\notin\Omega} |\rho(t,x)-\rho(t,y)|\, K_h(x-y)\, dx\, dy \\
&\quad \leq \frac{1}{\eta} \int_{\Pi^{2\,d}} |\rho(t,x)-\rho(t,y)|\, K_h(x-y)\, (w(t,x)+w(t,y))\, dx\, dy \\
&\quad \leq \frac{N}{\eta}\, (L + C\,|\log h|^{1/2}).
\end{aligned}$$

On the other hand, by symmetry

$$\begin{aligned}
\int_{\Omega^2} |\rho(t,x)-\rho(t,y)|\, K_h(x-y)\, dx\, dy &\leq C\,|\log h| \int_{\Omega} \rho(t,x)\, dx \\
&\leq C\, \frac{|\log h|}{|\log \eta|} \int_{\Pi^d} |\log w|\, \rho(t,x)\, dx \\
&\leq C\, \frac{|\log h|}{|\log \eta|}\, N,
\end{aligned}$$

by Lemma 1.12 which we may directly use as we chose $D = 0$.

Finally

$$\int_{\Pi^{2\,d}} |\rho(t,x)-\rho(t,y)|\, K_h(x-y)\, dx\, dy \leq C\, \frac{|\log h|}{|\log \eta|}\, N + \frac{N}{\eta}\, (L + C\,|\log h|^{1/2}),$$

which finishes the proof by choosing for example $\eta = |\log h|^{-1/4}$. □

3. Example of application: A coupled Stokes system

In this section, we want to describe an application in fluid mechanics where we can get compactness on the density using the quantitative regularity propagation technic previously described. In this case, the velocity field and the density are linked together through a PDE system composed by a transport equation and a momentum equation. In a first section we explain the recent result obtained by the two authors concerning the compressible Navier–Stokes equations with a non-monotone pressure law (see Theorem 3.1). Then we present a more simple PDE system namely the compressible Stokes system and propose a sketch of proof of Theorem 3.2.

3.1. The compressible Navier–Stokes system

The theory introduced in the last section of the previous section had in fact been developed in [17] for the study of the compressible Navier–Stokes system in various unstable regimes such as non-monotone pressure laws or anisotropic stress tensors.

In its simplest form the Navier–Stokes system reads

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) - \mu \Delta u - (\lambda + \mu) \nabla \operatorname{div} u + \nabla p(\rho) = \rho f, \end{cases} \tag{30}$$

with $2\,\mu/d + \lambda$, and p is the barotropic pressure law ($s \mapsto p(s)$ given) which is typically continuous on $[0, +\infty)$, and locally Lipschitz on $(0, +\infty)$. The initial condition reads

$$\rho|_{t=0} = \rho_0, \qquad (\rho u)|_{t=0} = m_0. \tag{31}$$

The main difficulty in obtaining global existence for system (30) is to prove compactness of the density ρ which exactly solves the continuity equation that was the object of our previous investigations.

The first global existence result has been obtained in [38], using the (non-quantitative) theory for renormalized solution introduced at the beginning of the second section. This was the start of many works, for instance [20, 48, 21], pushing the theory and in particular the required growth at infinity of the pressure. Those culminated in the estimates in [28, 31] and exposed at length in [29] (see also [40]). This also enabled us to treat the more physically realistic Navier–Stokes–Fourier system for which we refer to [30]. We also mention the recent [43] which is able to handle the isothermal system.

While system (30) is written with a constant viscosity, realistic physical settings often involve density-dependent viscosities. This requires a different type of approach with new regularity estimates exploited in [15], new integrability bounds in [39], and leading to the existence of weak solutions in this setting in [16, 49]. Those regularity estimates are based on a two-velocity interpretation of the Navier–Stokes system, which has several other applications as in [41].

The Navier–Stokes system is also a classical model for geophysical flows as illustrated in [27] and [47]. We finally refer to [13] for an example of recent important topics.

Because the classical theory of existence relies on non-quantitative regularity estimates for ρ, it requires pressure laws that are thermodynamically stable. We hence conclude this introduction by quoting, as an illustration, one of the results from [17].

H) *Hypothesis on the pressure*: The assumptions on the pressure are only to be continuous on $[0, +\infty)$, locally Lipschitz on $(0, +\infty)$ with $p(0) = 0$ and that there exists $C > 0$ with

$$C^{-1}\rho^\gamma - C \le p(\rho) \le C\rho^\gamma + C \tag{32}$$

and for all $s \ge 0$

$$|p'(s)| \le \bar{p} s^{\tilde{\gamma}-1}. \tag{33}$$

This allows oscillating pressure laws, alternating stable and unstable regions. Nevertheless this still leads to global existence as by

Theorem 3.1. *Assume that the initial data m_0 and ρ_0 satisfies the bound*

$$E_0 = \int_{\Pi^d} \big(\frac{1}{2}\big|\frac{m_0}{\sqrt{\rho_0}}\big|^2 + \rho_0 e(\rho_0)\big)\, dx < +\infty.$$

Let the pressure law p satisfies Hypothesis H) *with* (32) *and* (33) *with*

$$\gamma > \big(\max(2,\tilde{\gamma}) + 1\big)\,\frac{d}{d+2}. \tag{34}$$

Then there exists a global weak solution of (30)–(31).

3.2. The result on the Stokes system

In the rest of this section, we mostly follow the presentation in [18] and focus on an example of application, namely the coupled Stokes system

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ -\mu\Delta u + \alpha u + \nabla p(\rho) = S, \end{cases} \tag{35}$$

with $\mu, \alpha > 0$ endowed with the following initial condition

$$\rho|_{t=0} = \rho_0. \tag{36}$$

In addition to be a limit of the compressible Navier–Stokes system (30) in some regime, system (35) (with many variants) is commonly used to model various biological systems, tumor for example in [14, 26, 42].

One can then use the linear theory that we previously developed to prove

Theorem 3.2. *Assume that $S \in L^2(0,T;\ H^{-1}(\Pi^d))$ and the initial data ρ_0 satisfies the bound*

$$\rho^0 \geq 0, \qquad 0 < M_0 = \int_{\Pi^d} \rho^0 < +\infty, \qquad E_0 = \int_{\Pi^d} \rho^0\, e(\rho^0)\, dx < +\infty,$$

where $e(\rho) = \int_{\rho^\star}^{\rho} p(s)/s^2 ds$ with $\rho^\star$ a constant reference density. Let the pressure law p satisfies hypothesis H) *with* (32) *and* (33) *with $\gamma > 1$. Then there exists a global weak solution (ρ, u) of the compressible system* (35)–(36) *with*

$$\rho \in L^\infty(0,T; L^\gamma(\Pi^d)) \cap L^{2\gamma}((0,T)\times\Pi^d), \qquad u \in L^2(0,T; H^1(\Pi^d)).$$

Remark. As noted in [18], the regularity of S is not optimized and could be decreased.

3.3. Sketch of the proof of Theorem 3.2

Proofs of existence of global weak solutions of PDEs are usually divided into three steps:

- *A priori* energy estimates,
- Stability of weak sequences: Compactness,
- Construction of approximate solutions.

We mostly focus on the first two points here as they best illustrate the main ideas. We refer to [17, 18] for more technical precisions.

3.3.1. Construction of approximate solutions. To keep our analysis simple, we in fact consider a sequence $\rho_k,\ u_k$ of solutions to the exact system (35) and will prove that the limit of the sequence is also a solution to (35). Even though it is not a complete proof, such a result of weak stability gives the main ideas behind Theorem 3.2.

We only briefly indicate in this subsection what would be the approximate system from which (35) is obtained, namely

$$\begin{cases} \partial_t \rho_k + \operatorname{div}(\rho_k u_k) = \alpha_k \Delta \rho_k, \\ -\mu \Delta u_k + \alpha u_k + \nabla p_\varepsilon(\rho_k) + \alpha_k \nabla \rho_k \cdot \nabla u_k = S_k, \end{cases} \tag{37}$$

with the source term S_k and the fixed initial data

$$\rho_k|_{t=0} = \rho^0. \tag{38}$$

The pressure p_ϵ is defined as follows:

$$p_\varepsilon(\rho) = p(\rho) \text{ if } \rho \le c_{0,\varepsilon}, \qquad p_\varepsilon(\rho) = p(C_{0,\varepsilon}) + C(\rho - c_{0,\varepsilon})^\beta \text{ if } \rho \ge c_{0,\varepsilon},$$

with β large enough. We refer to [17, 18] and the references therein for the existence of such an approximate system.

3.3.2. Energy estimates. Let us start with the basic kinetic energy estimate. Multiply the Stokes equation by u_k and integrate by parts,

$$\mu \int_{\Pi^d} |\nabla u_k|^2 + \alpha \int_{\Pi^d} |u_k|^2 + \int_{\Pi^d} \nabla p(\rho_k) \cdot u = \int_{\Pi^d} S_k \cdot u_k.$$

Now we write the equation satisfied by $\rho_k e(\rho_k)$ where $e(\rho_k) = \int_{\rho_{\text{ref}}}^{\rho_k} p(s)/s^2 ds$, with ρ_{ref} a constant reference density,

$$\partial_t(\rho_k e(\rho_k)) + \operatorname{div}(\rho e(\rho_k) u_k) + p(\rho_k) \operatorname{div} u_k = 0.$$

Integrating in space and adding to the first equation we get

$$\frac{d}{dt} \int_{\Pi^d} \rho_k e(\rho_k) + \alpha \int_{\Pi^d} |u_k|^2 + \mu \int_{\Pi^d} |\nabla u_k|^2 = \int_{\Pi^d} S_k \cdot u_k.$$

One only needs $S_k \in L^2([0,\ T],\ H^{-1}(\Pi^d))$, and using the behavior of p, then we get the uniform bound

$$\rho_k^\gamma \in L^\infty(0, T; L^1(\Pi^d)), \qquad u_k \in L^2(0, T; H^1(\Pi^d)).$$

When now considering the compressible system (35), the divergence $\operatorname{div} u_k$ is given by

$$\operatorname{div} u_k = \frac{1}{\mu} p(\rho_k) + \frac{1}{\mu} \Delta^{-1} \operatorname{div} R_k$$

with $R_k = S - \alpha u_k$. Therefore, since $\rho_k \in L^\infty(0,T;L^\gamma(\Pi^d))$, if we multiply by ρ_k^θ, we obtain

$$I = \int_0^T \int_{\Pi^d} p(\rho_k)\rho_k^\theta = \mu \int_0^T \int_{\Pi^d} \operatorname{div} u_k \rho_k^\theta - \int_0^T \int_{\Pi^d} \Delta^{-1} \operatorname{div} R_k \, \rho_k^\theta,$$

which is easily bounded as follows

$$I \leq \left[\mu \| \operatorname{div} u_k \|_{L^2((0,T)\times\Pi^d)} + \|\Delta^{-1} \operatorname{div} R_k\|_{L^2((0,T)\times\Pi^d)}\right] \|\rho_k^\theta\|_{L^2((0,T)\times\Pi^d)}$$

Thus using the behavior of p and information on u_k and R_k, we get for large density

$$\int_0^T \int_{\Pi^d} (\rho^{\gamma+\theta}) \leq C + \varepsilon \int_0^T \int_{\Pi^d} (\rho^{2\theta}).$$

Thus we get a control on $\rho_k^{\gamma+\theta}$ if $\theta \leq \gamma$. Therefore, we get $\rho_k \in L^p((0,T)\times\Pi^d)$ for any $p \leq 2\,\gamma$ and in particular some $p > 2$ if $\gamma > 1$.

3.3.3. Stability of weak sequences: Compactness. From the energy estimates we can extract converging subsequences

$$\rho_k \longrightarrow \rho \text{ weak-* in } L^\infty(0,T;L^\gamma(\Pi^d)) \text{ and weak in } L^{2\gamma}((0,T)\times\Pi^d)$$
$$u_k \longrightarrow \rho \text{ weak-* in } L^2(0,T;H^1(\Pi^d)).$$

This is enough to pass to the limit in every term of system (35) except for $p(\rho_k)$. This requires the strong compactness of ρ_k for which we prove the following result which is the main part of the proof

Proposition 3.3. *Assume that (ρ_k, u_k) are weak solutions to system* (35) *with a pressure law satisfying* (32)–(33) *and with the following uniform bounds*

$$\sup_k \|\rho_k^\gamma\|_{L^\infty(0,T;L^1(\Pi^d))} < \infty, \qquad \sup_k \|\rho_k\|_{L^p((0,T)\times\Pi^d)} < \infty \quad \textit{with } p \leq 2\gamma,$$

and

$$\sup_k \|u_k\|_{L^2(0,T;H^1(\Pi^d))} < \infty.$$

Assume moreover that the source term S_k is compact in $L^2(0,T;H^{-1}(\Pi^d))$ and that the initial density sequence $(\rho_k)_0$ is compact in $L^1(\Pi^d)$ and hence satisfies

$$\limsup_k \left[\frac{1}{|\log h|} \int_{\Pi^{2d}} K_h(x-y)|(\rho_k^x)_0 - (\rho_k^y)_0|\right] = \epsilon(h) \to 0 \text{ as } h \to 0,$$

then ρ_k is compact in $L^p((0,T)\times\Pi^d)$ for all $p < 2\gamma$.

Remark 3.4. Here and in the following, we use the convenient notation $\rho_k^x = \rho_k(t,x)$, $\rho_k^y = \rho_k(t,y)$ and $(\rho_k^x)_0 = \rho_k(t=0,x)$, $(\rho_k^y)_0 = \rho_k(t=0,y)$. Similarly $w_k^x = w_k(t,x)$, $u_k^x = u_k(t,x)$, $w_k^y = w_k(t,y)$, $u_k^y = u_k(t,y)$.

Proof. Let us introduce as before the auxiliary equation on the weight w_k

$$\partial_t w_k + u_k \cdot \nabla w_k = -\lambda M |\nabla u| w - (1+\rho_k^\gamma) w. \tag{39}$$

We start by using Proposition 2.13 from the last section to find that

$$\begin{aligned}\int_{\Pi^{2d}} K_h(x-y) |\rho_k^x - \rho_k^y| (w_k^x + w_k^y) dx dy \\ \leq |\log h| \epsilon(h) + C |\log h|^{1/2} N + A \\ - 2 \int_0^t \int_{\Pi^{2d}} K_h(x-y) (1 + (\rho_k^x)\gamma) |\rho_k^x - \rho_k^y| w_k^x dx dy dt\end{aligned} \tag{40}$$

where

$$N = \sup_k \|\rho_k\|_{L^{2\gamma}((0,T)\times\Pi^d)} \|u_k\|_{L^2(0,T;H^1(\Pi^d))},$$

$$A = -2 \int_0^t \int_{\Pi^{2d}} (\operatorname{div} u_k(t,x) - \operatorname{div} u_k(t,y)) K_h(x-y) w_k^x (\rho_k^x + \rho_k^y) s_k(x,y) dx dy,$$

with $s_k(x,y) = \operatorname{sign}(\rho_k^x - \rho_k^y)$. Let us use the relation between $\operatorname{div} u_k^x$ (respectively $\operatorname{div} u_k^y$) and ρ_k^x (respectively ρ_k^y), to obtain

$$A = -2 \int_0^t \int_{\Pi^{2d}} K_h(x-y) (p(\rho_k^x) - p(\rho_k^y)) (\rho_k^x + \rho_k^y) s_k w^x dx dy dt - \frac{2}{\mu} Q_h,$$

where

$$Q_h = \int_0^t \int_{\Pi^{2d}} K_h(x-y) (\Delta^{-1} \operatorname{div} R_k(t,x) - \Delta^{-1} \operatorname{div} R_k(t,y)) (\rho_k^x + \rho_k^y) s_k w^x dx dy dt,$$

encodes the compactness in space of $\Delta^{-1} \operatorname{div} R_k$ and therefore has the right behavior. Indeed, in particular

$$\frac{1}{|\log h|} \int_0^t \int_{\Pi^{2d}} K_h(x-y) (\Delta^{-1} \operatorname{div} R_k(t,x) - \Delta^{-1} \operatorname{div}\ R_k(t,y)) dx dy dt \to 0,$$

as $h \to 0$ since R_k is compact in $L^2(0,T;H^{-1}(\Pi^d))$ and hence $\Delta^{-1} \operatorname{div} R_k$ is compact in $L^2((0,T)\times\Pi^d)$ by the gain of one derivative.

However, the "bad" term $p(\rho_k^y) w_k^x$ cannot a priori be bounded directly with weights, again because it mixes points x and y. We review the various configurations

First note that we have $\rho_k^x + \rho_k^y \geq |\rho_k^x - \rho_k^y|$.

– Case 1: $(p(\rho_k^x) - p(\rho_k^y))(\rho_k^x - \rho_k^y) \geq 0$. We then directly have that $(p(\rho_k^x) - p(\rho_k^y)) s_k$ and this yields the right sign and a dissipation term in A.

– Case 2: $p(\rho_k^x) - p(\rho_k^y))(\rho_k^x - \rho_k^y) < 0$ and $\rho_k^y \leq \rho_k^x/2$ or $\rho_k^y \geq 2\rho_k^x$.

Assume that we are in the case $\rho_k^y \geq 2\rho_k^x$, then

$$(p(\rho_k^x) - p(\rho_k^y))(\rho_k^x + \rho_k^y) s_k \geq -C (\rho_k^x)^\gamma |\rho_k^x - \rho_k^y|,$$

since $p(\xi) \leq p(0) + C\xi^{\gamma-1}\xi \leq C\xi^\gamma$. If we now look at the cases $p(\rho_k^x) \leq p(\rho_k^y)$ and $\rho_k^y \leq \rho_k^x/2$, then we again bound

$$(p(\rho_k^x) - p(\rho_k^y))(\rho_k^x + \rho_k^y)s_k \geq -C\,(\rho_k^x)^\gamma\,|\rho_k^x - \rho_k^y|.$$

– Case 3: The case where $p(\rho_k^x) - p(\rho_k^y)$ and $\rho_k^x - \rho_k^y$ have different signs but $\rho_k^x/2 \leq \rho_k^y \leq 2\rho_k^x$. Then it is easy to get again

$$(p(\rho_k^x) - p(\rho_k^y))(\rho_k^x + \rho_k^y)s_k \geq -C\,(1 + (\rho_k^x)^\gamma)\,|\rho_k^x - \rho_k^y|.$$

Therefore combining all three cases, we obtain

$$A \leq C\int_0^t \int_{\Pi^{2d}} K_h(x-y)\,(1 + (\rho_k^x)^\gamma))\,|\rho_k^x - \rho_k^y|\,w_k^x\,dx\,dy\,dt - \frac{2}{\mu}\,Q_h,$$

with $Q_h/|\log h| \to 0$. Inserting this in (40), we deduce that

$$\int_{\Pi^{2d}} K_h(x-y)\,|\rho_k^x - \rho_k^y|\,(w_k^x + w_k^y)\,dx\,dy \leq |\log h|\,\tilde\epsilon(h), \tag{41}$$

with

$$\tilde\epsilon(h) = \epsilon(h) + C\,|\log h|^{-1/2}\,N - \frac{2}{\mu}\,\frac{Q_h}{|\log h|} \longrightarrow 0, \quad \text{as } h \to 0.$$

We now need to remove the weight just as in the proof of Theorem 2.14. First of all since Equation (39) has an additional term with respect to (14), we remark that we have an easy extension of Lemma 1.12, namely

Lemma 3.5. *Assume that $u_k \in L^2(0,T;H^1(\Pi^{2d}))$ and that $\rho_k \in L^{2\gamma}((0,T)\times\Pi^{2d}))$ with $\gamma > 1$. Then, if w_k solves* (39),

$$\int_{\Pi^d} |\log w_k(t,x)|\,\rho_k(t,x)\,dx \\ \leq C\,(\|u_k\|_{L^2(0,T;H^1(\Pi^d))} + \|\rho_k\|_{L^{2\gamma}((0,T)\times\Pi^d)})\,\|\rho_k\|_{L^{2\gamma}((0,T)\times\Pi^d)}.$$

The proof of Lemma 3.5 is essentially identical to the one of Lemma 1.12 and we skip it here. Using the same decomposition as in the proof of Theorem 2.14, we obtain from (41) that

$$\int_{\Pi^{2d}} K_h(x-y)\,|\rho_k^x - \rho_k^y|\,(w_k^x + w_k^y)\,dx\,dy \leq C\,\frac{|\log h|}{|\log\eta|}\,N^2 + C\,|\log h|\,\tilde\epsilon(h),$$

which finishes the proof by optimizing in η. We use Proposition 1.8 to conclude on the strong compactness of ρ_k in $L^1((0,T)\times\Pi^d)$ and therefore in $L^p((0,T)\times\Pi^d)$ using the uniform bound on ρ_k. □

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Didier Bresch
Univ. Grenoble Alpes
Univ. Savoie Mont Blanc
CNRS, LAMA
F-73000 Chambéry, France
e-mail: didier.bresch@univ-smb.fr

Pierre-Emmanuel Jabin
CSCAMM and Dept. of Mathematics
University of Maryland
College Park, MD 20742, USA
e-mail: pjabin@cscamm.umd.edu

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Fully Resolved Compressible Two-Phase Flow: Modelling, Analytical and Numerical Issues

Christian Rohde

Abstract. Mathematical models for compressible two-phase flow of homogeneous fluids that occur in a liquid and a vapour phase can be classified as either belonging to the class of sharp interface models or to the class of diffuse interface models. Sharp interface models display the phase boundary as a sharp front separating two bulk model domains while diffuse interface models consist of a single model on the complete domain of interest such that phase boundaries are represented as transition zones. This contribution is devoted to a self-consistent introduction to both model classes.

Sharp interface models are analyzed within the theory of hyperbolic conservation laws with special focus on the Riemann problem. Based on the thermodynamically consistent solution of the Riemann problem a multidimensional finite volume method is introduced. For the associated diffuse interface ansatz the focus is on Navier–Stokes–Korteweg-type models. Several new variants are introduced which enable in particular thermodynamically consistent and asymptotically-preserving numerical discretizations. For all models it is assumed that the relevant spatial scale corresponds to fully resolved phase boundaries.

Mathematics Subject Classification (2010). 76T10, 76N10, 35L65.

Keywords. Compressible two-phase flow, liquid-vapour phase transition, the two-phase Riemann problem, moving mesh finite volume method, Navier–Stokes–Korteweg equations, entropy-dissipative methods, asymptotic preserving schemes.

Preface

The analysis and numerics for compressible one-phase flow have by now reached a rather mature status. Despite the fact that there are still unsolved fundamental questions in the analysis, the numerical simulation has become a routine tool for many flow regimes. This situation is quite different for compressible multi-phase flow. The continuum-mechanical description depends crucially on micro-scale effects such that even the mathematical modelling is still a controversial field of

research. Of course this affects directly the analysis and numerics for compressible multi-phase flow.

In this treatise we consider the compressible free flow of homogeneous fluids that occur in two phases: a liquid and a vapour phase. Phase change phenomena are our main issue and we are interested in the fully resolved situation where single phase boundaries exist as flow pattern. Homogenized scenarios are not considered. For the mathematical description of a homogeneous compressible fluid with liquid-vapour phase transitions one uses either models which display the phase boundary as a sharp front (sharp interface, SI, see Figure 0.1) or as a steep transition, smeared out over a small-scale distance (diffuse interface, DI, see Figure 0.1). Both model concepts are closely related via the SI limit when the sequence of solutions of the DI model are supposed to tend to a solution of an associated SI model. Therefore we try to describe both classes carving out their advantages and disadvantages.

The SI concept is discussed in Section 1. We set up a thermodynamical framework which is chosen to be isothermal. This restricts the physical relevance but in this case the mathematical theory is most evolved. The mathematical model can be understood as a free boundary value problem with appropriate coupling conditions across the interface, and evolution equations in the bulk domains. In the inviscid setting the Euler equations of hydromechanics govern the flow and Rankine–Hugoniot conditions drive the interface. However, these balance conditions do not

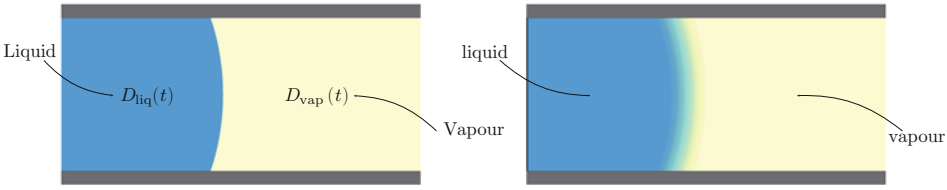


FIG. 0.1. Sketch of an SI solution (left) and a DI solution (right).

suffice to control the interfacial dynamics completely if mass is transferred across the interface. As for other phase transformation processes an additional condition is needed. This condition – called kinetic relation in the mathematical context – involves micro-scale information in the form of an algebraic constitutive relation. It is exactly this continuum-mechanical closure which is heavily debated in the literature. We will analyze choices of kinetic relations which allow at least a thermodynamically consistent well-posedness theory. Thermodynamical consistency is the major guiding principle for these notes, precisely we mean that an entropy inequality should be satisfied in an appropriate sense for the complete (bulk and interfacial) process. In particular we will focus on the planar Riemann problem which can be completely solved using recent progress in the theory of hyperbolic conservation laws. These results appear then to be only a weak base for (multi-dimensional) numerics but it turns out that an appropriate Riemann solver is a

key to construct moving-mesh finite volume methods for the thermodynamically consistent tracking of interfaces with mass transfer. The complete setting allows to apply the numerical method successfully to tackle, e.g., the dynamics of single droplets and bubbles.

What is not covered by the SI concept, that are all interface motions that undergo a topological change of the interface. Examples are the merging of droplets/bubbles or nucleation phenomena. The description of a three-phase contact line that shows up for the interaction of a liquid-vapour interface with a solid wall is quite complex in the SI world. In passing we note also that any rigorous upscaling for flow in a porous medium on the base of an SI model appears to be very complicated.

All these arguments motivate the advancement of DI ideas, which we discuss in Section 2. In compressible two-phase flow two different DI model classes are dominating. The first one is the Navier–Stokes–Korteweg (NSK) theory which can be considered as a classical second gradient theory. The free energy functional is extended to account for (generalized) Van-der-Waals contributions, and by classical concepts like least-action principles thermodynamically consistent dynamical models can be derived. The additional free energy term – physically motivated or not – leads to a regularized model such that one set of equations governs the two-phase process on the entire domain. For various instances of the NSK family basic well-posedness results are available. For the numerical discretization two major challenges occur. The extension of the free energy leads to non-standard (higher-order, nonlocal,...) contributions to the stress tensor.

Whereas thermodynamical consistency on the analytical level is straightforward the design of numerical approximations that obey an entropy inequality is a widely open issue. In this treatise we will mainly focus on the NSK class but in the last decade phase field systems became a quite popular alternative. With this ansatz additional phase field equations for an artificial order parameter are introduced. The phase field parameters also enter the stress tensor of the hydromechanical system leading to a strongly coupled nonlinear system. However, the approach makes it possible to prescribe independently the interfacial width and physical properties like, e.g., surface tension.

These notes base on the pertinent mathematical literature on compressible phase transition dynamics, without claiming to be complete but rather representing the point of view of the author. The treatise makes strong reference to publications of the author and/or co-authors. This is indicated in all places where it applies.

1. The sharp interface approach

In the first part of the notes we focus on sharp interface (SI) models. The SI ansatz is the most classical one in fluid mechanics (see, e.g., [31] for the equilibrium theory). Using the inviscid compressible Euler equations as the bulk state model we address modelling, analytical and numerical issues in Sections 1.1, 1.2, 1.3, respectively. The guiding line of this section is to understand SI models within the theory of first-order hyperbolic conservation laws. Therefore we will deal with the one-phase case and the two-phase case. In the first case hydrodynamical shock waves act as relevant interfaces and in the latter case we understand phase boundaries as non-standard shock-like discontinuous waves. The theory of conservation laws is strongly linked to thermodynamical modelling. In the last decades lots of techniques have been developped to integrate and exploit the second law of thermodynamics for modelling, analysis and numerics (see [21]). This path will be followed here, towards a deeper understanding of a two-phase flow. The first section is devoted to the set-up of the SI model for hydro-mechanical shock waves and phase boundaries in the form of a free boundary value problem. In Section 1.2 we will then present a complete thermodynamically consistent solution, in particular for the two-phase Riemann problem. In the last Section 1.3 we will use the analytical results to design a numerical method that can solve the SI model in multiple space dimensions. The method relies on a classical moving mesh ansatz and guarantees not only conservation on the discrete level but also permits thermodynamically consistent computations. The section displays apart from the pertinent literature direct material from the papers [14, 74] with several extensions.

1.1. The Euler equations for one- and two-phase flow

1.1.1. Thermodynamical framework. We fix the temperature $\Theta > 0$ for an isothermal set-up and denote the density by $\varrho \in \tilde{\mathcal{A}} := (0, \bar{\varrho})$ with $\bar{\varrho}$ being the excluded volume. The thermodynamical framework is presented in terms of specific volume $\tau = 1/\varrho > \bar{\tau} := 1/\bar{\varrho}$ but it is convenient to consider dependencies also in terms of ϱ. Then we will use the same symbol headed by a tilde.

The functions $p = p(\tau)$, $\psi = \psi(\tau)$, $\mu = \mu(\tau)$ are assumed to be smooth with

$$p(\tau) = -\psi'(\tau) \text{ and } \mu(\tau) = \psi(\tau) + p(\tau)\,\tau. \tag{1.1.1}$$

They are called *pressure*, *specific Helmholtz free energy* and *specific Gibbs free energy*, respectively. The derivative $c(\tau) := \sqrt{-p'(\tau)}$ is the *speed of sound*. As said before we will understand all these quantities and others also as functions of density ϱ, i.e., $\tilde{p} = \tilde{p}(\varrho)$, $\tilde{\psi} = \tilde{\psi}(\varrho)$, $\tilde{\mu} = \tilde{\mu}(\varrho)$.

The pressure for homogeneous fluids in one phase. Our thermodynamical framework is completely given by the pressure function, in particular we have for a simple one-phase pressure

Definition 1.1.1 (One-phase thermodynamics). The function $p = p(\tau) : (\bar{\tau}, \infty) \to (0, \infty)$ is called *one-phase pressure*, if the following conditions hold

$$p', -p'' < 0 \text{ in } (\bar{\tau}, \infty), \tag{1.1.2}$$

$$p(\tau) \to \infty \text{ for } \tau \to \bar{\tau}, \tag{1.1.3}$$

$$\lim_{R\to\infty} \int_{\bar{\tau}}^{R} c(\tau)\, d\tau = \infty. \tag{1.1.4}$$

For a one-phase pressure we define $\mathcal{A} := (\bar{\tau}, \infty)$.

The convexity condition in (1.1.2) is only imposed to facilitate the analysis later on, in this way more complicated (attached) wave configurations in the construction of solutions for the Riemann problem are avoided in Section 1.2. The other conditions allow a global solution of the Riemann problem. In particular, (1.1.4) excludes the case of vacuum which is out of our interests. Hypothesis (1.1.3) is naturally defining a minimal molecular distance, where the fluid cannot be compressed further. The most simple choice for a one-phase pressure function is the *ideal gas law*

$$p(\tau) = \frac{R\Theta}{\tau - \bar{\tau}}, \tag{1.1.5}$$

with $R > 0$ being the universal gas constant formed of the product of the Boltzmann and the Avogadro constant. Note that the non-homogeneous denominator (1.1.5) takes into account already volume occupation by the gas molecules.

To account for attractive forces between the fluid's molecules a correction term proportional to the negative of the square of density is added. This leads to the general *van-der-Waals equations of state*, given by

$$p(\tau) = \frac{R\,\Theta}{\tau - \bar{\tau}} - \frac{a}{\tau^2},\ a > 0. \tag{1.1.6}$$

It constitutes a one-phase pressure as long as the fixed temperature is larger than the critical temperature Θ_c, i.e,

$$\Theta > \Theta_c := \frac{8a}{27R\bar{\tau}}.$$

We refer to Figure 1.1 for an illustration of the one-phase set-up.

The pressure for homogeneous fluids in a liquid and a vapour phase. Proceeding to a two-phase situation requires a split specific volume state space $\mathcal{A} = \mathcal{A}_{\text{liq}} \cup \mathcal{A}_{\text{liq}}$ according to

$$\mathcal{A}_{\text{liq}} := (\bar{\tau}, \tau_{\text{liq}}^{\max}), \quad \mathcal{A}_{\text{spinodal}} := [\tau_{\text{liq}}^{\max}, \tau_{\text{vap}}^{\min}] \quad \text{and} \quad \mathcal{A}_{\text{vap}} := (\tau_{\text{vap}}^{\min}, \infty).$$

Here the numbers $\tau_{\text{liq}}^{\max}$, $\tau_{\text{vap}}^{\min}$ satisfy $\bar{\tau} < \tau_{\text{liq}}^{\max} < \tau_{\text{vap}}^{\min}$. The two intervals $\mathcal{A}_{\text{liq}}$ and $\mathcal{A}_{\text{vap}}$ define the *liquid* and the *vapour* phase. To be precise we introduce

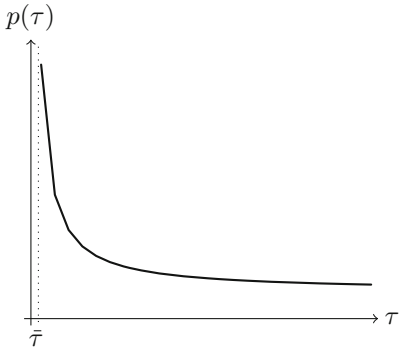


FIG. 1.1. Prototypical example of a one-phase pressure.

Definition 1.1.2. The function $p = p(\tau) : \mathcal{A}_{\text{liq}} \cap \mathcal{A}_{\text{vap}} \to (0, \infty)$ is called *two-phase pressure for an SI model*, if the following conditions hold:

$$p', -p'' < 0 \text{ in } \mathcal{A}_{\text{liq}} \cup \mathcal{A}_{\text{vap}}, \tag{1.1.7}$$

$$\exists\, \tau_{\text{liq}}^{\text{sat}} \in \mathcal{A}_{\text{liq}}, \tau_{\text{vap}}^{\text{sat}} \in \mathcal{A}_{\text{vap}} : \begin{cases} p(\tau_{\text{vap}}^{\text{sat}}) - p(\tau_{\text{liq}}^{\text{sat}}) = 0, \\ \mu(\tau_{\text{vap}}^{\text{sat}}) - \mu(\tau_{\text{liq}}^{\text{sat}}) = 0, \end{cases} \tag{1.1.8}$$

$$p(\tau) \to \infty \text{ for } \tau \to \bar{\tau}, \tag{1.1.9}$$

$$p'(\tau_{\text{liq}}) < p'(\tau_{\text{vap}}) \quad \forall\, \tau_{\text{liq}} \in \mathcal{A}_{\text{liq}}, \tau_{\text{vap}} \in \mathcal{A}_{\text{vap}}, \tag{1.1.10}$$

$$\lim_{R \to \infty} \int_{\tau_{\text{vap}}^{\min}}^{R} c(\tau)\, d\tau = \infty. \tag{1.1.11}$$

Note that p is monotone decreasing and convex in both phases, see Figure 1.2 (right) for some illustration. The spinodal or elliptic set $\mathcal{A}_{\text{spinodal}}$ is excluded from our studies as a set of unphysical states (see the discussion on phase boundaries in Section 1.2.2). The states $\left(\tau_{\text{liq}}^{\text{sat}}, \tau_{\text{vap}}^{\text{sat}}\right) \in \mathcal{A}_{\text{liq}} \times \mathcal{A}_{\text{vap}}$ in hypothesis (1.1.8) are called *saturation or Maxwell states*. These states are associated with the *thermodynamic equilibrium*. The sets $(\tau_{\text{liq}}^{\text{sat}}, \tau_{\text{liq}}^{\max})$ and $(\tau_{\text{vap}}^{\min}, \tau_{\text{vap}}^{\text{sat}})$ are called *metastable liquid* and *metastable vapour phases*, while the sets $(\bar{\tau}, \tau_{\text{liq}}^{\text{sat}}]$, $[\tau_{\text{vap}}^{\text{sat}}, \infty)$ are called *stable (liquid/vapour) phases*. Hypotheses (1.1.7), (1.1.9) and (1.1.10) limit again the amount of possible wave configurations for the solution of the Riemann problem. But as we will see in Section 1.2 the convexity condition in (1.1.7) does not prevent to have attached wave patterns in the two-phase case. Finally, the condition (1.1.10) corresponds to the fact that the sound speed $c = c(\tau)$ in the liquid phase of a fluid is usually much higher than in the vapour phase. Equations of state have to be determined, e.g., by experimental measurements. However, for a simple model fluid, that occur in a liquid and a vapour phase, we may consider the following explicit form, such that all conditions of Definition 1.1.2 are satisfied.

Example 1.1.3 (A two-phase pressure). The van-der-Waals equations of state (1.1.6) lead to a two-phase pressure for $\Theta \in (0, \Theta_c)$. The function is monotonically increasing for $\tau \in \mathcal{A}_{\text{spinodal}}$ and decreasing in $\mathcal{A}_{\text{liq/vap}}$, see Figure 1.2. The

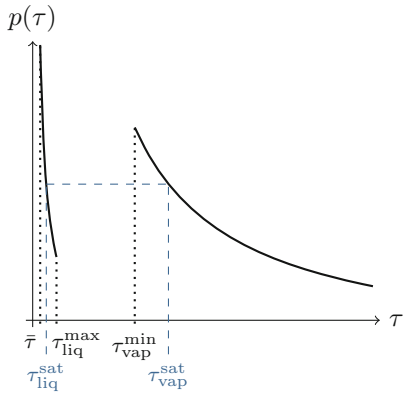


FIG. 1.2. Prototypical example of a two-phase pressure for an SI model. The dashed blue line connects the two saturation states with the identical pressure (and Gibbs free energy) value.

parameters for the graphs in Figure 1.2 and most of the figures and the numerical experiments in the sequel of the paper refer to $\Theta = 0.85$ and

$$a = 3,\ \bar{\tau} = \frac{1}{3},\ R = \frac{8}{3}. \tag{1.1.12}$$

For these numbers, the critical temperature is actually $\Theta_{\text{c}} = 1$ and we get $\tau \in \mathcal{A}_{\text{liq}} \cup \mathcal{A}_{\text{vap}}$ with $\mathcal{A}_{\text{liq}} = (1/3, 0.6)$ and $\mathcal{A}_{\text{vap}} = (2.5, \infty)$. Other parameter choices will be indicated explicitly.

1.1.2. Isothermal flow. We assume that the thermodynamic framework as described in Section 1.1.1 holds for the rest of the paper. In particular, we let some pressure function p be given.

Let $D \subset \mathbb{R}^d$ with $d \in \mathbb{N}$ be an open set. For any time $t \in [0, T]$, $T > 0$, we assume that D is portioned into the union of two open sets $D_+(t)$, $D_-(t)$, which contain two bulk phases, and a hyper-surface $\Gamma(t)$ – the *sharp interface* (*SI*) –, that separates the two spatial bulk sets. For the moment we consider both situations, the one-phase and the two-phase setting, and we do not specify the physical meaning of the sharp interface and the bulk sets (see Figure 1.3 for the geometrical setting).

To describe the fluid's dynamics we neglect higher-order effects like viscosity in the bulk and get in the spatial-temporal bulk sets $\{(\boldsymbol{x}, t) | \boldsymbol{x} \in D_-(t),\ t \in (0, T)\}$ and $\{(\boldsymbol{x}, t) | \boldsymbol{x} \in D_+(t),\ t \in (0, T)\}$ the Euler equations as a first-order system of

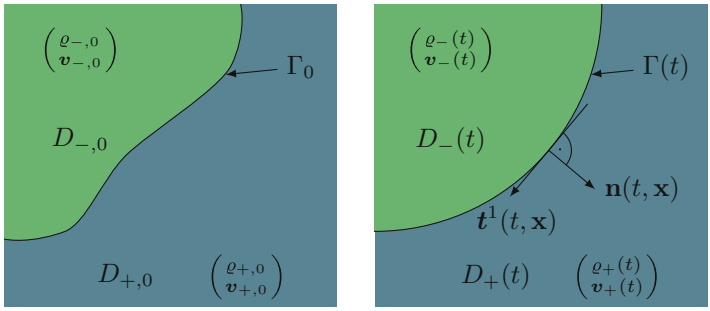


FIG. 1.3. Sketch of the initial configuration and a temporal snapshot for $d = 2$.

conservation laws, i.e.,

$$\begin{aligned} \varrho_t + \qquad \operatorname{div}(\varrho\,\boldsymbol{v}) \qquad &= 0, \\ (\varrho\,\boldsymbol{v})_t + \operatorname{div}\left(\varrho\,\boldsymbol{v}\otimes\boldsymbol{v} + \tilde{p}(\varrho)\,\boldsymbol{I}\right) &= \mathbf{0}. \end{aligned} \tag{1.1.13}$$

Here, $\varrho = \varrho(\boldsymbol{x},t) > 0$ denotes the unknown density field and $\boldsymbol{v} = \boldsymbol{v}(\boldsymbol{x},t) = (v_1(\boldsymbol{x},t),\dots,v_d(\boldsymbol{x},t))^T \in \mathbb{R}^d$ the unknown velocity field. By $\boldsymbol{I} \in \mathbb{R}^{d\times d}$ we denote the d-dimensional unit matrix.

Remark 1.1.4. If viscous effects are not neglected, the Euler system (1.1.13) can be substituted by the compressible Navier–Stokes equations. An SI theory can of course also be formulated in terms of the Navier–Stokes equations but it must be noted that the numerical methods in Section 1.3 cannot be used anymore. The discretization approach relies on the explicit determination of self-similar solutions of Riemann problems which do not exist for the Navier–Stokes equations.

As the next step coupling conditions at the free boundary $\Gamma(t)$ have to be provided. Let some $\boldsymbol{\xi} \in \Gamma(t)$ be given. We denote the speed of $\Gamma(t)$ in the normal direction $\boldsymbol{n} = \boldsymbol{n}(\boldsymbol{\xi},t) \in \mathbb{S}^{d-1}$ by $\sigma = \sigma(\boldsymbol{\xi},t) \in \mathbb{R}$. Throughout the paper the direction of the normal vector is always chosen, such that $\boldsymbol{n}(\cdot,t)$ points into the domain $D_+(t)$.

For some thermodynamical quantity a we define its trace jump $[\![a]\!]$ and its mean $\{a\}$ across some interface by

$$[\![a]\!] := \bar{a}_+ - \bar{a}_-, \quad \{a\} := \frac{1}{2}\big(\bar{a}_+ + \bar{a}_-\big),$$

with the trace

$$\bar{a}_\pm := \lim_{\varepsilon\to 0,\varepsilon>0} a(\boldsymbol{\xi} \pm \varepsilon\,\boldsymbol{n}). \tag{1.1.14}$$

The vectors $\boldsymbol{t}^1,\dots,\boldsymbol{t}^{d-1} \in \mathbb{S}^{d-1}$ are supposed to be a complete set of vectors tangential to $\boldsymbol{n}$.

Across the interface the following $d+1$ trace conditions are posed, the first two representing the conservation of mass and momentum:

$$\begin{aligned} [\![\varrho\,(\boldsymbol{v}\cdot\boldsymbol{n}-\sigma)]\!] &= 0,\\ [\![\varrho\,(\boldsymbol{v}\cdot\boldsymbol{n}-\sigma)\,\boldsymbol{v}\cdot\boldsymbol{n}+\tilde{p}(\varrho)]\!] &= 0,\\ [\![\boldsymbol{v}\cdot\boldsymbol{t}^l]\!] &= 0 \qquad (l=1,\dots,d-1). \end{aligned} \tag{1.1.15}$$

The conditions $(1.1.15)_3$ on the tangential velocities are not a consequence of the momentum balance but imposed as most simple choice in the inviscid case.

The density field, the velocity field and the interface Γ are initially determined by

$$\varrho(\cdot,0)=\varrho_0,\ \boldsymbol{v}(\cdot,0)=\boldsymbol{v}_0,\ \Gamma(0)=\Gamma_0. \tag{1.1.16}$$

For $D\neq\mathbb{R}^d$ it remains to fix appropriate boundary conditions on ∂D which will be done when considering specific examples. Altogether the equations (1.1.13) with coupling conditions (1.1.15) provide for $t\in[0,T]$ a free boundary value problem for the interface $\Gamma(t)$ and

$$\varrho(t,\cdot):D_-(t)\cup D_+(t)\to(0,\bar{\varrho}),\qquad \boldsymbol{v}(t,\cdot):D_-(t)\cup D_+(t)\to\mathbb{R}^d.$$

With $\boldsymbol{u}=(\varrho,\varrho\boldsymbol{v}^T)$ and

$$\begin{aligned} \boldsymbol{f}_1(\boldsymbol{u}) &= (\varrho v_1,\varrho v_1^2+\tilde{p}(\varrho),\varrho v_1v_2,\dots,\varrho v_1v_d)^T,\\ &\vdots\\ \boldsymbol{f}_d(\boldsymbol{u}) &= (\varrho v_d,\varrho v_1v_d,\dots,\varrho v_{d-1}v_d,\varrho v_d^2+\tilde{p}(\varrho))^T \end{aligned}$$

we can rewrite the Euler system (1.1.13) in the conservative form

$$\boldsymbol{u}_t+\boldsymbol{f}_1(\boldsymbol{u})_{x_1}+\cdots+\boldsymbol{f}_d(\boldsymbol{u})_{x_d}=\boldsymbol{0}. \tag{1.1.17}$$

If we fix $\mathbf{n}\in\mathbb{S}^{d-1}$ and let $\boldsymbol{F}(\boldsymbol{u};\mathbf{n})=n_1\boldsymbol{f}_1(\boldsymbol{u})+\cdots+n_d\boldsymbol{f}_d(\boldsymbol{u})$, the eigenvalues of the Jacobian $D\boldsymbol{F}(\boldsymbol{u};\mathbf{n})$ are given by

$$\begin{aligned} \lambda_1(\boldsymbol{u};\mathbf{n}) &= \boldsymbol{v}\cdot\mathbf{n}-\sqrt{\tilde{p}'(\varrho)},\\ \lambda_2(\boldsymbol{u};\mathbf{n}) &= \cdots=\lambda_d(\boldsymbol{u};\mathbf{n})=\boldsymbol{v}\cdot\mathbf{n},\\ \lambda_{d+1}(\boldsymbol{u};\mathbf{n}) &= \boldsymbol{v}\cdot\mathbf{n}+\sqrt{\tilde{p}'(\varrho)}. \end{aligned} \tag{1.1.18}$$

The associated eigenvectors are

$$\begin{aligned} \mathbf{r}_1(\boldsymbol{u};\mathbf{n}) &= \Big(1,\big(\boldsymbol{v}-\sqrt{-p'(\tau)}\mathbf{n}\big)^T\Big)^T,\\ \mathbf{r}_2(\boldsymbol{u};\mathbf{n}) &= \big(0,\boldsymbol{t}^{2,T}\big)^T,\dots,\mathbf{r}_d(\boldsymbol{u};\mathbf{n})=\big(0,\boldsymbol{t}^{d,T}\big)^T,\\ \mathbf{r}_{d+1}(\boldsymbol{u};\mathbf{n}) &= \Big(1,\big(\boldsymbol{v}+\sqrt{-p'(\tau)}\mathbf{n}\big)^T\Big)^T. \end{aligned} \tag{1.1.19}$$

Then we observe easily with Definition 1.1.1, that for a one-phase pressure the system (1.1.13) is hyperbolic in the state space

$$\tilde{\mathcal{U}}:=(\tilde{\mathcal{A}}:=(0,\bar{\varrho}))\times\mathbb{R}^d.$$

Definition 1.1.2 ensures for a two-phase pressure for an SI model that (1.1.13) is hyperbolic if and only if

$$\boldsymbol{u} \in \tilde{\mathcal{U}} := (\tilde{\mathcal{A}} := \tilde{\mathcal{A}}_{\rm liq} \cup \tilde{\mathcal{A}}_{\rm vap}) \times \mathbb{R}^d$$

holds. In both cases the extreme characteristic fields are genuinely nonlinear (that is $\nabla\lambda_i(\boldsymbol{u};\mathbf{n}) \cdot \mathbf{r}_i(\boldsymbol{u};\mathbf{n}) \neq 0$ for $i = 1, d+1$) whereas the fields $\lambda_2, \ldots, \lambda_d$ are all linearly degenerate (that is $\nabla\lambda_i(\boldsymbol{u};\mathbf{n}) \cdot \mathbf{r}_i(\boldsymbol{u};\mathbf{n}) = 0$ for $i = 2, \ldots, d$).

The hyperbolic structure does not suffice at all to ensure well-posedness of the free boundary value problem for (1.1.13). Thermodynamically consistency puts further constraints on the solutions. We follow in this contribution the classical entropy concept for conservation laws [21]. Let the total energy be defined by

$$W(\boldsymbol{u}) := \varrho\,\tilde{\psi}(\varrho) + \frac{1}{2}\,\varrho\,|\boldsymbol{v}|^2\,,$$

using the Helmholtz free energy $\tilde{\psi}$. It must be noted that Definition 1.1.1 and Definition 1.1.2 ensure that $W = W(\boldsymbol{u})$ is strictly convex in the respective state spaces $\tilde{\mathcal{U}}$. Therefore it can act as a (mathematical) entropy for (1.1.13). Together with an entropy flux $\boldsymbol{Q} = (Q_1, \ldots, Q_d)^T$ we can define the entropy-entropy flux pair

$$(W, \boldsymbol{Q}), \quad \boldsymbol{Q}(\boldsymbol{u}) = (W(\boldsymbol{u}) + \tilde{p}(\varrho))\boldsymbol{v}, \tag{1.1.20}$$

satisfying the compatibility condition $\nabla W(\boldsymbol{u})^T D\boldsymbol{f}_i(\boldsymbol{u}) = \nabla q_i(\boldsymbol{u})^T$ for $\boldsymbol{u} \in \tilde{\mathcal{U}}$. It is well known that the existence of such an entropy-entropy flux pair implies already the hyperbolicity of (1.1.17) [21].

Altogether we seek for functions $\boldsymbol{u} = (\varrho, \varrho\boldsymbol{v}^T)$ that satisfy the entropy condition

$$W(\boldsymbol{u})_t + \operatorname{div} \boldsymbol{Q}(\boldsymbol{u}) \leq 0 \tag{1.1.21}$$

in the distributional sense in the bulk regions and

$$-\sigma\,[\![W(\boldsymbol{u})]\!] + [\![\boldsymbol{Q}(\boldsymbol{u}) \cdot \boldsymbol{n}]\!] \leq 0 \tag{1.1.22}$$

at the interface. In this way solutions are thermodynamically consistent in the entire domain D.

1.1.3. Sharp interface solutions. In this section we propose solution concepts for the free boundary value problem with different choices for the coupling conditions and interpretations of the interface Γ. According to the splitting $D = D_-(t) \cup D_+(t)$ we introduce the notations

$$\boldsymbol{u}_\pm(\mathbf{x},t) = \boldsymbol{u}(\mathbf{x},t),\ \varrho_\pm(\mathbf{x},t) = \varrho(\mathbf{x},t),\ \boldsymbol{v}_\pm(\mathbf{x},t) = \boldsymbol{v}(\mathbf{x},t) \quad \text{for } (\mathbf{x} \in D_\pm(t))$$

and start with

Definition 1.1.5 (General entropic SI solutions). Let $D = \mathbb{R}^d$. A family of manifolds $\{\Gamma(t)\}_{t\in[0,T]}$, $\varrho \in L^\infty(D)$ and $\boldsymbol{v} \in (L^\infty(D))^d$ is called an *entropic sharp-interface* **(*SI*)** *solution* $(\Gamma, \boldsymbol{u}_\pm)$ of (1.1.13), (1.1.15), (1.1.16) if

(i) $\boldsymbol{u}_\pm(\cdot,t) \in \tilde{\mathcal{U}}$ a.e. in $D_\pm(t)$ for $t \in [0,T]$,

(ii) the Euler system (1.1.13) with initial datum (1.1.16) is satisfied in the distributional sense in D,
(iii) the entropy condition (1.1.21) is satisfied in the distributional sense in D,
(iv) the trace conditions (1.1.15), (1.1.22) with traces $\bar{\boldsymbol{u}}_\pm(\cdot, t)$ (as defined in (1.1.14)) hold across $\Gamma(t)$ for $t \in (0, T]$.

Remark 1.1.6.

(a) Regardless of the physical situation and the considered interface there is for $d > 1$ no global well-posedness result for the Euler problem (1.1.13), (1.1.16). In fact, it is by now clear that the chosen entropy selection criterion in Definition 1.1.5 does not imply uniqueness [22].
(b) The condition (iii) requires indirectly more regularity for an entropic SI solution. Evaluation of the traces could be ensured, e.g., in the sense of L^1-functions on the manifold if the bulk states belong to the space of functions of bounded variation.
(c) Definition 1.1.5 of an entropic SI solution excludes solutions which allow topological changes of the sharp interface Γ. Note that the hyper-surface Γ is supposed to be a manifold. Without further coupling conditions any sharp interface concept must fail in this situation. This is one of the motivations to consider DI models.
(d) The Rankine–Hugoniot conditions (1.1.15) are not supposed to hold for $t = 0$ such that the setting for a Riemann-type problem is included.

In the remainder of this section we want to consider wave-type SI solutions that correspond to the one-phase or the two-phase pressure case. The system (1.1.13) is a first-order system of conservation laws such that analyzing the existence and uniqueness of entropic SI solutions requires to determine the possible characteristic structure at the interface $\Gamma(t)$.

SI solutions and shock waves in one-phase flow. Let us consider an entropic SI solution $(\Gamma, \bar{\boldsymbol{u}}_\pm)$ for a one-phase pressure. The sharp interface $\Gamma(t)$ is then naturally interpreted as either a hydrodynamical shock wave or a contact discontinuity. Let us restrict ourselves here to shock waves. For $s \in [0, T]$ and $\boldsymbol{\xi} \in \Gamma(s)$ let the traces $\bar{\boldsymbol{u}}_\pm(\boldsymbol{\xi}, s)$ be given. Then we call the associated planar discontinuous wave

$$\mathbf{U}(\mathbf{x}, t; \boldsymbol{\xi}, s) = \begin{cases} \bar{\boldsymbol{u}}_-(\boldsymbol{\xi}, s) & : \quad \mathbf{x} \cdot \mathbf{n}(\boldsymbol{\xi}, s) - \sigma(\boldsymbol{\xi}, s)t < 0, \\ \bar{\boldsymbol{u}}_+(\boldsymbol{\xi}, s) & : \quad \mathbf{x} \cdot \mathbf{n}(\boldsymbol{\xi}, s) - \sigma(\boldsymbol{\xi}, s)t > 0 \end{cases}$$

a Laxian or hydro-mechanical shock wave if either

$$\begin{aligned} &\lambda_1(\bar{\boldsymbol{u}}_+; \mathbf{n}) < \sigma < \lambda_2(\bar{\boldsymbol{u}}_+; \mathbf{n}),\ \lambda_1(\bar{\boldsymbol{u}}_-; \mathbf{n}) > \sigma \qquad \text{(1-shock wave) or} \\ &\lambda_d(\bar{\boldsymbol{u}}_-; \mathbf{n}) < \sigma < \lambda_{d+1}(\bar{\boldsymbol{u}}_-; \mathbf{n}),\ \lambda_{d+1}(\bar{\boldsymbol{u}}_+; \mathbf{n}) < \sigma \quad \text{((}d+1\text{)-shock wave)} \end{aligned} \tag{1.1.23}$$

are satisfied (see Figure 1.4 for some illustration for a $(d+1)$-shock wave). For $i \in \{1, d+1\}$ the characteristics of the ith field λ_i impinge into the i-shock wave line.

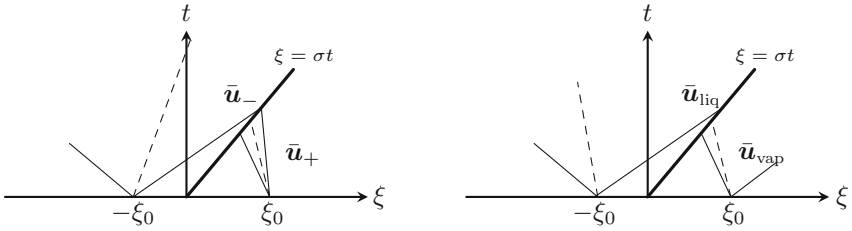


FIG. 1.4. Characteristic structure of a supersonic wave (left) versus subsonic wave (right). For some $\xi_0 > 0$ and $i = 1, \dots, d+1$, the lines $\{(\xi, t) \,|\, \xi + \xi_0 = \lambda_i(\bar{\boldsymbol{u}}_-; \mathbf{n})t, t \geq 0\}$ indicate the characteristics for the left state $\bar{\boldsymbol{u}}_-$ and $\{(\xi, t) \,|\, \xi + \xi_0 = \lambda_i(\bar{\boldsymbol{u}}_-; \mathbf{n})t, t \geq 0\}$ for the right state $\bar{\boldsymbol{u}}_+$ in the left graph (analogously with states $\bar{\boldsymbol{u}}_{\mathrm{liq/vap}}$ for the right graph). The dashed lines represent the contact line of multiplicity $d-1$.

Note that a hydro-mechanical shock wave is supersonic and itself an SI solution for the simple family of manifolds $\{\mathbf{x} \in \mathbb{R}^d \,|\, \mathbf{x} \cdot \mathbf{n}(\boldsymbol{\xi}, s) = \sigma(\boldsymbol{\xi}, s)t\}_{t \in [0,T]}$.

Definition 1.1.7 (Entropic SI solution for one-phase pressure). Let $D = \mathbb{R}^d$ and let a one-phase pressure p be given. A family of manifolds $\{\Gamma(t)\}_{t\in[0,T]}$, $\varrho \in L^\infty(D)$ and $\boldsymbol{v} \in (L^\infty(D))^d$ are called an *entropic SI solution* $(\Gamma, \boldsymbol{u}_\pm)$ *of* (1.1.13), (1.1.15), (1.1.16) *for a one-phase pressure* if $(\Gamma, \boldsymbol{u}_\pm)$ is an entropic SI solution and if for any $s \in (0, T]$, $\xi \in \Gamma(s)$ the associated planar discontinuous wave $\mathbf{U}(\cdot, \cdot; \boldsymbol{\xi}, s)$ is a hydro-mechanical shock wave (either a 1- or $(d+1)$-shock wave).

The local well-posedness of classical entropic SI solutions $(\Gamma, \boldsymbol{u}_\pm)$ of (1.1.13), (1.1.15), (1.1.16) for a one-phase pressure has been shown in [65]. By "classical" we mean the setting that $\boldsymbol{u}_\pm(\cdot, t)$ are smooth functions in $D_\pm(t)$.

SI solutions and phase boundaries in two-phase flow. Let us now turn to the more complex situation for an entropic SI solution $(\Gamma, \boldsymbol{u}_\pm)$ for a two-phase pressure. We associate $D_-(t)$ with a liquid phase domain $D_{\mathrm{liq}}(t)$ and $D_{\mathrm{vap}}(t)$ with a vapour phase domain $D_{\mathrm{vap}}(t)$. This means in particular $\varrho(\mathbf{x}, t) \in \tilde{\mathcal{A}}_{\mathrm{liq/vap}}$ for $\mathbf{x} \in D_{\mathrm{liq/vap}}(t)$.

As the sharp interface $\Gamma(t)$ we consider now phase boundaries. For traces $\bar{\boldsymbol{u}}_{\mathrm{vap/liq}} \in \tilde{\mathcal{A}}_{\mathrm{vap/liq}} \times \mathbb{R}^d$ the planar discontinuous wave

$$\mathbf{U}(\mathbf{x}, t; \boldsymbol{\xi}, s) = \begin{cases} \bar{\boldsymbol{u}}_{\mathrm{liq}}(\boldsymbol{\xi}, s) & : \quad \mathbf{x} \cdot \mathbf{n}(\boldsymbol{\xi}, s) - \sigma(\boldsymbol{\xi}, s)t < 0, \\ \bar{\boldsymbol{u}}_{\mathrm{vap}}(\boldsymbol{\xi}, s) & : \quad \mathbf{x} \cdot \mathbf{n}(\boldsymbol{\xi}, s) - \sigma(\boldsymbol{\xi}, \boldsymbol{s})t > 0 \end{cases} \tag{1.1.24}$$

is a called a subsonic phase boundary if either

$$\lambda_1(\bar{\boldsymbol{u}}_{\mathrm{vap}}; \mathbf{n}) < \sigma < \lambda_2(\bar{\boldsymbol{u}}_{\mathrm{vap}}; \mathbf{n}),\ \lambda_2(\bar{\boldsymbol{u}}_{\mathrm{liq}}; \mathbf{n}) > \sigma > \lambda_1(\bar{\boldsymbol{u}}_{\mathrm{liq}}; \mathbf{n}) \qquad \text{or}$$

$$\lambda_d(\bar{\boldsymbol{u}}_{\mathrm{liq}}; \mathbf{n}) < \sigma < \lambda_{d+1}(\bar{\boldsymbol{u}}_{\mathrm{liq}}; \mathbf{n}),\ \lambda_{d+1}(\bar{\boldsymbol{u}}_{\mathrm{vap}}; \mathbf{n}) < \sigma < \lambda_{d+1}(\bar{\boldsymbol{u}}_{\mathrm{vap}}; \mathbf{n})$$

are satisfied (we skipped the argument $(\boldsymbol{\xi}, s)$ for brevity, see Figure 1.4 for some illustration of subsonic phase boundaries). In contrast to a hydro-mechanical shock

wave for subsonic waves only one characteristic line of the ith field λ_i ($i \in \{1, d+1\}$) impinges into the phase boundary line. In the mathematical literature such phase boundaries are entitled as undercompressive waves [56].

Remark 1.1.8. In our setting we can also consider discontinuous waves connecting two states in different phases that satisfy (1.1.23). These supersonic phase boundaries lead to very high speeds σ. They will be discussed in more detail in Section 1.2.

A particular interesting situation is given by planar equilibrium solutions, i.e., phase boundaries with vanishing mass flux $j = \bar{\varrho}_{\mathrm{liq/vap}}(\bar{\boldsymbol{v}}_{\mathrm{liq/vap}} \cdot \mathbf{n} - \sigma)$ (cf. $(1.1.15)_1$!). A quick check of the Rankine–Hugoniot conditions in (1.1.15) shows that there are multiple equilibrium solutions, precisely all waves $\mathbf{U}$ with

$$\mathbf{U}(\mathbf{x}, t; \xi, s) = \begin{cases} \bar{\boldsymbol{u}}_{\mathrm{liq}}(\boldsymbol{\xi}, s) = (\bar{\varrho}_{\mathrm{liq}}, \bar{\varrho}_{\mathrm{liq}} \bar{\boldsymbol{v}}_{\mathrm{liq}}^T)^T(\boldsymbol{\xi}, s) & : \mathbf{x} \cdot \mathbf{n}(\boldsymbol{\xi}, s) - \sigma(\boldsymbol{\xi}, s)t < 0, \\ \bar{\boldsymbol{u}}_{\mathrm{vap}}(\boldsymbol{\xi}, s) = (\bar{\varrho}_{\mathrm{vap}}, \bar{\varrho}_{\mathrm{vap}} \bar{\boldsymbol{v}}_{\mathrm{vap}}^T)^T(\boldsymbol{\xi}, s) & : \mathbf{x} \cdot \mathbf{n}(\boldsymbol{\xi}, s) - \sigma(\boldsymbol{\xi}, s)t > 0, \end{cases} \tag{1.1.25}$$

connecting states $\bar{\boldsymbol{u}}_{\mathrm{liq}}$, $\bar{\boldsymbol{u}}_{\mathrm{vap}}$ with $p(\bar{\tau}_{\mathrm{liq}}) = p(\bar{\tau}_{\mathrm{vap}})$, $\bar{\boldsymbol{v}}_{\mathrm{liq}} = \bar{\boldsymbol{v}}_{\mathrm{vap}} = \boldsymbol{v} \in \mathbb{R}^d$ and $\sigma = \boldsymbol{v} \cdot \mathbf{n}$.

It is well known that the only physically acceptable states are the Maxwell states $\tau_{\mathrm{liq/vap}}^{\mathrm{sat}}$ that are characterized additionally by equal specific Gibbs energy, cf. Definition 1.1.2, [31]. Non-uniqueness appears for dynamical interfaces, which is also suggested by the under-determined characteristic structure as displayed in Figure 1.4, right. These mathematical arguments suggest to add an additional coupling condition to (1.1.15). In the mathematical literature this condition is known as *kinetic relation* (see, e.g., [1, 82]) and can be physically understood as a Gibbs–Thomson-like condition.

For relative flux $j = \bar{\varrho}_{\mathrm{liq/vap}}(\bar{\boldsymbol{v}}_{\mathrm{liq/vap}} \cdot \mathbf{n} - \sigma)$ and the so-called *driving force* $\mathcal{K} : \mathbb{R} \to \mathbb{R}$ it is given by

$$\left[\!\!\left[\tilde{\mu}(\varrho) + \frac{j}{2\varrho} \right]\!\!\right] = -\mathcal{K}(j). \tag{1.1.26}$$

Multiplying (1.1.26) with j and using the coupling conditions (1.1.15) leads to the identity

$$-\mathcal{K}(j)j = \left[\!\!\left[\tilde{\mu}(\varrho) + \frac{j}{2\varrho} \right]\!\!\right] j = -\sigma \left[\!\!\left[W(\boldsymbol{u}) \right]\!\!\right] + \left[\!\!\left[\boldsymbol{Q}(\boldsymbol{u}) \cdot \boldsymbol{n} \right]\!\!\right].$$

As a consequence of this computation subsonic phase boundaries that satisfy (1.1.26) are thermodynamically consistent if and only if the driving force satisfies the condition

$$\mathcal{K}(j)j \geq 0. \tag{1.1.27}$$

Another simple constraint is put on kinetic relations by the request that it must admit the equilibrium solutions from (1.1.25). In view of (1.1.26) and the definition of the saturation states in Definition 1.1.2 the driving force $\mathcal{K}$ must obey

$$\mathcal{K}(0) = 0. \tag{1.1.28}$$

We introduce some specific choices for the driving force as constitutive relation.

Remark 1.1.9 (Some kinetic relations).

(a) Let a mobility constant $\alpha \in [0, \infty)$ be given. A linear ansatz for $\mathcal{K} = \mathcal{K}_1^\alpha$ that trivially satisfies (1.1.27) is

$$\mathcal{K}_1^\alpha(j) = \alpha j.$$

This choice has been considered in, e.g., [1]. The material-dependent mobility constant has to be determined from experiments or from first-principle ideas. Density functional theory is used, e.g., in [52]. Particularly interesting is the choice $\alpha = 0$. Then, the phase boundaries do not dissipate entropy and are represented as reversible processes.

(b) From the mathematical point of view the choice

$$\mathcal{K}_2^\alpha(j) = \alpha j|j|^p$$

satisfies (1.1.27) for any $p > 0$.

(c) In view of (1.1.26) the limit $\alpha \to \infty$ in the conditions (i) and (ii) would (formally!) result in the condition $j = 0$, i.e., there is no mass flux (and entropy release) across the interface. This approach can be interpreted as a model for two compressible but immiscible fluids.

Having fixed a kinetic relation we can now present a notion of solution for homogeneous two-phase flow.

Definition 1.1.10 (Entropic SI solution for two-phase pressure). Let $D = \mathbb{R}^d$. Consider a two-phase pressure p for an SI model and some driving force $\mathcal{K}$ that satisfies (1.1.27).

A family of manifolds $\{\Gamma(t)\}_{t\in[0,T]}$, $\varrho \in L^\infty(D)$ and $\boldsymbol{v} \in (L^\infty(D))^d$ is called an *entropic SI solution of* (1.1.13), (1.1.26), (1.1.16) *for a two-phase pressure* if $(\Gamma, \bar{\boldsymbol{u}}_\pm)$ is an entropic SI solution and if for any $s \in (0, T]$, $\xi \in \Gamma(s)$ the associated planar discontinuous wave $\mathbf{U}(\cdot, \cdot; \boldsymbol{\xi}, s)$ is a subsonic phase boundary that obeys (1.1.26).

It satisfies

$$\tau_{\text{vap/liq}}(\cdot, t) \in \mathcal{A}_{\text{vap/liq}} \text{ a.e. in } D_{\text{vap/liq}}(t).$$

Remark 1.1.11. In our SI theory any kind of surface effects related to Γ are ignored. In presence of capillary forces induced by surface tension the relation

$$[\![\varrho\,(\boldsymbol{v}\cdot\boldsymbol{n} - \sigma)\,\boldsymbol{v}\cdot\boldsymbol{n} + \tilde{p}(\varrho)]\!] = (d-1)\zeta\,\kappa, \tag{1.1.29}$$

extends the condition (1.1.15) on momentum. We denote by $\kappa = \kappa(\boldsymbol{\xi}, t) \in \mathbb{R}$ the mean curvature of $\Gamma(t)$ associated with orientation given through the choice of the normal $\boldsymbol{n}$. The surface tension coefficient $\zeta \geq 0$ is assumed to be constant. The condition (1.1.29) reduces to the classical Young–Laplace law for the static case.

For a capillary fluid the condition (1.1.22) changes accordingly and we have then the generalized entropy inequality

$$-\sigma\,([\![W(\boldsymbol{u})]\!] + (d-1)\zeta\,\kappa) + [\![(W(\boldsymbol{u}) + \tilde{p}(\varrho))\,\boldsymbol{v}\cdot\boldsymbol{n}]\!] \leq 0 \tag{1.1.30}$$

across $\Gamma(t)$. For jump conditions in much more general continuum theories we refer to [2].

There is not much known about the global well-posedness of entropic SI solutions for two-phase pressure. Even local well-posedness is only known if the regularizing effect of surface tension is added and $D = \mathbb{R}^d$ holds. In view of Remark 1.1.11 we report on local well-posedness of entropic SI solution for a two-phase pressure in the case where (1.1.29) substitutes the second condition in (1.1.15), i.e., curvature effects are taken into account. To introduce an appropriate notion of classical solution let us assume that we have such an entropic SI solution $(\Gamma, \boldsymbol{u}_\pm)$ of (1.1.13), (1.1.26), (1.1.16). The theorem applies for a mildly curved interface that can in particular be represented as a graph of a function $X \in C^2(\mathbb{R}^{d-1} \times [0,T])$. Without loss of generality let us assume for fixed $t \in [0,T]$, that

$$\Gamma(t) = \{\mathbf{x} = (x_1, \dots, x_d)^T \in D \,|\, x_d = X(\mathbf{y}, t),\, \mathbf{y} := (x_1, \dots, x_{d-1})\}$$

holds. This assumption is used to perform a transformation, such that the solutions $\boldsymbol{u}_\pm(\cdot, t)$ on the time-dependent domains $D_\pm(t)$ are defined as shifted SI solutions $\underline{\boldsymbol{u}}(\cdot, t)$ on a fixed half-space, i.e.,

$$\underline{\mathbf{u}}(\cdot,t) : \begin{cases} \mathbb{R}^{d-1} \times \mathbb{R}_+ & \to & (\tilde{\mathcal{A}}_{\mathrm{liq}} \times \mathbb{R}^d) \times (\tilde{\mathcal{A}}_{\mathrm{vap}} \times \mathbb{R}^d) \\ (\mathbf{y}, z) & \mapsto & (\mathbf{u}(\mathbf{y}, z - X(t,\mathbf{y}), t), \mathbf{u}(\mathbf{y}, z + X(t,\mathbf{y}), t)). \end{cases}$$

It is interesting to note that the jump condition (1.1.29) after transformation into the half-space takes the form

$$[\![\varrho(u - X_t - \boldsymbol{w} \cdot \nabla_{\mathbf{y}} X)u - p(\varrho)]\!] = \zeta \Delta_{\mathbf{y}} X.$$

Here u ($\boldsymbol{w}$) represents the velocity component (components) normal (tangential) to $\{x_d = 0\}$ in the shifted solution. In this setting one observes after proper linearization around a reference state (see below) that the evolution of X is governed by a parabolic equation. It is exactly its regularizing effect which permits the stability estimates in order to prove Theorem 1.1.12 below.

Next, let us fix a reference wave $\mathbf{U} = \mathbf{U}(\mathbf{x}, t)$ as in (1.1.24), such that it is a subsonic phase boundary. As we noted above, $\mathbf{U}$ can also be understood as an entropic SI solution for a two-phase pressure and thus one can define in the same way as for $\boldsymbol{u}$ the shifted wave $\underline{\mathbf{U}}$. By a classical entropic SI solution we mean now an entropic SI solution such that the functions $\boldsymbol{u}_\pm$ are smooth in $\{(\mathbf{x}, t) \,|\, \mathbf{x} \in D_\pm(t), t \in [0,T]\}$.

Theorem 1.1.12 (Local well-posedness). *Let $k > \frac{d+3}{2}$ and $(\Gamma_0, \boldsymbol{u}_0)$ be given such that Γ_0 can be represented by a graph $X_0 \in H^{s+3/2}(\mathbb{R}^{d-1})$ and such that $\underline{\boldsymbol{u}}_0 - \underline{\mathbf{U}}(0,\cdot) \in H^{s+1/2}(\mathbb{R}^{d-1} \times \mathbb{R}_+)$ holds. Furthermore consider a two-phase pressure p for an SI model and the driving force $\mathcal{K} = \mathcal{K}_1^\alpha$ from Remark 1.1.9.*

Then, there are constants $\bar{\alpha}, \bar{\zeta}, \bar{\delta} > 0$, $\overline{T} \in (0,T)$, such that for $\alpha \in (0, \bar{\alpha})$, $\zeta \in (0, \bar{\zeta})$ and

$$\|\underline{\boldsymbol{u}}_0 - \underline{\mathbf{U}}(0,\cdot)\|_{H^{s+1/2}} + \|X_0\|_{H^{s+3/2}} < \delta$$

there is a classical entropic SI solution $(\Gamma, \boldsymbol{u}_\pm)$ *of* (1.1.13), (1.1.26), (1.1.16) *for a two-phase pressure on* $[0, \overline{T})$, *with* (1.1.29) *being valid across* Γ. *Moreover,* $\Gamma(t)$ *is represented as a graph for* $t \in (0, \bar{T})$.

In other words, for data close to a planar subsonic phase boundary we can guarantee the existence of a classical entropic SI solution. The proof extends the work of [20, 65] and the stability analysis in [4, 49]. The details can be found in the forthcoming paper [50]. It must be outlined that we have skipped in the formulation of Theorem 1.1.12 some assumptions on trace compatibility across Γ_0. To avoid the instantaneous spreading of discontinuous waves into the bulk domains (as in a Riemann problem, see Section 1.2) the initial data have to be close to a single wave (here the reference wave $\mathbf{U}$). Since also the derivatives of the initial datum are transported by (derived) nonlinear transport equations a classical solution concept must suppress this effect. This can be done exactly by additional conditions on the initial trace states, see [50] for the detailed statement.

1.2. The Riemann problem

As mentioned above there is almost no well-posedness result for entropic SI solutions in multiple space dimensions regardless whether we choose a one-phase or a two-phase pressure for an SI model. However, a complete theory is by now available for the one-dimensional case with constant initial datum in the two bulk phases: the Riemann problem. Irrespective of the analytical value, the thermodynamically consistent solution of the Riemann problem gives considerable insight into the possible choices for the driving force $\mathcal{K}$ in the kinetic relation (1.1.26). It is also of fundamental importance for the construction of numerical schemes for rotationally invariant systems like (1.1.13), see Section 1.3 below. For two-phase pressures we will introduce a numerical scheme that relies on an interface solver $\mathcal{R}$, see (1.2.2). By an interface solver we mean a mapping which determines from some (Riemann) states the adjacent states and the speed of a phase boundary as it appears, e.g., in the solution of the Riemann problem.

We will in this section first recall the standard approach to solve a one-phase Riemann problem to motivate then the generalization to the considerably more complex two-phase case. The content of this section follows the lines of [17] and in particular the recent work [74].

Riemann problems for general hyperbolic-elliptic systems and systems that admit undercompressive waves have been intensively studied in the last two decades, see [57] for a general theory. In the context of compressible multi-phase flow we refer to, e.g., [19, 28, 36, 40, 41, 43, 46, 60, 67, 69].

1.2.1. The rotated Riemann problem and Lagrangian setting. In view of the tracking algorithm in Section 1.3 we want to solve a planar problem at each point $\boldsymbol{\xi} \in \Gamma$ where Γ is a given manifold with normal $\mathbf{n} = \mathbf{n}(\boldsymbol{\xi}) \in \mathcal{S}^{d-1}$. Let $x = (\boldsymbol{x} - \boldsymbol{\xi}) \cdot \boldsymbol{n}$ for $\mathbf{x} \in \mathbb{R}^d$ and states $\boldsymbol{u}_{L/R} \in \tilde{\mathcal{A}} \times \mathbb{R}^d$ be given.

We consider the Riemann problem for the rotated system

$$\mathbf{U}_t + \boldsymbol{F}(\mathbf{U};\mathbf{n})_x = \mathbf{0} \text{ in } \mathbb{R}\times(0,\infty), \tag{1.2.1}$$
$$\mathbf{U}(x,0) = \begin{cases} \mathbf{U}_L = \left(\varrho_{\mathrm{L}}, \boldsymbol{v}_{\mathrm{L}}\cdot\mathbf{n}, \boldsymbol{v}_{\mathrm{L}}\cdot\boldsymbol{t}^1, \dots, \boldsymbol{v}_{\mathrm{L}}\cdot\boldsymbol{t}^{d-1}\right)^T & : x<0, \\ \mathbf{U}_R = \left(\varrho_{\mathrm{R}}, \boldsymbol{v}_{\mathrm{R}}\cdot\mathbf{n}, \boldsymbol{v}_{\mathrm{R}}\cdot\boldsymbol{t}^1, \dots, \boldsymbol{v}_{\mathrm{R}}\cdot\boldsymbol{t}^{d-1}\right)^T & : x>0. \end{cases}$$

Here $\mathbf{U} = \mathbf{U}(x,t) \in \tilde{\mathcal{A}} \times \mathbb{R}^d$ denotes the (rotated) unknown (using the same notation as for discontinuous waves (1.1.24), which shouldn't be mixed up).

Let us consider for a moment the two-phase case with $\mathbf{U}_L \in \tilde{\mathcal{A}}_{\mathrm{liq}} \times \mathbb{R}^d$ and $\mathbf{U}_R \in \tilde{\mathcal{A}}_{\mathrm{vap}} \times \mathbb{R}^d$. In Theorem 1.2.9 we will prove that the Riemann problem (1.2.1) is solvable and contains exactly one phase boundary connecting a state $\mathbf{U}_{\mathrm{liq}}$ with the state $\mathbf{U}_{\mathrm{vap}}$ and moving with speed $\mathfrak{s}$. Then we can define an *interface solver*

$$\mathcal{R}_{\boldsymbol{F}(\cdot;\mathbf{n})} : \begin{cases} (\tilde{\mathcal{A}}_{\mathrm{liq}} \times \mathbb{R}^d) \times (\tilde{\mathcal{A}}_{\mathrm{vap}} \times \mathbb{R}^d) & \to & \mathbb{R} \times (\tilde{\mathcal{A}}_{\mathrm{liq}} \times \mathbb{R}^d) \times (\tilde{\mathcal{A}}_{\mathrm{vap}} \times \mathbb{R}^d) \\ (\mathbf{U}_L, \mathbf{U}_R) & \mapsto & (\mathfrak{s}, \mathbf{U}_{\mathrm{liq}}, \mathbf{U}_{\mathrm{vap}}). \end{cases} \tag{1.2.2}$$

Let us note that the idea of an interface solver does not necessarily require to solve a continuum-mechanical Riemann problem. In view of the modelling problems associated with the kinetic relation other more microscopic models could provide even more accurate information (see [64] for a molecular-dynamical approach).

Let us proceed with the general case. The tangential part of the velocity field $\boldsymbol{v}$ is independent of the field in normal direction and is just transported. Consequently it is possible to neglect the tangential components and focus on a problem only for the unknowns density ϱ and normal velocity $v = \boldsymbol{v}\cdot\mathbf{n}$. Moreover the solution of the Riemann problem gets much more easy if we transform the Euler system (1.2.1) in Eulerian variables (x,t) into Lagrangian coordinates which we denote by (ξ,t).

Therefore we are lead to consider the following Riemann problem. Let $\mathbf{w}_{L/R} \in \mathcal{A}\times\mathbb{R}$ be given. We search for a self-similar entropy solution $\mathbf{w} = (\tau, v)^T : \mathbb{R}\times[0,\infty) \to \mathcal{A}\times\mathbb{R}$ of

$$\begin{aligned} \begin{pmatrix}\tau\\ v\end{pmatrix}_t + \begin{pmatrix}-v\\ p(\tau)\end{pmatrix}_\xi &= \begin{pmatrix}0\\0\end{pmatrix} \text{ in } \mathbb{R}\times(0,\infty), \\ \begin{pmatrix}\tau\\ v\end{pmatrix}(\xi,0) &= \begin{cases} \mathbf{w}_L = (\tau_{\mathrm{L}}, v_{\mathrm{L}})^T & \text{for } \xi<0, \\ \mathbf{w}_R = (\tau_{\mathrm{R}}, v_{\mathrm{R}})^T & \text{for } \xi>0. \end{cases} \end{aligned} \tag{1.2.3}$$

An entropy solution $\mathbf{w}$ of (1.2.3) has to satisfy the entropy condition

$$\left(\psi(\tau) + \frac{1}{2}v^2\right)_t + \left(p(\tau)\,v\right)_\xi \le 0$$

in the distributional sense (cf. (1.1.21) for the equivalent Eulerian formulation). To simplify the following analysis in the two-phase case we will always assume that we have $\tau_{\mathrm{L}} \in \mathcal{A}_{\mathrm{liq}}$ and $\tau_{\mathrm{R}} \in \mathcal{A}_{\mathrm{vap}}$.

System (1.2.3) is of course also a system of hyperbolic conservation laws. Using the notations as in the Eulerian case the real eigenvalues of the Jacobian of the flux are

$$\lambda_1(\tau) = -c(\tau), \quad \lambda_2(\tau) = c(\tau), \tag{1.2.4}$$

where $c = c(\tau)$ is the sound speed in Lagrangian coordinates (see (1.1.11)). The coordinate transformation does also not change the type of the characteristic fields: both characteristic fields are genuinely nonlinear in the state spaces for the one-phase and the two-phase pressure.

1.2.2. Elementary waves and phase boundaries. The solution of the Riemann problem will consist of a wave pattern with different types of waves. For a one-phase pressure elementary waves (rarefaction and Laxian shock waves) suffice, for a two-phase pressure additionally several types of phase boundaries are needed. We first introduce all these waves. The waves connect given left states $\boldsymbol{w}_\mathrm{l} = (\tau_\mathrm{l}, v_\mathrm{l})^T \in \mathcal{A} \times \mathbb{R}$ with right states $\boldsymbol{w}_\mathrm{r} = (\tau_\mathrm{r}, v_\mathrm{r})^T \in \mathcal{A} \times \mathbb{R}$. For more background we refer to standard text books like [21, 35].

Rarefaction wave. A left state $\boldsymbol{w}_\mathrm{l}$ and a right state $\boldsymbol{w}_\mathrm{r}$ are connected by an *i-rarefaction wave* ($i = 1, 2$) if

$$\boldsymbol{w}(\xi, t) = \begin{cases} \boldsymbol{w}_\mathrm{l} & \text{for } \xi < \lambda_i(\tau_\mathrm{l})\, t, \\ \bar{\boldsymbol{w}}(\xi/t) & \text{for } \lambda_i(\tau_\mathrm{l})\, t < \xi < \lambda_i(\tau_\mathrm{r})\, t, \\ \boldsymbol{w}_\mathrm{r} & \text{for } \xi > \lambda_i(\tau_\mathrm{r})\, t > 0 \end{cases}$$

is a continuous weak solution of system $(1.2.3)_1$ for some smooth function $\bar{\boldsymbol{w}} : \mathbb{R} \to \mathcal{A} \times \mathbb{R}$. For the velocity component of a 1/2-rarefaction wave we have $v_\mathrm{r} = v_\mathrm{l} \pm R(\tau_\mathrm{l}, \tau_\mathrm{r})$ with

$$R(\tau_\mathrm{l}, \tau_\mathrm{r}) := \int_{\tau_\mathrm{l}}^{\tau_\mathrm{r}} \sqrt{-p'(\tau)}\, d\tau.$$

Laxian shock wave. A (*Laxian*) *i-shock wave* is a discontinuous wave connecting $\boldsymbol{w}_l$ with $\boldsymbol{w}_r$ with speed $\mathfrak{s}_i \in \mathbb{R}$ such that

$$\lambda_i(\tau_\mathrm{l}) > \mathfrak{s}_i > \lambda_i(\tau_\mathrm{r}) \tag{1.2.5}$$

holds. Note that (1.2.5) corresponds to (1.1.23) in the Eulerian case. We will use the term shock wave in this section only for waves in the bulk, that means, for τ_l and τ_r in the same phase for a two-phase pressure.

Recall that any discontinuous wave satisfies the Rankine–Hugoniot conditions (1.1.15), which are simplified in the Lagrangian setting to be

$$\mathfrak{s}\, [\![\tau]\!] + [\![v]\!] = 0, \qquad -\mathfrak{s}\, [\![v]\!] + [\![p(\tau)]\!] = 0, \tag{1.2.6}$$

where $\mathfrak{s}$ denotes now the speed of the interface in Lagrangian coordinates. As a consequence the propagation speed for a 1/2-shock wave is

$$\mathfrak{s}_{1/2} = \mathfrak{s}_{1/2}(\tau_\mathrm{l}, \tau_\mathrm{r}) = \mp \sqrt{-\frac{p(\tau_\mathrm{r}) - p(\tau_\mathrm{l})}{\tau_\mathrm{r} - \tau_\mathrm{l}}}.$$

For some 1/2-shock wave with $\tau_\mathrm{l} \neq \tau_\mathrm{r}$ we have $v_\mathrm{r} = v_\mathrm{l} \pm S(\tau_\mathrm{l}, \tau_\mathrm{r})$ with

$$S(\tau_\mathrm{l}, \tau_\mathrm{r}) = \operatorname{sign}(\tau_\mathrm{r} - \tau_\mathrm{l}) \sqrt{-(\tau_\mathrm{r} - \tau_\mathrm{l})\ (p(\tau_\mathrm{r}) - p(\tau_\mathrm{l}))}.$$

The entropy condition for a discontinuous wave in the Lagrangian framework writes as

$$-\mathfrak{s} f \leq 0, \quad f(\tau_\mathrm{l}, \tau_\mathrm{r}) := [\![\psi(\tau)]\!] + [\![\tau]\!]\,\{p(\tau)\}\,. \tag{1.2.7}$$

Note that (1.2.7) is the interfacial entropy condition (1.1.22) in Lagrangian coordinates. It is well known that a Laxian i-shock wave satisfies the entropy condition (1.2.7).

Rarefaction and Laxian shock waves are summarized as elementary waves using the term *i-elementary wave* for an i-shock wave and an i-rarefaction wave. They obey the following compact representation.

$$v_\mathrm{r} = \begin{cases} v_\mathrm{l} + E(\tau_\mathrm{l}, \tau_\mathrm{r}) & \text{if } i = 1, \\ v_\mathrm{l} - E(\tau_\mathrm{l}, \tau_\mathrm{r}) & \text{if } i = 2 \end{cases} \quad \text{for} \quad E(\tau_\mathrm{l}, \tau_\mathrm{r}) = \begin{cases} R(\tau_\mathrm{l}, \tau_\mathrm{r}) & \text{if } i = 1 \text{ and } \tau_\mathrm{l} < \tau_\mathrm{r}, \\ S(\tau_\mathrm{l}, \tau_\mathrm{r}) & \text{if } i = 1 \text{ and } \tau_\mathrm{l} > \tau_\mathrm{r}, \\ R(\tau_\mathrm{l}, \tau_\mathrm{r}) & \text{if } i = 2 \text{ and } \tau_\mathrm{l} > \tau_\mathrm{r}, \\ S(\tau_\mathrm{l}, \tau_\mathrm{r}) & \text{if } i = 2 \text{ and } \tau_\mathrm{l} < \tau_\mathrm{r}. \end{cases}$$

We proceed to define phase boundaries. The definitions apply only for the choice of a two-phase pressure for an SI model. Let the states satisfy $\tau_\mathrm{l} \in \mathcal{A}_\mathrm{liq}$, $\tau_\mathrm{r} \in \mathcal{A}_\mathrm{vap}$.

Phase boundary. A discontinuous wave, that connects a left state $\boldsymbol{w}_\mathrm{l}$ and a right state $\boldsymbol{w}_\mathrm{r}$ in different phases, and that satisfies the entropy condition (1.2.7) is called *phase boundary*. It follows from (1.2.6) that phase transitions propagate either with speed

$$\mathfrak{s}_\mathrm{e}(\tau_\mathrm{l}, \tau_\mathrm{r}) = -\sqrt{-\frac{p(\tau_\mathrm{r}) - p(\tau_\mathrm{l})}{\tau_\mathrm{r} - \tau_\mathrm{l}}} \quad \text{or} \quad \mathfrak{s}_\mathrm{c}(\tau_\mathrm{l}, \tau_\mathrm{r}) = +\sqrt{-\frac{p(\tau_\mathrm{r}) - p(\tau_\mathrm{l})}{\tau_\mathrm{r} - \tau_\mathrm{l}}}. \tag{1.2.8}$$

A phase boundary that travels with speed $\mathfrak{s}_\mathrm{e}$ ($\mathfrak{s}_\mathrm{c}$) is called evaporation (condensation) wave. Indeed, for $\boldsymbol{w}_\mathrm{l} = (\tau_\mathrm{l}, v_\mathrm{l})^T \in \mathcal{A}_\mathrm{liq} \times \mathbb{R}$ and $\boldsymbol{w}_\mathrm{r} = (\tau_\mathrm{r}, v_\mathrm{r})^T \in \mathcal{A}_\mathrm{vap} \times \mathbb{R}$, a phase boundary with negative speed is always an evaporation wave and a phase boundary with positive speed is always a condensation wave.

We have for evaporation waves $v_\mathrm{r} = v_\mathrm{l} + P(\tau_\mathrm{l}, \tau_\mathrm{r})$ and for condensation waves $v_\mathrm{r} = v_\mathrm{l} - P(\tau_\mathrm{l}, \tau_\mathrm{r})$, with

$$P(\tau_\mathrm{l}, \tau_\mathrm{r}) = \operatorname{sign}(\tau_\mathrm{r} - \tau_\mathrm{l}) \sqrt{(\tau_\mathrm{r} - \tau_\mathrm{l})\ (p(\tau_\mathrm{l}) - p(\tau_\mathrm{r}))}.$$

Subsonic phase boundary. An evaporation wave (A condensation wave) is called *subsonic* if

$$|\mathfrak{s}_\mathrm{e}(\tau_\mathrm{l}, \tau_\mathrm{r})| < c(\tau_\mathrm{l}),\, c(\tau_\mathrm{r}) \qquad \big(|\mathfrak{s}_\mathrm{c}(\tau_\mathrm{l}, \tau_\mathrm{r})| < c(\tau_\mathrm{l}),\, c(\tau_\mathrm{r})\big) \tag{1.2.9}$$

holds. Due to $\tau_\mathrm{l} \in \mathcal{A}_\mathrm{liq}$ and the assumptions on the pressure derivatives in Definition 1.1.2 $|\mathfrak{s}_\mathrm{e}(\tau_\mathrm{l}, \tau_\mathrm{r})| < c(\tau_\mathrm{l})$ and $|\mathfrak{s}_\mathrm{c}(\tau_\mathrm{l}, \tau_\mathrm{r})| < c(\tau_\mathrm{l})$ are automatically satisfied.

Sonic and supersonic phase boundary. An evaporation wave (A condensation wave) is called *supersonic* if

$$|\mathfrak{s}_\mathrm{e}(\tau_\mathrm{l}, \tau_\mathrm{r})| > c(\tau_\mathrm{r}) \qquad \big(|\mathfrak{s}_\mathrm{c}(\tau_\mathrm{l}, \tau_\mathrm{r})| > c(\tau_\mathrm{r})\big) \tag{1.2.10}$$

holds. Note that these waves fulfill the Lax condition (1.2.5). Phase boundaries are called *sonic* if equality holds in (1.2.10). Both types fulfill the entropy condition (1.2.7).

The functions R, S, E and P are monotone decreasing with respect to the first argument and monotone increasing with respect to the second argument. This will become necessary in order to determine unique solutions of the Riemann problem in Section 1.2.3.

The subsonic phase boundaries for a two-phase pressure require further specifications. As has been discussed in Section 1.1.3 subsonic phase boundaries are constrained by a kinetic relation as in (1.1.26). The kinetic relation (1.1.26) can be rewritten in the Lagrangian setting (see (1.2.7)) as

$$K(f(\tau_\mathrm{l}, \tau_\mathrm{r}), \mathfrak{s}(\tau_\mathrm{l}, \tau_\mathrm{r})) := f(\tau_\mathrm{l}, \tau_\mathrm{r}) - \mathcal{K}(\mathfrak{s}(\tau_\mathrm{l}, \tau_\mathrm{r})) = 0. \tag{1.2.11}$$

Let us denote the domain of possible states $\tau_\mathrm{l}, \tau_\mathrm{r}$ that can be connected by an entropy-consistent subsonic phase boundary (i.e., a subsonic phase boundary that satisfies (1.2.7)) by $\mathcal{A}_{pb}$. For some specific pressure choice we refer to Figure 1.5. The intersection of the null clines of $K(f(\cdot,\cdot), \mathfrak{s}(\cdot,\cdot))$ with $\mathcal{A}_{pb}$ determine those states $\tau_\mathrm{l}, \tau_\mathrm{r}$, which lead to subsonic phase boundaries that can be admitted in the solution of the Riemann problem (1.2.3). Because subsonic phase boundaries correspond to either an evaporation and/or a condensation wave the null cline should split into two corresponding curves. Unique solvability of the Riemann problem requires that these two curves are graphs of two monotone functions. Therefore we will rely in Section 1.2.4 on the following assumption.

Assumption 1.2.1. Let a two-phase pressure p for an SI model and a driving force $\mathcal{K}$ be given.

There are numbers $\tau_\mathrm{liq}^\mathrm{sc} \in (\tau_\mathrm{liq}^\mathrm{min}, \tau_\mathrm{liq}^\mathrm{sat})$, $\tau_\mathrm{vap}^\mathrm{se} \in (\tau_\mathrm{vap}^\mathrm{min}, \infty)$ and monotone decreasing, differentiable functions

$$k_\mathrm{c} : [\tau_\mathrm{liq}^\mathrm{sc}, \tau_\mathrm{liq}^\mathrm{sat}] \to \mathcal{A}_\mathrm{vap} \text{ and } k_\mathrm{e} : [\tau_\mathrm{vap}^\mathrm{sat}, \tau_\mathrm{vap}^\mathrm{se}] \to \mathcal{A}_\mathrm{liq},$$

that satisfy

$$k_\mathrm{c}(\tau_\mathrm{liq}^\mathrm{sat}) = \tau_\mathrm{vap}^\mathrm{sat}, \quad k_\mathrm{c}(\tau_\mathrm{liq}^\mathrm{sc}) = \tau_\mathrm{vap}^\mathrm{se}, \quad \big|\mathfrak{s}_\mathrm{c}(\tau_\mathrm{liq}^\mathrm{sc}, \tau_\mathrm{vap}^\mathrm{sc})\big| = c(\tau_\mathrm{vap}^\mathrm{sc}), \tag{1.2.12}$$

$$k_\mathrm{e}(\tau_\mathrm{vap}^\mathrm{sat}) = \tau_\mathrm{liq}^\mathrm{sat}, \quad k_\mathrm{e}(\tau_\mathrm{vap}^\mathrm{se}) = \tau_\mathrm{liq}^\mathrm{sc}, \quad \big|\mathfrak{s}_\mathrm{e}(\tau_\mathrm{liq}^\mathrm{se}, \tau_\mathrm{vap}^\mathrm{se})\big| = c(\tau_\mathrm{vap}^\mathrm{se}), \quad k_\mathrm{e}'(\tau_\mathrm{vap}^\mathrm{se}) = 0 \tag{1.2.13}$$

and

$$(\tau_\mathrm{liq}, k_c(\tau_\mathrm{liq}) \in \mathcal{A}_{pb} \qquad \forall \tau_\mathrm{liq} \in [\tau_\mathrm{liq}^\mathrm{sc}, \tau_\mathrm{liq}^\mathrm{sat}]$$

and

$$(k_e(\tau_{\text{vap}}), \tau_{\text{vap}}) \in \mathcal{A}_{pb} \quad \forall\, \tau_{\text{vap}} \in [\tau_{\text{vap}}^{\text{sat}}, \tau_{\text{vap}}^{\text{se}}],$$

such that we have for K from (1.2.11)

$$K(f(\tau_{\text{liq}}, k_{\text{c}}(\tau_{\text{liq}})), \mathfrak{s}_{\text{c}}(\tau_{\text{liq}}, k_{\text{c}}(\tau_{\text{liq}}))) = 0$$

and

$$K(f(k_e(\tau_{\text{vap}}), \tau_{\text{vap}}), \mathfrak{s}_{\text{e}}(k_{\text{e}}(\tau_{\text{vap}}), \tau_{\text{vap}})) = 0.$$

The functions pair $(k_{\text{c}}, k_{\text{e}})$ is called pair of *kinetic functions*.

The first block of conditions (1.2.12) ensures that Maxwell equilibria (see (1.1.8)) are admitted as subsonic phase boundaries. Note that for $t \geq 0$ the (standing) discontinuous wave

$$\mathbf{w}(\xi, t) = \begin{cases} (\tau_{\text{liq}}^{\text{sat}}, 0)^T : \xi < 0, \\ (\tau_{\text{vap}}^{\text{sat}}, 0)^T : \xi > 0 \end{cases} \tag{1.2.14}$$

is a subsonic phase boundary that satisfies (1.2.7) with equality. Since the mass flux j in Eulerian coordinates is linked by $\mathfrak{s} = -j$ to the Lagrangian speed we recover all relevant equilibria in the Eulerian case.

The remaining conditions in (1.2.12) guarantee that solutions of the Riemann problem vary continuously (under, e.g., variation of one of the end states) from subsonic via sonic to supersonic phase transition pattern. The situation is illustrated in Figure 1.5.

Before we will proceed with the solutions of the Riemann problem we provide a note on the relation between the presented driving forces from Remark 1.1.9 and pairs of kinetic functions.

Remark 1.2.2 (Driving force and kinetic functions).

(a) For the linear ansatz $\mathcal{K}_1^\alpha$ only the choice $\alpha = 0$ corresponds to a (identical!) pair of kinetic functions. The graph of the null clines of the corresponding function K_1^0 is plotted in Figure 1.5. The null clines for some K_1^α with $\alpha \neq 0$, which are derived from $\mathcal{K}_1^\alpha$, are also plotted. Obviously, there is no associated pair of kinetic functions.

(b) If α is small enough the driving forces $\mathcal{K}_2^\alpha$ lead to pairs of (different) kinetic functions. Some example with $p = 1$ is presented in Figure 1.5.

(c) In Note 1.1.9(c) we discussed also the choice $j = \mathfrak{s} = 0$. This corresponds to a pair of kinetic functions such that the null clines of the corresponding function K_0 form the upper boundary of $\mathcal{A}_{pb}$.

1.2.3. The Riemann problem for one-phase flow. Let us consider a one-phase pressure from Definition 1.1.1. The study of the Riemann problem in this case is a classical topic (see, e.g., [35]). We present the main results to compare it to the two-phase case in Section 1.2.4.

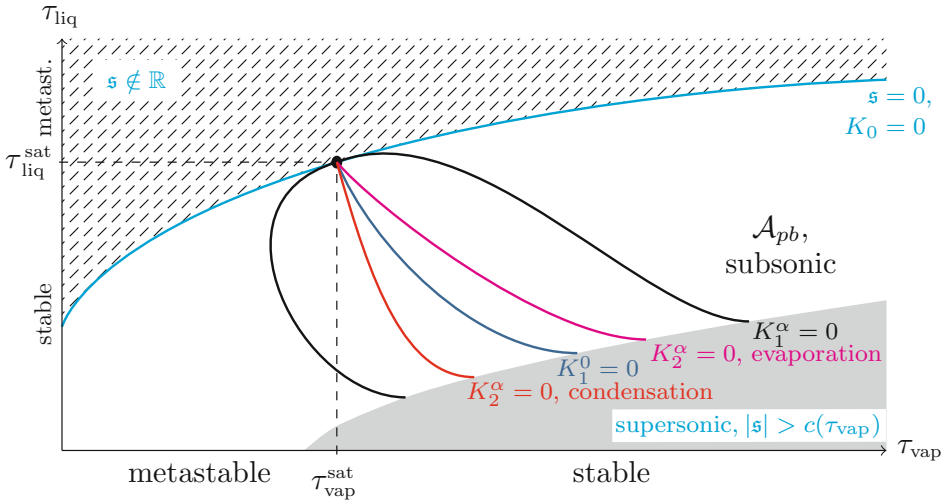


FIG. 1.5. The figure displays the position of the set $\mathcal{A}_{pb}$ together with the graphs of a possible pairs of kinetic functions corresponding to null clines of K, see Remark 1.2.2.

type	τ_{L}	τ^*	composition	$\mathcal{L}_1(\tau_{\text{L}}, \tau^*)$
1_L	$\mathcal{A}$	$\mathcal{A}$	1E	$E(\tau_L, \tau^*)$

type	τ^*	τ_{R}	composition	$\mathcal{L}_2(\tau^*, \tau_{\text{R}})$
1_{R}	$\mathcal{A}$	$\mathcal{A}$	2E	$E(\tau^*, \tau_{\text{R}})$

TABLE 1. Definition of the maps $\mathcal{L}_1$, $\mathcal{L}_2$ that determine the Lax curves. The function E is given in Section 1.2.2.

Let initial states $\mathbf{w}_{\text{L}} = (\tau_{\text{L}}, v_{\text{L}})^T \in \mathcal{A} \times \mathbb{R}$, $\mathbf{w}_{\text{R}} = (\tau_{\text{R}}, v_{\text{R}})^T \in \mathcal{A} \times \mathbb{R}$ be given. We define a forward 1-Lax curve

$$\boldsymbol{L}_1[\boldsymbol{w}_{\text{L}}] = \{\mathbf{w}^* = (\tau^*, v^*)^T \,|\, v^* = v_{\text{L}} + \mathcal{L}_1(\tau_{\text{L}}, \tau^*)\},$$

using a map $\mathcal{L}_1 : \mathcal{A} \times \mathcal{A} \to \mathbb{R}$. All states in $\boldsymbol{L}_1[\boldsymbol{w}_{\text{L}}]$ can be connected by a 1-elementary wave to $\mathbf{w}_{\text{L}}$. In the same way we use a map $\mathcal{L}_2 : \mathcal{A} \times \mathcal{A} \to \mathbb{R}$ to fix a backward 2-Lax curve

$$\boldsymbol{L}_2[\boldsymbol{w}_{\text{R}}] = \{\mathbf{w}^* = (\tau^*, v^*)^T \,|\, v^* = v_{\text{R}} + \mathcal{L}_2(\tau^*, \tau_{\text{R}})\}.$$

All states in $\boldsymbol{L}_2[\boldsymbol{w}_{\text{R}}]$ can be connected by a 2-elementary wave to $\mathbf{w}_{\text{R}}$. The definitions of the maps $\mathcal{L}_1, \mathcal{L}_2$ can be found in Table 1.

Proposition 1.2.3 (Properties of the 1-Lax curve). *The maps $\mathcal{L}_1, \mathcal{L}_2 : \mathcal{A} \to \mathbb{R}$ satisfy the following properties.*

(i) *The maps* $\mathcal{L}_1, \mathcal{L}_2$ *are differentiable.*
(ii) *The map*

$$\mathcal{A} \to \mathbb{R}, \qquad \tau^* \mapsto v^* = v_{\mathrm{L}} + \mathcal{L}_1(\tau_{\mathrm{L}}, \tau^*)$$

is differentiable and strictly monotone increasing in $\mathcal{A}$*. The map*

$$\mathcal{A} \to \mathbb{R}, \qquad \tau^* \mapsto v^* = v_{\mathrm{R}} + \mathcal{L}_2(\tau^*, \tau_{\mathrm{R}})$$

is differentiable and strictly monotone decreasing in $\mathcal{A}$*.*

Proof. The statements (i), (ii) can be readily checked using the definition of E. Note that Definition 1.1.1 ensures as in the Eulerian setting that the characteristic fields associated to the flux in $(1.2.3)_1$ are genuinely nonlinear. □

The solution of the Riemann problem 1.2.3 can now be obtained by analyzing the two Lax curves on intersection points.

Theorem 1.2.4 (Existence). *For any states* $\mathbf{w}_{\mathrm{L}}, \mathbf{w}_{\mathrm{R}} \in \mathcal{A} \times \mathbb{R}$ *the curves* $\mathrm{L}_1[\mathbf{w}_{\mathrm{L}}]$ *and* $\mathrm{L}_2[\mathbf{w}_{\mathrm{R}}]$ *have a unique intersection point* $(\tau^*, v^*)^T \in \mathcal{A} \times \mathbb{R}$*.*

The Riemann problem 1.2.3 *has a self-similar entropy solution* $\boldsymbol{w} = \boldsymbol{w}(\xi, t) \in \mathcal{A} \times \mathbb{R}$ *that is composed of a* 1*-elementary wave connecting the left initial state with the intersection point* $(\tau^*, v^*)^T$ *and a wave connecting* $(\tau^*, v^*)^T$ *to the right initial state by a* 2*-elementary wave, with* $v^* = v_{\mathrm{L}} + \mathcal{L}_1(\tau_{\mathrm{L}}, \tau^*) = v_{\mathrm{R}} + \mathcal{L}_2(\tau^*, \tau_{\mathrm{R}})$*.*

The function $\mathbf{w}$ is by construction a self-similar entropy solution to Riemann problem 1.2.3 because elementary waves obey the entropy condition.

Proof of Theorem 1.2.4. By Definition 1.1.1 the maps associated to the Lax curves satisfy

$$\lim_{\tau \to \bar{\tau}} \mathcal{L}_1(\tau_{\mathrm{L}}, \tau) = -\infty, \quad \lim_{\tau \to \bar{\tau}} \mathcal{L}_2(\tau, \tau_{\mathrm{R}}) = \infty,$$
$$\lim_{\tau \to \infty} \mathcal{L}_1(\tau_{\mathrm{L}}, \tau) = \infty, \quad \lim_{\tau \to \infty} \mathcal{L}_2(\tau, \tau_{\mathrm{R}}) = -\infty.$$

Proposition 1.2.3 ensures with these properties that the function

$$f(\tau) = v_{\mathrm{R}} - v_{\mathrm{L}} + \mathcal{L}_2(\tau, \tau_{\mathrm{R}}) - \mathcal{L}_1(\tau_{\mathrm{L}}, \tau)$$

is continuous, strictly monotone decreasing, and onto $\mathbb{R}$. Thus, a unique number $\tau^* \in (\bar{\tau}, \infty)$ exists such that $f(\tau^*) = 0$. This implies the statement of the theorem. □

1.2.4. The Riemann problem for two-phase flow. Let us now consider a two-phase pressure p for an SI model from Definition 1.1.2 and a kinetic relation (1.2.11). For the rest of Section 1.2.4 we suppose that Assumption 1.2.1 holds true with a pair of kinetic functions $(k_{\mathrm{c}}, k_{\mathrm{e}})$. We will analyze the Riemann problem (1.2.3) with specific volume states $\tau_{\mathrm{L}} \in \mathcal{A}_{\mathrm{liq}}$ and $\tau_{\mathrm{R}} \in \mathcal{A}_{\mathrm{vap}}$. The strategy for solving is similar as in the one-phase case. We will construct a forward and a backward curve, looking for a unique intersection. The curves will be more complicated because they will be composed of several waves including phase boundaries. An additional difficulty stems from the fact that the state space consists of two open sets and the specific

volume components of all waves are forbidden to take values in the spinodal region $\mathcal{A}_{\text{spinodal}}$.

We start with the forward 1-Lax curve emitting from the left state $\boldsymbol{w}_{\text{L}} = (\tau_{\text{L}}, v_{\text{L}})^T \in \mathcal{A}_{\text{liq}} \times \mathbb{R}$. The definition of the (generalized) forward 1-Lax curve

$$\boldsymbol{L}_1[\boldsymbol{w}_{\text{L}}] = \{\mathbf{w}^* = (\tau^*, v^*)^T \,|\, v^* = v_{\text{L}} + \mathcal{L}_1(\tau_{\text{L}}, \tau^*)\}$$

is again based on a map $\mathcal{L}_1 : \mathcal{A}_{\text{liq}} \times \mathcal{A}_{\text{liq}} \cup [\tau_{\text{vap}}^{\text{sat}}, \infty) \to \mathbb{R}$ (for the sake of simplicity we use the same notation as in Section 1.2.3). The definition of $\mathcal{L}_1$ can be found in Table 2.

type	τ_{L}	τ^*	composition	$\mathcal{L}_1(\tau_{\text{L}}, \tau^*)$
1_{L}	$\mathcal{A}_{\text{liq}}$	$\mathcal{A}_{\text{liq}}$	1E	$E(\tau_{\text{L}}, \tau^*)$
2_{L}	$\mathcal{A}_{\text{liq}}$	$[\tau_{\text{vap}}^{\text{sat}}, \tau_{\text{vap}}^{\text{se}}]$	1E-UE	$E(\tau_{\text{L}}, k_{\text{e}}(\tau^*)) \to P(k_{\text{e}}(\tau^*), \tau^*)$
3_{L}	$\mathcal{A}_{\text{liq}}$	$(\tau_{\text{vap}}^{\text{se}}, \infty)$	1E-UE-1R	$E(\tau_{\text{L}}, \tau_{\text{liq}}^{\text{se}}) \to P(\tau_{\text{liq}}^{\text{se}}, \tau_{\text{vap}}^{\text{se}}) \to R(\tau_{\text{vap}}^{\text{se}}, \tau^*)$

TABLE 2. Definition of the map $\mathcal{L}_1$ that determines the 1-Lax curve $\boldsymbol{L}_1[\boldsymbol{w}_{\text{L}}]$. The resulting (multiple) waves for left and right trace specific volume values τ_{L} and τ^* are composed of the waves given in the fourth column (from left to right): 1E stands for 1-elementary wave, 1R for 1-rarefaction wave, UE for subsonic (undercompressive) evaporation wave. The functions E, P, S and R are given in Section 1.2.2.

We summarize the main properties of the 1-Lax curve in a proposition. Note that the statements (iv),(vi) verify the well-posedness of the composite structures in Table 2.

Proposition 1.2.5 (Properties of the generalized 1-Lax curve). *Let a left state $(\tau_{\text{L}}, v_{\text{L}})^T \in \mathcal{A}_{\text{liq}} \times \mathbb{R}$ and the map $\mathcal{L}_1 : \mathcal{A}_{\text{liq}} \times \mathcal{A}_{\text{liq}} \cup [\tau_{\text{vap}}^{\text{sat}}, \infty) \to \mathbb{R}$ of Table 2 be given. Then the following properties hold.*

(i) *The map $\mathcal{L}_1$ is continuous.*

(ii) *The map*

$$\mathcal{A}_{\text{liq}} \cup [\tau_{\text{vap}}^{\text{sat}}, \infty) \to \mathbb{R}, \qquad \tau^* \mapsto v^* = v_{\text{L}} + \mathcal{L}_1(\tau_{\text{L}}, \tau^*)$$

is differentiable and strictly monotone increasing in $\mathcal{A}_{\text{liq}}$ and in $[\tau_{\text{vap}}^{\text{sat}}, \infty)$.

(iii) *It holds that $\mathcal{L}_1(\tau_{\text{L}}, \tau_{\text{liq}}^{\text{sat}}) = \mathcal{L}_1(\tau_{\text{L}}, \tau_{\text{vap}}^{\text{sat}})$.*

(iv) *All propagation speeds are negative. For waves of type 2_{L} and type 3_{L} the phase transition propagates faster than the elementary wave in the liquid phase and slower than the rarefaction wave connecting to τ^* in wave type 3_{L}.*

(v) *Evaporation waves are either subsonic or sonic.*

(vi) *The speed of an evaporation wave is limited by the sound speed $-c(\tau_{\text{vap}}^{\text{se}})$.*

Proof. (i) By definition, the map $\mathcal{L}_1$ is piecewise continuous. It is readily checked with Table 2, that also the transition from one domain of definition to another is continuous.

(ii) Note that $\mathcal{L}_1$ is piecewise smooth. The critical point is $\tau^* = \tau^{\mathrm{se}}_{\mathrm{vap}}$. A short calculation gives

$$\lim_{\tau^*\to\tau^{\mathrm{se}}_{\mathrm{vap}}} \frac{dS}{d\tau^*}(\tau_{\mathrm{L}}, k_{\mathrm{e}}(\tau^*)) = 0, \qquad \lim_{\tau^*\to\tau^{\mathrm{se}}_{\mathrm{vap}}} \frac{dR}{d\tau^*}(\tau_{\mathrm{L}}, k_{\mathrm{e}}(\tau^*)) = 0 \text{ with } k'_{\mathrm{e}}(\tau^{\mathrm{se}}_{\mathrm{vap}}) = 0,$$

$$\lim_{\tau^*\to\tau^{\mathrm{se}}_{\mathrm{vap}}} \frac{dS}{d\tau^*}(\tau^{\mathrm{se}}_{\mathrm{vap}}, \tau^*) = c(\tau^{\mathrm{se}}_{\mathrm{vap}}), \quad \lim_{\tau^*\to\tau^{\mathrm{se}}_{\mathrm{vap}}} \frac{dR}{d\tau^*}(\tau^{\mathrm{se}}_{\mathrm{vap}}, \tau^*) = c(\tau^{\mathrm{se}}_{\mathrm{vap}}) \text{ and}$$

$$\lim_{\tau^*\to\tau^{\mathrm{se}}_{\mathrm{vap}}} \frac{dP}{d\tau}(k_{\mathrm{e}}(\tau^*), \tau^*) = c(\tau^{\mathrm{se}}_{\mathrm{vap}}) \text{ with } k'_{\mathrm{e}}(\tau^{\mathrm{se}}_{\mathrm{vap}}) = 0 \text{ and } \left|\mathfrak{s}_{\mathrm{e}}(\tau^{\mathrm{sc}}_{\mathrm{liq}}, \tau^{\mathrm{sc}}_{\mathrm{vap}})\right| = c(\tau^{\mathrm{sc}}_{\mathrm{vap}})$$

for the functions S, R and P, introduced in Section 1.2.2. Thus, the derivatives of a wave of type 2_{L} and type 3_{L} coincide in $\tau^{\mathrm{se}}_{\mathrm{vap}}$. The functions S and R are strictly monotone increasing with respect to the second argument. A short calculation shows that $\mathcal{L}_1$ is strictly monotone increasing also for a wave of type 2_{L}, since $k'_{\mathrm{e}} < 0$.

(iii) The condition holds, since $P(\tau^{\mathrm{sat}}_{\mathrm{liq}}, \tau^{\mathrm{sat}}_{\mathrm{vap}}) = 0$.

(iv)–(vi) By definition, all waves of the first family have non-positive propagation speeds. The speed of the evaporation wave is between $-c(\tau^{\mathrm{se}}_{\mathrm{vap}})$ and 0. Due to the properties of a two-phase pressure waves in the liquid phase propagate faster (in absolute values) than the vapour sound speed. The phase transition in wave type 3_{L} is sonic and the vapour rarefaction wave is attached. □

The backward curve may contain a condensation wave. Condensation waves change from subsonic to supersonic or vice versa in the point $\tau^{\mathrm{sc}}_{\mathrm{liq}}$. Therefore the structure of the generalized 2-Lax curve

$$\boldsymbol{L}_2[\boldsymbol{w}_{\mathrm{R}}] = \{\mathbf{w}^* = (\tau^*, v^*)^T \,|\, v^* = v_{\mathrm{R}} + \mathcal{L}_2(\tau_{\mathrm{R}}, \tau^*)\},$$

which is defined from some map $\mathcal{L}_2 : (\tau^{\min}_{\mathrm{liq}}, \tau^{\mathrm{sat}}_{\mathrm{liq}}] \cup \mathcal{A}_{\mathrm{vap}} \times \mathcal{A}_{\mathrm{vap}} \to \mathbb{R}$ is more intricate.

The next statements introduce further switching states in $\mathcal{A}$. The proofs are simple consequences of Definition 1.1.2. In particular we introduce values $\hat{\tau}$, $\check{\tau}$ and a function g_s. The value $\hat{\tau}$ is such that the pressure function has the same slope in τ_{R} as the chord from $(\hat{\tau}, p(\hat{\tau}))$ to $(\tau_{\mathrm{R}}, p(\tau_R))$. The value $\check{\tau}$ is such that the points $(\check{\tau}, p(\check{\tau}))$, $(\tau^{\mathrm{sat}}_{\mathrm{vap}}, p(\tau^{\mathrm{sat}}_{\mathrm{vap}}))$, $(\tau_{\mathrm{R}}, p(\tau_{\mathrm{R}}))$ lie on one straight line. The function g_s is determined such that the pressure function has the same slope in $g_s(\tau)$ as the chord from $(\tau, p(\tau))$ to $(g_s(\tau), p(g_s(\tau)))$.

Lemma 1.2.6 (The values $\hat{\tau}$ and $\check{\tau}$). *For a fixed $\tau_{\mathrm{R}} \in (\tau^{\min}_{\mathrm{vap}}, \tau^{\mathrm{sc}}_{\mathrm{vap}}]$ there exists a unique $\hat{\tau} \in \mathcal{A}_{\mathrm{liq}}$, such that*

$$p'(\tau_{\mathrm{R}}) = \frac{p(\tau_{\mathrm{R}}) - p(\hat{\tau})}{\tau_{\mathrm{R}} - \hat{\tau}}, \tag{1.2.15}$$

or equivalently $\lambda_2(\tau_{\mathrm{R}}) = \mathfrak{s}_{\mathrm{c}}(\hat{\tau}, \tau_{\mathrm{R}})$ holds. Moreover, $\hat{\tau} \in (\tau^{\min}_{\mathrm{liq}}, \tau^{\mathrm{sc}}_{\mathrm{liq}}]$.

type	τ^*	τ_{R}	composition	$\mathcal{L}_2(\tau^*, \tau_{\mathrm{R}})$
1_{R}	$\mathcal{A}_{\mathrm{vap}}$	$\mathcal{A}_{\mathrm{vap}}$	2E	$E(\tau^*, \tau_{\mathrm{R}})$
2_{R}	$(\tau_{\mathrm{liq}}^{\min}, \hat{\tau}]$	$(\tau_{\mathrm{vap}}^{\min}, \tau_{\mathrm{vap}}^{\mathrm{sc}}]$	SSC	$P(\tau^*, \tau_{\mathrm{R}})$
3_{R}	$(\hat{\tau}, \tau_{\mathrm{liq}}^{\mathrm{sc}})$	$(\tau_{\mathrm{vap}}^{\min}, \tau_{\mathrm{vap}}^{\mathrm{sc}}]$	SC-2R	$P(\tau^*, g_s(\tau^*)) \to R(g_s(\tau^*), \tau_{\mathrm{R}})$
4_{R}	$[\tau_{\mathrm{liq}}^{\mathrm{sc}}, \tau_{\mathrm{liq}}^{\mathrm{sat}}]$	$(\tau_{\mathrm{vap}}^{\min}, \tau_{\mathrm{vap}}^{\mathrm{sc}}]$	KC-2E	$P(\tau^*, k_{\mathrm{c}}(\tau^*)) \to E(k_{\mathrm{c}}(\tau^*), \tau_{\mathrm{R}})$
5_{R}	$(\tau_{\mathrm{liq}}^{\min}, \check{\tau}]$	$(\tau_{\mathrm{vap}}^{\mathrm{sc}}, \infty)$	SSC	$P(\tau^*, \tau_{\mathrm{R}})$
6_{R}	$(\check{\tau}, \tau_{\mathrm{liq}}^{\mathrm{sat}}]$	$(\tau_{\mathrm{vap}}^{\mathrm{sc}}, \infty)$	KC-2S	$P(\tau^*, k_{\mathrm{c}}(\tau^*)) \to S(k_{\mathrm{c}}(\tau^*), \tau_{\mathrm{R}})$

TABLE 3. Definition of $\mathcal{L}_2$ that determines $\boldsymbol{L}_2[\boldsymbol{w}_{\mathrm{R}}]$. The waves for left and right trace specific volume values τ^* and τ_{R} are composed of the waves given in the fourth column (from left to right): 2E stands for 2-elementary wave, SC for sonic condensation, SSC for supersonic condensation and KC for stands for a subsonic condensation wave. The functions E, P, R are given in Section 1.2.2.

On the other hand, for fixed $\tau_{\mathrm{R}} > \tau_{\mathrm{vap}}^{\mathrm{sc}}$, there exists a unique $\check{\tau} \in \mathcal{A}_{\mathrm{liq}}$, such that

$$\frac{p(k_{\mathrm{c}}(\check{\tau})) - p(\check{\tau})}{k_{\mathrm{c}}(\check{\tau}) - \check{\tau}} = \frac{p(\tau_{\mathrm{R}}) - p(\check{\tau})}{\tau_{\mathrm{R}} - \check{\tau}},$$

or equivalently $\mathfrak{s}_{\mathrm{c}}(\check{\tau}, k_{\mathrm{c}}(\check{\tau})) = \mathfrak{s}_2(k_{\mathrm{c}}(\check{\tau}), \tau_{\mathrm{R}})$ holds. Moreover, $\check{\tau} \in (\tau_{\mathrm{liq}}^{\mathrm{sc}}, \tau_{\mathrm{liq}}^{\mathrm{sat}})$.

At the value $\hat{\tau}$, a supersonic condensation wave (see wave of type 2_{R} in Table 3) splits up into a sonic condensation wave and a 2-rarefaction wave. At the value $\check{\tau}$, a supersonic condensation wave (see wave of type 5_{R}) breaks into a subsonic condensation wave and a 2-shock wave. Waves of type 3_{R} are composed of a sonic condensation wave and an attached 2-rarefaction wave, cf. Table 3. The following lemma is helpful to find the characteristic point.

Lemma 1.2.7 (The function g_s). *For any given $\tau_{\mathrm{R}} \in (\tau_{\mathrm{vap}}^{\min}, \tau_{\mathrm{vap}}^{\mathrm{sc}}]$, equation (1.2.15) provides a unique $\hat{\tau} \in (\tau_{\mathrm{liq}}^{\min}, \tau_{\mathrm{liq}}^{\mathrm{sc}}]$. There exists a continuous monotone increasing function $g_s : [\hat{\tau}, \tau_{\mathrm{liq}}^{\mathrm{sc}}] \to [\tau_{\mathrm{R}}, \tau_{\mathrm{vap}}^{\mathrm{sc}}]$, $\tau \mapsto g_s(\tau)$ such that*

$$p'(g_s(\tau)) = \frac{p(g_s(\tau)) - p(\tau)}{g_s(\tau) - \tau},$$

or equivalently $\lambda_2(g_s(\tau)) = \mathfrak{s}_{\mathrm{c}}(g_s(\tau), \tau)$ holds.

Finally, we can define the map $\mathcal{L}_2$, see Table 3 below. Analogously to Proposition 1.2.5 we have

Proposition 1.2.8 (Properties of the generalized Lax curve $\mathcal{L}_2$). *Let a right state $(\tau_{\mathrm{R}}, v_{\mathrm{R}})^T \in \mathcal{A}_{\mathrm{vap}} \times \mathbb{R}$ and the map $\mathcal{L}_2 : (\tau_{\mathrm{liq}}^{\min}, \tau_{\mathrm{liq}}^{\mathrm{sat}}] \cup \mathcal{A}_{\mathrm{vap}} \times \mathcal{A}_{\mathrm{vap}} \to \mathbb{R}$ of Table 3 be given. Then the following properties hold.*

(i) *The map $\mathcal{L}_2$ is continuous.*
(ii) *The map*

$$(\tau_{\mathrm{liq}}^{\min}, \tau_{\mathrm{liq}}^{\max}) \cup \mathcal{A}_{\mathrm{vap}} \to \mathbb{R}, \qquad \tau^* \mapsto v^* = v_{\mathrm{R}} + \mathcal{L}_2(\tau^*, \tau_{\mathrm{R}})$$

is differentiable and strictly monotone decreasing in $(\tau_{\mathrm{liq}}^{\min}, \tau_{\mathrm{liq}}^{\max})$ and in $\mathcal{A}_{\mathrm{vap}}$.
(iii) *It holds that $\mathcal{L}_2(\tau_{\mathrm{liq}}^{\mathrm{sat}}, \tau_{\mathrm{R}}) = \mathcal{L}_2(\tau_{\mathrm{vap}}^{\mathrm{sat}}, \tau_{\mathrm{R}})$.*
(iv) *All propagation speeds are positive. In wave 3_{R}, 4_{R} and 6_{R}, the phase transition propagates slower than the elementary wave in the vapour phase.*

Proof. (i) The map $\mathcal{L}_2$ is piecewise continuous and it is readily checked with Table 3, that also the transition from one domain of definition to another one is continuous.

(ii) Note that $\mathcal{L}_2$ is piecewise smooth. The critical point in the transition of wave type 2_{R} to wave type 3_{R} is $\tau^* = \hat{\tau}$, in the transition of wave type 3_{R} to wave type 4_{R} it is $\tau^* = \tau_{\mathrm{liq}}^{\mathrm{se}}$ and from type 5_{R} to type 6_{R} it is $\tau^* = \check{\tau}$. For later use we derive

$$\begin{aligned}
\frac{dP}{d\tau}(\tau, g(\tau)) &= \frac{(g'(\tau) - 1)\, \mathfrak{s}_{\mathrm{c}}(\tau, g(\tau))}{2} + \frac{c^2(g(\tau))\, g'(\tau) - c^2(\tau)}{2\, \mathfrak{s}_{\mathrm{c}}(\tau, g(\tau))}, \\
\frac{dS}{d\tau}(g(\tau), \tau_{\mathrm{R}}) &= -\frac{g'(\tau)\, \mathfrak{s}_2(g(\tau), \tau_{\mathrm{R}})}{2} - \frac{c^2(g(\tau))\, g'(\tau)}{2\, \mathfrak{s}_2(g(\tau), \tau_{\mathrm{R}})}
\end{aligned}$$

for some smooth function g with $\tau < g(\tau) < \tau_{\mathrm{R}}$, the sound speed c in (1.1.11) and propagation speeds in Section 1.2.2. Furthermore, there holds $\frac{dR}{d\tau}(g(\tau), \tau_{\mathrm{R}}) = -c(g(\tau))\, g'(\tau)$.

We first check the limit $\tau^* \to \hat{\tau}$ and $\tau_{\mathrm{R}} \in (\tau_{\mathrm{vap}}^{\min}, \tau_{\mathrm{vap}}^{\mathrm{sc}}]$. Note that $g_s(\hat{\tau}) = \tau_{\mathrm{R}}$ and $\mathfrak{s}_{\mathrm{c}}(\hat{\tau}, \tau_{\mathrm{R}}) = c(\tau_{\mathrm{R}})$ with Lemma 1.2.7. With the choice $g = g_s$ we find

$$\begin{aligned}
\lim_{\tau^* \to \hat{\tau}} \frac{dP}{d\tau^*}(\tau^*, \tau_{\mathrm{R}}) &= \frac{-1}{2} \left(c(\tau_{\mathrm{R}}) + \frac{c^2(\hat{\tau})}{c(\tau_{\mathrm{R}})} \right), \\
\lim_{\tau^* \to \hat{\tau}} \frac{dR}{d\tau^*}(g_s(\tau^*), \tau_{\mathrm{R}}) &= -g_s'(\hat{\tau})\, c(\tau_{\mathrm{R}}), \\
\lim_{\tau^* \to \hat{\tau}} \frac{dP}{d\tau^*}(\tau^*, g_s(\tau^*)) &= \frac{-1}{2} \left(c(\tau_{\mathrm{R}}) + \frac{c^2(\hat{\tau})}{c(\tau_{\mathrm{R}})} \right) + g_s'(\hat{\tau})\, c(\tau_{\mathrm{R}}).
\end{aligned}$$

Thus, the derivatives of a wave of type 2_{R} and a wave of type 3_{R} coincide in $\tau^* = \hat{\tau}$.

Now we check the limit $\tau^* \to \tau_{\mathrm{liq}}^{\mathrm{sc}}$ at $\tau_{\mathrm{R}} \in (\tau_{\mathrm{vap}}^{\min}, \tau_{\mathrm{vap}}^{\mathrm{sc}}]$. Here, it holds $k_{\mathrm{c}}(\tau_{\mathrm{liq}}^{\mathrm{sc}}) = g_s(\tau_{\mathrm{liq}}^{\mathrm{sc}}) = \tau_{\mathrm{vap}}^{\mathrm{sc}}$ and $\mathfrak{s}_{\mathrm{c}}(\tau_{\mathrm{liq}}^{\mathrm{sc}}, \tau_{\mathrm{vap}}^{\mathrm{sc}}) = c(\tau_{\mathrm{vap}}^{\mathrm{sc}})$ with Assumption 1.2.1. We find

$$\begin{aligned}
\lim_{\tau^* \to \tau_{\mathrm{liq}}^{\mathrm{sc}}} \frac{dP}{d\tau^*}(\tau^*, k_{\mathrm{c}}(\tau^*)) &= \frac{-1}{2} \left(c(\tau_{\mathrm{vap}}^{\mathrm{sc}}) + \frac{c^2(\tau_{\mathrm{liq}}^{\mathrm{sc}})}{c(\tau_{\mathrm{vap}}^{\mathrm{sc}})} \right) + c(\tau_{\mathrm{vap}}^{\mathrm{sc}})\, k_{\mathrm{c}}'(\tau_{\mathrm{liq}}^{\mathrm{sc}}), \\
\lim_{\tau^* \to \tau_{\mathrm{liq}}^{\mathrm{sc}}} \frac{dR}{d\tau^*}(k_{\mathrm{c}}(\tau^*), \tau_{\mathrm{R}}) &= \lim_{\tau^* \to \tau_{\mathrm{liq}}^{\mathrm{sc}}} \frac{dS}{d\tau^*}(k_{\mathrm{c}}(\tau^*), \tau_{\mathrm{R}}) = -c(\tau_{\mathrm{vap}}^{\mathrm{sc}})\, k_{\mathrm{c}}'(\tau_{\mathrm{liq}}^{\mathrm{sc}}).
\end{aligned}$$

The term with k_c' cancels in wave type 4_R such that the result is independent of the function k_c. The same can be done for wave type 3_R replacing k_c by g_s. This leads to the same result and thus the derivatives coincide in $\tau^* = \tau_{liq}^{sc}$.

Finally, we have to check the limit $\tau^* \to \check{\tau}$ and $\tau_R \in (\tau_{vap}^{sc}, \infty)$. With Lemma 1.2.6, it holds $\mathfrak{s}_c(\check{\tau}, k_c(\check{\tau})) = \mathfrak{s}_2(k_c(\check{\tau}), \tau_R) = \mathfrak{s}_c(\check{\tau}, \tau_R)$. With the above derivatives, we find that the limits from both sides (type 5_R and type 6_R) are

$$\lim_{\tau^* \to \check{\tau}} \frac{d\mathcal{L}_2}{d\tau^*}(\tau^*, \tau_R) = \frac{-1}{2}\left(\mathfrak{s}_c(\check{\tau}, \tau_R) + \frac{-c^2(\check{\tau})}{\mathfrak{s}_c(\check{\tau}, \tau_R)}\right).$$

Monotonicity: the functions E and P are strictly decreasing with respect to the first argument, thus for wave type 1_R, type 2_R and type 5_R, there is nothing to do.

Consider $\frac{d\mathcal{L}_2}{d\tau^*}(\tau^*, \tau_R)$ in case of wave type 3_R. All terms with g_s' cancel out since $\mathfrak{s}_c(\tau^*, g_s(\tau^*)) = c(g_s(\tau^*))$ holds. The remaining terms are negative such that $\mathcal{L}_2(\cdot, \tau_R)$ is a strictly decreasing function. The same holds for wave type 4_R with $k_c(\tau^*) > \tau_R$. The wave is composed of a condensation wave and an attached 2-rarefaction wave, cf. wave type 3_R, and all terms with k_c' cancel out.

In wave type 4_R with $k_c(\tau^*) < \tau_R$ and type 6_R, the function k_c is monotonously decreasing and the term $\mathfrak{s}_c + c^2(\tau^*)/\mathfrak{s}_c$ is positive. Thus, it remains to demonstrate that

$$\mathfrak{s}_c(\tau^*, k_c(\tau^*)) + \frac{c^2(k_c(\tau^*))}{\mathfrak{s}_c(\tau^*, k_c(\tau^*))} - \mathfrak{s}_2(k_c(\tau^*), \tau_R^*) - \frac{c^2(k_c(\tau^*))}{\mathfrak{s}_2(k_c(\tau^*), \tau_R)} \geq 0.$$

We skip the dependencies and re-arrange the inequality: $(\mathfrak{s}_2 - \mathfrak{s}_c)\left(\frac{c^2}{\mathfrak{s}_c\,\mathfrak{s}_2} - 1\right) \geq 0$. This is true since the speeds in waves of type 4_R and type 6_R satisfy $c > \mathfrak{s}_2 \geq \mathfrak{s}_c$. Thus, $\mathcal{L}_2(\cdot, \tau_R)$ is a strictly decreasing function.

(iii) The condition holds due to $P(\tau_{liq}^{sat}, \tau_{vap}^{sat}) = 0$ and (iv) is obvious. □

The solution of the Riemann problem (1.2.3) can now be obtained by analyzing the two generalized Lax curves on intersection points.

Theorem 1.2.9 (Existence). *Consider a two-phase pressure p for an SI model and a driving force $\mathcal{K}$ such that Assumption* 1.2.1 *holds.*

For any states $\mathbf{w}_L \in \mathcal{A}_{liq} \times \mathbb{R}$ and $\mathbf{w}_R \in \mathcal{A}_{vap} \times \mathbb{R}$ the curves $\boldsymbol{L}_1[\mathbf{w}_L]$ and $\boldsymbol{L}_2[\mathbf{w}_R]$ have a unique intersection point $(\tau^*, v^*)^T \in ((\tau_{liq}^{min}, \tau_{liq}^{sat}] \cup (\tau_{vap}^{sat}, \infty)) \times \mathbb{R}$.

The Riemann problem (1.2.3) *has a self-similar entropy solution $\boldsymbol{w} = \boldsymbol{w}(\xi, t) \in \mathcal{A} \times \mathbb{R}$ that is composed of a wave connecting the left initial state with the intersection point $(\tau^*, v^*)^T$ according to Table 2 and a wave connecting $(\tau^*, v^*)^T$ to the right initial state according to Table 3, with $v^* = v_L + \mathcal{L}_1(\tau_L, \tau^*) = v_R + \mathcal{L}_2(\tau^*, \tau_R)$.*

The function $\mathbf{w}$ is a self-similar entropy solution of Riemann problem (1.2.3) by definition: any wave is entropy consistent as explained in Section 1.2.2. It contains exactly one admissible phase transition. Note that $\mathbf{w}$ provides also an entropic SI solution in the sense of Definition 1.1.10 for the case $d = 1$ if the phase boundary is subsonic. The global solution of the Riemann problem in Theorem 1.2.9 requires

also sonic and even supersonic phase boundaries. This fact might indicate that the constraints on the kinetic relation are still not realistic. Note that in our approach subsonic phase transitions are preferred, whenever this is possible.

Proof of Theorem 1.2.9. First we see that $\tau^* \notin (\tau_{\text{liq}}^{\text{sat}}, \tau_{\text{vap}}^{\text{sat}})$, such that we can exclude this interval from our consideration.

The maps associated to the generalized Lax curves satisfy

$$\lim_{\tau \to \tau_{\text{liq}}^{\min}} \mathcal{L}_1(\tau_{\text{L}}, \tau) = -\infty, \quad \lim_{\tau \to \tau_{\text{liq}}^{\min}} \mathcal{L}_2(\tau, \tau_{\text{R}}) = +\infty,$$
$$\lim_{\tau \to \infty} \mathcal{L}_1(\tau_{\text{L}}, \tau) = +\infty, \quad \lim_{\tau \to \infty} \mathcal{L}_2(\tau, \tau_{\text{R}}) = -\infty.$$

Set $\Delta = \tau_{\text{vap}}^{\text{sat}} - \tau_{\text{liq}}^{\text{sat}}$. Proposition 1.2.5 and Proposition 1.2.8 ensure, that the function

$$f(\tau) = \begin{cases} v_{\text{R}} - v_{\text{L}} + \mathcal{L}_2(\tau, \tau_{\text{R}}) - \mathcal{L}_1(\tau_{\text{L}}, \tau) & \text{for } \tau \leq \tau_{\text{liq}}^{\text{sat}} \\ v_{\text{R}} - v_{\text{L}} + \mathcal{L}_2(\tau - \Delta, \tau_{\text{R}}) - \mathcal{L}_1(\tau_{\text{L}}, \tau - \Delta) & \text{for } \tau > \tau_{\text{vap}}^{\text{sat}} \end{cases}$$

is continuous and strictly monotone decreasing from $+\infty$ to $-\infty$. Thus, $\tau^* \in (\bar{\tau}, \infty)$ exists such that $f(\tau^*) = 0$. If $\tau \neq \tau_{\text{liq}}^{\text{sat}}$ then τ^* resp. $\tau^* + \Delta$ is the unique solution of $v_{\text{L}} + \mathcal{L}_1(\tau_{\text{L}}, \tau^*) = v_{\text{R}} + \mathcal{L}_2(\tau^*, \tau_{\text{R}})$. If $\tau = \tau_{\text{liq}}^{\text{sat}}$, then also $\tau = \tau_{\text{vap}}^{\text{sat}}$ solves this relation. □

We conclude the section with an example that illustrates the complex wave fan of a two-phase Riemann problem and the effect of different kinetic relations.

Example 1.2.10. In this example we display a prototype solution of (1.2.3) for two different kinetic relations $\mathcal{K}_2^1$ and $\mathcal{K}_1^0$, both corresponding to a pair of kinetic functions (cf. Remark 1.2.2).

We use again the van-der-Waals pressure of Example 1.1.3 and as choice for the initial condition

$$\boldsymbol{w}_{\text{L}} = (0.57, 0)^T \in \mathcal{A}_{\text{liq}} \times \mathbb{R}, \; \boldsymbol{w}_{\text{R}} = (50, 0)^T \in \mathcal{A}_{\text{vap}} \times \mathbb{R},$$

such that the liquid state is in fact metastable. The solid lines in Figure 1.6 show the solutions of (1.2.3). Both solutions are composed of a shock wave followed by an evaporation wave with attached rarefaction wave and a shock wave. In terms of Table 2 and Table 3 the solution is composed of wave type 3_{L} and type 6_{R}. We see that the pressure in the liquid phase is higher for more entropy dissipation while the propagation speed gets slower.

1.3. A finite volume moving mesh method

The numerical approximation of entropic SI solutions of (1.1.13), (1.1.26), (1.1.16) for a two-phase pressure is complicated due to several reasons. As a free boundary value problem it does not only suffice to approximate the bulk quantities but also the interface itself as a geometrical object. The numerical method has to be designed in a way such that the values from the spinodal set $\mathcal{A}_{\text{spinodal}}$ are excluded. In fact, any time step control fails in the presence of spinodal states which produce complex transport speeds, see (1.1.18). The kinetic relation as an extra coupling

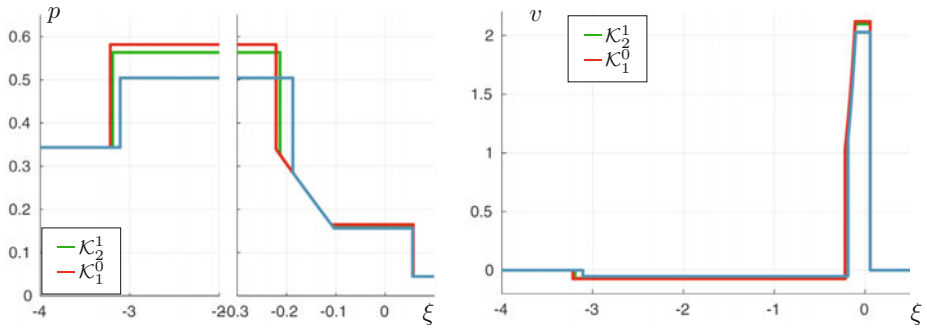


FIG. 1.6. Solution of a two-phase Riemann problem with two different kinetic relations. The left figure shows the pressure and the right one the velocity as functions of the Lagrangian space variable at time $t = 1$.

condition has to be directly integrated into the scheme. Finally, the occurrence of shock waves in the bulk regions requires an upwinding concept for stabilization.

As a first approach let us mention an idea which requires an approximating DI model (as, e.g., considered in Section 2). The regularization parameter is substituted by the mesh parameter. In this way the complexity of the SI ansatz is reduced and one can use quite standard (finite-volume, finite-difference,...) schemes for the numerics. Representative works are, e.g., [58, 59]. The major drawback of this ansatz is that the kinetic relation can only be addressed in an indirect way and it is hardly possible to adjust the mobility constant in Remark 1.1.9 for a specific fluid. Moreover the time step control can usually only be done in an ad hoc manner. The Glimm front tracking/random-choice method is frequently used for tracking fronts in hyperbolic conservation laws, see [12, 55]. For deterministic versions that use an extra tracking of the sharp interface one can consult [9, 10, 28, 67, 85]. The major drawback of all these schemes is the fact that they are not locally conservative. Moving mesh methods can overcome this drawback and this is exactly the strategy we want to present here. Whereas moving meshes have been routinely used for a long time in (compressible) fluid mechanics the computation for phase boundaries has started with [11] in one spatial dimension. We focus here on the multi-dimensional case, most of the material taken from the recent paper [14].

We conclude by mentioning papers on related applications like on the evolution of phase boundaries in solids in [45, 66] or the tracking of undercompressive overshoot waves in porous media [51], where mixed phase volumes are allowed like in the moving-mesh approach in [16] for compressible multiphase flow.

1.3.1. Finite volume schemes on moving meshes. Up to our knowledge finite volume moving mesh methods for conservation laws in one space dimension have been presented firstly by Harten and Hyman [42]. They have chosen the motion of the mesh points in a way such that the numerical diffusion (caused by averaging in the Godunov method) is minimized. In [77] moving mesh methods were also used and

a convergence analysis for one-dimensional moving mesh methods can be found in [75]. The extension of the method to the two-dimensional case is due to [80] and afterwards by many others. None of these works deals with explicit solving a free boundary value problem like we pursue here.

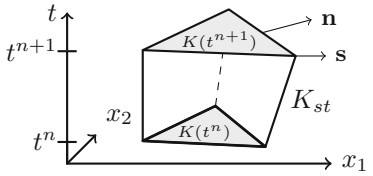


FIG. 1.7. Space-time cell K_{st} for the case $d = 2$.

Let for $N \in \mathbb{N}$ a time partition $t^0 = 0 < t^1 < \cdots < t^N = T$ of $[0,T]$ be given. For the sake of simplicity we assume $D = \mathbb{R}^d$ in Section 1.3.1. We fix some $n \in \{0,\dots,N-1\}$. Consider a prism-like space-time cell $[0,T] \times \Omega \supset K_{st} = \{(t,\mathbf{x}) \,|\, t^n \le t \le t^{n+1}, \mathbf{x} \in K(t)\}$, see Figure 1.7, with $0 \le t^n < t^{n+1} \le T$, $K(t) := \mathrm{conv}(\mathbf{p}_0(t), \mathbf{p}_1(t), \mathbf{p}_2(t))$ for $t \in [t^n, t^{n+1}]$ and time-dependent points $\mathbf{p}_0 = \mathbf{p}_0(t), \dots, \mathbf{p}_d = \mathbf{p}_d(t) \in \mathbb{R}^d$ which evolve linearly in time with speed $\mathbf{s}_0, \dots, \mathbf{s}_d \in \mathbb{R}^d$ according to

$$\mathbf{p}_l(t) = \mathbf{p}_l(t^n) + (t - t^n)\mathbf{s}_l \qquad (l = 0,\dots,d). \tag{1.3.1}$$

Let us first consider the Euler system (1.1.13) in the general conservation law form (1.1.17). Suppose that we have a function $\boldsymbol{u} \in C^1(\mathbb{R}^d \times [0,T), \tilde{\mathcal{U}})$, that satisfies (1.1.17) in the classical sense and let us ignore the interface Γ. Integration of (1.1.17) with respect to the space-time cell K_{st} leads to the component equations

$$\int_{t^n}^{t^{n+1}} \int_{K(t)} \boldsymbol{u}_t \, d\mathbf{x}\, dt + \int_{t^n}^{t^{n+1}} \int_{\partial K(t)} \boldsymbol{F}(\boldsymbol{u};\mathbf{n}) \, d\boldsymbol{z}\, dt = \mathbf{0}.$$

Evaluating the evolution term involves a geometric transport term, i.e.,

$$\begin{aligned} &\int_{t^n}^{t^{n+1}} \int_{K(t)} \boldsymbol{u}_t \, d\mathbf{x}\, dt \\ &\quad = \int_{K(t^{n+1})} \boldsymbol{u}(\cdot, t^{n+1}) \, d\mathbf{x} - \int_{K(t^n)} \boldsymbol{u}(\cdot, t^n) \, d\mathbf{x} - \int_{t^n}^{t^{n+1}} \int_{\partial K(t)} (\mathbf{s}\cdot\mathbf{n}) \boldsymbol{u} \, d\boldsymbol{z}\, dt. \end{aligned} \tag{1.3.2}$$

Here $\mathbf{s} : \partial K(t) \to \mathbb{R}^d$ is the speed of a point $\mathbf{x}(t) = \lambda \mathbf{p}_i(t) + (1-\lambda)\mathbf{p}_j(t) \in \partial K(t)$ with $i \ne j \in \{0,\dots,d-1\}$ and $\lambda \in [0,1]$, i.e., $\mathbf{s}(\mathbf{x}) = \lambda \mathbf{s}_i + (1-\lambda)\mathbf{s}_j$, and $\mathbf{n}(t) : \partial K(t) \to \mathbb{S}^{d-1}$ is the outer unit normal at the boundary of $K(t)$ (for simplicity using the same notation as for the interface). Altogether we get the generalized conservative integral form. This gives the conservation form of (1.1.17) on K_{st},

i.e.,

$$\int_{K(t^{n+1})} \boldsymbol{u}(\cdot, t^{n+1})\, d\mathbf{x} - \int_{K(t^n)} \boldsymbol{u}(\cdot, t^n)\, d\mathbf{x} + \int_{t^n}^{t^{n+1}} \int_{\partial K(t)} \boldsymbol{F}(u; \mathbf{n}) - (\mathbf{s} \cdot \mathbf{n})\boldsymbol{u}\, d\boldsymbol{z}\, dt = \mathbf{0}. \tag{1.3.3}$$

Based on the conservation form (1.3.3) on a space-time cell K_{st} we will derive a finite volume scheme on moving meshes. Let P_d denote the set all d-polygons with positive volume. By a *fixed mesh* with index set $I \subset \mathbb{Z}$ we mean the set $\tau = \{K_i \,|\, K_i \in P_d, i \in I\}$, such that

$$\mathbb{R}^2 = \bigcup_{i \in I} K_i, \quad \mathring{K}_i \cap \mathring{K}_j = \emptyset \,\forall\, i, j \in I, i \neq j,$$

and the following property holds: either $\left|\overline{K_i} \cap \overline{K_j}\right|_{d-1} = 0$ or $\left|\overline{K_i} \cap \overline{K_j}\right|_{d-1} > 0 \Rightarrow \overline{K_i} \cap \overline{K_j}$ is a line segment between two common vertices of K_i and K_j. Here $|\cdot|_e$ denotes the e-dimensional Hausdorff measure for $e = 1, \ldots, d$. We define $S_{i,j} = \overline{K_i} \cap \overline{K_j}$ and call it hyper-surface, if $|S_{i,j}|_{d-1} > 0$, $i \neq j$. The index set of all hyper-surfaces is defined as

$$E = \big\{(i, j) \in I \times I \,|\, |S_{i,j}|_{d-1} > 0\big\}.$$

For $i \in I$, the index set of all neighbors of K_i is given as $N(i) = \{j \in I \,|\, |S_{i,j}|_{d-1} > 0\}$. For each edge $S_{i,j}$ we define $\mathbf{n}_{i,j} \in \mathbb{S}^{d-1}$ as the outer unit vector at $S_{i,j}$ w.r.t. K_i. The mesh width h is defined as

$$h := \max_{(i,j) \in E} \big\{\, |S_{i,j}|_{d-1} \,\big\}.$$

Definition 1.3.1 (Moving mesh). Let a mesh $\tau = \{K_i \,|\, K_i \in P_d, i \in I\}$ on $\mathbb{R}^d$ with an index set I and some interval $[t_1, t_2]$ be given. Assume that for each $i \in I$ there is a continuous function

$$\Phi_i : [t_1, t_2] \to \mathcal{L}_a(K_i, \mathbb{R}^d), t \mapsto \Phi_i^t,$$

with $\Phi_i(t_1) = \mathrm{id}$. Here $\mathcal{L}_a(K_i, \mathbb{R}^d)$ denotes the space of affine mappings from K_i to $\mathbb{R}^d$.
We call $\mathcal{T} = (\tau, \{\Phi_i\}_{i \in I})$ a *moving mesh* for $[t_1, t_2]$, if

$$\forall\, t \in [t_1, t_2] : \{\Phi_i^t(K_i)\}_{i \in I} =: \tau(t) \text{ is a mesh with index set } I.$$

We define the time-dependent elements $K_i(t)$ and the time-dependent hyper-surfaces $S_{i,j}(t)$ of the moving mesh $\mathcal{T} = (\tau, \{\Phi_i\}_{i \in I})$ by

$$K_i(t) := \Phi_i^t(K_i) \text{ and } S_{i,j}(t) := \Phi_i^t(S_{i,j}) = \Phi_j^t(S_{i,j}).$$

Note that Definition 1.3.1 implies that the index set I remains invariant in time.
Suppose we have a moving mesh $(\tau, \{\Phi_i\}_{i \in I})$ of $\mathbb{R}^2$ with time-dependent elements $K_i(t)$, $i \in I$. Then we must introduce (approximate) fluxes to compute the surface integrals in (1.3.3).

Definition 1.3.2 (Numerical and geometrical flux functions). Let $(\tau, \{\Phi_i\}_{i\in I})$ be a moving mesh for the time interval $[t^n, t^{n+1}]$ and denote by $\mathbf{s}^{i,j}$ the speed of the midpoint of the edge $S_{i,j}$.

The Lipschitz continuous functions $\mathbf{g}^n_{i,j} = \mathbf{g}^n_{i,j}(\boldsymbol{u}, \boldsymbol{v})$ and $\mathbf{h}^n_{i,j} = \mathbf{h}^n_{i,j}(\boldsymbol{u}, \boldsymbol{v})$ are called *numerical flux function* and *geometrical flux function* for the system (1.1.17) if they fulfill the following properties.

(i) $\forall\, \boldsymbol{u} \in \mathcal{U}\ \forall\, i \in I\ \forall\, j \in N(i)$:

$$\mathbf{g}^n_{i,j}(\boldsymbol{u}, \boldsymbol{u}) = \boldsymbol{f}(\boldsymbol{u})\cdot\mathbf{n}_{i,j}(t^{n+1/2}) \text{ and } \mathbf{h}^n_{i,j}(\boldsymbol{u}, \boldsymbol{u}) = -\mathbf{n}_{i,j}(t^{n+1/2})\cdot\mathbf{s}^{i,j}\,\boldsymbol{u} \quad \text{(Consistency)} \tag{1.3.4}$$

(ii) $\forall\, \boldsymbol{u}, \boldsymbol{v} \in \mathcal{U}\ \forall\, i \in I\ \forall\, j \in N(i)$:

$$\mathbf{g}^n_{i,j}(\boldsymbol{u}, \boldsymbol{v}) + \mathbf{h}^n_{i,j}(\boldsymbol{u}, \boldsymbol{v}) = -\Big(\mathbf{g}^n_{j,i}(\boldsymbol{v}, \boldsymbol{u}) + \mathbf{h}^n_{j,i}(\boldsymbol{v}, \boldsymbol{u})\Big) \quad \text{(Conservation)}. \tag{1.3.5}$$

Common choices for the numerical flux are for example the Lax–Friedrichs flux, the Godunov-(type) flux or the Roe flux, see, e.g., [54]. The geometrical flux $\mathbf{h}^n_{i,j}$ can be treated in exactly the same manner as the numerical flux with the corresponding flux function $\mathbf{h}(\boldsymbol{u}) = -(\mathbf{n}_{i,j} \cdot \mathbf{s}^{i,j})\boldsymbol{u}$.

Definition 1.3.3 (Finite volume step on moving meshes). Let $\mathcal{T} = (\tau, \{\Phi_i\})$ be a moving mesh with index set I for the time interval $[t^n, t^{n+1}]$ of length $\Delta t^n := t^{n+1} - t^n$. For $\{\boldsymbol{u}^n_i \in \mathcal{U}\}_{i\in I} \subset \mathcal{U}$ the mapping

$$\mathrm{FVS} : (\{\boldsymbol{u}^n_i\}_{i\in I}, \mathcal{T}) \mapsto \{\boldsymbol{u}^{n+1}_i\}_{i\in I}$$

is called *finite volume step* for (1.1.17), if the values $\boldsymbol{u}^{n+1}_i$ are computed from

$$\begin{aligned} &\big|K_i(t^{n+1})\big|\,\boldsymbol{u}^{n+1}_i \\ &\quad = |K_i(t^n)|\,\boldsymbol{u}^n_i - \Delta t^n \sum_{j\in N(i)} \Big|S_{i,j}\left(t^{n+1/2}\right)\Big| \Big(\mathbf{g}^n_{i,j}(\boldsymbol{u}^n_i, \boldsymbol{u}^n_j) + \mathbf{h}^n_{i,j}(\boldsymbol{u}^n_i, \boldsymbol{u}^n_j)\Big), \end{aligned} \tag{1.3.6}$$

with $t^{n+1/2} = t^n + 0.5\Delta t^n$. Moreover, $\mathbf{g}^n_{i,j}$ is a numerical flux function and $\mathbf{h}^n_{i,j}$ is a geometrical flux function.

The resulting scheme on moving meshes is summarized as the following algorithm.

Algorithm 1.3.4 (Finite volume scheme on moving meshes for (1.1.17)**).**

1: **procedure** FINITE VOLUME SCHEME($\boldsymbol{u}_0, T, \mathcal{T}$)
2: $t^0 = 0$, $n = 0$
3: $\{\boldsymbol{u}^0_i\}_{i\in I} = \left\{\frac{1}{|K_i(0)|}\int_{K_i(0)} \boldsymbol{u}_0\, d\mathbf{x}\right\}_{i\in I}$
4: **while** $t^n < T$ **do**
5: $\Delta t^n = \min\limits_{i\in I}\Big(|K_i(t^n)| \cdot |\partial K_i(t^n)|^{-1} \cdot \lambda(\boldsymbol{u}^n_i)^{-1}\Big)$
6: $\{\boldsymbol{u}^{n+1}_i\}_{i\in I} = \mathrm{FVS}(\{\boldsymbol{u}^n_i\}_{i\in I}, \mathcal{T})$
7: $t^{n+1} = t^n + \Delta t^n$, $n = n + 1$

By $\lambda(\boldsymbol{u})$ we denote a positive number that scales with the maximum of the spectrum of $D\boldsymbol{F}(\boldsymbol{u};\mathbf{n})$. It realizes a CFL condition to stabilize the explicit finite volume scheme. We define the approximate solution computed within Algorithm 1.3.4 as the piecewise constant function

$$\boldsymbol{u}_h(\mathbf{x},t) = \boldsymbol{u}_i^n \qquad \text{if } t \in [t^n, t^{n+1}) \text{ and } \mathbf{x} \in K_i(t).$$

In passing we note that the finite volume scheme preserves mass by its construction, precisely we have

$$\int_{\mathbb{R}^d} (\boldsymbol{u}_h(\mathbf{x},t^n) - \boldsymbol{u}_0(\mathbf{x}))\, d\mathbf{x} = 0.$$

Up to now we have not taken care about the approximation of the phase boundary $\Gamma = \Gamma(t)$. Therefore we combine the moving mesh method with a special tracking of the phase boundary.

1.3.2. Finite volume schemes on moving meshes with interface tracking. The basic idea of the final method is to track a discrete interface that consists of hypersurfaces of the moving mesh (e.g., edges for $d = 2$). If we move interface edges such that the position of the phase boundary is tracked, we can treat the interface separately, using a different numerical flux as in the bulk domains. In this way we can avoid any smearing across the interface with values in the spinodal set $\mathcal{A}_{\text{spinodal}}$ due to averaging in standard numerical fluxes.

The computation of the approximate location of the interface is illustrated in Figure 1.8 and will define the moving mesh $\mathcal{T} = (\tau, \{\Phi_i\})$ we will need as an input for the finite volume step in Definition 1.3.3. Therefore we introduce the set of interface edges in addition to a mesh.

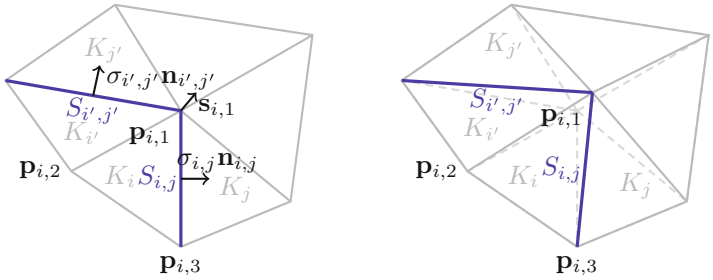


FIG. 1.8. Motion of the interface curve: Mesh at time $t = t^n$ (left figure) and at time $t = t^{n+1}$.

Definition 1.3.5 (Interface edges, approximate interface). For $n \in \{0,\dots,N-1\}$ let a fixed mesh τ^n with index set I and a set $\{\boldsymbol{u}_i^n \in \mathcal{U}\}_{i\in I}$ be given. We define the *interface edge indices* as

$$\mathcal{E} = \mathcal{E}(\tau^n, \{\boldsymbol{u}_i\}_{i\in I}) = \{(i,j) \in E \,|\, \text{The states } \boldsymbol{u}_i^n \text{ and } \boldsymbol{u}_j^n \text{ are in different phases}\}$$

and the *approximate interface*

$$\Gamma_h^n = \Gamma_h^n(\tau^n, \{\boldsymbol{u}_i^n\}_{i\in I}) = \bigcup_{(i,j)\in\mathcal{E}(\tau^n,\{\boldsymbol{u}_i^n\})} S_{i,j}^n.$$

An approximate interface is called *admissible* if it consists of one or more closed curves without any (self-)intersections.

We know that if we have an admissible approximate interface $\Gamma_h^n(\tau^n, \{\boldsymbol{u}_i^n\}_{i\in I})$ each vertex that is part of the interface curve has exactly two incident edges in the mesh τ^n. The key tool is now the interface solver $\mathcal{R}$ for (1.1.13) as defined in Section 1.2. It integrates the information of the kinetic relation into the algorithm. The evolution of the approximate interface Γ_h^n can be described by the following procedure. For each interface edge $S_{i,j}^n$, $(i,j) \in \mathcal{E}$ with $\boldsymbol{u}_i^n \in \tilde{\mathcal{A}}_{\text{liq}} \times \mathbb{R}^d$ and $\boldsymbol{u}_j \in \tilde{\mathcal{A}}_{\text{vap}} \times \mathbb{R}^d$ we consider the Riemann problem (1.2.1) with the choices

$$\mathbf{n} := \mathbf{n}_{i,j}^n,\ \boldsymbol{u}_{\text{L}} := \boldsymbol{u}_i^n, \boldsymbol{u}_{\text{R}} := \boldsymbol{u}_j^n.$$

Using the interface solver from (1.2.2) let us denote its output by by $\sigma_{i,j}^n$, $\mathbf{U}_{i,j}^n$, $\mathbf{V}_{i,j}^n$, i.e.,

$$\big(\sigma_{i,j}^n(\boldsymbol{u}_i^n, \boldsymbol{u}_j^n), \mathbf{U}_{i,j}^n(\boldsymbol{u}_i^n, \boldsymbol{u}_j^n), \mathbf{V}_{i,j}^n(\boldsymbol{u}_i^n, \boldsymbol{u}_j^n)\big) = \mathcal{R}_{\boldsymbol{F}(\cdot;\mathbf{n}_{i,j}^n)}(\boldsymbol{u}_i^n, \boldsymbol{u}_j^n). \tag{1.3.7}$$

Suppose $\mathbf{p}_{i,k}^n$ $(i \in I,\ k = 1, \dots, d)$ being the kth vertex of K_i^n in a mesh τ^n and being part of the approximate interface Γ_h^n. Then $\mathbf{p}_{i,k}^n$ has exactly two incident edges in the mesh, which separate two phases, since Γ_h^n is a set of closed, non-intersecting curves. We call these two incident edges $S_{i,j}^n$ and $S_{i',j'}^n$, see Figure 1.8. We define the speed of $\mathbf{p}_{i,k}^n$ as the weighted average speed computed at $S_{i,j}^n$ and $S_{i',j'}^n$

$$\mathbf{s}_{i,k}^n := \frac{\mathbf{n}_{i,j}^n \sigma_{i,j}^n(\boldsymbol{u}_i^n, \boldsymbol{u}_j^n) \left|S_{i,j}^n\right| + \mathbf{n}_{i',j'}^n \sigma_{i',j'}^n(\boldsymbol{u}_{i'}^n, \boldsymbol{u}_{j'}^n) \left|S_{i',j'}^n\right|}{\left|S_{i,j}^n\right| + \left|S_{i',j'}^n\right|} \tag{1.3.8}$$

and the time-dependent point $\mathbf{p}_{i,k}^n(t)$ as

$$\mathbf{p}_{i,k}^n(t) := \mathbf{p}_{i,k}^n(t^n) + (t - t^n)\mathbf{s}_{i,k}^n.$$

The moving mesh is then defined as $(\tau, \{\Phi_i\}_{i\in I})$ with

$$\Phi_i^t(\mathbf{x}) := \mathbf{x} + (t - t^n)\Big(\mathbf{s}_{i,d}^n + \sum_{k=1,\dots,d-1} \lambda_{i,k}(\mathbf{x})(\mathbf{s}_{i,k}^n - \mathbf{s}_{i,d}^n)\Big), \tag{1.3.9}$$

where $\lambda_{i,k}^n(\mathbf{x})$ denote the barycentric coordinates of $\mathbf{x}$ in K_i^n. We summarize the overall process with

Definition 1.3.6 (Interface motion function). Let τ^n be a mesh with index set I and let $\{\boldsymbol{u}_i^n \in \mathcal{U}\}_{i\in I}$ be a set with elements in $\mathcal{U}$, such that the approximate interface $\Gamma_h^n(\tau^n, \{\boldsymbol{u}_i^n\})$ is admissible.
The mapping

$$\text{IMF} : (t^n, \Delta t^n, \tau^n, \{\boldsymbol{u}_i^n\}_{i\in I}) \mapsto (\tau^n, \{\Phi_i\}_{i\in I})$$

is called *interface motion function*, if Φ_i, $i \in I$, is computed from formulas (1.3.7), (1.3.8) and (1.3.9).

It remains to make precise the choice of the numerical fluxes at the edges of the mesh. Using the interface solver as in (1.3.7) we define

$$\mathbf{g}^n_{i,j}(\boldsymbol{u},\boldsymbol{v}) = \begin{cases} \boldsymbol{F}(\mathbf{U}^n_{i,j}(\boldsymbol{u},\boldsymbol{v}))\cdot \mathbf{n}_{i,j} & \text{if } \boldsymbol{u} \in \tilde{\mathcal{A}}_{\mathrm{liq}} \times \mathbb{R}^d, \boldsymbol{v} \in \tilde{\mathcal{A}}_{\mathrm{vap}} \times \mathbb{R}^d, \\ \boldsymbol{F}(\mathbf{U}^n_{i,j}(\boldsymbol{v},\boldsymbol{u}))\cdot \mathbf{n}_{i,j} & \text{if } \boldsymbol{u} \in \tilde{\mathcal{A}}_{\mathrm{vap}} \times \mathbb{R}^d, \boldsymbol{v} \in \tilde{\mathcal{A}}_{\mathrm{liq}} \times \mathbb{R}^d, \\ \text{any consistent flux} & \text{otherwise,} \end{cases}$$

$$\mathbf{h}^n_{i,j}(\boldsymbol{u},\boldsymbol{v}) = \begin{cases} -\sigma^n_{i,j}(\boldsymbol{u},\boldsymbol{v})\mathbf{U}^n_{i,j}(\boldsymbol{u},\boldsymbol{v}) & \text{if } \boldsymbol{u} \in \tilde{\mathcal{A}}_{\mathrm{liq}} \times \mathbb{R}^d, \boldsymbol{v} \in \tilde{\mathcal{A}}_{\mathrm{vap}} \times \mathbb{R}^d, \\ +\sigma^n_{i,j}(\boldsymbol{v},\boldsymbol{u})\mathbf{U}^n_{i,j}(\boldsymbol{v},\boldsymbol{u}) & \text{if } \boldsymbol{u} \in \tilde{\mathcal{A}}_{\mathrm{vap}} \times \mathbb{R}^d, \boldsymbol{v} \in \tilde{\mathcal{A}}_{\mathrm{liq}} \times \mathbb{R}^d, \\ \text{any consistent flux} & \text{otherwise.} \end{cases} \tag{1.3.10}$$

Note that we used the abbreviation $\boldsymbol{F} = \boldsymbol{F}(\cdot;\mathbf{n}^n_{i,j})$. Obviously the only special choice is done at the discrete interface using there an exact Godunov flux.

Now all tools for the tracking algorithm are given, see Algorithm 1.3.7 below. However, the motion of the interface may cause narrow or even degenerate cells. This will affect the size of the time step and may cause that the method does not reach the final time T. Therefore an additional post-processing step is necessary to produce a new mesh, which conserves the approximate interface but consists of more or less regularly shaped volumes. We will not describe this re-meshing procedure here and refer to [14] and references therein for further informations on this important issue.

Algorithm 1.3.7 (Tracking algorithm).

procedure TRACKING-TYPE ALGORITHM($\boldsymbol{u}_0, T, \tau^0$)
  $t^0 = 0$, $n = 0$
  $\{\boldsymbol{u}^0_i\}_{i\in I^0} = \left\{ \frac{1}{|K_i(0)|} \int_{K_i(0)} \boldsymbol{u}_0 \, d\mathbf{x} \right\}_{i\in I^0}$
  while $t^n < T$ **do**
    $\Delta t^n = \min\limits_{i\in I^n} \left(\min\limits_{i\in I} \left(|K_i(t^n)| \cdot |\partial K_i(t^n)|^{-1} \cdot \lambda(\boldsymbol{u}^n_i)^{-1} \right) \right)$
    $\tau^n = \mathrm{IMF}(t^n, \Delta t, \tau^n, \{\boldsymbol{u}^n_i\}_{i\in I^n})$
    $\{\boldsymbol{u}^{n+1}_i\}_{i\in I^n} = \mathrm{FVS}(\{\boldsymbol{u}^n_i\}_{i\in I^n}, \mathcal{T}^n)$
    $t^{n+1} = t^n + \Delta t$, $n = n+1$

We conclude the section with a formal definition of the approximate solution as discrete counterpart of the sharp-interface solution from Definition 1.1.10.

Definition 1.3.8. Let bounded initial data $\boldsymbol{u}_0$, a time $T > 0$ and a (fixed) mesh τ^0 be given and assume that $\mathcal{T}^n$ are the moving meshes and $\{\boldsymbol{u}^n_i\}_{i\in I^n}$ the values computed within Algorithm 1.3.7.

Then, we define the *numerical approximation* $\boldsymbol{u}_h$ given by Algorithm 1.3.7 as piecewise constant function on the time-dependent elements $K_i(t)$, that is

$$\boldsymbol{u}_h(t,x) = \boldsymbol{u}_i^n \qquad \text{if } t \in [t^n, t^{n+1}) \text{ and } \mathbf{x} \in K_i(t) \text{ for } i \in I^n$$

and the *approximate interface* Γ_h as

$$\Gamma_h(t) = \Gamma_h(\mathcal{T}^n(t), \{\boldsymbol{u}_i^n\}_{i\in I^n}) \qquad \text{if } t \in [t^n, t^{n+1}).$$

1.3.3. Thermodynamical consistency in one spatial dimension. Like for the analytical setting the numerical approximations should satisfy a discrete form of the second law of thermodynamics. Usually this is the first step towards a convergence analysis for conservation laws. We cannot present such results for Algorithm 1.3.7 in arbitrary space dimensions. However a rigorous analytical result is possible for the one-dimensional case using the Godunov flux as a numerical flux (see Theorem 1.3.9 below). For the multi-dimensional case we present numerical evidence in Section 1.3.4.

We consider the planar case given by the (rotated) system (1.2.1) in $D = \mathbb{R}$ with flux $\boldsymbol{F}(\cdot) := \boldsymbol{F}(\cdot;\mathbf{n})$. The associated entropy-entropy flux pair (W, Q) has been introduced in (1.1.20) (for $d = 1$ the entropy flux is a scalar-valued function).

For a given moving mesh $\mathcal{T} = (\tau, \{\Phi_i\}_{i\in I})$ and some time $t \in [0, T]$ the fixed mesh $\tau(t) = \{\Phi_i^t(K_i)\}_{i\in I}$ consists of intervals $K_i(t)$. For the sake of simplicity we choose $I = \mathbb{Z}$ and introduce the geometrical notations

$$(x_{i-1/2}^n, x_{i+\frac12}^n) := K_i(t^n), \quad \mathcal{S}_{i+\frac12}^n := \{(z,t)) \,|\, z \in \partial K_i(t) \cap \partial K_{i+1}(t),\, t \in (t^n, t^{n+1})\}.$$

Furthermore let $\sigma_{i+\frac12}^n = \sigma_{i,i+1}^n$ for the slope of $\mathcal{S}_{i+\frac12}^n$ in the (t,x)-frame. For the one-dimensional setting there is a unique $\iota \in \mathbb{Z}$ such that $\mathcal{S}_{\iota+\frac12}^n$ coincides with the approximate interface Γ_h for all $n \in \mathbb{N}$. For fixed $n \in \mathbb{N}$ we can assume without loss of generality that all other segments $\mathcal{S}_{i+\frac12}^n$ are parallel to the t-axis.

The finite volume moving mesh method as explained in Algorithm 1.3.7 computes the numerical approximation $\boldsymbol{u}_h$ through the family $\{\boldsymbol{u}_i^n\}_{i\in\mathbb{Z}}$ and is given for the planar case by the iteration

$$\begin{aligned} \big|K_i(t^{n+1})\big|\, \boldsymbol{u}_i^{n+1} &= |K_i(t^n)|\, \boldsymbol{u}_i^n - \Delta t^n \Big(\mathbf{g}_{i+\frac12}^n + \mathbf{h}_{i+\frac12}^n - \mathbf{g}_{i-\frac12}^n - \mathbf{h}_{i+\frac12}^n\Big), \\ \boldsymbol{u}_i^0 &= \frac{1}{|K_i(t^n)|} \int_{x_{i-\frac12}^0}^{x_{i+\frac12}^0} \boldsymbol{u}_0(x)\, dx. \end{aligned} \tag{1.3.11}$$

As the numerical flux for $\boldsymbol{F}$ we choose the Godunov flux. Using the self-similarity of the Riemann problem (1.2.1) let $\mathbf{U}[x/t; \boldsymbol{u}_L, \boldsymbol{v}_R] := \mathbf{U}(x,t)$ denote the entropy solution of (1.2.1) which exists due to Theorem 1.2.9. Then the general Godunov flux is given for $s \in \mathbb{R}$ and $\boldsymbol{u}, \boldsymbol{v} \in \tilde{\mathcal{A}} \times \mathbb{R}^d$ by

$$\mathbf{g}(s, \boldsymbol{u}, \boldsymbol{v}) = \boldsymbol{F}\big(\mathbf{U}[s; \boldsymbol{u}, \boldsymbol{v}]\big), \tag{1.3.12}$$

i.e., it evaluates the constant value of the exact flux on the space-time segment $\mathcal{S}_{i+\frac12}^n$. Note that (1.3.12) is well posed: even if the solution of the Riemann problem

jumps across the edge $\mathcal{S}^n_{i+\frac12}$, the Rankine–Hugoniot conditions (1.1.15) hold and make the flux unique. With definition (1.3.12) we determine the specific fluxes in (1.3.11) by

$$\mathbf{g}^n_{i+\frac12} = \mathbf{g}(\sigma^n_{i+\frac12} = 0; \boldsymbol{u}^n_i, \boldsymbol{u}^n_{i+1}) \text{ for } i \in \mathbb{Z} \setminus \{\iota\}, \quad \mathbf{g}^n_{\iota+\frac12} = \mathbf{g}(\sigma^n_{\iota+\frac12}; \boldsymbol{u}^n_\iota, \boldsymbol{u}^n_{\iota+1}),$$
$$\mathbf{h}^n_{i+\frac12} = \mathbf{0} \text{ for } i \in \mathbb{Z} \setminus \{\iota\}, \qquad \mathbf{h}^n_{\iota+\frac12} = -\sigma^n_{\iota+\frac12}\mathbf{U}[\sigma^n_{\iota+\frac12}; \boldsymbol{u}^n_\iota, \boldsymbol{u}^n_{\iota+1}].$$

Recall that only $\mathcal{S}^n_{\iota+\frac12}$ might not be parallel to the t-axis.

It is the particular property of the numerical approximation $\boldsymbol{u}_h$ produced from the Godunov flux (1.3.12), that it can be characterized for each time level as an averaged entropy solution for $(1.2.1)_1$. Let us define the function

$$\bar{\boldsymbol{u}}_h(x,t) := \mathbf{U}\Big[\big(x - x_{i+\frac12}\big)/(t-t^n); \boldsymbol{u}^n_i, \boldsymbol{u}^n_{i+1}\Big] \qquad (x \in K_i(t), t \in [t^n, t^{n+1}]). \tag{1.3.13}$$

This function is composed of single entropy solutions of the Riemann problem (1.2.1). It is well defined as long as Δt^n is small enough such that the wave pattern do not cross each other. Then $\bar{\boldsymbol{u}}_h$ solves $(1.2.1)_1$ for the initial datum $\bar{\boldsymbol{u}}_h(\cdot, t^n) = \boldsymbol{u}_h(\cdot, t^n)$ and obeys thus the entropy inequality

$$\int_{t^n}^{t}\int_{\mathbb{R}} W(\bar{\boldsymbol{u}}_h(x,s))\Phi_t(x,s) + Q(\bar{\boldsymbol{u}}_h(x,s))\Phi_x(x,s)\,dxds \geq 0 \tag{1.3.14}$$

for all non-negative test functions Φ with compact support in $\mathbb{R} \times [t^n, s]$ for $s \in [t^n, t^{n+1}]$. The function $\boldsymbol{u}_h$ satisfies for $x \in K_i(t^{n+1})$ by construction

$$\boldsymbol{u}_h(x, t^{n+1}) = \frac{1}{|K_i(t^{n+1})|}\int_{x^{n+1}_{i-\frac12}}^{x^{n+1}_{i+\frac12}} \bar{\boldsymbol{u}}_h(x, t^{n+1})\,dx. \tag{1.3.15}$$

With these preparations we can now proceed to the main result of this section.

Theorem 1.3.9 (Cell entropy inequality for the finite volume moving mesh method). *Let $\{\boldsymbol{u}^n_i\}_{i\in\mathbb{Z}}$ be given for some $n \in \mathbb{N}$. Assume for some $\iota \in \mathbb{Z}$ that*

$$\boldsymbol{u}^n_i \in \tilde{\mathcal{A}}_{\mathrm{liq}} \times \mathbb{R}^d \text{ for } i \leq \iota \text{ and } \boldsymbol{u}^n_i \in \tilde{\mathcal{A}}_{\mathrm{vap}} \times \mathbb{R}^d \text{ for } i > \iota. \tag{1.3.16}$$

If Δt^n is small enough, the following statements hold.

(i) *The phase separation property* (1.3.16) *holds also for $t = t^{n+1}$.*

(ii) *There is a function $\mathbf{q} = \mathbf{q}(s; \boldsymbol{u}, \boldsymbol{v})$ that is consistent with the function $Q - sW$ for all $s \in \mathbb{R}$ and such that the finite volume moving mesh method* (1.3.11) *with the Godunov flux satisfies the discrete entropy inequality*

$$\begin{aligned} &|K_i(t^{n+1})|W(\boldsymbol{u}^{n+1}_i) \\ &\quad \leq |K_i(t^n)|W(\boldsymbol{u}^n_i) - \Delta t^n\big(\mathbf{q}(s^n_{i+\frac12}; \boldsymbol{u}^n_i, \boldsymbol{u}^n_{i+1}) - \mathbf{q}(s^n_{i-\frac12}; \boldsymbol{u}^n_{i-1}, \boldsymbol{u}^n_i)\big) \end{aligned} \tag{1.3.17}$$

for all $i \in \mathbb{Z}$.

Before we give the proof let us mention that cell entropy inequalities for finite volume schemes on fixed meshes using the Godunov flux can be found in any text book, e.g., [61]. The novelty of this result is the moving mesh aspect and the

application to a conservation law with separated state space. Let us also highlight property (i). Except the non-conservative ghost-fluid methods we are not aware of any (practically usable) method that avoids spinodal states.

Proof of Theorem 1.3.9. The statement (i) is a consequence of (1.3.15). The entropy solution $\bar{\boldsymbol{u}}_h(\cdot, t^{n+1})$ in $K_i(t^{n+1})$ is in the convex space $\tilde{\mathcal{A}}_{\text{liq}} \times \mathbb{R}^d$ for $i \le \iota$. The phase boundary is exactly $\mathcal{S}^n_{\iota+\frac{1}{2}}$. Therefore also the averaged values $\boldsymbol{u}_h(x, t^{n+1})$ keep the phase. The same applies for the cells right to $\mathcal{S}^n_{\iota+\frac{1}{2}}$ with values in the vapour state space.

For (ii) we exploit the entropy inequality (1.3.14) by choosing a sequence of test functions that converge towards the characteristic function of the truncated cone $\{(x,t) \,|\, x \in K_i(t), t \in [t^n, t^{n+1}]\}$ and obtain the trace relation

$$\begin{aligned}
\int_{K_i(t^{n+1})} W(\bar{\boldsymbol{u}}_h(x,t^{n+1}))\,dx \le \int_{K_i(t^n)} W(\bar{\boldsymbol{u}}_h(x,t^n))\,dx \\
- \frac{\Delta t^n}{|\mathcal{S}^n_{i+\frac{1}{2}}|} \int_{\mathcal{S}^n_{i+\frac{1}{2}}} \frac{1}{2}\Big(Q(\bar{\boldsymbol{u}}_{h,-}(z,t)) - \sigma^n_{i+\frac{1}{2}} W(\bar{\boldsymbol{u}}_{h,-}(z,t))\Big)\,dz \\
- \frac{\Delta t^n}{|\mathcal{S}^n_{i+\frac{1}{2}}|} \int_{\mathcal{S}^n_{i+\frac{1}{2}}} \frac{1}{2}\Big(Q(\bar{\boldsymbol{u}}_{h,+}(z,t)) - \sigma^n_{i+\frac{1}{2}} W(\bar{\boldsymbol{u}}_{h,+}(z,t))\Big)\,dz \\
+ \frac{\Delta t^n}{|\mathcal{S}^n_{i-\frac{1}{2}}|} \int_{\mathcal{S}^n_{i-\frac{1}{2}}} \frac{1}{2}\Big(Q(\bar{\boldsymbol{u}}_{h,-}(z,t)) - \sigma^n_{i-\frac{1}{2}} W(\bar{\boldsymbol{u}}_{h,-}(z,t))\Big)\,dz \\
+ \frac{\Delta t^n}{|\mathcal{S}^n_{i-\frac{1}{2}}|} \int_{\mathcal{S}^n_{i-\frac{1}{2}}} \frac{1}{2}\Big(Q(\bar{\boldsymbol{u}}_{h,+}(z,t)) - \sigma^n_{i-\frac{1}{2}} W(\bar{\boldsymbol{u}}_{h,+}(z,t))\Big)\,dz.
\end{aligned}$$

Here we have used the left-hand/right-hand traces $\bar{\boldsymbol{u}}_{h,\pm}(z,t) = \lim_{\varepsilon\to 0,\varepsilon>0} \bar{\boldsymbol{u}}_h(z \pm \varepsilon, t)$, see also below for the Riemann solution $\mathbf{U}[\cdot; \boldsymbol{u}^n_i, \boldsymbol{u}^n_{i+1}]$. Now we relate $\boldsymbol{u}_h$ to $\bar{\boldsymbol{u}}_h$ via (1.3.15). Since the left-hand/right-hand traces of the Riemann solutions for the entropy and the entropy fluxes are constant along $\mathcal{S}^n_{i+\frac{1}{2}}$ we get after evaluating the surface integrals

$$\begin{aligned}
\frac{1}{|K_i(t^n)|} \int_{K_i(t^{n+1})} W(\bar{\boldsymbol{u}}_h(x,t^{n+1}))\,dx \le \frac{1}{|K_i(t^n)|} \int_{K_i(t^n)} W(\boldsymbol{u}_h(x,t^n))\,dx \\
- \frac{\Delta t^n}{|K_i(t^n)|} \frac{1}{2}\Big(Q(\mathbf{U}_-[\sigma^n_{i+\frac{1}{2}}; \boldsymbol{u}^n_i, \boldsymbol{u}^n_{i+1}]) - \sigma^n_{i+\frac{1}{2}} W(\mathbf{U}_-[\sigma^n_{i+\frac{1}{2}}; \boldsymbol{u}^n_i, \boldsymbol{u}^n_{i+1}])\Big) \\
- \frac{\Delta t^n}{|K_i(t^n)|} \frac{1}{2}\Big(Q(\mathbf{U}_+[\sigma^n_{i+\frac{1}{2}}; \boldsymbol{u}^n_i, \boldsymbol{u}^n_{i+1}]) - \sigma^n_{i+\frac{1}{2}} W(\mathbf{U}_+[\sigma^n_{i+\frac{1}{2}}; \boldsymbol{u}^n_i, \boldsymbol{u}^n_{i+1}])\Big) \\
+ \frac{\Delta t^n}{|K_i(t^n)|} \frac{1}{2}\Big(Q(\mathbf{U}_-[\sigma^n_{i-\frac{1}{2}}; \boldsymbol{u}^n_{i-1}, \boldsymbol{u}^n_i]) - \sigma^n_{i-\frac{1}{2}} W(\mathbf{U}_-[\sigma^n_{i-\frac{1}{2}}; \boldsymbol{u}^n_{i-1}, \boldsymbol{u}^n_i])\Big) \\
+ \frac{\Delta t^n}{|K_i(t^n)|} \frac{1}{2}\Big(Q(\mathbf{U}_+[\sigma^n_{i-\frac{1}{2}}; \boldsymbol{u}^n_{i-1}, \boldsymbol{u}^n_i]) - \sigma^n_{i-\frac{1}{2}} W(\mathbf{U}_+[\sigma^n_{i-\frac{1}{2}}; \boldsymbol{u}^n_{i-1}, \boldsymbol{u}^n_i])\Big).
\end{aligned}$$

With obvious definition of the numerical entropy flux $\mathbf{q} = \mathbf{q}(\cdot, \boldsymbol{u}, \boldsymbol{v})$ we rewrite the last inequality as

$$\begin{aligned}&\frac{1}{|K_i(t^n)|}\int_{K_i(t^{n+1})} W(\bar{\boldsymbol{u}}_h(x, t^{n+1}))\,dx\\ &\quad\le W(\boldsymbol{u}_i^n) - \frac{\Delta t^n}{|K_i(t^n)|}(\mathbf{q}(s^n_{i+\frac12}; \boldsymbol{u}_i^n, \boldsymbol{u}_{i+1}^n) - \mathbf{q}(s^n_{i-\frac12}; \boldsymbol{u}_{i-1}^n, \boldsymbol{u}_i^n)).\end{aligned}$$

Applying once again (1.3.15) and Jensen's inequality for the convex entropy W and values in one of the two convex sets $\tilde{\mathcal{A}}_{\mathrm{liq/vap}} \times \mathbb{R}^d$ (see (i)) implies the statement (1.3.17) of the theorem. □

1.3.4. Numerical results for a single bubble. We conclude Section 1 with one numerical experiment taken from [14] that uses Algorithm 1.3.7. Other material can be found in [13, 14]. Precisely we consider the interaction of a phase boundary with an elementary wave. We take as initial condition a vapour bubble surrounded by liquid in Maxwell equilibrium together with a discontinuity at the left, which will result in a classical wave propagating to the right, i.e., in direction of the bubble.

$$\begin{aligned}&(\varrho_0, \varrho_0 v_{1,0}, \varrho_0 v_{2,0})(\mathbf{x})\\ &\quad= \begin{cases}(\varrho_{\mathrm{vap}}^{\mathrm{sat}} = 0.3207, 0, 0) & \left\|\mathbf{x} - (0.3, 0)^T\right\|_2^2 < 0.1,\\ (\varrho_{\mathrm{liq}}^{\mathrm{sat}} = 1.8071, 0, 0) & \left\|\mathbf{x} - (0.3, 0)^T\right\|_2^2 > 0.1 \ \text{ and } \ x_1 > -0.5,\\ (1.7010, -0.4, 0) & \text{else.}\end{cases}\end{aligned}$$

At the boundary we choose reflecting boundary conditions.

The results of the computation are displayed in Figure 1.9. First of all it is a remarkable property of the algorithm that the spherical Maxwell equilibrium is perfectly preserved as long as no perturbation enters the position of the droplet. This has to be compared to the situation for DI models which account for curvature effects, see Section 2.4 below.

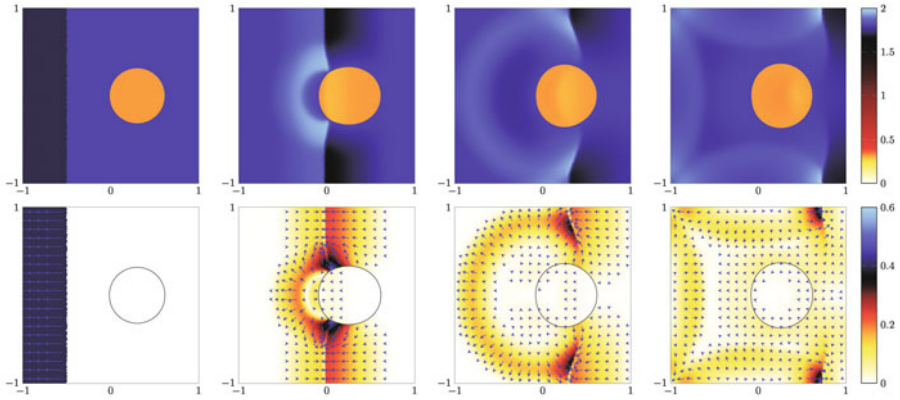
After it has reached the interface, the phase boundary states correspond not anymore to Maxwell equilibria, and the interface starts moving. Also, after the collision one can see the reflected waves in the bulk.

We know that for any entropic SI solution $\boldsymbol{u} = (\varrho, \varrho \boldsymbol{v}^T)^T$ the total energy $W = W(\boldsymbol{u})$ must not increase in time. Taking into account the boundary condition this means for the example that the quantity

$$S(t) = \int_D W(\boldsymbol{u}(\mathbf{x}, t))\,d\mathbf{x} + \int_0^t \int_{\partial D} \mathbf{m} \cdot \boldsymbol{Q}(\boldsymbol{u}(\boldsymbol{z}, s))\,d\boldsymbol{z}\,ds \tag{1.3.18}$$

is non-increasing. Here $\mathbf{m}$ denotes the outer normal of ∂D.

It can be seen in Figure 1.10 that the algorithm leads to a perfectly monotone decreasing total entropy.



(a) Initial state **(b)** t=0.6 **(c)** t=0.8 **(d)** t=1.0

FIG. 1.9. Numerical solution for a shock moving towards a phase transition in Maxwell equilibrium. The initial mesh has 45162 elements. The number of elements during runtime ranges between 45100 to 45162 due to the re-meshing process.

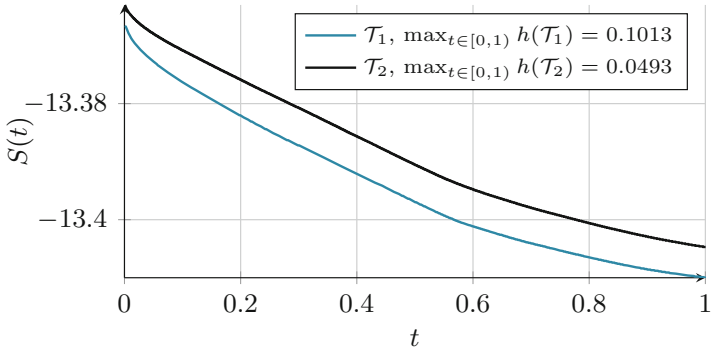


FIG. 1.10. Evolution of $S = S(t)$ from (1.3.18).

2. The diffuse interface approach

The SI approach from Section 1 is not only numerically quite complex. It is already problematic on the modelling level: phenomena like merging of bubbles/droplets or the nucleation of new phase states are excluded from the start. For these reasons various diffuse interface (DI) models have been suggested to describe the dynamics of a compressible fluid with liquid-vapour phase transition. The DI models split into at least two sub-groups: first, that ones that use a second-gradient-like approach as, e.g., the Navier–Stokes–Korteweg (NSK) models (see [3] and the literature in the up-coming text). Second, there are the phase field models

(e.g., [5, 24, 30, 63, 84]). Models from the NSK class start from an extended energy functional that accounts for more complex interactions between fluid particles close to a spatial region where phase change takes place. As a consequence the evolution equations are equipped with nonlocal terms. These provide a regularization effect such that the NSK models result in one system governing the fluid's dynamics in the entire domain D. The same applies for the phase field models. But they introduce an additional evolution equation for a phase field parameter which usually realizes a phase separation effect of Allen–Cahn or Cahn–Hilliard type. Regardless of the chosen DI model all of them contain a small interfacial parameter (denoted by $\varepsilon > 0$ in the sequel), that controls the width of the smeared-out phase boundary layer. The sharp interface (SI) limit $\varepsilon \to 0$ acts as a validation criterion for all DI models. In this limit they must recover solutions of the SI model (1.1.13) and a hyper-surface such that across this hyper-surface the coupling conditions (1.1.15) hold.

In this second section we will restrict ourselves for the sake of brevity to some instances of the NSK class. To proceed as in Section 1 we will start with the one-phase case in Section 2.1, which just leads to the compressible Navier–Stokes equations. Section 2.2 is then concerned with the classical Navier–Stokes–Korteweg model. We will review its thermodynamical consistency and discuss it from the numerical point of view. Whereas the analytical situation is quite clear the classical Navier–Stokes–Korteweg model suffers from various shortcomings in the discretization process. To overcome some of these problems we will present variations of the classical Navier–Stokes–Korteweg model in Sections 2.3, 2.4. In any case we will carefully analyze the thermodynamical consistency of all models. In particular we will introduce a new entropy-consistent discretization. For parts of the presentation we will use material including figures from the publications [23, 70].

2.1. The Navier–Stokes equations for one-phase flow

2.1.1. Modelling and thermodynamical consistency. Like in Section 1 we start this modeling section with a DI model for one-phase flow: the Navier–Stokes equations. Let the pressure function $\tilde{p} = \tilde{p}(\varrho)$ satisfy Assumption 1.1.1 for one-phase flow. A DI model that fits to the SI model (1.1.13) is given by the compressible *Navier–Stokes equations* [62]

$$\begin{aligned} \varrho_t &+ \operatorname{div}(\varrho \mathbf{v}) &&= 0, \\ (\varrho \mathbf{v})_t &+ \operatorname{div}(\varrho \mathbf{v} \otimes \mathbf{v} + p(\varrho)\boldsymbol{I}) &&= \operatorname{div}(\mathbf{T}) \end{aligned} \qquad \text{in } D \times (0,T). \tag{2.1.1}$$

The unknowns are density $\varrho = \varrho(\mathbf{x},t) : D \times [0,T] \to (0,\bar{\varrho})$ and velocity $\mathbf{v} = \mathbf{v}(\mathbf{x},t) : D \times [0,T] \to \mathbb{R}^d$.

The matrix $\mathbf{T} = \mathbf{T}(\mathbf{x},t) \in \mathbb{R}^{d\times d}$ in (2.1.1) stands for the viscous part of the stress tensor which is given for $\lambda = \lambda(\varepsilon), \mu = \mu(\varepsilon) \in \mathbb{R}$ with $\mu \geq 0$ and

$3\lambda + 2\mu > 0$ by

$$\mathbf{T}_{ij} := \lambda \operatorname{div}(\mathbf{v})\delta_{ij} + 2\mu \mathbf{D}_{ij}, \quad \mathbf{D}_{ij} := \frac{1}{2}\Big(v_{j,x_i} + v_{i,x_j}\Big) \qquad (i,j \in \{1,2\}). \quad (2.1.2)$$

The initial conditions for system (2.1.1) are

$$\varrho(\cdot,0) = \varrho_0,\ \boldsymbol{v}(\cdot,0) = \boldsymbol{v}_0 \text{ in } D \quad (2.1.3)$$

and as boundary conditions one might assume for $t \in [0,T]$

$$\boldsymbol{v}(\cdot,t) = \mathbf{0} \text{ on } \partial D. \quad (2.1.4)$$

Remark 2.1.1. In the Navier–Stokes equations (2.1.1) we have skipped a body force for the sake of simplicity. In the current Section 2.1 and Sections 2.2, 2.3 a body force can be simply added without changing any of the results. This is different in Section 2.4 on the computations of equilibrium or close-to-equilibrium solutions. Inserting, e.g., gravitational forces would not only alter the topology of the equilbria but of course also the acting forces at the interface. The chosen ansatz for stabilization has then also account for the contribution of the body force to the complete energy balance term that is added to the continuity equation, see $(2.4.2)_1$.

For pertinent results for global weak solutions of (2.1.1), (2.1.3), (2.1.4) we refer to, e.g., [29, 62] for the isothermal and non-isothermal case. With $\lambda(\varepsilon), \mu(\varepsilon) = o(1)$ the SI parameter $\varepsilon > 0$ controls via vanishing viscosity the SI limit in (2.1.1). In the SI limit (say for simplicity on $D = \mathbb{R}^d$) it is expected that solutions of (2.1.1), (2.1.3), (2.1.4) realize weak solutions of (1.1.13), in fact the dissipative structure of $\mathbf{T}$ ensures that the SI limit – if it exists – satisfies the entropy criterium (1.1.21) (see Proposition 2.1.2). Although it is widely believed that solutions of (2.1.1) converge to an entropic SI solution of (1.1.13), (1.1.15), (1.1.16) there are very less rigorous results on the existence of the SI limit. In particular the multi-dimensional case is widely open.

The generic statement on thermodynamical consistency for the one-phase case is expressed in the following theorem.

Proposition 2.1.2. *Let* $(\varrho, \mathbf{v})$ *be a classical solution of* (2.1.1), (2.1.3), (2.1.4). *Then we have for all* $t \in [0,T]$

$$\begin{aligned} &\frac{d}{dt}\Big(\int_D \frac{1}{2}\varrho(\mathbf{x},t)|\mathbf{v}(\mathbf{x},t)|^2 + \tilde{\psi}(\varrho(\mathbf{x},t))\, d\mathbf{x}\Big) \\ &\quad + \int_D 2\mu \mathbf{D}(\mathbf{v}(\mathbf{x},t)) : \mathbf{D}(\mathbf{v}(\mathbf{x},t)) + \lambda(\operatorname{div}(\mathbf{v}(\mathbf{x},t)))^2\, d\mathbf{x} = 0. \end{aligned} \quad (2.1.5)$$

Proof. Multiply (2.1.1) with the gradient of the entropy W from (1.1.20), that is $(-|\boldsymbol{v}|^2/2 + \frac{d}{d\varrho}\tilde{\psi}(\varrho), \boldsymbol{v}^T)^T$. Integration with respect to space and usage of the boundary conditions (2.1.4) implies (2.1.5). □

2.1.2. A thermodynamically consistent finite-volume scheme. In the remainder of this section we discuss the construction of a finite volume scheme that preserves the thermodynamical consistency property from Proposition 2.1.2 also on the discrete level. Our tool is the use of entropy-conservative schemes for systems of hyperbolic conservation laws as it has been originally introduced by Tadmor [78]. We review this approach here to prepare the design of finite volume methods for a DI model for two-phase flow in Section 2.3.2.

We consider the Navier–Stokes equations (2.1.1) for a one-phase pressure satisfying Assumption 1.1.1. For the sake of simplicity we restrict the construction to the one-dimensional version

$$\begin{pmatrix} \varrho \\ \varrho v \end{pmatrix}_t + \begin{pmatrix} \varrho v \\ \varrho v^2 + \tilde{p}(\varrho) \end{pmatrix}_x = \begin{pmatrix} 0 \\ \nu v_{xx} \end{pmatrix} \qquad \text{in } \mathbb{R} \times (0,T). \tag{2.1.6}$$

In (2.3.10) we have chosen $\nu = \lambda + 2\mu$. Due to Assumption 1.1.1 the first-order part of (2.1.6), i.e., the Euler system

$$\mathbf{U}_t + \boldsymbol{F}(\mathbf{U})_x = \mathbf{0} \tag{2.1.7}$$

is hyperbolic in $\tilde{\mathcal{U}} := (0, \bar{\varrho}) \times \mathbb{R}$. Here we used (similarly but slightly different as before in Section 1.2) the notations $\mathbf{U} = (\varrho, m := \varrho v)^T$ and $\boldsymbol{F}(\mathbf{U}) = (\varrho v, \varrho v^2 + \tilde{p}(\varrho))^T$. If we recall the discussion from Section 1.1.2 we observe that the pair (W, Q) given by

$$W(\varrho, m) = \tilde{\psi}(\varrho) + \frac{m^2}{2\varrho}, \qquad Q(\varrho, m) = \frac{m}{\varrho}\Big(\tilde{\psi}(\varrho) + \tilde{p}(\varrho)\Big) \tag{2.1.8}$$

is an entropy-entropy flux pair for (2.1.7), i.e., W is convex in $\tilde{\mathcal{U}}$, and we have in particular the compatibility relation $(\nabla W(\mathbf{U}))^T D\boldsymbol{F}(\mathbf{U}) = (\nabla Q(\mathbf{U}))^T$ for $\mathbf{U} \in \tilde{\mathcal{U}}$. As a consequence of the entropy's convexity (and $\tilde{\mathcal{U}}$ being convex) the mapping $\mathbf{U} \mapsto \mathbf{V}(\mathbf{U})$ from $\tilde{\mathcal{U}}$ to $\tilde{\mathcal{V}} := \mathbf{V}(\tilde{\mathcal{U}})$ with

$$\begin{aligned} \mathbf{V}(\mathbf{U}) &= (V_1(\mathbf{U}), V_2(\mathbf{U}))^T = \nabla W(\mathbf{U})^T \\ &= \left(\frac{d}{d\varrho}\tilde{\psi}(\varrho) - \frac{m^2}{2\varrho^2}, \frac{m}{\varrho} \right)^T = \left(\frac{d}{d\varrho}\tilde{\psi}(\varrho) - \frac{v^2}{2}, v \right)^T \end{aligned}$$

is one-to-one. With an appropriate flux function $\boldsymbol{G} = \boldsymbol{G}(\mathbf{V})$ the system (2.1.7) can then be rewritten equivalently in terms of the entropy variable $\mathbf{V}$, that is

$$\mathbf{U}(\mathbf{V})_t + \boldsymbol{G}(\mathbf{V})_x = 0. \tag{2.1.9}$$

Furthermore the flux $\boldsymbol{G}$ can be represented as the gradient of the potential function (cf. [78])

$$\Psi(\mathbf{V}) = \mathbf{V} \cdot \boldsymbol{G}(\mathbf{V}) - Q(\mathbf{U}(\mathbf{V})). \tag{2.1.10}$$

In our case we compute

$$\Psi(\mathbf{V}) = V_2 \left((V_1 + V_2^2) \left(\frac{d}{d\varrho}\tilde{\psi} \right)^{-1} \left(V_1 + \frac{V_2^2}{2} \right) - W(\mathbf{U}(\mathbf{V})) \right). \tag{2.1.11}$$

The numerical approach uses so-called entropy-conservative finite difference schemes which have been originally introduced by Tadmor in [78]. Let a uniform (fixed) mesh with cells

$$K_j = (x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}),\ x_{j+\frac{1}{2}} = \left(j + \frac{1}{2}\right) h,\ j \in \mathbb{Z},$$

and mesh width $h = x_{j+\frac{1}{2}} - x_{j-\frac{1}{2}}$ be given. Now let a numerical flux function $\mathbf{g}^* : \tilde{\mathcal{V}} \times \tilde{\mathcal{V}} \to \mathbb{R}^2$ with $\mathbf{g}^*(\mathbf{V}, \mathbf{V}) = \boldsymbol{G}(\mathbf{V})$ for $\mathbf{V} \in \tilde{\mathcal{V}}$ be given. We consider a semi-discrete finite volume scheme for (2.1.7). It takes for $t \in (0, T)$ and $j \in \mathbb{Z}$ the form

$$\begin{pmatrix} \varrho_j'(t) \\ m_j'(t) \end{pmatrix} = -\frac{1}{h} \begin{pmatrix} g^*_{j+\frac{1}{2},1}(t) - g^*_{j-\frac{1}{2},1}(t) \\ g^*_{j+\frac{1}{2},2}(t) - g^*_{j-\frac{1}{2},2}(t) \end{pmatrix} \Leftrightarrow \mathbf{U}_j'(t) = -\frac{1}{h}\left(\mathbf{g}^*_{j+\frac{1}{2}}(t) - \mathbf{g}^*_{j-\frac{1}{2}}(t)\right), \tag{2.1.12}$$

such that $\mathbf{g}^*_{j+\frac{1}{2}}(t) = \mathbf{g}^*(\mathbf{V}_j(t), \mathbf{V}_{j+1}(t))$. The following result can be found in [79, Theorem 3.1].

Theorem 2.1.3. *Consider the scheme* (2.1.12) *and assume that*

$$(\mathbf{V} - \mathbf{Z}) \cdot \mathbf{g}^*(\mathbf{V}, \mathbf{Z}) = \Psi(\mathbf{V}) - \Psi(\mathbf{Z}) \tag{2.1.13}$$

is valid for $\mathbf{V}, \mathbf{Z} \in \tilde{\mathcal{V}}$.

Then and only then the numerical flux $\mathbf{g}^*$ *is* entropy-conservative*, i.e., there is a scalar function* $q^* = q^*(\mathbf{V}, \mathbf{Z})$ *such that* $q^*(\mathbf{V}, \mathbf{V}) = Q(\mathbf{U}(\mathbf{V}))$ *and*

$$W(\mathbf{U}_j(t))' = -\frac{1}{h}\Big(q^*(\mathbf{V}_j(t), \mathbf{V}_{j+1}(t)) - q^*(\mathbf{V}_{j-1}(t), \mathbf{V}_j(t))\Big)$$

hold for all $\mathbf{V}, \mathbf{Z} \in \mathcal{V}$, $t \in (0, T)$ *and* $j \in \mathbb{Z}$.

The choice of an entropy-conservative numerical flux is not unique. The existence of a whole family of entropy fluxes plays no essential role for the one-phase pressure case, but is essential in Section 2.3. Therefore we give the full representation already now. Dealing with the two-dimensional system (2.1.7) let $\{\mathbf{r}^1, \mathbf{r}^2\}, \{\mathbf{l}^1, \mathbf{l}^2\}$ be sets of linear independent vectors in $\mathbb{R}^2$ with $\mathbf{r}^k \cdot \mathbf{l}^k = \delta_{kl}$. Then we get for $\overline{\mathbf{VZ}} := \mathbf{V} + \big(\mathbf{l}^1 \cdot (\mathbf{Z} - \mathbf{V})\big)\mathbf{r}^1$

$$\overline{\mathbf{VZ}} + \big(\mathbf{l}^2 \cdot (\mathbf{Z} - \mathbf{V})\big)\mathbf{r}^2 = \mathbf{V} + \big(\mathbf{l}^1 \cdot (\mathbf{Z} - \mathbf{V})\big)\mathbf{r}^1 + \big(\mathbf{l}^2 \cdot (\mathbf{Z} - \mathbf{V})\big)\mathbf{r}^2 = \mathbf{Z}, \tag{2.1.14}$$

that is an affine path connecting $\mathbf{V}$ and $\mathbf{Z}$. In [79, Theorem 6.1] it is shown that any numerical flux of the form

$$\mathbf{g}^*(\mathbf{V}, \mathbf{Z}) = \frac{\Psi\left(\overline{\mathbf{VZ}}\right) - \Psi\left(\mathbf{V}\right)}{\mathbf{l}^1 \cdot (\mathbf{V} - \mathbf{Z})}\mathbf{l}^1 + \frac{\Psi\left(\mathbf{Z}\right) - \Psi\left(\overline{\mathbf{VZ}}\right)}{\mathbf{l}^2 \cdot (\mathbf{V} - \mathbf{Z})}\mathbf{l}^2 \tag{2.1.15}$$

is entropy conservative (and consistent with $\mathbf{g}$).

Now we can present our semi-discrete finite volume scheme for (2.1.6) that is given for $j \in \mathbb{Z}$ by a solution $(\varrho_j, m_j) : [0,T) \to \tilde{\mathcal{U}}$ of the initial value problem

$$\begin{pmatrix} \varrho_j'(t) \\ m_j'(t) \end{pmatrix} = -\frac{1}{h} \begin{pmatrix} g^*_{j+\frac{1}{2},1}(t) - g^*_{j-\frac{1}{2},1}(t) \\ g^*_{j+\frac{1}{2},2}(t) - g^*_{j-\frac{1}{2},2}(t) \end{pmatrix} + \frac{\nu}{h^2} \begin{pmatrix} 0 \\ v_{j+1}(t) - 2v_j(t) + v_{j-1}(t) \end{pmatrix} (t \in (0,T)) \tag{2.1.16}$$

and

$$\varrho_j(0) = \int_{K_j} \varrho_0(x)\,dx, \quad m_j(0) = \int_{K_j} \varrho_0(x) v_0(x)\,dx. \tag{2.1.17}$$

It is easy to check that this scheme leads to

Theorem 2.1.4 (Discrete energy inequality for one-phase flow). *Let $\varrho_0 v_0^2$, ϱ_0^2, $\tilde{\psi}(\varrho_0) \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$. For $j \in \mathbb{Z}$ let $(\varrho_j, m_j) : [0,T] \to (0,\bar{\varrho}) \times \mathbb{R}$ be a solution of* (2.1.16), (2.1.17).

Then we have for each $t \in [0,T)$ the discrete energy inequality

$$\sum_{j\in\mathbb{Z}} \left(\frac{(m_j(t))^2}{2\varrho_j(t)} + \tilde{\psi}(\varrho_j) \right) \le \int_{\mathbb{R}} W(\varrho_0(x), \varrho_0(x)v_0(x))\,dx. \tag{2.1.18}$$

Proof. We multiply the scheme (2.1.16) by $\mathbf{V}_j(t)$. Arguing as in [79, p. 463] we obtain from Theorem 2.1.3 functions $q^* = q^*(\mathbf{V}, \mathbf{Z})$ such that

$$W(\mathbf{U}_j(t))' + \frac{1}{h}\left(q^*(\mathbf{V}_j(t), \mathbf{V}_{j+1}(t) - q^*(\mathbf{V}_{j-1}(t), \mathbf{V}_j(t))\right) = \frac{\nu}{h^2} \begin{pmatrix} 0 \\ v_{j+1}(t) - 2v_j(t) + v_{j-1}(t) \end{pmatrix} \cdot \mathbf{V}_j(t).$$

Using $V_2 = v$ we obtain after adding-up with respect to $j \in \mathbb{Z}$ and summation by parts

$$\sum_{j\in\mathbb{Z}} W(\mathbf{U}_j(t))' = -\frac{\nu}{h^2} \sum_{j\in\mathbb{Z}} \left(v_j(t) - v_{j-1}(t)\right)^2 \le 0. \qquad \square$$

The inequality (2.1.18) is the complete discrete analogue of the inequality in Proposition 2.1.2. In fact it must be noted that entropy-disspative schemes which rely on entropy-conservative numerical fluxes are by now frequently used in the direct numerical solution of compressible flow problems, we refer to [32, 48] for a more general approach that in particular includes multiple spatial dimensions.

2.2. The classical Navier–Stokes–Korteweg equations for two-phase flow

2.2.1. Modelling and thermodynamical consistency. Next, we present DI models for two-phase flow, starting with the classical Navier–Stokes–Korteweg system. To do this we will extend the pressure function $\tilde{p} : (\tilde{\mathcal{A}}_{\rm vap} \cup \tilde{\mathcal{A}}_{\rm liq}) \to (0,\infty)$ from Definition 1.1.2 artificially to a function $\tilde{p} : (0,\bar{\varrho}) \to (0,\infty)$. Since solutions of a DI model are expected to be smooth it is necessary to evaluate $\tilde{p}$ also for values in $\tilde{\mathcal{A}}_{\rm spinodal}$ like in Figure 2.1.

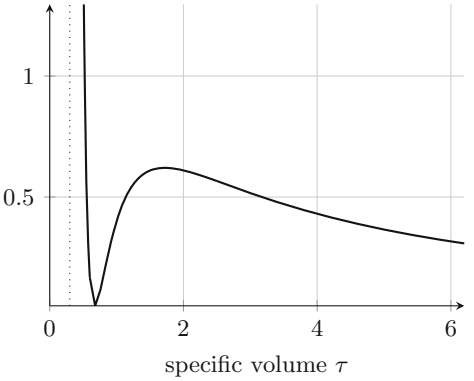


FIG. 2.1. Graph of a sample two-phase pressure for a DI model on $(\bar{\tau}, \infty)$ as a function of specfic volume. The dashed line indicates the asymptote at $\tau = \bar{\tau}$.

Definition 2.2.1. The function $p : (\bar{\tau}, \infty) \to (0, \infty)$ is called (*two-phase*) *pressure for a DI model*, if it is a two-phase pressure for an SI model in the sense of Definition 1.1.2 that satisfies additionally

$$p' > 0 \text{ in } \mathcal{A}_{\text{spinodal}}.$$

The extension of the pressure functions in the spinodal region is completely artificial. We mention in passing that this freedom has been used to trigger SI limits for DI models (e.g., in [47]). Let now a two-phase pressure for a DI model be given.

For the unknowns density $\varrho = \varrho(\mathbf{x}, t) : D \times [0, T] \to (0, \bar{\varrho})$, velocity $\mathbf{v} = \mathbf{v}(\mathbf{x}, t) : D \times [0, T] \to \mathbb{R}^d$ the classical *Navier–Stokes–Korteweg* (*NSK*) *system* is given by [3, 26, 44, 53]

$$\begin{aligned} \varrho_t \ &+ \ \operatorname{div}(\varrho \mathbf{v}) \ &= \ 0, \\ (\varrho \mathbf{v})_t \ &+ \ \operatorname{div}(\varrho \mathbf{v} \otimes \mathbf{v} + p(\varrho) \boldsymbol{I}) \ &= \ \operatorname{div}(\mathbf{T}) + \gamma \varrho \nabla \Delta \varrho \end{aligned} \quad \text{in } D \times (0, T). \tag{2.2.1}$$

The initial conditions for system (2.2.1) are as for the one-phase case

$$\varrho(\cdot, 0) = \varrho_0, \ \boldsymbol{v}(\cdot, 0) = \boldsymbol{v}_0 \text{ in } D. \tag{2.2.2}$$

For a bounded domain D boundary conditions for (2.2.1) are given for $t \in [0, T]$ and $\mathbf{m} \in \mathbb{S}^{d-1}$ being the outer normal of ∂D by

$$\mathbf{v}(\cdot, t) = \mathbf{0}, \quad \mathbf{m} \cdot \nabla \varrho(\cdot, t) = 0 \text{ in } \partial D. \tag{2.2.3}$$

Note that the choice $(2.2.3)_2$ will induce a 90° degree contact angle between the phases at the boundary. Other choices are of course possible but we note that the correct description of the three-phase contact line within DI models is still under discussion.

Local well-posedness of classical solutions for (2.2.1), (2.2.2), (2.2.3) has been verified in [26]. The global well-posedness of weak solutions for (2.2.1), (2.2.2), (2.2.3) is the subject of [8], with a proof given for a one-phase pressure.

Unlike in the one-phase case one can scale λ, μ and γ in different ways with respect to ε. Counting derivatives a natural scaling appears to be $\lambda(\varepsilon), \mu(\varepsilon) = \mathcal{O}(\varepsilon)$ and $\gamma(\varepsilon) = \mathcal{O}(\varepsilon^2)$. This we will use in our numerical experiments, but see the formal asymptotic analysis in [25] for various different scalings and consequently very different SI limit conditions. As in the one-phase case the rigorous analysis of the SI-limit $\varepsilon \to 0$ is widely open in multiple space dimensions. For $d = 1$ there is extensive work on traveling waves and associated SI limits towards subsonic phase boundaries (1.1.24), starting with [38].

Thermodynamical consistency requires for the two-phase regime a generalized energy with non-local contributions. Precisely we have

Proposition 2.2.2. *Let $(\varrho, \mathbf{v})$ be a classical solution of* (2.2.1), (2.2.2), (2.2.3). *Then we have for all $t \in [0, T]$*

$$\begin{aligned} &\frac{d}{dt}\Big(\int_D \frac{1}{2}\varrho(\mathbf{x},t)|\mathbf{v}(\mathbf{x},t)|^2 + \tilde{\psi}(\varrho(\mathbf{x},t)) + \frac{\gamma}{2}|\nabla\varrho(\mathbf{x},t)|^2 \, d\mathbf{x}\Big) \\ &\quad + \int_D 2\mu \mathbf{D}(\mathbf{v}(\mathbf{x},t)) : \mathbf{D}(\mathbf{v}(\mathbf{x},t)) + \lambda(\mathrm{div}(\mathbf{v}(\mathbf{x},t)))^2 \, d\mathbf{x} = 0. \end{aligned} \tag{2.2.4}$$

Proof. As in the proof of Proposition 2.1.2 we multiply (2.2.1) with the gradient of the entropy W from (1.1.20), that is $(-|\boldsymbol{v}|^2/2 + \frac{d}{d\varrho}\tilde{\psi}(\varrho), \boldsymbol{v}^T)^T$ and integrate with respect to space. This gives all the expressions from (2.1.5) in (2.2.4). The only new contribution shows up from the third-order term in (2.2.1). For this term we compute with the boundary conditions (2.2.3) and the mass conservation equation

$$\begin{aligned} \gamma \int_D \varrho(\mathbf{x},t)\boldsymbol{v}(\mathbf{x},t) \cdot \nabla\Delta\varrho(\mathbf{x},t) \, d\mathbf{x} &= -\gamma \int_D \mathrm{div}(\varrho(\mathbf{x},t)\boldsymbol{v}(\mathbf{x},t))\Delta\varrho(\mathbf{x},t) \, d\mathbf{x} \\ &= \gamma \int_D \varrho_t(\mathbf{x},t)\Delta\varrho(\mathbf{x},t) \, d\mathbf{x} \\ &= -\frac{\gamma}{2}\frac{d}{dt}\int_D |\nabla\varrho(\mathbf{x},t)|^2 \, d\mathbf{x}. \end{aligned}$$

Thus we obtain the complete statement in (2.2.4). □

Let us consider equilibrium solutions for (2.2.1). For vanishing velocity, i.e., static conditions, the generalized energy in (2.2.4) reduces to the van-der-Waals energy

$$F^\varepsilon[\varrho] = \int_D \tilde{\psi}(\varrho(\mathbf{x})) + \frac{\gamma}{2}|\nabla\varrho(\mathbf{x})|^2 \, d\mathbf{x}. \tag{2.2.5}$$

With other words, the equilibrium solutions should be minimizers of the functional F^ε. If we fix for some $m > 0$ the mass in D, that is

$$\int_D \varrho(\mathbf{x}) \, dx = m, \tag{2.2.6}$$

there exists some constant C_ε such that the minimizers obey the elliptic equation

$$-\Delta\varrho + \frac{d}{d\varrho}\tilde{\psi}(\varrho) = C_\varepsilon. \tag{2.2.7}$$

The SI limit $\varepsilon \to 0$ should identify the static equilibrium (1.1.25) for (1.1.13). In fact, this has been proven for the one-dimensional case in [37]. Let $m/|D| \in \mathcal{A}_{\text{spinodal}}$. Then it turns out that the global minimizer is piecewise constant taking as values exactly the saturation states $\varrho_{\text{vap}}^{\text{sat}} = 1/\tau_{\text{vap}}^{\text{sat}}$ and $\varrho_{\text{liq}}^{\text{sat}} = 1/\tau_{\text{liq}}^{\text{sat}}$ from Definiton 1.1.2. The multidimensional case is technically much more evolved but the corresponding result can be found in the seminal work of [68].

2.2.2. A numerical illustration. The DI approach leads to one system on the entire domain D, which shares many properties with the one-phase compressible Navier–Stokes equations. As a consequence one might choose any appropriate scheme for the Navier–Stokes equations for the numerical discretization. For the NSK class we refer to [6, 18, 23, 34, 39, 47, 81], suggesting discontinuous Galerkin methods as well as finite-element or finite volume schemes. Let us conclude this section with a numerical example to illustrate the multi-phasic behaviour of the solutions.

Example 2.2.3. The example addresses the classical Navier–Stokes–Korteweg equations (2.2.1) in $D = (0,1)^2$. The computations were done with the discontinuous Galerkin method as described in [23]. The initial velocity vanishes, and the initial density field can be observed in the first picture of Figure 2.2. Note that blue colour corresponds to a low value of density (bubble) and red colour to a high value (surrounding liquid). The evolution of density is shown, after $t = 100.0$ equilibrium seems to be reached because the remaining bubble is located at the boundary with spherical shape in the bulk. Note that attaching to the boundary is energetically favourized in view of the free energy in Proposition 2.2.2. The parameters are chosen according to

$$\varepsilon = \sqrt{0.0001},\ \mu(\varepsilon) = \varepsilon,\ \nu(\varepsilon) = -\frac{2}{3}\mu,\ \gamma(\varepsilon) = \varepsilon^2.$$

Regardless of these achievements there remain typical numerical difficulties with the NSK systems (or more general DI models). The interfacial width of a diffuse phase boundary has to be resolved completely by the mesh to enable stable computations such that local mesh adaption has to be employed. The strong coupling of the interfacial parameter to physical quantitites might even force to take very small values for the interfacial parameter. The mixed hyperbolic-elliptic structure of the classical NSK models prevents the use of stabilised discretizations to cover convection-dominated flow regimes. Whereas thermodynamical consistency on the level of equations is straightforward (see Proposition 2.2.2), it is not on the discrete level.

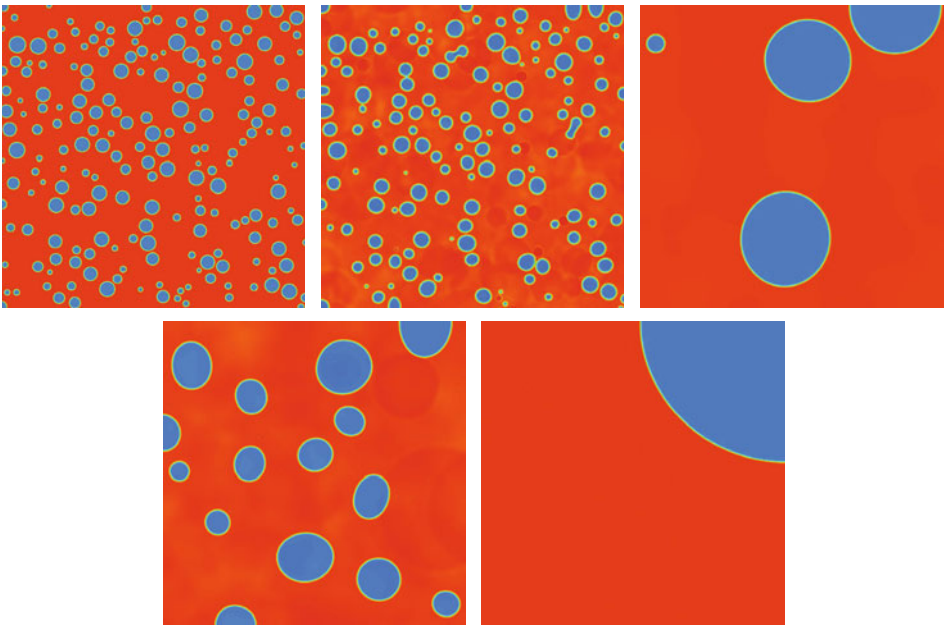


FIG. 2.2. Density distribution at times $t = 0.0, 0.2, 1.0, 4.0, 100.0$.

2.3. Relaxed Navier–Stokes–Korteweg equations for two-phase flow

2.3.1. Modelling and thermodynamical consistency. The classical NSK system involves third-order derivatives which makes it numerically quite complicated. Moreover the first-order part of the classical NSK system coincides with the hyperbolic-elliptic Euler system. Recall that the state space for the density components now includes $\tilde{\mathcal{A}}_{\text{spinodal}}$. As a consequence it is not possible to construct schemes that are stable as long as the mesh does not resolve the interfacial width scaling with the SI parameter ε.

Example 2.3.1. We consider the system (2.2.1) for $d = 1$ with $\gamma = \varepsilon^2$, $\lambda, \mu = \varepsilon$ and initial conditions

$$\varrho_0(x) = \begin{cases} 1.8, & x \in (0.3, 0.6) \\ 0.3, & \text{else} \end{cases}$$
$$v_0(x) = 0.$$

The system is solved with a discontinuous Galerkin approach (see [70] for details) using a uniform mesh with $h = 0.005$. In view of the mixed hyperbolic-elliptic structure which does not allow to use any more sophisticated flux function we choose a simple Lax–Friedrichs flux for the first-order part.

As solution we expect the evolution towards a two-phase equilibrium. We want to investigate if the discretization method can deal with tiny interfaces which appear in the SI limit ($\varepsilon \to 0$) for fixed h.

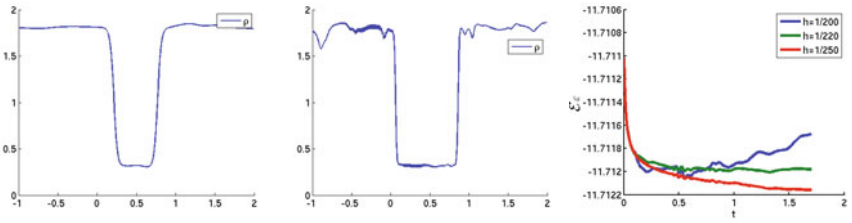


FIG. 2.3. We see the numerical density approximation at $t = 1.72$ for the NSK system for two different values of ε. The interface parameter ε is equal to $\varepsilon = 0.01$ and $\varepsilon = 0.001$. We observe severe oscillations for the NSK system. In the right figure the total energy evolution from Proposition 2.2.2 is plotted for various values of h, being oscillating in contrast to the analytical solution's behaviour. A computation with $\varepsilon = 0.0001$ failed due to the occurrence of negative density values.

Figure 2.3 shows, that the method for the NSK system (2.2.1) is not able to deal with phase transitions for $\varepsilon < 0.001 < h$.

Having in mind the shortcomings of the classical NSK system we discuss in this section a relaxed NSK system that has been suggested in modified form in [70, 73].

Let the numbers $\alpha, \beta > 0$ be given which we will refer to as the *Korteweg parameters*. We consider the *relaxed Navier–Stokes–Korteweg* (*R-NSK*) *system*

$$\begin{aligned} \varrho_t + \quad & \operatorname{div}(\varrho \mathbf{v}) \quad = 0, \\ (\varrho \mathbf{v})_t + \operatorname{div}(\varrho \mathbf{v} \otimes \mathbf{v} + \tilde{p}(\varrho)\boldsymbol{I}) &= \operatorname{div}(\mathbf{T}[v]) + \alpha \varrho \nabla(c - \varrho), \quad \text{in } D \times (0,T). \\ \beta c_t - \quad & \gamma \Delta c \quad = \alpha(\varrho - c) \end{aligned} \tag{2.3.1}$$

The system (2.3.1) extends the classical NSK system by a screened heat equation for the additional unknown $c = c(\mathbf{x}, t) \in \mathbb{R}$. The unknown c should be close to ϱ such that the initial conditions are chosen as

$$\varrho(\cdot, 0) = c(\cdot, 0) = \varrho_0, \, \boldsymbol{v}(\cdot, 0) = \boldsymbol{v}_0 \text{ in } D. \tag{2.3.2}$$

Note that (2.3.1) does not contain higher derivatives on ϱ. With $\mathbf{m} \in \mathbb{S}^{d-1}$ being the outer normal of ∂D, the boundary conditions from (2.2.3) with a Neumann condition on ϱ change to

$$\mathbf{v}(\cdot, t) = \mathbf{0}, \quad \mathbf{m} \cdot \nabla c(\cdot, t) = 0 \text{ in } \partial D. \tag{2.3.3}$$

Local well-posedness of classical solutions for (2.3.1), (2.3.2), (2.3.3) can be derived with standard contraction techniques, see, e.g., [73]. It is remarkable that there is a global existence result for weak solutions that applies also for two-phase pressure [15]. Before we go on to discuss the relation between the R-NSK system and the NSK system (2.2.1) let us present the following result on thermodynamical consistency for (2.3.1).

Proposition 2.3.2. *Let* $(\varrho, \mathbf{v}, c)$ *be a classical solution of* (2.3.1), (2.3.2), (2.3.3). *Then we have for* $t \in [0, T)$

$$\frac{d}{dt}\Big(\int_D \frac{1}{2}\varrho(\mathbf{x},t)|\mathbf{v}(\mathbf{x},t)|^2 + \tilde{\psi}(\varrho(\mathbf{x},t)) + \frac{\alpha}{2}(\varrho(\mathbf{x},t) - c(\mathbf{x},t))^2 + \frac{\gamma}{2}|\nabla c(\mathbf{x},t)|^2 \, d\mathbf{x}\Big)$$
$$+ \int_D 2\mu \mathbf{D}(\mathbf{v}(\mathbf{x},t)) : \mathbf{D}(\mathbf{v}(\mathbf{x},t)) + \lambda(\operatorname{div}(\mathbf{v}(\mathbf{x},t)))^2 \, d\mathbf{x} + \int_D c_t(\mathbf{x},t)^2 \, d\mathbf{x} = 0. \tag{2.3.4}$$

Proof. Multiply the first $d+1$ equations in (2.3.1) with $-\frac{1}{2}|\mathbf{v}|^2 + \frac{d}{d\varrho}\tilde{\psi}(\varrho)$, $v_1, \dots, v_d$ respectively, add up and integrate with respect to space. Using the first boundary condition in (2.3.3) we derive as in the other propositions

$$\begin{aligned}
&\frac{d}{dt}\int_D \frac{1}{2}\varrho(\mathbf{x},t)|\mathbf{v}(\mathbf{x},t)|^2 + W(\varrho(\mathbf{x},t)) \, d\mathbf{x} \\
&\qquad + \int_D 2\mu \mathbf{D}(\mathbf{v}(\mathbf{x},t)) : \mathbf{D}(\mathbf{v}(\mathbf{x},t)) + \lambda(\operatorname{div}(\mathbf{v}(\mathbf{x},t)))^2 \, d\mathbf{x} \\
&= \alpha \int_D \varrho(\mathbf{x},t)\mathbf{v}(\mathbf{x},t) \cdot \nabla(c(\mathbf{x},t) - \varrho(\mathbf{x},t)) \, d\mathbf{x} \\
&= -\alpha \int_D \operatorname{div}(\varrho(\mathbf{x},t)\mathbf{v}(\mathbf{x},t))(c(\mathbf{x},t) - \varrho(\mathbf{x},t)) \, d\mathbf{x} \\
&= \alpha \int_D \varrho_t(\mathbf{x},t)(c(\mathbf{x},t) - \varrho(\mathbf{x},t)) \, d\mathbf{x}.
\end{aligned} \tag{2.3.5}$$

For the last line we used the first equation in (2.3.1). The parabolic equation for c in (2.3.1) and the second condition in (2.3.3) yield after multiplication with c_t the relation

$$\begin{aligned}
\int_D (c_t(\mathbf{x},t))^2 \, d\mathbf{x} &= \int_D \gamma c_t(\mathbf{x},t)\Delta c(\mathbf{x},t) + \alpha c_t(\mathbf{x},t)(\varrho(\mathbf{x},t) - c(\mathbf{x},t)) \, d\mathbf{x} \\
&= -\gamma \frac{d}{dt}\int_D \frac{1}{2}|\nabla c|^2(\mathbf{x},t) \, d\mathbf{x} + \int_D \alpha c_t(\mathbf{x},t)(\varrho(\mathbf{x},t) - c(\mathbf{x},t)) \, d\mathbf{x}.
\end{aligned}$$

Thus we obtain with (2.3.5) the equation (2.3.4). □

Static equilibrium solutions of (2.3.1) are provided by minimizers of the functional

$$F^{\varepsilon,\gamma}[\varrho, c] = \int_D \tilde{\psi}(\varrho(\mathbf{x})) + \frac{\alpha}{2}(\varrho(\mathbf{x}) - c(\mathbf{x}))^2 + \frac{\varepsilon^2}{2}|\nabla c(\mathbf{x})|^2 \, d\mathbf{x}, \tag{2.3.6}$$

where we have put $\gamma(\varepsilon) = \varepsilon^2$. With other words, the equilibrium solutions should be minimizers of the functional $F^{\varepsilon,\alpha}$ if we prescribe mass as in (2.2.6). This functional has been suggested in [7] to approximate minimizers of the original van-der-Waals functional F^{ε} in (2.2.5). Being regular enough the minimizers satisfy for some constant $C_{\varepsilon,\alpha}$ the following Euler–Lagrange system

$$\frac{d}{d\varrho}\tilde{\psi}(\varrho) + \alpha(\varrho - c) = C_{\varepsilon,\alpha}, \qquad \varepsilon^2 \Delta c = \alpha(\varrho - c). \tag{2.3.7}$$

The SI limit $\varepsilon \to 0$ should identify again the static equilibrium (1.1.25) for (1.1.13). This is also part of the work in [7]. Let $m/|D| \in \mathcal{A}_{\text{spinodal}}$. Then in complete analogy to [68] the SI limit in multiple space dimensions has been analyzed in [76]. In this work it is also proven that minimizers of (2.3.6) tend for Korteweg parameter $\alpha \to \infty$ towards minimizers of the van-der-Waals functional (2.2.5). For the evolutionary system (2.3.1) similar results can be found in [15, 27, 33].
One might interpret the R-NSK system by having a different physical meaning than the original NSK system in the sense that it models different long-range interactions between the fluid particles, see also Remark 2.3.3 below. Here we view it as an approximate system for the NSK system which is numerically much more tractable: The R-NSK system contains only low-order local differential operators. The price to pay is an additional equation for the artificial unknown c. But this equation is a simple linear parabolic equation which can be solved extremely fast numerically, at least if a fixed mesh is used. But there is another issue which makes (2.3.1) attractive from the numerical point of view. Setting $\lambda = \mu = 0$ one can rewrite the momentum equations in (2.3.1) as

$$(\varrho \mathbf{v})_t + \operatorname{div}(\varrho \mathbf{v} \otimes \mathbf{v} + \tilde{p}_\alpha(\varrho)\boldsymbol{I}) = \alpha \varrho \nabla c, \tag{2.3.8}$$

with

$$\tilde{p}_\alpha(\varrho) = \tilde{p}(\varrho) + \frac{\alpha}{2}\varrho^2. \tag{2.3.9}$$

If the Korteweg parameter α satisfies

$$2\alpha > \left| \min \left\{ \frac{d^2}{d\varrho^2}\tilde{\psi}(r) \,\middle|\, r \in \tilde{\mathcal{A}}_{\text{spinodal}} \right\} \right|,$$

the relations (1.1.1) show that $\tilde{p}'_\alpha$ is monotone increasing. Using the splitting from (2.3.8) one can then modify the first-order part of the R-NSK system to become hyperbolic which is not possible for the NSK system. This property paves the way to use various numerical techniques from the theory of hyperbolic conservation laws. After a short remark on the physical interpretation of (2.3.1) we present two examples to underline this claim.

Remark 2.3.3. For $\beta = 0$ the equation $(2.3.1)_3$ for the unknown c is a linear elliptic equation with constant coefficients. It can be solved by convolution of the source term with the Green kernel. For, e.g., $d = 1$ one obtains

$$c(x,t) = [K^{\sqrt{\frac{\gamma}{\alpha}}} * \varrho(\cdot,t)](x),$$

where $K^\delta(x) = \frac{1}{2\delta}\exp(-|x|/\delta)$ is the kernel function. This expression can be used to substitute c into the momentum equation, leading to a nonlocal NSK system for unknowns density and velocity alone, i.e., the momentum equation writes as

$$(\varrho v)_t + \big(\varrho(v)^2 + \tilde{p}(\varrho)\big)_x = (\lambda + 2\nu)v_{xx} + \alpha\varrho\big([K^{\sqrt{\frac{\gamma}{\alpha}}} * \varrho(\cdot,t)] - \varrho\big)_x.$$

Nonlocal NSK systems starting from nonlocal van-der-Waals energies have been also suggested in [72]. In fact, the local van-der-Waals energy is only considered in the original work of van der Waals [83] as an approximation of the nonlocal ones.

This re-formulation as a nonlocal model can provide a physical interpretation of the R-NSK model, relating the Korteweg parameter α to micro-scale dimensions. For the equilibrium case this is discussed in [7].

Example 2.3.4. We perform the computations from Example 2.3.1 now for the R-NSK system (2.3.1) with the same parameter settings. For the Korteweg parameters we have chosen $\beta = 0, \alpha = 100$.

Using the re-formulation (2.3.8) we apply an upwind Roe-like flux for the first-order Euler sub-system. This enables us to perform computations for under-resolved interfaces without having instabilities like in Example 2.3.1.

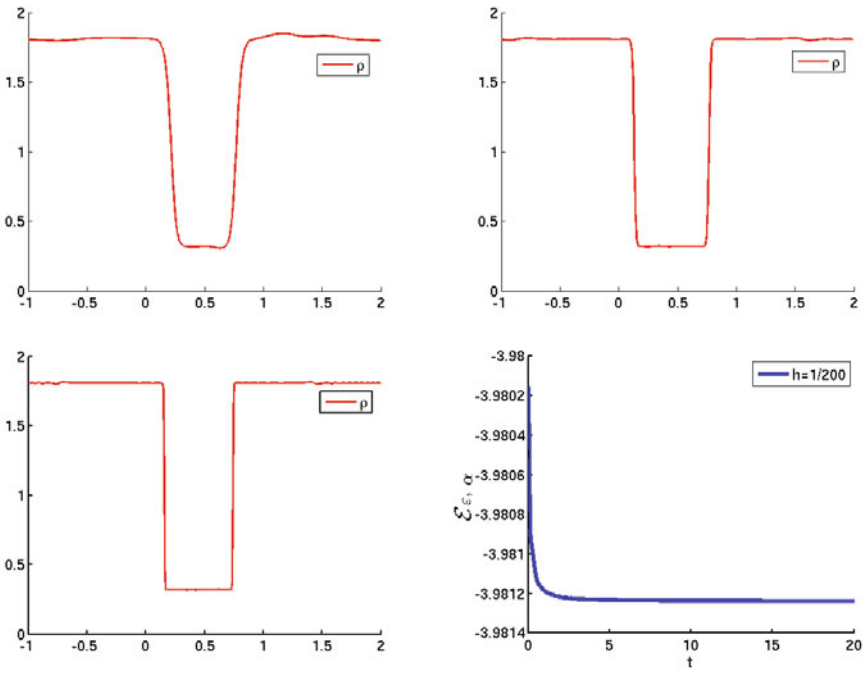


FIG. 2.4. We see the density at $t = 1.72$ for the R-NSK system for three different values of ε ($\varepsilon = 0.01, 0.001, 0.0001$). Although the diffuse interface cannot be resolved on the given mesh the computations remain stable. In the last figure the total energy evolution from Proposition 2.3.2 is plotted, providing now a monotone decreasing behaviour as predicted for the analytical solution.

Example 2.3.5. We compute numerical solutions to the R-NSK system (2.3.1) in two space dimensions to illustrate the effect of merging bubbles. This scenario can be described with the DI ansatz. We start with two spherical vapour bubbles with radii 0.3, 0.2, placed initially close to the center of the domain $D = (-1, 1)^2$,

which is otherwise filled with liquid. Figure (2.5) shows the density distribution at different times. This test case indicates that the R-NSK system is also able to describe the qualitative behaviour of a compressible two-phase flow correctly.

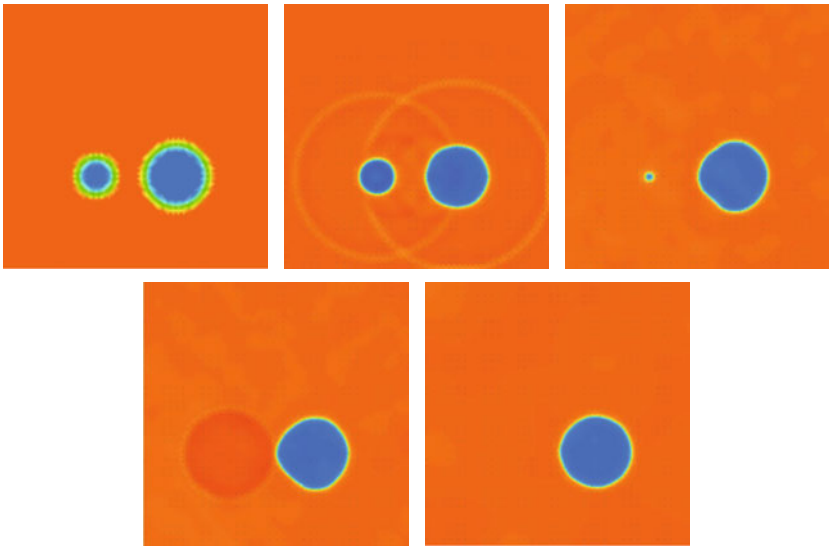


FIG. 2.5. Density distribution: The density varies between 0.3 (blue) and 1.8 (red). The figures show the density field at times $t = 0.25, 2.0, 2.3, 40$. At $t = 0.25$, two shock wave layer run off from the two bubbles. Then the smaller bubble shrinks and the bigger bubble grows. At $t = 2.3$ the small bubble collapses and emits a shock wave. At $t = 40$ the material seems to be close to a spherical state.

2.3.2. A thermodynamically consistent finite-volume scheme. The numerical computations from Examples 2.3.4, 2.3.5 appear to be quite promising. From the mathematical point of view it would be preferable to design a scheme with rigorously proven properties. We transfer the ideas for hyperbolic conservation laws from Section 2.1.2 to the R-NSK model (2.3.1). The hyperbolic structure of (2.3.1), see (2.3.8), is the most important ingredient in this case.

As in Section 2.1 we consider the one-dimensional situation, i.e.,

$$\begin{pmatrix} \varrho \\ \varrho v \end{pmatrix}_t + \begin{pmatrix} \varrho v \\ \varrho v^2 + \tilde{p}_\alpha(\varrho) \end{pmatrix}_x = \begin{pmatrix} 0 \\ \nu v_{xx} + \alpha \varrho c_x \end{pmatrix}, \quad \text{in } \mathbb{R} \times (0,T). \tag{2.3.10}$$
$$\beta c_t - \gamma c_{xx} = \alpha(c - \varrho)$$

In (2.3.10) we have used again $\nu = \lambda + 2\mu$. Now, let the Korteweg parameter $\alpha > 0$ be chosen such that $\tilde{p}'_\alpha(\varrho) > 0$ holds for all $\varrho \in (0, \bar{\varrho})$. Then the first-order conservation law

$$\mathbf{U}_t + \boldsymbol{F}_\alpha(\mathbf{U})_x = 0 \tag{2.3.11}$$

becomes hyperbolic in $\tilde{\mathcal{U}} = (0, \bar{\varrho}) \times \mathbb{R}$. Here we used as above $\mathbf{U} = (\varrho, m = \varrho v)^T$ but $\boldsymbol{F}_\alpha(\mathbf{U}) = (\varrho v, \varrho v^2 + \tilde{p}_\alpha(\varrho))^T$. With other words, for this sub-system of (2.3.10) we have exactly the same structure as for (2.1.7). Let us also define the corresponding entropy-entropy flux pair function

$$W_\alpha(\varrho, m) = \tilde{\psi}_\alpha(\varrho) + \frac{m^2}{2\varrho}, \quad Q_\alpha(\varrho, m) = \frac{m}{\varrho}\Big(\tilde{\psi}_\alpha(\varrho) + \tilde{p}_\alpha(\varrho)\Big), \tag{2.3.12}$$

which substitutes (W, Q) from (2.1.8). The entropy induces a potential Ψ_α and thus we can define a family of entropy-conservative numerical fluxes $\mathbf{g}^*_\alpha$ for (2.3.11) by (2.1.15).

With these preparations we construct now a numerical scheme for the relaxed NSK system (2.3.10).

For $\mathbf{V}, \mathbf{Z} \in \tilde{\mathcal{V}}$ let

$$h^*_\alpha(\mathbf{V}, \mathbf{Z}) = \begin{cases} \frac{g^*_{\alpha,1}(\mathbf{V},\mathbf{Z})}{V_2} & : \quad V_2 \neq 0, \\ \psi'^{-1}_\alpha\left(V_1 + \frac{V_2}{2}\right) & : \quad V_2 = 0. \end{cases} \tag{2.3.13}$$

For $j \in \mathbb{Z}$ we seek for the functions $(\varrho_j, m_j, c_j) : [0, T) \to \mathcal{U} \times \mathbb{R}$, solving the initial value problem

$$\begin{aligned} \begin{pmatrix} \varrho_j'(t) \\ m_j'(t) \end{pmatrix} = & -\frac{1}{h}\begin{pmatrix} g^*_{\alpha,j+\frac{1}{2},1}(t) - g^*_{\alpha,j-\frac{1}{2},1}(t) \\ g^*_{\alpha,j+\frac{1}{2},2}(t) - g^*_{\alpha,j-\frac{1}{2},2}(t) \end{pmatrix} \\ & + \frac{\nu}{h^2}\begin{pmatrix} 0 \\ v_{j+1}(t) - 2v_j(t) + v_{j-1}(t) \end{pmatrix} \qquad (t \in (0, T)) \\ & + \frac{\alpha}{h}\begin{pmatrix} 0 \\ h^*_\alpha(\mathbf{V}_j(t), \mathbf{V}_{j+1}(t))\big(c_{j+1}(t) - c_j(t)\big) \end{pmatrix}, \\ \beta c_j'(t) - \frac{\gamma}{h^2}\left(c_{j+1}(t) - 2c_j(t) + c_{j-1}(t)\right) = & \alpha(\varrho_j(t) - c_j(t)) \end{aligned} \tag{2.3.14}$$

and

$$\varrho_j(0) = c_j(0) = \int_{K_j} \varrho_0(x)\, dx, \quad m_j(0) = \int_{K_j} \varrho_0(x) v_0(x)\, dx. \tag{2.3.15}$$

For this construction we can determine an entropy conservative numerical flux $\mathbf{g}^*_\alpha$ such that we obtain

Theorem 2.3.6 (Discrete energy inequality for two-phase flow). *Let $\varrho_0 v_0^2$, ϱ_0^2, $W(\varrho_0) \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$. For $j \in \mathbb{Z}$ let a solution $(\varrho_j, m_j, c_j) : [0, T] \to (0, \bar{\varrho}) \times \mathbb{R} \times \mathbb{R}$ of* (2.3.15), (2.3.15) *be given with*

$$\mathbf{r}^1 = (1, 0)^T,\ \mathbf{r}^2 = (-1, 1),\ \mathbf{l}^1 = (1, 1)^T,\ \mathbf{l}^2 = (0, 1)^T \tag{2.3.16}$$

in the flux formula (2.1.15).

Then we have for each $t \in [0,T)$ the discrete energy inequality

$$\sum_{j\in\mathbb{Z}} \left(\frac{(m_j(t))^2}{2\varrho_j(t)} + \tilde{\psi}(\varrho_j(t)) + \frac{\alpha}{2}\left((\varrho_j(t) - c_j(t))^2 + \frac{\gamma}{2h^2}\left(c_{j+1}(t) - c_j(t)\right)^2\right)\right.$$
$$\left. \leq \sum_{j\in\mathbb{Z}} \left(\frac{(v_j(0))^2}{2} + \tilde{\psi}(\varrho_j(0)) + \frac{\gamma}{2h^2}\left(c_{j+1}(0) - c_j(0)\right)^2\right)\right. . \tag{2.3.17}$$

Proof. Consider the first component of the numerical flux $\mathbf{g}^*_\alpha(\mathbf{V},\mathbf{Z})$ for $\mathbf{V},\mathbf{Z} \in \tilde{\mathcal{V}}$ with $V_2 = 0$. Then we have from (2.1.15) with the choice (2.3.16), in particular for $\mathbf{l}^2$

$$\begin{aligned} g^*_{\alpha,1}(\mathbf{V},\mathbf{Z}) &= \frac{\tilde{\Psi}_\alpha\left(\overline{\mathbf{VZ}}\right) - \Psi_\alpha(\mathbf{V})}{\mathbf{l}^1\cdot(\mathbf{V}-\mathbf{Z})}\mathbf{l}^1_1 + \frac{\Psi_\alpha(\mathbf{Z}) - \Psi_\alpha\left(\bar{\mathbf{V}}\mathbf{Z}\right)}{\mathbf{l}^2\cdot(\mathbf{V}-\mathbf{Z})}\mathbf{l}^2_1 \\ &= \frac{\Psi_\alpha\left(\bar{\mathbf{V}}\mathbf{Z}\right) - \Psi_\alpha(\mathbf{V})}{\mathbf{l}^1\cdot(\mathbf{V}-\mathbf{Z})}\mathbf{l}^1_1. \end{aligned}$$

Moreover we observe from the path definition (2.1.14) and $\mathbf{r}^1 = (1,0)^T$ that $\overline{WZ}_2 = W_2 = 0$. Then the definition of the potential Ψ_α from (2.1.11) implies

$$V_2 = 0 \Rightarrow g^*_{\alpha,1}(\mathbf{V},\mathbf{Z}) = 0. \tag{2.3.18}$$

Now we consider the scheme (2.3.2) and multiply the two evolution equations by $\mathbf{V}_j(t)$. Similarly, as in Theorem 2.1.4, Theorem 2.1.3 ensures the existence of functions $q^*_\alpha = q^*_\alpha(\mathbf{V},\mathbf{Z})$ such that

$$\begin{aligned} &W_\alpha(\mathbf{U}_j(t))' + \frac{1}{h}\left(q^*_\alpha(\mathbf{V}_j(t),\mathbf{V}_{j+1}(t) - q^*_\alpha(\mathbf{V}_{j-1}(t),\mathbf{V}_j(t))\right) \\ &\quad = \frac{\varepsilon\beta}{h^2}\begin{pmatrix} 0 \\ v_{j+1}(t) - 2v_j(t) + v_{j-1}(t) \end{pmatrix}\cdot\mathbf{V}_j(t) \\ &\qquad + \frac{\alpha}{h}\begin{pmatrix} 0 \\ h^*_\alpha(\mathbf{V}_j(t),\mathbf{V}_{j+1}(t))\big(c_{j+1}(t) - c_j(t)\big) \end{pmatrix}\cdot\mathbf{V}_j(t) \\ &\quad = \frac{\varepsilon\beta}{h^2}\begin{pmatrix} 0 \\ v_{j+1}(t) - 2v_j(t) + v_{j-1}(t) \end{pmatrix}\cdot\mathbf{V}_j(t) \\ &\qquad + \frac{\alpha}{h}\begin{pmatrix} 0 \\ g^*_{\alpha,1}(\mathbf{V}_j(t),\mathbf{V}_{j+1}(t))\big(c_{j+1}(t) - c_j(t)\big) \end{pmatrix}\cdot\mathbf{V}_j(t). \end{aligned}$$

The last line follows from the definition (2.3.13) of h^*_α and in case of $V_{j,2} = 0$ from (2.3.18) above.

In the next step we sum up with respect to $j \in \mathbb{Z}$ and obtain with summation by parts

$$\begin{aligned}\sum_{j\in\mathbb{Z}} W_\alpha(\mathbf{U}_j(t))' &\le \frac{\alpha}{h}\sum_{j\in\mathbb{Z}} g^*_{\alpha,j+\frac12,1}(t)(c_{j+1}(t)-c_j(t))\\ &= -\frac{\alpha}{h}\sum_{j\in\mathbb{Z}}\Big(g^*_{\alpha,j+\frac12,1}(t)-g^*_{\alpha,j-\frac12,1}(t)\Big)c_j(t)\\ &= \alpha\sum_{j\in\mathbb{Z}}\varrho_j'(t)c_j(t).\end{aligned} \tag{2.3.19}$$

For the last equality we used the evolution equation for ϱ_j in (2.3.2). We turn to the equation for c_j which we multiply with c_j' and add up with respect to $j \in \mathbb{Z}$. This gives again using summation by parts and shifting indices

$$\beta\sum_{j\in\mathbb{Z}}(c_j'(t))^2 + \frac{\gamma}{2h^2}\sum_{j\in\mathbb{Z}}\bigg(\Big(c_{j+1}(t)-c_j(t)\Big)^2\bigg)' = \frac{\alpha}{2}\sum_{j\in\mathbb{Z}}\Big(2\varrho_j(t)c_j'(t)-((c_j(t))^2)'\Big). \tag{2.3.20}$$

Adding up inequality (2.3.19) and equation (2.3.20) we get the result (2.3.17). Note that we have skipped in the result all dissipative contributions. □

Example 2.3.7. In this experiment we verify the statement from Theorem 2.3.6 for the semi-discrete finite volume scheme (2.3.15), (2.3.15) that provides approximate solutions for the R-NSK system (2.3.1). In the semi-discrete case the relevant free energy from Proposition 2.3.2 is not monotone increasing in time.

We consider the system (2.3.1) for $D = (0,1)$ with $\gamma = 0.001$, $\lambda, \mu = 0.1$ and $\alpha = 100, \beta = 0$. The initial conditions are

$$\begin{aligned}\varrho_0(x) &= \begin{cases} 1.8, & x \in (0.15, 0.55) \cup (0.665, 0.735)\\ 0.3, & \text{else}\end{cases}\\ v_0(x) &= 0.\end{aligned}$$

The system is solved for periodic boundary conditions, using an explicit Euler stepping in time on a uniform mesh with mesh parameter $h = 0.005$. The numerical solutions for different times are displayed in Figure 2.6. One observes the evolution towards a two-phase situation with exactly two phase boundaries. In Figure 2.7 the temporal evolution of the total energy quantity

$$S(t) = \int_D \frac12\varrho(x,t)|v(x,t)|^2 + \tilde\psi(\varrho(x,t)) + \frac{\alpha}{2}(\varrho(x,t)-c(x,t))^2 + \frac{\gamma}{2}|\nabla c(x,t)|^2\,dx$$

is shown. Although Theorem 2.3.6 does not cover the fully-discrete scheme the energy dissipation is clearly observed.

2.4. Well-balanced Navier–Stokes–Korteweg equations for two-phase flow

We have motivated the NSK system (2.3.1) by numerical arguments. As an approximation of the original NSK system (2.2.1) it has a clear physical meaning. The next instance of a Navier–Stokes–Korteweg model, it is called well-balanced Navier–Stokes–Korteweg (WB-NSK) system, is solely introduced to cure a numer-

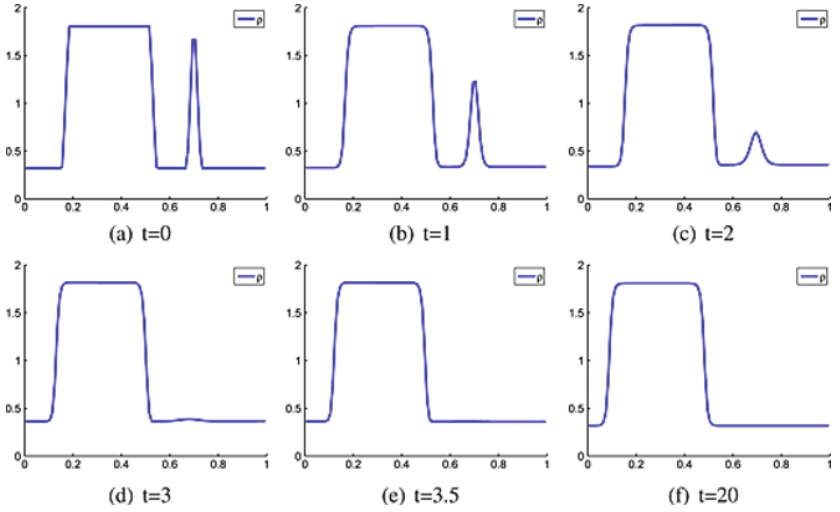


FIG. 2.6. Evolution of the density component $\varrho_h(\cdot, t)$.

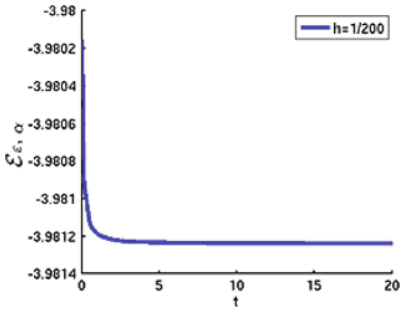


FIG. 2.7. Evolution of the total energy $S(t)$.

ical problem that comes with the original NSK system: the occurrence of parasitic currents.

Example 2.4.1. We consider a standard scheme for the NSK system (2.2.1) with $d = 2$, in fact a finite-volume scheme as in [23] (choosing the polynomial degree of the ansatz spaces in the paper to be 0). In Figure 2.8 we see the results of a computation of an initially static bubble of quadratic shape towards a spherical configuration. The latter one is close to a static equilibrium. One observes clearly parasitic currents close to the phase boundary layer. In other words, the analytical equilibrium is not an equilibrium on the discrete level. Let us however point out that the oscillating behaviour would vanish under refinement of the mesh.

What is the reason for the parasitic current behaviour? We conjecture that the parasitic currents are driven by the numerical dissipation that is inherent in

any scheme. Let us assume that the numerical dissipation has the form of artificial diffusion. Then the modified equation for the density component might be typically of the form

$$\varrho_t + \operatorname{div}(\varrho \mathbf{v}) = \delta(h)\Delta\varrho + o(h), \tag{2.4.1}$$

with $\delta(h) = \mathcal{O}(h)$ for a first-order scheme. If this holds true there is the following explanation for the parasitic currents. Smooth static analytical equilibria of (2.2.1) obey the elliptic equation (2.2.7). In this situation the leading term on the right-hand side of (2.4.1) cannot vanish: it triggers always variations in the density field, and thus in the momentum field. Let us conclude this example with the remark

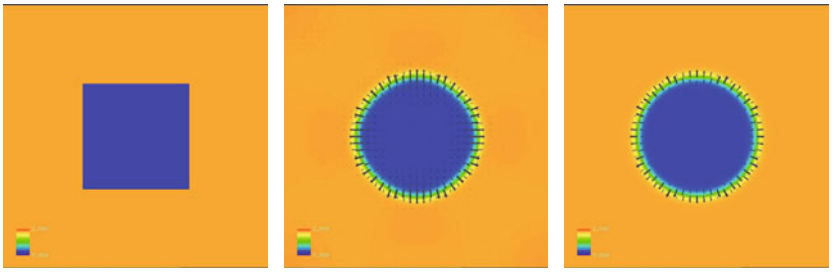


FIG. 2.8. Density fields for $t = 0, 5.0, 10.0$, computed by a standard scheme for (2.2.1).

that curing of parasitic currents is a major topic in numerical two-phase flow, see the recent review article [71].

If the arguments from Example 2.4.1 are correct, this suggests also a cure of the parasitic current problem: tune the numerical diffusion in a way such that the numerical dissipation term vanishes close to an equilibrium. This is exactly the idea of the WB-NSK system. For the unknowns density $\varrho = \varrho(\mathbf{x}, t) : D \times [0, T] \to (0, \bar{\varrho})$, velocity $\mathbf{v} = \mathbf{v}(\mathbf{x}, t) : D \times [0, T] \to \mathbb{R}^d$ the *well-balanced Navier–Stokes–Korteweg* (*WB-NSK*) system is given for some $\delta > 0$ by

$$\begin{aligned} \varrho_t + \qquad \operatorname{div}(\varrho \mathbf{v}) \qquad &= \delta\Delta\Big(-\gamma\Delta\varrho - \tfrac{1}{2}|\boldsymbol{v}|^2 + \tfrac{d}{d\varrho}\tilde{\psi}(\varrho)\Big), \\ (\varrho\mathbf{v})_t + \operatorname{div}(\varrho\mathbf{v}\otimes\mathbf{v} + p(\varrho)\boldsymbol{I}) &= \operatorname{div}(\mathbf{T}) + \gamma\varrho\nabla\Delta\varrho \end{aligned} \quad \text{in } D \times (0, T). \tag{2.4.2}$$

The initial conditions for system (2.4.2) are

$$\varrho(\cdot, 0) = \varrho_0,\ \boldsymbol{v}(\cdot, 0) = \boldsymbol{v}_0 \text{ in } D. \tag{2.4.3}$$

For a bounded domain D boundary conditions for the fourth-order system (2.4.2) are chosen to be

$$\mathbf{v}(\cdot, t) = \mathbf{0}, \quad \mathbf{m} \cdot \nabla\varrho(\cdot, t), \quad \mathbf{m} \cdot \nabla\Delta\varrho(\cdot, t) = 0 \text{ in } \partial D. \tag{2.4.4}$$

Thermodynamical consistency requires for the two-phase regime a generalized energy with non-local contributions. Precisely we have

Proposition 2.4.2. *Let* $(\varrho, \mathbf{v})$ *be a classical solution of* (2.4.2), (2.4.3), (2.4.4). *Then we have for all* $t \in [0,T]$

$$\begin{aligned}
&\frac{d}{dt}\Big(\int_D \frac{1}{2}\varrho(\cdot,t)|\mathbf{v}(\cdot,t)|^2 + \tilde{\psi}(\varrho(\cdot,t)) + \frac{\gamma}{2}|\nabla\varrho(\cdot,t)|^2\,d\mathbf{x}\Big)\\
&\quad + \int_D 2\mu\mathbf{D}(\mathbf{v}(\cdot,t)) : \mathbf{D}(\mathbf{v}(\cdot,t)) + \lambda(\mathrm{div}(\mathbf{v}(\cdot,t)))^2\,d\mathbf{x}\\
&\qquad = -\int_D \delta\Big|\nabla\Big(-\gamma\Delta\varrho(\cdot,t) - \frac{1}{2}|\boldsymbol{v}(\cdot,t)|^2 + \frac{d}{d\varrho}\tilde{\psi}(\varrho(\cdot,t))\Big)\Big|^2\,d\mathbf{x}.
\end{aligned} \tag{2.4.5}$$

Proof. Once again we multiply now (2.4.2) with the gradient of the entropy W from (1.1.20), that is $(-|\boldsymbol{v}|^2/2 + \frac{d}{d\varrho}\psi(\varrho), \boldsymbol{v}^T)^T$ and integrate with respect to space. This gives all the expressions from (2.1.5) in (2.2.4) and two different terms. From the multiplication of $(2.4.2)_1$ with $-|\boldsymbol{v}|^2/2 + \frac{d}{d\varrho}\tilde{\psi}(\varrho)$ we have the additional term

$$\begin{aligned}
&\delta\int_D \Big(-\frac{|\boldsymbol{v}|^2}{2} + \frac{d}{d\varrho}\tilde{\psi}(\varrho)\Big)\Delta\Big(-\gamma\Delta\varrho - \frac{1}{2}|\boldsymbol{v}|^2 + \frac{d}{d\varrho}\tilde{\psi}(\varrho)\Big)\,d\mathbf{x}\\
&\quad = \delta\int_D -\Big|\nabla\Big(-\frac{|\boldsymbol{v}|^2}{2} + \frac{d}{d\varrho}\tilde{\psi}(\varrho)\Big)\Big|^2 - \gamma\Big(-\frac{|\boldsymbol{v}|^2}{2} + \frac{d}{d\varrho}\tilde{\psi}(\varrho)\Big)\Delta\Delta\varrho\,d\mathbf{x}.
\end{aligned} \tag{2.4.6}$$

Note that we have used here the boundary conditions (2.4.4) during the partial integration. For the third-order term in (2.4.2) we get also a more complex contribution. For this term we compute with the boundary conditions (2.4.4) and the extended mass conservation equation $(2.4.2)_1$

$$\begin{aligned}
\gamma\int_D \varrho\boldsymbol{v}\cdot\nabla\Delta\varrho\,d\mathbf{x} &= -\gamma\int_D \mathrm{div}(\varrho\boldsymbol{v})\Delta\varrho\,d\mathbf{x}\\
&= \quad\gamma\int_D \varrho_t\Delta\varrho\,d\mathbf{x}\\
&\quad -\gamma\int_D \delta\Delta\Big(-\gamma\Delta\varrho - \frac{1}{2}|\boldsymbol{v}|^2 + \frac{d}{d\varrho}\tilde{\psi}(\varrho)\Big)\Delta\varrho\,d\mathbf{x}\\
&= -\frac{\gamma}{2}\frac{d}{dt}\int_D |\nabla\varrho(\mathbf{x},t)|^2\,d\mathbf{x}\\
&\quad -\gamma\int_D \delta\Delta\Big(-\gamma\Delta\varrho - \frac{1}{2}|\boldsymbol{v}|^2 + \frac{d}{d\varrho}\tilde{\psi}(\varrho)\Big)\Delta\varrho\,d\mathbf{x}.
\end{aligned}$$

Combining the last formula with the term in (2.4.6) gives in particular a perfect square as on the right-hand side of (2.4.5), the proposition is proven. □

Proposition 2.4.2 shows that at least classical solutions of the WB-NSK system (2.4.2) dissipate the same entropy term as the solutions of NSK system (2.2.1), albeit with a different entropy dissipation rate. If we discretize now (2.4.2) with a numerical scheme of higher order (such that the numerical diffusion scales in higher order with respect to h) and let depend $\delta = \delta(h)$ in an appropriate way on the mesh parameter we could hope that the right-hand side term of (2.4.2)

dominates the diffusion in the scheme. By construction it should also vanish for equilibrium solutions.

Example 2.4.3. We repeat the set-up of Example 2.4.1 with a higher-order method, discretizing now the system (2.4.2) with $\delta = h$. The resulting scheme looses its high order but it is able to damp parasitic currents. That is clearly demonstrated in Figure (2.9).

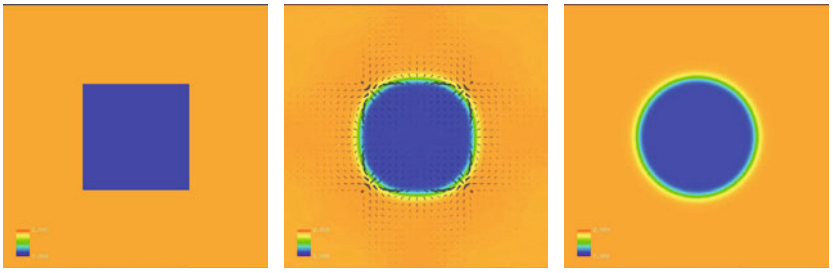


FIG. 2.9. Density fields for $t = 0, 5.0, 10.0$ computed by a scheme for 2.4.2. Parasitic currents are suppressed.

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Christian Rohde
Universität Stuttgart
Institut für Angewandte Analysis
und Numerische Simulation
Pfaffenwaldring 57
D-70569 Stuttgart, Germany
e-mail: christian.rohde@mathematik.uni-stuttgart.de