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Supersymmetry in Mathematics and Physics

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Editors

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Preface

The articles in this collection have grown out of the talks given at a 2-day workshop on supersymmetry held at UCLA in February 2010. The main theme of the Conference was supersymmetry in mathematics and physics. More precisely, the talks at the conference were dedicated to supersymmetry as a connecting theme between geometry, group theory, and fundamental physics. In addition to the speakers there were a few posters displayed in a poster session whose contents are also presented here.

In an introduction to this volume a brief survey of supersymmetry and its many applications as developed in the articles of this volume is given. So we limit ourselves to thanking the various people and organizations that have helped us and made the workshop and this volume possible.

First and foremost we thank the Department of Mathematics at UCLA for providing all kinds of help and support, organizational, technical, and personal, for the conference. To Chair Professor Sorin Popa and Chief administrative person Judith Levin goes our gratitude. We would like to thank Babette Dalton who took care of all the details of the conference; without her wholehearted cooperation and tireless work the workshop could not have been arranged. We thank Jacquie Bauwens for help during the 2 days of the workshop. We are also grateful to the participants, whose lively interest and interactions with one another made the workshop a great success. Last but not least, we thank profusely the people at Springer who were most forthcoming with their advice, technical and personal, as well as their cooperation, during the preparation of this volume.

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Finally, we thank the participants who provided us with the actual articles and who responded to all our requests, large and small, with infinite patience and courtesy.

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Introduction

V.S. Varadarajan

There is still no direct evidence that supersymmetry is a symmetry of the physical world, that elementary particles arrange themselves in supermultiplets of spin differing by half a unit. It must be broken, since if unbroken it would predict that the particles in a supermultiplet have the same mass. Finding the correct breaking mechanism is probably still the basic unsolved problem.

Bruno Zumino

Superalgebras serve as the basis for the construction of geometric objects, such as superprojective spaces and supermanifolds. The theory has applications to supergravitation in physics and it is studied by supermathematicians.

Igor Shafarevich

The axiomatic basis of theoretical physics cannot be extracted from experience but must be freely invented.

Albert Einstein

1 Some General Remarks

The purpose of this brief introduction is to give a bird's eye view of supersymmetry for a general mathematical audience. In particular it will include a brief outline of some of the themes that have emerged in recent (and not so recent) work in

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supersymmetry. Some of these themes are addressed in the contributions to the workshop that are contained in the following pages. Others are briefly mentioned in the last part of the introduction as part of a look into the future. No completeness is aimed for in these remarks. In view of the stated aim we shall also discuss ideas that are well understood by specialists but may not be familiar to the general mathematician. I feel this is necessary to create a greater appreciation of supersymmetric themes in mathematics and physics among mathematicians.

Supersymmetry was invented by the physicists to provide a unified way of understanding the behavior of the two basic constituents of the physical world, the *fermions* and the *bosons*. A unique and very striking feature of the evolution of supersymmetry is that it was not based on any experimental results; only theoretical considerations guided its entire development. Indeed, whether or not supersymmetry is an actual symmetry of the physical world is still an undecided issue. The reason for this is that the energy scale at which supersymmetry might have been broken is still very much higher than what is currently attainable in the laboratories. Nevertheless there are some grounds [1, 2] for hoping that the 10–15 Tev range reached by the new LHC (Large Hadron Collider) at CERN may allow some of the predictions of supersymmetry to be checked, and especially, for hoping that one of its most striking predictions, the existence of super partners of the known elementary particles, may be experimentally verified.

The ideas of the physicists led to the creation of supersymmetric electrodynamics [3] and supersymmetric Yang-Mills theory [4], and even to the creation of a supersymmetric version of Einstein gravity [5,6]. In their hands supersymmetry led to the discovery of a new type of geometrical object, namely, a *supermanifold* [7]. The next step was for mathematicians to formalize these discoveries, and then, as a natural generalization, to introduce the concept of a *superscheme*. The superscheme brings a deep unification of the Grothendieck theory of geometry with modern physics at its deepest level. Viewed from this perspective a superscheme is the natural end point of the historical evolution of geometry that originally started with Euclid, and was continued by the discoveries of Bolyai-Lobachevsky, Riemann, and eventually Grothendieck. However it did not stop there and was continued into the super world by the ideas of Salaam–Strathdee, Berezin, Kostant, Manin, Deligne, Leites, and many others. One may consult [8] for a more detailed account of this evolution and additional references. From this point of view, super Lie groups and super algebraic groups are the group objects in the category of supermanifolds and superschemes, and the automorphisms of supergeometric objects are the supersymmetries. The symmetries of the super world are described by unitary representations of super Lie groups, and one can even ask if the supersymmetric world can allow deformations that describe even more general worlds. These are the themes that are the concern of the articles collected together in this volume.

The beauty and esthetic completeness of supergeometry, supergroup theory, and the supersymmetric field theories, are compelling enough for us to believe strongly in them, in spite of the uncertainty surrounding the presence of supersymmetry in Nature. In some sense this may be viewed as contrary to usual modes of thought in physics where the experiments have generally guided the theory. However, starting with Einstein's ideas on space-time and the famous Heisenberg-Dirac insight that the physical quantities are represented by elements of a non-commutative algebra, it has become more and more clear that the mathematical concepts must first be introduced before the dictionary with the physical world is established. One of the most striking instances of this is the idea of Dirac on magnetic monopoles. This way of proceeding has been described by the famous physicist Y. Nambu as the *Dirac mode of thought*[1].

2 Bosons and Fermions: The Emergence of a $\mathbf{Z}_2$-Graded World

In classical mechanics particles are treated as points. However in quantum mechanics, particles have internal structures. *Spin* is an example of such a structure. It is determined by a representation of SU(2). If d is the dimension of the representation, the spin is the half integer

$$j = \frac{d-1}{2}.$$

For historical reasons particles are called *bosons* (after Satyendranath Bose, the Indian physicist) or *fermions* (after Enrico Fermi, the Italian physicist) according as the spin is an integer or a half integer. Electrons and positrons are fermions while photons are bosons. The spin dictates the behavior of aggregates of like particles. If $\mathcal{H}$ is the Hilbert space of one particle states, the Hilbert space of N identical particles is the symmetric product $S^N(\mathcal{H})$ if the particles are bosons, and the exterior product $\Lambda^N(\mathcal{H})$ if the particles are fermions. The behavior of bulk matter is thus completely different for the two types. The relation

$$\xi \wedge \xi = 0$$

[1]The full quotation from Nambu is as follows.

The Dirac mode is to invent, so to speak, a new mathematical concept or framework first, and then try to find its relevance in the real world, with the expectation that (in a distorted paraphrasing of Dirac) a mathematically beautiful idea must have been adopted by God. Of course the question of what constitutes a beautiful and relevant idea is where physics begins to become an art.

I think this second mode is unique to physics among the natural sciences, being most akin to the mode practiced by the mathematicians. Particle physics, in particular, has thrived on the interplay of these two modes. Among examples of this second approach, one may cite such concepts as

Magnetic monopole
Non-Abelian gauge theory
Supersymmetry

On rare occasions, these two modes can become one and the same, as in the cases of Einstein gravity and the Dirac equation [9].

A similar point of view is echoed in the quote from Albert Einstein at the beginning of the article, cited by Julian Schwinger in his paper *A Theory of the fundamental interactions* in Ann. Phys. 2, 407–434 (1957).

codifies the *Pauli exclusion principle*. The full Hilbert space of one particle states is thus $\mathbf{Z}_2$-graded, the even part representing bosonic states, and the odd part, the fermionic states. So certainly, at the quantum level, everything has to be done in a $\mathbf{Z}_2$-graded category.

Why should one aim for a unified treatment of bosons and fermions? According to our current understanding, processes that are taking place in the sub atomic regimes involve constant creation and annihilation of particles. Examples ($e^- =$ electron, $e^+ =$ positron, $\gamma =$ photon) are

$$e^- + e^+ \longrightarrow \gamma, \qquad \gamma \longrightarrow e^- + e^+$$

which are among the simplest instances where bosons get transformed into fermions and vice versa. It is clear therefore that any treatment of these fundamental processes should be based on a unified way of treating bosons and fermions. This means that one has to work in the category of vector spaces, algebras, even Hilbert spaces, which are $\mathbf{Z}_2$-graded.

2.1 Structure of Spacetime in the Small

It was therefore natural, as was initially done, to think of supersymmetries as transformations in the quantum $\mathbf{Z}_2$-graded Hilbert space that exchange bosonic and fermionic states. Eventually, the idea, pioneered by Salaam and Strathdee [7], that one has to make a fundamental change even at the classical geometric level took hold. They introduced the concept of a supermanifold whose local coordinates are both commutative and non commutative, more precisely grassmannian. This was in line with the emergence of the idea, prevalent among many physicists, that the micro-structure of spacetime itself should be the subject of investigation as a possible source of the singularities that were plaguing quantum field theories. The idea was that the usual models of spacetime used in quantum electrodynamics and other theories assume an unlimited extrapolation of the structure of spacetime down to zero distances while the experiments do not go that far, and so it is compatible with known facts to have a spacetime that has additional features in the ultra small regions. The supermanifold model for spacetime is one of the possibilities, and it leads to field theories with softer divergences (see [10], p 217).

In actual fact, to be honest, the idea that one should investigate radical new models for space or spacetime is not really new; as is the case with most ideas in geometry, it goes back to Riemann who made prophetic observations on the nature of space in his celebrated 1854 inaugural address at Göttingen [11]:

> Now it seems that the empirical notions on which the metric determinations of Space are based, the concept of a solid body and a light ray, lose their validity in the infinitely small; it is therefore quite definitely conceivable that the metric relations of Space in the infinitely small do not conform to the hypotheses of geometry; and in fact, one ought to assume this as soon as it permits a simpler way of explaining phenomena.

Here Riemann's phrase *do not conform to the hypotheses of geometry* almost certainly means that the structure of space in the infinitely small is *not* a manifold. But *how small is small?* There is a natural scale, called the *Planck scale*, which emerges when the Schwarzschild radius and Compton wave length are identified, so that both General Relativity and Quantum Theory become significant at the same time. No measurements are possible below this range. The Planck length is $\approx 10^{-33}$ cm and Planck time is $\approx 10^{-43}$ s. String theorists operate in this regime where there are no points and the geometry is therefore non-commutative. I should perhaps amplify my comment since there is no direct evidence of non-commutative geometry in string theory calculations. In any theory, string or otherwise, there must be an algebra of local observables, the so-called local rings of the mathematician. Now any commutative ring has a spectrum whose elements are the it points. In as much as the basic objects in string theory are extended, the local rings *cannot be commutative*. So if there is a coherent geometric formulation of string theory it must be based on non-commutative geometry. I do not claim that there is a direct connection between string arguments and non-commutative geometry but there must be a non-commutative geometry in the background.

However it is believed [1, 2] that supersymmetry is observable in regimes of energies much less than Planckian. Whether supersymmetry is observable in the 10–15 TeV range of the new collider (LHC) at CERN is the outstanding question. As I have mentioned earlier, much evidence for expecting an affirmative answer is given in [1, 2].

The basic assumption in supersymmetric physics is that the geometry of space-time is that of a supermanifold described locally by a set of coordinates consisting of the usual ones supplemented by a set of anticommuting grassmann coordinates. The grassmann coordinates model the Pauli exclusion principle for the fermions in an embryonic form. Such a space is nowadays called a *supermanifold*. Its automorphisms are supersymmetries which form a super Lie group. This is thus the final link in the line of thought that culminated in the principle that the world is supercommutative at the fundamental level.

The model of spacetime that results from the assumption of supersymmetry is the so-called super Minkowski spacetime whose automorphisms constitute the super Poincaré group. Super particles correspond to the unitary irreducible representations of this super Lie group, and the theory leads to the fact that these arrange themselves in multiplets. All the particles in a given multiplet are associated to the *same* classical orbit. Hence the super partners of the particles we see are not to be found among the known particles; they have to be entirely new ones. The theory makes this very clear.

For a profound physical, mathematical, and philosophical analysis of the notion of elementary particle, both as themselves and as the quanta of the various fields, see the beautiful book [1] of Kobzarev and Manin.

The best reference for supersymmetric physics (for both mathematicians and physicists) is the collection edited by Ferrara [10]. For the mathematical aspects of supermanifolds see [8, 12–16] and the references cited in them.

3 Foundations

We begin with the definition of a supermanifold.

- A *supermanifold* M of dimension $p|q$ is a smooth manifold (second countable) M_0 of dimension p together with a sheaf $\mathcal{O}_M$ of $\mathbf{Z}_2$-graded supercommuting algebras on M_0 that looks locally like the supermanifold $\mathbf{R}^{p|q}$ whose sheaf is

$$C^\infty(\mathbf{R}^p)[\theta^1, \theta^2, \ldots, \theta^q] \qquad (\theta_i \theta_j = -\theta_j \theta_i)$$

  and such that M_0 is obtained by putting all the grassmann variables to 0. We should note that M_0 is *imbedded in* M.

  A supercommutative algebra is a $\mathbf{Z}_2$-graded algebra such that

$$ab = (-1)^{p(a)p(b)} ba,$$

  where p is the parity function characteristic of the $\mathbf{Z}_2$-grading.
- *Intuitive picture.* The intuitive picture of M is that of a classical manifold M_0 surrounded by a grassmannian cloud [16]. The cloud cannot be seen: in any measurement with values in a field, the odd variables will go to 0 because their squares are 0. Thus measurement sees only the underlying classical manifold M_0. Nevertheless the presence of the cloud eventually has consequences that are striking.

  In the above discussion measurement is meant in the mathematical sense-namely a homomorphism of the local ring into a field (which is a purely even algebra). Hence it will map odd elements to zero as required by the general principles of the theory of superalgebras.
- Unlike classical geometry the local ring of a supermanifold contains *nilpotents*, for instance the odd coordinate variables whose squares are 0. So there is a deep analogy with a *Grothendieck scheme.* Physicists often refer to the sections of the structure sheaf as *superfields.*

  A *Supersymmetry* is just a diffeomorphism between supermanifolds. The diffeomorphism

$$\mathbf{R}^{1|2} \simeq \mathbf{R}^{1|2} : t^1 \longmapsto t^1 + \theta^1\theta^2, \quad \theta^\alpha \longmapsto \theta^\alpha$$

  is a typical supersymmetry. Note how the morphism mixes odd and even variables. This is a basic example of how the grassmann cloud interacts with the classical manifold underlying the supermanifold. This example also shows that we cannot think of a supermanifold as a type of exterior bundle on a classical manifold: there are more symmetries in the super case.
- In this example the morphism is specified by describing what it does to t, θ_1, θ_2. This is a consequence of a general fact that morphisms can be specified in this manner. For instance, in the above example, the morphism takes $f \in C^\infty(\mathbf{R})$ into $f(t + \theta_1\theta_2) =: f(t) + \theta_1\theta_2 f'(t)$, by formal Taylor expansion. The

additional terms are 0 because $\theta_j^2 = 0$. Since we cannot limit the order of the Taylor expansions a priori, we see why the underlying smooth manifold of a supermanifold cannot be C^k for finite k.

3.1 Super Lie Groups and Their Super Lie Algebras

It is simplest and most natural to define super Lie groups as group objects in the category of super manifolds. As in the theory of ordinary Lie groups one can define the super Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$ of a super Lie group G. The even part of the super Lie algebra $\mathfrak{g}$ is an ordinary Lie algebra $\mathfrak{g}_0$ which is the Lie algebra of the classical Lie group $G_0 = |G|$, and the super Lie group G (resp. algebra $\mathfrak{g}$) may be viewed as a supersymmetric enlargement of G_0 (resp. $\mathfrak{g}_0$). One thus obtains a pair $(G_0, \mathfrak{g})$ associated to the super Lie group G, called the *super Harish–Chandra pair*.

More precisely, a super Harish–Chandra pair is a pair $(G_0, \mathfrak{g})$ such that

(a) G_0 is a Lie group, $\mathfrak{g}$ is a super Lie algebra with $\mathfrak{g}_0 = \mathrm{Lie}(G_0)$a ,nd G_0 acts on $\mathfrak{g}$
(b) $\mathfrak{g}$ is a G_0-module, and the action of $\mathfrak{g}_0$ on $\mathfrak{g}$ is the differential of the action of G_0 on $\mathfrak{g}$

The super Harish–Chandra pairs form a category in an obvious manner. It is a fundamental theorem that for G a super Lie group, $\mathfrak{g} = \mathrm{Lie}(G)$, and G_0 the classical Lie group underlying G, $(G_0, \mathfrak{g})$ is a super Harish–Chandra pair, and the functor

$$G \longmapsto (G_0, \mathfrak{g})$$

is an equivalence of categories This allows for a very convenient way to treat super Lie groups.

Here are some examples of super Lie groups.

- $G = \mathbf{R}^{p|q}$, with addition. Here $G_0 = \mathbf{R}^p$ and $\mathfrak{g} = G$.
- $G = \mathrm{GL}(p|q)$ with the pair $G_0 = \mathrm{GL}(p) \times \mathrm{GL}(q), \mathfrak{g} = \mathfrak{gl}(p|q)$. Then $\mathrm{GL}(p|q)(\mathbf{R})$ is an open sub super manifold of $\mathbf{R}^{p^2+q^2|2pq}$.

The form of the super Lie group $\mathrm{GL}(p|q)$ suggests a generalization. Let R be a supercommutative ring and let $\mathrm{GL}(p|q)(R)$ be the group of matrices

$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where the entries of the matrix are in R, with those of A, D in R_0 (even) and the entries of B, C in R_1 (odd), and

$$\det(A), \det(D) \in R_0^\times.$$

Here $R_0^\times$ is the group of units of R_0, i.e., the group of invertible elements of R_0. So

$$R \longmapsto \mathrm{GL}(p|q)(R)$$

is a functor over the category of supercommuting rings R. It is usual to say that $\mathrm{GL}(p|q)(R)$ is the group of R-points of this functor. One can then extend this definition with almost no extra effort to define a super matrix group scheme over a base supercommuting ring R by restricting the functor to be defined over the category of supercommuting rings over R. Of course not all functors are allowed; only those that are *representable* in a natural sense [13].

There are additional examples, namely the super versions of the classical Lie groups. The simplest are the super Lie groups $\mathrm{SL}(p|q)$ with Lie algebras $\mathfrak{sl}(p|q)$ which are the subalgebras of $\mathfrak{gl}(p|q)$ consisting of elements whose supertrace vanish [12].

One can define a super Lie group through a functorial approach, similar to the above example. If G is a super Lie group, then for any supermanifold T, the set of morphisms $T \longrightarrow G$ is equipped with a natural group structure. Let us call this group $G(T)$. Then $T \longmapsto G(T)$ is a group-valued functor on the category of supermanifolds. This functor is contravariant, and $G(T)$ is said to be the group of T-points of G. If $T \longmapsto g(T)$ is a contravariant group-valued functor, it is said to be *representable* if there is a supermanifold G such that $g(T)$ is the set of T-points of g. It is then possible to show that G can be equipped with the structure of a super Lie group, and $G(T) = g(T)$ for all T, so that $g(T)$ is the group of T-points of G. In the example of $\mathrm{GL}(p|q)$ discussed above, if we take R to be the supercommuting algebra of global sections of T we get $\mathrm{GL}(p|q)$ realized as a super Lie group. However the functor $\mathrm{GL}(p|q)$ is defined on the larger category of *all* supercommuting rings, and so we have, in that case, not only a super Lie group, but an affine algebraic supergroup scheme.

3.2 Brief History

Gol'fand-Likhtman and Volkov-Akulov discovered the minimal supersymmetric extension of the Poincaré algebra in the early 1970s. Wess-Zumino discovered a little later, in 1974, the first example of a *simple* super Lie algebra, namely the minimal supersymmetric extension of the conformal Lie algebra ($\mathfrak{sl}(4) = \mathfrak{so}(4,2)$). In 1975 Kac formally defined super Lie algebras and carried out the super version of the Cartan-Killing classification of simple Lie algebras over **C** which includes the super Lie algebras listed above (see [8] for references). The modern theory of representations of super Lie algebras is a very active subject developed by Kac, Serganova, and many others [17–19]. See also the article of Serganova in this volume where some long-standing conjectures on finite dimensional supermodules are discussed. For the unitary case when *infinite dimensional representations* are involved, Jakobsen's work [20] is extremely interesting. I shall come to this later.

3.3 Additional Remarks on the Concept of a Super Lie Algebra

Super Lie algebras can be defined via a super version of the Jacobi identity. An alternative way is as follows.

- A super Lie algebra is a super ($\mathbf{Z}_2$-graded) vector space

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$$

with a bilinear bracket $[\cdot, \cdot]$ such that

(a) $\mathfrak{g}_0$ is a Lie algebra
(b) $\mathfrak{g}_1$ is a $\mathfrak{g}_0$-module for the action $a \longmapsto [a, b]$
(c) $a \otimes b \longmapsto [a, b]$ is a symmetric $\mathfrak{g}_0$-module map from $\mathfrak{g}_1 \otimes \mathfrak{g}_1$ into $\mathfrak{g}_0$
(d) For all $a \in \mathfrak{g}_1$, we have $[a, [a, a]] = 0$.

The only non-linear part of these conditions is (d). If we arrange so that $[\mathfrak{g}_1, \mathfrak{g}_1]$ is such that it acts trivially on $\mathfrak{g}_1$, then (d) is automatic, and the problem becomes linear. This is true in many, but not all, cases.

4 Generalizations: Super Riemann Surfaces and Super Schemes

Because of its physical origins the category of supermanifolds is an enlargement of the category of C^∞ manifolds (classical). However one can clearly formulate the concept of a supermanifold in other categories as well. Thus a *complex supermanifold* is a classical complex manifold M_0 on which is given a sheaf of supercommuting complex algebras whose quotient modulo the odd elements is M_0, such that locally the sheaf is like $\mathbf{C}^{p|q}$. In particular one has the notion of a *super Riemann surface*. This is physically interesting as it models the world sheet of a superstring. In the case of dimension 1 there is a more refined notion of a susy curve. For instance, a *susy curve* is a complex supermanifold of dimension $1|1$ equipped with a maximally non integrable odd distribution (=a locally free locally direct factor subsheaf of the tangent sheaf of X) [21, 22]. Very interesting questions arise in this context. One can ask for a theory of superelliptic curves and super theta functions, the Picard of a susy curve, and a super generalization of the theory of vector bundles on a classical curve. The article of Rabin in this volume addresses some of these questions (see the references in that article); see also the contribution of Kwok. The moduli space of susy curves of a given genus and its compactification are natural objects of study. I understand from Pierre Deligne that he has studied these questions. For $g > 1$ the moduli space has dimension $3g - 3|2g - 2$ and has a compactification [23] where the divisor at infinity has normal crossing. For other references on moduli of super Riemann surfaces, see [24, 25].

A second direction in which the concept of a supermanifold can be generalized is in the direction of superschemes. I will not go into this in any detail but only mention briefly a natural outgrowth of it, namely the theory of algebraic matrix supergroups, in the next section.

5 Simple Supergroups

I have already alluded to the circumstance that matrix supergroups can be defined as representable group-valued functors on the category of supercommutative rings (over some base ring). Clearly the question of obtaining a classification of simple supergroups and Chevalley supergroups parallel to the classification of simple super Lie algebras is of great interest. The article of Fioresi and Gavarini in this volume treats the construction of Chevalley supergroups [26].

6 Unitary Representations of Super Lie Groups

When special relativity and quantum physics are combined, one is led to the classification of free elementary particles which correspond to irreducible unitray representations of the Poincaré group. From the very beginning the physicists were interested in constructing the minimal supersymmetric extension of the Poincaré group and classifying its irreducible unitary representations. Salam-Strathdee already did this very early [27] and the subject has been revisited many times since. The basic picture is that the super particles come in *multiplets* of ordinary particles, leading, in the simplest case, to the idea that the particles we know about exhaust only a half of the particles, the full set consisting of known particles and their *superpartners* [27]. The extra particles which are more massive (because of supersymmetry breaking) may (conjecturally) explain the abundance of *dark matter* in the physical universe [1, 2].

The physicists essentially proceeded infinitesimally with one added ansatz, namely that the little group picture continues to hold in the super case. However the absence of the global susy transformations and lack of proof of the ansatz invited closer analysis. This was provided more recently [28] where the concept of a unitary representation of a super Lie group was subjected to a very precise mathematical analysis. The analysis in [28] also revealed the existence of super particles whose multiplets consist of massless particles of infinite (continuous) spin, a fact missed in the analysis of the physicists.

The concept of a unitary representation of a super Lie group. For an ordinary Lie group the unitary representations correspond infinitesimally to *skew-symmetric* representations of the Lie algebra. For a super Lie group it is possible to work with a super version of this. We identify the super Lie group with its corresponding super

Harish–Chandra pair. Imitating the definition of a super Harish–Chandra pair we think of a unitary representation of a super Harish–Chandra pair $(G_0, \mathfrak{g})$ as a pair (π, ρ) such that

- π is an even unitary representation of G_0 in a super Hilbert space
- ρ is a super skew-symmetric representation of $\mathfrak{g}$.
- π and ρ are compatible.

We have the relation

$$-id\pi([X, X]) = 2\rho(X)^2 \qquad (X \in \mathfrak{g}_1).$$

If π is infinite dimensional, $id\pi(Y)$ will be in general unbounded and so the operators $\rho(X)$ will in general be unbounded. This is the main technical problem in working with unitary representations of super Lie groups. We make the following definition and discuss its essential uniqueness.

- For any unitary representation λ of G_0, $C^\infty(\lambda)$ is the space of smooth vectors for λ. Then URs of $(G_0, \mathfrak{g})$ are pairs (π, ρ) with
  - π an even unitary representation of G_0 in $\mathcal{H}$
  - $\rho : \mathfrak{g}_1 \longrightarrow \mathbf{End}(C^\infty(\pi))_1$ linear and symmetric
  - $-id\pi([X, Y]) = \rho(X)\rho(Y) + \rho(Y)\rho(X)$ for $X, Y \in \mathfrak{g}_1$
  - $\rho(gX) = \pi(g)\rho(X)\pi(g)^{-1}$ for $g \in G_0, X \in \mathfrak{g}_1$.

  What makes this definition go is the following
- (*Key lemma*) $\rho(X)$ for $X \in \mathfrak{g}_1$, is *essentially self-adjoint* on $C^\infty(\pi)$.

This result gives a great rigidity to the concept of a unitary representation of a super Lie group and leads to the conclusion that there is essentially only one way to define the concept, as I shall explain now.

6.1 Essential Uniqueness

It may appear that the choice of $C^\infty(\pi)$ in the above definition, while natural and canonical, is still somewhat arbitrary; for instance we could have chosen the space of *analytic vectors* for π in its place. It turns out that all such choices are essentially equivalent in the sense that for any variant of the above definition which uses a different subspace than $C^\infty(\pi)$, the operators $\rho(X)$ can be extended all the way to $C^\infty(\pi)$ so that we obtain a unitary representation in the sense we have defined it, and moreover, the $\rho(X)$ will all be self-adjoint with $C^\infty(\pi)$ as a core. This remark makes it clear that the concept of a unitary representation of a super Lie group is extremely stable and robust.

7 Classification of Super Particles

I mentioned earlier that elementary particles in the supersymmetric world are described using irreducible unitary representations of the super Poincaré group. I shall now briefly outline this aspect of supersymmetry.

Minkowski Superspacetime and the Super Poincaré Group Let $\mathfrak{t}_0$ be the affine Minkowski spacetime of signature $(1, n)$. By a *Minkowski superspacetime* is meant a super Lie group whose even part is $\mathfrak{t}_0$ identified with its group of translations. The corresponding super Lie algebra $\mathfrak{t}$ has the grading

$$\mathfrak{t} = \mathfrak{t}_0 \oplus \mathfrak{t}_1.$$

We have the Lorentz group $\mathrm{SO}(\mathfrak{t}_0)^0 \simeq \mathrm{SO}(1, n)^0$ (the exponent 0 refers to the connected component) acting on $\mathfrak{t}_0$ and we require that there is an action of its two-fold spin covering group $\mathrm{Spin}(\mathfrak{t}_0) \simeq \mathrm{Spin}(1, n)$, on $\mathfrak{t}_1$. Physical interpretations lead to the requirement that this action be a so-called *spin module*. This means that when complexified, this action is a direct sum of the spin representations (see [8, 12] for a detailed discussion of the spin representations, their structure, reality, and other properties). In this case, at least when $\mathfrak{t}_1$ is irreducible, there is a *projectively unique* symmetric bilinear form

$$\mathfrak{t}_1 \otimes \mathfrak{t}_1 \longrightarrow \mathfrak{t}_0$$

compatible with the actions of the spin group on $\mathfrak{t}_0$ and $\mathfrak{t}_1$. If we choose this to be the supercommutator of odd elements we may regard $\mathfrak{t} = \mathfrak{t}_0 \oplus \mathfrak{t}_1$ as a super Lie algebra with an action of the group $\mathrm{Spin}(\mathfrak{t}_0)$. The super Lie group T of $\mathfrak{t}$ is flat *super Minkowski spacetime.* The semi direct product

$$G = T \times' \mathrm{Spin}(1, n)$$

is then the *super Poincaré group.* The uniqueness of the odd commutator means that we have an essentially unique supersymmetric version of spacetime and the Poincaré group.

The irreducible unitary representations of the super Poincaré group can be written down and they classify elementary super particles. Each super particle, when viewed as an even unitary representation of the underlying classical Poincaré group, is the direct sum of a collection of ordinary particles, called a *multiplet.* The members of a multiplet are called *super partners* of each other. All the particles in a given multiplet correspond to the *same classical orbit* and so the super partners cannot be identified with other known particles, a fact that I have mentioned earlier.

- *Unlike the classical case, the positivity of energy is a consequence of supersymmetry.*
  Indeed, supersymmetry forces the selection of positive energy orbits
- *The existence of the superpartners of the known particles is the biggest prediction of supersymmetry.*

It may be hoped that the new super collider LHC operating at CERN will create the super partners of the usual elementary particles. This is not certain because one does not know exactly the scale at which supersymmetry is broken (see however [1, 2] for the argument that this is in the range 10–15 TeV).

8 Unitary Representations of Super Lie Groups

The theory in [28] is actually valid for the *entire class of super semidirect products*, thus generalizing the entire Mackey theory (Mackey machine) of unitary representations of semidirect products to the super case. The super Poincaré groups form only a small part of this class. This leads to the following problems.

- To construct a theory of unitary representations of simple super Lie groups.
- To extend the general correspondence between orbits and representations to the super case. In general it appears that only those orbits are selected which have a natural positive Clifford structure.

Salmasian has obtained the definitive theory of unitary representations of nilpotent super Lie groups [29]. The article of Karl-Hermann Neeb and Salmasian in this volume discusses this theory and some natural outgrowths of it.

Not all super Lie groups have non-trivial unitary representations, in sharp contrast to the classical case. The main difficulty is that in constructing unitary representations associated with a super homogeneous space, essential use is made of the fact that the space is purely even, both in [28] and [29]. In the general case it appears difficult to construct invariant super hermitian scalar products. Hence it may be better to study *Frechet representations* which exist always. The article of Carmeli and Cassinelli in this volume is a beginning in understanding this issue.

9 Deformations

Quantum theory can be viewed as a *deformation* of classical mechanics. The ideas behind this view point are now well-known. One can ask about deformations of supersymmetric theories. I will not go into any detail but mention the article of Cervantes, Fioresi and Lledo in this volume. The deformations lead to non-commutative geometry and so must have contact with string theory at some level.

The article of Schwarz and Movshev also deals with deformations, this time of supersymmetric gauge theories.

10 Supergravity

I have not touched upon supergravity in this introduction because I am not very familiar with it. The article of Ferrara and Marrani is a contribution to this area. For some of the basic material see [22].

11 Some Problems

The following questions arise naturally from the themes addressed by the conference and the papers in this volume.

- To study super Riemann surfaces beyond $1|1$ susy curves.
- To extend supersymmetry to schemes and study super group schemes, over an arbitrary field, not necessarily algebraically closed. This is very important for physics where typically one wants a theory over **R**.
- To study complex super homogeneous spaces of real simple super Lie groups, the vector bundles on them and link these with the super highest weight modules of Jakobsen [20].
- To extend supersymmetry to p-adic Lie groups. Since the super Harish–Chandra pair is now defined over a p-adic field, one is led naturally, *not* to unitary representations but to p-adic representations. So a link with the work of Peter Schneider and his collaborators appears inevitable. The classical analog is the work of Schneider [30].

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Black Holes and First Order Flows in Supergravity

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Abstract We review the description of static, spherically symmetric, asymptotically-flat black holes in four dimensional supergravity in terms of an autonomous Hamiltonian system. A special role in this analysis is played by the so called *fake superpotential* W, which is identified with a particular solution to the Hamilton-Jacobi equation. This function defines a first order, gradient-flow, description of the radial flow of the scalar fields, coupled to the solution, and of the red-shift factor. Identifying W with the Liapunov's function, we can make the general statement that critical points of W are asymptotically stable equilibrium points of the corresponding first order dynamical system (in the sense of Liapunov). Such equilibrium points may only exist for extremal regular solutions and define their near horizon behavior. Thus the fake superpotential provides an alternative characterization of the attractor phenomenon. We focus on extremal black holes and deduce very general properties of the fake superpotential from its duality invariance. In particular we shall show that W has, along the entire radial flow, the same flat directions which exist at the attractor point. This allows to study properties of the ADM mass also for small black holes where in fact W has no critical points at finite distance in moduli space. In particular the W function for small non-BPS black holes can always be computed analytically, unlike for the large black hole case.

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1 Introduction

One of the most intriguing aspects of spherically symmetric, asymptotically flat black hole solutions in supergravity is their coupling with the scalar fields. The radial flow of these fields and of the metric warp-factor is governed by an autonomous Hamiltonian system and can be described, for N-extended theories based on symmetric coset manifolds G/H, in terms of a *fake superpotential function* [2,4,5,14,16–18],[1] solution to the Hamilton-Jacobi equation [4,6,29].[2] Of particular interest are regular extremal solutions, which exhibit the attractor mechanism at the horizon. The "fake" superpotential W describing this class of solutions is defined by particular boundary conditions which make it duality invariant. It is totally specified by the *duality orbit* [8, 9] of the dyonic charge vector $\mathcal{P} = (p^\Lambda, q_\Lambda)$ $(\Lambda = 1, \cdots, n_V)$ and the asymptotic values at radial infinity of the scalars of the theory: $(\phi_0^r) \in G/H$. The values $\phi_* \equiv (\phi_*^r)$ of the scalar fields at the horizon correspond to an extremum of W,

$$\left.\frac{\partial W}{\partial \phi^r}\right|_{\phi^r=\phi_*^r} = 0, \tag{1}$$

where space-time is $AdS_2 \times S_2$. In fact ϕ_* defines an equilibrium point of the autonomous dynamical system describing the radial flow of ϕ^r. It has not been appreciated enough that many properties of the W function are not only true at the horizon but in fact they hold on the entire radial flow and in particular at spatial infinity where space-time is flat and W coincides with the ADM mass of the solution. All these properties naturally follow from the identification of the W-function with Hamilton's characteristic function, which defines the *gradient-flow* equations for the scalar fields and the red-shift factor. From this characterization some important general properties of W follow:

- W is a positive definite function on the moduli space;
- The derivative of W along the flow, moving from the horizon to radial infinity, is always positive;
- W for extremal solutions is duality invariant.

The latter property was originally conjectured in [2] and later proven in [4] (see also [5]). Eventually, in [14, 17, 18] the explicit construction of W in terms of duality invariants was completed.

As it was proven in [5], the above properties of W, in the presence of a critical point ϕ_* of the first order system, promote W to a Liapunov's function (see for instance [25,30]), allowing to make precise statements about the *asymptotic stability* of ϕ_* (namely that ϕ_* is not just an attractive equilibrium point, but it is also stable), with no need of computing the Hessian of the potential. The existence of W, even

[1]The idea of "fake" supersymmetry was first introduced in [22].

[2]See also [13,23,26,27,31–33] for related independent works and [19] for a detailed discussion of this issue in relation to the Liouville integrability of models based on symmetric manifolds.

in a neighborhood of the critical point, provides an alternative (and more powerful) characterization of its attractiveness and stability properties.

Of particular interest are orbits of extremal, large (i.e. regular), black holes[3] in which the critical points are not isolated. This feature is related, in the symmetric models, to the existence of flat directions of the scalar potential V [21]. We shall prove in full generality, using the duality invariance of W, that W and V have the same symmetry properties, and thus that they also have the same flat directions. These flat directions, which have an intrinsic group theoretical characterization in terms of the stabilizer of the duality orbit of the quantized charges [8, 9], are also a feature of the central and matter charges.

An other consequence of the general properties of W is that the functional form of $W(i_n, I_4)$, where i_n are the H-invariant combinations of the moduli ϕ^r and charges $\mathcal{P}$, can also be calculated for $I_4 = 0$, in which case the *classical* horizon area vanishes and (1) has no solutions (in the interior of the moduli space). More precisely, for $I_4 = 0$, (1) has a runaway solution $W = 0$ at the boundary of the moduli space where some $\phi^r \to \infty$ [11, 12].

It is the aim of this analysis to further specify general properties of the W function for large and small black holes, such as their moduli spaces and symmetries. Moreover, depending on the number N of supersymmetries, $W(I_4 = 0)$ can be obtained by a suitable limit of large black hole solutions (where $I_4 \neq 0$), in such a way that W is always given by a calculable algebraic function of the H-invariants. The way the limit is performed also allows us to simply understand the interplay of BPS properties of small black holes versus large solutions.

The paper is organized as follows: In Sect. 2 we review and outline the above mentioned properties of the W superpotential. In particular, following [5], we give the general form of W for extremal solutions and address the issue of asymptotic stability of the critical points on W, by showing that W can be identified with a Liapunov's function. We also prove that the potential V, the superpotential W, together with the central and matter charges, have the same flat directions defined by the G-orbit of the quantized charges $\mathcal{P}$. In Sect. 3 we give a short account of the classification of small black hole solutions to N = 8 supergravity. The Appendices contain the proof of statements made in the text and some technical details.

2 Some General Properties of the W Function

Let us review some general properties of the fake superpotential W associated with U-duality orbits of static, extremal, asymptotically flat black hole solutions in an extended supergravity theory with a symmetric scalar manifold $\mathscr{M}_{scal} = \frac{G}{H}$.

[3]Large black holes are solutions for which a certain (quartic) duality-invariant expression of the charge vector $\mathcal{P}$, called $I_4(\mathcal{P})$ does not vanish. For small black holes, on the other hand, $I_4(\mathcal{P}) = 0$. A definition of $I_4(\mathcal{P})$,and its G-invariant form for symmetric geometries G/H is summarized in Sect. 2.1

Let us consider an extended supergravity describing n real scalar fields, spanning the manifold $\mathscr{M}_{scal}$ and n_V vector fields A^Λ_μ. The ansatz for the metric and the vector field strengths $F^\Lambda_{\mu\nu}$, for the kind of black holes we are considering, is:

$$d\,s^2 = -e^{2\,U}\,d\,t^2 + e^{-2\,U}\left[\frac{d\,\tau^2}{\tau^4} + \frac{1}{\tau^2}\,(d\,\theta^2 + \sin(\theta)\,d\,\varphi^2)\right],$$

$$\mathbb{F} = \begin{pmatrix} F^\Lambda_{\mu\nu} \\ G_{\Lambda\,\mu\nu} \end{pmatrix}\frac{dx^\mu\wedge dx^\nu}{2} = e^{2U}\,\mathbb{C}\cdot\mathcal{M}(\phi^r)\cdot\mathcal{P}dt\wedge d\tau + \mathcal{P}\,\sin(\theta)d\theta\wedge d\varphi, \tag{2}$$

where the coordinate $\tau = -1/r$ runs from 0, at radial infinity, to $-\infty$ at the horizon, where $e^{U(\tau)}$ vanishes. The scalar fields are taken to be functions of τ only: $\phi^r = \phi^r(\tau)$. The magnetic field strength $G_{\Lambda\,\mu\nu}$ in (2) is defined, as usual, as: $G_{\Lambda\,\mu\nu} \propto \epsilon_{\mu\nu\rho\sigma}\,\delta\mathcal{L}/\delta F^\Lambda_{\rho\sigma}$, $\mathcal{L}$ being the Lagrangian of the theory. The last equation in (2) is written in a manifestly symplectic covariant form, namely as an equality between two $2\,n_V$ dimensional symplectic vectors, where $\mathbb{C}_{MN}$, $M,\ N = 1,\ldots,2\,n_V$ is the $\mathrm{Sp}(2\,n_V,\mathbb{R})$-invariant matrix:

$$\mathbb{C} = \begin{pmatrix} \mathbf{0} & -\mathbb{1} \\ \mathbb{1} & \mathbf{0} \end{pmatrix}. \tag{3}$$

The vector $\mathcal{P} \equiv (p^\Lambda, q_\Lambda)$ consists of the quantized electric and magnetic charges. Finally the $2\,n_V \times 2\,n_V$ symmetric, negative definite, matrix $\mathcal{M}(\phi^r)_{MN} \equiv -(\mathbb{L}\,\mathbb{L}^T)_{MN}$, $\mathbb{L}(\phi^r)$ being the $\mathscr{M}_{scal}$ coset representative in the fundamental of $\mathrm{Sp}(2\,n_V,\mathbb{R})$, can also be written in the familiar form [3, 7, 10]:

$$\mathcal{M}(\phi^r) = \begin{pmatrix} I + R\,I^{-1}\,R & -R\,I^{-1} \\ -I^{-1}\,R & I^{-1} \end{pmatrix}, \tag{4}$$

where $I_{\Lambda\Sigma} \equiv \mathrm{Im}(\mathcal{N})_{\Lambda\Sigma} < 0$ is the vector kinetic matrix while $R_{\Lambda\Sigma} \equiv \mathrm{Re}(\mathcal{N})_{\Lambda\Sigma}$ defines the generalized theta-term. From its definition it follows that $\mathcal{M}(\phi^r)$ is symplectic, namely that $\mathcal{M}\mathbb{C}\mathcal{M} = \mathbb{C}$.

Once the electric and magnetic charges of the solution are assigned, the radial evolution of the $n+1$ fields $U(\tau)$, $\phi^r(\tau)$ is described by the effective action:

$$S_{eff} = \int \mathcal{L}_{eff}\,d\,\tau = \int\left(\dot{U}^2 + \frac{1}{2}\,G_{rs}(\phi)\,\dot{\phi}^r\,\dot{\phi}^s + e^{2\,U}\,V(\phi,\mathcal{P})\right)d\,\tau, \tag{5}$$

together with the Hamiltonian constraint, representing the extremality condition[4]:

[4]For non extremal black holes the value of the Hamiltonian on a solution coincides with the square of the extremality parameter. Notice that the Hamiltonian is not positive definite, being expressed as the difference of a "kinetic" and a positive "potential" term (this is in turn due to the fact that the role of the time variable is played by a radial coordinate τ). As a consequence of this we can have

$$\mathcal{H}_{eff} = \dot{U}^2 + \frac{1}{2}\, G_{rs}(\phi)\, \dot{\phi}^r\, \dot{\phi}^s - e^{2\,U}\, V(\phi, \mathcal{P}) = 0, \tag{6}$$

the effective potential being given by $V(\phi, \mathcal{P}) \equiv -\frac{1}{2}\, \mathcal{P}^T\, \mathcal{M}(\phi)\, \mathcal{P} > 0$ and the dot represents the derivative with respect to τ. The radial evolution of the $n+1$ fields $U(\tau)$, $\phi^r(\tau)$ in the solution admits a first order description [2, 4, 16] in terms of a fake superpotential $W(\phi, \mathcal{P})$:

$$\dot{U} = e^U\, W, \quad \dot{\phi}^r = 2\, e^U\, G^{rs}\, \frac{\partial W}{\partial \phi^s}. \tag{7}$$

If we interpret the fields $U(\tau)$, $\phi^r(\tau)$ as coordinates of a Hamiltonian system in which the radial variable plays the role of time, the first order description (7) is equivalent to solving the Hamilton-Jacobi problem with Hamilton's characteristic function

$$\mathcal{W}(U, \phi) \equiv 2\, e^U\, W(\phi). \tag{8}$$

Indeed, in terms of $W(\phi, \mathcal{P})$, the Hamilton-Jacobi equation has the form:

$$W^2 + 2\, G^{rs}\, \frac{\partial W}{\partial \phi^r}\, \frac{\partial W}{\partial \phi^s} = V, \tag{9}$$

which can also be derived from the Hamiltonian constraint (6) using (7).[5] We are not interested here in the most general solution to (9), nor to address the issue of its existence (see [4] for a discussion on this point). We are interested, instead, in the W functions associated with classes of extremal solutions whose general properties are in principle known. They are completely characterized by the set of quantized charges $\mathcal{P}$ and the values of the fields at radial infinity:

$$U(\tau = 0) = 0, \quad \phi^r(\tau = 0) = \phi_0^r. \tag{10}$$

We shall therefore simply denote them by: $U = U(\tau; \phi_0)$ and $\phi^r = \phi^r(\tau; \phi_0)$. The ADM mass and the scalar charges at infinity are given by:

$$\begin{aligned} M_{ADM}(\phi_0, \mathcal{P}) &= \dot{U}(\tau = 0) = W(\phi_0, \mathcal{P}), \\ \Sigma^r(\phi_0, \mathcal{P}) &= \dot{\phi}^r(\tau = 0) = 2\, G^{rs}(\phi_0)\, \frac{\partial W}{\partial \phi^r}(\phi_0, \mathcal{P}). \end{aligned} \tag{11}$$

non-trivial solutions on which the Hamiltonian vanishes. These correspond to the extremal black holes.

[5] In the case of non-extremal solutions the Hamilton-Jacobi equation reads $\left(\frac{\partial \mathcal{W}}{\partial U}\right)^2 + 2\, G^{rs}(\phi)\, \frac{\partial \mathcal{W}}{\partial \phi^r}\, \frac{\partial \mathcal{W}}{\partial \phi^s} = 4\, e^{2U}\, V + 4\, c^2$, and the corresponding first order equations have the form $\dot{U} = \frac{1}{2}\, \frac{\partial \mathcal{W}}{\partial U}$, $\dot{\phi}^r = G^{rs}(\phi)\, \frac{\partial \mathcal{W}}{\partial \phi^s}$. If $c \neq 0$ however, as it is apparent from the Hamilton-Jacobi equation, the dynamical system can have no equilibrium point $\frac{\partial \mathcal{W}}{\partial U} = \frac{\partial \mathcal{W}}{\partial \phi^r} = 0$.

Regular (*large*) extremal black holes have finite horizon area A_H and thus near the horizon ($\tau \to -\infty$) e^U has the following behavior: $e^{-2U} \sim \frac{A_H}{4\pi}\,\tau^2$, where $A_H = A_H(\mathcal{P})$ is a function of the quantized charges only. In fact $\mathcal{P}$ transforms under duality (see Sect. 2.2) in a symplectic representation of G and A_H, as a function of $\mathcal{P}$, is expressed in terms of the quartic invariant of G in this representation: $A_H(\mathcal{P}) = 4\pi\,\sqrt{|I_4(\mathcal{P})|}$ (here we use the units $c = \hbar = G = 1$, so that the Plank length is one.). Using (7) we see that W computed on the solution evolves, in the near horizon limit, towards $\sqrt{\frac{A_H}{4\pi}}$.

As far as the scalar fields are concerned, due to the attractor mechanism some of them are fixed at the horizon to values which are totally determined in terms of the quantized charges, while other scalar fields, which are flat directions of the potential, are not. That is, in the presence of flat directions, in the near horizon limit $\tau \to -\infty$ the non flat scalars evolve towards values which are totally fixed in terms of quantized charges, while the flat directions still depend, in general, on the boundary values ϕ_0^r taken at radial infinity ($\tau = 0$). Since only the scalars parametrizing the flat directions may depend at the horizon on ϕ_0^r, the near horizon geometry, which is determined in terms of the potential, will only depend on the quantized charges, consistently with the attractor mechanism. Summarizing, for large black holes, we have:

$$\lim_{\tau\to-\infty} e^{-2U} = \sqrt{|I_4(\mathcal{P})|}\,\tau^2, \quad \lim_{\tau\to-\infty} \phi^r(\tau) = \phi_*^r,$$
$$\lim_{\tau\to-\infty} W^2(\phi(\tau;\phi_0),\mathcal{P}) = W^2(\phi_*,\mathcal{P}) = V(\phi_*,\mathcal{P}) = \sqrt{|I_4(\mathcal{P})|}.$$

Small black holes are characterized by vanishing horizon area, i.e. by quantized charges for which $I_4(\mathcal{P}) = 0$. For $\tau \to -\infty$ the warp factor has the following behavior: $e^{-2U} \sim \tau^\alpha$, $\alpha < 2$. In the same limit scalar fields typically flow to values which are at the boundary of the scalar manifold. Either for large or for small solutions, from the first of (7) we deduce the following boundary condition for W:

$$\lim_{\tau\to-\infty} e^{U(\tau;\phi_0)}\,W(\phi(\tau;\phi_0),\mathcal{P}) = \lim_{\tau\to-\infty} \dot{U} = 0. \tag{12}$$

This allows us to write $W(\phi,\mathcal{P})$ for the two kinds of solutions in the following form (see [4]):

$$W(\phi_0,\mathcal{P}) = \int_{-\infty}^{0} e^{2\,U(\tau;\phi_0)}\,V(\phi(\tau;\phi_0),\mathcal{P})\,d\tau. \tag{13}$$

It should be stressed that the above expression allows to write the W function for a given class of solutions as a free function of the point ϕ_0 on the scalar manifold and of the quantized charges: Given a charge vector $\mathcal{P}$ and a point $\phi_0 = (\phi_0^r)$ in $\mathscr{M}_{scal}$, the corresponding value of W is given by the integral over τ of $e^{2U}\,V$, computed along the unique solution originating at infinity in ϕ_0.

2.1 I_4 Invariant for $\mathcal{N} = 2, 4, 8$ Supergravities

In $\mathcal{N} > 2$ theories and in $\mathcal{N} = 2$ theories based on symmetric spaces for the (vector multiplet) scalar fields, the entropy *area law* reads (the Boltzmann constant k_B being one in our units):

$$S = \frac{A_H}{4} = \pi\,\sqrt{|I_4(\mathcal{P})|}, \tag{14}$$

where, as anticipated in the previous section, $I_4(\mathcal{P})$ is a certain quartic invariant of the dyonic charge vector $\mathcal{P}$ and depends on the particular theory under consideration. Since $I_4(\mathcal{P})$ is moduli-independent, it can be expressed either in terms of the quantized charges $\mathcal{P}$ or in terms of the (dressed) central and matter charges $Z_{AB}(\phi, \mathcal{P})$, $Z_I(\phi, \mathcal{P})$ (see Sect. 2.3 for a precise definition of the latter). For our convenience we recall here the actual form of $I_4(\mathcal{P})$ in terms of the central and matter charges.

For $\mathcal{N} = 2$ theories, based on special geometry, we can define five H-invariant quantities i_n, as follows [17]:

$$\begin{aligned}
i_1 &\equiv Z\,\overline{Z},\\
i_2 &\equiv g^{i\bar{j}}\,Z_i\,\overline{Z}_{\bar{j}},\\
i_3 &\equiv \frac{1}{3}\,\mathrm{Re}\left(Z\,N_3(\overline{Z}_{\bar{\imath}})\right),\\
i_4 &\equiv -\frac{1}{3}\,\mathrm{Im}\left(Z\,N_3(\overline{Z}_{\bar{\imath}})\right),\\
i_5 &\equiv g^{i\bar{\imath}}\,C_{ijk}\,\overline{C}_{\bar{\imath}\bar{j}\bar{k}}\,\overline{Z}^j\,\overline{Z}^k\,Z^{\bar{j}}\,Z^{\bar{k}},
\end{aligned}$$

where $N_3(\overline{Z}_{\bar{\imath}}) \equiv C_{ijk}\,\overline{Z}^i\,\overline{Z}^j\,\overline{Z}^k$, $Z_i \equiv D_i Z$ and $Z^{\bar{\imath}} \equiv g^{\bar{\imath}i}\,Z_i$. In terms of these quantities the quartic invariant reads:

$$I_4 = (i_1 - i_2)^2 + 4\,i_4 - i_5 = I_4(\mathcal{P}), \tag{15}$$

where, as anticipated in the previous subsection, $\mathcal{P}$ transforms in a symplectic representation of G and $I_4(\mathcal{P})$ is the only non-vanishing invariant quantity built out of the charge vector. Note that, for the quadratic series ($C_{ijk} = 0$) we have: $I_4 = I_2^2$, where $I_2 \equiv |i_1 - i_2|$.

For $\mathcal{N} = 4$, we can define two SU(4) $\times$ SO(n) invariants:

$$\begin{aligned}
S_1 &\equiv \frac{1}{2}\,Z_{AB}\,\overline{Z}^{AB} - Z_I\,\overline{Z}_{\bar{J}}\,\delta^{I\bar{J}},\\
S_2 &\equiv \frac{1}{4}\,\epsilon^{ABCD}\,Z_{AB}\,Z_{CD} - Z_I\,Z_J\,\delta^{IJ},
\end{aligned}$$

in terms of the central charges $Z_{AB} = -Z_{BA}$, $A, B = 1, \ldots, 4$, and the n matter charges Z_I, $I = 1, \ldots, n$. Then the unique quartic $G = \mathrm{SL}(2, \mathbb{R}) \times \mathrm{SO}(6, \mathrm{n})$-invariant reads:

$$I_4^{(\mathcal{N}=4)}(\mathcal{P}) \equiv S_1^2 - |S_2|^2, \tag{16}$$

and the black hole potential is:

$$V^{(\mathcal{N}=4)}(\phi, \mathcal{P}) = \frac{1}{2} Z_{AB}\, \overline{Z}^{AB} + Z_I\, \overline{Z}^I. \tag{17}$$

Finally, in the $\mathcal{N} = 8$ theory the Cartan $G = \mathrm{E}_{7(7)}$-quartic invariant is given by the expression [20]:

$$I_4^{(\mathcal{N}=8)}(\mathcal{P}) \equiv \mathrm{Tr}[(\mathbb{Z}\,\mathbb{Z}^\dagger)^2] - [\mathrm{Tr}(\mathbb{Z}\,\mathbb{Z}^\dagger)]^2 + 8\,\mathrm{Re}[Pf(\mathbb{Z})], \tag{18}$$

where $\mathbb{Z} \equiv (Z_{AB}) = -\mathbb{Z}^T$, $A, B = 1, \ldots, 8$, is the complex central charge matrix [28]. In terms of the four skew-eigenvalues z_i, $i = 1, \ldots, 4$, of Z_{AB}, $I_4^{(\mathcal{N}=8)}$ reads:

$$I_4^{(\mathcal{N}=8)}(\mathcal{P}) \equiv \sum_{i=1}^{4} |z_i|^4 - 2\sum_{i<j} |z_i|^2\, |z_j|^2 + 4\,(z_1 z_2 z_3 z_4 + \bar{z}_1\, \bar{z}_2\, \bar{z}_3\, \bar{z}_4). \tag{19}$$

The black hole effective potential has the following form:

$$V^{(\mathcal{N}=8)}(\phi, \mathcal{P}) = \frac{1}{2} Z_{AB}\, \overline{Z}^{AB} = \sum_{i=1}^{4} |z_i|^2. \tag{20}$$

In any extended supergravity, BPS solutions are described by $W = |z_h|$, where z_h is the highest skew-eigenvalue (i.e. eigenvalue with highest modulus) of the central charge matrix Z_{AB} (for $\mathcal{N}=2$, $Z_{AB} = Z\,\epsilon_{AB}$, $A, B = 1, 2$, and $z_h = Z$). Therefore it is also true that:

$$V = |z_h|^2 + 2\,G^{rs}\,\partial_r |z_h|\,\partial_s |z_h|. \tag{21}$$

If however $\mathcal{P}$ is not in a BPS orbit, the flow defined by $W = |z_h|$ does not correspond to a physically acceptable solution and a different W-function should be used.

In particular, in the $\mathcal{N} = 8$ case for non-BPS configurations the corresponding W-function satisfies the following inequalities:

$$|z_h|^2 < W^2 \leq 4\,|z_h|^2, \tag{22}$$

the lower bound being saturated only for BPS solutions. The upper bound originates from the general property: $W^2 \leq V \leq 4\,|z_h|^2$. For non-BPS large black holes, it can be proven that, at the attractor point, $|z_i| = \rho = |z_h|$ and the upper bound is saturated: $W = 2\rho$.

2.2 The W Function and Duality

It is known that the on-shell global symmetries of an extended supergravity, at the classical level, are encoded in the isometry group G of the scalar manifold (if non-empty), whose action on the scalar fields is associated with a simultaneous linear symplectic action on the field strengths F^Λ and their duals G_Λ. This duality action of G is defined by a symplectic representation D of G:

$$g \in G : \begin{cases} \phi^r \to \phi^{r\,\prime} = g \star \phi^r \\ \begin{pmatrix} F^\Lambda \\ G_\Lambda \end{pmatrix} \to D(g) \cdot \begin{pmatrix} F^\Lambda \\ G_\Lambda \end{pmatrix}, \end{cases} \tag{23}$$

where $g\star$ denotes the non-linear action of g on the scalar fields and $D(g)$ is the $2\,n_v \times 2\,n_v$ symplectic matrix associated with g. The matrix $\mathcal{M}(\phi)$ transforms under G as follows:

$$\mathcal{M}(g \star \phi) = D(g)^{-T}\, \mathcal{M}(\phi)\, D(g)^{-1}. \tag{24}$$

A duality transformation $g \in G$ maps a black hole solution $U(\tau), \phi^r(\tau)$ with charges $\mathcal{P}$ into a new solution $U'(\tau) = U(\tau)$, $\phi'^r(\tau) = g \star \phi^r(\tau)$ with charges $\mathcal{P}' = D(g)\,\mathcal{P}$. More specifically, if $U(\tau), \phi^r(\tau)$ is defined by the boundary condition ϕ_0 for the scalar fields, $U'(\tau) = U(\tau)$, $\phi'^r(\tau)$ is the *unique solution*, within our class, with charges $\mathcal{P}'$ defined by the boundary condition $\phi'_0 = g \star \phi_0$

$$g \in G : \begin{cases} U(\tau;\, \phi_0) \\ \phi(\tau;\, \phi_0) \\ \mathcal{P} \end{cases} \longrightarrow \begin{cases} U'(\tau;\, g \star \phi_0) = U(\tau;\, \phi_0) \\ \phi'(\tau;\, g \star \phi_0) = g \star \phi(\tau;\, \phi_0)\,. \\ \mathcal{P}' = D(g)\,\mathcal{P} \end{cases} \tag{25}$$

Using (24) and (25), we see that the effective potential is invariant if we act on ϕ^r and $\mathcal{P}$ by means of G simultaneously:

$$V(\phi, \mathcal{P}) = V(g \star \phi, D(g)\,\mathcal{P}). \tag{26}$$

This implies that V, as a function of the scalar fields and quantized charges, is G-invariant. From this property of V it follows that the effective action (5) and the extremality constraint (6) are manifestly duality invariant. Let us show now that the W function shares with V the same symmetry property (26), namely that it is G-invariant as well:

$$W(\phi, \mathcal{P}) = W(g \star \phi, D(g)\,\mathcal{P}). \tag{27}$$

This is easily shown using the general form (13) and (25):

$$\begin{aligned} W(g \star \phi_0,\ D(g)\,\mathcal{P}) &= \int_{-\infty}^{0} e^{2\,U'(\tau;\,g\star\phi_0)}\, V(\phi'(\tau; g \star \phi_0), D(g)\,\mathcal{P})\, d\tau \\ &= \int_{-\infty}^{0} e^{2\,U(\tau;\,\phi_0)}\, V(g \star \phi(\tau; \phi_0), D(g)\,\mathcal{P})\, d\tau \\ &= \int_{-\infty}^{0} e^{2\,U(\tau;\,\phi_0)}\, V(\phi(\tau; \phi_0), \mathcal{P})\, d\tau = W(\phi_0, \mathcal{P}). \end{aligned} \quad (28)$$

Being the ADM mass expressed in terms of W, see (11), it is a G-invariant quantity as well:

$$M_{ADM}(\phi_0, \mathcal{P}) = M_{ADM}(g \star \phi_0, D(g)\,\mathcal{P}). \quad (29)$$

Extremal black holes can be grouped into orbits with respect to the duality action (25) of G. These orbits are characterized in terms of G-invariant functions of the scalar fields and the quantized charges, which are expressed in terms of H-invariant functions of the central and matter charges. One of these is the scalar-independent quartic invariant $I_4(\mathcal{P})$ of G which defines the area of the horizon for large black holes. Small black holes, on the other hand, belong to the orbits in which $I_4(\mathcal{P}) = 0$.

2.3 The Issue of Stability: Asymptotic Stability of the Critical Points

Let us notice, from (13), that W is always positive definite, since the effective potential is. Moreover its derivative along the solution $\phi^r(\tau)$ is positive definite as well (except in ϕ_* where it vanishes):

$$\frac{dW}{d\tau} = \dot{\phi}^r\, \partial_r W = e^{-U}\, G_{rs}(\phi)\, \dot{\phi}^r\, \dot{\phi}^s > 0. \quad (30)$$

We see that, if ϕ_* is isolated, W has the properties of a Liapunov's function and thus, in virtue of Liapunov's theorem, ϕ_* is a *stable attractor point* (we refer the reader to Appendix B for a brief review of the notion of asymptotic stability in the sense of Liapunov and of Liapunov's theorem, see also standard books like [25, 30]). This conclusion extends to models based on a generic (not necessarily homogeneous) scalar manifold: The very existence of the W-function (i.e. of a solution to the Hamilton-Jacobi equation) even just in a neighborhood of an isolated critical point ϕ_* is enough to guarantee asymptotic stability of ϕ_*, and thus that the horizon is a stable attractor. Let us emphasize that in this case we need not evaluate the Hessian of the potential on ϕ_*. In other words the (local) existence of W can be taken as an alternative and more powerful characterization of the attractiveness and stability properties of the horizon point ϕ_*.

There is a class of large extremal solutions, however, in which the critical points, defining the near-horizon behavior of the scalar fields, are not isolated but rather span a hypersurface $\mathcal{C}$ of the scalar manifold. This is the case of the non-BPS solutions with $I_4 < 0$ in the symmetric models. As we are going to show below, in full generality, the existence of this locus of critical points is related to the existence of $n_f < n$ *flat directions* φ^α, $\alpha = 1, \ldots, n_f$, of *both* the scalar potential V and the W function. The critical hypersurface $\mathcal{C}$ has in this case dimension n_f and is spanned by (φ^α). As far as the global behavior of the flows is concerned, the analysis of the simple STU model (see [24] for a discussion on this point) suggests a general property: The scalar manifold can be decomposed in hypersurfaces $\mathscr{M}_{(\alpha)}$ of dimension $n - n_f$ which intersect the hypersurface of critical points $\mathcal{C}$ in a single point $\phi_*|_\alpha$ characterized by fixed values φ^α of the flat directions. The hypersurfaces $\mathscr{M}_{(\alpha)}$ have the property of being *invariant* with respect to the flow, namely that, choosing the initial point ϕ_0 on a given $\mathscr{M}_\alpha$, the entire flow will be contained within the same hypersurface. Within each $\mathscr{M}_{(\alpha)}$ the critical point $\phi_*|_\alpha$ is isolated and Liapunov's theorem applies, implying it is asymptotically stable or, equivalently, a stable attractor.

2.4 The Issue of Flat Directions

Let us denote by $G_0 \subset G$ the *little group* (or stabilizer) of the orbit of the quantized charges $\mathcal{P}$ under the action of G [8, 9, 15]:

$$g_0 \in G_0 : \quad D(g_0)\,\mathcal{P} = \mathcal{P}. \tag{31}$$

Of course the embedding of G_0 within G depends in general on $\mathcal{P}$. Let us show that the scalar fields φ^α spanning the submanifold G_0/H_0, H_0 being the maximal compact subgroup of G_0, are *flat directions* of the potential and of the W-function, namely that neither V nor W, depend on φ^α. Since we are interested in the part of the little group which has a free action on the moduli, we shall define G_0 modulo compact group-factors. For instance if the little group is $\mathrm{SU}(3)\times\mathrm{SU}(2,1)$, we define G_0 to be $\mathrm{SU}(2,1)$ and thus $H_0 = \mathrm{U}(2)$. For a summary of the orbits of regular extremal black holes in the various theories and of the corresponding moduli spaces G_0/H_0 see Table 1.

To prove that φ^α are flat directions of both V and W, let us decompose the n scalar fields ϕ^r into the φ^α scalars parametrizing the submanifold G_0/H_0 and scalars φ^k, which can be chosen to transform linearly with respect to H_0. Let us stress at this point that the coordinates $\varphi^\alpha, \varphi^k$ will in general depend on the original ones ϕ^r and on the electric and magnetic charges, namely:

$$\varphi^\alpha = \varphi^\alpha(\phi^r, p^\Lambda, q_\Lambda), \quad \varphi^k = \varphi^k(\phi^r, p^\Lambda, q_\Lambda). \tag{32}$$

Table 1 Summary of regular, extremal black hole orbits in the various supergravities. The symbols I, II, III denote the $\frac{1}{\mathcal{N}}$-BPS, the non-BPS ($I_4 > 0$) and the non-BPS ($I_4 < 0$) orbits respectively. For those solutions with non-trivial moduli spaces $\frac{G_0}{H_0}$ (i.e. G_0 non-compact), the representations $\mathbf{R}_0$, $\mathbf{R}_1$ of H_0, pertaining to the flat and to the non-flat directions respectively, are given. The symbol "c.r." stands for *conjugate representations*

$\mathcal{N}$	$\frac{G}{H}$	Orbit	$\frac{G_0}{H_0}$	R_0	R_1
		I	$\frac{E_{6(2)}}{SU(2)\times SU(6)}$	$(\mathbf{2},\mathbf{20})$	$(\mathbf{1},\mathbf{15})$+c.r.
8	$\frac{E_{7(7)}}{SU(8)}$	III	$\frac{E_{6(6)}}{USp(8)}$	$\mathbf{42}$	$\mathbf{1}+\mathbf{27}$
		I	$\frac{SU(4,2)}{S[U(4)\times U(2)]}$	$(\mathbf{4},\mathbf{2})_{-3}$+c.r.	$(\mathbf{6},\mathbf{1})_{+2}+(\mathbf{1},\mathbf{1})_{-4}$+c.r.
		II	–		
6	$\frac{SO^*(12)}{U(6)}$	III	$\frac{SU^*(6)}{USp(6)}$	$\mathbf{14}$	$2\times\mathbf{1}+\mathbf{14}$
5	$\frac{SU(5,1)}{U(5)}$	I	$\frac{SU(2,1)}{U(2)}$	$\mathbf{2}_{+3}$+c.r.	$3\times\mathbf{1}_{-2}$+c.r.
		I	$\frac{SO(4,n)}{SO(4)\times SO(n)}$	$(\mathbf{4},\mathbf{n})$	$2\times[(\mathbf{1},\mathbf{1})+(\mathbf{1},\mathbf{n})]$
		II	$\frac{SO(6,n-2)}{SO(6)\times SO(n-2)}$	$(\mathbf{6},\mathbf{n}-\mathbf{2})$	$2\times[(\mathbf{1},\mathbf{1})+(\mathbf{6},\mathbf{1})]$
4	$\frac{SL(2,\mathbb{R})}{SO(2)}\times\frac{SO(6,n)}{SO(6)\times SO(n)}$	III	$SO(1,1)\times\frac{SO(5,n-1)}{SO(5)\times SO(n-1)}$	$(\mathbf{1},\mathbf{1})+(\mathbf{5},\mathbf{n}-\mathbf{1})$	$2\times(\mathbf{1},\mathbf{1})+(\mathbf{5},\mathbf{1})$ $+(\mathbf{1},\mathbf{n}-\mathbf{1})$
		I	$\frac{SU(2,n)}{S[U(2)\times U(n)]}$	$(\mathbf{2},\mathbf{n})_{n+2}$+c.r.	$(\mathbf{1},\mathbf{n})_{-2}$+c.r.
3	$\frac{SU(3,n)}{S[U(3)\times U(n)]}$	II	$\frac{SU(3,n-1)}{S[U(3)\times U(n-1)]}$	$(\mathbf{3},\mathbf{n}-\mathbf{1})_{n+2}$+c.r.	$(\mathbf{3},\mathbf{1})_{1-n}$+c.r.
		I	–		
	$\frac{SU(1,n+1)}{U(n+1)}$	II	$\frac{SU(1,n)}{U(n)}$	$\mathbf{n}_{n+1}$+c.r.	$\mathbf{1}_{-n}$+c.r.
		I	–		
	$\frac{SL(2,\mathbb{R})}{SO(2)}\times\frac{SO(2,n+2)}{SO(2)\times SO(n+2)}$	II	$\frac{SO(2,n)}{SO(2)\times SO(n)}$	$(\mathbf{2},\mathbf{n})$	$2\times[(\mathbf{2},\mathbf{1})+(\mathbf{1},\mathbf{1})]$
		III	$SO(1,1)\times\frac{SO(1,n+1)}{SO(n+1)}$	$\mathbf{1}+(\mathbf{n}+\mathbf{1})$	$3\times\mathbf{1}+(\mathbf{n}+\mathbf{1})$
		I	–		
		II	$\frac{SU(2,1)}{U(2)}$	$\mathbf{2}_{-3}$+c.r.	$\mathbf{1}_{-4}+\mathbf{3}_{+2}$+c.r.
	$\frac{Sp(6)}{U(3)}$	III	$\frac{SL(3,\mathbb{R})}{SO(3)}$	$\mathbf{5}$	$2\times\mathbf{1}+\mathbf{5}$
		I	–		
		II	$\left(\frac{SU(2,1)}{U(2)}\right)^2$	$(\mathbf{2},\mathbf{1})_{3,0}+(\mathbf{1},\mathbf{2})_{0,3}$+c.r.	$(\mathbf{2},\mathbf{2})_{1,-1}+(\mathbf{1},\mathbf{1})_{-2,2}$+c.r.
2	$\frac{SU(3,3)}{S[U(3)\times U(3)]}$	III	$\frac{SL(3,\mathbb{C})}{SU(3)}$	$\mathbf{8}$	$2\times\mathbf{1}+\mathbf{8}$
		I	–		
		II	$\frac{SU(4,2)}{S[U(4)\times U(2)]}$	$(\mathbf{4},\mathbf{2})_{-3}$+c.r.	$(\mathbf{6},\mathbf{1})_{+2}+(\mathbf{1},\mathbf{1})_{-4}$+c.r.
	$\frac{SO^*(12)}{U(6)}$	III	$\frac{SU^*(6)}{USp(6)}$	$\mathbf{14}$	$2\times\mathbf{1}+\mathbf{14}$
		I	–		
		II	$\frac{E_{6(-14)}}{U(1)\times SO(10)}$	$\overline{\mathbf{16}}_{+3}$+c.r.	$\mathbf{1}_{+4}+\mathbf{10}_{-2}$+c.r.
	$\frac{E_{7(-25)}}{U(1)\times E_6}$	III	$\frac{E_{6(-26)}}{F_4}$	$\mathbf{26}$	$2\times\mathbf{1}+\mathbf{26}$

Let us choose, for convenience, a basis of coordinates in the moduli space such that the first n_f components of ϕ^r coincide with the φ^α, the others being φ^k, that is $\phi^\alpha = \varphi^\alpha$, $\phi^k = \varphi^k$. We can move along the ϕ^α direction through the action of isometries in G_0. We shall consider infinitesimal isometries in G_0 whose effect is to shift the α-scalars only:

$$g_0 \in G_0 : \phi^r \to (g_0 \star \phi)^r = \phi^r + \delta^r_\alpha\, \delta\phi^\alpha, \quad \mathcal{P} \to \mathcal{P}' = \mathcal{P} + \delta\mathcal{P} = \mathcal{P}, \tag{33}$$

where we have used the definition of G_0, (31). Let us now use (26) and (27) to evaluate the corresponding infinitesimal variations of V and W:

$$\begin{aligned} V(\phi^r, \mathcal{P}) &= V(\phi^r + \delta\phi^r, \mathcal{P} + \delta\mathcal{P}) = V(\phi^k, \phi^\alpha + \delta\phi^\alpha, \mathcal{P}). \\ W(\phi^r, \mathcal{P}) &= W(\phi^r + \delta\phi^r, \mathcal{P} + \delta\mathcal{P}) = W(\phi^k, \phi^\alpha + \delta\phi^\alpha, \mathcal{P}). \end{aligned} \tag{34}$$

We conclude that $\frac{\partial V}{\partial \phi^\alpha} = \frac{\partial W}{\partial \phi^\alpha} = 0$, namely that ϕ^α are flat direction of both functions. Using (11) we see that the same property holds for the ADM mass: $\frac{\partial}{\partial \phi^\alpha} M_{ADM} = 0$. Let us now give a general characterization of the W-function in terms of the central and matter charges. We can write the coset representative $\mathbb{L}(\phi^r)$ of $\mathscr{M}_{scal}$ as the product of the G_0/H_0 coset representative $\mathbb{L}_0(\phi^\alpha)$ times a matrix $\mathbb{L}_1(\phi^k)$ depending on the remaining scalars:

$$\mathbb{L}(\phi^r) = \mathbb{L}(\phi^\alpha, \phi^k) = \mathbb{L}_0(\phi^\alpha)\, \mathbb{L}_1(\phi^k). \tag{35}$$

We can write $\mathbb{L}(\phi^r)$ as a $2\,n_V \times 2\,n_V$ matrix $\mathbb{L}(\phi^r)^M{}_{\hat{N}}$, where M is an index in the real symplectic representation, while $\hat{N}$ spans a complex basis in which the action of H is block-diagonal. We can obtain $\mathbb{L}(\phi^r)^M{}_{\hat{N}}$ from the coset representative in the real symplectic representation $\mathbb{L}_{Sp}(\phi^r)^M{}_N$ using the Cayley matrix:

$$\mathbb{L}(\phi^r) = \mathbb{L}_{Sp}(\phi^r)\, \mathcal{A}^\dagger \quad \text{where} \quad \mathcal{A} \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} \mathbb{1} & i\, \mathbb{1} \\ \mathbb{1} & -i\, \mathbb{1} \end{pmatrix}. \tag{36}$$

The central and matter charges Z_{AB}, Z_I of the theory can be arranged, together with their complex conjugates, in a $(2\,n_V)$-vector $Z_{\hat{M}}$ defined as follows:

$$Z_{\hat{M}}(\phi^r, \mathcal{P}) = \begin{pmatrix} Z_{AB} \\ Z_I \\ \bar{Z}^{AB} \\ \bar{Z}^I \end{pmatrix} = -\mathbb{L}(\phi^r)^T\, \mathbb{C}\, \mathcal{P} = -\mathbb{L}_1(\phi^k)^T\, \mathbb{L}_0(\phi^\alpha)^T\, \mathbb{C}\, \mathcal{P}, \tag{37}$$

Now we can use the property of $\mathbb{L}_0(\phi^\alpha)$ of being an element of G_0 in the symplectic representation, so that $\mathbb{L}_0^T\, \mathbb{C}\, \mathcal{P} = \mathbb{C}\, \mathbb{L}_0^{-1}\, \mathcal{P} = \mathbb{C}\, \mathcal{P}$ and write:

$$Z_{\hat{M}}(\phi^\alpha, \phi^k, \mathcal{P}) = -\mathbb{L}_1(\phi^k)^T\, \mathbb{C}\, \mathcal{P} = Z_{\hat{M}}(0, \phi^k, \mathcal{P}), \tag{38}$$

that is the central and matter charges do not depend on ϕ^α at all:

$$\frac{\partial}{\partial \phi^\alpha} Z_{AB} = \frac{\partial}{\partial \phi^\alpha} Z_I = 0. \tag{39}$$

Let us now describe the effect of a generic transformation g_0 in G_0 on the central charges. From the general properties of coset representatives we know that $D(g_0)\,\mathbb{L}_0(\phi^\alpha) = \mathbb{L}_0(g_0 \star \phi^\alpha)\,D(h_0)$, $D(h_0)$ being a compensator in H_0 depending on g_0 and ϕ^α. Now, using the property that ϕ^k transform in a linear representation of H_0, we can describe the action of g_0 on a generic point ϕ as follows:

$$\begin{aligned} D(g_0)\,\mathbb{L}(\phi^r) &= D(g_0)\,\mathbb{L}_0(\phi^\alpha)\mathbb{L}_1(\phi^k) = \mathbb{L}_0(g_0 \star \phi^\alpha)D(h_0)\mathbb{L}_1(\phi^k)D(h_0)^{-1}D(h_0) \\ &= \mathbb{L}_0(g_0 \star \phi^\alpha)\,\mathbb{L}_1(\phi'^k)\,D(h_0) = \mathbb{L}(g_0 \star \phi^r)\,D(h_0), \end{aligned} \tag{40}$$

where ϕ'^k is the transformed of ϕ^k by h_0, and $(g_0 \star \phi^\alpha, \phi'^k)$ define the transformed $g_0 \star \phi^r$ of ϕ^r by g_0. From (40) and the definition (37) we derive the following property:

$$\forall g_0 \in G_0 : Z_{\hat{M}}(g_0 \star \phi^r, \mathcal{P}) = [D(h_0)^{-T}]_{\hat{M}}{}^{\hat{N}}\, Z_{\hat{N}}(\phi^r, \mathcal{P}) = h_0 \star Z_{\hat{M}}(\phi^r, \mathcal{P}), \tag{41}$$

where, to simplify notations we have denoted by $h_0 \star Z$ the vector $[D(h_0)^{-T}]_{\hat{M}}{}^{\hat{N}}\, Z_{\hat{N}}$. Now consider the W function as a function of ϕ^r and $\mathcal{P}$ through the central and matter charges $Z_{\hat{M}}$:

$$W(\phi^r, \mathcal{P}) = \widehat{W}[Z_{\hat{M}}(\phi^r, \mathcal{P})]. \tag{42}$$

From the duality-invariance of W it follows that, for any $g_0 \in G_0$ we have

$$W(\phi^r, \mathcal{P}) = W(g_0 \star \phi^r, D(g_0)\,\mathcal{P}) = W(g_0 \star \phi^r, \mathcal{P}). \tag{43}$$

Furthermore, using (41) and (42) we find:

$$\begin{aligned} \widehat{W}[Z(\phi^r, \mathcal{P})] &= W(\phi^r, \mathcal{P}) = W(g_0 \star \phi^r, \mathcal{P}) = \widehat{W}[Z(g_0 \star \phi^r, \mathcal{P})] \\ &= \widehat{W}[h_0 \star Z_{\hat{N}}(\phi^r, \mathcal{P})]. \end{aligned} \tag{44}$$

The above equality holds for any $g_0 \in G_0$ and thus for any $h_0 \in H_0$. We conclude from this that W *can be characterized, for a given orbit of solutions, as an* H_0*-invariant function of the central and matter charges*. This is consistent with what was found in [4, 14]. Let us stress once more that we have started from a generic charge vector $\mathcal{P}$, so that the definition of G_0, and thus of H_0, is charge dependent. We could have started from a given G_0 inside G and worked out the representative $\mathcal{P}_0$ of the G-orbit having G_0 as manifest little group. In this case, by construction, the (ϕ^α, ϕ^k) parametrization is charge-independent.

2.4.1 A Detailed Analysis

Let us exploit now, for the BPS and non-BPS extremal, regular solutions, the symmetry properties of the W function discussed in the previous sections, to study general aspects of the evolution of the flat and non-flat directions.

We start computing the Killing vectors associated with the G_0-transformations and write the condition that W be G_0-invariant in the form of differential equations. To this aim, we will first compute the general expression for the vielbein of $\mathscr{M}_{scal}$ in the parametrization (35). Let us denote by $\{T_{\mathcal{A}}\}$, $\mathcal{A} = 1, \dots, \dim(G_0)$, the generators of G_0. We can perform the Cartan decomposition of the Lie algebras $\mathfrak{g}$ and $\mathfrak{g}_0$ generating G and G_0, respectively, with respect to their maximal compact subalgebras $\mathfrak{h}$, $\mathfrak{h}_0$:

$$\mathfrak{g} = \mathfrak{K} \oplus \mathfrak{h}, \quad \mathfrak{g}_0 = \mathfrak{K}_0 \oplus \mathfrak{h}_0. \tag{45}$$

Under the adjoint action of H_0, the space $\mathfrak{K}$ splits into subspaces $\mathfrak{K}_0$ and $\mathfrak{K}_1$ transforming in the representations $\mathbf{R}_0$, $\mathbf{R}_1$ of H_0. The non-compact generators $\{\mathbf{K}_{\hat{r}}\}$, $\hat{r} = 1 \dots, n$, of $\mathfrak{K}$ (the indices $\hat{r}$, $\hat{s}$ label basis elements of the tangent space to the manifold) split into the generators $\{\mathbf{K}_a\}$, $a, b = 1, \dots, n_f$, of $\mathfrak{K}_0$, belonging to the tangent space of the submanifold G_0/H_0, and the remaining $n-n_f$ generators $\{\mathbf{K}_{\hat{k}}\}$ of $\mathfrak{K}_1$. The Lie algebra $\mathfrak{h}_0$ of H_0 is generated by $\{\mathbf{H}_u\}$, $u = 1 \dots, \dim(H_0)$. As far as the choice of the parametrization is concerned, for the BPS and non-BPS ($I_4 > 0$) solutions, we choose the coset representative as follows:

$$\mathbb{L}(\phi^r) = \mathbb{L}_0(\phi^\alpha)\, \mathbb{L}_1(\phi^k) \in e^{\mathfrak{K}_0} \cdot e^{\mathfrak{K}_1}, \tag{46}$$

that is $\mathbb{L}_0(\phi^\alpha)$ is an element of $e^{\mathfrak{K}_0} \equiv G_0/H_0$ and $\mathbb{L}_1(\phi^k)$ is an element of $e^{\mathfrak{K}_1}$. This in particular implies that ϕ^α and ϕ^k transform in the representations $\mathbf{R}_0$, $\mathbf{R}_1$ of H_0, respectively (see Table 1 for a list of these representations). For the non-BPS ($I_4 < 0$) solutions, it is more convenient to adopt a parametrization of the coset which is different from (46), in which ϕ^k can be defined to transform linearly with respect to the whole G_0. For this class of solutions, see below, we define ϕ^k to be parameters of a solvable algebra $\{s_k\} = \{s_\Lambda,\, s_0\}$, generated by $n - n_f - 1$ nilpotent generators s_Λ and a Cartan generator s_0. As we shall see, for the standard choice of the charges $\mathcal{P}_0$, ϕ^k consist in $n - n_f - 1$ axions originating from the $D = 5$ vector fields and a dilaton describing the modulus of the internal radius in the $D = 5 \to D = 4$ dimensional reduction.

We want to compute the components of the vielbein of $\mathscr{M}_{scal}$ in the basis (46). To start with, the G_0/H_0 left-invariant 1-form reads:

$$\Omega_0(\phi^\alpha) = L_0^{-1} dL_0 = \Omega_0^{\mathcal{A}}\, T_{\mathcal{A}} = d\phi^\beta\, \Omega_{0\,\beta}{}^{\mathcal{A}}(\phi^\alpha)\, T_{\mathcal{A}}. \tag{47}$$

If we split the G_0-generators $T_{\mathcal{A}}$ into generators of G_0/H_0 ($\mathbf{K}_a$) and of H_0 ($\mathbf{H}_u$), that is $\mathcal{A} \to (a, u)$, then $\Omega_{0\,\beta}{}^b(\phi^\alpha) \equiv V_\beta{}^b(\phi^\alpha)$ defines the vielbein of G_0/H_0. Moreover, let us introduce the left-invariant 1-form:

$$\Omega_1(\phi^k) \equiv \mathbb{L}_1^{-1} d\mathbb{L}_1 = d\phi^k\, V_k{}^{\hat{k}}(\phi^k) \mathbf{K}_{\hat{k}} + \text{connection}, \tag{48}$$

where $d\phi^k\, V_k{}^{\hat{k}}$ will define the vielbein 1-forms along the directions $\mathbf{K}_{\hat{k}}$ of the tangent space.[6]

In terms of the above quantities, we can now compute the left-invariant 1-form of $\mathscr{M}_{scal}$ in the basis (46):

$$\begin{aligned}\Omega(\phi^r) &\equiv \mathbb{L}^{-1}\, d\mathbb{L} = \mathbb{L}_1{}^{-1}\,\Omega_0\,\mathbb{L}_1 + \mathbb{L}_1{}^{-1}\, d\mathbb{L}_1 = \Omega_0^{\mathcal{A}}(\phi^\alpha)\,\mathbb{L}_1^{-1} T_{\mathcal{A}}\mathbb{L}_1 + \Omega_1(\phi^k)\\ &= d\phi^\beta\,\Omega_{0\,\beta}{}^{\mathcal{A}}(\phi^\alpha)\,\mathbb{L}_{1\,\mathcal{A}}{}^{\hat{r}}(\phi^k)\,\mathbf{K}_{\hat{r}} + d\phi^k\, V_k{}^{\hat{k}}\,\mathbf{K}_{\hat{k}} + \text{connection},\end{aligned} \tag{49}$$

where we have written $\mathbb{L}_1{}^{-1}\, T_{\mathcal{A}}\,\mathbb{L}_1 = \mathcal{L}_{1\,\mathcal{A}}{}^{\hat{r}}(\phi^k)\,\mathbf{K}_{\hat{r}}$ + compact generators. Similarly we will also write $\mathbb{L}_0^{-1}\, T_{\mathcal{A}}\,\mathbb{L}_0 = \mathcal{L}_{0\,\mathcal{A}}{}^{\mathcal{B}}(\phi^\alpha)\, T_{\mathcal{B}}$. The non-vanishing components of the vielbein $\mathcal{V}_r{}^{\hat{s}}$ of $\mathscr{M}_{scal}$ are now readily computed:

$$\mathcal{V}_\beta{}^b = \Omega_{0\,\beta}{}^{\mathcal{A}}(\phi^\alpha)\mathcal{L}_{1\,\mathcal{A}}{}^b(\phi^k),\ \mathcal{V}_\beta{}^{\hat{k}} = \Omega_{0\beta}{}^{\mathcal{A}}(\phi^\alpha)\mathcal{L}_{1\,\mathcal{A}}{}^{\hat{k}}(\phi^k),\ \mathcal{V}_k{}^{\hat{k}} = V_k{}^{\hat{k}}(\phi^k). \tag{50}$$

Note that for all regular extremal black holes, our choice of parametrization is such that the vielbein matrix has a vanishing off-diagonal block $V_k{}^a$, see Appendix C.

The non-vanishing blocks of the inverse vielbein $\mathcal{V}^{-1}{}_{\hat{r}}{}^r$ are:

$$\mathcal{V}^{-1}{}_a{}^\beta,\ \mathcal{V}^{-1}{}_{\hat{k}}{}^k,\ \mathcal{V}^{-1}{}_a{}^k = -\mathcal{V}^{-1}{}_a{}^\beta\,\mathcal{V}_\beta{}^{\hat{k}}\,\mathcal{V}^{-1}{}_{\hat{k}}{}^k\,, \tag{51}$$

where $\mathcal{V}^{-1}{}_a{}^\beta$, $\mathcal{V}^{-1}{}_{\hat{k}}{}^k$ are the inverses of the diagonal blocks $\mathcal{V}_\alpha{}^b$, $\mathcal{V}_k{}^{\hat{k}}$, respectively.

Consider now an infinitesimal G_0-transformation $g_0 \sim \mathbf{1} + \epsilon^{\mathcal{A}}\, T_{\mathcal{A}}$, $\epsilon^{\mathcal{A}} \sim 0$, and write $(g_0 \star \phi)^r \sim \phi^r + \epsilon^{\mathcal{A}}\, k^r_{\mathcal{A}}(\phi)$. The Killing vectors $k^r_{\mathcal{A}}(\phi)$ are computed, in the parametrization (46), to be:

$$k^r_{\mathcal{A}} = \mathcal{L}_{0\,\mathcal{A}}{}^{\mathcal{B}}(\phi^\alpha)\,\mathcal{L}_{1\mathcal{B}}{}^{\hat{r}}(\phi^k)\,\mathcal{V}^{-1}{}_{\hat{r}}{}^r. \tag{52}$$

The G_0-invariance of the W-function ($W(g_0 \star \phi, \mathcal{P}) = W(\phi, \mathcal{P})$) can now be expressed in the following way:

$$k^r_{\mathcal{A}}\,\frac{\partial W}{\partial\phi^r} = 0 \Leftrightarrow \mathcal{L}_{1\,\mathcal{A}}{}^{\hat{r}}(\phi^k)\,\mathcal{V}^{-1}{}_{\hat{r}}{}^r\,\frac{\partial W}{\partial\phi^r} = 0, \tag{53}$$

where we have used the property that $\mathcal{L}_{0\,\mathcal{A}}{}^{\mathcal{B}}(\alpha)$ is non-singular. Using the expression of the vielbein, it will be useful to write the first-order flow-equations for the scalar fields in the following form:

$$\dot{\phi}^r\,\mathcal{V}_r{}^{\hat{r}} = e^U\,\mathcal{V}^{-1\,\hat{r}\,s}\,\frac{\partial W}{\partial\phi^s} \Leftrightarrow \begin{cases}\dot{\phi}^\beta\,\mathcal{V}_\beta{}^a = e^U\,\mathcal{V}^{-1\,a\,r}\,\frac{\partial W}{\partial\phi^r}\\ \dot{\phi}^r\,\mathcal{V}_r{}^{\hat{k}} = e^U\,\mathcal{V}^{-1\,\hat{k}\,k}\,\frac{\partial W}{\partial\phi^k}\end{cases}. \tag{54}$$

[6]The reason why the left-invariant 1-form Ω_1 in (48) does not expand on the generators $\mathbf{K}_a$ will be clarified in Appendix C.

We shall illustrate the implications of the above formula in two relevant cases: The BPS solution and the non-BPS one with $I_4 < 0$.

2.4.2 The BPS Black Holes

For the sake of concreteness we shall consider the supersymmetric regular solutions ($\frac{1}{8}$-BPS) in the maximal theory $\mathcal{N} = 8$, although our discussion is easily extended to non-maximal theories. In this case $G_0 = \mathrm{E}_{6(+2)}$ and $H_0 = \mathrm{SU}(2) \times \mathrm{SU}(6) \subset \mathrm{SU}(8) = \mathrm{H}$. With respect to the adjoint action of H_0, the coset space $\mathfrak{K}$, in the **70** of SU(8), splits into the subspaces $\mathfrak{K}_0 = \{\mathbf{K}_a\}$ in the $(\mathbf{2}, \mathbf{20})$ and $\mathfrak{K}_1 = \{\mathbf{K}_{\hat{k}}\}$ in the $(\mathbf{1}, \mathbf{15}) \oplus (\mathbf{1}, \overline{\mathbf{15}})$ of H_0, according to the branching:

$$\mathbf{70} \to (\mathbf{2}, \mathbf{20}) \oplus (\mathbf{1}, \mathbf{15}) \oplus (\mathbf{1}, \overline{\mathbf{15}}). \tag{55}$$

The parametrization (35) amounts to the following choice of the coset representative:

$$\mathbb{L} = \mathbb{L}_0(\phi^\alpha)\,\mathbb{L}_1(\phi^k), \quad \mathbb{L}_0(\phi^\alpha) \in e^{\mathfrak{K}_0}, \quad \mathbb{L}_1(\phi^k) \in e^{\mathfrak{K}_1}. \tag{56}$$

Since the index a spans a SU(2)-doublet ($a = (A, \lambda)$, $A = 1, 2$, $\lambda = [mnp] = 1, \ldots, 20$, $m, n, p = 1, \ldots, 6$), while $\hat{k}$ only SU(2)- singlets, being ϕ^k themselves SU(2)-singlets, the non vanishing components of the matrix $\mathcal{L}_{1\mathcal{A}}{}^{\hat{r}}$ are: $\mathcal{L}_{1a}{}^{b}(\phi^k)$, $\mathcal{L}_{1u}{}^{\hat{k}}(\phi^k)$. Consider now the implications of the G_0-invariance of W, as expressed by (53). The $H_0 = \mathrm{SU}(2) \times \mathrm{SU}(6)$-invariance corresponds to the $\mathcal{A} = u$ component of the equation, and implies

$$\mathcal{L}_{1u}{}^{\hat{k}}(\phi^k)\,\mathcal{V}^{-1}{}_{\hat{k}}{}^{k}(\phi^k)\,\frac{\partial W}{\partial \phi^k} = 0. \tag{57}$$

The invariance of W under G_0/H_0-transformations, on the other hand, implies, using (51) and (50):

$$\begin{aligned}
0 &= \mathcal{L}_{1a}{}^{b}(\phi^k)\,\mathcal{V}^{-1}{}_{b}{}^{r}\,\frac{\partial W}{\partial \phi^r} = \mathcal{L}_{1a}{}^{b}(\phi^k)\left[\mathcal{V}^{-1}{}_{b}{}^{\gamma}\,\frac{\partial W}{\partial \phi^\gamma} + \mathcal{V}^{-1}{}_{b}{}^{k}\,\frac{\partial W}{\partial \phi^k}\right] \\
&= \mathcal{L}_{1a}{}^{b}(\phi^k)\left[\mathcal{V}^{-1}{}_{b}{}^{\gamma}\,\frac{\partial W}{\partial \phi^\gamma} - \mathcal{V}^{-1}{}_{b}{}^{\gamma}\,\Omega_{0\gamma}{}^{u}\,\mathcal{L}_{1u}{}^{\hat{k}}\,\mathcal{V}^{-1}{}_{\hat{k}}{}^{k}\,\frac{\partial W}{\partial \phi^k}\right] \\
&= \mathcal{L}_{1a}{}^{b}(\phi^k)\,\mathcal{V}^{-1}{}_{b}{}^{\gamma}\,\frac{\partial W}{\partial \phi^\gamma} \quad \Rightarrow \quad \frac{\partial W}{\partial \phi^\alpha} = 0,
\end{aligned} \tag{58}$$

where we have used (57) and the property that the block $\mathcal{L}_{1a}{}^{b}(\phi^k)$ is non-singular. The above equation expresses the ϕ^α-independence of W, which we had proven before in a different way. Finally, consider the evolution of the ϕ^α-scalars as described in (54). From (58) it follows that:

$$\dot{\phi}^{\beta}\,\mathcal{V}_{\beta}{}^{a} = e^{U}\,\mathcal{V}^{-1\,a\,r}\,\frac{\partial W}{\partial \phi^{r}} = 0, \tag{59}$$

namely *the flat directions* ϕ^{α} *are constant along the flow*. This is consistent with the $\mathcal{N} = 2$ supersymmetry of the solution, since the variation of the fermions λ^{mnp} (the hyperinos in the $\mathcal{N} = 2$ truncation, in the **20** of SU(6)) on the solution reads:

$$\delta\lambda^{mnp} \propto \dot{\phi}^{\alpha}\mathcal{V}_{\alpha}{}^{A,\,mnp}\,\epsilon_{A} = 0, \tag{60}$$

where, as usual, we have written $a = (A, mnp)$.

As far as the non-BPS black holes with $I_4 > 0$ are concerned, the analysis is analogous to the BPS case illustrated above.

2.4.3 Non-BPS Black Holes with $I_4 < 0$

In this case the little group G_0 of the charge vector is the duality group of the five-dimensional parent theory (for the $\mathcal{N} = 8$ case $G_0 = \mathrm{E}_{6(6)}$), so that the flat directions (ϕ^{α}) spanning G_0/H_0 are the five-dimensional scalar fields. We can use the solvable parametrization for $\mathscr{M}_{scal}$ by writing $\mathscr{M}_{scal} = \exp(Solv)$, where $Solv$ is the solvable Lie algebra defined by the Iwasawa decomposition of G with respect to H. Let moreover $Solv_0$ be the solvable Lie algebra generating the submanifold spanned by the flat directions: $G_0/H_0 \equiv \exp(Solv_0)$.

In the solvable parametrization the moduli ϕ^{α} are parameters of the generators s_{α} of $Solv_0$.[7] We can decompose the scalars ϕ^{r} into ϕ^{α} and ϕ^{k} by decomposing $Solv$ with respect to $Solv_0$:

$$Solv = \mathfrak{o}(1,1) \oplus Solv_0 \oplus \mathbf{R}_{-2}, \tag{61}$$

where the $\mathfrak{o}(1,1)$ generator s_0 is parametrized by the modulus σ_0 of the radius of the fifth dimension and the abelian subalgebra $\mathbf{R}_{-2} = \{s_{\Lambda}\}$ is parametrized by the axions σ^{Λ} originating from the five-dimensional vector fields and transforming according to the representation $\bar{\mathbf{R}}$ of G_0 with O(1, 1)-grading +2 (in the maximal theory $\mathbf{R} = \mathbf{27}$). The decomposition (61) originates from the general branching rule of G with respect to G_0

$$\mathrm{Adj}(G) = \mathbf{1}_0 \oplus \mathrm{Adj}(G) \oplus \mathbf{R}_{-2} \oplus \bar{\mathbf{R}}_{+2}, \tag{62}$$

[7] Note that, in contrast to the parametrization used for the other classes of black holes, neither ϕ^{α} nor the corresponding solvable generators s_{α} transform, under the adjoint action of H_0, in a linear representation.

The non-flat directions ϕ^k therefore consist of σ_0 and σ^Λ, which transform in a representation of G_0. The following commutation relations hold:

$$[T_{\mathcal{A}}, s_0] = 0, \quad [s_0, s_\Lambda] = +2\, s_\Lambda, \quad [T_{\mathcal{A}}, s_\Lambda] = -T_{\mathcal{A}\,\Lambda}{}^{\Sigma}\, s_\Sigma. \tag{63}$$

We shall write $\mathbb{L}_1(\phi^k) = \mathbb{L}(\sigma^\Lambda)\, e^{\sigma_0\, s_0}$.

Note that the coset parametrizations that we are using throughout this section, defined in (46), differ from the standard parametrization of $\mathscr{M}_{scal}$, which originates from the $D = 5 \rightarrow D = 4$ reduction (like, for instance, the special coordinate parametrization of the special Kähler manifold in the $\mathcal{N} = 2$ theory). The standard parametrization corresponds indeed to the following choice of the coset representative:

$$\mathbb{L}(\phi^r) = \mathbb{L}(\tilde{\sigma}^\Lambda)\, e^{\sigma_0\, s_0}\, \mathbb{L}_0(\phi^\alpha). \tag{64}$$

The prescription (35), that we are using here, yields instead a different parametrization in which the order of the factors in the coset representative is different: $\mathbb{L}(\phi^r) = \mathbb{L}_0(\phi^\alpha)\, \mathbb{L}(\sigma^\Lambda)\, e^{\sigma_0\, s_0}$. The two parametrizations are related by a redefinition of the axions:

$$\tilde{\sigma}^\Lambda = \mathbb{L}_0^{-1}{}_{\Sigma}{}^{\Lambda}(\phi^\alpha)\, \sigma^\Sigma, \tag{65}$$

where $\mathbb{L}_0{}_\Sigma{}^\Lambda(\phi^\alpha)$ is the matrix form of $\mathbb{L}_0(\phi^\alpha)$ in the **R** representation: $\mathbb{L}_0(\phi^\alpha)^{-1}\, s_\Sigma\, \mathbb{L}_0(\phi^\alpha) = \mathbb{L}_0{}_\Sigma{}^\Lambda(\phi^\alpha)\, s_\Lambda$. The vielbein 1-forms $d\phi^r\, \mathcal{V}_r{}^{\hat{r}}$ are defined, as usual, as the components of the left-invariant 1-form along the non compact generators $\mathbf{K}_{\hat{k}} \propto (s_r + s_r^\dagger)$. The non-vanishing components of the vielbein matrix $\mathcal{V}_r{}^{\hat{r}}$ and of its inverse $\mathcal{V}^{-1}{}_{\hat{r}}{}^{r}$ are readily computed to be:

$$\begin{aligned}
&\mathcal{V}_\alpha{}^b(\phi^\alpha), \quad \mathcal{V}_\alpha{}^{\hat{\Lambda}} = -e^{-2\sigma_0}\, V_\alpha{}^b(\phi^\alpha)\, s_{b\,\Sigma}{}^{\hat{\Lambda}}\, \sigma^\Sigma, \quad \mathcal{V}_\Lambda{}^{\hat{\Sigma}} = e^{-2\sigma_0}\, \delta_\Lambda{}^{\hat{\Sigma}}, \quad \mathcal{V}_0{}^{\hat{0}} = 1,\\
&\mathcal{V}^{-1}{}_a{}^\beta(\phi^\alpha), \quad \mathcal{V}^{-1}{}_a{}^\Lambda = s_{a\,\Sigma}{}^\Lambda\, \sigma^\Sigma, \quad \mathcal{V}^{-1}{}_{\hat{\Lambda}}{}^\Sigma = e^{2\sigma_0}\, \delta_{\hat{\Lambda}}{}^\Sigma, \quad \mathcal{V}^{-1}{}_{\hat{0}}{}^0 = 1,
\end{aligned} \tag{66}$$

where $s_{a\,\Sigma}{}^\Lambda$ is the matrix form of the generator s_a of $Solv_0$ in the representation **R**. Consider now the G_0-invariance condition on W, as expressed by (53) and use the following property:

$$\mathbb{L}_1{}^{-1}\, T_{\mathcal{A}}\, \mathbb{L}_1 = T_{\mathcal{A}} - e^{-2\sigma_0}\, T_{\mathcal{A}\,\Sigma}{}^\Lambda\, s_\Lambda\, \sigma^\Sigma = \mathcal{L}_{\mathcal{A}}{}^{\hat{r}}\, \mathbf{K}_{\hat{r}} + \text{compact generators}. \tag{67}$$

After some algebra we find that the H_0-invariance of W (component $\mathcal{A} = u$ of (53)) implies :

$$T_{u\,\Sigma}{}^\Lambda\, \sigma^\Sigma\, \frac{\partial W}{\partial \sigma^\Lambda} = 0, \tag{68}$$

while the invariance with respect to G_0/H_0 (component $\mathcal{A} = a$ of the same equation) implies:

$$\frac{\partial W}{\partial \phi^\alpha} = 0, \tag{69}$$

that is W must be α-independent, as expected by other arguments. Let us note however that now the ϕ^α are evolving since:

$$\dot{\phi}^\alpha \, \mathcal{V}_{\alpha\, a} = e^U \, s_{a\,\Sigma}{}^\Lambda \, \sigma^\Sigma \, \frac{\partial W}{\partial \sigma^\Lambda} \neq 0, \tag{70}$$

since the right hand side represents the variation of W corresponding to an infinitesimal G_0/H_0 transformation of σ^Λ and W is invariant only with respect to H_0-transformations of σ^Λ (see (68)). One can easily verify that the flow of the non-flat scalars $(\sigma_0,\, \sigma^\Lambda)$ is described by an α-independent dynamical system which has an equilibrium point for $\frac{\partial W}{\partial \sigma^\Lambda} = \frac{\partial W}{\partial \sigma_0} = 0$, at which, by virtue of (70), also $\dot{\alpha} = 0$. Indeed, using (54) and the explicit form of the vielbein matrix and of its inverse (66), we can substitute in the equations for ϕ^k the expression of $\dot{\alpha}^a$ and find for the non-flat directions the following equations:

$$\dot{\sigma}^\Lambda = e^U \, \left(e^{4\sigma_0}\, \delta^{\Lambda\Sigma} + s^a{}_\Delta{}^\Lambda \, s_{a\,\Gamma}{}^\Sigma \, \sigma^\Delta \, \sigma^\Gamma\right) \, \frac{\partial W}{\partial \sigma^\Sigma}, \quad \dot{\sigma}_0 = e^U \, \frac{\partial W}{\partial \sigma_0}. \tag{71}$$

According to the above equations, the non-flat directions σ^Λ, σ_0 evolve towards fixed values at the horizon which depend only on the quantized charges and solve the equilibrium conditions $\frac{\partial W}{\partial \sigma^\Lambda} = \frac{\partial W}{\partial \sigma_0} = 0$. Only the flat directions can depend at the horizon on the values of the scalar fields at radial infinity, but this is not in contradiction with the attractor mechanism since the near horizon geometry only depends on the corresponding values of σ^Λ, σ_0, through V or W.

Let us finally give an example of the $(\phi^\alpha,\, \phi^k)$-parametrization in the STU model, in the case $I_4(\mathcal{P}) < 0$, and show that the central and matter charges do not depend on α. The STU model is a $\mathcal{N} = 2$ supergravity with $n = 6$ real scalar fields (i.e. 3 complex ones $\{s, t, u\} \equiv \{z_1,\, z_2,\, z_3\}$) belonging to three vector multiplets. The number of vector fields is $n_V = 4$. The scalar manifold has the following form:

$$\mathscr{M}_{STU} = \left(\frac{\mathrm{SL}(2,\mathbb{R})}{\mathrm{SO}(2)}\right)_s \times \left(\frac{\mathrm{SL}(2,\mathbb{R})}{\mathrm{SO}(2)}\right)_t \times \left(\frac{\mathrm{SL}(2,\mathbb{R})}{\mathrm{SO}(2)}\right)_u, \tag{72}$$

where each factor is parametrized by the complex scalars $s = a'_1 - i\, e^{\varphi_1}$, $t = a'_2 - i\, e^{\varphi_2}$, $u = a'_3 - i\, e^{\varphi_3}$. The eight quantized charges transform in the $(\mathbf{2}, \mathbf{2}, \mathbf{2})$ of the isometry group $G = \mathrm{SL}(2,\mathbb{R})^3$ and in this representation the coset representative is the tensor product of the coset representatives of each factor in (72) in the fundamental representation of $\mathrm{SL}(2,\mathbb{R})$:

$$\mathbb{L}(z_i) = \mathbb{L}_1(z_1) \otimes \mathbb{L}_2(z_2) \otimes \mathbb{L}_3(z_3), \tag{73}$$

where each 2×2 matrix has the following form:

$$\mathbb{L}_i(z_i) = \begin{pmatrix} 1 & 0 \\ -a_i' & 1 \end{pmatrix} \begin{pmatrix} e^{-\frac{\varphi_i}{2}} & 0 \\ 0 & e^{\frac{\varphi_i}{2}} \end{pmatrix}. \tag{74}$$

In this case the σ'^{Λ} axions are nothing but a_1', a_2', a_3'. The little group of the $I_4(\mathcal{P}) < 0$ orbit is $G_0 = \mathrm{O}(1,1)^2$. For generic charges, like for instance those corresponding to the $\overline{D0},\ D4$ system (q_0, p^i), the action of G_0 is rather involved and depends on the charges themselves. We can consider however, as representative of the same G-orbit, the charges corresponding to the $D0 - D6$ system (p^0, q_0). In this case G_0 is parametrized by two combinations of the dilatons φ_i: $\{\phi^\alpha\}_{\alpha=1,2} = \{\phi^1 = \frac{1}{\sqrt{2}}(\varphi_1 - \varphi_2),\ \phi^2 = \frac{1}{\sqrt{6}}(\varphi_1 + \varphi_2 - 2\,\varphi_3)\}$. According to the general prescription (35), the part $\mathbb{L}_0$ of the coset representative depending on the flat directions $\phi^1,\ \phi^2$, should be the left factor of the product. This corresponds to bringing the diagonal dilatonic factor in (18) to the left and redefining the axion:

$$\mathbb{L}_i(z_i) = \begin{pmatrix} e^{-\frac{\varphi_i}{2}} & 0 \\ 0 & e^{\frac{\varphi_i}{2}} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -a_i & 1 \end{pmatrix}, \tag{75}$$

where $a_i = a_i'\, e^{-\varphi_i}$. The three complex scalar fields, in this new parametrization, read: $z_i = e^{\varphi_i}\,(a_i - i)$. The central and matter charges have the following form:

$$Z = \frac{e^{\frac{\sqrt{3}}{2}\sigma_0}}{2\sqrt{2}}\,[q_0 + p^0\, e^{\sqrt{3}\,\sigma_0}(a_1 - i)(a_2 - \mathrm{i})(a_3 - i)], \tag{76}$$

$$Z_1 = \frac{e^{\frac{\sqrt{3}}{2}\sigma_0}}{2\sqrt{2}}\,[q_0 + p^0\, e^{\sqrt{3}\,\sigma_0}(a_1 + \mathrm{i})(a_2 - \mathrm{i})(a_3 - \mathrm{i})], \tag{77}$$

$$Z_2 = \frac{e^{\frac{\sqrt{3}}{2}\sigma_0}}{2\sqrt{2}}\,[q_0 + p^0\, e^{\sqrt{3}\,\sigma_0}(a_1 - \mathrm{i})(a_2 + \mathrm{i})(a_3 - \mathrm{i})], \tag{78}$$

$$Z_3 = \frac{e^{\frac{\sqrt{3}}{2}\sigma_0}}{2\sqrt{2}}\,[q_0 + p^0\, e^{\sqrt{3}\,\sigma_0}(a_1 - \mathrm{i})(a_2 - \mathrm{i})(a_3 + \mathrm{i})], \tag{79}$$

where $\sigma_0 \equiv \frac{1}{\sqrt{3}}(\varphi_1 + \varphi_2 + \varphi_3)$. We observe that none of the central and matter charges depend on the scalars $\{\phi^\alpha\} = \{\phi^1,\ \phi^2\}$, but only on the remaining scalar fields $\{\phi^k\}$, $k = 3, \ldots, 6$, defined as follows:

$$\{\phi^k\} = \{\sigma_0,\ a_i \equiv a_i'\, e^{-\varphi_i}\}. \tag{80}$$

The scalars ϕ^α are then flat directions of any function of the central and matter charges, including V and W.

3 Small Black Holes in the $\mathcal{N} = 8$ Theory

$\mathcal{N} = 8$ supergravity admits two orbits for "large" extremal black holes (one of which is 1/4-BPS and the other a non-BPS one) and three orbits for "small" extremal black holes (all of them BPS, preserving 1/8, 1/4, and 1/2 supersymmetry respectively).

Following the analysis of [15], the ADM mass for the three small orbits is given by the largest eigenvalue of the central charge matrix Z_{AB}. Its eigenvalues for 1/8 and 1/4 BPS solutions are given by the quartic and quadratic roots of the secular equation

$$\prod_{i=1}^{4}(\lambda - \lambda_i) = 0 \qquad\qquad (\lambda_i = \rho_i^2), \tag{81}$$

ρ_i being the skew-eigenvalue of Z_{AB} when written in normal form. In particular we have:

- For 1/8 BPS: $\lambda_1 > \lambda_2 \geq \lambda_3 \geq \lambda_4$
- For 1/4 BPS: $\lambda_1 = \lambda_2 > \lambda_3 = \lambda_4$
- For 1/2 BPS: $\lambda_i = \lambda \; \forall \; i = 1, \cdots, 4$.

The five $\mathcal{N} = 8$ orbits preserve, respectively, the following symmetries:

- large: $\begin{cases} \text{1/8 BPS:} & \mathrm{SU}(2) \times \mathrm{SU}(6) \\ \text{non-BPS:} & \mathrm{USp}(8) \end{cases}$
- small $\begin{cases} \text{1/8 BPS:} & \mathrm{USp}(2) \times \mathrm{USp}(6) \\ \text{1/4 BPS:} & \mathrm{SU}(4) \times \mathrm{USp}(4) \\ \text{1/2 BPS:} & \mathrm{USp}(8) \end{cases}$

The superpotential W, for all the BPS orbits, is given by the highest eigenvalue of the central charge matrix Z_{AB}, however one can also get small orbits from the large non-BPS orbit, in the limiting procedure $I_4 \to 0$. Indeed, in this limit the non-BPS orbit becomes supersymmetric, the fraction of supersymmetry preserved depending on whether further constraints on I_4 are imposed. For example, let us start with a non-BPS black hole with charges (p^0, q_0) turned on. It has $I_4 = -(p_0 q^0)^2$ and symmetry USp(8). The limit $I_4 = 0$, obtained for $p_0 q^0 = 0$, gives a 1/2 BPS black home which has the same USp(8) symmetry. For the most general W of a non-BPS configuration, as defined in [14], the $I_4 = 0$ limit just gives back (81) with $\lambda = W^2$.

A Proof of (13)

Equation (13), as shown in [4], is a particular form of the general solution to the Hamilton-Jacobi equation. In what follows we shall tailor the formal proof given in [4] to the class of extremal solutions we are considering, without making use of the Hamilton-Jacobi formalism.

Consider the extremal solutions $U(\tau;\phi_0,U_0)$ and $\phi(\tau;\phi_0,U_0)$, for a given charge vector $\mathcal{P}$, within the interval $\tau_* < \tau < \tau_0$, where now U_0, ϕ_0 denote the values of the fields computed at τ_0: $U_0 = U(\tau_0;\phi_0,U_0)$, $\phi_0 = \phi(\tau_0;\phi_0,U_0)$. The values of the fields at τ_*, for our family of solutions, is completely fixed in terms of $(U_0,\ \phi_0)$ and $\mathcal{P}$. Let us perform an infinitesimal variation of the boundary conditions: $U_0 \to U_0 + \delta U_0$ and $\phi_0 = \phi_0 + \delta\phi_0$. This will determine a new solution within the same class:

$$\begin{aligned} U(\tau;\phi_0+\delta\phi_0,U_0+\delta U_0) &= U(\tau;\phi_0,U_0)+\delta U(\tau),\\ \phi(\tau;\phi_0+\delta\phi_0,U_0+\delta U_0) &= \phi(\tau;\phi_0,U_0)+\delta\phi(\tau). \end{aligned} \tag{82}$$

Now we write a seemingly more general ansatz for W than the one in (13):

$$e^{U_0}\, W(\phi_0,\mathcal{P}) = e^{U_*}\, W(\phi_*,\mathcal{P}) + \int_{\tau_*}^{\tau_0} e^{2\,U(\tau;U_0,\phi_0)}\, V(\phi(\tau;U_0,\phi_0),\mathcal{P})\, d\tau. \tag{83}$$

As we shall see, the result of this integral does not depend on the choice of τ_0. For the sake of simplicity we shall suppress the dependence on τ and on the boundary values of the fields in the integrand. Since the integral is computed along solutions, we can use the Hamiltonian constraint (6) to rewrite W as follows:

$$\begin{aligned} e^{U_0}\, W(\phi_0,\mathcal{P}) &= e^{U_*}\, W(\phi_*,\mathcal{P}) + \frac{1}{2}\int_{\tau_*}^{\tau_0}\left[e^{2\,U}\, V(\phi,\mathcal{P}) + \dot{U}^2 + \frac{1}{2}\, G_{rs}\,\dot{\phi}^r\,\dot{\phi}^s\right] d\tau\\ &= e^{U_*}\, W(\phi_*,\mathcal{P}) + \frac{1}{2}\int_{\tau_*}^{\tau_0} \mathcal{L}_{\mathit{eff}}(U,\phi,\dot{U},\dot{\phi})\, d\tau. \end{aligned}$$

Now perform the variation (82), integrate by parts and use the equations of motion:

$$\begin{aligned} &\delta U_0\, e^{U_0}\, W(\phi_0,\mathcal{P}) + e^{U_0}\,\partial_r W(\phi_0,\mathcal{P})\,\delta\phi_0^r = \delta(e^{U_*}\, W(\phi_*,\mathcal{P}))\\ &\quad + \frac{1}{2}\int_{\tau_*}^{\tau_0}\left[\left(\frac{\partial}{\partial U}\mathcal{L}_{\mathit{eff}} - \frac{d}{d\tau}\frac{\partial}{\partial \dot{U}}\mathcal{L}_{\mathit{eff}}\right)\delta U + \left(\frac{\partial}{\partial \phi^r}\mathcal{L}_{\mathit{eff}} - \frac{d}{d\tau}\frac{\partial}{\partial \dot{\phi}^r}\mathcal{L}_{\mathit{eff}}\right)\delta\phi^r\right]\\ &\quad + \left.(\dot{U}\,\delta U + \frac{1}{2} G_{rs}\,\dot{\phi}^s\delta\phi^r)\right|_{\tau_*}^{\tau_0} = \delta(e^{U_*}\, W(\phi_*,\mathcal{P})) + \left.(\dot{U}\,\delta U + \frac{1}{2} G_{rs}\,\dot{\phi}^s\,\delta\phi^r)\right|_{\tau_*}^{\tau_0}, \end{aligned} \tag{84}$$

where we have used the short-hand notation $\partial_r W \equiv \frac{\partial W}{\partial \phi^r}$. We can choose $\tau_* = -\infty$, so that all terms computed at τ_* in the above equation vanish. Equating the variations at τ_0 on both sides we find:

$$\dot{U}(\tau_0) = e^{U_0}\, W(\phi_0,\mathcal{P}), \quad \dot{\phi}^s(\tau_0) = 2\, e^{U_0}\, G^{rs}(\phi_0)\,\partial_r W(\phi_0,\mathcal{P}). \tag{85}$$

Being τ_0 generic, we find that W defines the first order equations (7) for the fields and thus it is a solution to the Hamilton-Jacobi equation.

Note however that W, as defined in (83), may in principle depend on the chosen value of τ_0, that is $W = W(U_0, \phi_0, \tau_0, \mathcal{P})$. Let us show that this is not the case, namely that $W(U_0, \phi_0, \tau_0 + \delta\tau, \mathcal{P}) = W(U_0, \phi_0, \tau_0, \mathcal{P})$, for a generic $\delta\tau$. To do this we vary $\tau_0 \to \tau_0 + \delta\tau$, *keeping the boundary values of the fields fixed*. This requires to change the solution on which the integral is computed from $U(\tau), \phi(\tau)$ to $U'(\tau)$, $\phi'(\tau)$ such that:

$$U'(\tau_0 + \delta\tau) = U(\tau_0) = U(\tau_0 + \delta\tau) - \dot{U}(\tau_0)\,\delta\tau,$$
$$\phi'(\tau_0 + \delta\tau) = \phi(\tau_0) = \phi(\tau_0 + \delta\tau) - \dot{\phi}(\tau_0)\,\delta\tau. \tag{86}$$

and thus amounts to performing, along the flow, the transformation $U \to U - \dot{U}\,\delta\tau$, $\phi \to \phi - \dot{\phi}\,\delta\tau$, besides changing the domain of integration, $\delta\tau$ being chosen along the flow so that $\delta\tau_* = 0$. After some straightforward calculations we find:

$$e^{U_0}\left(W(U_0, \phi_0, \tau_0 + \delta\tau, \mathcal{P}) - W(U_0, \phi_0, \tau_0, \mathcal{P})\right) = \mathcal{H}_{\mathit{eff}}|_{\tau_0}\,\delta\tau = 0, \tag{87}$$

in virtue of the Hamiltonian constraint. Since the function W of the moduli space, as defined by (83), does not depend on the choice of τ_0, we can choose $\tau_0 = 0$, where $U_0 = 0$ and then find (13).

B Stability and Asymptotic Stability in the Sense of Liapunov

Let us briefly recall the notion of stability (in the sense of Liapunov) and of attractiveness of an equilibrium point. Given an autonomous dynamical system:

$$\dot{\phi}^r = f^r(\phi), \tag{88}$$

an equilibrium point ϕ_* ($f^r(\phi_*) = 0$), is *attractive* (or an attractor), for $\tau \to -\infty$, if there exist a neighborhood $\mathcal{I}_{\phi_*}$ of ϕ_*, such that all trajectories $\phi^r(\tau, \phi_0)$ originating at $\tau = 0$ in $\phi_0 \in \mathcal{I}_{\phi_*}$ evolve towards ϕ_* as $\tau \to -\infty$:

$$\lim_{\tau \to -\infty} \phi^r(\tau, \phi_0) = \phi_*^r, \quad \forall \phi_0 \in \mathcal{I}_{\phi_*}. \tag{89}$$

An equilibrium point ϕ_* (not necessarily attractive) is *stable* (in the sense of Liapunov) if, for any $\epsilon > 0$, there exist a ball $\mathcal{B}_\delta(\phi_*)$ of radius $\delta > 0$ centered in ϕ_*, such that:

$$\forall \phi_0 \in \mathcal{B}_\delta(\phi_*), \quad \forall \tau < 0 : \phi(\tau, \phi_0) \in \mathcal{B}_\epsilon(\phi_*), \tag{90}$$

that is, provided we take the starting point ϕ_0 sufficiently close to ϕ_*, the entire solution will stay, for all $\tau < 0$, in any given, whatever small, neighborhood of ϕ_*.

Finally an equilibrium point is *asymptotically stable* (in the sense of Liapunov) if it is attractive and stable.

Liapunov's Theorem: If there exist a function $v(\phi)$ which is positive definite in a neighborhood of ϕ_* (that is positive in a neighborhood of ϕ_* and $v(\phi_*) = 0$) and such that also the derivative of v along the solution, in the same neighborhood, is positive definite[8]: $\frac{dv}{d\tau} = \dot{\phi}^r \partial_r v > 0$, then ϕ_* is an asymptotically stable equilibrium point or, equivalently, a stable attractor.

For large extremal black holes such function is $v(\phi) = W(\phi) - W(\phi_*) = W(\phi) - |I_4|^{\frac{1}{4}}$.

C Properties of the Vielbein on $\mathscr{M}_{scal}$

Let us briefly motivate why, for all regular extremal black holes, our choice of parametrization is such that the vielbein matrix has a vanishing off-diagonal block $V_k{}^a$.

The reason is purely group theoretical. As far as the BPS and non-BPS ($I_4 > 0$) orbits are concerned, taking into account that ϕ^k belong to $\mathbf{R}_1$ and the index a label the $\mathbf{R}_0$ representation, $d\phi^k\, V_k{}^a(\phi^k)$ can be different from zero only if $\mathbf{R}_0$ is contained in the tensor product of a number of $\mathbf{R}_1$ representations. As the reader can ascertain from Table 1, this is never the case. For example in the case of regular BPS black holes, for $\mathcal{N} > 2$, $\mathbf{R}_0$ is a doublet with respect to an SU(2) subgroup of H_0, while $\mathbf{R}_1$ is a singlet with respect to the same group. If we think of the $\mathcal{N} = 2$ truncation of the original theory of which the same black hole is a 1/2-BPS solution, this SU(2) group is the quaternionic structure of a quaternionic Kähler submanifold of the scalar manifold spanned by the scalars ϕ^α which in fact are the hypermultiplets' scalars in the N=2 truncation under consideration (see [1]). On the other hand, as far as the non-BPS solutions with $I_4 < 0$ are concerned, the above argument does not apply in the coset parametrization (46), but choosing instead the solvable parametrization one finds $d\phi^k\, V_k{}^a(\phi^k) = 0$ since $\mathbb{L}_1{}^{-1}\, d\mathbb{L}_1$ belongs to the same solvable algebra spanned by ϕ^k, which is orthogonal to $\mathfrak{K}_0$.

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[8] Here we require positive definiteness because our critical point is located at $\tau \to -\infty$ and not at $+\infty$ as in standard textbooks.

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Representations of Super Lie Groups: Some Remarks

Claudio Carmeli and Gianni Cassinelli

Abstract We give a quick review of the basic aspects of the theory of representations of super Lie groups on finite-dimensional vector spaces. In particular, the various possible approaches to representations of super Lie groups, super Harish–Chandra pairs and actions are analyzed. A sketch of a general setting for induced representation is also presented and some basic examples of induced representations (i.e., special and odd induction) are given.

In this paper we briefly review the basic aspects of the representation theory of super Lie groups (SLG). We also review the basic aspects of super geometry needed for the theory. The approach to super geometry we adopt is the one used by Kostant in his seminal paper [10], and then adopted by many others (e.g., [5, 6, 12, 13, 16]). In particular we use the explicit realization of the sheaf of a super Lie group in terms of the corresponding super Harish–Chandra pair, as given by Koszul in [11]. This has the advantage that many constructions become more transparent and easy to prove.

In the first section we briefly recall the basic definitionsand results on super Lie groups and super Harish–Chandra pairs. In particular we state the precise link between them. This is the main ingredient of all subsequent results.

In Sect. 2, we shortly discuss the relationship existing between the notion of a finite-dimensional representation of a super Lie group, of a super Harish–Chandra pair, and the notion of a linear action of a super Lie group on a super vector space. We devote some effort in order to give the explicit formulae that allow to switch from one picture to the other.

This allows us to pass in an almost straightforward way to the notion of an infinite-dimensional representation of a super Lie group (Sect. 3). This is in fact

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formulated in terms of super Harish–Chandra pairs. The geometrical framework underlying one of the main sources of representations (G-super vector bundles) is briefly introduced and a quick introduction to basic aspects of the theory of induced representations is also given. The paper ends with the calculation of some of the basic inducing procedures used in super geometry. In order to reduce the technical details needed, we have adopted some simplifying assumptions. In particular, we will limit ourselves to the case in which the inducing module is finite-dimensional. Nevertheless, it will be clear from our discussion that all the results can be properly generalized to the infinite-dimensional setting.

1 Super Manifolds, Super Lie Groups, Super Harish–Chandra Pairs, and Actions

1.1 Smooth Super Manifolds

A (smooth) *super manifold* M of dimension $p|q$ is a second countable, Hausdorff topological space $|M|$ endowed with a sheaf $\mathcal{O}_M$ of super algebras, locally isomorphic to $\mathcal{C}^\infty(\mathbb{R}^p) \otimes \Lambda(\theta_1, \ldots, \theta_q)$. A morphism $\psi : M \to N$ between super manifolds is a pair of morphisms $(|\psi|, \psi^*)$ where $|\psi| : |M| \to |N|$ is a continuous map and $\psi^* : \mathcal{O}_N \to \mathcal{O}_M$ is a sheaf morphism above $|\psi|$.

Remark 1. We will consider only smooth super manifolds. It can be proved that in this category a morphism of super manifolds is determined once we know the corresponding morphism on the global sections (see, for example [2, 10]). In other words, a morphism $\psi : M \to N$ can be identified with a super algebra map $\psi^* : \mathcal{O}_N(|N|) \to \mathcal{O}_M(|M|)$. We will tacitly use this fact several times. Moreover, in the following, we will denote with $\mathcal{O}(M)$ the super algebra of global sections $\mathcal{O}_M(|M|)$. $\mathcal{O}(M)$ is a Fréchet super vector algebra (see, for example [5, 10]).

Suppose now U is an open subset of $|M|$ and let $\mathcal{J}_M(U)$ be the ideal of the nilpotent elements of $\mathcal{O}_M(U)$. It is possible to prove that $\mathcal{O}_M/\mathcal{J}_M$ defines a sheaf of purely even algebras over $|M|$ locally isomorphic to $\mathcal{C}^\infty(\mathbb{R}^p)$. Therefore $\widetilde{M} := (|M|, \mathcal{O}_M/\mathcal{J}_M)$ defines a classical manifold, called the *reduced manifold* associated to M. The projection $s \mapsto \widetilde{s} := s + \mathcal{J}_M(U)$, with $s \in \mathcal{O}_M(U)$, is the pullback of the embedding $j : \widetilde{M} \to M$. In the following we denote with $\mathrm{ev}_x(s) := \widetilde{s}(x)$ the *evaluation* of s at $x \in U$. It is also possible to check that, given a morphism $\psi : M \to N$, $|\psi|^*(\widetilde{s}) = \widetilde{\psi^*(s)}$, so that the map $|\psi|$ is automatically smooth. As a consequence, it is possible to associate a *reduced map* $\widetilde{\psi} = |\psi| : \widetilde{M} \to \widetilde{N}$ to each super manifold morphism ψ.

If M_1 and N_1 are two supermanifolds, the sheaf $\mathcal{O}_{M_1 \times N_1}$ can be identified with the topological completion $\mathcal{O}_{M_1} \hat{\otimes} \mathcal{O}_{N_1}$. If $\psi_1 : M_1 \to N_1$ and $\psi_2 : M_2 \to N_2$ are super manifold morphisms , then they determine a morphism $\psi_1 \hat{\otimes} \psi_2 : M_1 \times M_2 \to N_1 \times N_2$ (see [5] and [9]).

Remark 2. If M is a (smooth) super manifold, its sheaf of sections is isomorphic (in a non canonical way) to the sheaf of sections of an exterior vector bundle over $\widetilde{M}$ (see [3]). If such a vector bundle is trivial, the super manifold M is said to be *globally splitting*.

An important and very useful tool in working with super manifolds is the functor of points. Given a super manifold M one can construct the functor

$$M(\cdot)\colon \mathbf{SMan}^{\mathrm{op}} \longrightarrow \mathbf{Set}$$

from the opposite of the category of super manifolds to the category of sets defined by $S \mapsto M(S) := \mathrm{Hom}(S, M)$ and called the *functor of points* of M. In particular, for example, $M(\mathbb{R}^{0|0}) \cong |M|$ as sets. Each super manifold morphism $\psi\colon M \to N$ defines the natural transformation $\psi(\cdot)\colon M(\cdot) \to N(\cdot)$ given by $[\psi(S)](x) := \psi \circ x$. Due to Yoneda's lemma, each natural transformation between $M(\cdot)$ and $N(\cdot)$ arises from a unique morphism of super manifolds in the way just described. The category of super manifolds can thus be embedded into a full subcategory of the category $[\mathbf{SMan}^{\mathrm{op}}, \mathbf{Set}]$ of functors from the opposite of the category of super manifolds to the category of sets. Let

$$\begin{aligned} \mathcal{Y}\colon \mathbf{SMan} &\longrightarrow [\mathbf{SMan}^{\mathrm{op}}, \mathbf{Set}] \\ M &\longmapsto M(\cdot) \end{aligned}$$

denote such embedding. It is a fact that the image of **SMan** under $\mathcal{Y}$ is strictly smaller than $[\mathbf{SMan}^{\mathrm{op}}, \mathbf{Set}]$. The elements of $[\mathbf{SMan}^{\mathrm{op}}, \mathbf{Set}]$ isomorphic to elements in the image of $\mathcal{Y}$ are called *representable*. Super manifolds can thus be thought as the representable functors in $[\mathbf{SMan}^{\mathrm{op}}, \mathbf{Set}]$. For all the details we refer to [6, 10, 12, 13, 16].

Example 1. $\mathbb{R}^{p|q}$ is the super manifold whose reduced manifold is $\mathbb{R}^p$, and with the sheaf of sections given by the restriction of $\mathcal{C}^\infty(\mathbb{R}^p) \otimes \Lambda(\mathbb{R}^q)$. Notice that $\mathbb{R}^{p|q}$ denotes also the super vector space $\mathbb{R}^{p|q} = \mathbb{R}^p \oplus \mathbb{R}^q$. Using the functor of points one can prove that the two concepts can be identified (in the sheaf-theoretical approach, the linear structure of $\mathbb{R}^{p|q}$ is encoded by the linear sections $(\mathbb{R}^{p|q})^*$).

1.2 Super Lie Groups

Super Lie groups (SLG) are, by definition, group objects in the category of super manifolds. This means that morphisms μ, i, and e are defined satisfying the usual commutative diagrams for multiplication, inverse, and unit respectively. From this, it follows easily that the reduced morphisms $\widetilde{\mu}, \widetilde{i}$, and $\widetilde{e}$ endow $\widetilde{G}$ with a Lie group structure. $\widetilde{G}$ is called the *reduced (Lie) group* associated with G. $\widetilde{G}$ acts in a natural way on G. In particular, in the following, we will denote by

$$r_g := \mu \circ \langle \mathbb{1}_G, \hat{g} \rangle : G \longrightarrow G \qquad \ell_g := \mu \circ \langle \hat{g}, \mathbb{1}_G \rangle : G \longrightarrow G$$

the right and left translations by the element $g \in |G|$, respectively.[1]

Many classical constructions carry over to the super setting. For example it is possible to define *left-invariant vector fields* and to prove that they form a super Lie algebra $\mathfrak{g}$, isomorphic to the super tangent space at the identity of G. Moreover the even subspace $\mathfrak{g}_0$ of the super Lie algebra $\mathfrak{g}$ identifies with $\mathrm{Lie}(\widetilde{G})$. (see, for example, [5, 10, 16]).

In the spirit of the functor of points, one can think of a SLG as a representable functor from $\mathbf{SMan}^{\mathrm{op}}$ to the category **Grp** of set theoretical groups. The SLG structure imposes severe restrictions on the structure of the super manifold carrying it. In the next section, we want to briefly discuss this point.

Example 2. As an example of a SLG, we consider the super general linear group $\mathbf{GL}(V)$. Here $V = V_0 \oplus V_1$ denotes, as usual, a super vector space. $\mathbf{GL}(V)$ is defined as the super manifold whose underlying reduced manifold is $\mathrm{GL}(V_0) \times \mathrm{GL}(V_1)$, and whose sheaf of sections is the restriction of the sheaf over the super manifold $\underline{\mathrm{End}}(V)$. The super Lie group operations are then defined using the functor of points approach, as detailed for example in [5, 16]. The super Lie algebra associated with $\mathbf{GL}(V)$ is the super vector space of full endomorphisms $\underline{\mathrm{End}}(V)$ endowed with the canonical super bracket.

1.3 Koszul Super Manifolds and Super Harish–Chandra Pairs

Let now $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a super Lie algebra (SLA) and let $\mathfrak{U}(\mathfrak{g})$ be the corresponding universal enveloping algebra. $\mathfrak{U}(\mathfrak{g})$ is a graded connected Hopf algebra. Let $\mathfrak{M}$ be a graded left $\mathfrak{U}(\mathfrak{g}_0)$-module. Since $\mathfrak{U}(\mathfrak{g})$ is also a graded left $\mathfrak{U}(\mathfrak{g}_0)$-module we can consider the set

$$\mathfrak{M}^{\mathfrak{g}_0} := \underline{\mathrm{Hom}}_{\mathfrak{U}(\mathfrak{g}_0)}(\mathfrak{U}(\mathfrak{g}), \mathfrak{M})$$

of left $\mathfrak{U}(\mathfrak{g}_0)$-module morphisms. $\mathfrak{M}^{\mathfrak{g}_0}$ is clearly a graded vector subspace of the full $\underline{\mathrm{Hom}}(\mathfrak{U}(\mathfrak{g}), \mathfrak{M})$. It also carries a natural left $\mathfrak{U}(\mathfrak{g})$-module structure given by

$$(X.\phi)(u) := (-1)^{|X|(|\phi|+|u|)} \phi(uX)$$

[1]Some explanations of the notations used: given two morphisms $\alpha\colon X \to Y$ and $\beta\colon X \to Z$,

$$\langle \alpha, \beta \rangle\colon X \longrightarrow Y \times Z$$

is the morphism that composed with the projection on the first (resp. second) component gives α (resp. β); if $x \in |X|$, the map

$$\hat{x}\colon T \longrightarrow X$$

is the constant map obtained composing the unique map $T \to \mathbb{R}^{0|0}$ with the embedding $\mathbb{R}^{0|0} \to X$ whose image is x.

The $\mathfrak{U}(\mathfrak{g})$-module just defined is said to be *co-induced* from the $\mathfrak{U}(\mathfrak{g}_0)$-module $\mathfrak{M}$.

It is possible to identify $\mathfrak{M}^{\mathfrak{g}_0}$ in a very precise way. The key result is the following simple lemma (see [11]).

Lemma 1. *i) The antisymmetrization map*

$$\begin{aligned} \widehat{\gamma} :\ \Lambda(\mathfrak{g}_1) &\longmapsto \mathfrak{U}(\mathfrak{g}) \\ X_1 \cdots X_n &\longmapsto \frac{1}{n!} \sum_{\sigma \in S_n} (-1)^{|\sigma|} X_{\sigma(1)} \cdots X_{\sigma(n)} \end{aligned} \tag{1}$$

is a morphism of left $\mathfrak{U}(\mathfrak{g}_0)$-modules.

ii) The map

$$\begin{aligned} \gamma :\ \mathfrak{U}(\mathfrak{g}_0) \otimes \Lambda(\mathfrak{g}_1) &\longrightarrow \mathfrak{U}(\mathfrak{g}) \\ X \otimes Z &\longmapsto X \cdot \widehat{\gamma}(Z) \end{aligned}$$

is a $\mathfrak{U}(\mathfrak{g}_0)$-modules isomorphism.

It is clear that using this isomorphism we have the identification

$$\begin{aligned} t : \mathfrak{M}^{\mathfrak{g}_0} &\longrightarrow \underline{\mathrm{Hom}}(\Lambda(\mathfrak{g}_1), \mathfrak{M}) \\ \psi &\longmapsto \psi \circ \widehat{\gamma} \end{aligned}$$

Suppose now that a classical manifold M_0 is given together with an infinitesimal action of the even part $\mathfrak{g}_0$ of a super Lie algebra $\mathfrak{g}$:

$$\rho :\ \mathfrak{g}_0 \longrightarrow \mathrm{Vec}(M_0)$$

If U denotes an open subset of M_0, both $\mathfrak{U}(\mathfrak{g})$ and $\mathcal{C}^\infty(U)$ are left $\mathfrak{U}(\mathfrak{g}_0)$-modules.

Hence it makes sense to consider the coinduced module

$$\mathcal{O}_M(U) := \mathcal{C}^\infty_M(U)^{\mathfrak{g}_0} = \underline{\mathrm{Hom}}_{\mathfrak{U}(\mathfrak{g}_0)}(\mathfrak{U}(\mathfrak{g}), \mathcal{C}^\infty(U))$$

The following proposition is not surprising

Proposition 1. *$(M_0, \mathcal{O}_M)$ is a super manifold.*

Definition 1. Using the notations above, we call the super manifold $(M_0, \mathcal{O}_M)$ the *Koszul super manifold* associated to $(M_0, \mathfrak{g}, \rho)$.

The above analysis shows that Koszul super manifolds share the following distinctive properties:

1. They are globally splitting
2. The splitting is canonical

In other words for Koszul super manifolds the sheaf $\mathcal{O}_M$ carries a natural $\mathbb{Z}$-gradation.

Example 3. Suppose G is a SLG. In this case, the reduced manifold $\widetilde{G}$ is canonically endowed with a classical Lie group structure, and if $\mathfrak{g}$ denotes the super Lie algebra of G we have an infinitesimal action of $\mathfrak{g}_0 \simeq \mathrm{Lie}(\widetilde{G})$ on $\widetilde{G}$. The pair $(\widetilde{G}, \mathfrak{g})$ is called the *super Harish–Chandra pair* associated with G.

The previous example suggests to give the following definition.

Definition 2. A super Harish–Chandra pair (SHCP) is a pair consisting of a Lie group G_0, and a super Lie algebra $\mathfrak{g}$, with the following compatibility requirements:

(1) $\mathfrak{g}_0 \simeq \mathrm{Lie}(G_0)$
(2) There is an action σ of G_0 on $\mathfrak{g}$ such that the differential of the action is (equivalent to) the adjoint representation of $\mathfrak{g}_0$ on $\mathfrak{g}$.

Morphisms are defined in the natural way

Definition 3. If $(G_0, \mathfrak{g}, \sigma)$ and $(H_0, \mathfrak{h}, \tau)$ are SHCP, a morphism between them is a pair of morphisms

$$\psi_0 \colon G_0 \longrightarrow H_0$$
$$\rho_\psi \colon \mathfrak{g} \longrightarrow \mathfrak{h}$$

satisfying the compatibility conditions

(1) $\rho_{\psi}{}_{|\mathfrak{g}_0} \cong (d\psi_0)_e$;
(2) $\rho_\psi \circ \sigma(g) = \tau\big(\psi_0(g)\big) \circ \rho_\psi$ for all $g \in G_0$.

Remark 3. In the following, if $g \in |G|$ and $X \in \mathfrak{g}$, $\sigma(g)(X)$ will be shortened with $g.X$.

Example 4. If $\psi \colon G \to H$ is a SLG morphism, then the corresponding morphism between the associated SHCP is given by $\psi_0 = \widetilde{\psi}$ and $\rho_\psi = d\psi$.

Definitions 2 and 3 allow to define the category **SHCP** of super Harish–Chandra pairs. Moreover the above examples shows that the correspondence

$$\begin{aligned} \mathbf{SGrp} &\longrightarrow \mathbf{SHCP} \\ G &\longmapsto \big(\widetilde{G}, \mathrm{Lie}(G), \mathrm{Ad}\big) \end{aligned} \tag{2}$$

is functorial.

The following is a crucial result in the development of the theory (see [5,6,10]).

Theorem 1 (B. Kostant). *The functor* (2) *defines an equivalence of categories.*

Remark 4. In [10], the result is formulated in terms of super Lie-Hopf algebras. These are immediately identified with the super Harish–Chandra pairs.

The non trivial part of the above result is the reconstruction of the super Lie group G in terms of the corresponding SHCP. The sheaf is explicitly reconstructed using Koszul's recipe, the super Lie group morphisms are derived in [1, 11], and are collected in the following table.

Operation	Formula
Multiplication map	$\mu^*(\phi)(X,Y)(g,h) = \phi((h^{-1}.X)Y)(gh)$
Inverse map	$\left[i^*(\phi)(X)\right](g^{-1}) = \left[\phi(g^{-1}.\overline{X})\right](g)$
Unit	$e^*(\phi) = \left[\phi(1)\right](e)$
Evaluation map	$\widetilde{\phi} = \phi(1)$
Left translation	$\left[\ell_h^*(\phi)\right](X) = \widetilde{\ell_h^*}\big(\phi(X)\big)$
Right translation	$\left[r_h^*(\phi)\right](X) = \widetilde{r_h^*}\big(\phi(h^{-1}.X)\big)$
Left invariant vector fields	$(D_X^L\phi)(Y) = (-1)^{\lvert X\rvert}\phi(YX)$
Right invariant vector fields	$\left[(D_X^R\phi)(Y)\right](g) = (-1)^{\lvert X\rvert\lvert\phi\rvert}\phi\big((g^{-1}.X)Y\big)(g)$

In this framework the reconstruction of morphisms is also very natural. Suppose indeed that a morpshim $F : (G_0, \mathfrak{g}) \to (H_0, \mathfrak{h})$ of super Harish Chandra pairs is given, we want to reconstruct the corresponding morphism of super Lie groups.

Proposition 2. *The map*

$$f^* : \mathrm{Hom}_{\mathfrak{U}(\mathfrak{h}_0)}\left(\mathfrak{U}(\mathfrak{h}), \mathcal{C}^\infty(H_0)\right) \longrightarrow \mathrm{Hom}_{\mathfrak{U}(\mathfrak{g}_0)}\left(\mathfrak{U}(\mathfrak{g}), \mathcal{C}^\infty(G_0)\right) \qquad (3)$$
$$\phi \longmapsto f^*(\phi) := \tilde{f}^* \circ \phi \circ \rho^f$$

defines a morphism of super Lie groups whose reduced morphism is $\tilde{f}$ and whose differential at the identity is ρ^f.

1.4 Actions of SLG on Super Manifolds

Let M be a super manifold and let G denote a super Lie group.

Definition 4. A morphism of super manifolds

$$a : G \times M \longrightarrow M$$

is called an *action* of G on M if it satisfies

$$a \circ (\mu \times \mathbb{1}_M) = a \circ (\mathbb{1}_G \times a)$$
$$a \circ (\hat{e} \times \mathbb{1}_M) = \mathbb{1}_M$$

If an action a of G on M is given, then we say that G acts on M, or that M is a G-super manifold.

Since the category of super Lie groups is equivalent to the category of super Harish–Chandra pairs (see Theorem 1), one could ask whether there is an equivalent notion of action of a super Harish–Chandra pair on a super manifold. The answer is affirmative and it is given in the next proposition (see [1, 5, 6]).

Proposition 3. *Suppose G acts on a super manifold M, then there are*

i. *An action*

$$\underline{a} : \widetilde{G} \times M \longrightarrow M$$

of the reduced Lie group $\widetilde{G}$ on the super manifold M.

ii. *A representation*

$$\rho_a : \mathfrak{g} \longrightarrow Vec\,(M)^{op} \tag{4}$$
$$X \longmapsto \left(X_e \hat{\otimes} \mathbb{1}_M^*\right) \circ a^* \tag{5}$$

of the super Lie algebra $\mathfrak{g}$ of G on the opposite of the Lie algebra of vector fields over M.

The above two maps satisfy the following compatibility relation

$$\rho_{a\,|\mathfrak{g}_0}(X) = \left(X_e \hat{\otimes} \mathbb{1}_M^*\right) \circ \underline{a}^* \quad \forall X \in \mathfrak{g}_0 \tag{6}$$

Conversely, let $\left(\widetilde{G}, \mathfrak{g}\right)$ be the SHCP associated with the super Lie group G and let maps $\underline{a}$ and ρ like in points 1 and 2 above satisfying condition (6) be given. The map

$$\mathcal{O}_M(M) \longrightarrow \mathrm{Hom}_{\mathfrak{U}(\mathfrak{g}_0)}\left(\mathfrak{U}(\mathfrak{g}), \mathcal{C}^\infty(\widetilde{G}) \hat{\otimes} \mathcal{O}_M(M)\right) \tag{7}$$
$$s \longmapsto \left(X \longmapsto \left(\mathbb{1}_{\mathcal{C}^\infty(G_0)} \hat{\otimes} \rho(X)\right) \circ \tilde{a}(s)\right) \tag{8}$$

defines uniquely an action of the super Lie group G on M whose reduced and infinitesimal actions are the given ones.

1.5 Stability Sub Super Lie Group

Let G be a super Lie group, M a super manifold and $a \colon G \times M \to M$ an action of G on M. If $p \in |M|$, let p denote also the constant map $G \xrightarrow{!} \{\bullet\} \xrightarrow{p} M$ and let a_p be the map $G \xrightarrow{\langle \mathbb{1}_G, p \rangle} G \times M \xrightarrow{a} M$.

Since a_p has constant rank (see, for example, [5, 12]), there is a unique closed sub super manifold G_p of G that is the equalizer of

$$G_p \xrightarrow{i} G \underset{p}{\overset{a_p}{\rightrightarrows}} M$$

G_p is a sub super Lie group of G, said *stability sub super Lie group* of p. We also have the following characterization in terms of SHCP.

Proposition 4. *The super Harish–Chandra pair associated to a super Lie group G_p is $\left(\widetilde{G}_p, \mathfrak{g}_p\right)$, where $\widetilde{G}_p \subseteq \widetilde{G}$ is the classical stability subgroup of p and $\mathfrak{g}_p = \ker\left(da_p\right)_e$.*

For a complete discussion of these facts we refer to, for example, to [1], and [5].

1.6 Transitive Actions

Let M be a super manifold and let G be a SLG acting on M through

$$a : G \times M \longrightarrow M \tag{9}$$

Exactly as in the classical case we give the following definition

Definition 5. We say that the action (9) is transitive if there exists $p \in |M|$ such that

$$a_p : G \longrightarrow M \tag{10}$$

is a surjective submersion.

The following proposition establishes that each super manifold endowed with a transitive G-action is isomorphic to a homogeneous super space (see [1, 5]).

Proposition 5. *Let M be a super manifold endowed with a transitive G-action. Fix $p \in |M|$ and denote by G_p the stability sub super Lie group at p. Then*

$$M \simeq G/G_p \tag{11}$$

2 Finite-Dimensional Representations of Super Lie Groups

2.1 Finite Dimensional Linear Actions and Representations

In this section G denotes a SLG, $(\widetilde{G}, \mathfrak{g})$ the associated SHCP, and V a complex finite-dimensional super vector space. We want to establish the equivalence and the precise link between the notions of

– A *representation of a super Lie group* G in a super vector space V;
– A *representation of a super Harish Chandra pair* $(\widetilde{G}, \mathfrak{g})$ associated with G in a super vector space V;
– A *linear action* of G on V:

We start with the corresponding definitions.

Definition 6. (1) A *representation* of a SLG G on the super vector space $V = V_0 \oplus V_1$ is a SLG morphism

$$\pi : G \to \mathbf{GL}(V)$$

(2) A representation Π of the SHCP $(\widetilde{G}, \mathfrak{g})$ on the super vector space $V = V_0 \oplus V_1$ is a morphism between $(\widetilde{G}, \mathfrak{g})$ and the SHCP associated with $\mathbf{GL}(V)$. This means that the following morphisms are given

(a) A Lie group representation

$$\widetilde{\pi} : \widetilde{G} \to \mathrm{GL}(V_0) \times \mathrm{GL}(V_1)$$

(b) A representation of super Lie algebras

$$\rho^{\pi} : \mathfrak{g} \to \underline{\mathrm{End}}(V)$$

such that $d\widetilde{\pi} \simeq \rho^{\pi}_{|\mathfrak{g}_0}$ and

$$\rho^{\pi}(\mathrm{Ad}(g)X) \simeq \widetilde{\pi}(g)\rho^{\pi}(X)\widetilde{\pi}(g)^{-1}$$

(3) A *linear action* of G on V is a an action

$$a : G \times V \to V$$

preserving linear functionals on V, i.e.,

$$a^*(V^*) \subseteq \mathcal{O}(G) \otimes V^*$$

Remark 5. Notice that here V appears as a super manifold.

Proposition 6. *Suppose G is a SLG, let $(\widetilde{G}, \mathfrak{g})$ be the corresponding SHCP, and denote with $V = V_0 \oplus V_1$ a super vector space. There is a bijective correspondence between representations of G on V, representations of the corresponding SHCP $(\widetilde{G}, \mathfrak{g})$ on V, and linear actions of G on V.*

Proof. (Sketch). The main part of the proof consists in giving the explicit formulae that allow to switch from one picture to the other.

(1) $\Rightarrow$ (2)
Suppose that a representation π of G on V is given according to (1). Since V finite dimensional $\mathbf{GL}\,(V)$ is a super Lie group. Passing to the reduced manifolds, we obtain a representation

$$\tilde{\pi} : \widetilde{G} \longrightarrow \widetilde{\mathbf{GL}\,(V)} \simeq \mathrm{GL}\,(V_0) \times \mathrm{GL}\,(V_1) \tag{12}$$

of the underlying Lie groups. On the other hand taking the differential at the identity, we get the super Lie algebra morphism

$$\begin{aligned} (\mathrm{d}\pi)_e : \mathfrak{g} &\longrightarrow \underline{\mathrm{End}}\,(V) \\ X &\longmapsto X \circ \pi^* \end{aligned}$$

so that $(\tilde{\pi}, (\mathrm{d}\pi)_e)$ is the required morphism of SHCP.

(2) $\Rightarrow$ (1)
Conversely suppose that the SHCP representation

$$\begin{aligned} \widetilde{\pi} &: \widetilde{G} \to \mathrm{GL}(V_0) \times \mathrm{GL}(V_1) \\ \rho^{\pi} &: \mathfrak{g} \to \underline{\mathrm{End}}(V) \end{aligned}$$

is given. We have shown in Proposition 2, how to reconstruct the super Lie group representation. Explicitly we have to construct a SLG morphism π^* from

$$\underline{\mathrm{Hom}}_{\mathfrak{U}(\underline{\mathrm{End}}(V_0)\oplus\underline{\mathrm{End}}(V_1))}\left(\mathfrak{U}(\underline{\mathrm{End}}(V)), \mathcal{C}^{\infty}(\mathrm{GL}(V_0) \times \mathrm{GL}(V_1))\right)$$

to $\underline{\mathrm{Hom}}_{\mathfrak{U}(\mathfrak{g}_0)}\left(\mathfrak{U}(\mathfrak{g}), \mathcal{C}^{\infty}(\widetilde{G})\right)$. This is given by

$$\phi \to \widetilde{\pi}^* \circ \phi \circ \hat{\rho^{\pi}},$$

where $\hat{\rho^{\pi}}$ denotes the extension of the representation of ρ^{π} to the corresponding super enveloping algebra morphism.

(2) $\Rightarrow$ (3)
Let us now suppose that a representation of the super Harish–Chandra pair $\left(\widetilde{G}, \mathfrak{g}\right)$ on V is given. Using results from the beginning of the section, we show that it is possible to reconstruct the linear action of G on V in a very explicit way. Indeed, according to Definition 6, we want to construct a map

$$a^* : V^* \longrightarrow \mathcal{O}\,(G) \otimes V^* \tag{13}$$

satisfying the required commutative diagrams. Let hence V denote a $(l_0, \mathfrak{g})$-module, i.e.,

$$\widetilde{\pi} : \widetilde{G} \longrightarrow \mathrm{GL}(V_0) \times \mathrm{GL}(V_1) \tag{14}$$

$$\rho^{\pi} : \mathfrak{g} \longrightarrow \underline{\mathrm{End}}\,(V) \tag{15}$$

We want to apply the reconstruction formula given by (7) The pull-back map associated with (14)

$$\tilde{a} : V^* \longrightarrow \mathcal{C}^{\infty}(\widetilde{G}, V^*)$$

is defined by

$$\langle \left(\tilde{a}^* \omega\right)(g), v\rangle := \langle \omega, \widetilde{\pi}(h)v\rangle = \langle \widetilde{\pi}(h)^* \omega, v\rangle$$

so that

$$\langle (a^* \omega)(X)(g), v\rangle = \langle \rho^{\pi}(X)^* \widetilde{\pi}(g)^* \omega, v\rangle$$

All the remaining formulae can be obtained by combining the previous ones $\square$

2.2 Contragredient Representation

It is natural to define the contragredient representation π_c of $\left(\widetilde{G}, \mathfrak{g}\right)$

$$\pi_c : \begin{cases} \widetilde{G} \times V^* \longrightarrow & V^* \\ \quad \mathfrak{g} \quad \longrightarrow & \underline{\mathrm{End}}\,(V^*) \end{cases}$$

through

$$\langle \widetilde{\pi}_c(g)\,(\omega)\,, X\rangle := \langle \omega, \widetilde{\pi}\left(g^{-1}\right)(X)\rangle \tag{16}$$

$$\langle \rho_c^{\pi}\,(Z)\,(\omega), X\rangle := (-1)^{|Z||\omega|}\langle \omega, \rho^{\pi}(\overline{Z})X\rangle \tag{17}$$

Following the recipe given in Proposition 3 it is then easy to obtain the explicit form of the contragredient action and G-representation. For example we get

$$a_c^* : V \longrightarrow \underline{\mathrm{Hom}}_{\mathfrak{U}(\mathfrak{g}_0)}\left(\mathfrak{U}\,(\mathfrak{g})\,, \mathcal{C}^{\infty}(\widetilde{G}) \otimes V\right) \tag{18}$$

$$v \longmapsto \left(X \longmapsto \rho\left(\overline{X}\right)\widetilde{\pi}(\cdot)^{-1} v\right)$$

2.3 Coefficients of Representations

The previous discussion allows to define in a simple and explicit way the coefficients of a given representation.

Definition 7. Suppose $\Pi = (\widetilde{\pi}, \rho^{\pi})$ is a finite-dimensional representation of the SHCP $(\widetilde{G}, \mathfrak{g})$ on the finite-dimensional super vector space V. We define the *coefficient* $c_{\omega,v} \in \underline{\mathrm{Hom}}_{\mathfrak{U}(\mathfrak{g}_0)}\left(\mathfrak{U}(\mathfrak{g}), \mathcal{C}^{\infty}(\widetilde{G})\right)$ of the representation as

$$[c_{\omega,v}(X)](g) := (-1)^{|X||\omega|}\langle \omega, \widetilde{\pi}(g)\rho^{\pi}(X)v\rangle$$

3 Infinite-Dimensional Representations and Induced Representations of Super Lie Groups

In the previous section we described the concept of a linear representation of a SLG on a finite-dimensional super vector space. Nevertheless, as in the classical case, the representations usually arising in practice are defined on properly defined spaces of sections over a given super manifold (e.g., the structural sheaf, super vector bundles,...). It is hence natural to look for a generalization of Definition 6 to the infinite-dimensional setting. It is also clear that only the approach through SHCP remains meaningful in the infinite-dimensional context. Moreover, as we shall see more accurately in the following sections, most of the representations that can be constructed are not unitary (in any simple sense). We are hence lead to consider the case of a representation of a SLG G on a Hausdorff locally convex complete super vector space.

Let us recall some classical notions (for more details and proofs, see [17]). Let $\widetilde{G}$ be a Lie group and let V denote a complex complete Hausdorff locally convex vector space. Denote with $\mathrm{Aut}(V)$ the group of topological automorphism of V, we say that a map $\pi : \widetilde{G} \longrightarrow \mathrm{Aut}(V)$ is representation of $\widetilde{G}$ on V, if it is a group homomorphism and the map $\widetilde{G} \times V \longrightarrow V$ is continuous. This condition can be reformulated as follows

i. For each $v \in V$, the map $\widetilde{G} \longrightarrow V$ given by $g \mapsto \pi(g)v$, is continuous
ii. For each compact subset $K \subseteq G$, the set of operators $\pi(K)$ is equicontinuous

Remark 6. If V is Fréchet condition (ii) is redundant.

Recall that if V is a locally convex vector space (see, for example, [15]), $\mathcal{C}^{\infty}(\widetilde{G}, V)$ denotes the space of smooth functions from $\widetilde{G}$ to V. These are functions whose partial derivatives exist and are continuous. A vector $v \in V$ is then said to be *smooth* for the representation π, if the map

$$\begin{aligned} \widetilde{G} &\longrightarrow V \\ g &\longmapsto \pi(g)v \end{aligned} \tag{19}$$

is smooth. The set of smooth vectors of the representation π is usually denoted by $\mathcal{C}^{\infty}(\pi)$ or, when there is no ambiguity, by V^{∞}. It is clearly a vector subspace of V and it is possible to prove that it is a dense subspace of V (see, for example, [17]). Using (19), V^{∞} can be embedded in $\mathcal{C}^{\infty}(\widetilde{G}, V)$ and endowed with the relative topology. It can be shown that with respect to this topology (which is finer than the initial one) it becomes a closed subspace of $\mathcal{C}^\infty(\widetilde{G}, V)$.

Definition 8. Suppose $V = V_0 \oplus V_1$ is a super vector space. We say that V is a complete locally convex super vector space if each V_i $(i = 0, 1)$ is a complete locally convex vector space.

Moreover if $\widetilde{G}$ is a Lie group, a representation of $\widetilde{G}$ on the complete locally convex super vector space $V = V_0 \oplus V_1$ is a direct sum of continuous representations $\pi_0 \oplus \pi_1$.

Definition 9. Suppose G is a SLG with associated SHCP $(\widetilde{G}, \mathfrak{g})$ and let $V = V_0 \oplus V_1$ a complete locally convex super vector space. A representation of G on V is given by

1. A continuous representation of $\widetilde{G}$ on V
2. A representation

$$\rho^\pi : \mathfrak{g} \to \underline{\mathrm{End}}(\mathcal{C}^\infty(\pi))$$

such that the compatibility relations

– $\rho^\pi_{|_{\mathfrak{g}_0}} \simeq \mathrm{d}\widetilde{\pi}$
– $\rho^\pi(\mathrm{Ad}(g)X) = \pi(g)\rho^\pi(X)\pi(g)^{-1}$

are satisfied.

We call *differentiable representation* associated with Π the representation Π^∞ defined on V^∞ according to

i. $\pi^\infty := \pi_{|V^\infty}$
ii. $\rho^{\pi,\infty} := \rho^\pi$

Next definition is natural

Definition 10. Let Π and Σ be representations of $\left(\widetilde{G}, \mathfrak{g}\right)$ on the locally convex super vector spaces V and W respectively. We say that a continuous linear operator $A : V \longrightarrow W$ intertwines Π with Σ if

i. $A\pi(g) = \sigma(g)A$ for all $g \in \widetilde{G}$
ii. $A\rho^\pi(X) = \rho^\sigma(X)\, A$ for all $X \in \mathfrak{g}$

Remark 7. The previous definition is well posed, since it is easy to see that if A is an intertwining operator, then it preserves the space of smooth vectors.

4 G-Super Vector Bundles

In this section we review a geometrical construction that allows to construct many infinite-dimensional representations of SLG.

Definition 11. Let M be a super manifold and let V be a finite-dimensional super vector space. We say that $E = (\Gamma, M, V)$ is a super vector bundle over M of rank $\dim V$, if

i. Γ is an $\mathcal{O}_M$-module
ii. for each $x \in |M|$ there is an open neighborhood $U \ni x$ such that

$$\Gamma(U) \simeq \mathcal{O}_M(U) \otimes V \tag{20}$$

The space V is called the typical fiber of the bundle and M the base space.

We now introduce the *fiber over* x. For each open subset U containing x, consider the subsets

$$J_x^E(U) = \{s \mid s \in \Gamma(U) \text{ and } \tilde{s}(x) = 0\} \tag{21}$$

It is easy to see that $J_x^E(U)$ is an $\mathcal{O}_M(U)$ submodule of $\Gamma(U)$.

Lemma 2. *i. For each U, $\Gamma(U)/J_x^E(U)$ is a super vector space of dimension* $\dim V$.
ii. For each U and V containing x, $\Gamma(U)/J_x^E(U) \simeq \Gamma(V)/J_x^E(V)$
iii. The quotient space $V_x := \Gamma(M)/J_x^E$ is a vector space isomorphic to V.

Proof. Left to the reader. □

Definition 12. V_x is called the fiber of E over x.

Suppose now that G is a super Lie group. We introduce the notion of G-super vector bundle.

Definition 13. Let G be a super Lie group and let $E = (\Gamma, M, V)$ be a super vector bundle. We say that E is a G-super vector bundle if

i. G acts on M as described in Proposition 3 that is:

$$\underline{a} : \widetilde{G} \times M \longrightarrow M \tag{22}$$

$$\underline{\rho} : \mathfrak{g} \longrightarrow \mathrm{Vec}(M)^{op} \tag{23}$$

If g and X belong to $\widetilde{G}$ and $\mathfrak{g}$ respectively we will indicate the corresponding actions on $f \in \mathcal{O}(M)$ with $g.f$ and Xf respectively.
ii. There is a smooth representation $(\widetilde{\pi}, \rho^\pi)$ of the super Harish–Chandra pair $(\widetilde{G}, \mathfrak{g})$ on $\Gamma(M)$.

The action and the representation obey the following compatibility relations:

$$g.(fs) = g.f\ g.s \tag{24}$$

$$\rho^\pi(X)(fs) = X(f)s + (-1)^{p(X)p(f)} f\rho^\pi(X)s \tag{25}$$

for all $f \in \mathcal{O}_M(M_0)$ and $s \in \Gamma(M)$

If G acts transitively on M, the super vector bundle Γ is said to be a homogeneous super vector bundle.

4.1 Representation of the Stability Sub Super Lie Group

Fix now $x \in |M|$ and let G_x denote the stability sub super Lie group of G. The ideal J_x^E is clearly preserved by G_x. Indeed, suppose $s \in J_x^E$ and let g be in G_x

$$\widetilde{(g.s)}(x) = \widetilde{s}(g.x) = \tilde{s}(x) = 0$$

If $X \in \mathfrak{g}_x$ and $s \in J_x^E$, then

$$\widetilde{Xs}(x) = \mathrm{ev}_x Xs = X_x s = 0$$

We then easily obtain a representation of $\widetilde{G}_x$ on $\Gamma(M)/J_x^E =: V_x$ and a homomorphism

$$\mathfrak{g}_x \longrightarrow \underline{\mathrm{End}}\left(\Gamma(M)/J_x^E\right)$$

Our discussion can then be summarized by the following proposition.

Proposition 7. *Let E be a G-super vector bundle and denote by V_x the fiber over x. The action of G on E induces a natural action of G_x on V_x.*

We end this section describing briefly a framework appropriate for induced representations of SLG. It is the natural generalization of the classical construction.

Suppose hence $H \subseteq G$ is a closed sub SLG of G (*closed* here means that $|H|$ is closed in $|G|$), and let (σ, ρ^σ) be a finite-dimensional representation of H in the super vector space W. In order to avoid any confusion, we denote with $p : G \to G/H$ the canonical submersion. We want to define the super vector bundle over G/H of H-covariant sections over G.

Definition 14. Denote with $\mu_{G,H}$ the morphism given by the composition of the canonical injection of $G \times H$ into $G \times G$ with the multiplication $\mu : G \times G \to G$, and suppose U is an open subset of $\widetilde{G}/\widetilde{H}$, then we define:

$$\begin{aligned}((p_*\mathcal{O}_G)(U) \otimes W)^H &:= \{f \in (p_*\mathcal{O}_G)(U) \otimes W | (\mu^*_{G,H} \otimes \mathbb{1}_W)(f) \\ &= (\mathbb{1}_{(p_*\mathcal{O}_G)(U)} \hat{\otimes} a_c^*)(f)\},\end{aligned}$$

where p_* denotes the push-forward of the sheaf, and a_c^* denotes the pull-back of the contragradient representation as given by (18)

Remark 8. The above definition is the literal translation of the classical covariance condition

$$f(gh) = \sigma(h)^{-1} f(g)$$

Notice also that in the equation of Definition 14, the local identification

$$\mathcal{O}_G\left(|p|^{-1}(U)\right) \simeq \mathcal{O}_{G/H}(U)\hat{\otimes}\mathcal{O}(H)$$

is assumed (see [7] for the proof of this local splitting).

Next proposition spells the equivariance condition in terms of the SHCP $(\widetilde{H}, \mathfrak{h})$.

Proposition 8. *For each open U in $\widetilde{G}/\widetilde{H}$, we have the identification of* $((p_*\mathcal{O}_G)(U) \otimes W)^H$ *with*

$$\left\{ f \in (p_*\mathcal{O}_G)\,(U)\otimes W \text{ such that } \begin{matrix} \left(r_h^*\otimes \mathbb{1}_W - \mathbb{1}_{(p_*\mathcal{O}_G)(U)}\otimes\sigma(h)^{-1}\right) f = 0 \;\; \forall h\in\widetilde{H} \\ \left(D_X^L\otimes \mathbb{1}_W + \mathbb{1}_{(p_*\mathcal{O}_G)(U)}\otimes\rho^{\sigma}(X)\right) f = 0 \;\forall X\in\mathfrak{h}_1 \end{matrix} \right\}$$

Proof. Using the results of Sect. 1, it is not difficult to check that the two covariance conditions are equivalent. It is in fact clear that H-covariance implies $(\widetilde{H}, \mathfrak{h})$ covariance, let us hence consider the converse. Suppose f to be $(\widetilde{H}, \mathfrak{h})$ covariant, then

$$\begin{aligned} \left(\mu_{G,H}^* \otimes \mathbb{1}_W\right) f\,(X,Y)\,(g,h) &= f\left((h^{-1}.X)Y\right)(gh) \\ &= \left[\left(R_h^* D_Y^L f\right)(X)\right](g) \\ &= \left[\rho^{\sigma}\left(\overline{X}\right)\sigma\,(h)^{-1}\, f(X)\right](g) \\ &= (\mathbb{1}_G\hat{\otimes}a_c^*) f\,(X,Y)\,(g,h), \end{aligned}$$

where we have used (18) □

Proposition 9. *1. The assignment $U \to ((p_*\mathcal{O}_G)\,(U) \otimes W)^H$ is a super vector bundle of rank* dim W.
2. Moreover the module of global sections $(\mathcal{O}_G(\widetilde{G}) \otimes W)^H$ carries a natural representation of $(\widetilde{G}, \mathfrak{g})$.

Proof. The fact that each $((p_*\mathcal{O}_G)\,(U) \otimes W)^H$ is a $\mathcal{O}_{G/H}(U)$-module is easily proved. It hence only remains to prove the local triviality of the bundle. As in the classical case this is an easy consequence of the existence of local sections of the canonical submersion $p : G \to G/H$ (for the existence of such sections see for example [7]).

For point (2) one notices that $\widetilde{G}$ acts on $((p_*\mathcal{O}_G)\,(U) \otimes W)^H$ through left translation. Explicitly

$$[\widetilde{\pi}(g) f] := (\ell_{g^{-1}}^* \otimes \mathbb{1}_W) f$$

It is possible to prove that such a representation is smooth, i.e., the space of smooth vectors of the representation $\widetilde{\pi}$ coincides with the space $((p_*\mathcal{O}_G)\,(U) \otimes W)^H$ itself. The proof is a non completely trivial calculation but we omit it in order to keep the discussion in few lines.

The action of $\mathfrak{g}$ is finally defined according to

$$\rho^{\pi}(X)f := -(D^R_X \otimes \mathbb{1}_W)f,$$

where D^R_X denotes the right-invariant vector field associated with $X \in \mathfrak{g}$. □

Definition 15. The representation of G defined in the previous proposition is said to be induced from the H-module (σ, ρ^{σ}).

5 Examples of Induced Representations of Super Lie Groups

5.1 *Special Induction*

Let $G = (\widetilde{G}, \mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1)$ be a super Lie group and let $H = (\widetilde{H}, \mathfrak{h} = \mathfrak{h}_0 \oplus \mathfrak{h}_1)$ be a sub super Lie group.

Definition 16. We say that a sub super Lie group H of G is a *special* sub super Lie group if $\mathfrak{h}_1 = \mathfrak{g}_1$.

As could be expected, we have

Proposition 10. *Let G be a super Lie group and let H a special sub super Lie group. The quotient super manifold G/H is isomorphic to the quotient manifold $\widetilde{G}/\widetilde{H}$.*

Proof. By its very definition, see for example [7], we have

$$\mathcal{O}_{G/M}(U) = \left\{ f \in \mathcal{O}_G\left(p^{-1}(U)\right) \mid \begin{array}{c} R^*_h(f) = f \quad \forall h \in \widetilde{H} \\ D^L_X f = 0 \quad \forall X \in \mathfrak{h}_1 \end{array} \right\}$$

The infinitesimal invariance under $\mathfrak{h}_1 = \mathfrak{g}_1$ ensures that a section $f \in \mathcal{O}_{G/M}(U)$ can be identified with the reduced section $\phi(1)$. The $\widetilde{H}$ invariance allows to get the desire identification of the sheaf $\mathcal{O}_{G/H}$ with $\mathcal{C}^{\infty}_{\widetilde{G}/\widetilde{H}}$. □

Let $\Sigma = (\sigma, \rho^{\sigma})$ denote a representation of the special sub super Lie group H acting on the super space V. In this section we give an alternative description of the associated super vector bundle $B^{\Sigma} \simeq \left(\mathcal{O}_G\left(p^{-1}(U)\right) \otimes V\right)^H$. This description was used in [4], for classifying the irreducible unitary representations of the super Poincarè groups, and in [14] in the classification of the irreducible unitary representations of nilpotent super Lie groups.

Proposition 11. *Using the notations of the above paragraph, B^{Σ} is isomorphic to the vector bundle B^{σ} associated with the representation σ of $\widetilde{H}$. The isomorphism becomes an isomorphism of G-super vector bundles, if we define the action of G on B^{σ} according to:*

$$(\pi_0(g_0)f)(g) = f\left(g_0^{-1}g\right)$$
$$(X.f)(g) = -\left[\mathbb{1}_{\mathcal{C}^\infty(\widetilde{G})} \otimes \rho^\sigma\left(g^{-1}.X\right) f\right](g)$$

for all $f \in B^\sigma$, $g, g_o \in \widetilde{G}$ *and* $X \in \mathfrak{g}_1$.

Proof. Define

$$A : (\mathcal{O}(G) \otimes V)^H \longrightarrow \left(\mathcal{C}^\infty(\widetilde{G}) \otimes V\right)^{\widetilde{H}}$$
$$f \longmapsto \left(j^* \otimes \mathbb{1}_V\right) f,$$

where $j : \widetilde{G} \to G$ is the canonical injection.

We claim that A is an isomorphism of $\mathcal{C}^\infty(\widetilde{G}/\widetilde{H})$-modules. The fact that it is a $\mathcal{C}^\infty(\widetilde{G}/\widetilde{H})$-modules morphism is clear.

Injectivity
Let $f \neq 0$ and suppose $(j^* \otimes \mathbb{1}_V) f = 0$. Then we have, for each $X \in \mathfrak{g}_1$

$$0 = (\mathbb{1}_{\mathcal{C}^\infty(\widetilde{G})} \otimes \rho^\sigma(X))(j^* \otimes \mathbb{1}_V) f$$
$$= (-j^* \otimes \mathbb{1}_V)(D_X^L \otimes \mathbb{1}_V) f$$

Hence,

$$(\mathrm{ev}_g \otimes \mathbb{1}_V) f(g.X) = 0 \quad \forall X \in \mathfrak{g}_1 \forall g \in \widetilde{G}$$

From this injectivity easily follows.

Surjectivity
In order to prove surjectivity we define the inverse of A. Let $f \in (\mathcal{C}^\infty(\widetilde{G}) \otimes V)^{\widetilde{H}}$ and, for each $Z \in \Lambda(\mathfrak{g}_1)$ define the element of $\underline{\mathrm{Hom}}\left(\Lambda(\mathfrak{g}_1), \mathcal{C}^\infty(\widetilde{G}) \otimes V\right)$

$$(\mathrm{ev}_g \otimes \mathbb{1}_V)\hat{f}(Z) := -\left[\mathbb{1}_{\mathcal{C}^\infty(\widetilde{G})} \otimes \rho^\sigma\left(g^{-1}.Z\right) f\right](g)$$

Finally define the map

$$\mathrm{Hom}\left(\Lambda(\mathfrak{g}_1), \mathcal{C}^\infty(\widetilde{G})\right) \longrightarrow \underline{\mathrm{Hom}}_{\mathfrak{U}(\mathfrak{g}_0)}\left(\mathfrak{U}(\mathfrak{g}_0) \otimes \Lambda(\mathfrak{g}_1), \mathcal{C}^\infty(\widetilde{G})\right)$$
$$\hat{\phi}_f(X \otimes Z)(g) := D_X^L(\hat{f}(Z))(g)$$

It is an easy calculation to check that this map is the desired inverse of A.

Action
Using A and its inverse is clearly possible to give a G-module structure to B^σ by completing the following diagram

$$\begin{array}{ccc} B^{\sigma} & \xrightarrow{A^{-1}} & B^{\Sigma} \\ \downarrow & & \downarrow \\ B^{\sigma} & \xleftarrow[A]{} & B^{\Sigma} \end{array}$$

Explicitly

$$\pi(g_0)^* f := A\ell^*_{g_0^{-1}} A^{-1} f$$

$$D^R_X f := A D^R_X A^{-1} f$$

Hence

1. If $g_0 \in \widetilde{G}$

$$\begin{aligned} \left(\pi(g_0)^* f\right)(g) &:= (A\ell^*_{g_0^{-1}} A^{-1} f)(g) \\ &= (\mathrm{ev}_g \otimes \mathbb{1}_V)(j^* \otimes \mathbb{1}_V)\ell^*_{g_0^{-1}} A^{-1} f \\ &= (\mathrm{ev}_{g_0^{-1}g} \otimes \mathbb{1}_V)\left(A^{-1} f\right)(1) \\ &= f\left(g_0^{-1} g\right) \end{aligned}$$

2. If $X \in \mathfrak{g}_1$

$$\begin{aligned} (D^R_X f)(g) &= \mathrm{ev}_g (D^R_X A^{-1} f)(1) \\ &= \mathrm{ev}_g \left(A^{-1} f\right)(X) \\ &= -\left[\mathbb{1}_{\mathcal{C}^\infty(\widetilde{G})} \otimes \rho^\sigma \left(g^{-1}.X\right) f\right](g) \end{aligned}$$

□

Remark 9. It is not fortuitous that unitary representations appear naturally in the case of special induction. Indeed in this case, being the quotient space a classical manifold, it makes sense to define Radon measures over G/H. This allows to construct inner-product spaces and finally to construct unitary representations.

5.2 Odd Induction

We now consider a kind of induction that is, in some sense, opposite to the one considered above. The importance of such induction procedure has been emphasized in [8]. In the case we are going to treat, the inducing sub super Lie group is the whole Lie group $\widetilde{G}$. As could be expected we have the following proposition.

Proposition 12. *Let G be a SLG and let $\widetilde{G}$ denote the corresponding reduced Lie group. The homogeneous space $G/\widetilde{G}$ canonically identifies with the super manifold $\left(\{\mathrm{pt}\}, \Lambda(\mathfrak{g}_1)^*\right)$.*
The projection $p : G \to G/\widetilde{G}$

$$p^* : \Lambda(\mathfrak{g}_1)^* \longrightarrow \mathcal{O}_G \simeq \underline{\mathrm{Hom}}_{\mathfrak{U}(\mathfrak{g}_0)}\left(\mathfrak{U}(\mathfrak{g}_0) \otimes \Lambda(\mathfrak{g}_1), \mathcal{C}^\infty(G)\right) \tag{26}$$
$$\Xi \longmapsto \Phi_\Xi$$

is uniquely determined by

$$\Phi_\Xi(1 \otimes Z)(g) := \Xi\left(g.\overline{Z}\right)$$

The fibration admits a global section defined by:

$$s^* : \underline{\mathrm{Hom}}_{\mathfrak{U}(\mathfrak{g}_0)}\left(\mathfrak{U}(\mathfrak{g}), \mathcal{C}^\infty(G)\right) \longrightarrow \Lambda(\mathfrak{g}_1)^* \tag{27}$$
$$\phi \longmapsto \mathrm{ev}_e \circ \phi_{|\Lambda(\mathfrak{g}_1)}$$

Proof. Left to the reader. □

Let now denote with σ a representation of $\widetilde{G}$ on $V = V_0 \oplus V_1$.

The associated vector bundle $B^\sigma = (\mathcal{O}_G \otimes V)^{\widetilde{G}}$ is based over a point and the associated fiber is given by the tensor product $\Lambda(\mathfrak{g}_1)^* \otimes V$.

We want to construct an isomorphism $A : \Lambda(\mathfrak{g}_1)^* \otimes V \to (\mathcal{O}(G) \otimes V)^{\widetilde{G}}$. Exactly as in the classical case we need to define the morphism

$$h : G \xrightarrow{\delta} G \times G \xrightarrow{i \circ s \circ p \times 1} G \times G \xrightarrow{\mu} \widetilde{G},$$

where $\delta : G \to G \times G$ denotes the diagonal map, p and s have been defined in the previous proposition, μ and i are respectively the multiplication and the inverse map of G. In terms of the functor of points h can be written as

$$h(g) := s(p(g))^{-\mathbb{1}} \cdot g$$

Hence we can define

$$A : \Lambda(\mathfrak{g}_1)^* \otimes V \longrightarrow (\mathcal{O}(G) \otimes V)^{\widetilde{G}}$$
$$\Xi \otimes v \longmapsto p^*(\Xi) \cdot [(h^* \otimes \mathbb{1}_V) \circ a_c^*](v)$$

Explicitly, we have

$$[A(\Xi \otimes v)](Z)(g) = \Xi(g.Z) \otimes \sigma(g)^{-1} v$$

The inverse of the above morphism is given by

$$
\begin{aligned}
A^{-1} : (\mathcal{O}(G) \otimes V)^{\widetilde{G}} &\longrightarrow \Lambda(\mathfrak{g}_1)^* \otimes V \\
\phi &\longmapsto (Z \mapsto \mathrm{ev}_e(\phi(1 \otimes Z)))
\end{aligned}
$$

We want to compute the G-module structure induced on $\Lambda(\mathfrak{g}_1)^* \otimes V$. As usual, for this it is enough to consider the diagram

$$
\begin{tikzcd}
\Lambda(\mathfrak{g}_1)^* \otimes V \arrow[r, "A"] \arrow[d, dashed] & (\mathcal{O}(G) \otimes V)^{\widetilde{G}} \arrow[d, "\ell^*_{g^{-1}},\, D^R_X"] \\
\Lambda(\mathfrak{g}_1)^* \otimes V & (\mathcal{O}(G) \otimes V)^{\widetilde{G}} \arrow[l, "A^{-1}"]
\end{tikzcd}
$$

Hence

$$
\begin{aligned}
\ell^*_{g^{-1}}(\Xi \otimes v) &:= A^{-1}\ell^*_{g^{-1}}A(\Xi \otimes v) \\
D^R_X(\Xi \otimes v) &:= A^{-1}D^R_X A(\Xi \otimes v)
\end{aligned}
$$

Proposition 13. *As a $\widetilde{G}$- module, $\Lambda(\mathfrak{g}_1)^* \otimes V$ is equivalent to the tensor product* $\mathrm{Ad}^* \otimes \sigma$ *of the contragredient of the adjoint representation with the inducing representation.*

The proof is an easy calculation and it is left to the reader.

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On Chiral Quantum Superspaces

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Abstract We give a quantum deformation of the chiral Minkowski superspace in 4 dimensions embedded as the big cell into the chiral conformal superspace. Both deformations are realized as quantum homogeneous superspaces: we deform the ring of regular functions together with a coaction of the corresponding quantum supergroup.

1 Introduction

In his foundational work on supergeometry [23] Manin realized the Minkowski superspace as the big cell inside the flag supermanifold of $2|0$ and $2|1$ superspaces in the superspace of dimension $4|1$.

In his construction however, the actions of the Poincaré and the conformal supergroups on the super Minkowski and its compactification were left in the background and did not play a crucial role. Moreover there was no explicit construction of the coordinate rings associated with the Minkowski superspace and the conformal superspace together with their embedding into a suitable projective superspace. Such coordinate rings are necessary in order to construct a quantum deformation.

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Our intention is to fill this gap, by bringing the supergroup action to the center of the stage so that we can give explicitly the coordinate rings of the Minkowski and conformal superspaces together with their embeddings into projective superspace. This will be our starting point to build a quantum deformation of them. We shall concentrate our attention in realizing the chiral Minkowski superspace as the big cell in the Grassmannian supervariety of $2|0$ superspaces in $\mathbb{C}^{4|1}$ (the chiral conformal superspace). This is not precisely the same supervariety that Manin considers in his work; the Grassmannian is a simpler one, but it also has a physical meaning. Our choice is motivated because in some supersymmetric theories *chiral superfields* appear naturally. Chiral superfields, in our approach, are identified with elements of the coordinate superalgebra of the above mentioned Grassmannian. If one wants to formulate certain supersymmetric field theories in a noncommutative superspace one needs to have the notion of quantum chiral superfields. It is not obvious in other approaches how to construct a quantum chiral superalgebra without losing other properties, as the action of the group, for example. In our construction the quantum chiral superfields appear naturally together with the supergroup action.

We plan to explore in a forthcoming paper Manin's construction in this new framework.

We shall not go into the details of the proofs of all of our statements, since an enlarged version of part of this work is available in [3]; nevertheless we shall make a constant effort to convey the key ideas and steps of our constructions.

This is the content of the present paper.

In Sect. 2 we briefly outline few key facts of supergeometry, favouring intuition over rigorous definitions. Our main reference will be [2].

In Sect. 3 we discuss the chiral conformal superspace as an homogeneous superspace identified with the Grassmannian supervariety of $2|0$ superspaces in the complex vector superspace of dimension $4|1$. We also provide an explicit projective embedding of the super Grassmannian into a suitable projective superspace.

In Sect. 4 we give an equivalent approach via invariant theory to the theory discussed in Sect. 3.

In Sect. 5 we introduce the complex super Minkowski space as the big cell in the chiral conformal superspace. We also provide an explicit description of the action of the super Poincaré group.

In Sects. 6 and 7 we build a quantum deformation of the Minkowski superspace and its compactification together with a coaction of the quantum Poincaré and conformal supergroups.

Finally in Sect. 8 we discuss some relevant physical applications of the theory developed so far.

2 Basic Concepts in Supergeometry

Supergeometry is essentially $\mathbb{Z}_2$-graded geometry: any geometrical object is given a $\mathbb{Z}_2$-grading in some natural way and the morphisms are the maps respecting the geometric structure and the $\mathbb{Z}_2$-grading.

For instance, a super vector space V is a vector space where we establish a $\mathbb{Z}_2$-grading by giving a splitting $V_0 \oplus V_1$. The elements in V_0 are called *even* and the elements in V_1 are called *odd*. Hence we have a function p called the *parity* defined only on homogeneous elements. A *superalgebra* A is a super vector space with multiplication preserving parity. The *reduced superalgebra* associated with A is $A_r := A/I_{\mathrm{odd}}$, where I_{odd} is the ideal generated by the odd elements which are nilpotent. Notice that the reduced superalgebra A_r may have even nilpotents, thus making the terminology a bit awkward.

A superalgebra A is *commutative* if

$$xy = (-1)^{p(x)p(y)} yx$$

for all x, y homogeneous elements in A. From now on we assume all superalgebras to be commutative unless otherwise specified and their category is denoted with (salg). We also need to introduce the notion of *affine superalgebra*. This is a finitely generated superalgebra such that A_r has no nilpotents. In ordinary algebraic geometry such A_r's are associated bijectively to affine algebraic varieties.

The most interesting objects in supergeometry are the *algebraic supervarieties* and the *differential supermanifolds*. Both these concepts are encompassed by the idea of *superspace*.

Definition 2.1. We define *superspace* the pair $S = (|S|, \mathcal{O}_S)$ where $|S|$ is a topological space and $\mathcal{O}_S$ is a sheaf of superalgebras such that the stalk at a point $x \in |S|$ denoted by $\mathcal{O}_{S,x}$ is a local superalgebra for all $x \in |S|$.

A *morphism* $\phi : S \longrightarrow T$ of superspaces is given by $\phi = (|\phi|, \phi^{\#})$, where $\phi : |S| \longrightarrow |T|$ is a map of topological spaces and $\phi^{\#} : \mathcal{O}_T \longrightarrow \phi_* \mathcal{O}_S$ is a sheaf morphism such that $\phi^{\#}_x(\mathbf{m}_{|\phi|(x)}) = \mathbf{m}_x$ where $\mathbf{m}_{|\phi|(x)}$ and $\mathbf{m}_x$ are the maximal ideals in the stalks $\mathcal{O}_{T,|\phi|(x)}$ and $\mathcal{O}_{S,x}$ respectively.

Let us see an important example.

Example 2.2. The superspace $\mathbb{R}^{p|q}$ is the topological space $\mathbb{R}^p$ endowed with the following sheaf of superalgebras. For any $U \subset_{\mathrm{open}} \mathbb{R}^p$

$$\mathcal{O}_{\mathbb{R}^{p|q}}(U) = C^{\infty}(\mathbb{R}^p)(U) \otimes \mathbb{R}[\xi^1, \dots, \xi^q],$$

where $\mathbb{R}[\xi_1, \dots, \xi_q]$ is the exterior algebra (or *Grassmann algebra*) generated by the q variables $\xi_1, \dots, \xi_q$.

Definition 2.3. A *supermanifold* of dimension $p|q$ is a superspace $M = (|M|, \mathcal{O}_M)$ which is locally isomorphic to the superspace $\mathbb{R}^{p|q}$, i.e. for all $x \in |M|$ there exist an open set $V_x \subset |M|$ and $U \subset \mathbb{R}^{p|q}$ such that:

$$\mathcal{O}_M|_{V_x} \cong \mathcal{O}_{\mathbb{R}^{p|q}}|_U.$$

We shall now concentrate on the study of algebraic supervarieties, since we use the algebraic approach to quantum deformations.

There are two equivalent and quite different approaches to both, algebraic supervarieties and differential supermanifolds: the sheaf theoretic and the functor of points categorical approach. In the first of these approaches, an algebraic supervariety (resp. a supermanifold) is to be understood as a superspace, that is, a pair consisting of a topological space and a sheaf of superalgebras. In the special cases of an affine algebraic supervariety (resp. a differential supermanifold), the superalgebra of global sections of the sheaf allows us to reconstruct the whole sheaf and the underlying topological space (see [2, Chaps. 4 and 10]). Consequently an affine supervariety (resp. a differential supermanifold) can be effectively identified with a commutative superalgebra.

This is the super counterpart to the well known result of ordinary complex algebraic geometry: affine varieties are in one-to-one correspondence with their coordinate rings, in other words, we associate the zeros of a set of polynomials into some affine space to the ideal generated by such polynomials. For example we associate to the complex sphere in $\mathbb{C}^3$, the coordinate ring $\mathbb{C}[x, y, z]/(x^2 + y^2 + z^2 - 1)$.

We also say that there is an equivalence of categories between the category of affine supervarieties and the category of affine superalgebras. Besides the above mentioned correspondence, this amounts to the fact that morphisms of affine varieties correspond to morphisms of the correspondent coordinate rings.

We can take the same point of view in supergeometry and give the following definition.

Definition 2.4. Let $\mathcal{O}(X)$ be an affine superalgebra. We define the *affine supervariety* X associated with $\mathcal{O}(X)$ as the superspace $(|X|, \mathcal{O}_X)$, where $|X|$ is the topological space of an ordinary affine variety, while $\mathcal{O}_X$ is the (unique) sheaf of superalgebras, whose global sections coincide with $\mathcal{O}(X)$, and there exists an open cover U_i of $|X|$ such that

$$\mathcal{O}_X(U_i) = \mathcal{O}(X)_{f_i} = \left\{ \frac{g}{f_i} \,\middle|\, g \in \mathcal{O}(X) \right\}$$

for suitable $f_i \in \mathcal{O}(X)_0$. (for more details see [8, Chap. II] and [2, Chap. 10]).

A *morphism* of affine supervarieties is a morphism of the underlying superspaces, though one readily see it corresponds (contravariantly) to a morphism of the corresponding coordinate superalgebras:

$$\{\text{morphisms } X \longrightarrow Y\} \qquad \longleftrightarrow \qquad \{\text{morphisms } \mathcal{O}(Y) \longrightarrow \mathcal{O}(X)\}$$

We define an *algebraic supervariety* as a superspace which is locally isomorphic to an affine supervariety. □

Example 2.5. 1. The affine superspace. We define the *polynomial superalgebra* as:

$$\mathbb{C}[x^1, \ldots, x^p, \theta^1, \ldots, \theta^q] := \mathbb{C}[x^1, \ldots, x^p] \otimes \Lambda(\theta^1, \ldots, \theta^q).$$

We want to interpret this superalgebra as the coordinate superring of a supervariety that we call the *affine superspace* of superdimension $p|q$, and we shall denote it with the symbol $\mathbb{C}^{p|q}$ or $\mathbf{A}^{p|q}$. The underlying topological space is $\mathbf{A}^p$, that is $\mathbb{C}^p$ with the Zariski topology, while the sheaf is:

$$\mathcal{O}_{\mathbf{A}^{p|q}}(U) := \mathcal{O}_{\mathbf{A}^p}(U) \otimes \Lambda(\theta^1 \dots \theta^q).$$

2. *The supersphere.* The superalgebra $\mathbb{C}[x_1, x_2, x_3, \eta_1, \eta_2, \eta_3]/(x_1^2+x_2^2+x_3^2+\eta_1 x+\eta_2 x_2+\eta_3 x_3-1)$ is the superalgebra of the global sections of an affine supervariety whose underlying topological space is the unitary sphere in $\mathbf{A}^3$.

The first important example of a supervariety which is not affine is given by the *projective superspace*.

Example 2.6. 1. Projective superspace. Consider the $\mathbb{Z}$-graded superalgebra $S= \mathbb{C}[x_0 \dots x_m, \xi_1 \ \dots \xi_n]$. For each r, $0 \leq r \leq m$, we consider the graded superalgebra

$$S[r] = \mathbb{C}[x_0, \dots, x_m, \xi_1, \dots, \xi_n][x_r^{-1}], \qquad \deg(x_r^{-1}) = -1.$$

The subalgebra $S[r]^0 \subset S[r]$ of $\mathbb{Z}$-degree 0 is

$$S[r]^0 \approx \mathbb{C}[u_0, \dots, \hat{u}_r, \cdots, u_m, \eta_1, \dots \eta_n], \qquad u_s = \frac{x_s}{x_r}, \ \eta_\alpha = \frac{\xi_\alpha}{x_r}, \tag{1}$$

(the ‘ ˆ ’ means that this generator is omitted). This is an affine superalgebra and it corresponds to an affine superspace, (see Example 2.5) whose topological space we denote with $|U_r|$ and the corresponding sheaf with $\mathcal{O}_{U_r}$. Notice that the topological spaces $|U_r|$ form an affine open cover of $|\mathbf{P}^m|$, the ordinary projective space of dimension m.

A direct calculations shows that:

$$\mathcal{O}_{U_r}|_{|U_r| \cap |U_s|} = \mathcal{O}_{U_s}|_{|U_r| \cap |U_s|},$$

so we conclude that there exists a unique sheaf on the topological space $|\mathbf{P}^m|$, that we denote as $\mathcal{O}_{\mathbf{P}^{m|n}}$, whose restriction to $|U_i|$ is $\mathcal{O}_{U_i}$. Hence we have defined a supervariety that we denote with $\mathbf{P}^{m|n}$ and call the *projective superspace* of dimension $m|n$.

2. *Projective supervarieties.* Let $I \subset S = \mathbb{C}[x_1 \dots x_m, \xi_1 \dots \xi_n]$ be a homogeneous ideal; then S/I is also a graded superalgebra and we can repeat the same construction as above. First of all, we notice that the reduced algebra $(S/I)_r$ corresponds to an ordinary projective variety, whose topological space we denote with $|X|$, embedded into a projective superspace $|X| \subset |\mathbf{P}^m|$. Consider the superalgebra of $\mathbb{Z}$-degree zero elements in $(S/I)[x_i^{-1}]$ (this is called *projective localization*):

$$\left(\frac{\mathbb{C}[x_0,\dots x_m,\xi_1\dots\xi_n]}{I}[x_i^{-1}]\right)_0 \cong \frac{\mathbb{C}[u_0,\dots,\hat{u}_i,\dots u_m,\eta_1\dots\eta_n]}{I_{\mathrm{loc}}},$$

where I_{loc} are the even elements of $\mathbb{Z}$-degree zero in $I[x_i^{-1}]$.

Again this affine superalgebra defines an affine supervariety with topological space $|V_i| \subset |U_i| \subset |\mathbf{P}^m|$ and sheaf $\mathcal{O}_{V_i}$. One can check that the supersheaves $\mathcal{O}_{V_i}$ are such that $\mathcal{O}_{V_i}|_{|V_i|\cap|V_j|} = \mathcal{O}_{V_j}|_{|V_i|\cap|V_j|}$, so they glue to give a sheaf on $|X|$. Hence as before there exists a supervariety corresponding to the homogeneous superring S/I. This supervariety comes equipped with a projective embedding, encoded by the morphism of graded superalgebra $S \longrightarrow S/I$, hence $(|X|,\mathcal{O}_X)$ is called a *projective supervariety.* □

It is very important to remark that, contrary to the affine case, there is no coordinate superring associated intrinsically to a projective supervariety, but there is a coordinate superring associated with the projective supervariety and its projective embedding. In other words we can have the same projective variety admitting non isomorphic coordinate superrings with respect to two different projective embeddings.

We now want to introduce the functor of points approach to the theory of supervarieties.

Classically we can examine the points of a variety over different fields and rings. For example we can look at the rational points of the complex sphere described above. They are in one to one correspondence with the morphisms: $\mathbb{C}[x,y,z]/(x^2+y^2+z^2-1) \longrightarrow \mathbb{Q}$. In fact each such morphism is specified by the knowledge of the images of the generators. The idea behind the functor of points is to extend this and consider *all* morphisms from the coordinate ring of the affine supervariety to *all* superalgebras at once.

Definition 2.7. Let $\mathcal{A} \in$ (salg), the category of commutative superalgebras. We define the $\mathcal{A}$-points of an affine supervariety X as the (superalgebra) morphisms $\mathrm{Hom}(\mathcal{O}(X),\mathcal{A})$. We define the functor of points of X as:

$$h_X : (\mathrm{salg}) \longrightarrow (\mathrm{sets}), \qquad h_X(\mathcal{A}) = \mathrm{Hom}(\mathcal{O}(X),\mathcal{A}).$$

In other words $h_X(\mathcal{A})$ are the $\mathcal{A}$-points of X, for all commutative superalgebras $\mathcal{A}$.

Example 2.8. If $\mathcal{A}$ is a generic (commutative) superalgebra, an $\mathcal{A}$-point of $\mathbb{C}^{p|q}$ (see Example 2.5) is given by a morphism $\mathbb{C}[x^1,\dots,x^p,\theta^1,\dots,\theta^q] \longrightarrow \mathcal{A}$, which is determined once we know the image of the generators

$$(x^1,\dots,x^p,\theta^1,\dots,\theta^q) \longrightarrow (a^1,\dots,a^p,\alpha^1,\dots,\alpha^q),$$

with $a^i \in \mathcal{A}_0$ and $\alpha^j \in \mathcal{A}_1$. Notice that the $\mathbb{C}$-points of $\mathbb{C}^{p|q}$ are given by $(k_1\dots k_p,0\dots0)$ and coincide with the points of the affine space $\mathbb{C}^p$. In this example it is clear that the knowledge of the points over a field is by no means sufficient to describe the supergeometric object.

Remark 2.9. It is important at this point to notice that just giving a functor from (salg) to (sets), does not guarantee that it is the functor of points of a supervariety. A set of conditions to establish this is given in [2, Chap. 10].

The functor of points for projective supervarieties is more complicated and we are unable to give a complete discussion here. We shall nevertheless discuss the functor of points of the projective space and superspace.

Example 2.10. Let us consider the functor: $h : \text{(alg)} \longrightarrow \text{(sets)}$, where $h(\mathcal{A})$ are the projective $\mathcal{A}$-modules of rank one in $\mathcal{A}^n$.

Equivalently $h(\mathcal{A})$ consists of the pairs (L, ϕ), where L is a projective $\mathcal{A}$-module of rank one, and ϕ is a surjective morphisms $\phi : \mathcal{A}^{n+1} \longrightarrow L$. These pairs are taken modulo the equivalence relation

$$(L, \phi) \approx (L', \phi') \Leftrightarrow L \overset{a}{\approx} L', \qquad \phi' = a \circ \phi,$$

If $\mathcal{A} = \mathbb{C}$, then projective modules are free and a morphism

$$\phi : \mathbb{C}^{n+1} \to \mathbb{C}$$

is specified by a n-tuple, $(a^1, \dots a^{n+1})$, with $a^i \in \mathbb{C}$, not all of the $a^i = 0$. The equivalence relation becomes

$$(a^1, \dots, a^{n+1}) \sim (b^1, \dots b^{n+1}) \Leftrightarrow (a^1, \dots, a^{n+1}) = \lambda(b^1, \dots, b^{n+1}),$$

with $\lambda \in \mathbb{C}^\times$ understood as an automorphism of $\mathbb{C}$. It is clear then that $h(\mathbb{C})$ consists of all the lines through the origin in the vector space $\mathbb{C}^{n+1}$, thus recovering the usual definition of complex projective space.

Projective modules are free over local rings. We then have a situation similar to the field setting: equivalence classes are lines in the $\mathcal{A}$-module $\mathcal{A}^{n+1}$.

Using the Representability Theorem (see [2]) one can show that the functor h is the functor of points of a variety that we call the projective space and whose geometric points coincide with the projective space $\mathbf{P}^n$ over the field k as we usually understand it. □

This example can be easily generalized to the supercontext: we consider the functor $h_{\mathbf{P}^{m|n}} : \text{(salg)} \longrightarrow \text{(sets)}$, where $h_{\mathbf{P}^{m|n}}(\mathcal{A})$ is defined as the set the projective $\mathcal{A}$-modules of rank one in $\mathcal{A}^{m|n} := \mathcal{A} \otimes \mathbb{C}^{m|n}$. This is the functor of points of the *projective superspace* described in Example 2.6.

The next question that we want to tackle is how we can define an embedding of a (super)variety into the projective (super)space using the functor of points notation.

Let X be a projective supervariety and $\Phi : X \longrightarrow \mathbf{P}^{m|n}$ be an injective morphism. As we discussed in Example 2.6 this embedding is encoded by a surjective morphism:

$$\mathbb{C}[x_1, \dots, x_m, \xi_1 \dots, \xi_n] \longrightarrow \mathbb{C}[x_1, \dots x_m, \xi_1 \dots, \xi_n]/(f_1, \dots, f_r)$$

In the notation of the functor of points, Φ is a *natural transformation* between the two functors h_X and $h_{\mathbf{P}^{m|n}}$, given by

$$\Phi_{\mathcal{A}} : h_X(\mathcal{A}) \longrightarrow h_{\mathbf{P}^{m|n}}(\mathcal{A})$$

with $\Phi_{\mathcal{A}}$ injective.

If $\mathcal{A}$ is a local superalgebra, then an $\mathcal{A}$-point $(a_1 \dots, a_m, \alpha_1 \dots, \alpha_n) \in h_{\mathbf{P}^{m|n}(\mathcal{A})}$ is in $\phi_{\mathcal{A}}(h_X(\mathcal{A}))$ if and only if it satisfies the homogeneous polynomial relations

$$\begin{gathered} f_1(a_1 \dots a_m, \alpha_1 \dots, \alpha_n) = 0, \\ \vdots \\ f_r(a_1 \dots a_m, \alpha_1 \dots, \alpha_n) = 0. \end{gathered}$$

(See [3] for more details).

In summary, to determine the coordinate superalgebra of a projective supervariety with respect to a certain projective embedding, we need to check the relations satisfied by the coordinates *just on local superalgebras*. This will be our starting point when we shall determine the coordinate superalgebra of the Grassmannian supervariety with respect to its Plücker embedding.

3 The Chiral Conformal Superspace

We are interested in the super Grassmannian of (2|0)-planes inside the superspace $\mathbb{C}^{4|1}$, that we denote with Gr. This will be our chiral conformal superspace once we establish an action of the conformal supergroup on it.

Gr is defined via its functor of points. For a generic superalgebra $\mathcal{A}$, the $\mathcal{A}$-points of Gr consist of the projective modules of rank 2|0 in $\mathcal{A}^{4|1} := \mathcal{A} \otimes \mathbb{C}^{4|1}$. It is not immediately clear that this is the functor of points of a supervariety, however a fully detailed proof of this fact is available in [3], Appendix A. Another important issue is the fact that once a supervariety is given, its functor of points is completely determined just by looking at the *local* superalgebras, and similarly the natural transformations are determined if we know them for local superalgebras. This is a well known fact that can be found for example in [16], Appendix A.

On a local superalgebra $\mathcal{A}$, $h_{\mathrm{Gr}}(\mathcal{A})$ consists of free submodules of rank 2|0 in $\mathcal{A}^{4|1}$ (on local superalgebras, projective modules are free). One such module can be specified by a couple of independent even vectors, a and b, which in the canonical basis $\{e_1, e_2, e_3, e_4, \mathcal{E}_5\}$ are given by two column vectors that span the subspace

$$\pi = \langle a, b \rangle = \left\langle \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ \alpha_5 \end{pmatrix}, \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ \beta_5 \end{pmatrix} \right\rangle, \tag{2}$$

with $a_i, b_i \in \mathcal{A}_0$ and $\alpha_5, \beta_5 \in \mathcal{A}_1$. Let

$$h_{\mathrm{GL}(4|1)}(\mathcal{A}) = \left\{ \begin{pmatrix} c_{11} & c_{12} & c_{13} & c_{14} & \rho_{15} \\ c_{21} & c_{22} & c_{23} & c_{24} & \rho_{25} \\ c_{31} & c_{32} & c_{33} & c_{34} & \rho_{35} \\ c_{41} & c_{42} & c_{43} & c_{44} & \rho_{45} \\ \delta_{51} & \delta_{52} & \delta_{53} & \delta_{54} & d_{55} \end{pmatrix} \right\}, \tag{3}$$

define the functor of points of the supergroup GL(4|1), where $c_{ij}, d_{55} \in \mathcal{A}_0$ and $\rho_{i5}, \delta_{5i} \in \mathcal{A}_1$. We can describe the action of the supergroup GL(4|1) over Gr as a natural transformation of the functors (for $\mathcal{A}$ local),

$$\begin{aligned} h_{\mathrm{GL}(4|1)}(\mathcal{A}) \times h_{\mathrm{Gr}}(\mathcal{A}) &\longrightarrow \quad h_{\mathrm{Gr}}(\mathcal{A}) \\ g, \langle a, b\rangle \quad &\longmapsto \langle g\cdot a,\, g\cdot b\rangle. \end{aligned}$$

Let $\pi_0 = \langle e_1, e_2\rangle \in h_{\mathrm{Gr}}(A)$. The stabilizer of this point in GL(4|1) is the upper parabolic super subgroup P_u, whose functor of points is

$$h_{P_u}(\mathcal{A}) = \left\{ \begin{pmatrix} c_{11} & c_{12} & c_{13} & c_{14} & \rho_{15} \\ c_{21} & c_{22} & c_{23} & c_{24} & \rho_{25} \\ 0 & 0 & c_{33} & c_{34} & \rho_{35} \\ 0 & 0 & c_{43} & c_{44} & \rho_{45} \\ 0 & 0 & \delta_{53} & \delta_{54} & d_{55} \end{pmatrix} \right\} \subset h_{\mathrm{GL}(4|1)}(A).$$

Then, the Grassmannian is identified with the quotient

$$h_{\mathrm{Gr}}(\mathcal{A}) = h_{\mathrm{GL}(4|1)}(\mathcal{A})/h_{P_u}(\mathcal{A}).$$

We want now to work out the expression for the *Plücker embedding*, It is important to stress that, contrary to what happens in the non super setting, in the super context we have that a generic Grassmannian supervariety does not admit a projective embedding. However for this particular Grassmannian such embedding exists, as we are going to show presently.

We want to give a natural transformation among the functors

$$p : h_{\mathrm{Gr}} \to h_{\mathbf{P}(E)},$$

where E is the super vector space $E = \wedge^2\mathbb{C}^{4|1} \approx \mathbb{C}^{7|4}$. Given the canonical basis for $\mathbb{C}^{4|1}$ we construct a basis for E

$$\begin{aligned} &e_1\wedge e_2, e_1\wedge e_3, e_1\wedge e_4, e_2\wedge e_3, e_2\wedge e_4, e_3\wedge e_4, \mathcal{E}_5\wedge\mathcal{E}_5, && \text{(even)} \\ &e_1\wedge\mathcal{E}_5, e_2\wedge\mathcal{E}_5, e_3\wedge\mathcal{E}_5, e_4\wedge\mathcal{E}_5, && \text{(odd)} \end{aligned} \tag{4}$$

As in the super vector space case, if L is a $\mathcal{A}$-module, for $\mathcal{A} \in$ (salg), we can construct $\wedge^2 L$

$$\Lambda^2 L = L \otimes L / \langle u \otimes v + (-1)^{|u||v|} v \otimes u \rangle, \qquad u, v \in L.$$

If $L \in h_{\mathrm{Gr}}(\mathcal{A})$, then $\wedge^2 L \subset \wedge^2 \mathcal{A}^{4|1}$. It is clear that if L is a projective $\mathcal{A}$-module of rank $2|0$, then $\wedge^2 L$ is a projective $\mathcal{A}$-module of rank $1|0$. In other words it is an element of $h_{\mathbf{P}(E)}(\mathcal{A})$, for $E = \wedge^2 \mathbb{C}^{4|1}$. Hence we have defined a natural transformation:

$$\begin{aligned} h_{\mathrm{Gr}}(\mathcal{A}) &\xrightarrow{\;p\;} h_{\mathbf{P}(E)}(\mathcal{A}) \\ L &\longrightarrow \wedge^2 L. \end{aligned}$$

Once we have the natural transformation defined, we can again restrict ourselves to work only on local algebras.

Let a, b be two even independent vectors in $\mathcal{A}^{4|1}$. For any superalgebra $\mathcal{A}$, they generate a free submodule of $\mathcal{A}^{4|1}$ of rank $2|0$. The natural transformation described above is as follows.

$$\begin{aligned} h_{\mathrm{Gr}}(A) &\xrightarrow{\;p_{\mathcal{A}}\;} h_{\mathbf{P}(E)}(\mathcal{A}) \\ \langle a, b \rangle_{\mathcal{A}} &\longrightarrow \langle a \wedge b \rangle. \end{aligned}$$

The map $p_{\mathcal{A}}$ is clearly injective. The image $p_{\mathcal{A}}(h_{\mathrm{Gr}}(\mathcal{A}))$ is the subset of even elements in $h_{\mathbf{P}(E)}(\mathcal{A})$ decomposable in terms of two even vectors of $\mathcal{A}^{4|1}$. We are going to find the necessary and sufficient conditions for an even element $Q \in h_{\mathbf{P}(E)}(\mathcal{A})$ to be decomposable. Let

$$\begin{aligned} Q &= q + \lambda \wedge \mathcal{E}_5 + a_{55} \mathcal{E}_5 \wedge \mathcal{E}_5, \quad \text{with} \\ q &= q_{12} e_1 \wedge e_2 + \cdots + q_{34} e_3 \wedge e_4, \quad q_{ij} \in \mathcal{A}_0, \\ \lambda &= \lambda_1 e_1 + \cdots + \lambda_4 e_4, \quad \lambda_i \in \mathcal{A}_1. \end{aligned} \tag{5}$$

Q is decomposable if and only if

$$\begin{aligned} Q &= (r + \xi \mathcal{E}_5) \wedge (s + \theta \mathcal{E}_5) \quad \text{with} \\ r &= r_1 e_1 + \cdots r_4 e_4, \quad s = s_1 e_1 + \cdots s_4 e_4, \quad r_i, s_i \in \mathcal{A}_0 \quad \xi, \theta \in \mathcal{A}_1, \end{aligned}$$

which means

$$Q = r \wedge s + (\theta r - \xi s) \wedge \mathcal{E}_5 + \xi \theta \mathcal{E}_5 \wedge \mathcal{E}_5 \text{equivalent to } q = r \wedge s, \;\; \lambda = \theta r - \xi s, \;\; a_{55} = \xi \theta.$$

These are equivalent to the following:

$$q \wedge q = 0, \qquad q \wedge \lambda = 0, \qquad \lambda \wedge \lambda = 2 a_{55} q \qquad \lambda a_{55} = 0.$$

Plugging (5) we obtain

$$\begin{aligned}
&q_{12}q_{34} - q_{13}q_{24} + q_{14}q_{23} = 0, && \text{(classical Plücker relation)}\\
&q_{ij}\lambda_k - q_{ik}\lambda_j + q_{jk}\lambda_i = 0, && 1 \leq i < j < k \leq 4\\
&\lambda_i\lambda_j = a_{55}q_{ij} && 1 \leq i < j \leq 4\\
&\lambda_i a_{55} = 0. &&
\end{aligned} \tag{6}$$

These are the *super Plücker relations*. As we shall see in the next section the superalgebra

$$\mathcal{O}(\mathrm{Gr}) = k[q_{ij}, \lambda_k, a_{55}]/\mathcal{I}_P, \tag{7}$$

is associated to the supervariety Gr in the Plücker embedding described above, where $\mathcal{I}_P$ denotes the ideal of the super Plücker relations (6). In other words $\mathcal{I}_P$ contains all the relations involving the coordinates q_{ij}, λ_k and a_{55}.

Remark 3.1. The superalgebra $\mathcal{O}(\mathrm{Gr})$ is a sub superalgebra (though not a Hopf sub superalgebra) of $\mathcal{O}(\mathrm{GL}(4|1))$. It is in fact the superalgebra generated by the corresponding minors, and the Plücker relations are all the relations satisfied by these minors in $\mathcal{O}(\mathrm{GL}(4|1))$.

4 The Super Grassmannian via Invariant Theory

In this section we propose an alternative and equivalent way to construct the super Grassmannian Gr as a complex supervariety and we give the coordinate superring associated to the super Grassmannian in the Plücker embedding, thus completing the discussion initiated in the previous section.

As we have seen in Sect. 2, the super Grassmannian can be equivalently understood as a pair consisting of the underlying topological space $G(2,4)$, and a sheaf of superalgebras conveniently chosen that we shall describe presently.

We recall first what happens in the ordinary case. Let the set S be

$$S = \{(v,w) \in \mathbb{C}^4 \oplus \mathbb{C}^4 \;/\; \mathrm{rank}(v,w) = 2\},$$

and consider the equivalence relation

$$(v,w) \sim (v',w') \Leftrightarrow \mathrm{span}\{v,w\} = \mathrm{span}\{v',w'\},$$

or equivalently

$$(v,w) \sim (v',w') \Leftrightarrow \exists\, g \in \mathrm{GL}(2,\mathbb{C}) \text{ such that } (v',w') = (v,w)g.$$

Then we have that $G(2,4) = S/\sim$.

We consider now the set of polynomials on S, $\mathrm{Pol}(S)$, and the subset of such polynomials that is semi-invariant under the transformation of $\mathrm{GL}(2, \mathbb{C})$, that is

$$f(v', w') = f(u, v)\lambda(g), \qquad \lambda(g) \in \mathbb{C}, \quad f \in \mathrm{Pol}(S).$$

This defines the homogeneous ring of $G(2, 4)$, which is generated by the six determinants [19].

$$y_{ij} = v_i w_j - v_j w_i, \quad \text{with } i < j \ \text{ and } \ \lambda = \det g.$$

These are not all independent, they satisfy the Plücker relation

$$y_{12}y_{34} + y_{23}y_{14} + y_{31}y_{24} = 0.$$

Let $\mathcal{O}$ be the sheaf of polynomials on S, so for each open set in $\tilde{U} \subset S$, $\mathcal{O}(\tilde{U}) = \mathrm{Pol}(\tilde{U})$ and $\mathcal{O}^{\mathrm{inv}}$ the subsheaf of $\mathcal{O}$ corresponding to the semi-invariant polynomials.

Let $\pi : S \to G(2, 4)$ be the natural projection. It is clear that for $U \subset_{\mathrm{open}} G(2, 4)$, then $\tilde{U} = \pi^{-1}(U) \subset S$ is also open in S. We can define the following sheaf over $G(2, 4)$:

$$\mathcal{O}(U) = \mathcal{O}^{\mathrm{inv}}(\pi^{-1}(U)).$$

This is the structural sheaf of the projective variety $G(2, 4)$ with respect to the Plücker embedding.

Now we turn to the super setting and we want to define the sheaf of superalgebras generalizing the non super construction to the super Grassmannian. We define the superalgebra

$$\mathcal{F}(S) := \mathrm{Pol}(S) \otimes \Lambda[\xi_1, \xi_2].$$

Let $(v, w) \in S$ and consider the (5×2) matrix

$$\begin{pmatrix} v & w \\ \xi_1 & \xi_2 \end{pmatrix} = \begin{pmatrix} v_1 & w_1 \\ \vdots & \vdots \\ v_4 & w_4 \\ \xi_1 & \xi_2 \end{pmatrix}.$$

The group $\mathrm{GL}(2, \mathbb{C})$ acts on the right on these matrices

$$\begin{pmatrix} v' & w' \\ \xi_1' & \xi_2' \end{pmatrix} = \begin{pmatrix} v & w \\ \xi_1 & \xi_2 \end{pmatrix} \cdot g, \qquad g \in \mathrm{GL}(2, \mathbb{C}).$$

We will write an element $f(v, w, \xi) \in \mathcal{F}(S)$ as

$$f(v, w, \xi) = \sum_{i,j=0,1} f_{ij}(v, w)\xi_1^i \xi_2^j.$$

We will refer to the elements of $\mathcal{F}(S)$ as 'functions', being this customary in the physics literature. We now consider the set of semi-invariant functions

$$f(v', w', \xi') = f(v, w, \xi)\lambda(g), \qquad \lambda(g) \in \mathbb{C}, \quad f \in \mathcal{F}(S).$$

The following functions are semi-invariant:

$$y_{ij} = v_i w_j - v_j w_i, \quad \theta_i = v_i \xi_2 - w_i \xi_1, \quad a = \xi_1 \xi_2, \tag{8}$$

with $\lambda(g) = \det g$ but they are not all independent. They satisfy the *super Plücker relations* (6)

$$\begin{aligned}
&y_{12}y_{34} - y_{13}y_{24} + y_{14}y_{23} = 0, && \text{(standard Plücker relation)}\\
&y_{ij}\theta_k - y_{ik}\theta_j + y_{jk}\theta_i = 0 && 1 \le i < j < k \le 4\\
&\theta_i\theta_j = a y_{ij} && 1 \le i < j \le 4\\
&\theta_i a = 0 && 1 \le i \le 4 = 0.
\end{aligned}$$

We want to show that the elements in (8) generate the ring of semi-invariants and that (6) are all the relations among these generators.

Proposition 4.1. *Let f be a homogeneous semi-invariant function, so*

$$f(v', w', \xi') = f(v, w, \xi)\lambda(g)$$

with

$$\begin{pmatrix} v' & w' \\ \xi_1' & \xi_2' \end{pmatrix} = \begin{pmatrix} v & w \\ \xi_1 & \xi_2 \end{pmatrix} \cdot g, \qquad g \in \mathrm{GL}(2, \mathbb{C}).$$

Then in the decomposition

$$f(v, w, \xi) = f_0(v, w) + \sum_i f_i(v, w)\xi_i + f_{12}(v, w)\xi_1\xi_2, \tag{9}$$

one has that $f_0(v, w)$ and $f_{12}(v, w)$ are standard (non-super) semi-invariants and

$$\sum_i f_i(v, w)\xi_i = \sum_i h_i(v, w)\theta_i,$$

with $h_i(v, w)$ also a standard semi-invariant.

Proof. Let us take

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \text{so} \quad \begin{pmatrix} v' & w' \\ \xi_1' & \xi_2' \end{pmatrix} = \begin{pmatrix} va + wc & vb + wd \\ \xi_1 a + \xi_2 c & \xi_1 b + \xi_2 d \end{pmatrix}.$$

Then we can see immediately that each term in (9) has to be a semi-invariant, so

$$f_0(v', w') = \lambda(g) f_0(v, w), \qquad \sum_i f_i(v', w')\xi_i' = \lambda(g) \sum_i f_i(v, w)\xi_i,$$
$$f_{12}(v', w')\xi_1'\xi_2' = f_{12}(v, w)\xi_1\xi_2.$$

We have that f_0 is an ordinary semi-invariant transforming with $\lambda(g)$, and since $\xi_1'\xi_2' = \xi_1\xi_2 \det g$, $f_{12}(v, w)$ is a ordinary semi-invariant transforming with $\lambda(g) \det g^{-1}$. The odd terms θ^i are of the same form as the ordinary invariants y_{ij}, since the fact that ξ_i is odd plays no particular role here (recall that we are considering the action of an ordinary group, namely $\mathrm{GL}(2, \mathbb{C})$). So by the same argument we have in the ordinary case, there are no other odd invariants, besides those we have already found, that are linear in the odd variables ξ_1 and ξ_2. Then

$$\sum_i f_i \xi_i = \sum_i h(v, w)_i \theta_i,$$

where $h(v, w)_i$ transforms with $\lambda(g) \det g^{-1}$. □

We now wish to give a result that describes completely the relations among the invariants.

Consider the polynomial superalgebra $\mathbb{C}[a_{ib}]$, $1 \le i \le 5$, $1 \le b \le 2$, with the parity defined as

$$p(a_{ij}) = p(i) + p(j), \quad \text{with} \quad p(k) = 0 \text{ if } 0 \le k \le 4 \text{ and } p(5) = 1.$$

On $\mathbb{C}[a_{ij}]$ there exists the following action of $\mathrm{GL}(2, \mathbb{C})$:

$$\begin{array}{ccc} \mathbb{C}[a_{ib}] \times \mathrm{GL}(2, \mathbb{C}) & \longrightarrow & \mathbb{C}[a_{ib}] \\ (a_{ia}, g^{-1}) & \longrightarrow & \sum_k a_{ib} g^{-1}_{ba} \end{array}$$

We have just proven that the semi-invariants are generated by the polynomials

$$d_{ij} = a_{i1}a_{j2} - a_{i2}a_{j1}, \quad 1 \le i < j \le 5,$$
$$d_{55} = a_{51}a_{52}.$$

We have the following proposition:

Proposition 4.2. *Let $\mathcal{O}(\mathrm{Gr})$ be the subring of $\mathbb{C}[a_{ib}]$ generated by the determinants $d_{ij} = a_{i1}a_{j2} - a_{j1}a_{i2}$ and $d_{55} = a_{51}a_{52}$. Then $\mathcal{O}(\mathrm{Gr}) \cong \mathbb{C}[a_{ib}]/I_P$, where I_P is the ideal of the super Plücker relations (6). In other words I_P contains all the possible relations satisfied by d_{ij} and d_{55}.*

Proof. It is easy to verify that d_{ij} and d_{55} satisfy all the above relations, the problem is to prove that these are the only relations.

The proof of this fact is the same as in the classical setting. Let us briefly sketch it. Let $I_1, \dots, I_r$ be multiindices organized in a tableau. We say that a tableau is *superstandard* if it is strictly increasing along rows with the exception of the number 5 (that can be repeated) and weakly increasing along columns. A *standard monomial* in $\mathcal{O}(\mathrm{Gr})$ is a monomial $d_{I_1}, \cdots, d_{I_r}$ where the indices $I_1, \dots, I_r$ form a superstandard tableau. Using the super Plücker relation one can verify that any monomial in $\mathcal{O}(\mathrm{Gr})$ can be written as a linear combination of standard ones. This can be done directly or using the same argument for the classical case (see [19, p. 110] for more details). The standard monomials are also linearly independent, hence they form a basis for $\mathcal{O}(\mathrm{Gr})$ as $\mathbb{C}$-vector space. Again this is done with the same argument as in [19, p. 110]. So given a relation in $\mathcal{O}(\mathrm{Gr})$, once we write each term as a standard monomial we obtain that either the relation is identically zero (hence it is a relation in the Plücker ideal) or it gives a relation among the standard monomials, which gives a contradiction. □

In the end we summarize the main results of Sects. 3 and 4 with a corollary.

Corollary 4.3. *1. Let* Gr *be the Grassmannian of* $2|0$ *spaces in* $\mathbb{C}^{4|1}$. *Then* $\mathrm{Gr} \subset \mathbf{P}^{7|4}$, *that is* Gr *is a projective supervariety. Such embedding is encoded by the superring* $\mathcal{O}(\mathrm{Gr})$ *described above.*

2. $\mathcal{O}(\mathrm{Gr})$ *is isomorphic to the ring generated by the determinants* d_{ij}, d_{55}, *inside* $O(\mathrm{GL}(4|1))$.

5 The Chiral Minkowski Superspace

In this section we concentrate our attention to determine the big cell inside the Grassmannian supervariety that we have discussed in the previous sections. We shall identify such big cell with the chiral Minkowski superspace.

As in the ordinary setting, the super Grassmannian Gr admits an open cover in terms of affine superspaces: topologically the two covers are the same.

We want to describe the functor of points of the *big cell* U_{12} inside Gr. This is the open affine functor corresponding to the points in which the coordinate q_{12} is invertible.

First of all, we write an element of $h_{\mathrm{GL}(4|1)}(\mathcal{A})$ in blocks as (see (3))

$$\begin{pmatrix} C_1 & C_2 & \rho_1 \\ C_3 & C_4 & \rho_2 \\ \delta_1 & \delta_2 & d_{55} \end{pmatrix}.$$

Assuming that $\det C_1$ is invertible, we can bring this matrix, with a transformation of $h_{P_u}(\mathcal{A})$, to the form

$$\begin{pmatrix} C_1 & C_2 & \rho_1 \\ C_3 & C_4 & \rho_2 \\ \delta_1 & \delta_2 & d_{55} \end{pmatrix} h_{P_u}(\mathcal{A}) = \begin{pmatrix} 1\!\!1_2 & 0 & 0 \\ A & 1\!\!1_2 & 0 \\ \alpha & 0 & 1 \end{pmatrix} h_{P_u}(\mathcal{A}) \in h_{\mathrm{GL}(4|1)}(\mathcal{A}) \big/ h_{P_u}(\mathcal{A}) \qquad (10)$$

Consider the subspace $\pi = \text{span}\{a, b\}$ in $h_{\text{Gr}}(\mathcal{A})$ for $\mathcal{A}$ local. Recall that in Sect. 3 we made the identification: $h_{\text{Gr}}(\mathcal{A}) \cong h_{\text{GL}(4|1)}(\mathcal{A}) \,/\, h_{P_u}(\mathcal{A})$. Hence:

$$\pi = \text{span}\{a, b\} \approx \begin{pmatrix} C_1 & C_2 & \rho_1 \\ C_3 & C_4 & \rho_2 \\ \delta_1 & \delta_2 & d_{55} \end{pmatrix} h_{P_u}(\mathcal{A}) \in h_{\text{GL}(4|1)}(\mathcal{A}) \,/\, h_{P_u}(\mathcal{A})$$

with $\det C_1$ invertible. Then, by a change of coordinate (10) we can bring this matrix to the standard form detailed above

$$\pi \approx \begin{pmatrix} 1\!\!1_2 & 0 & 0 \\ A & 1\!\!1_2 & 0 \\ \alpha & 0 & 1 \end{pmatrix} h_{P_u}(\mathcal{A}), \qquad A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \qquad \alpha = (\alpha_1, \alpha_2),$$

with the entries of A in $\mathcal{A}_0$ and the entries of α in $\mathcal{A}_1$. Its column vectors generate also the submodule $\langle a, b\rangle$.

The assumption that $\det C_1$ is invertible is equivalent to assume to be in the topological open set $|U_{12}| = |\text{Gr}| \cap |V_{12}|$, where V_{12} is the affine open set corresponding to the topological open set $|V_{12}|$ defined by taking in $\mathbf{P}(E)$ the coordinate q_{12} to be invertible. Consequently the coordinate superring of the affine open subvariety U_{12} of Gr corresponds to the projective localization of the Grassmannian superring in the coordinate q_{12}. In other words it consists of the elements of degree zero in

$$\mathbb{C}[q_{ij}q_{12}^{-1}, \lambda_j q_{12}^{-1}, a_{55}q_{12}^{-1}] \subset \mathcal{O}(\text{Gr})[q_{12}^{-1}].$$

As one can readily check, there are no relations among these generators so that the big cell U_{12} of Gr is the affine superspace with coordinate ring

$$\mathcal{O}(U_{12}) = \mathbb{C}[x_{ij}, \xi_j] \approx \mathbb{C}^{4|2}. \tag{11}$$

where we set $x_{ij} = q_{ij}q_{12}^{-1}$, $x_{55} = a_{55}q_{12}^{-1}$, $\xi_j = \lambda_j q_{12}^{-1}$.

We are now interested in the super subgroup of GL(4|1) that preserves the big cell U_{12}. This the lower parabolic sub-supergroup P_l (see [3]), whose functor of points is given in suitable coordinates as

$$h_{P_l}(\mathcal{A}) = \left\{ \begin{pmatrix} x & 0 & 0 \\ tx & y & y\eta \\ d\tau & d\xi & d \end{pmatrix} \right\} \subset h_{\text{GL}(4|1)}(\mathcal{A}),$$

where x and y are even, invertible 2×2 matrices, t is an even, arbitrary 2×2 matrix, η a 2×1 odd matrix, τ, ξ are 1×2 odd matrices and d is an invertible even element.

The action of the supergroup P_l on the big cell U_{12} is as follows,

$$\begin{array}{ccc} h_{P_l}(\mathcal{A}) \times h_{U_{12}}(\mathcal{A}) & \longrightarrow & h_{U_{12}}(\mathcal{A}) \\ \left(\begin{pmatrix} x & 0 & 0 \\ tx & y & y\eta \\ d\tau & d\xi & d \end{pmatrix}, \begin{pmatrix} 1\!\!1_2 \\ A \\ \alpha \end{pmatrix} \right) & \longrightarrow & \begin{pmatrix} 1\!\!1_2 \\ A' \\ \alpha' \end{pmatrix}, \end{array}$$

where, using a transformation of $h_{P_u}(\mathcal{A})$ to revert the resulting matrix to the standard form (10), we have

$$\begin{pmatrix} 1\!\!1_2 \\ A' \\ \alpha' \end{pmatrix} = \begin{pmatrix} 1\!\!1_2 \\ y(A+\eta\alpha)x^{-1} + t \\ d(\alpha + \tau + \xi A)x^{-1} \end{pmatrix}. \tag{12}$$

The subgroup with $\xi = 0$ is the super Poincaré group times dilations (compare with (14) in [18]). In that case

$$d = \det x \det y.$$

6 Quantum Chiral Conformal Superspace

In this section we give a quantum deformation of $\mathcal{O}(\mathrm{Gr})$, discussed in the previous sections. This will yield a quantum deformation of the chiral conformal superspace together with the natural coaction of the conformal supergroup on it.

Definition 6.1. Let us define following Manin [24] the quantum matrix superalgebra.

$$M_q(m|n) =_{def} \mathbb{C}_q < a_{ij} > / I_M,$$

where $\mathbb{C}_q < a_{ij} >$ denotes the free algebra over $\mathbb{C}_q = \mathbb{C}[q, q^{-1}]$ generated by the homogeneous variables a_{ij} and the ideal I_M is generated by the relations [24]:

$$a_{ij}a_{il} = (-1)^{\pi(a_{ij})\pi(a_{il})} q^{(-1)^{p(i)+1}} a_{il}a_{ij}, \quad j < l$$

$$a_{ij}a_{kj} = (-1)^{\pi(a_{ij})\pi(a_{kj})} q^{(-1)^{p(j)+1}} a_{kj}a_{ij}, \quad i < k$$

$$a_{ij}a_{kl} = (-1)^{\pi(a_{ij})\pi(a_{kl})} a_{kl}a_{ij}, \quad i < k, j > l \quad or \quad i > k, j < l$$

$$a_{ij}a_{kl} - (-1)^{\pi(a_{ij})\pi(a_{kl})} a_{kl}a_{ij}$$

$$= (-1)^{\pi(a_{ij})\pi(a_{kj})} (q^{-1} - q) a_{kj}a_{il} \quad i < k, j < l,$$

where $p(i) = 0$ if $1 \leq i \leq m$, $p(i) = 1$ otherwise and $\pi(a_{ij}) = p(i) + p(j)$ denotes the parity of a_{ij}.

$M_q(m|n)$ is a bialgebra with the usual comultiplication and counit:

$$\Delta(a_{ij}) = \sum a_{ik} \otimes a_{kj}, \qquad \mathcal{E}(a_{ij}) = \delta_{ij}.$$

We are ready to define the general linear supergroup which will be most interesting for us.

Definition 6.2. We define *quantum general linear supergroup*

$$\mathrm{GL}_q(m|n) =_{def} M_q(m|n)\langle D_1{}^{-1}, D_2{}^{-1}\rangle,$$

where $D_1{}^{-1}$, $D_2{}^{-1}$ are even indeterminates such that:

$$D_1 D_1^{-1} = 1 = D_1{}^{-1} D_1, \qquad D_2 D_2{}^{-1} = 1 = D_2{}^{-1} D_2$$

and

$$D_1 =_{def} \textstyle\sum_{\sigma\in S_m} (-q)^{-l(\sigma)} a_{1\sigma(1)} \dots a_{m\sigma(m)}$$

$$D_2 =_{def} \textstyle\sum_{\sigma\in S_n} (-q)^{l(\sigma)} a_{m+1,m+\sigma(1)} \dots a_{m+n,m+\sigma(n)}$$

are the quantum determinants of the diagonal blocks.

$\mathrm{GL}_q(m|n)$ is a Hopf algebra, where the comultiplication and counit are the same as in $M_q(m|n)$, while the antipode S is detailed in [15].

We now give the central definition in analogy with the ordinary setting (compare with Prop. 4.3).

Definition 6.3. Let the notation be as above. We define *quantum super Grassmannian* of $2|0$ planes in $4|1$ dimensional superspace as the non commutative superalgebra Gr_q generated by the following quantum super minors in $\mathrm{GL}_q(4|1)$:

$$D_{ij} = a_{i1}a_{j2} - q^{-1}a_{i2}a_{j1}, \qquad 1 \le i < j \le 4, \qquad D_{55} = a_{51}a_{52}$$

$$D_{i5} = a_{i1}a_{52} - q^{-1}a_{i2}a_{51}, \qquad 1 \le i \le 4.$$

For clarity, let us write all the generators:

$$D_{12}, \quad D_{13}, \quad D_{14}, \quad D_{23}, \quad D_{24}, \quad D_{34}, \quad D_{55}, \quad D_{15}, \quad D_{25}, \quad D_{35}, \quad D_{45}$$

Notice that when $q = 1$ this is the coordinate ring of the super Grassmannian.

We need to work out the commutation relations and the quantum Plücker relations in order to be able to give a presentation of the quantum Grassmannian in terms of generators and relations.

Let us start with the commutation relations. With very similar calculations to the ones in [11] one finds the following relations:

- If i, j, k, l are *not* all distinct we have ($1 \leq i, j, k, l \leq 5$):

$$D_{ij} D_{kl} = q^{-1} D_{kl} D_{ij}, \qquad (i, j) < (k, l)$$

  where $<$ refers to the lexicographic ordering.
- If i, j, k, l are instead all distinct we have:

$$D_{ij} D_{kl} = q^{-2} D_{kl} D_{ij}, \qquad 1 \leq i < j < k < l \leq 5$$

$$D_{ij} D_{kl} = q^{-2} D_{kl} D_{ij} - (q^{-1} - q) D_{ik} D_{jl}, \qquad 1 \leq i < k < j < l \leq 5$$

$$D_{ij} D_{kl} = D_{kl} D_{ij}, \qquad 1 \leq i < k < l < j \leq 5$$

- The only commutation relations that we are left to be shown are the following:

$$D_{ij} D_{55}, \qquad D_{i5} D_{j5}, \qquad D_{i5} D_{55}$$

  After some computations one gets:

$$D_{ij} D_{55} = q^{-2} D_{55} D_{ij}, \qquad 1 \leq i < j \leq 4$$

$$D_{i5} D_{j5} = -q^{-1} D_{j5} D_{i5} - (q^{-1} - q) D_{ij} D_{55} \qquad 1 \leq i < j \leq 4$$

$$D_{i5} D_{55} = D_{55} D_{i5} = 0, \qquad 1 \leq i \leq 4.$$

This concludes the discussion of the commutation relations. As for the Plücker relations, using the result for the non super setting (refer to [11]) we have

$$D_{12} D_{34} - q^{-1} D_{13} D_{24} + q^{-2} D_{14} D_{23} = 0$$

$$D_{ij} D_{k5} - q^{-1} D_{ik} D_{j5} + q^{-2} D_{i5} D_{jk} = 0, \qquad 1 \leq i < j < k \leq 4$$

To this we must add the relations, which can be computed directly:

$$D_{i5} D_{j5} = q D_{ij} D_{55}, \qquad 1 \leq i < j \leq 4.$$

The next proposition summarizes all of our calculations and the proof can be found in [3].

Proposition 6.4.

- *The quantum Grassmannian ring is given in terms of generators and relations as:*

$$Gr_q = \mathbb{C}_q \langle X_{ij} \rangle / I_{Gr},$$

where I_{Gr} is the two-sided ideal generated by the commutations and Plücker relations in the indeterminates X_{ij}. Moreover $Gr_q/(q-1) \cong \mathcal{O}(\mathrm{Gr})$ (see Sect. 3).

- *The quantum Grassmannian ring is the free ring over $\mathbb{C}_q$ generated by the monomials in the quantum determinants:*

$$D_{i_1 j_1}, \dots, D_{i_r j_r},$$

where $(i_1, j_1), \dots, (i_r, j_r)$ form a semistandard tableau (for its definition we refer to [3]).

The quantum Grassmannian that we have constructed admits a coaction of the quantum supergroup $\mathrm{GL}_q(4|1)$. The proof of the following proposition amounts to a direct check (we refer again to [3] for more details).

Proposition 6.5. *Gr_q is a quantum homogeneous superspace for the quantum supergroup $\mathrm{GL}_q(4|1)$, i.e. we have a coaction given via the restriction of the comultiplication of $\mathrm{GL}_q(4|1)$:*

$$\Delta|_{Gr_q} : Gr_q \longrightarrow \mathrm{GL}_q(4|1) \otimes Gr_q.$$

7 Quantum Minkowski Superspace

We now turn to the quantum deformation of the big cell inside Gr_q; it will be our model for the quantum Minkowski superspace.

In Sect. 5 we wrote the action of the lower parabolic supergroup P_l using the functor of points (12). We want now to translate it into the coaction language in order to make the generalization to the quantum setting.

Let $\mathcal{O}(P_l)$ be the superalgebra:

$$\mathcal{O}(P_l) := \mathcal{O}(\mathrm{GL}(4|1))/\mathcal{I},$$

where $\mathcal{I}$ is the (two-sided) ideal generated by

$$g_{1j}, g_{2j}, \quad \text{for} \quad j = 3, 4 \quad \text{and} \quad \gamma_{15}, \gamma_{25}.$$

This is the Hopf superalgebra coordinate superring of the lower parabolic subgroup P_l, with comultiplication naturally inherited by $\mathcal{O}(\mathrm{GL}(4|1))$.

In matrix form, for $\mathcal{A}$ local, we have

$$h_{P_l}(\mathcal{A}) = \left\{ \begin{pmatrix} g_{11} & g_{12} & 0 & 0 & 0 \\ g_{21} & g_{22} & 0 & 0 & 0 \\ g_{31} & g_{32} & g_{33} & g_{34} & \gamma_{35} \\ g_{41} & g_{42} & g_{43} & g_{44} & \gamma_{45} \\ \gamma_{51} & \gamma_{52} & \gamma_{53} & \gamma_{54} & g_{55} \end{pmatrix} \right\} \subset h_{\mathrm{GL}(m|n)}(\mathcal{A}). \tag{13}$$

The superalgebra representing the big cell U_{12} can be realized as a subalgebra of $\mathcal{O}(P_l)$. In order to see this better, let us make the following two different changes of variables in P_l:

$$\begin{pmatrix} g_{11} & g_{12} & 0 & 0 & 0 \\ g_{21} & g_{22} & 0 & 0 & 0 \\ g_{31} & g_{32} & g_{33} & g_{34} & \gamma_{35} \\ g_{41} & g_{42} & g_{43} & g_{44} & \gamma_{45} \\ \gamma_{51} & \gamma_{52} & \gamma_{53} & \gamma_{54} & g_{55} \end{pmatrix} = \begin{pmatrix} x & 0 & 0 \\ tx & y & y\eta \\ \tilde{\tau}x & d\xi & d \end{pmatrix} = \begin{pmatrix} x & 0 & 0 \\ tx & y & y\eta \\ d\tau & d\xi & d \end{pmatrix} \tag{14}$$

Notice that the only difference between the two sets of variables is that we replace τ with $\tilde{\tau}$ and we have:

$$d\tau = \tilde{\tau}x, \tag{15}$$

The next proposition tells us that these are sets of generators for $\mathcal{O}(P_l)$ and that having $\tilde{\tau}$ is essential to describe the big cell. Again for the proof we refer the reader to [3], while the explicit expressions for the generators come from a direct calculation.

Proposition 7.1. *1. The Hopf superalgebra $\mathcal{O}(P_l)$ is generated by the following sets of variables:*

- *x, y, t, $\tilde{\tau}$, ξ, η and d;*
- *x, y, t, τ, ξ, η and d*

defined as

$$x = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}, \qquad y = \begin{pmatrix} g_{33} & g_{34} \\ g_{43} & g_{44} \end{pmatrix},$$

$$t = \begin{pmatrix} -d_{23}d_{12}^{-1} & d_{13}d_{12}^{-1} \\ -d_{24}d_{12}^{-1} & d_{14}d_{12}^{-1} \end{pmatrix} \qquad d = g_{55}$$

$$\tilde{\tau} = (-d_{25}d_{12}^{-1}, d_{15}d_{12}^{-1}) \quad \tau = (g_{55}^{-1}\gamma_{51}, g_{55}^{-1}\gamma_{52}) \tag{16}$$

$$\eta = \begin{pmatrix} d_{34}^{34\,-1}\gamma_{35} \\ d_{34}^{34\,-1}\gamma_{45} \end{pmatrix} \qquad \xi = \begin{pmatrix} g_{55}^{-1}\gamma_{53} & g_{55}^{-1}\gamma_{54} \end{pmatrix},$$

where for $1 \le i < j \le 4$

$$d_{ij} = g_{i1}g_{j2} - g_{j1}g_{i2}, \qquad d_{i5} = g_{i1}\gamma_{52} - \gamma_{51}g_{i2}, \qquad d_{34}^{34} = g_{33}g_{44} - g_{34}g_{43}.$$

2. *The subalgebra of $\mathcal{O}(P_l)$ generated by $(t, \tilde{\tau})$ coincides with the big cell superring $\mathcal{O}(U_{12})$ as defined in (11). It is given by the projective localization of $\mathcal{O}(\mathrm{Gr})$ with respect to d_{12}.*

3. *There is a well defined coaction* $\tilde{\Delta}$ *of* $\mathcal{O}(P_l)$ *on* $\mathcal{O}(U_{12})$ *induced by the coproduct in* $\mathcal{O}(P_l)$*,*
$$\tilde{\Delta} : \mathcal{O}(U_{12}) \xrightarrow{\ \tilde{\Delta}\ } \mathcal{O}(P_l) \otimes \mathcal{O}(U_{12}),$$
which explicitly takes the form:
$$\begin{aligned}\tilde{\Delta} t_{ij} &= t_{ij} \otimes 1 + y_{ia} S(x)_{bj} \otimes t_{ab} + y_i \eta_a S(x)_{bj} \otimes \tilde{\tau}_{jb},\\ \tilde{\Delta} \tilde{\tau}_j &= (d \otimes 1)(\tau_a \otimes 1 + \xi_b \otimes t_{ba} + 1 \otimes \tilde{\tau}_a)(S(x)_{aj} \otimes 1),\end{aligned}$$
The reader should notice right away that this is the dual to the expression (12).

We now turn to the quantum setting. In order to keep our notation minimal, we use the same letters as in the classical case to denote the generators of the quantum big cell and the quantum supergroups.

Let $\mathcal{O}(P_{l,q})$ be the superalgebra:
$$\mathcal{O}(P_{l,q}) := \mathcal{O}(\mathrm{GL}_q(4|1))/\mathcal{I}_q,$$
where $\mathcal{I}_q$ is the (two-sided) ideal in $\mathcal{O}(\mathrm{GL}_q(4|1))$ generated by
$$g_{1j}, g_{2j}, \quad \text{for} \quad j = 3, 4 \quad \text{and} \quad \gamma_{15}, \gamma_{25}. \tag{17}$$
This is the Hopf superalgebra of the lower parabolic subgroup, again with comultiplication the one naturally inherited from $\mathcal{O}(\mathrm{GL}_q(4|1))$.

As in the classical case, it is convenient to change coordinates exactly in the same way (see (14)), this time, however, paying extra attention to the order in which we take the variables. We can write the new coordinates for $\mathcal{O}(P_{l,q})$ explicitly:
$$x = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}, \qquad t = \begin{pmatrix} -q^{-1} D_{23} D_{12}^{-1} & D_{13} D_{12}^{-1} \\ -q^{-1} D_{24} D_{12}^{-1} & D_{14} D_{12}^{-1} \end{pmatrix}$$
$$y = \begin{pmatrix} g_{33} & g_{34} \\ g_{43} & g_{44} \end{pmatrix}, \qquad d = g_{55},$$
$$\tilde{\tau} = \begin{pmatrix} -q^{-1} D_{25} D_{12}^{-1} & D_{15} D_{12}^{-1} \end{pmatrix}, \qquad \xi = \begin{pmatrix} g_{55}^{-1} \gamma_{53} & g_{55}^{-1} \gamma_{54} \end{pmatrix}$$
$$\eta = y^{-1} \begin{pmatrix} \gamma_{35} \\ \gamma_{45} \end{pmatrix} = (D_{34}^{34})^{-1} \begin{pmatrix} g_{44} & -q^{-1} g_{34} \\ -q g_{43} & g_{33} \end{pmatrix} = \begin{pmatrix} -q^{-1} D_{34}^{34\,-1} D_{34}^{45} \\ D_{34}^{34\,-1} D_{34}^{35} \end{pmatrix}$$
It is not hard to see that $\mathcal{O}(P_{l,q})$ is also generated by x, y, d, η, ξ and $\tilde{\tau}$.

Remark 7.2. The *quantum Poincaré supergroup times dilations* is the quotient of $\mathcal{O}(P_{l,q})$ by the ideal $\xi = 0$. In fact as one can readily check with a simple calculation, if $\mathcal{O}(P_o)$ denotes the function algebra of the super (unquantized) Poincaré groups times dilations, we have that

$$\left(\mathcal{O}(P_{l,q}) \,/\, (\xi)\right) / (q-1) \cong \mathcal{O}(P_o).$$

One can also easily check that (ξ) is a Hopf ideal, so the comultiplication goes to the quotient. The quantum Poincaré supergroup times dilations is then generated by the images in the quotient of x, y, d, η and $\tilde{\tau}$. In matrix form, one has

$$\begin{pmatrix} x & 0 & 0 \\ tx & y & y\eta \\ \tilde{\tau}x & 0 & d \end{pmatrix}.$$

Explicitly in these coordinates its presentation is given as follows:

$$\mathcal{O}(P_{l,q}) \,/\, (\xi) = \mathbb{C}_q < t, x, y, \eta, \tau > / I_{P_o,q},$$

where $I_{P_o,q}$ is the ideal generated by the following relations. The indeterminates x and y behave respectively as quantum (even) matrices, that is, their entries are subject to the relations 6.1. In other words we have for x (and similarly for y):

$$x_{11}x_{12} = q^{-1}x_{12}x_{11}, \quad x_{11}x_{21} = q^{-1}x_{21}x_{11}, \quad x_{21}x_{22} = q^{-1}x_{22}x_{21}$$
$$x_{12}x_{22} = q^{-1}x_{22}x_{12}, \quad x_{12}x_{21} = x_{21}x_{12}, \quad x_{11}x_{22} - x_{22}x_{11} = (q^{-1} - q)x_{12}x_{21}$$

Moreover the entries in x and y commute with each other. x and t, $\tilde{\tau}$ commute in the following way. Let $i = 1, 2$, $j = 3, 4$.

$$x_{1i}t_{32} = t_{32}x_{1i}, \qquad x_{1i}t_{42} = t_{42}x_{1i}, \qquad x_{1i}\tilde{\tau}_{52} = \tilde{\tau}_{52}x_{1i}$$
$$x_{1i}t_{31} = q^{-1}t_{31}x_{1i},\, x_{1i}t_{41} = q^{-1}t_{41}x_{1i},\, x_{1i}\tilde{\tau}_{51} = q^{-1}\tilde{\tau}_{51}x_{1i}$$

$$x_{2i}t_{j2} = q^{-1}t_{j2}x_{2i} + q(q^{-1} - q)x_{1i}t_{j1}$$
$$x_{2i}\tilde{\tau}_{52} = q^{-1}\tilde{\tau}_{52}x_{2i} + q(q^{-1} - q)x_{1i}\tilde{\tau}_{51}$$

x commutes with η and d. y, t and $\tilde{\tau}$ satisfy similar relations as x, t and τ that we leave to the reader as an exercise (the rows are exchanged with the columns). y and η commute as follows. Let $j = 3, 4$.

$$y_{j3}\eta_{35} = q^{-1}\eta_{35}y_{33}, \qquad\qquad y_{j3}\eta_{45} = \eta_{45}y_{j3}$$
$$y_{j4}\eta_{35} = \eta_{35}y_{j4}, \quad y_{j4}\eta_{45} = q^{-1}\eta_{45}y_{j4} - (q^{-1} - q)q^{-1}y_{j3}\eta_{35}$$

y and d commute. The commutation among t and $\tilde{\tau}$ are expressed in Proposition 7.4. t and η commute as follows:

$$t_{3i}\eta_{35} = q^{-1}\eta_{35}t_{3i},\, t_{3i}\eta_{45} = q^{-1}\eta_{45}t_{3i}$$
$$t_{4i}\eta_{35} = q^{-1}\eta_{35}t_{4i},\, t_{4i}\eta_{45} = q^{-1}\eta_{45}t_{4i}$$

t, τ and d satisfy the following relations. Let $i = 1, 2$.

$$\begin{aligned} dt_{3i} &= t_{3i}d + (q^{-1} - q)\tilde{\tau}_{5i}[y_{33}\eta_{35} + y_{34}\eta_{45}] \\ dt_{4i} &= t_{4i}d + (q^{-1} - q)\tilde{\tau}_{5i}[y_{43}\eta_{35} + y_{44}\eta_{45}] \end{aligned}$$

$$\tilde{\tau}_{5j}d = d\tilde{\tau}_{5j}$$

$\tilde{\tau}$ and η commute with each other, while finally

$$\eta_{j5}d = q^{-1}d\eta_{j5}.$$

In analogy with the classical (non quantum) supersetting, we give the following definition.

Definition 7.3. We define the *quantum big cell* $\mathcal{O}_q(U_{12})$ as the subring of $\mathcal{O}(P_{l,q})$ generated by t and $\tilde{\tau}$.

We compute now the quantum commutation relations among the generators of the quantum big cell $\mathcal{O}_q(U_{12})$, which is our chiral Minkowski superspace, and see that the quantum big cell admits a well defined coaction of the quantum supergroup $\mathcal{O}(P_{l,q})$.

Proposition 7.4. *The quantum big cell superring $\mathcal{O}_q(U_{12})$ has the following presentation:*

$$\mathcal{O}_q(U_{12}) := \mathbb{C}_q\langle t_{ij}, \tilde{\tau}_{5j}\rangle \,/\, I_U, \qquad 3 \le i \le 4,\ j = 1, 2$$

where I_U is the ideal generated by the relations:

$$\begin{gathered} t_{i1}t_{i2} = q\, t_{i2}t_{i1}, \qquad t_{3j}t_{4j} = q^{-1}\, t_{4j}t_{3j}, \qquad 1 \le j \le 2, \quad 3 \le i \le 4 \\ t_{31}t_{42} = t_{42}t_{31}, \qquad t_{32}t_{41} = t_{41}t_{32} + (q^{-1} - q)t_{42}t_{31}, \\ \tilde{\tau}_{51}\tilde{\tau}_{52} = -q^{-1}\tilde{\tau}_{52}\tilde{\tau}_{51}, \qquad t_{ij}\tilde{\tau}_{5j} = q^{-1}\tilde{\tau}_{5j}t_{ij}, \qquad 1 \le j \le 2 \\ t_{i1}\tilde{\tau}_{52} = \tilde{\tau}_{52}t_{i1}, \qquad t_{i2}\tilde{\tau}_{51} = \tilde{\tau}_{51}t_{i2} + (q^{-1} - q)t_{i1}\tilde{\tau}_{52}. \end{gathered}$$

As in the classical setting we have the following proposition.

Proposition 7.5. *The quantum big cell $\mathcal{O}_q(U_{12})$ admits a coaction of $\mathcal{O}(P_{l,q})$ obtained by restricting suitably the comultiplication in $\mathcal{O}(P_{l,q})$. In other words we have a well defined morphism:*

$$\tilde{\Delta} : \mathcal{O}_q(U_{12}) \longrightarrow \mathcal{O}(P_{l,q}) \otimes \mathcal{O}_q(U_{12})$$

satisfying the coaction properties and give explicitly by: (see Proposition 7.1),

$$\tilde{\Delta}t_{ij} = t_{ij} \otimes 1 + y_{ia}S(x)_{bj} \otimes t_{ab} + y_i\eta_a S(x)_{bj} \otimes \tilde{\tau}_{jb},$$

$$\tilde{\Delta}\tilde{\tau}_j = (d \otimes 1)(\tau_a \otimes 1 + \xi_b \otimes t_{ba} + 1 \otimes \tilde{\tau}_a)(S(x)_{aj} \otimes 1) \qquad .$$

by choosing as before generators x, y, t, d, τ, η, ξ for $\mathcal{O}(P_{l,q})$ and t, $\tilde{\tau}$ for $\mathcal{O}_q(U_{12})$ with $d\tau = \tilde{\tau}x$.

Furthermore, this coaction goes down to a well defined coaction for the quantization of the super Poincaré group (see Remark 7.2).

To compare with other deformations of the Minkowski space, we write here the even part of $\mathcal{O}_q(U_{12})$ in terms of the more familiar generators

$$t = x^\mu \sigma_\mu = \begin{pmatrix} x^0 + x^3 & x^1 - i x^2 \\ x^1 + i x^2 & x^0 + x^3 \end{pmatrix}.$$

The commutation relations of the generators x^μ are then [4]

$$\begin{aligned}
x^0 x^1 &= \frac{2}{q^{-1}+q} x^1 x^0 + i \frac{q^{-1}-q}{q^{-1}+q} x^0 x^2, \\
x^0 x^2 &= \frac{2}{q^{-1}+q} x^2 x^0 - i \frac{q^{-1}-q}{q^{-1}+q} x^0 x^1, \\
x^0 x^3 &= x^3 x^0, \\
x^1 x^2 &= \frac{i(q^{-1}+q)}{2} \left(-(x^0)^2 + (x^3)^2 + x^3 x^0 - x^0 x^3 \right), \\
x^1 x^3 &= \frac{2}{q^{-1}+q} x^3 x^1 - i \frac{q^{-1}-q}{q^{-1}+q} x^2 x^3, \\
x^2 x^3 &= \frac{2}{q^{-1}+q} x^3 x^2 + i \frac{q^{-1}-q}{q^{-1}+q} x^1 x^3.
\end{aligned}$$

8 Chiral Superfields in Minkowski Superspace

In this section we wish to motivate the importance of the chiral conformal superspace and its quantum deformation in physics. We introduce chiral superfields in Minkowski superspace as they are used in physics. We start by introducing the complexified Minkowski space: the chiral superfields are a sub superalgebra of the coordinate superalgebra of Minkowski space. They can also be seen as the coordinate superalgebra of the chiral Minkowski superspace, which is complex.

8.1 Definitions

We consider the complexified Minkowski space $\mathbb{C}^4$. The $N = 1$ *scalar superfields* on the complexified Minkowski space are elements of the commutative superalgebra

$$\mathcal{O}(\mathbb{C}^{4|4}) \equiv C^\infty(\mathbb{C}^4) \otimes \Lambda[\theta^1, \theta^2, \bar\theta^1, \bar\theta^2], \tag{18}$$

where $\Lambda[\theta^1, \theta^2, \bar{\theta}^1, \bar{\theta}^2]$ is the Grassmann (or exterior) algebra generated by the odd variables $\theta^1, \theta^2, \bar{\theta}^1, \bar{\theta}^2$.

We will denote the coordinates (or generators) of the superspace as

$$\begin{aligned} &x^\mu, && \mu = 0, 1, 2, 3 \quad \text{(even coordinates)}, \\ &\theta^\alpha, \bar{\theta}^{\dot\alpha}, && \alpha, \dot\alpha = 1, 2 \quad \text{(odd coordinates)}, \end{aligned}$$

and a superfield, in terms of its *field components*, as

$$\begin{aligned} \Psi(x, \theta, \bar\theta) = {} & \psi_0(x) + \psi_\alpha(x)\theta^\alpha + \psi'_{\dot\alpha}(x)\bar\theta^{\dot\alpha} + \psi_{\alpha\beta}(x)\theta^\alpha\theta^\beta + \psi_{\alpha\dot\beta}(x)\theta^\alpha\bar\theta^{\dot\beta} \\ & + \psi'_{\dot\alpha\dot\beta}(x)\bar\theta^{\dot\alpha}\bar\theta^{\dot\beta} + \psi_{\alpha\beta\dot\gamma}(x)\theta^\alpha\theta^\beta\bar\theta^{\dot\gamma} + \psi'_{\alpha\dot\beta\dot\gamma}(x)\theta^\alpha\bar\theta^{\dot\beta}\bar\theta^{\dot\gamma} \\ & + \psi_{\alpha\beta\dot\gamma\dot\delta}(x)\theta^\alpha\theta^\beta\bar\theta^{\dot\gamma}\bar\theta^{\dot\delta}. \end{aligned}$$

8.1.1 Action of the Lorentz Group SO(1,3)

There is an action of the double covering of the complexified Lorentz group, $\mathrm{Spin}(1,3)^c \approx \mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C})$ over $\mathbb{C}^{4|4}$. The even coordinates x^μ transform according to the fundamental representation of $\mathrm{SO}(1,3)$ (V),

$$x^\mu \mapsto \Lambda^\mu{}_\nu x^\nu,$$

while θ and $\bar\theta$ are Weyl spinors (or half spinors). More precisely, the coordinates θ transform in one of the spinor representations, say $S^+ \approx (1/2, 0)$ and $\bar\theta$ transform in the opposite chirality representation, $S^- \approx (0, 1/2)$,

$$\theta^\alpha \mapsto S^\alpha{}_\beta \theta^\beta, \qquad \bar\theta^{\dot\alpha} \mapsto \tilde{S}^{\dot\alpha}{}_{\dot\beta}\theta^{\dot\beta}.$$

The scalar superfields are invariant under the action of the Lorentz group,

$$\Psi(x, \theta, \bar\theta) = (R\Psi)(\Lambda^{-1}x, S^{-1}\theta, \tilde{S}^{-1}\bar\theta),$$

where $R\Psi$ is the superfield obtained by transforming the field components

$$R\psi_0(x) = \psi_0(x), \quad R\psi_\alpha(x) = S_\alpha{}^\beta \psi_\beta(x), \quad \dots$$

The hermitian matrices

$$\sigma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

define a Spin(1, 3)-morphism

$$\begin{array}{ccc} S^+ \otimes S^- & \longrightarrow & V \\ s^\alpha \otimes t^{\dot\alpha} & \longrightarrow & s^\alpha \sigma^\mu_{\alpha\dot\alpha} t^{\dot\alpha}. \end{array}$$

8.1.2 Derivations

A *left derivation* of degree $m = 0, 1$ of a super algebra $\mathcal{A}$ is a linear map $D^L : \mathcal{A} \mapsto \mathcal{A}$ such that

$$D^L(\Psi \cdot \Phi) = D^L(\Psi) \cdot \Phi + (-1)^{mp_\Psi} \Psi \cdot D^L(\Phi).$$

Graded left derivations span a $\mathbb{Z}_2$-graded vector space (or *supervector space*).

In general, linear maps over a supervector space are also a $\mathbb{Z}_2$-graded vector space. A map has degree 0 if it preserves the parity and degree 1 if it changes the parity. For the case of derivations of a commutative superalgebra, an even derivation has degree 0 as a linear map and an odd derivation has degree 1 as a linear map.

In the same way one defines *right derivations*,

$$D^R(\Psi \cdot \Phi) = (-1)^{mp_\Phi} D^R(\Psi) \cdot \Phi + \Psi \cdot D^R(\Phi).$$

Notice that derivations of degree zero are both, right and left derivations. Moreover, given a left derivation D^L of degree m one can define a right derivation D^R also of degree m in the following way

$$D^R\Psi = (-1)^{m(p_\Psi+1)} D^L\Psi. \tag{19}$$

Let us now focus on the commutative superalgebra $\mathcal{O}(\mathbb{C}^{4|4})$. We define the standard left derivations

$$\partial^L_\alpha \Psi = \psi_\alpha + 2\psi_{\alpha\beta}\theta^\beta + \psi_{\alpha\dot\beta}\bar\theta^{\dot\beta} + 2\psi_{\alpha\beta\dot\gamma}\theta^\beta\bar\theta^{\dot\gamma} + \psi'_{\alpha\dot\beta\dot\gamma}\bar\theta^{\dot\beta}\bar\theta^{\dot\gamma} + 2\psi_{\alpha\beta\dot\gamma\dot\delta}\theta^\beta\bar\theta^{\dot\gamma}\bar\theta^{\dot\delta},$$

$$\partial^L_{\dot\alpha} \Psi = \psi'_{\dot\alpha} - \psi_{\beta\dot\alpha}\theta^\beta + 2\psi'_{\dot\alpha\dot\beta}\bar\theta^{\dot\beta} + \psi_{\gamma\beta\dot\alpha}\theta^\gamma\theta^\beta - 2\psi'_{\beta\dot\alpha\dot\gamma}\theta^\beta\bar\theta^{\dot\gamma} + 2\psi_{\gamma\beta\dot\alpha\dot\delta}\theta^\gamma\theta^\beta\bar\theta^{\dot\delta}.$$

Also, using (19) one can define $\partial^R_\alpha, \partial^R_{\dot\alpha}$.

We consider now the odd left derivations

$$Q^L_\alpha = \partial^L_\alpha - \mathrm{i}\sigma^\mu_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\partial_\mu, \qquad \bar Q^L_{\dot\alpha} = -\partial^L_{\dot\alpha} + \mathrm{i}\theta^\alpha\sigma^\mu_{\alpha\dot\alpha}\partial_\mu.$$

They satisfy the anticommutation rules

$$\{Q^L_\alpha, \bar Q^L_{\dot\alpha}\} = 2\mathrm{i}\sigma^\mu_{\alpha\dot\alpha}\partial_\mu, \qquad \{Q^L_\alpha, Q^L_\beta\} = \{\bar Q^L_{\dot\alpha}, \bar Q^L_{\dot\beta}\} = 0,$$

with $\partial_\mu = \partial/\partial x^\mu$. Q^L and $\bar Q^L$ are the *supersymmetry charges* or *supercharges*. Together with

$$P^\mu = -\mathrm{i}\partial_\mu,$$

they form a Lie superalgebra, the *supertranslation algebra*, which then acts on the superspace $\mathbb{C}^{4|4}$.

Let us define another set of (left) derivations,

$$D^L_\alpha = \partial_\alpha + \mathrm{i}\sigma^\mu_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\partial_\mu, \qquad \bar{D}^L_{\dot{\alpha}} = -\partial_{\dot{\alpha}} - \mathrm{i}\theta^\alpha\sigma^\mu_{\alpha\dot{\alpha}}\partial_\mu,$$

with anticommutation rules

$$\{D^L_\alpha, \bar{D}^L_{\dot{\alpha}}\} = -2\mathrm{i}\sigma^\mu_{\alpha\dot{\alpha}}\partial_\mu, \qquad \{D^L_\alpha, D^L_\beta\} = \{\bar{D}^L_{\dot{\alpha}}, \bar{D}^L_{\dot{\beta}}\} = 0.$$

They also form a Lie superalgebra, isomorphic to the supertranslation algebra. This can be seen by taking

$$Q^L \to -D^L, \qquad \bar{Q}^L \longrightarrow \bar{D}^L.$$

It is easy to see that the supercharges anticommute with the derivations D^L and $\bar{D}^L$. For this reason, D^L and $\bar{D}^L$ are called *supersymmetric covariant derivatives* or simply *covariant derivatives*, although they are not related to any connection form.

We go now to the central definition.

Definition 8.1. A *chiral superfield* is a superfield Φ such that

$$\bar{D}^L_{\dot{\alpha}}\Phi = 0. \tag{20}$$

Because of the anticommuting properties of $D's$ and $Q's$, we have that

$$\bar{D}^L_{\dot{\alpha}}\Phi = 0 \quad \Rightarrow \quad \bar{D}^L_{\dot{\alpha}}(Q^L_\beta\Phi) = 0, \quad \bar{D}^L_{\dot{\alpha}}(\bar{Q}^L_{\dot{\beta}})\Phi = 0.$$

This means that the supertranslation algebra acts on the space of chiral superfields.

On the other hand, due to the derivation property,

$$\bar{D}^L_{\dot{\alpha}}(\Phi\Psi) = \bar{D}^L_{\dot{\alpha}}(\Phi)\Psi + (-1)^{p_\Phi}\Phi\bar{D}^L_{\dot{\alpha}}(\Psi),$$

we have that the product of two chiral superfields is again a chiral superfield.

8.2 Shifted Coordinates

One can solve the constraint (20). Notice that the quantities

$$y^\mu = x^\mu + \mathrm{i}\theta^\alpha\sigma^\mu_{\alpha\dot{\alpha}}\theta^{\dot{\alpha}}, \qquad \theta^\alpha \tag{21}$$

satisfy

$$\bar{D}^L_{\dot{\alpha}}y^\mu = 0, \qquad \bar{D}^L_{\dot{\alpha}}\theta^\alpha = 0.$$

Using the derivation property, any superfield of the form

$$\Phi(y^\mu, \theta), \qquad \text{satisfies} \quad \bar{D}^L_{\dot\alpha}\Phi = 0$$

and so it is a chiral superfield. This is the general solution of (20).

We can make the change of coordinates

$$x^\mu,\ \theta^\alpha,\ \bar\theta^{\dot\alpha} \ \longrightarrow\ y^\mu = x^\mu + i\theta^\alpha\sigma^\mu_{\alpha\dot\alpha}\bar\theta^{\dot\alpha},\ \theta^\alpha,\ \bar\theta^{\dot\alpha}.$$

A superfield may be expressed in both coordinate systems

$$\Phi(x, \theta, \bar\theta) = \Phi'(y, \theta, \bar\theta).$$

The covariant derivatives and supersymmetry charges take the form

$$D^L_\alpha\Phi' = \frac{\partial^L\Phi'}{\partial\theta^\alpha} + 2i\sigma^\mu_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\frac{\partial^L\Phi'}{\partial y^\mu} \qquad \bar{D}^L_{\dot\alpha}\Phi' = -\frac{\partial^L\Phi'}{\partial\bar\theta^{\dot\alpha}},$$
$$\bar{Q}^L_{\dot\alpha}\Phi' = -\frac{\partial^L\Phi'}{\partial\bar\theta^{\dot\alpha}} + 2i\theta^\alpha\sigma^\mu_{\alpha\dot\alpha}\frac{\partial^L\Phi'}{\partial y^\mu} \qquad Q^L_\alpha\Phi' = \frac{\partial^L\Phi'}{\partial\theta^\alpha}.$$

In the new coordinate system the chirality condition is simply

$$\frac{\partial^L\Phi'}{\partial\bar\theta^{\dot\alpha}} = 0,$$

so it is similar to a holomorphicity condition on the θ's.

This shows that chiral scalar superfields are elements of the commutative superalgebra $\mathcal{O}(\mathbb{C}^{4|2}) = \mathbb{C}^\infty(\mathbb{C}^4)\otimes\Lambda[\theta^1, \theta^2]$. In the previous sections we realized this superspace as the big cell inside the chiral conformal superspace, which is the Grassmannian of $2|0$-subspaces of $\mathbb{C}^{4|1}$.

The complete (non chiral) conformal superspace is in fact the flag space of $2|0$-subspaces inside $2|1$-subspaces of $\mathbb{C}^{4|1}$. On this supervariety one can put a reality condition, and the real Minkowski space is the big cell inside the flag. It is instructive to compare (21) with the incidence relation for the big cell of the flag manifold in (12) of [18]. We can then be convinced that the Grassmannian that we use to describe chiral superfields is inside the (complex) flag.

There are supersymmetric theories in physics (like Wess-Zumino models, or super Yang-Mills) that include in the formulation chiral superfields. In previous approaches it has been difficult to formulate them on non commutative superspaces (with non trivial commutation relations of the odd coordinates). The reason was that the covariant derivatives are not anymore derivations of the noncommutative superspace, and the chiral superfields do not form a superalgebra [9, 10]. Some proposals to solve these problems include the partial (explicit) breaking of supersymmetry [10, 26]. In our approach to quantization of superspace, the quantum chiral ring appears in a natural way, thus making possible the formulation of supersymmetric theories in non commutative superspaces. Also, the super variety and the supergroup acting on it become non commutative, the group law is not changed, so the physical symmetry principle remains intact. This is a virtue of the deformation based on quantum matrix groups.

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On the Construction of Chevalley Supergroups

R. Fioresi and F. Gavarini

Abstract We give a description of the construction of Chevalley supergroups, providing some explanatory examples. We avoid the discussion of the $A(1,1)$, $P(3)$ and $Q(n)$ cases, for which our construction holds, but the exposition becomes more complicated. We shall not in general provide complete proofs for our statements, instead we will make an effort to convey the key ideas underlying our construction. A fully detailed account of our work is scheduled to appear in R. Fioresi, F. Gavarini, *Chevalley Supergroups*, preprint arXiv:0808.0785 Memoirs of the AMS (2008) (to be published).

1 Introduction

The notion of Chevalley group, introduced by Chevalley in 1955, provided a unified combinatorial construction of all simple algebraic groups over a generic field k. The consequences of Chevalley's work were many and have had tremendous impact in the following decades. His construction was motivated by issues linked to the problem of the classification of semisimple algebraic groups: he provided an existence theorem for such groups, essentially exhibiting an example of simple group for each of the predicted possibility. In the course of this discussion, he discovered new examples of finite simple groups, which had escaped to the group theorists up to then. Later on, in the framework of a modern treatment of algebraic

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geometry, his work was instrumental to show that all simple algebraic groups are algebraic schemes over $\mathbb{Z}$ and to study arithmetic questions over arbitrary fields.

We may say that we have similar motivations: we want a unified approach to describe all algebraic supergroups, which have Lie superalgebras of classical type and we also want to give new examples of supergroups, over arbitrary fields. For instance, our discussion enables us to provide an explicit construction of algebraic supergroups associated with the exceptional and the strange Lie superalgebras. To our knowledge these supergroups have not been examined before, though an approach in the differential setting can be very well carried through the language of *super Harish–Chandra pairs*. In such approach a supergroup is understood as a pair $(G_0, \mathfrak{g})$, consisting of an ordinary group G_0 and a super Lie algebra $\mathfrak{g}$, with even part $\mathfrak{g}_0 = \mathrm{Lie}(G_0)$, together with some natural compatibility conditions involving the adjoint action of the group G_0 on $\mathfrak{g}$. It is clear that in positive characteristic this method shows severe limitations.

In the present work we outline the construction of the Chevalley supergroups associated with Lie superalgebras of classical type. We shall not present complete proofs for our statements, they will appear in [6], however we shall concentrate on the key ideas and examples that will help to understand our construction.

In our statements, we shall leave out the strange Lie superalgebra $Q(n)$ and some low dimensional cases, which can be treated very well with the same method, with minor modifications, but present extra difficulties that make our construction and notation opaque.

Essentially, we are going to follow Chevalley's recipe and push it as far as we can, before resorting to more sophisticated algebraic geometry techniques, when the supergeometric nature of our objects forces us to do so.

We start with a complex Lie superalgebra of classical type $\mathfrak{g}$, together with a fixed Cartan subalgebra $\mathfrak{h}$, and we define the *Chevalley basis* of $\mathfrak{g}$. This is an homogeneous basis of $\mathfrak{g}$, as super vector space, whose elements have the brackets expressed as a linear combination of the basis elements with just *integral* coefficients. Consequently they give us an integral form of $\mathfrak{g}$, that we call $\mathfrak{g}_{\mathbb{Z}}$ the *Chevalley Lie superalgebra* associated with $\mathfrak{g}$ and $\mathfrak{h}$. Such integral form gives raise to the Kostant integral form $K_{\mathbb{Z}}(\mathfrak{g})$ of the universal enveloping superalgebra $U(\mathfrak{g})$ of $\mathfrak{g}$. $K_{\mathbb{Z}}(\mathfrak{g})$ is free over $\mathbb{Z}$ with basis given by the ordered monomials in the divided powers of the root vectors and the binomial coefficients in the generators of $\mathfrak{h}$ in the Chevalley basis: $X^m/m!$, $\binom{H_i}{n}$, $\alpha \in \Delta$ (root system) and $m, n \in \mathbb{N}$.

Next, we look at a faithful rational representation of $\mathfrak{g}$ in a finite dimensional complex vector space V. Inside V we can find an *integral lattice* M which is invariant under the action of $K_{\mathbb{Z}}(\mathfrak{g})$ and its stabilizer $\mathfrak{g}_V$ in $\mathfrak{g}$ defines an integral form of $\mathfrak{g}$. In complete analogy with Chevalley, for an arbitrary field k, we can give the following key definitions:

$$V_k := k \otimes_{\mathbb{Z}} M, \qquad \mathfrak{g}_k := k \otimes_{\mathbb{Z}} \mathfrak{g}_V, \qquad U_k := k \otimes_{\mathbb{Z}} K_{\mathbb{Z}}(\mathfrak{g}).$$

We could even take k to be a commutative ring, however for the scope of the present work and to stress the analogy with Chevalley's construction, we prefer the restrictive hypothesis of k to be a field.

This is the point where our construction departs dramatically from Chevalley's one. In fact, starting from the faithful representation V_k of $\mathfrak{g}_k$, Chevalley defines the Chevalley group G_V as generated by the exponentials $exp(tX_\alpha) := 1 + tX_\alpha + (t^2/2)X_\alpha^2 + \dots$, for $t \in k$ and X_α the root vector corresponding to the root α in the Chevalley basis. Such an expression makes sense since the X_α's act as nilpotent elements. If we were to repeat without changes this construction in the super setting, we shall find only ordinary groups over k associated with the Lie algebra $\mathfrak{g}_0$, the even part of $\mathfrak{g}$. This is because over a field, we cannot see any supergeometric behaviour; the only thing we can recapture is the underlying classical object. For this reason, we need to go beyond Chevalley's construction and build our supergroups as *functors*.

We define $\mathbf{G}$ the *Chevalley supergroup* associated with $\mathfrak{g}$ and the faithful representation V, as the functor $\mathbf{G} : (\text{salg}) \longrightarrow (\text{sets})$, with $\mathbf{G}(A)$ the subgroup of $\mathrm{GL}(A \otimes V_k)$ generated by $\mathbf{G}_0(A)$ and the elements $1 + \theta_\beta X_\beta$, for $\beta \in \Delta_1$. In other words we have:

$$\mathbf{G}_V(A) = \langle \mathbf{G}_0(A), 1 + \theta_\beta X_\beta \rangle \subset \mathrm{GL}(A \otimes V_k), \qquad A \in (\text{salg}), \qquad \theta_\beta \in A_1$$

where (salg) and (sets) are the categories of commutative superalgebras and sets respectively and (as always) we use X_β to denote also the image of the root vector X_β in the chosen faithful representation V_k. $\mathbf{G}_0$ is the functor of points of the (reductive) algebraic supergroup associated to $\mathfrak{g}_0$ and the representation V_k.

This is a somehow natural generalization of what Chevalley does in his original construction: he provides the k-points of the algebraic group scheme constructed starting from a complex semisimple Lie algebra and a faithful representation, for all the fields k, while we give the A-points of the supergroup scheme for any commutative k-superalgebra A.

Once this definition is properly established, we need to show that $\mathbf{G}$ is the functor of points of an algebraic supergroup, in other words, that it is representable. This is the price to pay when we employ the language of the functor of points: it is much easier to define geometric objects, however we need to prove representability in order to speak properly of supergroup schemes. As customary, we use the same letter to denote both the superscheme and its functor of points.

We shall obtain the representability of $\mathbf{G}$ by showing that

$$\mathbf{G} \cong \mathbf{G}_0 \times \mathbf{A}^{0|N},$$

where $\mathbf{A}^{0|N}$ is the functor of points of an affine superspace of dimension $0|N$. Once this isomorphism is established the representability follows at once, since both $\mathbf{G}_0$ and $\mathbf{A}^{0|N}$ are representable, i.e., they are the functors of points of superschemes, hence their product is.

The next question we examine is how much our construction depends on the chosen representation. In complete analogy to Chevalley approach, we show that if we have two representations V and V', with weight lattices $L_V \subset L_{V'}$, then there is a surjective morphism $\mathbf{G}_{V'} \longrightarrow \mathbf{G}_V$, with kernel in the center of $\mathbf{G}_{V'}$. This implies right away that our construction depends only on the weight lattice of the chosen representation V and in particular it shows that it is independent from the choice of the lattice M inside V.

This paper is organized as follows.

In Sect. 2 we review quickly some facts of algebraic supergeometry and the theory of Lie superalgebras. For more details see [1,2,7,12,13].

In Sects. 3 and 4 we go to the heart of the construction of Chevalley's supergroups going through all the steps detailed above.

Finally in Sect. 5 we provide some insight into our construction with some examples and observations.

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2 Supergeometry in the Algebraic Setting

Let k be the ground field.

A *super vector space* V is a vector space with $\mathbb{Z}_2$ grading: $V = V_0 \oplus V_1$, the elements in V_0 are called *even* and the elements in V_1 are called *odd*. Hence we have a function p called the *parity* defined only on homogeneous elements. A *superalgebra* A is a super vector space with multiplication preserving parity; A is *commutative* if $xy = (-1)^{p(x)p(y)} yx$ for all x, y homogeneous elements in A. All superalgebras are assumed to be commutative unless otherwise specified and their category is denoted with (salg).

Definition 2.1. A *superspace* $S = \big(|S|, \mathcal{O}_S\big)$ is a topological space $|S|$ endowed with a sheaf of commutative superalgebras $\mathcal{O}_S$ such that the stalk $\mathcal{O}_{S,x}$ is a local superalgebra for all $x \in |S|$.

A *morphism* $\phi : S \longrightarrow T$ of superspaces consists of a pair $\phi = \big(|\phi|, \phi^*\big)$, where $\phi : |S| \longrightarrow |T|$ is a morphism of topological spaces and $\phi^* : \mathcal{O}_T \longrightarrow \phi_* \mathcal{O}_S$ is a sheaf morphism such that $\phi_x^*\big(\mathbf{m}_{|\phi|(x)}\big) = \mathbf{m}_x$ where $\mathbf{m}_{|\phi|(x)}$ and $\mathbf{m}_x$ are the maximal ideals in the stalks $\mathcal{O}_{T,|\phi|(x)}$ and $\mathcal{O}_{S,x}$ respectively and ϕ_x^* is the morphism induced by ϕ^* on the stalks and $\phi_* \mathcal{O}_S$ is the sheaf on $|T|$ defined as $\phi_* \mathcal{O}_S(V) := \mathcal{O}_S(\phi^{-1}(V))$.

The next example of superspace turns out to be extremely important, as $\underline{Spec}\, A$, for a commutative superalgebra A, is the local model for superschemes, very much in the same way as $\underline{Spec}\, A_0$ is the local model for ordinary schemes for A_0 a commutative algebra.

Example 2.2. Let $A \in$ (salg) and let $\mathcal{O}_{A_0}$ be the structural sheaf of the ordinary scheme $\underline{Spec}\,(A_0) = \big(Spec(A_0), \mathcal{O}_{A_0}\big)$, where $Spec(A_0)$ denotes the prime spectrum of the commutative ring A_0. Now A is a module over A_0, so we have a sheaf $\mathcal{O}_A$ of $\mathcal{O}_{A_0}$-modules over $Spec(A_0)$ with stalk A_p, the p-localization of the A_0-module A, at the prime $p \in Spec(A_0)$.

$\underline{Spec}\,(A) := \big(Spec(A_0), \mathcal{O}_A\big)$ is a superspace, as one can readily check.

Given $f : A \longrightarrow B$ a superalgebra morphism, one can define $\underline{Spec}\, f : \underline{Spec}\, B \longrightarrow \underline{Spec}\, A$ in a natural way, very similarly to the ordinary setting, thus making $\underline{Spec}$ a functor $\underline{Spec} : (\text{salg}) \longrightarrow (\text{sets})$, where (salg) is the category of superalgebras and (sets) the category of sets (see [1] chap. 5 or [5] chap. 1, for more details).

Definition 2.3. Given a superspace X, we say it is an *affine superscheme* if it is isomorphic to $\underline{Spec}\,(A)$ for some commutative superalgebra A. We say that X is a *superscheme* if it is locally isomorphic to an affine superscheme.

Example 2.4. The *affine superspace* $\mathbb{A}_{\Bbbk}^{p|q}$, also denoted $\Bbbk^{p|q}$, is defined as

$$\mathbb{A}_{\Bbbk}^{p|q} := \Bbbk[x_1 \dots x_p] \otimes \wedge(\xi_1 \dots \xi_q),$$

where $\wedge(\xi_1 \dots \xi_q)$ is the exterior algebra generated by the indeterminates $\xi_1, \dots, \xi_q$.

The formalism of the functor of points that we borrow from algebraic geometry allows us to handle supergeometric objects that would be otherwise very difficult to treat using just the superschemes language.

Definition 2.5. Let X be a superscheme. Its *functor of points* is the functor defined on the objects as

$$h_X : (\text{salg}) \longrightarrow (\text{sets})\ , \qquad h_X(A) := \text{Hom}\big(\underline{Spec}\,(A), X\big)$$

and on the arrows as $h_X(f)(\phi) := \phi \circ \underline{Spec}\,(f)$.

Since the category of affine superschemes is equivalent to the category of commutative superalgebras ([1, 5]) we have that, when X is affine, its functor of points is equivalently defined as follows:

$$h_X(A) = \text{Hom}\big(\mathcal{O}(X), A\big), \qquad h_X(f)(\phi) = f \circ \phi,$$

where $\mathcal{O}(X)$ is the superalgebra of global sections of the structure sheaf on X.

If h_X is group valued, i.e., it is valued in the category (groups) of groups, we say that X is a *supergroup*. When X is affine, this is equivalent to the fact that $\mathcal{O}(X)$ is a (commutative) *Hopf superalgebra*. More in general, we call *supergroup functor* any functor $G : (\text{salg}) \longrightarrow (\text{groups})$.

Any representable supergroup functor is the same as an affine supergroup. Following a customary abuse of notation, we shall then use the same letter to denote both the superscheme X and its functor of points h_X.

As always, Yoneda's lemma plays a crucial role, allowing us to use natural transformations between the functors of points of superschemes and the morphisms of the superschemes themselves interchangeably.

Proposition 2.6. *(Yoneda's Lemma) Let $\mathcal{C}$ be a category, and let R, S be two objects in $\mathcal{C}$. Consider the two functors $h_R, h_S : \mathcal{C} \longrightarrow$ (sets) defined on the objects by $h_R(A) := \mathrm{Hom}\big(R, A\big)$, $h_S(A) := \mathrm{Hom}\big(S, A\big)$ and on the arrows by $h_R(f)(\phi) := f \circ \phi$, $h_S(f)(\psi) := f \circ \psi$.*

Then there exists a one-to-one correspondence between the natural transformations and the morphisms

$$\big\{h_R \longrightarrow h_S\big\} \longleftrightarrow \mathrm{Hom}(R, S)\,.$$

This has an immediate corollary.

Corollary 2.7. *Two affine superschemes are isomorphic if and only if their functors of points are isomorphic.*

The next examples turn out to be very important in the sequel.

Example 2.8. (1) *Super vector spaces as superschemes*. Let V be a super vector space. For any superalgebra A we define $V(A) := (A \otimes V)_0 = A_0 \otimes V_0 \oplus A_1 \otimes V_1$. This is a representable functor in the category of superalgebras, whose representing object is $\mathrm{Pol}(V)$, the algebra of polynomial functions on V. Hence any super vector space can be equivalently viewed as an affine superscheme. If $V = k^{m|n}$, that is $V_0 \cong k^p$ and $V_1 \cong k^q$, V is the functor of points of the affine superspace described in Example 2.4.

(2) $\mathrm{GL}(V)$ *as an algebraic supergroup*. Let V be a finite dimensional super vector space of dimension $p|q$. For any superalgebra A, let $\mathrm{GL}(V)(A) := \mathrm{GL}\big(V(A)\big)$ be the set of isomorphisms $V(A) \longrightarrow V(A)$. If we fix a homogeneous basis for V, we see that $V \cong k^{p|q}$. In this case, we also denote $\mathrm{GL}(V)$ with $\mathrm{GL}(p|q)$. Now, $\mathrm{GL}(p|q)(A)$ is the group of invertible matrices of size $(p+q)$ with diagonal block entries in A_0 and off-diagonal block entries in A_1. It is well known that the functor $\mathrm{GL}(V)$ is representable; see (e.g.), [18], Chap. 3, for further details.

We end our minireview of supergeometry by introducing the concept of *Lie superalgebra* and stating the Kac's classification theorem for Lie superalgebras of classical type.

We assume now $char(k) \neq 2, 3$.

Definition 2.9. Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a super vector space. We say that $\mathfrak{g}$ is a Lie superalgebra, if we have a bracket $[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$ which satisfies the following properties (as usual for all x, $y \in \mathfrak{g}$ homogeneous):

- *Anti-symmetry:*

$$[x, y] + (-1)^{p(x)p(y)}[y, x] = 0$$

- *Jacobi identity:*

$$(-1)^{p(x)p(z)}[x,[y,z]] + (-1)^{p(y)p(x)}[y,[z,x]] + (-1)^{p(z)p(y)}[z,[x,y]] = 0.$$

The standard example is $\mathrm{End}(V)$ the endomorphisms of the super vector space V, with $\mathrm{End}(V)_0$ the endomorphisms preserving parity and $\mathrm{End}(V)_1$ the endomorphisms reversing parity. The bracket is defined as:

$$[X, Y] := XY - (-1)^{|X||Y|}YX.$$

If $V := k^{p|q} = k^p \oplus k^q$, with $V_0 := k^p$ and $V_1 := k^q$, we write $\mathrm{End}\big(k^{p|q}\big) := \mathrm{End}(V)$ or $\mathfrak{gl}(p\,|q) := \mathrm{End}(V)$. In this case $\mathrm{End}(V)_0$ consists of diagonal block matrices, while $\mathrm{End}(V)_1$ consists of off diagonal block matrices all with entries in k.

In $\mathrm{End}(k^{p|q})$ we can define the *supertrace* as follows:

$$\mathrm{str}\begin{pmatrix} A & B \\ C & D \end{pmatrix} := \mathrm{tr}(A) - \mathrm{tr}(D).$$

There is an important class of Lie superalgebras, namely the *simple Lie superalgebras* that have been classified by Kac (see [11]).

Definition 2.10. A non abelian Lie superalgebra $\mathfrak{g}$ is *simple* if it has no nontrivial homogeneous ideals. A Lie superalgebra $\mathfrak{g}$ is called of *classical type* if it is simple and $\mathfrak{g}_1$ is completely reducible as a $\mathfrak{g}_0$-module. Furthermore, $\mathfrak{g}$ is said to be *basic* if, in addition, it admits a non-degenerate, invariant bilinear form.

We now give a list of Lie superalgebras of classical type, sending the reader to [11, 14] for the details.

Example 2.11. (1) $\mathfrak{sl}(m|\,n)$. Define $\mathfrak{sl}(m|n)$ as the subset of $\mathfrak{gl}(m|n)$ consisting of all matrices with supertrace zero. This is a Lie subalgebra of $\mathfrak{gl}(m|n)$, with the following $\mathbb{Z}_2$-grading:

$$\mathfrak{sl}(m|\,n)_0 = \mathfrak{sl}(m) \oplus \mathfrak{sl}(n) \oplus \mathfrak{gl}(1)\ , \qquad \mathfrak{sl}(m|\,n)_1 = f_m \otimes f_n' \oplus f_m' \otimes f_n,$$

where f_r is the defining representation of $\mathfrak{sl}(r)$ and f_r' is its dual (for any r). When $m \neq n$ this is a Lie superalgebra of *classical type*.

(2) $\mathfrak{osp}(p\,|q)$. Let ϕ denote the standard nondegenerate consistent supersymmetric bilinear form in $V := k^{p|q}$. This means that V_0 and V_1 are mutually orthogonal and the restriction of ϕ to V_0 is a symmetric and to V_1 a skewsymmetric form (in particular, $q = 2n$ is even). We define in $\mathfrak{gl}(p\,|q)$ the subalgebra $\mathfrak{osp}(p\,|q) := \mathfrak{osp}(p,|q)_0 \oplus \mathfrak{osp}(p\,|q)_1$ by setting, for all $s \in \{0,1\}$,

$$\mathfrak{osp}(p\,|q)_s := \Big\{\ell \in \mathfrak{gl}(p\,|q) \,\Big|\, \phi\big(\ell(x), y\big) = -(-1)^{s\,|x|}\phi\big(x, \ell(y)\big)\,\forall x, y \in k^{p|q}\Big\}$$

and we call $\mathfrak{osp}(p\,|q)$ the *orthosymplectic* Lie superalgebra. Note that $\mathfrak{osp}(0|\,q)$ is the symplectic Lie algebra $\mathfrak{sp}(q)$, while $\mathfrak{osp}(p|\,0)$ is the orthogonal Lie algebra $\mathfrak{so}(p)$.

Again, all the $\mathfrak{osp}(p|\,q)$'s are Lie superalgebras *of classical type*. Moreover, if $m, n \geq 2$, we have:

$$\mathfrak{osp}(2m+1|2n)_0 = \mathfrak{so}(2m+1) \oplus \mathfrak{sp}(2n), \quad \mathfrak{osp}(2m|2n)_0 = \mathfrak{so}(2m) \oplus \mathfrak{sp}(2n)$$
$$\mathfrak{osp}(p\,|2n)_1 = f_p \otimes f_{2n} \quad \forall\, p > 2, \qquad \mathfrak{osp}(2|2n)_1 = f_{2n}^{\oplus 2}$$

We now introduce some terminology in order to be able to state the classification theorem.

Definition 2.12. Define the following Lie superalgebras:

(1) $A(m,n) := \mathfrak{sl}(m+1|\,n+1)$, $A(n,n) := \mathfrak{sl}(n+1|\,n+1) \Big/ k I_{2n}$, $\forall m \neq n$;
(2) $B(m,n) := \mathfrak{osp}(2m+1|\,2n)$, $\quad \forall\ m \geq 0, n \geq 1$;
(3) $C(n) := \mathfrak{osp}(2|\,2n-2)$, $\quad$ for all $n \geq 2$;
(4) $D(m,n) := \mathfrak{osp}(2m|\,2n)$, $\quad$ for all $m \geq 2, n \geq 1$;
(5) $P(n) := \left\{ \begin{pmatrix} A & B \\ C & -A^t \end{pmatrix} \in \mathfrak{gl}(n+1|\,n+1) \,\middle|\, \begin{matrix} A \in \mathfrak{sl}(n+1) \\ B^t = B, C^t = -C \end{matrix} \right\}$
(6) $Q(n) := \left\{ \begin{pmatrix} A & B \\ B & A \end{pmatrix} \in \mathfrak{gl}(n+1|\,n+1) \,\middle|\, B \in \mathfrak{sl}(n+1) \right\} \Big/ k I_{2(n+1)}$.

Theorem 2.13. *Let k be an algebraically closed field of characteristic zero. Then the Lie superalgebras of classical type are either isomorphic to a simple Lie algebra or to one of the following Lie superalgebras:*

$$A(m,n),\ m \geq n \geq 0, m+n>0; \quad B(m,n),\ m \geq 0, n \geq 1; \quad C(n),\ n \geq 3$$
$$D(m,n),\ m \geq 2, n \geq 1; \qquad P(n),\ n \geq 2; \quad Q(n),\ n \geq 2$$
$$F(4); \qquad G(3); \qquad D(2,1;a),\ a \in k \setminus \{0,-1\}.$$

For the definition of $F(4)$, $G(3)$, $D(2,1;a)$, and for the proof, we refer to [11].

3 Chevalley Basis and Kostant Integral Form

The main ingredient to construct a Chevalley supergroup starting from a complex Lie superalgebra $\mathfrak{g}$ of classical type is the *Chevalley basis*. This is an homogeneous basis for $\mathfrak{g}$, consisting of elements that have brackets expressed as *integral* combinations of the basis elements. Consequently a Chevalley basis determines what is called the *Chevalley Lie algebra* $\mathfrak{g}_{\mathbb{Z}}$ of $\mathfrak{g}$, which is an integral form of $\mathfrak{g}$.

Assume $\mathfrak{g}$ to be a Lie superalgebra of classical type different from $A(1,1)$, $P(3)$, $Q(n)$. We want to leave out these pathological cases for which our construction holds, but with a more complicated set of statements and proofs. We invite the reader to go to [6] for a complete and unified treatment of all of these cases. We also consider $D(2,1;a)$ for only integral values for the coefficient a.

Let us fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, that is a maximal solvable Lie subalgebra of $\mathfrak{g}$. The adjoint action of $\mathfrak{h}$ on $\mathfrak{g}$ gives the usual *root space* decomposition of $\mathfrak{g}$:

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha,$$

where

$$\mathfrak{g}_\alpha := \big\{ x \in \mathfrak{g} \big| \, [h,x] = \alpha(h)x, \forall h \in \mathfrak{h} \big\}$$

and $\Delta = \Delta_0 \cup \Delta_1$ with

$$\Delta_0 := \big\{ \alpha \in \mathfrak{h}^* \setminus \{0\} \big| \, \mathfrak{g}_\alpha \cap \mathfrak{g}_0 \neq \{0\} \big\} = \{ \textit{even roots} \text{ of } \mathfrak{g} \}.$$

$$\Delta_1 := \big\{ \alpha \in \mathfrak{h}^* \big| \, \mathfrak{g}_\alpha \cap \mathfrak{g}_1 \neq \{0\} \big\} = \{ \textit{odd roots} \text{ of } \mathfrak{g} \}.$$

As in the ordinary setting we shall call Δ *root system* and the $\mathfrak{g}_\alpha$'s the *root spaces*. If we fix a simple system (see [11] for its definition) the root system splits into positive and negative roots, exactly as in the ordinary setting:

$$\Delta = \Delta^+ \coprod \Delta^-, \qquad \Delta_0 = \Delta_0^+ \coprod \Delta_0^-, \qquad \Delta_1 = \Delta_1^+ \coprod \Delta_1^-.$$

Observation 3.1. 1. Notice that the definition allows $\Delta_0 \cap \Delta_1 \neq \emptyset$, as in fact happens for $\mathfrak{g} = Q(n)$, where the roots are simultaneously all even and odd and the root spaces have all dimension $1|1$.
2. Δ_0 is the root system of the reductive Lie algebra $\mathfrak{g}_0$, while Δ_1 is the set of weights of the representation of $\mathfrak{g}_0$ in $\mathfrak{g}_1$.

If $\mathfrak{g}$ is not of type $P(n)$ or $Q(n)$, there is an even non-degenerate, invariant bilinear form on $\mathfrak{g}$, whose restriction to $\mathfrak{h}$ is in turn an invariant bilinear form on $\mathfrak{h}$. On the other hand, if $\mathfrak{g}$ is of type $P(n)$ or $Q(n)$, then such a form on $\mathfrak{h}$ exists because $\mathfrak{g}_0$ is simple (of type A_n), though it does not come by restricting an invariant form on the whole $\mathfrak{g}$.

If (x,y) denotes such form, we can identify $\mathfrak{h}^*$ with $\mathfrak{h}$, via $H'_\alpha \mapsto (H'_\alpha,)$. We can then transfer $(,)$ to $\mathfrak{h}^*$ in the natural way: $(\alpha,\beta) = (H'_\alpha, H'_\beta)$. Define $H_\alpha := 2\frac{H'_\alpha}{(H'_\alpha, H'_\alpha)}$ when the denominator is non zero. When $(H'_\alpha, H'_\alpha) = 0$ such renormalization can be found in detail in [9]. We call H_α the *coroot* associated with α.

We summarize in the next proposition all the relevant properties of the root system, sending the reader to [11, 14, 15] for the complete story.

Proposition 3.2. *Let* $\mathfrak{g}$ *be a Lie superalgebra of* classical type*, as above, that is* $\mathfrak{g} \neq A(1,1)$*,* $P(3)$*,* $Q(n)$*, and let* $n \in \mathbb{N}$*.*

(a) $\Delta_0 \cap \Delta_1 = \emptyset$*,*
(b) $-\Delta_0 = \Delta_0$*,* $-\Delta_1 \subseteq \Delta_1$*. If* $\mathfrak{g} \neq P(n)$*, then* $-\Delta_1 = \Delta_1$*.*
(c) Let $\mathfrak{g} \neq P(2)$*, and* $\alpha, \beta \in \Delta$*,* $\alpha = c\beta$*, with* $c \in \mathbb{K} \setminus \{0\}$*. Then*

$$\alpha, \beta \in \Delta_r \quad (r = 0, 1) \Rightarrow c = \pm 1 \;, \qquad \alpha \in \Delta_r, \beta \in \Delta_s, r \neq s \Rightarrow c = \pm 2 \;.$$

(d) $\dim_{\mathbb{K}}(\mathfrak{g}_\alpha) = 1$ *for each* $\alpha \in \Delta$*.*

We are finally ready to give the definition of *Chevalley basis.*

Definition 3.3. We define a *Chevalley basis* of a Lie superalgebra $\mathfrak{g}$ as above any homogeneous basis

$$B = \{H_1 \dots H_l, \; X_\alpha, \; \alpha \in \Delta\}$$

of $\mathfrak{g}$ as complex vector space, with the following requirements:

(a) $\{H_1, \dots, H_\ell\}$ is a basis of the complex vector space $\mathfrak{h}$. Moreover

$$\mathfrak{h}_{\mathbb{Z}} := Span_{\mathbb{Z}}\{H_1, \dots, H_\ell\} = Span_{\mathbb{Z}}\{H_\alpha \,\big|\, \alpha \in \Delta\}.$$

(b) $[H_i, H_j] = 0, \quad [H_i, X_\alpha] = \alpha(H_i) X_\alpha, \qquad \forall i, j \in \{1, \dots, \ell\}, \alpha \in \Delta;$
(c) $[X_\alpha, X_{-\alpha}] = \sigma_\alpha H_\alpha \qquad \forall\, \alpha \in \Delta \cap (-\Delta)$ with H_α as after 3.2, and $\sigma_\alpha := -1$ if $\alpha \in \Delta_1^-$, $\sigma_\alpha := 1$ otherwise;
(d) $[X_\alpha, X_\beta] = c_{\alpha,\beta} X_{\alpha+\beta} \qquad \forall\, \alpha, \beta \in \Delta : \alpha \neq -\beta$, with $c_{\alpha,\beta} \in \mathbb{Z}$.

More precisely,

- If $(\alpha, \alpha) \neq 0$, or $(\beta, \beta) \neq 0$, $c_{\alpha,\beta} = \pm(r+1)$ or (only if $\mathfrak{g} = P(n)$), $c_{\alpha,\beta} = \pm(r+2)$, where r is the length of the α-string through β.
- If $(\alpha, \alpha) \neq 0 = (\beta, \beta) = 0$, $c_{\alpha,\beta} = \beta(\alpha)$.

Notice that this definition clearly extends to direct sums of finitely many of the $\mathfrak{g}$'s under the above hypotheses.

Definition 3.4. If B is a Chevalley basis of a Lie superalgebra $\mathfrak{g}$ as above, we set

$$\mathfrak{g}_{\mathbb{Z}} := \mathrm{span}_{\mathbb{Z}}\{B\} \subset \mathfrak{g}$$

and we call it the *Chevalley superalgebra* of $\mathfrak{g}$.

Observe that $\mathfrak{g}_{\mathbb{Z}}$ is a Lie superalgebra over $\mathbb{Z}$ inside $\mathfrak{g}$. Since a Chevalley basis B of $\mathfrak{g}$ is unique up to a choice of a sign for each root vector and the choice of the H_i's we have that $\mathfrak{g}_{\mathbb{Z}}$ is independent of the choice of B (but of course depends on the choice of $\mathfrak{h}$ the Cartan subalgebra).

The existence of a Chevalley basis for the families A, B, C, D is a known result; for example an almost explicit Chevalley basis for types B, C and D is in [16], while for A is a straightforward calculation. More in general, an abstract existence result, with a uniform proof, is given in [9] for all *basic* types. In [6] we provide an existence theorem for *all* cases giving both a case by case analysis, comprehending all Lie superalgebras of classical type and a uniform proof, that however leaves out the $P(n)$ case.

We now turn to another important ingredient for our construction: the *Kostant* $\mathbb{Z}$*-form*.

Definition 3.5. Let $\mathfrak{g}$ be a complex Lie superalgebra of classical type over $\mathbb{C}$ and let $B = \{H_1 \dots H_\ell, X_\alpha, \alpha \in \Delta\}$ be a Chevalley basis. We define the *Kostant superalgebra*, $K_{\mathbb{Z}}(\mathfrak{g})$, the $\mathbb{Z}$-superalgebra inside $U(\mathfrak{g})$, generated by

$$X_\alpha^{(n)}, \quad X_\gamma, \quad \binom{H_i}{n} \qquad \forall\, \alpha \in \Delta_0,\ n \in \mathbb{N},\ \gamma \in \Delta_1,\ i = 1, \dots, \ell,$$

where

$$X_\alpha^{(n)} := X_\alpha^n / n! \qquad \binom{H}{n} := \frac{H(H-1)\cdots(H-n+1)}{n!} \in U(\mathfrak{g})$$

for all H in $\mathfrak{h}$. These are called respectively *divided powers* and *binomial coefficients*.

Notice that we can remove all the binomial coefficients corresponding to coroots H_i's relative to even roots and still generate the superalgebra $K_{\mathbb{Z}}(\mathfrak{g})$. In fact a classical result (see [17] p. 9) tells us that the even divided powers generate all such binomial coefficients. Unfortunately we cannot obtain the odd coroot binomial coefficients and this is because the X_γ, for $\gamma \in \Delta_1$ appear only in degree one.

As in the ordinary setting (see [17] p. 7) we have a PBW type of result for $K_{\mathbb{Z}}(\mathfrak{g})$ providing us with a $\mathbb{Z}$-basis for the Kostant superalgebra. The proof is very similar to the ordinary setting and we send the reader to [6] for more details.

Theorem 3.6. *The Kostant superalgebra $K_{\mathbb{Z}}(\mathfrak{g})$ is a free $\mathbb{Z}$-module. For any given total order $\preceq$ of the set $\Delta \cup \{1, \dots, \ell\}$, a $\mathbb{Z}$-basis of $K_{\mathbb{Z}}(\mathfrak{g})$ is the set of ordered "PBW-like monomials," i.e., all products without repetitions: of factors of type:*

$$X_\alpha^{(n_\alpha)}, \quad \binom{H_i}{n_i}, \quad X_\gamma$$

$\alpha \in \Delta_0$, $i \in \{1, \dots, \ell\}$, $\gamma \in \Delta_1$ and n_α, $n_i \in \mathbb{N}$ – taken in the right order with respect to $\preceq$.

4 Chevalley Supergroups

This section is devoted to the construction of *Chevalley supergroups* and to prove they are supergroup schemes.

Let $\mathfrak{g}$ be a complex Lie superalgebra of classical type, $B = \{H_1 \dots H_\ell,\ X_\alpha,\ \alpha \in \Delta\}$ a Chevalley basis of $\mathfrak{g}$ and $K_{\mathbb{Z}}(\mathfrak{g})$ its Kostant superalgebra. We start with a finite dimensional complex representation V of $\mathfrak{g}$ and the notion of *admissible lattice* in V.

Definition 4.1. Let V be a complex finite dimensional representation for $\mathfrak{g}$. We say that V is *rational* if $\mathfrak{h}_{\mathbb{Z}} := Span_{\mathbb{Z}}\big(H_1, \dots, H_\ell\big)$ acts diagonally on V with integral eigenvalues.

Notice that this condition is automatic for semisimple Lie algebras, while it is actually restrictive for some Lie superalgebras as the next example shows.

Example 4.2. Let $\mathfrak{g} = \mathfrak{sl}(m|n)$ and $\mathfrak{h}$ the diagonal matrices, so that $\mathfrak{h}_{\mathbb{Z}} = Span_{\mathbb{Z}}\big\{E_{m,m} + E_{m+1,m+1}, E_{ii} - E_{i+1,i+1}, i \neq m\big\}$, where E_{ij} denotes an elementary matrix. Let V be a representation with highest weight $\Lambda = \lambda_1\epsilon_1 + \dots + \lambda_m\epsilon_m + \mu_1\delta_1 + \dots \mu_n\delta_n$, where $\epsilon_i : \mathfrak{h} \longrightarrow \mathbb{C}$, $\epsilon_i(E_{jj}) = \delta_{ij}$ and similarly for δ_k. We have that (see [11]) V is finite dimensional if and only if $\lambda_i - \lambda_{i+1}$, $\mu_j - \mu_{j+1} \in \mathbb{Z}^+$, $i = 1 \dots m-1$, $j = 1 \dots n-1$, in other words if and only if $\Lambda(H_i) \in \mathbb{Z}^+$ for $i \neq m$. There are hence no conditions on $\Lambda(H_m) = \lambda_m + \mu_1$. Consequently if we pick any (non integral) complex number for such a sum and we build the induced module, we shall obtain a finite dimensional representation for $\mathfrak{g}$ where H_m acts diagonally, with a complex, non integral eigenvalue.

Let us now fix V a finite dimensional rational complex semisimple representation of $\mathfrak{g}$.

We say that an integral lattice M in V is *admissible* if it is $K_{\mathbb{Z}}(\mathfrak{g})$-stable.

As in the ordinary setting any rational complex finite dimensional semisimple representation of $\mathfrak{g}$ admits an admissible lattice M, which is generated by the highest weight vector v and it is the sum of its weight components M_μ. In particular if V is simple we have:

$$M = K_{\mathbb{Z}}(\mathfrak{g}) \cdot v, \qquad M = \oplus M_\mu .$$

The next proposition establishes the existence of an integral form of $\mathfrak{g}$ stabilizing the admissible lattice M inside the representation V. We send the reader to [6] Sect. 5 for the proof.

Theorem 4.3. *Let* $\mathfrak{g}$, V *and* M *as above. Define:*

$$\mathfrak{g}_V = \big\{ X \in \mathfrak{g} \,\big|\, X.M \subseteq M \big\}.$$

If V *is faithful, then*

$$\mathfrak{g}_V = \mathfrak{h}_V \bigoplus \left(\oplus_{\alpha\in\Delta} \mathbb{Z} X_\alpha \right), \qquad \mathfrak{h}_V := \big\{ H \in \mathfrak{h} \big|\ \mu(H) \in \mathbb{Z}, \forall \mu \in \Lambda \big\},$$

where Λ *is the set of all weights of* V. *In particular,* $\mathfrak{g}_V$ *is a lattice in* $\mathfrak{g}$, *and it is independent of the choice of the admissible lattice* M *(but not of course of* V*).*

We end this discussion by saying that $\mathfrak{g}_{\mathbb{Z}}$ corresponds to the adjoint representation of $\mathfrak{g}$ and that in general all the integral forms $\mathfrak{g}_V$ lie between the two integral forms $\mathfrak{g}_{roots}$ and $\mathfrak{g}_{weights}$ corresponding respectively to the root and the fundamental weight representations:

$$\mathfrak{g}_{roots} \subset \mathfrak{g}_V \subset \mathfrak{g}_{weights}.$$

We now start the construction of the Chevalley supergroup associated with the data $\mathfrak{g}$ and V.

Let k be a generic field.

Definition 4.1 and Theorem 4.3 allow us to move from the complex field to a generic field quite easily as the next definition shows.

Definition 4.4. Let $\mathfrak{g}$ be a complex Lie superalgebra of classical type (as usual $\mathfrak{g} \neq A(1,1)$, $P(2)$, $Q(n)$). Let V be a faithful rational complex representation of $\mathfrak{g}$, M an admissible lattice in V.

Define:

$$\mathfrak{g}_k := k \otimes_{\mathbb{Z}} \mathfrak{g}_V, \qquad V_k := k \otimes_{\mathbb{Z}} M, \qquad U_k(\mathfrak{g}) := k \otimes_{\mathbb{Z}} K_{\mathbb{Z}}(\mathfrak{g}).$$

We are now ready to define the super equivalent of the one-parameter subgroups in the classical theory. As we shall see, homogeneous one-parameter subgroups appear in the super setting with three different dimensions: $1|0$, $0|1$ and $1|1$. In order to keep the analogy with the ordinary setting, we nevertheless have preferred to keep the terminology *one-parameter subgroup*, though in the supersetting the term “one” can be misleading.

Definition 4.5. Let X_α, X_β, X_γ be root vectors in the Chevalley basis, $\alpha \in \Delta_0$, β, $\gamma \in \Delta_1$, with $[X_\beta, X_\beta] = 0$, $[X_\gamma, X_\gamma] \neq 0$.

We define *homogeneous one-parameter subgroups* the following supergroup functors from the categories of superalgebras to the category of sets:

$$\begin{aligned}
x_\alpha(A) &:= \left\{ \exp\bigl(tX_\alpha\bigr) \big| t \in A_0 \right\} \\
&= \left\{ \bigl(1 + tX_\alpha + t^2 \tfrac{X_\alpha^2}{2} + \cdots\bigr) \big| t \in A_0 \right\} \subset \mathrm{GL}(V_k)(A), \\
x_\beta(A) &:= \left\{ \exp\bigl(\vartheta X_\beta\bigr) \big| \vartheta \in A_1 \right\} = \left\{ \bigl(1 + \vartheta X_\beta\bigr) \big| \vartheta \in A_1 \right\} \subset \mathrm{GL}(V_k)(A), \\
x_\gamma(A) &:= \left\{ \exp\bigl(\vartheta X_\gamma + tX_\gamma^2\bigr) \big| \vartheta \in A_1, t \in A_0 \right\} \\
&= \left\{ \bigl(1 + \vartheta X_\gamma\bigr) \exp\bigl(tX_\gamma^2\bigr) \big| \vartheta \in A_1, t \in A_0 \right\} \subset \mathrm{GL}(V_k)(A).
\end{aligned}$$

Notice that the infinite sums reduce to finite ones since X_α and X_β act as nilpotent operators on V_k. As usual we identify a generic root vector X_α with its image under the representation of $U_k(\mathfrak{g})$ in V_k (the divided powers come at hand exactly at this point).

One can readily see that the functors x_α, x_β, x_γ are representable, hence they are algebraic supergroups in the sense of Sect. 2 and their representing Hopf superalgebras are respectively $k[x]$, $k[\xi]$ and $k[x, \xi]$, where as usual the roman letters correspond to even elements while the greek letters to odd ones. The comultiplication is coadditive except for the element x in $k[x, \xi]$: $x \mapsto 1 \otimes x +$

$x \otimes 1 + \xi \otimes \xi$. It is very clear by looking at the Hopf superalgebras representing x_α, x_β and x_γ that the superdimensions of these supergroups are respectively $1|0$, $0|1$ and $1|1$. It is not hard to see that these are all of the allowed superdimensions for homogeneous one-parameter subgroups (see [6] for more details).

As an abuse of notation we shall sometimes write for $t \in A_0$ and $\theta \in A_1$:

$$\begin{aligned} x_\alpha(t) &:= \exp\big(tX_\alpha\big), \\ x_\beta(\theta) &:= \exp\big(\vartheta X_\beta\big) = 1 + \vartheta X_\beta \\ x_\gamma(\mathbf{t}) &:= \exp\big(\vartheta X_\gamma + tX_\gamma^2\big), \qquad \mathbf{t} = (t, \theta). \end{aligned}$$

We now turn to the definition of the generators of what classically is the *maximal torus.*

Definition 4.6. For any $\alpha \in \Delta \subseteq \mathfrak{h}^*$, let $H_\alpha \in \mathfrak{h}_\mathbb{Z}$ as in 3.2. Let $V = \oplus_\mu V_\mu$ be the splitting of V into weight spaces. As V is rational, we have $\mu(H_\alpha) \in \mathbb{Z}$ for all $\alpha \in \Delta$. Define:

$$h_\alpha(t).v := t^{\mu(H_\alpha)} v \in V_k(A) \ \forall v \in (V_k)_\mu, \mu \in \mathfrak{h}^* \quad t \in A^\times, \quad A \in (\text{salg})$$

Notice that this defines an operator $h_\alpha(t) \in \mathrm{GL}(V_k)(A)$. Hence we can define:

$$h_H(t) := \prod h_\alpha^{a_\alpha}(t) \in \mathrm{GL}(V_k)(A), \qquad H = \sum a_\alpha H_\alpha.$$

We have immediately that h_H defines a supergroup functor:

$$h_H : (\text{salg}) \longrightarrow (\text{sets}), \qquad h_H(A) := \big\{ h_H(t) \quad \big| \quad t \in A^\times \big\}$$

which is clearly representable, its Hopf superalgebra being given by $k[x, x^{-1}]$ with comultiplication $x \mapsto x \otimes x$.

We now want to define an ordinary algebraic group associated with the ordinary Lie algebra $\mathfrak{g}_0$, the even part of $\mathfrak{g}$. One must exert some care at this point, since a Chevalley basis for $\mathfrak{g}_0$ is not in general the even part or even a subset of a Chevalley basis for $\mathfrak{g}$, even if $\mathfrak{h} = \mathfrak{h}_0$, that is the Cartan subalgebras for $\mathfrak{g}_0$ and $\mathfrak{g}$ coincide. Let us illustrate these phenomenons with simple example. Let us look at $A(2,1)$. $\mathfrak{h}_\mathbb{Z} = (\mathfrak{h}_\mathbb{Z})_0 = \mathrm{Span}_\mathbb{Z}\{H_1, H_2, H_3, H_4\}$, with $H_i = E_{ii} - E_{i+1,i+1}$, $i \neq 3$, $H_3 = E_{33} + E_{44}$, where the E_{ij}'s are the elementary matrices with 1 in the (i, j)th position and zero elsewhere. We have only one odd coroot H_3. This is an even vector, that we would however miss if we were to consider just the even coroots, that is the coroots corresponding to even root spaces. These are the coroots of the roots in Δ_0 the root system associated with $A_2 \oplus A_1$ the *simple* even part of $\mathfrak{g}$, which in this case is not the same as $\mathfrak{g}_0 = A_2 \oplus A_1 \oplus \mathbb{C}$, which is *reductive*. To produce an instance of the other phenomenon is more complicated. The point is that we can have that the span of the odd coroots may contain some of the even coroots and consequently we can omit those even coroots, so that the Chevalley basis will not be a subset of the Chevalley basis of the even part. This happens for example in the $D(m,n)$ case.

It is possible to construct a reductive algebraic group $\mathbf{G}_0$ overcoming these difficulties. $\mathbf{G}_0$ will encode also the information contained in the extra odd coroot (it is in fact possible always to reduce to the case of just one odd coroot) and such construction is explained in detail in [6]. The group $\mathbf{G}_0$ is constructed following Chevalley's philosophy, but taking into account the extra odd coroot, which would be otherwise missing. On local superalgebras $\mathbf{G}_0$ is described as follows. Let $\mathbf{G}_0'$ be the ordinary algebraic group scheme associated with the semisimple part of $\mathfrak{g}$ (which could be smaller than $\mathfrak{g}_0$ as in the $A(m,n)$ case) and let $T : (\text{salg}) \longrightarrow (\text{sets})$, $T(A) = \langle h_H(A) \quad | \quad H \in \mathfrak{h}_{\mathbb{Z}} \rangle$. This is in general larger than the maximal torus T_0 in G_0, since it contains the extra odd root (though one must be aware of some exceptions as we detail in the observation below). If A is a local superalgebra we define:

$$\mathbf{G}_0(A) = \langle \mathbf{G}_0'(A), T(A) \rangle.$$

It is possible to show that this definition extends to any superalgebra A and that the functor so obtained is representable (see [6] Sect. 5).

Observation 4.7. We want to observe that there are cases in which the missing odd root can be somehow recovered without extra work. Let us look at the example of $\mathfrak{osp}(1|2)$. The roots are α, 2α and the corresponding coroots are $H_\alpha = 2H_{2\alpha}$, $H_{2\alpha}$ (the relation between coroots depends on the chosen normalization). Consequently, we have that by taking just the even coroot $H_{2\alpha}$, we can get both the coroots H_α and $H_{2\alpha}$, so in this case it is not necessary to add anything more, in other words $\mathbf{G}_0 = \mathbf{G}_0'$. Clearly this phenomenon is observed for all the superalgebras $B(m,n)$. Notice that the even coroot 2α corresponds to the adjoint representation of the even part $\mathfrak{sl}_2$ of $\mathfrak{osp}(1|2)$. This fact has consequences on the questions regarding which supergroups can be built using our method and we plan to fully explore this in a forthcoming paper.

We are finally ready for the definition of *Chevalley supergroup functor*.

Definition 4.8. Let $\mathfrak{g}$ be a complex Lie superalgebra of classical type and V a faithful rational complex representation of $\mathfrak{g}$. We call the *Chevalley supergroup*, associated to $\mathfrak{g}$ and V, the functor $\mathbf{G} : (\text{salg}) \longrightarrow (\text{grps})$ defined as:

$$\begin{aligned}\mathbf{G}(A) &:= \Big\langle \mathbf{G}_0(A), x_\beta(A) \Big| \beta \in \Delta_1 \Big\rangle \\ &= \Big\langle \mathbf{G}_0(A), 1 + \theta_\beta X_\beta \Big| \beta \in \Delta_1, \theta_\beta \in A_1 \Big\rangle \subset \mathrm{GL}\big(V_k(A)\big).\end{aligned}$$

In other words $\mathbf{G}(A)$ is the subgroup of $\mathrm{GL}\big(V_k(A)\big)$ generated by $\mathbf{G}_0(A)$ described above and the $0|1$ one-parameter subgroups $x_\beta(A)$ with $\beta \in \Delta_1$. $\mathbf{G}$ is defined on the arrows in the natural way, since $\mathbf{G}(A)$ is a subgroup of $\mathrm{GL}\big(V_k(A)\big)$.

From the classical theory (see [4] Sect. 5.7)we know that on local algebras, since $\mathbf{G}_0$ is reductive:

$$\mathbf{G}_0(A) = \Big\langle x_\alpha(A), h_i(A) \Big| \, i = 1, \ldots, \ell, \alpha \in \Delta_0 \Big\rangle, \quad h_i = h_{H_i}.$$

Consequently on local superalgebras we then have:

$$\mathbf{G}(A) = \Big\langle x_\alpha(A), h_i(A) \Big| \, i = 1, \ldots, \ell, \alpha \in \Delta \Big\rangle.$$

We call *Chevalley supergroup functor* the functor $G : (\text{salg}) \longrightarrow (\text{grps})$ defined as:

$$G(A) = \Big\langle x_\alpha(A), h_i(A) \Big| \, i = 1, \ldots, \ell, \alpha \in \Delta \Big\rangle.$$

In [6] we explore more deeply the relation between the two functors $\mathbf{G}$ and G and we show that $\mathbf{G}$ is the *sheafification* of G. This important property sheds light on our construction and it is actually needed in the key proofs, since it provides a more explicit way to handle the Chevalley supergroups. Nevertheless, given the scope of the present work, we shall not give the definition of sheafification of a functor, in order to avoid the technicalities involved, that are not adding any insight into our construction. For all the details we send the reader to the appendix in [6] and, for the ordinary setting to [19], where the sheafification of functors is fully explained and to [4] Sect. 5.7.6for its application to reductive groups.

The fact that we have defined the Chevalley supergroup $\mathbf{G}$ as a functor does not automatically imply that it is *representable*, in other words, that it is the functor of points of an algebraic supergroup scheme. This is a new question specific to the supersetting, in fact in the ordinary setting, the definition of Chevalley group is given only on fields, the group is exhibited an abstract group and only later one shows it is has an algebraic scheme structure. On the other hand in the supergeometric environment looking at superobjects on fields only, will not give us much information since the odd coordinates disappear when we look at points over a field, thus leaving us with just the underlying ordinary group. In other words $\mathbf{G}(k) = \mathbf{G}_0(k)$ for all fields k, since the θ_β's in Definition 4.8 are nilpotent.

In order to prove the representability of $\mathbf{G}$, we shall give a series of lemmas regarding G, which is more accessible than $\mathbf{G}$, since we know its generators for all $A \in (\text{salg})$. As in the ordinary setting the key to the theory are the explicit formulas for the commutators. The proof is a straightforward generalization of the corresponding proofs for the ordinary setting (see [17] Sect. 3), which we state as (1) of Lemma 4.11.

Before this, in order to properly state our results and the intermediate steps to obtain them, we need to define the following auxiliary sets.

Definition 4.9. For any $A \in (\text{salg})$, we define the subsets of $G(A)$

$$
\begin{aligned}
G_1(A) &:= \left\{ \prod_{i=1}^{n} x_{\gamma_i}(\vartheta_i) \,\middle|\, n \in \mathbb{N}, \gamma_1, \dots, \gamma_n \in \Delta_1, \vartheta_1, \dots, \vartheta_n \in A_1 \right\} \\
G_0^{\pm}(A) &:= \left\{ \prod_{i=1}^{n} x_{\alpha_i}(t_i) \,\middle|\, n \in \mathbb{N}, \alpha_1, \dots, \alpha_n \in \Delta_0^{\pm}, t_1, \dots, t_n \in A_0 \right\} \\
G_1^{\pm}(A) &:= \left\{ \prod_{i=1}^{n} x_{\gamma_i}(\vartheta_i) \,\middle|\, n \in \mathbb{N}, \gamma_1, \dots, \gamma_n \in \Delta_1^{\pm}, \vartheta_1, \dots, \vartheta_n \in A_1 \right\} \\
G^{\pm}(A) &:= \left\{ \prod_{i=1}^{n} x_{\beta_i}(\mathbf{t}_i) \,\middle|\, n \in \mathbb{N}, \beta_1, \dots, \beta_n \in \Delta^{\pm}, \mathbf{t}_1, \dots, \mathbf{t}_n \in A_0 \times A_1 \right\} \\
&= \left\langle G_0^{\pm}(A), G_1^{\pm}(A) \right\rangle
\end{aligned}
$$

Moreover, fixing any total order $\preceq$ on $\Delta_1^{\pm}$, and letting $N_{\pm} = \left|\Delta_1^{\pm}\right|$, we set

$$
G_1^{\pm,<}(A) := \left\{ \prod_{i=1}^{N_{\pm}} x_{\gamma_i}(\vartheta_i) \,\middle|\, \gamma_1 \prec \cdots \prec \gamma_{N_{\pm}} \in \Delta_1^{\pm}, \vartheta_1, \dots, \vartheta_{N_{\pm}} \in A_1 \right\}
$$

and for any total order $\preceq$ on Δ_1, and letting $N := \left|\Delta\right| = N_+ + N_-$, we set

$$
G_1^{<}(A) := \left\{ \prod_{i=1}^{N} x_{\gamma_i}(\vartheta_i) \,\middle|\, \gamma_1 \prec \cdots \prec \gamma_N \in \Delta_1, \vartheta_1, \dots, \vartheta_N \in A_1 \right\}
$$

Note that for special choices of the order, one has $G_1^{<}(A) = G_1^{-,<}(A) \cdot G_1^{+,<}(A)$ or $G_1^{<}(A) = G_1^{+,<}(A) \cdot G_1^{-,<}(A)$.

Remark 4.10. Note that $G_1(A)$, $G_0^{\pm}(A)$, $G_1^{\pm}(A)$ and $G^{\pm}(A)$ are subgroups of $G(A)$, while $G_1^{\pm,<}(A)$ and $G_1^{<}(A)$ instead are *not*, in general.

Lemma 4.11. *1. Let $\alpha, \beta \in \Delta_0$, $A \in (\mathrm{salg})$ and $t, u \in A_0$. Then there exist $c_{ij} \in \mathbb{Z}$ such that*

$$
\left(x_\alpha(t), x_\beta(u)\right) = \prod x_{i\alpha+j\beta}\left(c_{ij}\, t^i u^j\right) \in G_0(A).
$$

2. Let $\alpha \in \Delta_0$, $\gamma \in \Delta_1$, $A \in (\mathrm{salg})$ and $t \in A_0$, $\vartheta \in A_1$. Then there exist $c_s \in \mathbb{Z}$ such that

$$
\left(x_\gamma(\vartheta), x_\alpha(t)\right) = \prod_{s>0} x_{\gamma+s\alpha}\left(c_s\, t^s \vartheta\right) \in G_1(A),
$$

(the product being finite). More precisely, with $\varepsilon_k = \pm 1$ and $r \in \mathbb{Z}$,

$$
\left(1 + \vartheta X_\gamma, x_\alpha(t)\right) = \prod_{s>0}\left(1 + \prod_{k=1}^{s} \varepsilon_k \cdot \binom{s+r}{r} \cdot t^s \vartheta X_{\gamma+s\alpha}\right),
$$

where the factors in the product are taken in any order (as they do commute).

3. Let $\gamma, \delta \in \Delta_1$, $A \in (\mathrm{salg})$, $\vartheta, \eta \in A_1$. Then (notation of Definition 3.3)

$$
\left(x_\gamma(\vartheta), x_\delta(\eta)\right) = x_{\gamma+\delta}\left(-c_{\gamma,\delta}\vartheta\eta\right) = \left(1 - c_{\gamma,\delta}\vartheta\eta\, X_{\gamma+\delta}\right) \in G_0(A)
$$

if $\delta \neq -\gamma$; otherwise, for $\delta = -\gamma$, we have

$$\big(x_\gamma(\vartheta), x_{-\gamma}(\eta)\big) = \big(1 - \vartheta\,\eta H_\gamma\big) = h_\gamma\big(1 - \vartheta\,\eta\big) \in G_0(A).$$

4. *Let $\alpha, \beta \in \Delta$, $A \in (\mathrm{salg})$, $t \in U(A_0)$, $\mathbf{u} \in A_0 \times A_1 = A$. Then*

$$h_\alpha(t) x_\beta(\mathbf{u}) h_\alpha(t)^{-1} = x_\beta\big(t^{\beta(H_\alpha)}\mathbf{u}\big) \in G_{p(\beta)}(A),$$

where $p(\beta)$ denotes as usual the parity of a root β, that is $p(\beta) = 0$ if $\beta \in \Delta_0$ and $p(\beta) = 1$ if $\beta \in \Delta_1$.

We are still under the simplifying assumption $\mathfrak{g} \neq Q(n)$ hence $\Delta_0 \cap \Delta_1 = \emptyset$. We stress that our results hold for *all* Lie superalgebras of classical type, but we choose in the present work for clarity of exposition to restrict ourselves to $\mathfrak{g} \neq A(1,1), P(3), Q(n)$.

As a direct consequence of the commutation relations, we have the following proposition involving the sets we have introduced: $G^{\pm}$, etc. The proof is a simple exercise.

Theorem 4.12. *Let $A \in (\mathrm{salg})$. There exist set-theoretic factorizations*

$$G(A) = G_0(A) G_1(A) = G_1(A) G_0(A)$$

$$G^{\pm}(A) = G_0^{\pm}(A) G_1^{\pm}(A) = G_1^{\pm}(A) G_0^{\pm}(A).$$

This decomposition has a further refinement that we state down below, whose proof is harder and we send the reader to [6] Sect. 5.3 for the details.

Theorem 4.13. *For any $A \in (\mathrm{salg})$ we have*

$$G(A) = G_0(A) G_1^{<}(A) = G_1^{<}(A) G_0(A)$$

From the previous results we have that a generic $g \in G(A)$ can be factorized (once we choose a suitable ordering on the roots):

$$g = g_0 g_1^+ g_1^-, \qquad g_0 \in G_0(A), \quad g_1^{\pm} = G_1^{\pm,<}(A).$$

The next theorem gives us the key to the representability of $\mathbf{G}$, by stating the uniqueness of the above decomposition. Again for the proof see [6], Sect. 5.3.

Theorem 4.14. *Let the notation be as above. For any $A \in (\mathrm{salg})$, the group product gives the following bijection:*

$$G_0(A) \times G_1^{-,<}(A) \times G_1^{+,<}(A) \hookrightarrow\!\!\!\!\rightarrow G(A)$$

and all the similar bijections obtained by permuting the factors $G_1^{\pm,<}(A)$ and the factor $G_0(A)$.

As one can readily see, the functors $G_1^{\pm,<} : (\mathrm{salg}) \longrightarrow (\mathrm{sets})$ are representable and they are the functor of points of an odd dimensional affine super space: $G_1^{\pm,<} \cong \mathbb{A}^{0|N^\pm}$, for $N^\pm = |\Delta_1^\pm|$. Then this, together with the definition of $\mathbf{G}$ gives:

$$\mathbf{G} \cong \mathbf{G}_0 \times G_1^{-,<} \times G_1^{+,<} = \mathbf{G}_0 \times \mathbf{A}^{0|N}$$

for $N = N^+ + N^-$. Consequently $\mathbf{G}$ is representable, since it is the direct product of representable functors. We have sketched the proof of the main result of the paper:

Theorem 4.15. *The Chevalley supergroup* $\mathbf{G} : (\mathrm{salg}) \longrightarrow (\mathrm{sets})$*,*

$$\mathbf{G}(A) := \left\langle \mathbf{G}_0(A), x_\beta(A) \middle| \beta \in \Delta_1 \right\rangle$$

is representable.

The next proposition establishes how much the Chevalley supergroup scheme $\mathbf{G}$ we have built depends on the chosen representation. It turns out that two different complex $\mathfrak{g}$-representations V and V' (as in beginning of Sect. 4), with weight lattices $L_{V'} \subset L_V$ of the same complex Lie superalgebra $\mathfrak{g}$ of classical type give raise to a morphism between the corresponding Chevalley supergroups, with kernel inside the center of $\mathbf{G}$, as it happens in the ordinary setting. This is actually expected, since the kernel is related with the fundamental group, which is a topological invariant, unchanged by the supergeneralization.

Theorem 4.16. *Let* $\mathbf{G}$ *and* $\mathbf{G}'$ *be two Chevalley supergroups constructed using faithful complex representations* V *and* V' *of the same complex Lie superalgebra of classical type* $\mathfrak{g}$*. Let* L_V*,* $L_{V'}$ *be the corresponding lattices of weights. If* $L_V \supseteq L_{V'}$*, then there exists a unique morphism* $\phi : \mathbf{G} \longrightarrow \mathbf{G}'$ *such that* $\phi_A\big(1 + \vartheta X_\alpha\big) = 1 + \vartheta X'_\alpha$*, and Ker* $(\phi_A) \subseteq Z\big(\mathbf{G}(A)\big)$*, for every local algebra* A*. Moreover,* ϕ *is an isomorphism if and only if* $L_V = L_{V'}$*.*

We observe that this theorem tells us that our construction of $\mathbf{G}$ does not depend on the chosen representation V, but only on the weight lattice of V. In particular $\mathbf{G}$ is independent of the choice of an admissible lattice.

In the end we want to ask the following question: does our construction provide all the algebraic supergroups whose Lie superalgebra is of classical type? The answer to this question is positive and we plan to explore furtherly the topics in a forthcoming paper.

5 Examples and Further Topics

In this final section we want to discuss some examples and to indicate possible further developments and applications of the theory we have described.

We start by discussing how our construction can be generalized to other Lie superalgebras, provided some conditions are satisfied.

We list down below some requirements a Lie superalgebra must satisfy so that we can try to replicate our construction.

We start from a complex Lie superalgebra $\mathfrak{g} = \langle X_a \big| a \in \mathcal{A} \rangle$ generated (as Lie superalgebra) by the homogeneous elements X_a, where $a \in \mathcal{A}$ a finite set of indices, and a complex finite dimensional representation V.

We assume the following:

1. $\mathfrak{g}$ admits a basis $B \supset \{X_a\}_{a \in \mathcal{A}}$ and an integral form $\mathfrak{g}_{\mathbb{Z}} = \mathrm{span}_{\mathbb{Z}}\{B\}$ in which all the brackets are integral combinations of elements in B;
2. There exists a suitable $\mathbb{Z}$-subalgebra of $U(\mathfrak{g})$ denoted by $U_{\mathbb{Z}}(\mathfrak{g}) \subset U(\mathfrak{g})$ admitting a PBW theorem. In other words, $U_{\mathbb{Z}}(\mathfrak{g})$ is a free $\mathbb{Z}$-module with a basis consisting of suitable monomials, which form also a basis for $U(\mathfrak{g})$.
3. V contains an integral lattice M stable under $U_{\mathbb{Z}}(\mathfrak{g})$;
4. There is well defined algebraic group $\mathbf{G}_0$ over $\mathbb{Z}$, whose k-points embed into $\mathrm{GL}(k \otimes M)$ and whose corresponding Lie algebra is $\mathfrak{g}_0$. This will allow us to consider its functor of points $\mathbf{G}_0 : (\mathrm{salg}) \longrightarrow (\mathrm{sets})$.

If the requirements listed above are satisfied, then we can certainly give the same definition as in 4.8 and define the *Chevalley-type supergroup functor.*

Notice that the first part, up to Sect. 3 is devoted to prove (1)–(3) for $\mathfrak{g}$ of classical type, (4) for the classical type is discussed in [6] Sect. 4.

Definition 5.1. Let $\mathfrak{g}$ and V be as above. Define as before (compare before Definition 4.5):

$$\mathfrak{g}_k := k \otimes \mathfrak{g}_V, \qquad V_k := k \otimes M, \qquad U_k(\mathfrak{g}) := k \otimes U_{\mathbb{Z}}(\mathfrak{g}).$$

Define the *Chevalley-type supergroup functor* as the functor $\mathbf{G} : (\mathrm{salg}) \longrightarrow (\mathrm{sets})$ given on the objects by:

$$\mathbf{G}(A) = \big\langle \mathbf{G}_0(A), \quad 1 + \theta X_b, \quad X_b \quad \text{odd}, \quad \theta \in A_1 \big\rangle \subset \mathrm{GL}(V_k)(A).$$

In other words $\mathbf{G}(A)$ is the subgroup of $\mathrm{GL}(V_k)$ generated by the A-points of $\mathbf{G}_0(A)$ and the elements $1 + \theta X_b$. Again we identify an element X_b with its image in the representation V_k.

As we have already remarked after Definition 4.8, this definition does not ensure $\mathbf{G}$ to be representable, hence to be rightfully called a supergroup scheme, and in fact a key role in the proof of the representability of this functor in the case of $\mathfrak{g}$ of classical type, is played by the commutation relations between the elements generating the group $G(A)$.

Before we go to the representability issues, let us give an example of Lie superalgebra together with a class of representations, which is *not* of classical type and yet it satisfies the requirements listed above, hence it admits a Chevalley-type supergroup functor.

Example 5.2. Let us consider the Heisenberg Lie superalgebra $\mathcal{H}$, which is generated by an even generator e and by $2n$ odd generators a_i, b_i, $i = 1 \dots n$, with the only non zero brackets:

$$[a_i, b_j] = \delta_{ij} e, \qquad i, j = 1 \dots n.$$

Define the following irreducible faithful complex representation (see [11] Sect. 1.1). Let $V = \wedge(\xi_1 \dots \xi_n)$, the complex exterior algebra with generators $\xi_1 \dots \xi_n$. V is a complex representation for $\mathcal{H}$ by setting:

$$e \cdot u = \alpha u, \quad a_i \cdot u = \frac{\partial u}{\partial \xi_i}, \quad b_i \cdot u = \alpha \xi_i u, \qquad \alpha \in \mathbb{C}\{0\}.$$

Assume we take $\alpha \in \mathbb{Z}\{0\}$.

If we set $\{X_a\}_{a=1,\dots,2n} = \{a_i, b_i, i = 1 \dots n\}$ and $B = \{e, a_i, b_i, i = 1 \dots n\}$ we have immediately satisfied item (1). As for item (2), we have that:

$$\{\binom{e}{n} a_{i_1} \dots a_{i_p} b_{j_1} \dots b_{j_q}\}, \qquad 1 \leq i_1 < \dots < i_p \leq n, \quad 1 \leq j_1 < \dots < j_q$$

is a $\mathbb{Z}$-basis of

$$U_{\mathbb{Z}}(\mathcal{H}) := \left\langle \binom{e}{n}, a_i, b_j \right\rangle \subset U(\mathcal{H}).$$

V contains the following integral lattice stable under $U_{\mathbb{Z}}(\mathcal{H})$:

$$M = span_{\mathbb{Z}} \left\{ \xi_{i_1} \dots \xi_{i_m} \quad \middle| \quad 1 \leq i_1 < \dots < i_m \leq n \right\}$$

Finally certainly a_i and b_i act as nilpotent operators and consequently also item (3) is satisfied.

As for item (4), we have immediately that $\mathbf{G}_0 \cong k$ is an algebraic group, the additive group of the affine line. In the representation V_k the elements in $\mathbf{G}_0(k)$ act as follows:

$$h_e(t) \cdot u = t^\alpha u$$

hence $\mathbf{G}_0(k)$ is embedded into $\mathrm{GL}(V_k)$ as the diagonal matrices:

$$\mathbf{G}_0(k) = \begin{pmatrix} t^\alpha & 0 \dots & 0 \\ \vdots & & \vdots \\ 0 & 0 \dots & t^\alpha \end{pmatrix} \subset \mathrm{GL}(V_k).$$

Its functor of points is hence given simply by taking $t \in A_0$:

$$G_0(A) = \begin{pmatrix} t^\alpha & 0 \dots & 0 \\ \vdots & & \vdots \\ 0 & 0 \dots & t^\alpha \end{pmatrix} \subset \mathrm{GL}(V_k), \qquad t \in A_0.$$

We now go to the problem of representability of the functor $\mathbf{G}$ we have defined. Besides the technical problems involved, our proof of the representability issue for the Chevalley supergroup functor discussed in 4 relies on the following facts:

1. $\mathbf{G}(A) = \mathbf{G}_0(A) G_1^<(A)$, where $G_1^<(A) = \{(1 + \theta_{a_1} X_{a_1}) \dots (1 + \theta_{a_n} X_{a_n})\}$ where $a_i < a_{i+1}$ in an order on $\mathcal{A}$, the index set of the indices a_i.
2. The above decomposition is unique, that is $\mathbf{G}(A) = \mathbf{G}_0(A) \times G_1^<(A)$ is unique.
3. $G_1^< \cong \mathbf{A}^{0|N}$ for a suitable N.

Clearly this leads immediately to the representability of the functor $\mathbf{G}$, since it is the direct product of two representable functors.

Coming back to our example of the Heisenberg superalgebra, by a direct calculation very similar to the one in Sect. 4 one sees that the commutator:

$$(1 + \theta X_a, 1 + \eta X_b) = 1 + c\theta\eta e, \quad c \in \mathbb{Z}.$$

Notice that $1 + c\theta\eta e$ acts on $u \in V_k$ as a diagonalmatrix with entries $1 + c\theta\eta t^\alpha$. This is an element in $G_0(A)$ since $(1 + c\theta\eta t)^\alpha = 1 + c\theta\eta t^\alpha$. By repeating the reordering arguments as in [6], Sects. 5.15 and 5.16 one can show that properties (1)–(3) are satisfied, hence giving us the representability of the Chevalley-type supergroup functor for the Heisenberg Lie superalgebra. Consequently we have define the Heisenberg supergroup associated to the Heisenberg Lie superalgebra in the following way:

$$\mathbf{G}(A) = \langle \mathbf{G}_0(A), 1 + \theta_i a_i, 1 + \eta_i b_i \rangle \subset \mathrm{GL}(V)(A).$$

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Indecomposable Finite-Dimensional Representations of a Class of Lie Algebras and Lie Superalgebras

Hans Plesner Jakobsen

1 Introduction

The topic of indecomposable finite-dimensional representations of the Poincaré group was first studied in a systematic way by Paneitz [5, 6]. In these investigations only representations with one source were considered, though by duality, one representation with two sources was implicitly present.

The idea of nilpotency was mentioned indirectly in Paneitz's articles, but a more down-to-earth method was chosen there.

The results form a part of a major investigation by Paneitz and Segal into physics based on the conformal group. Induction from indecomposable representations plays an important part in this theory. See [7] and references cited therein.

The defining representation of the Poincaré group, when given as a subgroup of SU(2,2) (see below), is indecomposable. This representation was studied by the present author prior to the articles by Paneitz in connection with a study of special aspects of Dirac operators and positive energy representations of the conformal group [4].

Indecomposable representations in theoretical physics have also been used in a major way in a study by Cassinelli, Olivieri, Truini, and Varadarajan [1]. The main object is the Poincaré group. In an appendix to the article, the indecomposable representations of the 2-dimensional Euclidean group are considered, and many results are obtained. This group can also be studied by our method, but we will not pursue this here. One small complication is that the circle group is abelian.

In the article at hand, we wish to sketch how, by utilizing nilpotency to its fullest extent while using methods from the theory of universal enveloping algebras, a

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complete description of the indecomposable representations may be reached. In practice, the combinatorics is still formidable, though.

It turns out that the method applies to both a class of ordinary Lie algebras and to a similar class of Lie superalgebras.

Besides some examples, due to the level of complexity we will only describe a few precise results. One of these is a complete classification of which ideals can occur in the enveloping algebra of the translation subgroup of the Poincaré group. Equivalently, this determines all indecomposable representations with a single, 1-dimensional source. Another result is the construction of an infinite-dimensional family of inequivalent representations already in dimension 12. This is much lower than the 24-dimensional representations which were thought to be the lowest possible. The complexity increases considerably, though yet in a manageable fashion, in the supersymmetric setting. Besides a few examples, only a subclass of ideals of the enveloping algebra of the super Poincaré algebra will be determined in the present article.

2 Finite-Dimensional Indecomposable Representations of the Poincaré Group

We are here only interested in what happens on the level of the Lie algebra. Equivalently, we consider a double covering of the Poincaré group given by

$$P = \left\{ \begin{pmatrix} a & 0 \\ k & {a^\star}^{-1} \end{pmatrix} \mid a \in SL(2,\mathbb{R})\ ; k^\star = a^\star k a^{-1} \in gl(2,\mathbb{C}) \right\}.$$

This is a subgroup of $SU(2,2)$ when the latter is defined by the hermitian form

$$\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}.$$

For our purposes, we may equivalently even consider the group

$$P = \left\{ \begin{pmatrix} u & 0 \\ z & v \end{pmatrix} \mid u, v \in SL(2,\mathbb{C})\ ; z \in gl(2,\mathbb{C}) \right\}.$$

Let G_0 denote the group $SL(2,\mathbb{C}) \times SL(2,\mathbb{C})$. Thus,

$$G_0 = \left\{ \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} \mid u, v \in SL(2,\mathbb{C}) \right\}.$$

For what we shall be doing, it does not matter if we work with this group, its Lie algebra, or with $SU(2) \times SU(2)$.

It is important to consider the abelian Lie algebra

$$\mathfrak{p}^- = \left\{ \begin{pmatrix} 0 & 0 \\ z & 0 \end{pmatrix} \mid z \in gl(2, \mathbb{C}) \right\}.$$

along with its enveloping algebra

$$\mathcal{U}(\mathfrak{p}^-) = \mathcal{S}(\mathfrak{p}^-) = \mathbb{C}[z_1, z_2, z_3, z_4].$$

The last equality comes from writing the 2×2 matrix z above as

$$z = \begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix}.$$

In passing we make the important observation that the polynomial $\det z = z_1 z_4 - z_2 z_3$ is invariant in the sense that $\det uzv = \det z$ for all $u, v \in SL(2, \mathbb{C})$.

We make the basic assumption that all representations and equivalences are over $\mathbb{C}$. This has the powerful consequence that the abelian algebra $\mathfrak{p}^-$ acts nilpotently.

The general setting is the following: We consider a reductive Lie algebra $\mathfrak{g}_0$ in which the elements of the abelian ideals are given by semi-simple elements and a nilpotent Lie algebra $\mathfrak{n}$ together with a Lie algebra homomorphism α of $\mathfrak{g}_0$ into the derivations $Der(\mathfrak{n})$ of $\mathfrak{n}$. This gives rise, in the usual fashion, to the semi-direct product

$$\mathfrak{g} = \mathfrak{g}_0 \times_\alpha \mathfrak{n}.$$

In this situation a well-known result from algebra [2,3] can easily be generalized to include the $\mathfrak{g}_0$ invariance.

Recall that a flag in a vector space V is a sequence of subspaces $0 = W_0 \subsetneq W_1 \subsetneq \cdots \subsetneq W_r = V$.

Theorem 2.1. *Suppose given a representation of* $\mathfrak{g}$ *in some finite-dimensional vector space* V*. Then there is a flag of subspaces such that* $\mathfrak{n}$ *maps* W_i *into* W_{i-1} *for* $i = 1, \ldots, r$ *and such that each* W_i *is invariant and completely decomposable under* $\mathfrak{g}_0$.

We associate a graph to the indecomposable representation V as follows: The nodes are the $\mathfrak{g}_0$ irreducible representations that occur. Two nodes, labeled by irreducibles V_1 and V_2, are connected by an arrow pointing from V_1 to V_2 if $V_2 \subset \mathfrak{n}^- \cdot V_1$ inside V. If there are multiplicities in the isotypic components the situation becomes more complicated. If the multiplicity at the node i is n_i one can simply place n_i black dots at the node. They can be placed in a stack or in a circle. In case $n_i > 1, n_j > 1$ there may also be a number $a_{i,j} > 1$ of arrows from i to j, and this needs also to be indicated. The simplest way is just to attach the n_i to each node and to attach the $a_{i,j}$ to the arrow from i to j with the further stipulation that only numbers strictly greater that 1 need to be given. We shall not pursue such details here; see, however, the third of the simple examples below.

The theorem above has the immediate consequence that there are no closed paths in this graph.

Remark 2.2. The assumption of finite dimension is essential here. Already on the level of the polynomial algebra $\mathbb{C}[z_1, z_2, z_3, z_4]$, if one quotients out by the ideal generated by an inhomogeneous polynomial in $\det z$ there will be closed loops, but the resulting module is infinite-dimensional. If one insists on finite-dimensionality, one must have all homogeneous polynomials in the quotient after a certain degree. Thus $\det z^n$ for some n must be in the ideal. This precludes an inhomogeneous polynomial in $\det z$ since in $\mathbb{C}[T]$ (where $T = \det z$), any inhomogeneous polynomial $p(T)$ is relatively prime to T^n for any $n = 0, 1, 2, \ldots$.

Given any such directed graph, any node with arrows only pointing out is as usual called a source. The opposite is called a sink. There is a simple way whereby one may reverse all arrows, thereby turning sources into sinks, and vice versa: Replace V by its dual module V'.

Simple situations:

$$\begin{tikzcd} & V_2 & & & V_2' \arrow[dl] \\ \text{One generator } V_1 \arrow[ur] \arrow[dr] & & & \text{(or its dual...) } V_1' & \\ & V_3 & & & V_3' \arrow[ul] \end{tikzcd}$$

This leads to decomposable representations if the targets (sinks) are equal (respectively if the origins (sources) are equivalent). Otherwise they are indecomposable.

$$\begin{tikzcd} & \bullet_{(sink)} & & & \bullet_{(source)} \arrow[dl] \\ \text{One source: } \bullet_{(source)} \arrow[ur] \arrow[dr] & & & \text{(or its dual...) } \bullet_{(sink)} & \\ & \bullet_{(sink)} & & & \bullet_{(source)} \arrow[ul] \end{tikzcd}$$

$$\begin{tikzcd} \text{Two sources, three sinks} & & \bullet^2 \arrow[dl] \arrow[d] \arrow[dr] & \\ & \bullet & \bullet & \bullet \end{tikzcd}$$

$$\begin{tikzcd} & & \bullet & \\ \text{Two generators - two sinks } & \bullet \arrow[ur] \arrow[dr] & & \bullet \arrow[ul] \arrow[dl] \\ & & \bullet & \end{tikzcd}$$

2.1 One Generator

We consider only the Poincaré algebra. Let V_0 denote an irreducible finite-dimensional representation of $\mathfrak{g}_0 = sl(2,\mathbb{C}) \times sl(2,\mathbb{C})$, given by non-negative integers (n,m) so that the spins are $(\frac{n}{2}, \frac{m}{2})$ and the dimension is $(n+1)(m+1)$.

Let Π denote an indecomposable finite-dimensional representation of $\mathcal{S}(\mathfrak{p}^-) \times_s \mathfrak{g}_0$ in a space V_Π, generated by a $\mathfrak{g}_0$ invariant source V_0. Let $\mathcal{P}(V_0)$ denote the space of polynomials in the variables z_1, z_2, z_3, z_4 with values in V_0. This is generated by polynomials of the form $p_0(z_1, z_2, z_3, z_4) \cdot v$ for $v \in V_0$ and p_0 a complex polynomial. We consider this a left $\mathcal{S}(\mathfrak{p}^-) \times_s \mathfrak{g}_0$ module in the obvious way. The map

$$p_0(z_1, z_2, z_3, z_4) \cdot v \mapsto \pi(p_0)v \tag{1}$$

is clearly $\mathcal{S}(\mathfrak{p}^-) \times_s \mathfrak{g}_0$ equivariant.

The decomposition of the $\mathfrak{g}_0$ module $\mathcal{S}(\mathfrak{p}^-)$ is well-known and is given by the representations spin$(\frac{n}{2}, \frac{n}{2})$ for $n = 0, 1, 2, \ldots$. Each occurs with infinite multiplicity due to the invariance of $\det z$ under $\mathfrak{g}_0$. We will describe these representations in detail below.

The decomposition of $\mathcal{P}(V_0)$ into irreducible $\mathfrak{g}_0$ representations follows easily from this using the well-known decomposition of the tensor product of two irreducible representations of $su(2)$. The decomposition of $\mathcal{P}(V_0)$ is in general more degenerate than what results from the invariance of $\det z$.

It is clear that there exists a sub-module $\mathcal{I} \subseteq \mathcal{P}(V_0)$ such that

$$\mathcal{P}(V_0)/\mathcal{I} \equiv V_\Pi.$$

The finite-dimensionality assumption on V_Π then implies that $\mathcal{I}$ contains all homogeneous polynomials of a degree greater than or equal to some fixed degree, say d_0. Since there are only a finite number of linearly independent homogeneous polynomials in $\mathcal{P}(V_0)$ of degree d_0, it follows that there exists a finite number of elements $p_1, p_2, \ldots, p_j$ in $\mathcal{P}(V_0)$ (these may be chosen for instance as highest weight vectors) such that if $\mathcal{I}\langle p_1, p_2, \ldots, p_j\rangle$ denotes the $\mathcal{S}(\mathfrak{p}^-) \times_s \mathfrak{g}_0$ submodule generated by these elements, then

$$V_\Pi \equiv \mathcal{P}(V_0)/\mathcal{I}\langle p_1, p_2, \ldots, p_j\rangle.$$

We assume that the number j of polynomials is minimal.

Once the elements $p_1, p_2, \ldots, p_j$ are known, it is possible to construct the whole graph as above. In particular, **the sinks in V_Π can be directly determined from this**. See Proposition 2.4 below for a simple example that indicates how. In case $\dim V_0$ is large the task, of course, will be more cumbersome.

Example 2.3. As is well known, we have that

$$\mathfrak{p}^- \otimes \mathrm{spin}(\frac{1}{2}, 0) = \mathrm{spin}(1, \frac{1}{2}) \oplus \mathrm{spin}(0, \frac{1}{2}).$$

If we mod out by all second order polynomials, and possibly one of the first order polynomial representations, we get the following three indecomposable representations:

- $\mathrm{spin}(\frac{1}{2}, 0) \rightarrow \mathrm{spin}(0, \frac{1}{2})$. This 4-dimensional representation comes from the defining representation.
- $\mathrm{spin}(\frac{1}{2}, 0) \rightarrow \mathrm{spin}(1, \frac{1}{2})$. This is an 8-dimensional representation.
- $\mathrm{spin}(\frac{1}{2}, 0) \rightarrow \mathrm{spin}(0, \frac{1}{2}), \mathrm{spin}(1, \frac{1}{2})$. This is a 10-dimensional representation which includes the two former.

Proceeding analogously,

$$\mathfrak{p}^- \otimes \mathrm{spin}(1, 0) = \mathrm{spin}(\frac{3}{2}, \frac{1}{2}) \oplus \mathrm{spin}(\frac{1}{2}, \frac{1}{2})$$

leads to inequivalent representations in dimensions 7,11,15.

Similarly,

$$\mathfrak{p}^- \otimes \mathrm{spin}(\frac{1}{2}, \frac{1}{2}) = \mathrm{spin}(0, 0) \oplus \mathrm{spin}(1, 0) \oplus \mathrm{spin}(0, 1) \oplus \mathrm{spin}(1, 1)$$

leads to indecomposable representations in dimensions 5, 7, 8, 10, 11, 13, 16, 17, 19, 20. Several dimensions here carry a number of inequivalent representations.

Together with duals of these or versions obtained as mirror images by interchanging the spins, these exhaust all the known representations in dimensions less than or equal to 8 with the exception of a 6-dimensional representation which we describe in Example 2.5.

2.2 Special Case: Ideals in $\mathcal{U}(\mathfrak{p}^-)$

It is well-known that there is a decomposition

$$\mathcal{U}(\mathfrak{p}^-) = \oplus W_{r,s}$$

into $\mathfrak{g}_0$ representations, where the subspace $W_{r,s}$ may be defined through its highest weight vector, say $z_1^r \det z^s$. This is possible since each representation occurs with multiplicity one. Denote this representation simply by $[r, s]$. It is elementary to see that the action of $\mathfrak{p}^-$ on the left on $\mathcal{U}(\mathfrak{p}^-)$, when expressed in terms of representations, is given as follows. All arrows represent non-trivial maps.

$$
\begin{array}{ccccccc}
\downarrow &&&&&& \\
[1,0] &&&&&& \\
\downarrow & \searrow &&&&& \\
[2,0] && [0,1] &&&& \\
\downarrow & \searrow & \downarrow &&&& \\
[3,0] && [1,1] &&&& \\
\downarrow & \searrow & \downarrow & \searrow &&& \\
[4,0] && [2,1] && [0,2] && \\
\downarrow & \searrow & \downarrow & \searrow & \downarrow && \\
[5,0] && [3,1] && [1,2] && \\
\downarrow & \searrow & \downarrow & \searrow & \downarrow & \searrow & \\
[6,0] && [4,1] && [2,2] && [0,3] \\
\vdots && \vdots && \vdots && \vdots
\end{array} \tag{2}
$$

Any ideal $\mathcal{I} \subseteq \mathcal{U}(\mathfrak{p}^-)$ that has finite codimension must clearly contain some $z_1^{r_1}$ for some minimal $r_1 \in \mathbb{N}$ (we omit the trivial case of codimension 0). Since we are assuming that the ideals are $\mathfrak{g}_0$ invariant, if some other element $z_1^{r_2} p(\det z)$ is in the ideal then first of all we can assume $r_2 < r_1$ and secondly we can assume that the polynomial p is homogeneous; $p(\det z) = \det z^{s_2}$ for some $s_2 > 0$. The latter inequality follows by the minimality of r_1.

Thus the following is clear:

Proposition 2.4. *Any $\mathfrak{g}_0$ invariant ideal $\mathcal{I} \subseteq \mathcal{U}(\mathfrak{p}^-)$ of finite codimension is of the form*

$$\mathcal{I} = \mathfrak{g}_0 \cdot \langle z_1^{r_1} \det z^{s_1}, z_1^{r_2} \det z^{s_2}, \cdots, z_1^{r_t} \det z^{s_t} \rangle \tag{3}$$

for some positive integers $r_1 > r_2 > \cdots > r_t$ and integers $0 = s_1 < s_2 < \cdots < s_t$. If the set is minimal, then furthermore

$$\forall j = 2, 3, \ldots, t : s_1 + s_2 + \cdots + s_j \leq r_1 - r_j .$$

Any set of such integers determine an invariant ideal of finite codimension. The sinks in the quotient module are

$$z_1^{r_1 - s_2} \det z^{s_2 - 1}, z_1^{r_1 - s_2 - s_3} \det z^{s_3 - 1} \ldots z_1^{r_1 - s_2 - \cdots - s_t} \det z^{s_t - 1}, \textit{ and } \det z^{s_t + r_t - 1}.$$

Example 2.5. If we mod out by all second order polynomials, that is by z_1^2 and $\det z$, we get the 5-dimensional indecomposable representation

$$\mathrm{spin}(0,0) \to \mathrm{spin}(\frac{1}{2}, \frac{1}{2}).$$

If we instead mod out by z_1^2 and $z_1 \det z$ we get the 6-dimensional indecomposable representation

$$\mathrm{spin}(0,0) \to \mathrm{spin}(\frac{1}{2}, \frac{1}{2}) \to \mathrm{spin}(0,0).$$

Example 2.6. The representations determined by ideals are easily written down, though some finer details from the representation theory of $su(2)$ will have to be invoked to get the precise form. In simple examples like the last in the previous example, everything follows immediately since there is no need to be precise about the relative scales in the 3 spaces:

$$\mathfrak{p}^- \ni \underline{p} = (p_1, p_2, p_3, p_4) \mapsto \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ p_1 & 0 & 0 & 0 & 0 & 0 \\ p_2 & 0 & 0 & 0 & 0 & 0 \\ p_3 & 0 & 0 & 0 & 0 & 0 \\ p_4 & 0 & 0 & 0 & 0 & 0 \\ 0 & p_4 & -p_3 & -p_2 & p_1 & 0 \end{pmatrix}. \tag{4}$$

An element (u, v) in the diagonal subgroup G_0 (see Sect. 2) acts as $0 \oplus u \otimes (v^t)^{-1} \oplus 0$.

Notice that the matrix with just p_1 corresponds to a map which sends the constant 1 to the polynomial $p_1 z_1$ and sends the polynomial z_4 to $p_1 \det z$ and all other first order polynomials z_1, z_2, z_3 to 0.

2.3 Two Sources and Two Sinks

We here consider the Poincaré algebra.

Consider the situation previously depicted under 'Simple situations' where there is one source and two sinks. The resulting representations may be written as

$$\begin{pmatrix} 0 & 0 \\ w & 0 \end{pmatrix} \mapsto \begin{pmatrix} 0 & 0 & 0 \\ F(w) & 0 & 0 \\ G(w) & 0 & 0 \end{pmatrix}, \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} \mapsto \begin{pmatrix} \tau_1(u,v) & 0 & 0 \\ 0 & \tau_2(u,v) & 0 \\ 0 & 0 & \tau_3(u,v) \end{pmatrix}. \tag{5}$$

With this one can easily write down a representation with two sources and two sinks, indeed a 4-parameter family given by elements $(\alpha, \beta, \gamma, \delta) \in \mathbb{C}^4$:

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \alpha \cdot F(w) & \beta \cdot F(w) & 0 & 0 \\ \gamma \cdot G(w) & \delta \cdot G(w) & 0 & 0 \end{pmatrix} \text{ resp. } \begin{pmatrix} \tau_1(u,v) & 0 & 0 & 0 \\ 0 & \tau_1(u,v) & 0 & 0 \\ 0 & 0 & \tau_2(u,v) & 0 \\ 0 & 0 & 0 & \tau_3(u,v) \end{pmatrix}. \tag{6}$$

In this case, there is a continuum of inequivalent representations and they are generically indecomposable. This lead to a continuum already in dimension 12 where the two sources are equal and 2-dimensional – say spin($\frac{1}{2}$,0), and the two sinks are spin(1,$\frac{1}{2}$) and spin(0,$\frac{1}{2}$) or, also in dimension 12, the 2 sources are the

4-dimensional spin($\frac{1}{2}$,$\frac{1}{2}$), and the sinks are spin(1,0) and spin(0,0), or in dimension 16 where one is the 2-dimensional spin ($\frac{1}{2}$,0) and the other is the 6-dimensional spin($\frac{1}{2}$,1) and the targets are spin(0,$\frac{1}{2}$) and spin(1,$\frac{1}{2}$). The moduli space in these cases is $\mathbb{CP}^1$. Specifically, the indecomposable is determined by a point $(p, q) \in \mathbb{CP}^1 \times \mathbb{CP}^1$ giving relative scales on the arrows. Here, $p \equiv (\alpha, \beta)$ and $q \equiv (\gamma, \delta)$ in the above representation. Two such points (p_1, q_1) and (p_2, q_2), are equivalent if there is an element $g \in GL(2, \mathbb{C})$ such that $(p_2, q_2) = (gp_1, gp_2)$.

3 Supersymmetry

The previous considerations are now extended to the supersymmetric setting as follows: Let H_1, H_2, and H_3 be reductive matrix Lie groups with Lie algebras $\mathfrak{h}_1, \mathfrak{h}_2$, and $\mathfrak{h}_3$, respectively. We assume that possible abelian ideals are represented by semi-simple elements. We consider an irreducible representation of each of these Lie algebras; V_1, V_2, and V_3. We identify the representation with the space in which it acts. We denote furthermore the dual representation of a representation V by V' (this is the $\mathbb{C}$ linear dual). Let

$$W_1 = \hom(V_1, V_2) \equiv V_1' \otimes V_2 \; ; \; W_2 = \hom(V_2, V_3) \equiv V_2' \otimes V_3 \quad (7)$$

$$\text{and} \qquad Z = \hom(V_1, V_3) \equiv V_1' \otimes V_3. \quad (8)$$

The Lie superalgebra $\mathfrak{g}_{super} = \mathfrak{g}_{super}(\mathfrak{h}_1, \mathfrak{h}_2, \mathfrak{h}_3, V_1, V_2, V_3)$ is defined as

$$\mathfrak{g}_{super} = \quad (9)$$

$$\left\{ \begin{pmatrix} a & 0 & 0 \\ w_1 & g & 0 \\ z & w_2 & b \end{pmatrix} \mid a \in \mathfrak{h}_1 \,, g \in \mathfrak{h}_2 \,, b \in \mathfrak{h}_3 \,, w_1 \in W_1 \,, w_2 \in W_2 \,, \text{ and } z \in Z \right\}.$$

The odd part is given as

$$\mathfrak{g}^1_{super} = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ w_1 & 0 & 0 \\ 0 & w_2 & 0 \end{pmatrix} \mid w_1 \in W_1 \,, \text{ and } w_2 \in W_2 \right\}.$$

Let

$$\mathfrak{n}_{super} = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ w_1 & 0 & 0 \\ z & w_2 & 0 \end{pmatrix} \mid w_1 \in W_1 \,, w_2 \in W_2 \,, \text{ and } z \in Z \right\}.$$

and

$$\mathfrak{g}^r_{super} = \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & g & 0 \\ 0 & 0 & b \end{pmatrix} \mid a \in \mathfrak{h}_1 \,, g \in \mathfrak{h}_2 \,, \text{ and } b \in \mathfrak{h}_3 \right\}.$$

Obviously, $\mathfrak{n}_{super}$ is a maximal nilpotent ideal and $\mathfrak{g}^r_{super}$ is the reductive part.

We let

$$\mathfrak{p}^- = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ z & 0 & 0 \end{pmatrix} \mid z \in Z \right\}.$$

Then $\mathfrak{g}^0_{super} = \mathfrak{g}^r_{super} \oplus \mathfrak{p}^-$.

We then have the following super algebraic generalization of Theorem 2.1:

Theorem 3.1. *Consider a finite-dimensional representation V_{super} of $\mathfrak{g}_{super}$. Then there is a flag of subspaces $\{0\} = W_0 \subsetneq W_1 \subsetneq \cdots \subsetneq W_{r-1} \subsetneq W_r = V_{super}$ such that each W_i is invariant and completely reducible under $\mathfrak{g}^0_{super}$ and such that $\mathfrak{n}_{super}$ maps W_i into W_{i-1} for $i = 1, \ldots, r$.*

Thus, the previous treatment with directed graphs and ideals carry over. Naturally, the picture gets more complicated with the odd generators.

The most simple thing to consider would be the $\mathfrak{g}_{super}$ module $\mathcal{U}(\mathfrak{n}_{super})$, but even here the situation is complex, though in principle tractable.

Consider as an example the case of the simplest super Poincaré algebra,

$$\mathfrak{g}^P_{super} \tag{10}$$

$$= \left\{ \begin{pmatrix} a & 0 & 0 \\ w_1 & 0 & 0 \\ z & w_2 & b \end{pmatrix} \mid a, b \in sl(2,\mathbb{C}), w_1 \in M(1,2), w_2 \in M(2,1), \text{ and } z \in M(2,2) \right\}.$$

Let $W_1 = M(1,2)$ and $W_2 = M(2,1)$. We number the spaces

$$\begin{matrix} 1 & W_2 & W_2 \wedge W_2 \\ W_1 & W_1 \wedge W_2 & W_1 \wedge W_2 \wedge W_2 \\ W_1 \wedge W_1 & W_1 \wedge W_1 \wedge W_2 & W_1 \wedge W_1 \wedge W_2 \wedge W_2 \end{matrix} = \begin{matrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{matrix}.$$

We then have that

$$\mathcal{U}(\mathfrak{n}_{super}) = \sum_{i=1}^{9} \mathcal{U}(\mathfrak{p}^-) \otimes \mathcal{U}(\mathfrak{q}^1_{super})_i \,. \tag{11}$$

Each of the nine summands is invariant under $\mathfrak{g}^r_{super}$. The representations corresponding to this are given right below. Here, n, d are independent non-negative integers (in a few obvious cases, n must furthermore be non-zero). The labels $\uparrow$

and $\downarrow$ may be taken just as part of a short hand notation that are defined by the stated equations. They are listed here for convenience even though they are not used directly. One can ascertain useful information from them about how the various pieces occur in the tensor products of $su(2)$ representations.

$$
\begin{aligned}
&1\,[n,n,d] \oplus \\
2\uparrow[n,d] = 2\,[n-1,n,d] &\oplus 2\downarrow[n,d] = 2\,[n+1,n,d] \\
&3\,[n,n,d] \oplus \\
4\uparrow[n,d] = 4\,[n,n-1,d] &\oplus 4\downarrow[n,d] = 4\,[n,n+1,d] \\
5\uparrow\uparrow[n,d] = 5\,[n-1,n-1,d] &\oplus 5\uparrow\downarrow[n,d] = 5\,[n-1,n+1,d] \\
5\downarrow\uparrow[n,d] = 5\,[n+1,n-1,d] &\oplus 5\downarrow\downarrow[n,d] = 5\,[n+1,n+1,d] \\
6\uparrow[n,d] = 6\,[n,n-1,d] &\oplus 6\downarrow[n,d] = 6\,[n+1,n,d] \\
&7\,[n,n,d] \oplus \\
8\uparrow[n,d] = 6\,[n-1,n,d] &\oplus 8\downarrow[n,d] = 8\,[n+1,n,d] \\
&\oplus 9\,[n,n,d]
\end{aligned}
$$

A further complication is that there are representations in different spaces that are equivalent under $\mathfrak{g}^r_{super}$:

$$
\begin{aligned}
8\uparrow[n,d] &\leftrightarrow 4\uparrow[n-1,d+1] \\
8\downarrow[n,d] &\leftrightarrow 4\,[n+1,d] \\
5\downarrow\downarrow[n,d] &\leftrightarrow 1\,[n+1,d] \\
5\uparrow\uparrow[n,d] &\leftrightarrow 1\,[n-1,d+1] \\
5\downarrow\downarrow[n-1,d+1]\,, 5\uparrow\uparrow[n+1,d] &\leftrightarrow 9\,[n,d] \\
5\downarrow\downarrow[n-1,d+1]\,, 5\downarrow\downarrow[n+1,d] &\leftrightarrow 9\,[n,d] \\
6\uparrow[n,d] &\leftrightarrow 2\downarrow[n-1,d+1] \\
6\uparrow[n,d] &\leftrightarrow 2\downarrow[n-1,d+1] \\
6\downarrow[n,d] &\leftrightarrow 2\uparrow[n+1,d] \\
6\downarrow[n,d] &\leftrightarrow 2\uparrow[n+1,d]
\end{aligned}
$$

To each finite-dimensional representation V_r of $\mathfrak{g}^r_{super}$ (may be reducible), the general object of interest is the left module

$$
\mathcal{U}(\mathfrak{n}_{super}) \cdot V_r = \sum_{i=1}^{n} \mathcal{U}(\mathfrak{p}^-) \otimes \mathcal{U}(\mathfrak{q}^1_{super})_i \cdot V_r. \tag{12}
$$

To further analyze this we have to choose a PBW-type basis. We will do this in the indicated fashion with $\mathcal{U}(\mathfrak{p}^-)$ to the left and with furthermore W_1 always to the left of W_2.

Example 3.2. Assume that $\mathcal{U}(\mathfrak{p}^-)$ acts trivially on the space V_r. The resulting module is then

$$\bigwedge(\mathfrak{g}^1_{super}) \cdot V_r, \tag{13}$$

or some of the subrepresentations thereof. The exterior algebra $\bigwedge(\mathfrak{g}^1_{super})$ occurs because the W_1, W_2 anticommute in the considered quotient.

Observe that in the sum (12) the summand

$$\mathcal{U}_{2,3,5,6,8,9}(\mathfrak{n}_{super}) \cdot V_r = \sum_{i=2,3,5,6,8,9} \mathcal{U}(\mathfrak{p}^-) \otimes \mathcal{U}(\mathfrak{q}^1_{super})_i \cdot V_r \tag{14}$$

is invariant. We may then pass to a general subclass of indecomposable modules by first taking the quotient by this. The vector space that results is

$$\mathcal{U}_{rest}(\mathfrak{n}_{super}) \cdot V_r = \sum_{i=1,4,7} \mathcal{U}(\mathfrak{p}^-) \otimes \mathcal{U}(\mathfrak{q}^1_{super})_i \cdot V_r. \tag{15}$$

If we let $\mathcal{U}(\mathfrak{p}^-)_{\geq s}$ be the ideal generated by all homogeneous elements of degree s, it is easy to see that for each $s = 0, 1, 2, \ldots,$ the space

$$\mathcal{U}^s_{rest}(\mathfrak{n}_{super}) \cdot V_r = \mathcal{U}(\mathfrak{p}^-)_{s+2} \cdot V_r \oplus \mathcal{U}(\mathfrak{p}^-)_{s+1} \cdot W_1 \cdot V_r \oplus \mathcal{U}(\mathfrak{p}^-)_s \cdot (W_1 \wedge W_1) \cdot V_r \tag{16}$$

is invariant.

Example 3.3. Let V^1_r be the irreducible 2-dimensional representation which is only non-trivial on the $\mathfrak{h}_1$ piece of the reductive part. The defining representation of $\mathfrak{g}^P_{super}$ is a subrepresentation of the quotient

$$\mathcal{U}_{rest}(\mathfrak{n}_{super}) \cdot V^1_r / \mathcal{U}^0_{rest}(\mathfrak{n}_{super}) \cdot V^1_r.$$

Indeed, we just have to limit ourselves further by removing two appropriate $\mathfrak{g}^r_{super}$ representations.

Returning to the more general situation, let us assume from now on that V_r is the trivial 1-dimensional module. We are thus left with the space

$$\mathcal{U}_{rest}(\mathfrak{n}_{super}) = \mathcal{U}(\mathfrak{p}^-) \oplus \mathcal{U}(\mathfrak{p}^-) \cdot W_1 \oplus \mathcal{U}(\mathfrak{p}^-) \cdot (W_1 \wedge W_1). \tag{17}$$

The general form of the representation is (we give only the W_1, W_2 operators)

$$\begin{pmatrix} 0 & (w_2^1 p_{11} + w_2^2 p_{21})1_{\mathcal{P}} & (w_2^1 p_{12} + w_2^2 p_{22})1_{\mathcal{P}} & 0 \\ w_1^1 1_{\mathcal{P}} & 0 & 0 & -(w_2^1 p_{12} + w_2^2 p_{22})1_{\mathcal{P}} \\ w_1^2 1_{\mathcal{P}} & 0 & 0 & (w_2^1 p_{11} + w_2^2 p_{21})1_{\mathcal{P}} \\ 0 & -w_1^2 1_{\mathcal{P}} & w_1^1 1_{\mathcal{P}} & 0 \end{pmatrix}. \tag{18}$$

Here, each block corresponds to a space of polynomials. The operators $p_{i,j}$ are multiplication operators in $\mathcal{U}(\mathfrak{p}^-)$ – and hence also such operators in the given space. Notice that they increase the degree of the target by 1. The symbol $1_{\mathcal{P}}$ denotes the identity operator.

The finer details are given as follows, where the arrows point upwards come from W_2 and those pointing downwards come from W_1.

$$\begin{array}{ccccc} & & 1\,[n,d+1] & & \\ & \nearrow & & \nwarrow & \\ 4\downarrow[n-1,d+1] & & & & 4\uparrow[n+1,d] \\ & \nwarrow & & \nearrow & \\ & & 7\,[n,d] & & \end{array} \tag{19}$$

$$\begin{array}{ccccc} & & 1\,[n,d] & & \\ & \swarrow & & \searrow & \\ 4\downarrow[n,d] & & & & 4\uparrow[n,d] \\ & \nwarrow & & \nearrow & \\ & & 7\,[n,d] & & \end{array} \tag{20}$$

Any invariant ideal $\mathcal{I}_{super}$ in $\mathcal{U}_{rest}(\mathfrak{n}_{super})$ contains a sum of the form

$$\mathcal{I}_1(\mathfrak{p}^-) \oplus \mathcal{I}_4(\mathfrak{p}^-) \cdot W_1 \oplus \mathcal{I}_7(\mathfrak{p}^-) \cdot (W_1 \wedge W_1), \tag{21}$$

where $\mathcal{I}_1(\mathfrak{p}^-) \subseteq \mathcal{I}_4(\mathfrak{p}^-) \subseteq \mathcal{I}_7(\mathfrak{p}^-) \subseteq \mathcal{U}(\mathfrak{p}^-)$ are $\mathfrak{p}^-$ ideals. These are precisely the ideals determined in Sect. 1.

Proposition 3.4. *If we have a representation* $7\,[n,d] \in \mathcal{I}_7(\mathfrak{p}^-)$ *then* $4\,[n,d+1] \in \mathcal{I}_4(\mathfrak{p}^-)$ *and* $1\,[n,d+1] \in \mathcal{I}_1(\mathfrak{p}^-)$. *Furthermore,* $4\uparrow[n,d], 4\downarrow[n,d] \in \mathcal{I}_{super}$. *In particular,*

$$\mathfrak{p}^- \cdot \mathfrak{p}^- \cdot \mathcal{I}_7(\mathfrak{p}^-) \subseteq \mathcal{I}_1.$$

This result in principle solves the problem but there are still extremely many cases – even if we start with an ideal $\mathcal{I}_7$ and ask for how many configurations of the ideals $\mathcal{I}_1, \mathcal{I}_4$ that are possible. We refrain from pursuing this further and just give a low-dimensional example.

Example 3.5. Let $\mathcal{I}_7 = \mathcal{I}\langle z_1\rangle$. The following list is exhaustive and each case occurs.

- $\mathcal{I}_1 = \mathcal{I}\langle z_1\rangle$ then $\mathcal{I}_4 = \mathcal{I}\langle z_1\rangle$.
- $\mathcal{I}_1 = \mathcal{I}\langle z_1^2\rangle$ then either $\mathcal{I}_4 = \mathcal{I}\langle z_1^2\rangle, \mathcal{I}_4 = \mathcal{I}\langle z_1^2, \det z\rangle$, or $\mathcal{I}_4 = \mathcal{I}_7$.

- $\mathcal{I}_1 = \mathcal{I}\langle z_1^2, \det z\rangle$ then $\mathcal{I}_4 = \mathcal{I}\langle z_1^2, \det z\rangle$.
- $\mathcal{I}_1 = \mathcal{I}\langle z_1^3, \det z\rangle$ then $\mathcal{I}_4 = \mathcal{I}\langle z_1^2, \det z\rangle$.
- $\mathcal{I}_1 = \mathcal{I}\langle z_1^3, z_1 \det z\rangle$ then $\mathcal{I}_4 = \mathcal{I}\langle z_1^2, \det z\rangle$ or $\mathcal{I}_4 = \mathcal{I}_1^2$.

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On the Geometry of Super Riemann Surfaces

Stephen Kwok

1 Introduction

Super Riemann surfaces-$1|1$ complex supermanifolds with a SUSY-1 structure-furnish a rich field of study in algebraic supergeometry.

Let us recall that a $1|1$ *super Riemann surface* is a pair $(X, \mathcal{D})$, where X is a $1|1$-dimensional complex supermanifold, and $\mathcal{D}$ is a locally direct (and consequently locally free, by Nakayama's lemma) rank $0|1$ subsheaf of the tangent sheaf $\mathcal{T}X$ which is as "non-integrable as possible," in the sense that the *Frobenius map*:

$$\mathcal{D} \otimes \mathcal{D} \to \mathcal{T}X/\mathcal{D}$$
$$Y \otimes Z \mapsto [Y, Z] \ (mod \ \mathcal{D})$$

is an isomorphism of sheaves. The distinguished subsheaf $\mathcal{D}$ is called a *SUSY-1 structure* on X, and $1|1$ super Riemann surfaces are thus alternatively referred to as *SUSY-1 curves*. One may readily define families of $1|1$ super Riemann surfaces over a complex base supermanifold B by replacing all objects in the previous definition with their relative counterparts over B.

Many of the basic properties of super Riemann surfaces are given in a paper of Rosly, Schwarz and Voronov [22]. The study of the moduli space of super Riemann surfaces is still quite open. Some results along these lines appear in [10] where the (uncompactified) moduli space of compact SUSY-1 curves with marked points and level-n structures is constructed. Deligne, in a series of unpublished letters to Manin [7], has compactified the moduli space of genus g compact SUSY-1 curves,

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for $g \geq 2$, as well as proving a uniformization theorem for super Riemann surfaces (see also [6]).

Rabin has done extensive work in this area, as noted below in many of the references, as well as the related area of $1|1$ super curves (i.e., $1|1$ complex supermanifolds): results on the KP hierarchy on super Jacobians of $1|1$ super curves [3], as well as the theory of $\mathfrak{D}$-modules on $1|1$ super curves [21].

In a more analytical vein, a Selberg supertrace formula for super Riemann surfaces is presented in [2]; Grosche has made a detailed investigation of these ideas in a series of papers [12–14].

In the development of algebraic supergeometry, it turns out that line bundles (or their equivalent, invertible sheaves), which are central in classical algebraic geometry, are no longer so fundamental in algebraic supergeometry. For instance, there exist super Grassmannians (for instance, $Gr(1|1, 2|2)$) which possess no ample line bundles, and therefore cannot be embedded as sub-supermanifolds of super projective space $\mathbb{P}^{m|n}$ (see [4] and [17] for an example).

Manin [18] has suggested that a different concept, due to Skornyakov, should be a substitute for invertible sheaves in supergeometry: that of Π*-invertible sheaf*. These objects are pairs (S, ϕ), where S is a locally free sheaf of rank $1|1$ and ϕ is an odd endomorphism of S such that $\phi^2 = -1$. Their transition functions can be reduced to $\mathbb{G}_m^{1|1}$, a nonabelian supergroup analogous to the usual multiplicative group $\mathbb{G}_m$.

Deligne [8] has pointed out that $\mathbb{G}_m^{1|1}$ can be interpreted as the multiplicative group of the so called "super skew field." This point of view sheds considerable light on Π-projective geometry, and the study of Π-invertible sheaves on super Riemann surfaces may lead to a deeper understanding of these objects.

In the first section we shall concentrate an important family of super Riemann surfaces – the super elliptic curves, and discuss some results on super theta functions defined on these superelliptic curves. The second section is concerned with the ideas of Π-projective geometry: Π-invertible sheaves, Π-projective spaces.

In what follows, we will work over the field of complex numbers $\mathbb{C}$, except when otherwise stated.

2 Super Elliptic Curves

A $1|1$ super elliptic curve is a SUSY-1 curve whose underlying reduced Riemann surface is of genus 1. A family $X \to B$ of super elliptic curves over a base supermanifold B is then simply a map $X \to B$ such that the fibers are super elliptic curves.

Levin [16] has defined two main families of super elliptic curves, which we now review:

2.1 Even Family

There are three even families of SUSY-1 elliptic curves $X_0 \to \mathbb{H}$ defined over the complex upper half-plane $\mathbb{H} = \{\tau \in \mathbb{C} : Im(\tau) > 0\}$. For the sake of brevity, we shall only describe the first one, the others being essentially the same. X_0 is realized as a quotient of $\mathbb{C}^{1|1} \times \mathbb{H}$ by a group $G \cong \mathbb{Z}^2$ of transformations: $X_0 = (\mathbb{C}^{1|1} \times \mathbb{H})/G$. Here, $\mathbb{C}^{1|1}$ is endowed with the coordinates z, ζ, and the action of the group is then given by specifying the actions of generators S and T of G:

$$\begin{aligned} S &: z \mapsto z + 1 \quad \zeta \mapsto \zeta \\ T &: z \mapsto z + \tau \quad \zeta \mapsto -\zeta \end{aligned}$$

The vector field $\partial_\zeta + \zeta\partial_z$ on $\mathbb{C}^{1|1}$ is invariant under the action of G, and therefore defines a relative SUSY-1 structure on X_0.

2.2 Odd Family

There is also an odd family of curves $X_1 \to \mathbb{H} \times \mathbb{C}^{0|1}$. Again we let z, ζ be coordinates on $\mathbb{C}^{1|1}$, and τ, δ the coordinates on the base $\mathbb{H} \times \mathbb{C}^{0|1}$. Then we may define X_1 as the quotient of $\mathbb{C}^{1|1} \times \mathbb{H} \times \mathbb{C}^{0|1}$ by the $G \cong \mathbb{Z}^2$ action

$$\begin{aligned} S &: z \mapsto z + 1 &\quad \zeta &\mapsto \zeta \\ T &: z \mapsto z + \tau + \delta\zeta &\quad \zeta &\mapsto \zeta + \delta \end{aligned}$$

The vector field $\partial_\zeta + \zeta\partial_z$ on $\mathbb{C}^{1|1}$ is again invariant under the action of G, and therefore defines a relative SUSY-1 structure on X_1.

As an aside, we note that the base of the even family is purely even, while the base of the odd family has nonzero odd dimension. These families of super elliptic curves have been studied (from the physical point of view) in [11], where the theory of elliptic functions is generalized to superelliptic curves. The super analogues of the Weierstrass elliptic functions are defined and used to give explicit equations for the embedding of a superelliptic curve into superprojective space $\mathbb{P}^{2|3}$; see also [20] for a deeper study of the geometry of the odd family. In [19] seven families of SUSY-2 superelliptic curves are described; much work remains to be done on these objects.

2.3 Super Theta Functions

The Jacobi theta functions may also be generalized to superelliptic curves. In order to define them, one must first investigate two super Lie groups (see [18]).

$\underline{\mathbb{G}_{\mathbf{a}}^{\mathbf{1|1}}}$. This is a noncommutative analogue of the usual additive Lie group $\mathbb{G}_a$. The underlying supermanifold of $\mathbb{G}_a^{1|1}$ is $\mathbb{C}^{1|1}$, with coordinates (z, ζ). In terms of the functor of points, the group law on $\mathbb{G}_a^{1|1}$ is given by:

$$(z, \zeta) \cdot (z', \zeta') = (z + z' + \zeta\zeta', \zeta + \zeta')$$

$\mathbb{G}_a^{1|1}$ carries a natural invariant SUSY-1 structure, spanned by the odd vector field $\mathcal{D}_Z := \partial_\zeta + \zeta\partial_z$.

$\underline{\mathbb{G}_{\mathbf{m}}^{\mathbf{1|1}}}$. This is a noncommutative analogue of the multiplicative Lie group $\mathbb{G}_m$. The underlying supermanifold of $\mathbb{G}_m^{1|1}$ is $\mathbb{C}^{1|1}\backslash\{0\}$, with coordinates (w, η). The group law is given by:

$$(w, \eta) \cdot (w', \eta') = (ww' + \eta\eta', w\eta' + w'\eta).$$

$\mathbb{G}_m^{1|1}$ also carries a natural invariant SUSY-1 structure, spanned by the odd vector field $\mathcal{D}_V := w\partial_\eta + \eta\partial_w$.

$\mathbb{G}_m^{1|1}$ may also be characterized as the complex super Lie group represented by the functor of points $S \mapsto \mathcal{O}_S^*$. This means that the sheaf $\mathcal{O}^*$ may be naturally identified with the sheaf of groups $\underline{\mathbb{G}_m^{1|1}}$, a fact we will use later.

We may now define the super exponential map $Exp : \mathbb{G}_a^{1|1} \to \mathbb{G}_m^{1|1}$ by

$$\begin{aligned} Exp\,(z, \zeta) &:= e^{z+\zeta} \\ &= e^z(1 + \zeta) \end{aligned}$$

As noted in [18], we have the following:

Proposition. *Exp is a surjective morphism of complex super Lie groups, compatible with SUSY-structures, with kernel* $(2\pi i\mathbb{Z}, 0)$.

From this we see that there is a short exact sequence of super Lie groups:

$$0 \to \mathbb{Z} \xrightarrow{2\pi i} \mathbb{G}_a^{1|1} \xrightarrow{Exp} \mathbb{G}_m^{1|1} \to 0$$

completely analogous to the classical short exact sequence:

$$0 \to \mathbb{Z} \xrightarrow{2\pi i} \mathbb{G}_a \xrightarrow{exp} \mathbb{G}_m \to 0$$

Levin's super theta functions associated to the even family may now be defined:

$$\theta_{ab}^{ev}(z, \zeta\,|\tau) := \sum_{n\in\mathbb{Z}} Exp(\pi i(n + a)^2\tau + 2\pi i(n + a)(z + b), (-1)^{n+a}\zeta)$$

Here ab denotes the characteristic of the function: an ordered pair (a, b) with $a, b \in \{0, \frac{1}{2}\}$.

It follows easily from standard properties of classical theta functions that these series converge and define holomorphic functions on $\mathbb{G}_a^{1|1} \times \mathbb{H}$. As with their classical analogues, they satisfy quasiperiodicity conditions under the transformations S, T:

$$\theta_{ab}^{ev}(z+1, \zeta\,|\tau) = \theta_{ab}^{ev}(z, \zeta\,|\tau) \cdot e^{2\pi i a}$$

$$\theta_{ab}^{ev}(z+\tau, -\zeta\,|\tau) = \theta_{ab}^{ev}(z, \zeta\,|\tau) \cdot e^{-\pi i \tau - 2\pi i (z+b)}$$

The super theta functions associated to the odd family are defined by:

$$\theta_{ab}^{odd}(z, \zeta\,|\delta, \tau) := \sum_{n\in\mathbb{Z}} Exp(\pi i (n+a)^2 \tau + 2\pi i (n+a)(z+b) + 2\pi i \zeta\delta[\tfrac{A^2(n+a)}{6} + \tfrac{(n+a)^2}{2} - \tfrac{A^2(n+a)^3}{6}], \sqrt{2\pi i}[A(n+a)^2 \tfrac{\delta}{2} + A(n+a)\zeta])$$

Here for $\sqrt{2\pi i}$ we take the branch of the square root which is positive on $\mathbb{R}$, and A is an even function on $\mathbb{H} \times \mathbb{C}^{0|1}$ (see [16] for details). Again it is easily shown that these series converge and define holomorphic functions on $\mathbb{G}_a^{1|1} \times \mathbb{H} \times \mathbb{C}^{0|1}$. These also obey quasiperiodicity conditions:

$$\theta_{ab}^{odd}(z+1, \zeta\,|\tau, \delta) = \theta_{ab}^{odd}(z, \zeta\,|\tau, \delta) \cdot e^{2\pi i a}$$

$$\theta_{ab}^{odd}(z+\tau+\zeta\delta, \zeta+\delta\,|\tau, \delta) = \theta_{ab}^{odd}(z, \zeta\,|\delta, \tau) \cdot Exp(-\pi i(\tau+\zeta\delta) - 2\pi i(z+b), -\sqrt{2\pi i}A(\tfrac{\delta}{2}+\zeta))$$

In [16], it is noted that one may use the super theta functions to construct super analogues of the Jacobi elliptic functions. For the even family of Levin, these are:

$$Sn^{ev}(z, \zeta, \tau) := k_{ev}^{-1/2}(\tau) \frac{\theta_{11}^{ev}(z, \zeta|\tau)}{\theta_{01}^{ev}(z, \zeta|\tau)}$$

$$Cd^{ev}(z, \zeta, \tau) := k_{ev}^{-1/2}(\tau) \frac{\theta_{10}^{ev}(z, \zeta|\tau)}{\theta_{00}^{ev}(z, \zeta|\tau)}$$

Here $k_{ev}^{1/2} := \frac{\theta_{10}^{ev}(0,0|\tau)}{\theta_{00}^{ev}(0,0|\tau)}$ and $k_{ev}'^{1/2} := \frac{\theta_{01}^{ev}(0,0|\tau)}{\theta_{00}^{ev}(0,0|\tau)}$ are the "theta Nullwerte" or "theta-constants."

For the odd family, they are:

$$Sn^{odd}(z,\zeta|\tau,\delta) := k_{odd}^{-1/2}(\tau,\delta)\frac{\theta_{1,1}^{odd}(z,\zeta|\tau,\delta)}{\theta_{0,1}^{odd}(z,\zeta|\tau,\delta)}$$

$$Cd^{odd}(z,\zeta|\tau,\delta) := \frac{k_{odd}'^{1/2}(\tau,\delta)}{k_{odd}^{1/2}(\tau,\delta)}\frac{\theta_{1,0}^{odd}(z,\zeta|\tau,\delta)}{\theta_{0,1}^{odd}(z,\zeta|\tau,\delta)}$$

$$Dn^{odd}(z,\zeta|\tau,\delta) := k_{odd}'^{1/2}(\tau,\delta)\frac{\theta_{0,0}^{odd}(z,\zeta|\tau,\delta)}{\theta_{0,1}^{odd}(z,\zeta|\tau,\delta)}$$

Because of the quasiperiodicity properties of the super theta functions, these functions are superelliptic for the even (resp. odd) families, and may be used to embed the families into products of (relative) Π-projective spaces: the even family into $\mathbb{P}^1_\Pi \times \mathbb{P}^1_\Pi$, and the odd family into $\mathbb{P}^3_\Pi$ (see [16]).

2.4 Super Heisenberg Groups

There is a super analogue of the classical Heisenberg group, defined as follows.

Recall that a *super Harish–Chandra pair* is a pair $(G_0, \mathfrak{g})$ associated to any (real or complex) super Lie group G, where G_0 is the reduced Lie group of G, and $\mathfrak{g}$ the super Lie algebra of G. Conversely, given any pair $(G_0, \mathfrak{g})$ consisting of a Lie group G_0 and a super Lie algebra $\mathfrak{g}$ which satisfy certain compatibility conditions (see [4] for more details), there exists a unique super Lie group G whose associated SHCP is $(G_0, \mathfrak{g})$. A *morphism* between two SHCPs $(G_0, \mathfrak{g})$ and $(H_0, \mathfrak{h})$ is a pair consisting of a Lie group homomorphism $G_0 \to H_0$ and a super Lie algebra homomorphism $\mathfrak{g} \to \mathfrak{h}$ satisfying the requisite compatibility conditions.

With this definition of morphism, SHCPs form a category, and it is a theorem that the functor $G \mapsto (G_0, \mathfrak{g})$ defines an equivalence of categories between the category of super Lie groups and their morphisms and the category of super Harish–Chandra pairs and their morphisms.

The super Heisenberg group is then the (real) super Harish–Chandra pair

$$SH := (H_3, \mathfrak{sh}(2|1))$$

where H_3 is the classical Heisenberg group (realized here as a subgroup of $GL(3|1, \mathbb{R})$):

$$H_3 := \left\{ \left(\begin{array}{ccc|c} 1 & x & z & 0 \\ 0 & 1 & y & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array} \right) : x, y, z \in \mathbb{R} \right\}$$

and $\mathfrak{sh}(3|1)$ is the super Lie algebra of matrices:

$$\mathfrak{sh}(3|1) := \left\{ \left(\begin{array}{ccc|c} 0 & a & c & \alpha \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array} \right) : a, b, c, \alpha \in \mathbb{R} \right\}$$

The action of H_3 on $\mathfrak{sh}(3|1)$ is the adjoint action.

The Stone-Von Neumann theorem asserts that the classical Heisenberg group possesses a unique irreducible representation, up to choice of central character. There is a super analogue of this theorem for the super Heisenberg group; a proof may be found in [23].

In [15] the author constructs an explicit model $(\pi, \rho, \mathcal{H})$ for the irreducible representation whose existence is guaranteed by the super Stone-Von Neumann theorem. We will describe it briefly here. Fix $\tau \in \mathbb{H}$. Consider the super vector space $\mathcal{V}$ of holomorphic functions on $\mathbb{C}^{1|1}$.

Define a metric $\langle \cdot, \cdot \rangle$ on $\mathcal{V}$ by:

$$\begin{aligned} \langle f, g \rangle &= \int e^{\frac{-y^2}{Im\,\tau}} f \bar{g}\, dx\, dy \\ \langle \zeta f, \zeta g \rangle &= \int e^{\frac{-y^2}{Im\,\tau}} f \bar{g}\, dx\, dy \\ \langle v, w \rangle &= 0 \text{ if } v \text{ and } w \text{ have opposite parity.} \end{aligned}$$

Let $\mathcal{H} := \{s \in \mathcal{V} : \langle s, s \rangle < \infty\}$. $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ is a super Hilbert space. Define an action $\pi : H_3 \to \mathcal{H}$:

$$\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \cdot f = e^{2ic}[e^{\pi i a^2 \tau + 2\pi i a z} f(z + a\tau + b)] \text{ for } f \in \mathcal{H}_0$$

$$\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \cdot \zeta g = e^{2ic}[e^{\pi i a^2 \tau + 2\pi i a (z + \frac{1}{2})} \zeta g(z + a\tau + b)] \text{ for } \zeta g \in \mathcal{H}_1$$

We define an $\mathbb{R}$-linear map $\rho^{\pi} : (\mathfrak{sh}(3|1))_1 \to (End(\mathcal{H}))_1$; it suffices to specify the action of

$$Z := \left(\begin{array}{ccc|c} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array} \right)$$

We define $\rho^{\pi}(Z)$ by:

$$\begin{aligned} f(z) &\mapsto \zeta f(z + \tfrac{1}{2}) \\ \zeta g(z) &\mapsto g(z - \tfrac{1}{2}) \end{aligned}$$

Having constructed this model, the author proves:

Theorem. *The even super theta functions are the unique elements of this representation which are invariant under the integral subgroup of H_3, and which are projectively invariant, i.e., carried into a constant multiple of themselves, by the action of the subgroup $\mathbb{R}_m^{1|1}$.*

This hints at further connections between the super Heisenberg group and super theta functions.

3 Π-Projective Geometry

3.1 The Super Skew Field D

Let $D := \mathbb{C}[\theta]$, with θ odd, $\theta^2 = -1$. This object is a noncommutative superalgebra, and any nonzero homogeneous element is invertible; it is therefore called the *super skew field* over $\mathbb{C}$ [9]. It may be regarded as a super analogue of the quaternion algebra over the field of real numbers $\mathbb{R}$.

The multiplicative group D^* of D defines a complex super Lie group via the functor of points:

$$S \mapsto [\mathcal{O}_S \otimes_{\mathbb{C}} D]_0^*$$

In [8] it is noted that $\mathbb{G}_m^{1|1}$ may be naturally identified with D^* via the following isomorphism of functors of points:

$$a + \alpha \mapsto a + \alpha\theta$$

3.2 Π-Invertible Sheaves and Π-Projective Spaces

The quasiperiodicity properties of Levin's super theta functions may be interpreted as the assertion that they really descend from functions on $\mathbb{G}_a^{1|1} \times \mathbb{H}$ (resp. $\mathbb{G}_a^{1|1} \times \mathbb{H} \times \mathbb{C}^{0|1}$) to sections of vector bundles on the even (resp. odd) families. However they are not sections of line bundles, but rather of objects called Π-*invertible sheaves*.

Definition. Let X be a complex supermanifold (superscheme, etc.). A Π-*invertible sheaf* on X is a pair (S, ϕ) where S is a locally free sheaf of rank $1|1$, and ϕ is an odd endomorphism on S with $\phi^2 = -1$.

A *morphism* of Π-invertible sheaves (S, ϕ) and (S', ϕ') is simply a morphism of sheaves $f : S \to S'$ such that $f \circ \phi = \phi' \circ f$.

It is easily seen that a D-module structure on a super vector space (resp. free R-module, etc.) V is equivalent to an odd endomorphism ϕ on V such that $\phi^2 = -1$; this holds true e.g., for Va free module over a complex superalgebra as well. Therefore Π-invertible sheaves are simply $1|1$ locally free sheaves with a D-action. Hence Π-projective geometry might therefore more properly be regarded as "D-geometry."

The basic properties of Π-invertible sheaves are summarized in Sect. 4 of [24], and we shall review some of them here.

3.2.1 Sheaf-Cohomological Interpretation

Let X be a complex supermanifold. We shall prove that isomorphism classes of Π-invertible sheaves are in one-to-one correspondence with elements of $H^1(X, \mathcal{O}^*)$, just as isomorphism classes of $1|0$ line bundles on a supermanifold X are in one-to-one correspondence with the elements of $H^1(X, \mathcal{O}_0^*)$. (We emphasize that $H^1(X, \mathcal{O}^*)$ has only the structure of a pointed set, not an abelian group, since $\mathcal{O}^*$ is a sheaf of nonabelian groups. For the basic facts on the cohomology theory of sheaves of nonabelian groups, we refer the reader to part II, Sect. 1 of [1]).

First we show that the transition functions of a Π-invertible sheaf may be reduced from $\underline{GL(1|1)}$ to take values in $\underline{\mathbb{G}_m^{1|1}} = \mathcal{O}^*$. Here we note that $\mathbb{G}_m^{1|1}$ is naturally realized as a subgroup of $GL(1|1)$ via the homomorphism:

$$a + \alpha \mapsto \left(\begin{array}{c|c} a & \alpha \\ \hline \alpha & a \end{array}\right),$$

where this homomorphism is interpreted at the level of functors of points.

We begin by considering the situation where V is a rank $1|1$ free module over a complex superalgebra A, with an odd A-linear endomorphism ϕ such that $\phi^2 = -1$. One checks readily that $\phi^2 = -1 \iff$ any matrix representing ϕ has the form:

$$P := \left(\begin{array}{c|c} \alpha & a \\ \hline -a^{-1} & -\alpha \end{array}\right)$$

for some a a unit in A_0.

We apply a change of basis via the matrix:

$$B := \left(\begin{array}{c|c} ia^{-1} & i\alpha \\ \hline 0 & 1 \end{array}\right)$$

obtaining:

$$\begin{aligned} P' := BPB^{-1} &= \left(\begin{array}{c|c} ia^{-1} & i\alpha \\ \hline 0 & 1 \end{array}\right) \left(\begin{array}{c|c} \alpha & a \\ \hline -a^{-1} & -\alpha \end{array}\right) \left(\begin{array}{c|c} -ia & -a\alpha \\ \hline 0 & 1 \end{array}\right) \\ &= \left(\begin{array}{c|c} 0 & i \\ \hline i & 0 \end{array}\right) \end{aligned}$$

In other words, given a pair (V, ϕ) as above, there exists a homogeneous basis $\{v_0|v_1\}$ for V in which ϕ is represented by the matrix:

$$\left(\begin{array}{c|c} 0 & i \\ \hline i & 0 \end{array}\right)$$

We note that this argument relies only on the fact that the field over which we are working contains $\sqrt{-1}$.

Let (S, ϕ) be a Π-invertible sheaf. By the argument just given, there exists a trivialization of S with trivializing cover $\{U_j\}$, for which $\phi_j := \phi|_{U_j}$ is given as the matrix

$$\phi_j = \left(\begin{array}{c|c} 0 & i \\ \hline i & 0 \end{array}\right)$$

for all j. Let

$$A_{jk} := \left(\begin{array}{c|c} a_{jk} & \alpha_{jk} \\ \hline \beta_{jk} & b_{jk} \end{array}\right)$$

be the transition functions of S in this trivialization. The $\{A_{jk}\}$ are a 1-cocycle with values in $GL(1|1, \mathcal{O}(U_j \cap U_k))$. That the $\{\phi_j\}$ define an endomorphism of S is equivalent to the assertion that $A_{jk}\phi_k = \phi_j A_{jk}$ for all j, k, or:

$$\left(\begin{array}{c|c} a_{jk} & \alpha_{jk} \\ \hline \beta_{jk} & b_{jk} \end{array}\right)\left(\begin{array}{c|c} 0 & i \\ \hline i & 0 \end{array}\right) = \left(\begin{array}{c|c} 0 & i \\ \hline i & 0 \end{array}\right)\left(\begin{array}{c|c} a_{jk} & \alpha_{jk} \\ \hline \beta_{jk} & b_{jk} \end{array}\right)$$

from which it is obvious that $a_{jk} = b_{jk}, \alpha_{jk} = \beta_{jk}$ for all j, k. This proves the desired reduction of the transition functions to $\mathbb{G}_m^{1|1}$.

Conversely, given a 1-cocycle

$$A'_{jk} := \left(\begin{array}{c|c} a'_{jk} & \alpha'_{jk} \\ \hline \alpha'_{jk} & a'_{jk} \end{array}\right)$$

taking values in $\mathbb{G}_m^{1|1}$, the $\{A_{jk}\}$ define transition functions for a $1|1$ locally free sheaf S', and one may define an endomorphism ϕ' of S' by setting

$$\phi'_j := \left(\begin{array}{c|c} 0 & i \\ \hline i & 0 \end{array}\right)$$

for all j. It is readily seen that ϕ' so defined is an odd endomorphism of S' with $(\phi')^2 = -1$. This shows that a Π-invertible sheaf is the same thing as a 1-cocycle for the sheaf $\underline{\mathbb{G}_m^{1|1}}$.

Now suppose given an open cover $\{U_j\}$ of X, and recall that two 1-cocycles A_{jk}, A'_{jk} of a sheaf $\mathcal{F}$ of (not necessarily abelian) groups are *cohomologous* if there exists a collection $\{B_l\}$, with $B_l \in \mathcal{O}_{\mathcal{F}}(U_l)$, such that $A'_{jk} = B_j A_{jk} B_k^{-1}$ on $U_j \cap U_k$

for all j, k. It is easily checked that being cohomologous is an equivalence relation. We will now prove that two $\mathbb{G}_m^{1|1}$-cocycles A_{jk}, A'_{jk} define isomorphic Π-invertible sheaves if and only if A_{jk} is cohomologous to A'_{jk}.

Let $f : S \to S'$ be an isomorphism of Π-invertible sheaves. Let us take a pair of trivializations for S, S' in which both ϕ_j, ϕ'_j are both represented by the matrix:

$$\phi_j := \left(\begin{array}{c|c} 0 & i \\ \hline i & 0 \end{array}\right)$$

We may assume that the trivializations share a common open cover $\{U_j\}$ by taking a common refinement, if necessary. Let A_{jk}, A'_{jk} be the transition functions for S, S' in their respective trivializations. Then the isomorphism f is represented by a collection $\{f_l\}$, with $f_l \in GL(1|1)(\mathcal{O}_{U_l})$. The fact that f is a sheaf homomorphism is equivalent to the equation:

$$f_j A_{jk} = A'_{jk} f_k$$

and the fact that f is invertible means that:

$$A'_{jk} = f_j A_{jk} f_k^{-1}$$

It only remains to show that the $\{f_l\}$ lie in $\mathbb{G}_m^{1|1}$; this follows from the fact that f intertwines ϕ and ϕ' and that we have chosen trivializations where ϕ_l, ϕ'_l are represented by the matrix $\left(\begin{array}{c|c} 0 & i \\ \hline i & 0 \end{array}\right)$ for all l. This proves that A_{jk}, A'_{jk} are cohomologous.

The converse statement that two cohomologous 1-cocycles in $\mathbb{G}_m^{1|1}$ represent isomorphic Π-invertible sheaves follows by reversing each step of the argument just given: two such cocycles B_{jk}, B'_{jk} define Π-invertible sheaves, and if $B'_{jk} = g_j B_{jk} g_k^{-1}$ for some collection $\{g_l\}$, with $g_l \in \mathbb{G}_m^{1|1}(\mathcal{O}_{U_l})$, then $B'_{jk} g_k = g_j B_{jk}$, that is to say the collection $\{g_l\}$ defines an isomorphism of the corresponding Π-invertible sheaves. This completes the proof that isomorphism classes of Π-invertible sheaves on a supermanifold X correspond bijectively to elements of the cohomology set $H^1(X, \underline{\mathbb{G}_m^{1|1}}) = H^1(X, \mathcal{O}^*)$. We denote the pointed set $H^1(X, \mathcal{O}^*)$ by $Pic^{\Pi}(X)$, and its distinguished element is $\mathcal{O} \oplus \Pi\mathcal{O}$.

3.2.2 Relation to Line Bundles

Let $\mathcal{O}_1$ denote the sheaf of odd functions under addition. There is a natural short exact sequence:

$$0 \to \mathcal{O}_0^* \to \mathcal{O}^* \to \mathcal{O}_1 \to 0,$$

where the homomorphism $\mathcal{O}^* \to \mathcal{O}_1$ is given by $a + \alpha \mapsto a^{-1}\alpha$. We may take the sheaf cohomology of this exact sequence, obtaining the exact sequence of pointed sets:

$$Pic_0(X) \to Pic^{\Pi}(X) \to H^1(X, \mathcal{O}_1)$$

Here $Pic_0(X)$ is the usual Picard group of line bundles of rank $1|0$. It may be shown that the map $Pic_0(X) \to Pic^{\Pi}(X)$ is $L \mapsto L \oplus \Pi L$, where Π denotes the parity-reversal functor.

The map $Pic^{\Pi}(X) \to H^1(X, \mathcal{O}_1)$ is referred to in [24] as an "odd analogue of the Chern class," a characteristic class associated to Π-invertible sheaves, which has no counterpart in classical (even) geometry.

3.2.3 Π-Projective Spaces

The projective spaces associated to Π-invertible sheaves are called Π-projective spaces. They are constructed out of the data of a D-module (V, ϕ) and denoted by $\mathbb{P}_{\Pi}(V)$. In [17] they are realized as the subvariety of the super Grassmanian $Gr(1|1, V)$ corresponding to ϕ-invariant $1|1$ subspaces.

In [15], the author produces an explicit construction that shows that the complex Π-projective spaces are quotients of $\mathbb{C}^{n+1|n+1} - \{0\}$ by $\mathbb{G}_m^{1|1}$ (this fact is asserted in [16] without proof).

As we are working over $\mathbb{C}$, we may invoke a theorem of Carmeli and Fioresi [5] that asserts that over a field of characteristic zero, the functor $G \mapsto (G_0, \mathfrak{g})$ determines an equivalence between the category of affine algebraic supergroups and the category of algebraic super Harish Chandra pairs. The SHCP theory thereby becomes available to us in the complex superalgebraic category.

This allows us to define the notion of a supergroup G acting on a superalgebra A in a fashion that bypasses the need for the functor of points: a G-action on A becomes simply a pair consisting of an action of the reduced group G_0 and an action of $\mathfrak{g}$ on A which satisfy certain compatibility conditions (see [4] for details), where $(G_0, \mathfrak{g})$ is the SHCP associated to G. Then the sub-superalgebra A^G of G-invariants is easily defined as the intersection of the sub-superalgebras A^{G_0} and $A^{\mathfrak{g}}$.

We regard $\mathbb{C}^{n+1|n+1} - \{0\}$ as a complex algebraic supermanifold with coordinates $(z_0, \ldots, z_n \,|\, \zeta_0, \ldots, \zeta_n)$. We set $G := \mathbb{G}_m^{1|1}$, regarding $\mathbb{G}_m^{1|1}$ as an algebraic supergroup. A routine calculation shows that the SHCP associated to $\mathbb{G}_m^{1|1}$ is the pair $(\mathbb{C}^*, \mathfrak{g}_m^{1|1})$, where $\mathfrak{g}_m^{1|1}$ is the super Lie algebra with basis $\{C \mid Z\}$ and relations:

$$[C, C] = 0$$
$$[C, Z] = 0$$
$$[Z, Z] = -2C$$

and the action of $\mathbb{C}^*$ on $\mathfrak{g}_m^{1|1}$ is the trivial one. We define an action of $\mathbb{G}_m^{1|1}$ on $\mathbb{C}^{n+1|n+1} - \{0\}$ by letting $\mathbb{C}^*$ act via:

$$t \cdot (z_0, \ldots, z_n, \zeta_0, \ldots, \zeta_n) = (tz_0, \ldots, tz_n, t\zeta_0, \ldots, t\zeta_n)$$

and letting Z act via the vector field:

$$\sum_i z_i \partial_{\zeta_i} - \zeta_i \partial_{z_i}$$

The reduced space of $\mathbb{C}^{n+1|n+1} - \{0\}$ is $\mathbb{C}^{n+1} - \{0\}$, and there is a quotient map $\pi : \mathbb{C}^{n+1} - \{0\} \to \mathbb{CP}^n$. $\mathbb{CP}^n$ is covered by open sets $U_i = \{z_i \neq 0\}$. We set the reduced space of $\mathbb{CP}^n_\Pi$ to be $\mathbb{P}^n$ and then define the sheaf of functions on $\mathbb{CP}^n_\Pi$:

$$\mathcal{O}_{\mathbb{P}^n_\Pi}(U) := \mathcal{O}_{\mathbb{C}^{n+1|n+1}-\{0\}}(\pi^{-1}(U))^G,$$

where $\cdot^G$ denotes the subsuperalgebra of G-invariants. We pay particular attention to the invariant superalgebras $A_i := \mathcal{O}(\pi^{-1}(U_i))^G$. In [15] it is proven that these algebras A_i are free complex superalgebras on the $n|n$ generators

$$w_j := \frac{z_j}{z_i} - \frac{\zeta_i \zeta_j}{z_i^2}$$

$$\eta_j := \frac{\zeta_j}{z_i} - \frac{\zeta_i z_j}{z_i^2},$$

where j runs through $1, \ldots, \hat{i}, \ldots, n$, implying that $\mathbb{P}^n_\Pi$ is a smooth algebraic supervariety.

We show that $\mathbb{C}^{n+1|n+1} - \{0\}$ is a $\mathbb{G}_m^{1|1}$-principal bundle over $\mathbb{P}^n_\Pi$ so constructed, justifying the assertion that $\mathbb{P}^n_\Pi$ is the quotient of $\mathbb{C}^{n+1|n+1} - \{0\}$. Then we explicitly construct the tautological Π-invertible sheaf $\mathcal{O}_\Pi(1)$ on $\mathbb{P}^n_\Pi$, with its associated endomorphism ϕ such that $\phi^2 = -1$, and prove that it is locally free of rank $1|1$. Manin asserts in [18] that a morphism from a supervariety $f : X \to \mathbb{P}^n_\Pi$ is equivalent to a choice to a Π-invertible sheaf (S, ϕ) on X and a collection of sections $\{s_i, t_i\}$ of S such that $\phi(s_i) = -t_i, \phi(t_i) = s_i$, and such that the s_i, t_i have no common base points.

A key feature of line bundles is that they possess a tensor product such that the product of two line bundles is another line bundle; in particular, one can obtain a very ample line bundle by taking high tensor powers L^k of an ample line bundle on a complex manifold (resp. abstract variety) X, and thus embed X into a projective space, realizing it as a projective variety. As yet there appears to be no natural definition of the tensor product of two Π-invertible sheaves that yields another Π-invertible sheaf (but see [24] for a definition of a "tensor product" on Π-invertible sheaves which does *not* yield another Π-invertible sheaf). This would seem to indicate that Π-projective supervarieties are rather scarce, since, even if an ample Π-invertible sheaf exists on X, one cannot simply take high tensor powers of it to obtain a very ample Π-invertible sheaf.

It is therefore an intriguing phenomenon that these families of super elliptic curves possess natural maps, indeed imbeddings, into Π-projective spaces, given in terms of the super theta functions, as described above. This naturally leads to the question: do higher genus $1|1$ super Riemann surfaces (resp. families of super Riemann surfaces) also possess natural maps into Π-projective spaces, perhaps induced by natural sections of Π-invertible sheaves.

3.2.4 Relative Π-Projective Superspaces

Much of the above discussion can be generalized to the context of abstract algebraic geometry, i.e., that of superschemes. We briefly recall the basic notions of superschemes, starting with *Spec* of a supercommutative ring.

Let R be a supercommutative ring. We define $\underline{Spec}(R)$ to be the super ringed space $(Spec(R_0), \mathcal{O}_{\underline{Spec}(R)})$, where $Spec(R_0)$ denotes the topological space consisting of the set of all prime ideals of the even subring R_0. The closed sets of $Spec(R_0)$ are taken to be all sets of the form $V(\mathfrak{a}) := \{\mathfrak{p} \in Spec(R_0) \mid \mathfrak{p} \supseteq \mathfrak{a}\}$, for $\mathfrak{a}$ an ideal of R_0. A basis of this topology is given by the collection of principal open sets $\{D(f)\}_{f \in R_0}$, where $D(f)$ is the open complement of $V((f))$.

The structure sheaf $\mathcal{O}_{\underline{Spec}(R)}$ is defined as follows. Note that R is naturally an R_0-module. Therefore, given $f \in R_0$, the localization R_f makes sense as an R_0-module, and R_f possesses a natural structure of supercommutative ring induced from R. We then define

$$\mathcal{O}_{\underline{Spec}(R)}(D(f)) := R_f$$

Since the $D(f)$ form a basis of the topology of $Spec(R_0)$, this defines $\mathcal{O}_{\underline{Spec}(R)}$ as a sheaf on $Spec(R_0)$, and then one checks, as in the purely even case, that $\underline{Spec}(R)$ so defined is indeed a local super ringed space. Any local super ringed space which is isomorphic to $\underline{Spec}(R)$ for some supercommutative ring R is then called an *affine superscheme*.

Superschemes are then defined in complete analogy to ordinary schemes: a *superscheme* $X := (|X|, \mathcal{O}_X)$ is a local super ringed space which is locally isomorphic to affine superschemes, in the sense that every point of $|X|$ is contained in some open neighborhood $|U|$ such that $(|U|, \mathcal{O}_X|_U)$ is isomorphic to some affine superscheme.

The construction of $\mathbb{CP}^n_\Pi$ given in the previous section can be adapted to define relative Π-projective spaces $\mathbb{P}^n_{\Pi,A}$ over affine superschemes $\underline{Spec}(A)$. To this end, let A be an arbitrary supercommutative ring. Then we consider the polynomial A-superalgebra on $n+1|n+1$ variables $R := A[z_0, \ldots, z_n, \zeta_0, \ldots, \zeta_n]$. Let Z be the odd A-linear derivation

$$Z := \sum_i z_i \partial_{\zeta_i} - \zeta_i \partial_{z_i};$$

this is the algebraic analogue of the $\mathbb{G}_m^{1|1}$-action on R. Then we consider $R_i := (R[z_i^{-1}]^0)^Z$, the degree-zero elements of the localized ring $R[z_i^{-1}]$ that are annihilated by Z. Arguments similar to those employed in the case of Π-projective spaces over a field show that the R_i are isomorphic to polynomial A-algebras on the $n|n$ variables:

$$w_j := \frac{z_j}{z_i} - \frac{\zeta_i \zeta_j}{z_i^2}$$

$$\eta_j := \frac{\zeta_j}{z_i} - \frac{\zeta_i z_j}{z_i^2},$$

where j runs through $1, \dots, \hat{i}, \dots, n$, and that the $Spec(R_i)$ may be glued together to form a superscheme, which we denote by $\mathbb{P}^n_{\Pi,A}$.

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Charge Orbits and Moduli Spaces of Black Hole Attractors

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Abstract We report on the theory of "large" U-duality charge orbits and related "moduli spaces" of extremal black hole attractors in $\mathcal{N} = 2$, $d = 4$ Maxwell–Einstein supergravity theories with symmetric scalar manifolds, as well as in $\mathcal{N} \geqslant 3$-extended, $d = 4$ supergravities.

1 Introduction

The *Attractor Mechanism* (AM) [1–5] governs the dynamics in the scalar manifold of Maxwell–Einstein (super) gravity theories. It keeps standing as a crucial fascinating key topic within the international high-energy physics community. Along the last years, a number of papers have been devoted to the investigation of attractor configurations of extremal black p-branes in diverse space-time dimensions; for some lists of references, see e.g. [6–14].

The AM is related to dynamical systems with fixed points, describing the equilibrium state and the stability features of the system under consideration.[1] When

[1] We recall that a point x_{fix} where the phase velocity $v\left(x_{fix}\right)$ vanishes is called a *fixed* point, and it gives a representation of the considered dynamical system in its equilibrium state,

$$v\left(x_{fix}\right) = 0.$$

The fixed point is said to be an *attractor* of some motion $x\left(t\right)$ if

$$lim_{t\rightarrow\infty}x\left(t\right) = x_{fix}.$$

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the AM holds, the particular property of the long-range behavior of the dynamical flows in the considered (dissipative) system is the following: in approaching the fixed points, properly named *attractors*, the orbits of the dynamical evolution lose all memory of their initial conditions, but however the overall dynamics remains completely deterministic.

The first example of AM in supersymmetric systems was discovered in the theory of static, spherically symmetric, asymptotically flat extremal dyonic black holes in $\mathcal{N}=2$ Maxwell–Einstein supergravity in $d=4$ and 5 space-time dimensions (see the first two references of [1–5]). In the following, we will briefly present some basic facts about the $d=4$ case.

The multiplet content of a completely general $\mathcal{N}=2$, $d=4$ supergravity theory is the following (see e.g. [15], and references therein):

1. The *gravitational* multiplet

$$\left(V_{\mu}^{a}, \psi^{A}, \psi_{A}, A^{0}\right), \tag{1}$$

described by the *Vielbein* one-form V^a ($a=0,1,2,3$) (together with the spin-connection one-form ω^{ab}), the $SU(2)$ doublet of gravitino one-forms ψ^A, ψ_A ($A=1,2$, with the upper and lower indices respectively denoting right and left chirality, i.e. $\gamma_5\psi_A=-\gamma_5\psi^A=1$), and the graviphoton one-form A^0;

2. n_V *vector* supermultiplets

$$\left(A^{I}, \lambda^{iA}, \overline{\lambda}_{A}^{\bar{i}}, z^{i}\right), \tag{2}$$

each containing a gauge boson one-form A^I ($I=1,...,n_V$), a doublet of gauginos (zero-form spinors) $\lambda^{iA}, \overline{\lambda}_{A}^{\bar{i}}$, and a complex scalar field (zero-form) z^i ($i=1,...,n_V$). The scalar fields z^i can be regarded as coordinates on a complex manifold $\mathcal{M}_{n_V}$ ($dim_{\mathbb{C}}\mathcal{M}_{n_V}=n_V$), which is actually a *special Kähler* manifold;

3. n_H *hypermultiplets*

$$\left(\zeta_{\alpha}, \zeta^{\alpha}, q^{u}\right), \tag{3}$$

each formed by a doublet of zero-form spinors, that is the hyperinos $\zeta_\alpha, \zeta^\alpha$ ($\alpha=1,...,2n_H$), and four real scalar fields q^u ($u=1,...,4n_H$), which can be considered as coordinates of a quaternionic manifold $\mathcal{Q}_{n_H}$ ($dim_{\mathbb{H}}\mathcal{Q}_{n_H}=n_H$).

At least in absence of gauging, the n_H hypermultiplets are spectators in the AM. This can be understood by looking at the transformation properties of the Fermi fields: the hyperinos $\zeta_\alpha, \zeta^\alpha$'s transform independently on the vector fields, whereas the gauginos' supersymmetry transformations depend on the Maxwell vector fields. Consequently, the contribution of the hypermultiplets can be dynamically decoupled from the rest of the physical system; in particular, it is also completely independent from the evolution dynamics of the complex scalars z^i's coming from the vector multiplets (i.e. from the evolution flow in $\mathcal{M}_{n_V}$). By disregarding for simplicity's sake the fermionic and gauging terms, the supersymmetry transformations of hyperinos read (see e.g. [15], and references therein)

$$\delta\zeta_\alpha = i\mathcal{U}_u^{B\beta}\partial_\mu q^u \gamma^\mu \varepsilon^A \epsilon_{AB} \mathbb{C}_{\alpha\beta}, \tag{4}$$

implying the asymptotical configurations of the quaternionic scalars of the hypermultiplets to be unconstrained, and therefore to vary continuously in the manifold $\mathcal{Q}_{n_H}$ of the related quaternionic non-linear sigma model.

Thus, as far as ungauged theories are concerned, for the treatment of AM one can restrict to consider $\mathcal{N} = 2$, $d = 4$ Maxwell–Einstein supergravity, in which n_V vector multiplets (2) are coupled to the gravity multiplet (1). The relevant dynamical system to be considered is the one related to the radial evolution of the configurations of complex scalar fields of such n_V vector multiplets. When approaching the event horizon of the black hole, the scalars dynamically run into fixed points, taking values which are only function (of the ratios) of the electric and magnetic charges associated to Abelian Maxwell vector potentials under consideration.

The inverse distance to the event horizon is the fundamental evolution parameter in the dynamics towards the fixed points represented by the *attractor* configurations of the scalar fields. Such near-horizon configurations, which "attracts" the dynamical evolutive flows in $\mathcal{M}_{n_V}$, are completely independent on the initial data of such an evolution, i.e. on the spatial asymptotical configurations of the scalars. Consequently, for what concerns the scalar dynamics, the system completely loses memory of its initial data, because the dynamical evolution is "attracted" by some fixed configuration points, purely depending on the electric and magnetic charges.

Recently, intriguing connections with the (quantum) theory of information arose out [16–21].

In the framework of supergravity theories, extremal black holes can be interpreted as BPS (Bogomol'ny-Prasad-Sommerfeld)-saturated [22] interpolating metric singularities in the low-energy effective limit of higher-dimensional superstrings or M-theory [23]. Their asymptotically relevant parameters include the ADM mass [24], the electrical and magnetic charges (defined by integrating the fluxes of related field strengths over the 2-sphere at infinity), and the asymptotical values of the (dynamically relevant set of) scalar fields. The AM implies that the class of black holes under consideration loses all its "scalar hair" within the near-horizon geometry. This means that the extremal black hole solutions, in the near-horizon limit in which they approach the Bertotti-Robinson $AdS_2 \times S^2$ conformally flat metric [25, 26], are characterized only by electric and magnetic charges, but not by the continuously-varying asymptotical values of the scalar fields.

An important progress in the geometric interpretation of the AM was achieved in the last reference of [1–5], in which the attractor near-horizon scalar configurations were related to the critical points of a suitably defined black hole effective potential function V_{BH}, whose explicit form in maximal supergravity is e.g. given by (20) below. In general, V_{BH} is a positive definite function of scalar fields and electric and magnetic charges, and its non-degenerate critical points in $\mathcal{M}_{n_V}$

$$\forall i = 1, ..., n_V, \frac{\partial V_{BH}}{\partial z^i} = 0 : \quad V_{BH}|_{\frac{\partial V_{BH}}{\partial z}=0} > 0, \tag{5}$$

fix the scalar fields to depend only on electric and magnetic fluxes (charges). In the Einstein two-derivative approximation, the (semi)classical Bekenstein–Hawking entropy (S_{BH}) - area (A_H) formula [27–31] yields the (purely charge-dependent) black hole entropy S_{BH} to be

$$S_{BH} = \pi \frac{A_H}{4} = \pi \; V_{BH}|_{\frac{\partial V_{BH}}{\partial z}=0} = \pi \sqrt{|\mathcal{I}_4|}, \tag{6}$$

where $\mathcal{I}_4$ is the unique independent invariant homogeneous polynomial (quartic in charges) in the relevant representation $\mathbf{R}_V$ of G in which the charges sit (see (7) and discussion below). The last step of (6) does not apply to $d = 4$ supergravity theories with quadratic charge polynomial invariant, namely to the $\mathcal{N} = 2$ *minimally coupled* sequence [32] and to the $\mathcal{N} = 3$ [33] theory; in these cases, in (6) $\sqrt{|\mathcal{I}_4|}$ gets replaced by $|\mathcal{I}_2|$.

In presence of $n = n_V + 1$ Abelian vector fields, the fluxes sit in a $2n$-dimensional representation $\mathbf{R}_V$ of the U-duality group G, defining the embedding of G itself into $Sp(2n, \mathbb{R})$, which is the largest group acting linearly on the fluxes themselves:

$$G \overset{\mathbf{R}_V}{\subsetneq} Sp(2n, \mathbb{R}) . \tag{7}$$

It should be pointed out that we here refer to U-duality as the continuous version of the U-duality groups introduced in [34]. This is consistent with the assumed (semi-)classical limit of large charges, also indicated by the fact that we consider $Sp(2n, \mathbb{R})$, and not $Sp(2n, \mathbb{Z})$ (no Dirac–Schwinger–Zwanziger quantization condition is implemented on the fluxes themselves).

After [35–37], the $\mathbf{R}_V$-representation space of the U-duality group is known to exhibit a stratification into disjoint classes of orbits, which can be defined through invariant sets of constraints on the (lowest order, actually unique) G-invariant $\mathcal{I}$ built out of the symplectic representation $\mathbf{R}_V$. It is here worth remarking the crucial distinction between the "large" orbits and "small" orbits. While the former have $\mathcal{I} \neq 0$ and support an attractor behavior of the scalar flow in the near-horizon geometry of the extremal black hole background [1–5], for the latter the Attractor Mechanism does not hold, they have $\mathcal{I} = 0$ and thus they correspond to solutions with vanishing Bekenstein–Hawking [27–31] entropy (*at least* at the Einsteinian two-derivative level).

This short report, contributing to the Proceedings of the Workshop *"Supersymmetry in Mathematics and Physics"* (organized by Prof. R. Fioresi and Prof. V. S. Varadarajan), held on February 2010 at the Department of Mathematics of the University of California at Los Angeles, presents the main results of the theory of U-duality charge orbits and "moduli spaces" of extremal black hole attractor solutions in supergravity theories with $\mathcal{N} \geqslant 2$ supercharges in $d = 4$ space-time dimensions. In particular, $\mathcal{N} = 2$ Maxwell–Einstein theories with symmetric scalar manifolds will be considered.

The plan of this short review is as follows.

Section 2 introduces the "large" (i.e. attractor-supporting) charge orbits of the $\mathcal{N} = 2$, $d = 4$ *symmetric* Maxwell–Einstein supergravities, namely of those $\mathcal{N} = 2$ supergravity theories in which a certain number of Abelian vector multiplets is coupled to the gravity multiplet, and the corresponding complex scalars span a special Kähler manifold which is also a symmetric coset $\frac{G}{H_0 \times U(1)}$, where G is the U-duality group and $H_0 \times U(1)$ is its maximal compact subgroup.

Then, Sect. 3 is devoted to the analysis of the "large" charge orbits of the maximal $\mathcal{N} = 8$ supergravity theory. The non-compactness of the stabilizer groups of such (generally non-symmetric) coset orbits gives rise to the so-called "moduli spaces" of attractor solutions, namely proper subspaces of the scalar manifold of the theory in which the Attractor Mechanism is not active.

The "moduli spaces" of the various classes of non-supersymmetric attractors in $\mathcal{N} = 2$, $d = 4$ *symmetric* Maxwell–Einstein supergravities are then reported and discussed in Sect. 4.

The short Sect. 5 concludes the paper, analyzing the attractor-supporting orbits of $\mathcal{N} \geqslant 3$-extended "pure" and matter-coupled theories, whose scalar manifolds are all symmetric.

2 Charge Orbits of $\mathcal{N} = 2$, $d = 4$ Symmetric Maxwell–Einstein Supergravities

$\mathcal{N} = 2$, $d = 4$ Maxwell–Einstein supergravity theories [38–40] with homogeneous symmetric special Kähler vector multiplets' scalar manifolds $\frac{G}{H_0 \times U(1)}$ will be shortly referred to as *symmetric* Maxwell–Einstein supergravities. The various symmetric non-compact special Kähler spaces $\frac{G}{H_0 \times U(1)}$ (with $H_0 \times U(1)$ being the maximal compact subgroup with symmetric embedding (*mcs*) of G, the $d = 4$ U-duality group) have been classified in [41, 42] (see e.g. [43] for a recent account), and they are reported in Table 1.

All these theories can be obtained by dimensional reduction of the minimal $\mathcal{N} = 2$, $d = 5$ supergravities [38–40], and they all have cubic prepotential holomorphic functions. The unique exception is provided by the theories with $\mathbb{CP}^n$ scalar manifolds, describing the *minimal coupling* of n Abelian vector multiplets to the gravity multiplet itself [32] (see also [44, 45]); in this case, the prepotential is quadratic in the scalar fields, and thus $C_{ijk} = 0$.

By disregarding the $\mathbb{CP}^n$ sequence, the cubic prepotential of all these theories is related to the norm form of the Euclidean degree-3 Jordan algebra that defines them [38–40]. The reducible sequence in the third row of Table 1, usually referred to as

Table 1 Riemannian globally symmetric non-compact special Kähler spaces (*alias* vector multiplets' scalar manifolds of the *symmetric* $\mathcal{N}=2$, $d=4$ Maxwell Einstein supergravity theories). r denotes the rank of the manifold, whereas n_V stands for the number of vector multiplets

	$\frac{G}{H_0 \times U(1)}$	r	$\dim_{\mathbb{C}} \equiv n_V$
Minimal coupling $n \in \mathbb{N}$	$\mathbb{CP}^n \equiv \frac{SU(1,n)}{U(1)\times SU(n)}$	1	n
$\mathbb{R} \oplus \mathbf{\Gamma}_{1,n-1},\ n \in \mathbb{N}$	$\frac{SU(1,1)}{U(1)} \times \frac{SO(2,n)}{SO(2)\times SO(n)}$	$2\ (n=1)$ $3\ (n \geqslant 2)$	$n+1$
$J_3^{\mathbb{O}}$	$\frac{E_{7(-25)}}{E_{6(-78)}\times U(1)}$	3	27
$J_3^{\mathbb{H}}$	$\frac{SO^*(12)}{U(6)}$	3	15
$J_3^{\mathbb{C}}$	$\frac{SU(3,3)}{S\left(U(3)\times U(3)\right)}$	3	9
$J_3^{\mathbb{R}}$	$\frac{Sp(6,\mathbb{R})}{U(3)}$	3	6

the *generic Jordan family*, is based on the sequence of *reducible* Euclidean Jordan algebras $\mathbb{R} \oplus \mathbf{\Gamma}_{1,n-1}$, where $\mathbb{R}$ denotes the 1-dimensional Jordan algebra and $\mathbf{\Gamma}_{1,n-1}$ stands for the degree-2 Jordan algebra with a quadratic form of Lorentzian signature $(1, n-1)$, which is nothing but the Clifford algebra of $O(1, n-1)$ [46].

Then, four other theories exist, defined by the irreducible degree-3 Jordan algebras $J_3^{\mathbb{O}}$, $J_3^{\mathbb{H}}$, $J_3^{\mathbb{C}}$ and $J_3^{\mathbb{R}}$, namely the Jordan algebras of Hermitian 3×3 matrices over the four division algebras $\mathbb{O}$ (octonions), $\mathbb{H}$ (quaternions), $\mathbb{C}$ (complex numbers) and $\mathbb{R}$ (real numbers) [38–40, 46–49]. Because of their symmetry groups fit in the celebrated *Magic Square* of Freudenthal, Rozenfeld and Tits [50–52], these theories have been named *"magic."* By defining $A \equiv \dim_{\mathbb{R}}\mathbb{A}$ $(=8,4,2,1$ for $\mathbb{A}=\mathbb{O},\mathbb{H},\mathbb{C},\mathbb{R}$, respectively), the complex dimension of the scalar manifolds of the "magic" Maxwell–Einstein theories is $3\,(A+1)$. It should also be recalled that the $\mathcal{N}=2$ "magic" theory based on $J_3^{\mathbb{H}}$ shares the same bosonic sector with the $\mathcal{N}=6$ "pure" supergravity (see e.g. [53–55]), and accordingly in this case the attractors enjoy a "dual" interpretation [44]. Furthermore, it should also be remarked that $J_2^{\mathbb{A}} \sim \Gamma_{1,A+1}$ (see e.g. the eighth reference of [6–14]).

Within these theories, the "large"charge orbits, i.e. the ones supporting extremal black hole attractors have a non-maximal (nor generally symmetric) coset structure. The results [44] are reported in Table 2. After [35], the charge orbit supporting ($\frac{1}{2}$-) BPS attractors has coset structure

$$\mathcal{O}_{BPS} = \frac{G}{H_0}, \text{ with } H_0 \times U(1) \overset{mcs}{\subsetneq} G. \tag{8}$$

Table 2 Charge orbits of attractors in *symmetric* $\mathcal{N} = 2$, $d = 4$ Maxwell–Einstein supergravities

	$\frac{1}{2}$-BPS orbit $\mathcal{O}_{\frac{1}{2}-BPS} = \frac{G}{H_0}$	nBPS $Z_H \neq 0$ orbit $\mathcal{O}_{nBPS,Z_H \neq 0} = \frac{G}{\widehat{H}}$	nBPS$Z_H = 0$ orbit $\mathcal{O}_{nBPS,Z_H = 0} = \frac{G}{\widetilde{H}}$
Minimal coupling $n \in \mathbb{N}$	$\frac{SU(1,n)}{SU(n)}$	–	$\frac{SU(1,n)}{SU(1,n-1)}$
$\mathbb{R} \oplus \mathbf{\Gamma}_{1,n-1} n \in \mathbb{N}$	$\frac{SU(1,1)}{U(1)} \times \frac{SO(2,n)}{SO(n)}$	$\frac{SU(1,1)}{SO(1,1)} \times \frac{SO(2,n)}{SO(1,n-1)}$	$\frac{SU(1,1)}{U(1)} \times \frac{SO(2,n)}{SO(2,n-2)}$
$J_3^{\mathbb{O}}$	$\frac{E_{7(-25)}}{E_6}$	$\frac{E_{7(-25)}}{E_{6(-26)}}$	$\frac{E_{7(-25)}}{E_{6(-14)}}$
$J_3^{\mathbb{H}}$	$\frac{SO^*(12)}{SU(6)}$	$\frac{SO^*(12)}{SU^*(6)}$	$\frac{SO^*(12)}{SU(4,2)}$
$J_3^{\mathbb{C}}$	$\frac{SU(3,3)}{SU(3) \times SU(3)}$	$\frac{SU(3,3)}{SL(3,\mathbb{C})}$	$\frac{SU(3,3)}{SU(2,1) \times SU(1,2)}$
$J_3^{\mathbb{R}}$	$\frac{Sp(6,\mathbb{R})}{SU(3)}$	$\frac{Sp(6,\mathbb{R})}{SL(3,\mathbb{R})}$	$\frac{Sp(6,\mathbb{R})}{SU(2,1)}$

As shown in [44], there are other two charge orbits supporting extremal black hole attractors, and they are both non-supersymmetric (not saturating the BPS bound [22]). One has non-vanishing $\mathcal{N}=2$ central charge at the horizon ($Z_H \neq 0$), with coset structure

$$\mathcal{O}_{nBPS,Z_H\neq 0}=\frac{G}{\widehat{H}}, \text{ with } \widehat{H}\times SO\left(1,1\right)\subsetneq G, \tag{9}$$

where $\widehat{H}$ denotes the $d=5$ U-duality group, and thus $SO\left(1,1\right)$ corresponds to the S^1-radius in the Kaluza-Klein reduction $d=5\rightarrow 4$. Also the remaining attractor-supporting charge orbit is non-supersymmetric, but it corresponds to $Z_H=0$; its coset structure reads

$$\mathcal{O}_{nBPS,Z_H=0}=\frac{G}{\widetilde{H}}, \text{ with } \widetilde{H}\times U\left(1\right)\subsetneq G. \tag{10}$$

It is worth remarking that $\widehat{H}$ and $\widetilde{H}$ are the only two non-compact forms of H_0 such that the group embedding in the right-hand side of (10) and (9) are both maximal and symmetric (see e.g. [56–58]).

Due to (8), H_0 is the maximal compact symmetry group of the particular class of non-degenerate critical points of the effective black hole potential V_{BH} corresponding to BPS attractors. On the other hand, the maximal compact symmetry group of the non-BPS $Z_H\neq 0$ and non-BPS $Z_H=0$ critical points of V_{BH} respectively is

$$\widehat{h}=mcs\left(\widehat{H}\right);\ \ \widetilde{h}=mcs\left(\widetilde{H}\right). \tag{11}$$

Actually, in the non-BPS $Z_H=0$ case, the maximal compact symmetry is $\widetilde{h}'\equiv\frac{\widetilde{h}}{U(1)}$; see e.g. [44] for further details.

General results on the rank $\mathfrak{r}$ of the $2n_V\times 2n_V$ Hessian matrix $\mathbf{H}$ of V_{BH} are known. Firstly, the BPS (non-degenerate) critical points of $V_{BH,\mathcal{N}=2}$ are stable, and thus $\mathbf{H}_{BPS}$ has no massless modes (see the fifth reference of [1–5]), and its rank is maximal: $\mathfrak{r}_{BPS}=2n_V$. Furthermore, the analysis of [44] showed that for the other two classes of (non-degenerate) non-supersymmetric critical points of $V_{BH,\mathcal{N}=2}$, the rank of $\mathbf{H}$ is model-dependent:

$$\mathbb{CP}^n:\mathfrak{r}_{nBPS,Z_H=0}=2; \tag{12}$$

$$\mathbb{R}\oplus\mathbf{\Gamma}_{1,n-1}:\left\{\begin{array}{l}\mathfrak{r}_{nBPS,Z_H\neq 0}=n+2;\\ \\ \mathfrak{r}_{nBPS,Z_H=0}=6;\end{array}\right. \tag{13}$$

$$J_3^{\mathbb{A}}:\left\{\begin{array}{l}\mathfrak{r}_{nBPS,Z_H\neq 0}=3A+4;\\ \\ \mathfrak{r}_{nBPS,Z_H=0}=2A+6.\end{array}\right. \tag{14}$$

3 $\mathcal{N} = 8$, $d = 4$ Supergravity

The analysis of extremal black hole attractors in the theory with the maximal number of supercharges, namely in $\mathcal{N} = 8$, $d = 4$ supergravity, provides a simpler, warm-up framework for the analysis and classification of the "moduli spaces" of the two classes ($Z_H \neq 0$ and $Z_H = 0$) of non-BPS attractors of quarter-minimal Maxwell–Einstein supergravities with symmetric scalar manifolds, which have been introduced in Sect. 2.

Maximal supergravity in four dimensions is based on the real, rank-7, 70-dimensional homogeneous symmetric manifold

$$\frac{G_{\mathcal{N}=8}}{H_{\mathcal{N}=8}} = \frac{E_{7(7)}}{SU(8)}, \tag{15}$$

where $SU(8) = mcs\left(E_{7(7)}\right)$. After [35–37,59,60], two classes of (non-degenerate) critical points of $V_{BH,\mathcal{N}=8}$ are known to exist:

- The $\frac{1}{8}$-BPS class, supported by the orbit

$$\mathcal{O}_{\frac{1}{8}-BPS,\mathcal{N}=8} \equiv \frac{G_{\mathcal{N}=8}}{\mathcal{H}_{\mathcal{N}=8}} = \frac{E_{7(7)}}{E_{6(2)}},\ E_{6(2)} \times U(1) \subsetneq E_{7(7)}; \tag{16}$$

- The non-BPS class, supported by the orbit

$$\mathcal{O}_{nBPS,\mathcal{N}=8} \equiv \frac{G_{\mathcal{N}=8}}{\widehat{\mathcal{H}}_{\mathcal{N}=8}} = \frac{E_{7(7)}}{E_{6(6)}},\ E_{6(6)} \times SO(1,1) \subsetneq E_{7(7)}. \tag{17}$$

Both charge orbits $\mathcal{O}_{\frac{1}{8}-BPS,\mathcal{N}=8}$ and $\mathcal{O}_{nBPS,\mathcal{N}=8}$ belong to the fundamental representation space **56** of the maximally non-compact (split) form $E_{7(7)}$ of the exceptional group E_7. The embeddings in the right-hand side of (16) and (17) are both maximal and symmetric (see e.g. [56, 58]). Among all non-compact forms of the exceptional Lie group E_6 (i.e. $E_{6(-26)}$, $E_{6(-14)}$, $E_{6(2)}$ and $E_{6(6)}$), $E_{6(2)}$ and $E_{6(6)}$ are the only two which are maximally and symmetrically embedded (through an extra group factor $U(1)$ or $SO(1,1)$) into $E_{7(7)}$.

In the maximal theory, the Hessian matrix $\mathbf{H}_{\mathcal{N}=8}$ of the effective potential $V_{BH,\mathcal{N}=8}$ is a square 70×70 symmetric matrix. At $\frac{1}{8}$-BPS attractor points, $\mathbf{H}_{\mathcal{N}=8}$ has rank 30, with 40 massless modes [61] sitting in the representation $(\mathbf{20}, \mathbf{2})$ of the enhanced $\frac{1}{8}$-BPS symmetry group $SU(6) \times SU(2) = mcs\left(\mathcal{H}_{\mathcal{N}=8}\right)$ [60]. Moreover, at non-BPS attractor points, $\mathbf{H}_{\mathcal{N}=8}$ has rank 28, with 42 massless modes sitting in the representation **42** of the enhanced non-BPS symmetry group $USp(8) = mcs\left(\widehat{\mathcal{H}}_{\mathcal{N}=8}\right)$ [60]. Actually, the massless modes of $\mathbf{H}_{\mathcal{N}=8}$ are "flat" directions of $V_{BH,\mathcal{N}=8}$ at the corresponding classes of its critical points. Thus, such "flat" directions of the critical $V_{BH,\mathcal{N}=8}$ span some "moduli spaces" of the attractor solutions [62], corresponding to the scalar degrees of freedom which are

Table 3 "Moduli spaces" of non-BPS $Z_H \neq 0$ critical points of $V_{BH,\mathcal{N}=2}$ in $\mathcal{N}=2$, $d=4$ *symmetric* Maxwell–Einstein supergravities. They are the $\mathcal{N}=2$, $d=5$ symmetric real special manifolds

	$\frac{\widehat{H}}{mcs(\widehat{H})}$	r	$\dim_{\mathbb{R}}$
$\mathbb{R} \oplus \mathbf{\Gamma}_{1,n-1},\ n \in \mathbb{N}$	$SO(1,1) \times \frac{SO(1,n-1)}{SO(n-1)}$	$1\ (n=1)$ $2\ (n \geqslant 2)$	n
$J_3^{\mathbb{O}}$	$\frac{E_{6(-26)}}{F_{4(-52)}}$	2	26
$J_3^{\mathbb{H}}$	$\frac{SU^*(6)}{USp(6)}$	2	14
$J_3^{\mathbb{C}}$	$\frac{SL(3,\mathbb{C})}{SU(3)}$	2	8
$J_3^{\mathbb{R}}$	$\frac{SL(3,\mathbb{R})}{SO(3)}$	2	5

not stabilized by the *Attractor Mechanism* [1–5] at the black hole event horizon. In the $\mathcal{N}=8$ case, such "moduli spaces" are the following real symmetric submanifolds of $\frac{E_{7(7)}}{SU(8)}$ itself [62]:

$$\frac{1}{8}\text{-BPS} : \mathcal{M}_{\frac{1}{8}-BPS} = \frac{\mathcal{H}_{\mathcal{N}=8}}{mcs\,(\mathcal{H}_{\mathcal{N}=8})} = \frac{E_{6(2)}}{SU(6) \times SU(2)},\ \dim_{\mathbb{R}} = 40, \text{rank} = 4; \tag{18}$$

$$\text{non-BPS} : \mathcal{M}_{nBPS} = \frac{\widehat{\mathcal{H}}_{\mathcal{N}=8}}{mcs\left(\widehat{\mathcal{H}}_{\mathcal{N}=8}\right)} = \frac{E_{6(6)}}{USp(8)},\ \dim_{\mathbb{R}} = 42, \text{rank} = 6. \tag{19}$$

It is easy to realize that $\mathcal{M}_{\frac{1}{8}-BPS}$ and $\mathcal{M}_{nBPS}$ are nothing but the cosets of the non-compact stabilizer of the corresponding supporting charge orbit ($E_{6(2)}$ and $E_{6(6)}$, respectively) and of its *mcs*. Actually, this is the very structure of all "moduli spaces" of attractors (see Sects. 4 and 5). Moreover, $\mathcal{M}_{nBPS}$ is nothing but the scalar manifold of $\mathcal{N}=8$, $d=5$ supergravity. This holds more in general, and, as given by the treatment of Sect. 4 (see also Table 3), the "moduli space" of $\mathcal{N}=2$, $d=4$ non-BPS $Z_H \neq 0$ attractors is nothing but the scalar manifold of the $d=5$ uplift of the corresponding theory [62] (see also [63]).

Following [62] and considering the maximal supergravity theory, we now explain the reason why the "flat" directions of the Hessian matrix of the effective potential at its critical points actually span a "moduli space" (for a recent discussion, see also [64]).

Let us start by recalling that $V_{BH,\mathcal{N}=8}$ is defined as

$$V_{BH,\mathcal{N}=8} \equiv \frac{1}{2} Z_{AB}(\phi, Q) \overline{Z}^{AB}(\phi, Q), \tag{20}$$

where Z_{AB} is the antisymmetric complex $\mathcal{N} = 8$ central charge matrix [36]

$$Z_{AB}(\phi, Q) = \left(Q^T L(\phi)\right)_{AB} = \left(Q^T\right)_\Lambda L^\Lambda_{AB}(\phi). \tag{21}$$

ϕ denotes the 70 real scalar fields parametrising the aforementioned coset $\frac{E_{7(7)}}{SU(8)}$, Q is the $\mathcal{N} = 8$ charge vector sitting in the fundamental irrepr. **56** of the U-duality group $E_{7(7)}$. Moreover, $L^\Lambda_{AB}(\phi)$ is the ϕ-dependent coset representative, i.e. a local section of the principal bundle $E_{7(7)}$ over $\frac{E_{7(7)}}{SU(8)}$ with structure group $SU(8)$.

The action of an element $g \in E_{7(7)}$ on $V_{BH,\mathcal{N}=8}(\phi, Q)$ is such that

$$V_{BH,\mathcal{N}=8}(\phi, Q) = V_{BH,\mathcal{N}=8}\left(\phi_g, Q^g\right) = V_{BH,\mathcal{N}=8}\left(\phi_g, \left(g^{-1}\right)^T Q\right); \tag{22}$$

thus, $V_{BH,\mathcal{N}=8}$ is not $E_{7(7)}$-invariant, because its coefficients (given by the components of Q) do not in general remain the same. The situation changes if one restricts $g \equiv g_Q \in H_Q$ to belong to the stabilizer H_Q of one of the orbits $\frac{E_{7(7)}}{H_Q}$ spanned by the charge vector Q within the **56** representation space of $E_{7(7)}$ itself. In such a case:

$$Q^{g_Q} = Q \Rightarrow V_{BH,\mathcal{N}=8}(\phi, Q) = V_{BH,\mathcal{N}=8}\left(\phi_{g_Q}, Q\right). \tag{23}$$

Then, it is natural to split the 70 real scalar fields ϕ as $\phi = \left\{\phi_Q, \breve{\phi}_Q\right\}$, where $\phi_Q \in \frac{H_Q}{\mathrm{mcs}(H_Q)} \subsetneq \frac{E_{7(7)}}{SU(8)}$ and $\breve{\phi}_Q$ coordinatise the complement of $\frac{H_Q}{\mathrm{mcs}(H_Q)}$ in $\frac{E_{7(7)}}{SU(8)}$. By denoting with

$$V_{BH,\mathcal{N}=8,crit}\left(\phi_Q, Q\right) \equiv V_{BH,\mathcal{N}=8}(\phi, Q)\big|_{\frac{\partial V_{BH,\mathcal{N}=8}}{\partial \breve{\phi}_Q}=0} (\neq 0) \tag{24}$$

the values of $V_{BH,\mathcal{N}=8}$ along the equations of motion for the scalars $\breve{\phi}_Q$, the invariance of $V_{BH,\mathcal{N}=8,crit}\left(\phi_Q, Q\right)$ under H_Q directly follows from (23):

$$V_{BH,\mathcal{N}=8,crit}\left(\left(\phi_Q\right)_{g_Q}, Q\right) = V_{BH,\mathcal{N}=8,crit}\left(\phi_Q, Q\right). \tag{25}$$

Now, it is crucial to observe that H_Q generally is a *non-compact* Lie group; for instance, $H_Q = E_{6(2)} \equiv \mathcal{H}_{\mathcal{N}=8}$ for $Q \in \mathcal{O}_{\frac{1}{8}-BPS,\mathcal{N}=8}$ given by (16), and $H_Q = E_{6(6)} \equiv \widehat{\mathcal{H}}_{\mathcal{N}=8}$ for $Q \in \mathcal{O}_{nBPS,\mathcal{N}=8}$ given by (17). This implies $V_{BH,\mathcal{N}=8}$ to be independent *at its critical points* on the subset

$$\phi_Q \in \frac{H_Q}{\mathrm{mcs}\left(H_Q\right)} \subsetneq \frac{E_{7(7)}}{SU\left(8\right)}. \tag{26}$$

Thus, $\frac{H_Q}{\mathrm{mcs}\left(H_Q\right)}$ can be regarded as the "moduli space" of the attractor solutions supported by the charge orbit $\frac{E_{7(7)}}{H_Q}$. For $\mathcal{N} = 8$ non-degenerate critical points, supported by $\mathcal{O}_{\frac{1}{8}-BPS,\mathcal{N}=8}$ and $\mathcal{O}_{nBPS,\mathcal{N}=8}$, this reasoning yields to the "moduli spaces" $\mathcal{M}_{\frac{1}{8}-BPS}$ and $\mathcal{M}_{nBPS}$, respectively given by (18) and (17).

The results on $\mathcal{N} = 8$ theory are summarized in the last row of Tables 4 and 5.

The above arguments apply to a general, not necessarily supersymmetric, Maxwell–Einstein theory with scalars coordinatising an homogeneous (not necessarily symmetric) space. In particular, one can repeat the above reasoning for all supergravities with $\mathcal{N} \geqslant 1$ based on homogeneous (not necessarily symmetric) manifolds $\frac{G_{\mathcal{N}}}{H_{\mathcal{N}}} \equiv \frac{G_{\mathcal{N}}}{\mathrm{mcs}(G_{\mathcal{N}})}$, also in presence of matter multiplets. It is here worth recalling that theories with $\mathcal{N} \geqslant 3$ all have symmetric scalar manifolds (see e.g. [53]).

A remarkable consequence is the existence of "moduli spaces" of attractors is the following. By choosing Q belonging to the orbit $\frac{G_{\mathcal{N}}}{H_Q} \subsetneq \mathbf{R}_V\left(G_{\mathcal{N}}\right)$ and supporting a class of non-degenerate critical points of $V_{BH,\mathcal{N}}$, *up to some "flat" directions* (spanning the "moduli space" $\frac{H_Q}{\mathrm{mcs}\left(H_Q\right)} \subsetneq \frac{G_{\mathcal{N}}}{H_{\mathcal{N}}}$), *all* such critical points of $V_{BH,\mathcal{N}}$ in *all* $\mathcal{N} \geqslant 0$ Maxwell–Einstein (super)gravities with an homogeneous (not necessarily symmetric) scalar manifold (also in presence of matter multiplets) are *stable*, and thus they are *attractors* in a generalized sense. For $d = 4$ supergravities, $H_Q = \mathcal{H}, \widehat{\mathcal{H}}$ or $\widetilde{\mathcal{H}}$ (see e.g. Tables 4 and 5; see the third, fifth and seventh references of [6–14]).

All this reasoning can be extended to a number of space-time dimensions $d \neq 4$ (see e.g. [65–68]). As found in [69, 70] for "large" charge orbits of $\mathcal{N} = 2$, $d = 4$ *stu* model, and then proved in a model-independent way in [64], the "moduli spaces" of charge orbits are defined *all along the corresponding scalar flows*, and thus they can be interpreted as "moduli spaces" of unstabilized scalars at the event horizon of the extremal black hole, as well as "moduli spaces" of the ADM mass [24] of the extremal black hole at spatial infinity.

Remarkably, one can associate "moduli spaces" also to non-attractive, "small" orbits, namely to charge orbits supporting black hole configurations which have vanishing horizon area in the Einsteinian approximation [68, 71, 72]. Differently from "large" orbits, for "small" orbits there exists a "moduli space" also when the semi-simple part of H_Q is compact, and it has translational nature [68]. Clearly, in the "small" case the interpretation at the event horizon breaks down, simply because such an horizon does not exist at all, *at least* in Einsteinian supergravity approximation.

Table 4 Charge orbits supporting extremal black hole attractors in $\mathcal{N} \geqslant 3$-extended, $d = 4$ supergravities (n is the number of matter multiplets) (see the fifth reference of [6–14]. The related Euclidean degree-**3** Jordan algebra is also given (*if any*)

	$\frac{1}{\mathcal{N}}$-BPS orb $\frac{G_{\mathcal{N}}}{\mathcal{H}_{\mathcal{N}}}$	nBPS $Z_{AB,H} \neq 0$ orb $\frac{G_{\mathcal{N}}}{\widehat{\mathcal{H}}_{\mathcal{N}}}$	nBPS $Z_{AB,H} = 0$ orb $\frac{G_{\mathcal{N}}}{\widetilde{\mathcal{H}}_{\mathcal{N}}}$
$\mathcal{N} = 3$ $n \in \mathbb{N}$	$\frac{SU(3,n)}{SU(2,n)}$	–	$\frac{SU(3,n)}{SU(3,n-1)}$
$\mathcal{N} = 4$ $n \in \mathbb{N}$, $\mathbb{R} \oplus \mathbf{\Gamma}_{5,n-1}$	$\frac{SU(1,1)}{U(1)} \times \frac{SO(6,n)}{SO(4,n)}$	$\frac{SU(1,1)}{SO(1,1)} \times \frac{SO(6,n)}{SO(5,n-1)}$	$\frac{SU(1,1)}{U(1)} \times \frac{SO(6,n)}{SO(6,n-2)}$
$\mathcal{N} = 5$	$\frac{SU(1,5)}{SU(3) \times SU(2,1)}$	–	–
$\mathcal{N} = 6$ $J_3^{\mathbb{H}}$	$\frac{SO^*(12)}{SU(4,2)}$	$\frac{SO^*(12)}{SU^*(6)}$	$\frac{SO^*(12)}{SU(6)}$
$\mathcal{N} = 8$ $J_3^{\mathbb{O}_s}$	$\frac{E_{7(7)}}{E_{6(2)}}$	$\frac{E_{7(7)}}{E_{6(6)}}$	–

Table 5 "Moduli spaces" of black hole attractor solutions in $\mathcal{N} \geqslant 3$-extended, $d = 4$ supergravities. n is the number of matter multiplets (see the fifth reference of [6–14])

	$\frac{1}{\mathcal{N}}$-BPS "moduli space" $\frac{\mathcal{H}_{\mathcal{N}}}{mcs(\mathcal{H}_{\mathcal{N}})}$	nBPS $Z_{AB,H} \neq 0$ "moduli space" $\frac{\widehat{\mathcal{H}}_{\mathcal{N}}}{mcs(\widehat{\mathcal{H}}_{\mathcal{N}})}$	nBPS $Z_{AB,H} = 0$ "moduli space" $\frac{\widetilde{\mathcal{H}}_{\mathcal{N}}}{mcs(\widetilde{\mathcal{H}}_{\mathcal{N}})}$
$\mathcal{N} = 3$	$\frac{SU(2,n)}{SU(2) \times SU(n) \times U(1)}$	–	$\frac{SU(3,n-1)}{SU(3) \times SU(n-1) \times U(1)}$
$\mathcal{N} = 4$	$\frac{SO(4,n)}{SO(4) \times SO(n)}$	$SO(1,1) \times \frac{SO(5,n-1)}{SO(5) \times SO(n-1)}$	$\frac{SO(6,n-2)}{SO(6) \times SO(n-2)}$
$\mathcal{N} = 5$	$\frac{SU(2,1)}{SU(2) \times U(1)}$	–	–
$\mathcal{N} = 6$	$\frac{SU(4,2)}{SU(4) \times SU(2) \times U(1)}$	$\frac{SU^*(6)}{USp(6)}$	–
$\mathcal{N} = 8$	$\frac{E_{6(2)}}{SU(6) \times SU(2)}$	$\frac{E_{6(6)}}{USp(8)}$	–

4 "Moduli Spaces"of Attractors in $\mathcal{N} = 2, d = 4$ Symmetric Maxwell–Einstein Supergravities

The arguments outlined in Sect. 3 can be used to determine the "moduli spaces" of non-BPS attractors (with $Z_H \neq 0$ or $Z_H = 0$) for all $\mathcal{N} = 2$, $d = 4$ Maxwell–Einstein supergravities with symmetric scalar manifolds [62].

After the fifth reference of [1–5], it is known that, regardless of the geometry of the vector multiplets' scalar manifold, the BPS non-degenerate critical points of $V_{BH,\mathcal{N}=2}$ are *stable*, and thus define an attractor configuration in strict sense, in which all scalar fields are stabilized in terms of charges by the Attractor Mechanism [1–5]. This is ultimately due to the fact that the Hessian matrix $\mathbf{H}_{\frac{1}{2}-BPS}$ at such critical points has no massless modes at all. Therefore, as far as the metric of the scalar manifold is non-singular and positive-definite and no massless degrees of freedom appear in the theory, there is no "moduli space" for BPS attractors in *any* $\mathcal{N} = 2$, $d = 4$ Maxwell–Einstein supergravity theory.

This is an important difference with respect to $\frac{1}{\mathcal{N}}$-BPS attractors in $\mathcal{N} > 2$-extended supergravities (see the third, fifth and seventh references of [6–14]; for instance, in $\mathcal{N} = 8$ theory $\frac{1}{8}$-BPS attractors exhibit the "moduli space" $\mathcal{M}_{\frac{1}{8}-BPS}$ given by (18). From a group theoretical perspective, such a difference can be ascribed to the *compactness* of the stabilizer H_0 of the "large" BPS charge orbit $\mathcal{O}_{\frac{1}{2}-BPS,\mathcal{N}=2}$ in the $\mathcal{N} = 2$ symmetric case (see Table 3). From a supersymmetry perspective, such a difference can be traced back to the different degrees of supersymmetry preservation exhibited by attractor solutions in theories with a different number $\mathcal{N}$ of supercharges. Indeed, ($\frac{1}{2}$-)BPS attractors in theories with local $\mathcal{N} = 2$ supersymmetry are *maximally* supersymmetric (namely, they preserve the *maximum* number of supersymmetries out of the ones related to the asymptotical Poincaré background). On the other hand, in $\mathcal{N}$-extended ($2 < \mathcal{N} \leqslant 8$) supergravities BPS attractors correspond to $\frac{1}{\mathcal{N}}$-BPS configurations, which are *not maximally* supersymmetric. In these latter theories, the maximally supersymmetric configurations correspond to vanishing black hole entropy (at the two-derivative Einsteinian level).

Exploiting the observation below (17), it is possible to determine the "moduli spaces" of non-BPS critical points ($Z_H \neq 0$ or $Z_H = 0$) of $V_{BH,\mathcal{N}=2}$ for all $\mathcal{N} = 2$, $d = 4$ Maxwell–Einstein supergravities with symmetric scalar manifold. Consistent with the notation introduced in Sect. 2 (recall (11)), the $\mathcal{N} = 2$ non-BPS $Z_H \neq 0$ and $Z_H = 0$ "moduli spaces" are respectively denoted by (see [44, 62] for further details on notation)

$$\mathcal{M}_{nBPS,Z_H\neq 0} = \frac{\widehat{H}}{\mathrm{mcs}\left(\widehat{H}\right)} \equiv \frac{\widehat{H}}{\widehat{h}}; \tag{27}$$

$$\mathcal{M}_{nBPS,Z_H=0} = \frac{\widetilde{H}}{\mathrm{mcs}\left(\widetilde{H}\right)} \equiv \frac{\widetilde{H}}{\widetilde{h}} = \frac{\widetilde{H}}{\widetilde{h}' \times U(1)}. \tag{28}$$

Table 6 "Moduli spaces" of non-BPS $Z_H = 0$ critical points of $V_{BH,\mathcal{N}=2}$ in $\mathcal{N} = 2$, $d = 4$ *symmetric* Maxwell–Einstein supergravities. They are (non-special) symmetric Kähler manifolds

	$\frac{\widetilde{H}}{mcs(\widetilde{H})} \equiv \frac{\widetilde{H}}{\widetilde{h}' \times U(1)}$	r	$\dim_{\mathbb{C}}$
Minimal coupling $n \in \mathbb{N}$	$\frac{SU(1,n-1)}{U(1) \times SU(n-1)}$	1	$n-1$
$\mathbb{R} \oplus \mathbf{\Gamma}_{1,n-1}$, $n \in \mathbb{N}$	$\frac{SO(2,n-2)}{SO(2) \times SO(n-2)}$, $n \geqslant 3$	1 ($n = 3$) 2 ($n \geqslant 4$)	$n-2$
$J_3^{\mathbb{O}}$	$\frac{E_{6(-14)}}{SO(10) \times U(1)}$	2	16
$J_3^{\mathbb{H}}$	$\frac{SU(4,2)}{SU(4) \times SU(2) \times U(1)}$	2	8
$J_3^{\mathbb{C}}$	$\frac{SU(2,1)}{SU(2) \times U(1)} \times \frac{SU(1,2)}{SU(2) \times U(1)}$	2	4
$J_3^{\mathbb{R}}$	$\frac{SU(2,1)}{SU(2) \times U(1)}$	1	2

The results are reported in Tables 3 and 6 [62].

As observed below (19), the non-BPS $Z_H \neq 0$ "moduli spaces" are nothing but the scalar manifolds of minimal ($\mathcal{N} = 2$) Maxwell–Einstein supergravity in $d = 5$ space-time dimensions [38–40]. Their real dimension $\dim_{\mathbb{R}}$ (rank r) is the complex dimension $\dim_{\mathbb{C}}$ (rank r) of the $\mathcal{N} = 2$, $d = 4$ symmetric special Kähler manifolds listed in Table 1, minus one. With the exception of the $n = 1$ element of the generic Jordan family $\mathbb{R} \oplus \mathbf{\Gamma}_{1,n-1}$ (the so-called st^2 model) having $\frac{\widehat{H}}{\widehat{h}} = SO(1,1)$ with rank $r = 1$, all non-BPS $Z_H \neq 0$ "moduli spaces" have rank $r = 2$. The results reported in Table 3 are consistent with the "$n_V + 1$ / $n_V - 1$" mass degeneracy splitting of non-BPS $Z_H \neq 0$ attractors [44, 60, 73, 74], holding for a generic special Kähler cubic geometry of complex dimension n_V.

The non-BPS $Z_H = 0$ "moduli spaces," reported in Table 6, are symmetric (generally non-special) Kähler manifolds. Note that in the $n = 1$ and $n = 2$ elements of the generic Jordan family $\mathbb{R} \oplus \mathbf{\Gamma}_{1,n-1}$ (the so-called st^2 and stu models, respectively), there are no non-BPS $Z_H = 0$ "flat" directions at all (see Appendix II of [44], and [62]). By recalling the definition $A \equiv dim_{\mathbb{R}}\mathbb{A}$ given above, the results reported in Table 6 [62] imply that the non-BPS $Z_H = 0$ "moduli spaces" of $\mathcal{N} = 2$, $d = 4$ "magic" supergravities have complex dimension $2A$. As observed in [62], the non-BPS $Z_H = 0$ "moduli space" of $\mathcal{N} = 2$, $d = 4$ "magic" supergravity associated to $J_3^{\mathbb{O}}$ is the manifold $\frac{E_{6(-14)}}{SO(10) \otimes U(1)}$, which is related to another exceptional Jordan triple system over $\mathbb{O}$, as found long time ago in [38–40].

5 $\mathcal{N} \geqslant 3$-Extended, $d = 4$ Supergravities

As anticipated above, the scalar manifolds of all $d = 4$ supergravity theories with $\mathcal{N} \geqslant 3$ supercharges are symmetric spaces. Both $\frac{1}{\mathcal{N}}$-BPS and non-BPS attractors exhibit a related "moduli space." An example is provided by the maximal theory, already reviewed in Sect. 3. As mentioned above, the *non-compactness* of the stabilizer group of the corresponding supporting charge orbit is the ultimate reason of the existence of the "moduli spaces" of attractor solutions [60, 62] (see also the fifth reference of [6–14]).

By performing a supersymmetry truncation down to $\mathcal{N} = 2$ [60, 61, 75], the $\frac{1}{\mathcal{N}}$-BPS "flat" directions of $V_{BH,\mathcal{N}}$ can be interpreted in terms of left-over $\mathcal{N} = 2$ hypermultiplets' scalar degrees of freedom. As studied in [60], for non-BPS "flat" directions the situation is more involved, and an easy interpretation in terms of truncated-away hypermultiplets' scalars degrees of freedom is generally lost.

Tables 4 and 5 report all classes of charge orbits supporting attractor solutions in $\mathcal{N} \geqslant 3$-extended supergravity theories in $d = 4$ space-time dimensions (see the third, fifth and seventh references of [6–14]).

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Maximal Supersymmetry

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Abstract We have studied supersymmetric and super Poincaré invariant deformations of maximally supersymmetric gauge theories, in particular, of ten-dimensional super Yang-Mills theory and of its reduction to a point. We have described all infinitesimal super Poincaré invariant deformations of equations of motion and proved that all of them are Lagrangian deformations and all of them can be extended to formal deformations. Our methods are based on homological algebra, in particular, on the theory of L-infinity and A-infinity algebras. In this paper we formulate some of the results we have obtained, but skip all proofs. However, we describe (in Sects. 2 and 3) the results of the theory of L-infinity and A-infinity algebras that serve as the main tool in our calculations.

1 Supersymmetric Deformations

The superspace technique is a very powerful tool of construction of supersymmetric theories. However this technique does not work for theories with large number of supersymmetries. It is possible to apply methods of homological algebra and formal non-commutative geometry to prove existence of supersymmetric deformations of gauge theories and give explicit construction of them. We describe these methods (based on the theory of L_∞ and A_∞ algebras) in Sects. 2 and 3. (These sections do not depend on Sect. 1.) In Sect. 1 we discuss results obtained by such methods in the analysis of SUSY deformations of 10-dimensional SUSY YM-theory (SYM theory) and its dimensional reductions. These deformations are quite important from the viewpoint of string theory. It is well known that D-brane action in the first

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approximation is given by dimensional reduction of ten-dimensional SYM theory; taking into account the α' corrections we obtain SUSY deformation of this theory. (More precisely, we obtain a power series with respect to α' specifying a formal deformation of the theory at hand.) Our approach is closely related to pure spinors techniques; it seems that it could be quite useful to understand better the pure spinor formalism in string theory constructed by Berkovits [1]. Recall that in component form the action functional of 10-dimensional SUSY YM-theory looks as follows:

$$S_{SYM}(A,\chi) = \int \mathcal{L}_{SYM} d^{10}x = \int \mathrm{tr}\left(\frac{1}{4}F_{ij}F_{ij} + \frac{1}{2}\Gamma^i_{\alpha\beta}\chi^\alpha\nabla_i\chi^\beta\right)d^{10}x, \tag{1}$$

where $A_i(x)$ are gauge fields with values in the Lie algebra of the unitary group $\mathrm{U}(N)$, $\nabla_i = \frac{\partial}{\partial x_i} + A_i(x)$ are covariant derivatives, χ^α are chiral spinors with values in the adjoint representation, $F_{ij} = [\nabla_i, \nabla_j]$ is the curvature.[1] We consider deformations that can be described by action functionals of the form

$$\int \mathrm{tr}(Y)d^{10}x, \tag{2}$$

where $\mathrm{tr}(Y)$ is an arbitrary gauge invariant local expression in terms of gauge fields A_i and spinor fields χ^α. Here Y involves arbitrary product of covariant derivatives of the curvature F_{ij} and spinor fields χ^α. One can say that Y is gauge covariant local expression. The integrals in formulas (1) and (2) are understood as formal expressions. We completely ignore the issues of convergence. In this formal approach the integrals are invariant with respect to some field transformation iff the variation of the integrand is a total derivative. We consider only deformations that can be applied simultaneously to gauge theories with all gauge groups $\mathrm{U}(N)$ where N is an arbitrary positive integer. This remark is important because it is very likely that we miss some important deformations that are defined only for a finite range of N. It is also interesting to consider the dimensional reductions of 10-D SUSY YM theory; after reducing to dimension 4 we obtain $N = 4$ SUSY YM theory; reducing to dimension 1 leads to BFFS matrix model, reducing to dimension 0 leads to IKKT matrix model. Of course, reducing a deformation of 10-D SUSY YM-theory we obtain a deformation of the corresponding reduced theory. However the reduced theory can have more deformations. We will give a complete description of SUSY-deformations of 10-D SUSY YM theory and its reduction to $D = 0$ (of IKKT model). In the components the supersymmetry operators θ_α are equal to

$$\begin{aligned}\theta_\alpha\nabla_i &= \Gamma_{\alpha\beta i}\chi^\beta\\ \theta_\alpha\chi^\beta &= \Gamma^{\beta ij}_\alpha F_{ij}\end{aligned} \tag{3}$$

[1] In this text small Roman indices i, j run over $1,\dots,10$, Greek indices α,β,γ run over $1,\dots,16$. We do not distinguish lower and upper Roman indices because we assume that the ten-dimensional space is equipped with the Riemann metric $(dx^i)^2$.

Denote by D_i the lift of the space-time translation $\partial/\partial x^i$ to the space of the gauge fields and spinor fields. The lift is defined only up to gauge transformation. We fix the gauge freedom in a choice of D_i requiring that

$$\begin{aligned} D_i \nabla_j &= F_{ij} \\ D_i \chi^\alpha &= \nabla_i \chi^\alpha \end{aligned} \tag{4}$$

For fields obeying the equations of motion of S_{SYM} infinitesimal symmetries θ_α satisfy

$$\begin{aligned} [\theta_\alpha, \theta_\beta] &= \Gamma^i_{\alpha\beta} D_i \\ [\theta_\alpha, D_i] A_k &= -\Gamma_{\alpha\beta i} \nabla_k \chi^\beta \\ [\theta_\alpha, D_i] \chi^\gamma &= \Gamma_{\alpha\beta i} [\chi^\beta, \chi^\gamma] \end{aligned} \tag{5}$$

We see that on shell (on the space of solutions of the equations of motion where gauge equivalent solutions are identified) supersymmetry transformations commute with space-time translations:

$$[\theta_\alpha, D_i] = 0 \text{ on shell.} \tag{6}$$

Talking about SUSY-deformations we have in mind deformations of action functional and simultaneous deformation of these 16 supersymmetries. Notice that 10-D SUSY YM-theory has also 16 trivial supersymmetries, corresponding to constant shifts of fermion fields. The analysis of deformations preserving these symmetries was left out of scope of the present paper. We will work with Lagrangian densities $\mathcal{L}$ instead of action functionals $S = \int \mathcal{L} d^{10}x$. As a first approximation to the problem we would like to solve we will study infinitesimal supersymmetric (SUSY) deformations of equations of motion of ten-dimensional SUSY Yang-Mills theory. We reduce this problem to a question in homological algebra. The homological reformulation leads to highly nontrivial, but solvable problem. We will analyze also super Poincaré invariant (= supersymmetric +Lorentz invariant) infinitesimal deformations. We will prove that all of them are Lagrangian deformations of equations of motion (i.e. the deformed equations come from deformed Lagrangian). One of the tools that we are using is the theory of A_∞ and L_∞ algebras. The theory of L_∞ algebras is closely related to BV formalism. One can say that the theory of L_∞ algebras with invariant odd inner product is equivalent to classical BV-formalism if we are working at formal level. (This means that we are considering all functions at hand as formal power series). The theory of A_∞ algebras arises if we would like to consider Yang-Mills theory for all gauge groups $U(N)$ at the same time. Notice that the relation of deformations of A_∞ algebras to Hochschild cohomology and (in the case of algebras with invariant inner product) to cyclic cohomology was established in [15]; our methods are based on generalization of results of [15] to the case of algebras with symmetry. In Sects. 2 and 3 we review the main definitions and theorems of the theory of L_∞ and A_∞ that are used in our calculations. Recall that in BV-formalism (see for example [16]) the space of solutions to the equations of motion (EM) can be characterized as zero locus Sol of odd vector field Q obeying $[Q,Q]=0$.[2] It is convenient to work with the space $Sol/\sim$ obtained from zero locus Sol after identification of physically equivalent solutions. One can consider Q as a derivation of the algebra of functionals on the space of fields M. The space M is equipped with an odd symplectic structure; Q preserves this structure and therefore the corresponding derivation can be written in the form $Qf=\{S,f\}$ where $\{\cdot,\cdot\}$ stands for the odd Poisson bracket and S plays the role of the action functional in the BV formalism. A vector field q_0 on M is an infinitesimal symmetry of EM if $[Q,q_0]=0$. However, studying the symmetry Lie algebra we should disregard trivial symmetries (symmetries of the form $q_0=[Q,\rho_0]$). Hence, in BV formalism talking about symmetry Lie algebra $\mathfrak{g}$ with structure constants $f^{\tau_3}_{\tau_1\tau_2}$ we should impose the condition

$$[q_{\tau_1},q_{\tau_2}]=f^{\tau_3}_{\tau_1\tau_2}q_{\tau_3}+[Q,q_{\tau_1\tau_2}] \tag{7}$$

on the infinitesimal symmetries q_τ. We say in this case that $\mathfrak{g}$ acts weakly on the space of fields. However, it is more convenient to work with notion of L_∞ action of $\mathfrak{g}$. To define L_∞ action we should consider in addition to $q_\tau, q_{\tau_1\tau_2}$ also their higher analogs $q_{\tau_1,\dots,\tau_k}$ and impose some conditions generalizing (7). Introducing the generating function q we can represent these conditions in compact form:

$$d_{\mathfrak{g}}q+[Q,q]+\frac{1}{2}[q,q]=0.$$

Here

$$d_{\mathfrak{g}}=\frac{1}{2}f^{\tau_3}_{\tau_1\tau_2}c^{\tau_1}c^{\tau_2}\frac{\partial}{\partial c^{\tau_3}} \tag{8}$$

stands for the differential calculating the Lie algebra cohomology of $\mathfrak{g}$, c^τ are ghosts corresponding to the Lie algebra. This equation can be formulated also in Lagrangian BV formalism; then we should replace the supercommutators of vector fields by odd Poisson bracket of functionals depending of fields, antifields, ghosts and antifields for ghosts. Using (7) we can study the problem of classification of deformations preserving the given Lie algebra of symmetries. It is important to emphasize that we can start with an arbitrary BV formulation of the given theory and the answer does not depend on our choices. In the case of infinitesimal deformations the classification can be reduced to a homological problem (to the calculation of cohomology of the differential $d_{\mathfrak{g}}+[q,\cdot]$ acting on the space of vector fields depending on ghosts).

Our new paper concludes the series of papers devoted to the analysis of deformations of SYM theories [8–10, 12, 13]. It contains a review of most important

[2] We use a unified notation $[\cdot,\cdot]$ for the commutators and super-commutators.

results of these papers as well as some new constructions. We are planning to publish a detailed exposition of our results in Physics Reports. The first part of this long paper was posted already on the web [14]. In this part we apply the above ideas to the ten-dimensional SYM theory and to its reduction to a point. We describe in this language all infinitesimal super Poincaré invariant deformations. We show that almost all of them are given by a simple general formula (the corresponding Lagrangians are obtained by means of action of 16 supersymmetries). We sketch the proof of the fact that SUSY infinitesimal deformations can be extended to formal SUSY deformations (by definition a formal deformation is a deformation that can be written as a formal power series with respect to some parameter; in string theory the role of this parameter is played by α'). In the second part of the paper we are planning to report results about deformations of d-dimensional reduction of ten-dimensional SYM theory for the case when d is an arbitrary integer between 0 and 10 generalizing the results obtained in the first part for $d = 0$ and $d = 10$. In this part we give a complete calculation of Euler characteristics of all relevant cohomology groups and use this calculation to make a conjecture about the structure of these cohomology groups. For the cases $d = 0$ and $d = 10$ one can prove this conjecture. We show that the homology of the supersymmetry Lie algebras are related to supersymmetric deformations and analyze these homologies.

Our proofs are based on the results of the theory of L_∞ and A_∞ algebras described in Sects. 2 and 3.

2 L_∞ and A_∞ Algebras

Let us consider a supermanifold equipped with an odd vector field Q obeying $[Q, Q] = 0$ (a Q-manifold). Let us introduce a coordinate system in a neighborhood of a point of Q-manifold belonging to zero locus Q. Then the vector field Q considered as a derivation of the algebra of formal power series can be specified by its action on the coordinate functions z^A:

$$Q(z^A) = \sum_n \sum \pm \mu^A_{B_1,\ldots,B_n} z^{B_1} \ldots z^{B_n} \tag{9}$$

We can use tensors $\mu_n = \mu^A_{B_1,\ldots,B_n}$ to define a series of operations. The operation μ_n has n arguments; it can be considered as a linear map $V^{\otimes n} \to V$ (here V stands for the tangent space at $x = 0$). However, it is convenient to change parity of V and consider μ_n as a symmetric map $(\Pi V)^{\otimes n} \to \Pi V$. It is convenient to add some signs in the definition of μ_n. With appropriate choice of signs we obtain that operations μ_n obey some quadratic relations; by definition the operators μ_n obeying these relations specify a structure of L_∞ algebra on $W = \Pi V$. We see that a point of zero locus of the field Q specifies an L_∞ algebra; geometrically one can say that L_∞ algebra is a formal Q-manifold. (A formal manifold is a space whose algebra of functions can be identified with the algebra of formal power series. If the algebra is equipped with odd derivation Q, such that $[Q, Q] = 0$ we have a structure of formal Q manifold.) The considerations of our paper are formal. This means that we can interpret all functions of fields at hand as formal power series. Therefore instead of working with Q-manifolds we can work with L_∞ algebras.

On a Q-manifold with a compatible odd symplectic structure we can choose the coordinates $z^1, \dots, z^n$ as Darboux coordinates, i.e. we can assume that the coefficients of symplectic form do not depend on z. Then the L_∞ algebra is equipped with an invariant odd inner product.

Hence we can say that L_∞ algebra specifies a classical system and L_∞ algebra with invariant odd inner product specifies a Lagrangian classical system.

It is often important to consider $\mathbb{Z}$-graded L_∞-algebras (in BV-formalism this corresponds to the case when the fields are classified according to the ghost number). *We assume in this case that the derivation Q raises the grading (the ghost number) by one.*

An L_∞ algebra where all operations μ_n with $n \geq 3$ vanish can be identified with differential graded Lie algebra (the operation μ_1 is the differential, μ_2 is the bracket). An L_∞ algebra corresponding to Lie algebra with zero differential is $\mathbb{Z}$-graded.

For an L_∞ algebra $\mathfrak{g} = (W, \mu_n)$ one can define a notion of cohomology generalizing the standard notion of cohomology of a Lie algebra. For example, in the case of trivial coefficients we can consider cohomology of Q acting as a derivation of the algebra $\widehat{\mathrm{Sym}}(W^*)$ of formal functions on W (of the algebra of formal series). In the case when the L_∞ algebra corresponds to differential u_z graded Lie algebra $\mathfrak{g}$ this cohomology coincides with Lie algebra cohomology $H(\mathfrak{g}, \mathbb{C})$ (cohomology with trivial coefficients).Considering cohomology of Q acting on the space of vector fields (space of derivations of the algebra of functions) we get a notion generalizing the notion of cohomology $H(\mathfrak{g}, \mathfrak{g})$ (cohomology with coefficients in adjoint representation).[3]

Notice, that to every L_∞ algebra $\mathfrak{g} = (W, \mu_n)$ we can assign a supercommutative differential algebra $(\widehat{\mathrm{Sym}}(W^*), Q)$ that is in some sense dual to the original L_∞-algebra. If only a finite number of operations μ_n does not vanish the operator Q transforms a polynomial function into a polynomial function, hence we can consider also a free supercommutative differential algebra $(\mathrm{Sym}(W^*), Q)$ where $\mathrm{Sym}(W^*)$ stands for the algebra of polynomials on W^*. We will use the notations $(\mathrm{Sym}(W^*), Q) = C^\bullet(\mathfrak{g}), (\widehat{\mathrm{Sym}}(W^*), Q) = \hat{C}^\bullet(\mathfrak{g})$ and the notations $H(\mathfrak{g}, \mathbb{C})$, $\hat{H}(\mathfrak{g}, \mathbb{C})$ for corresponding cohomology.[4] Similarly for the cohomology in the space of derivations we use the notations $H(\mathfrak{g}, \mathfrak{g})$, $\hat{H}(\mathfrak{g}, \mathfrak{g})$.

[3]Usually the definition of Lie algebra cohomology is based on the consideration of polynomial functions of ghosts; using formal series we obtain a completion of cohomology.

[4]In the case of a Lie algebra the functor $C^\bullet$ coincides with Cartan-Eilenberg construction of differential algebra giving Lie algebra cohomology.

In the case when an L_∞ algebra is $\mathbb{Z}$-graded the cohomology $H(\mathfrak{g}, \mathbb{C})$ and $H(\mathfrak{g}, \mathfrak{g})$ are also $\mathbb{Z}$-graded.

One can consider intrinsic cohomology of an L_∞ algebra. They are defined as $\mathrm{Ker}\mu_1/\mathrm{Im}\mu_1$. One says that an L_∞ homomorphism, which is the same as Q-map in the language of Q-manifolds,[5] is a quasi-isomorphism if it induces an isomorphism of intrinsic cohomology. Notice, that in the case of $\mathbb{Z}$-graded L_∞ algebras L_∞ homomorphism should respect $\mathbb{Z}$ grading.

Every $\mathbb{Z}$-graded L_∞ algebra is quasi-isomorphic to an L_∞ algebra with $\mu_1 = 0$. (In other words every L_∞ algebra has a minimal model). Moreover, every $\mathbb{Z}$-graded L_∞ algebra is quasi-isomorphic to a direct product of a minimal L_∞ algebra and a trivial one. (We say that an L_∞ algebra is trivial if $\mathrm{Ker}\mu_1/\mathrm{Im}\mu_1 = 0$.)

The role of zero locus of Q is played by the space of solutions of Maurer-Cartan (MC) equation:

$$\sum_n \frac{1}{n!}\mu_n(a, ..., a) = 0.$$

To obtain a space of solutions $Sol/\sim$ we should factorize space of solutions Sol of MC in appropriate way or work with a minimal model of A.

Our main interest lies in gauge theories. We consider these theories for all groups $\mathrm{U}(n)$ at the same time. To analyze these theories it is more convenient to work with A_∞ instead of L_∞ algebras.

An A_∞ algebra can be defined as a formal non-commutative Q-manifold. In other words we consider an algebra of power series of several variables which do not satisfy any relations (some of them are even, some are odd). An A_∞ algebra is defined as an odd derivation Q of this algebra which satisfies $[Q, Q] = 0$.

More precisely we consider a $\mathbb{Z}_2$-graded vector space W with coordinates z^A. The algebra of formal noncommutative power series $\mathbb{C}\langle\langle z^A\rangle\rangle$ is a completion $\hat{T}(W^*)$ of the tensor algebra $T(W^*)$ (of the algebra of formal noncommutative polynomials). The derivation is specified by the action on z^A:

$$Q(z^A) = \sum_n \sum \pm\mu^A_{B_1,...,B_n} z^{B_1} \dots z^{B_n}. \tag{10}$$

We can use $\mu^A_{B_1,...,B_n}$ to specify a series of operations μ_n on the space ΠW as in L_∞ case. (In the case when W is $\mathbb{Z}$-graded instead parity reversal Π we should consider the shift of the grading by 1.) If Q defines an A_∞ algebra then the condition $[Q, Q] = 0$ leads to quadratic relations between operations; these relations can be used to give an alternative definition of A_∞ algebra. In this definition an A_∞ algebra is a $\mathbb{Z}_2$-graded or $\mathbb{Z}$- graded linear space, equipped with a series of maps $\mu_n : A^{\otimes n} \to A, n \geq 1$ of degree $2 - n$ that satisfy quadratic relations:

[5] Recall, that a map of Q-manifolds is a Q-map if it is compatible with Q.

$$\sum_{i+j=n+1}\sum_{0\le l\le i}\epsilon(l,j)\times$$
$$\mu_i(a_0,\dots,a_{l-1},\mu_j(a_l,\dots,a_{l+j-1}),a_{l+j},\dots,a_n)=0 \tag{11}$$

where $a_m \in A$, and

$$\epsilon(l,j)=(-1)^{j\sum_{0\le s\le l-1}deg(a_s)+l(j-1)+j(i-1)}.$$

In particular, $\mu_1^2=0$.

Notice that in the case when only finite number of operations μ_n do not vanish (the RHS of (10) is a polynomial) we can work with polynomial functions instead of power series. We obtain in this case a differential on the tensor algebra $(T(\Pi W^*),Q)$. The transition from A_∞ algebra $A=(W,\mu_n)$ to a differential graded algebra $\mathrm{cobar}A=(T(\Pi W^*),Q)$ is known as a co-bar construction. If we consider instead of tensor algebra its completion (the algebra of formal power series) we obtain the differential algebra $(\hat{T}(\Pi W^*),Q)$ as a completed co-bar construction $\widehat{\mathrm{cobar}}A$.

The cohomology of differential algebra $(T(\Pi W^*),Q)=\mathrm{cobar}A$ are called Hochschild cohomology of A with coefficients in trivial module $\mathbb{C}$; they are denoted by $HH(A,\mathbb{C})$. Using the completed co-bar construction we can give another definition of Hochschild cohomology of A_∞ algebra as the cohomology of $(\hat{T}(\Pi W^*),Q)=\widehat{\mathrm{cobar}}A$; this cohomology can be defined also in the case when we have infinite number of operations. It will be denoted by $\widehat{HH}(A,\mathbb{C})$. Under some conditions (for example, if μ_1 is equal to zero) one can prove that $\widehat{HH}(A,\mathbb{C})$ is a completion of $HH(A,\mathbb{C})$; in the case when $HH(A,\mathbb{C})$ is finite-dimensional this means that the definitions coincide. We will always assume that $\widehat{HH}(A,\mathbb{C})$ is a completion of $HH(A,\mathbb{C})$.

The theory of A_∞ algebras is very similar to the theory of L_∞ algebras. In particular μ_1 is a differential: $\mu_1^2=0$. It can be used to define intrinsic cohomology of A_∞ algebra. If $\mu_n=0$ for $n\ge 3$ then operations μ_1,μ_2 define a structure of differential associative algebra on W.

The role of equations of motion is played by so called MC equation

$$\sum_{n\ge 1}\mu_n(a,\dots,a)=0 \tag{12}$$

Again to get a space of solutions $Sol/\sim$ we should factorize solutions of MC equation in appropriate way or to work in a framework of minimal models, i.e. we should use the A_∞ algebra that is quasi-isomorphic to the original algebra and has $\mu_1=0$. (Every $\mathbb{Z}$-graded A_∞ algebra has a minimal model.)

We say that 1 is a unit element of A_∞ algebra if $\mu_2(1,a)=\mu_2(a,1)=a$ (i.e. 1 is the unit for binary operation) and all other operations with 1 as one of arguments give zero. For every A_∞ algebra A we construct a new A_∞ algebra $\tilde{A}$ adjoining a unit element.[6]

Having an A_∞ algebra A we can construct a series $L_N(A)$ of L_∞ algebras. If $N = 1$ it is easy to describe the corresponding L_∞ algebra in geometric language. There is a map from noncommutative formal functions on ΠA to ordinary (super) commutative formal functions on the same space. Algebraically it corresponds to imposing (super) commutativity relations among generators. Derivation Q is compatible with such modification. It results in $L_1(A)$. By definition $L_N(A) = L_1(A \otimes \mathrm{Mat}_N)$.

If A is an ordinary associative algebra, then $L_1(A)$ is in fact a Lie algebra- it has the same space and the operation is equal to the commutator $[a, b] = ab - ba$.

The use of A_∞ algebras in the YM theory is based on the remark that one can construct an A_∞ algebra $\mathcal{A}$ with inner product such that for every N the algebra $L_N(\tilde{\mathcal{A}})$ specifies YM theory with matrices of size $N \times N$ in BV formalism. (Recall, that we construct $\tilde{\mathcal{A}}$ adjoining a unit element to $\mathcal{A}$.) The construction of the A_∞ algebra $\tilde{\mathcal{A}}$ is very simple: in the formula for Q in BV-formalism of YM theory in component formalism we replace matrices with free variables. The operator Q obtained in this way specifies also differential algebras $\mathrm{cobar}\tilde{\mathcal{A}}$ and $\widehat{\mathrm{cobar}}\tilde{\mathcal{A}}$. To construct the A_∞ algebra $\mathcal{A}$ in the case of the reduced YM theory we notice that the elements of the basis of $\tilde{\mathcal{A}}$ correspond to the fields of the theory; the element corresponding to the ghost field c is the unit; remaining elements of the basis generate the algebra $\mathcal{A}$. In the case of the reduced theory the differential algebra $\mathrm{cobar}\mathcal{A}$ can be obtained from $\mathrm{cobar}\tilde{\mathcal{A}}$ by means of factorization with respect to the ghost field c; we denote this algebra by BV_0 and the original algebra $\mathcal{A}$ will be denoted by bv_0. The construction in unreduced case is similar. In this case the ghost field (as all other fields) is a function on ten-dimensional space; to obtain $\mathrm{cobar}\mathcal{A}$ (that will be denoted later by BV) we factorize $\mathrm{cobar}\tilde{\mathcal{A}}$ with respect to the ideal generated by the constant ghost field c. We will use the notation bv for the algebra $\mathcal{A}$ in unreduced case.

It is easy to reduce classification of deformations of A_∞ algebra A to a homological problem (see [15]). Namely it is clear that an infinitesimal deformation of Q obeying $[Q, Q] = 0$ is an odd derivation q obeying $[Q, q] = 0$. The operator Q specifies a differential on the space of all derivations by the formula

$$\widetilde{Q}q = [Q, q]. \tag{13}$$

We see that infinitesimal deformations correspond to cocycles of this differential. It is easy to see that two infinitesimal deformations belonging to the same cohomology class are equivalent (if $q = [Q, v]$ where v is a derivation then we can eliminate q by a change of variables $exp(\epsilon v)$, ϵ is the infinitesimal parameter). We see that the classes of infinitesimal deformations can be identified with homology

[6]Notice, that in our definition of Hochschild cohomology we should work with non-unital algebras; otherwise the result for the cohomology with coefficients in $\mathbb{C}$ would be trivial. In more standard approach one defines Hochschild cohomology of unital algebra using the augmentation ideal.

$H(Vect(\mathbb{V}), d)$ of the space of vector fields. (Vector fields on $\mathbb{V}$ are even and odd derivations of $\mathbb{Z}_2$-graded algebra of formal power series.) If the number of operations is finite we can restrict ourselves to polynomial vector fields (in other words, we can replace $Vect(\mathbb{V})$ with cobar$A \otimes A$).

The above construction is another particular case of Hochschild cohomology (the cohomology with coefficients in coefficients in $\mathbb{C}$ was defined in terms of cobar construction.) We denote it by $\widehat{HH}(A, A)$ (if we are working with formal power series) or by $HH(A, A)$ (if we are working with polynomials). Notice that these cohomologies have a structure of (super) Lie algebra induced by commutator of vector fields.

We will give a definition of Hochschild cohomology of differential graded associative algebra

$$(A, d_A), A = \bigoplus_{i \geq 0} A_i$$

with coefficients in a differential bimodule

$$(M, d_M), M = \bigoplus_{i} M_i$$

in terms of Hochschild cochains (multilinear functionals on A with values in M).

We use the standard notation for the degree $\bar{a} = i$ of a homogeneous element $a \in A_i$.

We first associate with the pair (A, M) a bicomplex $(C^{n,m}, D_I, D_{II}), n \geq 0$, $D_I : C^{n,m} \to C^{n+1,m}$,$D_{II} : C^{n,m} \to C^{n,m+1}$ as follows:

$$C^{n.m}(A, M) = \prod_{i_1,\dots,i_n} \mathrm{Hom}(A_{i_1} \otimes \cdots \otimes A_{i_n}, M_{m+i_1+\cdots i_n}) \tag{14}$$

and for $c \in C^{n,m}$

$$\begin{aligned} D_I c &= a_0 c(a_1, \dots, a_n) + \sum_{i=0}^{n-1} (-1)^{i+1} c(a_0, \dots, a_i a_{i+1}, \dots, a_n) \\ &\quad + (-1)^{m\bar{a}_n + n} c(a_0, \dots, a_{n-1}) a_n \\ D_{II} c &= \sum_{i=1}^{n} (-1)^{1+\bar{a}_1+\cdots a_{i-1}} c(a_1, \dots, d_A(a_i), \dots, a_n) + (-1)^k d_M c(a_1, \dots, a_n) \end{aligned} \tag{15}$$

Clearly

$$D_I^2 = 0, D_{II}^2 = 0, D_I D_{II} + D_{II} D_I = 0$$

We define the space of Hochschild i-th cochains as

$$\widehat{C}^i(A, M) = \prod_{n+m=i} C^{n,m}(A, M). \tag{16}$$

Then $\widehat{C}^\bullet(A, M)$ is the complex $(\prod C^i(A, M), D)$ with $D = D_I + D_{II}$. The operator D can also be considered as a differential on the direct sum $C(A, M) = \bigoplus_i C^i(A, M)$ with direct products in (14) and (16) replaced by the direct sums (on the space of non-commutative polynomials on ΠA with values in M). Similarly $\widehat{C}(A, M)$ gets interpreted as the space of formal power series on A with values in M. We define the Hochschild cohomology $HH(A, M)$ and $\widehat{HH}(A, M)$ as the cohomology of this differential. Again under certain conditions that will be assumed in our consideration the second group is a completion of the first one; the groups coincide if $HH(A, M)$ is finite-dimensional.

Notice that $C(A, M)$ can be identified with the tensor product cobar$A \otimes M$ with a differential defined by the formula

$$D(c \otimes m) = (d_{\text{cobar}} + d_M)c \otimes m + [e, c \otimes m], \tag{17}$$

where e is the tensor of the identity map id $\in \text{End}(A) \cong \Pi A^* \otimes A \subset \text{cobar}(A) \otimes A$.

A similar statement is true for $\hat{C}(A, M)$.

Notice that we can define the total grading of Hochschild cohomology $HH^i(A, M)$ where i stands for the total grading defined in terms of A, M and the ghost number (the number of arguments).

In the case when M is the algebra $A(\mu_1 = 0)$ considered as a bimodule the elements of $HH^2(A, A)$ label infinitesimal deformations of associative algebra A and the elements of $HH^\bullet(A, A)$ label infinitesimal deformations of A into an A_∞ algebra. Derivations of A specify elements of $HH^1(A, A)$ (more precisely, a derivation can be considered as one-dimensional Hochschild cocycle; inner derivations are homologous to zero).

We can define Hochschild homology $HH_\bullet$ considering Hochschild chains (elements of $A \otimes ... \otimes A \otimes M$). If A and M are finite-dimensional (or graded with finite-dimensional components) we can define homology by means of dualization of cohomology

$$HH_i(A, M) = HH^i(A, M^*)^*.$$

Let us assume that the differential bimodule M is equipped with bilinear inner product of degree n[7] that descends to non-degenerate inner product on homology. This product generates a quasi-isomorphism $M \to M^*$ and therefore an isomorphism between $HH_i(A, M)$ and $HH^{n-i}(A, M)$ (Poincaré isomorphism). Let us suppose now that our A_∞ algebra has a Lie algebra of symmetries $\mathfrak{g}$ and we are interested in deformations of this algebra preserving the symmetries.

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[7] This means that the inner product does not vanish only if the sum of degrees of arguments is equal to n.

This problem appears if we consider YM theory for all groups U(n) at the same time and we would like to deform the equations of motion preserving the symmetries of the original theory (however we do not require that the deformed equations come from an action functional).

When we are talking about symmetries of A_∞ algebra A we have in mind derivations of the algebra $\widehat{\mathrm{cobar}A} = (\hat{T}(W^*), Q)$ (vector fields on a formal non-commutative manifold) that commute with Q; see (10). We say that symmetries $q_1, \ldots, q_k$ form Lie algebra $\mathfrak{g}$ if they satisfy commutation relations of $\mathfrak{g}$ up to Q-exact terms. These symmetries determine a homomorphism of Lie algebra $\mathfrak{g}$ into Lie algebra $\widehat{HH}(A, A)$. We will say that this homomorphism specifies weak action of $\mathfrak{g}$ on A.

In the case when A_∞ algebra is $\mathbb{Z}$-graded we can impose the condition that the symmetry is compatible with the grading.

Another way to define symmetries of an A_∞ algebra is to identify them with L_∞ actions of Lie algebra $\mathfrak{g}$ on this algebra, i.e. with L_∞ homomorphisms of $\mathfrak{g}$ into differential Lie algebra of derivations *Vect* of the algebra $\widehat{\mathrm{cobar}A}$ (the differential acts on *Vect* as (super)commutator with Q). More explicitly L_∞ action is defined as a linear map

$$q : \mathrm{Sym}\Pi\mathfrak{g} \to \Pi \mathit{Vect} \tag{18}$$

or as an element of odd degree

$$q \in C^{\bullet}(\mathfrak{g}) \otimes \mathit{Vect} \tag{19}$$

obeying

$$d_{\mathfrak{g}} q + [Q, q] + \frac{1}{2}[q, q] = 0, \tag{20}$$

where $d_{\mathfrak{g}}$ is a differential entering the definition of Lie algebra cohomology. We can write q in the form

$$q = \sum \frac{1}{r!} q_{\alpha_1,\ldots,\alpha_r} c^{\alpha_1} \cdots c^{\alpha_r},$$

where c^α are ghosts of the Lie algebra; here $d_{\mathfrak{g}} = \frac{1}{2} f^{\alpha}_{\beta\gamma} c^\beta c^\gamma \frac{\partial}{\partial c^\alpha}$ where $f^{\gamma}_{\alpha\beta}$ denote structure constants of $\mathfrak{g}$.

One can represent (20) as an infinite sequence of equations for the coefficients; the first of these equations has the form

$$[q_\alpha, q_\beta] = f^{\gamma}_{\alpha\beta} q_\gamma + [Q, q_{\alpha\beta}].$$

We see that q_α satisfy commutation relations of $\mathfrak{g}$ up to Q-exact terms (as we have said this means that they specify a weak action of $\mathfrak{g}$ on A and a homomorphism $\mathfrak{g} \to \widehat{HH}(A, A)$).

In the remaining part of this section we use the notation HH instead of $\widehat{HH}$.

Let us consider now an A_∞ algebra A equipped with L_∞ action of Lie algebra $\mathfrak{g}$. To describe infinitesimal deformations of A preserving the Lie algebra of symmetries we should find solutions of (20) and $[Q, Q] = 0$ where Q is replaced by $Q + \delta Q$ and q by $q + \delta q$. After appropriate identifications these solutions can be described by elements of cohomology group that will be denoted by $HH_{\mathfrak{g}}(A, A)$. To define this group we introduce ghosts c^{α}. In other words we multiply $Vect(\mathbb{V})$ by $\Lambda(\Pi\mathfrak{g}^*)$ and define the differential by the formula

$$d = \widetilde{Q} + \frac{1}{2} f^{\alpha}_{\beta\gamma} c^{\beta} c^{\gamma} \frac{\partial}{\partial c^{\alpha}} + q_{\alpha} c^{\alpha} + \dots \tag{21}$$

The dots denote the terms having higher order with respect to c^{α}. They should be included to satisfy $d^2 = 0$ if q_{α} obey commutations of $\mathfrak{g}$ up to Q-exact term. They can be expressed in terms of $q_{\alpha_1,\dots,\alpha_r}$:

$$d = \widetilde{Q} + \frac{1}{2} f^{\alpha}_{\beta\gamma} c^{\beta} c^{\gamma} \frac{\partial}{\partial c^{\alpha}} + \sum_{r \geq 1} \frac{1}{r!} c^{\alpha_1} \cdots c^{\alpha_r} q_{\alpha_1,\dots,\alpha_r} \tag{22}$$

One can say that $HH_{\mathfrak{g}}(A, A)$ is the Lie algebra cohomology of $\mathfrak{g}$ with coefficients in the L_{∞} $\mathfrak{g}$ -module $(Vect(\mathbb{V}), \widetilde{Q})$:

$$HH_{\mathfrak{g}}(A, A) = H(\mathfrak{g}, (Vect(\mathbb{V}), \widetilde{Q})). \tag{23}$$

From the other side in the case of trivial $\mathfrak{g}$ we obtain Hochschild cohomology. Therefore we will use the term Lie-Hochschild cohomology for the group (23).

Every deformation of an A_{∞} algebra A induces a deformation of the algebra $\tilde{A}$ and of the corresponding L_{∞} algebra $L_N(\tilde{A})$; if an A_{∞} algebra has Lie algebra of symmetries $\mathfrak{g}$ then the same is true for this L_{∞} algebra. Deformations of A_{∞} algebra preserving the symmetry algebra $\mathfrak{g}$ induce symmetry preserving deformations of the L_{∞} algebra.[8] This remark permits us to say that the calculations of symmetry preserving deformations of A_{∞} algebra A corresponding to YM theory induces symmetry preserving deformations of EM for YM theories with gauge group $U(N)$ for all N.

The calculation of cohomology groups $HH_{\mathfrak{susy}}(YM, YM)$ permits us to describe SUSY-invariant deformations of EM. However we would like also to characterize Lagrangian deformations of EM. This problem also can be formulated in terms of homology. Namely we should consider A_{∞} algebras with invariant inner product and their deformations. We say that an A_{∞} algebra A is equipped with an odd invariant nondegenerate inner product $\langle .,. \rangle$ if $\langle a_0, \mu_n(a_1, \dots, a_n) \rangle = (-1)^{n+1} \langle a_n, \mu_n(a_0, \dots, a_{n-1}) \rangle$. It is obvious that the corresponding L_{∞} algebras $L_N(A)$ are equipped with odd invariant inner product. Therefore the corresponding vector field Q comes from a solution of a Master equation $\{S, S\} = 0$ (i.e. we have Lagrangian equations of motion). We will check that the deformations of A_{∞}

[8]At the level of cohomology groups it means that we have a series of homomorphisms $HH_{\mathfrak{g}}(A, A) \to H_{\mathfrak{g}}(L_N(\widetilde{A}), L_n(\widetilde{A}))$.

algebra preserving invariant inner product are labeled by cyclic cohomology of the algebra [15].

As we have seen the deformations of an A_∞ algebra are labeled by Hochschild cohomology cocycles of differential $\widetilde{Q}$ (see formula (13)) acting on the space of derivations $Vect(\mathbb{V})$.

A derivation ρ is uniquely defined by its values on generators of the basis of vector space W^*(on generators of algebra $\hat{T}(W^*)$). Let us introduce notations $\rho(z^i) = \rho^i(z^1,\dots,z^n)$. The condition that ρ specifies a cocycle of d means that it specifies a Hochschild cocycle with coefficients in A. The condition that ρ preserves the invariant inner product is equivalent the cyclicity condition on $\rho_{i_0,i_1\dots i_n}$, where $\rho_{i_0}(z^1,\dots,z^n) = \sum \rho_{i_0,i_1\dots i_k} z^{i_1}\dots z^{i_k}$. (We lower the upper index in ρ using the invariant inner product.) The cyclicity condition has the form

$$\rho_{i_0,i_1\dots i_k} = (-1)^{k+1}\rho_{i_k,i_0\dots i_{k-1}} \tag{24}$$

We say that $\rho_{i_0,i_1,\dots,i_k}$ obeying formula (24) is a cyclic cochain. To define cyclic cohomology we use Hochschild differential on the space of cyclic cochains.[9]

If we consider deformations of an A_∞ algebra with inner product and a Lie algebra $\mathfrak{g}$ of symmetries and we are interested in deformations of A to an algebra that also has invariant inner product and the same algebra of symmetries we should consider cyclic cohomology $HC_{\mathfrak{g}}(A)$. The definition of this cohomology can be obtained if we modify the definition of $HC(A)$ in the same way as we modified the definition of $HH(A,A)$ to $HH_{\mathfrak{g}}(A,A)$.

It is obvious that there exist a homomorphism from $HC(A)$ to $HH(A,A)$ and from $HC_{\mathfrak{g}}(A)$ to $HH_{\mathfrak{g}}(A,A)$ (every deformation preserving inner product is a deformation).[10] Our main goal is to calculate the image of $HC_{\mathfrak{g}}(A)$ in $HH_{\mathfrak{g}}(A,A)$ for the A_∞ algebra of YM theory, i.e. we would like to describe all supersymmetric deformations of YM that come from a Lagrangian.

Cyclic cohomology are related to Hochschild cohomology by Connes exact sequence:

$$\dots \to HC^n(A) \to HH^n(A,A^*) \to HC^{n-1}(A) \to HC^{n+1}(A) \to \dots$$

Similar sequence exists for Lie-cyclic cohomology.

[9]One can say that the vector field ρ preserving inner product is a Hamiltonian vector field. The cyclic cochain $\rho_{i_0,i_1\dots i_k}$ can be considered as its Hamiltonian. The differential (13) acts on the space of Hamiltonian vector fields. The cohomology of corresponding differential acting on the space of Hamiltonians is called cyclic cohomology.

[10]Notice that we have assumed that A is equipped with non-degenerate inner product. The definition of cyclic cohomology does not require the choice of inner product; in general there exists a homomorphism $HC(A) \to HH(A,A^*)$. The homomorphism $HC(A) \to HH(A,A)$ can be obtained as a composition of this homomorphism with a homomorphism $HH(A,A^*) \to HH(A,A)$ induced by a map $A^* \to A$.

To define the cyclic homology $HC_\bullet(A)$ we work with cyclic chains (elements of $A \otimes ... \otimes A$ factorized with respect to the action of cyclic group). The natural map of Hochschild chains with coefficients in A to cyclic chains commutes with the differential and therefore specifies a homomorphism $HH_k(A, A) \xrightarrow{l} HC_k(A)$. This homomorphism enters the homological version of Connes exact sequence

$$... \to HC_{n-1}(A) \xrightarrow{b} HH_n(A, A) \xrightarrow{I} HC_n(A) \xrightarrow{S} HC_{n-2}(A) \to ...$$

We define the differential $B : HH_n(A, A) \to HH_{n+1}(A, A)$ as a composition $b \circ I$.

An interesting refinement of Connes exact sequence exists in the case when A is the universal enveloping of a Lie algebra $\mathfrak{g}$ over $\mathbb{C}$. In this case cyclic homology get an additional index: $HC_{k,j}(A)$. Such groups fit into the long exact sequence [7]:

$$\begin{aligned} ... \to HC_{n-1,i}(U(\mathfrak{g})) &\xrightarrow{b_{n-1,i}} HH_n(U(\mathfrak{g}), \mathrm{Sym}^i(\mathfrak{g})) \xrightarrow{I_{n,i}} HC_{n,i+1}(U(\mathfrak{g})) \\ &\xrightarrow{S_{n,i+1}} HC_{n-2,i}(U(\mathfrak{g})) \to ... \end{aligned}$$

The differential

$$B_i : HH_n(U(\mathfrak{g}), \mathrm{Sym}^i(\mathfrak{g})) \to HH_{n+1}(U(\mathfrak{g}), \mathrm{Sym}^{i-1}(\mathfrak{g}))$$

is defined as a composition $b_{n,i+1} \circ I_{n,i}$. Finally if the Lie algebra $\mathfrak{g}$ is graded then all homological constructs acquire an additional bold index: $HH_{n\mathbf{l}}(U(\mathfrak{g}), \mathrm{Sym}^i(\mathfrak{g}))$, $HC_{n,i,\mathbf{l}}(U(\mathfrak{g}))$. This index is preserved by the differential in the above sequence.

It is worthwhile to mention that all natural constructions that exist in cyclic homology can be extended to Lie-cyclic homology.

It is important to emphasize that homology and cohomology theories we considered in this section are invariant with respect to quasi-isomorphism (under certain conditions that are fulfilled in our situation).[11]

According to [5] a quasi-isomorphism of two algebras $A \to B$ induces an isomorphism in Hochschild cohomology $HH^\bullet(A, A) \cong HH^\bullet(B, B)$. As we have mentioned Hochschild cohomology $HH^\bullet(A, A)$ is equipped with a structure of super Lie algebra, the isomorphism is compatible with this structure.

This theorem guarantees that quasi-isomorphism $A \to B$ allows us to translate a weak $\mathfrak{g}$ action from A to B.

We have defined L_∞ action as an L_∞ homomorphism of Lie algebra $\mathfrak{g}$ into differential Lie algebra of derivations $Vect(A)$. It follows from the results of [5] that a quasiisomorphism $\phi : A \to B$ induces a quasi- isomorphism $\tilde{\phi} : Vect(A) \to$

[11]The most general results and precise formulation of this statement can be found in [5] for Hoschschild cohomology and in [4] for cyclic cohomology.

$Vect(B)$ compatible with L_∞ structure.[12] We obtain that L_∞ action on A can be transferred to an L_∞ action on quasi-isomorphic algebra B.

The calculation of cohomology groups we are interested in is a difficult problem. To solve this problem we apply the notion of duality of associative and A_∞ algebras.

3 Duality

We define a pairing of two differential graded augmented[13] algebras A and B as a degree one element $e \in A \otimes B$ that satisfies Maurer-Cartan equation

$$(d_A + d_B)e + e^2 = 0 \tag{25}$$

Here we understand $A \otimes B$ as a completed tensor product.

Example 1. Let $x_1, \dots, x_n$ be the generating set of the quadratic algebra A. The set $\xi^1, \dots, \xi^n$ generates the dual quadratic algebra $A^!$ (see preliminaries). The element $e = x_i \otimes \xi^i$ has degree one, provided x_i and ξ^i have degrees two and minus one. The element e satisfies $e^2 = 0$ – a particular case of (25) for algebras with zero differential and therefore specifies a pairing between A and $A^!$.

Remark. Many details of the theory depend on the completion of the tensor product, mentioned in the definition of e. We, however, chose to completely ignore this issue because the known systematic way to deal with it requires introduction of a somewhat artificial language of co-algebras.[14]

We call a non-negatively (non-positively) graded differential algebra $A = \bigoplus_i A_i$ connected, if $A_0 \cong \mathbb{C}$. Such algebra is automatically augmented $\epsilon : A \to A_0$. We call a non-negatively graded connected algebra A simply-connected if $A_1 \cong 0$.

Let us consider a differential graded algebra $\mathrm{cobar}A = (T(\Pi A^*), d)$ where A is an associative algebra and d is the Hochschild differential. In other words we consider the co-bar construction for the algebra A.

Proposition 2. *The pairing e defines the map*

$$\rho : \mathrm{cobar}(A) \to B$$

of differential graded algebras.

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[12] In fact the structure of $Vect(A)$ is richer: it is a B_∞ algebra (see [5] for details), but we will use only L_∞ (Lie) structure. One of the results of [5] asserts that $\tilde{\phi}$ is compatible with B_∞ structure. As a corollary it induces a quasi-isomorphism of L_∞ structures.

[13] A differential graded algebra A is called augmented if it is equipped with a d-invariant homomorphism $\epsilon : A \to \mathbb{C}$ of degree zero. We assume that the algebras at hand are $\mathbb{Z}$-graded and graded components are finite-dimensional.

[14] One can define the notion of duality between algebra and co-algebra. This notion has better properties than the duality between algebras.

Proof. The algebra cobar(A) is generated by elements of ΠA^*. The value of the map ρ on $l \in \Pi A^*$ is equal to

$$\rho(l) = \langle l, f_i \rangle g^i,$$

where $e = f_i \otimes g^i \in A \otimes B$. The compatibility of ρ with the differential follows automatically from (25). (Notice, that for graded spaces we always consider the dual as graded dual, i.e. as a direct sum of dual spaces to the graded components.) $\square$

Similarly the element e defines a map cobar$(B) \to A$.

Definition 3. The differential algebras A and B are dual if there exists a pairing (A, B, e) such that the maps cobar$(A) \to B$ and cobar$(B) \to A$ are quasi-isomorphisms.[15]

Notice that duality is invariant with respect to quasi-isomorphism.

If an ordinary algebra A is quadratic then A is dual to $A^!$ iff A is a Koszul algebra.

If a differential graded algebra A has a dual algebra, then A is dual to cobar$A$. If A is a connected and simply-connected differential graded algebra, i.e. $A = \bigoplus_{i \geq 0} A_i$ and $A_0 = \mathbb{C}$ and $A_1 = 0$, then A and cobar$A$ are dual.

If differential graded algebras A and B are dual it is clear that Hochschild cohomology $HH(A, \mathbb{C})$ of A with trivial coefficients coincide with intrinsic cohomology of B. This is because B is quasi-isomorphic to cobar(A). One say also that

$$HH(A, A) = HH(B, B), \tag{26}$$

This is clear because these cohomology can be calculated in terms of complex $A \otimes B$, that is quasi-isomorphic both to $A \otimes$ cobar$A$ and cobar$B \otimes B$.

This statement can be generalised to Hochschild cohomology of A with coefficients in any bimodule M. Namely, we should introduce in $B \otimes M$ a differential by the formula

$$d(b \otimes m) = (d_B + d_M) b \otimes m + [e, b \otimes m] \tag{27}$$

Proposition 4. *Let A be a connected and simply-connected differential graded algebra, i.e. $A = \bigoplus_{i \geq 0} A_i$ and $A_0 = \mathbb{C}$ and $A_1 = 0$. Then the Hochschild cohomology $HH(A, M)$ coincide with the cohomology of $B \otimes M$ with respect to differential (27).*

To prove this statement we notice that the quasi-isomorphism cobar$A \to B$ induces a homomorphism $C(A, M) =$ cobar$A \otimes M \to B \otimes M$; it follows from (17) that this homomorphism commutes with the differentials and therefore induces a homomorphism on homology. The induced homomorphism is an isomorphism; this can be derived from the fact that the map cobar$A \to B$ is a quasi-isomorphism. (The derivation is based on the techniques of spectral sequences; the condition on algebra A guarantees the convergence of spectral sequence.)

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[15] Very similar notion of duality was suggested independently by Kontsevich [6].

The above proposition can be applied to the case when A is a Koszul quadratic algebra and $B = A^!$ is the dual quadratic algebra. We obtain the following useful statement.

Proposition 5. *If differential graded algebra A is dual to B and quasi-isomorphic to the universal enveloping algebra $U(\mathfrak{g})$ of Lie algebra $\mathfrak{g}$ then B is quasi-isomorphic to the super-commutative differential algebra $C^\bullet(\mathfrak{g})$.*

This statement follows from the fact that the cohomology of $C^\bullet(\mathfrak{g})$ (=Lie algebra cohomology of $\mathfrak{g}$) coincides with Hochschild cohomology of $U(\mathfrak{g})$ with trivial coefficients.

It turns out that it is possible to calculate cyclic and Hochschild cohomology of A in terms of suitable homological constructions for a dual algebra B.

Let A and B be dual differential graded algebras. Let us assume that A and B satisfy assumptions of Proposition 4.

Proposition 6. *Under above assumptions there is a canonical isomorphism*

$$HC_{-1-n}(A) \cong HC^n(B),$$

where $HC^n(HC_n)$ stands for ith cohomology(resp. homology) of an algebra.

Proposition 7. *Under the above assumptions there is an isomorphism*

$$HH^n(A, A^*) = HH_{-n}(B, B),$$

where $HH^n(HH_n)$ stands for nth Hochschild cohomology (resp. homology).

For the case when A and B are quadratic algebras these two propositions were proven in [3]. The proof in general case is similar. It can be based on results of [2] or [7].

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Lie Supergroups, Unitary Representations, and Invariant Cones

Karl-Hermann Neeb and Hadi Salmasian

1 Introduction

The goal of this article is twofold. First, it presents an application of the theory of invariant convex cones of Lie algebras to the study of unitary representations of Lie supergroups. Second, it provides an exposition of recent results of the second author on the classification of irreducible unitary representations of nilpotent Lie supergroups using the method of orbits.

In relation to the first goal, it is shown that there is a close connection between unitary representations of Lie supergroups and dissipative unitary representations of Lie groups (in the sense of [20]). It will be shown that for a large class of Lie supergroups the only irreducible unitary representations are highest weight modules in a suitable sense. This circle of ideas leads to explicit necessary conditions for determining when a Lie supergroup has faithful unitary representations. These necessary conditions are then used to analyze the situation for simple and semisimple Lie supergroups.

Pertaining to the second goal, the main results in [27] are explained in a more reader friendly style. Complete proofs of the results are given in [27], and will not be repeated. However, wherever appropriate, ideas of the proofs are sketched.

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2 Algebraic Background

We start by introducing the notation and stating several facts which are used in this article. The reader is assumed to be familiar with basics of the theory of superalgebras, and therefore this section is rather terse. For more detailed accounts of the subject the reader is referred to [1, 13, 28].

Let $\mathbb{S}$ be an arbitrary associative unital ring. A possibly nonassociative $\mathbb{S}$-algebra $\mathfrak{s}$ is called a *superalgebra* if it is $\mathbb{Z}_2$-graded, i.e., $\mathfrak{s} = \mathfrak{s}_{\overline{0}} \oplus \mathfrak{s}_{\overline{1}}$ where $\mathfrak{s}_i \mathfrak{s}_j \subseteq \mathfrak{s}_{i+j}$. The degree of a homogeneous element $a \in \mathfrak{s}$ is denoted by $|a|$.

A superalgebra $\mathfrak{s}$ is called *supercommutative* if

$$ab = (-1)^{|a|\cdot|b|} ba$$

for every two homogeneous elements $a, b \in \mathfrak{s}$.

A *Lie superalgebra* is a superalgebra whose multiplication, usually called its *superbracket*, satisfies graded analogues of antisymmetry and the Jacobi identity. This means that if A, B, C are homogeneous elements of a Lie superalgebra, then

$$[A, B] = -(-1)^{|A|\cdot|B|}[B, A]$$

and

$$(-1)^{|A|\cdot|C|}[A, [B, C]] + (-1)^{|B|\cdot|A|}[B, [C, A]] + (-1)^{|C|\cdot|B|}[C, [A, B]] = 0.$$

Let $\mathbb{K}$ be a field and $\mathfrak{g}$ be a Lie superalgebra over $\mathbb{K}$. If $\mathfrak{h}$ is a Lie subsuperalgebra of $\mathfrak{g}$ then $\mathscr{Z}_{\mathfrak{g}}(\mathfrak{h})$ denotes the *supercommutant* of $\mathfrak{h}$ in $\mathfrak{g}$, i.e.,

$$\mathscr{Z}_{\mathfrak{g}}(\mathfrak{h}) = \{ X \in \mathfrak{g} \mid [\mathfrak{h}, X] = \{0\} \}.$$

The *center* of $\mathfrak{g}$ is the supercommutant of $\mathfrak{g}$ in $\mathfrak{g}$ and is denoted by $\mathscr{Z}(\mathfrak{g})$. The *universal enveloping algebra* of $\mathfrak{g}$, which is defined in [13, Sect. 1.1.3], is denoted by $\mathscr{U}(\mathfrak{g})$. The group of $\mathbb{K}$-linear (even) automorphisms of $\mathfrak{g}$ is denoted by $\mathrm{Aut}(\mathfrak{g})$. Finally, recall that the definitions of nilpotent and solvable Lie superalgebras are the same as the ones for Lie algebras (see [13, Sect. 1]).

2.1 Centroid, Derivations, and Differential Constants

Let $\mathbb{K}$ be an arbitrary field and $\mathfrak{s}$ be a finite dimensional superalgebra over $\mathbb{K}$. The *multiplication algebra* of $\mathfrak{s}$, denoted by $\mathscr{M}(\mathfrak{s})$, is the associative unital superalgebra over $\mathbb{K}$ which is generated by the elements R_x and L_x of $\mathrm{End}_{\mathbb{K}}(\mathfrak{s})$, for all homogeneous $x \in \mathfrak{s}$, where

$$\mathrm{L}_x(y) = xy \text{ and } \mathrm{R}_x(y) = (-1)^{|x|\cdot|y|}yx \text{ for every homogeneous } y \in \mathfrak{s}.$$

As usual, the superbracket on $\mathrm{End}_{\mathbb{K}}(\mathfrak{s})$ is defined by

$$[A, B] = AB - (-1)^{|A|\cdot|B|}BA$$

for homogeneous elements $A, B \in \mathrm{End}_{\mathbb{K}}(\mathfrak{s})$, and is then extended to $\mathrm{End}_{\mathbb{K}}(\mathfrak{s})$ by linearity. The *centroid* of $\mathfrak{s}$, denoted by $\mathscr{C}(\mathfrak{s})$, is the supercommutant of $\mathscr{M}(\mathfrak{s})$ in $\mathrm{End}_{\mathbb{K}}(\mathfrak{s})$, i.e,

$$\mathscr{C}(\mathfrak{s}) = \left\{ A \in \mathrm{End}_{\mathbb{K}}(\mathfrak{s}) \;\middle|\; [A, B] = 0 \text{ for every } B \in \mathscr{M}(\mathfrak{s}) \right\}.$$

Obviously $\mathscr{C}(\mathfrak{s})$ is a unital associative superalgebra over $\mathbb{K}$. If $\mathfrak{s}^2 = \mathfrak{s}$ then $\mathscr{C}(\mathfrak{s})$ is supercommutative (see [6, Proposition 2.1] for a proof).

If $s \in \{\overline{0}, \overline{1}\}$, a *homogeneous derivation of degree* s of $\mathfrak{s}$ is an element $D \in \mathrm{End}_{\mathbb{K}}(\mathfrak{s})$ such that for every two homogeneous elements $a, b \in \mathfrak{s}$,

$$D(ab) = D(a)b + (-1)^{|a|\cdot s}aD(b).$$

The subspace of $\mathrm{End}_{\mathbb{K}}(\mathfrak{s})$ which is spanned by homogeneous derivations of $\mathfrak{s}$ is a Lie superalgebra over $\mathbb{K}$ and is denoted by $\mathrm{Der}_{\mathbb{K}}(\mathfrak{s})$. The *ring of differential constants*, denoted by $\mathscr{R}(\mathfrak{s})$, is the supercommutant of $\mathrm{Der}_{\mathbb{K}}(\mathfrak{s})$ in $\mathscr{C}(\mathfrak{s})$.

Suppose that $\mathfrak{s}$ is *simple*, i.e., $\mathfrak{s}^2 \neq \{0\}$ and $\mathfrak{s}$ does not have proper two-sided ideals. By Schur's Lemma every nonzero homogeneous element of $\mathscr{C}(\mathfrak{s})$ is invertible. Since $\mathfrak{s}^2$ is always a two-sided ideal, $\mathfrak{s}^2 = \mathfrak{s}$ and therefore $\mathscr{C}(\mathfrak{s})$ is supercommutative. It follows that $\mathscr{C}(\mathfrak{s})_{\overline{1}} = \{0\}$, $\mathscr{C}(\mathfrak{s})_{\overline{0}}$ is a field, and $\mathscr{R}(\mathfrak{s})$ is a subfield of $\mathscr{C}(\mathfrak{s})_{\overline{0}}$ containing $\mathbb{K}$.

2.2 Derivations of Base Extensions

Let $\mathbb{K}$ be a field of characteristic zero and $\mathbf{\Lambda}(n, \mathbb{K})$ be the *Graßmann superalgebra over* $\mathbb{K}$ *in* n *indeterminates*, i.e., the associative unital superalgebra over $\mathbb{K}$ generated by odd elements $\xi_1, \ldots, \xi_n$ modulo the relations

$$\xi_i\xi_j + \xi_j\xi_i = 0 \text{ for every } 1 \leq i, j \leq n.$$

Let $\mathfrak{s}$ be a superalgebra over $\mathbb{K}$. The tensor product $\mathfrak{s} \otimes_{\mathbb{K}} \mathbf{\Lambda}(n, \mathbb{K})$ is a superalgebra over $\mathbb{K}$. Note that since $\mathbf{\Lambda}(n, \mathbb{K})$ is supercommutative, if $\mathfrak{s}$ is a Lie superalgebra then so is $\mathfrak{s} \otimes_{\mathbb{K}} \mathbf{\Lambda}(n, \mathbb{K})$.

It is proved in [6, Proposition 7.1] that

$$\mathrm{Der}_{\mathbb{K}}\big(\mathfrak{s} \otimes_{\mathbb{K}} \mathbf{\Lambda}(n, \mathbb{K})\big) = \mathrm{Der}_{\mathbb{K}}(\mathfrak{s}) \otimes_{\mathbb{K}} \mathbf{\Lambda}(n, \mathbb{K}) + \mathscr{C}(\mathfrak{s}) \otimes_{\mathbb{K}} \mathbf{W}(n, \mathbb{K}), \quad (1)$$

where

$$\mathbf{W}(n,\mathbb{K}) = \mathrm{Der}_{\mathbb{K}}\big(\mathbf{\Lambda}(n,\mathbb{K})\big).$$

The right hand side of (1) acts on $\mathfrak{s} \otimes_{\mathbb{K}} \mathbf{\Lambda}(n,\mathbb{K})$ via

$$(D_{\mathfrak{s}} \otimes_{\mathbb{K}} a)(X \otimes_{\mathbb{K}} b) = (-1)^{|a|\cdot|X|} D_{\mathfrak{s}}(X) \otimes_{\mathbb{K}} ab$$

and

$$(T \otimes_{\mathbb{K}} D_{\mathbf{\Lambda}})(X \otimes_{\mathbb{K}} a) = (-1)^{|D_{\mathbf{\Lambda}}|\cdot|X|} T(X) \otimes_{\mathbb{K}} D_{\mathbf{\Lambda}}(a).$$

Note that the right hand side of (1) is indeed a direct sum of the two summands. This follows from the fact that every element of $\mathscr{C}(\mathfrak{s}) \otimes_{\mathbb{K}} \mathbf{W}(n,\mathbb{K})$ vanishes on $\mathfrak{s} \otimes_{\mathbb{K}} 1_{\mathbf{\Lambda}(n,\mathbb{K})}$, but an element of $\mathrm{Der}_{\mathbb{K}}(\mathfrak{s}) \otimes_{\mathbb{K}} \mathbf{\Lambda}(n,\mathbb{K})$ which vanishes on $\mathfrak{s} \otimes_{\mathbb{K}} 1_{\mathbf{\Lambda}(n,\mathbb{K})}$ must be zero.

2.3 Cartan Subsuperalgebras

Let $\mathbb{K}$ be a field of characteristic zero and $\mathfrak{g}$ be a finite dimensional Lie superalgebra over $\mathbb{K}$. A Lie subsuperalgebra of $\mathfrak{g}$ which is nilpotent and self normalizing is called a *Cartan subsuperalgebra*.

An important property of Cartan subsuperalgebras of $\mathfrak{g}$ is that they are uniquely determined by their intersections with $\mathfrak{g}_{\bar{0}}$. Our next goal is to state this fact more formally.

For every subset $\mathcal{W}_{\bar{0}}$ of $\mathfrak{g}_{\bar{0}}$, let

$$\mathscr{N}_{\mathfrak{g}}(\mathcal{W}_{\bar{0}}) = \big\{ X \in \mathfrak{g} \,\big|\, \text{for every } W \in \mathcal{W}_{\bar{0}}, \text{ if } k \gg 0 \text{ then } \mathrm{ad}(W)^k(X) = 0 \big\}.$$

One can easily prove that $\mathscr{N}_{\mathfrak{g}}(\mathcal{W}_{\bar{0}})$ is indeed a subsuperalgebra of $\mathfrak{g}$. The next proposition is stated in [29, Proposition 1] (see also [24, Proposition 1]).

Proposition 2.3.1. *If $\mathfrak{h} = \mathfrak{h}_{\bar{0}} \oplus \mathfrak{h}_{\bar{1}}$ is a Cartan subsuperalgebra of $\mathfrak{g}$ then $\mathfrak{h}_{\bar{0}}$ is a Cartan subalgebra of $\mathfrak{g}_{\bar{0}}$. Conversely, if $\mathfrak{h}_{\bar{0}}$ is a Cartan subalgebra of $\mathfrak{g}_{\bar{0}}$ then $\mathscr{N}_{\mathfrak{g}}(\mathfrak{h}_{\bar{0}})$ is a Cartan subsuperalgebra of $\mathfrak{g}$. The correspondence*

$$\mathfrak{h}_{\bar{0}} \longleftrightarrow \mathscr{N}_{\mathfrak{g}}(\mathfrak{h}_{\bar{0}})$$

is a bijection between Cartan subalgebras of $\mathfrak{g}_{\bar{0}}$ and Cartan subsuperalgebras of $\mathfrak{g}$.

2.4 Compactly Embedded Subalgebras

Let $\mathfrak{g}$ be a finite dimensional Lie superalgebra over $\mathbb{R}$. The group $\mathrm{Aut}(\mathfrak{g})$ is a (possibly disconnected) Lie subgroup of $\mathrm{GL}(\mathfrak{g})$, the group of invertible elements of $\mathrm{End}_{\mathbb{R}}(\mathfrak{g})$. The subgroup of $\mathrm{Aut}(\mathfrak{g})$ generated by $e^{\mathrm{ad}(\mathfrak{g}_{\bar{0}})}$ is denoted by $\mathrm{Inn}(\mathfrak{g})$.

If $\mathfrak{h}_{\bar{0}}$ is a Lie subalgebra of $\mathfrak{g}_{\bar{0}}$ then $\mathrm{INN}_{\mathfrak{g}}(\mathfrak{h}_{\bar{0}})$ denotes the closure in $\mathrm{Aut}(\mathfrak{g})$ of the subgroup generated by $e^{\mathrm{ad}(\mathfrak{h}_{\bar{0}})}$. When $\mathrm{INN}_{\mathfrak{g}}(\mathfrak{h}_{\bar{0}})$ is compact $\mathfrak{h}_{\bar{0}}$ is said to be *compactly embedded in* $\mathfrak{g}$.

Cartan subalgebras of $\mathfrak{g}_{\bar{0}}$ which are compactly embedded in $\mathfrak{g}$ are especially interesting because they yield root decompositions of the complexification of $\mathfrak{g}$. The next proposition states this fact formally. In the next proposition, let τ denote the usual complex conjugation of elements of $\mathfrak{g}^{\mathbb{C}} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$, i.e., $\tau(X+iY) = X - iY$ for every $X, Y \in \mathfrak{g}$.

Proposition 2.4.1. *Let* $\mathfrak{t}_{\bar{0}}$ *be a Cartan subalgebra of* $\mathfrak{g}_{\bar{0}}$ *which is compactly embedded in* $\mathfrak{g}$. *Then the following statements hold.*

(i) $\mathfrak{t}_{\bar{0}}$ *is abelian.*
(ii) *One can decompose* $\mathfrak{g}^{\mathbb{C}}$ *as*

$$\mathfrak{g}^{\mathbb{C}} = \bigoplus_{\alpha \in \Delta} \mathfrak{g}^{\mathbb{C},\alpha}, \tag{2}$$

where

$$\Delta = \left\{ \alpha \in \mathfrak{t}_{\bar{0}}^{*} \,\middle|\, \mathfrak{g}^{\mathbb{C},\alpha} \neq \{0\} \right\}$$

and

$$\mathfrak{g}^{\mathbb{C},\alpha} = \left\{ X \in \mathfrak{g}^{\mathbb{C}} \,\middle|\, [H,X] = i\alpha(H)X \textit{ for every } H \in \mathfrak{t}_{\bar{0}} \right\}.$$

(iii) *If* $\alpha \in \Delta$ *then* $-\alpha \in \Delta$ *as well, and if* $X \in \mathfrak{g}^{\mathbb{C},\alpha}$ *then* $\tau(X) \in \mathfrak{g}^{\mathbb{C},-\alpha}$.
(iv) $\mathfrak{g}_{\bar{0}} = \mathfrak{t}_{\bar{0}} \oplus [\mathfrak{t}_{\bar{0}}, \mathfrak{g}_{\bar{0}}]$.

Proof. The proof of [20, Theorem VII.2.2] can be adapted to prove (i), (ii), and (iii). Part (iv) can be proved using the fact that $\mathfrak{t}_{\bar{0}} = \mathscr{Z}_{\mathfrak{g}_{\bar{0}}}(\mathfrak{t}_{\bar{0}})$ (see [3, Chap. VII]).

More generally, if $\mathfrak{g}_{\bar{0}}$ has a Cartan subalgebra which is compactly embedded in $\mathfrak{g}$, then any Cartan subalgebra of $\mathfrak{g}^{\mathbb{C}}$ yields a root decomposition. This is the content of the following proposition.

Proposition 2.4.2. *Assume that* $\mathfrak{g}_{\bar{0}}$ *has a Cartan subalgebra which is compactly embedded in* $\mathfrak{g}$. *If* $\mathfrak{h}^{\mathbb{C}}$ *is an arbitrary Cartan subalgebra of* $\mathfrak{g}^{\mathbb{C}}$ *then* $\mathfrak{h}_{\bar{0}}^{\mathbb{C}}$ *is abelian and there exists a root decomposition of* $\mathfrak{g}^{\mathbb{C}}$ *associated to* $\mathfrak{h}^{\mathbb{C}}$, *i.e.,*

$$\mathfrak{g}^{\mathbb{C}} = \bigoplus_{\alpha \in \Delta(\mathfrak{h}^{\mathbb{C}})} \mathfrak{g}^{\mathbb{C},\alpha},$$

where

$$\Delta(\mathfrak{h}^{\mathbb{C}}) = \left\{ \alpha \in (\mathfrak{h}_{\bar{0}}^{\mathbb{C}})^{*} \mid \mathfrak{g}^{\mathbb{C},\alpha} \neq \{0\} \right\}$$

and

$$\mathfrak{g}^{\mathbb{C},\alpha} = \{ X \in \mathfrak{g}^{\mathbb{C}} \mid [H,X] = i\alpha(H)X \ \textit{ for every } H \in \mathfrak{h}_{\bar{0}}^{\mathbb{C}} \}.$$

Moreover, $\Delta(\mathfrak{h}^{\mathbb{C}}) = -\Delta(\mathfrak{h}^{\mathbb{C}})$.

Proof. Let $\mathfrak{t}_{\bar{0}}$ be a Cartan subalgebra of $\mathfrak{g}_{\bar{0}}$ which is compactly embedded in $\mathfrak{g}$. Then $\mathfrak{t}_{\bar{0}}^{\mathbb{C}}$ is a Cartan subalgebra of $\mathfrak{g}_{\bar{0}}^{\mathbb{C}}$, and by Proposition 2.3.1 it corresponds to a Cartan subsuperalgebra $\mathfrak{t}^{\mathbb{C}}$ of $\mathfrak{g}^{\mathbb{C}}$. Proposition 2.4.1 implies that there is a root decomposition of $\mathfrak{g}^{\mathbb{C}}$ associated to $\mathfrak{t}^{\mathbb{C}}$, and if $\Delta(\mathfrak{t}^{\mathbb{C}})$ denotes the corresponding set of roots then $\Delta(\mathfrak{t}^{\mathbb{C}}) = -\Delta(\mathfrak{t}^{\mathbb{C}})$. It is known that any two Cartan subalgebras of $\mathfrak{g}_{\bar{0}}^{\mathbb{C}}$ are conjugate under inner automorphisms of $\mathfrak{g}_{\bar{0}}^{\mathbb{C}}$. Using Proposition 2.3.1 one can show that any two Cartan subsuperalgebras of $\mathfrak{g}^{\mathbb{C}}$ are conjugate under the group of $\mathbb{C}$-linear automorphisms of $\mathfrak{g}^{\mathbb{C}}$ generated by $e^{\mathrm{ad}(\mathfrak{g}_{\bar{0}}^{\mathbb{C}})}$. By conjugacy, the root decomposition associated to $\mathfrak{t}^{\mathbb{C}}$ turns into one associated to $\mathfrak{h}^{\mathbb{C}}$.

2.5 Simple and Semisimple Lie Superalgebras

The classification of finite dimensional complex simple Lie superalgebras and their real forms is known from [13] and [31]. Every complex simple Lie superalgebra is isomorphic to one of the following types.

(i) A Lie superalgebra of *classical* type, i.e., $\mathbf{A}(m|n)$ where $m,n > 0$, $\mathbf{B}(m|n)$ where $m \geq 0$ and $n > 0$, $\mathbf{C}(n)$ where $n > 1$, $\mathbf{D}(m|n)$ where $m > 1$ and $n > 0$, $\mathbf{G}(3)$, $\mathbf{F}(4)$, $\mathbf{D}(2|1,\alpha)$ where $\alpha \in \mathbb{C} \setminus \{0,-1\}$, $\mathbf{P}(n)$ where $n > 1$, or $\mathbf{Q}(n)$ where $n > 1$.

(ii) A Lie superalgebra of *Cartan type,* i.e., $\mathbf{W}(n)$ where $n \geq 3$, $\mathbf{S}(n)$ where $n \geq 4$, $\widetilde{\mathbf{S}}(n)$ where n is even and $n \geq 4$, or $\mathbf{H}(n)$ where $n \geq 5$.

(iii) A complex simple Lie algebra.

Let $\mathfrak{s}$ be a finite dimensional real simple Lie superalgebra with nontrivial odd part, i.e., $\mathfrak{s}_{\bar{1}} \neq \{0\}$. Since $\mathscr{C}(\mathfrak{s})$ is a finite dimensional field extension of $\mathbb{R}$, we have $\mathscr{C}(\mathfrak{s}) = \mathbb{R}$ or $\mathscr{C}(\mathfrak{s}) = \mathbb{C}$. If $\mathscr{C}(\mathfrak{s}) = \mathbb{C}$, then $\mathfrak{s}$ is a complex simple Lie superalgebra which is considered as a real Lie superalgebra. If $\mathscr{C}(\mathfrak{s}) = \mathbb{R}$, then $\mathfrak{s}$ is a real form of the complex simple Lie superalgebra $\mathfrak{s} \otimes_{\mathbb{R}} \mathbb{C}$. The classification of these real forms is summarized in Table 1.

A Lie superalgebra is called *semisimple* if it has no nontrivial solvable ideals. Semisimple Lie superalgebras are not necessarily direct sums of simple Lie superalgebras. In fact the structure theory of semisimple Lie superalgebras is rather complicated. The following statement can be obtained by a slight modification of the arguments in [6].

Theorem 2.5.1. *If a real Lie superalgebra $\mathfrak{g}$ is semisimple then there exist real simple Lie superalgebras $\mathfrak{s}_1, \ldots, \mathfrak{s}_k$ and nonnegative integers $n_1, \ldots, n_k$ such that*

$$\bigoplus_{i=1}^{k} \left(\mathfrak{s}_i \otimes_{\mathbb{K}_i} \mathbf{\Lambda}(n_i, \mathbb{K}_i)\right) \subseteq \mathfrak{g} \subseteq \bigoplus_{i=1}^{k} \left(\mathrm{Der}_{\mathbb{K}_i}(\mathfrak{s}_i) \otimes_{\mathbb{K}_i} \mathbf{\Lambda}(n_i, \mathbb{K}_i) + \mathbb{L}_i \otimes_{\mathbb{K}_i} \mathbf{W}(n_i, \mathbb{K}_i)\right)$$

where $\mathbb{K}_i = \mathscr{R}(\mathfrak{s}_i)$ and $\mathbb{L}_i = \mathscr{C}(\mathfrak{s}_i)$ for every $1 \leq i \leq k$.

Table 1 Simple real Lie superalgebras with nontrivial odd part

$\mathfrak{s} \otimes_{\mathbb{R}} \mathbb{C}$	$\mathfrak{s}$		$\mathfrak{s}_{\overline{0}}/\mathrm{rad}(\mathfrak{s}_{\overline{0}})$
	$\mathfrak{su}(p,q\|r,s)$	$(p+q=m, r+s=n)$	$\mathfrak{su}(p,q) \oplus \mathfrak{su}(r,s)$
$\mathbf{A}(m-1\|n-1)$	$\mathfrak{su}^*(2p\|2q)$	$(m=2p, n=2q$ even)	$\mathfrak{su}^*(2p) \oplus \mathfrak{su}^*(2q)$
$m>n>1$	$\mathfrak{sl}(m\|n,\mathbb{R})$		$\mathfrak{sl}(m,\mathbb{R}) \oplus \mathfrak{sl}(n,\mathbb{R})$
	$\mathfrak{psu}(p,q\|r,s)$	$(p+q=r+s=m)$	$\mathfrak{su}(p,q) \oplus \mathfrak{su}(r,s)$
	$\mathfrak{psu}^*(2p\|2p)$	$(m=2p$ even)	$\mathfrak{su}^*(2p) \oplus \mathfrak{su}^*(2p)$
	$\mathfrak{psl}(m\|m,\mathbb{R})$		$\mathfrak{sl}(m,\mathbb{R}) \oplus \mathfrak{sl}(m,\mathbb{R})$
$\mathbf{A}(m-1\|m-1)$	$\mathfrak{p}\overline{\mathfrak{q}}(m)$		$\mathfrak{sl}(m,\mathbb{C})$
$n>1$	$\mathfrak{usp}(m)$		$\mathfrak{sl}(m,\mathbb{C})$
	$\mathfrak{osp}(p,q\|2n)$	$(p+q=2m+1)$	$\mathfrak{so}(p,q) \oplus \mathfrak{sp}(2n,\mathbb{R})$
$\mathfrak{osp}(m\|2n,\mathbb{C})$	$\mathfrak{osp}^*(m\|p,q)$	$(p+q=n)$	$\mathfrak{so}^*(m) \oplus \mathfrak{sp}(p,q)$
	$D(2\|1,\alpha,2)$	$\alpha \in \mathbb{R}$	$\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R})$
	$D(2\|1,\alpha,0)$	$\alpha \in \mathbb{R}$	$\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(2)$
$\mathbf{D}(2\mid 1,\alpha)$	$D(2\|1,\frac{1}{\alpha},0)$	$\alpha \in \mathbb{R}$	$\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(2)$
$\alpha=\overline{\alpha}$ or	$D(2\|1,-\frac{\alpha}{1+\alpha},0)$	$\alpha \in \mathbb{R}$	$\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(2)$
$\alpha=-1-\overline{\alpha}$	$D(2\|1,\alpha,1)$	$\alpha=-1-\overline{\alpha}$	$\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{C})$
	$\mathbf{F}(4,0)$		$\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{so}(7)$
	$\mathbf{F}(4,1)$		$\mathfrak{su}(2) \oplus \mathfrak{so}(1,6)$
	$\mathbf{F}(4,2)$		$\mathfrak{su}(2) \oplus \mathfrak{so}(2,5)$
$\mathbf{F}(4)$	$\mathbf{F}(4,3)$		$\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{so}(3,4)$
	$\mathbf{G}(3,1)$		$\mathfrak{sl}(2,\mathbb{R}) \oplus \mathrm{Der}_{\mathbb{R}}(\mathbb{O})$
$\mathbf{G}(3)$	$\mathbf{G}(3,2)$		$\mathfrak{sl}(2,\mathbb{R}) \oplus \mathrm{Der}_{\mathbb{R}}(\mathbb{O}_{\mathsf{split}})$
	$\mathfrak{sp}(n,\mathbb{R})$		$\mathfrak{sl}(n,\mathbb{R})$
$\mathbf{P}(n-1)$	$\mathfrak{sp}^*(n)$	(n even)	$\mathfrak{su}^*(n)$
	$\mathfrak{psq}(n,\mathbb{R})$		$\mathfrak{sl}(n,\mathbb{R})$
	$\mathfrak{psq}(p,q)$	$(p+q=n)$	$\mathfrak{su}(p,q)$
$\mathbf{Q}(n-1)$	$\mathfrak{psq}^*(n)$	(n even)	$\mathfrak{su}^*(n)$
$\mathbf{W}(n)$	$\mathbf{W}(n,\mathbb{R})$		$\mathfrak{gl}(n,\mathbb{R})$
$\mathbf{S}(n)$	$\mathbf{S}(n,\mathbb{R})$		$\mathfrak{sl}(n,\mathbb{R})$
$\widetilde{\mathbf{S}}(n)$ n even	$\widetilde{\mathbf{S}}(n,\mathbb{R})$		$\mathfrak{sl}(n,\mathbb{R})$
$\mathbf{H}(n)$	$\mathbf{H}(p,q)$	$(p+q=n)$	$\mathfrak{so}(p,q)$

3 Geometric Background

Since we are interested in studying unitary representations from an analytic viewpoint, we need to realize them as representations of Lie supergroups on $\mathbb{Z}_2$-graded Hilbert spaces. To this end, we first need to make precise what we mean by Lie supergroups.

One can define Lie supergroups abstractly as *group objects* in the category of supermanifolds. To give sense to this definition, one needs to define the category of supermanifolds. It will be seen below that this can be done by means of sheaves and ringed spaces.

Nevertheless, the above abstract definition of Lie supergroups is not well-suited for the study of unitary representations, and a more explicit description of Lie supergroups is necessary. The aim of this section is to explain the latter description, which is based on the notion of Harish–Chandra pairs, and to clarify the relation between Harish–Chandra pairs and the categorical definition of Lie supergroups.

This section starts with a quick review of the theory of supermanifolds. The reader who is not familiar with the basics of this subject and is interested in further detail is referred to [9, 16–18, 33].

We remind the reader that in the study of unitary representations only the simple point of view of Harish–Chandra pairs will be used. Therefore the reader may also skip the review of supergeometry and continue reading from Sect. 3.4, where Harish–Chandra pairs are introduced.

3.1 Supermanifolds

Let p and q be nonnegative integers, and let $\mathscr{O}_{\mathbb{R}^p}$ denote the sheaf of smooth real valued functions on $\mathbb{R}^p$. The *smooth* $(p|q)$*-dimensional superspace* $\mathbb{R}^{p|q}$ is the ringed space $(\mathbb{R}^p, \mathscr{O}_{\mathbb{R}^{p|q}})$ where $\mathscr{O}_{\mathbb{R}^{p|q}}$ is the sheaf of smooth *superfunctions* in q odd coordinates. The latter statement simply means that for every open $U \subseteq \mathbb{R}^p$ one has

$$\mathscr{O}_{\mathbb{R}^{p|q}}(U) = \mathscr{O}_{\mathbb{R}^p}(U) \otimes_{\mathbb{R}} \mathbf{\Lambda}(q, \mathbb{R})$$

and the restriction maps of $\mathscr{O}_{\mathbb{R}^{p|q}}$ are obtained by base extensions of the restriction maps of $\mathscr{O}_{\mathbb{R}^p}$.

The ringed space $(\mathbb{R}^p, \mathscr{O}_{\mathbb{R}^{p|q}})$ is an object of the category $\mathsf{Top}_{\mathsf{s-alg}}$ of topological spaces which are endowed with sheaves of associative unital superalgebras over $\mathbb{R}$. If $\mathcal{X} = (\mathcal{X}_\circ, \mathscr{O}_{\mathcal{X}})$ and $\mathcal{Y} = (\mathcal{Y}_\circ, \mathscr{O}_{\mathcal{Y}})$ are objects in $\mathsf{Top}_{\mathsf{s-alg}}$ then a morphism $\varphi : \mathcal{X} \to \mathcal{Y}$ is a pair $(\varphi_\circ, \varphi^\#)$ such that $\varphi_\circ : \mathcal{X}_\circ \to \mathcal{Y}_\circ$ is a continuous map and

$$\varphi^\# : \mathscr{O}_{\mathcal{Y}} \to (\varphi_\circ)_* \mathscr{O}_{\mathcal{X}}$$

is a morphism of sheaves of associative unital superalgebras over $\mathbb{R}$, where $(\varphi_\circ)_* \mathscr{O}_{\mathcal{X}}$ is the direct image[1] of $\mathscr{O}_{\mathcal{X}}$.

An object of $\mathsf{Top}_{\mathsf{s-alg}}$ is called a *supermanifold* of dimension $(p|q)$ if it is locally isomorphic to $\mathbb{R}^{p|q}$. supermanifolds constitute objects of a full subcategory of $\mathsf{Top}_{\mathsf{s-alg}}$.

3.2 Some Basic Constructions for Supermanifolds

If $\mathcal{M} = (\mathcal{M}_\circ, \mathscr{O}_{\mathcal{M}})$ is a supermanifold of dimension $(p|q)$ then the nilpotent sections of $\mathscr{O}_{\mathcal{M}}$ generate a sheaf of ideals $\mathscr{I}_{\mathcal{M}}$. Indeed the underlying space $\mathcal{M}_\circ$

[1] In [9] the authors define morphisms based on pullback. Since pullback and direct image are adjoint functors, the definition of [9] is equivalent to the definition given in this article, which is also used in [17].

is an ordinary smooth manifold whose sheaf of smooth functions is $\mathscr{O}_{\mathcal{M}}/\mathscr{I}_{\mathcal{M}}$. One can also show that if $\mathcal{M} = (\mathcal{M}_\circ, \mathscr{O}_{\mathcal{M}})$ and $\mathcal{N} = (\mathcal{N}_\circ, \mathscr{O}_{\mathcal{N}})$ are two supermanifolds and $\varphi : \mathcal{M} \to \mathcal{N}$ is a morphism then the map $\varphi_\circ : \mathcal{M}_\circ \to \mathcal{N}_\circ$ is smooth (see [17, Sect. 2.1.5]).

Locally, $\mathscr{O}_{\mathcal{M}}/\mathscr{I}_{\mathcal{M}}$ is isomorphic to $\mathscr{O}_{\mathbb{R}^p}$. Therefore, if $U \subseteq \mathcal{M}_\circ$ is an open set, then for every section $f \in \mathscr{O}_{\mathcal{M}}(U)$ and every point $m \in U$ the value $f(m)$ is well defined. In this fashion, from any section f one obtains a smooth map

$$\widetilde{f} : U \to \mathbb{R}.$$

Nevertheless, because of the existence of nilpotent sections, f is not uniquely determined by $\widetilde{f}$.

Supermanifolds resemble ordinary manifolds in many ways. For example, one can prove the existence of finite direct products in the category of supermanifolds. Moreover, for a supermanifold $\mathcal{M}$ of dimension $(p|q)$ the sheaf $\mathrm{Der}_{\mathbb{R}}(\mathscr{O}_{\mathcal{M}})$ of $\mathbb{R}$-linear derivations of the structural sheaf $\mathscr{O}_{\mathcal{M}}$ is a locally free sheaf of $\mathscr{O}_{\mathcal{M}}$-modules of rank $(p|q)$. Sections of the latter sheaf are called *vector fields* of $\mathcal{M}$. The space of vector fields is closed under the superbracket induced from $\mathrm{End}_{\mathbb{R}}(\mathscr{O}_{\mathcal{M}})$.

If $\mathcal{M} = (\mathcal{M}_\circ, \mathscr{O}_{\mathcal{M}})$ is a supermanifold and $m \in \mathcal{M}_\circ$, then there exists an obvious morphism

$$\delta_m : \mathbb{R}^{0|0} \to \mathcal{M},$$

where $(\delta_m)_\circ : \mathbb{R}^0 \to \mathcal{M}_\circ$ maps the unique point of $\mathbb{R}^0$ to m, and for every open set $U \subseteq \mathcal{M}_\circ$ if $f \in \mathscr{O}_{\mathcal{M}}(U)$ then

$$(\delta_m)^\#(f) = \begin{cases} \widetilde{f}(m) & \text{if } m \in U, \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, $\mathbb{R}^{0|0}$ is a terminal object in the category of supermanifolds. Indeed for every supermanifold $\mathcal{M} = (\mathcal{M}_\circ, \mathscr{O}_{\mathcal{M}})$ there exists a morphism

$$\kappa_{\mathcal{M}} : \mathcal{M} \to \mathbb{R}^{0|0}$$

such that $(\kappa_{\mathcal{M}})_\circ : \mathcal{M}_\circ \to \mathbb{R}^0$ maps every point of $\mathcal{M}_\circ$ to the unique point of $\mathbb{R}^0$ and for every $t \in \mathscr{O}_{\mathbb{R}^{0|0}}(\mathbb{R}^0) \simeq \mathbb{R}$ one has $(\kappa_{\mathcal{M}})^\#(t) = t \cdot 1_{\mathcal{M}}$.

3.3 Lie Supergroups and Their Lie Superalgebras

Recall that by a *Lie supergroup* we mean a group object in the category of supermanifolds. In other words, a supermanifold $\mathcal{G} = (\mathcal{G}_\circ, \mathscr{O}_{\mathcal{G}})$ is a Lie supergroup if there exist morphisms

$$\mu : \mathcal{G} \times \mathcal{G} \to \mathcal{G}, \quad \varepsilon : \mathbb{R}^{0|0} \to \mathcal{G}, \quad \text{and } \iota : \mathcal{G} \to \mathcal{G},$$

which satisfy the standard relations that describe associativity, existence of an identity element, and inversion. It follows that $\mathcal{G}_\circ$ is a Lie group whose multiplication is given by $\mu_\circ : \mathcal{G}_\circ \times \mathcal{G}_\circ \to \mathcal{G}_\circ$.

To a Lie supergroup $\mathcal{G}$ one can associate a Lie superalgebra $\mathrm{Lie}(\mathcal{G})$ which is the subspace of $\mathrm{Der}_{\mathbb{R}}(\mathscr{O}_{\mathcal{G}})$ consisting of left invariant vector fields of $\mathcal{G}$. The only subtle point in the definition of $\mathrm{Lie}(\mathcal{G})$ is the definition of left invariant vector fields. Left invariant vector fields can be defined in several ways. For example, in [9] the authors use the functor of points. We would like to mention a different method which is also described in [4]. For every $g \in \mathcal{G}_\circ$, one can define left translation morphisms $\lambda_g : \mathcal{G} \to \mathcal{G}$ by

$$\lambda_g = \mu \circ ((\delta_g \circ \kappa_{\mathcal{G}}) \times \mathrm{id}_{\mathcal{G}}),$$

where $\mathrm{id}_{\mathcal{G}} : \mathcal{G} \to \mathcal{G}$ is the identity morphism. Similarly, one can define right translation morphisms

$$\rho_g = \mu \circ (\mathrm{id}_{\mathcal{G}} \times (\delta_g \circ \kappa_{\mathcal{G}})).$$

A vector field D is called *left invariant* if it commutes with left translation, i.e.,

$$(\lambda_g)^\# \circ D = D \circ (\lambda_g)^\#.$$

It is easily checked that $\mathrm{Lie}(\mathcal{G})$, the space of left invariant vector fields of $\mathcal{G}$, is closed under the superbracket of $\mathrm{Der}_{\mathbb{R}}(\mathscr{O}_{\mathcal{M}})$. Moreover, there is an action of $\mathcal{G}_\circ$ on $\mathrm{Lie}(\mathcal{G})$ given by

$$D \mapsto (\rho_g)^\# \circ D \circ (\rho_{\iota_\circ(g)})^\#. \tag{3}$$

Because of Part (ii) of Proposition 3.3.1 below it is natural to denote this action by $\mathrm{Ad}(g)$.

Proposition 3.3.1. *For a Lie supergroup $\mathcal{G} = (\mathcal{G}_\circ, \mathscr{O}_{\mathcal{G}})$ the following statements hold.*

(i) $\mathrm{Lie}(\mathcal{G}) = \mathrm{Lie}(\mathcal{G})_{\overline{0}} \oplus \mathrm{Lie}(\mathcal{G})_{\overline{1}}$ *is a Lie superalgebra over* $\mathbb{R}$.

(ii) *The action of* $\mathcal{G}_\circ$ *on* $\mathrm{Lie}(\mathcal{G})$ *given by* (3) *yields a smooth homomorphism of Lie groups*

$$\mathrm{Ad} : \mathcal{G}_\circ \to \mathrm{GL}(\mathrm{Lie}(\mathcal{G}))$$

such that $\mathrm{Ad}(\mathcal{G}_\circ) \subseteq \mathrm{End}_{\mathbb{R}}(\mathrm{Lie}(\mathcal{G}))_{\overline{0}}$.

(iii) $\mathrm{Lie}(\mathcal{G})_{\overline{0}}$ *is the Lie algebra of* $\mathcal{G}_\circ$ *and if* $\mathsf{d}(\mathrm{Ad})$ *denotes the differential of the above map* Ad*, then*

$$\mathsf{d}(\mathrm{Ad})(X)(Y) = \mathrm{ad}(X)(Y)$$

for every $X \in \mathrm{Lie}(\mathcal{G})_{\overline{0}}$ *and every* $Y \in \mathrm{Lie}(\mathcal{G})$*, where*

$$\mathrm{ad}(X)(Y) = [X, Y].$$

3.4 Harish–Chandra Pairs

Proposition 3.3.1 states that to a Lie supergroup $\mathcal{G}$ one can associate an ordered pair $(\mathcal{G}_\circ, \mathrm{Lie}(\mathcal{G}))$, where $\mathcal{G}_\circ$ is a real Lie group and $\mathrm{Lie}(\mathcal{G})$ is a Lie superalgebra over $\mathbb{R}$, which satisfy certain properties. Such an ordered pair is a *Harish–Chandra pair*.

Definition 3.4.1. A Harish–Chandra pair is a pair $(G, \mathfrak{g})$ consisting of a Lie group G and a Lie superalgebra $\mathfrak{g}$ which satisfy the following properties.

(i) $\mathfrak{g}_{\overline{0}}$ is the Lie algebra of G.
(ii) G acts on $\mathfrak{g}$ smoothly by $\mathbb{R}$-linear automorphisms.
(iii) The differential of the action of G on $\mathfrak{g}$ is equal to the adjoint action of $\mathfrak{g}_{\overline{0}}$ on $\mathfrak{g}$.

One can prove that

$$\mathcal{G} \mapsto (\mathcal{G}_\circ, \mathrm{Lie}(\mathcal{G}))$$

is an equivalence of categories from the category of Lie supergroups to the category of Harish–Chandra pairs. Under this equivalence of categories, a morphism $\psi : \mathcal{G} \to \mathcal{H}$ in the category of Lie supergroups corresponds to a pair $(\psi_\circ, \psi_{\mathrm{Lie}})$ where $\psi_\circ : \mathcal{G}_\circ \to \mathcal{H}_\circ$ is a homomorphism of Lie groups,

$$\psi_{\mathrm{Lie}} : \mathrm{Lie}(\mathcal{G}) \to \mathrm{Lie}(\mathcal{H})$$

is a homomorphism of Lie superalgebras, and

$$\mathsf{d}\psi_\circ = \psi_{\mathrm{Lie}}\big|_{\mathrm{Lie}(\mathcal{G})_{\overline{0}}}.$$

Remark 3.4.2. Using Harish–Chandra pairs one can study Lie supergroups and their representations without any reference to the structural sheaves. In the rest of this article, Lie supergroups will always be realized as Harish–Chandra pairs.

4 Unitary Representations

According to [9, Sect. 4.4] one can define a finite dimensional super Hilbert space as a finite dimensional complex $\mathbb{Z}_2$-graded vector space which is endowed with an even super Hermitian form. Nevertheless, since the even super Hermitian form is generally indefinite, in the infinite dimensional case one should address the issues of topological completeness and separability. For the purpose of studying unitary representations it would be slightly more convenient to take an equivalent approach which is more straightforward, but less canonical.

4.1 Super Hilbert Spaces

A *super Hilbert space* is a $\mathbb{Z}_2$-graded complex Hilbert space $\mathscr{H} = \mathscr{H}_{\overline{0}} \oplus \mathscr{H}_{\overline{1}}$ such that $\mathscr{H}_{\overline{0}}$ and $\mathscr{H}_{\overline{1}}$ are mutually orthogonal closed subspaces. If $\langle \cdot, \cdot \rangle$ denotes the inner product of $\mathscr{H}$, then for every two homogeneous elements $v, w \in \mathscr{H}$ the even super Hermitian form (v, w) of $\mathscr{H}$ is defined by

$$(v,w) = \begin{cases} 0 & \text{if } v \text{ and } w \text{ have opposite parity,} \\ \langle v, w\rangle & \text{if } v \text{ and } w \text{ are even,} \\ i\,\langle v, w\rangle & \text{if } v \text{ and } w \text{ are odd.} \end{cases}$$

One can check that $(\cdot,\cdot)$ satisfies the properties stated in [9, Sect. 4.4]. In this article the latter sesquilinear form will not be used.

4.2 The Definition of a Unitary Representation

In order to obtain an analytic theory of unitary representations of Lie supergroups one should deal with the same sort of analytic difficulties that exist in the case of Lie groups. One of the main difficulties is that in general one cannot define the differential of an infinite dimensional representation of a Lie group on the entire representation space. However, one can always define the differential on certain invariant dense subspaces, such as the space of *smooth vectors*.

In the rest of this article, the reader is assumed to be familiar with classical results in the theory of unitary representations of Lie groups. For a detailed and readable treatment of this subject see [32].

If $\mathscr{H}$ is a (possibly $\mathbb{Z}_2$-graded) complex Hilbert space, the group of unitary operators of $\mathscr{H}$ is denoted by $\mathbf{U}(\mathscr{H})$. As usual, if $\pi : G \to \mathbf{U}(\mathscr{H})$ is a unitary representation of a Lie group G, then the space of smooth vectors (respectively, analytic vectors) of $(\pi, \mathscr{H})$ is denoted by $\mathscr{H}^\infty$ (respectively, $\mathscr{H}^\omega$).

Definition 4.2.1. Let $(G, \mathfrak{g})$ be a Lie supergroup. A unitary representation of $(G, \mathfrak{g})$ is a triple $(\pi, \rho^\pi, \mathscr{H})$ satisfying the following properties.

(i) $\mathscr{H} = \mathscr{H}_{\bar 0} \oplus \mathscr{H}_{\bar 1}$ is a super Hilbert space.
(ii) $(\pi, \mathscr{H})$ is a unitary representation of G and $\pi(g) \in \mathrm{End}_{\mathbb{C}}(\mathscr{H})_{\bar 0}$ for every $g \in G$.
(iii) $\rho^\pi : \mathfrak{g} \to \mathrm{End}_{\mathbb{C}}(\mathscr{H}^\infty)$ is an $\mathbb{R}$-linear $\mathbb{Z}_2$-graded map, where $\mathscr{H}^\infty$ denotes the space of smooth vectors of $(\pi, \mathscr{H})$. Moreover, for every $X, Y \in \mathfrak{g}_{\bar 1}$,

$$\rho^\pi(X)\rho^\pi(Y) + \rho^\pi(Y)\rho^\pi(X) = -i\rho^\pi([X, Y]).$$

(iv) $\rho^\pi(X) = \mathsf{d}\pi(X)\big|_{\mathscr{H}^\infty}$ for every $X \in \mathfrak{g}_{\bar 0}$.
(v) For every $X \in \mathfrak{g}_{\bar 1}$ the operator $\rho^\pi(X)$ is symmetric, i.e., if $v, w \in \mathscr{H}^\infty$ then

$$\langle \rho^\pi(X)v, w\rangle = \langle v, \rho^\pi(X)w\rangle.$$

(vi) For every $g \in G$ and every $X \in \mathfrak{g}$,

$$\rho^\pi\big(\mathrm{Ad}(g)(X)\big) = \pi(g)\rho^\pi(X)\pi(g)^{-1}.$$

Remark 4.2.2. It is easy to see that by letting an element $X_{\bar{0}} + X_{\bar{1}} \in \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ act on $\mathscr{H}^\infty$ as

$$\rho^\pi(X_{\bar{0}}) + e^{\frac{\pi}{4}i}\rho^\pi(X_{\bar{1}})$$

one obtains from ρ^π a homomorphism of Lie superalgebras from $\mathfrak{g}$ into $\mathrm{End}_{\mathbb{C}}(\mathscr{H}^\infty)$.

Remark 4.2.3. Subrepresentations, irreducibility, and unitary equivalence of unitary representations of Lie supergroups are defined similar to unitary representations of Lie groups (see [5]). Note that in the definition of unitary equivalence, intertwining operators are assumed to preserve the grading. This means that in general a unitary representation is not necessarily unitarily equivalent to the one obtained by parity change.

Lemma 4.2.4. For each $X \in \mathfrak{g}$, the operator $\rho^\pi(X)\colon \mathscr{H}^\infty \to \mathscr{H}^\infty$ is continuous with respect to the Fréchet topology on $\mathscr{H}^\infty$. Moreover, the bilinear map

$$\mathfrak{g} \times \mathscr{H}^\infty \to \mathscr{H}^\infty, \quad (X, v) \mapsto \rho^\pi(X)v \tag{4}$$

is continuous.

Proof. Since $\mathfrak{g}$ is finite dimensional, it suffices to show that each operator $\rho^\pi(X)$ is continuous. For $X \in \mathfrak{g}_{\bar{0}}$, this follows from the definition of the Fréchet topology on $\mathscr{H}^\infty$.

For $X \in \mathfrak{g}_{\bar{1}}$, the operator $\rho^\pi(X)$ on $\mathscr{H}^\infty$ is symmetric (see Definition 4.2.1(v)), hence the graph of $\rho^\pi(X)$ is closed. Now the Closed Graph Theorem for Fréchet spaces (see [26, Theorem 2.15]) implies its continuity. ☐

From now on we assume that $\mathscr{H}$ is separable. Although this assumption is not needed in Definition 4.2.1, it helps in avoiding technical conditions in various constructions, e.g., when induced representations are defined. Note that if $(\pi, \rho^\pi, \mathscr{H})$ is irreducible then $\mathscr{H}$ is separable.

In Definition 4.2.1 the fact that $\mathscr{H}^\infty$ is chosen as the space of the representation of $\mathfrak{g}$ is not a limitation. In fact it is shown in [5, Proposition 2] that in some sense any reasonable choice of the space of the representation of $\mathfrak{g}$, i.e., one which is dense in $\mathscr{H}$ and satisfies natural invariance properties under the actions on G and $\mathfrak{g}$, would yield a definition equivalent to the one given above. This fact also plays a role in showing that restriction and induction functors are well defined. Another useful fact, which follows from [5, Proposition 3], is that the space $\mathscr{H}^\omega$ of analytic vectors of $(\pi, \mathscr{H})$ is invariant under $\rho^\pi(\mathfrak{g})$.

4.3 Restriction and Induction

Suppose that $\mathcal{G} = (G, \mathfrak{g})$ is a Lie supergroup, and $\mathcal{H} = (H, \mathfrak{h})$ is a Lie subsupergroup of $\mathcal{G}$. Let $(\pi, \rho^\pi, \mathscr{H})$ be a unitary representation of $\mathcal{G}$. A priori it is not clear how to restrict $(\pi, \rho^\pi, \mathscr{H})$ to $\mathcal{H}$. The difficulty is that in general the space of smooth vectors of the restriction of $(\pi, \mathscr{H})$ to H will be larger than $\mathscr{H}^\infty$. To circumvent this issue one can use [5, Proposition 2] to show that the action of $\mathcal{H}$ on $\mathscr{H}^\infty$ determines a unique unitary representation of $\mathcal{H}$ on $\mathscr{H}$. This representation is called the restriction of $(\pi, \rho^\pi, \mathscr{H})$ to $\mathcal{H}$, and is denoted by

$$\mathrm{Res}^{\mathcal{G}}_{\mathcal{H}}(\pi, \rho^\pi, \mathscr{H}).$$

Inducing from $\mathcal{H}$ to $\mathcal{G}$ is more delicate. Let $(\sigma, \rho^\sigma, \mathscr{K})$ be a unitary representation of $\mathcal{H}$. The first step towards defining a representation $(\pi, \rho^\pi, \mathscr{H})$ of $\mathcal{G}$ that is induced from $(\sigma, \rho^\sigma, \mathscr{K})$ is to identify the super Hilbert space $\mathscr{H}$. By analogy with the case of Lie groups one expects the super Hilbert space $\mathscr{H}$ to be a space of $\mathscr{K}$-valued functions on $\mathcal{G}$ which satisfy an equivariance property with respect to the left regular action of $\mathcal{H}$. One can then describe the action of $\mathcal{G}$ by formal relations, hoping that a unitary representation, as defined in Definition 4.2.1, is obtained. This formal approach leads to technical complications and it is not clear how to get around some of them. Nevertheless, at least in the special case that the homogeneous super space $\mathcal{H}\backslash\mathcal{G}$ is purely even, i.e., when $\dim \mathfrak{g}_{\bar{1}} = \dim \mathfrak{h}_{\bar{1}}$, it is shown in [5, Sect. 3] that the induced representation can be defined rigorously. In this article, only the special case when both G and H are unimodular groups is used, and in this case the induced representation is defined as follows. Since the homogeneous space $\mathcal{H}\backslash\mathcal{G}$ is purely even, there is a natural isomorphism $\mathcal{H}\backslash\mathcal{G} \simeq H\backslash G$. Choose an invariant measure μ on $H\backslash G$, and let $\mathscr{H}$ be the space of measurable functions $f : G \to \mathscr{K}$ which satisfy the following properties.

(i) $f(hg) = \sigma(h)f(g)$ for every $g \in G$ and every $h \in H$.
(ii) $\int_{H\backslash G} ||f||^2 d\mu < \infty$

The action of G on $\mathscr{H}$ is the right regular representation, i.e.,

$$\big(\pi(g)f\big)(g_1) = f(g_1 g) \quad \text{for every} \quad g, g_1 \in G,$$

and one can easily check that it is unitary with respect to the standard inner product of $\mathscr{H}$. The most natural way to define the action of an element $X \in \mathfrak{g}_{\bar{1}}$ on an element $f \in \mathscr{H}^\infty$ is via the formula

$$\big(\rho^\pi(X)f\big)(g) = \rho^\sigma\big(\mathrm{Ad}(g)(X)\big)\big(f(g)\big). \tag{5}$$

It is known that every $f \in \mathscr{H}^\infty$ is a smooth function from G to $\mathscr{K}$ and $f(g) \in \mathscr{K}^\infty$ for every $g \in G$ [25, Theorem 5.1]. Consequently, the right hand side of (5) is well defined. However, a priori it is not obvious why for an element $X \in \mathfrak{g}_{\bar{1}}$ the right hand side of (5) belongs to $\mathscr{H}^\infty$. One can prove the weaker statement that $\rho^\pi(X)f \in \mathscr{H}$ using a trick which is based on the ideas used in [5]. Since this trick sheds some light on the situation, it may be worthwhile to mention it. One can prove that the operator $\rho^\pi(X)$ is essentially self-adjoint. Let $\overline{\rho^\pi(X)}$ denote the closure of $\rho^\pi(X)$. The operator $I + \overline{\rho^\pi(X)}^2$ has a bounded inverse whose domain is all of $\mathscr{H}$

(this follows for instance from [7, Chap. X, Proposition 4.2]). For every $f \in \mathscr{H}^\infty$,

$$\rho^\pi(X)f = \overline{\rho^\pi(X)}f = \overline{\rho^\pi(X)}(I + \overline{\rho^\pi(X)}^2)^{-1}(I + \overline{\rho^\pi(X)}^2)f.$$

Using the spectral theory of self-adjoint operators one can show that the operator $\overline{\rho^\pi(X)}(I + \overline{\rho^\pi(X)}^2)^{-1}$ is bounded. Moreover,

$$(I + \overline{\rho^\pi(X)}^2)f = \big(I - \frac{i}{2}\mathsf{d}\pi([X, X])\big)f \in \mathscr{H}^\infty.$$

Finally, boundedness of $\overline{\rho^\pi(X)}(I + \overline{\rho^\pi(X)}^2)^{-1}$ implies that $\rho^\pi(X)f \in \mathscr{H}$.

To prove that indeed $\rho^\pi(X)f \in \mathscr{H}^\infty$ requires more effort. This is proved in [5, Sect. 3] in an indirect way. The idea of the proof is to find a dense subspace $\mathscr{B} \subseteq \mathscr{H}^\infty$ such that $\rho^\pi(\mathfrak{g})\mathscr{B} \subseteq \mathscr{B}$. As shown in [5, Sect. 3], one can take $\mathscr{B}$ to be the subspace of $\mathscr{H}^\infty$ consisting of functions from G to $\mathscr{K}$ with compact support modulo H. That $(\pi, \rho^\pi, \mathscr{H})$ is well defined then follows from [5, Proposition 2].

The representation $(\pi, \rho^\pi, \mathscr{H})$ induced from $(\sigma, \rho^\sigma, \mathscr{K})$ is denoted by

$$\mathrm{Ind}_{\mathcal{H}}^{\mathcal{G}}(\sigma, \rho^\sigma, \mathscr{K}).$$

It can be shown [27, Proposition 3.2.1] that induction may be done in stages, i.e., if $\mathcal{H}$ is a Lie subsupergroup of $\mathcal{G}$, $\mathcal{K}$ is a Lie subsupergroup of $\mathcal{H}$, and $(\sigma, \rho^\sigma, \mathscr{K})$ is a unitary representation of $\mathcal{K}$, then

$$\mathrm{Ind}_{\mathcal{H}}^{\mathcal{G}}\mathrm{Ind}_{\mathcal{K}}^{\mathcal{H}}(\sigma, \rho^\sigma, \mathscr{K}) \simeq \mathrm{Ind}_{\mathcal{K}}^{\mathcal{G}}(\sigma, \rho^\sigma, \mathscr{K}).$$

5 Invariant Cones in Lie Algebras

The goal of this section is to take a brief look at convex cones in finite dimensional real Lie algebras which are invariant under the adjoint action. A natural reduction to the case where the cone is pointed and generating leads to an interesting class of Lie algebras with a particular structure that will be discussed below.

A closed convex cone C in a finite dimensional vector space V is said to be *pointed* if $C \cap -C = \{0\}$, i.e., if C contains no affine lines. It is said to be *generating* if $C - C = V$ or equivalently if $Int(C)$ is nonempty, where $Int(C)$ denotes the set of interior points of C. If C is a cone in a finite dimensional vector space V then $C^\star$ denotes the cone in V^* consisting of all $\lambda \in V^*$ such that $\lambda(x) \geq 0$ for every $x \in C$.

5.1 Pointed Generating Invariant Cones

Let $\mathfrak{g}$ be a finite dimensional Lie algebra over $\mathbb{R}$. A cone $C \subseteq \mathfrak{g}$ is called *invariant* if it is closed, convex, and invariant under $\mathrm{Inn}(\mathfrak{g})$.

Suppose that C is an invariant cone in $\mathfrak{g}$ and set $\mathrm{H}(C) = C \cap -C$ and $\mathfrak{g}(C) = C - C$. The subspaces $\mathrm{H}(C)$ and $\mathfrak{g}(C)$ are ideals of $\mathfrak{g}$ and $C/\mathrm{H}(C)$ is a pointed generating invariant cone in the quotient Lie algebra $\mathfrak{g}(C)/\mathrm{H}(C)$. The main concern of the theory of invariant cones is to understand the situation when C is pointed and generating.

The existence of pointed generating invariant cones in a Lie algebra has the following simple but useful consequence.

Lemma 5.1.1. Let C be a pointed generating invariant cone in $\mathfrak{g}$. If $\mathfrak{a}$ is an abelian ideal of $\mathfrak{g}$ then $\mathfrak{a} \subseteq \mathscr{Z}(\mathfrak{g})$.

Proof. If $X \in Int(C)$, then $C \supseteq e^{\mathrm{ad}(\mathfrak{a})}X = X + [\mathfrak{a}, X]$. Since C contains no affine lines, $[\mathfrak{a}, X] = \{0\}$. Since $X \in Int(C)$ is arbitrary, $\mathfrak{a} \subseteq \mathscr{Z}(\mathfrak{g})$. □

To study invariant cones further, we need the following lemma.

Lemma 5.1.2. Let V be a finite dimensional vector space, $S \subseteq V$ be a convex subset, and $K \subseteq \mathrm{GL}(V)$ be a subgroup which leaves S invariant. Suppose that the closure of K in $\mathrm{GL}(V)$ is compact. If S is open or closed, then it contains K-fixed points.

Proof. Let $\overline{K}$ be the closure of K, and $\mu_{\overline{K}}$ be a normalized Haar measure on $\overline{K}$. For every $v \in S$, the point $v_\circ = \int_{\overline{K}}(k \cdot v)\, d\mu_{\overline{K}}(k)$ is K-fixed, and it is easily verified that $v_\circ \in S$. □

The preceding lemma has the following interesting consequence for invariant cones.

Lemma 5.1.3. Let $C \subseteq \mathfrak{g}$ be a pointed generating invariant cone. Then a subalgebra $\mathfrak{k} \subseteq \mathfrak{g}$ is compactly embedded in $\mathfrak{g}$ if and only if $\mathscr{Z}_\mathfrak{g}(\mathfrak{k}) \cap Int(C) \neq \emptyset$.

Proof. If $\mathfrak{k} \subseteq \mathfrak{g}$ is compactly embedded in $\mathfrak{g}$ then Lemma 5.1.2 implies that $Int(C)$ contains fixed points for $\mathrm{Inn}_\mathfrak{g}(\mathfrak{k})$, i.e.,

$$\mathscr{Z}_\mathfrak{g}(\mathfrak{k}) \cap Int(C) \neq \emptyset. \tag{6}$$

Conversely, if $\mathscr{Z}_\mathfrak{g}(\mathfrak{k}) \cap Int(C) \neq \emptyset$ then set $K = \mathrm{Inn}_\mathfrak{g}(\mathfrak{k})$ and observe that K is a subgroup of $\mathrm{Inn}(\mathfrak{g})$ with a fixed point $X_0 \in Int(C)$. The set $C \cap (X_0 - C)$ is a compact K-invariant subset of $\mathfrak{g}$ with interior points. This implies that K is bounded in $\mathrm{GL}(\mathfrak{g})$ and therefore it has compact closure in $\mathrm{Aut}(\mathfrak{g})$. □

5.2 Compactly Embedded Cartan Subalgebras

Let $\mathfrak{g}$ be a finite dimensional Lie algebra over $\mathbb{R}$. Our next goal is to show that the existence of a pointed generating invariant cone in $\mathfrak{g}$ implies that $\mathfrak{g}$ has compactly embedded Cartan subalgebras. The next lemma shows how such a Cartan subalgebra can be obtained explicitly.

Lemma 5.2.1. Let $C \subseteq \mathfrak{g}$ be a pointed generating invariant cone. Suppose that $Y \in Int(C)$ is a regular element of $\mathfrak{g}$, i.e., the subspace

$$\mathscr{N}_{\mathfrak{g}}(Y) = \bigcup_n \ker(\mathrm{ad}(Y)^n)$$

has minimal dimension. If $\mathfrak{t} = \ker(\mathrm{ad}(Y))$, then $\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{g}$ which is compactly embedded in $\mathfrak{g}$.

Proof. For any such Y, the subspace $\mathfrak{t} = \mathscr{N}_{\mathfrak{g}}(Y)$ is a Cartan subalgebra of $\mathfrak{g}$ (see [3, Chap. VII]). Since $Y \in \mathscr{Z}_{\mathfrak{g}}(\mathscr{Z}_{\mathfrak{g}}(\mathbb{R}Y))$, Lemma 5.1.3 implies that $\mathscr{Z}_{\mathfrak{g}}(\mathbb{R}Y)$ is compactly embedded in $\mathfrak{g}$. It follows immediately that $\mathbb{R}Y$ is compactly embedded in $\mathfrak{g}$. Therefore the endomorphism $\mathrm{ad}(Y) : \mathfrak{g} \to \mathfrak{g}$ is semisimple and

$$\mathfrak{t} = \ker(\mathrm{ad}(Y)) = \mathscr{Z}_{\mathfrak{g}}(Y)$$

from which it follows that $\mathfrak{t}$ is compactly embedded in $\mathfrak{g}$. $\square$

Remark 5.2.2. It is known that the set of regular elements of $\mathfrak{g}$ is dense (see [3, Chap. VII]). Since $Int(C) \neq \emptyset$, the intersection of $Int(C)$ with the set of regular elements of $\mathfrak{g}$ is nonempty.

5.3 Characterization of Lie Algebras with Invariant Cones

The material in this section is meant to shed light on the connection between invariant cones and Hermitian Lie algebras. The reader is assumed to be familiar with the classification of real semisimple Lie algebras.

The study of invariant cones in finite-dimensional Lie algebras was initiated by Kostant, Segal and Vinberg [30, 34]. A structure theory of invariant cones in general finite dimensional Lie algebras was developed by Hilgert and Hofmann in [11]. The characterization of those finite dimensional Lie algebras containing pointed generating invariant cones was obtained in [19] in terms of certain symplectic modules called of convex type, whose classification can be found in [22]. A self-contained exposition of this theory is available in [20], where the Lie algebras $\mathfrak{g}$ for which there exist pointed generating invariant cones in $\mathfrak{g} \oplus \mathbb{R}$ are called *admissible*.

Example 5.3.1. (*cf.* [34]) Suppose that $\mathfrak{g}$ is a real simple Lie algebra with a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Since $\mathfrak{p}$ is a simple nontrivial $\mathfrak{k}$-module, $\mathscr{Z}_{\mathfrak{g}}(\mathfrak{k}) = \mathscr{Z}(\mathfrak{k})$. If C is a pointed generating invariant cone in $\mathfrak{g}$, then from Lemma 5.1.3 it follows that

$$Int(C) \cap \mathscr{Z}(\mathfrak{k}) \neq \emptyset.$$

In particular $\mathscr{Z}(\mathfrak{k}) \neq \{0\}$, i.e., $\mathfrak{g}$ is *Hermitian*. Conversely, assume that $\mathfrak{g}$ is Hermitian and $0 \neq Z \in \mathscr{Z}(\mathfrak{k})$. If $(\cdot,\cdot)$ denotes the Killing form of $\mathfrak{g}$, then from the Cartan decomposition $\mathrm{Inn}(\mathfrak{g}) = \mathrm{Inn}(\mathfrak{k})e^{\mathrm{ad}(\mathfrak{p})}$ it follows that

$$(\mathrm{Inn}(\mathfrak{g})Z, Z) = (e^{\mathrm{ad}(\mathfrak{p})}Z, Z).$$

If $P \in \mathfrak{p}$ then $(e^{\mathrm{ad}(P)}Z, Z) < 0$ because

$$(e^{\mathrm{ad}(P)}Z, Z) = \sum_{n=0}^{\infty} \frac{(\mathrm{ad}(P)^{2n}(Z), Z)}{(2n)!}$$

and the linear transformations $\mathrm{ad}(P)^{2n} : \mathfrak{k} \to \mathfrak{k}$ are positive definite with respect to $(\cdot,\cdot)$. It follows that $\mathrm{Inn}(\mathfrak{g})Z$ lies in a proper invariant cone $C \subseteq \mathfrak{g}$. Since $\mathfrak{g}$ is simple, C is pointed and generating.

A slight refinement of the above arguments shows that a reductive Lie algebra $\mathfrak{g}$ is admissible if and only if $\mathscr{Z}_{\mathfrak{g}}(\mathscr{Z}(\mathfrak{k})) = \mathfrak{k}$ holds for a maximal compactly embedded subalgebra $\mathfrak{k}$ of $\mathfrak{g}$. Lie algebras satisfying this property are called *quasihermitian*. This is equivalent to all simple ideals of $\mathfrak{g}$ being either compact or Hermitian. A reductive admissible Lie algebra contains pointed generating invariant cones if and only if it is not compact semisimple. This clarifies the structure of reductive Lie algebras with invariant cones.

Below we shall need the following lemma.

Lemma 5.3.2. Let $\mathfrak{g}$ be a quasihermitian Lie algebra, $\mathfrak{k} \subseteq \mathfrak{g}$ be a maximal compactly embedded subalgebra of $\mathfrak{g}$, and $p_{\mathfrak{z}}\colon \mathfrak{g} \to \mathscr{Z}(\mathfrak{k})$ be the fixed point projection for the compact group $e^{\mathrm{ad}\,\mathfrak{k}}$. Then every closed invariant convex subset $C \subseteq \mathfrak{g}$ satisfies $p_{\mathfrak{z}}(C) = C \cap \mathscr{Z}(\mathfrak{k})$.

Proof. Let $\mathfrak{p} \subseteq \mathfrak{g}$ be a $\mathfrak{k}$-invariant complement and recall that $\mathfrak{g}$ is said to be quasihermitian if $\mathfrak{k} = \mathscr{Z}_{\mathfrak{g}}(\mathscr{Z}(\mathfrak{k}))$. This condition implies in particular that $\mathfrak{p}$ contains no non-zero trivial $\mathfrak{k}$-submodule, so that $\mathscr{Z}_{\mathfrak{g}}(\mathfrak{k}) = \mathscr{Z}(\mathfrak{k})$. The assertion now follows from the proof of Lemma 5.1.2. □

In the case of an arbitrary Lie algebra $\mathfrak{g}$ having a pointed generating invariant cone, one can use Lemma 5.1.1 to show that the maximal nilpotent ideal $\mathfrak{n}$ of $\mathfrak{g}$ is two-step nilpotent, i.e., a generalized Heisenberg algebra. Moreover, $\mathfrak{n}$ clearly contains $\mathscr{Z}(\mathfrak{g})$, which is contained in any compactly embedded Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{g}$. Let $\mathfrak{a} \subseteq \mathfrak{t}$ be a complement to $\mathscr{Z}(\mathfrak{g})$ and $\mathfrak{s}$ be a $\mathfrak{t}$-invariant Levi complement to $\mathfrak{n}$ in $\mathfrak{g}$ (which always exists), and set $\mathfrak{l} = \mathfrak{a} \oplus \mathfrak{s}$. Then $\mathfrak{l}$ is reductive, $\mathfrak{g} = \mathfrak{l} \ltimes \mathfrak{n}$, and $\mathfrak{l}$ is an admissible reductive Lie algebra (see [20, Proposition VII.1.9]). At this point the structure of $\mathfrak{n}$ and $\mathfrak{l}$ is quite clear. However, to derive a classification of Lie algebras with invariant cones from this semidirect decomposition, one has to analyze the possibilities for the $\mathfrak{l}$-module structure on $\mathfrak{n}$ in some detail. This is done in [19] and [22].

6 Unitary Representations and Invariant Cones

A Lie supergroup $\mathcal{G} = (G, \mathfrak{g})$ is called $\star$-*reduced* if for every nonzero $X \in \mathfrak{g}$ there exists a unitary representation $(\pi, \rho^{\pi}, \mathcal{H})$ of $\mathcal{G}$ such that $\rho^{\pi}(X) \neq 0$. Note that when $\mathfrak{g}$ is simple, $\mathcal{G}$ is $\star$-reduced if and only if it has a nontrivial unitary representation. In this section we study properties of $\star$-reduced Lie supergroups via methods based on the theory of invariant cones. We obtain necessary conditions for a Lie supergroup $\mathcal{G}$ to be $\star$-reduced. It turns out that these necessary conditions are strong enough for the classification of $\star$-reduced simple Lie supergroups.

Let $\mathcal{G} = (G, \mathfrak{g})$ be an arbitrary Lie supergroup, and let $(\pi, \rho^\pi, \mathscr{H})$ be a unitary representation of $\mathcal{G}$. Fix an element $X \in \mathfrak{g}_{\overline{1}}$. From

$$\rho^\pi([X, X]) = i[\rho^\pi(X), \rho^\pi(X)] = 2i\rho^\pi(X)^2$$

and the fact that the operator $\rho^\pi(X)$ is symmetric it follows that

$$\langle i\rho^\pi([X, X])v, v\rangle \leq 0 \text{ for every } v \in \mathscr{H}^\infty.$$

Let $Cone(\mathcal{G})$ denote the invariant cone in $\mathfrak{g}_{\overline{0}}$ which is generated by elements of the form $[X, X]$ where $X \in \mathfrak{g}_{\overline{1}}$. Linearity of ρ^π implies that

$$\langle i\rho^\pi(Y)v, v\rangle \leq 0 \text{ for every } v \in \mathscr{H}^\infty \text{ and every } Y \in Cone(\mathcal{G}). \tag{7}$$

This means that π is $Cone(\mathcal{G})$-dissipative in the sense of [20].

6.1 Properties of $\star$-Reduced Lie Supergroups

Unlike Lie groups, which are known to have faithful unitary representations, certain Lie supergroups do not have such representations. The next proposition, which is given in [27, Lemma 4.1.1], shows how this can happen. The proof of this proposition is based on the fact that for every $X \in \mathfrak{g}_1$, the spectrum of $-i\rho^\pi([X, X])$ is nonnegative, so that a sum of such operators vanishes if and only if all summands vanish.

Proposition 6.1.1. *Let $(\pi, \rho^\pi, \mathscr{H})$ be a unitary representation of $\mathcal{G} = (G, \mathfrak{g})$. Suppose that elements $X_1, \ldots, X_m \in \mathfrak{g}_{\overline{1}}$ satisfy*

$$[X_1, X_1] + \cdots + [X_m, X_m] = 0.$$

Then $\rho^\pi(X_1) = \cdots = \rho^\pi(X_m) = 0$.

The next proposition provides necessary conditions for a Lie supergroup to be $\star$-reduced.

Proposition 6.1.2. *If $\mathcal{G} = (G, \mathfrak{g})$ is $\star$-reduced, then the following statements hold.*

(i) *$Cone(\mathcal{G})$ is pointed.*
(ii) *For every $\lambda \in Int(Cone(\mathcal{G})^\star)$, the symmetric bilinear form*

$$\Omega_\lambda : \mathfrak{g}_{\bar{0}} \times \mathfrak{g}_{\bar{0}} \to \mathbb{R} \quad \textit{defined by} \quad \Omega_\lambda(X, Y) = \lambda([X, Y])$$

is positive definite .

(iii) *Let $\mathfrak{k}_{\bar{0}}$ be a Lie subalgebra of $\mathfrak{g}_{\bar{0}}$. If $\mathfrak{k}_{\bar{0}}$ is compactly embedded in $\mathfrak{g}_{\bar{0}}$, then $\mathfrak{k}_{\bar{0}}$ is compactly embedded in $\mathfrak{g}$ as well.*

(iv) *If $\mathfrak{g}_{\bar{0}} = [\mathfrak{g}_{\bar{1}}, \mathfrak{g}_{\bar{1}}]$ then $\mathfrak{g}_{\bar{0}}$ has a Cartan subalgebra which is compactly embedded in $\mathfrak{g}$.*

(v) *Assume that there exists a Cartan subalgebra $\mathfrak{h}_{\bar{0}}$ of $\mathfrak{g}_{\bar{0}}$ which is compactly embedded in $\mathfrak{g}$. Let $\mathbf{p} : \mathfrak{g}_{\bar{0}} \to \mathfrak{h}_{\bar{0}}$ be the projection map corresponding to the decomposition*

$$\mathfrak{g}_{\bar{0}} = \mathfrak{h}_{\bar{0}} \oplus [\mathfrak{h}_{\bar{0}}, \mathfrak{g}_{\bar{0}}]$$

(see Proposition 2.4.1) and $\mathbf{p}^* : \mathfrak{h}_{\bar{0}}^* \to \mathfrak{g}_{\bar{0}}^*$ be the corresponding dual map. Then

$$Int(Cone(\mathcal{G})^\star) \cap \mathbf{p}^*(\mathfrak{h}_{\bar{0}}^*) \neq \emptyset.$$

Proof. (i) Suppose, on the contrary, that $Y, -Y \in Cone(\mathcal{G})$ for some nonzero Y. Let $(\pi, \rho^\pi, \mathscr{H})$ be a unitary representation of $\mathcal{G}$. For every $v \in \mathscr{H}^\infty$,

$$0 \leq \langle i\rho^\pi(Y)v, v\rangle \leq 0,$$

which implies that $\langle i\rho^\pi(Y)v, v\rangle = 0$. Therefore for every $v, w \in \mathscr{H}^\infty$ and every $z \in \mathbb{C}$,

$$\begin{aligned} 0 &= \langle \big(i\rho^\pi(Y)\big)(v + zw), v + zw\rangle \\ &= \langle i\rho^\pi(Y)v, v\rangle + \bar{z}\langle i\rho^\pi(Y)v, w\rangle + z\langle i\rho^\pi(Y)w, v\rangle + |z|^2\langle i\rho^\pi(Y)w, w\rangle \\ &= \bar{z}\langle i\rho^\pi(Y)v, w\rangle + z\langle i\rho^\pi(Y)w, v\rangle \end{aligned}$$

and since z is arbitrary, $\langle i\rho^\pi(Y)v, w\rangle = 0$ for every $v, w \in \mathscr{H}^\infty$. This means that $\rho^\pi(Y) = 0$, hence $Y = 0$ because $\mathcal{G}$ is $\star$-reduced.

(ii) That Ω_λ is positive semidefinite is immediate from the definition of $Cone(\mathcal{G})^\star$. If $X \in \mathfrak{g}_1$ satisfies $\Omega_\lambda(X, X) = 0$ then from $\lambda \in Int(Cone(\mathcal{G})^\star)$ it follows that $[X, X] = 0$. Since $\mathcal{G}$ is $\star$-reduced, Proposition 6.1.1 implies that $X = 0$.

(iii) Part (i) implies that $Cone(\mathcal{G})$ is pointed, and therefore $Int(Cone(\mathcal{G})^\star)$ is nonempty [20, Proposition V.1.5]. The action of the compact group $\mathrm{INN}_{\mathfrak{g}_{\bar{0}}}(\mathfrak{k}_{\bar{0}})$ on $Cone(\mathcal{G})^\star$ leaves $Int(Cone(\mathcal{G})^\star)$ invariant. By Lemma 5.1.2, this action has a fixed point $\lambda \in Int(Cone(\mathcal{G})^\star)$. Therefore the symmetric bilinear form Ω_λ of Part (ii) is positive definite and invariant with respect to $\mathrm{INN}_{\mathfrak{g}}(\mathfrak{k}_{\bar{0}})$. From the inclusion $\mathrm{Aut}(\mathfrak{g}) \subseteq \mathrm{Aut}(\mathfrak{g}_{\bar{0}}) \times \mathrm{GL}(\mathfrak{g}_{\bar{1}})$ it follows that $\mathrm{INN}_{\mathfrak{g}}(\mathfrak{k}_{\bar{0}})$ is compact.

(iv) By Part (iii) it is enough to prove the existence of a Cartan subalgebra which is compactly embedded in $\mathfrak{g}_{\bar{0}}$. Part (i) implies that $Cone(\mathcal{G})$ is pointed. The equality $\mathfrak{g}_{\bar{0}} = [\mathfrak{g}_{\bar{1}}, \mathfrak{g}_{\bar{1}}]$ means that $Cone(\mathcal{G})$ is generating. Therefore Lemma 5.2.1 completes the proof.

(v) Part (i) implies that $Int(Cone(\mathcal{G})^\star) \neq \emptyset$. Since $\mathrm{INN}_{\mathfrak{g}_{\overline{0}}}(\mathfrak{h}_{\overline{0}})$ is compact and leaves $Int(Cone(\mathcal{G})^\star)$ invariant, Lemma 5.1.2 implies that there exists a $\mu \in Int(Cone(\mathcal{G})^\star)$ which is fixed by $\mathrm{INN}_{\mathfrak{g}_{\overline{0}}}(\mathfrak{h}_{\overline{0}})$, i.e., contained in $\mathbf{p}^*(\mathfrak{h}_{\overline{0}}^*)$. □

Proposition 6.1.3. *Suppose that $\mathcal{G} = (G, \mathfrak{g})$ is a $\star$-reduced Lie supergroup. Let*

(i) $\mathfrak{h}_{\overline{0}}$ *be a Cartan subalgebra of $\mathfrak{g}_{\overline{0}}$ which is compactly embedded in $\mathfrak{g}$,*
(ii) Δ *be the root system associated to $\mathfrak{h}_{\overline{0}}$ (see Proposition 2.4.1),*
(iii) $\mu \in Int(Cone(\mathcal{G})^\star) \cap \mathbf{p}^*(\mathfrak{h}_{\overline{0}}^*)$, *where $\mathbf{p}^*$ is the map defined in the statement of Proposition 6.1.2.*

Then for every nonzero $\alpha \in \Delta$ the Hermitian form

$$\langle \cdot, \cdot \rangle_\alpha : \mathfrak{g}_{\overline{1}}^{\mathbb{C},\alpha} \times \mathfrak{g}_{\overline{1}}^{\mathbb{C},\alpha} \to \mathbb{C}$$

defined by $\langle X, Y \rangle_\alpha = \mu([X, \overline{Y}])$ is positive definite.

Proof. Let

$$\Omega_\mu : \mathfrak{g}_{\overline{1}} \times \mathfrak{g}_{\overline{1}} \to \mathbb{R}$$

be the symmetric bilinear form defined by

$$\Omega_\mu(X, Y) = \mu([X, Y]).$$

By Proposition 6.1.2(ii) the form Ω_μ is positive definite. If $X \in \mathfrak{g}_{\overline{1}}^{\mathbb{C},\alpha}$ then $\overline{X} \in \mathfrak{g}_{\overline{1}}^{\mathbb{C},-\alpha}$ and

$$\begin{aligned}\Omega_\mu(X + \overline{X}, X + \overline{X}) &= \mu([X + \overline{X}, X + \overline{X}]) \\ &= \mu([X, X]) + \mu([X, \overline{X}]) + \mu([\overline{X}, X]) + \mu([\overline{X}, \overline{X}]).\end{aligned}$$

But $[X, X] \in \mathfrak{g}_{\overline{0}}^{\mathbb{C},2\alpha}$ and $[\overline{X}, \overline{X}] \in \mathfrak{g}_{\overline{0}}^{\mathbb{C},-2\alpha}$, and from $\mu \in \mathbf{p}^*(\mathfrak{h}_{\overline{0}}^*)$ and $\alpha \neq 0$ it follows that

$$\mu([X, X]) = \mu([\overline{X}, \overline{X}]) = 0.$$

Consequently

$$\mu([X, \overline{X}]) = \frac{1}{2}\Omega_\mu(X + \overline{X}, X + \overline{X}) \geq 0,$$

and $\mu(X, \overline{X}]) = 0$ implies that $X = 0$.

Moreover, if $\mu([X, \overline{X}]) = 0$ then $\Omega_\mu(X + \overline{X}, X + \overline{X})$ from which it follows that $X + \overline{X} = 0$. This means that $iX \in \mathfrak{g}$, hence $[\mathfrak{h}_{\overline{0}}, iX] \subseteq \mathfrak{g}$. However, if $H \in \mathfrak{h}_{\overline{0}}$ is chosen such that $\alpha(H) \neq 0$, then

$$[H, iX] = i[H, X] = i^2\alpha(H)X = -\alpha(H)X$$

and this yields a contradiction because clearly $-\alpha(H)X \notin \mathfrak{g}$. □

6.2 Application to Real Simple Lie Superalgebras

Let $\mathcal{G} = (G, \mathfrak{g})$ be a Lie supergroup such that G is connected and $\mathfrak{g}$ is a real simple Lie superalgebra with nontrivial odd part. Assume that $\mathcal{G}$ has nontrivial unitary representations. The goal of this section is use the necessary conditions obtained in Sect. 6.1 to obtain strong conditions on $\mathfrak{g}$.

Since $\mathfrak{g}$ is simple, $\mathcal{G}$ will be $\star$-reduced and Proposition 6.1.2(iv) implies that $\mathfrak{g}_{\overline{0}}$ contains a compactly embedded Cartan subalgebra. In particular, since complex simple Lie algebras do not have compactly embedded Cartan subalgebras, $\mathfrak{g}$ should be a real form of a complex simple Lie superalgebra. However, as Theorem 6.2.1 below shows, for a large class of these real forms there are no nontrivial unitary representations. For simplicity, we exclude the real forms of exceptional cases $\mathbf{G}(3)$, $\mathbf{F}(4)$ and $\mathbf{D}(2|1, \alpha)$.

Theorem 6.2.1. *If $\mathfrak{g}$ is one of the following Lie superalgebras then $\mathcal{G}$ does not have any nontrivial unitary representations.*

(i) $\mathfrak{sl}(m|n, \mathbb{R})$ *where* $m > 2$ *or* $n > 2$.
(ii) $\mathfrak{su}(p, q|r, s)$ *where* $p, q, r, s > 0$.
(iii) $\mathfrak{su}^*(2p, 2q)$ *where* $p, q > 0$ *and* $p + q > 2$.
(iv) $\mathfrak{p}\overline{\mathfrak{q}}(m)$ *where* $m > 1$.
(v) $\mathfrak{usp}(m)$ *where* $m > 1$.
(vi) $\mathfrak{osp}^*(m|p, q)$ *where* $p, q, m > 0$.
(vii) $\mathfrak{osp}(p, q|2n)$ *where* $p, q, n > 0$.
(viii) *Real forms of* $\mathbf{P}(n)$, $n > 1$.
(ix) $\mathfrak{psq}(n, \mathbb{R})$ *where* $n > 2$, $\mathfrak{psq}^*(n)$ *where* $n > 2$, *and* $\mathfrak{psq}(p, q)$, *where* $p, q > 0$.
(x) *Real forms of* $\mathbf{W}(n)$, $\mathbf{S}(n)$, *and* $\widetilde{\mathbf{S}}(n)$.
(xi) $\mathbf{H}(p, q)$ *where* $p + q > 4$.

Proof. Throughout the proof, for every n we denote the $n \times n$ identity matrix by I_n, and set

$$\mathrm{I}_{p,q} = \begin{bmatrix} \mathrm{I}_p & 0 \\ 0 & -\mathrm{I}_q \end{bmatrix} \quad \text{and} \quad \mathrm{J}_n = \begin{bmatrix} 0 & \mathrm{I}_n \\ -\mathrm{I}_n & 0 \end{bmatrix}.$$

(i) Since $[\mathfrak{g}_{\overline{1}}, \mathfrak{g}_{\overline{1}}] \cong \mathfrak{sl}(m, \mathbb{R}) \oplus \mathfrak{sl}(n, \mathbb{R})$ has no compactly embedded Cartan subalgebra, this follows from Proposition 6.1.2(iv).
(ii) In the standard realization of $\mathfrak{sl}(p + q|r + s, \mathbb{C})$ as quadratic matrices of size $p + q + r + s$, $\mathfrak{su}(p, q|r, s)$ can be described as

$$\left\{ \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathfrak{sl}(p + q|r + s, \mathbb{C}) \,\middle|\, \begin{bmatrix} -\mathrm{I}_{p,q} A^* \mathrm{I}_{p,q} & i\mathrm{I}_{p,q} C^* \mathrm{I}_{r,s} \\ i\mathrm{I}_{r,s} B^* \mathrm{I}_{p,q} & -\mathrm{I}_{r,s} D^* \mathrm{I}_{r,s} \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \right\}.$$

Suppose, on the contrary, that $\mathcal{G}$ is $\star$-reduced. Proposition 6.1.2(iii) implies that the diagonal matrices in $\mathfrak{su}(p, q|r, s)$ constitute a Cartan subalgebra of $\mathfrak{su}(p, q|r, s)_{\overline{0}}$ which is compactly embedded in $\mathfrak{su}(p, q|r, s)$. Let μ be chosen as in Proposition 6.1.3. For every $a \leq r$ and $b \leq p$, the matrix

$$X_{a,b} = \begin{bmatrix} 0 & 0 \\ \mathrm{E}_{a,b} & 0 \end{bmatrix}$$

is a root vector. Let τ denote the complex conjugation corresponding to the above realization of $\mathfrak{su}(p,q|r,s)$. One can easily check that

$$\tau(X_{a,b}) = \begin{bmatrix} 0 & i\,\mathrm{E}_{b,a} \\ 0 & 0 \end{bmatrix}.$$

Set $H_{a,b} = [X_{a,b}, \tau(X_{a,b})]$. It is easily checked that

$$H_{a,b} = \begin{bmatrix} i\,\mathrm{E}_{b,b} & 0 \\ 0 & i\,\mathrm{E}_{a,a} \end{bmatrix}.$$

For a and b there are three other possibilities to consider. If $a \leq r$ and $b > p$, or if $a > r$ and $b \leq p$, then

$$H_{a,b} = \begin{bmatrix} -i\,\mathrm{E}_{b,b} & 0 \\ 0 & -i\,\mathrm{E}_{a,a} \end{bmatrix},$$

and if $a > r$ and $b > p$ then

$$H_{a,b} = \begin{bmatrix} i\,\mathrm{E}_{b,b} & 0 \\ 0 & i\,\mathrm{E}_{a,a} \end{bmatrix}.$$

Proposition 6.1.3 implies that $\mu(H_{a,b}) > 0$ for every $1 \leq a \leq p+q$ and every $1 \leq b \leq r+s$. However, from the assumption that p, q, r, and s are all positive, it follows that the zero matrix lies in the convex hull of the $H_{a,b}$'s, which is a contradiction. Therefore $\mathcal{G}$ cannot be $\star$-reduced.

(iii) Note that $\mathfrak{su}^*(2p|2q)_{\overline{0}} \simeq \mathfrak{su}^*(2p) \oplus \mathfrak{su}^*(2q)$. The maximal compact subalgebra of $\mathfrak{su}^*(2n)$ is $\mathfrak{sp}(n)$, which has rank n. The rank of the complexification of $\mathfrak{su}^*(2n)$, which is $\mathfrak{sl}(2n,\mathbb{C})$, is $2n-1$. If $n > 1$, then $2n-1 > n$ implies that $\mathfrak{su}^*(2n)$ does not have a compactly embedded Cartan subalgebra. Now use Proposition 6.1.2(i) and Lemma 5.2.1.

(iv) This Lie superalgebra is a quotient of $\overline{\mathfrak{q}}(m)$ by its center, where $\overline{\mathfrak{q}}(m)$ is defined in the standard realization of $\mathfrak{sl}(m|m,\mathbb{C})$ by

$$\overline{\mathfrak{q}}(m) = \left\{ \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathfrak{sl}(m|m,\mathbb{C}) \;\middle|\; \begin{bmatrix} \overline{D} & \overline{C} \\ \overline{B} & \overline{A} \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \right\}.$$

One can now use Proposition 6.1.2(iv) because $\overline{\mathfrak{q}}(m)_{\overline{0}} \cong \mathfrak{sl}(m,\mathbb{C}) \oplus \mathbb{R}$ contains no compactly embedded Cartan subalgebra.

(v) This Lie superalgebra is a quotient of $\mathfrak{up}(m)$ by its center, where $\mathfrak{up}(m)$ is defined in the standard realization of $\mathfrak{sl}(m|m,\mathbb{C})$ by

$$\mathfrak{up}(m) = \left\{ \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathfrak{sl}(m|m,\mathbb{C}) \;\middle|\; \begin{bmatrix} -D^* & B^* \\ -C^* & -A^* \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \right\}.$$

This implies that $\mathfrak{up}(m)_{\overline{0}} \cong \mathrm{sl}(m,\mathbb{C}) \oplus \mathbb{R}$. Since this Lie algebra has no compactly embedded Cartan subalgebra, the assertion follows from Proposition 6.1.2(iv).

(vi) From Sect. 5.3 it follows that $\mathfrak{osp}^*(m|p,q)_{\overline{0}} \simeq \mathfrak{so}^*(m) \oplus \mathfrak{sp}(p,q)$ has pointed generating invariant cones if and only if $p = 0$ or $q = 0$. One can now use Proposition 6.1.2(i).

(vii) The argument for this case is quite similar to the one given for $\mathfrak{su}(p,q|r,s)$, i.e., the idea is to find root vectors $X_\alpha \in \mathfrak{g}_{\overline{1}}^{\mathbb{C},\alpha}$ such that the convex hull of the $[X_\alpha, \tau(X_\alpha)]$'s contains the origin. The details are left to the reader, but it may be helpful to illustrate how one can find the root vectors. The complex simple Lie superalgebra $\mathfrak{osp}(m|2n,\mathbb{C})$ can be realized inside $\mathfrak{sl}(m|2n,\mathbb{C})$ as

$$\mathfrak{osp}(m|2n,\mathbb{C}) = \left\{ \begin{bmatrix} A & B \\ C & D \end{bmatrix} \;\middle|\; \begin{bmatrix} -A^t & -C^t \mathrm{J}_n \\ -\mathrm{J}_n B^t & \mathrm{J}_n D^t \mathrm{J}_n \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \right\}.$$

If p and q are nonnegative integers satisfying $p + q = m$ then $\mathfrak{osp}(p,q|2n)$ is the set of fixed points of the map

$$\tau : \mathfrak{osp}(m|2n,\mathbb{C}) \to \mathfrak{osp}(m|2n,\mathbb{C})$$

defined by

$$\tau\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right) = \begin{bmatrix} \mathrm{I}_{p,q}\overline{A}\mathrm{I}_{p,q} & \mathrm{I}_{p,q}\overline{B} \\ \overline{C}\mathrm{I}_{p,q} & \overline{D} \end{bmatrix}.$$

Moreover, $\mathfrak{osp}(p,q|2n)_{\overline{0}} \simeq \mathfrak{so}(p,q) \oplus \mathfrak{sp}(2n,\mathbb{R})$ consists of block diagonal matrices, i.e., matrices for which B and C are zero.

Assume that $\mathfrak{osp}(p,q|2n)$ is $\star$-reduced. Then the span of

$$\left\{ \mathrm{E}_{j,p+1-j} - \mathrm{E}_{p+1-j,j} \mid 1 \le j \le \lfloor \frac{p}{2} \rfloor \right\}$$

and

$$\left\{ \mathrm{E}_{p+j,p+q+1-j} - \mathrm{E}_{p+q+1-j,p+j} \mid 1 \le j \le \lfloor \frac{q}{2} \rfloor \right\}$$

is a compactly embedded Cartan subalgebra of $\mathfrak{so}(p,q)$, and the span of

$$\{ \mathrm{E}_{p+q+j,p+q+n+j} - \mathrm{E}_{p+q+n+j,p+q+j} \mid 1 \le j \le n \}$$

is a compactly embedded Cartan subalgebra of $\mathfrak{sp}(2n,\mathbb{R})$.

Fix $1 \leq b \leq n$. For every $a \leq p$ we can obtain two root vectors as follows. If we set

$$B_{a,b} = \mathrm{E}_{a,b} + i\mathrm{E}_{a,b+n} + i\mathrm{E}_{p+1-a,b} - \mathrm{E}_{p+1-a,b+n}$$

and

$$C_{a,b} = -i\mathrm{E}_{b,a} + \mathrm{E}_{b,p+1-a} + \mathrm{E}_{b+n,a} + i\mathrm{E}_{b+n,p+1-a},$$

then the matrix

$$X_{a,b} = \begin{bmatrix} 0 & B_{a,b} \\ C_{a,b} & 0 \end{bmatrix}$$

is a root vector, and $H_{a,b} = [X_{a,b}, \tau(X_{a,b})]$ is given by

$$H_{a,b} = \begin{bmatrix} A_{a,b} & 0 \\ 0 & D_{a,b} \end{bmatrix},$$

where $A_{a,b} = 2\mathrm{E}_{a,p+1-a} - 2\mathrm{E}_{p+1-a,a}$ and $D_{a,b} = -2\mathrm{E}_{b,b+n} + 2\mathrm{E}_{b+n,b}$. Similarly, setting

$$B_{a,b} = \mathrm{E}_{a,b} - i\mathrm{E}_{a,b+n} + i\mathrm{E}_{p+1-a,b} + \mathrm{E}_{p+1-a,b+n}$$

and

$$C_{a,b} = i\mathrm{E}_{b,a} - \mathrm{E}_{b,p+1-a} + \mathrm{E}_{b+n,a} + i\mathrm{E}_{b+n,p+1-a}$$

yields another root vector $X_{a,b}$, and in this case for the corresponding $H_{a,b}$ we have

$$A_{a,b} = -2\mathrm{E}_{a,p+1-a} + 2\mathrm{E}_{p+1-a,a}$$

and

$$D_{a,b} = -2\mathrm{E}_{b,b+n} + 2\mathrm{E}_{b+n,b}.$$

Moreover, when p is odd, setting

$$B_{\lceil \frac{p+1}{2} \rceil,b} = \mathrm{E}_{\lceil \frac{p+1}{2} \rceil,b} + i\mathrm{E}_{\lceil \frac{p+1}{2} \rceil,b+n}$$

and

$$C_{\lceil \frac{p+1}{2} \rceil,b} = -i\mathrm{E}_{b,\lceil \frac{p+1}{2} \rceil} + \mathrm{E}_{b+n,\lceil \frac{p+1}{n} \rceil}$$

yields a root vector $X_{\lceil \frac{p+1}{2} \rceil,b}$, and $H_{\lceil \frac{p+1}{2} \rceil,b}$ is given by

$$A_{\lceil \frac{p+1}{2} \rceil,b} = 0$$

and

$$D_{\lceil \frac{p+1}{2} \rceil,b} = -2\mathrm{E}_{b,b+n} + 2\mathrm{E}_{b+n,b}.$$

The case $p < a \leq p + q$ is similar.

(viii) Follows from Proposition 2.4.2, as the root system of $\mathbf{P}(n)$ is not symmetric.
(ix) For $\mathfrak{psq}(n,\mathbb{R})$ and $\mathfrak{psq}^*(n)$, use Proposition 6.1.2(iv) and the fact that

$$\mathfrak{psq}(n,\mathbb{R})_{\overline{0}} \simeq \mathfrak{sl}(n,\mathbb{R}) \quad \text{and} \quad \mathfrak{psq}^*(n)_{\overline{0}} \simeq \mathfrak{su}^*(n).$$

For $\mathfrak{psq}(p,q)$ and $p,q > 0$, we observe that it is a quotient of the subsuperalgebra $\widetilde{\mathfrak{g}}$ of $\mathfrak{sl}(p+q|p+q,\mathbb{C})$ given by

$$\widetilde{\mathfrak{g}} = \mathfrak{sq}(p,q) = \left\{ \begin{bmatrix} A & B \\ B & A \end{bmatrix} \;\middle|\; \begin{bmatrix} -\mathrm{I}_{p,q}A^*\mathrm{I}_{p,q} & i\mathrm{I}_{p,q}B^*\mathrm{I}_{p,q} \\ i\mathrm{I}_{p,q}B^*\mathrm{I}_{p,q} & -\mathrm{I}_{p,q}A^*\mathrm{I}_{p,q} \end{bmatrix} = \begin{bmatrix} A & B \\ B & A \end{bmatrix} \right\}.$$

Let $\zeta \in \mathbb{C}$ be a squareroot of i. Then the maps

$$\mathfrak{u}(p,q) \to \widetilde{\mathfrak{g}}_{\overline{0}}, \quad A \mapsto \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}$$

and

$$\mathfrak{u}(p,q) \to \widetilde{\mathfrak{g}}_{\overline{1}}, \quad B \mapsto \begin{bmatrix} 0 & \zeta^{-1}B \\ \zeta^{-1}B & 0 \end{bmatrix}$$

are linear isomorphisms. Note that $\mathfrak{k}_{\overline{0}} = \mathfrak{u}(p) \oplus \mathfrak{u}(q)$ is a maximal compactly embedded subalgebra of $\widetilde{\mathfrak{g}}_{\overline{0}}$. Its center is

$$\mathscr{Z}(\mathfrak{k}_{\overline{0}}) = \mathbb{R}i\mathrm{I}_p \oplus \mathbb{R}i\mathrm{I}_q$$

and $\widetilde{\mathfrak{g}}_{\overline{0}}$ is quasihermitian. The projection $p_{\mathfrak{z}}\colon \mathfrak{u}(p,q) \to \mathscr{Z}(\mathfrak{k}_{\overline{0}})$ is simply given by

$$p_{\mathfrak{z}} \begin{bmatrix} a & b \\ b^* & d \end{bmatrix} = \begin{bmatrix} \frac{1}{p}\operatorname{tr}(a)\mathrm{I}_p & 0 \\ 0 & \frac{1}{q}\operatorname{tr}(d)\mathrm{I}_q \end{bmatrix}.$$

Let $C \subseteq \widetilde{\mathfrak{g}}_{\overline{0}}$ be the closed convex cone generated by $[X,X]$, $X \in \widetilde{\mathfrak{g}}_{\overline{1}}$. Since $\widetilde{\mathfrak{g}}_{\overline{0}}$ is quasihermitian, Lemma 5.3.2 implies that $p_{\mathfrak{z}}(C) = C \cap \mathscr{Z}(\mathfrak{k}_{\overline{0}})$.

Next we observe that

$$\begin{bmatrix} 0 & \zeta^{-1}B \\ \zeta^{-1}B & 0 \end{bmatrix}^2 = \begin{bmatrix} -iB^2 & 0 \\ 0 & -iB^2 \end{bmatrix} \quad \text{for every} \quad B \in \mathfrak{u}(p,q).$$

For $B = \begin{bmatrix} a & b \\ b^* & d \end{bmatrix}$ we have

$$B^2 = \begin{bmatrix} a & b \\ b^* & d \end{bmatrix}^2 = \begin{bmatrix} a^2 + bb^* & ab + bd \\ b^*a + ab^* & b^*b + d^2 \end{bmatrix},$$

so that

$$p_{\mathfrak{z}}(-iB^2) = -i \begin{bmatrix} \frac{1}{p}\operatorname{tr}(a^2 + bb^*) & 0 \\ 0 & \frac{1}{q}\operatorname{tr}(b^*b + d^2) \end{bmatrix}.$$

Applying this to positive multiples of matrices where only the a, b or d-component is non-zero, we see that the closed convex cone $p_{\mathfrak{z}}(C)$ contains the elements

$$Z_1 = \begin{bmatrix} i\mathrm{I}_p & 0 \\ 0 & 0 \end{bmatrix}, \quad Z_2 = \begin{bmatrix} 0 & 0 \\ 0 & i\mathrm{I}_q \end{bmatrix} \quad \text{and} \quad Z_3 = -\begin{bmatrix} \frac{1}{p} i\mathrm{I}_p & 0 \\ 0 & \frac{1}{q} i\mathrm{I}_q \end{bmatrix}.$$

This implies that $p_{\mathfrak{z}}(C) = \mathscr{Z}(\mathfrak{k}_{\overline{0}}) \subseteq C$.

We conclude that $\mathscr{Z}(\widetilde{\mathfrak{g}_{\overline{0}}}) = i\mathbb{R}\mathrm{I}_{p+q} \subseteq C$, so that $C = (\widetilde{\mathfrak{g}_{\overline{0}}}) \oplus C_1$, where $C_1 = C \cap [\widetilde{\mathfrak{g}_{\overline{0}}}, \widetilde{\mathfrak{g}_{\overline{0}}}]$ is a non-pointed non-zero invariant closed convex cone in a simple Lie algebra isomorphic to $\mathfrak{su}(p,q)$. This leads to $C_1 = [\widetilde{\mathfrak{g}_{\overline{0}}}, \widetilde{\mathfrak{g}_{\overline{0}}}]$. We conclude that $C = \widetilde{\mathfrak{g}_{\overline{0}}}$ and the same holds also for the quotient $\mathfrak{psq}(p,q)$.

(x) Follows from Proposition 2.4.2, as the root systems of these complex simple Lie superalgebras are not symmetric (see [23, Appendix A]).

(xi) Suppose, on the contrary, that $\mathcal{G}$ is $\star$-reduced. Proposition 6.1.2(i) and Lemma 5.1.1 imply that every abelian ideal of $\mathfrak{g} = \mathbf{H}(p,q)$ lies in its center. The standard $\mathbb{Z}$-grading of $\mathbf{H}(p+q)$ (see [13, Proposition 3.3.6]) yields a grading of $\mathbf{H}(p+q)_{\overline{0}}$, i.e.,

$$\mathbf{H}(p+q)_{\overline{0}} = \mathbf{H}(p+q)_{\overline{0}}^{(0)} \oplus \mathbf{H}(p+q)_{\overline{0}}^{(2)} \oplus \cdots \oplus \mathbf{H}(p+q)_{\overline{0}}^{(k)},$$

where $k = p+q-3$ if $p+q$ is odd and $k = p+q-4$ otherwise. This grading is consistent with the real form $\mathbf{H}(p,q)_{\overline{0}}$. Since $\mathbf{H}(p,q)_{\overline{0}}^{(k)}$ is an abelian ideal of $\mathbf{H}(p,q)_{\overline{0}}$, it should lie in the center of $\mathbf{H}(p,q)_{\overline{0}}$. It follows that $\mathbf{H}(p+q)_{\overline{0}}^{(k)}$ lies in the center of $\mathbf{H}(p+q)_{\overline{0}}$. However, this is impossible because it is known (see [13, Proposition 3.3.6]) that $\mathbf{H}(p+q)_{\overline{0}}^{(0)} \simeq \mathfrak{so}(p+q,\mathbb{C})$ and the representation of $\mathbf{H}(p+q)_{\overline{0}}^{(0)}$ on $\mathbf{H}(p+q)_{\overline{0}}^{(k)}$ is isomorphic to $\wedge^{k+2}\mathbb{C}^{p+q}$, from which it follows that

$$[\mathbf{H}(p+q)_{\overline{0}}^{(0)}, \mathbf{H}(p+q)_{\overline{0}}^{(k)}] \neq \{0\}. \qquad \square$$

Remark 6.2.1. In classical cases, Theorem 6.2.1 can be viewed as a converse to the classification of highest weight modules obtained in [12]. From Theorem 6.2.1 it also follows that for the nonclassical cases, unitary representations are rare.

Remark 6.2.2. The results of [12] imply that real forms of $\mathbf{A}(m|m)$ do not have any unitarizable highest weight modules. However, $\mathbf{A}(m|m)$ is a quotient of $\mathfrak{sl}(m|m,\mathbb{C})$, and there exist unitarizable modules of $\mathfrak{su}(p,m-p|m,0)$ which do not factor to the simple quotient. For instance, the standard representation is a finite dimensional unitarizable module of $\mathfrak{su}(m,0|m,0)$ with this property.

6.3 Application to Real Semisimple Lie Superalgebras

Although real semisimple Lie superalgebras may have a complicated structure, those which have faithful unitary representations are relatively easy to describe.

Given a finite dimensional real Lie superalgebra $\mathfrak{g}$, let us call it $\star$-reduced if there exists a $\star$-reduced Lie supergroup $\mathcal{G} = (G, \mathfrak{g})$.

Theorem 6.3.1. *Let $\mathcal{G} = (G, \mathfrak{g})$ be a $\star$-reduced Lie supergroup. If $\mathfrak{g}$ is a real semisimple Lie superalgebra then there exist $\star$-reduced real simple Lie superalgebras $\mathfrak{s}_1, \ldots, \mathfrak{s}_k$ such that*

$$\mathfrak{s}_1 \oplus \cdots \oplus \mathfrak{s}_k \subseteq \mathfrak{g} \subseteq \mathrm{Der}_{\mathbb{R}}(\mathfrak{s}_1) \oplus \cdots \oplus \mathrm{Der}_{\mathbb{R}}(\mathfrak{s}_k).$$

Proof. We use the description of $\mathfrak{g}$ given in Theorem 2.5.1. First note that for every i we have $n_i = 0$. To see this, suppose on the contrary that $n_i > 0$ for some i, and let $\xi_1, \ldots, \xi_{n_i}$ be the standard generators of $\mathbf{\Lambda}_{\mathbb{K}_i}(n_i)$. For every nonzero $X \in (\mathfrak{s}_i)_{\overline{0}}$ have $X \otimes \xi_1 \in (\mathfrak{s}_i)_{\overline{1}}$ and

$$[X \otimes \xi_1, X \otimes \xi_1] = 0.$$

Proposition 6.1.1 implies that $X \otimes \xi_1$ lies in the kernel of every unitary representation of $\mathcal{G}$, which is a contradiction.

From the fact that all of the n_i, $1 \leq i \leq k$, are zero it follows that

$$\mathfrak{s}_1 \oplus \cdots \oplus \mathfrak{s}_k \subseteq \mathfrak{g} \subseteq \mathrm{Der}_{\mathbb{K}_1}(\mathfrak{s}_1) \oplus \cdots \oplus \mathrm{Der}_{\mathbb{K}_k}(\mathfrak{s}_k)$$

and from $\mathfrak{s}_i \subseteq \mathfrak{g}$ it follows that every $\mathfrak{s}_i$ is $\star$-reduced. □

6.4 Application to Nilpotent Lie Supergroups

Another interesting by-product of the results of Sect. 6.1 is the following statement about unitary representations of nilpotent Lie supergroups. (A Lie supergroup $\mathcal{G} = (G, \mathfrak{g})$ is called *nilpotent* if $\mathfrak{g}$ is nilpotent.)

Theorem 6.4.1. *If $(\pi, \rho^\pi, \mathscr{H})$ is a unitary representation of a nilpotent Lie supergroup $(G, \mathfrak{g})$ then $\rho^\pi([\mathfrak{g}_{\overline{1}}, [\mathfrak{g}_{\overline{1}}, \mathfrak{g}_{\overline{1}}]]) = \{0\}$.*

Proof. By passing to a quotient one can see that it suffices to show that if $(G, \mathfrak{g})$ is nilpotent and $\star$-reduced then $[\mathfrak{g}_{\overline{1}}, [\mathfrak{g}_{\overline{1}}, \mathfrak{g}_{\overline{1}}]] = \{0\}$. Without loss of generality one can assume that $\mathfrak{g}_{\overline{0}} = [\mathfrak{g}_{\overline{1}}, \mathfrak{g}_{\overline{1}}]$. By Proposition 6.1.2(iv) there exists a Cartan subalgebra $\mathfrak{h}_{\overline{0}}$ of $\mathfrak{g}_{\overline{0}}$ which is compactly embedded in $\mathfrak{g}$. As $\mathfrak{g}_{\overline{0}}$ is nilpotent, we have $\mathfrak{g}_{\overline{0}} = \mathfrak{h}_{\overline{0}}$. Proposition 2.4.1 implies that $\mathfrak{g}_{\overline{0}}$ acts semisimply on $\mathfrak{g}$. Nevertheless, since $\mathfrak{g}$ is nilpotent, for every $X \in \mathfrak{g}_{\overline{0}}$ the linear map

$$\mathrm{ad}(X) : \mathfrak{g} \to \mathfrak{g}$$

is nilpotent. It follows that $[\mathfrak{g}_{\overline{0}}, \mathfrak{g}] = \{0\}$. In particular, $[\mathfrak{g}_{\overline{1}}, [\mathfrak{g}_{\overline{1}}, \mathfrak{g}_{\overline{1}}]] = \{0\}$. □

7 Highest Weight Theory

In this section, we analyse the structure of the irreducible unitary representations of Lie supergroups whose Lie algebra $\mathfrak{g}$ is generated by its odd part. The main result is Theorem 7.3.2 which asserts that this structure is quite similar to the structure of highest weight modules of Lie algebras. The crucial difference is that the highest weight space is an irreducible representation of a Clifford Lie superalgebra and therefore it can have dimension higher than one.

7.1 *A Fréchet Space of Analytic Vectors*

Let G be a connected Lie group with Lie algebra $\mathfrak{g}$. Let $\mathfrak{t} \subseteq \mathfrak{g}$ be a compactly embedded Cartan subalgebra, and $T = \exp(\mathfrak{t})$ be the corresponding subgroup of G. Then $\mathfrak{g}^{\mathbb{C}}$ carries a norm $\|\cdot\|$ which is invariant under $\mathrm{Ad}(T)$. In particular, for each $r > 0$, the open ball $B_r = \{X \in \mathfrak{g}^{\mathbb{C}} : \|X\| < r\}$ is an open subset which is invariant under $\mathrm{Ad}(T)$.

Let $(\pi, \mathscr{H})$ be a unitary representation of G. A smooth vector $v \in \mathscr{H}^\infty$ is *analytic* if and only if there exists an $r > 0$ such that the power series

$$f_v : B_r \to \mathscr{H}, \quad f_v(X) = \sum_{n=0}^{\infty} \frac{1}{n!} \mathsf{d}\pi(X)^n v \tag{8}$$

defines a holomorphic function on B_r. In fact, if the series (8) converges on some B_r, then it defines a holomorphic function, and the theory of analytic vectors for unitary one-parameter groups implies that $f_v(X) = \pi(\exp(X))v$ for every $X \in B_r \cap \mathfrak{g}$. Therefore the orbit map of v is analytic.

If the series (8) converges on B_r, it converges uniformly on B_s for every $s < r$ ([2, Proposition 4.1]). This means that the seminorms

$$q_n(v) = \sup\{\|\mathsf{d}\pi(X)^n v\| \;\Big|\; \|X\| \le 1 \,,\; X \in \mathfrak{g}^{\mathbb{C}}\}$$

satisfy

$$\sum_{n=0}^{\infty} \frac{s^n}{n!} q_n(v) < \infty \text{ for every } s < r.$$

Note that the seminorms q_n define the topology of $\mathscr{H}^\infty$ (cf. [21, Proposition 4.6]).

For every $r > 0$, let $\mathscr{H}^{\omega,r}$ denote the set of all analytic vectors for which (8) converges on B_r, so that

$$\mathscr{H}^\omega = \bigcup_{r>0} \mathscr{H}^{\omega,r}.$$

If $v \in \mathscr{H}^{\omega,r}$ and $s < r$, set

$$p_s(v) = \sum_{n=0}^{\infty} \frac{s^n}{n!} q_n(v)$$

and note that this is a norm on $\mathscr{H}^{\omega,r}$.

Lemma 7.1.1. The norms p_s, $s < r$, turn $\mathscr{H}^{\omega,r}$ into a Fréchet space.

Proof. Since $p_s < p_t$ for $s < t < r$, the topology on $\mathscr{H}^{\omega,r}$ is defined by the sequence of seminorms $(p_{s_n})_{n\in\mathbb{N}}$ for any sequence (s_n) with $s_n \to r$. Therefore $\mathscr{H}^{\omega,r}$ is metrizable and we have to show that it is complete.

If (v_n) is a Cauchy sequence in $\mathscr{H}^{\omega,r}$ then for every $s < r$ the sequence $f_{v_n} : B_r \to \mathscr{H}$ of holomorphic functions converges uniformly on each B_s to some function $f : B_r \to \mathscr{H}$, which implies that f is holomorphic.

Fix $X \in \mathfrak{g}^{\mathbb{C}}$ and let $v = f(0)$. For each $k \in \mathbb{N}$, $\mathsf{d}\pi(X)^k v_n$ is a Cauchy sequence in $\mathscr{H}$. This implies that $v \in \mathscr{H}^{\infty}$. Observe that for every $k \in \mathbb{N}$, the operator $\mathsf{d}\pi(X)^k$ has a densely defined adjoint and therefore it is closable. Consequently, $\mathsf{d}\pi(X)^k v_n \to \mathsf{d}\pi(X)^k v$ for every $k \in \mathbb{N}$. Therefore $f = f_v$ on B_r ([2, Proposition 3.1]), and this means that $v \in \mathscr{H}^{\omega,r}$ with $v_n \to v$ in the topology of $\mathscr{H}^{\omega,r}$.

Lemma 7.1.2. If $K \subseteq G$ is a subgroup leaving the norm $\|\cdot\|$ on $\mathfrak{g}^{\mathbb{C}}$ invariant, then the norms p_s, $s < r$, on $\mathscr{H}^{\omega,r}$ are K-invariant and the action of K on $\mathscr{H}^{\omega,r}$ is continuous. In particular, the action of K on $\mathscr{H}^{\omega,r}$ integrates to a representation of the convolution algebra $L^1(K)$ on $\mathscr{H}^{\omega,r}$.

Proof. Since K preserves the defining family of norms, continuity of the K-action on $\mathscr{H}^{\omega,r}$ follows if we show that all orbit maps are continuous at $\mathbf{1}_K$, where $\mathbf{1}_K$ denotes the identity element of K. Let $v \in \mathscr{H}^{\omega,r}$ and suppose that $k_m \to \mathbf{1}_K$ in K. Then

$$p_s(\pi(k_m)v - v) = \sum_{n=0}^{\infty} \frac{s^n}{n!} q_n(\pi(k_m)v - v)$$

and

$$q_n(\pi(k_m)v - v) \leq q_n(\pi(k_m)v) + q_n(v) = 2q_n(v).$$

Since K acts continuously on $\mathscr{H}^{\infty}$, $q_n(\pi(k_m)v - v) \to 0$ for every $n \in \mathbb{N}$, and since $p_s(v) < \infty$, the Dominated Convergence Theorem implies that $p_s(\pi(k_n)v - v) \to 0$.

The fact that $\mathscr{H}^{\omega,r}$ is complete implies that it can be considered as a subspace of the product space $\prod_{s<r} \mathscr{V}_s$, where $\mathscr{V}_s$ denotes the completion of $\mathscr{H}^{\omega,r}$ with respect to the norm p_s. We thus obtain continuous isometric representations of K on the Banach spaces $\mathscr{V}_s$, which leads by integration to representations of $L^1(K)$ on these spaces (see [10, (40.26)]). Finally, since $\mathscr{H}^{\omega,r} \subseteq \prod_{s<r} \mathscr{V}_s$ is closed by completeness (Lemma 7.1.1) and K-invariant, it is also invariant under $L^1(K)$. □

From now on assume that r is small enough such that the exponential function of the simply connected Lie group $\widetilde{G}^{\mathbb{C}}$ with Lie algebra $\mathfrak{g}^{\mathbb{C}}$ maps B_r diffeomorphically onto an open subset of $\widetilde{G}^{\mathbb{C}}$. For every $X \in \mathfrak{g}^{\mathbb{C}}$ the corresponding left and right invariant vector fields define differential operators on $\exp(B_r)$ by

$$(L_X f)(g) = \frac{d}{dt}\bigg|_{t=0} f(g\exp(tX)) \quad \text{and} \quad (R_X f)(g) = \frac{d}{dt}\bigg|_{t=0} f(\exp(tX)g).$$

Define similar operators L_X^* and R_X^* on B_r by

$$L_X^*(f \circ \exp|_{B_r}) = (L_X f) \circ \exp|_{B_r} \quad \text{and} \quad R_X^*(f \circ \exp|_{B_r}) = (R_X f) \circ \exp|_{B_r}.$$

One can see that

$$L_X^* f_v = f_{\mathrm{d}\pi(X)v} \quad \text{and} \quad R_X^* f_v = \mathrm{d}\pi(X) \circ f_v. \tag{9}$$

If $\mathcal{H}ol(B_r, \mathscr{H})$ denotes the Fréchet space of holomorphic $\mathscr{H}$-valued functions on B_r, then the subspace $\mathcal{H}ol(B_r, \mathscr{H})^{\mathfrak{g}}$ defined by

$$\mathcal{H}ol(B_r, \mathscr{H})^{\mathfrak{g}} = \{f \in \mathcal{H}ol(B_r, \mathscr{H}) \mid R_X^* f = \mathrm{d}\pi(X) \circ f \text{ for every } X \in \mathfrak{g}\},$$

is a closed subspace, hence a Fréchet space. Therefore the map

$$\mathrm{ev}_0 : \mathcal{H}ol(B_r, \mathscr{H})^{\mathfrak{g}} \to \mathscr{H}, \quad f \mapsto f(0) \tag{10}$$

is a continuous linear isomorphism onto $\mathscr{H}^{\omega,r}$, hence a topological isomorphism by the Open Mapping Theorem (see [26, Theorem 2.11]).

This implies in particular that

Lemma 7.1.3. The subspace $\mathscr{H}^{\omega,r} \subseteq \mathscr{H}$ is invariant under $\mathscr{U}(\mathfrak{g}^{\mathbb{C}})$.

7.2 *Spectral Theory for Analytic Vectors*

We have already seen in Lemma 7.1.2 that if $(\pi, \mathscr{H})$ is a unitary representation of G then the subspaces $\mathscr{H}^{\omega,r}$ are invariant under the action of the convolution algebras of certain subgroups $K \subseteq G$. As a consequence, we shall now derive that elements of spectral subspaces of certain unitary one-parameter groups can be approximated by analytic vectors.

We begin by a lemma about the relation between one-parameter groups and spectral measures. Let $\mathfrak{B}(\mathbb{R})$ denote the space of Borel measurable functions on $\mathbb{R}$ and $\mathscr{S}(\mathbb{R})$ denote the Schwartz space of $\mathbb{R}$.

Lemma 7.2.1. Let $\gamma : \mathbb{R} \to \mathrm{U}(\mathscr{H})$ be a unitary representation of the additive group of $\mathbb{R}$ and $A = A^* = -i\gamma'(0)$ be its self-adjoint generator, so that $\gamma(t) = e^{itA}$ in terms of measurable functional calculus. Then the following assertions hold.

(i) For each $f \in L^1(\mathbb{R}, \mathbb{C})$, we have $\gamma(f) = \widehat{f}(A)$, where

$$\widehat{f}(x) = \int_{\mathbb{R}} e^{ixy} f(y)\, dy$$

is the Fourier transform of f.

(ii) Let $P : \mathfrak{B}(\mathbb{R}) \to \mathcal{L}(\mathscr{H})$ be the unique spectral measure with $A = P(\mathrm{id}_{\mathbb{R}})$. Then for every closed subset $E \subseteq \mathbb{R}$ the condition $v \in P(E)\mathscr{H}$ is equivalent to $\gamma(f)v = 0$ for every $f \in \mathscr{S}(\mathbb{R})$ with $\widehat{f}\big|_E = 0$.

Proof. Since the unitary representation $(\gamma, \mathscr{H})$ is a direct sum of cyclic representations, it suffices to prove the assertions for cyclic representations. Every cyclic representation of $\mathbb{R}$ is equivalent to the representation on some space $\mathscr{H} = L^2(\mathbb{R}, \mu)$, where μ is a Borel probability measure on $\mathbb{R}$ and $(\gamma(t)\xi)(x) = e^{itx}\xi(x)$ (see [20, Theorem VI.1.11]).

(i) This means that $(A\xi)(x) = x\xi(x)$, so that $\widehat{f}(A)\xi(x) = \widehat{f}(x)\xi(x)$. For every $f \in L^1(\mathbb{R}, \mathbb{C})$ the equalities

$$(\gamma(f)\xi)(x) = \int_{\mathbb{R}} f(t)e^{itx}\xi(x)\, dt = \widehat{f}(x)\xi(x)$$

hold in the space $\mathscr{H} = L^2(\mathbb{R}, \mu)$.

(ii) In terms of functional calculus, we have $P(E) = \chi_E(A)$, where χ_E is the characteristic function of E. If $\widehat{f}\big|_E = 0$, then (i) and the fact that $\widehat{f}\chi_E = 0$ imply that

$$0 = (\widehat{f} \cdot \chi_E)(A) = \widehat{f}(A)\chi_E(A) = \gamma(f)P(E).$$

Conversely, suppose that $v \in \mathscr{H}$ satisfies $\gamma(f)v = 0$ for every $f \in \mathscr{S}(\mathbb{R})$ with $\widehat{f}\big|_E = 0$. If $v \notin P(E)\mathscr{H}$, then $P(E^c)v \neq 0$, and since E^c is open and a countable union of compact subsets, there exists a compact subset $B \subseteq E^c$ with $P(B)v \neq 0$. Let $\psi \in C_c^\infty(\mathbb{R})$ be such that $\psi\big|_B = 1$ and $\mathrm{supp}(\psi) \subseteq E^c$. Then

$$0 \neq P(B)v = \chi_B(A)v = (\chi_B \cdot \psi)(A)v = \chi_B(A)\psi(A)v$$

implies that $\psi(A)v \neq 0$. Since the Fourier transform defines a bijection $\mathscr{S}(\mathbb{R}) \to \mathscr{S}(\mathbb{R})$ ([26]), there exists an $f \in \mathscr{S}(\mathbb{R})$ with $\widehat{f} = \psi$. Then $\gamma(f)v = \widehat{f}(A)v = \psi(A)v \neq 0$, contradicting our assumption. This implies that $v \in P(E)\mathscr{H}$.

Proposition 7.2.1. *Let $(\pi, \mathscr{H})$ be a unitary representation of the Lie group G and $X \in \mathfrak{g}$ such that the group $e^{\mathbb{R}\mathrm{ad}(X)}$ preserves a norm $\|\cdot\|$ on $\mathfrak{g}^{\mathbb{C}}$. If $P : \mathfrak{B}(\mathbb{R}) \to \mathcal{L}(\mathscr{H})$ is the spectral measure of the unitary one-parameter group $\pi_X(t) = \pi(\exp(tX))$ then for every open subset $E \subseteq \mathbb{R}$ the subspace $(P(E)\mathscr{H}) \cap \mathscr{H}^{\omega,r}$ is dense in $P(E)\mathscr{H}^{\omega,r}$.*

Proof. On $\mathscr{H}^{\omega,r}$ we consider the Fréchet topology defined by the seminorms $(p_s)_{s<r}$ in Lemma 7.1.1. Applying Lemma 7.1.2 to $K = \exp(\mathbb{R}X)$ implies that

all of these seminorms are invariant under $\pi_X(\mathbb{R})$ and π_X defines a continuous representation of $\mathbb{R}$ on $\mathscr{H}^{\omega,r}$ which integrates to a representation

$$\widetilde{\pi}_X : (L^1(\mathbb{R},\mathbb{C}), *) \to \mathrm{End}_{\mathbb{C}}(\mathscr{H}^{\omega,r})$$

of the convolution algebra that is given by

$$\widetilde{\pi}_X(f) = \int_{\mathbb{R}} f(t)\pi_X(t)\,dt.$$

This essentially means that the operators $\widetilde{\pi}_X(f)$ of the integrated representation $L^1(\mathbb{R}) \to \mathcal{L}(\mathscr{H})$ preserve the subspace $\mathscr{H}^{\omega,r}$.

Next we write the open set E as the union of the compact subsets

$$E_n := \left\{ t \in E \,\middle|\, |t| \leq n, \mathrm{dist}(t, E^c) \geq \frac{1}{n} \right\}$$

and observe that $\bigcup_n P(E_n)\mathscr{H}$ is dense in $P(E)\mathscr{H}$. For every n, there exists a compactly supported function $h_n \in C_c^\infty(\mathbb{R},\mathbb{R})$ such that $\mathrm{supp}(h_n) \subseteq E$, $0 \leq h_n \leq 1$, and $h_n\big|_{E_n} = 1$. Let $f_n \in \mathscr{S}(\mathbb{R})$ with $\widehat{f}_n = h_n$. Then

$$\widetilde{\pi}_X(f_n) = \widehat{f}_n(-i\,\mathsf{d}\pi(X)) = h_n(-i\,\mathsf{d}\pi(X))$$

and consequently

$$P(E_n)\mathscr{H} \subseteq \widetilde{\pi}_X(f_n)\mathscr{H} \subseteq P(E)\mathscr{H}.$$

If $w = P(E)v$ for some $v \in \mathscr{H}^{\omega,r}$ then

$$\widetilde{\pi}_X(f_n)w = \widetilde{\pi}_X(f_n)P(E)v = \widetilde{\pi}_X(f_n)v \in \mathscr{H}^{\omega,r}$$

and

$$\|\widetilde{\pi}_X(f_n)w - w\|^2 = \|h_n(-i\,\mathsf{d}\pi(X))w - w\|^2 \leq \|P(E\setminus E_n)w\|^2 \to 0$$

from which it follows that $\widetilde{\pi}_X(f_n)w \to w$. $\square$

Proposition 7.2.2. *If $Y \in \mathfrak{g}^{\mathbb{C}}$ satisfies $[X,Y] = i\mu Y$ then for every open subset $E \subseteq \mathbb{R}$ the spectral measure of π_X satisfies*

$$\mathsf{d}\pi(Y)\big(P(E)\mathscr{H} \cap \mathscr{H}^\infty\big) \subseteq P(E+\mu)\mathscr{H}. \tag{11}$$

Proof. To verify this relation, we first observe that

$$\pi_X(t)\mathsf{d}\pi(Y)v = \mathsf{d}\pi(e^{t\,\mathrm{ad}X}(Y))\pi_X(t)v = e^{it\mu}\mathsf{d}\pi(Y)\pi_X(t)v$$

for every $v \in \mathscr{H}^\infty$. For $f \in \mathscr{S}(\mathbb{R})$, the continuity of the map

$$\mathscr{S}(\mathbb{R}) \to \mathscr{H}^\infty, \quad f \mapsto \widetilde{\pi}_X(f)v$$

leads to

$$\widetilde{\pi}_X(f)\mathsf{d}\pi(Y)v = \mathsf{d}\pi(Y)\int_{\mathbb{R}} f(t)e^{it\mu}\pi_X(t)v = \mathsf{d}\pi(Y)\widetilde{\pi}_X(f\cdot e_\mu)v,$$

where $e_\mu(t) = e^{it\mu}$. If $v \in P(E)\mathscr{H}$ and $\widehat{f}$ vanishes on $E+\mu$ then the function $(e_\mu f)\widehat{} = \widehat{f}(\mu+\cdot)$ vanishes on E, and Lemma 7.2.1(ii) implies that $\widetilde{\pi}_X(f\cdot e_\mu)v = 0$. By applying Lemma 7.2.1(ii) one more time, we obtain that $\mathsf{d}\pi(Y)v \in P(E+\mu)\mathscr{H}$. □

7.3 Application to Irreducible Unitary Representations of Lie Supergroups

Let $(\pi, \rho^\pi, \mathscr{H})$ be an irreducible unitary representation of the Lie supergroup $\mathcal{G} = (G, \mathfrak{g})$. Before we turn to the fine structure of such a representation, we verify that Lemma 7.1.3 generalizes to the super context.

Lemma 7.3.1. The subspace $\mathscr{H}^{\omega,r} \subseteq \mathscr{H}$ is invariant under $\mathscr{U}(\mathfrak{g}^{\mathbb{C}})$.

Proof. In view of Lemma 7.1.3, it only remains to show that, for every $Y \in \mathfrak{g}_{\overline{1}}$ and $v \in \mathscr{H}^{\omega,r}$, we have $\rho^\pi(Y)v \in \mathscr{H}^{\omega,r}$. For every $X \in \mathfrak{g}_{\overline{0}} \cap B_r$, we have the relation

$$\pi(\exp X)\rho^\pi(Y)v = \rho^\pi(e^{\operatorname{ad} X}Y)\pi(\exp X)v = \rho^\pi(e^{\operatorname{ad} X}Y)f_v(X). \tag{12}$$

The complex bilinear map

$$\mathfrak{g}_{\overline{1}}^{\mathbb{C}} \times \mathscr{H}^\infty \to \mathscr{H}^\infty, \quad (Z,v) \mapsto \rho^\pi(Z)v$$

is continuous by Lemma 4.2.4 and therefore holomorphic. Moreover, the map

$$\mathfrak{g}_{\overline{0}}^{\mathbb{C}} \to \mathfrak{g}_{\overline{1}}^{\mathbb{C}}, \quad X \mapsto e^{\operatorname{ad} X}Y$$

is holomorphic. Since compositions of holomorphic maps are holomorphic, it therefore suffices to show that $f_v(B_r) \subseteq \mathscr{H}^\infty$ and that the map $f_v\colon B_r \to \mathscr{H}^\infty$ is holomorphic. In fact, this implies that the map

$$\mathfrak{g}_{\overline{0}} \cap B_r \to \mathscr{H}, \quad X \mapsto \pi(\exp X)\rho^\pi(Y)v$$

extends holomorphically to B_r, i.e., $\rho^\pi(Y)v \in \mathscr{H}^{\omega,r}$.

We recall the topological isomorphism

$$\mathrm{ev}_0\colon \mathcal{H}ol(B_r, \mathscr{H})^{\mathfrak{g}} \to \mathscr{H}^{\omega,r}, \quad f \mapsto f(0).$$

By definition of $\mathcal{H}ol(B_r, \mathscr{H})^{\mathfrak{g}}$, we have for each $X \in \mathfrak{g}_{\overline{0}}$ the relation

$$\mathsf{d}\pi(X) \circ f_v = R_X^* f_v,$$

showing in particular that $\mathsf{d}\pi(X) \circ f_v\colon B_r \to \mathscr{H}$ is a holomorphic function. From the definition of the topology on $\mathscr{H}^\infty$, it therefore follows that f_v is holomorphic as a map $B_r \to \mathscr{H}^\infty$. □

The following theorem clarifies the key features of the $\mathfrak{g}$-representation on $\mathscr{H}^\infty$.

Theorem 7.3.2. *Let $(\pi, \rho^\pi, \mathscr{H})$ be an irreducible unitary representation of the Lie supergroup $\mathcal{G} = (G, \mathfrak{g})$ which is $\star$-reduced and satisfies*

$$\mathfrak{g}_{\overline{0}} = [\mathfrak{g}_{\overline{1}}, \mathfrak{g}_{\overline{1}}].$$

Pick a regular element $X_0 \in Int(Cone(\mathcal{G}))$ and let $\mathfrak{t} = \mathfrak{t}_{\overline{0}} \oplus \mathfrak{t}_{\overline{1}}$ be the corresponding Cartan subsuperalgebra of $\mathfrak{g}$ (see Lemma 5.2.1 and Proposition 2.3.1). Suppose that no root vanishes on X_0. Then the following assertions hold.

(i) *$\mathfrak{t}_{\overline{0}}$ is compactly embedded and $\Delta^+ = \{\, \alpha \in \Delta \mid \alpha(X_0) > 0 \,\}$ satisfies $\Delta \backslash \{0\} = \Delta^+ \dot{\cup} - \Delta^+$.*
(ii) *The space $\mathscr{H}^{\mathfrak{t}}$ of $\mathfrak{t}$-finite elements in $\mathscr{H}^\infty$ is an irreducible $\mathfrak{g}$-module which is a $\mathfrak{t}_{\overline{0}}$-weight module and dense in $\mathscr{H}$.*
(iii) *The maximal eigenspace $\mathscr{V}$ of $i\rho^\pi(X_0)$ is an irreducible finite dimensional $\mathfrak{t}$-module on which $\mathfrak{t}_{\overline{0}}$ acts by some weight $\lambda \in \mathfrak{t}_{\overline{0}}^*$. It generates the $\mathfrak{g}$-module $\mathscr{H}^{\mathfrak{t}}$ and all other $\mathfrak{t}_{\overline{0}}$-weights in this space are of the form*

$$\lambda - m_1\alpha_1 - \cdots - m_k\alpha_k, \quad \alpha_j \in \Delta^+, \;\; k \in \mathbb{N}, \, m_1, \ldots, m_k \in \mathbb{N} \cup \{0\}.$$

(iv) *Two representations $(\pi, \rho^\pi, \mathscr{H})$ and $(\pi', \rho^{\pi'}, \mathscr{H}')$ of $\mathcal{G}$ are isomorphic if and only if the corresponding $\mathfrak{t}$-representations on $\mathscr{V}$ and $\mathscr{V}'$ are isomorphic.*

Proof. (i) Proposition 6.1.2 implies that $Cone(\mathcal{G})$ is a pointed generating invariant cone and $\mathfrak{g}_{\overline{0}}$ has a Cartan subalgebra $\mathfrak{t}_{\overline{0}}$ which is compactly embedded in $\mathfrak{g}$. Then the corresponding Cartan supersubalgebra is given by its centralizer $\mathfrak{t} = \mathscr{Z}_{\mathfrak{g}}(\mathfrak{t}_{\overline{0}})$. Pick a regular element $X_0 \in \mathfrak{t}_{\overline{0}} \cap Int(Cone(\mathcal{G}))$, so that Δ^+ satisfies $\Delta \backslash \{0\} = \Delta^+ \dot{\cup} - \Delta^+$.

(ii) Recall from (7) that $i\rho^\pi(X_0) \leq 0$. We want to prove the existence of an eigenvector of maximal eigenvalue for $i\rho^\pi(X_0)$. Let

$$\delta = \min\{\alpha(X_0) | \alpha \in \Delta^+\}$$

and note that $\delta > 0$. Let $P([a,b])$, $a \leq b \in \mathbb{R}$, denote the spectral projections of the self-adjoint operator $i\overline{\rho^\pi(X_0)}$ and put

$$\lambda = \sup(\mathrm{Spec}(i\overline{\rho^\pi(X_0)})) \leq 0.$$

Since $(\pi, \rho^\pi, \mathscr{H})$ is irreducible and the space $\mathscr{H}^\omega$ of analytic vectors is dense, there exists an $r > 0$ with $\mathscr{H}^{\omega,r} \neq \{0\}$. Then the invariance of $\mathscr{H}^{\omega,r}$ under $\mathscr{U}(\mathfrak{g}^\mathbb{C})$ (Lemma 7.3.1) implies that $\mathscr{H}^{\omega,r}$ is dense in $\mathscr{H}$. Hence Proposition 7.2.1 implies that, for every $\varepsilon > 0$, the intersection

$$P(]\mu - \varepsilon, \mu])\mathscr{H} \cap \mathscr{H}^{\omega,r}$$

is dense in $P(]\mu - \varepsilon, \mu])\mathscr{H}$. In particular, it contains a non-zero vector v_0. With Proposition 7.2.2 for $\varepsilon < \delta$ and $\alpha \in \Delta^+$ we obtain

$$\rho(\mathfrak{g}^{\mathbb{C},\alpha})v_0 \subseteq P(]\mu, \infty[)\mathscr{H} = \{0\}.$$

In view of the Poincaré–Birkhoff–Witt Theorem, this leads to

$$\mathscr{U}(\mathfrak{g}^\mathbb{C})v_0 = \mathscr{U}(\mathfrak{g}^- \rtimes \mathfrak{t}^\mathbb{C})v_0.$$

Since $\mathfrak{t}^\mathbb{C}$ commutes with $\mathfrak{t}_{\overline{0}}$, the subspace $\mathscr{U}(\mathfrak{t}^\mathbb{C})v_0$ is contained in $P([\mu-\varepsilon, \mu])\mathscr{H}$, so that Proposition 7.2.2 yields

$$\mathscr{U}(\mathfrak{g}^\mathbb{C})v_0 \subseteq P(]-\infty, \mu - \delta])\mathscr{H} + P([\mu - \varepsilon, \mu])\mathscr{H}$$

for every $\varepsilon > 0$. As $\mathscr{U}(\mathfrak{g}^\mathbb{C})v_0$ is dense in $\mathscr{H}$, for every sufficiently small $\varepsilon > 0$ we have $P([\mu - \varepsilon, \mu]) = P(\{\mu\})$. Hence $i\rho(X_0)v_0 = \mu v_0$. Since $\mathfrak{g}^\mathbb{C}$ is spanned by $\mathrm{ad}(X_0)$-eigenvectors, the same holds for $\mathscr{U}(\mathfrak{g}^\mathbb{C})$, and hence for $\mathscr{U}(\mathfrak{g}^\mathbb{C})v_0$. Since $\mathscr{U}(\mathfrak{g}^\mathbb{C})v_0$ is dense in $\mathscr{H}$, this means that $\mathscr{H}$ is a direct sum of weight spaces for $\exp(\mathbb{R}X_0)$. Repeating the same argument for other regular elements in $\mathfrak{t}_{\overline{0}} \cap \mathit{Int}(\mathit{Cone}(\mathcal{G}))$ forming a basis of $\mathfrak{t}_{\overline{0}}$, we conclude that $\mathscr{H}$ is an orthogonal direct sum of weight spaces for the group $T = \exp(\mathfrak{t}_{\overline{0}})$.

Let $\mathscr{V} = P(\{\mu\})\mathscr{H}$ be the maximal eigenspace of $i\rho^\pi(X_0)$. Then Proposition 7.2.1 applied to sets of the form $E =]\mu - \varepsilon, \mu + \varepsilon[$ implies that $\mathscr{H}^{\omega,r} \cap \mathscr{V}$ is dense in $\mathscr{V}$. Further $\mathscr{V}$ is T-invariant, hence an orthogonal direct sum of T-weight spaces. From Lemma 7.1.2, applied to $K = T$, we now derive that in each T-weight space $\mathscr{V}^\alpha(T)$, the intersection with $\mathscr{H}^{\omega,r}$ is dense.

Let $v_\alpha \in \mathscr{V}^\alpha \cap \mathscr{H}^{\omega,r}$ be a T-eigenvector. From the density of $\mathscr{U}(\mathfrak{g}^\mathbb{C})v_\alpha = \mathscr{U}(\mathfrak{g}^-)\mathscr{U}(\mathfrak{t}^\mathbb{C})v_\alpha$ in $\mathscr{H}$ we then derive as above that

$$\mathscr{U}(\mathfrak{t}^\mathbb{C})v_\alpha \subseteq \mathscr{V}^\alpha$$

is dense in $\mathscr{V}$. The left hand side of the latter inclusion is finite dimensional, and therefore $\mathscr{V} = \mathscr{V}^\alpha$ is also finite dimensional and contained in $\mathscr{H}^{\omega,r}$.

Since all $\mathfrak{t}_{\overline{0}}$-weight spaces in $\mathscr{U}(\mathfrak{g}^-)$ are finite dimensional, we conclude that $\mathscr{U}(\mathfrak{g}^{\mathbb{C}})\mathscr{V}$ is a locally finite $\mathfrak{t}$-module with finite $\mathfrak{t}_{\overline{0}}$-multiplicities. In view of the finiteness of multiplicities, density of $\mathscr{U}(\mathfrak{g}^{\mathbb{C}})\mathscr{V}$ in $\mathscr{H}$ leads to the equality $\mathscr{H}^{\mathfrak{t}} = \mathscr{U}(\mathfrak{g}^{\mathbb{C}})\mathscr{V}$. As this $\mathfrak{g}$-module consists of analytic vectors, its irreducibility follows from the irreducibility of the $\mathcal{G}$-representation on $\mathscr{H}$.

(iii) If $\mathscr{V}' \subseteq \mathscr{V}$ is a non-zero $\mathfrak{t}$-submodule, then $\mathscr{U}(\mathfrak{t})\mathscr{V}'$ is dense in $\mathscr{V}$ and orthogonal to the subspace $\mathscr{V}'' = (\mathscr{V}')^{\perp}$, which leads to $\mathscr{V}'' = \{0\}$. Therefore the $\mathfrak{t}$-module $\mathscr{V}$ is irreducible. All other assertions have already been verified above.

(iv) Clearly, the equivalence of the $\mathcal{G}$-representations implies equivalence of the $\mathfrak{t}$-representations on $\mathscr{V}$ and $\mathscr{V}'$.

Suppose, conversely, that there exists a $\mathfrak{t}$-isomorphism $\phi\colon \mathscr{V} \to \mathscr{V}'$. We consider the direct sum representation $\mathscr{K} = \mathscr{H} \oplus \mathscr{H}'$ of $\mathcal{G}$, for which

$$\mathscr{K}^{\mathfrak{t}} = \mathscr{H}^{\mathfrak{t}} \oplus (\mathscr{H}')^{\mathfrak{t}}$$

as $\mathfrak{g}$-modules. Consider the $\mathfrak{g}$-submodule $W \subseteq \mathscr{K}^{\mathfrak{t}}$ generated by the $\mathfrak{t}$-submodule

$$\Gamma(\phi) = \{(v, \phi(v))\colon v \in \mathscr{V}\} \subseteq \mathscr{V} \oplus \mathscr{V}'.$$

Since $\Gamma(\phi)$ is annihilated by $\mathfrak{g}^+$, the PBW Theorem implies that

$$W = \mathscr{U}(\mathfrak{g})\Gamma(\phi) = \mathscr{U}(\mathfrak{g}^-)\mathscr{U}(\mathfrak{t})\mathscr{U}(\mathfrak{g}^+)\Gamma(\phi) = \mathscr{U}(\mathfrak{g}^-)\Gamma(\phi).$$

It follows that

$$W \cap (\mathscr{V} \oplus \mathscr{V}') = \Gamma(\phi)$$

is the maximal eigenspace for iX_0 on W.

As W consists of analytic vectors, its closure $\overline{W}$ is a proper G-invariant subspace of $\mathscr{K}$, so that we obtain a unitary $\mathcal{G}$-representation on this space.

If the two $\mathcal{G}$ representations $(\pi, \rho^{\pi}, \mathscr{H})$ and $(\pi', \rho^{\pi'}, \mathscr{H}')$ are not equivalent, then Schur's Lemma implies that $\mathscr{H}$ and $\mathscr{H}'$ are the only non-trivial $\mathcal{G}$-invariant subspaces of $\mathscr{K}$, contradicting the existence of $\overline{W}$. ☐

Remark 7.3.3. (a) The preceding theorem suggests to call the $\mathfrak{g}$-representation on $\mathscr{H}^{\mathfrak{t}}$ a *highest weight representation* because it is generated by a weight space of $\mathfrak{t}_{\overline{0}}$ which is an irreducible $\mathfrak{t}$-module, hence a (finite dimensional) irreducible module of the Clifford Lie superalgebra $\mathfrak{t}_{\overline{1}} + [\mathfrak{t}_{\overline{1}}, \mathfrak{t}_{\overline{1}}]$.

(b) Suppose that $\mathfrak{g}$ is $\star$-reduced with $\mathfrak{g}_{\overline{0}} = [\mathfrak{g}_{\overline{1}}, \mathfrak{g}_{\overline{1}}]$. Let $\mathscr{H}$ be a complex Hilbert space and $\mathscr{D} \subseteq \mathscr{H}$ a dense subspace on which we have a unitary representation $(\rho, \mathscr{D})$ of $\mathfrak{g}$ in the sense that (i), (iii), (v) in Definition 4.2.1 are satisfied.

Suppose further that the action of $\mathfrak{t}_{\overline{0}}$ on $\mathscr{D}$ is diagonalizable with finite dimensional weight spaces. Then the $\mathfrak{g}$-module $\mathscr{D}$ is semisimple, hence irreducible if it is generated by a $\mathfrak{t}_{\overline{0}}$-weight space V on which $\mathfrak{t}$ acts irreducibly.

The finite dimensionality of the $\mathfrak{t}_{\overline{0}}$-weight spaces on $\mathscr{D}$ also implies the semisimplicity of $\mathscr{D}$ as a $\mathfrak{g}_{\overline{0}}$-module. Hence, as $i\rho(X_0) \leq 0$, an argument as in the proof of Theorem 7.3.2 implies that each simple submodule of $\mathscr{D}$ is a unitary highest weight module, hence integrable by [20, Corollary XII.2.7]. We conclude in particular that the $\mathfrak{g}_0$-representation on $\mathscr{D}$ is integrable with $\mathscr{D}$ consisting of analytic vectors.

8 The Orbit Method and Nilpotent Lie Supergroups

One of the most elegant and powerful ideas in the theory of unitary representations of Lie groups since the early stages of its development is the *orbit method*. The basic idea of the orbit method is to attach unitary representations to special homogeneous symplectic manifolds, such as the coadjoint orbits, in a natural way. One of the goals of the orbit method is to obtain a concrete realization of the representation and to extract information about the representation (e.g., its distribution character) from this realization.

Recall that a Lie supergroup $\mathcal{G} = (G, \mathfrak{g})$ is called *nilpotent* when the Lie superalgebra $\mathfrak{g}$ is nilpotent. In this article the orbit method is only studied for nilpotent Lie supergroups. It is known that among Lie groups, the orbit method works best for the class of nilpotent ones. For further reading on the subject of the orbit method, the reader is referred to [15] and [35].

8.1 Quantization and Polarizing Subalgebras

All of the irreducible unitary representations of nilpotent Lie groups can be classified by the orbit method. Let G be a nilpotent real Lie group and $\mathfrak{g}$ be its Lie algebra. For simplicity, G is assumed to be simply connected. In this case, there exists a bijective correspondence between coadjoint orbits (i.e., G-orbits in $\mathfrak{g}^*$) and irreducible unitary representations of G. In some sense the correspondence is surprisingly simple. To construct a representation $\pi_{\mathcal{O}}$ of G which corresponds to a coadjoint orbit $\mathcal{O} \subseteq \mathfrak{g}^*$, one first chooses an element $\lambda \in \mathcal{O}$ and considers the skew symmetric form

$$\Omega_\lambda : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R} \ \text{ defined by } \ \Omega_\lambda(X, Y) = \lambda([X, Y]). \tag{13}$$

It can be shown that there exist maximal isotropic subspaces of Ω_λ which are also subalgebras of $\mathfrak{g}$. Such subalgebras are called *polarizing subalgebras*. For a given polarizing subalgebra $\mathfrak{m}$ of $\mathfrak{g}$, one can consider the one dimensional representation of the subgroup $M = \exp(\mathfrak{m})$ of G given by

$$\chi_\lambda(m) = e^{i\lambda(\log(m))} \quad \text{for } m \in M.$$

The unitary representation of G corresponding to $\mathcal{O}$ is $\pi_{\mathcal{O}} = \mathrm{Ind}_M^G \chi_\lambda$. Of course one needs to prove that the construction is independent of the choices of λ and

$\mathfrak{m}$, the representation $\pi_{\mathcal{O}}$ is irreducible, and the correspondence is bijective. These statements are proved in [14]. Many other proofs have been found as well.

8.2 Heisenberg–Clifford Lie Supergroups

Heisenberg groups play a distinguished role in the harmonic analysis of nilpotent Lie groups. Therefore it is natural to expect that the analogues of Heisenberg groups in the category of Lie supergroups play a similar role in the representation theory of nilpotent Lie supergroups. These analogues, which deserve to be called *Heisenberg–Clifford* Lie supergroups, can be described as follows. Let (W, Ω) be a finite dimensional real *super symplectic* vector space. This means that $\mathsf{W} = \mathsf{W}_{\overline{0}} \oplus \mathsf{W}_{\overline{1}}$ is endowed with a bilinear form

$$\Omega : \mathsf{W} \times \mathsf{W} \to \mathbb{R}$$

that satisfies the following properties.

(i) $\Omega(\mathsf{W}_{\overline{0}}, \mathsf{W}_{\overline{1}}) = \Omega(\mathsf{W}_{\overline{1}}, \mathsf{W}_{\overline{0}}) = \{0\}$.
(ii) The restriction of Ω to $\mathsf{W}_{\overline{0}}$ is a symplectic form.
(iii) The restriction of Ω to $\mathsf{W}_{\overline{1}}$ is a nondegenerate symmetric form.

The Heisenberg–Clifford Lie supergroup corresponding to (W, Ω) is the super Harish–Chandra pair $(H^{\mathsf{W}}, \mathfrak{h}^{\mathsf{W}})$ where

(i) $\mathfrak{h}^{\mathsf{W}}_{\overline{0}} = \mathsf{W}_{\overline{0}} \oplus \mathbb{R}$ and $\mathfrak{h}^{\mathsf{W}}_{\overline{1}} = \mathsf{W}_{\overline{1}}$ (as vector spaces).
(ii) For every $X, Y \in \mathsf{W}$ and every $a, b \in \mathbb{R}$, the superbracket of $\mathfrak{h}^{\mathsf{W}}$ is defined by

$$[(X, a), (Y, b)] = (0, \Omega(X, Y)).$$

(iii) H^{W} is the simply connected Lie group with Lie algebra $\mathfrak{h}^{\mathsf{W}}_{\overline{0}}$.

When $\dim \mathsf{W}_{\overline{1}} = 0$ the Lie supergroup $(H^{\mathsf{W}}, \mathfrak{h}^{\mathsf{W}})$ is purely even, i.e., it is a Lie group. In this case, it is usually called a *Heisenberg Lie group*. When $\dim \mathsf{W}_{\overline{0}} = 0$ the Lie supergroup $(H^{\mathsf{W}}, \mathfrak{h}^{\mathsf{W}})$ is called a *Clifford Lie supergroup*.

Irreducible unitary representations of Heisenberg Lie groups are quite easy to classify. One can use the orbit method of Sect. 8.1 to classify them, but their classification was known as a consequence of the Stone–von Neumann Theorem long before the orbit method was developed. The Stone–von Neumann Theorem implies that there exists a bijective correspondence between infinite dimensional irreducible unitary representations of a Heisenberg Lie group and nontrivial characters (i.e., one dimensional unitary representations) of its center.

For Heisenberg–Clifford Lie supergroups there is a similar classification of representations. Let $(\pi, \rho^\pi, \mathscr{H})$ be an irreducible unitary representation of $(H^{\mathsf{W}}, \mathfrak{h}^{\mathsf{W}})$. By a super version of Schur's Lemma, for every $Z \in \mathscr{Z}(\mathfrak{h}^{\mathsf{W}})$ the action of $\rho^\pi(Z)$

is via multiplication by a scalar $c_{\rho^\pi}(Z)$. If $c_{\rho^\pi}(Z) = 0$ for every $Z \in \mathscr{Z}(\mathfrak{h}^{\mathsf{W}})$, then $\mathscr{H}$ is one dimensional, and essentially obtained from a unitary character of $\mathsf{W}_{\bar{0}}$. The irreducible unitary representations for which $\rho^\pi(\mathscr{Z}(\mathfrak{h}^{\mathsf{W}})) \neq \{0\}$ are classified by the following statement (see [27, Theorem 5.2.1]).

Theorem 8.2.1. *Let* S *be the set of unitary equivalence classes of irreducible unitary representations* $(\pi, \rho^\pi, \mathscr{H})$ *of* $(H^{\mathsf{W}}, \mathfrak{h}^{\mathsf{W}})$ *for which* $\rho^\pi(\mathscr{Z}(\mathfrak{h}^{\mathsf{W}})) \neq \{0\}$. *Then* S *is nonempty if and only if the restriction of* Ω *to* $\mathsf{W}_{\bar{1}}$ *is (positive or negative) definite. Moreover, the map*

$$[(\pi, \rho^\pi, \mathscr{H})] \mapsto c_{\rho^\pi}$$

yields a surjection from S *onto the set of* $\mathbb{R}$*-linear functionals* $\gamma : \mathscr{Z}(\mathfrak{h}^{\mathsf{W}}) \to \mathbb{R}$ *which satisfy*

$$i\gamma([X, X]) < 0 \text{ for every } 0 \neq X \in \mathsf{W}_{\bar{1}}.$$

When $\dim \mathsf{W}_{\bar{1}}$ *is odd the latter map is a bijection, and when* $\dim \mathsf{W}_{\bar{1}}$ *is even it is two-to-one, and the two representations in the fiber are isomorphic via parity change.*

Every irreducible unitary representation of a Clifford Lie supergroup is finite dimensional (see [27, Sect. 4.5]). In fact the theory of Clifford modules implies that the only possible values for the dimension of such a representation are one or

$$2^{\left(\dim \mathsf{W}_{\bar{1}} - \lfloor \frac{\dim \mathsf{W}_{\bar{1}}}{2} \rfloor\right)}.$$

It will be seen below that Clifford Lie supergroups are used to define analogues of polarizing subalgebras for Lie supergroups.

8.3 *Polarizing Systems and a Construction*

In order to construct the irreducible unitary representations of a nilpotent Lie supergroup using the orbit method, first we need to generalize the notion of polarizing subalgebras. What makes the case of Lie supergroups more complicated than the case of Lie groups is the fact that irreducible unitary representations of nilpotent Lie supergroups are not necessarily induced from one dimensional representations. However, it will be seen that they are induced from certain finite dimensional representations which are obtained from representations of Clifford Lie supergroups.

Let $(G, \mathfrak{g})$ be a Lie supergroup. Associated to every $\lambda \in \mathfrak{g}_{\bar{0}}^*$ there exists a skew symmetric bilinear form Ω_λ on $\mathfrak{g}_{\bar{0}}$ which is defined in (13). There is also a symmetric bilinear form

$$\Omega_\lambda : \mathfrak{g}_{\bar{1}} \times \mathfrak{g}_{\bar{1}} \to \mathbb{R} \tag{14}$$

associated to λ, which is defined by $\Omega_\lambda(X, Y) = \lambda([X, Y])$.

Definition 8.3.1. Let $\mathcal{G} = (G, \mathfrak{g})$ be a nilpotent Lie supergroup. A polarizing system in $(G, \mathfrak{g})$ is a pair $(\mathcal{M}, \lambda)$ satisfying the following properties.

(i) $\lambda \in \mathfrak{g}_{\bar{0}}^*$ and Ω_λ is a positive semidefinite form.
(ii) $\mathcal{M} = (M, \mathfrak{m})$ is a Lie subsupergroup of $\mathcal{G}$ and $\dim \mathfrak{m}_{\bar{1}} = \dim \mathfrak{g}_{\bar{1}}$.
(iii) $\mathfrak{m}_{\bar{0}}$ is a polarizing subalgebra of $\mathfrak{g}_{\bar{0}}$ with respect to λ, i.e., a subalgebra of $\mathfrak{g}_{\bar{0}}$ which is also a maximal isotropic subspace with respect to Ω_λ.

Given a polarizing system $(\mathcal{M}, \lambda)$, one can construct a unitary representation of $\mathcal{G}$ as follows. Let

$$\mathfrak{j} = \ker \lambda \oplus \mathrm{rad}(\Omega_\lambda), \tag{15}$$

where $\mathrm{rad}(\Omega_\lambda)$ denotes the radical of the symmetric form Ω_λ. One can show that $\mathfrak{j}$ is an ideal of $\mathfrak{m}$ that corresponds to a Lie subsupergroup $\mathcal{J} = (J, \mathfrak{j})$ of $\mathcal{M}$, and the quotient $\mathcal{M}/\mathcal{J}$ is a Clifford Lie supergroup. Let $\mathscr{Z}(\mathfrak{m}/\mathfrak{j})$ denote the center of $\mathfrak{m}/\mathfrak{j}$. Since Ω_λ is positive semidefinite, from Theorem 8.2.1 it follows that up to parity and unitary equivalence there exists a unique unitary representation $(\sigma, \rho^\sigma, \mathscr{K})$ of $\mathcal{M}/\mathcal{J}$ such that for every $Z \in \mathscr{Z}(\mathfrak{m}/\mathfrak{j})$, the operator $\rho^\sigma(Z)$ acts via multiplication by $i\lambda(Z)$. Clearly $(\sigma, \rho^\sigma, \mathscr{K})$ can also be thought of as a representation of $\mathcal{M}$, and one can consider the induced representation

$$(\pi, \rho^\pi, \mathscr{H}) = \mathrm{Ind}_{\mathcal{M}}^{\mathcal{G}}(\sigma, \rho^\sigma, \mathscr{K}). \tag{16}$$

8.4 *Existence of Polarizing Systems*

Throughout this section $\mathcal{G} = (G, \mathfrak{g})$ will be a nilpotent Lie supergroup such that G is simply connected.

It is natrual to ask for which $\lambda \in \mathfrak{g}_{\bar{0}}^*$ such that Ω_λ is positive semidefinite there exists a polarizing system $(\mathcal{M}, \lambda)$ in the sense of Definition 8.3.1. It turns out that for all such λ the answer is affirmative. The latter statement can be proved as follows. Fix such a $\lambda \in \mathfrak{g}_{\bar{0}}^*$. Proving the existence of a polarizing system $(\mathcal{M}, \geq)$ amounts to showing that there exists a polarizing subalgebra $\mathfrak{m}_{\bar{0}}$ of $\mathfrak{g}_{\bar{0}}$ such that $\mathfrak{m}_{\bar{0}} \supseteq [\mathfrak{g}_{\bar{1}}, \mathfrak{g}_{\bar{1}}]$. Since $[\mathfrak{g}_{\bar{1}}, \mathfrak{g}_{\bar{1}}]$ is an ideal of the Lie algebra $\mathfrak{g}_{\bar{0}}$ and $\mathfrak{g}_{\bar{0}}$ is nilpotent, one can find a sequence of ideal of $\mathfrak{g}_{\bar{0}}$ such as

$$\{0\} = \mathfrak{i}^{(0)} \subseteq \mathfrak{i}^{(1)} \subseteq \cdots \subseteq \mathfrak{i}^{(k-1)} \subseteq \mathfrak{i}^{(k)} = [\mathfrak{g}_{\bar{1}}, \mathfrak{g}_{\bar{1}}] \subseteq \mathfrak{i}^{k+1} \subseteq \cdots \subseteq \mathfrak{i}^{(r)} = \mathfrak{g}_{\bar{0}}$$

where for every $0 \leq s \leq r-1$, the codimension of $\mathfrak{i}^{(s)}$ in $\mathfrak{i}^{(s+1)}$ is equal to one. For every $0 \leq s \leq r-1$, let

$$\Omega_\lambda^{(s)} : \mathfrak{i}^{(s)} \times \mathfrak{i}^{(s)} \to \mathbb{R}$$

be the skew symmetric form defined by

$$\Omega_\lambda^{(s)}(X, Y) = \lambda([X, Y])$$

and let $\mathrm{rad}(\Omega_\lambda^{(s)})$ denote the radical of $\Omega_\lambda^{(s)}$. It is known that the subspace of $\mathfrak{g}_{\overline{0}}$ defined by

$$\mathrm{rad}(\Omega_\lambda^{(1)}) + \cdots + \mathrm{rad}(\Omega_\lambda^{(s)})$$

is a polarizing subspace of $\mathfrak{g}_{\overline{0}}$ corresponding to λ (see [8, Theorem 1.3.5]). To prove that

$$[\mathfrak{g}_{\overline{1}}, \mathfrak{g}_{\overline{1}}] \subseteq \mathrm{rad}(\Omega_\lambda^{(1)}) + \cdots + \mathrm{rad}(\Omega_\lambda^{(s)})$$

it suffices to show that $\mathrm{rad}(\Omega_\lambda^{(k)}) = \mathfrak{i}^{(k)}$. This is where one needs the fact that Ω_λ is positive semidefinite. The proof is by a backward induction on the dimension of $\mathcal{G}$. Details of the argument appear in [27, Sect. 6.3].

8.5 A Bijective Correspondence

Throughout this section $\mathcal{G} = (G, \mathfrak{g})$ will be a nilpotent Lie supergroup such that G is simply connected.

One can check easily that the set

$$\mathscr{P}(\mathcal{G}) = \left\{ \lambda \in \mathfrak{g}_{\overline{0}}^* \;\middle|\; \Omega_\lambda \text{ is positive semidefinite} \right\}$$

is an invariant cone in $\mathfrak{g}_{\overline{0}}^*$. Section 8.4 shows that for every $\lambda \in \mathscr{P}(\mathcal{G})$ one can find a polarizing system $(\mathcal{M}, \lambda)$. Therefore the construction of Sect. 8.3 yields a unitary representation $(\pi_\lambda, \rho^{\pi_\lambda}, \mathscr{H}_\lambda)$ of $\mathcal{G}$ which is given by (16). The main result of [27] can be stated as follows.

Theorem 8.5.1. *The map which takes a $\lambda \in \mathscr{P}(\mathcal{G})$ to the representation $(\pi_\lambda, \rho^{\pi_\lambda}, \mathscr{H}_\lambda)$ results in a bijective correspondence between G-orbits in $\mathscr{P}(\mathcal{G})$ and irreducible unitary representations of $\mathcal{G}$ up to unitary equivalence and parity change.*

To prove Theorem 8.5.1 one needs to show that the construction given in Sect. 8.3 yields an irreducible representation and is independent of the choice of λ in a G-orbit or the polarizing system. One also has to show that if λ and λ' are not in the same G-orbit then inducing from polarizing systems $(\mathcal{M}, \lambda)$ and $(\mathcal{M}', \lambda')$ does not lead to representations which are identical up to parity or unitary equivalence. The proofs of all of these facts are given in [27, Sect. 6]. To some extent, the method of proof is similar to the original proof of the Lie group case in [14], where induction on the dimension is used. In the Lie group case, what makes the inductive argument work is the existence of three dimensional Heisenberg subgroups in any nilpotent Lie group of dimension bigger than one with one dimensional center. For Lie supergroups a similar statement only holds under extra assumptions. The next proposition shows that it suffices to assume that the corresponding Lie superalgebra has no self-commuting odd elements.

Proposition 8.5.2. *Let $\mathcal{G} = (G, \mathfrak{g})$ be as above. Assume that there are no nonzero $X \in \mathfrak{g}_{\overline{1}}$ such that $[X, X] = 0$. If $\dim \mathscr{Z}(\mathfrak{g}) = 1$ then either $\mathcal{G}$ is a Clifford Lie supergroup, or it has a Heisenberg Lie subsupergroup of dimension* $(3|0)$.

Using Proposition 6.1.1 one can pass to a quotient and reduce the analysis of the general case to the case where the assumptions of Proposition 8.5.2 are satisfied. Proposition 8.5.2 makes induction on the dimension of $\mathfrak{g}$ possible.

Although the proof of Theorem 8.5.1 is inspired by the methods and arguments in [14] and [8], one must tackle numerous additional analytic technical difficulties which emerge in the case of Lie supergroups. This is because many facts in the theory of unitary representations of Lie supergroups are generally not as powerful as their analogues for Lie groups. For instance to prove that $(\pi_\lambda, \rho^{\pi_\lambda}, \mathscr{H}_\lambda)$ is irreducible one cannot use Mackey theory and needs new ideas.

8.6 Branching to the Even Part

Let $\mathcal{G} = (G, \mathfrak{g})$ be as in Sect. 8.5. For every $\lambda \in \mathscr{P}(\mathcal{G})$ let $(\pi_\lambda, \rho^{\pi_\lambda}, \mathscr{H}_\lambda)$ be the representation of $\mathcal{G}$ associated to λ in Sect. 8.5. As an application of Theorem 8.5.1 one can obtain a simple decomposition formula for the restriction of $(\pi_\lambda, \rho^{\pi_\lambda}, \mathscr{H}_\lambda)$ to G.

Recall that $(\pi_\lambda, \rho^{\pi_\lambda}, \mathscr{H}_\lambda)$ is induced from a polarizing system $(\mathcal{M}, \lambda)$. Let $\mathfrak{m}$ be the Lie superalgebra of $\mathcal{M}$ and $\mathfrak{j}$ be defined as in (15).

Corollary 8.6.1. *The representation $(\pi_\lambda, \mathscr{H}_\lambda)$ of G decomposes into a direct sum of $2^{\dim \mathfrak{m} - \dim \mathfrak{j}}$ copies of the irreducible unitary representation of G which is associated to the coadjoint orbit containing λ (in the sense of Sect. 8.1).*

9 Conclusion

In this article we discussed irreducible unitary representations of Lie supergroups in some detail for the case where $\mathcal{G}$ is either nilpotent or $\mathfrak{g}$ is $\star$-reduced and satisfies $\mathfrak{g}_{\overline{0}} = [\mathfrak{g}_{\overline{1}}, \mathfrak{g}_{\overline{1}}]$. The overlap between these two classes is quite small because for any nilpotent Lie superalgebra satisfying the latter conditions $\mathfrak{g}_{\overline{0}}$ is central, so that it essentially is a Clifford–Lie superalgebra, possibly with a multidimensional center, and in this case the irreducible unitary representations are the well-known spin representations. Precisely these representations occur as the $\mathfrak{t}$-modules on the highest weight space $\mathscr{V}$ in the other case.

Clearly, the condition of being $\star$-reduced is natural if one is interested in unitary representations. The requirement that $\mathfrak{g}_{\overline{0}} = [\mathfrak{g}_{\overline{1}}, \mathfrak{g}_{\overline{1}}]$ is more serious, as we have seen in the nilpotent case. In general one can consider the ideal $\mathfrak{g}_c = [\mathfrak{g}_{\overline{1}}, \mathfrak{g}_{\overline{1}}] \oplus \mathfrak{g}_{\overline{1}}$ and our results show that the irreducible unitary representations of this ideal are highest weight representations. For nilpotent Lie supergroups, how to use them to parametrize the irreducible unitary representations of $\mathcal{G}$ was explained in Sect. 8. It is conceivable that other larger classes of groups could be studied by combining tools from the Orbit Method, induction procedures and highest weight theory.

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Geometry of Dual Pairs of Complex Supercurves

Jeffrey M. Rabin

Abstract Supercurves are a generalization to supergeometry of Riemann surfaces or algebraic curves. I review the definitions, examples, key results, and open problems in this area.

1 Introduction

The notion of a super Riemann surface is an outgrowth of string theory: just as the world sheet of a bosonic string carries the structure of a Riemann surface, the world sheet in superstring theory is a super Riemann surface. The geometry of super Riemann surfaces and their moduli spaces was studied intensively in the 1980s to provide tools for perturbative string theory computations. Gradually it became clear that a super Riemann surface is the special "self-dual" case of the more general notion of supercurve, and this generalization makes the theory clearer and more elegant. Supercurves have additional applications, for example to supersymmetric integrable systems.

In this article I sketch the current understanding of (smooth, $N = 1$) supercurves, how they generalize both Riemann surfaces and super Riemann surfaces, and some recent results and open problems. Section 2 gives the definition and two classes of examples: split supercurves, and super elliptic curves. Section 3 introduces divisors and the duality they lead to: supercurves naturally occur in pairs such that the points of one are the irreducible divisors of the other. Section 4 explains contour integration of differentials on supercurves, and the resulting theory of periods, Jacobians and the Abel map. Section 5 summarizes recent results about $\mathcal{D}$-modules on supercurves, which in the simplest instance are line bundles having constant transition functions.

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Section 6 presents some open problems, including Hodge decomposition and theta functions for supercurves.

As noted, super Riemann surfaces were studied prior to more general supercurves. Here is a brief and selective sketch of the history. Super Riemann surfaces, also known as SUSY-curves or superconformal manifolds, were introduced by Baranov and Schwarz [1], and by Friedan [8]. Since perturbative string theory involves integration over moduli spaces of super Riemann surfaces, the study of these moduli spaces was an early priority; see [3, 12, 22]. Rosly et al. generalized many of the classical results about differentials and line bundles on a Riemann surface to the super context, including divisors, the Jacobian variety and Abel's theorem [20]. Dual supercurves and the universal divisor were introduced by Dolgikh et al. [7]. Super elliptic curves (in the super Riemann surface context) were introduced by Levin [13] and studied extensively by me in [18]. The work of Rosly et al. cited above was further extended from super Riemann surfaces to general supercurves (along with their duals) by Bergvelt and myself [2]. We also discussed super theta functions and applications to supersymmetric KP equations. Unknown to us, many of our results on super theta functions were anticipated by Tsuchimoto [21].

I want to particularly acknowledge the many contributions of Albert Schwarz to this subject, which are evident from these citations.

2 Definitions and Examples

I will assume general familiarity with supermanifolds; see for example [4, 14]. Fix a complex exterior algebra $\Lambda = \mathbb{C}[\beta_1, \beta_2, \dots, \beta_n]$. The generators β_i are considered to have odd parity, and the algebra is thereby $\mathbb{Z}_2$-graded. For us, a (smooth) supercurve X will be a family of $1|1$-dimensional complex supermanifolds over Spec $\Lambda = (\mathrm{pt}, \Lambda)$. (More general families are possible, but this already displays the characteristic "super" phenomena and is consistent with the viewpoint of physicists.) That is, X is a Riemann surface X_{red} with a sheaf $\mathcal{O}$ of functions locally isomorphic to $\mathcal{O}_{\mathrm{red}} \otimes \Lambda[\theta]$, where θ is an additional odd generator. More explicitly, on each open set U, $\mathcal{O}(U)$ consists of functions $F(z,\theta) = f(z) + \theta\phi(z)$. Here we show explicitly the dependence on the coordinates z, θ while hiding that on the parameters β_i. This is in keeping with the viewpoint of physicists that z, θ are true (even and odd) variables while the β_i are merely "anticommuting constants."

The global structure of X is described by invertible (parity-preserving) transition functions on chart overlaps, having the form $\tilde{z} = F(z,\theta)$, $\tilde{\theta} = \Psi(z,\theta)$. Here the reduced part of $F(z,\theta)$, namely $f_{\mathrm{red}}(z)$, is the transition function for X_{red} on the same overlap.

Two distinct viewpoints on the transition functions are useful and important. First, they describe the restriction maps on the structure sheaf. Suppose two charts overlap, and we wish to compare functions $G(z,\theta)$ and $H(\tilde{z},\tilde{\theta})$ in those charts, on the common overlap. For this we use the restriction map described by the transition functions to express H in terms of the coordinates z, θ. Since the transition functions are holomorphic in z, compositions like $H(F(z,\theta), \Psi(z,\theta))$ make sense via a Taylor expansion in all nilpotent quantities, which eventually terminates.

The transition functions may also be viewed as the transformation law for Λ-valued points of X. A Λ-valued point in some chart U is a parity-preserving Λ-algebra homomorphism that evaluates functions on U to give elements of Λ. The "constants" β_i must of course evaluate to themselves. Since z and θ are themselves local functions, we give such a homomorphism by first specifying the elements of Λ to which they evaluate, say z_0 and θ_0. The reduced part of z_0 defines the underlying reduced point of X_{red}. A general function $G(z,\theta)$ must then evaluate to $G(z_0,\theta_0)$, so a Λ-valued point may indeed be identified with a pair of Λ-valued coordinates (z_0,θ_0) in each chart. When charts overlap, their Λ-valued points are identified if they give the same evaluation of every function on the overlap. This defines a transformation rule of their coordinates (z_0,θ_0). Not surprisingly, the transition functions give this transformation rule. Physicists tend to think of supermanifolds in the familiar terms of their Λ-valued points.

The simplest examples of supercurves are the *split* supercurves. To construct one, choose a Riemann surface to serve as X_{red}. Fix some line bundle $\mathcal{L}$ on X_{red} and define X by transition functions

$$\tilde{z} = f(z), \quad \tilde{\theta} = \theta g(z),$$

where $f(z)$ are transition functions for X_{red} and $g(z)$ are transition functions for $\mathcal{L}$. In effect, X becomes the total space of the dual bundle, with θ as (odd) fiber coordinate. For example, if X_{red} is the complex plane $\mathbb{C}$ and $\mathcal{L}$ is the trivial line bundle, then X is the affine superspace $\mathbb{C}^{1|1}$.

A nontrivial set of examples that we will use in the sequel are the super elliptic curves. Fix an even element $\tau \in \Lambda$ with $\operatorname{Im} \tau_{\text{red}} > 0$, and two odd elements $\epsilon, \delta \in \Lambda$. X will be $\mathbb{C}^{1|1}/G$, where the group $G \cong \mathbb{Z} \times \mathbb{Z}$ has generators A, B acting on $\mathbb{C}^{1|1}$ by

$$A(z,\theta) = (z+1,\theta), \quad B(z,\theta) = (z+\tau+\theta\epsilon, \theta+\delta). \tag{1}$$

Then X_{red} is the torus with parameter τ_{red}. Associated to a supercurve X there is always a split supercurve X/Λ, obtained by "setting the β_i equal to zero," and in this case it is the torus with the trivial line bundle on it.

We use these examples to highlight some differences in the behavior of cohomology for ordinary curves and supercurves. For a split supercurve, it is easy to see that the global functions are $H^0(X,\mathcal{O}) = (\mathbb{C}|\Gamma(\mathcal{L})) \otimes \Lambda$. This notation indicates the even and odd subspaces of a super vector space over Λ. That is, the "even functions" of the form $f(z)$ are the even constants from Λ as expected, but there are also "odd" global holomorphic functions $\theta s(z)$ coming from the global sections $s(z)$ of $\mathcal{L}$, if any. Of course, one can take Λ-linear combinations of these, respecting parity, as well. The presence of nonconstant global functions is a counterintuitive feature of supergeometry.

For a super elliptic curve, it is not hard to see that global functions are either constants a or of the form $\theta\alpha$ with α constant, but not all of the latter are G-invariant, because of the action $\theta \mapsto \theta + \delta$ of the generator B. In this way one computes that

$$H^0(X, \mathcal{O}) = \{a + \theta\alpha : \alpha\delta = 0\}. \tag{2}$$

Because of the restriction on α, the cohomology is not freely generated as a Λ-module. This is typical for nonsplit supercurves and is a major complication in dealing with them. It means, for example, that there is no simple result like the Riemann-Roch theorem that characterizes cohomology modules by computing their ranks.

Fortunately Serre duality does work for supercurves: $H^1(X, \mathcal{O}) \cong H^0(X, \mathrm{Ber})^*$ as Λ-modules, as shown in [2]. Here the dual space consists of the Λ-linear functionals on $H^0(X, \mathrm{Ber})$. Earlier work had established Serre duality in the sense of $\mathbb{C}$-linear functionals on individual supermanifolds rather than families [11, 16]

Here the dualizing Berezinian or "canonical" sheaf Ber is the line bundle (see Sect. 4) on X with transition functions

$$\mathrm{ber}\begin{bmatrix} \partial_z F & \partial_z \Psi \\ \partial_\theta F & \partial_\theta \Psi \end{bmatrix} = \frac{\partial_z F - \partial_z \Psi (\partial_\theta \Psi)^{-1} \partial_\theta F}{\partial_\theta \Psi}. \tag{3}$$

Serre duality is parity-reversing: even elements of $H^1(X, \mathcal{O})$ correspond to odd linear functionals.

In the split case, $\mathrm{Ber} = K\mathcal{L}^{-1}|K$ (we omit the $\otimes\Lambda$ by abuse of notation). That is, the sections of Ber are generated by even sections $f(z)$ of $K\mathcal{L}^{-1}$, where K is the canonical bundle of differentials on X_{red}, and odd sections having the form $\theta s(z)$ with $s(z)$ itself a differential on X_{red}.

For a super elliptic curve, Ber is trivial, and one finds $H^1(X, \mathcal{O}) = (\Lambda/\delta\Lambda)|\Lambda$. That is, $H^1(X, \mathcal{O})$ consists by Serre duality of parity-reversed linear functionals on sections of the trivial bundle, which is to say on the global functions we have already determined. A linear functional on the constants a is determined by the element of Λ that is the image of 1. A linear functional on $\theta\alpha$, where $\alpha\delta = 0$, is given by an element that should be "the image of θ," but only defined modulo δ. In this way $H^1(X, \mathcal{O})$ also is a non-free Λ-module. The simple behavior of linear functionals on Λ is due to the fact that an exterior algebra is a self-injective ring [6].

The behavior observed in this example generalizes in the following way [2]. $H^0(X, \mathcal{O})$, respectively $H^1(X, \mathcal{O})$, is always a submodule, respectively a quotient, of a free Λ-module. The free modules in question are isomorphic to the cohomologies of the associated split supercurve, and their ranks can be found from the Riemann-Roch theorem applied to X_{red} and $\mathcal{L}$.

3 Divisors and the Dual Curve

We use the standard basis for vector fields on a supercurve, $\partial = \partial_z$, $D = \partial_\theta + \theta\partial_z$, and observe that $D^2 = \frac{1}{2}[D, D] = \partial$. A divisor on X is a subvariety of dimension $0|1$, locally given by an even equation $G(z, \theta) = 0$ with G_{red} not identically zero. For example, $z - z_0 - \theta\theta_0 = 0$ locally defines a divisor. The reduced divisor is simply the point with coordinate $(z_0)_{\mathrm{red}}$ on X_{red}, but functions $F(z, \theta)$ restrict to this subvariety as $F(z_0 + \theta\theta_0, \theta)$. In general, near a simple zero of G_{red}, $G(z, \theta)$ contains a factor $z - z_0 - \theta\theta_0$ with the parameters z_0, θ_0 determined by the conditions

$$G(z_0, \theta_0) = DG(z_0, \theta_0) = 0. \tag{4}$$

This follows from the Taylor series expansion in the form

$$G(z, \theta) = \sum_{j=0}^{\infty} \frac{1}{j!}(z - z_0 - \theta\theta_0)^j [\partial^j G(z_0, \theta_0) + (\theta - \theta_0) D\partial^j G(z_0, \theta_0)]. \tag{5}$$

Although irreducible divisors depend on two parameters (z_0, θ_0) just like Λ-valued points, a crucial observation is that they are *not* points. To see this, we ask how the parameters of the same divisor are related in two overlapping charts. This is easily computed by using the transition functions to write

$$\tilde{z} - \tilde{z}_0 - \tilde{\theta}\tilde{\theta}_0 = F(z, \theta) - \tilde{z}_0 - \Psi(z, \theta)\tilde{\theta}_0, \tag{6}$$

and applying the conditions (4) to this function G to obtain

$$\tilde{z}_0 = F(z_0, \theta_0) + \frac{DF(z_0, \theta_0)}{D\Psi(z_0, \theta_0)}\Psi(z_0, \theta_0), \quad \tilde{\theta}_0 = \frac{DF(z_0, \theta_0)}{D\Psi(z_0, \theta_0)}. \tag{7}$$

Thus the parameters of a divisor have their own transformation rule distinct from that of points. It is automatic that these new transition functions satisfy a cocycle condition and thus they define a new supercurve denoted $\hat{X}$ and called the dual to X. It has the same reduced curve, and due to the symmetry of the function $z - z_0 - \theta\theta_0$ between (z, θ) and (z_0, θ_0), the dual of $\hat{X}$ is necessarily X again. Thus, supercurves naturally occur in pairs, with the points of each representing the irreducible divisors of the other. Not only does either supercurve determine the other, but a chosen atlas on one determines an associated atlas with the same collection of charts on the other.

We easily determine the duals of our basic examples of supercurves. For split X, we find $\hat{X} = (X_{\mathrm{red}}, K\mathcal{L}^{-1})$. That is, this duality simply acts as Serre duality on the line bundle characterizing X. The dual of the super elliptic curve X with parameters τ, ϵ, δ is again a super elliptic curve, with parameters $\tau + \epsilon\delta, \delta, \epsilon$. Note in particular the interchange $\epsilon \leftrightarrow \delta$.

Many aspects of the relationship between X and $\hat{X}$ are clarified by introducing a supermanifold Δ that fibers over both. In $X \times \hat{X}$, we define the universal divisor or superdiagonal Δ as the subvariety $z - \hat{z} - \theta\hat{\theta} = 0$. It has dimension $1|2$, and by choosing as the even coordinate either z or $\hat{z}$ we see that it fibers over both X and $\hat{X}$ with fibers of dimension $0|1$:

$$X \overset{\pi}{\leftarrow} \Delta \overset{\hat{\pi}}{\rightarrow} \hat{X}.$$

One can show that it is "Calabi-Yau" in the sense of having trivial canonical bundle. It is also an example of an "$N = 2$ super Riemann surface." One can introduce Δ first and construct both X and $\hat{X}$ from it in a symmetric and coordinate-free manner, as in [7, 15].

Riemann surfaces are special among algebraic varieties in that their irreducible divisors coincide with their points. We have seen that general supercurves do not share this property. The super-analog of a Riemann surface would thus be a self-dual supercurve. These are the "super Riemann surfaces" introduced in connection with string theory in the 1980s. From (7) we find that the transition functions of a super Riemann surface are "superconformal," meaning that $DF = \Psi D\Psi$. For split X this means $\mathcal{L}^2 = K$, so that the Serre self-dual line bundle $\mathcal{L}$ defines a spin structure on X_{red}. For super elliptic curves self-duality means $\epsilon = \delta$.

A Riemann surface structure can be defined in differential-geometric terms as a conformal equivalence class of metrics on a smooth surface. Similarly, a super Riemann surface can be introduced as a superconformal equivalence class of supergravity geometries on a smooth supermanifold, a viewpoint that grows out of the worldsheet formulation of superstring theory [5].

4 Differentials, Integration, Line Bundles

The fundamental exact sequence underlying contour integration theory for supercurves is

$$0 \to \Lambda \to \mathcal{O} \overset{D}{\to} \hat{\mathrm{Ber}} \to 0. \tag{8}$$

It is the analog of the sequence

$$0 \to \mathbb{C} \to \mathcal{O} \overset{d}{\to} \Omega^1 \to 0 \tag{9}$$

on a Riemann surface. That is, given representatives $F(z, \theta)$ of a function in some local charts on X, one can check that the derivatives $DF(z, \theta)$ transform as local sections of the canonical bundle $\hat{\mathrm{Ber}}$ of the dual curve $\hat{X}$ [following the cosmetic replacement of the arguments (z, θ) by $(\hat{z}, \hat{\theta})$]. Sections $\hat{\omega}$ of $\hat{\mathrm{Ber}}$ should be viewed as "holomorphic differentials" on $\hat{X}$, and locally have antiderivatives with respect to D, which are functions on X determined up to a constant. An antiderivative of $f(\hat{z}) + \hat{\theta}\phi(\hat{z})$ is $\theta f(z) + \int^z \phi$. Note that integration is parity-reversing, in addition to mapping between a curve and its dual. Once we have local antiderivatives, contour integrals of the form $\int_P^Q \hat{\omega}$ make sense, as follows. If the points P and Q of X lie in a single (contractible) chart, and F is an antiderivative of $\hat{\omega}$ in this chart, then the integral is defined to be $F(Q)-F(P)$. More generally, we define a super contour C as the pair of points P, Q together with a contour from $P_{\rm red}$ to $Q_{\rm red}$ on $X_{\rm red}$, and we choose a sequence of points $P = P_1, P_2, \ldots, P_k = Q$ along this contour such that each consecutive pair lies in a common chart. Then the contour integral is defined to be

$$\int_C \hat{\omega} = \sum_{i=1}^{k-1} \int_{P_i}^{P_{i+1}} \hat{\omega}. \tag{10}$$

As in the Riemann surface case, this is independent of the choice of intermediate points.

Similarly, periods and residues of a meromorphic differential make sense: the former is the integral around a nontrivial homology cycle (for example, one of the basis A and B cycles) and the latter is the integral around a closed contour encircling a pole. In each case the integral is independent of any choice of base point on the closed contour. Among the classical facts about Riemann surfaces which generalize to this context, I point out the Riemann bilinear period relation for holomorphic differentials, which here takes the form

$$\sum_{i=1}^{g} [A_i(\omega) B_i(\hat{\omega}) - B_i(\omega) A_i(\hat{\omega})] = 0. \tag{11}$$

Here g is the genus of the (reduced) curve, ω and $\hat{\omega}$ are arbitrary and independent holomorphic differentials on X and $\hat{X}$ respectively, and the notation $A_i(\omega)$ denotes the period of ω around the cycle A_i. On a Riemann surface, this relation is responsible for the symmetry of the period matrix. In the super context it relates the periods of holomorphic differentials on X and $\hat{X}$.

Due to the non-freeness of cohomology, there is generally no basis of holomorphic differentials having periods normalized in the familiar fashion with $A_i(\omega_j) = \delta_{ij}$. The super elliptic curve illustrates this. In this case the bundle Ber is trivial, so differentials are simply functions. The function 1 on X has antiderivative $\hat{\theta}$ on $\hat{X}$, and its periods are the changes in $\hat{\theta}$ under the group generators A, B. That is, $A(1) = 0$ and $B(1) = \epsilon$. Similarly, the function $\theta\alpha$, where $\alpha\delta = 0$, has antiderivative $\alpha\hat{z}$ and periods $A(\theta\alpha) = \alpha, \quad B(\theta\alpha) = \alpha\tau$. There is no holomorphic differential with A-period 1, because we cannot choose $\alpha = 1$.

As usual, a line bundle on X is defined by even, invertible transition functions $g_{ij}(z,\theta)$ in overlaps $U_i \cap U_j$, satisfying a cocycle condition, and line bundles are therefore classified by $H^1(X, \mathcal{O}_{\rm ev}^\times)$. The usual exponential exact sequence

$$0 \to \mathbb{Z} \to \mathcal{O}_{\rm ev} \xrightarrow{\exp 2\pi i\cdot} \mathcal{O}_{\rm ev}^\times \to 0 \tag{12}$$

holds, and shows that degree-zero bundles are classified by the component of the Picard group $\mathrm{Pic}^0(X) = H^1(X, \mathcal{O}_{\rm ev})/H^1(X, \mathbb{Z})$. By means of Serre and Poincaré duality, this is isomorphic to the Jacobian $\mathrm{Jac}(X) = H^0(X, \mathrm{Ber})^*_{\rm odd}/H_1(X, \mathbb{Z})$.

This isomorphism is via the *Abel map*: a degree-zero bundle on X can be described by the divisor $\sum_i n_i \hat{P}_i$ of a meromorphic section, and corresponds to the odd linear functional on holomorphic differentials (on X) given by

$$\sum_i n_i \int_{\hat{P}_0}^{\hat{P}_i}$$

modulo periods. Here $\sum_i n_i = 0$, and $\hat{P}_0$ is an arbitrary basepoint on $\hat{X}$. Abel's Theorem is due to [20] in the (free) super Riemann surface case, and to [2] in general.

We can compute the Jacobians of our example supercurves. For the split supercurve $(X_{\text{red}}, \mathcal{L})$, we find $\text{Jac}(X) = [\text{Jac}(X_{\text{red}}), \mathbb{C}^{h^1(\mathcal{L})}]$. That is, the Jacobian is also a split supermanifold, whose reduced space is the Jacobian of the reduced curve and whose odd vector bundle is trivial of rank $h^1(\mathcal{L})$. On a super elliptic curve X, we describe line bundles by their *multipliers*, trivial around the A cycle but given by $\exp(a + \theta\alpha)$ around the B cycle, where $a \in \Lambda_{\text{ev}},\ \alpha \in \Lambda_{\text{odd}}$. (That is, sections lift to the covering space $\mathbb{C}^{1|1}$ as functions invariant under the A transformation but changing by the multiplier under B.) The group operation in the Picard group is simply vector addition on the coordinates (a, α). The multipliers representing trivial bundles correspond to parameter values (a, α) that are integer linear combinations of $(1, 0)$, (τ, ϵ), plus arbitrary (odd) multiples of $(\delta, 0)$.

Jacobians as defined here are simply sets of Λ-valued points. They are not (representable by) supermanifolds in general, but can be viewed as nilpotent quotients thereof. In particular, the Jacobian of the super elliptic curve X is a nilpotent quotient of $\hat{X}$. Recall that $\hat{X}$ is the quotient of $\mathbb{C}^{1|1}$ by the group having the two generators

$$A(z, \theta) = (z + 1, \theta), \quad B(z, \theta) = (z + \tau + \epsilon\delta + \theta\delta, \theta + \epsilon). \tag{13}$$

According to the result above, $\text{Jac}(X)$ is also a quotient of $\mathbb{C}^{1|1}$, by a group having two similar generators,

$$A(z, \theta) = (z + 1, \theta), \quad B(z, \theta) = (z + \tau, \theta + \epsilon), \tag{14}$$

and a further nilpotent identification of all multiples of $(\delta, 0)$ to zero. In the presence of this nilpotent identification, the two B actions agree, so indeed $\text{Jac}(X)$ is the quotient of $\hat{X}$ by the nilpotent identification.

By computing the map $H^1(X, \Lambda) \to H^1(X, \mathcal{O})$, we find that line bundles on a super elliptic curve having constant transition functions correspond to multipliers $\exp(a + \theta\alpha)$ such that $\alpha \in \epsilon\Lambda_{\text{ev}}$. We conclude that the bundles having constant transition functions form a *proper* subset of the degree-zero bundles. In contrast, on a Riemann surface *every* degree-zero bundle can be presented with constant transition functions [10]. The proof makes use of the Hodge decomposition. It has long been known that this does not hold on super Riemann surfaces [9].

5 Line Bundles and $\mathcal{D}$-Modules on X, $\hat{X}$, and Δ

It is interesting to explore the relationships between line bundles on the three supermanifolds X, $\hat{X}$, and Δ. Bundles on X or $\hat{X}$ can be pulled back to Δ, but there are no natural maps between X and $\hat{X}$. However, a bundle having constant transition functions can be viewed as living on either X or $\hat{X}$. It is possible for such a bundle to be nontrivial on one curve but trivial on the other (and on Δ). For example, the bundle having multiplier $\exp a$ for $a \in \epsilon\Lambda_{\text{odd}}$ is trivial on $\hat{X}$ but nontrivial on X. Recall that $H^1(\hat{X}, \mathcal{O}) = (\Lambda/\epsilon\Lambda)|\Lambda$, whereas $H^1(X, \mathcal{O}) = (\Lambda/\delta\Lambda)|\Lambda$.

A line bundle having constant transition functions can be more invariantly described as a bundle equipped with a flat connection, namely a $\mathcal{D}$-module. To any $\mathcal{D}$-module $\mathcal{F}$ on X there corresponds a dual $\mathcal{D}$-module $\hat{\mathcal{F}}$ on $\hat{X}$. In fact, the categories of $\mathcal{D}$-modules on X, $\hat{X}$, and Δ are equivalent [19].

Beginning with the case of a trivial $\mathcal{D}$-module, functions $F(z, \theta)$ on X map locally, and coordinate-dependently, to functions $F(\hat{z}, \hat{\theta})$ on $\hat{X}$. This mapping can be expressed in terms of a differential operator, which allows generalization to arbitrary $\mathcal{D}$-modules:

$$F(\hat{z}, \hat{\theta}) = F(z - \theta\hat{\theta}, \hat{\theta}) = (1 - \theta\partial_\theta + \hat{\theta}\partial_\theta - \theta\hat{\theta}\partial_z)F. \tag{15}$$

Similarly, to a local section F of $\mathcal{F}$ there corresponds a local section

$$\hat{F} = (1 - \theta\nabla_\theta + \hat{\theta}\nabla_\theta - \theta\hat{\theta}\nabla_z)F \tag{16}$$

of $\hat{\mathcal{F}}$.

A more invariant description of this correspondence is as follows. Recall that Δ fibers over both X and $\hat{X}$:

$$X \overset{\pi}{\leftarrow} \Delta \overset{\hat{\pi}}{\rightarrow} \hat{X}. \tag{17}$$

Given a $\mathcal{D}$-module $\mathcal{F}$ on X, pull it back to $\pi^{-1}\mathcal{F}$ on Δ. Then take the direct image $\hat{\pi}_+\pi^{-1}\mathcal{F}$ on $\hat{X}$, namely those sections annihilated by the vertical vector field $\partial_\theta + \hat{\theta}\partial_z$.

Given line bundles on X and $\hat{X}$, the tensor product of their pullbacks to Δ gives a line bundle there. Does this construction give all line bundles on Δ? That is, can each line bundle on Δ be factored into such a tensor product? Ongay and I showed that there is an obstruction to such a factorization, which is an element of $H^2(X_{\text{red}}, \Lambda_{\text{ev}}^\times)$ [17].

6 Open Problems

Most of the classical theory of Riemann surfaces was extended to super Riemann surfaces during the 1980s, at least under the simplifying assumption that relevant cohomology groups were free modules. Much has now been further extended to general supercurves, and without restriction on the cohomology, but many interesting questions remain open. Many of the results would be closely related to the Hodge decomposition of cohomology on a Riemann surface, but on a supercurve this cannot hold in its naive form, e.g. $H^1(X, \Lambda) = H^{1,0}(X) \oplus H^{0,1}(X)$, because the left side is free while the right side may not be. Is there a generalization or substitute for Hodge decomposition?

There should be further relationships between line bundles on X, $\hat{X}$, and Δ. To what extent does the Jacobian of one of these curves determine those of the others?

Abel's Theorem for supercurves was proved in [2]. It states that a degree-zero divisor $\sum_i n_i \hat{P}_i$ is that of a meromorphic function F iff the associated linear functional $\sum_i n_i \int_{\hat{P}_0}^{\hat{P}_i}$ vanishes up to periods. Rothstein and I (work in progress) are working out the alternate proof along the lines of [10] by constructing the meromorphic differential that would be $D \log F$. This leads to greater insight on the existence of meromorphic and holomorphic differentials having prescribed periods on a supercurve. We also hope to prove a Jacobi Inversion theorem for supercurves. Naively this would say that every point in the Jacobian is the image under the Abel map of a divisor $\sum_{i=1}^{g}(\hat{P}_i - \hat{P}_0)$ with g the genus of X. This is not true without some conditions on the bundle $\mathcal{L}$: for example, the odd dimension of the Jacobian (of the underlying split supercurve), namely $h^1(X_{\mathrm{red}}, \mathcal{L})$, must not exceed g.

Should the Jacobian be a supermanifold? Because $H^1(X, \mathcal{O})$ is a quotient of a free module, the moduli space of line bundles is not a supermanifold in general. We have described it simply as a set of Λ-valued points, but this is not the set of Λ-valued points of a supermanifold. Tsuchimoto [21] showed one way around this problem: a moduli space of suitably framed line bundles is free, leading to a modified Jacobian which is a supermanifold but does not contain the full information as to which bundles are isomorphic. Is there a supervariety, or some sort of equivariant object, that can represent this functor of points?

Theta functions for supercurves need to be better understood. Such theta functions exist when the Jacobian is free, and are related to the super tau functions associated to supersymmetric integrable systems [2, 21]. They can also be constructed on super elliptic curves, for example

$$H(z, \theta) = \sum_{n \in \mathbb{Z}} \exp \pi i (2nz + n^2\tau + n\theta\epsilon + n^2\theta\epsilon + \frac{1}{3}n^3\delta\epsilon) \tag{18}$$

is such a theta function. By this I mean that it is invariant under the A transformation but acquires a phase linear in the coordinates under B:

$$H(z + \tau + \theta\epsilon, \theta + \delta) = H(z, \theta) \exp -\pi i (2z + \tau + 2\theta\epsilon + \frac{1}{3}\delta\epsilon). \tag{19}$$

This theta function is related to the Riemann theta function $\Theta(z; \tau)$ by

$$H(z, \theta) = \Theta(z + \frac{1}{2}\theta\epsilon; \tau + \theta\epsilon) - \frac{\delta\epsilon}{24\pi^2} \partial_z^3 \Theta(z; \tau). \tag{20}$$

One can define a theta subvariety of the Jacobian as the image by the Abel map of $(g-1)$-point divisors. Assuming free cohomology, it would be expected to have codimension $1|0$, making it a true theta divisor, if $h^1(X_{\mathrm{red}}, \mathcal{L}) = g-1$. Its properties are completely unexplored.

We have restricted ourselves to smooth supercurves in this paper. Very little is known about the classification or resolution of singular supercurves, let alone invertible sheaves on them.

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On the Superdimension of an Irreducible Representation of a Basic Classical Lie Superalgebra

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Abstract In this paper we prove the Kac-Wakimoto conjecture that a simple module over a basic classical Lie superalgebra has non-zero superdimension if and only if it has maximal degree of atypicality. The proof is based on the results of [Duflo and Serganova, On associated variety for Lie superalgebras, math/0507198] and [Gruson and Serganova, Proceedings of the London Mathematical Society, 101(3), 852–882, (2010)]. We also prove the conjecture in [Duflo and Serganova, On associated variety for Lie superalgebras, math/0507198] about the associated variety of a simple module and the generalized Kac-Wakimoto conjecture in [Geer, Kujawa and Patureau-Mirand, Generalized trace and modified dimension functions on ribbon categories, Selecta Math. **17**(2), 453–504 (2011)] for the general linear Lie superalgebra.

1 Preliminaries

In this paper all superalgebras are over the field $\mathbb{C}$ of complex numbers, $\mathfrak{g} = \mathfrak{gl}(m,n)$, $\mathfrak{osp}(2m+1,2n)$ or $\mathfrak{osp}(2m,2n)$. By $(\ ,\ )$ we denote the non-degenerate invariant symmetric form on $\mathfrak{g}$ defined by $(x,y) = \operatorname{str} xy$. This form identifies $\mathfrak{g}$ with $\mathfrak{g}^*$.

Fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ (it coincides with some Cartan subalgebra of $\mathfrak{g}_0$). The restriction of $(\ ,\ )$ to $\mathfrak{h}$ is non-degenerate and therefore we have a non-degenerate form on $\mathfrak{h}^*$ which we denote by the same symbol. We have $\dim\mathfrak{h} = m+n$. One can choose a basis $\varepsilon_1,...,\varepsilon_m,\delta_1,...,\delta_n$ of $\mathfrak{h}^*$ such that

$$(\varepsilon_i,\varepsilon_j) = \delta_{ij},\ (\varepsilon_i,\delta_j) = 0,\ (\delta_i,\delta_j) = -\delta_{ij}.$$

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Recall that $\mathfrak{g}$ has a *root decomposition*

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha,$$

where

$$\mathfrak{g}_\alpha = \{x \in \mathfrak{g} | [h, x] = \alpha(h)x \text{ for any } h \in \mathfrak{h}\}.$$

The set Δ is called the set of *roots*. Every root space $\mathfrak{g}_\alpha$ has dimension $(1|0)$ or $(0|1)$. Depending on the parity of $\mathfrak{g}_\alpha$ we call a root α even or odd. A root α is called *isotropic* if $(\alpha, \alpha) = 0$.

If $\mathfrak{g} = \mathfrak{gl}(m, n)$, then

$$\Delta_0 = \{\varepsilon_i - \varepsilon_j | 1 \le i, j \le m, i \ne j\} \cup \{\delta_i - \delta_j | 1 \le i, j \le n, i \ne j\},$$

$$\Delta_1 = \{\pm(\varepsilon_i - \delta_j) | 1 \le i \le m, 1 \le j \le n\}.$$

If $\mathfrak{g} = \mathfrak{osp}(2m, 2n)$, then

$$\begin{aligned}
\Delta_0 &= \{\pm(\varepsilon_i - \varepsilon_j), \pm(\varepsilon_i + \varepsilon_j) | 1 \le i < j \le m\} \\
&\quad \cup \{\pm(\delta_i - \delta_j), \pm(\delta_i + \delta_j), \pm 2\delta_i | 1 \le i < j \le n\}, \\
\Delta_1 &= \{\pm(\varepsilon_i - \delta_j), \pm(\varepsilon_i + \delta_j) | 1 \le i \le m, 1 \le j \le n\}.
\end{aligned}$$

If $\mathfrak{g} = \mathfrak{osp}(2m+1, 2n)$, then

$$\begin{aligned}
\Delta_0 &= \{\pm(\varepsilon_i - \varepsilon_j), \pm(\varepsilon_i + \varepsilon_j), \pm\varepsilon_i | 1 \le i < j \le m\} \\
&\quad \cup \{\pm(\delta_i - \delta_j), \pm(\delta_i + \delta_j), \pm 2\delta_i | 1 \le i < j \le n\}, \\
\Delta_1 &= \{\pm(\varepsilon_i - \delta_j), \pm(\varepsilon_i + \delta_j), \pm\delta_i | 1 \le i \le m, 1 \le j \le n\}.
\end{aligned}$$

We call

$$\Lambda = \bigoplus_{i=1}^{m} \mathbb{Z}\varepsilon_i \oplus \bigoplus_{j=1}^{n} \mathbb{Z}\delta_j$$

the *weight lattice* of $\mathfrak{g}$. Define a homomorphism $p : \Lambda \to \mathbb{Z}_2$ by putting $p(\varepsilon_i) = 0$ and $p(\delta_j) = 1$. Note that $\Delta \subset \Lambda$ and p is compatible with the parity of the roots.

A $\mathfrak{g}$-module M is a *weight module* if

$$M = \bigoplus_{\mu \in \Lambda} M_\mu,$$

where

$$M_\mu = \{m \in M | hm = \mu(h)m \text{ for all } h \in \mathfrak{h}\}.$$

Let $\mathcal{F}$ denote the category of all finite-dimensional weight modules M satisfying the additional condition

$$(M_{\mu})_{1} = 0, \text{ if } p(\mu) = 0, \ (M_{\mu})_{0} = 0, \text{ if } p(\mu) = 1. \tag{1}$$

The category $\mathcal{C}$ of all finite-dimensional weight $\mathfrak{g}$-modules is equivalent to the category of finite-dimensional modules of the algebraic supergroup $GL(m,n)$, $OSP(2m+1,2n)$ or $OSP(2m,2n)$ with Lie algebra $\mathfrak{g}$. It is not hard to see that $\mathcal{C} = \mathcal{F} \oplus \Pi(\mathcal{F})$, where Π is the change of parity functor. Usually this choice of parity of each weight space is not very important but in this paper it plays a crucial role in the proof of our main result.

In order to describe all simple objects of $\mathcal{F}$ we fix a decomposition $\Delta = \Delta^{+} \cup \Delta^{-}$ and the corresponding triangular decomposition

$$\mathfrak{g} = \mathfrak{n}^{-} \oplus \mathfrak{h} \oplus \mathfrak{n}^{+},$$

where

$$\mathfrak{n}^{\pm} = \bigoplus_{\alpha \in \Delta^{\pm}} \mathfrak{g}_{\alpha}.$$

Recall that $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^{+}$ is called a Borel subalgebra. Any root $\alpha \in \Delta^{+}$ can be written uniquely in the form

$$\alpha = \sum_{i=1}^{r} n_i \alpha_i,$$

where $\{\alpha_1, \ldots \alpha_r\}$ is the set of simple roots and $n_i \in \mathbb{Z}_{\geq 0}$. By $\leq$ we denote the partial order on Λ defined by

$$\mu \leq \lambda \text{ if } \lambda - \mu = \sum_{i=1}^{r} n_i \alpha_i, \ n_i \in \mathbb{Z}_{\geq 0}.$$

Let $\rho \in \mathfrak{h}^{*}$ be such that $2(\rho, \alpha_i) = (\alpha_i, \alpha_i)$ for all simple α_i. If

$$\rho_0 = \frac{1}{2} \sum_{\alpha \in \Delta_0^{+}} \alpha \text{ and } \rho_1 = \frac{1}{2} \sum_{\alpha \in \Delta_1^{+}} \alpha,$$

one can set $\rho = \rho_0 - \rho_1$, but if $\mathfrak{g} = \mathfrak{gl}(m,n)$ the choice of ρ is not unique.

For any $\lambda \in \Lambda$ let C_{λ} be the simple $\mathfrak{b}$-module with character λ of dimension $(1|0)$ if $p(\lambda) = 0$ and $(0|1)$ if $p(\lambda) = 1$. A *Verma module* with highest weight λ is by definition the induced module

$$M(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} C_{\lambda}.$$

It has a unique irreducible quotient $L(\lambda)$. We call λ *dominant* if $L(\lambda)$ is finite-dimensional. Every simple object of $\mathcal{F}$ is isomorphic to $L(\lambda)$ or $\Pi(L(\lambda))$ for some λ.

Let $Z(\mathfrak{g})$ denote the center of the universal enveloping algebra $U(\mathfrak{g})$. Fix a triangular decomposition

$$\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+.$$

By the Poincare–Birkhoff–Witt theorem

$$U(\mathfrak{g}) = U(\mathfrak{n}^-) \otimes U(\mathfrak{h}) \otimes U(\mathfrak{n}^+).$$

Let $\phi : U(\mathfrak{g}) \to U(\mathfrak{h})$ denote the projection with kernel $\mathfrak{n}^- U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{n}^+$. Then the restriction of ϕ to $Z(\mathfrak{g})$ is an injective homomorphism of rings. It is called the Harish–Chandra homomorphism. Since $\mathfrak{h}$ is an abelian Lie algebra, $U(\mathfrak{h}) \simeq S(\mathfrak{h})$ can be considered as the algebra of polynomial functions on $\mathfrak{h}^*$. For any $\lambda \in \mathfrak{h}^*$ define the homomorphism $\chi_\lambda : Z(\mathfrak{g}) \to \mathbb{C}$ by

$$\chi_\lambda(z) = \phi(z)(\lambda).$$

It is easy to see that for any $\lambda \in \mathfrak{h}^*$, the center $Z(\mathfrak{g})$ acts via χ_λ on the Verma module $M(\lambda)$, i.e.

$$zm = \chi_\lambda(z)m$$

for any $z \in Z(\mathfrak{g})$ and $m \in M(\lambda)$.

In order to describe the image of ϕ we define

$$Z(\lambda) = \{\mu \in \mathfrak{h}^* | \chi_\lambda = \chi_\mu\}.$$

Let $A(\lambda)$ be a maximal set of mutually orthogonal linearly independent isotropic roots α such that $(\lambda + \rho, \alpha) = 0$. Let W denote the Weyl group of $\mathfrak{g}_0$ and the shifted action of W on $\mathfrak{h}^*$ be defined by

$$w \cdot \lambda = w(\lambda + \rho) - \rho.$$

The following theorem follows from the results of [5, 8, 11].

Theorem 1.1.

$$Z(\lambda) = \cup_{w \in W} w \cdot (\lambda + \oplus_{\alpha \in A(\lambda)} \mathbb{C}\alpha).$$

Let $\chi : Z(\mathfrak{g}) \to \mathbb{C}$ be some central character. Denote by $\mathcal{F}^\chi$ (resp. $\mathcal{C}^\chi$) the subcategory of $\mathcal{F}$ (resp. $\mathcal{C}$) consisting of all modules which admit the generalized central character χ. We have the decompositions

$$\mathcal{F} = \bigoplus \mathcal{F}^\chi, \ \mathcal{C} = \bigoplus \mathcal{C}^\chi.$$

The *defect* (notation $\operatorname{def} \mathfrak{g}$) is the maximal number of linearly independent mutually orthogonal isotropic roots. (In our case $\operatorname{def} \mathfrak{g} = \min\{m, n\}$.)

For any weight λ the *degree of atypicality* (notation $\operatorname{at}(\lambda)$) is the cardinality of $A(\lambda)$. Obviously $\operatorname{at}(\lambda) \leq \operatorname{def} \mathfrak{g}$. If $\mu \in Z(\lambda)$ then $\operatorname{at}(\lambda) = \operatorname{at}(\mu)$. Hence for any

$\chi : Z(\mathfrak{g}) \to \mathbb{C}$, $\mathrm{at}(\chi)$ is well-defined. A weight λ and a central character χ are *typical* if $\mathrm{at}(\lambda) = 0$ and $\mathrm{at}(\chi) = 0$.

For any vector superspace V set $\mathrm{sdim}(V) = \dim V_0 - \dim V_1$.

Conjecture 1.2. ([9] Conjecture 3.1). Let $\lambda \in \Lambda$ be a dominant weight. Then $\mathrm{sdim}(L(\lambda)) \neq 0$ if and only if $\mathrm{at}(\lambda) = \mathrm{def}\,\mathfrak{g}$.

2 The Fiber Functor

Here we recall some results of [2]. Let

$$X = \{x \in \mathfrak{g}_1 \mid [x, x] = 0\}$$

be the cone of self-commuting elements. Let G_0 denote an algebraic group with Lie algebra $\mathfrak{g}_0$. Clearly X is invariant under the adjoint action of G_0. It was shown in [2] that for any $x \in X$ there exist $g \in G_0$ and isotropic mutually orthogonal linearly independent roots $\alpha_1, \dots, \alpha_k$ such that $\mathrm{Ad}_g(x) = x_1 + \dots + x_k$, with $x_i \in \mathfrak{g}_{\alpha_i}$. The number k does not depend on the choice of g, we call it *the rank* of x (notation $\mathrm{rk}\,(x) = k$). Let

$$X_k = \{x \in X \mid \mathrm{rk}\,(x) = k\}\,.$$

If s is the defect of $\mathfrak{g}$, then $X = X_0 \cup \dots \cup X_s$ is a stratification of X such that $\bar{X}_k = X_0 \cup \dots \cup X_k$.

Let M be an arbitrary $\mathfrak{g}$-module. Define the *fiber* M_x as the cohomology of x in M

$$M_x = \mathrm{Ker}\,x / \mathrm{Im}\,x.$$

The *associated variety* X_M is by definition the set of all $x \in X$ such that $M_x \neq 0$.

Let $\mathfrak{g}_x = C_{\mathfrak{g}}(x)/[x, \mathfrak{g}]$, where $C_{\mathfrak{g}}(x)$ denotes the centralizer of x in $\mathfrak{g}$. Since $\mathrm{Ker}\,x$ is $C_{\mathfrak{g}}(x)$-invariant and $[x, \mathfrak{g}]\,\mathrm{Ker}\,x \subset \mathrm{Im}\,x$, M_x is a $\mathfrak{g}_x$-module. Thus, $M \to M_x$ defines a functor from the category of $\mathfrak{g}$-modules to the category of $\mathfrak{g}_x$-modules which we call the *fiber functor*.

Let $x = x_1 + \dots + x_k$ as above. If $\mathfrak{h}_\alpha = [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ then

$$\mathfrak{h}_x = (\ker\alpha_1 \cap \dots \cap \ker\alpha_k)/(\mathfrak{h}_{\alpha_1} \oplus \dots \oplus \mathfrak{h}_{\alpha_k}) \tag{2}$$

is a Cartan subalgebra of $\mathfrak{g}_x$ and

$$\Delta_x = \{\alpha \in \Delta | (\alpha, \alpha_i) = 0, \alpha \neq \pm\alpha_i, i = 1, \dots, k\}$$

is the set of roots of $\mathfrak{g}_x$.

If $\mathrm{rk}(x) = \mathrm{rk}(y)$, then $\mathfrak{g}_x$ and $\mathfrak{g}_y$ are isomorphic. If $\mathrm{rk}(x) = k$ and $\mathfrak{g} = \mathfrak{gl}(m, n)$, $\mathfrak{osp}(2m+1, 2n)$ or $\mathfrak{osp}(2m, 2n)$ then $\mathfrak{g}_x \simeq \mathfrak{gl}(m-k, n-k)$, $\mathfrak{osp}(2(m-k)+1, 2(n-k))$ or $\mathfrak{osp}(2(m-k), 2(n-k))$ respectively.

Let $U(\mathfrak{g})^x$ be the subalgebra of ad_x-invariants in $U(\mathfrak{g})$. Then $[x, U(\mathfrak{g})]$ is an ideal in $U(\mathfrak{g})^x$ and we have the canonical isomorphism

$$U(\mathfrak{g}_x) \simeq U(\mathfrak{g})^x / [x, U(\mathfrak{g})].$$

Denote by θ the natural projection: $U(\mathfrak{g})^x \to U(\mathfrak{g}_x)$. If $y \in U(\mathfrak{g})^x$ and $m \in \operatorname{Ker} x$, then $xym = yxm = 0$. Hence $\operatorname{Ker} x$ is $U(\mathfrak{g})^x$-invariant. Moreover, if $y = [x, z]$ for some $z \in U(\mathfrak{g})$, then $ym = xzm$. Hence $[x, U(\mathfrak{g})] \operatorname{Ker} x \subset \operatorname{Im} x$. Therefore we have

$$ym = \theta(y)\, m \mod \operatorname{Im} x,$$

for any $y \in U(\mathfrak{g})^x$ and $m \in \operatorname{Ker} x$. Let $Z(\mathfrak{g})$, $Z(\mathfrak{g}_x)$ be the centers of $U(\mathfrak{g})$ and $U(\mathfrak{g}_x)$ respectively. The restriction $\theta : Z(\mathfrak{g}) \to Z(\mathfrak{g}_x)$ is a homomorphism of rings. Denote by $\check{\theta}$ the dual map $\operatorname{Hom}(Z(\mathfrak{g}_x), \mathbb{C}) \to \operatorname{Hom}(Z(\mathfrak{g}), \mathbb{C})$. Suppose that M admits a central character χ. Then for any $z \in Z(\mathfrak{g})$ and $m \in \operatorname{Ker} x$ we have

$$\chi(z)\, m = zm = \theta(z)\, m \mod \operatorname{Im} x.$$

Hence if M_x contains a submodule which admits a central character $\xi \in \operatorname{Hom}(Z(\mathfrak{g}_x), \mathbb{C})$, then $\check{\theta}(\xi) = \chi$.

Now we are going to describe $\check{\theta}$. For this we again assume that $x = x_1 + \cdots + x_k$. It is not difficult to check case by case that one can always find a Borel subalgebra such that $\alpha_1, \ldots, \alpha_k$ are simple. We use the Harish–Chandra homomorphisms $\phi : Z(\mathfrak{g}) \to S(\mathfrak{h})$ and $\phi_x : Z(\mathfrak{g}_x) \to S(\mathfrak{h}_x)$ associated to the given choice of Borel subalgebra. Note that (2) implies

$$\mathfrak{h}_x^* = (\mathbb{C}\alpha_1 \oplus \cdots \oplus \mathbb{C}\alpha_k)^{\perp} / (\mathbb{C}\alpha_1 \oplus \cdots \oplus \mathbb{C}\alpha_k).$$

Let

$$\pi : (\mathbb{C}\alpha_1 \oplus \cdots \oplus \mathbb{C}\alpha_k)^{\perp} \to \mathfrak{h}_x^*$$

denote the natural projection. By Theorem 1.1 if $\nu, \nu' \in \pi^{-1}(\mu)$ then $\chi_\nu = \chi_{\nu'}$. We claim that $\check{\theta}(\chi_\mu) = \chi_\nu$ for $\nu \in \pi^{-1}(\mu)$. Indeed, let M be the quotient of the Verma module $M(\nu)$ by the submodule generated by $\mathfrak{g}_{-\alpha_1} v, \ldots, \mathfrak{g}_{-\alpha_k} v$ (v stands for the highest vector). Then $v \in M_x$ and therefore M_x contains the Verma module over $\mathfrak{g}_x$ with highest weight μ. Since this Verma module admits the central character χ_μ, we have $\chi_\nu = \check{\theta}(\chi_\mu)$.

The above implies in particular that

$$\operatorname{at}\left(\check{\theta}(\xi)\right) = \operatorname{at}(\xi) + k. \tag{3}$$

Thus, if $\operatorname{at}(\chi) < k$, $\check{\theta}^{-1}(\chi) = \varnothing$. Therefore if M admits a central character with degree of atypicality less than $\operatorname{def} \mathfrak{g}$, then $M_x = 0$ for $x \in X$ of maximal rank. Since obviously $\operatorname{sdim} M_x = \operatorname{sdim} M$ we obtain a proof of Conjecture 1.2 in one direction.

All above also implies the following

Theorem 2.1. *If* $\operatorname{at}(\chi) < \operatorname{rk}(x)$*, then the restriction of the fiber functor to* $\mathcal{F}^{\chi}$ *is zero. Otherwise the fiber functor maps* $\mathcal{F}^{\chi}$ *to the direct sum* $\oplus\mathcal{C}^{\xi}(\mathfrak{g}_x)$*, where* ξ *runs over the set* $\check{\theta}^{-1}(\chi)$ *and* $\mathcal{C}^{\xi}(\mathfrak{g}_x)$ *is the obvious analogue of* $\mathcal{C}^{\chi}$ *for* $\mathfrak{g}_x$ *.*

Now assume that $\operatorname{at}(\chi) = \operatorname{rk}(x) = k$. If $\mathfrak{g} = \mathfrak{gl}(m,n)$ or $\mathfrak{g} = \mathfrak{osp}(2m+1,2n)$ the preimage $\check{\theta}^{-1}(\chi)$ is a single point set [2].

Let $\mathfrak{g} = \mathfrak{osp}(2m,2n)$. Denote by σ the involutive automorphism which preserves the Cartan subalgebra and which acts on the weights by

$$\sigma(\varepsilon_m) = -\varepsilon_m,\ \sigma(\varepsilon_i) = \varepsilon_i \text{ for } i \le m-1$$

and

$$\sigma(\delta_j) = \delta_j \text{ for all } j \le n.$$

A weight

$$\lambda = a_1\varepsilon_1 + \cdots a_m\varepsilon_m + b_1\delta_1 + \cdots + b_n\delta_n$$

is called *positive* if $a_i, b_j \ge 0$. If λ is integral dominant, then λ or $\sigma(\lambda)$ is positive. It is also clear that the action of σ extends to the category $\mathcal{F}$ and to the set of central characters. In the case $\mathfrak{g} = \mathfrak{osp}(2m,2n)$, the preimage $\check{\theta}^{-1}(\chi) = \{\xi, \sigma(\xi)\}$ may have two elements.

By Theorem 2.1 if ξ is typical, then $\mathcal{F}^{\xi}$ is semisimple and has one up to isomorphism simple object. Hence we obtain the following corollary.

Corollary 2.2. *Let* $\operatorname{at}(\chi) = \operatorname{rk}(x)$.

Let $\mathfrak{g} \neq \mathfrak{osp}(2m|2n)$*. Then there exists a typical simple* $\mathfrak{g}_x$*-module* $L_{\mathfrak{g}_x}(\mu(\chi))$ *such that for any* M *in* $\mathcal{F}^{\chi}$*,*

$$M_x \simeq L_{\mathfrak{g}_x}(\mu(\chi)) \otimes C_x(M),$$

where $C_x(M)$ *is a superspace with trivial* $\mathfrak{g}_x$*-action.*

Let $\mathfrak{g} = \mathfrak{osp}(2m|2n)$*, then there exists a typical simple* $\mathfrak{g}_x$*-module* $L_{\mathfrak{g}_x}(\mu(\chi))$ *with positive* $\mu(\chi)$ *such that*

$$M_x \simeq L_{\mathfrak{g}_x}(\mu(\chi)) \otimes C'_x(M) \oplus L^{\sigma}_{\mathfrak{g}_x}(\mu(\chi))) \otimes C''_x(M),$$

where $C'_x(M)$*,* $C''_x(M)$ *are superspaces with trivial* $\mathfrak{g}_x$*-action.*

Note that the weight $\mu(\chi)$ is always a typical dominant weight of $\mathfrak{g}_x$. We call it the *core* of the central character χ.

If $\mathfrak{g} = \mathfrak{osp}(2m|2n)$, set $C_x(M) = C'_x(M) \oplus C''_x(M)$. Our goal is to prove the following

Theorem 2.3. *Let* $\operatorname{at}(\chi) = \operatorname{rk}(x)$ *and* M *be a simple* $\mathfrak{g}$*-module with central character* χ*, then* $\operatorname{sdim}(C_x(M)) \neq 0$.

We have shown already that Conjecture 1.2 holds in one direction. Now we will show that Theorem 2.3 implies Conjecture 1.2 in the other direction.

Lemma 2.4. *If Theorem 2.3 is true then* $\operatorname{sdim}(L(\lambda)) \neq 0$ *for any dominant weight* λ *whose degree of atypicality equals* $\operatorname{def}\mathfrak{g}$.

Proof. If $\operatorname{rk}(x) = \operatorname{def}\mathfrak{g}$, then $\mathfrak{g}_x$ does not have isotropic roots. Therefore $\mathfrak{g}_x$ is either a Lie algebra or is isomorphic to $\mathfrak{osp}(1, 2q)$. In both cases $\operatorname{sdim}(L_{\mathfrak{g}_x}(\mu(\chi)) \neq 0$ and the statement follows from Corollary 2.2. Let us mention that in the typical case Theorem 2.3 follows Proposition 2.10 in [7]. In particular, it implies the result for $\mathfrak{osp}(1, 2q)$. □

Note also that Theorem 2.3 implies Conjecture 5.5 in [2].

Corollary 2.5. *Let* M *be a simple* $\mathfrak{g}$*-module and* k *be the degree of atypicality of* M*. Then* $X_M = \bar{X}_k$.

3 Translation Functors and Weight Diagrams

In the rest of the paper we fix a choice of a Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$. We list below the simple roots in each case.

- If $\mathfrak{g} = \mathfrak{gl}(m, n)$, $m \geq n$, the simple roots are

$$\varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3, \ldots, \varepsilon_m - \delta_1, \delta_1 - \delta_2, \ldots, \delta_{n-1} - \delta_n,$$

$$\rho = m\varepsilon_1 + \cdots + \varepsilon_m - \delta_1 - \cdots - n\delta_n;$$

- If $\mathfrak{g} = \mathfrak{osp}(2m + 1, 2n)$ and $m \geq n$, the simple roots are

$$\varepsilon_1 - \varepsilon_2, \ldots, \varepsilon_{m-n+1} - \delta_1, \delta_1 - \varepsilon_{m-n+2}, \ldots, \varepsilon_m - \delta_n, \delta_n,$$

$$\rho = -\frac{1}{2}\sum_{i=1}^{m} \varepsilon_i + \frac{1}{2}\sum_{j=1}^{n} \delta_j + \sum_{i=1}^{m-n} (m - n - i + 1)\varepsilon_i;$$

- If $\mathfrak{g} = \mathfrak{osp}(2m + 1, 2n)$ and $m < n$, the simple roots are

$$\delta_1 - \delta_2, \ldots, \delta_{n-m} - \varepsilon_1, \varepsilon_1 - \delta_{n-m+1}, \ldots, \varepsilon_m - \delta_n, \delta_n,$$

$$\rho = -\frac{1}{2}\sum_{i=1}^{m} \varepsilon_i + \frac{1}{2}\sum_{j=1}^{n} \delta_j + \sum_{j=1}^{n-m} (n - m - j)\delta_j;$$

- If $\mathfrak{g} = \mathfrak{osp}(2m, 2n)$ and $m > n$, the simple roots are

$$\varepsilon_1 - \varepsilon_2, \ldots, \varepsilon_{m-n} - \delta_1, \delta_1 - \varepsilon_{m-n+1}, \ldots, \delta_n - \varepsilon_m, \delta_n + \varepsilon_m,$$

$$\rho = \sum_{i=1}^{m-n} (m - n - i)\varepsilon_i;$$

- If $\mathfrak{g} = \mathfrak{osp}(2m, 2n)$ and $m \leq n$, the simple roots are

$$\delta_1 - \delta_2, ..., \delta_{n-m+1} - \varepsilon_1, \varepsilon_1 - \delta_{n-m+2}, ..., \delta_n - \varepsilon_m, \delta_n + \varepsilon_m,$$

$$\rho = \sum_{i=1}^{n-m} (n - m - i + 1)\delta_i.$$

It is convenient to describe dominant weights in terms of weight diagrams. The language of weight diagrams was introduced in [1] for $\mathfrak{gl}(m, n)$ and was extended to the orthosymplectic case in [6]. Let $\mathbb{T} \subset \mathbb{R}$ be a set, $X = (x_1, ..., x_m) \in \mathbb{T}^m$, $Y = (y_1, ..., y_n) \in \mathbb{T}^n$. A diagram $f_{X,Y}$ is a function defined on $\mathbb{T}$ whose values are multisets with elements $<, >, \times$ according to the following algorithm.

- Put the symbol $>$ in position t for all i such that $x_i = t$.
- Put the symbol $<$ in position t for all i such that $y_i = t$.
- If there are both $>$ and $<$ in the same position replace them by the symbol $\times$, repeat if possible.

By definition $f_{X,Y}(t)$ may contain at most one of the two symbols $>, <$. We represent $f_{X,Y}$ by the picture with 0 standing in position t whenever $f(t)$ is an empty set.

Let $\mathfrak{g} = \mathfrak{gl}(m, n)$. Let λ be a dominant integral weight such that

$$\lambda + \rho = a_1\varepsilon_1 + ... + a_m\varepsilon_m + b_1\delta_1 + ... + b_n\delta_n.$$

Set $\mathbb{T} = \mathbb{Z}$,

$$X_\lambda = (a_1, ..., a_m), Y_\lambda = (-b_1, ..., -b_n).$$

The diagram $f_\lambda = f_{X_\lambda, Y_\lambda}$ is called the *weight diagram* of λ.

A diagram is the weight diagram of some dominant weight if and only if $f(t)$ is empty or is a single element set since both sequences $a_1, ..., a_m$ and $b_1, ..., b_n$ are strictly decreasing and hence do not have repetitions.

Each dominant weight is uniquely determined by its weight diagram. The number of $<$ is n, the number of $>$ is m (counting $\times$ as both $<$ and $>$).

Now let $\mathfrak{g} = \mathfrak{osp}(2m, 2n)$. Set $\mathbb{T} = \mathbb{Z}_{\geq 0}$. For a dominant weight λ such that $\lambda + \rho = a_1\varepsilon_1 + ... + a_m\varepsilon_m + b_1\delta_1 ... + b_n\delta_n$ let

$$X_\lambda = (|a_1|, ..., |a_m|), Y_\lambda = (b_1, ..., b_n), f_\lambda = f_{X_\lambda, Y_\lambda}.$$

Dominance of λ implies that $|a_1| \geq ... \geq |a_m|$ and $b_1 \geq ... \geq b_n$. It is not difficult to see that f_λ is a weight diagram of a dominant λ if and only if

- For any $t \neq 0$, $f_\lambda(t)$ is empty or a single element set;
- The multiset $f_\lambda(0)$ does not contain $<$, contains $>$ with multiplicity at most 1 (it may contain any number of $\times$).

Any integral dominant weight is determined by its weight diagram uniquely up to the action of σ.

Finally let $\mathfrak{g} = \mathfrak{osp}(2m+1, 2n)$. All coordinates a_i, b_j of $\lambda + \rho$ belong to $-\frac{1}{2} + \mathbb{Z}_{\geq 0}$. Let $\mathbb{T} = \frac{1}{2} + \mathbb{Z}_{\geq 0}$ and define X_λ, Y_λ and f_λ as in the case $\mathfrak{g} = \mathfrak{osp}(2m, 2n)$. The dominance condition is equivalent to the following condition on a weight diagram f

- $f(t)$ is empty or a single element set for any $t \neq \frac{1}{2}$;
- $f(\frac{1}{2})$ may contain at most one of $<$ or $>$ and any number of $\times$.

As in the previous case, it is possible that two dominant weights have the same weight diagram. That may happen if $f(\frac{1}{2})$ does not contain $>$ or $<$ and has at least one $\times$. For example, the diagram with two $\times$ at $\frac{1}{2}$ corresponds to $(\frac{1}{2}, -\frac{1}{2}|\frac{1}{2}, \frac{1}{2})$ and to $(-\frac{1}{2}, -\frac{1}{2}|\frac{1}{2}, \frac{1}{2})$. So if the weight diagram has at least one $\times$ and no $<, >$ at the position $\frac{1}{2}$ we put an indicator (which we sometimes refer to as "sign") $\pm$ before the weight diagram in parentheses. Its value is $+$ if the corresponding weight has the form

$$\lambda + \rho = (a_1, ..., a_{m-s}, \frac{1}{2}, -\frac{1}{2}, ..., -\frac{1}{2}|b_1, ..., b_n),$$

and $-$ if the corresponding weight has the form

$$\lambda + \rho = (a_1, ..., a_{m-s}, -\frac{1}{2}, -\frac{1}{2}, ..., -\frac{1}{2}|b_1, ..., b_n),$$

where s is the number of crosses at the position $\frac{1}{2}$.

The number of $\times$ in f_λ equals the degree of atypicality at(λ). Replacing all $\times$ in the diagram by zeros gives a diagram of the weight $\mu(\chi_\lambda) - \rho'$, where ρ' is is defined by the conditions $(\rho', \varepsilon_i) = (\rho, \varepsilon_{i+k})$, $(\rho', \delta_i) = (\rho, \delta_{i+k})$, and $k = \text{at}(\lambda)$. In particular, two dominant weights have the same central character iff their diagrams coincide after replacing all $\times$ by zeros.

Example 3.1. (1) Let $\mathfrak{g} = \mathfrak{gl}(3,2)$ and $\lambda = 0$. Then

$$f_\lambda = \cdots, 0, \times, \times, >, 0, \cdots,$$

where the position of the left $\times$ is 1.

(2) Let $\mathfrak{g} = \mathfrak{osp}(5,4)$ and $\lambda = 0$. Then

$$f_\lambda = (-)^{\times}_{\times}, 0, \cdots,$$

where the position of the left $\times$ is $\frac{1}{2}$.

Let E be the natural or conatural representation of $\mathfrak{g}$. A *translation functor* $T_{\chi,\eta} : \mathcal{F}^\chi \to \mathcal{F}^\eta$ is defined by

$$T_{\chi,\eta}(M) = (M \otimes E)^\eta,$$

where $(V)^\eta$ denote the projection of V to the block $\mathcal{F}^\eta$. The left adjoint of $T_{\chi,\eta}$ is $T_{\eta,\chi}$ defined by

$$T_{\eta,\chi}(M) = (M \otimes E^*)^\chi.$$

The following theorem is a slight generalization of Lemma 11 in [6].

Theorem 3.2. *Let χ and η be central characters with the same degree of atypicality. Assume that $f_{\mu(\eta)}$ is obtained from $f_{\mu(\chi)}$ by moving one symbol $<$ or $>$ from position t to position $t+1$.*

(i) Assume that $t \neq 0$ if $\mathfrak{g} = \mathfrak{osp}(2m, 2n)$ and $t \neq \frac{1}{2}$ if $\mathfrak{g} = \mathfrak{osp}(2m+1, 2n)$. Then $T_{\chi,\eta}$ is an equivalence of categories with the inverse functor $T_{\eta,\chi}$. Furthermore $T_{\chi,\eta}(L(\lambda)) = L(\nu)$ such that f_ν is obtained from f_λ by exchanging symbols in positions t and $t+1$.

(ii) If $\mathfrak{g} = \mathfrak{osp}(2m+1, 2n)$ and $f_{\mu(\eta)}$ is obtained from $f_{\mu(\chi)}$ by moving $>$ or $<$ from $\frac{1}{2}$ to $\frac{3}{2}$, then $T_{\chi,\eta}$ is an equivalence of categories ($T_{\eta,\chi}$ being the inverse functor), and $T_{\chi,\eta}(L(\lambda)) = L(\nu)$ where f_ν is obtained from f_λ by exchanging symbols in positions $\frac{1}{2}$ and $\frac{3}{2}$ and the sign of f_ν is $+$ if $f_\lambda(\frac{3}{2}) = \times$ and $-$ if $f_\lambda(\frac{3}{2}) = 0$.

(iii) If $\mathfrak{g} = \mathfrak{osp}(2m, 2n)$ and $f_{\mu(\eta)}$ is obtained from $f_{\mu(\chi)}$ by moving $>$ from position 0 to position 1, then $T_{\chi,\eta}(L(\lambda)) = L(\nu)$ or $L(\nu) \oplus L^\sigma(\nu)$, where f_ν is obtained from f_λ by exchanging symbols in positions 0 and 1. For the left adjoint functor $T_{\eta,\chi}$ we have $T_{\eta,\chi}(L(\nu)) = T_{\eta,\chi}(L^\sigma(\nu)) = L(\lambda)$.

Proof. The first two statements are proven in [6]. To prove (iii) let q be the number of $\times$ at position 0 in f_λ. Consider two cases $f_\lambda(1) = \times$ and $f_\lambda(1) = 0$. In the first case γ is a weight of E, $\lambda + \gamma$ is dominant and $\chi_{\lambda+\gamma} = \eta$ iff $\gamma = -\varepsilon_{m-q-1}$. So in this case by Lemma 10 from [6] we deduce that $T_{\chi,\eta}(L(\lambda)) = L(\nu)$ with $\nu = \lambda + \gamma$. In the second case if $q \neq 0$ then $\lambda + \gamma$ is dominant and $\chi_{\lambda+\gamma} = \eta$ iff $\gamma = \varepsilon_{m-q}$ and we again have $T_{\chi,\eta}(L(\lambda)) = L(\nu)$ with $\nu = \lambda + \gamma$. If $q = 0$ then γ takes two values $\pm\varepsilon_m$. Since $L(\lambda)^\sigma = L(\lambda)$, $L^\sigma(\lambda + \varepsilon_m) = L(\lambda - \varepsilon_m)$ and $E^\sigma = E$, the selfduality of $T_{\chi,\eta}(L(\lambda))$ implies either $T_{\chi,\eta}(L(\lambda)) = 0$ or

$$T_{\chi,\eta}(L(\lambda)) = L(\lambda - \varepsilon_m) \oplus L(\lambda + \varepsilon_m).$$

It remains to show that $T_{\chi,\eta}(L(\lambda)) \neq 0$. That can be done by repeating the argument in the proof of Lemma 10 in [6]. The statement about the adjoint functor can be proved by the same argument. □

Example 3.3. Let $\mathfrak{g} = \mathfrak{osp}(6, 4)$ and $\lambda = (3, 2, 0|3, 2)$. Then

$$f_\lambda = \ >, 0, \times, \times, 0, \cdots .$$

Moving $>$ one position to the right corresponds to the translation functor $T_{\chi,\eta}(L(\lambda)) = L(\nu_1) \oplus L(\nu_2)$ with $\nu_1 = (3, 2, 1|3, 2)$ and $\nu_2 = (3, 2, -1|3, 2)$.

On the other hand, if $\lambda = (2, 1, 0|2, 1)$ then

$$f_\lambda = \ >, \times, \times, 0, \cdots .$$

If $T_{\chi,\eta}$ is the translation functor that corresponds to moving the $>$ one position to the right, then $T_{\chi,\eta}(L(\lambda)) = L(\nu)$ with $\nu = (2, 1, 0|2, 0)$.

4 Reduction to the Equal Rank Case

Suppose that at $\chi = k$. Let

- $\mathfrak{g}_k = \mathfrak{gl}(k|k)$ if $\mathfrak{g} = \mathfrak{gl}(m|n)$;
- $\mathfrak{g}_k = \mathfrak{osp}(2k+1|2k)$ if $\mathfrak{g} = \mathfrak{osp}(2m+1|2n)$;
- $\mathfrak{g}_k = \mathfrak{osp}(2k|2k)$ if $\mathfrak{g} = \mathfrak{osp}(2m|2n)$.

Note that if $\mathfrak{g} = \mathfrak{g}_k$ and $x \in X$ has rank k, then $\mathfrak{g}_x = 0$.

It is not difficult to check that for any root preserving embedding $\mathfrak{g}_k \subset \mathfrak{g}$ any G_0-orbit of X_k meets $\mathfrak{g}_k$.

Let λ be an integrable dominant weight. We call λ and $L(\lambda)$ *stable* if all $\times$ in f_λ stand to the left of all symbols $<$ and $>$.

The following statement is a direct consequence of Theorem 3.2.

Corollary 4.1. *For any $L(\lambda) \in \mathcal{F}^\chi$ there exists a central character ζ and a functor $T : \mathcal{F}^\chi \to \mathcal{F}^\zeta$ which is a composition of translation functors satisfying the conditions of Theorem 3.2 such that $T(L(\lambda))$ is a stable simple module or a direct sum of two stable simple modules (invariant under the action of σ).*

Lemma 4.2. *Let λ be stable and λ' be the weight of $\mathfrak{g}_k$ whose weight diagram is obtained from that of λ by removing all symbols $>$ and $<$. If we denote by $L_k(\lambda')$ the simple $\mathfrak{g}_k$-module with highest weight λ', then for any $x \in X_k \cap \mathfrak{g}_k$*

$$C_x(L(\lambda)) = C_x(L_k(\lambda')).$$

Proof. For short put $\lambda = \chi_\lambda$, $L = L(\lambda)$. The stability of λ implies that $A(\lambda) = \{\alpha_1, \dots, \alpha_k\}$ where each α_i is a root of $\mathfrak{g}_k$.

Let $x = x_1 + \dots + x_k$ with $x_i \in \mathfrak{g}_{\alpha_i} \subset \mathfrak{g}_k$. If $y_i \in \mathfrak{g}_{-\alpha_i}$, $y = y_1 + \cdots + y_k$, $h = [x, y]$, then x, y and h span a subalgebra isomorphic to $\mathfrak{sl}(1, 1)$. One can choose y_i in generic way so that $\nu(h) = 0$ implies $(\nu, \alpha_1) = \cdots = (\nu, \alpha_k) = 0$ for all $\nu \in \Lambda$. If $m \in \operatorname{Ker} x$ and $hm = cm$ for some $c \neq 0$, then $m = x(\frac{ym}{c}) \in \operatorname{Im} x$. Therefore $L_x \simeq L_x^h$, where $L^h = \operatorname{Ker} h$.

Let $m \in L_x$ be a highest vector with respect to $\mathfrak{b} \cap \mathfrak{g}_x$. Let

$$\pi : (\mathbb{C}\alpha_1 \oplus \cdots \oplus \mathbb{C}\alpha_k)^\perp \to \mathfrak{h}_x^*, \ \ \zeta : \operatorname{Ker} x \to L_x$$

be the natural projections. Then by above one can find

$$\tilde{m} \in \bigoplus_{\nu \in \pi^{-1}(\mu(\chi))} L_\nu$$

such that $\zeta(\tilde{m}) = m$.

Since $A(\lambda) = \{\alpha_1, \ldots, \alpha_k\}$ we have

$$\mu(\chi) = \pi(\lambda + \rho) - \pi(\rho).$$

Hence

$$\tilde{m} \in \bigoplus_{\nu \in \lambda + \mathbb{C}\alpha_1 + \cdots + \mathbb{C}\alpha_k} L_\nu.$$

That implies $\tilde{m} \in U(\mathfrak{g}_k)v$, where v is the highest vector of L. Therefore we obtain

$$L_x \simeq L_k(\lambda')_x \otimes L_{\mathfrak{g}_x}(\mu(\chi)).$$

Hence the statement. ◻

The following Lemma is obvious. We leave its proof as an exercise.

Lemma 4.3. *For any $\mathfrak{g}$-modules M and N we have $(M \otimes N)_x = M_x \otimes N_x$.*

It has the following important corollary.

Corollary 4.4. *Let $T_{\chi,\eta} : \mathcal{F}^\chi \to \mathcal{F}^\eta$ be a translation functor then for any $M \in \mathcal{F}^\chi$*

$$(T_{\chi,\eta}(M))_x = (M_x \otimes E_x)^{\check{\theta}^{-1}(\eta)},$$

where $V^{\check{\theta}^{-1}(\eta)}$ denotes the projection of a $\mathfrak{g}_x$-module V onto the sum

$$\bigoplus_{\xi \in \check{\theta}^{-1}(\eta)} \mathcal{C}^\xi.$$

In particular, if $\mathrm{at}(\chi) = \mathrm{at}(\eta) = \mathrm{rk}(x)$ *and* $(T_{\chi,\eta}(M)) = N$*, where both M and N are simple, then*

$$C_x(M) \simeq C_x(N). \tag{4}$$

The following lemma reduces the proof of Theorem 2.3 to the case $\mathfrak{g} = \mathfrak{g}_k$.

Lemma 4.5. *If Theorem 2.3 holds for $\mathfrak{g}_k$ when $\mathrm{at}\chi = k$, then it holds for all $\mathfrak{g}$.*

Proof. Lemma 4.2 implies the statement in the case when $L(\lambda)$ is stable. Corollary 4.4 and Corollary 4.1 imply the statement for all λ. ◻

5 Proof in the Equal Rank Case

We assume now that $\mathfrak{g} = \mathfrak{g}_k$, at $(\chi) = k$ and $x \in X_k$. In this case $\chi = \chi_0$ and for any $M \in \mathcal{F}^\chi$ we have

$$M_x = C_x(M).$$

Since sdim M =sdim M_x in order to prove Theorem 2.3 we have to show that sdim $L(\lambda) \neq 0$ for any $L(\lambda) \in \mathcal{F}^\chi$.

Let $\mathfrak{p}$ be any parabolic subalgebra of $\mathfrak{g}$ containing $\mathfrak{b}$ and P denote the algebraic supergroup with Lie superalgebra $\mathfrak{p}$.

For a P-module V we denote by the calligraphic letter $\mathcal{V}$ the vector bundle $G \times_P V$ over the generalized grassmannian G/P. Note that the space of sections of $\mathcal{V}$ on any open set has a natural structure of a $\mathfrak{g}$-module, in other words the sheaf of sections of $\mathcal{V}$ is a $\mathfrak{g}$-sheaf. Therefore the cohomology groups $H^i(G/P, \mathcal{V})$ are $\mathfrak{g}$-modules. For details see [6].

Define a derived functor Γ_i from the category of $\mathfrak{p}$-modules to the category of $\mathfrak{g}$-modules by

$$\Gamma_i(G/P, V) := (H^i(G/P, \mathcal{V}^*))^*.$$

In what follows we use the following result which is an immediate consequence of Proposition 1 in [6].

Lemma 5.1. *Let*

$$D_0 = \prod_{\alpha \in \Delta_0^+} (e^{\alpha/2} - e^{-\alpha/2}),$$

For any finite-dimensional P-module V we have

$$\sum_i (-1)^i \operatorname{ch} \Gamma_i(G/P, V) = \frac{1}{D_0} \sum_{w \in W} \varepsilon(w) w(e^{\rho_0} \operatorname{ch} V \prod_{\alpha \in \Delta_1 \setminus \Delta_1(\mathfrak{p})} (1 + e^\alpha)). \tag{5}$$

We choose a parabolic subalgebra $\mathfrak{p}$ so that its reductive part is isomorphic $\mathfrak{g}_{k-1} \oplus \mathbb{C}^2$. (In notations of [6] $\mathfrak{p} = \mathfrak{p}^1$.) By $L_\mathfrak{p}(\lambda)$ we denote the irreducible finite-dimensional $\mathfrak{p}$-module with highest weight λ. Finally by ω we denote the highest weight of the natural representation E.

Lemma 5.2. *If $L(\lambda) \in \mathcal{F}^\chi$, then*

$$\sum_i (-1)^i \operatorname{sdim} \Gamma_i(G/P, L_\mathfrak{p}(\lambda)) = 0. \tag{6}$$

Proof. Note that for any $t \in \mathbb{Z}_{\geq 0}$ the weight $\lambda + t\omega$ is dominant integral. If $t > 0$ then $\lambda + t\omega$ is $\mathfrak{p}$-typical, i.e. it satisfies the conditions of Lemma 5 in [6]. Hence we have

$$\Gamma_i(G/P, L_\mathfrak{p}(\lambda)) = \begin{cases} 0 \textit{ if } i > 0 \\ L_\lambda \textit{ if } i = 0 \end{cases}$$

The degree of atypicality of $\lambda + t\omega$ is $k-1$. Therefore for $t > 0$ we have

$$\sum_i (-1)^i \operatorname{sdim} \Gamma_i(G/P, L_{\mathfrak{p}}(\lambda + t\omega)) = \operatorname{sdim} L(\lambda + t\omega) = 0.$$

On the other hand,

$$\operatorname{ch} L_{\mathfrak{p}}(\lambda + t\omega) = e^{t\omega} \operatorname{ch} L_{\mathfrak{p}}(\lambda).$$

Therefore (5) implies that

$$\sum_i (-1)^i \operatorname{sdim} \Gamma_i(G/P, L_{\mathfrak{p}}(\lambda + t\omega)) = p(t)$$

for some polynomial $p(t)$. But we have $p(t) = 0$ for any $t \in \mathbb{Z}_{>0}$. Hence $p(0) = 0$. $\square$

We assume now that $\lambda \neq 0$ and ω (in case $\mathfrak{g} = \mathfrak{osp}(2k+1, 2k)$). The structure of $\Gamma_i(G/P, L_{\mathfrak{p}}(\lambda))$ is described completely in [6] and in [10] in terms of weight diagrams. We briefly repeat it here.

We need to introduce some notations. Observe that in our case the weight diagram f_λ has only $\times$ and 0. For any weight diagram f and $s < t$ let $l_f(s,t)$ be the number of $\times$ minus the number of 0 strictly between s and t. Set $s_0 = 0$ if $\mathfrak{g} = \mathfrak{osp}(2k, 2k)$ and $s_0 = 1/2$ if $\mathfrak{g} = \mathfrak{osp}(2k+1, 2k)$. Denote by $|f|$ twice the number of $\times$ at s_0.

If a weight diagram f is obtained from g by moving one $\times$ from position s to $t > s$ so that $l_f(s,t') \geq 0$ for any $s < t' \leq t$ we say that f is obtained from g by a *legal move* of degree $l = l_f(s,t)$. If the diagrams f and g have signs, then they are the same unless $s = s_0$. If $s = s_0$ they are opposite.

If a weight diagram f is obtained from g by moving one $\times$ from position s_0 to $t > s_0$ so that $l_f(s_0, t') + |f| \geq 0$ for any $s_0 < t' \leq t$ (and the signs of both diagrams are the same) then we say that f is obtained from g by a *tail move* of degree $l_f(s,t) + |f|$.

Finally, if a weight diagram f is obtained from g by moving two $\times$ from s_0 to the positions $t > s > s_0$ so that $l_f(a,s) \leq 0$ for all $a < s$, $l_f(s_0,s) + |f|$ is a positive odd number and $l_f(s,t') \geq 0$ for all $s < t' \leq t$, (and the sign is not changed) then we say that f is obtained from g by an *exceptional move* of degree $l_f(s,t)$.

There are no tail or exceptional moves in the case $\mathfrak{g} = \mathfrak{gl}(k,k)$.

Let t be the position of the rightmost $\times$ in f_λ, and let $\mathcal{M}_l(\lambda)$ be the set of all $\nu \in \Lambda$ such that f_λ is obtained from f_ν by a legal move or a tail move of degree l of one $\times$ from some $s < t$ to t, or by an exceptional move of degree l of two $\times$ from s_0 to $s < t$.

The following is a slight reformulation of Proposition 7 in [6].

Lemma 5.3. *Assume that* $\lambda \neq 0$ *if* $\mathfrak{g} = \mathfrak{osp}(2k+1, 2k)$ *or* $\mathfrak{osp}(2k, 2k)$ *and* $\lambda \neq \omega$ *if* $\mathfrak{g} = \mathfrak{osp}(2k+1, 2k)$.

If $i > 0$, then in the Grothendick group of $\mathcal{F}$

$$[\Gamma_i(G/P, L_{\mathfrak{p}}(\lambda))] = \sum_{\nu\in\mathcal{M}_i(\lambda)} [L(\nu)],$$

and

$$[\Gamma_0(G/P, L_{\mathfrak{p}}(\lambda))] = [L(\lambda)] + \sum_{\nu\in\mathcal{M}_0(\lambda)} [L(\nu)].$$

Example 5.4. Let $\mathfrak{g} = \mathfrak{osp}(4,4)$ and $\lambda = (2,1|2,1)$. Then $t = 2$. One can check that $\mathcal{M}_i(\lambda)$ is not empty only for $i = 1$ and

$$\mathcal{M}_1(\lambda) = \{(1,0|1,0)\}.$$

The corresponding move

$$\times, \times, 0, \cdots \to 0, \times, \times, \cdots$$

is a legal move of degree 1.

If $\mu = (1,0|1,0)$, then $t = 1$ and $\mathcal{M}_i(\mu)$ is not empty for $i = 0$ and 2 and

$$\mathcal{M}_0(\mu) = \mathcal{M}_2(\mu) = \{(0,0|0,0\}.$$

The corresponding move

$$\substack{\times\\ \times}, 0, \cdots \to \times, \times, \cdots$$

have two meanings. It is a legal move of degree 0 and a tail move of degree 2.

Now let $\mathfrak{g} = \mathfrak{osp}(6,6)$ and $\lambda = (3,2,0|3,2,0)$. In this case $t = 3$ and we have

$$\mathcal{M}_0(\lambda) = \{(0,0,0|0,0,0)\},$$

corresponding to the exceptional move

$$\times^{\times^{\times}}, 0, 0, 0, \cdots \to \times, 0, \times, \times, \cdots$$

of degree 0 and

$$\mathcal{M}_0(\lambda) = \{(2,1,0|2,1,0)\},$$

corresponding to the legal move

$$\times, \times, \times, 0, \cdots \to \times, 0, \times, \times, \cdots$$

of degree 1.

Lemma 5.2 and Lemma 5.3 imply

Corollary 5.5.

$$\operatorname{sdim} L(\lambda) = \sum_i (-1)^{i+1} \sum_{\nu \in \mathcal{M}_i(\lambda)} \operatorname{sdim} L(\nu).$$

Lemma 5.6. *Let λ satisfy the assumptions of Lemma 5.3. If $\nu \in \mathcal{M}_i(\lambda)$, then $p(\nu - \lambda) = i + 1$ mod (2).*

Proof. Let $t_i(\lambda)$ be the position of the ith $\times$ in f_λ counting from the right and $t(\lambda) = \sum_{i=1}^{k} t_i(\lambda)$. One can see easily that $p(\nu - \lambda) = t(\nu) - t(\lambda)$ mod (2). Now the statement follows immediately from the definition of moves. □

Theorem 5.7. *For any $L(\lambda) \in \mathcal{F}^\chi$ we have* $\operatorname{sdim}(L(\lambda)) > 0$ *if* $p(\lambda) = 0$ *and* $\operatorname{sdim}(L(\lambda)) < 0$ *if* $p(\lambda) = 1$.

Proof. If $\mathfrak{g} = \mathfrak{osp}(2k, 2k)$ or $\mathfrak{osp}(2k + 1, 2k)$ the statement follows easily from Corollary 5.5 by induction on the position t of the rightmost $\times$ in f_λ with base of induction $t = s_0$. For the base of induction we have $\lambda = 0$ (and $\lambda = \omega$ if $\mathfrak{g} = \mathfrak{osp}(2k + 1, 2k)$). Note $\operatorname{sdim}(L_0) = 1$ (and $\operatorname{sdim}(L_\omega) = 1$ if $\mathfrak{g} = \mathfrak{osp}(2k + 1, 2k)$).

When $\mathfrak{g} = \mathfrak{gl}(k, k)$ the set $\mathbb{T}$ does not have a minimal element. In this case we use a more complicated induction. For a weight diagram f we call a *clique* a sequence of adjacent $\times$ bounded by 0 on the left and on the right. Let $d(\lambda)$ be the distance between the leftmost and the rightmost $\times$ in f_λ and $a(\lambda)$ be the length of the rightmost clique. Assume that $k \geq 2$. The case $k = 1$ is trivial. If $d(\lambda) = a(\lambda) = k$, then $L(\lambda)$ is a tensor power of the supertrace representation, it has superdimension ± 1 depending on $p(\lambda)$. If $\nu \in \mathcal{M}_i(\lambda)$ then $d(\nu) \leq d(\lambda)$. If $d(\nu) = d(\lambda)$ then the leftmost $\times$ is moved to the right of the rightmost clique of f_ν and $a(\nu) + 1 = a(\lambda)$ unless $a(\nu) = k$. We define a new order on the set of weights. We say that ν is less than λ if $d(\nu) < d(\lambda)$ or $d(\nu) = d(\lambda)$ and $a(\nu) > a(\lambda)$. Then we can prove the statement using Corollary 5.5 and induction on this new order. □

Theorem 2.3 follows from Theorem 5.7 and Lemma 4.5.

6 Modified Dimension and Ribbon Categories

In this section we discuss connections with [4] and [3]. In these papers the category $\mathcal{C}$ is studied as a ribbon Ab-category.

Let $x \in X$ be a self-commuting element. Then it is easy to see that the fiber functor $\mathcal{C} \to \mathcal{C}(\mathfrak{g}_x)$ is a functor of ribbon categories, i.e. it respects all additional structures: duality, braiding, twist.

For every $M \in \mathcal{C}$ we denote by $\mathcal{I}_M$ the ideal in $\mathcal{C}$ generated by M. More precisely $\mathcal{I}_M$ can be defined as the full subcategory of $\mathcal{C}$ of all directs summands in $M \otimes N$ for all $N \in \mathcal{C}$.

Lemma 6.1. *Let $M \in \mathcal{C}$ be simple with degree of atypicality k, and $x \in X$ have rank k. If $N \in \mathcal{I}_M$ then N_x is projective in $\mathcal{C}(\mathfrak{g}_x)$.*

Proof. By Lemma 4.3 the fiber functor maps objects of $\mathcal{I}_M$ to the objects of $\mathcal{I}_{M_x}$ in $\mathcal{C}(\mathfrak{g}_x)$. It was proven in [3] that $\mathcal{I}_{M_x}$ coincides with the ideal of projective modules in $\mathcal{C}(\mathfrak{g}_x)$. Hence the statement. □

Lemma 6.2. *Assume that λ and μ are both dominant, $\mu \leq \lambda$, $\chi_\lambda = \chi_\mu$ and λ is stable. Then $\lambda - \mu$ is a sum of positive roots of the superalgebra $\mathfrak{g}_k$, where $k = \mathrm{at}(\lambda)$.*

Proof. If $\chi_\lambda = \chi_\mu$ and $\mu \leq \lambda$, then f_μ is obtained from f_λ by moving some $\times$ to the left. By stability of λ, all $\times$ stand to the left of the symbols $>$ and $<$. Therefore moving $\times$ to the left corresponds to subtracting positive roots of $\mathfrak{g}_k$. □

Lemma 6.3. *Let $k \leq \operatorname{def}\mathfrak{g}$. There exists a simple M with degree of atypicality k such that any simple N of degree of atypicality k belongs to $\mathcal{I}_M$.*

Proof. Let μ be a stable dominant weight with degree of atypicality k such that all $\times$ in f_μ are in positions $1, 2, \dots, k$ if $\mathfrak{g} = \mathfrak{gl}(m,n)$, all $\times$ are in position s_0 if $\mathfrak{g} = \mathfrak{osp}(2m,2n)$ or $\mathfrak{osp}(2m+1,2n)$. In the case $\mathfrak{g} = \mathfrak{osp}(2m+1,2n)$ we also assume that f_μ has negative sign. Set $M = L(\mu)$

Let λ be a stable dominant weight with degree of atypicality k and such that $\nu = \lambda - \mu$ is again dominant stable with degree of atypicality k. Let $L = L(\lambda)$ and $\chi = \chi_\lambda$. We claim that $(L(\nu) \otimes M)^\chi = L$, and hence $L \in \mathcal{I}_M$. Indeed, if κ is the weight of some $\mathfrak{n}^+$-invariant vector $v \in (L(\nu) \otimes M)^\chi$, then $\kappa \leq \lambda$. On the other hand $\chi_\kappa = \chi_\lambda$, therefore by Lemma 6.2 $\lambda - \kappa$ is a sum of positive roots of $\mathfrak{g}_k$. Hence any $\mathfrak{n}^+$-invariant vector v with weight κ must belong to $L_{\mathfrak{l}}(\mu) \otimes L_{\mathfrak{l}}(\nu)$, where $\mathfrak{l} = \mathfrak{h} + \mathfrak{g}_k$. But $L_{\mathfrak{l}}(\mu)$ is one-dimensional, therefore $L_{\mathfrak{l}}(\mu) \otimes L_{\mathfrak{l}}(\nu) = L_{\mathfrak{l}}(\lambda)$. Hence v is the highest vector of $L(\lambda)$. That shows $L \in \mathcal{I}_M$.

Finally, let N be an arbitrary simple module with degree of atypicality k. We will show using Theorem 3.2 that N can be obtained by a sequence of translation functors from some L satisfying the conditions of the previous paragraph. Indeed, it is easy to see that λ satisfies the conditions of the previous paragraph if in the weight diagram f_λ all $\times$ lies to the left of all $<$ and $>$ and the distances between non-empty positions in the core part and the distance between the core part and the rightmost $\times$ is sufficiently large (for example greater than the distance between the first and the last non-empty position in f_μ). Using translation functors we can move all $<$ and $>$ as far to the right as we want, and the statement follows. Thus, we have that $N \in \mathcal{I}_L$. Hence $N \in \mathcal{I}_M$. □

Recall some definitions from [4]. Let $b_M : \mathbb{C} \to M \otimes M^*$, $b'_M : \mathbb{C} \to M^* \otimes M$, $d_M : M^* \otimes M \to \mathbb{C}$ and $d'_M : M \otimes M^* \to \mathbb{C}$ be the natural morphisms of $\mathfrak{g}$-modules. Let M be simple. For any $f \in \operatorname{End}_{\mathfrak{g}}(M \otimes N)$ we have

$$(\mathrm{Id}_M \otimes d'_N) \circ (f \otimes \mathrm{Id}_{N^*}) \circ (\mathrm{Id}_M \otimes b_N) = \mathrm{tr}_{R,M}(f)\, \mathrm{Id}_M,$$

for some $= \mathrm{tr}_{R,M}(f) \in \mathbb{C}$. We call a simple M *ambidextrous* if for any $f \in \mathrm{End}_{\mathfrak{g}}(M \otimes M)$

$$(\mathrm{Id}_M \otimes d'_M) \circ (f \otimes \mathrm{Id}_{M^*}) \circ (\mathrm{Id}_M \otimes b_M) = (d_M \otimes \mathrm{Id}_M) \circ (\mathrm{Id}_{M^*} \otimes f) \circ (b'_M \otimes \mathrm{Id}_M), \tag{7}$$

and $\mathrm{tr}_{R,M}(f) \neq 0$ for some $f \in \mathrm{End}_{\mathfrak{g}}(M \otimes M)$.

Let M be an ambidextrous simple module. If $U \in \mathcal{I}_M$, then there exist $N \in \mathcal{C}$, $p : U \to M \otimes N$ and $q : M \otimes N \to U$ such that $q \circ p = \mathrm{Id}_U$. We define a *modified dimension* $d_M : \mathcal{I}_M \to \mathbb{C}$ by the formula

$$d_M(U) = \mathrm{tr}_{R,M}(p \circ q).$$

The modified dimension $d_M(U)$ does not depend on the choice of N, p, q.

The modified dimension $d_M : \mathcal{I}_M \to \mathbb{C}$ satisfies many natural properties (see [4] Sect. 4). It was proven in [4] (Theorem 6.2.1) that if $\mathfrak{g} = \mathfrak{gl}(m,n)$ then any typical simple module M is ambidextrous. If one defines

$$d(\lambda) = \prod_{\alpha \in \Delta_0^+} \frac{(\lambda + \rho, \alpha)}{(\rho, \alpha)} / \prod_{\alpha \in \Delta_1^+} (\lambda + \rho, \alpha),$$

then for any two simple typical M and N one has

$$d_M(N) = \pm \frac{d(\nu)}{d(\mu)}, \tag{8}$$

where μ and ν are the highest weights of M and N respectively and the sign depends on the parity of the highest vectors.

Lemma 6.4. *Let $\mathfrak{g} = \mathfrak{gl}(m,n)$, χ be a central character, $x \in X$ be such that* $\mathrm{at}(\chi) = \mathrm{rk}(x)$. *Assume that the double core $2\mu(\chi)$ is a typical weight of $\mathfrak{g}_x$. Let $M \in \mathcal{C}^\chi$ be such that M_x is a simple $\mathfrak{g}_x$-module. Then M is ambidextrous and for any $N \in \mathcal{I}_M$ we have*

$$d_M(N) = d_{M_x}(N_x).$$

Proof. The natural map $\mathrm{End}_{\mathfrak{g}}(M) \to \mathrm{End}_{\mathfrak{g}_x}(M_x)$ is a homomorphism of algebras. Therefore for any $f \in \mathrm{End}_{\mathfrak{g}}(M \otimes N)$ we have

$$\mathrm{tr}_{R,M}(f) = \mathrm{tr}_{R,M_x}(f_x). \tag{9}$$

By [4] M_x is ambidextrous. Hence (7) holds for M.

Next we will construct $f \in \mathrm{End}_{\mathfrak{g}}(M \otimes M)$ such that $\mathrm{tr}_{R,M}(f) \neq 0$. Since $2\mu(\chi)$ is typical we have

$$(M \otimes M)_x^\eta = (M_x \otimes M_x)^\eta = L_{\mathfrak{g}_x}(2\mu(\chi)),$$

where η is the central character of $L_{\mathfrak{g}_x}(2\mu(\chi))$. Let $\chi = \check{\theta}(\eta)$, and $f \in \mathrm{End}_{\mathfrak{g}}(M \otimes M)$ be the projection onto $(M \otimes M)^{\chi}$. Then $f_x \in \mathrm{End}_{\mathfrak{g}_x}(M_x \otimes M_x)$ is the projection onto $L_{\mathfrak{g}_x}(2\mu(\chi))$ and

$$\mathrm{tr}_{R,M}(f) = \mathrm{tr}_{R,M_x}(f_x) = d_{M_x}(L_{\mathfrak{g}_x}(2\mu(\chi))) \neq 0$$

by (8). This proves that M is ambidextrous.

The dimension formula follows from (9). □

Corollary 6.5. *Let $\mathfrak{g} = \mathfrak{gl}(m,n)$, $k \leq \operatorname{def}\mathfrak{g}$. There exists in the category $\mathcal{C}$ an ambidextrous simple M with degree of atypicality k such that any simple N of degree of atypicality k belongs to $\mathcal{I}_M$.*

Proof. Let $\mu = (n+1)(\varepsilon_1 + \cdots + \varepsilon_{m-k+1})$ and $M = L(\mu)$. Then M satisfies the conditions in the proof of Lemma 6.3. Therefore any simple N with degree of atypicality k belongs to $\mathcal{I}_M$.

On the other hand, μ is stable and $L_k(\mu')$ is trivial. Hence by Lemma 4.2 M_x is a simple $\mathfrak{g}_x$-module for any $x \in X_k$. Finally, we see that $2\mu(\chi_\mu)$ is typical. Hence M is ambidextrous. □

Corollary 6.6. *If $\mathfrak{g} = \mathfrak{gl}(m,n)$, then any simple $\mathfrak{g}$-module in $\mathcal{C}$ is ambidextrous.*

Proof. Let N be a simple module and k be its degree of atypicality. Let M be as in the proof of Corollary 6.5. Then $N \in \mathcal{I}_M$ and $d_M(N) \neq 0$ by Lemma 6.4, (8) and Theorem 2.3. Theorem 4.2.1 in [4] implies that $\mathcal{I}_M = \mathcal{I}_N$ and Theorems 3.3.1 and 3.3.2 in [4] imply that N is ambidextrous. □

Note that Theorem 2.3 and Lemma 6.4 imply Conjecture 6.3.2 in [4] for $\mathfrak{g} = \mathfrak{gl}(m,n)$.

Corollary 6.7. *Let $\mathfrak{g} = \mathfrak{gl}(m,n)$, M be a simple $\mathfrak{g}$-module in $\mathcal{C}$ and $L \in \mathcal{I}_M$ be another simple module. Then $\mathrm{at}(L) \leq \mathrm{at}(M)$ and $d_M(L) \neq 0$ iff $\mathrm{at}(L) = \mathrm{at}(M)$.*

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