UNITEXT for Physics

Vladimir Pletser

Lagrangian and Hamiltonian Analytical Mechanics: Forty Exercises Resolved and Explained

Springer

UNITEXT for Physics

UNITEXT for Physics series, formerly UNITEXT Collana di Fisica e Astronomia, publishes textbooks and monographs in Physics and Astronomy, mainly in English language, characterized of a didactic style and comprehensiveness. The books published in UNITEXT for Physics series are addressed to graduate and advanced graduate students, but also to scientists and researchers as important resources for their education, knowledge and teaching.

More information about this series at http://www.springer.com/series/13351

Vladimir Pletser

Lagrangian and Hamiltonian Analytical Mechanics: Forty Exercises Resolved and Explained

Springer

Vladimir Pletser
PITC, Technology and Engineering
Centre for Space Utilization
Chinese Academy of Sciences
Beijing, China

ISSN 2198-7882 ISSN 2198-7890 (electronic)
UNITEXT for Physics
ISBN 978-981-13-3025-4 ISBN 978-981-13-3026-1 (eBook)
https://doi.org/10.1007/978-981-13-3026-1

Library of Congress Control Number: 2018958950

This Springer imprint is published by the registered company Springer Nature Singapore Pte Ltd.
The registered company address is: 152 Beach Road, #21-01/04 Gateway East, Singapore 189721, Singapore

Preface

The teaching of rational analytical mechanics is supported by the learning of solving examples, exercises and classical and more recent problems. This collection of forty solved exercises is intended to be a pedagogical tool that explains step by step the resolution of the forty exercises carefully chosen for their importance in classical mechanics, celestial mechanics and quantum mechanics.

This collection of exercises comprises six chapters:

1. Lagrange Equations
2. Hamilton Equations
3. First Integral and Variational Principle
4. Canonical Transformations
5. Hamilton–Jacobi Equations
6. Phase Integral and Angular Frequencies

Each chapter begins with a brief theoretical reminder before the proposed exercises, which are solved in detail. We particularly emphasized the last two chapters because of the importance and flexibility of Hamilton–Jacobi's method in solving many mechanical problems in classical mechanics as well as quantum and celestial mechanics.

The forty proposed and solved exercises and problems address the following themes:
in classical mechanics:

- the harmonic oscillator with one dimension (Exercises 7, 20, 21, 34) and three dimensions (36)
- the double (1, 27) and simple (35) pendulum
- particles subjected to different potentials and constrains (2, 6, 24, 25, 29)
- free particles (9, 22, 23)
- movement of solids (3, 4)
- sliding and rotating masses, the Watt regulator (5, 8)

- minimization problems (10–12)
- canonical transformations (13–20)
- unconventional mechanics (26)

in electromagnetism:

- Stark effect (31, 33)
- double Coulomb field (32)

in celestial mechanics:

- the classical (28, 38) and relativistic (39) Kepler's problem
- the problem of Mercury's perihelion advance (40)

in quantum mechanics:

- Schrödinger equation (30)
- the Bohr atom (37)

This collection of exercises gathers for the most part exercises given at the beginning of the 1980s in the Physics Department of the Faculty of Sciences of the University of Kinshasa, Congo, and complemented by other more recent exercises.

This collection of exercises is intended for students in the second year of their bachelor's and first year of their master's studies in Faculties of Sciences and Polytechnic Schools, who are taking or have taken a course in "Analytical Mechanics". A basic knowledge of integral calculus is a prerequisite. However, the method of resolution of integrals is indicated and the reader is referred to classical tables of integrals.

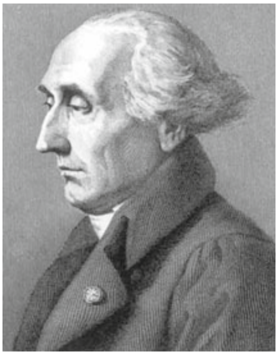
It is a pleasure to thank Profs. A. Deprit, N. Rouche, P. Y. Willems and D. Johnson from the Catholic University of Louvain, Louvain-la-Neuve, Belgium; Prof. D. Huylebrouck from the Catholic University of Leuven, Belgium; and Prof. H. Pollack from the University of Kinshasa, Congo.

Beijing, China
2018

Vladimir Pletser
Assistant Professor, Department of Physics
Faculty of Sciences, University of Kinshasa, Congo
Catholic University of Louvain
Louvain-la-Neuve, Belgium (1982–85)
Senior Physicist—Engineer, European Space Research
and Technology Centre (ESTEC)
European Space Agency (ESA)
Noordwijk, The Netherlands (1985–2016)
Visiting Professor, Technology and Engineering
Centre for Space Utilization (CSU)
Chinese Academy of Sciences (CAS)
Beijing, China (2016–2018)

Contents

Chapter 1
Lagrange Equations



Joseph-Louis Lagrange, Comte (born Turin, Italy, January 25, 1736 – died Paris, France, April 10, 1813) was a Franco-Italian mathematician and astronomer. A brilliant self-taught, he was appointed a professor in 1755, aged 19. In 1766, on the recommendation of Euler and d'Alembert, Lagrange succeeded Euler as the director of mathematics at the Prussian Academy of Sciences in Berlin, Prussia, where he stayed for over twenty years, producing volumes of work and winning several prizes of the French Academy of Sciences. He made significant contributions to the fields of analysis, number theory, and both classical and celestial mechanics. Lagrange's treatise on analytical mechanics (*Mécanique analytique*, 1788–1789), written in Berlin presented the most comprehensive treatment of classical mechanics since Newton and formed a basis for the development of mathematical physics in the nineteenth century. After moving from Berlin to Paris in 1787, he became a member of the French Academy of Sciences. He became the first professor of analysis at the *École Polytechnique* upon

V. Pletser, *Lagrangian and Hamiltonian Analytical Mechanics: Forty Exercises Resolved and Explained*, UNITEXT for Physics,
https://doi.org/10.1007/978-981-13-3026-1_1

its opening in 1794, was a founding member of the *Bureau des Longitudes*, and became a Senator in 1799. Within mathematical analysis, Lagrange researched extensively into the calculus of variations, and in the process, invented the variation of parameters. He also devised ways of using differential calculus to solve problems pertaining to theory of probabilities.

1.1 Reminder

1.1.1 Generalized Coordinates

For a system of N particles, the coordinates $q_1, q_2, \ldots, q_n$, where n is the degree of freedom, are the generalized coordinates that are independent of each other.

1.1.2 Kinetic Energy

If the system is such that time t does not intervene in the transformation equations, the kinetic energy T reads as the sum for the N particles of the product of the mass of each particle m_ν by the square of the generalized velocity $\dot{q}_\nu$ of this particle

$$T = \frac{1}{2} \sum_{\nu=1}^{N} m_\nu \dot{q}_\nu^2 \tag{1.1}$$

where $\dot{q}_\nu = dq_\nu/dt$ is the generalized velocity corresponding to the generalized coordinate q_ν. Note that $\dot{q}_\nu^2$ may also sometimes be written as the product of two different generalized velocities $\dot{q}_\alpha \dot{q}_\beta$.

If the system is such that the time t intervenes in the transformation equations, the expression of kinetic energy (1.1) will contain linear terms in $\dot{q}_\nu$.

1.1.3 Generalized Forces

Let F_ν be forces acting on the ν particles with radius vectors r_ν. One calls

$$\phi_\alpha = \sum_{\nu=1}^{N} F_\nu \frac{\partial r_\nu}{\partial q_\alpha} \tag{1.2}$$

the generalized force associated with the generalized coordinate q_α.

1.1.4 Lagrange Equations

Generally speaking, Lagrange's equations read as follows

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_\alpha}\right) - \frac{\partial T}{\partial q_\alpha} = \phi_\alpha \tag{1.3}$$

If the system is conservative, i.e. if all forces derive from a potential V, the generalized forces ϕ_α read

$$\phi_\alpha = -\frac{\partial V}{\partial q_\alpha} \tag{1.4}$$

Conversely, the potential V can obviously be written

$$V = -\int \phi_\alpha dq_\alpha \tag{1.5}$$

One introduces then the Lagrangian L

$$L = T - V \tag{1.6}$$

and Lagrange's equations become simpler,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\alpha}\right) - \frac{\partial L}{\partial q_\alpha} = 0 \tag{1.7}$$

If only part of the forces derives from a potential V and that other forces ϕ'_α are not conservative (for example, friction forces, or in general, all forces proportional to velocity[1]), Lagrange equations become

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\alpha}\right) - \frac{\partial L}{\partial q_\alpha} = \phi'_\alpha \tag{1.8}$$

1.1.5 Generalized Moment

One defines the conjugate moment or generalized moment associated with the generalized coordinate q_α

$$p_\alpha = \frac{\partial T}{\partial \dot{q}_\alpha} \tag{1.9}$$

[1]Except the Coriolis force.

If the system is conservative, i.e. if the forces are derived from a potential, then we have

$$p_\alpha = \frac{\partial L}{\partial \dot{q}_\alpha} \tag{1.10}$$

1.1.6 Lagrange Equations for Systems with Constraints

If there are constraints on a system, these must be taken into account in the Lagrange equations. Suppose there is a number c of constraints C, with $c < n$ for the system not to be blocked (n is the number of degrees of freedom of the system), and that these c constraints can be written as c constraint equations

$$\sum_{\alpha=1}^{n} C_{\mu,\alpha} dq_\alpha + C_{\mu,t} dt = 0 \tag{1.11}$$

with $1 \le \mu \le c$. Lagrange equations read then

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_\alpha}\right) - \frac{\partial T}{\partial q_\alpha} = \phi_\alpha + \sum_{\mu=1}^{c} \lambda_\mu C_{\mu,\alpha} \tag{1.12}$$

where the c parameters λ_μ are called Lagrange's multipliers.

If the system is conservative, Lagrange equations (1.7) are used with the Lagrangian

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\alpha}\right) - \frac{\partial L}{\partial q_\alpha} = \sum_{\mu=1}^{c} \lambda_\mu C_{\mu,\alpha} \tag{1.13}$$

Physically, Lagrange multipliers are associated with constraint forces acting on the system. When determining Lagrange multipliers, the effect of constraint forces is essentially taken into account without explicitly calculating them.

1.1.7 Lagrange Equations with Impulse Forces

If the forces F_ν acting on a system are such that

$$\lim_{\tau \to 0} \int_0^\tau F_\nu dt = I_\nu \tag{1.14}$$

where τ represents an interval of time during which the forces F_ν are applied to the system. One calls the forces F_ν impulse forces and I_ν impulse.

Lagrange's equations then become

$$\left(\frac{\partial T}{\partial \dot{q}_\alpha}\right)_p - \left(\frac{\partial T}{\partial \dot{q}_\alpha}\right)_a = \mathcal{F}_\alpha \tag{1.15}$$

where the instants p (posterior) and a (anterior) refer respectively to after and before the shock and $\mathcal{F}_\alpha$are generalized impulses

$$\mathcal{F}_\alpha = \sum_{\nu=1}^{N} I_\nu \frac{\partial r_\nu}{\partial q_\alpha} \tag{1.16}$$

1.2 Exercises

1.2.1 Exercise 1 : Double Pendulum

> Find Lagrange's equations of the motion of a double pendulum oscillating in a plane in a uniform gravity field.

Proof The system has two degrees of freedom, $n = 2$, and is without constraints ($c = 0$). The generalized coordinates are the angles φ_1 and φ_2 relative to the local vertical of rigid rods of negligible mass and length l_1 and l_2 (Fig. 1.1).

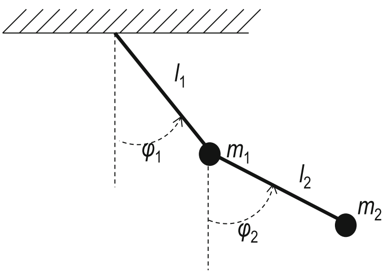
For the point of mass m_1, the kinetic energy and potential read

$$T_1 = \frac{m_1 l_1^2 \dot{\varphi}_1^2}{2} \tag{1.17}$$

$$V_1 = -m_1 g\, l_1 \cos\varphi_1 \tag{1.18}$$

For the point of mass m_2, the square of its velocity v_2 read

Fig. 1.1 Double pendulum



$$v_2^2 = \left[\frac{d}{dt}\left(l_1 \sin\varphi_1 + l_2 \sin\varphi_2\right)\right]^2 + \left[\frac{d}{dt}\left(l_1 \cos\varphi_1 + l_2 \cos\varphi_2\right)\right]^2$$
$$= \left[l_1 \cos\varphi_1 \dot{\varphi}_1 + l_2 \cos\varphi_2 \dot{\varphi}_2\right]^2 + \left[-l_1 \sin\varphi_1 \dot{\varphi}_1 - l_2 \sin\varphi_2 \dot{\varphi}_2\right]^2$$
$$= l_1^2 \dot{\varphi}_1^2 + l_2^2 \dot{\varphi}_2^2 + 2 l_1 l_2 \cos\left(\varphi_1 - \varphi_2\right) \dot{\varphi}_1 \dot{\varphi}_2$$

and its kinetic energy and potential read

$$T_2 = \frac{m_2}{2}\left(l_1^2 \dot{\varphi}_1^2 + l_2^2 \dot{\varphi}_2^2 + 2 l_1 l_2 \cos\left(\varphi_1 - \varphi_2\right) \dot{\varphi}_1 \dot{\varphi}_2\right) \tag{1.19}$$
$$V_2 = -m_2 g\left(l_1 \cos\varphi_1 + l_2 \cos\varphi_2\right) \tag{1.20}$$

The Lagrangian is

$$\begin{aligned} L &= \left(T_1 + T_2\right) - \left(V_1 + V_2\right) \\ &= \left(\frac{m_1 + m_2}{2}\right) l_1^2 \dot{\varphi}_1^2 + \frac{m_2}{2} l_2^2 \dot{\varphi}_2^2 + m_2 l_1 l_2 \cos\left(\varphi_1 - \varphi_2\right) \dot{\varphi}_1 \dot{\varphi}_2 \\ &\quad + \left(m_1 + m_2\right) g\, l_1 \cos\varphi_1 + m_2 g\, l_2 \cos\varphi_2 \end{aligned} \tag{1.21}$$

Lagrange's first equation is written

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\varphi}_1}\right) - \frac{\partial L}{\partial \varphi_1} = 0$$

with

$$\begin{aligned} \frac{\partial L}{\partial \dot{\varphi}_1} &= \left(m_1 + m_2\right) l_1^2 \dot{\varphi}_1 + m_2 l_1 l_2 \cos\left(\varphi_1 - \varphi_2\right) \dot{\varphi}_2 \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\varphi}_1}\right) &= \left(m_1 + m_2\right) l_1^2 \ddot{\varphi}_1 + m_2 l_1 l_2 \left[\cos\left(\varphi_1 - \varphi_2\right) \ddot{\varphi}_2 \right. \\ &\qquad \left. - \sin\left(\varphi_1 - \varphi_2\right) \dot{\varphi}_2 \left(\dot{\varphi}_1 - \dot{\varphi}_2\right)\right] \\ \frac{\partial L}{\partial \varphi_1} &= -m_2 l_1 l_2 \dot{\varphi}_1 \dot{\varphi}_2 \sin\left(\varphi_1 - \varphi_2\right) - \left(m_1 + m_2\right) g\, l_1 \sin\varphi_1 \end{aligned}$$

yielding

$$\left(m_1 + m_2\right) l_1 \ddot{\varphi}_1 + m_2 l_2 \left[\cos\left(\varphi_1 - \varphi_2\right) \ddot{\varphi}_2 + \sin\left(\varphi_1 - \varphi_2\right) \dot{\varphi}_2^2 + \left(m_1 + m_2\right) g \sin\varphi_1\right] = 0 \tag{1.22}$$

Lagrange's second equation is written

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\varphi}_2}\right) - \frac{\partial L}{\partial \varphi_2} = 0$$

with

$$\frac{\partial L}{\partial \dot{\varphi}_2} = m_2 l_2^2 \dot{\varphi}_2 + m_2 l_1 l_2 \cos(\varphi_1 - \varphi_2)\, \dot{\varphi}_1$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\varphi}_2}\right) = m_2 l_2^2 \ddot{\varphi}_2 + m_2 l_1 l_2 [\cos(\varphi_1 - \varphi_2)\, \ddot{\varphi}_1 - \sin(\varphi_1 - \varphi_2)\, \dot{\varphi}_1 (\dot{\varphi}_1 - \dot{\varphi}_2)]$$

$$\frac{\partial L}{\partial \varphi_2} = m_2 l_1 l_2 \dot{\varphi}_1 \dot{\varphi}_2 \sin(\varphi_1 - \varphi_2) - m_2 g\, l_2 \sin \varphi_2$$

yielding

$$l_2 \ddot{\varphi}_2 + l_1 \left[\cos(\varphi_1 - \varphi_2)\, \ddot{\varphi}_1 - \sin(\varphi_1 - \varphi_2)\, \dot{\varphi}_1^2\right] + g \sin \varphi_2 = 0 \qquad (1.23)$$

□

1.2.2 Exercise 2: Particle on a Paraboloid

In a uniform gravity field, a particle of mass m moves without friction on the inner surface of a paraboloid of revolution $x^2 + y^2 = az$. Find the motion equations.

Proof The system has three degrees of freedom, $n = 3$. The system configuration and axial symmetry with respect to the Z axis lead to the choice of cylindrical coordinates (ρ, φ, z) as generalized coordinates (Fig. 1.2). The system is subject to

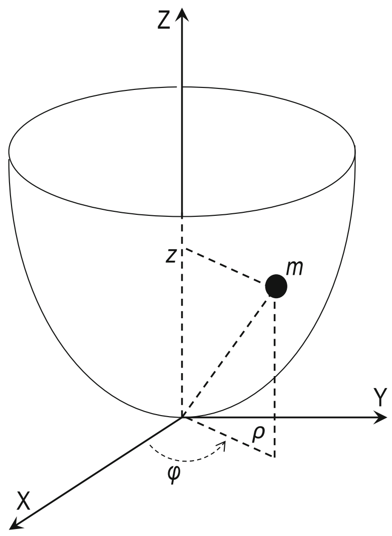


Fig. 1.2 Particle inside a paraboloid

one constraint ($c = 1$): the mass point m must move on the inner surface of the paraboloid of equation $x^2 + y^2 = az$ in Cartesian coordinates or, as $\rho^2 = x^2 + y^2$

$$\rho^2 = az \tag{1.24}$$

in cylindrical coordinates. Differentiating (1.24) yields

$$2\varrho\,\delta\varrho - a\,\delta z = 0 \tag{1.25}$$

which, from (1.11), gives $C_{1,\rho} = 2\rho$, $C_{1,\varphi} = 0$, $C_{1,z} = -a$ et $C_{1,t} = 0$.

The kinetic energy, potential and Lagrangian read respectively

$$\begin{aligned} T &= \frac{m}{2}\left(\dot{\rho}^2 + \rho^2\dot{\varphi}^2 + \dot{z}^2\right) \\ V &= mgz \\ L &= \frac{m}{2}\left(\dot{\rho}^2 + \rho^2\dot{\varphi}^2 + \dot{z}^2\right) - mgz \end{aligned}$$

Lagrange equations (1.13) are then written

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\alpha}\right) - \frac{\partial L}{\partial q_\alpha} = \lambda_1 C_{1,\alpha} \tag{1.26}$$

For $q_\alpha = \rho$,

$$\begin{aligned} \frac{\partial L}{\partial \dot{\rho}} &= m\dot{\rho} \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\rho}}\right) &= m\ddot{\rho} \\ \frac{\partial L}{\partial \varrho} &= m\rho\dot{\varphi}^2 \end{aligned}$$

yielding

$$m\ddot{\rho} - m\rho\dot{\varphi}^2 = 2\rho\lambda_1 \tag{1.27}$$

For $q_\alpha = \varphi$,

$$\begin{aligned} \frac{\partial L}{\partial \dot{\varphi}} &= m\rho^2\dot{\varphi} \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\varphi}}\right) &= m\rho^2\ddot{\varphi} + 2m\rho\dot{\rho}\dot{\varphi} \\ \frac{\partial L}{\partial \varphi} &= 0 \end{aligned}$$

yielding

$$m\rho^2\ddot{\varphi} + 2m\rho\dot{\rho}\dot{\varphi} = 0 \tag{1.28}$$

For $q_\alpha = z$,

$$\frac{\partial L}{\partial \dot{z}} = m\dot{z}$$
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}}\right) = m\ddot{z}$$
$$\frac{\partial L}{\partial \varphi} = -mg$$

yielding

$$m\ddot{z} + mg = -a\lambda_1 \tag{1.29}$$

To these three Lagrange equations with four variables, one adds the constraint equation (1.24) derived with respect to time

$$2\rho\dot{\rho} - a\dot{z} = 0 \tag{1.30}$$

This gives four equations in ρ, φ, z et λ_1. □

1.2.3 Exercise 3: Sphere Rolling on Another Sphere

A sphere of radius a and mass m is on top of another sphere of radius b. The first sphere is moved slightly and starts rolling without slipping. Find the motion equations.

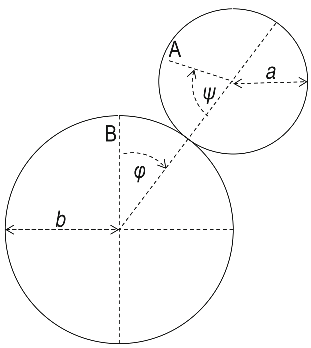


Fig. 1.3 Sphere rolling without slipping on another sphere

Proof The system has two degrees of freedom ($n = 2$). Given the system configuration, one chooses as generalized coordinates the two angles φ and ψ defined as follows (Fig. 1.3). At the initial instant, the sphere of radius a is on top of the sphere of radius b. The contact point is called A on the sphere of radius a and B on the sphere of radius b. After some time, the line passing through the centres of the two spheres shifted by an angle φ with respect to the radius reaching the initial contact point B on the sphere of radius b and from an angle of ψ with respect to the radius reaching the initial point of contact A on the sphere of radius a. The angles φ and ψ are counted positively in the clockwise direction.

The system has one constraint, rolling without slipping, which results in the equation

$$b\dot{\varphi} = a\dot{\psi} \tag{1.31}$$

or

$$b\varphi = a\psi \tag{1.32}$$

as $\varphi = 0$ and $\psi = 0$ at the initial instant when points A and B were combined.

Kinetic energy includes two terms: the first for the movement of rotation of the centre of mass of the sphere of radius a around the sphere of radius b and the second term for the rotation of the sphere of radius a around its centre, yielding

$$T = \frac{1}{2}m\,(a+b)^2\,\dot{\varphi}^2 + \frac{1}{2}I\,\omega^2 \tag{1.33}$$

where I is the moment of inertia of the sphere equal to $I = \frac{2}{5}ma^2$ and ω is the instantaneous rotation velocity of the sphere of radius a whose value is $\omega = \dot{\varphi} + \dot{\psi}$. The kinetic energy is then

$$T = \frac{1}{2}m\,(a+b)^2\,\dot{\varphi}^2 + \frac{1}{5}ma^2\left(\dot{\varphi} + \dot{\psi}\right)^2 \tag{1.34}$$

Taking as a reference the horizontal plane passing through the centre of the sphere of radius b, the potential is

$$V = mg\,(a+b)\cos\varphi \tag{1.35}$$

The Lagrangian reads

$$L = \frac{1}{2}m\,(a+b)^2\,\dot{\varphi}^2 + \frac{1}{5}ma^2\left(\dot{\varphi} + \dot{\psi}\right)^2 - mg\,(a+b)\cos\varphi \tag{1.36}$$

Lagrange equations (1.13) are then written

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\alpha}\right) - \frac{\partial L}{\partial q_\alpha} = \lambda_1 C_{1,\alpha} \tag{1.37}$$

The constraint equation (1.32) gives after differentiation

$$b\,\partial\varphi - a\,\partial\psi = 0 \tag{1.38}$$

which leads to $C_{1,\varphi} = b$ et $C_{1,\psi} = -a$.

For $q_\alpha = \varphi$

$$\begin{aligned}
\frac{\partial L}{\partial \dot\varphi} &= m\,(a+b)^2\,\dot\varphi + \frac{2}{5}ma^2\left(\dot\varphi + \dot\psi\right)\\
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\varphi}\right) &= m\,(a+b)^2\,\ddot\varphi + \frac{2}{5}ma^2\left(\ddot\varphi + \ddot\psi\right)\\
\frac{\partial L}{\partial \varphi} &= mg\,(a+b)\sin\varphi
\end{aligned}$$

Lagrange's first equation reads

$$m\,(a+b)^2\,\ddot\varphi + \frac{2}{5}ma^2\left(\ddot\varphi + \ddot\psi\right) - mg\,(a+b)\sin\varphi = b\lambda_1 \tag{1.39}$$

For $q_\alpha = \psi$

$$\begin{aligned}
\frac{\partial L}{\partial \dot\psi} &= \frac{2}{5}ma^2\left(\dot\varphi + \dot\psi\right)\\
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\psi}\right) &= \frac{2}{5}ma^2\left(\ddot\varphi + \ddot\psi\right)\\
\frac{\partial L}{\partial \psi} &= 0
\end{aligned}$$

Lagrange's second equation reads

$$\frac{2}{5}ma^2\left(\ddot\varphi + \ddot\psi\right) = -a\lambda_1 \tag{1.40}$$

Deriving the constraint equation (1.31) with respect to time, we have $\ddot\psi = \frac{b}{a}\ddot\varphi$, that is replaced in (1.40) to obtain

$$\lambda_1 = -\frac{2}{5}m\,(a+b)\,\ddot\varphi \tag{1.41}$$

By substituting by (1.40) and (1.41) in (1.39), one obtains finally

$$\ddot\varphi = \frac{5g}{7\,(a+b)}\sin\varphi \tag{1.42}$$

□

1.2.4 Exercise 4: Truck Descending a Slope

A truck with four identical wheels runs down a slope freewheeling , i.e. with the motor disengaged, without braking or slipping. Each wheel is considered to be a homogeneous disc of radius r and mass m. The rest of the truck, i.e. wheels not included, has a mass M. To a point A of the truck is suspended an object of non-negligible mass m', and of moment of inertia I with respect to a horizontal axis perpendicular to the velocity of A and passing through A. All considered movements are frictionless and parallel to the plane of the Figure.
(1) Give the Lagrange equations of the system.
(2) Give a first integral of the system.
(3) Show that the system can take, for appropriate initial conditions, a movement in which the inclination of the object of mass $m'$and the truck acceleration remain both constant.
(4) Calculate this angle of inclination in case 3 and show that it is not zero or equal to a right angle if the road followed by the truck is neither horizontal nor vertical.
(5) Show that, if the wheel mass can be neglected, the angle of inclination in case 3 can be equal to the slope inclination.

Proof (1) Let γ be the slope angle with the horizontal, and l the distance from the centre of inertia G of the object of mass m' to the suspension point A (Fig. 1.4).

The system has two degrees of freedom ($n = 2$): the linear displacement of the truck and the oscillation of the suspended object. Given the system configuration, one chooses as generalized coordinates x, the abscissa C of the hub of a rear wheel along an axis parallel to the slope and counted from a point O of reference, and θ, the angle made by AG with the downward vertical.

The kinetic energy consists of three terms: the first for the rotational movement of the four wheels of the truck, the second for the linear displacement of the truck and the third for the oscillation movement of the suspended object, i.e.

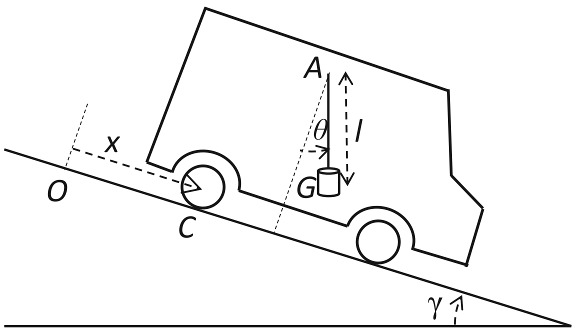


Fig. 1.4 Truck descending a slope

$$T = T_{4\,wheels} + T_{truck} + T_{object} \tag{1.43}$$

The kinetic energy of the four wheels is obviously four times the kinetic energy of a wheel, which consists of two terms: the first due to the inertia of the wheel in its rotation movement around the hub and the second due to the linear displacement of the wheel (more precisely, the linear displacement of the centre of mass of the wheel supposed to be in the middle of the wheel), i. e.

$$T_{4\,wheels} = 4T_{1\,wheel} \tag{1.44}$$

$$= 4\left(\frac{I_{wheel}\dot{\omega}^2}{2} + \frac{m\dot{x}^2}{2}\right) \tag{1.45}$$

where $\dot{\omega}$ is the instantaneous angular velocity of the wheel rotation and I_{wheel} is the moment of inertia with respect to the centre of the wheel considered as a homogeneous disc of radius r and mass m, i.e. $I_{wheel} = \frac{mr^2}{2}$. It then comes from (1.45)

$$T_{4\,wheels} = 4\left(\frac{mr^2\dot{\omega}^2}{4} + \frac{m\dot{x}^2}{2}\right) \tag{1.46}$$

$$= 4\left(\frac{m\dot{x}^2}{4} + \frac{m\dot{x}^2}{2}\right) \tag{1.47}$$

$$= 3m\dot{x}^2 \tag{1.48}$$

where one used in (1.46) the fact that $\dot{x} = r\dot{\omega}$, which is the condition of rolling without slipping.

The truck's kinetic energy is simply

$$T_{truck} = \frac{M\dot{x}^2}{2} \tag{1.49}$$

and the kinetic energy of the suspended object also comprises two terms: the first due to the inertia of the object in its oscillation movement and the second due to the linear displacement of the centre of inertia of the object, i.e.

$$T_{object} = \frac{I\dot{\theta}^2}{2} + \frac{m' v_{object}^2}{2} \tag{1.50}$$

The velocity v_{object} of the object is that of the object's centre of mass which can be assumed to be identical to the centre of inertia G of the object. It is found by deriving with respect to time the coordinate along the X axis of the G point, i.e.

$$v_{object} = \frac{d\,x_{OG}}{dt} = \frac{d}{dt}\left(x_{OC} + x_{CA} + x_{AG}\right) \tag{1.51}$$
$$= \frac{d}{dt}\left(x + x_{CA} + l\sin\left(\theta + \gamma\right)\right) \tag{1.52}$$
$$= \dot{x} + l\dot{\theta}\cos\left(\theta + \gamma\right) \tag{1.53}$$

where the distance x_{CA} between the rear wheel hub and the suspension point of the object is assumed to be constant. It then comes from (1.50) with (1.53)

$$T_{objet} = \frac{I\dot{\theta}^2}{2} + \frac{m'}{2}\left(\dot{x} + l\dot{\theta}\cos\left(\theta + \gamma\right)\right)^2 \tag{1.54}$$

which gives for total kinetic energy (1.43)

$$T = 3m\dot{x}^2 + \frac{M\dot{x}^2}{2} + \frac{I\dot{\theta}^2}{2} + \frac{m'}{2}\left(\dot{x} + l\dot{\theta}\cos\left(\theta + \gamma\right)\right)^2 \tag{1.55}$$
$$= \left(\frac{6m + M + m'}{2}\right)\dot{x}^2 + m'l\dot{x}\dot{\theta}\cos\left(\theta + \gamma\right) + \left(\frac{I + m'l^2\cos^2\left(\theta + \gamma\right)}{2}\right)\dot{\theta}^2 \tag{1.56}$$

The potential has also three terms and is written

$$V = \left(V_{4\,wheels} + V_{truck}\right) + V_{object} \tag{1.57}$$
$$= -\left(4m + M\right)gx\sin\gamma - m'g\left(x\sin\gamma + l\cos\theta\right) \tag{1.58}$$
$$= -\left(4m + M + m'\right)gx\sin\gamma - m'gl\cos\theta \tag{1.59}$$

The Lagrangian reads then

$$L = \left(\frac{6m + M + m'}{2}\right)\dot{x}^2 + m'l\dot{x}\dot{\theta}\cos\left(\theta + \gamma\right) + \left(\frac{I + m'l^2\cos^2\left(\theta + \gamma\right)}{2}\right)\dot{\theta}^2$$
$$+ \left(4m + M + m'\right)gx\sin\gamma + m'gl\cos\theta \tag{1.60}$$

For the generalized coordinate x, we find

$$\frac{\partial L}{\partial \dot{x}} = \left(6m + M + m'\right)\dot{x} + m'l\dot{\theta}\cos\left(\theta + \gamma\right) \tag{1.61}$$
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \left(6m + M + m'\right)\ddot{x} + m'l\ddot{\theta}\cos\left(\theta + \gamma\right) - m'l\dot{\theta}^2\sin\left(\theta + \gamma\right) \tag{1.62}$$
$$\frac{\partial L}{\partial x} = \left(4m + M + m'\right)g\sin\gamma \tag{1.63}$$

Lagrange's first equation reads

$$\left(6m + M + m'\right)\ddot{x} + m'l\ddot{\theta}\cos\left(\theta + \gamma\right) - m'l\dot{\theta}^2\sin\left(\theta + \gamma\right) - \left(4m + M + m'\right)g\sin\gamma = 0 \tag{1.64}$$

For the generalized coordinate θ, we find

$$\frac{\partial L}{\partial \dot{\theta}} = m' l \dot{x} \cos(\theta + \gamma) + \left(I + \dot{\theta}^2 m' l^2 \cos^2(\theta + \gamma)\right)\dot{\theta} \tag{1.65}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = m' l \ddot{x} \cos(\theta + \gamma) - m' l \dot{x}\dot{\theta} \sin(\theta + \gamma) \tag{1.66}$$

$$+ \left(I + m' l^2 \cos^2(\theta + \gamma)\right)\ddot{\theta} - 2m' l^2 \cos(\theta + \gamma)\sin(\theta + \gamma)\,\dot{\theta}^2 \tag{1.67}$$

$$\frac{\partial L}{\partial \theta} = -m' l \dot{x}\dot{\theta} sin(\theta + \gamma) - m' l^2 \dot{\theta}^2 \cos(\theta + \gamma)\sin(\theta + \gamma) - m' g l \sin\theta \tag{1.68}$$

Lagrange's second equation reads

$$m' l \ddot{x} \cos(\theta + \gamma) + \left(I + m' l^2 \cos^2(\theta + \gamma)\right)\ddot{\theta} \tag{1.69}$$

$$-m' l^2 \cos(\theta + \gamma)\sin(\theta + \gamma)\,\dot{\theta}^2 + m' g l \sin\theta = 0 \tag{1.70}$$

(2) A first integral of the movement is given by $T + V = E$ where E is a constant, the total energy of the system as there is no friction or other energy losses, which yields

$$\left(\tfrac{6m+M+m'}{2}\right)\dot{x}^2 + m' l \dot{x}\dot{\theta}\cos(\theta + \gamma) + \left(\tfrac{I+m' l^2 \cos^2(\theta+\gamma)}{2}\right)\dot{\theta}^2 \tag{1.71}$$

$$-\left(4m + M + m'\right) g x \sin\gamma - m' g l \cos\theta = E \tag{1.72}$$

(3) For θ constant, i.e. $\dot{\theta} = \ddot{\theta} = 0$, Lagrange's first equation (1.64) yields

$$\ddot{x} = \left(\frac{4m + M + m'}{6m + M + m'}\right) g \sin\gamma \tag{1.73}$$

The acceleration of the truck and the inclination of the object can therefore be constant together.
(4) In this case 3, Lagrange's second equation reduces to

$$\ddot{x}\cos(\theta + \gamma) + g\sin\theta = 0 \tag{1.74}$$

By replacing $\ddot{x}$ by (1.73), one obtains

$$\left(\frac{4m + M + m'}{6m + M + m'}\right)\sin\gamma\cos(\theta + \gamma) + \sin\theta = 0 \tag{1.75}$$

or

$$\theta = \arctan\left(\frac{\sin\gamma\cos\gamma}{\sin^2\gamma - \left(\frac{6m+M+m'}{4m+M+m'}\right)}\right) \tag{1.76}$$

As the road taken by the truck is neither horizontal nor vertical, i.e. $0 < \gamma < \frac{\pi}{2}$, θ can't be nil or equal to $\frac{\pi}{2}$.
(5) If the wheels have a negligible mass in front of those of the truck and the suspended object, i.e. $m \ll M + m'$, we have from (1.76)

$$\tan\theta \approx \frac{\sin\gamma\cos\gamma}{\sin^2\gamma - 1} = -\tan\gamma \tag{1.77}$$

and $\theta \approx -\gamma$. The inclination of the object can therefore be (almost) equal to the slope. □

1.2.5 Exercise 5: Sliding and Rotating Masses

A mass m is attached to two rigid bars of the same length l and without mass. The top bar is connected to a fixed point of a vertical axis and the bottom bar is attached to another mass m sliding freely and without friction on the vertical axis. All connections are ideal, i.e. without friction. The entire system is rotating at constant angular velocity ω around the vertical axis in a uniform gravity field. What are the equilibrium positions of the system?

Proof Let's call the two masses m respectively P_1 rotating around the axis and P_2 sliding on the axis (Fig. 1.5). There is only one degree of freedom ($n = 1$) as both masses are linked and connected to the vertical axis. The only generalized coordinate is the angle θ made by the two bars with the vertical axis.[2]

One first looks for Lagrange's equations of the system. In Cartesian coordinates with the Z axis along the downward vertical, the coordinates of the positions of the two masses are respectively

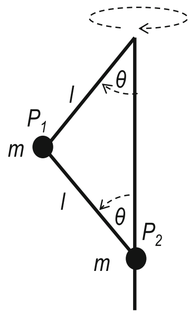


Fig. 1.5 Masses sliding and rotating around a vertical axis

[2]One notices that the two angles θ can only be equal as long as they are comprised between 0 and $\frac{\pi}{2}$, $0 \leq \theta \leq \frac{\pi}{2}$.

$$(x_1,\ y_1,\ z_1) = (l\sin\theta\cos(\omega t)\,,\ l\sin\theta\sin(\omega t)\,,\ l\cos\theta) \tag{1.78}$$
$$(x_2,\ y_2,\ z_2) = (0,\ 0,\ 2l\cos\theta) \tag{1.79}$$

where the angle ωt is counted positively from an X axis horizontal (not shown in the figure). The velocity coordinates are then written as follows:

$$(\dot{x}_1,\ \dot{y}_1,\ \dot{z}_1) = \Big(\big(l\dot{\theta}\cos\theta\cos(\omega t) - l\omega\sin\theta\sin(\omega t)\big)\,,$$
$$\big(l\dot{\theta}\cos\theta\sin(\omega t) + l\omega\sin\theta\cos(\omega t)\big)\,,\ -l\dot{\theta}\sin\theta\Big) \tag{1.80}$$
$$(\dot{x}_2,\ \dot{y}_2,\ \dot{z}_2) = \big(0,\ 0,\ -2l\dot{\theta}\sin\theta\big) \tag{1.81}$$

The velocity squares of the two masses read then $v_1^2 = l^2\left(\dot{\theta}^2 + \omega^2\sin^2\theta\right)$ and $v_2^2 = 4l^2\dot{\theta}^2\sin^2\theta$. The kinetic energy, potential and Lagrangian read

$$T = \frac{m}{2}\left(v_1^2 + v_2^2\right) = \frac{ml^2}{2}\left(\dot{\theta}^2 + \left(\omega^2 + 4\dot{\theta}^2\right)\sin^2\theta\right) \tag{1.82}$$
$$V = -mg\,(z_1 + z_2) = -3mgl\cos\theta \tag{1.83}$$
$$L = \frac{ml^2}{2}\left(\dot{\theta}^2 + \left(\omega^2 + 4\dot{\theta}^2\right)\sin^2\theta\right) + 3mgl\cos\theta \tag{1.84}$$

With

$$\frac{\partial L}{\partial\dot{\theta}} = ml^2\dot{\theta}\left(1 + 4\sin^2\theta\right) \tag{1.85}$$
$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) = ml^2\left(\ddot{\theta}\left(1 + 4\sin^2\theta\right) + 8\dot{\theta}^2\sin\theta\cos\theta\right) \tag{1.86}$$
$$\frac{\partial L}{\partial\theta} = ml^2\left(\omega^2 + 4\dot{\theta}^2\right)\sin\theta\cos\theta - 3mgl\sin\theta \tag{1.87}$$

Lagrange's equation reads after simplification,

$$\ddot{\theta}\left(1 + 4\sin^2\theta\right) + \left(4\dot{\theta}^2 - \omega^2\right)\sin\theta\cos\theta + 3\frac{g}{l}\sin\theta = 0 \tag{1.88}$$

At equilibrium, angular velocities and accelerations must be nil, $\ddot{\theta} = \dot{\theta} = 0$, which replaced in (1.88) yields

$$\left(-\omega^2\cos\theta + 3\frac{g}{l}\right)\sin\theta = 0 \tag{1.89}$$

which gives the equation of dynamic equilibrium in θ which has three solutions: $\theta = 0$ or π for $\sin\theta = 0$, and $\theta = \arccos\left(\frac{3g}{\omega^2 l}\right)$ for the other term.

The first solution $\theta = 0$ corresponds to a stable equilibrium with the two masses aligned along the vertical axis. The second solution $\theta = \pi$ corresponds to an unstable equilibrium with mass P_2 at the upper fixed attachment point and mass P_1 above it.

Here the two angles θ are different, that of P_2 is nil and that of P_1 is equal to π. These two solutions are independent of the rotation velocity ω and mass m.

As the function arccos x is only defined for arguments x such that $-1 \leq x \leq 1$, for the third solution to be possible, one must have

$$\omega \geq \sqrt{\frac{3g}{l}} \tag{1.90}$$

which means that for slow rotations such as $\omega < \sqrt{\frac{3g}{l}}$, there are only the first two dynamic equilibrium solutions for $\theta = 0$ and π that exist. For faster rotations such as condition (1.90) is fulfilled, there is a third dynamic equilibrium position whose angular value θ depends on the speed of rotation ω but always independent of the mass m. When the rotation velocity ω increases indefinitely, the angle θ tends towards $\frac{\pi}{2}$, i.e. $\omega \to \infty \Rightarrow \theta \to \frac{\pi}{2}$. □

Chapter 2
Hamilton Equations

Sir William Rowan Hamilton (born 4 August 1805 and died 2 September 1865 in Dublin, Ireland) was an Irish mathematician, astronomer, and mathematical physicist, who made important contributions to classical mechanics, optics, geometry and algebra. Raised by his uncle from the age of three, Hamilton learnt several foreign languages and became first a linguist. Hamilton was then part of a school of mathematicians associated with Trinity College in Dublin, which he entered at age 18. He studied both classics and mathematics, and was appointed Professor of Astronomy just prior to his graduation in 1827, at age 22. He then took up residence at Dunsink Observatory where he spent the rest of his life. His studies of mechanical and optical systems led him to discover new mathematical concepts and techniques. His best-known contribution to mathematical physics is the reformulation of Newtonian mechanics, now called Hamiltonian mechanics. This work has proven central to the modern study of classical field theories such as electromagnetism, and to the development of quantum mechanics. Many of the fundamental concepts used in quantum mechanics have been

V. Pletser, *Lagrangian and Hamiltonian Analytical Mechanics: Forty Exercises Resolved and Explained*, UNITEXT for Physics,
https://doi.org/10.1007/978-981-13-3026-1_2

named "Hamiltonian" in his honor. His most important discovery was the algebra of quaternions in 1843 and later on, of the biquaternion algebra, which provided representational tools for Minkowski space and the Lorentz group early in the twentieth century.

2.1 Reminder

2.1.1 *Hamiltonian*

One defines the Hamiltonian H as

$$H = \sum_{\alpha=1}^{n} p_\alpha \dot{q}_\alpha - L \tag{2.1}$$

H is therefore a function $H(p_\alpha, q_\alpha, t)$ of generalized coordinates q_α and of generalized moments $p_\alpha = \frac{\partial L}{\partial \dot{q}_\alpha}$. In other words, all generalized velocities $\dot{q}_\alpha$ of the Lagrangian are replaced by the generalized moments in the Hamiltonian.

2.1.2 *Hamilton Equations*

Hamilton equations read

$$\dot{p}_\alpha = -\frac{\partial H}{\partial q_\alpha} \tag{2.2}$$

$$\dot{q}_\alpha = \frac{\partial H}{\partial p_\alpha} \tag{2.3}$$

So we have $2n$ equations whose degree does not exceed $\dot{q}$ while we had n equations of Lagrange whose maximum degree was $\ddot{q}$.

2.1.3 *Conservative System*

If the system is conservative, i.e. all forces derive from a potential, then H represents the total energy of the system

$$H = T + V \tag{2.4}$$

2.1.4 Expression of the Hamiltonian in Different Coordinate Systems

In spherical coordinates (r, θ, φ)

$$H = \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\varphi^2}{r^2 \sin^2\theta}\right) + V(r, \theta, \varphi) \tag{2.5}$$

In cylindrical coordinates (ρ, φ, z)

$$H = \frac{1}{2m}\left(p_\rho^2 + \frac{p_\varphi^2}{\rho^2} + p_z^2\right) + V(\rho, \varphi, z) \tag{2.6}$$

In parabolic coordinates (ξ, η, φ)

$$H = \frac{2}{m}\left(\frac{\xi p_\xi^2 + \eta p_\eta^2}{\xi + \eta} + \frac{p_\varphi^2}{4\xi\eta}\right) + V(\xi, \eta, \varphi) \tag{2.7}$$

In elliptical coordinates (ξ, η, φ)

$$H = \frac{1}{2m\sigma^2\left(\xi^2 - \eta^2\right)}\left(\left(\xi^2 - 1\right)p_\xi^2 + \left(1 - \eta^2\right)p_\eta^2 + \left(\frac{1}{\xi^2 - 1} + \frac{1}{1 - \eta^2}\right)p_\varphi^2\right) + V(\xi, \eta, \varphi) \tag{2.8}$$

where 2σ is the distance between the two foci or the two attractive points.

2.2 Exercises

2.2.1 Exercise 6: Particle in a Plane with Central Force

A particle of mass m is moving in the plane (X, Y) under the influence of a central force depending only on its distance to the origin. Find Hamilton equations of movement.

Proof The system has two degrees of freedom ($n = 2$). One chooses polar coordinates (r, θ) as generalized coordinates (see Fig. 2.1). The kinetic energy reads

$$T = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) \tag{2.9}$$

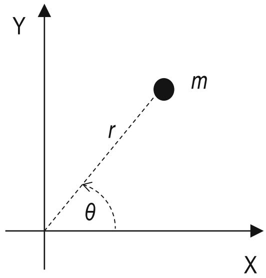


Fig. 2.1 Particle in a plane with central force

The potential due to the central force is isotropic in the plane (X, Y) and is written generally $V(r)$. The Lagrangian reads

$$L = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - V(r) \tag{2.10}$$

The generalized moments are derived as follows

$$p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r} \Rightarrow \dot{r} = \frac{p_r}{m} \tag{2.11}$$

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} \Rightarrow \dot{\theta} = \frac{p_\theta}{mr^2} \tag{2.12}$$

The Hamiltonian (2.1) reads

$$H = \left(p_r\dot{r} + p_\theta\dot{\theta}\right) - \left[\frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - V(r)\right] \tag{2.13}$$

Replacing $\dot{r}$ and $\dot{\theta}$ by (2.11) and (2.12) in (2.13) yields

$$H = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} + V(r) \tag{2.14}$$

Hamilton's equations (2.2), (2.3) are then written for $q_\alpha = r$

$$\dot{r} = \frac{\partial H}{\partial p_r} = \frac{p_r}{m} \tag{2.15}$$

$$\dot{p}_r = -\frac{\partial H}{\partial r} = \frac{p_\theta^2}{mr^3} - \frac{\partial V(r)}{\partial r} \tag{2.16}$$

and for $q_\alpha = \theta$

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{mr^2} \tag{2.17}$$

$$\dot{p}_\theta = -\frac{\partial H}{\partial \theta} = 0 \tag{2.18}$$

We deduce from (2.18) that p_θ is a constant of the movement and that θ is a cyclic coordinate (see next chapter). □

2.2.2 Exercise 7: Harmonic Oscillator

The harmonic oscillator is a mass m attached to a massless spring of stiffness k and attached to a fixed point. The mass slides without friction on a horizontal support. Give Hamilton equations of movement.

Proof The system has one degree of freedom ($n = 1$). One chooses as the generalized coordinate the horizontal distance q between the current position x and the position at rest x_0 of the centre of mass of the mass m, i.e. $q = x - x_0$ (see Fig. 2.2). The spring restoring force is $-kq$.

The kinetic energy, potential and Lagrangian read

$$T = \frac{1}{2}m\dot{q}^2 \tag{2.19}$$

$$V = \frac{1}{2}kq^2 \tag{2.20}$$

$$L = \frac{1}{2}\left(m\dot{q}^2 - kq^2\right) \tag{2.21}$$

The generalized moment p is found by

$$p = \frac{\partial L}{\partial \dot{q}} = m\dot{q} \Rightarrow \dot{q} = \frac{p}{m} \tag{2.22}$$

The Hamiltonian reads

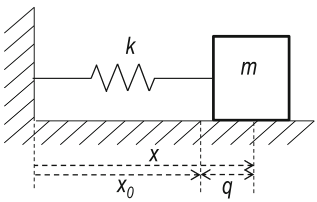


Fig. 2.2 Harmonic oscillator

$$H = p\dot{q} - \frac{1}{2}\left(m\dot{q}^2 - kq^2\right) \tag{2.23}$$

$$= \frac{p^2}{2m} + \frac{kq^2}{2} \tag{2.24}$$

Hamilton equations are

$$\dot{q} = \frac{\partial H}{\partial p} = \frac{p}{m} \tag{2.25}$$

$$\dot{p} = -\frac{\partial H}{\partial q} = -kq \tag{2.26}$$

The movement equation of the harmonic oscillator is deduced from (2.25), (2.26)

$$\ddot{q} = -\frac{k}{m}q \tag{2.27}$$

□

Chapter 3
First Integral and Variational Principle

3.1 Reminder

3.1.1 Cyclic Coordinate

A generalized coordinate q_α is cyclic or ignorable when

$$\frac{\partial L}{\partial q_\alpha} = 0 \tag{3.1}$$

i.e. the coordinate does not appear in the Lagrangian (but the generalized velocity $\dot{q}_\alpha$ can appear in L). Lagrange's equations then reduce to

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\alpha}\right) = 0 \tag{3.2}$$

or

$$\frac{\partial L}{\partial \dot{q}_\alpha} = p_\alpha = \text{constant} \tag{3.3}$$

and, by definition, the conjugated moment p_α is constant. In Exercise 6, p_θ is constant and one verifies that θ does not appear in L. The constant conjugated moment p_α is called a constant of movement. This allows to directly determine a first integral of the movement, i.e. that it is easy to write the movement trajectory equation.

3.1.2 Poisson Brackets

For two functions of two variables p and q and of time t, $f\,(p, q, t)$ and $g\,(p, q, t)$, one defines the Poisson brackets of these two functions by

V. Pletser, *Lagrangian and Hamiltonian Analytical Mechanics: Forty Exercises Resolved and Explained*, UNITEXT for Physics,
https://doi.org/10.1007/978-981-13-3026-1_3

$$[f, g] = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} \right) \tag{3.4}$$

The Poisson bracket has the following algebraic properties:

$$[f, g] = -[g, f] \quad \text{(anticommutativity)} \tag{3.5}$$
$$[f, f] = 0 \tag{3.6}$$
$$[f, k] = 0 \quad \text{if } k = \text{constant} \tag{3.7}$$
$$[(f_1 + f_2), g] = [f_1, g] + [f_2, g] \tag{3.8}$$
$$[(f_1 f_2), g] = f_1 [f_2, g] + f_2 [f_1, g] \tag{3.9}$$
$$[(kf), g] = k [f, g] \quad \text{if } k = \text{constant} \tag{3.10}$$
$$\frac{\partial}{\partial t} [f, g] = \left[\frac{\partial f}{\partial t}, g \right] + \left[f, \frac{\partial g}{\partial t} \right] \tag{3.11}$$
$$[f, [g, h]] + [g, [h, f]] + [h, [f, g]] = 0 \quad \text{(Jacobi identity)} \tag{3.12}$$

that can be demonstrated directly by application of the definition (3.4).

The total time derivative of a function f can be written in function of the Hamiltonian H as

$$\frac{df}{dt} = [f, H] + \frac{\partial f}{\partial t} \tag{3.13}$$

The function $f(q, p, t)$ is a first integral if and only if

$$\frac{df}{dt} = 0 \Longleftrightarrow [f, H] + \frac{\partial f}{\partial t} = 0 \tag{3.14}$$

The equations of movement can be written as

$$[q_i, H] = \frac{\partial H}{\partial p_i} \tag{3.15}$$
$$[p_i, H] = -\frac{\partial H}{\partial q_i} \tag{3.16}$$

because one has $\dot{q}_i = [q_i, H]$ et $\dot{p}_i = [p_i, H]$.

3.1.3 Theorem of Poisson

Theorem. If $f(q, p, t)$ and $g(q, p, t)$ are two first integrals of the movement, their Poisson bracket $[f, g]$ is also a first integral of the movement.
This theorem leads to the following corollaries:

- if f is a first integral of the movement, $\frac{\partial f}{\partial t}$ is also a first integral of the movement.
- if f and all following derivatives up to the order $(n-1)$ are first integrals of the movement, then $\frac{\partial^n f}{\partial t^n}$ is also an first integral of the movement.

It should be noted that this process does not work indefinitely; it soon appears that the new first integrals are no longer independent of each other.

3.1.4 Euler Equation

A common problem in mathematics is to find a curve $y = Y(x)$ joining two points $x = a$ and $x = b$ such that the integral $\int_a^b F\left(x, y, y'\right) dx$ (with $y' = \frac{dy}{dx}$) is either maximum or minimum. When this condition is met, it is said that the curve $Y(x)$ is extremal. This condition is shown to be equivalent to

$$\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial y} = 0 \tag{3.17}$$

which is Euler's equation.

Similarly in mechanics, one considers the Lagrangian of a system $L = T - V$ as a function of which one would like to determine the extremal curve between two instants t_1 and t_2, i.e. such that the integral $\int_{t_1}^{t_2} L\left(q, \dot{q}, t\right) dt$ is maximum or minimum. This condition is equivalent to the Euler equation applied to the Lagrangian, which yields Lagrange equation (1.7) of the movement.

3.1.5 Variational Principle

This finding prompted Hamilton to state the Variational Principle:
"*A conservative mechanical system evolves from instant t_1 to instant t_2 in such a way that the action integral $\int_{t_1}^{t_2} L\left(q, \dot{q}, t\right) dt$ has an extremal value*".
In most problems, the extremal value will be minimal and this principle is often noted in the form of $\delta\left(\int_{t_1}^{t_2} L\left(q, \dot{q}, t\right) dt\right) = 0$.

3.1.6 Application in Optics: Fermat Principle

All geometric optics can be based on Fermat Principle:
"*Light travels from one point to another over trajectories such that the travel time is minimal locally*" (i.e., minimal for each element of the trajectory).

This Fermat Principle is a variational principle such as time $\mathcal{T}$ taken by a light beam between two points x_1 and x_2 is minimum, i.e. $\delta\left(\mathcal{T}\right) = \delta\left(\int_{x_1}^{x_2} \mathcal{L}\, dt\right) = 0$, where $\mathcal{L}$ is here the "optical" Lagrangian, equivalent to the mechanical Lagrangian (1.6). The trajectories of the light rays follow equations similar to those of Euler and Lagrange (see Exercise 12).

3.2 Exercises

3.2.1 Exercise 8: Watt Regulator

In a uniform gravity field, two equal masses m are attached at the ends of two rigid bars of same length l revolving around a vertical axis. The inclination of these two bars with respect to the vertical axis can vary by means of a slide (sliding without friction on the vertical axis) and to which are fixed two small support bars mounted on free articulations. This system is called a Watt regulator and allows to stabilize the rotation velocity of the vertical axis.
(1) Give the first integrals of movement.
(2) How does this system stabilize the rotation velocity of the vertical axis?

Proof (1) The system has two degrees of freedom ($n = 2$). One chooses as generalized coordinates the two angles, respectively θ, the inclination of each of the two bars with respect to the vertical axis and φ, the rotation angle of the plane of the two bars with respect to a fixed direction (see Fig. 3.1).

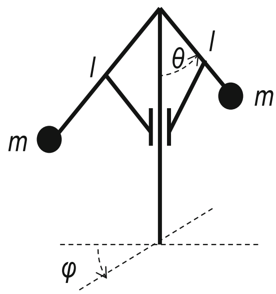
The kinetic energy, potential and Lagrangian read

$$T = ml^2\left(\dot{\theta}^2 + \dot{\varphi}^2 \sin^2\theta\right) \tag{3.18}$$

$$V = -2mgl\ \cos\theta \tag{3.19}$$

$$L = ml^2\left(\dot{\theta}^2 + \dot{\varphi}^2 \sin^2\theta\right) + 2mgl\ \cos\theta \tag{3.20}$$

Fig. 3.1 Watt regulator



The generalized coordinate φ does not appear explicitly in the Lagrangian, so it is cyclic. The first integral of movement is given by

$$\frac{\partial L}{\partial \dot{\varphi}} = 2ml^2 \dot{\varphi} \sin^2 \theta = \text{constant} = p_\varphi \tag{3.21}$$

which corresponds to the projection of the kinetic moment on the vertical axis (i.e. the product of the distance of the mass m to the vertical axis by the mass m and by the linear velocity of rotation).

As the system is conservative (there is no friction), the second integral of the movement is given by the energy conservation

$$E = T + V = \text{constant} \tag{3.22}$$

$$= ml^2 \left(\dot{\theta}^2 + \dot{\varphi}^2 \sin^2 \theta\right) - 2mgl \cos \theta \tag{3.23}$$

which yields

$$\dot{\theta}^2 + \dot{\varphi}^2 \sin^2 \theta - 2\frac{g}{l} \cos \theta = \frac{E}{ml^2} = \text{constant} \tag{3.24}$$

(2) The Eq. (3.21) explains the dynamic stability of the system: if the inclination θ is constant, the rotation velocity $\dot{\varphi}$ remains constant. If $\dot{\varphi}$ increases, then $\sin^2 \theta$ decreases; if $\dot{\varphi}$ decreases, then $\sin^2 \theta$ increases. □

3.2.2 Exercise 9: First Integral of a Free Material Point

Consider a free material point whose Hamiltonian in Cartesian coordinates is

$$H = \frac{1}{2m} \sum_{i=1}^{3} p_i^2 \tag{3.25}$$

with $p_i = m\dot{x}_i$.
(1) Show that the function $f(p_1, x_1, t) = \frac{p_1 t}{m} - x_1$ is a first integral of movement.
(2) Show that successive derivatives of f are also first integrals of movement.
(3) Show that this process is not infinite.

Proof (1) Suppose that $f(p_1, x_1, t)$ is a first integral; then by (3.14), $[f, H] + \frac{\partial f}{\partial t} = 0$, with

$$[f, H] = \sum_{i=1}^{3} \left(\frac{\partial f}{\partial x_i} \frac{\partial H}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial H}{\partial x_i} \right) = -\frac{p_1}{m} \tag{3.26}$$

$$\frac{\partial f}{\partial t} = \frac{p_1}{m} \tag{3.27}$$

The sum of (3.26) et (3.27) is nil, so $\frac{df}{dt} = 0$ and f is indeed a first integral.
(2) One knows two first integrals of movement: the function f and the total energy E or Hamiltonian H for a conservative system. One can assume that the system is conservative in this case. So, by Poisson's theorem, the bracket $[f, H]$ is also a first integral.

Now, it has been shown that $[f, H] + \frac{\partial f}{\partial t} = 0$ or $\frac{\partial f}{\partial t} = -[f, H]$, so $\frac{\partial f}{\partial t}$ is also a first integral. One has then the relations $\left[\frac{\partial f}{\partial t}, H\right] + \frac{\partial^2 f}{\partial t^2} = 0$, and if $\frac{\partial f}{\partial t}$ is a first integral, then $\frac{\partial^2 f}{\partial t^2}$ will be too. And so forth, $\frac{\partial^n f}{\partial t^n}$ will be a first integral because $\frac{\partial^n f}{\partial t^n} = -\left[\frac{\partial^{n-1} f}{\partial t^{n-1}}, H\right]$.
(3) This process is not infinite because already in the case of a free particle $\frac{\partial^2 f}{\partial t^2} = \frac{\partial}{\partial t}\left(\frac{\partial f}{\partial t}\right) = \frac{\partial}{\partial t}\left(\frac{p_1}{m}\right) = 0$. All subsequent first integrals will also be nil. □

3.2.3 Exercise 10: Brachistochrone Problem

In a uniform gravity field, a particle of mass m glides without friction on a curve in a vertical plane.
(1) Find the time taken by the particle to travel along the curve between two points, if the initial state is at rest.
(2) Determine the form of the curve so that this time is minimum.

Proof Let C be the curve to be determined, P the particle of mass m and coordinates (x, y) and s the curvilinear abscissa of P. One takes two points on the curve such that the first one is the origin O and the second is point A of coordinates (x_f, y_f) (see Fig. 3.2).
(1) A first integral of movement is given by the total energy conservation, i.e. the sum of potential and kinetic energies at points O and P are equal

$$mgy_f + 0 = mg\left(y_f - y\right) + \frac{m}{2}\left(\frac{ds}{dt}\right)^2 \tag{3.28}$$

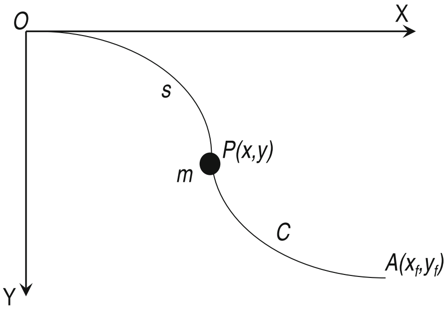


Fig. 3.2 Particle on a curve

where we took the horizontal plane passing through A as a reference for the potential energy and where $\frac{ds}{dt}$ is the instantaneous velocity of P at time t. From (3.28), we find successively

$$\frac{ds}{dt} = \sqrt{2gy} \tag{3.29}$$

$$dt = \frac{ds}{\sqrt{2gy}} \tag{3.30}$$

where the + sign is chosen in front of the radical in (3.29) because s increases when t increases. One finds the time t_f taken by the particle to go from O to A by integrating (3.30)

$$t_f = \int_{s=O}^{s=A} \frac{ds}{\sqrt{2gy}} \tag{3.31}$$

As $ds = \sqrt{(dx)^2 + (dy)^2} = \sqrt{1 + y'^2}dx$ with $y' = \frac{dy}{dx}$, which introduced in (3.31) yields

$$t_f = \frac{1}{\sqrt{2g}} \int_0^{x_f} \sqrt{\frac{1 + y'^2}{y}} dx \tag{3.32}$$

(2) For the time t_f to be minimum, (3.17) must be verified, i.e. $\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = \frac{\partial F}{\partial y}$ with $F = \sqrt{\frac{1+y'^2}{y}}$, yielding

$$1 + y'^2 + 2yy'' = 0 \tag{3.33}$$

with $y'' = \frac{d^2y}{dx^2}$. To find the form of the curve, one must solve the Eq. (3.33). Let $y' = u$, yielding $y'' = \frac{du}{dx} = \frac{du}{dy}\frac{dy}{dx} = \frac{du}{dy}u$. The Eq. (3.33) becomes successively after transformation,

$$1 + u^2 + 2yu\frac{du}{dy} = 0 \tag{3.34}$$

$$\frac{2u\,du}{1+u^2} + \frac{dy}{y} = 0 \tag{3.35}$$

One integrates the equation (3.35) with the integration constant equal to $\ln b$ where b is a constant, giving successively

$$\ln\left(1+u^2\right) + \ln y = \ln b \tag{3.36}$$

$$\left(1+u^2\right) y = b \tag{3.37}$$

$$u = \sqrt{\frac{b-y}{y}} \tag{3.38}$$

$$dx = \sqrt{\frac{y}{b-y}}dy \tag{3.39}$$

where u has been replaced by $u = y' = \frac{dy}{dx}$. One integrates the equation (3.39) with the variable change $y = b\sin^2\theta$ with the integration constant equal to c. One finds then the parametric equations of the curve

$$x = \frac{b}{2}\left(2\theta - \sin 2\theta\right) + c \tag{3.40}$$

$$y = \frac{b}{2}\left(1 - \cos 2\theta\right) \tag{3.41}$$

The geometrical conditions on the curve are such that (1) $c = 0$ because the curve passes through the origin; (2) by taking $\varphi = 2\theta$ and $a = b/2$, a is determined by the condition of the curve passing by the point $A\left(x_f, y_f\right)$. One finally finds

$$x = a\left(\varphi - \sin\varphi\right) \tag{3.42}$$

$$y = a\left(1 - \cos\varphi\right) \tag{3.43}$$

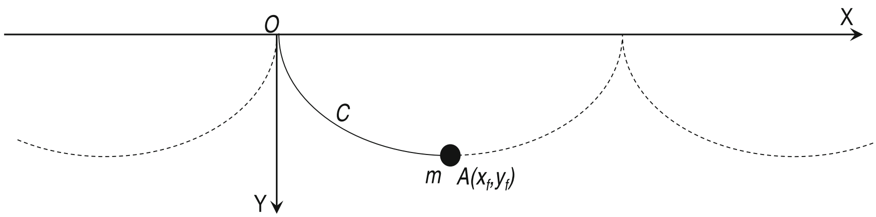


Fig. 3.3 Brachistochrone and cycloid

which is the parametric equation of a cycloid, a curve described by a fixed point on a circle when this circle rolls without slipping on a straight line. The brachistochrone curve is a portion of this cycloid (Fig. 3.3). □

3.2.4 *Exercise 11: Minimum Surface of Revolution*

A complete revolution is described around a X axis by a segment of curve C between the two points $P\,(x_1, y_1)$ and $P'\,(x_2, y_2)$ such that the obtained revolution surface S is minimal.
(1) Show that the surface S is expressed by

$$S = 2\pi \int_{x_1}^{x_2} y\sqrt{1 + y'^2}dx \tag{3.44}$$

(2) Show that the differential equation of the curve C is

$$yy'' = 1 + y'^2 \tag{3.45}$$

(3) Determine the shape of the curve C so that the surface S is minimal.

Proof Let C the curve segment between the points $P\,(x_1, y_1)$ and $P'\,(x_2, y_2)$. The X axis is selected as the axis of rotation (see Fig. 3.4).
(1) The surface S of revolution generated by the rotation of the curve segment C around the X axis is given by Theorem of Guldin

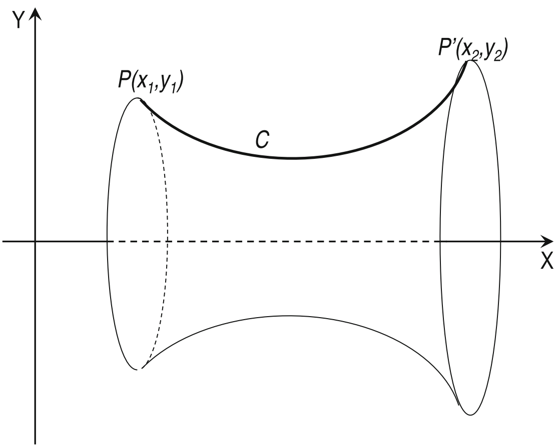


Fig. 3.4 Surface of revolution

$$S = 2\pi\, l_s\, r_s \tag{3.46}$$

where s, l_s and r_s are respectively the curvilinear abscissa, the length of the curve segment C and the distance of the centre of gravity of the curve segment C to the X axis. With $ds = \sqrt{(dx)^2 + (dy)^2} = \sqrt{1 + y'^2}dx$ where $y' = \frac{dy}{dx}$, one has by definition

$$l_s = \int_{s=P}^{s=P'} ds = \int_{x_1}^{x_2} \sqrt{1 + y'^2}dx \tag{3.47}$$

$$r_s = \frac{\int_{s=P}^{s=P'} y\, ds}{\int_{s=P}^{s=P'} ds} = \frac{\int_{x_1}^{x_2} y\sqrt{1 + y'^2}dx}{\int_{x_1}^{x_2} \sqrt{1 + y'^2}dx} \tag{3.48}$$

By replacing in (3.46), one obtains (3.44).
(2) For the surface S to be minimal, (3.17) must be verified, or $\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = \frac{\partial F}{\partial y}$ with $F = y\sqrt{1 + y'^2}$, which yields immediately (3.45) with $y'' = \frac{d^2y}{dx^2}$.
(3) To find the form of the curve, the Eq. (3.45) must be solved. Let $y' = u$, which yields $y'' = \frac{du}{dx} = \frac{du}{dy}\frac{dy}{dx} = \frac{du}{dy}u$. The Eq. (3.45) becomes successively after transformation,

$$1 + u^2 - yu\frac{du}{dy} = 0 \tag{3.49}$$

$$\frac{u\, du}{1 + u^2} - \frac{dy}{y} = 0 \tag{3.50}$$

The Eq. (3.50) is integrated with the integration constant equal to $\ln b$ where b is a constant, which gives successively

$$\frac{1}{2}\ln\left(1 + u^2\right) - \ln y = \ln b \tag{3.51}$$

$$\sqrt{1 + u^2} = by \tag{3.52}$$

$$u = \sqrt{b^2y^2 - 1} \tag{3.53}$$

$$dx = \frac{dy}{\sqrt{b^2y^2 - 1}} \tag{3.54}$$

where u has been replaced by $u = y' = \frac{dy}{dx}$. The Eq. (3.54) is integrated with the integration constant equal to c. Then one obtains successively

$$x = \frac{1}{b} \ln \left(by + \sqrt{b^2 y^2 - 1} \right) + c \tag{3.55}$$

$$e^{b(x-c)} = by + \sqrt{b^2 y^2 - 1} \tag{3.56}$$

$$\sqrt{b^2 y^2 - 1} = e^{b(x-c)} - by \tag{3.57}$$

$$b^2 y^2 - 1 = \left(e^{b(x-c)} - by \right)^2 \tag{3.58}$$

$$y = \frac{e^{b(x-c)} + e^{-b(x-c)}}{2b} = a \cosh \left(\frac{x-c}{a} \right) \tag{3.59}$$

where $a = \frac{1}{b}$ as been posed in (3.59). The two constants a and c can be determined by the condition that the curve C passes through the two end points $P\,(x_1, y_1)$ et $P'\,(x_2, y_2)$. The integration constant c can also be eliminated by translating the origin of the axes along the X axis of a length equal to c, which finally gives

$$y = a \cosh \left(\frac{x}{a} \right) \tag{3.60}$$

that is the equation of a catenary, i.e. a set of small chains connected to each other and attached by the extremities at a certain height above the ground (the attachment points can be at different heights as in our case). The revolution surface is called a catenoid and is a minimal surface of revolution. □

It should be noted that this minimal surface of revolution is also the one formed by a soap film obtained after dipping two parallel circular loops in soapy water.

3.2.5 Exercise 12: Optical Path and Fermat Principle

(1) Calculate generally the optical path of a light beam in a medium of refractive index η.
(2) Show that the optical path is a straight line in a medium of constant refractive index, $\eta =$ constant.
(3) Calculate the optical path in the atmosphere above a flat, sandy and very hot desert, where the refractive index depends on the altitude y above ground and can be represented by the relation $\eta = \eta_0\,(1 - ay)$, with a a constant characteristic of the medium and whose unit is the inverse of a distance.
(4) Deduce the distance between an observer lost in the desert and an oasis he sees from an angle $\arctan\left(\frac{1}{2}\right)$ above the horizontal (one neglects Earth's curvature).

Proof (1) The velocity v of light in a medium of refractive index η and the velocity c of light in vacuum are linked by the relation

$$v = \frac{c}{\eta} \tag{3.61}$$

with obviously $\eta > 1$.

Suppose that the light beam must propagate from a point P_1 to a point P_2 and that the path of the light beam is in a plane containing P_1 and P_2,[1] i.e. that the optical trajectory is a plane curve of equation $y = y(x)$ passing through P_1 and P_2. Calling ds the length of the infinitesimal trajectory along $y(x)$, described in an infinitesimal time dt, the time $\mathcal{T}_{12}$ taken by the light between the two points P_1 and P_2 of coordinates respectively (x_1, y_1) and (x_2, y_2), reads generally

$$\mathcal{T}_{12} = \int_{P_1}^{P_2} dt = \int_{P_1}^{P_2} \frac{ds}{v} = \frac{1}{c} \int_{P_1}^{P_2} \eta ds \tag{3.62}$$

As $ds = \sqrt{(dx)^2 + (dy)^2} = \sqrt{1 + y'^2} dx$ where $y' = \frac{dy}{dx}$, (3.62) becomes

$$\mathcal{T}_{12} = \frac{1}{c} \int_{x_1}^{x_2} \eta \sqrt{1 + y'^2} dx \tag{3.63}$$

and the "optical" Lagrangian $\mathcal{L}$ reads

$$\mathcal{L} = \frac{\eta \sqrt{1 + y'^2}}{c} \tag{3.64}$$

Note that the refractive index η may not be constant and may depend on the optical path, i.e. of x and y, $\eta = \eta(x, y)$.

Applying the Fermat Principle, one minimizes the travel time of the light beam $\mathcal{T}_{12}$, which is given by Euler equation (3.17) with the "optical" Lagrangian $\mathcal{L}$ (3.64). By replacing by

$$\frac{\partial \mathcal{L}}{\partial y'} = \frac{\eta y'}{c\sqrt{1 + y'^2}} \tag{3.65}$$

$$\frac{d}{dx}\left(\frac{\partial \mathcal{L}}{\partial y'}\right) = \frac{1}{c}\left[\frac{\left(\left(\frac{\partial \eta}{\partial x} + y'\frac{\partial \eta}{\partial y}\right) y' + \eta y''\right)}{\sqrt{1 + y'^2}} - \frac{\eta y'^2 y''}{\sqrt{\left(1 + y'^2\right)^3}}\right] \tag{3.66}$$

$$\frac{\partial \mathcal{L}}{\partial y} = \frac{\frac{\partial \eta}{\partial y}\sqrt{1 + y'^2}}{c} \tag{3.67}$$

[1] If this is not the case, the real non-planar trajectory can be broken down into a series of trajectory elements that can be locally approximated by plane trajectory elements in planes tangential to the real trajectory element.

Equation (3.17) becomes, after simplification,

$$y'' + \frac{y'^2 + 1}{\eta} \left(\frac{\partial \eta}{\partial x} y' - \frac{\partial \eta}{\partial y} \right) = 0 \tag{3.68}$$

(2) In a medium of constant refractive index η, one has $\frac{\partial \eta}{\partial x} = \frac{\partial \eta}{\partial y} = 0$, and (3.68) reduces to $y'' = 0$, so y' is constant and $y(x) = c_1 x + c_2$ where c_1 et c_2 are integration constants. The optical path is therefore a straight line in a medium of constant refractive index η.
(3) If $\eta = \eta_0 (1 - ay)$, one has $\frac{\partial \eta}{\partial x} = 0$, $\frac{\partial \eta}{\partial y} = -\eta_0 a$ and (3.68) becomes

$$y'' + \frac{a\left(y'^2 + 1\right)}{(1 - ay)} = 0 \tag{3.69}$$

After changing variable $u = \frac{1}{a} - y$, which gives successively $y = \frac{1}{a} - u$, $y' = -u'$ et $y'' = -u''$, (3.69) becomes

$$-u'' + \frac{\left(u'^2 + 1\right)}{u} = 0 \tag{3.70}$$

or

$$uu'' - u'^2 - 1 = 0 \tag{3.71}$$

This non-linear second degree differential equation is solved (see 6.111, Kamke, 1943) in

$$u = \frac{1}{c_1} \cosh(c_1 x + c_2) \tag{3.72}$$

or

$$y = \frac{1}{a} - \frac{1}{c_1} \cosh(c_1 x + c_2) \tag{3.73}$$

where c_1 and c_2 are integration constants. The optical path is therefore described by a hyperbolic function.
(4) Let the lost observer in the desert be in $x = 0$ and the oasis in $x = L$. As the oasis is viewed by the observer under an angle arctan $\left(\frac{1}{2}\right)$, the boundary conditions on the optical path $y(x)$ are respectively $y_{(x=0)} = 0$, $y_{(x=L)} = 0$ and $y'_{(x=0)} = \frac{1}{2}$. From (3.73), one has

$$y' = -\sinh(c_1 x + c_2) \tag{3.74}$$

For $y'_{(x=0)} = \frac{1}{2}$, one finds $y'_{(x=0)} = -\sinh(c_2) = \frac{1}{2}$, i.e. $c_2 = -\operatorname{arcsin h}(\frac{1}{2}) = -\ln\left(\frac{1+\sqrt{5}}{2}\right) \approx -0.48121$. For the condition $y_{(x=0)} = 0$, one finds from (3.73) $y_{(x=0)} = \frac{1}{a} - \frac{1}{c_1}\cosh(c_2) = 0$, i.e.. $c_1 = a\cosh\left(\ln\left(\frac{1+\sqrt{5}}{2}\right)\right) = \frac{a\sqrt{5}}{2}$. Replacing in (3.73), one finds the expression of the optical path

$$y = \frac{1}{a}\left(1 - \frac{2\sqrt{5}}{5}\cosh\left(\frac{a\sqrt{5}}{2}x - \ln\left(\frac{1+\sqrt{5}}{2}\right)\right)\right) \tag{3.75}$$

The last condition $y_{(x=L)} = 0$ in (3.75) yields the distance L

$$y_{(x=L)} = \frac{1}{a}\left(1 - \frac{2\sqrt{5}}{5}\cosh\left(\frac{a\sqrt{5}}{2}L - \ln\left(\frac{1+\sqrt{5}}{2}\right)\right)\right) = 0 \tag{3.76}$$

which yields $\cosh\left(\frac{a\sqrt{5}}{2}L - \ln\left(\frac{1+\sqrt{5}}{2}\right)\right) = \frac{\sqrt{5}}{2}$, or $L = \frac{2\sqrt{5}}{5a}\left(\operatorname{arccos h}\left(\frac{\sqrt{5}}{2}\right) + \ln\left(\frac{1+\sqrt{5}}{2}\right)\right) = \frac{4\sqrt{5}}{5a}\ln\left(\frac{1+\sqrt{5}}{2}\right)$. □

If $a = 1$/km, the oasis is at $L \approx 860$ m; if $a = 0.5$/km, $L \approx 1722$ m; if $a = 0.25$/km, $L \approx 3443$ m.

Chapter 4
Canonical Transformations or Contact Transformations

4.1 Reminder

4.1.1 Canonical Transformations

The ease with which mechanical problems can be solved depends on the choice of the generalized coordinates used. Therefore, it is interesting to examine the transformations of a system of coordinates and moments to another system.

If we call p_α and q_α on one hand and P_α and Q_α on the other hand respectively old and new moments and coordinates, the transformation is $P_\alpha = P_\alpha(p_\alpha, q_\alpha, t)$ and $Q_\alpha = Q_\alpha(p_\alpha, q_\alpha, t)$. One considers only the transformations, called canonical transformations or contact transformations, for which there is a function $\mathcal{H}$, called Hamiltonian in the new coordinates such as

$$\dot{P}_\alpha = -\frac{\partial \mathcal{H}}{\partial Q_\alpha} \tag{4.1}$$

$$\dot{Q}_\alpha = \frac{\partial \mathcal{H}}{\partial P_\alpha} \tag{4.2}$$

where P_α and Q_α are the canonical moments and coordinates. The Lagrangian and Hamiltonian in the old and new coordinates are respectively

$$L(p_\alpha, q_\alpha, t) \ \text{and} \ H = \sum_{\alpha=1}^{n} p_\alpha \dot{q}_\alpha - L \tag{4.3}$$

$$\mathcal{L}(P_\alpha, Q_\alpha, t) \ \text{and} \ \mathcal{H} = \sum_{\alpha=1}^{n} P_\alpha \dot{Q}_\alpha - \mathcal{L} \tag{4.4}$$

V. Pletser, *Lagrangian and Hamiltonian Analytical Mechanics: Forty Exercises Resolved and Explained*, UNITEXT for Physics,
https://doi.org/10.1007/978-981-13-3026-1_4

4.1.2 Condition for a Transformation to be Canonical

Theorem *Transformation* $P_\alpha = P_\alpha\,(p_\alpha, q_\alpha, t)$ *and* $Q_\alpha = Q_\alpha\,(p_\alpha, q_\alpha, t)$ *is canonical if* $\left(\sum_{\alpha=1}^{n}[p_\alpha dq_\alpha] - \sum_{\alpha=1}^{n}[P_\alpha dQ_\alpha]\right)$ *is an exact differential.*

To recall, $(Adp + Bdq)$ is an exact differential if and only if $\frac{\partial A}{\partial q} = \frac{\partial B}{\partial p}$.

4.1.3 Generating Functions

By Hamilton's Variational Principle, the canonical transformations $P_\alpha = P_\alpha\,(p_\alpha, q_\alpha, t)$ and $Q_\alpha = Q_\alpha\,(p_\alpha, q_\alpha, t)$ must be such that the integrals $\int_{t_1}^{t_2} L\,dt$ and $\int_{t_1}^{t_2} \mathcal{L}\,dt$ are both extremal, i.e. that one needs to have simultaneously $\delta \int_{t_1}^{t_2} L\,dt = 0$ and $\delta \int_{t_1}^{t_2} \mathcal{L}\,dt = 0$, which is satisfied if there is a generating function $\mathcal{G}$ such that $\frac{d\mathcal{G}}{dt} = L - \mathcal{L}$.

One supposes that $\mathcal{G}$ is a function of time and two of the old and new coordinates and moments. Let's take for example the old coordinates q_α and new moments P_α; one has $\mathcal{G} = \mathscr{T}\,(q_\alpha, P_\alpha, t)$. It can be demonstrated that

$$p_\alpha = \frac{\partial \mathscr{T}}{\partial q_\alpha} \tag{4.5}$$

$$Q_\alpha = \frac{\partial \mathscr{T}}{\partial P_\alpha} \tag{4.6}$$

$$\mathcal{H} = \frac{\partial \mathscr{T}}{\partial t} + H \tag{4.7}$$

with

$$\dot{P}_\alpha = -\frac{\partial \mathcal{H}}{\partial Q_\alpha} \tag{4.8}$$

$$\dot{Q}_\alpha = \frac{\partial \mathcal{H}}{\partial P_\alpha} \tag{4.9}$$

Other groupings may be taken, e. g. $\mathscr{U} = \mathscr{U}\,(q_\alpha, Q_\alpha, t)$, or $\mathscr{V} = \mathscr{V}\,(Q_\alpha, p_\alpha, t)$, or $\mathscr{W} = \mathscr{W}\,(P_\alpha, p_\alpha, t)$, etc.

4.2 Exercises

4.2.1 Exercise 13: Canonical Transformation 1

Demonstrate that the transformation

$$P = \frac{1}{2}\left(p^2 + q^2\right) \tag{4.10}$$

$$Q = \arctan\left(\frac{q}{p}\right) \tag{4.11}$$

is canonical.

Proof This can be demonstrated by two methods.
(1) First method:
Let $H\,(p, q)$ and $\mathcal{H}\,(P, Q)$ the two Hamiltonians. The moments and coordinates p and q are canonical coordinates, so

$$\dot{p} = -\frac{\partial H}{\partial q} \tag{4.12}$$

$$\dot{q} = \frac{\partial H}{\partial p} \tag{4.13}$$

One can also write

$$\dot{p} = \frac{\partial p}{\partial P}\dot{P} + \frac{\partial p}{\partial Q}\dot{Q} \tag{4.14}$$

$$\dot{q} = \frac{\partial q}{\partial P}\dot{P} + \frac{\partial q}{\partial Q}\dot{Q} \tag{4.15}$$

and

$$\frac{\partial H}{\partial q} = \frac{\partial \mathcal{H}}{\partial P}\frac{\partial P}{\partial q} + \frac{\partial \mathcal{H}}{\partial Q}\frac{\partial Q}{\partial q} \tag{4.16}$$

$$\frac{\partial H}{\partial p} = \frac{\partial \mathcal{H}}{\partial P}\frac{\partial P}{\partial p} + \frac{\partial \mathcal{H}}{\partial Q}\frac{\partial Q}{\partial p} \tag{4.17}$$

From (4.10) and (4.11), one has respectively

$$\frac{\partial P}{\partial q} = q \text{ and } \frac{\partial P}{\partial p} = p \tag{4.18}$$

$$\frac{\partial Q}{\partial q} = \frac{p}{\left(p^2 + q^2\right)} \text{ and } \frac{\partial Q}{\partial p} = \frac{-q}{\left(p^2 + q^2\right)} \tag{4.19}$$

Taking the partial derivative of (4.10) and (4.11) with respect to P, one obtains respectively

$$1 = p\frac{\partial p}{\partial P} + q\frac{\partial q}{\partial P} \tag{4.20}$$

$$0 = \frac{p\frac{\partial q}{\partial P} - q\frac{\partial p}{\partial P}}{\left(p^2 + q^2\right)} \tag{4.21}$$

Similarly, deriving partially (4.10) and (4.11) with respect to Q, one obtains respectively

$$0 = p\frac{\partial p}{\partial Q} + q\frac{\partial q}{\partial Q} \tag{4.22}$$

$$1 = \frac{p\frac{\partial q}{\partial Q} - q\frac{\partial p}{\partial Q}}{\left(p^2 + q^2\right)} \tag{4.23}$$

From (4.20), one has

$$\frac{\partial q}{\partial P} = \frac{1 - p\frac{\partial p}{\partial P}}{q} \tag{4.24}$$

that is replaced in (4.21) to obtain, after simplification by $q\left(p^2 + q^2\right)$,

$$\frac{\partial p}{\partial P} = \frac{p}{\left(p^2 + q^2\right)} \tag{4.25}$$

Again, from (4.20), one has

$$\frac{\partial p}{\partial P} = \frac{1 - q\frac{\partial q}{\partial P}}{p} \tag{4.26}$$

that is replaced in (4.21) to obtain, after simplification by $p\left(p^2 + q^2\right)$,

$$\frac{\partial q}{\partial P} = \frac{q}{\left(p^2 + q^2\right)} \tag{4.27}$$

From (4.22), one has

$$\frac{\partial p}{\partial Q} = -\frac{q}{p}\frac{\partial q}{\partial Q} \tag{4.28}$$

that is replaced in (4.23), to obtain

$$\frac{\partial q}{\partial Q} = p \tag{4.29}$$

Again, from (4.22), one has

$$\frac{\partial q}{\partial Q} = -\frac{p}{q}\frac{\partial p}{\partial Q} \tag{4.30}$$

that is replaced in (4.23), to obtain

$$\frac{\partial p}{\partial Q} = -q \tag{4.31}$$

From (4.14) and (4.15), and replacing by (4.25), (4.31), (4.27), and (4.29), one obtains respectively

$$\dot{p} = \frac{p}{\left(p^2+q^2\right)}\dot{P} - q\dot{Q} \tag{4.32}$$

$$\dot{q} = \frac{q}{\left(p^2+q^2\right)}\dot{P} + p\dot{Q} \tag{4.33}$$

Similarly, from (4.16) and (4.17), and replacing by (4.18) and (4.19), one obtains respectively

$$\frac{\partial H}{\partial q} = q\frac{\partial \mathcal{H}}{\partial P} + \frac{p}{\left(p^2+q^2\right)}\frac{\partial \mathcal{H}}{\partial Q} \tag{4.34}$$

$$\frac{\partial H}{\partial p} = p\frac{\partial \mathcal{H}}{\partial P} - \frac{q}{\left(p^2+q^2\right)}\frac{\partial \mathcal{H}}{\partial Q} \tag{4.35}$$

Replacing respectively (4.32) and (4.34) on one hand, and (4.33) and (4.35) on the other hand, in the definition of the canonical coordinates (4.12) and (4.13), one obtains

$$\frac{p}{\left(p^2+q^2\right)}\dot{P} - q\dot{Q} = -q\frac{\partial \mathcal{H}}{\partial P} - \frac{p}{\left(p^2+q^2\right)}\frac{\partial \mathcal{H}}{\partial Q} \tag{4.36}$$

$$\frac{q}{\left(p^2+q^2\right)}\dot{P} + p\dot{Q} = p\frac{\partial \mathcal{H}}{\partial P} - \frac{q}{\left(p^2+q^2\right)}\frac{\partial \mathcal{H}}{\partial Q} \tag{4.37}$$

Multiplying (4.36) by p and (4.37) by q and adding the resulting equations on one hand, and multiplying (4.36) by q and (4.37) by p and subtracting the equations thus obtained on the other hand, one finds

$$\dot{P} = -\frac{\partial \mathcal{H}}{\partial Q} \tag{4.38}$$

$$\dot{Q} = \frac{\partial \mathcal{H}}{\partial P} \tag{4.39}$$

which clearly shows that the coordinates and transformation are canonical.
(2) Second method:

By the above theorem, the transformation is canonical if $\left(\sum_{\alpha=1}^{n} p_\alpha dq_\alpha - \sum_{\alpha=1}^{n} P_\alpha dQ_\alpha\right)$ is an exact differential. In this case, by differentiating (4.11) and replacing by (4.10), one finds

$$p\,dq - P\,dQ = p\,dq - \frac{1}{2}\left(p^2+q^2\right)\left(\frac{p\,dq - q\,dp}{p^2+q^2}\right) \tag{4.40}$$

$$= p\,dq - \frac{1}{2}p\,dq + \frac{1}{2}q\,dp = d\left(\frac{1}{2}pq\right) \tag{4.41}$$

which is indeed an exact differential, showing that the transformation is canonical. □

4.2.2 *Exercise 14: Canonical Transformation 2*

Demonstrate that the transformation

$$Q = \log\left(\frac{\sin p}{q}\right) \tag{4.42}$$

$$P = q\cot p \tag{4.43}$$

is canonical.

Proof With $dQ = \cot p\,dp - dq$, let us show that $p\,dq - P\,dQ$ is an exact differential

$$p\,dq - P\,dQ = p\,dq - q\cot p\,(\cot p\,dp - dq) \tag{4.44}$$

$$= (p + \cot p)\,dq - \left(q\cot^2 p\right)dp \tag{4.45}$$

$$= d\left(q\,(p + \cot p)\right) \tag{4.46}$$

which is indeed an exact differential and the transformation is canonical. □

4.2.3 *Exercise 15: Canonical Transformation 3*

Demonstrate that the transformation

$$Q = p \tag{4.47}$$

$$P = -q \tag{4.48}$$

is canonical.

Proof Immediate as $P\,dQ = -q\,dp$, one has $p\,dq - P\,dQ = p\,dq + q\,dp = d\,(pq)$. □

4.2.4 Exercise 16: Canonical Transformation 4

Demonstrate that the transformation

$$Q = q \tan p \tag{4.49}$$
$$P = \ln(\sin p) \tag{4.50}$$

is canonical.

Proof As $dQ = \tan p\,dq + \frac{q}{\cos^2 p}dp$, one has

$$p\,dq - P\,dQ = p\,dq - \ln(\sin p)\left(\tan p\,dq + \frac{q}{\cos^2 p}dp\right) \tag{4.51}$$
$$= (p - \ln(\sin p)\tan p)\,dq - \left(\frac{q\ln(\sin p)}{\cos^2 p}\right)dp \tag{4.52}$$
$$= d\,(q\,(p - \ln(\sin p)\tan p)) \tag{4.53}$$

which is an exact differential and the transformation is canonical. □

4.2.5 Exercise 17: Canonical Transformation 5

Is the transformation

$$Q = \exp(p) \tag{4.54}$$
$$P = \exp(q) \tag{4.55}$$

canonical?

Proof As $dQ = \exp(p)\;dp$, one has $p\,dq - P\,dQ = p\,dq - \exp(p+q)\;dp$, which is not an exact differential and the transformation is not canonical. □

4.2.6 Exercise 18: Canonical Transformation 6

(1) Is the transformation

$$Q = \log\left(1 + \sqrt{q}\cos p\right) \tag{4.56}$$
$$P = 2\sqrt{q}\sin p\left(1 + \sqrt{q}\cos p\right) \tag{4.57}$$

canonical?
(2) If so, give a generating function.

Proof (1) As $dQ = \left(\frac{\cos p}{2\sqrt{q}}dq - \sqrt{q}\sin p\, dp\right) / \left(1 + \sqrt{q}\cos p\right)$, one has

$$p\,dq - P\,dQ = p\,dq - 2\sqrt{q}\sin p\left(1 + \sqrt{q}\cos p\right)\left(\frac{\frac{\cos p}{2\sqrt{q}}dq - \sqrt{q}\sin p\, dp}{1 + \sqrt{q}\cos p}\right) \tag{4.58}$$
$$= \left(p - \sin p\cos p\right)dq + 2q\sin^2 p\, dp \tag{4.59}$$
$$= d\left(q\left(p - \sin p\cos p\right)\right) \tag{4.60}$$

which is an exact differential and the transformation is canonical.
(2) If dU is an exact differential, then

$$dU\left(p, q\right) = \frac{\partial U}{\partial p}dp + \frac{\partial U}{\partial q}dq \tag{4.61}$$
$$\frac{\partial^2 U}{\partial p\partial q} = \frac{\partial^2 U}{\partial q\partial p} \tag{4.62}$$

Identifying (4.61) to (4.59) and replacing $\sin p\cos p = \frac{1}{2}\sin\left(2p\right)$, one obtains

$$\frac{\partial U}{\partial p} = 2q\sin^2 p \tag{4.63}$$
$$\frac{\partial U}{\partial q} = p - \frac{\sin\left(2p\right)}{2} \tag{4.64}$$

From (4.63), one has

$$\partial U = 2q\sin^2 p\, dp \tag{4.65}$$

that can be integrated

$$U = \int 2q\sin^2 p\, dp \tag{4.66}$$
$$= qp - q\frac{\sin\left(2p\right)}{2} + \alpha\left(q\right) \tag{4.67}$$

where $\alpha(q)$ is a function of q only and independent from p. Differentiating (4.67) with respect to q, one obtains

$$\frac{\partial U}{\partial q} = p - \frac{\sin(2p)}{2} + \alpha'(q) \tag{4.68}$$

that can be identified to (4.64), which yields $\alpha'(q) = 0$ or $\alpha(q) = c$, where c is a constant. One finds then the generating function

$$U = q\left(p - \frac{\sin(2p)}{2}\right) + c \tag{4.69}$$

□

4.2.7 *Exercise 19: Canonical Transformation 7*

For which values of α and β is the following transformation canonical?

$$Q = q^{\alpha}\cos(\beta p) \tag{4.70}$$
$$P = q^{\alpha}\sin(\beta p) \tag{4.71}$$

Proof As $dQ = \alpha q^{\alpha-1}\cos(\beta p)\, dq - \beta q^{\alpha}\sin(\beta p)\, dp$, one has

$$p\,dq - P\,dQ = p\,dq - q^{\alpha}\sin(\beta p)\left(\alpha q^{\alpha-1}\cos(\beta p)\, dq - \beta q^{\alpha}\sin(\beta p)\, dp\right) \tag{4.72}$$

$$= \Big(\underbrace{p - \alpha q^{2\alpha-1}\sin(\beta p)\cos(\beta p)}_{A}\Big)\, dq + \Big(\underbrace{\beta q^{2\alpha}\sin^2(\beta p)}_{B}\Big)\, dp \tag{4.73}$$

The condition $\frac{\partial A}{\partial p} = \frac{\partial B}{\partial q}$ for (4.73) to be an exact differential yields

$$1 - \alpha\beta q^{2\alpha-1}\left(\cos^2(\beta p) - \sin^2(\beta p)\right) = 2\alpha\beta q^{2\alpha-1}\sin^2(\beta p) \tag{4.74}$$

that reduces to

$$\alpha\beta q^{2\alpha-1} = 1 \tag{4.75}$$

from which one finds $\alpha = \frac{1}{2}$ et $\beta = 2$. □

4.2.8 Exercise 20: Canonical Transformation 8 and Harmonic Oscillator 2

(1) For which value of α is the following transformation canonical?

$$Q = \arcsin\left(q\left(\frac{P}{\alpha}A^{-\alpha}\right)^{-\frac{1}{2}}\right) \tag{4.76}$$

$$p = \left(\frac{P}{\alpha}A^{\alpha}\right)^{\frac{1}{2}} \cos Q \tag{4.77}$$

where A is a constant.
(2) Show that a generating function yielding this canonical transformation is

$$S = \alpha A^{\alpha} q^2 \cot Q \tag{4.78}$$

(3) If $A = km$, apply this transformation to the Hamiltonian of the one dimension harmonic oscillator, where m is the mass and k the spring stiffness. What can you say about the new coordinate Q?

Proof (1) By the theorem of Sect. 4.1.2, the transformation (4.76), (4.77) is canonical if $p\,dq - P\,dQ$ is an exact differential. Expressions for P and dQ need to be found first.

From (4.76) and (4.77), one has successively

$$q = \sqrt{\frac{PA^{-\alpha}}{\alpha}} \sin Q \tag{4.79}$$

$$p = \sqrt{\frac{PA^{\alpha}}{\alpha}} \cos Q \tag{4.80}$$

Dividing (4.79) by (4.80), it follows successively

$$\frac{q}{p} = A^{-\alpha} \tan Q \tag{4.81}$$

$$Q = \arctan\left(A^{\alpha}\frac{q}{p}\right) \tag{4.82}$$

$$dQ = A^{\alpha}\left(\frac{p\,dq - q\,dp}{p^2 + A^{2\alpha}q^2}\right) \tag{4.83}$$

Similarly, adding the squares of q (4.79) multiplied by A^{α} and of p (4.80), one has

$$P = \alpha A^{-\alpha}\left(A^{2\alpha}q^2 + p^2\right) \tag{4.84}$$

Replacing P and dQ by (4.84) and (4.83) in $p\,dq - P\,dQ$, one obtains

$$p\,dq - P\,dQ = p\,dq - \left(\alpha A^{-\alpha}\left(A^{2\alpha}q^2 + p^2\right)\right)\left(A^{\alpha}\left(\frac{p\,dq - q\,dp}{p^2 + A^{2\alpha}q^2}\right)\right) \tag{4.85}$$

$$= p\,dq - \alpha\,(p\,dq - q\,dp) \tag{4.86}$$

$$= (1-\alpha)\,p\,dq + \alpha q\,dp \tag{4.87}$$

One must have $(1-\alpha) = \alpha$, i.e. $\alpha = \frac{1}{2}$ for

$$p\,dq - P\,dQ = \frac{p\,dq + q\,dp}{2} = d\left(\frac{pq}{2}\right) \tag{4.88}$$

to be an exact differential, showing that the transformation is canonical.
(2) For $S\,(q, Q)$ (4.78) to be a generating function, (4.5) and (4.6) must be verified for $S\,(q, Q)$ to be of the form $\mathscr{U} = \mathscr{U}\,(q_\alpha, Q_\alpha, t)$, i.e. here $p = \frac{\partial S}{\partial q}$ and $P = \frac{\partial S}{\partial Q}$. Replacing α by $\frac{1}{2}$, one finds

$$p = \frac{\partial S}{\partial q} = A^{\frac{1}{2}} q \cot Q \tag{4.89}$$

$$P = \frac{\partial S}{\partial Q} = -\frac{A^{\frac{1}{2}} q^2}{2\sin^2 Q} \tag{4.90}$$

If dS is an exact differential, then

$$\frac{\partial^2 S}{\partial q \partial Q} = \frac{\partial^2 S}{\partial Q \partial q} \tag{4.91}$$

Relation (4.91) is verified as it can be seen by deriving partially (4.89) with respect to Q and (4.90) with respect to q

$$\frac{\partial^2 S}{\partial q \partial Q} = \frac{\partial^2 S}{\partial Q \partial q} = -\frac{A^{\frac{1}{2}} q}{\sin^2 Q} \tag{4.92}$$

and S is indeed a generating function.
(3) From Exercise 7, the Hamiltonian of the one dimension harmonic oscillator reads $H = \frac{p^2}{2m} + \frac{kq^2}{2}$ (2.24). If $A = km$, the generating function (4.78) becomes $S = \frac{\sqrt{km}}{2} q^2 \cot Q$. The new Hamiltonian (4.7) found with this generating function S reads

$$\mathcal{H} = \frac{\partial S}{\partial t} + \frac{p^2}{2m} + \frac{kq^2}{2} \tag{4.93}$$

$$= \frac{\sqrt{km}}{2}\left(2q\dot{q}\cot Q - \frac{q^2\dot{Q}}{\sin^2 Q}\right) + \frac{p^2}{2m} + \frac{kq^2}{2} \tag{4.94}$$

with the time derivative of the new coordinate (4.9) and the new moment (4.8) being

$$\dot{Q} = \frac{\partial \mathcal{H}}{\partial P} = 0 \tag{4.95}$$

$$\dot{P} = -\frac{\partial \mathcal{H}}{\partial Q} = \frac{\sqrt{2km}q\dot{q}}{\sin^2 Q} \tag{4.96}$$

From $\dot{Q} = 0$ in (4.95), one deduces that the new coordinate Q is constant. □

Chapter 5
Hamilton–Jacobi Equations

5.1 Reminder

5.1.1 Hamilton–Jacobi Equations

If one can find a canonical transformation leading to $\mathcal{H} = 0$, then the equations

$$\dot{P}_\alpha = -\frac{\partial \mathcal{H}}{\partial Q_\alpha} \tag{5.1}$$

$$\dot{Q}_\alpha = \frac{\partial \mathcal{H}}{\partial P_\alpha} \tag{5.2}$$

yield that P_α and Q_α are constant, i.e. that P_α and Q_α are ignorable coordinates (to recall, $1 \leq \alpha \leq n$ where n is the degree of freedom). Through this transformation, one can find p_α and q_α and thereby determine the movement of the system. All come down to find the right generating function.

From Eq. (4.7) $\mathcal{H} = \frac{\partial \mathscr{S}}{\partial t} + H$, it can be seen that if $\mathcal{H} = 0$, this generating function must satisfy the partial differential equation

$$\frac{\partial \mathscr{S}}{\partial t} + H\left(p_\alpha, q_\alpha, t\right) = 0 \tag{5.3}$$

or, as $p_\alpha = \frac{\partial \mathscr{S}}{\partial q_\alpha}$ (4.5),

$$\frac{\partial \mathscr{S}}{\partial t} + H\left(\frac{\partial \mathscr{S}}{\partial q_\alpha}, q_\alpha, t\right) = 0 \tag{5.4}$$

This Eq. (5.4) is called the Hamilton–Jacobi equation.

V. Pletser, *Lagrangian and Hamiltonian Analytical Mechanics: Forty Exercises Resolved and Explained*, UNITEXT for Physics,
https://doi.org/10.1007/978-981-13-3026-1_5

5.1.2 Solution of Hamilton–Jacobi Equations

Equation (5.4) contains $(n+1)$ independent variables $q_1, q_2, \ldots, q_n$ and t. One of the solutions is called the complete solution that contains $(n+1)$ constants.

By omitting an arbitrary additive constant (which can always be done, for example an integration constant of the movement that will be determined by the initial conditions) and designating the n remaining constants by $\beta_1, \beta_2, \ldots, \beta_n$ (none of these constants is additive), the solution can be written

$$\mathscr{S} = S\left(q_1, q_2, \ldots, q_n, \beta_1, \beta_2, \ldots, \beta_n, t\right) \tag{5.5}$$

When this solution is obtained, we can determine the old moments by $p_\alpha = \frac{\partial \mathscr{S}}{\partial q_\alpha}$ (4.5). Similarly, if we identify the new moments P_α with the constants β_α, then

$$P_\alpha = \beta_\alpha \tag{5.6}$$

$$Q_\alpha = \frac{\partial \mathscr{S}}{\partial \beta_\alpha} = \gamma_\alpha \tag{5.7}$$

where γ_α are constants, which is obvious with $\mathcal{H} = 0$ since the transformation is canonical, (5.2) yields that Q_α are constants.

Using this method, we will find the q_α as function of the β_α, γ_α and t, which will give the movement of the system.

5.1.3 Time Independent Hamiltonian

This case corresponds to the case where H is constant and is the total energy of the system.

To find the complete solution of the Hamilton–Jacobi equation in this case, it is often better to write the solution in the form of

$$\mathscr{S} = S_1\left(q_1\right) + S_2\left(q_2\right) + \cdots + S_n\left(q_n\right) + S_t\left(t\right) \tag{5.8}$$

where each function $S_1, S_2, \ldots, S_n, S_t$ is only dependent on one variable only (method of variable separation). When the Hamiltonian is independent of time, one finds that

$$S_t\left(t\right) = -E\,t \tag{5.9}$$

and the time-independent part of $\mathscr{S}$ can be written

$$S = S_1\left(q_1\right) + S_2\left(q_2\right) + \cdots + S_n\left(q_n\right) \tag{5.10}$$

The Hamilton–Jacobi equation reduces then from (5.4) to

$$H\left(\frac{\partial \mathscr{S}}{\partial q_\alpha}, q_\alpha\right) = E \tag{5.11}$$

where E is a constant that represents the total energy of the system. This Eq. (5.11) can be found again by posing that a generating function S is independent of time, which yields

$$p_\alpha = \frac{\partial S}{\partial q_\alpha} \tag{5.12}$$

$$Q_\alpha = \frac{\partial S}{\partial P_\alpha} \tag{5.13}$$

5.2 Exercises

5.2.1 Exercise 21: Harmonic Oscillator 3

Give the Hamilton–Jacobi equation of the one dimension harmonic oscillator with a mass m and spring stiffness k.

Proof The one dimension harmonic oscillator was already treated in Exercise 7. The kinetic energy, potential, Lagrangian, moment and Hamiltonian are given by Eqs. (2.19)–(2.24). The Hamiltonian $H = \frac{p^2}{2m} + \frac{kq^2}{2}$ (2.24) is independent of time and the system is conservative, H is therefore the total energy of the system. The Hamilton–Jacobi equation (5.4) reduces here to

$$\frac{\partial \mathscr{S}}{\partial t} + \frac{1}{2m}\left(\frac{\partial \mathscr{S}}{\partial q}\right)^2 + \frac{kq^2}{2} = 0 \tag{5.14}$$

From (5.8), let us pose as solution $\mathscr{S} = S_q(q) + S_t(t)$, which introduced in (5.14) yields

$$\frac{1}{2m}\left(\frac{dS_q}{dq}\right)^2 + \frac{kq^2}{2} = -\frac{dS_t}{dt} \tag{5.15}$$

One poses both sides of (5.15) equal to a constant β

$$\frac{1}{2m}\left(\frac{dS_q}{dq}\right)^2 + \frac{kq^2}{2} = \beta \tag{5.16}$$

$$\frac{dS_t}{dt} = -\beta \tag{5.17}$$

As the system is conservative, i.e. there is no loss of energy and the Hamiltonian does not depend on time, the constant β is the constant total energy E of the conservative system, $\beta = E$. Replacing in (5.16), the resolution with respect to S_q yields successively

$$dS_q = \sqrt{2m\left(E - \frac{kq^2}{2}\right)}dq \tag{5.18}$$

$$S_q = \int \sqrt{2m\left(E - \frac{kq^2}{2}\right)}dq \tag{5.19}$$

and, similarly, (5.17) is solved with respect to S_t, giving

$$S_t = -E\,t \tag{5.20}$$

and where integration constants were omitted in (5.19) and (5.20). The solution $\mathscr{S} = S_q\,(q) + S_t\,(t)$ is then

$$\mathscr{S} = \int \sqrt{2m\left(E - \frac{kq^2}{2}\right)}dq - E\,t \tag{5.21}$$

One identifies E with the new moment P and we then have for the new coordinate Q

$$P = E \tag{5.22}$$

$$Q = \frac{\partial \mathscr{S}}{\partial P} = \frac{\partial \mathscr{S}}{\partial E} \tag{5.23}$$

$$= \frac{\partial}{\partial E}\left(\int \sqrt{2m\left(E - \frac{kq^2}{2}\right)}dq - E\,t\right) \tag{5.24}$$

$$= \frac{\sqrt{2m}}{2}\int \frac{dq}{\sqrt{E - \frac{kq^2}{2}}} - t \tag{5.25}$$

Since the new coordinate Q is constant and set equal to γ, (5.25) becomes

$$\frac{\sqrt{2m}}{2}\int \frac{dq}{\sqrt{E - \frac{kq^2}{2}}} = t + \gamma \tag{5.26}$$

Integrating (5.26) (see 2.271-4, Gradshteyn and Ryzhik, 2007; 14.237, Spiegel, 1974), one obtains

$$\sqrt{\frac{m}{k}}\arcsin\left(q\sqrt{\frac{k}{2E}}\right) = t + \gamma \tag{5.27}$$

and solving (5.27) with respect to q, it comes

$$q = \sqrt{\frac{2E}{k}} \sin\left(\sqrt{\frac{k}{m}}(t+\gamma)\right) \tag{5.28}$$

The constants E and γ are to be determined by the initial conditions. □

5.2.2 Exercise 22: Free Falling Particle

Determine by Hamilton–Jacobi's method the motion of a particle of mass m initially at rest and falling freely vertically in a uniform gravity field.

Proof There is one degree of freedom ($n = 1$). One chooses the height z in Cartesian coordinates as the generalized coordinate. The potential V reads $V = mgz$. The Lagrangian and Hamiltonian read respectively

$$L = \frac{m\dot{z}^2}{2} - mgz \tag{5.29}$$

$$H = \frac{m\dot{z}^2}{2} + mgz \tag{5.30}$$

As $p_z = \frac{\partial L}{\partial \dot{z}}$, one finds from (5.29) $p_z = m\dot{z}$ and $\dot{z} = \frac{p_z}{m}$. The Hamiltonian becomes

$$H = \frac{p_z^2}{2m} + mgz \tag{5.31}$$

which is independent of time. So one has a conservative system and H is the total energy of the system. The Hamilton–Jacobi equation (5.4) reduces here to

$$\frac{\partial \mathscr{S}}{\partial t} + \frac{1}{2m}\left(\frac{\partial \mathscr{S}}{\partial z}\right)^2 + mgz = 0 \tag{5.32}$$

From (5.8), one writes the complete solution $\mathscr{S} = S_z(z) + S_t(t)$, which introduced in(5.32) gives

$$\frac{1}{2m}\left(\frac{dS_z}{dz}\right)^2 + mgz = -\frac{dS_t}{dt} \tag{5.33}$$

Both sides of (5.33) are set equal to a constant E, which is the total energy of the system, as it is conservative,

$$\frac{1}{2m}\left(\frac{dS_z}{dz}\right)^2 + mgz = E \tag{5.34}$$

$$\frac{dS_t}{dt} = -E \tag{5.35}$$

which yield respectively

$$S_z = \int \sqrt{2m\,(E - mgz)}dz \tag{5.36}$$

$$S_t = -Et \tag{5.37}$$

where integration constants were omitted. The solution $\mathscr{S} = S_z\,(z) + S_t\,(t)$ is then

$$\mathscr{S} = \int \sqrt{2m\,(E - mgz)}dz - E\,t \tag{5.38}$$

One identifies E with the new moment P and one then has for the new coordinate Q

$$P = E \tag{5.39}$$

$$Q = \frac{\partial \mathscr{S}}{\partial P} = \frac{\partial \mathscr{S}}{\partial E} \tag{5.40}$$

$$= \frac{\partial}{\partial E}\left(\sqrt{2m\,(E - mgz)}dz - E\,t\right) \tag{5.41}$$

$$= \sqrt{\frac{m}{2}} \int \frac{dz}{\sqrt{E - mgz}} - t \tag{5.42}$$

As the new coordinate Q is constant and set equal to γ, one obtains successively from (5.42) and with (2.242-1, Gradshteyn and Ryzhik, 2007; 14.84, Spiegel, 1974) for the integration,

$$\sqrt{\frac{m}{2}} \int \frac{dz}{\sqrt{E - mgz}} = t + \gamma \tag{5.43}$$

$$\sqrt{\frac{m}{2}}\left(\frac{2\sqrt{E - mgz}}{-mg}\right) = t + \gamma \tag{5.44}$$

One solves (5.44) for z which gives

$$z = -\left(\frac{gt^2}{2} + \underbrace{g\gamma}\,t + \underbrace{\left(\frac{g\gamma^2}{2} - \frac{E}{mg}\right)}\right) \tag{5.45}$$

$$= -\left(\frac{gt^2}{2} + \;v_0\;t + \qquad z_0 \qquad\right) \tag{5.46}$$

which is the classical equation of the uniformly accelerated rectilinear movement. As the particle is at rest before falling, the initial velocity $v_0 = g\gamma$ is nil, yielding $\gamma = 0$. (5.45) becomes then

$$z = -\frac{gt^2}{2} + \frac{E}{mg} \tag{5.47}$$

To find the value of E, one identifies $\frac{E}{mg}$ to the initial height z_0, which gives $E = mgz_0$ which is the potential energy and the total energy E of the system before the beginning of the fall. (5.47) becomes finally

$$z = -\frac{gt^2}{2} + z_0 \tag{5.48}$$

One notices the negative sign in front of the first term of (5.48) as the height z decreases when time t increases during the fall. □

5.2.3 *Exercise 23: Ballistic Flight of a Projectile*

(1) Determine by Hamilton–Jacobi's method the movement of a projectile of mass m in ballistic flight in a uniform gravity field and launched with an initial velocity v_0 and angle α on the horizontal.
(2) Give the value of the launch angle to obtain the maximum horizontal distance achieved by the projectile.

Proof (1) The problem has two degrees of freedom and one chooses as generalized coordinates the horizontal distance x and height y in Cartesian coordinates (see Fig. 5.1). The initial conditions for $t = 0$ are: $(x_0, y_0) = (0, 0)$ and $(\dot{x}_0, \dot{y}_0) = (v_0 \cos\alpha, v_0 \sin\alpha)$. The potential is $V = mgy$, and the Lagrangian and Hamiltonian read respectively

$$L = \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2\right) - mgy \tag{5.49}$$

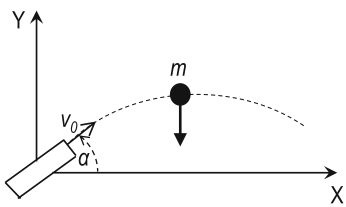


Fig. 5.1 Projectile launched with an angle α on the horizontal

$$H = \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2\right) + mgy \tag{5.50}$$

$$= \frac{1}{2m}\left(p_x^2 + p_y^2\right) + mgy \tag{5.51}$$

as $p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x}$ and $p_y = \frac{\partial L}{\partial \dot{y}} = m\dot{y}$ yielding respectively $\dot{x} = \frac{p_x}{m}$ and $\dot{y} = \frac{p_y}{m}$ in (5.51). As one has also $p_x = \frac{\partial \mathscr{S}}{\partial x}$ and $p_y = \frac{\partial \mathscr{S}}{\partial y}$, the Hamilton–Jacobi equation reads

$$\frac{\partial \mathscr{S}}{\partial t} + \frac{1}{2m}\left(\left(\frac{\partial \mathscr{S}}{\partial x}\right)^2 + \left(\frac{\partial \mathscr{S}}{\partial y}\right)^2\right) + mgy = 0 \tag{5.52}$$

The complete solution (5.8) is $\mathscr{S} = S_x(x) + S_y(y) + S_t(t)$ and (5.52) yields

$$\frac{1}{2m}\left(\left(\frac{dS_x}{dx}\right)^2 + \left(\frac{dS_y}{dy}\right)^2\right) + mgy = -\frac{dS_t}{dt} \tag{5.53}$$

The two sides of (5.53) are set equal to a constant E, yielding

$$\frac{1}{2m}\left(\frac{dS_y}{dy}\right)^2 + mgy = E - \frac{1}{2m}\left(\frac{dS_x}{dx}\right)^2 \tag{5.54}$$

$$S_t = -E\,t \tag{5.55}$$

where E is the total energy of the system, as the Hamiltonian does not depend on time, the system is conservative (any other loss of energy through friction is neglected). Both sides of (5.54) are set equal to a constant β, yielding

$$S_x = \int \sqrt{2m(E-\beta)}dx \tag{5.56}$$

$$S_y = \int \sqrt{2m(\beta - mgy)}dy \tag{5.57}$$

and the complete solution becomes

$$\mathscr{S} = \int \sqrt{2m(E-\beta)}dx + \int \sqrt{2m(\beta - mgy)}dy + -E\,t \tag{5.58}$$

The movement equations yield successively first for $\frac{\partial \mathscr{S}}{\partial E} = \gamma_1$,

$$\int \frac{m\,dx}{\sqrt{2m(E-\beta)}} - t = \gamma_1 \tag{5.59}$$

$$\sqrt{\frac{m}{2(E-\beta)}}x = t + \gamma_1 \tag{5.60}$$

$$x = \sqrt{\frac{2\,(E-\beta)}{m}}\,(t+\gamma_1) \tag{5.61}$$

and then for $\frac{\partial \mathscr{S}}{\partial \beta} = \gamma_2$,

$$-\int \frac{m\,dx}{\sqrt{2m\,(E-\beta)}} + \int \frac{m\,dy}{\sqrt{2m\,(\beta - mgy)}} = \gamma_2 \tag{5.62}$$

$$-\sqrt{\frac{m}{2\,(E-\beta)}}x - \sqrt{\frac{2}{m}}\frac{\sqrt{\beta - mgy}}{g} = \gamma_2 \tag{5.63}$$

Solving (5.63) for y, it comes

$$y = -\frac{mg}{4\,(E-\beta)}x^2 - g\gamma_2\sqrt{\frac{m}{2\,(E-\beta)}}x + \left(\frac{\beta}{mg} - \frac{g\gamma_2^2}{2}\right) \tag{5.64}$$

The initial condition $(x_0, y_0) = (0, 0)$ for $t = 0$ applied to (5.61) and (5.64) yields respectively $\gamma_1 = 0$ and $\gamma_2 = \sqrt{\frac{2\beta}{mg^2}}$, which simplifies the expression of y (5.64)

$$y = -\frac{mg}{4\,(E-\beta)}x^2 - \sqrt{\frac{\beta}{E-\beta}}x \tag{5.65}$$

The time derivative of (5.65) reads then

$$\dot{y} = -\left(\frac{mg}{2\,(E-\beta)}x + \sqrt{\frac{\beta}{E-\beta}}\right)\dot{x} \tag{5.66}$$

The initial conditions $x_0 = 0$ and $(\dot{x}_0, \dot{y}_0) = (v_0 \cos\alpha, v_0 \sin\alpha)$ applied to (5.66) yield $\beta = E\sin^2\alpha$ and relations (5.65) and (5.61) simplify in

$$y = -\frac{mg}{4E\cos^2\alpha}x^2 + \tan\alpha\, x \tag{5.67}$$

$$x = \sqrt{\frac{2E}{m}}\cos\alpha\, t \tag{5.68}$$

(5.67) and (5.68) give respectively y in function of x and x in function of t. Replacing x by (5.68) in (5.67), one obtains y in function of t

$$y = -\frac{gt^2}{2} + \sqrt{\frac{2E}{m}}\sin\alpha\, t \tag{5.69}$$

One can replace the constant total energy of the system E by the initial kinetic energy $E = \frac{mv_0^2}{2}$, which replaced in (5.67)–(5.69) yield finally

$$y = -\frac{g}{2v_0^2 \cos^2 \alpha} x^2 + \tan \alpha \, x \tag{5.70}$$

$$x = v_0 \cos \alpha \, t \tag{5.71}$$

$$y = -\frac{gt^2}{2} + v_0 \sin \alpha \, t \tag{5.72}$$

(2) To determine the value of α that maximizes the horizontal distance x, one searches for the values of x that intersect the X axis for $y = 0$. (5.70) yields

$$\left(-\frac{g}{2v_0^2 \cos^2 \alpha} x + \tan \alpha \right) x = 0 \tag{5.73}$$

that cancels out for $x = 0$ and $x = \frac{v_0^2}{g} \sin(2\alpha)$. This latter value is maximum for $\sin(2\alpha) = 1$, i.e. for $\alpha = 45°$. □

5.2.4 *Exercise 24: Particle Sliding on an Inclined Plane*

In a uniform gravity field, use Hamilton–Jacobi's method to determine the motion of a particle of mass m initially at rest and sliding without friction on an inclined plane forming an angle α with the horizontal.

Proof The problem has one degree of freedom ($n = 1$) and one chooses as the generalized coordinate the distance x from the starting point x_0 and along the inclined plane of length l (see Fig. 5.2). So, one has for initial conditions: $x_{(t=0)} = x_0$ and $\dot{x}_{(t=0)} = 0$. The potential is written $V = mg(l - x) \sin \alpha$, yielding the Lagrangian and Hamiltonian

$$L = \frac{m\dot{x}^2}{2} - mg(l - x) \sin \alpha \tag{5.74}$$

$$H = \frac{m\dot{x}^2}{2} + mg(l - x) \sin \alpha \tag{5.75}$$

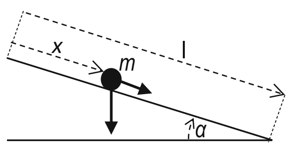


Fig. 5.2 Particle sliding on an inclined plane

$$= \frac{p_x^2}{2m} + mg\,(l - x)\sin\alpha \tag{5.76}$$

where in (5.76), $\dot{x}$ has been replaced by $\frac{p_x}{m}$ from $p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x}$. As one has also $p_x = \frac{\partial \mathscr{S}}{\partial x}$, the Hamilton–Jacobi equation reads

$$\frac{\partial \mathscr{S}}{\partial t} + \frac{1}{2m}\left(\frac{\partial \mathscr{S}}{\partial x}\right)^2 + mg\,(l - x)\sin\alpha = 0 \tag{5.77}$$

From (5.8), the complete solution becomes $\mathscr{S} = S_x\,(x) + S_t\,(t)$, which allows to rewrite (5.77) as

$$\frac{1}{2m}\left(\frac{dS_x}{dx}\right)^2 + mg\,(l - x)\sin\alpha = -\frac{dS_t}{dt} \tag{5.78}$$

Both sides of (5.78) are set equal to a constant E, which is the total energy of the system, as it is conservative (as one has a frictionless sliding),

$$\frac{1}{2m}\left(\frac{dS_x}{dx}\right)^2 + mg\,(l - x)\sin\alpha = E \tag{5.79}$$

$$\frac{dS_t}{dt} = -E \tag{5.80}$$

yielding successively

$$S_x = \int \sqrt{2m\,(E - mg\,(l - x)\sin\alpha)}dx \tag{5.81}$$

$$S_t = -E\,t \tag{5.82}$$

The complete solution becomes

$$\mathscr{S} = \int \sqrt{2m\,(E - mg\,(l - x)\sin\alpha)}dx - E\,t \tag{5.83}$$

and the equation of movement $\frac{\partial \mathscr{S}}{\partial E} = \gamma$ gives successively

$$\sqrt{\frac{m}{2}} \int \frac{dx}{\sqrt{E - mg\,(l - x)\sin\alpha}} = \gamma + t \tag{5.84}$$

$$\sqrt{\frac{m}{2}} \left(\frac{2\sqrt{E - mg\,(l - x)\sin\alpha}}{mg\sin\alpha}\right) = \gamma + t \tag{5.85}$$

with (2.242-1, Gradshteyn and Ryzhik, 2007; 14.84, Spiegel, 1974) for the integration. The expression of x is obtained from (5.85)

$$x = \frac{g \sin \alpha}{2} t^2 + \underbrace{g\gamma \sin \alpha}\, t + \underbrace{\left(\frac{g \sin \alpha \gamma^2}{2} + l - \frac{E}{mg \sin \alpha} \right)} \tag{5.86}$$

$$= \frac{g \sin \alpha}{2} t^2 + \quad v_0 \quad t + \qquad x_0 \tag{5.87}$$

As the particle is at rest initially, the initial velocity $v_0 = g\gamma \sin \alpha$ is nil which yields $\gamma = 0$. (5.86) then becomes

$$x = \frac{g \sin \alpha}{2} t^2 + l - \frac{E}{mg \sin \alpha} \tag{5.88}$$

To find the value of E, we identify $l - \frac{E}{mg \sin \alpha}$ as the initial position x_0, which gives $E = mg \sin \alpha\,(l - x_0)$ which is the potential energy and total energy E of the system before the start of the movement. (5.88) becomes finally

$$x = \frac{g \sin \alpha}{2} t^2 + x_0 \tag{5.89}$$

□

5.2.5 *Exercise 25: Connected Particles Sliding on Inclined Surfaces*

In a uniform gravity field, use Hamilton–Jacobi's method to determine the movement of two particles of masses m_1 and m_2, sliding without friction and connected by a rope of negligible mass passing through a pulley at the top of two inclined planes having angles α_1 and α_2 on the horizontal.

Proof Let the length of the rope connecting the two masses be l. So, there's only one degree of freedom ($n = 1$) as the two masses are connected together. One takes as generalized coordinate the position x of one of the two masses from the top as described in the Fig. 5.3.

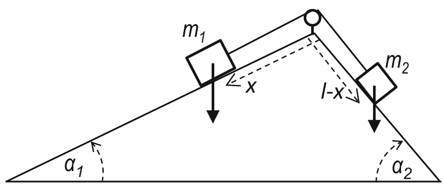


Fig. 5.3 Two particles connected and sliding without friction on two inclined planes

The initial condition for $t = 0$ is $x = 0$. The potential is

$$V = -m_1 g\, x \,\sin\alpha_1 - m_2 g\,(l - x)\sin\alpha_2 \tag{5.90}$$

and the kinetic energy is written $T = \frac{\dot{x}^2}{2}(m_1 + m_2)$, which gives for the Lagrangian and Hamiltonian

$$L = \frac{\dot{x}^2}{2}(m_1 + m_2) + m_1 g\, x \,\sin\alpha_1 + m_2 g\,(l - x)\sin\alpha_2 \tag{5.91}$$

$$H = \frac{\dot{x}^2}{2}(m_1 + m_2) - m_1 g\, x \,\sin\alpha_1 - m_2 g\,(l - x)\sin\alpha_2 \tag{5.92}$$

$$= \frac{p_x^2}{2(m_1 + m_2)} - m_1 g\, x \,\sin\alpha_1 - m_2 g\,(l - x)\sin\alpha_2 \tag{5.93}$$

as $p_x = \frac{\partial L}{\partial \dot{x}} = \dot{x}(m_1 + m_2)$, yielding $\dot{x} = \frac{p_x}{m_1 + m_2}$. As $p_x = \frac{\partial \mathscr{S}}{\partial x}$, the Hamilton equation-Jacobi reads

$$\frac{\partial \mathscr{S}}{\partial t} + \frac{1}{2(m_1 + m_2)}\left(\frac{\partial \mathscr{S}}{\partial x}\right)^2 - m_1 g\, x \,\sin\alpha_1 - m_2 g\,(l - x)\sin\alpha_2 = 0 \tag{5.94}$$

The complete solution (5.8) becomes $\mathscr{S} = S_x(x) + S_t(t)$ and (5.94) yields

$$\frac{1}{2(m_1 + m_2)}\left(\frac{dS_x}{dx}\right)^2 - m_1 g\, x \,\sin\alpha_1 - m_2 g\,(l - x)\sin\alpha_2 = -\frac{dS_t}{dt} \tag{5.95}$$

The two sides of (5.95) are set equal to a constant E, which gives

$$S_t = -E\,t \tag{5.96}$$

$$S_x = \int \sqrt{2(m_1 + m_2)(E + m_1 g\, x \,\sin\alpha_1 + m_2 g\,(l - x)\sin\alpha_2)}dx \tag{5.97}$$

where E is the constant total energy of the conservative system as sliding is without friction. The equation of movement $\frac{\partial \mathscr{S}}{\partial E} = \gamma$ yields successively

$$\sqrt{\frac{m_1 + m_2}{2}} \int \frac{dx}{\sqrt{x\,(m_1 g \,\sin\alpha_1 - m_2 g \,\sin\alpha_2) + (E + m_2 g\, l \,\sin\alpha_2)}} = t + \gamma \tag{5.98}$$

$$\sqrt{2(m_1 + m_2)}\frac{\sqrt{x\,(m_1 g \,\sin\alpha_1 - m_2 g \,\sin\alpha_2) + (E + m_2 g\, l \,\sin\alpha_2)}}{m_1 g \,\sin\alpha_1 - m_2 g \,\sin\alpha_2} = t + \gamma \tag{5.99}$$

$$2(m_1 + m_2)\,\frac{x\,(m_1 g \,\sin\alpha_1 - m_2 g \,\sin\alpha_2) + (E + m_2 g\, l \,\sin\alpha_2)}{(m_1 g \,\sin\alpha_1 - m_2 g \,\sin\alpha_2)^2} = (t + \gamma)^2 \tag{5.100}$$

which gives for x

$$x = \frac{(t+\gamma)^2\,(m_1 g\,\sin\alpha_1 - m_2 g\,\sin\alpha_2)}{2\,(m_1+m_2)} - \frac{E + m_2 g\,l\,\sin\alpha_2}{m_1 g\,\sin\alpha_1 - m_2 g\,\sin\alpha_2} \tag{5.101}$$

The initial condition $x = 0$ for $t = 0$ yields

$$\gamma = \frac{\sqrt{2\,(m_1+m_2)\,(E + m_2 g\,l\,\sin\alpha_2)}}{m_1 g\,\sin\alpha_1 - m_2 g\,\sin\alpha_2} \tag{5.102}$$

The constant total energy E can be replaced by the potential energy (5.95) at the initial moment $t = 0$ for $x = 0$, resulting in

$$E = V = -m_2 g\,l\,\sin\alpha_2 \tag{5.103}$$

which, replaced in (5.102), yields that $\gamma = 0$. (5.101) become finally

$$x = \frac{(m_1 g\,\sin\alpha_1 - m_2 g\,\sin\alpha_2)}{2\,(m_1+m_2)}\,t^2 \tag{5.104}$$

□

5.2.6 Exercise 26: Unconventional Mechanics

One considers a physical system whose dynamics is governed by the Hamiltonian

$$H(p,q,t) = \lambda p^2 q^2 \tag{5.105}$$

with $\lambda > 0$.
(1) Give Hamilton equations and find the Lagrangian from which this Hamiltonian derives.
(2) With $\Lambda(P,Q,t) = 0$, deduct from the solution of the Hamilton–Jacobi equation the generating function $\mathcal{G} = \mathcal{F}(q_\alpha, P_\alpha, t)$ such that (4.5) and (4.6) are satisfied and find the expressions of $q\,(P,Q,t)$ and of $p\,(P,Q,t)$.
(3) Verify that these solutions satisfy the equations of movement.

Proof (1) The system has clearly a single degree of freedom ($n = 1$) as a single coordinate q is given in the Hamiltonian (5.105).

Hamilton's equations (2.2, 2.3) read

$$\dot{p} = -\frac{\partial H}{\partial q} = -2\lambda p^2 q \tag{5.106}$$

$$\dot{q} = \frac{\partial H}{\partial p} = 2\lambda p q^2 \tag{5.107}$$

To find the Lagrangian, it suffices to reverse the relation (2.1) with $n = 1$

$$L = p(q, \dot{q})\,\dot{q} - H(p, q) \tag{5.108}$$
$$= \frac{\dot{q}^2}{2\lambda q^2} - \lambda (pq)^2 \tag{5.109}$$
$$= \frac{\dot{q}^2}{2\lambda q^2} - \lambda \left(\frac{\dot{q}}{2\lambda q}\right)^2 \tag{5.110}$$
$$= \frac{\dot{q}^2}{4\lambda q^2} \tag{5.111}$$

where (5.105) has replaced H and $\frac{\dot{q}}{2\lambda q^2}$ from (5.107) has replaced $p(q, \dot{q})$ in (5.108), and $\frac{\dot{q}}{2\lambda q}$ from (5.107) has replaced the product pq in (5.109).

(2) Hamilton–Jacobi equation (5.4) $\frac{\partial \mathscr{S}}{\partial t} + H = 0$ has the complete solution $\mathscr{S} = S_q(q) + S_t(t)$. The separation of variables gives $S_t = -\beta t$, with β constant. Although the Hamiltonian (5.105) does not directly dependent on time t, we know a priori nothing about the system. So, we cannot consider it conservative and β may not be the total energy of the system.The other term of the solution reads and yields successively

$$\lambda q^2 \left(\frac{dS_q}{dq}\right)^2 = \beta \tag{5.112}$$
$$\frac{dS_q}{dq} = \sqrt{\frac{\beta}{\lambda}}\frac{1}{q} \tag{5.113}$$
$$S_q = \sqrt{\frac{\beta}{\lambda}} \int \frac{dq}{q} = \sqrt{\frac{\beta}{\lambda}} \ln q \tag{5.114}$$

The complete solution therefore reads

$$\mathscr{S} = \sqrt{\frac{\beta}{\lambda}} \ln q - \beta t \tag{5.115}$$

One identifies the constant β to the new moment P and one obtains the generating function

$$\mathcal{G} = \mathscr{T}(q, P, t) = \sqrt{\frac{P}{\lambda}} \ln q - Pt \tag{5.116}$$

from which one deducts with (4.5) and (4.6)

$$p = \frac{\partial \mathscr{T}}{\partial q} = \sqrt{\frac{P}{\lambda}}\frac{1}{q} \tag{5.117}$$

$$Q = \frac{\partial \mathscr{T}}{\partial P} = \frac{\ln q}{2\sqrt{\lambda P}} - t \tag{5.118}$$

Inverting (5.118), one finds

$$q(t) = \exp\left(2\sqrt{\lambda P}(Q+t)\right) \tag{5.119}$$

which can be replaced in (5.117) to find out

$$p(t) = \sqrt{\frac{P}{\lambda}} \exp\left(-2\sqrt{\lambda P}(Q+t)\right) \tag{5.120}$$

(3) For verification, one calculates first $\dot{q}(t)$ and $\dot{p}(t)$ from (5.119) and (5.120)

$$\dot{q}(t) = 2\sqrt{\lambda P} q(t) \tag{5.121}$$
$$\dot{p}(t) = -2\sqrt{\lambda P} p(t) \tag{5.122}$$

One finds from (5.117),

$$\sqrt{P} = \sqrt{\lambda} p(t) q(t) \tag{5.123}$$

which is inserted in (5.121) and (5.122), to find

$$\dot{q}(t) = 2\lambda p(t) q(t)^2 \tag{5.124}$$
$$\dot{p}(t) = -2\lambda p(t)^2 q(t) \tag{5.125}$$

that are identical to the Hamilton equations of movement (5.107) et (5.106). □

5.2.7 *Exercise 27: Double Pendulum 2*

Show that the Hamilton–Jacobi method is longer and more complicated than the Lagrange method to calculate the movement of the double pendulum of same length and mass.

Proof Referring to Exercise 1, the variables and parameters are changed as follows: $\varphi_1 = \theta, \varphi_2 = \varphi, l_1 = l_2 = l$ and $m_1 = m_2 = m$. The kinetic energy and potential for both masses read respectively

$$T = \left(\frac{m}{2}l^2\dot{\theta}^2\right) + \left(\frac{m}{2}l^2\dot{\theta}^2 + \frac{m}{2}l^2\dot{\varphi}^2 + ml^2\dot{\theta}\dot{\varphi}\cos(\theta-\varphi)\right) \tag{5.126}$$

$$= ml^2\left(\dot{\theta}^2 + \frac{\dot{\varphi}^2}{2} + \dot{\theta}\dot{\varphi}\cos(\theta-\varphi)\right) \tag{5.127}$$

$$\begin{aligned} V &= (-mgl\cos\theta) + (-mgl(\cos\theta + \cos\varphi)) && (5.128)\\ &= -mgl(2\cos\theta + \cos\varphi) && (5.129)\end{aligned}$$

The Lagrangian is written then

$$L = ml^2\left(\dot\theta^2 + \frac{\dot\varphi^2}{2} + \dot\theta\dot\varphi\cos(\theta-\varphi)\right) + mgl(2\cos\theta + \cos\varphi) \tag{5.130}$$

leading to conjugated moments

$$p_\theta = \frac{\partial L}{\partial\dot\theta} = ml^2\left(2\dot\theta + \dot\varphi\cos(\theta-\varphi)\right) \tag{5.131}$$

$$p_\varphi = \frac{\partial L}{\partial\dot\varphi} = ml^2\left(\dot\varphi + \dot\theta\cos(\theta-\varphi)\right) \tag{5.132}$$

yielding respectively

$$\dot\theta = \frac{p_\theta}{2ml^2} - \frac{\dot\varphi\cos(\theta-\varphi)}{2} \tag{5.133}$$

$$\dot\varphi = \frac{p_\varphi}{ml^2} - \dot\theta\cos(\theta-\varphi) \tag{5.134}$$

Replacing $\dot\theta$ by (5.133) in (5.134), one obtains

$$\dot\varphi = \frac{2p_\varphi - p_\theta\cos(\theta-\varphi)}{ml^2\left(1+\sin^2(\theta-\varphi)\right)} \tag{5.135}$$

where the equality $2-\cos^2(\theta-\varphi) = 1+\sin^2(\theta-\varphi)$ was used. Replacing now $\dot\varphi$ by (5.135) in (5.133), one finds

$$\dot\theta = \frac{p_\theta - p_\varphi\cos(\theta-\varphi)}{ml^2\left(1+\sin^2(\theta-\varphi)\right)} \tag{5.136}$$

The Hamiltonian reads

$$H = ml^2\left(\dot\theta^2 + \frac{\dot\varphi^2}{2} + \dot\theta\dot\varphi\cos(\theta-\varphi)\right) - mgl(2\cos\theta + \cos\varphi) \tag{5.137}$$

Replacing $\dot\varphi$ et $\dot\theta$ by respectively (5.135) and (5.136) in (5.137), one finds

$$H = \frac{\frac{p_\theta^2}{2} + p_\varphi^2 - p_\theta p_\varphi\cos(\theta-\varphi)}{ml^2\left(1+\sin^2(\theta-\varphi)\right)} - mgl(2\cos\theta + \cos\varphi) \tag{5.138}$$

As $p_\theta = \frac{\partial\mathscr{S}}{\partial\theta}$ and $p_\varphi = \frac{\partial\mathscr{S}}{\partial\varphi}$, Hamilton–Jacobi equation (5.4) reads

$$\frac{\partial \mathscr{S}}{\partial t} + \frac{1}{ml^2\left(1+\sin^2(\theta-\varphi)\right)} \left(\frac{1}{2} \left(\frac{\partial \mathscr{S}}{\partial \theta}\right)^2 + \left(\frac{\partial \mathscr{S}}{\partial \varphi}\right)^2 - \frac{\partial \mathscr{S}}{\partial \theta} \frac{\partial \mathscr{S}}{\partial \varphi} \cos(\theta - \varphi) \right)$$
$$-mgl\,(2\cos\theta + \cos\varphi) = 0 \tag{5.139}$$

The complete solution (5.8) becomes $\mathscr{S} = S_\theta(\theta) + S_\varphi(\varphi) + S_t(t)$ and (5.139) yields

$$\frac{1}{ml^2\left(1+\sin^2(\theta-\varphi)\right)} \left(\frac{1}{2} \left(\frac{dS_\theta}{d\theta}\right)^2 + \left(\frac{dS_\varphi}{d\varphi}\right)^2 - \frac{dS_\theta}{d\theta} \frac{dS_\varphi}{d\varphi} \cos(\theta - \varphi) \right)$$
$$-mgl\,(2\cos\theta + \cos\varphi) = -\frac{dS_t}{dt} \tag{5.140}$$

One sets both sides of (5.140) equal to a constant E, yielding $S_t = -E\,t$ and

$$\frac{1}{ml^2\left(1+\sin^2(\theta-\varphi)\right)} \left(\frac{1}{2} \left(\frac{dS_\theta}{d\theta}\right)^2 + \left(\frac{dS_\varphi}{d\varphi}\right)^2 - \frac{dS_\theta}{d\theta} \frac{dS_\varphi}{d\varphi} \cos(\theta - \varphi) \right)$$
$$-mgl\,(2\cos\theta + \cos\varphi) = E \tag{5.141}$$

where E is the total energy of the conservative system. Then one has to separate the variables θ and φ in (5.141), but since both variables are taken in a multiplication of functions containing them in the term $\frac{dS_\theta}{d\theta}\frac{dS_\varphi}{d\varphi}\cos(\theta-\varphi)$, the variable separation method will not work here and one must change generalized coordinates.

It is then concluded that Hamilton–Jacobi's method for this simplified case of the double pendulum is longer and more complicated than Lagrange's method of Exercise 1. □

5.2.8 *Exercise 28: Classical Problem of Kepler*

> Use the Hamilton–Jacobi method to solve the problem of Kepler for a particle of mass m moving in an inverse square law force field.

Proof The problem is planar and one chooses the polar coordinates (r, θ) as generalized coordinates. The Hamiltonian has been calculated in Exercise 6, (2.14). The potential $V(r)$ is here $V(r) = -\frac{K}{r}$ where K is a constant dependent on the masses in presence. So one has

$$H = \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2}\right) - \frac{K}{r} \tag{5.142}$$

As $p_r = \frac{\partial \mathscr{S}}{\partial r}$ and $p_\theta = \frac{\partial \mathscr{S}}{\partial \theta}$ from (4.5), Hamilton–Jacobi equation (5.4) reads

$$\frac{\partial \mathscr{S}}{\partial t} + \frac{1}{2m}\left(\left(\frac{\partial \mathscr{S}}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial \mathscr{S}}{\partial \theta}\right)^2\right) - \frac{K}{r} = 0 \tag{5.143}$$

Writing the complete solution

$$\mathscr{S} = S_r(r) + S_\theta(\theta) + S_t(t) \tag{5.144}$$

for the variable separation method, the replacing in (5.143) yields

$$\frac{1}{2m}\left(\left(\frac{dS_r}{dr}\right)^2 + \frac{1}{r^2}\left(\frac{dS_\theta}{d\theta}\right)^2\right) - \frac{K}{r} = -\frac{dS_t}{dt} \tag{5.145}$$

One sets both sides of (5.145) equal to a constant β_1, yielding respectively

$$\frac{1}{2m}\left(\left(\frac{dS_r}{dr}\right)^2 + \frac{1}{r^2}\left(\frac{dS_\theta}{d\theta}\right)^2\right) - \frac{K}{r} = \beta_1 \tag{5.146}$$

$$\frac{dS_t}{dt} = -\beta_1 \;\Rightarrow\; S_t = -\beta_1 t = -E\,t \tag{5.147}$$

As the system is conservative, β_1 is the total energy E of the system. After multiplying (5.146) by $2mr^2$, one has

$$\left(\frac{dS_\theta}{d\theta}\right)^2 = r^2\left(2m\left(E + \frac{K}{r}\right) - \left(\frac{dS_r}{dr}\right)^2\right) \tag{5.148}$$

The left side (5.148) depends only on θ and the right side depends only on r; each side is then obviously constant. One sets them equal to a constant β_2. One obtains then

$$\left(\frac{dS_\theta}{d\theta}\right)^2 = \beta_2 \tag{5.149}$$

$$r^2\left(2m\left(E + \frac{K}{r}\right) - \left(\frac{dS_r}{dr}\right)^2\right) = \beta_2 \tag{5.150}$$

But as θ is a cyclic coordinate or ignorable (it does not appear in the expression of the Lagrangian (2.10)), one has $p_\theta = \frac{\partial \mathscr{S}}{\partial \theta} = \frac{dS_\theta}{d\theta}$, and (5.149) yields $\beta_2 = p_\theta^2$, which, replaced in (5.149) and (5.150), yields

$$\left(\frac{dS_\theta}{d\theta}\right) = \sqrt{\beta_2} = p_\theta \;\Rightarrow\; S_\theta = p_\theta \theta \tag{5.151}$$

$$r^2\left(2m\left(E + \frac{K}{r}\right) - \left(\frac{dS_r}{dr}\right)^2\right) = p_\theta^2 \tag{5.152}$$

From (5.152), it comes successively

$$dS_r = \sqrt{2m\left(E + \frac{K}{r}\right) - \frac{p_\theta^2}{r^2}}\,dr \tag{5.153}$$

$$S_r = \int \sqrt{2m\left(E + \frac{K}{r}\right) - \frac{p_\theta^2}{r^2}}\,dr \tag{5.154}$$

Replacing in (5.144), one obtains the complete solution

$$\mathscr{S} = \int \sqrt{2m\left(E + \frac{K}{r}\right) - \frac{p_\theta^2}{r^2}}\,dr + p_\theta \theta - E\,t \tag{5.155}$$

Then, one identifies p_θ and E with the new moments, respectively P_r and P_θ, one obtains then by (5.7) the new coordinates Q_r and Q_θ, that are set equal to two new constants, γ_1 and γ_2, which yields

$$Q_r = \frac{\partial \mathscr{S}}{\partial P_r} = \frac{\partial \mathscr{S}}{\partial p_\theta} = \frac{\partial}{\partial p_\theta}\left(\int \sqrt{2m\left(E + \frac{K}{r}\right) - \frac{p_\theta^2}{r^2}}\,dr\right) + \theta = \gamma_1 \tag{5.156}$$

$$Q_\theta = \frac{\partial \mathscr{S}}{\partial P_\theta} = \frac{\partial \mathscr{S}}{\partial E} = \frac{\partial}{\partial E}\left(\int \sqrt{2m\left(E + \frac{K}{r}\right) - \frac{p_\theta^2}{r^2}}\,dr\right) - t = \gamma_2 \tag{5.157}$$

After differentiation under the integral sign, (5.156) and (5.157) becomes respectively

$$\int \frac{p_\theta}{r^2\sqrt{2m\left(E + \frac{K}{r}\right) - \frac{p_\theta^2}{r^2}}}\,dr = \theta - \gamma_1 \tag{5.158}$$

$$\int \frac{m}{\sqrt{2m\left(E + \frac{K}{r}\right) - \frac{p_\theta^2}{r^2}}}\,dr = t + \gamma_2 \tag{5.159}$$

For the rest of the resolution, one considers three cases: (1) $E < 0$ [1], (2) $E = 0$, and (3) $E > 0$.
For case (2) with $E = 0$, solving (5.158) for r in function of θ gives successively (see 2.246, Gradshteyn and Ryzhik, 2007; 14.87, Spiegel, 1974)

[1] As long as the expression under the radical sign (5.158) and (5.159) remains non-negative (see further).

$$\int \frac{p_\theta}{r\sqrt{2mKr - p_\theta^2}}\,dr = \theta - \gamma_1 \tag{5.160}$$

$$2\arctan\left(\sqrt{\frac{2mKr}{p_\theta^2} - 1}\right) = \theta - \gamma_1 \tag{5.161}$$

$$r = \frac{\frac{p_\theta^2}{mK}}{1 + \cos(\theta - \gamma_1)} \tag{5.162}$$

The resolution of (5.159) for r in function of t yields successively (see 2.242-1, Gradshteyn and Ryzhik, 2007; 14.85, Spiegel, 1974)

$$\int \frac{m\,r}{\sqrt{2mKr - p_\theta^2}}\,dr = t + \gamma_2 \tag{5.163}$$

$$\frac{\left(mKr + p_\theta^2\right)\sqrt{2mKr - p_\theta^2}}{3mK^2} = t + \gamma_2 \tag{5.164}$$

One solves then (5.164) for r to obtain an expression of r in function of time t.
For cases (1) and (3) with $E \neq 0$, the resolution of (5.158) for r in function of θ gives successively (see 2.266, Gradshteyn and Ryzhik, 2007; 14.283, Spiegel, 1974) as r is always positive,

$$\int \frac{p_\theta}{r\sqrt{2mEr^2 + 2mKr - p_\theta^2}}\,dr = \theta - \gamma_1 \tag{5.165}$$

$$\arcsin\left(\frac{r - \frac{p_\theta^2}{mK}}{r\sqrt{1 + \frac{2p_\theta^2 E}{mK^2}}}\right) = \theta - \gamma_1 \tag{5.166}$$

$$r = \frac{\frac{p_\theta^2}{mK}}{1 + \sqrt{1 + 2\frac{p_\theta^2 E}{mK^2}}\cos\left(\theta - \gamma_1 + \frac{\pi}{2}\right)} \tag{5.167}$$

with the condition for case (1) where $E < 0$ that $|E| \leq \frac{mK^2}{2p_\theta^2}$ (where $|E|$ is the absolute value of E). The other Eq. (5.159) becomes successively (see 2.264-2, Gradshteyn and Ryzhik, 2007; 14.281, Spiegel, 1974)

$$\int \frac{m\,r}{\sqrt{2mEr^2 + 2mKr - p_\theta^2}}\,dr = t + \gamma_2 \tag{5.168}$$

$$\frac{\sqrt{2mEr^2 + 2mKr - p_\theta^2}}{2E} - \frac{mK}{2E} I(r) = t + \gamma_2 \tag{5.169}$$

where $I(r)$ is the integral

$$I(r) = \int \frac{dr}{\sqrt{2mEr^2 + 2mKr - p_\theta^2}} \tag{5.170}$$

For case (1) where $E < 0$, the integral $I(r)$ (5.170) becomes (see 2.264-1, Gradshteyn and Ryzhik, 2007; 14.280, Spiegel, 1974)

$$I(r) = \int \frac{dr}{\sqrt{-2m|E|r^2 + 2mKr - p_\theta^2}} \tag{5.171}$$

$$= -\frac{1}{\sqrt{2m|E|}} \arcsin\left(\frac{1 - \frac{2|E|}{K}r}{\sqrt{1 - \frac{2|E|p_\theta^2}{mK^2}}}\right) \tag{5.172}$$

and (5.169) becomes, after replacing E by $-|E|$,

$$-\frac{\sqrt{-2m|E|r^2 + 2mKr - p_\theta^2}}{2|E|} - \frac{1}{2\sqrt{\frac{2|E^3|}{mK^2}}} \arcsin\left(\frac{1 - \frac{2|E|}{K}r}{\sqrt{1 - \frac{2|E|p_\theta^2}{mK^2}}}\right) = t + \gamma_2 \tag{5.173}$$

that yields a relation between the radial distance r and time t.

For case (3) where $E > 0$, the integral $I(r)$ (5.170) yields (see 2.264-1, Gradshteyn and Ryzhik, 2007; 14.280, Spiegel, 1974)

$$I(r) = \frac{1}{\sqrt{2mE}} \ln\left(2\sqrt{2mE\left(2mEr^2 + 2mKr - p_\theta^2\right)} + 4mEr + 2mK\right) \tag{5.174}$$

$$= \frac{1}{\sqrt{2mE}} \ln\left(4mK\left(\sqrt{\frac{E}{K}\left(\frac{E}{K}r^2 + r - \frac{p_\theta^2}{2mK}\right)} + \frac{E}{K}r + \frac{1}{2}\right)\right) \tag{5.175}$$

which, replaced in (5.169), gives another relation the radial distance r and time t. □

5.2.9 Additional Note on the Classical Problem of Kepler

This exercise is especially important because it illustrates the problem of Kepler of two body movement. In the expression of the potential V, the constant K is $K = GMm$ where G is the constant of gravitation ($G \approx 6.674\times10{-}11\,\mathrm{Nm}^2/\mathrm{kg}^2$ or $\mathrm{m}^3/\mathrm{kg\,s}^2$) and M is the mass of the attractive body (e.g. the Sun) around which the particle of mass m (e.g. a planet) describes the Keplerian movement. Relations (5.162) and (5.167) are alike in their forms (except for an integration constant), i.e. the ratio of a constant to the sum of unity and the product of another constant by the cosine of the position angle. It is shown in celestial mechanics (see for example: Roy, 1988) that the equation of the orbit is a conic whose general form in polar coordinates is

$$r = \frac{p}{1 + e\,\cos\theta} \tag{5.176}$$

where the angular polar coordinate θ is called the true anomaly and is counted positively counter clockwise from a reference line passing through the pericentre[2] and the conic focus, and p and e are constants describing the orbit, specifically p here is the semi-latus rectum (or orbital parameter (*paramètre de l'orbite*) in French), equal to

$$p = \frac{p_\theta^2}{mK} \tag{5.177}$$

where p_θ is the angular momentum of the particle of mass m, and e is the eccentricity equal to

$$e = \sqrt{1 + 2\frac{Ep_\theta^2}{K^2 m}} \tag{5.178}$$

For the three cases of values of the total energy E of the above system:
(1) $-\frac{K^2 m}{2p_\theta^2} \le E < 0 \Rightarrow 0 \le e < 1$: the orbit is an ellipse.
(2) $E = 0 \Rightarrow e = 1$: the orbit is a parabola;
(3) $E > 0 \Rightarrow e > 1$: the orbit is a hyperbola.

Note that for $E = -\frac{K^2 m}{2p_\theta^2}$, i.e. $e = 0$, the orbit is a circle.

Relations (5.162) and (5.167) are then identical to (5.176) if one sets $\gamma_1 = 0$ for the case (2) of a parabolic orbit and $\gamma_1 = \frac{\pi}{2}$ for cases (1) and (3) of elliptic and hyperbolic orbits.

[2] Closest point of the orbit to the focus; in particular, perigee or perihelion in the case of an orbit around respectively the Earth or the Sun.

5.2.10 *Exercise 29: Particle and Potential in* $-\frac{K\cos\theta}{r^2}$

A particle of mass m moves in a force field whose potential in spherical coordinates is $V = -\frac{K\cos\theta}{r^2}$. Use the Hamilton–Jacobi method to find the movement equations.

Proof Choosing the spherical coordinates as generalized coordinates (see Fig. 5.4), positions and velocities read respectively

$$x = r\sin\theta\cos\varphi \tag{5.179}$$
$$y = r\sin\theta\sin\varphi \tag{5.180}$$
$$z = r\cos\theta \tag{5.181}$$

$$\dot{x} = \dot{r}\sin\theta\cos\varphi + r\dot{\theta}\cos\theta\cos\varphi - r\dot{\varphi}\sin\theta\sin\varphi \tag{5.182}$$
$$\dot{y} = \dot{r}\sin\theta\sin\varphi + r\dot{\theta}\cos\theta\sin\varphi + r\dot{\varphi}\sin\theta\cos\varphi \tag{5.183}$$
$$\dot{z} = \dot{r}\cos\theta - r\dot{\theta}\sin\theta \tag{5.184}$$

The kinetic energy, Lagrangian and Hamiltonian read respectively

$$T = \frac{m}{2}v^2 = \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) \tag{5.185}$$
$$= \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\dot{\varphi}^2\sin^2\theta\right) \tag{5.186}$$
$$L = T - V = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\dot{\varphi}^2\sin^2\theta\right) + \frac{K\cos\theta}{r^2} \tag{5.187}$$
$$H = T + V = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\dot{\varphi}^2\sin^2\theta\right) - \frac{K\cos\theta}{r^2} \tag{5.188}$$

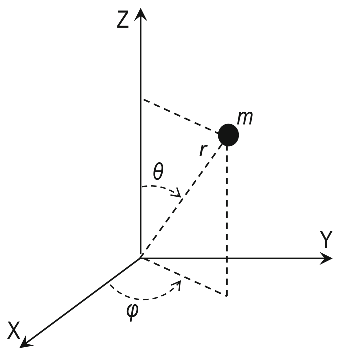


Fig. 5.4 Particle in spherical coordinates

The generalised moments $p_\alpha = \frac{\partial L}{\partial \dot{q}_\alpha}$ read and yield

$$p_r = m\dot{r} \Rightarrow \dot{r} = \frac{p_r}{m} \tag{5.189}$$

$$p_\theta = mr^2\dot{\theta} \Rightarrow \dot{\theta} = \frac{p_\theta}{mr^2} \tag{5.190}$$

$$p_\varphi = mr^2 \sin^2\theta\dot{\varphi} \Rightarrow \dot{\varphi} = \frac{p_\varphi}{mr^2 \sin^2\theta} \tag{5.191}$$

The Hamiltonian (5.188) becomes then

$$H = \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\varphi^2}{r^2 \sin^2\theta}\right) - \frac{K\cos\theta}{r^2} \tag{5.192}$$

As $p_r = \frac{\partial \mathscr{S}}{\partial r}$, $p_\theta = \frac{\partial \mathscr{S}}{\partial \theta}$ and $p_\varphi = \frac{\partial \mathscr{S}}{\partial \varphi}$ from (4.5), the Hamilton–Jacobi equation (5.4) becomes

$$\frac{\partial \mathscr{S}}{\partial t} + \frac{1}{2m}\left(\left(\frac{\partial \mathscr{S}}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial \mathscr{S}}{\partial \theta}\right)^2 + \frac{1}{r^2\sin^2\theta}\left(\frac{\partial \mathscr{S}}{\partial \varphi}\right)^2\right) - \frac{K\cos\theta}{r^2} = 0 \tag{5.193}$$

As the Hamiltonian is independent of time, one sets $\mathscr{S} = S_r(r) + S_\theta(\theta) + S_\varphi(\varphi) - E\,t$ where E is the total energy of the conservative system; (5.193) becomes

$$\frac{1}{2m}\left(\left(\frac{dS_r}{dr}\right)^2 + \frac{1}{r^2}\left(\frac{dS_\theta}{d\theta}\right)^2 + \frac{1}{r^2\sin^2\theta}\left(\frac{dS_\varphi}{d\varphi}\right)^2\right) - \frac{K\cos\theta}{r^2} = E \tag{5.194}$$

Multiplying by $2mr^2$ and rearranging (5.194), it comes

$$r^2\left(\frac{dS_r}{dr}\right)^2 - 2mEr^2 = -\left(\frac{dS_\theta}{d\theta}\right)^2 - \frac{1}{\sin^2\theta}\left(\frac{dS_\varphi}{d\varphi}\right)^2 + 2mK\cos\theta \tag{5.195}$$

One sets both sides of (5.195) equal to β_1, which yields

$$r^2\left(\frac{dS_r}{dr}\right)^2 - 2mEr^2 = \beta_1 \tag{5.196}$$

$$-\left(\frac{dS_\theta}{d\theta}\right)^2 - \frac{1}{\sin^2\theta}\left(\frac{dS_\varphi}{d\varphi}\right)^2 + 2mK\cos\theta = \beta_1 \tag{5.197}$$

Multiplying $\sin^2\theta$ and rearranging (5.197), it comes

$$\left(\frac{dS_\varphi}{d\varphi}\right)^2 = \sin^2\theta\left(2mK\cos\theta - \beta_1 - \left(\frac{dS_\varphi}{d\varphi}\right)^2\right) \tag{5.198}$$

One sets both sides of (5.198) equal to β_2, which gives

$$\left(\frac{dS_\varphi}{d\varphi}\right)^2 = \beta_2 \tag{5.199}$$

$$\sin^2\theta\left(2mK\cos\theta - \beta_1 - \left(\frac{dS_\theta}{d\theta}\right)^2\right) = \beta_2 \tag{5.200}$$

But as φ is a cyclic coordinate or ignorable (it does not appear in the expression of the Lagrangian (5.187)), one has $p_\varphi = \frac{\partial \mathscr{S}}{\partial\varphi} = \frac{dS_\varphi}{d\varphi}$, and (5.199) yields that $\beta_2 = p_\varphi^2$, which replaced in (5.200), yields after rearranging

$$\frac{dS_\theta}{d\theta} = \sqrt{2mK\cos\theta - \frac{p_\varphi^2}{\sin^2\theta} - \beta_1} \tag{5.201}$$

One obtains then respectively from (5.196), (5.201) and (5.199), omitting the integration constants,

$$S_r = \int\sqrt{2mE + \frac{\beta_1}{r^2}}dr \tag{5.202}$$

$$S_\theta = \int\sqrt{2mK\cos\theta - \frac{p_\varphi^2}{\sin^2\theta} - \beta_1 d\theta} \tag{5.203}$$

$$S_\varphi = p_\varphi\varphi \tag{5.204}$$

and the complete solution $\mathscr{S}$ reads

$$\mathscr{S} = \int\sqrt{2mE + \frac{\beta_1}{r^2}}dr + \int\sqrt{2mK\cos\theta - \frac{p_\varphi^2}{\sin^2\theta} - \beta_1 d\theta} + p_\varphi\varphi - Et \tag{5.205}$$

The equations of movement are found by deriving $\mathscr{S}$ (5.205) with respect to β_1 and p_φ, yielding successively on one hand

$$\frac{\partial\mathscr{S}}{\partial\beta_1} = \gamma_1 \tag{5.206}$$

$$\frac{\partial}{\partial\beta_1}\left(\int\sqrt{2mE + \frac{\beta_1}{r^2}}dr + \int\sqrt{2mK\cos\theta - \frac{p_\varphi^2}{\sin^2\theta} - \beta_1 d\theta}\right) = \gamma_1 \tag{5.207}$$

$$\int\frac{dr}{2r\sqrt{2mEr^2 + \beta_1}} + \int\frac{d\theta}{2\sqrt{2mK\cos\theta - \frac{p_\varphi^2}{\sin^2\theta} - \beta_1}} = \gamma_1 \tag{5.208}$$

and on the other hand

$$\frac{\partial \mathscr{S}}{\partial p_\varphi} = \gamma_2 \tag{5.209}$$

$$\frac{\partial}{\partial p_\varphi}\left(\int \sqrt{2mK\cos\theta - \frac{p_\varphi^2}{\sin^2\theta} - \beta_1}\, d\theta + p_\varphi \varphi\right) = \gamma_2 \tag{5.210}$$

$$\int \frac{d\theta}{\sin^2\theta\sqrt{2mK\cos\theta - \frac{p_\varphi^2}{\sin^2\theta} - \beta_1}} + \varphi = \gamma_2 \tag{5.211}$$

Solving simultaneously (5.208) and (5.211), one finds the expressions of r and of θ. □

5.2.11 Exercise 30: Schrödinger Equation

The Schrödinger equation generally reads

$$-\frac{\hbar^2}{2m}\triangle\psi + V\psi = i\,\hbar\,\frac{\partial\psi}{\partial t} \tag{5.212}$$

where $\triangle$ is the Laplacian operator, $V = mc^2$, m is the mass, c is the speed of light and ψ is a wave function of complex values of x, y, z and t

$$\psi = e^{\left(\frac{i}{\hbar}\right)S(x,y,z,t)} \tag{5.213}$$

(1) Which conditions the function $S\,(x, y, z, t)$ must satisfy in order to apply the method of Hamilton–Jacobi?
(2) Which solution of the time independent Schrödinger equation

$$\frac{\hbar^2}{2m}\triangle\psi + (E - V)\,\psi = 0 \tag{5.214}$$

leads under the same conditions to the Hamilton–Jacobi equation for the characteristic function of Hamilton?

Preliminary note

The Schrödinger equation (5.212) can be written successively

$$\frac{1}{2m}\left(-\hbar^2\frac{\partial^2\psi}{\partial x^2}-\hbar^2\frac{\partial^2\psi}{\partial y^2}-\hbar^2\frac{\partial^2\psi}{\partial z^2}\right)+V\psi=i\,\hbar\,\frac{\partial\psi}{\partial t} \quad (5.215)$$

$$\frac{1}{2m}\left(\left(\frac{\hbar}{i}\frac{\partial}{\partial x}\right)^2\psi+\left(\frac{\hbar}{i}\frac{\partial}{\partial y}\right)^2\psi+\left(\frac{\hbar}{i}\frac{\partial}{\partial z}\right)^2\psi\right)+V\psi=i\,\hbar\,\frac{\partial\psi}{\partial t} \quad (5.216)$$

One can now replaces the operators $\left(\frac{\hbar}{i}\frac{\partial}{\partial x_i}\right)$ of ψ by the conjugate moments p_{x_i} of the Hamilton–Jacobi equation, which yields

$$\frac{1}{2m}\left(p_x^2+p_y^2+p_z^2\right)+V\psi=i\,\hbar\,\frac{\partial\psi}{\partial t} \quad (5.217)$$

Proof (1) Let (5.213) be a solution. One has then successively

$$\frac{\partial\psi}{\partial t} = \frac{i}{\hbar}\psi\frac{\partial S}{\partial t} \quad (5.218)$$

$$\frac{\partial\psi}{\partial x} = \frac{i}{\hbar}\psi\frac{\partial S}{\partial x} \quad (5.219)$$

$$\frac{\partial^2\psi}{\partial x^2} = \frac{i}{\hbar}\left(\frac{\partial\psi}{\partial x}\frac{\partial S}{\partial x}+\psi\frac{\partial^2 S}{\partial x^2}\right) \quad (5.220)$$

$$= \frac{i}{\hbar}\left(\left(\frac{i}{\hbar}\psi\frac{\partial S}{\partial x}\right)\frac{\partial S}{\partial x}+\psi\frac{\partial^2 S}{\partial x^2}\right) \quad (5.221)$$

$$=\psi\left(-\frac{1}{\hbar^2}\left(\frac{\partial S}{\partial x}\right)^2+\frac{i}{\hbar}\frac{\partial^2 S}{\partial x^2}\right) \quad (5.222)$$

where (5.219) was introduced in (5.220). One replaces by (5.218) and (5.222) in Schrödinger equation (5.215), which yields successively

$$\frac{\psi}{2m}\left[\left(\left(\frac{\partial S}{\partial x}\right)^2-i\hbar\frac{\partial^2 S}{\partial x^2}\right)+\left(\left(\frac{\partial S}{\partial y}\right)^2-i\hbar\frac{\partial^2 S}{\partial y^2}\right)+\left(\left(\frac{\partial S}{\partial z}\right)^2-i\hbar\frac{\partial^2 S}{\partial z^2}\right)\right]+V\psi=-\psi\frac{\partial S}{\partial t} \quad (5.223)$$

$$\frac{1}{2m}\left[\left(\left(\frac{\partial S}{\partial x}\right)^2+\left(\frac{\partial S}{\partial y}\right)^2+\left(\frac{\partial S}{\partial z}\right)^2\right)-i\hbar\left(\frac{\partial^2 S}{\partial x^2}+\frac{\partial^2 S}{\partial y^2}+\frac{\partial^2 S}{\partial z^2}\right)\right]+V=-\frac{\partial S}{\partial t} \quad (5.224)$$

As V and S are real in (5.224), the coefficient of the imaginary part must be nil, i.e.

$$\triangle S=\frac{\partial^2 S}{\partial x^2}+\frac{\partial^2 S}{\partial y^2}+\frac{\partial^2 S}{\partial z^2}=0 \quad (5.225)$$

(2) Replacing by (5.222) in the time independent Schrödinger equation (5.214), one finds

$$\frac{1}{2m}\left[\left(\left(\frac{\partial S}{\partial x}\right)^2+\left(\frac{\partial S}{\partial y}\right)^2+\left(\frac{\partial S}{\partial z}\right)^2\right)-i\hbar\left(\frac{\partial^2 S}{\partial x^2}+\frac{\partial^2 S}{\partial y^2}+\frac{\partial^2 S}{\partial z^2}\right)\right]-(E-V)=0 \tag{5.226}$$

Under the same conditions (5.225), one obtains

$$\frac{1}{2m}(\nabla S)^2-E+V=0 \tag{5.227}$$

where ∇ is the gradient operator. As one has in this equation $-\frac{\partial S}{\partial t}=E$, i.e. the total energy of the system if it is conservative, one has that $H=T+V$, which yields

$$T=\frac{1}{2m}(\nabla S)^2 \tag{5.228}$$

□

5.2.12 Exercise 31: Stark Effect

An electron of charge e^- moves in the field of a nucleus of charge $+Ze$ where Z is the atomic number, and subject to the effect of an external constant electric field $\boldsymbol{E}$ (Stark effect).
(1) Give the Hamilton–Jacobi equation in spherical coordinates and show that a change of variables is inevitable.
(2) Give the Hamiltonian in generalized coordinates (q_1, q_2, q_3) defined by the transformation

$$q_1=r(1-\cos\theta) \tag{5.229}$$

$$q_2=r(1+\cos\theta) \tag{5.230}$$

$$q_3=\varphi \tag{5.231}$$

and give the general form of the equations of motion.
(3) Give the equations of movement by the Hamilton–Jacobi method using the parabolic coordinates as generalized coordinates.

Proof (1) Without loss of generality, one chooses the direction of the $\boldsymbol{E}$ field along the vertical axis Z. The potential V acting on the electron is the superposition of two potential due to the nucleus $-\frac{Ze^2}{r}$ and to the external electric field $+e\,\boldsymbol{E}\,z$ or $+e\,\boldsymbol{E}\,r\cos\theta$ in spherical coordinates and where the $+$ sign is due to the fact that the electrons go back up the electric field (see Fig. 5.5), yielding

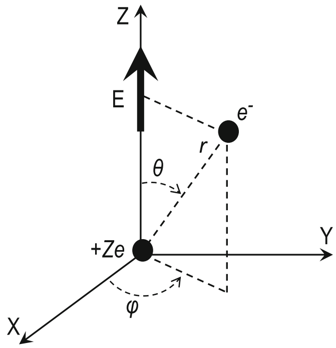


Fig. 5.5 Electron in movement around a nucleus and subject to an electric field $\boldsymbol{E}$

$$V = -\frac{Ze^2}{r} + e\,\boldsymbol{E}\,r\cos\theta \tag{5.232}$$

The Lagrangian and Hamiltonian in spherical coordinates (see also (2.5)) are written

$$L = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\varphi}^2\right) + \frac{Ze^2}{r} - e\,\boldsymbol{E}\,r\cos\theta \tag{5.233}$$

$$H = \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\varphi^2}{r^2\sin^2\theta}\right) - \frac{Ze^2}{r} + e\,\boldsymbol{E}\,r\cos\theta \tag{5.234}$$

The coordinate φ is ignorable as it does not appear explicitly in the Lagrangian. The conjugate moment p_φ is thus constant and a constant of movement. With $p_r = \frac{\partial \mathscr{S}}{\partial r}$, $p_\theta = \frac{\partial \mathscr{S}}{\partial \theta}$ and $p_\varphi = \frac{\partial \mathscr{S}}{\partial \varphi}$, the Hamilton–Jacobi equation reads

$$\frac{\partial \mathscr{S}}{\partial t} + \frac{1}{2m}\left(\left(\frac{\partial \mathscr{S}}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial \mathscr{S}}{\partial \theta}\right)^2 + \frac{1}{r^2\sin^2\theta}\left(\frac{\partial \mathscr{S}}{\partial \varphi}\right)^2\right) - \frac{Ze^2}{r} + e\,\boldsymbol{E}\,r\cos\theta = 0 \tag{5.235}$$

The complete solution (5.8) becomes $\mathscr{S} = S_r(r) + S_\theta(\theta) + S_\varphi(\varphi) + S_t(t)$ and (5.235) yields

$$\frac{1}{2m}\left(\left(\frac{dS_r}{dr}\right)^2 + \frac{1}{r^2}\left(\frac{dS_\theta}{d\theta}\right)^2 + \frac{1}{r^2\sin^2\theta}\left(\frac{dS_\varphi}{d\varphi}\right)^2\right) - \frac{Ze^2}{r} + e\,\boldsymbol{E}\,r\cos\theta = -\frac{dS_t}{dt} = E \tag{5.236}$$

where both sides were set equal to a constant E, the total energy of the conservative system, yielding $S_t = -E\,t$ and

$$\frac{1}{2m}\left(r^2\left(\frac{dS_r}{dr}\right)^2 + \left(\frac{dS_\theta}{d\theta}\right)^2 + \frac{1}{\sin^2\theta}\left(\frac{dS_\varphi}{d\varphi}\right)^2\right) - Ze^2 r + e\,\boldsymbol{E}\,r^3\cos\theta = E\,r^2 \tag{5.237}$$

As in Exercise 28, variable separation is not possible here because the last term of the left side of (5.237) contains a multiplication of functions of the two variables r and θ. Therefore, one must change generalized coordinates.

(2) One applies then the transformation of generalized coordinates (5.229)–(5.231) from spherical coordinates to these new coordinates. Taking the time derivative of (5.229)–(5.231), one obtains

$$\dot{q}_1 = \dot{r}\,(1-\cos\theta) + r\sin\theta\,\dot{\theta} \tag{5.238}$$

$$\dot{q}_2 = \dot{r}\,(1+\cos\theta) - r\sin\theta\,\dot{\theta} \tag{5.239}$$

$$\dot{q}_3 = \dot{\varphi} \tag{5.240}$$

From (5.238) and (5.239), one finds

$$\dot{r} = \frac{\dot{q}_1 + \dot{q}_2}{2} \tag{5.241}$$

Let us look for an expression for $\dot{\theta}$. From (5.229) and (5.230), one finds

$$r = \frac{q_1 + q_2}{2} \tag{5.242}$$

$$\cos\theta = \frac{q_2 - q_1}{2r} = \frac{q_2 - q_1}{q_1 + q_2} \tag{5.243}$$

Deriving (5.243), one has

$$-\sin\theta\,\dot{\theta} = \frac{2\,(q_1\dot{q}_2 - q_2\dot{q}_1)}{(q_1+q_2)^2} \tag{5.244}$$

Replacing $\sin\theta$ by $\sqrt{1-\cos^2\theta}$ with (5.243), one obtains successively

$$\sin\theta = \frac{2\sqrt{q_1q_2}}{q_1+q_2} \tag{5.245}$$

$$\dot{\theta} = \frac{(-q_1\dot{q}_2 + q_2\dot{q}_1)}{\sqrt{q_1q_2}\,(q_1+q_2)} \tag{5.246}$$

Replacing by (5.241), (5.246), (5.240), (5.245) and (5.242) in the Lagrangian (5.233), one finds

$$L = \frac{m}{2}\left(\left(\frac{\dot{q}_1+\dot{q}_2}{2}\right)^2 + \frac{(-q_1\dot{q}_2+q_2\dot{q}_1)^2}{4q_1q_2} + q_1q_2{\dot{q}_3}^2\right) + \frac{2Ze^2}{q_1+q_2} - e\,\boldsymbol{E}\left(\frac{q_2-q_1}{2}\right) \tag{5.247}$$

One has then the conjugate moments

$$p_1 = \frac{\partial L}{\partial \dot{q}_1} = \frac{m}{4}\left(\dot{q}_1 + \dot{q}_2 + \frac{(-q_1\dot{q}_2 + q_2\dot{q}_1)}{q_1}\right) = \frac{m\dot{q}_1}{4}\left(\frac{q_1+q_2}{q_1}\right) \quad (5.248)$$

$$p_2 = \frac{\partial L}{\partial \dot{q}_2} = \frac{m}{4}\left(\dot{q}_1 + \dot{q}_2 - \frac{(-q_1\dot{q}_2 + q_2\dot{q}_1)}{q_2}\right) = \frac{m\dot{q}_2}{4}\left(\frac{q_1+q_2}{q_2}\right) \quad (5.249)$$

$$p_3 = \frac{\partial L}{\partial \dot{q}_3} = m q_1 q_2 \dot{q}_3 \quad (5.250)$$

One remarks that the coordinate $q_3 = \varphi$ is cyclic and the conjugate moment p_3 is then constant and a constant of movement.

Inverting Eqs. (5.248)–(5.250), one finds respectively

$$\dot{q}_1 = \frac{4p_1}{m}\left(\frac{q_1}{q_1+q_2}\right) \quad (5.251)$$

$$\dot{q}_2 = \frac{4p_2}{m}\left(\frac{q_2}{q_1+q_2}\right) \quad (5.252)$$

$$\dot{q}_3 = \frac{p_3}{m q_1 q_2} \quad (5.253)$$

The Hamiltonian reads then

$$H = \frac{2}{m}\left(\frac{q_1 p_1^2 + q_2 p_2^2}{q_1+q_2}\right) + \frac{p_3^2}{2m q_1 q_2} - \frac{2Ze^2}{q_1+q_2} + e\,\boldsymbol{E}\left(\frac{q_2-q_1}{2}\right) \quad (5.254)$$

One writes the complete solution (5.8) $\mathscr{S} = S_1(q_1) + S_2(q_2) + S_3(q_3) + S_t(t)$ and (5.4) yields

$$\frac{2}{m(q_1+q_2)}\left(q_1\left(\frac{dS_1}{dq_1}\right)^2 + q_2\left(\frac{dS_2}{dq_2}\right)^2\right) + \frac{1}{2mq_1q_2}\left(\frac{dS_3}{dq_3}\right)^2 - \frac{2Ze^2}{q_1+q_2} + e\,\boldsymbol{E}\left(\frac{q_2-q_1}{2}\right) = -\frac{dS_t}{dt} \quad (5.255)$$

One sets both sides of (5.255) equal to a constant E, the total energy of the conservative system, yielding $S_t = -E\,t$. One remarks that the term which included the multiplication of functions of the two variables (the one with the electric field $\boldsymbol{E}$) in (5.237) and that did not allow the variable separation in the case of spherical coordinates is replaced here by a term with a difference of two variables in (5.255), which now renders the variable separation possible. One then isolates the term in q_3 in (5.255), yielding

$$\left(\frac{2}{m\,(q_1+q_2)}\left(q_1\left(\frac{dS_1}{dq_1}\right)^2+q_2\left(\frac{dS_2}{dq_2}\right)^2\right)-\frac{2Ze^2}{q_1+q_2}+e\,\boldsymbol{E}\left(\frac{q_2-q_1}{2}\right)-E\right)q_1q_2$$
$$=-\frac{1}{2m}\left(\frac{dS_3}{dq_3}\right)^2 \tag{5.256}$$

One sets both sides of (5.256) equal to a constant $-\beta_1$, yielding $S_3=\sqrt{2m\beta_1}q_3$ and as the conjugate moment $p_3=\frac{\partial\mathscr{S}}{\partial q_3}=\frac{\partial S_3}{\partial q_3}$ is constant, one has $\beta_1=\frac{p_3^2}{2m}$ and $S_3=p_3q_3$. On the other hand, one also has

$$\frac{2}{m(q_1+q_2)}\left(q_1\left(\frac{dS_1}{dq_1}\right)^2+q_2\left(\frac{dS_2}{dq_2}\right)^2\right)-\frac{2Ze^2}{q_1+q_2}+e\,\boldsymbol{E}\left(\frac{q_2-q_1}{2}\right)-E+\frac{\beta_1}{q_1q_2}=0 \tag{5.257}$$

$$\frac{2}{m}\left(q_1\left(\frac{dS_1}{dq_1}\right)^2+q_2\left(\frac{dS_2}{dq_2}\right)^2\right)-2Ze^2+e\,\boldsymbol{E}\left(\frac{q_2^2-q_1^2}{2}\right)-E\,(q_1+q_2)+(q_1+q_2)\,\frac{\beta_1}{q_1q_2}=0 \tag{5.258}$$

Separating the terms in q_1 from those in q_2 in (5.258) and setting the two sides equal to a new constant β_2 yield

$$\frac{2q_1}{m}\left(\frac{dS_1}{dq_1}\right)^2-\frac{e\,\boldsymbol{E}\,q_1^2}{2}-E\,q_1+\frac{\beta_1}{q_1}-2Ze^2=-\frac{2q_2}{m}\left(\frac{dS_2}{dq_2}\right)^2-\frac{e\,\boldsymbol{E}\,q_2^2}{2}+E\,q_2-\frac{\beta_1}{q_2}=\beta_2 \tag{5.259}$$

Solving for S_1 and S_2, one obtains

$$S_1=\int\sqrt{\frac{m}{2}\left(\frac{e\,\boldsymbol{E}\,q_1}{2}+E+\frac{\beta_2+2Ze^2}{q_1}-\frac{\beta_1}{q_1^2}\right)}dq_1 \tag{5.260}$$

$$S_2=\int\sqrt{\frac{m}{2}\left(-\frac{e\,\boldsymbol{E}\,q_2}{2}+E-\frac{\beta_2}{q_2}-\frac{\beta_1}{q_2^2}\right)}dq_2 \tag{5.261}$$

and the complete solution reads then

$$\mathscr{S}=\int\sqrt{\frac{m}{2}\left(\frac{e\,\boldsymbol{E}\,q_1}{2}+E+\frac{\beta_2+2Ze^2}{q_1}-\frac{\beta_1}{q_1^2}\right)}dq_1$$
$$+\int\sqrt{\frac{m}{2}\left(-\frac{e\,\boldsymbol{E}\,q_2}{2}+E-\frac{\beta_2}{q_2}-\frac{\beta_1}{q_2^2}\right)}dq_2+\sqrt{2m\beta_1}q_3-E\,t \tag{5.262}$$

The movement equations are $\frac{\partial\mathscr{S}}{\partial E}=\gamma_1$, $\frac{\partial\mathscr{S}}{\partial\beta_1}=\gamma_2$ et $\frac{\partial\mathscr{S}}{\partial\beta_2}=\gamma_3$, i.e.

$$\frac{\partial \mathscr{S}}{\partial E} = \frac{1}{2}\sqrt{\frac{m}{2}}\left(\int \frac{q_1 dq_1}{\sqrt{\frac{e\,\boldsymbol{E}\,q_1^3}{2} + E\,q_1^2 + \left(\beta_2 + 2Ze^2\right) q_1 - \beta_1}} + \int \frac{q_2 dq_2}{\sqrt{-\frac{e\,\boldsymbol{E}\,q_2^3}{2} + E\,q_2^2 - \beta_2 q_2 - \beta_1}}\right) - t = \gamma_1 \tag{5.263}$$

$$\frac{\partial \mathscr{S}}{\partial \beta_1} = \frac{1}{2}\sqrt{\frac{m}{2}}\left(-\int \frac{dq_1}{q_1\sqrt{\frac{e\,\boldsymbol{E}\,q_1^3}{2} + E\,q_1^2 + \left(\beta_2 + 2Ze^2\right) q_1 - \beta_1}} - \int \frac{dq_2}{q_2\sqrt{-\frac{e\,\boldsymbol{E}\,q_2^3}{2} + E\,q_2^2 - \beta_2 q_2 - \beta_1}} + \frac{2q_3}{\sqrt{\beta_2}}\right) = \gamma_2 \tag{5.264}$$

$$\frac{\partial \mathscr{S}}{\partial \beta_2} = \frac{1}{2}\sqrt{\frac{m}{2}}\left(\int \frac{dq_1}{\sqrt{\frac{e\,\boldsymbol{E}\,q_1^3}{2} + E\,q_1^2 + \left(\beta_2 + 2Ze^2\right) q_1 - \beta_1}} - \int \frac{dq_2}{\sqrt{-\frac{e\,\boldsymbol{E}\,q_2^3}{2} + E\,q_2^2 - \beta_2 q_2 - \beta_1}}\right) = \gamma_3 \tag{5.265}$$

with $\beta_1 = \frac{p_3^2}{2m}$.

(3) To write the potential $V = -\frac{Ze^2}{r} + e\,\boldsymbol{E}\,z$ (5.232) in parabolic coordinates, one applies the following transformations of Cartesian coordinates (x, y, z) in parabolic coordinates (ξ, η, φ)

$$(x,\ y,\ z) = \left(\sqrt{\xi\eta}\cos\varphi\,,\ \sqrt{\xi\eta}\sin\varphi\,,\ \frac{\xi - \eta}{2}\right) \tag{5.266}$$

and with

$$r = \sqrt{x^2 + y^2 + z^2} = \frac{\xi + \eta}{2} \tag{5.267}$$

one obtains

$$V = -\frac{2Ze^2}{\xi + \eta} + e\,\boldsymbol{E}\left(\frac{\xi - \eta}{2}\right) \tag{5.268}$$

The Hamiltonian (2.7) in parabolic coordinates reads then

$$H = \frac{2}{m}\left(\frac{\xi p_\xi^2 + \eta p_\eta^2}{\xi + \eta} + \frac{p_\varphi^2}{4\xi\eta}\right) - \frac{2Ze^2}{\xi + \eta} + e\,\boldsymbol{E}\left(\frac{\xi - \eta}{2}\right) \tag{5.269}$$

One notice that the coordinate φ is again cyclic. The Hamilton–Jacobi equation (5.4) reads

$$\frac{\partial \mathscr{S}}{\partial t}+\frac{2}{m}\left(\frac{\xi}{\xi+\eta}\left(\frac{\partial \mathscr{S}}{\partial \xi}\right)^2+\frac{\eta}{\xi+\eta}\left(\frac{\partial \mathscr{S}}{\partial \eta}\right)^2+\frac{1}{4\xi\eta}\left(\frac{\partial \mathscr{S}}{\partial \varphi}\right)^2\right)-\frac{2Ze^2}{\xi+\eta}+e\,\boldsymbol{E}\left(\frac{\xi-\eta}{2}\right)=0 \tag{5.270}$$

The complete solution (5.8) reads $\mathscr{S}=S_\xi(\xi)+S_\eta(\eta)+S_\varphi(\varphi)+S_t(t)$ and (5.270) becomes

$$\frac{2}{m}\left(\frac{\xi}{\xi+\eta}\left(\frac{dS_\xi}{d\xi}\right)^2+\frac{\eta}{\xi+\eta}\left(\frac{dS_\eta}{d\eta}\right)^2+\frac{1}{4\xi\eta}\left(\frac{dS_\varphi}{d\varphi}\right)^2\right)-\frac{2Ze^2}{\xi+\eta}+e\,\boldsymbol{E}\left(\frac{\xi-\eta}{2}\right)=-\frac{dS_t}{dt} \tag{5.271}$$

One sets both sides equal to a constant E, the total energy of the conservative system, giving $S_t=-E\,t$. One isolates then the term containing φ, yielding

$$\xi\eta\left(\frac{2}{m}\left(\frac{\xi}{\xi+\eta}\left(\frac{dS_\xi}{d\xi}\right)^2+\frac{\eta}{\xi+\eta}\left(\frac{dS_\eta}{d\eta}\right)^2\right)-\frac{2Ze^2}{\xi+\eta}+e\,\boldsymbol{E}\left(\frac{\xi-\eta}{2}\right)-E\right)=-\frac{1}{2m}\left(\frac{dS_\varphi}{d\varphi}\right)^2 \tag{5.272}$$

One sets both sides of (5.272) equal to a constant $-\beta_1$, yielding $S_\varphi=\sqrt{2m\beta_1}\varphi$ and as the conjugate moment $p_\varphi=\frac{\partial \mathscr{S}}{\partial \varphi}=\frac{\partial S_\varphi}{\partial \varphi}$ is constant, one has $\beta_1=\frac{p_\varphi^2}{2m}$ and $S_\varphi=p_\varphi q_\varphi$. On the other hand, one has also

$$\xi\eta\left(\frac{2}{m}\left(\frac{\xi}{\xi+\eta}\left(\frac{dS_\xi}{d\xi}\right)^2+\frac{\eta}{\xi+\eta}\left(\frac{dS_\eta}{d\eta}\right)^2\right)-\frac{2Ze^2}{\xi+\eta}+e\,\boldsymbol{E}\left(\frac{\xi-\eta}{2}\right)-E\right)+\beta_1=0 \tag{5.273}$$

The terms depending of variables ξ and η are separated to obtain

$$\frac{2}{m}\xi\left(\frac{dS_\xi}{d\xi}\right)^2-2Ze^2+e\,\boldsymbol{E}\frac{\xi^2}{2}-E\,\xi+\frac{\beta_1}{\xi}=-\frac{2}{m}\eta\left(\frac{dS_\eta}{d\eta}\right)^2+e\,\boldsymbol{E}\frac{\eta^2}{2}+E\,\eta-\frac{\beta_1}{\eta}=\beta_2 \tag{5.274}$$

where one sets both sides of (5.274) equal to a new constant β_2, yielding respectively

$$S_\xi=\int\sqrt{\frac{m}{2}\left(-e\,\boldsymbol{E}\,\frac{\xi}{2}+E+\frac{\beta_2+2Ze^2}{\xi}-\frac{\beta_1}{\xi^2}\right)}d\xi \tag{5.275}$$

$$S_\eta=\int\sqrt{\frac{m}{2}\left(e\,\boldsymbol{E}\,\frac{\eta}{2}+E-\frac{\beta_2}{\eta}-\frac{\beta_1}{\eta^2}\right)}d\eta \tag{5.276}$$

and finally

$$\mathscr{S}=\int\sqrt{\frac{m}{2}\left(-e\,\boldsymbol{E}\,\frac{\xi}{2}+E+\frac{\beta_2+2Ze^2}{\xi}-\frac{\beta_1}{\xi^2}\right)}d\xi$$
$$+\int\sqrt{\frac{m}{2}\left(e\,\boldsymbol{E}\,\frac{\eta}{2}+E-\frac{\beta_2}{\eta}-\frac{\beta_1}{\eta^2}\right)}d\eta+\sqrt{2m\beta_1}\varphi+-E\,t \tag{5.277}$$

The movement equations are $\frac{\partial \mathscr{S}}{\partial E}=\gamma_1$, $\frac{\partial \mathscr{S}}{\partial \beta_1}=\gamma_2$ and $\frac{\partial \mathscr{S}}{\partial \beta_2}=\gamma_3$, yielding

$$\frac{\partial \mathscr{S}}{\partial E} = \frac{1}{2}\sqrt{\frac{m}{2}}\left(\int \frac{\xi\, d\xi}{\sqrt{-\frac{e\boldsymbol{E}\xi^3}{2} + E\,\xi^2 + \left(\beta_2 + 2Ze^2\right)\xi - \beta_1}} + \int \frac{\eta\, d\eta}{\sqrt{\frac{e\boldsymbol{E}\eta^3}{2} + E\,\eta^2 - \beta_2\eta - \beta_1}}\right) - t = \gamma_1 \tag{5.278}$$

$$\frac{\partial \mathscr{S}}{\partial \beta_1} = \frac{1}{2}\sqrt{\frac{m}{2}}\left(-\int \frac{d\xi}{\xi\sqrt{-\frac{e\boldsymbol{E}\xi^3}{2} + E\,\xi^2 + \left(\beta_2 + 2Ze^2\right)\xi - \beta_1}} - \int \frac{d\eta}{\eta\sqrt{\frac{e\boldsymbol{E}\eta^3}{2} + E\,\eta^2 - \beta_2\eta - \beta_1}} + \frac{2\varphi}{\sqrt{\beta_1}}\right) = \gamma_2 \tag{5.279}$$

$$\frac{\partial \mathscr{S}}{\partial \beta_2} = \frac{1}{2}\sqrt{\frac{m}{2}}\left(\int \frac{d\xi}{\sqrt{-\frac{e\boldsymbol{E}\xi^3}{2} + E\,\xi^2 + \left(\beta_2 + 2Ze^2\right)\xi - \beta_1}} - \int \frac{d\eta}{\sqrt{\frac{e\boldsymbol{E}\eta^3}{2} + E\,\eta^2 - \beta_2\eta - \beta_1}}\right) = \gamma_3 \tag{5.280}$$

with $\beta_1 = \frac{p_\varphi^2}{2m}$.

Equations (5.278)–(5.280) are identical to equations (5.263)–(5.265) of the second case. The transformation (5.229)–(5.231) is in fact the transformation from spherical coordinates to parabolic coordinates. □

5.2.13 *Exercise 32: Particle in a Double Coulomb Field*

> Give the equations of movement by the method of Hamilton–Jacobi of a particle P of mass m moving in a Coulomb field of two fixed points at a distance 2σ from each other.

Proof Without loss of generality, let the two points A_1 and A_2 (A_1 above A_2) be on the Z axis and with cylindrical coordinates $(0, 0, \pm\sigma)$. The distances of the point P, of cylindrical coordinates (ρ, φ, z), to the two points A_1 and A_2 are respectively r_1 and r_2 (see Fig. 5.6).

The Coulomb field of two fixed points reads

$$V = \frac{\alpha_1}{r_1} + \frac{\alpha_2}{r_2} \tag{5.281}$$

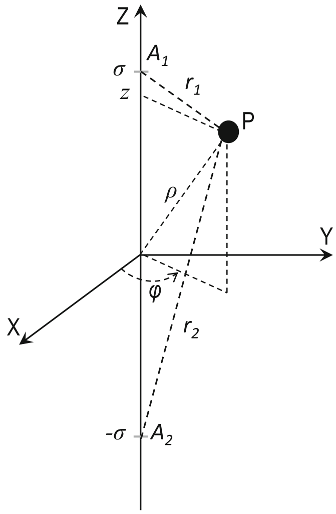


Fig. 5.6 Particle subjected to a Coulomb field of two fixed points

One chooses elliptic coordinates (ξ, η, φ) as generalized coordinates. One passes from cylindric coordinates (ρ, φ, z) to elliptic coordinates (ξ, η, φ) as follows:

$$\rho = \sigma\sqrt{\left(\xi^2 - 1\right)\left(1 - \eta^2\right)} \tag{5.282}$$

$$\varphi = \varphi \tag{5.283}$$

$$z = \sigma\xi\eta \tag{5.284}$$

The elliptic coordinates ξ and η vary between values $1 \le \xi < \infty$ and $-1 \le \eta \le +1$. Distances r_1 and r_2 of point P to the two fixed points A_1 and A_2 can be written in cylindric coordinates

$$r_1 = \sqrt{(z + \sigma)^2 + \rho^2} \tag{5.285}$$

$$r_2 = \sqrt{(z - \sigma)^2 + \rho^2} \tag{5.286}$$

Applying transformations (5.282)–(5.284), one obtains distances r_1 and r_2 in elliptic coordinates

$$r_1 = \sigma\left(\xi + \eta\right) \tag{5.287}$$

$$r_2 = \sigma\left(\xi - \eta\right) \tag{5.288}$$

The potential (5.281), with distances (5.287) and (5.288), reads

$$V = \frac{\alpha_1}{\sigma\,(\xi + \eta)} + \frac{\alpha_2}{\sigma\,(\xi - \eta)} \tag{5.289}$$

$$= \frac{(\alpha_1 + \alpha_2)\,\xi + (\alpha_2 - \alpha_1)\,\eta}{\sigma\left(\xi^2 - \eta^2\right)} \tag{5.290}$$

The Hamiltonian in elliptic coordinates is written (2.8) with the potential (5.290)

$$H = \frac{1}{2m\sigma^2\left(\xi^2 - \eta^2\right)}\left(\left(\xi^2 - 1\right)p_\xi^2 + \left(1 - \eta^2\right)p_\eta^2 + \left(\frac{1}{\xi^2 - 1} + \frac{1}{1 - \eta^2}\right)p_\varphi^2\right) + \frac{(\alpha_1 + \alpha_2)\,\xi + (\alpha_2 - \alpha_1)\,\eta}{\sigma\left(\xi^2 - \eta^2\right)} \tag{5.291}$$

One observes that the coordinate φ is cyclic. The Hamilton–Jacobi equation (5.4) $\frac{\partial \mathscr{S}}{\partial t} + H = 0$ has the complete solution $\mathscr{S} = S_\xi\,(\xi) + S_\eta\,(\eta) + S_\varphi\,(\varphi) + S_t\,(t)$. The first variable separation yields $S_t = -E\,t$, with E the constant total energy of the conservative system, and

$$\frac{1}{2m\sigma^2\left(\xi^2 - \eta^2\right)}\left(\left(\xi^2 - 1\right)\left(\frac{dS_\xi}{d\xi}\right)^2 + \left(1 - \eta^2\right)\left(\frac{dS_\eta}{d\eta}\right)^2 + \left(\frac{1}{\xi^2 - 1} + \frac{1}{1 - \eta^2}\right)\left(\frac{dS_\varphi}{d\varphi}\right)^2\right) + \frac{(\alpha_1 + \alpha_2)\,\xi + (\alpha_2 - \alpha_1)\,\eta}{\sigma\left(\xi^2 - \eta^2\right)} = E \tag{5.292}$$

One isolates the term in $\left(\frac{dS_\varphi}{d\varphi}\right)^2$ by multiplying (5.292) by $2m\sigma^2\left(\xi^2 - 1\right)\left(1 - \eta^2\right)$. As coordinate φ is cyclic, the conjugate moment $p_\varphi = \frac{dS_\varphi}{d\varphi}$ is constant, yielding $S_\varphi = p_\varphi \varphi$. The other part of (5.292) gives

$$\frac{(\xi^2-1)(1-\eta^2)}{(\xi^2-\eta^2)}\left(\left(\xi^2 - 1\right)\left(\frac{dS_\xi}{d\xi}\right)^2 + \left(1 - \eta^2\right)\left(\frac{dS_\eta}{d\eta}\right)^2 + 2m\sigma\left((\alpha_1 + \alpha_2)\,\xi + (\alpha_2 - \alpha_1)\,\eta\right)\right) - 2m\sigma^2 E\left(\xi^2 - 1\right)\left(1 - \eta^2\right) + p_\varphi^2 = 0 \tag{5.293}$$

One multiplies (5.293) by $\frac{(\xi^2-\eta^2)}{(\xi^2-1)(1-\eta^2)}$ and one separates the terms depending on ξ and on η to obtain

$$\left(\xi^2 - 1\right)\left(\frac{dS_\xi}{d\xi}\right)^2 + 2m\sigma\,(\alpha_1 + \alpha_2)\,\xi - 2m\sigma^2 E\xi^2 + \frac{p_\varphi^2}{(\xi^2-1)} = -\left[\left(1 - \eta^2\right)\left(\frac{dS_\eta}{d\eta}\right)^2 + 2m\sigma\,(\alpha_2 - \alpha_1)\,\eta + 2m\sigma^2 E\eta^2 + \frac{p_\varphi^2}{(1-\eta^2)}\right] = \beta \tag{5.294}$$

where both sides were set equal to a constant β. One finds then the solutions

$$S_\xi = \int \sqrt{\frac{2m\sigma^2 E\xi^2 - 2m\sigma\left(\alpha_1 + \alpha_2\right)\xi + \beta - \frac{p_\varphi^2}{(\xi^2-1)}}{\xi^2 - 1}} d\xi \tag{5.295}$$

$$S_\eta = \int \sqrt{\frac{-2m\sigma^2 E\eta^2 - 2m\sigma\left(\alpha_2 - \alpha_1\right)\eta - \beta - \frac{p_\varphi^2}{(1-\eta^2)}}{1 - \eta^2}} d\eta \tag{5.296}$$

The complete solution reads

$$\mathscr{S} = \int \sqrt{\frac{2m\sigma^2 E\xi^2 - 2m\sigma\left(\alpha_1 + \alpha_2\right)\xi + \beta - \frac{p_\varphi^2}{(\xi^2-1)}}{\xi^2 - 1}} d\xi \tag{5.297}$$

$$+ \int \sqrt{\frac{-2m\sigma^2 E\eta^2 - 2m\sigma\left(\alpha_2 - \alpha_1\right)\eta - \beta - \frac{p_\varphi^2}{(1-\eta^2)}}{1 - \eta^2}} d\eta + p_\varphi \varphi - E\, t \tag{5.298}$$

and one obtains the movement equations by deriving (5.297) with respect to E, then to p_φ and then to β by setting the derivatives equal to constants $\frac{\partial \mathscr{S}}{\partial E} = \gamma_1$, $\frac{\partial \mathscr{S}}{\partial p_\varphi} = \gamma_2$ and $\frac{\partial \mathscr{S}}{\partial \beta} = \gamma_3$. □

5.2.14 Exercise 33: Particle in Coulomb and Uniform Fields

Find the equations of movement by the method of Hamilton–Jacobi of a particle P of mass m moving in a field resulting from the superposition of a Coulomb field and a uniform field.

Proof Without loss of generality, let the Z axis be aligned along the uniform field in the opposite direction, and the fixed point producing the Coulomb field be at the axes origin. The distance of the point P of cylindrical coordinates (ρ, φ, z), to the fixed point is ρ. The potential resulting from the superposition of the two fields reads

$$V = \frac{\alpha}{\rho} - Fz \tag{5.299}$$

One chooses the parabolic coordinates as generalized coordinates and one can pass from cylindrical coordinates to parabolic coordinates with the transformation (5.266) and (5.267) $\rho = \frac{\xi+\eta}{2}$ and $z = \frac{\xi-\eta}{2}$, yielding

$$V = \frac{2\alpha}{\xi + \eta} - \frac{F\left(\xi - \eta\right)}{2} \tag{5.300}$$

The Hamiltonian in parabolic coordinates reads (2.7)

$$H = \frac{2}{m}\left(\frac{\xi p_\xi^2 + \eta p_\eta^2}{\xi+\eta} + \frac{p_\varphi^2}{4\xi\eta}\right) + \frac{2\alpha}{\xi+\eta} - \frac{F(\xi-\eta)}{2} \tag{5.301}$$

The Hamilton–Jacobi equation (5.4) $\frac{\partial \mathscr{S}}{\partial t} + H = 0$ has the complete solution $\mathscr{S} = S_\xi(\xi) + S_\eta(\eta) + S_\varphi(\varphi) + S_t(t)$. The first variable separation gives $S_t = -E\,t$, with constant E, the total energy of the conservative system, and

$$\frac{2}{m}\left(\frac{\xi}{\xi+\eta}\left(\frac{dS_\xi}{d\xi}\right)^2 + \frac{\eta}{\xi+\eta}\left(\frac{dS_\eta}{d\eta}\right)^2 + \frac{1}{4\xi\eta}\left(\frac{dS_\varphi}{d\varphi}\right)^2\right) + \frac{2\alpha}{\xi+\eta} - \frac{F(\xi-\eta)}{2} - E = 0 \tag{5.302}$$

One sees that the coordinate φ is cyclic, and thus $\left(\frac{dS_\varphi}{d\varphi}\right)^2 = p_\varphi^2$ yielding $S_\varphi = p_\varphi\varphi$. One multiplies (5.302) by $\frac{m(\xi+\eta)}{2}$ and one separates the terms in ξ and in η, yielding with β constant

$$\xi\left(\frac{dS_\xi}{d\xi}\right)^2 + \frac{p_\varphi^2}{4\xi} + m\alpha - \frac{mF}{4}\xi^2 - \frac{mE}{2}\xi = -\left(\eta\left(\frac{dS_\eta}{d\eta}\right)^2 + \frac{p_\varphi^2}{4\eta} + \frac{mF}{4}\eta^2 - \frac{mE}{2}\eta\right) = \beta \tag{5.303}$$

It comes further

$$S_\xi = \int \frac{1}{\xi}\sqrt{\frac{mF}{4}\xi^3 + \frac{mE}{2}\xi^2 + (\beta - m\alpha)\,\xi - \frac{p_\varphi^2}{4}}\,d\xi \tag{5.304}$$

$$S_\eta = \int \frac{1}{\eta}\sqrt{-\frac{mF}{4}\eta^3 + \frac{mE}{2}\eta^2 - \beta\eta - \frac{p_\varphi^2}{4}}\,d\eta \tag{5.305}$$

The complete solution $\mathscr{S}$ finally reads

$$\begin{aligned}\mathscr{S} = &\int \frac{1}{\xi}\sqrt{\frac{mF}{4}\xi^3 + \frac{mE}{2}\xi^2 + (\beta - m\alpha)\,\xi - \frac{p_\varphi^2}{4}}\,d\xi \\ &+ \int \frac{1}{\eta}\sqrt{-\frac{mF}{4}\eta^3 + \frac{mE}{2}\eta^2 - \beta\eta - \frac{p_\varphi^2}{4}}\,d\eta + p_\varphi\varphi - E\,t\end{aligned} \tag{5.306}$$

One obtains the movement equations by deriving (5.306) with respect to E, then to p_φ and then to β, setting the derivatives equal to constants $\frac{\partial \mathscr{S}}{\partial E} = \gamma_1$, $\frac{\partial \mathscr{S}}{\partial p_\varphi} = \gamma_2$ and $\frac{\partial \mathscr{S}}{\partial \beta} = \gamma_3$. □

Chapter 6
Phase Integral and Action-Angle Variables

6.1 Reminder

6.1.1 Phase Integral

Hamilton's method is useful in the search for solutions to a periodic mechanical system. In this case, the projection of the movement of the representative point in the phase space on any plane (p_α, q_α) is a closed curve.

The line integral

$$J_\alpha = \oint_{C_\alpha} p_\alpha dq_\alpha \tag{6.1}$$

is called phase integral or action variable, with $1 \leq \alpha \leq n$, where n is the number of degrees of freedom. One can show that $\mathscr{S} = \mathscr{S}\,(q_1, ..., q_n, J_1, ..., J_n)$, i.e. that the $J_1, ..., J_n$ are function of only the β_α, with $p_\alpha = \frac{\partial \mathscr{S}}{\partial q_\alpha}$ and $\omega_\alpha = \frac{\partial \mathscr{S}}{\partial J_\alpha}$, where one wrote ω_α instead of Q_α.

Hamilton equations (4.1) and (4.2) become

$$\dot{J}_\alpha = -\frac{\partial \mathcal{H}}{\partial \omega_\alpha} \tag{6.2}$$

$$\dot{\omega}_\alpha = \frac{\partial \mathcal{H}}{\partial J_\alpha} \tag{6.3}$$

with $\mathcal{H} = E$, the total energy of the conservative system which depends only on the constants J_α.

V. Pletser, *Lagrangian and Hamiltonian Analytical Mechanics: Forty Exercises Resolved and Explained*, UNITEXT for Physics,
https://doi.org/10.1007/978-981-13-3026-1_6

6.1.2 Frequency and Angular Variable

Equation (6.3) yields

$$\omega_\alpha = \frac{\partial \mathcal{H}}{\partial J_\alpha} t + c_\alpha \tag{6.4}$$

with c_α a constant. One sets

$$f_\alpha = \frac{\partial \mathcal{H}}{\partial J_\alpha} \tag{6.5}$$

where f_α is a constant frequency, yielding that ω_α is an angular variable. In addition, (6.2) is such that $\frac{\partial \mathcal{H}}{\partial \omega_\alpha} = 0$, yielding that J_α is a constant, that is the phase integral.

6.2 Exercises

6.2.1 Exercise 34: Harmonic Oscillator 4

Determine the frequencies of the harmonic oscillator.

Proof From Exercise 6, the Hamiltonian reads $H = \frac{p^2}{2m} + \frac{kq^2}{2}$ (2.24). The Hamilton–Jacobi equation (5.14) was found in Exercise 20. The complete solution (5.21) is $\mathscr{S} = \int \sqrt{2m\left(E - \frac{kq^2}{2}\right)} dq - E\,t$ and the solution in q is (5.28) $q = \sqrt{\frac{2E}{k}} \sin \left(\sqrt{\frac{k}{m}}\,(t+\gamma)\right)$, where E is the total energy of the conservative system.

A complete cycle is such as successively:

- $q = -\sqrt{\frac{2E}{k}}$ for $\sqrt{\frac{k}{m}}\,(t+\gamma) = -\frac{\pi}{2}$ or $\sin\left(\sqrt{\frac{k}{m}}\,(t+\gamma)\right) = -1$;
- $q = +\sqrt{\frac{2E}{k}}$ for $\sqrt{\frac{k}{m}}\,(t+\gamma) = \frac{\pi}{2}$ or $\sin\left(\sqrt{\frac{k}{m}}\,(t+\gamma)\right) = +1$;
- $q = -\sqrt{\frac{2E}{k}}$ for $\sqrt{\frac{k}{m}}\,(t+\gamma) = -\frac{\pi}{2}$ or $\sin\left(\sqrt{\frac{k}{m}}\,(t+\gamma)\right) = -1$.

The action variable (6.1) reads, with $p = \frac{\partial \mathscr{S}}{\partial q} = \sqrt{2m\left(E - \frac{kq^2}{2}\right)}$,

$$J = \oint \sqrt{2m\left(E - \frac{kq^2}{2}\right)} dq \tag{6.6}$$

$$= 2 \int_{-\sqrt{\frac{2E}{k}}}^{\sqrt{\frac{2E}{k}}} \sqrt{2m\left(E - \frac{kq^2}{2}\right)} dq \tag{6.7}$$

$$= 4\sqrt{mk} \int_0^{\sqrt{\frac{2E}{k}}} \sqrt{\frac{2E}{k} - q^2} dq \tag{6.8}$$

$$= 4\sqrt{mk} \left[\frac{q}{2}\sqrt{\frac{2E}{k} - q^2} + \frac{\beta}{k} \arcsin \frac{q}{\sqrt{\frac{2E}{k}}} \right]_0^{\sqrt{\frac{2\beta}{k}}} \tag{6.9}$$

$$= 4\sqrt{mk} \left(\frac{E}{k} \frac{\pi}{2} \right) = 2\pi E \sqrt{\frac{m}{k}} \tag{6.10}$$

where (2.271-3, Gradshteyn and Ryzhik, 2007; 14.244, Spiegel, 1974) was used to solve the integral in (6.8). One reverses (6.10) to find

$$E = \frac{J}{2\pi}\sqrt{\frac{k}{m}} \tag{6.11}$$

As $E = \mathcal{H}$, the total energy of the system, one finds the frequency (6.5) taking the total and not partial derivative of (6.11), as there is only one degree of freedom,

$$f = \frac{d\mathcal{H}}{dJ} = \frac{dE}{dJ} = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \tag{6.12}$$

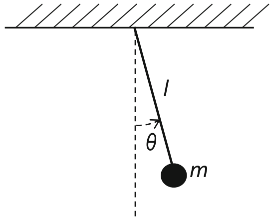
□

6.2.2 *Exercise 35: Small Oscillations of the Pendulum*

(1) Find the action variable or variables of the movement of the simple pendulum.
(2) In the case where the simple pendulum perform small oscillations, determine the frequency or frequencies by an approximate method and by an exact method.

Proof (1) The system has only one degree of freedom ($n = 1$) and the generalized coordinate θ is chosen as shown in the Fig. 6.1. The double pendulum had been treated in Exercise 1, where the first pendulum had as potential (1.18). The kinetic energy, potential, and Hamiltonian read respectively here

Fig. 6.1 Simple pendulum



$$T = \frac{ml^2\dot{\theta}^2}{2} \tag{6.13}$$

$$V = -mgl\cos\theta \tag{6.14}$$

$$H = \frac{ml^2\dot{\theta}^2}{2} - mgl\cos\theta \tag{6.15}$$

The conjugate moment is $p_\theta = \frac{\partial L}{\partial \dot{\theta}} = ml^2\dot{\theta}$, yielding $\dot{\theta} = \frac{p_\theta}{ml^2}$. The Hamiltonian (6.15) becomes

$$H = \frac{p_\theta^2}{2ml^2} - mgl\cos\theta \tag{6.16}$$

The Hamilton–Jacobi equation (5.4) $\frac{\partial \mathscr{S}}{\partial t} + H = 0$ becomes

$$\frac{\partial \mathscr{S}}{\partial t} + \frac{1}{2ml^2}\left(\frac{\partial \mathscr{S}}{\partial \theta}\right)^2 - mgl\cos\theta = 0 \tag{6.17}$$

and has the complete solution $\mathscr{S} = S_\theta(\theta) + S_t(t)$. The variable separation gives $S_t = -Et$, with constant E, the total energy of the conservative system, and

$$\frac{1}{2ml^2}\left(\frac{dS_\theta}{d\theta}\right)^2 - mgl\cos\theta - E = 0 \tag{6.18}$$

yielding

$$S_\theta = \int \sqrt{2ml^2(mgl\cos\theta + E)}d\theta \tag{6.19}$$

The action variable $J_\theta = \oint p_\theta d\theta$ (6.1) reads, with $p_\theta = \frac{\partial \mathscr{S}}{\partial \theta} = \sqrt{2ml^2(mgl\cos\theta + E)}$, and considering that on a full cycle, θ varies from θ_{max} to $-\theta_{max}$ and then from $-\theta_{max}$ back to θ_{max}, i.e. by symmetry of the movement, 4 times the part from 0 to θ_{max}

$$J_\theta = 4 \int_0^{\theta_{max}} \sqrt{2ml^2 (mgl \cos\theta + E)} d\theta \tag{6.20}$$

$$= 4ml\sqrt{2gl} \int_0^{\theta_{max}} \sqrt{\cos\theta + \frac{E}{mgl}} d\theta \tag{6.21}$$

As E is the total energy of the conservative system, at the instant when the pendulum is at its maximum elongation, $\theta = \theta_{max}$, the kinetic energy is nil and the total energy is the potential energy that is maximum, i.e.

$$E = -mgl \cos\theta_{max} \tag{6.22}$$

yielding

$$\frac{E}{mgl} = -\cos\theta_{max} \tag{6.23}$$

From (6.22), one sees that:

- for $\theta_{max} = 0$, the total energy E is minimal, $E = -mgl$, and the pendulum is at rest;
- for $\theta_{max} = \pi$, the total energy E is maximal, $E = mgl$, and the pendulum is in a vertical position in unstable equilibrium above the attachment point.

The action variable reads then

$$J_\theta = 4ml\sqrt{2gl} \int_0^{\theta_{max}} \sqrt{\cos\theta - \cos\theta_{max}} d\theta \tag{6.24}$$

(2) For small oscillations, one considers that $-\theta_{max} \leq \theta \leq \theta_{max}$ and $\theta_{max} \ll 1$, i.e. that the maximum amplitude of the oscillations is much smaller than 1 rad., say less than 10%, or $\theta_{max} \leq 0.1$ rad $\approx 5°$.
(2.1) In the approximate case, one develops $\cos\theta$ in Taylor series,

$$\cos\theta = \sum_{i=0}^{\infty} (-1)^i \frac{\theta^{2i}}{(2i)!} \tag{6.25}$$

$$\approx 1 - \frac{\theta^2}{2} \tag{6.26}$$

and one retains only the first two terms for small oscillations, $-\theta_{max} \leq \theta \leq \theta_{max}$ and $\theta_{max} \ll 1$, which gives (6.26), with an error of the order of θ^4. For $\theta = \theta_{max}$, it comes from (6.23) and (6.26)

$$\theta_{max} \approx \sqrt{2\left(\frac{E}{mgl} + 1\right)} \tag{6.27}$$

Replacing by (6.26) and (6.27) in (6.24), it comes successively (see 2.271-3, Gradshteyn and Ryzhik, 2007; 14.244, Spiegel, 1974),

$$J_\theta \approx 4ml\sqrt{gl}\int_0^{\theta_{max}} \sqrt{\theta_{max}^2 - \theta^2}d\theta \tag{6.28}$$

$$\approx 4ml\sqrt{gl}\left[\frac{\theta}{2}\sqrt{\theta_{max}^2 - \theta^2} + \frac{\theta_{max}^2}{2}\arcsin\left(\frac{\theta}{\theta_{max}}\right)\right]_0^{\theta_{max}} \tag{6.29}$$

$$\approx 4ml\sqrt{gl}\left[\frac{\theta_{max}^2}{2}\left(\arcsin\left(1\right) - \arcsin\left(0\right)\right)\right] \tag{6.30}$$

$$\approx 4ml\sqrt{gl}\left[\frac{\theta_{max}^2}{2}\frac{\pi}{2}\right] \tag{6.31}$$

$$\approx 4ml\sqrt{gl}\left[\frac{\pi}{2}\left(\frac{E}{mgl} + 1\right)\right] \tag{6.32}$$

$$\approx 2\pi\sqrt{\frac{l}{g}}\left(E + mgl\right) \tag{6.33}$$

where (6.27) was used in (6.31). Since

$$E = \mathcal{H} \approx \frac{J_\theta}{2\pi}\sqrt{\frac{g}{l}} - mgl \tag{6.34}$$

is the total energy of the system, one finds the frequency (6.5)

$$f_\theta = \frac{\partial\mathcal{H}}{\partial J_\theta} \approx \frac{1}{2\pi}\sqrt{\frac{g}{l}} \tag{6.35}$$

which is the value of the frequency of small isochronous oscillations of the pendulum, to a very good approximation.

(2.2) The complete resolution of the integral in (6.24), i.e. without the approximation (6.26), involves a change of variable and then two complete elliptic integrals. Replacing by (6.23) in (6.24) and substituting $\cos\theta$ by $\left(1 - 2\sin^2\left(\frac{\theta}{2}\right)\right)$ yield

$$J_\theta = 8ml\sqrt{gl}\sin\left(\frac{\theta_{max}}{2}\right)\int_0^{\theta_{max}} \sqrt{1 - \frac{\sin^2\left(\frac{\theta}{2}\right)}{\sin^2\left(\frac{\theta_{max}}{2}\right)}}d\theta \tag{6.36}$$

With the change of variable

$$\sin\varphi = \frac{\sin\left(\frac{\theta}{2}\right)}{\sin\left(\frac{\theta_{max}}{2}\right)} \tag{6.37}$$

or

$$\varphi = \arcsin\left(\frac{\sin\left(\frac{\theta}{2}\right)}{\sin\left(\frac{\theta_{max}}{2}\right)}\right) \tag{6.38}$$

one obtains successively

$$d\theta = 2\sin\left(\frac{\theta_{max}}{2}\right)\sqrt{\frac{1-\sin^2\varphi}{1-\sin^2\left(\frac{\theta}{2}\right)}}d\varphi \tag{6.39}$$

$$\theta = 0 \Rightarrow \varphi = 0 \tag{6.40}$$

$$\theta = \theta_{max} \Rightarrow \varphi = \frac{\pi}{2} \tag{6.41}$$

Replacing (6.37) and (6.39)–(6.41) and setting $p = \sin\left(\frac{\theta_{max}}{2}\right)$ in (6.36), one has

$$J_\theta = 16ml\sqrt{gl}\, p^2 \int_0^{\frac{\pi}{2}} \frac{\cos^2\varphi}{\sqrt{1-p^2\sin^2\varphi}}d\varphi \tag{6.42}$$

The integral I in (6.42) is solved after some algebraic manipulations

$$I = \int_0^{\frac{\pi}{2}} \frac{\cos^2\varphi}{\sqrt{1-p^2\sin^2\varphi}}d\varphi = \int_0^{\frac{\pi}{2}} \frac{1-\sin^2\varphi}{\sqrt{1-p^2\sin^2\varphi}}d\varphi \tag{6.43}$$

$$= \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{1-p^2\sin^2\varphi}}d\varphi + \int_0^{\frac{\pi}{2}} \frac{-\sin^2\varphi}{\sqrt{1-p^2\sin^2\varphi}}d\varphi \tag{6.44}$$

$$= \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{1-p^2\sin^2\varphi}}d\varphi + \frac{1}{p^2}\int_0^{\frac{\pi}{2}} \frac{-p^2\sin^2\varphi+1-1}{\sqrt{1-p^2\sin^2\varphi}}d\varphi \tag{6.45}$$

$$= \frac{\left(p^2-1\right)}{p^2}\int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{1-p^2\sin^2\varphi}}d\varphi + \frac{1}{p^2}\int_0^{\frac{\pi}{2}} \frac{1-p^2\sin^2\varphi}{\sqrt{1-p^2\sin^2\varphi}}d\varphi \tag{6.46}$$

$$= \frac{\left(p^2-1\right)}{p^2}\int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{1-p^2\sin^2\varphi}}d\varphi + \frac{1}{p^2}\int_0^{\frac{\pi}{2}} \sqrt{1-p^2\sin^2\varphi}\,d\varphi \tag{6.47}$$

$$= \frac{1}{p^2}\left[\left(p^2-1\right)\mathbf{F}(p;\varphi) + \mathbf{E}(p;\varphi)\right]_0^{\frac{\pi}{2}} \tag{6.48}$$

where $\mathbf{F}(p;\varphi)$ and $\mathbf{E}(p;\varphi)$ are the incomplete elliptic integrals of the first and second kinds defined by (see 5.111-1-2, Gradshteyn and Ryzhik, 2007; 34.1-3, Spiegel, 1974; 62.3.1-2, Spanier et Oldham, 1987)

$$\mathbf{F}(p;\varphi) = \int_0^{\varphi} \frac{d\vartheta}{\sqrt{1 - p^2 \sin^2 \vartheta}} \tag{6.49}$$

$$\mathbf{E}(p;\varphi) = \int_0^{\varphi} \sqrt{1 - p^2 \sin^2 \vartheta}\, d\vartheta \tag{6.50}$$

of module p and argument φ. For a nil argument, $\varphi = 0$, $\mathbf{F}(p;0) = \mathbf{E}(p;0) = 0$ and for an argument $\varphi = \frac{\pi}{2}$, $\mathbf{F}\left(p;\frac{\pi}{2}\right) = \mathbf{K}(p)$ and $\mathbf{E}\left(p;\frac{\pi}{2}\right) = \mathbf{E}(p)$ (see 62.7.1, Spanier et Oldham, 1987), where $\mathbf{K}(p)$ and $\mathbf{E}(p)$ are the complete elliptic integrals of the first and second kinds defined by the integrals from 0 to $\frac{\pi}{2}$ (see 5.111-1-2, Gradshteyn and Ryzhik, 2007; 34.2-4, Spiegel, 1974; 61.3.1-2, Spanier et Oldham, 1987)

$$\mathbf{K}(p) = \int_0^{\frac{\pi}{2}} \frac{d\vartheta}{\sqrt{1 - p^2 \sin^2 \vartheta}} \tag{6.51}$$

$$\mathbf{E}(p) = \int_0^{\frac{\pi}{2}} \sqrt{1 - p^2 \sin^2 \vartheta}\, d\vartheta \tag{6.52}$$

and by the series (see 61.6.1-2, Spanier et Oldham, 1987)

$$\mathbf{K}(p) = \frac{\pi}{2} \sum_{j=0}^{\infty} \left[\left(\frac{(2j-1)!!}{(2j)!!} p^j \right)^2 \right] \tag{6.53}$$

$$= \frac{\pi}{2} \left(1 + \frac{1}{4} p^2 + \frac{9}{64} p^4 + \frac{25}{256} p^6 + \cdots \right) = \frac{\pi}{2} s_{\mathbf{K}} \tag{6.54}$$

$$\mathbf{E}(p) = -\frac{\pi}{2} \sum_{j=0}^{\infty} \left[\frac{1}{2j-1} \left(\frac{(2j-1)!!}{(2j)!!} p^j \right)^2 \right] \tag{6.55}$$

$$= \frac{\pi}{2} \left(1 - \frac{1}{4} p^2 - \frac{3}{64} p^4 - \frac{5}{256} p^6 - \cdots \right) = \frac{\pi}{2} s_{\mathbf{E}} \tag{6.56}$$

where $(2n)!! = 2 \times 4 \times \cdots \times 2n$ is the double factorial of $2n$.

Replacing by (6.51) and (6.52) in (6.48) and then by (6.54) and (6.56), it comes successively

$$I = \frac{1}{p^2}\left(\mathbf{E}(p) + \left(p^2-1\right)\mathbf{K}(p)\right) \tag{6.57}$$

$$= \frac{1}{p^2}\left((\mathbf{E}(p) - \mathbf{K}(p)) + p^2\mathbf{K}(p)\right) \tag{6.58}$$

$$= \frac{1}{p^2}\left(\frac{\pi p^2}{4}\left(1 + \frac{1}{8}p^2 + \frac{3}{64}p^4 + \frac{25}{1024}p^6 + \cdots\right)\right) \tag{6.59}$$

$$= \frac{\pi}{4}s \tag{6.60}$$

where s is the series in (6.59). Replacing in the expression (6.42) of J_θ, it comes

$$J_\theta = 4\pi ml\sqrt{gl}\,p^2 s \tag{6.61}$$

$$= 2\pi ml\sqrt{gl}\left(\frac{E}{mgl} + 1\right)s \tag{6.62}$$

$$= 2\pi\sqrt{\frac{l}{g}}\,(E + mgl)\,s \tag{6.63}$$

where p^2 in (6.61) was replaced by $\frac{1}{2}\left(\frac{E}{mgl} + 1\right)$ from (6.23).

One finds then the total energy by reverting (6.63)

$$E = \mathcal{H} = \frac{J_\theta}{2\pi s}\sqrt{\frac{g}{l}} - mgl \tag{6.64}$$

and the frequency (6.5) by taking the total derivative of (6.64) and noting also that s depends indirectly of J_θ

$$f_\theta = \frac{d\mathcal{H}}{dJ_\theta} = \frac{1}{2\pi}\sqrt{\frac{g}{l}}\frac{d}{dJ_\theta}\left(\frac{J_\theta}{s}\right) \tag{6.65}$$

$$= \frac{1}{2\pi s}\sqrt{\frac{g}{l}}\left(1 - \frac{J_\theta\frac{ds}{dJ_\theta}}{s}\right) \tag{6.66}$$

$$= \frac{1}{2\pi s}\sqrt{\frac{g}{l}}\left(1 - \frac{J_\theta}{s\frac{dJ_\theta}{ds}}\right) \tag{6.67}$$

$$= \frac{1}{2\pi s}\sqrt{\frac{g}{l}}\left(1 - \frac{J_\theta\frac{ds}{dp}}{s\frac{dJ_\theta}{dp}}\right) \tag{6.68}$$

where one used $\frac{dJ_\theta}{ds} = \frac{dJ_\theta}{dp}\frac{dp}{ds}$ in (6.67). As one has from (6.61),

$$\frac{dJ_\theta}{dp} = 4\pi ml\sqrt{gl}\left(2ps + p^2\frac{ds}{dp}\right) \tag{6.69}$$

the substitution in (6.68) and the simplification with (6.61) yield

$$f_\theta = \frac{1}{2\pi s}\sqrt{\frac{g}{l}}\left(1-\frac{p^2 s\frac{ds}{dp}}{s\left(2ps+p^2\frac{ds}{dp}\right)}\right) \tag{6.70}$$

$$= \frac{1}{2\pi s}\sqrt{\frac{g}{l}}\left(1-\frac{p\frac{ds}{dp}}{2s+p\frac{ds}{dp}}\right) \tag{6.71}$$

$$= \frac{1}{2\pi}\sqrt{\frac{g}{l}}\left(\frac{1}{s+\frac{p}{2}\frac{ds}{dp}}\right) \tag{6.72}$$

From the expression (6.59) of s, one has

$$\frac{p}{2}\frac{ds}{dp} = \frac{p}{2}\left(\frac{1}{4}p+\frac{3}{16}p^3+\frac{75}{512}p^5+\cdots\right) \tag{6.73}$$

$$= \frac{1}{8}p^2+\frac{3}{32}p^4+\frac{65}{1024}p^6+\cdots \tag{6.74}$$

that, added to s (6.59), finally yields

$$s+\frac{p}{2}\frac{ds}{dp} = 1+\frac{1}{4}p^2+\frac{9}{64}p^4+\frac{25}{256}p^6+\cdots \tag{6.75}$$

which is the series $s_{\mathbf{K}}$ in (6.54) of the development in series of $\frac{2\mathbf{K}(p)}{\pi}$. The frequency f_θ can finally be written indistinctly

$$f_\theta = \frac{1}{2\pi}\sqrt{\frac{g}{l}}\left(\frac{\pi}{2\mathbf{K}(p)}\right) = \frac{1}{4\mathbf{K}(p)}\sqrt{\frac{g}{l}} \tag{6.76}$$

$$= \frac{1}{2\pi}\sqrt{\frac{g}{l}}\left(\frac{1}{\sum_{j=0}^{\infty}\left[\left(\frac{(2j-1)!!}{(2j)!!}p^j\right)^2\right]}\right) \tag{6.77}$$

$$= \frac{1}{2\pi}\sqrt{\frac{g}{l}}\left(\frac{1}{1+\frac{1}{4}p^2+\frac{9}{64}p^4+\frac{25}{256}p^6+\cdots}\right) \tag{6.78}$$

$$= \frac{1}{2\pi s_{\mathbf{K}}}\sqrt{\frac{g}{l}} \tag{6.79}$$

Replacing p by $\sin\left(\frac{\theta_{max}}{2}\right)$, the series $s_{\mathbf{K}}$ reads also as

$$s_{\mathbf{K}} = 1+\frac{1}{4}\sin^2\left(\frac{\theta_{max}}{2}\right)+\frac{9}{64}\sin^4\left(\frac{\theta_{max}}{2}\right)+\frac{25}{256}\sin^6\left(\frac{\theta_{max}}{2}\right)+\cdots \tag{6.80}$$

For $\theta_{max} \ll 1$, one has $\sin\left(\frac{\theta_{max}}{2}\right) \approx \frac{\theta_{max}}{2}$, and $s_{\mathbf{K}}$ can be approximated by

$$s_{\mathbf{K}} \approx 1 + \frac{1}{16}\theta_{max}^2 + \frac{9}{1024}\theta_{max}^4 + \frac{25}{16384}\theta_{max}^6 + \cdots \tag{6.81}$$

which remains all the more close to 1 that θ_{max} is small. By replacing $s_{\mathbf{K}}$ by 1 in (6.65), one finds the classic value of the period of small isochronous oscillations (6.35). □

The case of pendulum large oscillations is treated in several text books and articles (see e.g. Baker and Blackburn, 2005; Belendez et al. 2007; Fulcher and Davis, 1976). Illustrations can be found in (Belendez et al. 2007).

6.2.3 Exercise 36: Three Dimension Harmonic Oscillator

(1) Determine by the method of the angular variables the eigenfrequencies of the three dimension harmonic oscillator with three different restoring forces. One supposes that the weight of the harmonic oscillator is negligible in front of the vertical resultant of the spring forces.
(2) One defines an isotropic oscillator as a two or three dimension harmonic oscillator having equal oscillation frequencies. Express the energy of a three dimension isotropic oscillator in function of only one of the action variables.

Proof The one dimension harmonic oscillator was studied in the Exercises 6, 19, 20 and 33. Here, the system has three degrees of freedom ($n = 3$). It is assumed that there is no interaction between the movements of the three axes[1] One uses the

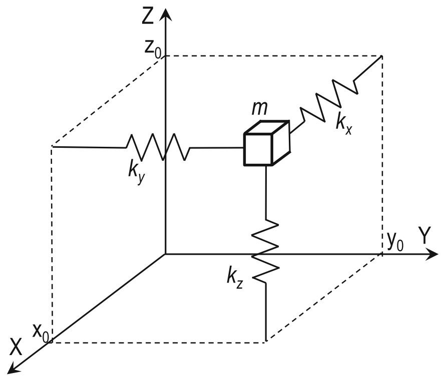


Fig. 6.2 Three dimension harmonic oscillator

[1] In the contrary case, see paragraph 23, pg 65 of (Landau et Lifshitz, 1969).

three Cartesian coordinates (x, y, z) as generalized coordinates, representing the differences between the components along the three axes of the current position and the position at rest at equilibrium (x_0, y_0, z_0) (see Fig. 6.2). The restoring constants or stiffnesses of the three springs are k_x, k_y and k_z.

By similarity with (2.24), the Hamiltonian is written

$$H = \frac{p_x^2 + p_y^2 + p_z^2}{2m} + \frac{k_x x^2 + k_y y^2 + k_z z^2}{2} \tag{6.82}$$

The Hamilton–Jacobi equation (5.4) $\frac{\partial \mathscr{S}}{\partial t} + H = 0$ reads with $p_\alpha = \frac{\partial \mathscr{S}}{\partial \alpha}$

$$\frac{\partial \mathscr{S}}{\partial t} + \frac{1}{2m}\left(\left(\frac{\partial \mathscr{S}}{\partial x}\right)^2 + \left(\frac{\partial \mathscr{S}}{\partial y}\right)^2 + \left(\frac{\partial \mathscr{S}}{\partial z}\right)^2\right) + \frac{k_x x^2 + k_y y^2 + k_z z^2}{2} = 0 \tag{6.83}$$

and has the complete solution $\mathscr{S} = S_x(x) + S_y(y) + S_z(z) + S_t(t)$. The first variable separation yields $S_t = -E\,t$, with constant E, the total energy of the conservative system. The other variable separations yield successively

$$\frac{1}{2m}\left(\frac{dS_x}{dx}\right)^2 + \frac{k_x x^2}{2} + \frac{1}{2m}\left(\frac{dS_y}{dy}\right)^2 + \frac{k_y y^2}{2} - E = -\left(\frac{1}{2m}\left(\frac{dS_z}{dz}\right)^2 + \frac{k_z z^2}{2}\right) = -\beta_1 \tag{6.84}$$

$$S_z = \int \sqrt{2m\left(\beta_1 - \frac{k_z z^2}{2}\right)}dz \tag{6.85}$$

$$\frac{1}{2m}\left(\frac{dS_x}{dx}\right)^2 + \frac{k_x x^2}{2} - E + \beta_1 = -\left(\frac{1}{2m}\left(\frac{dS_y}{dy}\right)^2 + \frac{k_y y^2}{2}\right) = -\beta_2 \tag{6.86}$$

$$S_y = \int \sqrt{2m\left(\beta_2 - \frac{k_y y^2}{2}\right)}dy \tag{6.87}$$

$$S_x = \int \sqrt{2m\left(E - \beta_1 - \beta_2 - \frac{k_x x^2}{2}\right)}dx \tag{6.88}$$

with constants β_1 and β_2. The complete solution $\mathscr{S}$ reads then

$$\begin{aligned}\mathscr{S} = &\int \sqrt{2m\left(E - \beta_1 - \beta_2 - \frac{k_x x^2}{2}\right)}dx + \int \sqrt{2m\left(\beta_2 - \frac{k_y y^2}{2}\right)}dy \\ &+ \int \sqrt{2m\left(\beta_1 - \frac{k_z z^2}{2}\right)}dz - E\,t\end{aligned} \tag{6.89}$$

The conjugate moments read

$$p_x = \frac{\partial \mathscr{S}}{\partial x} = \sqrt{2m\left(E - \beta_1 - \beta_2 - \frac{k_x x^2}{2}\right)} \tag{6.90}$$

$$p_y = \frac{\partial \mathscr{S}}{\partial y} = \sqrt{2m\left(\beta_2 - \frac{k_y y^2}{2}\right)} \tag{6.91}$$

$$p_z = \frac{\partial \mathscr{S}}{\partial z} = \sqrt{2m\left(\beta_1 - \frac{k_z z^2}{2}\right)} \tag{6.92}$$

One considers that a complete cycle corresponds to the movement from the position of equilibrium (at rest) to the maximum position, then back through the position of equilibrium to the minimum position and then back to the equilibrium position, i.e. by symmetry of movement, four times the part from the position of equilibrium to the maximum position. Action variables (6.1) then read

$$J_x = \oint p_x dx = 4 \int_0^{x_{max}} \sqrt{2m\left(E - \beta_1 - \beta_2 - \frac{k_x x^2}{2}\right)} dx \tag{6.93}$$

$$J_y = \oint p_y dy = 4 \int_0^{y_{max}} \sqrt{2m\left(\beta_2 - \frac{k_y y^2}{2}\right)} dy \tag{6.94}$$

$$J_z = \oint p_z dz = 4 \int_0^{z_{max}} \sqrt{2m\left(\beta_1 - \frac{k_z z^2}{2}\right)} dz \tag{6.95}$$

with the conditions to ensure that the integrals in (6.93)–(6.95) are real, i.e. that the term under the radical sign is non-negative at any time of the movement.

As there is no interaction between the movements along the three axes, one can consider that the total energy is the sum of the energies of each of the three movements along the three axes, i.e.

$$E = E_x + E_y + E_z \tag{6.96}$$

At maximum elongations, for $x = x_{max}$, $y = y_{max}$ and $z = z_{max}$, the kinetic energy is nil, the potential energy is maximum and one has

$$E_x = \frac{k_x x_{max}^2}{2} \tag{6.97}$$

$$E_y = \frac{k_y y_{max}^2}{2} \tag{6.98}$$

$$E_z = \frac{k_z z_{max}^2}{2} \tag{6.99}$$

In (6.93)–(6.95), $(E - \beta_1 - \beta_2)$, β_2 and β_1 are the energies respectively E_x, E_y and E_z of movements along the axes X, Y and Z.

The action variables (6.93)–(6.95) becomes

$$J_x = 4\int_0^{x_{max}} \sqrt{2m\left(\frac{k_x x_{max}^2}{2} - \frac{k_x x^2}{2}\right)}dx = 4\sqrt{mk_x}\int_0^{x_{max}} \sqrt{x_{max}^2 - x^2}dx \quad (6.100)$$

$$J_y = 4\int_0^{y_{max}} \sqrt{2m\left(\frac{k_y y_{max}^2}{2} - \frac{k_y y^2}{2}\right)}dy = 4\sqrt{mk_y}\int_0^{y_{max}} \sqrt{y_{max}^2 - y^2}dy \quad (6.101)$$

$$J_z = 4\int_0^{z_{max}} \sqrt{2m\left(\frac{k_z z_{max}^2}{2} - \frac{k_z z^2}{2}\right)}dz = 4\sqrt{mk_z}\int_0^{z_{max}} \sqrt{z_{max}^2 - z^2}dz \quad (6.102)$$

These three integrals are easily solved (see 2.271-3, Gradshteyn and Ryzhik, 2007; 14.244, Spiegel, 1974)

$$J_x = 4\sqrt{mk_x}\left[\frac{x\sqrt{x_{max}^2 - x^2}}{2} + \frac{x_{max}^2}{2}\arcsin\left(\frac{x}{x_{max}}\right)\right]_0^{x_{max}} \quad (6.103)$$

$$= 4\sqrt{mk_x}\left(\frac{x_{max}^2}{2}\arcsin(1)\right) \quad (6.104)$$

$$= 4\sqrt{mk_x}\left(\frac{E_x}{k_x}\frac{\pi}{2}\right) = 2\pi E_x\sqrt{\frac{m}{k_x}} \quad (6.105)$$

One finds similarly

$$J_y = 2\pi E_y\sqrt{\frac{m}{k_y}} \quad (6.106)$$

$$J_z = 2\pi E_z\sqrt{\frac{m}{k_z}} \quad (6.107)$$

Reversing (6.105)–(6.107), one finds the total energy (6.96)

$$E = \frac{1}{2\pi\sqrt{m}}\left(J_x\sqrt{k_x} + J_y\sqrt{k_y} + J_z\sqrt{k_z}\right) \quad (6.108)$$

As here $E = \mathcal{H}$, one finds the three frequencies (6.5)

$$f_x = \frac{\partial\mathcal{H}}{\partial J_x} = \frac{1}{2\pi}\sqrt{\frac{k_x}{m}} \quad (6.109)$$

$$f_y = \frac{\partial\mathcal{H}}{\partial J_y} = \frac{1}{2\pi}\sqrt{\frac{k_y}{m}} \quad (6.110)$$

$$f_z = \frac{\partial \mathcal{H}}{\partial J_z} = \frac{1}{2\pi}\sqrt{\frac{k_z}{m}} \tag{6.111}$$

(2) If the harmonic oscillator is isotropic, the three springs are identical, i.e. having the same stiffness $k = k_x = k_y = k_z$ and the same maximum elongation $x_{max} = y_{max} = z_{max}$, which makes the energies of each of the three movements along the three axes E_x (6.97), E_y (6.98) and E_z (6.99) equal, $E_x = E_y = E_z = \frac{E}{3}$, and three action variables J_x (6.100), J_y (6.101) and J_z (6.102) equal, $J_x = J_y = J_z = J$. Thus, one finds directly from (6.108) that the total energy E is

$$E = \frac{3J}{2\pi}\sqrt{\frac{k}{m}} \tag{6.112}$$

□

6.2.4 Exercise 37: Energy in a Bohr Atom

The classic model of an atom consists of a cloud of electron of charge $-e$ describing orbits in a central force field around a nucleus of charge Ze such as the force acting on a electron is

$$\overrightarrow{F} = -\frac{Ze^2\overrightarrow{r}}{r^3} \tag{6.113}$$

where $\overrightarrow{r}$ is the vector position of the electron with respect to the nucleus and Z is the atomic number.
In Bohr's atomic quantum theory, phase integrals are integer multiples of Planck constant h

$$\oint p_r dr = n_1 h \tag{6.114}$$

$$\oint p_\theta d\theta = n_2 h \tag{6.115}$$

and $n = n_1 + n_2$ $(n, n_1, n_2 \in \mathbb{N}_0)$ is called the orbital quantum number. The gravitational attraction between the nucleus and the electron and the weight of the electron are negligible.
(1) Show that the total energy of the system is represented by discrete values.
(2) What is the expression of this energy?
(3) Can we still calculate the angular variables and frequencies of the system? Why?

Proof (1–2) The movement in a central force field in $1/r^2$ is planar. So, there are two degrees of freedom ($n = 2$). One chooses as generalized coordinates the two dimension polar coordinates (r, θ). One finds from (6.113) the magnitude of the force $F = -\frac{Ze^2}{r^2}$, from which one obtains the potential $V = -\frac{Ze^2}{r}$. The Hamiltonian in polar coordinates reads from (2.5)

$$H = \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2}\right) - \frac{Ze^2}{r} \tag{6.116}$$

with m the electron mass.

As $p_r = \frac{\partial \mathscr{S}}{\partial r}$ et $p_\theta = \frac{\partial \mathscr{S}}{\partial \theta}$, the Hamilton–Jacobi equation (5.4) reads

$$\frac{\partial \mathscr{S}}{\partial t} + \frac{1}{2m}\left(\left(\frac{\partial \mathscr{S}}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial \mathscr{S}}{\partial \theta}\right)^2\right) - \frac{Ze^2}{r} = 0 \tag{6.117}$$

and has the complete solution $\mathscr{S} = S_r(r) + S_\theta(\theta) + S_t(t)$. The first variable separation yields $S_t = -E\,t$, with constant E, the total energy of the conservative system. As the generalized coordinate θ is ignorable (it does not appear in the Lagrangian), the conjugated moment $p_\theta = \frac{\partial \mathscr{S}}{\partial \theta}$ is constant and a constant of movement. The other variable separation yields successively

$$\left(\frac{dS_\theta}{d\theta}\right)^2 = r^2\left(2m\left(E + \frac{Ze^2}{r}\right) - \left(\frac{dS_r}{dr}\right)^2\right) = p_\theta^2 \tag{6.118}$$

$$S_\theta = p_\theta \theta \tag{6.119}$$

$$S_r = \int \sqrt{2m\left(E + \frac{Ze^2}{r}\right) - \frac{p_\theta^2}{r^2}}dr \tag{6.120}$$

and the complete solution reads

$$\mathscr{S} = \int \sqrt{2m\left(E + \frac{Ze^2}{r}\right) - \frac{p_\theta^2}{r^2}}dr + p_\theta\theta - E\,t \tag{6.121}$$

The phase integrals are

$$J_r = \oint p_r dr = \oint \sqrt{2m\left(E + \frac{Ze^2}{r}\right) - \frac{p_\theta^2}{r^2}}dr = n_1 h \tag{6.122}$$

$$J_\theta = \oint p_\theta d\theta = p_\theta \oint d\theta = 2\pi p_\theta = n_2 h \tag{6.123}$$

where the last equalities in (6.122) and (6.123) come from (6.114) and (6.115). From (6.123), one obtains

$$p_\theta = n_2 \frac{h}{2\pi} \tag{6.124}$$

A complete cycle is such that the coordinate r varies from its minimal value $r = r_{min}$ to its maximal value $r = r_{max}$ and then again to its minimal value $r = r_{min}$. By symmetry of movement, the complete cycle for r is twice the part from $r = r_{min}$ to $r = r_{max}$. From (6.122), one finds then

$$J_r = 2 \int_{r_{min}}^{r_{max}} \frac{\sqrt{2m\left(Er^2 + Ze^2 r\right) - p_\theta^2}}{r} dr = n_1 h \tag{6.125}$$

The values of r_{min} and r_{max} correspond to extremums, minimum and maximum, of the function $\int \frac{\sqrt{2m\left(Er^2+Ze^2r\right)-p_\theta^2}}{r} dr$ in (6.125), in other words, to the solutions of $\frac{\sqrt{2m\left(Er^2+Ze^2r\right)-p_\theta^2}}{r} = 0$. There are three possible solutions, the first two correspond to the zeros of the trinomial under the radical sign of the numerator,

$$r_{extr} = \frac{-mZe^2 \pm \sqrt{m^2 Z^2 e^4 + 2mEp_\theta^2}}{2mE} = \frac{Ze^2\left(-1 \pm \sqrt{1 + \frac{2Ep_\theta^2}{mZ^2e^4}}\right)}{2E} \tag{6.126}$$

and the third corresponds to the denominator tending to infinity, i.e. $r_{extr} \to \infty$. One can reject the latter possibility as we know that the atom in its normal state is stable, i.e. that the electrons are in orbits whose radius keeps finite value. As in Exercise 19, one considers that the electron is in an orbit of a general elliptic shape, i.e. with a negative total energy, such as

$$-\frac{Z^2 e^4 m}{2p_\theta^2} \le E < 0 \tag{6.127}$$

Writing $E = -|E|$ (where vertical bars denote the absolute value), (6.126) yields

$$r_{\frac{min}{max}} = \frac{Ze^2\left(1 \mp \sqrt{1 - \frac{2|E|p_\theta^2}{mZ^2e^4}}\right)}{2|E|} \tag{6.128}$$

with r_{min} (resp. r_{max}) corresponding to the $-$ (resp. $+$) sign in front of the radical sign. The integral of (6.125) is solved (see 2.267-1, Gradshteyn and Ryzhik, 2007; 14.288, Spiegel, 1974) as

$$J_r = 2\left(\left[\sqrt{2m\left(-|E|r^2 + Ze^2 r\right) - p_\theta^2}\right]_{r_{min}}^{r_{max}} + mZe^2 I_1 - p_\theta^2 I_2\right) \tag{6.129}$$

where integrals I_1 and I_2 are

$$I_1 = \int_{r_{min}}^{r_{max}} \frac{dr}{\sqrt{2m\left(-|E|\,r^2 + Ze^2 r\right) - p_\theta^2}} \tag{6.130}$$

$$I_2 = \int_{r_{min}}^{r_{max}} \frac{dr}{r\sqrt{2m\left(-|E|\,r^2 + Ze^2 r\right) - p_\theta^2}} \tag{6.131}$$

With the values (6.128) of r_{min} and r_{max}, the first term of (6.129) is nil and the action variable J_r (6.129) becomes

$$J_r = 2\left(mZe^2 I_1 - p_\theta^2 I_2\right) \tag{6.132}$$

The integrals (6.130) and (6.131) yield (see 2.264-1, Gradshteyn and Ryzhik, 2007; 14.280, Spiegel, 1974[2] and 2.266, Gradshteyn and Ryzhik, 2007; 14.283, Spiegel, 1974)

$$I_1 = -\frac{1}{\sqrt{2m\,|E|}}\left[\arcsin\left(\frac{1 - \frac{2|E|}{Ze^2}r}{\sqrt{1 - \frac{2|E|p_\theta^2}{mZ^2e^4}}}\right)\right]_{r_{min}}^{r_{max}} \tag{6.133}$$

$$= -\frac{1}{\sqrt{2m\,|E|}}\left[\arcsin\left(\frac{1 - \frac{2|E|}{Ze^2}r_{max}}{\sqrt{1 - \frac{2|E|p_\theta^2}{mZ^2e^4}}}\right) - \arcsin\left(\frac{1 - \frac{2|E|}{Ze^2}r_{min}}{\sqrt{1 - \frac{2|E|p_\theta^2}{mZ^2e^4}}}\right)\right] \tag{6.134}$$

$$= -\frac{1}{\sqrt{2m\,|E|}}\left[\arcsin(-1) - \arcsin(+1)\right] \tag{6.135}$$

$$= -\frac{1}{\sqrt{2m\,|E|}}\left(-\frac{\pi}{2} - \frac{\pi}{2}\right) = \frac{\pi}{\sqrt{2m\,|E|}} \tag{6.136}$$

$$I_2 = \frac{1}{p_\theta}\left[\arcsin\left(\frac{r - \frac{p_\theta^2}{mZe^2}}{r\sqrt{1 - \frac{2|E|p_\theta^2}{mZ^2e^4}}}\right)\right]_{r_{min}}^{r_{max}} \tag{6.137}$$

$$= \frac{1}{p_\theta}\left[\arcsin\left(\frac{1 - \frac{p_\theta^2}{mZe^2 r_{max}}}{\sqrt{1 - \frac{2|E|p_\theta^2}{mZ^2e^4}}}\right) - \arcsin\left(\frac{1 - \frac{p_\theta^2}{mZe^2 r_{min}}}{\sqrt{1 - \frac{2|E|p_\theta^2}{mZ^2e^4}}}\right)\right] \tag{6.138}$$

$$= \frac{1}{p_\theta}\left[\arcsin(+1) - \arcsin(-1)\right] = \frac{\pi}{p_\theta} \tag{6.139}$$

where r_{min} and r_{max} were replaced by (6.128) in (6.134) and (6.138).

[2] Remark that the coefficient $(-2m\,|E|)$ of the term in r^2 is negative.

The action variable J_r (6.132) reads then with (6.136) and (6.139)[3]

$$J_r = \pi Z e^2 \sqrt{\frac{2m}{|E|}} - 2\pi p_\theta \tag{6.142}$$

Substituting p_θ by its value (6.124), J_r by (6.114) and reversing (6.142), one finds the energy E expression

$$E = -\frac{2\pi^2 m Z^2 e^4}{n^2 h^2} \tag{6.143}$$

where $n = n_1 + n_2$ is a natural integer. The energy E is therefore represented by discrete values and is no longer continuous. One notes also that the condition (6.127) yields that $n \geq \frac{2\pi p_\theta}{h}$.

(3) One cannot calculate the frequencies because $f_\alpha = \frac{\partial \mathcal{H}}{\partial J_\alpha}$ (6.5) and that here, J_r and J_θ are represented by discrete and not continuous values. One cannot derive with respect to discrete non-continuous values. □

6.2.5 *Exercise 38: Classical Kepler Problem 2*

(1) Determine the frequencies of the Kepler problem for elliptic orbits.
(2) Is this method applicable to the cases of parabolic and hyperbolic orbits? Why?

Proof (1) The Kepler problem was studied in Exercise 22, where one found the Hamiltonian (5.142) $H = \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2}\right) - \frac{K}{r}$ and the complete solution (5.155) $\mathscr{S} = \int \sqrt{2m\left(E + \frac{K}{r}\right) - \frac{p_\theta^2}{r^2}}\,dr + p_\theta \theta - E\,t$, where E is the total energy of the conservative system and p_θ is the constant conjugated angular moment.

A complete cycle is such that the coordinate θ varies from 0 to 2π and that the coordinate r varies from its minimum value $r = r_{min}$[4] to its maximum value

---

[3]One notes that the value of J_r can be calculated faster and easier by using the method of the residue theory (Spiegel, 1964). From (6.122), one finds directly

$$J_r = 2\pi i \left(\sqrt{-p_\theta^2} + \frac{mZe^2}{\sqrt{2mE}}\right) \tag{6.140}$$

$$= -2\pi p_\theta + \frac{2\pi m Z e^2}{\sqrt{2m\,|E|}} \tag{6.141}$$

[4]Distance of the point of the orbit called pericentre; in particular, perigee or perihelion for orbits around the Earth or the Sun.

$r = r_{max}$[5] and then again to its minimum value $r = r_{min}$. By symmetry of movement, the complete cycle for r is twice the part from the pericentre to the apocentre.

The action variables (6.1) are written, with $p_r = \frac{\partial \mathscr{S}}{\partial r} = \sqrt{2m\left(E + \frac{K}{r}\right) - \frac{p_\theta^2}{r^2}}$ and $p_\theta = \frac{\partial \mathscr{S}}{\partial \theta}$,

$$J_\theta = \oint p_\theta d\theta = \int_0^{2\pi} p_\theta d\theta = 2\pi p_\theta \tag{6.144}$$

$$J_r = \oint \sqrt{2m\left(E + \frac{K}{r}\right) - \frac{p_\theta^2}{r^2}} dr \tag{6.145}$$

$$= 2\int_{r_{min}}^{r_{max}} \frac{\sqrt{2m\left(Er^2 + Kr\right) - p_\theta^2}}{r} dr \tag{6.146}$$

The values of r_{min} and r_{max} corresponding to extremums, minimum and maximum, of the function $\int \frac{\sqrt{2m\left(Er^2+Kr\right)-p_\theta^2}}{r} dr$ in (6.146), in other words to the solutions of $\frac{\sqrt{2m\left(Er^2+Kr\right)-p_\theta^2}}{r} = 0$. There are three possible solutions, the first two correspond to the zeros of the trinomial under the radical sign of the numerator,

$$r_{extr} = \frac{-mK \pm \sqrt{m^2K^2 + 2mEp_\theta^2}}{2mE} = \frac{K\left(-1 \pm \sqrt{1 + \frac{2Ep_\theta^2}{mK^2}}\right)}{2E} \tag{6.147}$$

and the third corresponds to the denominator tending to infinity, i.e.

$$r_{extr} \to \infty \tag{6.148}$$

Among these three possibilities, the two values of r that will be r_{min} and r_{max} will depend on the values of E.

One considers like in Exercise 22 the three cases:

(1) $-\frac{K^2m}{2p_\theta^2} \leq E < 0$: the orbit is an ellipse;
(2) $E = 0$: the orbit is a parabola;
(3) $E > 0$: the orbit is a hyperbola.

For the cases (1) and (3) where $E \neq 0$, the integral of (6.146) is solved as follows (see 2.267-1, Gradshteyn and Ryzhik, 2007; 14.288, Spiegel, 1974)

[5] Distance of the point of the orbit called apocentre; in particular, apogee or aphelion.

$$J_r = 2\left(\left[\sqrt{2m\left(Er^2 + Kr\right) - p_\theta^2}\right]_{r_{min}}^{r_{max}} + mK\, I_1 - p_\theta^2 I_2\right) \tag{6.149}$$

where the integrals I_1 and I_2 are

$$I_1 = \int_{r_{min}}^{r_{max}} \frac{dr}{\sqrt{2m\left(Er^2 + Kr\right) - p_\theta^2}} \tag{6.150}$$

$$I_2 = \int_{r_{min}}^{r_{max}} \frac{dr}{r\sqrt{2m\left(Er^2 + Kr\right) - p_\theta^2}} \tag{6.151}$$

For the case (1) of the elliptic orbit, one knows that the distance r is bounded and thus, the third solution $r_{extr} \to \infty$ is not possible, and only the two roots r_{extr} (6.147) are to be considered. As in this case, $-\frac{K^2 m}{2p_\theta^2} \leq E < 0$, one replaces E by $-|E|$ and the expression (6.147) of r_{extr} becomes

$$r_{\substack{min\\max}} = \frac{K\left(1 \mp \sqrt{1 - \frac{2|E|p_\theta^2}{mK^2}}\right)}{2\,|E|} \tag{6.152}$$

where the superior $-$ (respectively inferior $+$) sign in front of the radical sign corresponds to r_{min} (respectively r_{max}).

With these values (6.152) of r_{min} and r_{max}, the first term of (6.149) is nil and the action variable J_r (6.149) becomes

$$J_r = 2\left(mK\, I_1 - p_\theta^2 I_2\right) \tag{6.153}$$

The integrals (6.150) and (6.151) yield (see 2.264-1, Gradshteyn and Ryzhik, 2007; 14.280, Spiegel, 1974[6] and 2.266, Gradshteyn and Ryzhik, 2007; 14.283, Spiegel, 1974)

$$I_1 = -\frac{1}{\sqrt{2m\,|E|}}\left[\arcsin\left(\frac{1 - \frac{2|E|}{K}r}{\sqrt{1 - \frac{2|E|p_\theta^2}{mK^2}}}\right)\right]_{r_{min}}^{r_{max}} \tag{6.154}$$

$$= -\frac{1}{\sqrt{2m\,|E|}}\left[\arcsin\left(\frac{1 - \frac{2|E|}{K}r_{max}}{\sqrt{1 - \frac{2|E|p_\theta^2}{mK^2}}}\right) - \arcsin\left(\frac{1 - \frac{2|E|}{K}r_{min}}{\sqrt{1 - \frac{2|E|p_\theta^2}{mK^2}}}\right)\right] \tag{6.155}$$

$$= -\frac{1}{\sqrt{2m\,|E|}}\left[\arcsin\left(-1\right) - \arcsin\left(+1\right)\right] \tag{6.156}$$

[6]Note that the coefficient $(-2m\,|E|)$ of the term en r^2 is negative.

$$= -\frac{1}{\sqrt{2m\,|E|}}\left(-\frac{\pi}{2}-\frac{\pi}{2}\right) = \frac{\pi}{\sqrt{2m\,|E|}} \tag{6.157}$$

$$I_2 = \frac{1}{p_\theta}\left[\arcsin\left(\frac{r-\frac{p_\theta^2}{mK}}{r\sqrt{1-\frac{2|E|p_\theta^2}{mK^2}}}\right)\right]_{r_{min}}^{r_{max}} \tag{6.158}$$

$$= \frac{1}{p_\theta}\left[\arcsin\left(\frac{1-\frac{p_\theta^2}{mKr_{max}}}{\sqrt{1-\frac{2|E|p_\theta^2}{mK^2}}}\right) - \arcsin\left(\frac{1-\frac{p_\theta^2}{mKr_{min}}}{\sqrt{1-\frac{2|E|p_\theta^2}{mK^2}}}\right)\right] \tag{6.159}$$

$$= \frac{1}{p_\theta}\left[\arcsin(+1) - \arcsin(-1)\right] = \frac{\pi}{p_\theta} \tag{6.160}$$

where r_{min} and r_{max} were replaced in (6.155) and (6.159) by (6.152).

The action variable J_r (6.153) reads then with (6.157) and (6.160)[7]

$$J_r = \pi K\sqrt{\frac{2m}{|E|}} - 2\pi p_\theta \tag{6.163}$$

The sum of the action variables (6.163) and (6.144) is then

$$J_r + J_\theta = \pi K\sqrt{\frac{2m}{|E|}} \tag{6.164}$$

As

$$E = \mathcal{H} = -\frac{2\pi^2 mK^2}{(J_r+J_\theta)^2} \tag{6.165}$$

is the total energy of the conservative system, one finds the frequencies (6.5)

$$f_r = \frac{\partial\mathcal{H}}{\partial J_r} = \frac{4\pi^2 mK^2}{(J_r+J_\theta)^3} \tag{6.166}$$

$$f_\theta = \frac{\partial\mathcal{H}}{\partial J_\theta} = \frac{4\pi^2 mK^2}{(J_r+J_\theta)^3} \tag{6.167}$$

[7] One simplifies again the calculations by using the method of the residue theory (Spiegel, 1964). When applied to (6.145), it yields

$$J_r = 2\pi i\left(\sqrt{-p_\theta^2} + \frac{mK}{\sqrt{2mE}}\right) \tag{6.161}$$

$$= -2\pi p_\theta + \frac{2\pi mK}{\sqrt{2mE}} \tag{6.162}$$

These two frequencies are equal. One says then that the system is degenerated because the two frequencies are indistinguishable.
(2) For the case (3) of the hyperbolic orbit, as on the one hand, the coordinate r is not bounded, and on the other hand $E > 0$, the smaller of the two roots r_{extr} (6.147) is negative and since the polar coordinate r must always be positive, this negative root cannot be considered. One has then

$$r_{min} = \frac{K\left(-1+\sqrt{1+\frac{2Ep_\theta^2}{mK^2}}\right)}{2E} \tag{6.168}$$

$$r_{max} \to \infty \tag{6.169}$$

The integral (6.150) becomes (see 2.264-1, Gradshteyn and Ryzhik, 2007; 14.280, Spiegel, 1974[8])

$$I_1 = \frac{1}{\sqrt{2mE}}\left[\ln\left(4mE\left(\sqrt{r^2+\frac{K}{E}r-\frac{p_\theta^2}{2Em}}+r+\frac{K}{2E}\right)\right)\right]_{r_{min}}^{r_{max}} \tag{6.170}$$

The integral I_2 has the form (6.158). Replacing r_{max} by the limit (6.169), it comes that the action variable J_r (6.149) tends also toward infinity, $J_r \to \infty$.

For the case (2) of the parabolic orbit, as $E = 0$, (6.146) reduces to

$$J_r = 2\int_{r_{min}}^{r_{max}} \frac{\sqrt{2mKr - p_\theta^2}}{r}dr \tag{6.171}$$

where $r_{min} = \frac{p_\theta^2}{2mK}$ and $r_{max} \to \infty$. The integral in (6.171) is solved as (see 2.225-1, Gradshteyn and Ryzhik, 2007; 14.92, Spiegel, 1974 and 2.224-5, Gradshteyn and Ryzhik, 2007; 14.87, Spiegel, 1974[9])

$$J_r = 4\left[\sqrt{2mKr - p_\theta^2} - p_\theta \arctan\sqrt{\frac{2mK}{p_\theta^2}r - 1}\right]_{r_{min}}^{r_{max}} \tag{6.172}$$

and here also, it comes that the action variable J_r (6.172) tends toward infinity, $J_r \to \infty$.

In the two cases of hyperbolic and parabolic orbits, the action variables are infinite and frequencies are indeterminate. □

[8] Note that the coefficient $(2mE)$ of the term in r^2 is positive.

[9] Note that the independent term, i.e. $-p_\theta^2$, is negative.

6.2.6 Exercise 39: Relativistic Kepler Problem

If one applies the theory of relativity to the Keplerian movement of a particle of mass m moving in a central force field, the Hamiltonian is given by

$$H = \sqrt{\left(p_r^2 + \frac{p_\theta^2}{r^2}\right) c^2 + m^2c^4} - \frac{K}{r} \tag{6.173}$$

where c is the speed of light.
(1) Calculate the expression of the total energy of the system in function of action variables.
(2) Give the frequencies of the movement.
(3) The system established through the classical Kepler problem is said to be degenerate. Can we say the same in the relativistic case? Why?

Proof (1) With $p_r = \frac{\partial \mathscr{S}}{\partial r}$ and $p_\theta = \frac{\partial \mathscr{S}}{\partial \theta}$, the Hamilton–Jacobi equation (5.4) $\frac{\partial \mathscr{S}}{\partial t} + H = 0$ reads

$$\frac{\partial \mathscr{S}}{\partial t} + \sqrt{\left(\left(\frac{\partial \mathscr{S}}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial \mathscr{S}}{\partial \theta}\right)^2\right) c^2 + m^2c^4} - \frac{K}{r} = 0 \tag{6.174}$$

and has the complete solution $\mathscr{S} = S_r(r) + S_\theta(\theta) + S_t(t)$. The first variable separation yields $S_t = -E\,t$, with constant E, the total energy of the conservative system. The other variable separations yield successively

$$\sqrt{\left(\frac{dS_r}{dr}\right)^2 + \frac{1}{r^2}\left(\frac{dS_\theta}{d\theta}\right)^2 + m^2c^2} = \frac{\frac{K}{r} + E}{c} \tag{6.175}$$

$$\left(\frac{dS_r}{dr}\right)^2 + \frac{1}{r^2}\left(\frac{dS_\theta}{d\theta}\right)^2 = \left(\frac{\frac{K}{r} + E}{c}\right)^2 - m^2c^2 \tag{6.176}$$

$$\left(\frac{dS_\theta}{d\theta}\right)^2 = r^2\left(-\left(\frac{dS_r}{dr}\right)^2 + \left(\frac{\frac{K}{r} + E}{c}\right)^2 - m^2c^2\right) = p_\theta^2 \tag{6.177}$$

As the generalized coordinate θ is ignorable (it does not appear in the Lagrangian), the conjugated moment $p_\theta = \frac{\partial \mathscr{S}}{\partial \theta}$ is constant and a constant of movement, which yields successively

$$S_\theta = p_\theta \theta \tag{6.178}$$

$$\left(\frac{dS_r}{dr}\right)^2 = \left(\frac{\frac{K}{r}+E}{c}\right)^2 - m^2c^2 - \frac{p_\theta^2}{r^2} \tag{6.179}$$

$$S_r = \int \sqrt{\left(\frac{\frac{K}{r}+E}{c}\right)^2 - m^2c^2 - \frac{p_\theta^2}{r^2}}dr \tag{6.180}$$

The complete solution reads

$$\mathscr{S} = \int \sqrt{\left(\frac{\frac{K}{r}+E}{c}\right)^2 - m^2c^2 - \frac{p_\theta^2}{r^2}}dr + p_\theta\theta - E\,t \tag{6.181}$$

A complete cycle is such that the coordinate θ varies from 0 to 2π and that the coordinate r varies from $r = r_{min}$ to $r = r_{max}$ and then again to $r = r_{min}$. By symmetry of movement, the complete cycle corresponds to twice the part from $r = r_{min}$ to $r = r_{max}$.

The action variables (6.1) read, with $p_r = \frac{\partial \mathscr{S}}{\partial r} = \sqrt{\left(\frac{\frac{K}{r}+E}{c}\right)^2 - m^2c^2 - \frac{p_\theta^2}{r^2}}$,

$$J_\theta = \int_0^{2\pi} p_\theta d\theta = 2\pi p_\theta \tag{6.182}$$

$$J_r = \oint \sqrt{\left(\frac{\frac{K}{r}+E}{c}\right)^2 - m^2c^2 - \frac{p_\theta^2}{r^2}}dr \tag{6.183}$$

$$= \oint \sqrt{\left(\frac{K^2}{c^2} - p_\theta^2\right)\frac{1}{r^2} + \frac{2EK}{c^2}\frac{1}{r} + \left(\frac{E^2}{c^2} - m^2c^2\right)}dr \tag{6.184}$$

$$= \frac{2}{c}\int_{r_{min}}^{r_{max}} \frac{\sqrt{\left(E^2 - m^2c^4\right)r^2 + 2EKr + \left(K^2 - p_\theta^2c^2\right)}}{r}dr \tag{6.185}$$

Similarly to (6.147) and (6.148), the values of r_{min} and r_{max} correspond to solutions of $\frac{\sqrt{\left(E^2-m^2c^4\right)r^2+2EKr+\left(K^2-p_\theta^2c^2\right)}}{r} = 0$. There are three possible solutions, the first two correspond to the zeros of the trinomial under the radical sign of the numerator,

$$r_{extr} = \frac{EK \pm \sqrt{K^2m^2c^4 - \left(m^2c^4 - E^2\right)p_\theta^2c^2}}{m^2c^4 - E^2} \tag{6.186}$$

and the third corresponds to the denominator tending to infinity, i.e. $r_{extr} \to \infty$. Among these three possibilities, which of the two values of r will be r_{min} and r_{max} will depend on the values of E. As the total energy E of the system cannot be larger than

mc^2,[10] i.e. $E < mc^2$, the denominator is always positive. Furthermore, the expression under the radical sign must be positive, i.e. $E > mc^2\sqrt{1-\frac{K^2}{p_\theta^2 c^2}}$ with $\frac{K}{c} < p_\theta$ for the expression under the radical sign to be also positive. The two combined conditions yield

$$\sqrt{1-\frac{K^2}{p_\theta^2 c^2}} < \frac{E}{mc^2} < 1 \; ; \quad \frac{K}{c} < p_\theta \tag{6.187}$$

One obtains then the expressions of r_{min} and r_{max}

$$r_{min} = \frac{EK - \sqrt{K^2 m^2 c^4 - \left(m^2 c^4 - E^2\right) p_\theta^2 c^2}}{m^2 c^4 - E^2} \tag{6.188}$$

$$r_{max} = \frac{EK + \sqrt{K^2 m^2 c^4 - \left(m^2 c^4 - E^2\right) p_\theta^2 c^2}}{m^2 c^4 - E^2} \tag{6.189}$$

The two conditions (6.187) also ensure that r_{min} is always positive.

The integral of (6.185) is solved as follows (see 2.267-1, Gradshteyn and Ryzhik, 2007; 14.288, Spiegel, 1974)

$$J_r = \frac{2}{c}\left(\left[\sqrt{\left(E^2 - m^2c^4\right) r^2 + 2EKr + \left(K^2 - p_\theta^2 c^2\right)}\right]_{r_{min}}^{r_{max}} + EK\, I_1 + \left(K^2 - p_\theta^2 c^2\right) I_2\right) \tag{6.190}$$

where the integrals I_1 and I_2 are

$$I_1 = \int_{r_{min}}^{r_{max}} \frac{dr}{\sqrt{\left(E^2 - m^2c^4\right) r^2 + 2EKr + \left(K^2 - p_\theta^2 c^2\right)}} \tag{6.191}$$

$$I_2 = \int_{r_{min}}^{r_{max}} \frac{dr}{r\sqrt{\left(E^2 - m^2c^4\right) r^2 + 2EKr + \left(K^2 - p_\theta^2 c^2\right)}} \tag{6.192}$$

As r_{min} and r_{max} are the zeros of the trinomial under the radical sign of the first term of (6.190), this first term vanishes.

The integrals (6.191) and (6.192) give (see 2.264-1, Gradshteyn and Ryzhik, 2007; 14.280, Spiegel, 1974[11] and 2.266, Gradshteyn and Ryzhik, 2007; 14.283, Spiegel, 1974[12])

[10] The cases $E \geq mc^2$ will not be addressed.

[11] Note that the coefficient $\left(E^2 - m^2c^4\right)$ of the term in r^2 is negative.

[12] Note that the coefficient $\left(K^2 - p_\theta^2 c^2\right)$ of the independent term is negative.

$$I_1 = -\frac{1}{\sqrt{m^2c^4 - E^2}} \left[\arcsin\left(\frac{-\left(m^2c^4 - E^2\right) r + EK}{\sqrt{K^2m^2c^4 - \left(m^2c^4 - E^2\right) p_\theta^2 c^2}} \right) \right]_{r_{min}}^{r_{max}} \tag{6.193}$$

$$= -\frac{1}{\sqrt{m^2c^4 - E^2}} \left[\arcsin(-1) - \arcsin(+1)\right] \tag{6.194}$$

$$= -\frac{1}{\sqrt{m^2c^4 - E^2}} \left(-\frac{\pi}{2} - \frac{\pi}{2}\right) = \frac{\pi}{\sqrt{m^2c^4 - E^2}} \tag{6.195}$$

$$I_2 = \frac{1}{\sqrt{p_\theta^2 c^2 - K^2}} \left[\arcsin\left(\frac{EKr + \left(K^2 - p_\theta^2 c^2\right)}{r\sqrt{K^2m^2c^4 - \left(m^2c^4 - E^2\right) p_\theta^2 c^2}} \right) \right]_{r_{min}}^{r_{max}} \tag{6.196}$$

$$= \frac{1}{\sqrt{p_\theta^2 c^2 - K^2}} \left[\arcsin(+1) - \arcsin(-1)\right] = \frac{\pi}{\sqrt{p_\theta^2 c^2 - K^2}} \tag{6.197}$$

where r_{min} and r_{max} were replaced in (6.193) and (6.196) by (6.188) and (6.189).

The action variable J_r (6.190) reads then with (6.195) and (6.197)[13]

$$J_r = \frac{2\pi}{c} \left(\frac{EK}{\sqrt{m^2c^4 - E^2}} - \sqrt{p_\theta^2 c^2 - K^2} \right) \tag{6.200}$$

$$= \frac{2\pi EK}{c\sqrt{m^2c^4 - E^2}} - \sqrt{4\pi^2 p_\theta^2 - \frac{4\pi^2 K^2}{c^2}} \tag{6.201}$$

$$= \frac{2\pi EK}{c\sqrt{m^2c^4 - E^2}} - \sqrt{J_\theta^2 - \frac{4\pi^2 K^2}{c^2}} \tag{6.202}$$

where one used (6.182) in (6.201).

One finds the energy E from (6.202)

$$E = \frac{mc^2}{\sqrt{1 + \left(\frac{2\pi K}{c\left(J_r + \sqrt{J_\theta^2 - \frac{4\pi^2 K^2}{c^2}}\right)} \right)^2}} \tag{6.203}$$

(2) As $E = \mathcal{H}$, one finds the frequencies (6.5)

[13] Here also, J_r can be calculated easier and faster by the method of the residue theory (Spiegel, 1964). The lines from (6.185) to (6.197) can be replaced by: Applying the residue theory to (6.184), it comes that

$$J_r = 2\pi i \left(\sqrt{\left(\frac{K^2}{c^2} - p_\theta^2\right)} + \frac{\frac{EK}{c^2}}{\sqrt{\frac{E^2}{c^2} - m^2c^2}} \right) \tag{6.198}$$

$$= \frac{2\pi}{c} \left(-\sqrt{p_\theta^2 c^2 - K^2} + \frac{EK}{\sqrt{m^2c^4 - E^2}} \right) \tag{6.199}$$

$$f_r = \frac{\partial \mathcal{H}}{\partial J_r} = \frac{4\pi^2 m K^2}{\sqrt{\left(\left(J_r + \sqrt{J_\theta^2 - \frac{4\pi^2 K^2}{c^2}}\right)^2 + \frac{4\pi^2 K^2}{c^2}\right)^3}} \tag{6.204}$$

$$f_\theta = \frac{\partial \mathcal{H}}{\partial J_\theta} = \frac{4\pi^2 m K^2}{\sqrt{\left(\left(J_r + \sqrt{J_\theta^2 - \frac{4\pi^2 K^2}{c^2}}\right)^2 + \frac{4\pi^2 K^2}{c^2}\right)^3}} \frac{J_\theta}{\sqrt{J_\theta^2 - \frac{4\pi^2 K^2}{c^2}}} \tag{6.205}$$

(3) These frequencies are no more equal as for the classic Kepler problem in the previous Exercise 38. The system in this formalism is no longer degenerated because the orbit is not closed any more, although remaining constrained to a plane. One notes that, if the terms $\frac{4\pi^2 K^2}{c^2}$ in (6.204) and (6.205) are neglected, for example by making c to tend to infinity ($c \to \infty$), one finds again the frequencies (6.166) and (6.167) of the classic Kepler problem, which are equal. □

6.2.7 Exercise 40: Advance of Mercury Perihelion

Preliminary Note

In the Theory of general relativity of gravitation, Einstein set as a postulate that any inertial mass is also a gravitational mass and thus a source of gravitational force. By introducing the Riemannian geometry in physics, Einstein showed that one can bring together in one entity space, time and matter, which can be considered as a curved portion of a four dimension space.

In this formalism, a geodesic such as

$$\bar{ds}^2 = -c^2 dt^2 + dx^2 + dy^2 + dz^2 \tag{6.206}$$

is called a Minkowski line element and expresses the "distance" in a four dimension space, with time t and c the speed of light. Time became a geometric property of space, or more exactly, of a new entity, the "space-time continuum".

The principle of Least Action

$$\delta \int_{\tau_1}^{\tau_2} \bar{ds} = 0 \tag{6.207}$$

expresses the movement of a particle that is free from external forces.

One makes a point transformation to pass from rectangular coordinates to curvilinear coordinates

$$\bar{ds} = \sqrt{\sum_{i=0}^{3} g_{ik} dq_i dq_k} \tag{6.208}$$

with g_{ik} the ten elements of a symmetrical 4×4 matrix of the metric tensor. For example, for the Minkowski line element, one has

$$\begin{pmatrix} g_{00} & g_{01} & g_{02} & g_{03} \\ g_{01} & g_{11} & g_{12} & g_{13} \\ g_{02} & g_{12} & g_{22} & g_{23} \\ g_{03} & g_{13} & g_{23} & g_{33} \end{pmatrix} = \begin{pmatrix} -c^2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

For the movement of planets in the theory of general relativity, Einstein passes from a revolution in a central force field in a three-dimensional Newtonian space to a purely geodesic movement, i.e. a movement of a particle without forces in a four dimension space with a Riemannian structure.

Einstein's theory generalizes Newton gravitational potential to a system of ten field quantities, which are the ten components of the four dimension Riemann line element.

One can generalize the equation of Newton potential to Einstein field equations that gives the gravitational field of the Sun, as long as this field has spherical symmetry. This result is given by the line element of Schwarzschild in polar coordinates

$$\bar{ds}^2 = \left(1 - \frac{\alpha}{r}\right) dx_4^2 - \frac{dr^2}{1 - \frac{\alpha}{r}} - r^2 \left(d\theta^2 + \sin^2\theta \, d\varphi^2\right) \tag{6.209}$$

where α is a constant distance parameter to be determined. For $\alpha = 0$, one finds again the "flat" Minkowski element line (6.206) in polar coordinates (disregarding a convention $-$ sign).

From there on, the problem of the movement of planets under the action of a central body becomes equivalent to the evaluation of geodesics in a Riemannian space with the line item (6.209), which means in other words, to find the solution to a dynamic problem whose Hamiltonian reads

$$H = \frac{p_4^2}{1 - \frac{\alpha}{r}} - \left(1 - \frac{\alpha}{r}\right) p_r^2 - \frac{1}{r^2}\left(p_\theta^2 + \frac{p_\varphi^2}{\sin^2\theta}\right) \tag{6.210}$$

On the other hand, in the Poincaré-Minkowski theory, which is based on general relativity and adapted from Newton equations of planetary movement, the Hamiltonian reads

$$H = \frac{1}{\left(m + \frac{V}{c^2}\right)^2}\left(p_4^2 - p_r^2 - \frac{p_\theta^2}{r^2}\right) \tag{6.211}$$

where V is the scalar gravitational potential, $V = -\frac{GMm}{r}$, G the constant of gravitation, M and m the masses of respectively the Sun and of a planet.

One of the successes of Einstein's general relativity theory is to have brought in 1915 an elegant solution to the more than 60 years old problem of the advance of Mercury perihelion,[14] whose observed value is $\eta_{obs} = 43.1'' \pm 0.4$ /century (where " means arc seconds).

Exercise Statement

Knowing that the semi-major axis a and the eccentricity e of the orbit of Mercury are $a \approx 5.7909 \times 10^{10}$ m and $e \approx 0.20563$ and that Mercury describes 415 revolutions per century, calculate the advance of the of Mercury perihelion, according to the four following steps.
(1) Starting from the Hamiltonian (6.210), give an expression of the generalized coordinate θ in function of r in a planar approximation.
(2) Assuming that the constant distance parameter α is small with respect to planetary radial distances, $\alpha \ll r$, show that the approximate analytical expression of θ includes two movements, one yielding a cumulative precession effect on each revolution, the other yielding a small periodic effect.
(3) Comparing the Hamiltonian (6.210) in the planar approximation and the Hamiltonian (6.211), find an expression for the distance parameter α and show that it is indeed small with respect to planetary radial distances, $\alpha \ll r$.
(4) Calculate the advance of Mercury perihelion.

Proof (1) Due to the spherical symmetry, one can reduce the problem to a movement in a plane with $\varphi = \frac{\pi}{2}$. So, one can treat the problem with only three (instead of four) pairs of canonical variables. The Hamiltonian (6.210) becomes

$$H = \frac{p_4^2}{1 - \frac{\alpha}{r}} - \left(1 - \frac{\alpha}{r}\right) p_r^2 - \frac{p_\theta^2}{r^2} \tag{6.212}$$

The Hamiltonian does not explicitly depend on θ and x_4. These two coordinates are then ignorable and one can substitute the constant conjugated moments

$$p_\theta = -A \tag{6.213}$$

$$p_4 = AB \tag{6.214}$$

where A and B are constants, which replaced in (6.212) yields

$$H = \frac{A^2 B^2}{1 - \frac{\alpha}{r}} - \left(1 - \frac{\alpha}{r}\right) p_r^2 - \frac{A^2}{r^2} \tag{6.215}$$

[14]The perihelion is the point of the orbit where the radial distance r is minimal, or the point of the orbit at which a planet is closest to the Sun.

with only one variable r remaining. What interests us is not the movement in function of time, i.e. $r = r(t)$ or $\theta = \theta(t)$, but the geometric orbit $r = r(\theta)$. Therefore, the Hamilton equations (2.3) read

$$\dot{r} = \frac{\partial H}{\partial p_r} = -2\left(1 - \frac{\alpha}{r}\right) p_r \tag{6.216}$$

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = -\frac{2p_\theta}{r^2} = \frac{2A}{r^2} \tag{6.217}$$

Dividing (6.216) by (6.217) yields

$$\frac{dr}{d\theta} = -\frac{r^2}{A}\left(1 - \frac{\alpha}{r}\right) p_r \tag{6.218}$$

Another condition is for the system to be conservative, i.e. for the total energy to be constant, which gives another relation between r and p_r. Let us set this constant equal to unity $E = H = 1$ in (6.215), which yields

$$p_r = \sqrt{\frac{\frac{A^2B^2}{1-\frac{\alpha}{r}} - \frac{A^2}{r^2} - 1}{1 - \frac{\alpha}{r}}} \tag{6.219}$$

which, replaced in (6.218), gives

$$d\theta = -\frac{dr}{r^2\sqrt{\left(1 - \frac{\alpha}{r}\right)\left(\frac{B^2}{1-\frac{\alpha}{r}} - \frac{1}{r^2} - \frac{1}{A^2}\right)}} \tag{6.220}$$

where the sign in front of the radical sign is chosen such as θ increases for increasing r. After changing the variable $\rho = \frac{1}{r}$ in (6.220), it comes

$$\theta = \int \frac{d\rho}{\sqrt{(1 - \alpha\rho)\left(\frac{B^2}{1-\alpha\rho} - \rho^2 - \frac{1}{A^2}\right)}} \tag{6.221}$$

The denominator of (6.221) is the square root of a cubic function of ρ. The integration would lead to an elliptic integral.
(2) Assuming that the constant distance parameter α is small with respect to planetary radial distances, $\alpha \ll r$ or $\alpha\rho \ll 1$, the term $\left(\frac{1}{1-\alpha\rho}\right)$ in (6.221) can be developed in series of Taylor, neglecting terms of degree higher than the second for

$$\frac{B^2}{1 - \alpha\rho} \approx B^2\left(1 + \alpha\rho + (\alpha\rho)^2\right) \tag{6.222}$$

and of degree higher than the first for

$$\frac{1}{\sqrt{1-\alpha\rho}} \approx 1 + \frac{\alpha\rho}{2} \tag{6.223}$$

Their replacement in (6.221) yield

$$\theta \approx \int \frac{\left(1+\frac{\alpha\rho}{2}\right)d\rho}{\sqrt{-\left(1-\alpha^2B^2\right)\rho^2+\alpha B^2\rho-\left(\frac{1}{A^2}-B^2\right)}} = \int \frac{N}{D}d\rho \tag{6.224}$$

that is integrable in elementary functions instead of an elliptic integral. The trinomial under the radical to the denominator of (6.224) can be put in the form

$$D = \sqrt{-\left(1-\alpha^2B^2\right)(\rho-\rho_1)(\rho-\rho_2)} \tag{6.225}$$

where ρ_1 and ρ_2 are the zeros of the trinomial such as

$$\rho_1+\rho_2 = \frac{\alpha B^2}{1-\alpha^2B^2} \tag{6.226}$$

$$\rho_1\rho_2 = \frac{\frac{1}{A^2}-B^2}{1-\alpha^2B^2} \tag{6.227}$$

Let us set $\rho_0 = \frac{\rho_1+\rho_2}{2}$ and $b = \frac{\rho_1-\rho_2}{2}$ and let us change the variable again $\rho = \rho_0 + u$. The denominator (6.225) becomes

$$D = \sqrt{\left(1-\alpha^2B^2\right)\left(\left(\frac{\rho_1-\rho_2}{2}\right)^2-\left(\rho-\left(\frac{\rho_1+\rho_2}{2}\right)\right)^2\right)} \tag{6.228}$$

$$= \sqrt{\left(1-\alpha^2B^2\right)\left(b^2-u^2\right)} \tag{6.229}$$

Replacing in (6.224) yields

$$\theta \approx \int \frac{\left(1+\frac{\alpha\rho_0}{2}+\frac{\alpha u}{2}\right)du}{\sqrt{\left(1-\alpha^2B^2\right)\left(b^2-u^2\right)}} \tag{6.230}$$

$$\approx \frac{1+\frac{\alpha\rho_0}{2}}{\sqrt{1-\alpha^2B^2}}\int\frac{du}{\sqrt{b^2-u^2}} + \frac{\frac{\alpha}{2}}{\sqrt{1-\alpha^2B^2}}\int\frac{u\,du}{\sqrt{b^2-u^2}} \tag{6.231}$$

The term under the radical sign in the factor of (6.231) is close to 1, $\left(1-\alpha^2B^2\right) \approx 1$ as, with $\rho_1+\rho_2 = 2\rho_0$, multiplying (6.226) by α yields $\alpha^2B^2 = \frac{2\alpha\rho_0}{1+2\alpha\rho_0} \approx 2\alpha\rho_0$ $(1-2\alpha\rho_0) \approx 2\alpha\rho_0 \ll 1$. The factors in front of the two integrals of (6.231) can then be approximated by

$$\frac{1+\frac{\alpha\rho_0}{2}}{\sqrt{1-\alpha^2B^2}} \approx \frac{1+\frac{\alpha\rho_0}{2}}{\sqrt{1-2\alpha\rho_0}} \approx \frac{1+\frac{\alpha\rho_0}{2}}{1-\alpha\rho_0} \approx \left(1+\frac{\alpha\rho_0}{2}\right)(1+\alpha\rho_0) \tag{6.232}$$

$$\approx 1+\frac{3\alpha\rho_0}{2}+\frac{\alpha^2\rho_0^2}{2} \approx 1+\frac{3\alpha\rho_0}{2} \tag{6.233}$$

$$\frac{\frac{\alpha}{2}}{\sqrt{1-\alpha^2B^2}} \approx \frac{\alpha}{2}(1+\alpha\rho_0) \approx \frac{\alpha}{2}+\frac{\alpha^2\rho_0}{2} \approx \frac{\alpha}{2} \tag{6.234}$$

where the terms in α^2 are finally neglected in (6.233) and (6.234). After replacement in (6.231), it comes

$$\theta \approx \left(1+\frac{3\alpha\rho_0}{2}\right)\int\frac{du}{\sqrt{b^2-u^2}}+\frac{\alpha}{2}\int\frac{u\,du}{\sqrt{b^2-u^2}} \tag{6.235}$$

The evolution of the angle θ is thus the sum of two movements given by these two integrals. The first integral of (6.235) is solved (see 2.271-4, Gradshteyn and Ryzhik, 2007; 14.237, Spiegel, 1974) in

$$\theta \approx \left(1+\frac{3\alpha\rho_0}{2}\right)\arcsin\left(\frac{u}{b}\right) \tag{6.236}$$

Reversing (6.236) and returning to initial variables, i.e. replacing $u=\rho-\rho_0=\frac{1}{r}-\frac{1}{r_0}$, one obtains

$$\frac{1}{r} \approx \frac{1}{r_0}+b\sin\left(\frac{\theta}{1+\frac{3\alpha\rho_0}{2}}\right) \tag{6.237}$$

which is the focal equation of an ellipse, except for an adjustment factor ($\frac{3\alpha\rho_0}{2}$) which yields to a small precession of the ellipse in its own plane. This means that the angle between two successive perihelia is not 2π, but $2\pi+3\pi\alpha\rho_0$.

The second integral of (6.235) gives (see 2.271-7, Gradshteyn and Ryzhik, 2007; 14.238, Spiegel, 1974)

$$\theta \approx -\frac{\alpha}{2}\sqrt{b^2-u^2} \tag{6.238}$$

with $u=\rho-\rho_0=\frac{1}{r}-\rho_0$, which yields

$$\theta \approx -\frac{\alpha}{2}\sqrt{\left(b^2-\rho_0^2\right)+\frac{\rho_0}{r}-\frac{1}{r^2}} \tag{6.239}$$

This component of the evolution of θ yields only a periodic disturbance of the orbit too small to be observed and which has no cumulative effect, while the advance of the perihelion appears each revolution and is a cumulative effect, measurable after around a hundred revolutions.

(3) To find the value of the distance parameter α, one has to compare the two Hamiltonians (6.212) and (6.211). Setting the mass of the planet equal to unity, $m = 1$, it comes for the first term in p_4 in both Hamiltonians that

$$1 - \frac{\alpha}{r} = \left(1 + \frac{V}{c^2}\right)^2 = 1 - \frac{2GM}{rc^2} + \frac{G^2M^2}{r^2c^4} \tag{6.240}$$

or, neglecting the last second degree term,

$$\alpha \approx \frac{2GM}{c^2} \tag{6.241}$$

With the values $G \approx 6.672 \times 10^{-11}\,\mathrm{m^3/kg\,s^2}$, $M \approx 2 \times 10^{30}\,\mathrm{kg}$, $c \approx 3 \times 10^8\mathrm{m/s}$, one obtains $\alpha \approx 2.94 \times 10^3\,\mathrm{m} \approx 3\,\mathrm{km}$, which is called the gravitational radius or the Schwarzschild radius and which corresponds to the radius of a black hole that would have the mass of the Sun. This distance is very small compared to the radii of planetary orbits (typically, in the solar system from 5.8×10^{10} to $4.5 \times 10^{12}\,\mathrm{m}$ for Mercury to Neptune), the product $(\alpha\rho)$ is then of the order of 10^{-7} to 10^{-9}, which justifies the approximations made of the Taylor series expansions for $\left(1 - \frac{\alpha}{r}\right)$.

(4) In the case of Mercury, the cumulative effect of the perihelion advance becomes very pronounced. If a is the semi-major axis of Mercury's orbit and e his eccentricity, one has $\rho_0 = \frac{1}{a\left(1-e^2\right)}$, the inverse of the perihelion distance. The perihelion advance for each revolution is then

$$\eta = 3\pi\alpha\rho_0 \approx \frac{6\pi GM}{a\left(1-e^2\right)c^2} \tag{6.242}$$

With $a \approx 5.7909 \times 10^{10}\,\mathrm{m}$ and $e \approx 0.20563$ for Mercury, one finds approximately 0.1 s of arc per revolution. As Mercury describes 415 revolutions per century, the calculated perihelion advance is

$$\eta_{calc} = 43.03" \pm 0.03\,/\mathrm{century} \tag{6.243}$$

while the observed value is $\eta_{obs} = 43.1" \pm 0.4\,/\mathrm{century}$. □

This famous result, in very good agreement with the observed value for Mercury, was the first experimental proof of the general relativity theory.

One can also calculate the advance of the perihelion of Venus, Earth and Mars, which are also in good agreement with observations, respectively 8.62, 3.84 and 1.35 seconds of arc per century.

Selected Bibliography

Textbooks on Classical Mechanics

Arnold V.I., “Mathematical Methods of Classical Mechanics”, Springer Verlag, New York, 1978.
Brouwer D. and Clemence G.M., “Methods of Celestial Mechanics”, Academic Press, New York, 1961.
Deprit A. and Rouche N., “Mécanique Rationnelle”, Tomes 1 and 2, Vander Éditeur, Louvain, 1970.
Goldstein H., “Classical Mechanics”, Addison Wesley, 2nd edition, 1980.
Landau L.D. and Lifshitz E.M., “Mechanics”, Course of Theoretical Physics, Vol. 1, 2nd ed., Pergamon Press, Oxford, 1969.
Roy A.E., “Orbital motion”, 3rd ed., Adam Hilger, Bristol, 1988.
Spiegel M.R., “Theoretical Mechanics”, Schaum’s Outline séries in Sciences, McGraw-Hill, New York, 1967.
Texier C., “Mécanique Quantique”, 2nd edition, Dunod, Paris, 2014.

References and Mathematical Tables

Gradshteyn I.S., Ryzhik I.M., “Table of Integrals, Series, and Products”, 7th ed., Academic Press, Elsevier, London, 2007.
Kamke E., “Differentialgleichungen, Lösungsmethoden und Lösungen”, 2 Auflage, Akademische VerlagsGesellschaft Becker & Erler Kom. Ges., Leipzig, 1943.
Milne-Thomson L.M., “Elliptic Integrals”, in “Handbook of Mathematical Functions”, M. Abramowitz and I.A. Stegun eds., National Bureau of Standards, 10th ed., Washington, 587–626, 1972.
Spanier J. and Oldham K.B., “An Atlas of Functions”, Hemisphere Publ. Corp., Springer-Verlag, Berlin, 1987.

V. Pletser, *Lagrangian and Hamiltonian Analytical Mechanics: Forty Exercises Resolved and Explained*, UNITEXT for Physics,
https://doi.org/10.1007/978-981-13-3026-1

Spiegel M. R., “Formules et Tables de Mathématiques”, Série Schaum, McGraw-Hill, Paris, 1974.
Spiegel M. R., “Complex Variables”, Série Schaum, McGraw-Hill, 1964 (ISBN 2-7042-0020-3).

References on the Pendulum Problem

Baker G.L., Blackburn J.A., “The Pendulum: A Case Study in Physics”, Oxford, New York, 2005.
Belendez A., Pascual C., Mendez D.I., Belendez T., Neipp C., “Exact solution for the nonlinear pendulum”, Revista Brasileira de Ensino de Fisica, v. 29, n. 4, p. 645–648, 2007
(http://www.scielo.br/pdf/rbef/v29n4/a24v29n4.pdf).
Fulcher L.P., Davis B.F., “Theoretical and experimental study of the motion of the simple pendulum”, American Journal of Physics, 44, 51, 1976.

Index

V. Pletser, *Lagrangian and Hamiltonian Analytical Mechanics: Forty Exercises Resolved and Explained*, UNITEXT for Physics,
https://doi.org/10.1007/978-981-13-3026-1