Undergraduate Lecture Notes in Physics

Giampaolo Cicogna

Exercises and Problems in Mathematical Methods of Physics

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Preface

This book is a collection of 350 exercises and problems in Mathematical Methods of Physics: its peculiarity is that exercises and problems are proposed not in a "random" order, but having in mind a precise didactic scope. Each section and subsection starts with exercises based on first definitions, elementary notions and properties, followed by a group of problems devoted to some intermediate situations, and finally by problems which propose gradually more elaborate developments and require some more refined reasoning.

Part of the problems is unavoidably "routine", but several problems point out interesting nontrivial properties, which are often omitted or only marginally mentioned in the textbooks. There are also some problems in which the reader is guided to obtain some important results which are usually stated in textbooks without complete proofs: for instance, the classical "uncertainty principle" $\Delta t\ \Delta\omega \geq 1/2$, an introduction to Kramers–Kronig dispersion rules and their relation with causality principles, the symmetry properties of the hydrogen atom, and the harmonic oscillator in Quantum Mechanics.

In this sense, this book may be used as (or perhaps, to some extent, better than) a textbook. Avoiding unnecessary difficulties and excessive formalism, it offers indeed an alternative way to understand the mathematical notions on which Physics is based, proceeding in a carefully structured sequence of exercises and problems.

I believe that there is no need to emphasize that the best (or perhaps the unique) way to understand correctly Mathematics is that of facing and solving exercises and problems. This holds a fortiori for the present case, where mathematical notions and procedures become a fundamental tool for Physics. An example can illustrate perfectly the point. The definition of eigenvectors and eigenvalues of a linear operator needs just two or three lines in a textbook, and the notion is relatively simple and intuitive. But only when one tries to find explicitly eigenvectors and eigenvalues in concrete cases, then one realizes that a lot of different procedures are required and extremely various situations occur. This book offers a fairly exhaustive description of possible cases.

This book covers a wide range of topics useful to Physics: Chap. 1 deals with Hilbert spaces and linear operators. Starting from the crucial concept of complete system of vectors, many exercises are devoted to the fundamental tool provided by Fourier expansions, with several examples and applications, including some typical Dirichlet and Neumann Problems. The second part of the chapter is devoted to studying the different properties of linear operators between Hilbert spaces: their domains, ranges, norms, boundedness, and closedness, and to examining special classes of operators: adjoint and self-adjoint operators, projections, isometric and unitary operators, functionals, and time-evolution operators. Great attention is paid to the notion of eigenvalues and eigenvectors, with the various procedures and results encountered in their determination. Another frequently raised question concerns the different notions of convergence of given sequences of operators.

Chapter 2 starts with a survey of the basic properties of analytic functions of a complex variable, of their power series expansions (Taylor–Laurent series), and of their singularities, including branch points and cut lines. The evaluation of many types of integrals by complex variable methods is proposed. Some examples of conformal mappings are finally studied, in order to solve Dirichlet Problems; the results are compared with those obtained in other chapters with different methods, with a discussion about the uniqueness of the solutions.

The problems in Chap. 3 concern Fourier and Laplace transforms with their different applications. The physical meaning of the Fourier transform as “frequency analysis” is carefully presented. The Fourier transform is extended to the space of tempered distributions $\mathscr{S}'$, which include the Dirac delta, the Cauchy principal part, and other related distributions. Applications concern ordinary and partial differential equations (in particular the heat, d’Alembert, and Laplace equations, including a discussion about the uniqueness of solutions), and general linear systems. The important notion of Green function is considered in many details, together with the notion of causality. Various examples and applications of Laplace transform are proposed, also in comparison with Fourier transform.

The first problems in Chap. 4 deal with basic properties of groups and of group representations. Fundamental results following from Schur lemma are introduced since the beginning in the case of finite groups, with a simple application of character theory, in the study of vibrational levels of symmetric systems. Other problems concern the notion and the properties of Lie groups and Lie algebras, mainly oriented to physical examples: rotation groups SO_2, SO_3, SU_2, translations, Euclidean group, Lorentz transformations, dilations, Heisenberg group, and SU_3, with their physically relevant representations. The last section starts with some examples and applications of symmetry properties of differential equations, and then provides a group-theoretical description of some problems in quantum mechanics: the Zeeman and Stark effects, the Schrödinger equation of the hydrogen atom (the group SO_4), and the three-dimensional harmonic oscillator (the group U_3).

At the end of the book, there are the solutions to almost all problems. In particular, there is a complete solution of the more significant or difficult problems.

This book is the result of my lectures during several decades at the Department of Physics of the University of Pisa. I would like to acknowledge all my colleagues who helped me in the organization of the didactic activity, in the preparation of the problems and for their assistance in the examinations of my students. Special thanks are due to Prof. Giovanni Morchio, for his constant invaluable support: many of the problems, specially in Sect. 2 of Chap. 1, have been written with his precious collaboration. I am also grateful to Prof. Giuseppe Gaeta for his encouragement to write this book, which follows my previous lecture notes (in Italian) *Metodi Matematici della Fisica*, published by Springer-Verlag Italia in 2008 (second edition in 2015).

Finally, I would thank in advance the readers for their comments, and in particular those readers who will suggest improvements and amendments to all possible misprints, inaccuracies, and inadvertent mistakes (hopefully, not too serious) in this book, including also errors and imperfections in my English.

Pisa, Italy
January 2018

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Contents

Chapter 1
Hilbert Spaces

1.1 Complete Sets, Fourier Expansions

The argument of this section is the study of basic properties of Hilbert spaces, without involving the presence of linear operators.
Among other facts, the first problems in Sect. 1.1.1 emphasize the notion of dense *subspaces, and the difference between linear subspaces and* closed *(i.e., Hilbert) subspaces. The fundamental concept of* complete system *(or* complete set*) of vectors is then pointed out, clearly distinguishing between complete sets and* orthonormal (or orthogonal) complete sets *(to avoid confusion, the term "basis" is never used).*
The next subsection is devoted to the Fourier expansion, which is, as well known, a fundamental tool in calculations and applications. Many exercises are proposed in the context of "abstract" Hilbert spaces, in the space of sequences ℓ^2, and in the "concrete" space of square-integrable functions L^2 as well.
A special application of Fourier expansion concerns some examples of Dirichlet and Neumann Problems (Sect. 1.1.3).

1.1.1 Preliminary Notions, Subspaces, and Complete Sets

(1.1)
(1) Consider the sequence of functions defined in $[0, \pi]$

$$f_n(x) \ = \begin{cases} n \ \sin nx & \text{for } \ 0 \le x \le \pi/n \\ 0 & \text{for } \ \pi/n \le x \le \pi \end{cases} \quad , \qquad n = 1, 2, \ldots$$

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Show that $f_n(x) \to 0$ *pointwise for all* $x \in [0, \pi]$ as $n \to \infty$, but $\int_0^\pi f_n(x)\,dx$ does not tend to zero.

(2) Consider a sequence of functions of the form, with $x \in \mathbf{R}$,

$$f_n(x) = \begin{cases} c_n & \text{for} \quad 0 < x < n \\ 0 & \text{elsewhere} \end{cases}, \qquad n = 1, 2, \ldots$$

where c_n are constants. Choose c_n in such a way that $f_n(x) \to 0$ uniformly $\forall x \in \mathbf{R}$, but $\int_{-\infty}^{+\infty} f_n(x)\,dx$ does not tend to zero.[1]

(3) Consider sequences of functions of the same form as in (2). Choose (if possible) the constants c_n in such a way that

(a) $f_n(x) \to 0$ in the norm $L^2(\mathbf{R})$ but not in the norm $L^1(\mathbf{R})$;

(b) $f_n(x) \to 0$ in the norm $L^1(\mathbf{R})$ but not in the norm $L^2(\mathbf{R})$.

(4) Now consider a sequence of functions of the form, with $x \in \mathbf{R}$,

$$f_n(x) = \begin{cases} c_n & \text{for} \quad 0 < x < 1/n \\ 0 & \text{elsewhere} \end{cases}, \qquad n = 1, 2, \ldots$$

where c_n are constants:

(a) verify that $f_n(x) \to 0$ pointwise *almost everywhere*;

(b) the same questions (a), (b) as in (3).

(1.2)

(1) Show that if a function $f(x) \in L^2(I)$, where I is an interval $\big($of finite length $\mu(I)\big)$, then also $f(x) \in L^1(I)$. Is the converse true? What is the relationship between the norms $\|f\|_{L^1(I)}$ and $\|f\|_{L^2(I)}$?

(2) What changes if $I = \mathbf{R}$?

(3) Is it possible to find a function $f(x) \in L^2(I)$, where e.g., $I = (-1, 1)$, such that $\sup_{x\in I} |f(x)| = \varepsilon$ (where $\varepsilon \ll 1$) but $\|f\|_{L^2(I)} = 1$? Or such that $\|f\|_{L^2(I)} = \varepsilon$ but $\sup_{x\in I} |f(x)| = 1$?

(4) The same questions as in (3) if $I = \mathbf{R}$.

(1.3)

(1) Let $f(x) \in L^2(\mathbf{R})$ and let $f_n(x)$ be the "truncated" functions

$$f_n(x) = \begin{cases} f(x) & \text{for } |x| < n \\ 0 & \text{for } |x| > n \end{cases}, \qquad n = 1, 2, \ldots$$

[1] See the Introduction to Sect. 1.2 for the statement of the fundamental Lebesgue theorem about the convergence of the integrals of sequences of functions.

Show that $f_n \in L^1(\mathbf{R}) \cap L^2(\mathbf{R})$ and $\|f - f_n\| \to 0$ as $n \to \infty$. Conclude: is the subspace of the functions $f \in L^1(\mathbf{R}) \cap L^2(\mathbf{R})$ dense in $L^2(\mathbf{R})$? The same question for the subspace of functions $f \in L^2(\mathbf{R})$ having compact support. Are they Hilbert subspaces in $L^2(\mathbf{R})$?

(2) Is the subspace $\mathscr{S}$ of test functions for the tempered distributions (i.e., the subspace of the C^∞ functions rapidly going to zero with their derivatives as $|x| \to +\infty$) dense in $L^2(\mathbf{R})$?

(1.4)

(1) Let $g(x) \in L^1(\mathbf{R}) \cap L^2(\mathbf{R})$ be such that

$$\int_{-\infty}^{+\infty} g(x)\, dx = M \neq 0$$

introduce then the functions

$$w_n(x) = \begin{cases} M/n & \text{for } 0 < x < n \\ 0 & \text{elsewhere} \end{cases}, \qquad n = 1, 2, \ldots$$

and let $z_n(x) = g(x) - w_n(x)$. Show that $\int_{-\infty}^{+\infty} z_n(x)\, dx = 0$ and $\|g - z_n\| \to 0$ as $n \to \infty$.

(2) Conclude: is the set of functions $f(x) \in L^1(\mathbf{R}) \cap L^2(\mathbf{R})$ with zero mean value a *dense* subspace in $L^2(\mathbf{R})$? (for an alternative proof, see Problem 3.8).

(1.5)

(1) In $L^2(-a, a)$ $(a \neq \infty)$, consider the subset of the functions such that

$$\int_{-a}^{a} f(x)\, dx = 0$$

Is this a Hilbert subspace? What is its orthogonal complement, and what are their respective dimensions? Choose an orthonormal complete system in each one of these subspaces.

(2) What changes if $a = \infty$? (see previous problem).

(3) The same questions as in (1) and (2) for the subset of the *even* functions such that $\int_{-a}^{a} f(x)\, dx = 0$.

(1.6)

(1) Consider in the space $L^2(-1, 1)$ the function $u = u(x) = 1$ and consider the sequence of functions

$$g_n(x) = \begin{cases} n|x| & \text{for } \ |x| \leq 1/n \\ 1 & \text{for } \ 1/n \leq |x| \leq 1 \end{cases}, \qquad n = 1, 2, \ldots$$

Show that $\|g_n - u\|_{L^2} \to 0$.

(2) Show that the subspace of the functions $g(x) \in L^2(I)$ which are continuous in a neighborhood of a point $x_0 \in I$ and satisfy $g(x_0) = 0$ is dense in H. Show that the same is also true for the subspace of the functions which are C^∞ in a neighborhood of $x_0 \in I$ and satisfy $g^{(n)}(x_0) = 0$ for all $n \geq 0$.

(1.7)
(1) Consider the limit

$$\lim_{N\to\infty} \frac{1}{2N} \int_{-N}^{N} f(x)\, dx$$

Does it exist (and is the same) for all $f(x) \in L^2(\mathbf{R})$?

(2) Consider now the limit

$$\lim_{N\to\infty} \frac{1}{2N} \int_{-N}^{N} x f(x)\, dx$$

Is there a *dense* set of functions $\in L^2(\mathbf{R})$ such that this limit is zero? Find a function $\in L^2(\mathbf{R})$ such that this limit is equal to 1, a function $\in L^2(\mathbf{R})$ such that is $+\infty$; show finally that if $f(x) = \sin(x^{1/3})/x^{2/3}$ this limit does not exist.

(1.8)
(1) Construct a function $f(x) \in L^1(\mathbf{R})$ which *does not* vanish as $|x| \to \infty$. *Hint*: a simple construction is the following: consider a function which is equal to 1 on all intervals $(n,\ n + \delta_n),\ n \in \mathbf{Z},\ 0 < \delta_n < 1$, and equal to zero elsewhere; it is enough to choose suitably $\delta_n \ldots$. With a different choice of δ_n, it is also possible to construct a function $\in L^1(\mathbf{R})$ which is *unbounded* as $|x| \to \infty$.

(2) The same questions for functions $f(x) \in L^2(\mathbf{R})$. It should be clear that the above constructions can be modified in order to have continuous (or even C^∞) functions.

(3) Show that if both $f(x)$ and its derivative $f'(x)$ belong to $L^1(\mathbf{R})$, then $\lim_{|x|\to\infty} f(x) = 0$. *Hint*: it is clearly enough to show that $f(x)$ admits limit at $|x| \to \infty$; to this aim, apply Cauchy criterion: the limit exists if for any $\varepsilon > 0$ one has $|f(x_2) - f(x_1)| < \varepsilon$ for any $x_1,\ x_2$ large enough. But

$$f(x_2) - f(x_1) = \int_{x_1}^{x_2} f'(y)\, dy$$

then, ...

(4) Show that if both $f(x)$ and its derivative $f'(x)$ belong to $L^2(\mathbf{R})$, then $\lim_{|x|\to\infty} f(x) = 0$. *Hint*: use the same criterion $\big($assume for simplicity $f(x)$ real$\big)$:

$$f^2(x_2) - f^2(x_1)| = \int_{x_1}^{x_2} \frac{d}{dy} f^2(y)\, dy = \ldots$$

(1.9)
(1) Show that any sequence of *orthonormal* elements $\{x_n,\ n = 1, 2, \ldots\}$, in a Hilbert space, is not norm-convergent (check the Cauchy property) as $n \to \infty$, but weakly convergent (to what vector?).

(2) Let $\{x_n,\ n = 1, 2, \ldots\}$ be any sequence of vectors:

(a) show that if there is some $x \in H$ such that $\|x_n\| \to \|x\|$ and x_n weakly converges to x, then x_n is norm-convergent to x, i.e., $\|x_n - x\| \to 0$;

(b) show that if x_n is norm-convergent to x, then the sequence x_n is bounded, i.e., there is a positive constant M such that $\|x_n\| < M,\ \forall n$.

(1.10)
Consider the following linear subspaces of the Hilbert space $L^2(-1, 1)$:

$$V_1 = \{\text{the even polynomials, i.e., the polynomials of the form} \sum_{n=0}^{N} a_n x^{2n}\}\,;$$

$$V_2 = \{\text{the even C}^\infty \text{ functions}\}\,;$$

$$V_3 = \{\text{the functions } g(x) \text{ such that } \int_0^1 g(x)\, dx = 0\}\,;$$

$$V_4 = \{\text{the functions } g(x) \in C^0 \text{ such that } g(0) = 0\}\,.$$

What of these subspaces is a *Hilbert* subspace? and what are their orthogonal complementary subspaces?

(1.11)
Recalling that $\{x^n,\ n = 0, 1, 2, \ldots\}$ is a complete set in $L^2(-1, 1)$:

(1) Deduce: is the set of polynomials a dense subspace in $L^2(-1, 1)$? Is a Hilbert subspace?

(2) Show that $\{x^{2n}\}$ is a complete set in $L^2(0, 1)$. And $\{x^{2n+1}\}$?

(3) Show that also $\{x^N,\ x^{N+1},\ x^{N+2},\ \ldots\}$, where N is *any* fixed integer > 0, is a complete set in $L^2(-1, 1)$.

(1.12)
Let $\{e_n,\ n = 1, 2, \ldots\}$ be an orthonormal complete system in a Hilbert space H.

(1) Is the set $v_n = e_n - e_1,\ n = 2, 3, \ldots$ a complete set in H?

(2) Fixed any $w \in H$, is the set $v_n = e_n - w$ a complete set in H?

(3) Let w be any nonzero vector: for what sequences of complex numbers α_n is the set $v_n = e_n - \alpha_n w$ *not* a complete set in H?

(4) Under what condition on $\alpha,\ \beta \in \mathbf{C}$ is the set $v_n = \alpha e_n - \beta e_{n+1}$ a complete set in H?

(1.13)
Let $\{e_n,\ n \in \mathbf{Z}\}$ be an orthonormal complete system in a Hilbert space H.

(1) Is the set $v_n = e_n - e_{n+1}$ a complete set in H? And the set $w_n = \alpha e_n - \beta e_{n+1}$ where $\alpha,\ \beta \in \mathbf{C}$?

(2) Let now $H = L^2(0, 2\pi)$ and $e_n = \exp(inx)/\sqrt{2\pi}$: the sets $v_n,\ w_n$ acquire a "concrete" form. Confirm the results obtained above.

(1.14)

(1) Specify what among the following sets, with $n = 1, 2, \ldots$, are complete in $L^2(-\pi, \pi)$:

$$(a)\ \{x,\ x\cos nx,\ x\sin nx\}\ ;\ \ (b)\ \{P(x),\ P(x)\cos nx,\ P(x)\sin nx\}$$

where $P(x)$ is a polynomial $\big($does the answer depend on the form of $P(x)$?$\big)$;

$$(c)\ \{1,\ x\cos nx,\ \sin nx\}\ ;\ \ (d)\ \{x,\ \cos nx,\ \sin nx\}\ ;\ \ (e)\ \{x^2,\ \cos nx,\ \sin nx\}\ ;$$

$$(f)\ \{x,\ x\cos nx,\ \sin n|x|\}\ ;\ \ (g)\ \{x\cos nx,\ x\sin nx\}\ ;\ \ (h)\ \{x^{1/3}\cos nx,\ x^{1/3}\sin nx\}$$

(2) If $\{e_n(x)\}$ is a complete set in $H = L^2(I)$, under what conditions for the function $h(x)$ is the set $\{h(x)e_n(x)\}$ complete in H?

(1.15)
(1) Let $\{a_n,\ n = 1, 2, \ldots\}$ be a sequence of complex numbers $\in \ell^1$, i.e., such that $\sum_{n=1}^{\infty} |a_n| < \infty$. Show that also $\{a_n\} \in \ell^2$, i.e., $\sum_{n=1}^{\infty} |a_n|^2 < \infty$. Is the converse true?

(2) Show that the space ℓ^1 is a dense subspace in the Hilbert space ℓ^2.

(1.16)
In the space ℓ^2 consider the set, with $n = 1, 2, \ldots$,

$$w_1 = (1, -1, 0, 0, \ldots)/\sqrt{2}\quad ,\quad w_2 = (1, 1, -2, 0, 0, \ldots)/\sqrt{6}\quad ,\quad \ldots\ ,$$

$$w_n = (\underbrace{1, 1, \ldots, 1}_{n}, -n, 0, 0, \ldots)/\sqrt{n(n+1}\quad ,\quad \ldots$$

(1) Show that this is an orthonormal complete system in ℓ^2.

(2) Deduce that the subspace $\ell^{(0)} \subset \ell^2$ of the sequences such that $\sum_{n=1}^{\infty} a_n = 0$ is *dense* in ℓ^2.

(3) Show that

$$z_n = (1, \underbrace{-1/n, \ldots, -1/n}_{n}, 0, 0, \ldots) \in \ell^{(0)}$$

and that $z_n \to e_1$ as $n \to \infty$.

(1.17)
(1) In the space $H = L^2(0, +\infty)$ consider the set of orthonormal functions

$$u_n(x) = \begin{cases} 1 & \text{for} \quad n-1 < x < n \\ 0 & \text{elsewhere} \end{cases} \quad , \qquad n = 1, 2, \dots$$

Here are three possible answers to the question: Is this set a complete set in H? What is the correct answer?

(α) the condition $(u_n, f) = 0$, $\forall n$ is $\int_{n-1}^{n} f\, dx = 0$, $\forall n$, and this happens only if $f = 0$, then the set is complete.

(β) the function $f(x) = \sin 2\pi x$ if $x \geq 0$ satisfies $(u_n, f) = 0$, $\forall n$, then the set is not complete.

(γ) the function (e.g.,) $f(x) = \begin{cases} \sin 2\pi x & \text{for } 0 < x < 1 \\ 0 & \text{for } x > 1 \end{cases}$ satisfies $(u_n, f) = 0$, $\forall n$, then the set is not complete.

(2) In the same space, consider the set of orthogonal functions

$$v_n(x) = \begin{cases} \sin x & \text{for } (n-1)\pi \leq x \leq n\pi \\ 0 & \text{elsewhere} \end{cases} \quad , \qquad n = 1, 2, \dots$$

Is this set complete in H ?

(3) In the same space consider the set of functions

$$w_n(x) = \begin{cases} \sin\, nx & \text{for } 0 \leq x \leq n\pi \\ 0 & \text{for} \quad x \geq n\pi \end{cases} \quad , \qquad n = 1, 2, \dots$$

(a) Are the functions $w_n(x)$ orthogonal?

(b) Is the set $\{w_n(x)\}$ a complete set ?

(1.18)
(1) Is the set $\{x \ \sin nx,\ n = 1, 2, \dots\}$ a complete set in $L^2(0, \pi)$? And the subset $\{x \ \sin nx\}$ with $n = 2, 3, \dots$?

(2) The same questions for the set $\{x^2 \ \sin nx,\ n = 1, 2, \dots\}$ and respectively for the subset $\{x^2 \ \sin nx\}$ with $n = 2, 3, \dots$

(1.19)
(a) Is the set $\{\exp(-nx)\,, n = 1, 2, \dots\}$ a complete set in $L^2(0, +\infty)$? *Hint*: put $y = \exp(-x)$.

(b) The same for the set $\{\exp(inx),\ n \in \mathbf{Z}\}$ in $L^2(-2\pi, 2\pi)$.

(c) The same for the set $\{\exp(inx),\ n \in \mathbf{Z}\}$ in $L^2(0, \pi)$.

(d) The same for the set $\{\sin nx \ \sin ny\}$, $n = 1, 2, \dots$ in $L^2(Q)$, where Q is the square $0 \leq x \leq \pi$, $0 \leq y \leq \pi$.

(e) The same for the set $\{\exp(-nx)\ \sin ny\}$, $n = 1, 2, \dots$ in $L^2(\Omega)$, where Ω is the semi-infinite strip $0 \le x \le \pi$, $y \ge 0$.

(f) The same for the set $\{\exp(-x^2)\exp(inx),\ n \in \mathbf{Z}\}$ in $L^2(\mathbf{R})$.

(Another example of a complete set in $L^2(0, +\infty)$ will be proposed in Problem 3.131: the proof is based on properties of Laplace transform.)

1.1.2 Fourier Expansions

(1.20)

(1) Evaluate the Fourier expansion in terms of the orthonormal complete system in $L^2(-\pi, \pi)$

$$\frac{1}{\sqrt{2\pi}},\ \frac{1}{\sqrt{\pi}}\sin nx,\ \frac{1}{\sqrt{\pi}}\cos nx\,, \qquad n = 1, 2, \dots$$

of the following functions:

$$f_1(x) = \begin{cases} -1 \text{ for } -\pi < x < 0 \\ 1 \ \text{ for } \ \ 0 < x < \pi \end{cases} \quad ; \qquad f_2(x) = |x|$$

and discuss the convergence of the series.

(2) The same questions for the function in $L^2(0, \pi)$

$$f(x) = 1$$

in terms of the orthonormal complete system

$$\sqrt{\frac{2}{\pi}}\sin nx\,, \qquad n = 1, 2, \dots$$

Notice that the series is automatically defined $\forall x \in \mathbf{R}$, also out of the interval $0, \pi$: to what function does this series converge? Does it converge at the point $x = \pi$? to what value? and at the point $x = 3\pi/2$? to what value?

(1.21)

(1) Find the Fourier expansion in terms of the complete set $\{1,\ \cos nx,\ \sin nx\,, n = 1, 2, \dots\}$ in $L^2(-\pi, \pi)$ of the function $f_1(x) = x$ (with $-\pi < x < \pi$) and discuss the convergence of the series.

(2) The same for the function

$$f_2(x) = \begin{cases} x + \pi & \text{for } \ -\pi < x < 0 \\ x - \pi & \text{for } \ \ 0 < x < \pi \end{cases}$$

Recognize that the two functions $f_1(x)$ and $f_2(x)$ (or better, their periodic prolongations with period 2π) are actually the same function apart from a translation; accordingly, verify that their Fourier coefficients are related by a simple rule.

(1.22)
In the space $L^2(0,a)$ the following three sets are, as well known, orthogonal complete sets:

$$(i) \qquad 1,\ \cos\left(\frac{2n\pi}{a}x\right),\ \sin\left(\frac{2n\pi}{a}x\right), \qquad n = 1, 2, \dots;$$

$$(ii)\quad 1,\ \cos\left(\frac{n\pi}{a}x\right) \quad ; \quad (iii) \quad \sin\left(\frac{n\pi}{a}x\right), \qquad n = 1, 2, \dots$$

The series obtained as Fourier expansion of a function $f(x) \in L^2(0,a)$ with respect to the set i) is automatically extended to all $x \in \mathbf{R}$ and converges to a function $\widetilde{f}_1(x)$ with period a, whereas the series obtained as Fourier expansion with respect to the sets ii) and iii) converge to functions $\widetilde{f}_2(x)$ and $\widetilde{f}_3(x)$ with period .... Consider, for instance, the function $f(x) = x \in L^2(0,a)$: without evaluating the Fourier expansions, specify what are the functions $\widetilde{f}_1(x)$, $\widetilde{f}_2(x)$, $\widetilde{f}_3(x)$.

(1.23)
Consider the space $L^2(Q)$, where Q is the square $0 \le x \le \pi,\ 0 \le y \le \pi$.
(1) Evaluate the double Fourier expansion of the function

$$f(x,y) = 1$$

in terms of the orthonormal complete system

$$e_{n,m} = \frac{2}{\pi} \sin nx \ \sin my\,, \qquad n, m = 1, 2, \dots$$

The series is automatically defined $\forall x, y \in \mathbf{R}^2$: to what function $\widetilde{f}(x,y)$ does this series converge?
(2) The same questions for the function

$$f(x,y) = \sin x$$

(1.24)
(1) Show that if the coefficients a_n of a Fourier series in $L^2(0,2\pi)$ of the form

$$f(x) = \sum_{n=-\infty}^{+\infty} a_n \exp(inx)$$

satisfy $\sum_n |a_n| < \infty$, i.e., $\{a_n\} \in \ell^1$, then $f(x)$ is continuous.

(2) Generalize: assume that for some integer h one has

$$\sum_{n=-\infty}^{+\infty} |n^h a_n| < \infty$$

How many times (at least) is the function $f(x)$ continuously differentiable ?

(3) Assume that for some real α one has

$$|a_n| \le \frac{c}{|n|^\alpha} \quad \text{with} \quad \alpha > k + \frac{1}{2}$$

at least for $|n| > n_0$ where n_0 and k are given integers and c a constant. Show that $f(x) \in C^{k-1}$, i.e., $f(x)$ is $k-1$ times continuously differentiable, and that $f^{(k)}(x)$ is possibly not continuous but $\in L^2(0, 2\pi)$.

(4) Assume that the coefficients a_n satisfy a condition of the form, if $|n| > n_0$,

$$|a_n| \le \frac{c}{2^{|n|}}$$

what property of differentiability can be expected for the function $f(x)$? (Clearly, all the above results also hold for similar Fourier expansions where $\exp(inx)$ are replaced, e.g., by $\cos nx$ and/or $\sin nx$).

(1.25)

(1) Specify what properties can be deduced for the function $f(x) \in L^2(-\pi, \pi)$ if its Fourier series is

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{n^2(n^3+1)^{1/4}} \cos nx$$

(2) Show that any function $f(x)$ admitting a Fourier series of the following form

$$f(x) = \sum_{n=1}^{\infty} \frac{a_n}{n} \sin nx$$

where $a_n \in \ell^2$, is a continuous function (extensions to series of similar form are obvious).

(1.26)

In all the questions of this problem, do *not* try to evaluate the Fourier coefficients of the proposed functions. No calculations needed!

(1) In $H = L^2(-\pi, \pi)$, consider the function

$$f(x) = \begin{cases} 0 & \text{for } -\pi \le x \le 0 \\ x\sqrt{x} & \text{for } \quad 0 \le x \le \pi \end{cases}$$

(a) Is the Fourier expansion of $f(x)$ with respect to the complete set $\{\exp(inx),\ n \in \mathbf{Z}\}$ convergent at the point $x = \pi$? To what value? and at the point $x = 4$?

(b) Is it true that the Fourier coefficients c_n of the above expansion satisfy $c_n \in \ell^1$?

(2) In the same space H, let $f(x) = \sqrt{|x|}$. Is it true that the Fourier coefficients a_n of the expansion $f(x) = \sum_n a_n \cos nx$ satisfy $na_n \in \ell^2$?

(3) In the same space H, let $f(x) = \exp(x^2)$. Is it true that the Fourier coefficients a_n of the expansion $f(x) = \sum_n a_n \cos nx$ satisfy $na_n \in \ell^2$? and $na_n \in \ell^1$?

(1.27)
Let v_n be the orthonormal complete system in $L^2(0, \pi)$

$$v_n(x) = \sqrt{2/\pi}\,\sin nx\,, \qquad n = 1, 2, \ldots$$

(1) Consider the Fourier expansion of the function

$$f_1(x) = \Big|x - (\pi/2)\Big|$$

with respect to the subset $v_1, v_3, \ldots, v_{2m+1}, \ldots$: is this expansion convergent (with respect to the L^2 norm, of course)? to what function? (No calculation needed!)

(2) The same questions for the function $f_2(x) = x - (\pi/2)$.

(3) The same questions for the function $f_3(x) = x$.

(4) Is the subset $v_1, v_3, \ldots, v_{2m+1}, \ldots$ a complete system in the space $L^2(0, \pi/2)$?

(1.28)
Consider in the space $L^2(-\pi, \pi)$ the orthonormal *not* complete set

$$\frac{1}{\sqrt{2\pi}},\ \frac{1}{\sqrt{\pi}}\sin nx\,, \qquad n = 1, 2, \ldots$$

Find the functions which are obtained performing the Fourier expansion (no calculation needed !) of the following functions with respect to this set:

$$f_1(x) = 2 + \exp(2ix)\,, \quad f_2(x) = x \log|x|\,, \quad f_3(x) = \begin{cases} 1 & \text{for} \quad |x| < \pi/2 \\ 0 & \text{for} \quad \pi/2 < |x| < \pi \end{cases}$$

(1.29)
Consider in the space $L^2(0, 4\pi)$ the orthogonal *not* complete set $\{\exp(inx), n \in \mathbf{Z}\}$. Find the functions which are obtained performing the Fourier expansion (no calculation needed !) of the following functions with respect to this set:

$$f_1(x) = \begin{cases} 1 & \text{for } 0 < x < 2\pi \\ 0 & \text{for } 2\pi < x < 4\pi \end{cases} \quad ; \quad f_2(x) = |\sin(x/2)|$$

(1.30)
Consider in $L^2(0,\infty)$ the set

$$v_n(x) \;=\; \begin{cases} \sin\, nx \;\; \text{for } 0 \le x \le n\pi \\ 0 \qquad \text{for } x \ge n\pi \end{cases} \qquad , \qquad n = 1, 2, \dots$$

See Problem 1.17, q.(3) for the orthogonality and the non-completeness of this set. Find the functions which are obtained performing the Fourier expansion of the following functions with respect to this set:

$$f_1(x) = \begin{cases} 1 \;\; \text{for} \;\; 0 < x < \pi \\ 0 \;\; \text{for} \;\; x > \pi \end{cases} \quad ; \quad f_2(x) = \begin{cases} 1 \;\; \text{for} \;\; 0 < x < 2\pi \\ 0 \;\; \text{for} \;\; x > 2\pi \end{cases}$$

(1.31)
In the space $H = L^2(0,+\infty)$, consider the set of orthonormal functions $u_n(x)$

$$u_n(x) \;=\; \begin{cases} 1 \;\; \text{for} \;\; n-1 < x < n \\ 0 \;\; \text{elsewhere} \end{cases} \qquad , \qquad n = 1, 2, \dots$$

given in Problem 1.17, q.(1).

(1) What function is obtained performing the Fourier expansion of a function $f(x) \in L^2(0,\infty)$ with respect to the set $u_n(x)$?

(2) Is the sequence of the functions $u_n(x)$ pointwise convergent as $n \to \infty$? Is the convergence uniform? Is this sequence a Cauchy sequence (with respect to the $L^2(0,\infty)$ norm)? Is it *weakly* L^2-convergent $\big($i.e., does the numerical sequence (u_n, g) admit limit $\forall g \in L^2(0,\infty)\big)$?

In the three following problems, we will introduce as independent variable the time t —just to help the physical interpretation—instead of the "position" variable x. Accordingly, we will write, e.g., $u = u(t)$, $\dot{u} = du/dt$, etc.

(1.32)
(1) Consider the equation of the periodically forced harmonic oscillator

$$\ddot{u} + u \;=\; g(t)\,, \qquad u = u(t)$$

where $g(t)$ is a 2π-periodic given function $\in L^2(0, 2\pi)$, and look for 2π-periodic solutions $u(t)$. Write $g(t)$ as Fourier series with respect to the orthogonal complete system $\{\exp(int),\ n \in \mathbf{Z}\}$ in $L^2(0, 2\pi)$: $g(t) = \sum_n g_n \exp(int)$, and obtain the solution in the form of a Fourier series: $u(t) = \sum_n u_n \exp(int)$. Under what condition on $g(t)$ (or on its Fourier coefficients g_n) does this equation admit solution? and, when the solution exists, is it unique?

(2) The same questions for the equation

$$\ddot{u} + 2u = g(t)$$

(1.33)
(1) The same questions as in q. (1) of the above problem for the equation

$$\dot{u} + u = g(t)\,, \qquad u = u(t)$$

(this is, e.g., the equation of an electric series circuit of a resistance R and an inductance L (with $R = L = 1$), submitted to a periodic potential $g(t)$ where $u(t)$ is the electric current). As before, assume that $g(t)$ is a 2π-periodic given function $\in L^2(0, 2\pi)$ and look for 2π-periodic solutions $u(t)$. Introducing the orthogonal complete system $\{\exp(int),\ n \in \mathbf{Z}\}$ in $L^2(0, 2\pi)$, write in the form of a Fourier series the solution of this equation.

(2) Show that the solution $u(t)$ is a continuous function.

(1.34)
A doubt concerning the existence and uniqueness of solutions of the equations given in the two above problems. In Problem 1.32, q.(1) the conclusion was that the equation $\ddot{u} + u = g$ has no solution if the Fourier coefficients $g_{\pm 1}$ of $g(t)$ with respect the orthogonal complete set $\{\exp(int),\ n \in \mathbf{Z}\}$ are not zero. However, it is well known from elementary analysis that, e.g., the equation $\ddot{u} + u = \sin t$ admits the solution $u(t) = -(t/2)\cos t$ (this is the case of "resonance"). Explain why this solution does not appear in the present context of Fourier expansions. A related difficulty appears in Problems 1.32, q.(2) and 1.33, q.(1): the conclusion was that the solution is unique, but it is well known that the differential equations considered in these problems admit respectively ∞^2 and ∞^1 solutions: explain why these solutions do not appear in the above calculations. Similar apparent difficulties appear in many other cases: see e.g., Problems 1.84–1.90.

1.1.3 Harmonic Functions: Dirichlet and Neumann Problems

In this subsection, simple examples of Dirichlet and Neumann Problems will be proposed. The Dirichlet Problem amounts of finding a harmonic function $U = U(x, y)$ in some region $\Omega \subset R^2$ satisfying a given condition on the boundary of Ω, i.e.,

$$\Delta_2 U \equiv \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = 0 \quad \text{in} \quad \Omega, \quad \text{with} \quad U\Big|_{\partial\Omega} = F(x, y)$$

Neumann Problem amounts of finding a harmonic function in Ω when a condition is given on its normal derivative on the boundary, i.e., $\partial U/\partial n\Big|_{\partial\Omega} = G(x, y)$.
The Dirichlet Problem will be also reconsidered, with different methods, in Chap. 2, Sect. 2.3, and in Chap. 3, Problems 3.110, 3.111, 3.112, 3.115.
In the four following exercises, recall that the most general form of a harmonic function $U = U(r, \varphi)$ in the interior of the circle centered at the origin of radius R, in polar coordinates r, φ, is given by

$$U(r, \varphi) = a_0 + \sum_{n=1}^{\infty} r^n (a_n \cos n\varphi + b_n \sin n\varphi)$$

(1.35)
(1) Assume for simplicity $R = 1$. Solve the Dirichlet Problem for the circle, i.e., find $U(r, \varphi)$ for $r < 1$ if the boundary value $U(1, \varphi) = F(\varphi) \in L^2(0, 2\pi)$ is given:

(a) if $F(\varphi) = 1$ (trivial !);

(b) if $F(\varphi) = \cos^2 \varphi$ (nearly trivial !);

(c) obtain as a Fourier series $U(r, \varphi)$ if $F(r, \varphi) = \begin{cases} 1 & \text{for } 0 < \varphi < \pi \\ -1 & \text{for } \pi < \varphi < 2\pi \end{cases}$.

(2) Show that $U(r, \varphi)$ is a C^∞ function if $r < 1$.

(1.36)
Consider the case of a semicircle $0 \le \varphi \le \pi$ (radius $R = 1$) with the boundary conditions

$$U(r, 0) = U(r, \pi) = 0, \quad U(1, \varphi) = F(\varphi) \in L^2(0, \pi)$$

(1) Show that in this case the Dirichlet Problem can be solved with $a_0 = a_n = 0$ for all n.

(2) Let $F(\varphi) = 1$: the solution $U(r, \varphi)$ (written as a Fourier series) can be also extended to the semicircle with $\pi < \varphi < 2\pi$. What is the value of $U(1, 3\pi/2)$?

(3) Solve the Dirichlet Problem with the boundary conditions

$$U(r, 0) = U(r, \pi) = a \neq 0, \quad U(1, \varphi) = F(\varphi) \in L^2(0, \pi)$$

where a = const. *Hint*: solve first the problem with $U(1, \varphi) = F(\varphi) - a$ and $U(r, 0) = U(r, \pi) = 0$, then ….

(1.37)
Consider the case of a quarter-circle $0 \le \varphi \le \pi/2$ (radius $R = 1$) with the boundary conditions

$$U(r, 0) = U(r, \pi/2) = 0, \quad U(1, \varphi) = F(\varphi) \in L^2(0, \pi/2)$$

(1) Show that in this case the Dirichlet Problem can be solved with $a_0 = a_n = 0$ for all n, and $b_n = 0$ if n is odd.

(2) Let $F(\varphi) = 1$: the solution $U(r, \varphi)$ (written as a Fourier series) can be also extended to the whole circle. What is the value of $U(1, 3\pi/4)$? and $U(1, 5\pi/4)$? and $U(1, 7\pi/4)$?

(1.38)
(1) Show that the Neumann Problem for the circle, i.e., the problem of finding $U(r, \varphi)$ in the interior of the circle if the normal derivative at the boundary $\partial U/\partial r|_{r=R} = G(\varphi) \in L^2(0, 2\pi)$ is given, can be solved if (and only if) $G(\varphi)$ satisfies

$$g_0 = \frac{1}{2\pi} \int_0^{2\pi} G(\varphi)\, d\varphi = 0$$

Show also that the solution (when existing) is not unique.

(2) If $U(r, \varphi)$ is a two-dimensional electric potential, explain why the results obtained in (1) admit a clear physical interpretation.

(1.39)
(1) Consider the case of a rectangle in the (x, y) plane, say $0 \le x \le \pi$, $0 \le y \le h$, with boundary conditions

$$U(0, y) = U(\pi, y) = 0, \quad U(x, 0) = F_1(x), \quad U(x, h) = F_2(x)$$

Using the separation of variables $U(x, y) = X(x)Y(y)$, show that the solution of the Dirichlet Problem can be written in the form

$$U(x, y) = \sum_{n=1}^{\infty} \sin nx \Big(a_n \exp(ny) + b_n \exp(-ny) \Big)$$

where the coefficients a_n, b_n are uniquely determined by $F_1(x),\ F_2(x)$.

(2) Find $U(x, y)$ in the case $F_1(x) = \sin x,\ F_2(x) = \sin 2x$.

(3) What changes if $h = \infty$ (imposing that the solution belongs to L^2)?

(1.40)
Consider the Dirichlet Problem in a rectangle with nonzero boundary conditions on all the four sides of the rectangle. Show how the problem can be solved by a superposition of two problems similar to the previous one, q. (1).

(1.41)
(1) Consider the Dirichlet Problem in the annular region between the two circles centered at the origin with radius $R_1 < R_2$. Recalling that the most general form of the harmonic function in the region $R_1 < r < R_2$ can be written in polar coordinates r, φ as

$$U(r, \varphi) = a_0 + b_0 \log r + \sum_{n=\pm 1, \pm 2, \ldots} \exp(in\varphi)(a_n r^n + b_n r^{-n})$$

show that the Dirichlet Problem admits unique solution if the two boundary conditions

$$U(R_1, \varphi) = F_1(\varphi), \quad U(R_2, \varphi) = F_2(\varphi)$$

are given.

(2) Solve the problem in the (rather simple) cases

(a) $F_1(\varphi) = c_1,\ F_2(\varphi) = c_2 \neq c_1$, where $c_1,\ c_2$ are constants;

(b) $F_1(\varphi) = \cos\varphi$ with $R_1 = 1/2$ and $F_2(\varphi) = \cos\varphi$ with $R_2 = 2$;

(c) $F_1(\varphi) = \cos\varphi$ with $R_1 = 1$ and $F_2(\varphi) = \cos 2\varphi$ with $R_2 = 2$.

1.2 Linear Operators in Hilbert Spaces

This section is devoted to studying the different properties of linear operators between Hilbert spaces: their domains, ranges, norms, boundedness, closedness, and to examining special classes of operators: adjoint and self-adjoint operators, projections, isometric and unitary operators, functionals, and time-evolution operators.
Great attention is paid to the notion of eigenvalues and eigenvectors, due to its relevance in physical problems. Many exercises propose the different procedures needed for finding eigenvectors and the extremely various situations which can occur. According to the physicists use, the term "eigenvector" is used instead of the more correct "eigenspace", and "degeneracy" instead of "geometrical multiplicity" of the eigenvalue (i.e., the dimension of the eigenspace). The term "not degenerate" is also used instead of "degeneracy equal to 1". The notion of spectrum *is only occasionally mentioned.*

Another frequent question concerns the convergence of given sequences of operators in a Hilbert space H. *Let us recall that the convergence of a sequence* T_n *to* T *as* $n \to \infty$ *is said to be*
(i) *"in norm" if* $\|T_n - T\| \to 0$
(ii)*"strong" if* $\|(T_n - T)x\| \to 0\,, \forall x \in H$
(iii)*"weak" if* $\Big(y, (T_n - T)x\Big) \to 0\,, \forall x, y \in H$
Clearly, norm convergence implies strong and strong implies weak convergence, but the converse is not true. Many of the exercises proposed provide several examples of this. Similar definitions hold for families of operators T_a *depending on some continuous parameter* a.
Questions as "Study the convergence" or "Find the limit" of the given sequence T_n *(or family* T_a*) of operators are actually "cumulative" questions, which indeed include and summarize several aspects. A first aspect is to emphasize the fact that "convergence" (and the related notions of "approximation" and "neighborhood") is a "relative" notion, being strictly dependent on the definition of convergence which has been chosen. The next "operative" aspects are that, given the sequence of operators, one has to*
(a) *conjecture the possible limit* T *(this is usually rather easy)*
(b) *evaluate some norms of operators* $\|T_n - T\|$ *and/or of vectors* $\|(T_n - T)x\|$ *and so on, to decide what type of convergence is involved.*

Frequent use will be done in this section, and also in Chap. 3, of the Lebesgue-dominated convergence theorem *(briefly: Lebesgue theorem) concerning the convergence of integrals of sequences of functions. The statement of the theorem in a form convenient for our purposes is the following:*
Assume that a sequence of real functions $\{f_n(x)\} \in L^1(\mathbf{R})$ *satisfies the following hypotheses:*
(i) $f_n(x)$ *converges pointwise almost everywhere to a function* $f(x)$,
(ii)*there is a function* $g(x) \in L^1(\mathbf{R})$ *such that*

$$|f_n(x)| \leq g(x)$$

then
(a) $f(x) \in L^1(\mathbf{R})$
(b) $\displaystyle\lim_{n\to\infty} \int_{-\infty}^{+\infty} f_n(x)\,dx = \int_{-\infty}^{+\infty} f(x)\,dx$
The theorem is clearly also true if, instead of a sequence of functions depending on a integer index n, *one deals with family of functions* $\{f_a(x)\}$ *depending on a continuous parameter* a, *and–typically–one considers the limit as* $a \to 0$. *See Problem 1.1 for simple examples of sequences of functions not satisfying the assumptions of this theorem.*

Other examples of linear operators will be proposed in Sects. 3.1.2 and 3.2.2 in the context of Fourier transforms.

1.2.1 Linear Operators Defined Giving $T\, e_n = v_n$, and Related Problems

A common and convenient way to define a linear operator is that of assigning its values when applied to an orthonormal complete system $\{e_n\}$ in a Hilbert space, i.e., of giving $v_n = T e_n$. Some significant cases are proposed in this subsection; other examples can also be found in the following subsections.

The first problem is to check if the domain of these linear operators can be extended to the whole Hilbert space in such a way to obtain (whenever possible) a continuous operator. Let us start with the simplest cases in the two following problems, where the $\{e_n\}$ are eigenvectors of T.

(1.42)
Let $\{e_n\,,\, n = 1, 2, \ldots\}$ be an orthonormal complete system in a Hilbert space H, and

$$T\, e_n = c_n e_n\,, \qquad c_n \in \mathbf{C};\ \text{no sum over } n$$

In each one of the following cases

$$c_n = n\,; \quad c_n = 1/n\,, \qquad n = 1, 2, \ldots\,;$$

$$c_n = \exp(in\pi/7)\,; \quad c_n = \exp(in)\,; \quad c_n = \frac{n-i}{n+i}\,; \quad c_n = \frac{n^2}{n^2+1}\,, \qquad n \in \mathbf{Z}$$

(a) find the degeneracy of the eigenvalues, find $\|T\|$ (and specify if there is some $x_0 \in H$ such that $\|T x_0\| = \|T\|\|x_0\|$), or show that T is unbounded;

(b) find domain and range of T (check in particular if they coincide with the whole space H or—at least—if they are dense in it).

(1.43)
Let $\{e_n\,,\, n \in \mathbf{Z}\}$ and let T_N (where N is a fixed integer) be defined by

$$T_N\, e_n = e_n \quad \text{for} \quad |n| \le N \quad \text{and} \quad T_N\, e_n = 0 \quad \text{for} \quad |n| > N$$

(1) Show that T_N is a projection.[2] Is it compact?

(2) Study the convergence as $N \to \infty$ of the sequence of operators T_N to the operator T_∞= the identity.

(1.44)
Let $\{e_n\,,\, n \in \mathbf{Z}\}$ and let S_N (where N is a fixed integer) be defined by

[2] In this book, only *orthogonal* projections P will be considered, i.e., operators satisfying the properties $P^2 = P$ (idempotency) and $P^+ = P$ (Hermiticity).

$$S_N\, e_n = e_{-n} \text{ for } 1 \le |n| \le N, \quad S_N\, e_0 = e_0 \text{ and } S_N\, e_n = 0 \text{ for } |n| > N$$

(1) Find the eigenvectors and eigenvalues (with their degeneracy) of S_N. Is it compact?

(2) Consider the operator S_∞ defined by $S_\infty\, e_n = e_{-n}$ for all nonzero $n \in \mathbf{Z}$ and with $S_\infty e_0 = e_0$. Study the convergence as $N \to \infty$ of the sequence of operators T_N to the operator S_∞.

(3) If $e_n = \exp(inx)/\sqrt{2\pi}$ in $H = L^2(-\pi, \pi)$, show that the operator S_∞ takes a very simple form!

(1.45)
In a Hilbert space H with orthonormal complete system $\{e_n\,, n = 1, 2, \ldots\}$ consider the set of vectors

$$v_1 = \frac{e_1 + e_2}{\sqrt{2}}\,,\ v_2 = \frac{e_3 + e_4}{\sqrt{2}},\ \ldots, v_n = \frac{e_{2n-1} + e_{2n}}{\sqrt{2}},\ \ldots, \qquad n = 1, 2, \ldots$$

(1) Is $\{v_n\}$ a orthonormal set? a complete set?

(2) Let T be the linear operator defined by

$$T\, e_n = v_n$$

(a) Does T preserve scalar products? is it unitary?

(b) Show that $\mathtt{Ran}\, T$ is a Hilbert subspace of H; what is its orthogonal complementary subspace and the dimension of this subspace?

(c) Does T admit eigenvectors ? What is its kernel?

(1.46)
In a Hilbert space H with orthonormal complete system $\{e_n,\ n = 1, 2, \ldots\}$, consider the linear operators defined by

$$T\, e_n = e_{n+1} \quad \text{and} \quad \begin{cases} S\, e_1 = 0 \\ S\, e_n = e_{n-1} \text{ for } n > 1 \end{cases}$$

(1) Writing a generic vector $x \in H$ in the form $x = \sum_n a_n\, e_n \equiv (a_1, a_2, \ldots)$, obtain Tx and Sx (equivalently: choose $H = \ell^2$). Show that the domain of these operators is the whole Hilbert space.

(2) Is T injective? surjective? the same questions for S.

(3) Calculate $\|T\|$ and $\|S\|$.

(4) Show that $S = T^+$.

(5) Show that T is "isometric", i.e., preserves scalar products: $(x, y) = (Tx, Ty)$, $\forall x, y \in H$ but is not unitary. Study the operators TT^+ and T^+T. Show that TT^+ is a projection: on what subspace?

(6) Show that T has no eigenvectors, but $S = T^+$ has many (!) eigenvectors.

(7) Are T and S compact operators?

(1.47)

(1) What changes in the above problem if $T\,e_n = e_{n+1}$ is defined on an orthonormal complete system where now $n \in \mathbf{Z}$?

(2) A "concrete" version of this operator is the following: let $H = L^2(0, 2\pi)$ with $e_n = \exp(inx)/\sqrt{2\pi}$. Then T becomes simply $T\,f(x) = \exp(ix)f(x)$ for any $f \in L^2(0, 2\pi)$. Find again (and confirm) the results obtained before.

(1.48)

Although the operators proposed in this problem are not of the form $Te_n = v_n$ which is that considered in this subsection, they share similar properties with those of Problems 1.46 and 1.47; in particular they are respectively an isometric and an unitary operator.

(1) In $H = L^2(0, \infty)$ consider $T\,f(x) = f(x-1)$. To avoid difficulties with the restriction $x > 0$, it would be more convenient to write for clarity

$$T\,f(x) = f(x-1)\theta(x-1) \quad \text{and} \quad S\,f(x) = f(x+1)\theta(x)$$

where

$$\theta(x) = \begin{cases} 0 \text{ for } x < 0 \\ 1 \text{ for } x > 0 \end{cases}$$

The same questions (2)–(7) as in Problem 1.46. For what concerns the eigenvectors of S, consider only the functions $f(x) = \exp(-\alpha x),\ \alpha > 0$.

(2) What changes if $T\,f(x) = f(x-1)$ is defined in $L^2(\mathbf{R})$? (compare with Problem 1.47, q.(1)).

(1.49)

Another operator with similar properties as T in Problem 1.46:
In a Hilbert space H with orthonormal complete system $\{e_n,\ n = 1, 2, \ldots\}$, consider the operators

$$T\,e_n = e_{2n} \quad \text{and} \quad \begin{cases} S\,e_n = e_{n/2} \text{ for } n \text{ even} \\ S\,e_n = 0 \quad \text{for } n \text{ odd} \end{cases}$$

Exactly all the same questions (1)–(7) as in Problem 1.46. Show that S has many and many eigenvectors.

(1.50)
The same remark as for Problem 1.48. Another case:

Consider the operators T and S of $L^2(0, 1)$ in itself defined respectively by

$$T\,f(x) = g(x) \text{ where } g(x) = f(2x)$$

(warning: $f(x)$ is given only in $0 < x < 1$, so it defines $g(x)$ only if $0 < x < 1/2$; it is understood that $g(x)$ is put equal to zero if $1/2 < x < 1$); and let S be defined by

$$S\,f(x) = h(x) \text{ where } h(x) = (1/2)f(x/2)$$

The same questions (2)–(7) as in Problem 1.46. For what concerns the eigenvectors of S, consider only the functions $f(x) = x^\alpha$. Verify that the corresponding eigenvalues λ_α satisfy the condition $\|S\| \geq \sup|\lambda_\alpha|$.

(1.51)
Study the convergence as $N \to \infty$ of the sequences of operators T^N, S^N where T and S are some of the isometric and unitary operators considered in the previous Problems 1.46, 1.47, 1.48. Precisely:

(1) if T and S are given in Problem 1.48, q.(1), i.e., $T^N f(x) = f(x - N)\theta(x - N)$ in $L^2(0, \infty)$, etc.

(2) if T and S are given in Problem 1.48, q.(2), i.e., the same as in (1) but in $L^2(\mathbf{R})$ (use Fourier transform; see also Problem 3.20).

(3) if T and S are given in Problem 1.46, i.e., $T^N e_n = e_{n+N}$ etc. with $n = 1, 2, \ldots$.

(4) if T and S are given in Problem 1.47, q.(2), i.e., $T^N f(x) = \exp(iNx)f(x)$ in $L^2(0, 2\pi)$.

(5) if T and S are given in Problem 1.47, q.(1), i.e., the same as in (3), i.e., $T^N e_n = e_{n+N}$ but with $n \in \mathbf{Z}$.

(1.52)
(1) Let $v_n = e_n - e_{n-1}$, where $\{e_n,\ n \in \mathbf{Z}\}$ is an orthonormal complete system in a Hilbert space H and let T be the operator

$$T\,e_n = v_n$$

(a) Find $\mathtt{Ker}\,T$.

(b) Show that $\|T\| \leq 2$.

(c) Find $T(e_0+e_1+\cdots+e_k)$: what information can be deduced about the boundedness of T^{-1}?

(2) More in general, let $w_n = \alpha e_n - \beta e_{n-1}$ with nonzero $\alpha, \beta \in \mathbf{C}$, and let

$$T\,e_n = w_n$$

(a) Is the set w_n a complete set in H? deduce: is $\mathtt{Ran}\,T$ a dense subspace in H?

(b) Look for eigenvectors of T.

(3) If now $H = L^2(0, 2\pi)$ and $e_n = \exp(inx)/\sqrt{2\pi}$, the above operator acquires a concrete (possibly simpler) form: $Tf(x) = \varphi(x)f(x)$ where $f(x) \in L^2(0, 2\pi)$ and $\varphi(x) = \ldots$.

(a) Find $\|T\|$.

(b) Confirm the results seen in (2).

(c) Under what conditions about $\alpha,\ \beta$ does the operator T given in (2) admit bounded inverse? and does $\mathtt{Ran}\,T$ coincide with $H = L^2(0, 2\pi)$?

(d) In the case $\alpha = \beta = 1$, does the function $f(x) = 1$ belong to $\mathtt{Ran}\,T$? and the function $f(x) = \sin x$?

(1.53)
Consider the operator defined in a Hilbert space H with orthonormal complete system $\{e_n,\ n = 1, 2, \ldots\}$

$$T\,e_n \;=\; c_n x_0 + e_n\,, \qquad n = 1, 2, \ldots$$

where $c_n \in \ell^2$ and $x_0 \in H$ are given.

(1) Show that T is bounded.

(2) Find eigenvalues and eigenvectors of T. What condition on c_n and x_0 ensures that the eigenvectors provide a complete set for H?

(3) In the case $x_0 = \sum_m c_m e_m$, find $\|T\|$ and check if the eigenvectors provide a complete set for H.

(1.54)
Let $\{e_n,\ n \in \mathbf{Z}\}$ be an orthonormal complete system in a Hilbert space H and let T be the operator defined by

$$T\,e_n \;=\; \alpha_n e_{n+1}\,, \qquad \alpha_n \in \mathbf{C}$$

(1) For what choice of α_n:

(a) is T unitary?

(b) is T bounded?

(c) is T a projection?

(2) Find T^+.

(3) Let $\alpha_n = \exp(in\pi/2) + i$:

(a) find $\|T\|$;

(b) find $\mathtt{Ker}\,T$; are there vectors $\in \mathtt{Ker}\,T$ which also belong to $\mathtt{Ran}\,T$?

(1.55)
Let $\{e_n,\ n = 1, 2, \ldots\}$ be an orthonormal complete system in a Hilbert space H and let T the operator defined by

$$T\,e_n = x_0$$

where x_0 is a *fixed* nonzero vector.

(1) (a) is the domain of T the whole Hilbert space?

(b) is T a bounded operator?

(2) What is the kernel of T? (see Problem 1.16)

(3) Construct two sequences of vectors z_n and w_n both tending as $n \to \infty$ to e_1 (e.g.,), but such that $T(z_n) = 0$ and $T(w_n) \to e_1$.

(4) Conclude: is T a closed operator?

(1.56)
Consider the operator defined in a Hilbert space H with orthonormal complete system $\{e_n,\ n = 1, 2, \ldots\}$

$$T\,e_n = \alpha e_n + \beta e_1\,, \qquad \alpha, \beta \in \mathbf{C};\ \alpha, \beta \neq 0$$

(1) The same questions as in (1) of the previous problem.

(2) For what values of α, β does T admit a nontrivial kernel? of what dimension?

(3) Show that T coincides with a multiple of the identity operator in a dense subspace of H.

(4) Is T a closed operator?

(1.57)
Let $\{e_n,\ n \in \mathbf{Z}\}$ be an orthonormal complete system in a Hilbert space H and let T be the operator defined by

$$T\,e_0 = c_0 e_0 \quad , \quad T\,e_n = c_n e_{-n}\ , \qquad n \neq 0,\ c_n \in \mathbf{C}$$

(1) Give the conditions on the coefficients c_n in order to have

(a) T bounded, (b) T normal, (c) T Hermitian, (d) $T^2 = I$

(2) Show that the problem of looking for the eigenvectors of T reduces to simple problems in two-dimensional subspaces (or to the trivial one-dimensional case $T e_0 = c_0 e_0$). Is the set of the eigenvectors a complete set in H?

(3) Consider the particular cases:

(a) $c_n = -c_{-n} \neq 0$: find the eigenvectors and eigenvalues (with their degeneracy): can one expect that the eigenvectors are orthogonal? and the eigenvalues real?

(b) $c_n = \begin{cases} n & \text{for} \quad n \geq 0 \\ 1/n^2 & \text{for} \quad n < 0 \end{cases}$: find the eigenvectors and eigenvalues (with their degeneracy); the sequence of the eigenvalues is bounded and converges to 0 as $n \to \infty$; however, T is not compact (actually, it is unbounded), is this surprising?

(c) $c_n = \alpha^n$, $\alpha \in \mathbf{C}$: how can one choose α in order to have

(i) T bounded? (ii) T unitary? (iii) the image of T coinciding the whole space H?

(1.58)
Let $e_n = \exp(inx)(2\pi)^{-1/2}$, $n \in \mathbf{Z}$ in the space $L^2(-\pi, \pi)$, and let T be defined by

$$T\,e_0 = 0 \quad , \quad T\,e_n = n^2 e_{-n}$$

(1) Find eigenvectors and eigenvalues of T with their degeneracy. Do the eigenvectors provide a complete set for the space?

(2) Is the operator $T + I$ invertible?

(3) For what values of $c \in \mathbf{C}$ is the operator $T + cI$ invertible?

(4) Find $\|(T + 20I)^{-1}\|$, $\|(T + iI)^{-1}\|$, $\|\Big(T + (2 + i)\Big)^{-1}\|$

(1.59)
Let $e_n = \exp(inx)(2\pi)^{-1/2}$, $n \in \mathbf{Z}$ in the space $L^2(-\pi, \pi)$, and let T be defined by

$$T\,e_0 = 0 \quad , \quad T\,e_n = \frac{1}{n^2} e_{-n} \quad \text{for} \quad n \neq 0$$

(1) Find eigenvectors and eigenvalues of T with their degeneracy.

(2) What is $\mathtt{Ran}\,T$? Specify if it is a Hilbert subspace of $L^2(-\pi, \pi)$, or—alternatively—what is its closure.

Consider now the equation

$$T\,f = g$$

where $g = g(x) \in L^2(-\pi, \pi)$ is given and $f = f(x) \in L^2(-\pi, \pi)$ unknown.

(3) (a) Let $g(x) = \cos^2(x) - (1/2)$: does this equation admit solution? is the solution unique?

(b) Same questions if $g(x) = \cos^4(x)$ (no calculation needed !).

(c) Same questions if $g(x) = |x| - \pi/2$ (no calculation needed !).

(4) Show that the sequence of operators T^N is norm-convergent as $N \to \infty$: to what operator?

(1.60)
Let $\{e_n,\ n = 1, 2, \ldots\}$ be an orthonormal complete system in a Hilbert space H and let T be the operator defined by

$$T\,e_n = \alpha_n e_1 + \beta_n e_2$$

where $n = 1, 2, \ldots$ and α_n and β_n are given sequences of complex numbers.

(1) Let $\alpha_n = \beta_n = 1/2^n$. *Hint*: put $z = \sum_n e_n/2^n$, then ...:

(a) find $\|T\|$;

(b) find eigenvectors and eigenvalues of T;

(c) find T^+

(2) For what α_n and β_n is T bounded?

(3) (a) For what α_n and β_n is the range of T one-dimensional?

(b) For what α_n and β_n are the range and the kernel of T orthogonal?

(4) Let $\alpha_n = \beta_n = \begin{cases} 0 & \text{for} \quad n \le N \\ 1/2^{(n-N)} & \text{for } n > N \end{cases}$ and let T_N be the corresponding operator.

Study the convergence as $N \to \infty$ of the sequences of operators T_N and T_N^+. *Hint*: put $z_N = \sum_{n>N}^{\infty} e_n/2^{(n-N)}$, then $T_N x = \ldots$

(1.61)
Let $\{e_n\,, n \in \mathbf{Z}\}$ be an orthonormal complete system in a Hilbert space H and consider the operators defined by

$$S\, e_n = \frac{n^2}{1+n^4}\, e_n \quad , \quad T_N\, e_n = \begin{cases} e_n & \text{for} \quad |n| \le N \\ e_{-n} & \text{for} \quad |n| > N \end{cases} \qquad (N = 1, 2, \ldots)$$

(1) Fixed any integer N, find eigenvectors and eigenvalues (with their degeneracy) of S and of T_N. Is there an orthogonal complete set for H of simultaneous eigenvectors?

(2) Is the operator $T_N S$ compact? and $T_N\, S$?

(3) Study the convergence of the sequence of operators T_N , $T_N S$, $S T_N$ as $N \to \infty$.

1.2.2 Operators of the Form $T\,x = v(w, x)$ and $T\,x = \sum_n V_n(w_n, x)$

It can be noted that the operators of the previous subsection can be viewed as a special case of the operators considered here. Indeed, choosing $w_n = e_n$, where $\{e_n\}$ is a orthonormal complete system in the Hilbert space, one obtains just $T\,e_n = v_n$.

(1.62)
Consider in $L^2(-a, a)$ ($a > 0, \neq \infty$) the operator

$$T\,f(x) = h(x) \int_{-a}^{a} f(x)\,dx$$

where $h(x) \in L^2(-a, a)$ is a given function.

(1) Find the domain, the kernel, and the range of T, with their dimensions.

(2) Find the eigenvectors and eigenvalues of T with their degeneracy.

(3) Find $\|T\|$.

(4) Find the adjoint operator T^+, its eigenvectors , and eigenvalues with their degeneracy.

(5) Study the operator T^2.

(1.63)
The same as the above problem but with $a = \infty$, i.e.,

$$T\,f(x) = h(x) \int_{-\infty}^{+\infty} f(x)\,dx$$

(1) Find eigenvectors and eigenvalues of T with their degeneracy $\big($distinguish the cases $h(x) \in L^1(\mathbf{R}) \cap L^2(\mathbf{R})$ and $h(x) \notin L^2(\mathbf{R})$ or $h(x) \notin L^1(\mathbf{R})\big)$.

(2) Specify the domain and the kernel of T (see also Problem 1.4).

(3) Show that T is not closed (see also Problem 1.4).

(4) Show that if the kernel of an operator is *dense* in the Hilbert space, then the operator is not closed (cf., for a different example, Problem 1.55).

(1.64)

(1) Let I be an interval $I \subseteq \mathbf{R}$ and consider the operator

$$T\,f(x) = h(x) \int_I g(x)\,f(x)\,dx = h(g^*, f)$$

where h and g are given functions in $L^2(I)$. The same questions (1)–(5) as in Problem 1.62.

(2) The operator in (1) can be generalized in abstract setting in the form

$$T\,x = v(w, x)$$

where v, w are given vectors in a Hilbert space.

(a) The same questions (1)–(5) as in Problem 1.62.

(b) Is it possible to choose v, w in such a way that T is a projection?

(c) Show that it is possible to choose v, w in such a way that $T^2 = 0$ (but $T \neq 0$).

(1.65)
(1) For each fixed integer N, consider the operators in $L^2(0, 2\pi)$

$$T_N\, f(x) = \frac{1}{2\pi} \sum_{n=-N}^{N} \exp(inx) \int_0^{2\pi} \exp(-iny) f(y)\, dy$$

Recognize that these operators admit elementary properties …; study the convergence as $N \to \infty$ of the sequence of the operators T_N; see the following (2)(a).

(2) Consider now the operator

$$C\, f(x) = \frac{1}{2\pi} \sum_{n \in \mathbf{Z}} c_n \int_0^{2\pi} \exp\Big(in(x-y)\Big) f(y)\, dy$$

(a) let $c_n = 1$: then C becomes a trivial operator . . .;

(b) and if $c_n = in$?

(c) let instead $c_n = 1/2^{|n|}$: show that the functions $g(x) = Cf(x)$ in $\mathtt{Ran}\, C$ have special continuity properties.

(1.66)
Consider the following (apparently similar) operators defined in $L^2(0, 2\pi)$:

$$A_n^{(\pm)}\, f(x) = \frac{1}{2\pi} \int_0^{2\pi} \exp\Big(in(x \pm y)\Big) f(y)\, dy\,, \qquad n \in \mathbf{Z}$$

$$B_n^{(\pm)}\, f(x) = \frac{1}{\pi} \int_0^{2\pi} \sin\Big(n(x \pm y)\Big) f(y)\, dy\,, \qquad n = 1, 2, \ldots$$

$$C_n^{(\pm)}\, f(x) = \frac{1}{\pi} \int_0^{2\pi} \cos\Big(n(x \pm y)\Big) f(y)\, dy\,, \qquad n = 1, 2, \ldots$$

(1) What of these operators are projections? on what subspace?

(2) Find eigenvectors and eigenvalues with their degeneracy of all the above operators.

(3) Study the convergence of the sequences of these operators as $n \to \infty$.

(1.67)
In the space $L^2(-\pi, \pi)$, consider the operators

$$T_N f(x) = \frac{1}{\pi} \sin x \int_{-\pi}^{\pi} \sin y \, f(y)\, dy + \frac{1}{\pi} \sin(2x) \int_{-\pi}^{\pi} \sin(2y)\, f(y)\, dy + \dots$$
$$+ \frac{1}{\pi} \sin(Nx) \int_{-\pi}^{\pi} \sin(Ny)\, f(y)\, dy$$

where $N \geq 1$ is an integer.

(1) Show that T_N is a projection, find its eigenvectors and eigenvalues with their degeneracy.

(2) Study the convergence as $N \to \infty$ of the sequence of the operators T_N and show that also the limit operator T_∞ is a projection.

(3) Are the operators T_N compact operators? and the operator T_∞?

(4) A "variation" (and an abstract version) of this problem: in a Hilbert space H with orthonormal complete system $\{e_n,\ n \in \mathbf{Z}\}$, consider the operators

$$T_N x = \sum_{n=-N}^{N} e_n (e_n, x)\,, \qquad x \in H$$

The same questions (1), (2), (3). What is in this case the operator T_∞? Compare this problem with Problem 1.43.

(1.68)
This problem looks at first sight quite similar to the previous one. This is not the case (what is the main difference?): let $H = L^2(-1, 1)$ and let T_N be defined by (with $N \geq 1$)

$$T_N f(x) = \int_{-1}^{1} f(y)\, dy + x \int_{-1}^{1} y\, f(y)\, dy + \dots + x^N \int_{-1}^{1} y^N f(y)\, dy$$

(1) Find $\mathtt{Ran}\, T_N$ and $\mathtt{Ker}\, T_N$ with their dimensions.

(2) Fix $N = 2$: are the functions $f_0(x) = 1,\ f_1(x) = x,\ f_2(x) = x^2,\ f_3(x) = x^3$ eigenfunctions of T_2?

(3) Is T_2 (and in general T_N) a projection?

(4) Find the eigenvectors and eigenvalues of T_2.

(1.69)
For each integer N, let $\chi_N(x)$ be the characteristic function of the interval $(-N, N)$ i.e., $\chi_N(x) = \begin{cases} 1 \text{ for } |x| < N \\ 0 \text{ for } |x| > N \end{cases}$ and let T_N be the operators in $L^2(\mathbf{R})$

$$T_N\, f(x) = \chi_N(x) \int_{-N}^{N} f(x)\, dx\,, \qquad N = 1, 2, \dots$$

(1) Find T_N^2. Show that T_N is a projection apart from a factor c_N.

(2) Show that $T_M T_N \neq T_N T_M$ (if $N \neq M$, of course).

(3) Find eigenvectors and eigenvalues (with their degeneracy) of $T_1 T_N$ and of $T_N T_1$.

(4) Are there *even* functions $f(x) \in L^2(\mathbf{R})$ such that $T_N(f) = 0$ for all N?

(1.70)
In the space $L^2(0, \infty)$, let $\chi_n(x)$ be the characteristic function of the interval $(n - 1, n)$, $n = 1, 2, \ldots$

(1) Show that, for any $f(x) \in L^2(0, \infty)$, the sequence $c_n = (\chi_n, f) \in \ell^2$.

(2) Consider the operators, for each integer N,

$$T_N\, f(x) \;=\; \sum_{n=1}^{N} \chi_n(x) \int_0^{+\infty} \chi_n(y) f(y)\, dy$$

Fixed $N \geq 1$, find eigenvectors and eigenvalues of T_N, with their degeneracy.

(3) Consider the operator

$$T_\infty\, f(x) \;=\; \sum_{n=1}^{\infty} \chi_n(x) \int_0^{+\infty} \chi_n(y) f(y)\, dy$$

(a) is T_∞ defined in the whole space?

(b) find its norm;

(c) is it compact?

(4) Study the convergence as $N \to \infty$ of the sequence of operators T_N to T_∞.

(5) Consider now

$$S_N\, f \;=\; \sum_{n=1}^{N} \frac{1}{n}\, \chi_n(x) \int_0^{+\infty} \chi_n(y) f(y)\, dy$$

(a) study the convergence as $N \to \infty$ of the sequence of operators S_N to S_∞;

(b) is S_∞ compact?

(1.71)
Let T be the operator in $L^2(-\pi, \pi)$

$$T\ f(x) \;=\; \sum_{n=1}^{\infty} \frac{a_n}{n} \sin\, nx \quad \text{where} \quad a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, \sin nx\, dx$$

(1) Find $T(f)$ if $f(x) = \exp(2ix)$.

(2) Find eigenvectors and eigenvalues of T with their degeneracy. Is T compact?

(3) Is it possible to write T in the form $T = \sum_n \lambda_n P_n$, i.e., as a "discrete" spectral decomposition?

(4) Study $\texttt{Ran}\ T$; clearly, it is contained in the Hilbert subspace of *odd* functions: does it coincide with this subspace or—at least—is dense in it? Is it true that if $g(x) \in \texttt{Ran}\ T$ then $g(x)$ is a continuous function? Is the converse true?

(1.72)
In the space $L^2(-\pi, \pi)$, consider the operator

$$T\ f(x) = \sum_{n=1}^{\infty} \frac{1}{n\pi} \cos nx(\sin nx, f)$$

(1) Find $\|T\|$ and specify if there is some $f_0(x)$ such that $\|Tf_0\| = \|T\|\|f_0\|$.

(2) Find eigenvalues and eigenvectors with their degeneracy.

(3) For each $f(x) \in L^2(-\pi, \pi)$, is the function $g(x) = Tf(x)$ a continuous function? Is its derivative a continuous function? a function in L^2? Is the operator $\frac{d}{dx}T$ a bounded operator?

(4) Does the equation $\big($the unknown is $f(x)\big)$

$$T\ f(x) = \alpha + \beta x^3 + |x|$$

admit solution $\big($it is not requested to obtain $f(x)\big)$ for some values of the constants α, β?

(1.73)
In the space $L^2(0, \pi)$ consider, for each fixed $n = 1, 2, \ldots$, the operator

$$T_n\ f(x) = \cos nx \int_0^{\pi} \Big(\cos ny - \frac{1}{\sqrt{2}}\Big) f(y)\, dy$$

(1) Find $\|T_n\|$. *Hint*: introducing the orthonormal complete system $e_0 = 1/\sqrt{\pi},\ e_n = \sqrt{2/\pi} \cos nx$, write the operator in a more convenient form …

(2) Find $\texttt{Ker}\ T_n$ and $\texttt{Ran}\ T_n$: are they orthogonal?

(3) Find eigenvectors and eigenvalues of T_n. Is it true in this example that $\|T_n\| = \sup |\text{ eigenvalues }|$?

(4) Are there functions $f(x) \in L^2(0, \pi)$ such that $T_n(f) = 0$ for all n?

(5) Study the convergence as $n \to \infty$ of the sequence of operators T_n.

1.2.3 Operators of the Form $T\,f(x) = \varphi(x)\,f(x)$

(1.74)
(1) In the space $H = L^2(0, a)$ ($a \neq \infty$) consider the operator

$$T\,f(x) = x\,f(x)$$

(a) Find $\|T\|$. Show that there is no function $f_0(x) \in H$ such that $\|T\,f_0\| = \|T\|\,\|f_0\|$.

(b) Construct a family of functions $f_\varepsilon(x) \in H$ such that $\sup\limits_{\varepsilon\to 0} \|T\,f_\varepsilon\|/\|f_\varepsilon\| = \|T\|$.

(2) Does T admit eigenvectors? Is its kernel trivial?

(3) Find the spectrum of T (Recall: the *spectrum* of an operator T is the set of the numbers $\sigma \in \mathbf{C}$ such that $T - \sigma\,I$ does *not* admit bounded inverse, or—more explicitly—such that $T - \sigma\,I$ is either not invertible or admits *unbounded* inverse).

(1.75)
Consider the same operator as before in the space $H = L^2(\mathbf{R})$:

(1) Is its domain the whole space H, or—at least—dense in it?

(2) Show that this operator is unbounded: construct a sequence of functions $f_n(x) \in H$ such that $\sup\limits_n \|T\,f_n\|/\|f_n\| = \infty$.

(3) Is its range the whole space H, or—at least—dense in it?

(4) The same questions for the operators $T\,f(x) = x^a f(x)$ with $a > 0$.

(1.76)
Consider the operator in $L^2(\mathbf{R})$ with $\varphi = 1/x$ (which is the inverse of the operator of the previous problem), i.e.,

$$T\,f(x) = \frac{1}{x} f(x)\,, \qquad x \in \mathbf{R}$$

(1) Show that this operator is unbounded: construct a sequence of functions $f_n(x) \in H$ such that $\sup\limits_n \|T\,f_n\|/\|f_n\| = \infty$.

(2) Study the domain and the range of this operator (cf. the previous problem).

(1.77)
Find kernel, domain and range, specifying if domain and range coincide with $H = L^2(\mathbf{R})$, or at least are dense in it (cf. also Problems 1.75 and 1.6) of the operators $Tf = \varphi f$ in each one of the following cases:

$$(a)\ \varphi = \frac{1}{1+x^2}\,; \quad (b)\ \varphi = \frac{x+i}{x-i}\,; \quad (c)\ \varphi = x + |x|\,;$$

$$(d)\ \varphi = \sin x\,; \quad (e)\ \varphi = \sin x^4\,; \quad (f)\ \varphi = \exp(-x^2)\,; \quad (g)\ \varphi = \exp(-1/x^2)$$

(1.78)
In the space $H = L^2(\mathbf{R})$, consider the operator in the general form

$$T\,f(x) = \varphi(x)\,f(x)$$

where $\varphi(x)$ is a given (real or complex) function. Under what conditions on $\varphi(x)$:
(a) is the operator T bounded? Then find its norm;
(b) is a projection?
(c) does admit bounded inverse? Then find $\|T^{-1}\|$;
(d) is unitary?

(1.79)
Consider the operator in $H = L^2(\mathbf{R})$ with $\varphi = x^2/(1+x^2)$, i.e.,

$$T\,f(x) = \frac{x^2}{1+x^2}f(x)$$

(1) Find $\|T\|$. Is there some function $f_0(x) \in H$ such that $\|T\,f_0\| = \|T\|\|f_0\|$?
(2) Does T admit eigenvectors? Is it compact?
(3) Is $\mathtt{Ran}\ T$ coinciding with H or at least dense in it ? And its domain?
(4) Study the convergence as $N \to \infty$ of the sequence of operators T^N, i.e., of the operators $T^N\,f = \varphi^N\,f$.

(1.80)
The same questions as before for the operator $T\,f = \varphi\,f$ in $L^2(\mathbf{R})$ with

$$\varphi(x) = \begin{cases} 0 & \text{for } x < 0 \\ 1 & \text{for } 0 < x < 1 \\ 1/x & \text{for } x > 1 \end{cases}$$

In particular, is the image $\mathtt{Ran}\,T$ of T a Hilbert subspace of $L^2(\mathbf{R})$? is $\mathtt{Ran}\,T$ orthogonal to $\mathtt{Ker}\,T$? is it correct to say that $L^2(\mathbf{R}) = \mathtt{Ker}\,T \oplus \mathtt{Ran}\,T$?

(1.81)
(1) Consider in the space $H = L^2(0,\pi)$ the operator with $\varphi = \exp(ix)$, i.e.,

$$T\,f(x) = \exp(ix)f(x)$$

(a) For what values of $\rho \in \mathbf{C}$ does the operator $T - \rho I$ admit bounded inverse?
(b) Find the norm of $(T + 2iI)^{-1}$
(c) Study the convergence of the sequence of operators T^N as $N \to \infty$.
(2) The same questions if $H = L^2(0, 2\pi)$.
(3) The same questions if $H = L^2(\mathbf{R})$.

(1.82)
In the space $L^2(0, 2\pi)$, consider the operator

$$T\,f(x) \;=\; \Big(1 - \alpha \exp{(ix)}\Big) f(x)\,, \qquad \alpha \in \mathbf{C}$$

(1) Find the adjoint T^+ and specify if T is normal.

(2) Find $\|T\|$. For what values of $\alpha \in \mathbf{C}$ does T admit bounded inverse?

(3) For what values of $\alpha \in \mathbf{C}$ is the series of functions

$$\sum_{n=0}^{\infty} \alpha^n \exp{(inx)}$$

convergent in $L^2(0, 2\pi)$? What is its limit? Is the convergence uniform?

(4) Consider now the operator $S = I - T$, i.e., the operator $S\,f(x) = \alpha \exp(ix)\,f(x)$: show that for $|\alpha| < 1$ the sequence

$$A_N = T \sum_{n=0}^{N} S^n = (I - S) \sum_{n=0}^{N} S^n$$

is norm-convergent to the identity $\big($then one can write

$$\sum_{n=0}^{\infty} S^n = (I - S)^{-1}$$

and $T(I - S)^{-1} = I$, in agreement with the definition $T = I - S\big)$.

1.2.4 Problems Involving Differential Operators

(1.83)
Consider in $L^2(0, 1)$ the operator

$$T \;=\; i\,\frac{d}{dx} \quad \text{with the boundary condition} \quad f(1) = \alpha f(0) \quad (\alpha \in \mathbf{C},\ \alpha \neq 0)$$

(1) For what values of α is T Hermitian (in a suitable domain)?

(2) Find eigenvectors and eigenvalues of T (for generic α). Are the eigenvectors orthogonal? Are the eigenvectors a complete set in $L^2(0, 1)$?

(3) Is the domain of T dense in $L^2(0, 1)$?

(4) Check the correctness of the answer given to (1) by comparison with the properties obtained in (2) (the eigenvectors are orthogonal and the eigenvalues real if ...).

Compare with Problems 1.32, 1.33 and in particular 1.34 for a discussion about existence and uniqueness of the solutions of Problems 1.84–1.90. Clearly, the independent variable x can be replaced by the time variable t (as done in Problems 1.32–1.34), but to avoid too frequent changes in the notations, we continue to use here x as independent variable.

(1.84)
Consider in $H = L^2(-\pi, \pi)$ the operator

$$T = \frac{d}{dx} + \alpha I \quad \text{with periodic boundary conditions}: \ u(-\pi) = u(\pi)$$

with $\alpha \in \mathbf{C}$, $\alpha \neq 0$.

(1) Find eigenvalues and eigenvectors of T.

(2) Consider the equation $Tf = g$, i.e.,

$$f' + \alpha f = g$$

where $g(x) \in H$ is given and $f(x) \in H$ the unknown. Expand $f(x)$ and $g(x)$ in Fourier series with respect to the eigenvectors found in (1). Under what condition on the constant α does this equation admit solution? In this case, is the solution unique?

(3) Let $\alpha = 2i$: under what condition on the Fourier coefficients g_n of $g(x)$ does the equation admit solution? In this case, show that the solution is not unique and write as a Fourier series the most general solution.

(4) Show that T is invertible and find $\|T^{-1}\|$ in each one of the cases

$$\alpha = 1, \quad \alpha = 2 + i, \quad \alpha = 1 + 2i, \quad \alpha = 3i/2$$

(1.85)
Consider in the space $H = L^2(0, \pi)$ the operator

$$T = \frac{d^2}{dx^2} \quad \text{with vanishing boundary conditions}: f(0) = f(\pi) = 0$$

(1) Show that this operator is Hermitian, i.e., $(g, Tf) = (Tg, f)$ in the dense domain of doubly differentiable functions in L^2, satisfying the above boundary conditions.

(2) Find the adjoint operator T^+ and verify that its domain is larger than the one of T (the boundary conditions are different).

(3) Solve the equation $\big($where $g(x) \in H$ is given and $f(x) \in H$ the unknown$\big)$

$$\frac{d^2}{dx^2}f = g$$

expanding both $g(x)$ and $f(x)$ in Fourier series with respect to the complete set $\{\sin nx,\ n = 1, 2, \ldots\}$, and verify that the equation admits a unique solution.

(4) Compare with the next problem.

(1.86)
Consider in the space $H = L^2(-\pi, \pi)$ the operator

$$T = \frac{d^2}{dx^2} \quad \text{with periodic boundary conditions, i.e.,}$$

$$f(-\pi) = f(\pi),\ f'(-\pi) = f'(\pi)$$

(1) The same question (1) as in the above problem, now with periodic boundary conditions.

(2) What about T^+ and its domain?

(3) Solve the equation $\big(g(x) \in H$ is given, $f(x) \in H$ the unknown$\big)$

$$\frac{d^2}{dx^2}f = g$$

now expanding both $g(x)$ and $f(x)$ in Fourier series with respect to the complete set $\{e_n = \exp(inx),\ n \in \mathbf{Z}\}$, which is a complete set consistent with the periodic boundary conditions. What about the existence and uniqueness of the solution? What condition must be imposed to the function $g(x)$ in order that some solution exists?

(1.87)
In the space $H = L^2(0, \pi)$ consider the operator $T = -d^2/dx^2 + I$ and the equation $Tf = g$, i.e., $\big(g(x) \in H$ is given and $f(x) \in H$ the unknown$\big)$

$$-f'' + f = g$$

with vanishing boundary conditions $f(0) = f(\pi) = 0$. Expand $g(x)$ and $f(x)$ in Fourier series with respect to the orthogonal complete set $\{\sin nx,\ n = 1, 2, \ldots\}$.

(1) Show that the equation admits a unique solution for any $g(x) \in H$.

(2) Find $\|S\|$, where S is the operator defined by (S is the inverse of the above operator T)

$$S g = f$$

(3) Let $g(x) \in H$ and let g_N be a sequence of functions "approximating" $g(x)$ in the norm L^2 (i.e., $\|g - g_N\|_{L^2} \to 0$ as $N \to \infty$). Let $f(x)$ and $f_N(x)$ be the

corresponding solutions of the differential equation given before, i.e., $f(x) = Sg(x)$, $f_N(x) = Sg_N(x)$. Is it true that also $f_N(x)$ approximate $f(x)$ in the L^2 norm?

(4) If, in particular, g_N are obtained "truncating" the Fourier expansion of g (i.e., g_N is the Nth partial sum of the Fourier series), show that f_N converges "rapidly" to the solution f, meaning that there are constants $C_N \to 0$ as $N \to \infty$ such that

$$\|f - f_N\| \leq C_N \|g - g_N\|$$

(1.88)
In the space $L^2(-\pi, \pi)$ consider the operator

$$T\,f = \frac{d^2 f}{dx^2} + f \quad \text{with periodic boundary conditions}$$

as in Problem 1.86

(1) Find eigenvalues and eigenvectors of T with their degeneracy.

(2) Determine $\mathtt{Ker}\,T$.

(3) Consider the equation $\big(f(x)$ is the unknown and $g(x) \in L^2(-\pi, \pi)$ is given$\big)$

$$T\,f = g$$

Use expansions in Fourier series in terms of the eigenvectors found in (1). What condition must be imposed to the function $g(x)$ in order that some solution exists? For instance, does this equation admit solution if $g(x) = \cos^4 x$? and if $g(x) = \cos^3 x$? When the solution exists, is it unique? Write in the form of a Fourier series the most general solution.

(4) Considering more in general the operator

$$T_\alpha\,f = \frac{d^2 f}{dx^2} + \alpha f$$

for what $\alpha \in \mathbf{C}$ does T_α admit bounded inverse?

(5) Find $\|T_\alpha^{-1}\|$ if $\alpha = 1 + ia,\ a \in \mathbf{R}$.

(1.89)
In the space $H = L^2(0, 2\pi)$, consider the operator

$$T = \frac{d^2}{dx^2} \quad \text{with the boundary conditions } f'(0) = f'(2\pi) = 0$$

(1) Is T Hermitian (in a suitable domain)? Find eigenfunctions and eigenvalues of T. Show that the eigenfunctions provide an orthogonal complete system in H. Specify $\mathtt{Ker}\ T$ and $\mathtt{Ran}\ T$.

(2) Using the Fourier expansion in terms of the eigenfunctions of T, specify for what $g(x) \in H$ the equation

$$T f = g$$

admits solution $f(x) \in H$. Is the solution unique? Given a $g(x)$ such that this equation admits solution, write in the form of a Fourier series the most general solution $f(x)$.

(3) Is the solution obtained in (2) a continuous function? Is it possible to find a constant K such that, for any $x \in (0, 2\pi)$,

$$\left|\frac{df}{dx}\right| \le K\|g\|_{L^2} \ ?$$

(1.90)
In the space $H = L^2(0, \pi)$, consider the equation, where $f(x)$ is the unknown function and $\varphi(x)$ is given,

$$\frac{d^2 f}{dx^2} = \varphi(x) \quad \text{with vanishing boundary conditions}$$

(1) Write, using Fourier expansion in terms of the set $\{\sin nx,\ n = 1, 2, \ldots\}$, the (unique) solution $f(x)$. Find a constant C such that $\|f\| \le C\|\varphi\|$.

(2) Show that

$$\sup_{0\le x\le\pi} |f(x)| \le \left|(G, \Phi)\right|$$

where $G = G(x) \in H$ does not depend on φ, and $\Phi = \sum_{n\ge1} |\varphi_n| \sin nx$ where φ_n are defined by the Fourier expansion $\varphi = \sum_n \varphi_n \sin nx$.

(3) Let $\varphi_\alpha(x)$ be a family of functions $\in H$ and let $f_\alpha(x)$ be the corresponding solutions; assume that $\varphi_\alpha(x) \to \psi(x)$ in the norm L^2, i.e., $\|\varphi_\alpha - \psi\|_{L^2} \to 0$ as $\alpha \to 0$, and let $h(x)$ be the solution corresponding to $\psi(x)$, i.e., $d^2h/dx^2 = \psi(x)$. It is true that $f_\alpha(x) \to h(x)$, and in what sense?

(1.91)
Consider the operator in $L^2(0, \pi)$

$$T = -\frac{d^2}{dx^2} + c\frac{d}{dx} \quad (c \in \mathbf{R}) \quad \text{with the boundary conditions } u(0) = u(\pi) = 0$$

(1) Find the eigenvalues and eigenvectors of T. Are the eigenvectors a complete set in $L^2(0, \pi)$?

(2) Does T admit a bounded inverse? If yes, calculate $\|T^{-1}\|$.

(3) Show that the eigenvectors are orthogonal with respect to the weight function $\rho = \exp(-cx)$, i.e., with respect to the scalar product defined by

$$(u, v)_\rho = \int_0^\pi \rho(x) u^*(x) v(x) dx \ .$$

(4) Verify that the problem is actually a classical Sturm–Liouville Problem

$$-\frac{1}{\rho}\left[\frac{d}{dx}\left(p(x)\frac{d}{dx}\right) + q(x)\right] = \lambda\, u$$

with $p = q = \ldots$: this confirms the orthogonality property seen before.

(1.92)
In the Hilbert space $L^2(1, \sqrt{3})$, consider the operator

$$T = x\frac{d}{dx} \quad \text{with the boundary condition } u(1) = u(\sqrt{3})$$

(1) Find the eigenvalues λ_n and eigenvectors $u_n(x)$ of T.

(2) Show that the eigenvectors $u_n(x)$ of T are not orthogonal, but that they turn out to be orthogonal with respect to the modified scalar product obtained introducing the weight function $\rho(x) = x$:

$$(f,\ g)_\rho = \int_1^{\sqrt{3}} x\, f^*(x) g(x)\, dx$$

Are the eigenvectors a complete set in H?

(3) Considering the expansion

$$f(x) = \sum_n c_n u_n(x)$$

give the formula for obtaining the coefficients c_n $\big($use the results seen in (2)$\big)$.

(4) Calculate $\sum_n |c_n|^2$ if in the above expansion $f(x) = x$.

(5) Does the expansion given in (3) provide a periodic function out of the interval $(1,\ \sqrt{3})$? If this expansion at the point, e.g., $x = 3/2$, converges to some value, at what points, out of the interval, does it take the same value?

(1.93)
(1) (*a*) In the space $L^2(0, +\infty)$ find, using integration by parts, the adjoint of the operator

$$A = x\frac{d}{dx}$$

(in a suitable dense domain of smooth functions chosen in such a way that the "finite part" $\Big[\dots\Big]$ of the integral is zero).

(*b*) Show that the operator

$$\widetilde{A} = x\frac{d}{dx} + \frac{1}{2}I$$

is anti-Hermitian (i.e., $\widetilde{A}^+ = -\widetilde{A}$).

(2) Consider now the operator

$$T_\alpha f(x) = f\Big((\exp\alpha)x\Big) , \qquad \alpha \in \mathbf{R}$$

(*a*) Find $\|T_\alpha\|$.

(*b*) Show that one can find a real coefficient $c(\alpha)$ in such a way that the operator $\widetilde{T}_\alpha = c(\alpha)T_\alpha$ is unitary.

(3) Find the operators B and $\widetilde{B}$ defined by

$$\frac{d}{d\alpha}T_\alpha f\Big|_{\alpha=0} = Bf \quad \text{and} \quad \frac{d}{d\alpha}\widetilde{T}_\alpha f\Big|_{\alpha=0} = \widetilde{B}f$$

compare with the operators A and $\widetilde{A}$ defined in (1) and interpret the results in terms of Lie groups and algebras of transformations (see also Problem 4.19).

(1.94)
Consider in the space $L^2(Q)$, where $Q = \{0 \le x \le \pi,\ 0 \le y \le \pi\}$ is the square of side π, the Laplace operator

$$T = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

with vanishing boundary conditions on the four sides of the square.

(1) Find eigenvalues and eigenvectors of T with their degeneracy.

(2) Show that the operator is invertible and find $\|T^{-1}\|$.

(3) Does the equation $\big($where $g(x, y)$ is given and $f(x, y)$ the unknown$\big)$

$$T\,f = g$$

admit solution for any $g(x, y) \in L^2(Q)$? Is the solution unique? (Use the Fourier expansion in terms of the eigenvectors of T).

(4) If $g(x, y) = g(y, x)$, is the same property shared by the solution $f(x, y)$?

(5) Let S be the operator $Sf(x, y) = f(y, x)$. Find the common eigenvectors of T and S. Do they provide an orthogonal complete system in the space $L^2(Q)$?

(1.95)
Let Q be the square $\{0 \le x \le 2\pi,\ 0 \le y \le 2\pi\}$ and let T be the operator defined in $L^2(Q)$

$$T = \Big(\frac{\partial}{\partial x} - \alpha\Big)\Big(\frac{\partial}{\partial y} - \beta\Big), \qquad \alpha, \beta \in \mathbf{C}$$

with periodic boundary conditions:

$$u(x, 0) = u(x, 2\pi)\,; \qquad u(0, y) = u(2\pi, y)$$

(1) For what values of α, β is T Hermitian (in a suitable dense domain)?

(2) Using separation of the variables, find eigenvectors and eigenvalues of T (with arbitrary α, β); do the eigenvectors provide an orthogonal complete system for $L^2(Q)$?

(3) For what values of α, β does $\mathtt{Ker}\, T \neq \{0\}$?

(4) Fix now $\alpha = \beta = 0$. Using the results obtained in (2), specify for what $g(x, y) \in L^2(Q)$ the equation

$$Tf - f = g$$

admits solution $f(x, y) \in L^2(Q)$. Is the solution unique? Find the most general solution (if existing) with $g(x, y) = \sin 2(x + y)$.

(1.96)
Let Q be the square $\{0 \le x \le 2\pi,\ 0 \le y \le 2\pi\}$ and let T be the operator defined in $L^2(Q)$

$$T = \frac{\partial}{\partial x} + \alpha \frac{\partial}{\partial y}\,, \qquad \alpha \in \mathbf{R},\ \alpha \neq 0$$

with periodic boundary conditions (as in the previous problem).

(1) Using separation of the variables, find eigenvectors and eigenvalues of T; do the eigenvectors provide an orthogonal complete system for $L^2(Q)$?

(2) (a) Let $\alpha = 1$: find $\mathtt{Ker}\, T$, its dimension and find an orthogonal complete system for it.

(b) The same question if $\alpha = \sqrt{2}$.

(3) Using the results obtained in (2)(a), write, as a Fourier expansion, the solution of the equation

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0$$

satisfying the condition

$$u(x,0) = \begin{cases} 1 & \text{for} \quad 0 < x < \pi \\ 0 & \text{for} \quad \pi < x < 2\pi \end{cases}$$

Find in particular $u(x,\pi)$.

(4) Observing that the most general solution of the equation $u_x + u_y = 0$ is $u(x,y) = F(x-y)$ where F is arbitrary, it is easy to write the solution of the above equation satisfying the generic condition

$$u(x,0) = \varphi(x) \in L^2(0,2\pi).$$

1.2.5 Functionals

(1.97)
Study each one of the following functionals Φ: specify if the functional is bounded or not. In the case of bounded functionals, find their norm, the representative vector according to Riesz theorem, and the kernel. In the case of unbounded functionals, study their domain and kernel, which provide examples of *dense* subspaces in the Hilbert space. Therefore, these functionals are examples of not closed operators, see Problem 1.63, q. (4).

(1) In a Hilbert space with orthonormal complete system $\{e_n\,, n = 1, 2, \ldots\}$, and with $x = \sum_{n=1}^{\infty} a_n e_n$ consider

(a)

$$\Phi(x) = \sum_{n=1}^{N} a_n \quad (N \geq 1) \quad \text{and} \quad \Phi(x) = \sum_{n=1}^{\infty} a_n$$

Recall Problems 1.16 and 1.55 to verify explicitly that the second functional is not a closed operator.

(b) For what sequences of complex numbers c_n, is the functional

$$\Phi(x) = \sum_{n=1}^{\infty} c_n a_n$$

a bounded functional?

(2) In $L^2(a,b)$ with $-\infty < a < b < \infty$ consider

$$\Phi(f) = \int_a^b f(x)\,dx \quad \text{and} \quad \Phi(f) = \int_a^b \exp(x) f(x)\,dx$$

(3) In $L^2(\mathbf{R})$ consider

(a)

$$\Phi(f) = \int_{-\infty}^{+\infty} \exp(-|x|) f(x)\,dx$$

(b)

$$\Phi(f) = \int_{-\infty}^{+\infty} f(x)\,dx \quad \text{and} \quad \Phi(f) = \int_{-\infty}^{+\infty} \sin x\, f(x)\,dx$$

(recall Problems 1.4 and 3.8 for what concerns the kernels)

(4) In $L^2(-1,1)$ consider, for different values of $\alpha, \beta, \gamma \in \mathbf{R}$,

$$\Phi(f) = \int_{-1}^{1} |x|^\alpha f(x)\,dx\ ; \ \int_0^{+\infty} x^\beta f(x)\,dx\ ; \ \int_{-\infty}^{+\infty} |x|^\gamma f(x)\,dx$$

For what values of α, β, γ are these functionals bounded? (cf. Problem 1.75).

(5) In $L^2(I)$, where I is any interval, consider

$$\Phi(f) = f(x_0)$$

which is defined in the subspaces of the functions continuous in a neighborhood of $x_0 \in I$, see also Problem 1.6.

(1.98)
In a Hilbert space with orthonormal complete system $\{e_n\}$, $n = 1, 2, \ldots$, consider

$$\Phi_n(x) = a_n = (e_n, x)$$

Study the convergence as $n \to \infty$ of the sequence of functionals Φ_n.

(1.99)
Let $\Phi_a(f)$ be the functional defined in $L^2(0,1)$

$$\Phi_a(f) = \frac{1}{\sqrt{a}} \int_0^a f(x)\,dx\,, \qquad 0 < a < 1$$

(1) Find $\|\Phi_a\|$.

(2) Show that $\Phi_a \to 0$ strongly (but not in norm) as $a \to 0$. *Hint*: restrict first to the dense subspace of continuous functions, then ....

(1.100)
Let $\Phi_a(f)$ be the functional defined in $L^2(0,+\infty)$

$$\Phi_a(f) = \frac{1}{\sqrt{a}} \int_0^a f(x)\,dx\,, \qquad a > 0$$

(1) Find $\|\Phi_a\|$.

(2) Show that $\Phi_a \to 0$ strongly (but not in norm) as $a \to +\infty$. *Hint*: restrict first to the dense subspace of functions with compact support, then ….

1.2.6 Time-Evolution Problems: Heat Equation

(1.101)
Consider the heat equation, also called diffusion equation, in $L^2(0,\pi)$

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}\,, \qquad u = u(x,t)$$

with vanishing boundary conditions

$$u(0,t) = u(\pi,t) = 0$$

and the (generic) initial datum

$$u(x,0) = f(x) \in L^2(0,\pi)$$

(1) Find the time-evolution $u = u(x,t)$ for $t > 0$. *Hint*: write $f(x)$ and $u(x,t)$ as Fourier expansions in terms of the orthogonal complete system $\{\sin nx,\ n = 1,2,\ldots\}$.

(2) Show that the solution $u(x,t)$ tends to zero $\big($in the $L^2(0,\pi)$ norm$\big)$ as $t \to +\infty$. How "rapidly" does it tend to zero?

(3) Show that for any $t > 0$ the solution $u(x,t)$ is infinitely differentiable with respect to x and to t.

(1.102)
Considering the time-evolution problem proposed above, let E_t be the "time-evolution operator" defined by

$$E_t\ :\ u(x,0) \to u(x,t)\,, \qquad t > 0$$

(1) Find eigenvalues and eigenvectors of E_t.

(2) Calculate $\|E_t\|$ and verify that $\|E_t\| \to 0$ as $t \to +\infty$.

(3) Show that $E_t \to I$ in the strong sense, not in norm, as $t \to 0^+$.

(1.103)
Consider the heat equation in $L^2(-\pi, \pi)$, now with periodic boundary conditions

$$u(-\pi, t) = u(\pi, t), \quad u_x(-\pi, t) = u_x(-\pi, t)$$

and initial condition

$$u(x, 0) = f(x) \in L^2(-\pi, \pi)$$

(1) Find the time-evolution $u(x, t)$: use now Fourier expansions with respect to the orthogonal complete system $\{\exp(inx),\ n \in \mathbf{Z}\}$.

(2) What happens as $t \to +\infty$?

(3) If $\int_{-\pi}^{\pi} u(x, 0)dx = 0$, is this property preserved for all $t > 0$, i.e., $\int_{-\pi}^{\pi} u(x, t)dx = 0, \forall t > 0$?

(4) Study the convergence as $t \to +\infty$ and as $t \to 0^+$ of the operator E_t defined in the previous problem.

(1.104)
Consider the case of a nonhomogeneous equation in $L^2(-\pi, \pi)$ of the form

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + F(x)$$

with periodic boundary conditions as in previous problem, and initial datum

$$u(x, 0) = f(x)$$

where $F(x)$ and $f(x) \in L^2(-\pi, \pi)$. Find (in the form of a Fourier expansion) the time-evolution $u(x, t)$. *Hint*: write $F(x)$ and $f(x)$ in the form of a Fourier expansion: $F = \sum_n F_n \exp(nix)$, $f = \sum_n f_n \exp(nix)$ and look for the solution writing $u(x, t) = \sum_n a_n(t) \exp(nix)$, then deduce a differential equation for $a_n(t)$.

(1.105)
(1) Consider an orthonormal complete system $\{e_n,\ n = 1, 2, \ldots\}$ in a Hilbert space H and let T be the linear operator defined by

$$T e_n = -n^2 e_n$$

Find the time evolution of the problem for $v = v(t) \in H$

$$\frac{d}{dt} v = T v$$

with the generic initial condition $v(0) = v_0 \in H$.

(2) The same problem with $n \in \mathbf{Z}$.

(3) Study the properties of the time-evolution operator E_t defined by $E_t : v_0 \to v(t)$. (This is just the "abstract" version of Problems 1.101 and 1.103).

(1.106)
Consider the time-evolution problem, in a Hilbert space H where $\{e_n,\ n \in \mathbf{Z}\}$ is an orthonormal complete system

$$\frac{d}{dt}v = Tv\,, \qquad v = v(t) \in H$$

with the linear operator T defined by

$$T\,e_0 = e_0 \quad ; \quad T\,e_n = e_{-n}$$

(1) Find the time evolution if the initial condition is given by $v(0) = e_0$.

(2) The same if $v(0) = e_1$.

(3) Extend to the case where $v(0)$ is a generic vector $v \in H$.

(1.107)
The same problem as before with

$$T\,e_n = n\,e_{-n}$$

(1) Find the time evolution if the initial condition is given by $v(0) = e_0$.

(2) The same if $v(0) = e_n$ (with fixed $n \neq 0$).

(3) Extend to the case where $v(0)$ is a generic vector $v \in H$.

(1.108)
In the space $L^2(0, 2\pi)$, consider the operator

$$T = \frac{d}{dx} \quad \text{with the boundary condition} \quad f(2\pi) = -f(0)$$

(1) Find the eigenvectors $u_n(x)$ and the eigenvalues of T. Are the eigenvectors orthogonal? Show that the eigenvectors are a complete set in $L^2(0, 2\pi)$.

(2) Using the series expansion $\sum_n a_n(t)u_n(x)$, write in the form of series the solution of the equation

$$\frac{d}{dt}f = T\,f \quad \text{with the generic initial condition} \quad f(x,0) = f_0(x) \in L^2(0, 2\pi)$$

(3) Show that this solution is periodic in time: what is the period?

(4) Let E_t be the "time-evolution" operator

$$E_t \,:\, f(x,0) \to f(x,t)$$

verify that the eigenvectors of T are also eigenvectors of E_t. Is E_t unitary?
(5) Study the convergence of the operators E_t to the identity operator I as $t \to 0$.

(1.109)
(1) Let A be the 2×2 matrix

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

Evaluate $\exp(At)$ and solve the time-evolution problem

$$\dot{u} = Au, \qquad u = u(t) \in \mathbf{R}^2$$

with the generic initial condition $u(0) = a \in \mathbf{R}^2$.
(2) Generalize: consider a Hilbert space H with orthonormal complete system $\{e_n,\ n = 1, 2, \ldots\}$ and the linear operator B defined by

$$B\, e_n = \begin{cases} e_n + e_{n+1} & \text{for} \quad n = 1, 3, 5, \ldots \\ e_n \quad \text{for} \quad n = 2, 4, \ldots \end{cases}$$

(a) evaluate $\exp(Bt)$. *Hint*: it can be useful to write $B = I + \widetilde{B}$, then $\widetilde{B}^2 = \ldots$;
(b) find explicitly the solution of the time-evolution problem, where $u = u(t) \in H$,

$$\dot{u} = Bu$$

if $u(0) = u_0 = e_1$; if $u_0 = e_1$;
(c) write the solution with a generic initial condition $u(0) = u_0 \in H$.

(1.110)
Consider the heat equation in $L^2(0, \pi)$ with "wrong" sign:

$$\frac{\partial^2 u}{\partial x^2} = -\frac{\partial u}{\partial t}$$

(or, equivalently, the heat equation after time inversion) with vanishing boundary conditions $u(0, t) = u(\pi, t) = 0$. Let $f(x) \in L^2(0, \pi)$ denote the initial condition: $f(x) = u(x, 0)$.
(1) For what initial condition $f(x)$ does the L^2-norm of the solution $u(x, t)$ remain bounded for all $t > 0$? (i.e., does a constant C exist such that $\|u(x, t)\| < C,\ \forall t > 0$?)
(2) Show that there is a dense set of initial conditions $f(x)$ such that for each finite $t > 0$ the solution exists $\big($i.e., $u(x, t) \in L^2(0, \pi)$ for each fixed $t > 0\big)$.
(3) Give an example of a continuous $f(x)$ such that the solution does not exist $\big($i.e., $\notin L^2(0, \pi)\big)$ for any $t > 0$.

(4) Let $f_N(x)$ be a sequence of functions converging to zero in the L^2-norm, i.e., $\|f_N(x)\|_{L^2(0,\pi)} \to 0$ as $N \to \infty$. Does this imply that the same is true for the corresponding solutions $u_N(x,t)$ for each fixed $t > 0$?

(1.111)
Consider the d'Alembert equation, also called wave equation,

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}\,, \qquad u = u(x,t),\ 0 \le x \le \pi,\ t \in \mathbf{R}$$

describing, e.g., the displacements of a vibrating string. Assume vanishing boundary conditions

$$u(0,t) = u(\pi,t) = 0$$

and initial data of the form

$$u(x,0) = f(x) \in L^2(0,\pi) \quad \text{and} \quad u_t(x,0) = 0$$

(1) Show that the solution can be written in the form

$$u(x,t) \;=\; \sum_{n=1}^{\infty} a_n \sin nx\, \cos nt$$

where $a_n = \ldots$.

(2) Let now

$$u(x,0) \;=\; f(x) \;=\; \begin{cases} \sin 2x & \text{for } 0 \le x \le \pi/2 \\ 0 & \text{for } \pi/2 \le x \le \pi \end{cases}$$

Expanding $f(x)$ in Fourier series in terms of the orthogonal complete system $\{\sin nx,\ n = 1, 2, \ldots\}$, show that the Fourier expansion of the solution has the form

$$u(x,t) = (1/2)\sin 2x\, \cos 2t + u_1(x,t)$$

where $u_1(x,t) = \ldots$. Comparing with the graph of $u(x,0)$, it is easy to deduce the graph of $u_1(x,0)$. Show that $u_1(x,\pi) = -u_1(x,0)$ and deduce $u(x,\pi)$. Comparing then the graphs of $u(x,0)$ and of $u(x,\pi)$ confirm a well-known property of waves propagating in an elastic string with vanishing boundary conditions.

1.2.7 Miscellaneous Problems

(1.112)
Let $V \subset H$ be an invariant subspace under a (bounded) operator T in a Hilbert space, i.e., $T : V \to V$.

(1) Show that $T^+ : V^\perp \to V^\perp$, where $V^\perp$ is the orthogonal complementary subspace to V.

(2) Show that if T admits an eigenvector v, it is *not* true in general that v is also eigenvector of T^+ (a counterexample where T is a 2×2 matrix is enough).

(3) Show that if T is normal, i.e., if $TT^+ = T^+T$, then

(a) if T admits an eigenvector v, then v is also eigenvector of T^+ (with eigenvalue ...). *Hint*: start from $\|(T - \lambda I)v\| = 0$;

(b) if T admits two eigenvectors with different eigenvalues, then these eigenvectors are orthogonal.

(4) Assume that T admits a complete set of orthonormal eigenvectors: show that

(a) T is Hermitian if and only if the eigenvalues are real numbers;

(b) T is unitary if and only if the eigenvalues λ satisfy $|\lambda| = 1$.

(1.113)
(1) It is well known that unitary operators in any Hilbert space H map orthonormal complete systems into orthonormal complete systems. Conversely, show that an operator $T : H \to H$ is unitary if it maps an orthonormal complete system $\{e_n\}$ into an orthonormal complete system $\{v_n\}$ $\big($show first that the domain and the range of T coincide with H, and then that $(Tx, Ty) = (x, y)$, $\forall x, y \in H$, or—more simply, as well known—that $\|Tx\| = \|x\|\big)$.

(2) Let $x \in H$: is the series

$$\sum_n (e_n, x)\, v_n$$

convergent in H? to what vector?

(3) Let T be the operator which maps the canonical orthonormal complete system[3] $\{e_n,\ n = 1, 2, \ldots\}$ in ℓ^2 into the orthonormal complete system w_n defined in Problem 1.16. Find $T^{-1}e_n = T^+e_n$.

(1.114)
Let H_1, H_2 be two Hilbert subspaces of a Hilbert space and let P_1, P_2 be the corresponding projections.

(1) Under what condition on $P_1,\ P_2$ is $T = P_1 + P_2$ a projection? on what subspace?

(2) Under what condition on $P_1,\ P_2$ is $T = P_1 P_2$ a projection? on what subspace?

[3] I.e., $e_1 = (1, 0, 0, \ldots),\ e_2 = (0, 1, 0, 0, \ldots)$, etc.

(1.115)
(1) What condition ensures that a projection is a compact operator?

(2) Is it true that an operator having finite-dimensional range is compact? is bounded?

(3) Show that any bounded operator B maps any weakly convergent sequence of vectors into a weakly convergent sequence. If B is bounded and C is compact, is it true that BC and CB are compact operators?

(1.116)
Let T be a bounded operator in a Hilbert space admitting a "discrete" spectral decomposition

$$T = \sum_{n=1}^{\infty} \lambda_n P_n$$

with standard notations. Show that the series $\sum_{n=1}^{\infty} \lambda_n P_n$ (more correctly: the partial sum $T_N = \sum_{n=1}^{N} \lambda_n P_n$) is norm-convergent if the eigenvalues satisfy $|\lambda_n| \to 0$, and strongly convergent if λ_n are bounded: $|\lambda_n| < M$.

(1.117)
(1) Let A be an operator in a Hilbert space H and consider the bilinear form defined by

$$< u, v > = (u, Av), \qquad u, v \in H$$

Under what conditions on the operator A does this linear form define a scalar product in H?

(2) Assume that A, in addition to the conditions established in (1), is a bounded operator: show that if a sequence of vectors u_n is a Cauchy sequence with respect to the norm induced by the usual scalar product (,), then it is also a Cauchy sequence with respect to the norm induced by the scalar product defined by the bilinear form $< \ , \ >$ given in (1). Is the converse true? What changes if A is unbounded?

(1.118)
In the Hilbert space $L^2(\mathbf{R})$, consider the operator

$$T_a f(x) = \begin{cases} -f(x) & \text{for } x < a \\ f(x) & \text{for } x > a \end{cases}, \qquad a \in \mathbf{R}$$

(1) Find the eigenvalues and eigenvectors of T.

(2) Study the convergence of the family of operators T_a as $a \to \infty$.

(3) Let $a, b \in \mathbf{R}$ with $a \neq b$. Is it possible to write the operator $T_a T_b$ as a combination of projections?

(4) Is it possible to have an orthogonal complete system of simultaneous eigenvectors of T_a and T_b?

(1.119)
In the space $H = L^2(0, 2\pi)$ let $v_n = \exp(inx)$, $n \in \mathbf{Z}$, and let T be the operator

$$T\, f(x) \;=\; \frac{1}{\pi}\int_0^{\pi} f(x+y)\,dy$$

where the functions $f(x)$ are periodically prolonged with period 2π.

(1) Find $T(v_n)$, and then the eigenvectors and eigenvalues (with their degeneracy) of T (do not forget the case $n = 0$!)

(2) Show that $T(f)$ is a continuous function (see Problem 1.25).

(3) Is T compact?

(4) Find $\|T^N\|$ where N is any integer.

(5) Study the convergence of the sequence of operators T^N as $N \to \infty$.

(1.120)
Let T be the operator in $L^2(\mathbf{R})$

$$T\, f(x) \;=\; \alpha f(x) + \beta f(-x)\,, \qquad \alpha, \beta \in \mathbf{C}\,;\; \alpha, \beta \neq 0$$

(1) Find $\|T\|$

(2) Find eigenvalues and eigenvectors of T. Is there an orthogonal complete system of eigenvectors of T?

(3) Under what conditions on $\alpha,\ \beta$ is T unitary? Let, e.g., $\alpha = 1/\sqrt{2}$: find β in order to have T unitary.

(4) Let $\alpha = 2/3,\ \beta = 1/3$: study the convergence as $n \to \infty$ of the sequence of operators T^n.

(1.121)
In the Hilbert space $H = L^2(0, +\infty)$, consider the two operators

$$T_n \;=\; \frac{x}{1+x^2}\, P_n \quad \text{and} \quad S_n \;=\; \sin \pi x\; P_n\,, \qquad n = 1, 2, \ldots$$

where P_n is the projection on the Hilbert subspace $L^2(0, n)$.

(1) Specify if T_n and S_n are Hermitian. Find $\|T_n\|$ and $\|S_n\|$.

(2) (a) Fixed n, for what $g(x) \in H$ does the equation $\big($with $f(x) \in H$ the unknown$\big)$

$$T_n\, f \;=\; g$$

admit solution? Is the solution (when existing) unique?

(b) Is $\mathtt{Ran}\, T_n$, the image of T_n, a Hilbert subspace of H? Is it true that $H = \mathtt{Ker}\, T_n \oplus \mathtt{Ran}\, T_n$?

(3) The same question as in (2)(a) for the equation

$$T_n\, f \;=\; S_n\, g$$

(4) Study the convergence as $n \to \infty$ of the two sequences of operators T_n and S_n.

(1.122)
In the space $H = L^2(0, 2\pi)$, with the functions periodically prolonged with period 2π out of this interval, let T be the operator

$$T\, f(x) \;=\; f(x - \pi/2)$$

(1) Find T^4. What information can be deduced about the eigenvalues of T?

(2) Find eigenvalues and eigenvectors of T with their degeneracy. Is $|\sin 2x|$ an eigenfunction of T?

(3) Let $S = T + T^2$. Show that $S = I + T^{-1}$. Find $\mathtt{Ker}\, S$ and $\mathtt{Ran}\, S$ $\big($expand in terms of the eigenvectors obtained in (2)$\big)$. Is it true that $\mathtt{Ran}\, S$ is a Hilbert subspace of H and that $H = \mathtt{Ran}\, S \oplus \mathtt{Ker}\, S$?

(1.123)
Let T be the operator defined in the space $L^2(-\pi, \pi)$ $\big($the functions must be periodically prolonged with period 2π out of the interval $(-\pi, \pi)\big)$

$$T\, f(x) \;=\; \frac{1}{2i}\left(f\left(x + \frac{\pi}{2}\right) - f\left(x - \frac{\pi}{2}\right)\right)$$

(1) Show that T is a combination of two unitary operators.

(2) Find eigenvectors and eigenvalues (with their degeneracy) of T (use the complete system $\{e_n = \exp(inx),\ n \in \mathbf{Z}\}$).

(3) Show that T is a combination of two projections (on what subspaces?)

(4) Find $\|T\|$, $\|T + (1 + 2i)I\|$, and $\|\Big(T + (1 + 2i)I\Big)^{-1}\|$

(5) Find the eigenvalues and the norm of $T^m + T^{m+1}$ for each $m \geq 1$.

(1.124)
In the space $H = L^2(-1, 1)$, let T be the operator

$$T\, f(x) \;=\; \int_{-1}^{1} K(x, y)\, f(y)\, dy \quad \text{where} \quad K(x, y) = 1 + xy$$

(1) Find $\mathtt{Ran}\, T$, specify its dimension, and find an orthonormal complete system for it.

(2) Replace now (in this question) K with $K_{AB} = A + B\,xy$ where A, B are constants. With $B = 0$, find A in such a way that the corresponding operator T_{AB} is a projection; with $A = 0$, find B in such a way that T_{AB} is a projection. Is it possible to choose A, B both nonzero in such a way that T_{AB} is a projection?

(3) Find eigenvalues and eigenvectors of T, with their degeneracy.

(4) Find $\|T\|$.

(5) The same questions (1) and (2) with $K = 1 + x^2y^2$ and with $K_{AB} = A + Bx^2y^2$.

(1.125)
Consider the operator defined in $L^2(-\pi, \pi)$

$$T\,f(x) = \alpha f(x) + \beta \int_{-\pi}^{\pi} f(y)\,dy\,, \qquad \alpha, \beta \in \mathbf{C};\ \alpha, \beta \neq 0$$

(1) (a) Find eigenvalues and eigenvectors of T, with their degeneracy, and specify if the eigenvectors provide an orthogonal complete system for $L^2(-\pi, \pi)$.

(b) Find $\|T\|$.

(2) Let now $\alpha = 1$: is it possible choose β in such a way that T becomes a projection?

(3) Study the convergence as $n \to \infty$ of the sequence of operators T_n defined by

$$T_n\,f(x) = \alpha f(x) + \beta \exp(inx) \int_{-\pi}^{\pi} f(y)\,dy$$

(1.126)
Consider in the space $L^2(-\pi, \pi)$ the operator

$$T\,f(x) = \frac{1}{2a} \int_{x-a}^{x+a} f(y)\,dy\,, \qquad 0 < a < \pi$$

where the function f must be periodically prolonged with period 2π.

(1) Let D be the operator

$$D = \frac{d}{dx} \quad \text{with the periodic boundary condition } f(-\pi) = f(\pi)$$

(a) Find the eigenvalues and eigenvectors of D, with their degeneracy.

(b) Show that $TD = DT$ (in a dense domain).

(2) Using the results obtained in (1), find the eigenvalues and eigenvectors of T, with their degeneracy.

(3) Using the results obtained in (2), specify if T is Hermitian and find its norm.

(4) Is T compact?

(1.127)
Consider in $H = L^2(-\pi, \pi)$ the operator (which is the same as in the above problem)

$$T_a f(x) = \frac{1}{2a}\int_{-a}^{a} f(x+y)\,dy\,, \qquad 0 < a < \pi$$

with the function $f(x)$ periodically prolonged with period 2π.

(1) Writing $f(x)$ in the form $f(x) = \sum_n c_n \exp(inx)$, i.e., as Fourier series with respect to the complete set $\{\exp(inx)\,, n \in \mathbf{Z}\}$, show that $T_a f(x)$ can be written in the form

$$T_a f(x) = \sum_n c_n \lambda_n \exp(inx)$$

where $\lambda_n = \ldots$ are the eigenvalues of T_a.

(2) Let $a = \pi$: determine $\mathtt{Ran}\, T_a$ and find $T_a f$ for a generic $f(x) \in H$.

(3) Let $a = \pi/2$: for what (integer) values of m does the function $g(x) = 1 + \sin mx$ belong to $\mathtt{Ran}\, T_a$?

(4) Let $a = 1$: is $\mathtt{Ran}\, T_a$ a dense subspace in H? does it coincide with H? is T_a invertible?

(5) Study the convergence as $a \to 0$ of the family of operators T_a.

(1.128)
Consider the operator T_a defined in $L^2(0, 2\pi)$, with the functions periodically prolonged with period 2π out of this interval,

$$T_a\, f(x) = f(x+a) - f(x)\,, \qquad 0 < a < 2\pi$$

and the operator

$$D = \frac{d}{dx} \quad \text{with the periodic boundary condition } f(0) = f(2\pi)$$

(1) Find eigenvectors and eigenvalues (with their degeneracy) of the operator D.

(2) Observing that $DT_a = T_a D$ (in a dense domain) find eigenvectors and eigenvalues of T_a. What is the degeneracy of the eigenvalues if $a = \pi/2$? and if $a = 1$?

(3) Show that T_a is a normal operator for any a. Find $\|T_a\|$.

(4) For what choice of a does the operator $T_a - iI$ admit bounded inverse? Find $\|(T_a - iI)^{-1}\|$ if $a = \pi/2$ and if $a = 1$.

(1.129)
Consider the operator defined in $H = L^2(0, +\infty)$

$$T\, f(x) = \frac{1}{x} f\Big(\frac{1}{x}\Big)$$

(1) Find $\|T\|$ and T^{-1}.

(2) Is T unitary?

(3) Observing that $T^2 = \ldots$, it is easy to find the eigenvalues and construct the eigenvectors of T .... Compare with the next problem.

(1.130)
Let now T be the operator defined in $H = L^2(0, +\infty)$

$$T\, f(x) \;=\; f\Big(\frac{1}{x}\Big)$$

(1) Is T bounded?

(2) Specify what among the following functions $\in H$ belong to the domain of T:

$$f_1(x) = \begin{cases} 1 & \text{for} \quad 0 < x < 1 \\ 0 & \text{for} \quad x > 1 \end{cases} \quad ; \quad f_2(x) = \begin{cases} x & \text{for} \quad 0 < x < 1 \\ 0 & \text{for} \quad x > 1 \end{cases} \quad ;$$

$$f_3(x) = \exp(-x) \quad ; \quad f_4(x) = \begin{cases} x & \text{for} \quad 0 < x < 1 \\ 1/(1+x) & \text{for} \quad x > 1 \end{cases}$$

(3) As in the case of the operator considered in the problem above, one has $T^2 = I$. But in the present case it is not easy to find its eigenvectors, indeed only when $f(x)$ belongs to the domain of T, one can apply the usual argument, namely: if $g \equiv (T - I)f \neq 0$ then g is eigenvector, etc. Find explicitly at least some eigenvector of T.

(1.131)
(1) Let T be the operator defined in $L^2(0, 1)$

$$T\, f(x) \;=\; \int_0^x f(t)\, dt\,, \qquad 0 < x < 1$$

(a) Find the matrix elements $T_{nm} = (e_n, T e_m)$ of T with respect to the orthonormal complete system $\{e_n(x) = \exp(inx)/\sqrt{2\pi},\; n \in Z\}$.

(b) Is T bounded?

(c) Look for the eigenvalues of T (start differentiating both members of the equation $Tf = \lambda f$).

(2) The same questions $(b), (c)$ for the operator defined in $L^2(-\infty, 1)$

$$T\, f(x) \;=\; \int_{-\infty}^x f(t)\, dt\,, \qquad -\infty < x < 1$$

(3) The same questions $(b), (c)$ for the operator given in (2) but defined in $L^2(-\infty, +\infty)$.

(1.132)
Let T_n be the operators defined in $L^2(0, +\infty)$

$$T_n\, f(x) = \sqrt{n}\, f(nx)\,, \qquad n = 1, 2, \ldots$$

(1) Find $\|T_n\|$.

(2) Fix here, for instance and for simplicity, $n = 2$. Show that the functions with compact support cannot be eigenfunctions of T_2. Show that T_2 has no eigenfunctions $\big($start with an empirical argument: assume that $f(x)$ is an eigenfunction and that at some point $\overline{x}$ the function $f(x)$ has a value $\overline{a} \neq 0$, i.e., $f(\overline{x}) = \overline{a}$, then $f(x)$ is determined in all points $2\overline{x},\ 4\overline{x}, \ldots, \overline{x}/2$, etc., then ...$\big)$.

(3) Show that the sequence T_n tends weakly to zero (but not strongly). *Hint*: restrict first to the dense subspace of bounded functions with compact support, then .... If $f(x)$ has compact support K, what is the support of $f(nx)$?

Chapter 2
Functions of a Complex Variable

2.1 Basic Properties of Analytic Functions

(2.1)
(1) Evaluate

$$(1+i)^{100}\,; \quad \sqrt[3]{1-i}\,; \quad \Big(\frac{1}{1+i\sqrt{3}}\Big)^{21/2}$$

(2) Solve the equations, with $z \in \mathbf{C}$,

$$\sin z = 4i\,; \quad \cos z = 30\,; \quad \exp z = \pm 1/e\,; \quad \cosh z := \frac{\exp z + \exp(-z)}{2} = -1$$

(2.2)
The following functions of the complex variable $z \in \mathbf{C}$

$$z^2 \sin(1/z)\,; \quad \exp(-1/z^4)\,; \quad z\exp(-1/z^2)\,; \quad z\exp(i/z)$$

have, as well known, an essential singularity at $z = 0$, therefore their limit as $z \to 0$ does not exist. Verify explicitly that this limit does not exist showing that these functions assume different values approaching arbitrarily near $z = 0$ along different paths, or sequences of points.

(2.3)
(1) Write the most general function $f(z)$ which has a pole of order 2 at $z = i$ with residue $-3i$ and is analytic in all other points, including the point $z = \infty$. What changes without the assumption of analyticity at $z = \infty$?

(2) A function $f(z)$ is analytic $\forall z \in \mathbf{C}$ apart from the point $z = \infty$ and the point $z = 1$, where the residue is 1 and where

G. Cicogna, *Exercises and Problems in Mathematical Methods of Physics*, Undergraduate Lecture Notes in Physics,
https://doi.org/10.1007/978-3-319-76165-7_2

$$\lim_{z \to 1} (z-1)^3 f(z) = 2$$

(i) Consider the second derivative $f''(z)$: determine what is its singularity and its residue at $z = 1$. Are there any other singularities (apart from $z = \infty$)? And can $f''(z)$ be analytic at $z = \infty$?

(ii) What about the singularities of the primitive function $F(z)$ of $f(z)$ $\big($i.e. :$F'(z) = f(z)\big)$?

(2.4)
Let $f(z)$ be an analytic function for all complex z (apart from $z = \infty$).

(1) It is known that its real part $u(x, y) = \mathtt{Re}\, f$ has the form

$$u(x, y) = a(x) + b(y)$$

What is the most general $f(z)$ satisfying this condition?

(2) The same question if

$$u(x, y) = a(x)\, b(y)$$

(2.5)
Expand the function

$$f(z) = \frac{1}{1-z}$$

in Taylor–Laurent power series

(a) in a neighborhood of $z_0 = 0$

(b) in a neighborhood of $z_0 = 2i$

(c) in a neighborhood of $z_0 = \infty$

(d) in a neighborhood of $z_0 = 1$

and specify the region of convergence.

(2.6)
(1) What is the radius R of convergence of the series

$$\sum_{n=1}^{\infty} n^p z^n$$

where p is any fixed real number?

(2) Find the sum of the series

$$\sum_{n=1}^{\infty} n z^n$$

(3) Determine the annulus of convergence of the series

$$\sum_{n=0}^{\infty}\left(\frac{z}{3}\right)^n + \sum_{n=0}^{\infty}\left(\frac{2}{z}\right)^n$$

(2.7)
For what $z \in \mathbf{C}$ is the series

$$\sum_{n=0}^{\infty} \exp(-nz)$$

convergent? Find its sum $S(z)$ and determine the singularities (including the point $z = \infty$) of the function $S(z)$ extended to all the complex plane $\mathbf{C}$.

(2.8)
Determine the singularity at point $z = \infty$ of the function

$$f(z) = \frac{z \sin z}{a_4 z^4 + a_2 z^2 + a_0}, \qquad a_0, a_2, a_4 \neq 0$$

and find its residue at point $z = \infty$. *Hint*: $f(z)$ is an *even* function of z, then it contains only powers of $z^2, \ldots$.

(2.9)
Verify the validity of the l'Hôpital theorem in its simplest form in the following case:

$$\lim_{z \to z_0} \frac{f(z)}{g(z)}$$

where $f(z)$ and $g(z)$ are analytic in a neighborhood of z_0 and $f(z_0) = g(z_0) = 0$.

(2.10)

Determine the singularities of the function

$$f(z) = \frac{z^2 + \pi^2}{1 + \exp z}$$

and specify the radius R of convergence of the Taylor expansion of $f(z)$ around the origin $z_0 = 0$.

(2.11)
Determine the singularities in the complex plane (included the point $z = \infty$) of the functions

$$f(z) = \frac{1}{\sin z} \quad ; \quad f(z) = \frac{1}{\sin(1/z)}$$

(2.12)
(1) Find the coefficients $a_{-1},\ a_0,\ a_1$ and a_2 of the Taylor–Laurent expansion $\sum_n a_n z^n$ of the function

$$f(z) = \frac{1}{\sin z}$$

in the neighborhood of the origin $z_0 = 0$. *Hint*: find first a_{-1}, then the function $f(z) - (a_{-1}/z)$ is analytic around $z_0 = 0$.

(2) Show that the function

$$f(z) = \frac{1}{1 - \cos z} - \frac{2}{z^2}$$

is analytic in a neighborhood of $z_0 = 0$ and find the coefficients a_0 and a_1 of its Taylor expansion.

(2.13)
Determine the singularities in the complex plane z (included the point $z = \infty$) of the functions

$$f(z) = \frac{\sin^2 z}{z^4(z-\pi)(z+2\pi)^2}\ ; \quad f(z) = \frac{\sin z}{z^2} - \frac{\cos z}{z}$$

(2.14)
Determine the singularities in the complex plane z (included the point $z = \infty$) and the region of convergence of the Taylor–Laurent expansion around $z_0 = 1$ of the function

$$f(z) = \frac{\exp(z^2) + \exp(-z^2) - 2 - z^4}{z^n(z-1)}$$

depending on the values of the integer number n.

(2.15)
Determine the singularities in the complex plane (including the point $z = \infty$) of the following functions

$$f(z) = \sin\sqrt{z}\ ; \quad f(z) = \sin^2\sqrt{z}\ ; \quad f(z) = \cos\sqrt{z}\ ;$$

$$f(z) = \frac{\sin\sqrt{z}}{z}\ ; \quad f(z) = \frac{\sin\sqrt{z}}{\sqrt{z}}\ ; \quad f(z) = \frac{\cos\sqrt{z}}{\sqrt{z}}\ ;$$

$$\frac{\sin\sqrt{z}}{z^2} - \frac{\cos\sqrt{z}}{z}\ ; \quad f(z) = z\,\sin\frac{1}{\sqrt{z}}\ ; \quad f(z) = \sqrt{z}\,\sin\frac{1}{\sqrt{z}}$$

(2.16)
Find the branch points of the functions

$$f(z) = \log z - \log(z-1)\ ; \quad f(z) = \log z + \log(z-1)\ ;$$

$$f(z) = \sqrt{z(z-1)}\,; \quad f(z) = \sqrt[3]{z(z-1)}\,; \quad f(z) = \sqrt[3]{z^2(z-1)}$$

(2.17)
(1) For what positive integer n does the function

$$f_1(z) = \frac{\sin z - z + \sin(z^3/6)}{z^n(1-z^2)^2}$$

admit Taylor expansion (with positive powers of z) in the neighborhood of $z = 0$? What is the behavior of the function at $z = \infty$?

(2) What changes for the function

$$f_2(z) = \frac{\sin z - z + \sin(z^3/6)}{z^n\sqrt{1-z^2}} \; ?$$

(2.18)
(1) For what values of $\alpha \in \mathbf{C}$ (if any, $\alpha \neq 0$) is the function

$$f(z) = \frac{\exp(\alpha z) - \exp(-\alpha z)}{1-z^2}$$

analytic for all $z \in \mathbf{C}$ (apart from $z = \infty$)? What happens at $z = \infty$?

(2) The same question for the function

$$f(z) = \frac{\exp(\alpha z) - \exp(-\alpha z)}{\sqrt{1-z^2}}$$

(2.19)
Fix the cut line of the functions $\log z$ and $\sqrt{z}$ along the positive real axis.

(1) Check if the following identity is true

$$\log(z^3) = 3\log z$$

(2) Determine the singularities (apart from the cut line) of the functions

$$f(z) = \frac{i\pi + \log z}{z+1}\,; \qquad f(z) = \frac{i\pi - \log z}{z+1}$$

(3) The same question for the functions

$$f(z) = \frac{1}{\sqrt{z} - i}\,; \qquad f(z) = \frac{1}{\sqrt{z} + i}$$

(2.20)
Despite the presence of branch points of the following functions, show that it is

possible to choose conveniently the cut lines in such a way to have functions analytic at $z = 0$. Evaluate the first terms of their Taylor expansions:

$$f(z) = \sqrt{1 \pm z^2}\,; \quad f(z) = \log\frac{1+z}{1-z}\,; \quad f(z) = \log(1 \pm z^2)$$

2.2 Evaluation of Integrals by Complex Variable Methods

(2.21)
(1) Show that if a function $f(z)$ is analytic at the point $z = \infty$ where it has a zero of order ≥ 2 then its residue $\mathtt{R}(\infty)$ is zero.

(2) As well known, the integral of a rational function

$$\int_{-\infty}^{+\infty} \frac{P(x)}{Q(x)}\,dx,$$

where $P(x)$, $Q(x)$ are polynomials of the real variable x, exists if *(i)* $Q(x)$ has no (real) zeroes and *(ii)* the degrees n_P, n_Q of $P(x)$, $Q(x)$ satisfy the condition $n_Q \geq n_P + 2$. Show that this second condition implies that the residue $\mathtt{R}(\infty)$ at the point $z = \infty$ of the complex function $f(z) = P(z)/Q(z)$ is zero.

(3) To evaluate the above integral with the method of residues, one can consider a closed contour consisting of a segment $-R \leq x \leq R$ along the real axis and a semicircle of radius R, either in the *upper* or in the *lower* complex plane $\mathbf{C}$ (and then let $R \to \infty$). Show that the property $\mathtt{R}(\infty) = 0$ ensures that—as expected!—the result of the integration does not depend on the choice about the closing contour.

(2.22)
Evaluate the integrals

$$\int_{-\infty}^{+\infty} \frac{x^2}{(x^2+1)^2}\,dx\,; \qquad \int_{-\infty}^{+\infty} \frac{x}{(x+i)(x-2i)^2(x-3i)}\,dx$$

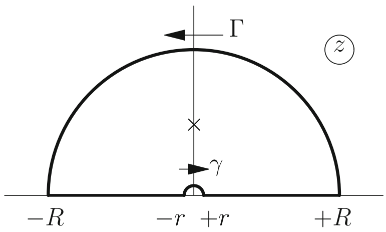
(2.23)
Evaluate the integrals

$$\oint_{|z|=2} \frac{\sin z}{(z-1)^5}\,dz\,; \qquad \oint_{|z-2|=3} \frac{z}{\sin^2 z}\,dz\,; \qquad \oint_{|z|=5} \frac{1-\exp z}{1+\exp z}\,dz$$

(2.24)
Evaluate the integrals

$$\oint_{|z|=1} \frac{z^2}{(2z-1)(z^2+2)}\,dz\,; \qquad \oint_{|z|=1} \frac{\exp z}{z}\,dz$$

Fig. 2.1 See Problems 2.28–2.31



Integrals of this type, which can be very easily evaluated in the complex plane, produce nontrivial results when z is replaced by $\exp(i\theta)$: verify!

(2.25)
Evaluate the integrals (put $\exp(i\theta) = z$ and transform the integrals into integrals along the circle $|z| = 1$ in the complex plane):

$$\int_0^{2\pi} \frac{1}{1+\cos^2\theta}\, d\theta\ ; \quad \int_0^{2\pi} \frac{\cos\theta}{2+\cos\theta}\, d\theta\ ; \quad \int_0^{2\pi} \frac{\sin\theta}{(2+\sin\theta)^2}\, d\theta$$

(2.26)
Evaluate the integrals, using Jordan lemma,

$$\int_{-\infty}^{+\infty} \frac{\exp(\pm iax)}{(x-i)^2}\, dx \quad (a>0)\ ; \qquad \int_{-\infty}^{+\infty} \frac{\exp(ix)}{(x^2+1)^2}\, dx$$

(2.27)
Evaluate the following integrals; here, the functions $\sin x$, $\cos x$ must be replaced by $\exp(\pm ix)$, in order that the semicircular closing contour of integration in the complex plane gives a vanishing contribution, according to Jordan lemma:

$$\int_{-\infty}^{+\infty} \frac{\cos x}{1+x^2}\, dx\ ; \quad \int_{-\infty}^{+\infty} \frac{\sin x}{1+x+x^2}\, dx$$

(2.28)
Evaluate the following integrals; in these cases, the use of Jordan lemma produces the appearance of singularities (simple poles) along the real axis, and therefore the necessity of introducing one or more "indentations" along the real axis: see Fig. 2.1, which refers to the first one of the following integrals

$$\int_{-\infty}^{+\infty} \frac{\sin x}{x(1+x^2)}\, dx\ ; \qquad \int_{-\infty}^{+\infty} \frac{\sin x}{x}\, dx\ ;$$

$$\int_{-\infty}^{+\infty} \frac{\cos(\pi x/2)}{(1-x)(x^2+1)}\, dx\ ; \qquad \int_{-\infty}^{+\infty} \frac{\sin(\pi x)}{x(1-x^2)}\, dx$$

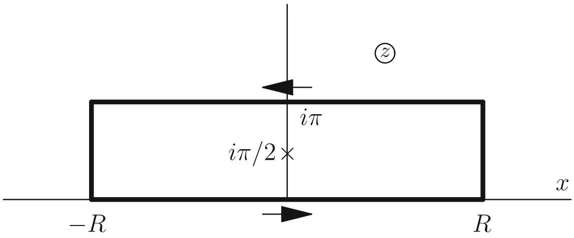


Fig. 2.2 See Problems 2.32, 2.39 and 2.40

(2.29)
Evaluate the integrals

$$\int_{-\infty}^{+\infty} \frac{\sin x}{x(x-i)}\,dx\,; \qquad \int_{-\infty}^{+\infty} \frac{\exp(ix)-\exp(-2ix)}{x(x-i)}\,dx$$

(2.30)
Evaluate the integral

$$\int_{-\infty}^{+\infty} \frac{\sin^2 x}{x^2}\,dx$$

Hint: Put $(\sin^2 x)/x^2 = \mathtt{Re}\big(1-\exp(2ix)/(2x^2)\big)$ in order to have a simple pole on the real axis. This integral can also be evaluated in a different way: see Problem 3.9, q. (1).

(2.31)
Evaluate the following integrals (containing a Cauchy principal part due to the presence of singular points on the real axis)

$$\mathtt{P}\int_{-\infty}^{+\infty} \frac{\exp(ix\pi)}{(x+1)(x+i)}\,dx\,; \quad \mathtt{P}\int_{-\infty}^{+\infty} \frac{\exp(ix)}{x(x-i)(x+2i)}\,dx$$

(2.32)
Evaluate the integral (use the closed contour in Fig. 2.2). *Hint*: $\cosh(x+\pi i) = \ \ldots$

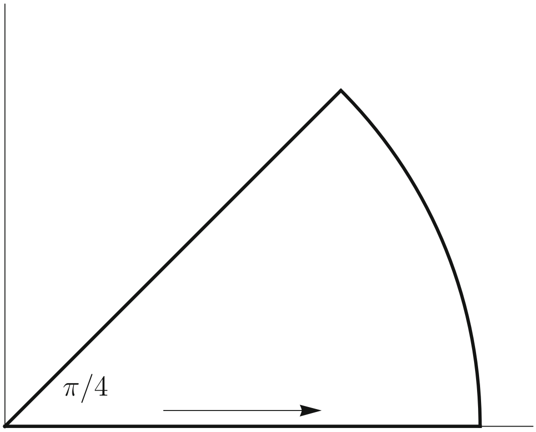
$$\int_{-\infty}^{+\infty} \frac{\cos(ax)}{\cosh x}\,dx \qquad (a \in \mathbf{R})$$

(2.33)
Evaluate the integral (use the contour in Fig. 2.3)

$$\int_{0}^{+\infty} \exp(i\,x^2)\,dx$$

and deduce the integrals

Fig. 2.3 See Problem 2.33



$$\int_0^{+\infty} \sin(x^2)\,dx \;=\; \int_0^{+\infty} \cos(x^2)\,dx$$

(2.34)
The integrals proposed in the following exercises of this subsection involve functions, as $\log z$ and z^α, which present branch points and cut lines. The following questions are in preparation of the evaluation of these integrals:

(1) Find the following residues, assuming that the cut line is placed along the positive real axis:

(a) residue at $z = \pm i$ of $f(z) = \dfrac{\sqrt{z}}{z \mp i}$

(b) residue at $z = \pm i$ of $f(z) = \dfrac{\log z}{z \mp i}$

(c) residue at $z = -1$ of $f(z) = \dfrac{\sqrt{z}}{z + 1}$

(d) residue at $z = -i$ of $f(z) = \dfrac{z^{4/3}}{(z + i)^2}$

(e) residue at $z = -i$ of $f(z) = \dfrac{\sin(\pi\sqrt{2z})}{z + i}$

(f) residue at $z = -i$ of $f(z) = \dfrac{\exp(\pi\sqrt{2z})}{z + i}$

(2) Find the following residues:

(a) residue at $z = i$ of $f(z) = \dfrac{\log(z^2 - 1)}{z - i}$ assuming that the cut is from $-\infty$ to -1 and from 1 to $+\infty$ in the real axis

(b) residue at $z = i$ of $f(z) = \dfrac{\sqrt{z^2-1}}{z-i}$ assuming that the cut is along the segment $[-1, 1]$ of the real axis
(c) residues at the (isolated) singular points of the function

$$f(z) = \frac{\sqrt{z-1}}{(z^2-1)(z^2+1)}$$

with the cut along the line $x \geq 1$ in the real axis

(3) Find the discontinuity presented by the following functions along the indicated branch cut (here, the discontinuity is defined as the value of the function at the *upper* margin of the cut minus the value at the *lower* margin):
(a) $f(z) = \sqrt{z^2-1}$ with the cut along the segment $[-1, 1]$ of the real axis
(b) $f(z) = (z-1)^\alpha$ with $\alpha \in \mathbf{R}$ ($\alpha \neq 0, \pm 1, \pm 2, \ldots$) with the cut from 1 to $+\infty$ in the real axis
(c) $f(z) = \log(z^2-1)$ with the cut from $-\infty$ to -1 and from 1 to $+\infty$ in the real axis
(d) $f(z) = \log \dfrac{z+1}{z-1}$ with the same cut as in (c).
(e) $f(z) = \log \dfrac{z+1}{z-1}$ with the cut along the segment $[-1, 1]$ of the real axis

(2.35)
Evaluate the integrals (use the closed contour in Fig. 2.4, where the cut is indicated by a dashed line)

$$\int_0^\infty \frac{x^{\pm 1/2}}{1+x^2}\,dx\,; \quad \int_0^\infty \frac{x^{\pm 1/3}}{1+x^2}\,dx\,; \quad \int_0^\infty \frac{\sqrt{x}}{(x+i)^2}\,dx$$

(2.36)
Evaluate the integrals (use the closed contour as in the problem above)

$$\int_0^{+\infty} \frac{x^a}{(1+x)^2}\,dx \quad (-1 < a < 1)\,; \qquad \int_0^{+\infty} \frac{x^b}{1+x^3}\,dx \quad (-1 < b < 2)$$

(with $a \neq 0$; $b \neq 0$ and $\neq 1$, otherwise there would be no cut line!)

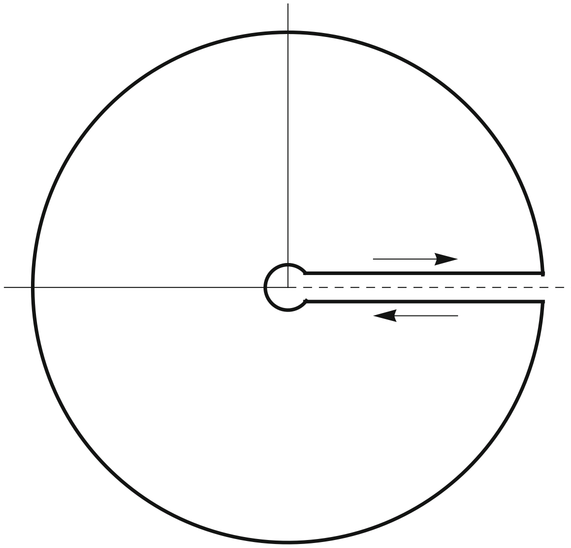
(2.37)
(1) Integrating the functions

$$f(z) = \frac{\log z}{1+z+z^2}\,; \qquad f(z) = \frac{\log z}{1+z^3}$$

along a closed contour as in Fig. 2.4, obtain the integrals

$$\int_0^{+\infty} \frac{1}{1+x+x^2}\,dx\,; \qquad \int_0^{+\infty} \frac{1}{1+x^3}\,dx$$

Fig. 2.4 See Problems 2.35–2.37



(2) Integrating the function

$$f(z) = \frac{(\log z)^2}{1+z^2}$$

along the same closed contour, obtain the integrals (which can also be obtained by different methods, see next problem for the first integral)

$$\int_0^{+\infty} \frac{\log x}{1+x^2}\, dx\,; \qquad \int_0^{+\infty} \frac{1}{1+x^2}\, dx$$

(2.38)
Evaluate the integrals

$$\int_0^{+\infty} \frac{\log x}{1+x^2}\, dx\,; \qquad \int_0^{+\infty} \frac{\log x}{1+x^4}\, dx$$

using for the first integral the closed contour shown in Fig. 2.5a. As suggested by the figure, the line along the positive real axis is chosen on the "upper margin" of the cut. *Hint*: if $x < 0$, then $\log x = \ldots$. For the second integral use the contour shown in Fig. 2.5b. *Hint*: if $y > 0$, then $\log(iy) = \ldots$

(2.39)
Evaluate the integrals

$$\int_{-\infty}^{+\infty} \frac{\exp(ax)}{1+\exp x}\, dx \quad (0 < a < 1)\,; \qquad \int_0^{+\infty} \frac{1}{x^b(1+x)}\, dx \quad (0 < b < 1)$$

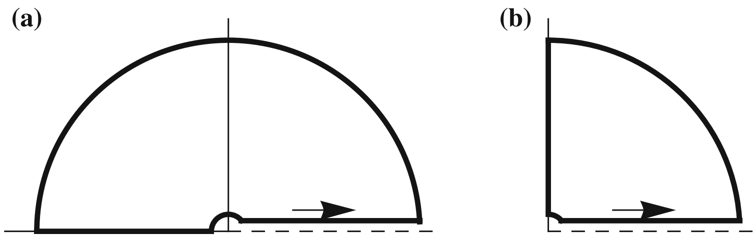


Fig. 2.5 See Problem 2.38

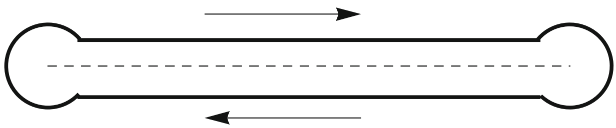


Fig. 2.6 See Problem 2.41

Putting $x' = \exp x$ the first integral is transformed into the second one, with $b = 1-a$. The first integral can be evaluated using a contour similar to that in Fig. 2.2, but with height $2\pi i$; the second one ….

(2.40)
Evaluate the integral

$$\int_0^{+\infty} \frac{\log^2 x}{1+x^2}\, dx$$

Put $x' = \log x$ and use the same contour as in Fig. 2.2.

(2.41)
Evaluate the integrals (use closed contour as in Fig. 2.6. *Hint*: do not forget the residue at $z = \infty$)

$$\int_{-1}^{1} \sqrt{\frac{1+x}{1-x}}\, dx\,; \qquad \int_{-1}^{1} \frac{\sqrt{1-x^2}}{2-x}\, dx\,; \qquad \int_0^1 \frac{1+x^2}{\sqrt{x(1-x)}}\, dx$$

(2.42)
Evaluate the integral (use the closed contour in Fig. 2.7)

$$\int_{-\infty}^{+\infty} \frac{\log(a^2+x^2)}{1+x^2}\, dx, \qquad (a > 1)$$

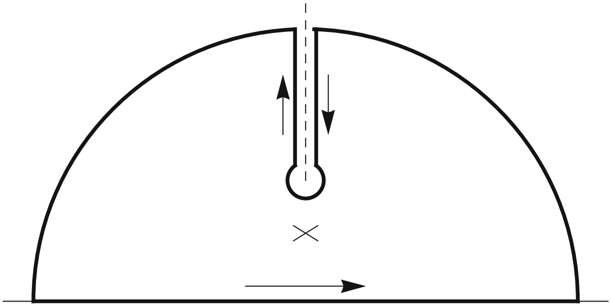


Fig. 2.7 See Problem 2.42

2.3 Harmonic Functions and Conformal Mappings

(2.43)
A harmonic function $u = u(x, y)$ can be interpreted, for instance—as well known—as a two-dimensional electric potential. Let $v = v(x, y)$ be its harmonic conjugate (unique apart from an additive constant), and let $f = u + iv$ be the corresponding analytic function. Then, the lines $u(x, y) = \text{const.}$ and $v(x, y) = \text{const.}$ represent the equipotential lines and respectively the lines of force of the electric field. Draw the lines $u(x, y) = \text{const.}$ and $v(x, y) = \text{const.}$ of the elementary analytic function $f(z) = z$ (trivial!), and of the functions

$$f(z) = z^2\,; \quad f(z) = \sqrt{z}\,; \quad f(z) = \log z$$

(2.44)
(1) Let D be the region in the complex plane z included between the positive real axis and the positive imaginary axis. Consider the Dirichlet Problem of finding a harmonic function $u(x, y)$ in D with the condition $u(x, y) = 0$ on the boundaries. Observing that the conformal map

$$z \to z' = \Phi(z) = z^2$$

maps D into ..., where the problem admits an elementary (linear: see Problem (2.43), but also (2.48)) solution, obtain a solution of the given Dirichlet Problem, and construct the lines $u(x, y) = \text{const.}$ and $v(x, y) = \text{const.}$

(2) Consider now the region D in the complex plane z included between the positive real axis and the half straight line ℓ starting from the origin and forming an angle α with the real axis, see Fig. 2.8a. Consider the Dirichlet Problem of finding a harmonic function $u(x, y)$ in D with the boundary conditions $u = 0$ on the real axis and $u = u_0 = \text{const} \neq 0$ on the line ℓ. Using the conformal map

$$z \to z' = \Phi(z) = \log z$$

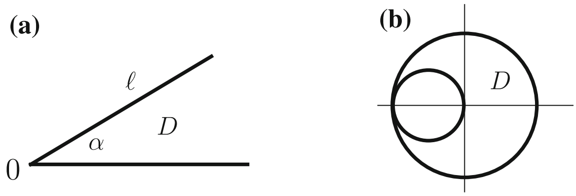


Fig. 2.8 See Problems 2.44 and 2.47

which maps D into ..., where the problem admits an elementary solution, solve the given Dirichlet Problem, and construct the lines $u(x, y)$ = const. and $v(x, y)$ = const.

(2.45)
(1) Show that the conformal mapping

$$z \to z' = \Phi(z) = \frac{i-z}{i+z}$$

transforms the half-plane $y = \mathtt{Im}\, z \geq 0$ into the circle $|z'| \leq 1$. *Hint*: $|z'| = 1$ means $|z - i| = |z + i|$, etc.

(2) Consider the Dirichlet Problem for the half-plane $y \geq 0$: i.e., find the harmonic function $u(x, y)$ satisfying a given boundary condition on the x axis

$$u(x, 0) = F(x)$$

Using the conformal map given in (1), with $z' = r' \exp(i\varphi')$, show that the problem becomes a Dirichlet Problem for the circle of radius $r' = 1$ centered at the origin of the plane z', where the boundary condition becomes

$$\widetilde{u}(r', \varphi') = \widetilde{F}(\varphi') = \widetilde{F}\big(\tan(\varphi'/2)\big), \qquad -\pi < \varphi' < \pi$$

(3) Using the result seen in (2) and recalling Sect. 1.1.3, solve the Dirichlet Problem for the half-plane $y \geq 0$ if

$$F(x) = \frac{1}{1+x^2}$$

Hint: $\widetilde{u}(r', \varphi') = (1 + r' \cos\varphi')/2 = \mathtt{Re}\,(1 + z')/2$, then ...

(4) The same as in (3) if

$$F(x) = \frac{x^2}{(1+x^2)^2}$$

(2.46)
Show that the conformal map given in Problem 2.45, q. (1) transforms the line $z = x + i$ into the circumference $|z' + 1/2| = 1/2$, and the half-plane $y = \mathtt{Im}\, z \geq 1$ into

the interior of this circle. What is the image in the plane z' of the strip $0 \le y \le 1$ in the plane z under the transformation given above?

(2.47)
Solve the Dirichlet Problem in the region D obtained excluding the circle $|z + 1/2| \le 1/2$ from the circle $|z| \le 1$, see Fig. 2.8b, with the boundary conditions $u = 0$ on the circumference $|z| = 1$ and $u = 1$ on the circumference $|z + 1/2| = 1/2$. To solve this problem, use the conformal map

$$z \to z' = \Phi(z) = i\frac{1 - z}{1 + z}$$

which is just the *inverse* of the map considered in Problems 2.45 and 2.46, and which transforms D into …(see Problem 2.46), where the problem becomes elementary. Verify explicitly that the solution satisfies the boundary conditions given for the region D.

(2.48)
(1) Show that the Dirichlet Problem for the half-plane $\mathtt{Im}\, z = y \ge 0$ does *not* admit unique solution, but actually admits infinitely many solutions: verify for instance that the real parts of the analytic functions iz, iz^2, etc., satisfy both $\Delta_2 u = 0$ and the vanishing boundary condition $u(x, 0) = 0$. What is the most general solution of this problem?

(2) The existence of infinitely many solutions of the Dirichlet Problem in the half-plane may appear surprising if compared with the Dirichlet Problem for the circle, recalling also that the two problems are connected by a conformal map (see Problem 2.45). To explain this fact, choose, e.g., the simplest solution $u(x, y) = y = \mathtt{Re}\,(-iz)$ to the Dirichlet Problem in the half-plane $\mathtt{Im}\, z \ge 0$ with the boundary condition $u(x, 0) = 0$, and find the corresponding solution $\widetilde{u}(r', \varphi')$ for the circle using the conformal map

$$z' \to z = \Psi(z') = i\frac{1 - z'}{1 + z'}$$

What is the singularity presented by this solution $\widetilde{u}(r', \varphi')$? Repeat calculations, for instance, for the solution in the half-plane $u = xy = \mathtt{Re}\,(-iz^2/2)$, and verify that the same situation occurs. Conclude: how can one recover the uniqueness of the solution of the Dirichlet Problem? See also the discussion in Problem 3.112.

Chapter 3
Fourier and Laplace Transforms. Distributions

3.1 Fourier Transform in $L^1(\mathbf{R})$ and $L^2(\mathbf{R})$

There is some arbitrariness in the notation and definition of the Fourier transform. First of all, the independent variable of the functions to be transformed is often chosen to be the time $t \in \mathbf{R}$*, and accordingly the independent variable of the transformed functions is the "frequency"* $\omega \in \mathbf{R}$*. This choice is due to the peculiar and characterizing physical interpretation of the Fourier transform, namely that of being the "frequency analysis". The notations frequently used in the following for the Fourier transform will then be*

$$\mathscr{F}\big(f(t)\big) = g(\omega) = \widehat{f}(\omega)$$

In the case that the function f *depends instead on the space position* $x \in \mathbf{R}$*, then its Fourier transform will depend on the "associate" physical variable* $k = 2\pi/\lambda$*, with usual notations; then we will write* $\mathscr{F}\big(f(x)\big) = g(k) = \widehat{f}(k)$*.*
However, when no specific physical interpretation is involved and only the mathematical properties are concerned, the independent variable of the Fourier transform of a function $f(x)$ *will be denoted either by* k *or by* ω*, or sometimes by a "generic" variable* $y \in \mathbf{R}$*.*
The definition of the Fourier transform (using the variables t *and* ω*) for functions* $f(t) \in L^1(\mathbf{R})$*, which will be adopted is*

$$\mathscr{F}\big(f(t)\big) = \int_{-\infty}^{+\infty} f(t)\exp(i\omega t)\, dt = g(\omega) = \widehat{f}(\omega)$$

and then for the inverse transformation

$$\mathscr{F}^{-1}\big(g(\omega)\big) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} g(\omega)\exp(-i\omega t)\, d\omega = f(t)$$

G. Cicogna, *Exercises and Problems in Mathematical Methods of Physics*, Undergraduate Lecture Notes in Physics,
https://doi.org/10.1007/978-3-319-76165-7_3

Other definitions are possible: the factor $\exp(i\omega t)$ *in the first integral may be changed into* $\exp(-i\omega t)$ *and correspondingly the factor* $\exp(-i\omega t)$ *in the second one must be changed into* $\exp(i\omega t)$.
The factor $\frac{1}{2\pi}$ *in the second integral may be "distributed" into two factors* $\frac{1}{\sqrt{2\pi}}$ *in front to both integrals. As well known, the Fourier transform and inverse Fourier transform defined in this way (using here the independent variables* x *and* y)

$$\widetilde{\mathscr{F}} = \frac{1}{\sqrt{2\pi}}\mathscr{F} \quad \text{, i.e.,} \quad \widetilde{\mathscr{F}}\big(f(x)\big) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(x)\exp(ixy)\,dx = g(y)$$

and

$$\widetilde{\mathscr{F}}^{-1} = \sqrt{2\pi}\mathscr{F}^{-1} \quad \text{, i.e.,} \quad \widetilde{\mathscr{F}}^{-1}\big(g(y)\big) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} g(y)\exp(-ixy)\,dy = f(x)$$

are unitary *operators in the Hilbert space* $L^2(\mathbf{R})$.
Notice that if $f(t) \in L^2(\mathbf{R})$ *but* $\notin L^1(\mathbf{R})$ *the correct notation for the Fourier transform would be*

$$\widehat{f}(\omega) = \mathtt{P}\int_{-\infty}^{+\infty} f(t)\exp(i\omega t)\,dt = \lim_{T\to+\infty}\int_{-T}^{T}\ldots$$

where the symbol P *denotes the Cauchy principal part of the integral (here, with respect to the infinity). For the sake of simplicity, the symbol* P *will be usually omitted. The same remark holds for the inverse Fourier transformation formula.*

Several examples of linear systems will be proposed: these are characterized by an applied "input" which produces an answer, or "output", depending linearly on the input. The connection is often expressed by means of a linear differential equation, or— more in general—by specifying a Green function. *In the latter case, assuming that the independent variable is the time* t, *denoting by* $a = a(t)$ *the input and by* $b = b(t)$ *the corresponding output, one can write*

$$b(t) = \int_{-\infty}^{+\infty} G(t, t')\,a(t')\,dt'$$

where $G = G(t, t')$ *is the Green function. In many cases one has* $G = G(t-t')$ *(thus the system has "time-invariant" properties), and* $b(t)$ *is expressed by a convolution product* $b(t) = \big(G * a\big)(t)$:

$$b(t) = \int_{-\infty}^{+\infty} G(t-t')\,a(t')\,dt' = \int_{-\infty}^{+\infty} G(\tau)\,a(t-\tau)\,d\tau$$

or equivalently $\widehat{b}(\omega) = \widehat{G}(\omega)\widehat{a}(\omega)$, where $\widehat{G}(\omega)$ is sometimes called transfer function. Then recall that the Green function is the answer to the Dirac delta input $a(t) = \delta(t)$, whereas $b(t) = a(t)$ for any $a(t)$ if $G(t) = \delta(t)$ (see Sect. 3.2). As well known, the Green function is said to be "causal" if $G(t) = 0$ for $t < 0$. This indeed guarantees that the answer $b(t)$ at any instant t depends only on the values of the input $a(t')$ at the "past" instants $t' < t$. Equivalently, assuming that there is some time t_0 such that the input $a(t)$ is equal to zero for any $t < t_0$, then also the corresponding solution, or answer, $b(t)$ is equal to zero for any $t < t_0$. In other words, the answer does not "precede" the input. For the case that the independent variable is not the time but a space variable x, see some comment before Problem 3.87.

It is certainly useful to prepare a list of "basic" Fourier transforms and inverse transforms. Remark that many transforms can be obtained by means of elementary integration: e.g., if

$$f(t) = \theta(\pm t)\exp(\mp t), \ \text{where}\, \theta(t) = \begin{cases} 0 & \text{for } t < 0 \\ 1 & \text{for } t > 0 \end{cases}, \ \text{then}\ \hat{f}(\omega) = \frac{1}{1 \mp i\omega}$$

or if

$$\widehat{f}(\omega) = \begin{cases} 1 & \text{for } |\omega| < 1 \\ 0 & \text{for } |\omega| > 1 \end{cases}, \ \text{then}\ f(t) = \frac{\sin t}{\pi t}$$

Other transforms require either the use of Jordan lemma, or— in some cases, more simply—the Fourier inversion theorem; e.g., using the above transforms, it is immediate to deduce

$$\mathscr{F}^{-1}\Big(\frac{1}{\omega + i}\Big) = \ldots \ \text{and}\ \mathscr{F}\Big(\frac{\sin t}{t}\Big) = \ldots$$

Elementary properties of transforms, as $\mathscr{F}\big(t^n f(t)\big) = \ldots$, the translation theorems, etc., can also be useful. To transform rational functions, one can either use Jordan lemma or the well-known decomposition of the function as a combination of simple fractions.

3.1.1 Basic Properties and Applications

(3.1)
Find the Fourier transform $\widehat{f}(\omega)$ of the "quasi-monochromatic" wave with fixed frequency ω_0 in the time interval: $-t_0 < t < t_0$:

$$f(t) = \begin{cases} \exp(-i\omega_0 t) & \text{for} \quad |t| < t_0 \\ 0 & \text{for} \quad |t| > t_0 \end{cases}$$

Draw $|\widehat{f}(\omega)|$ assuming "large" t_0; show that the principal contribution to $\widehat{f}(\omega)$ is centered around ω_0, with maximum value proportional to t_0, according to the physical interpretation of the Fourier transform as "frequency analysis". See also Problem 3.15. For the limit case $t_0 \to \infty$, see Problem 3.25, q. (1).

(3.2)
Repeat the same calculations and considerations as in the above problem for these other types of "quasi-monochromatic" waves:

$$f(t) = \exp(-i\omega_0 t)\frac{a^2}{a^2+t^2} \quad ; \quad f(t) = \exp(-i\omega_0 t)\,\exp(-t^2/a^2)$$

with "large" a. See also Problem 3.15. For the limit case $a \to \infty$, see Problem 3.25, q. (2).

(3.3)
Find the following Fourier and inverse Fourier transforms

$$\mathscr{F}\Big(\frac{1}{t^2+t+1}\Big) \quad ; \quad \mathscr{F}\Big(\frac{t}{(t^2+1)^2}\Big) \quad ; \quad \mathscr{F}\Big(\frac{\cos t}{1+t^2}\Big) \quad ;$$

$$\mathscr{F}^{-1}\Big(\frac{\omega\sin\omega}{1+\omega^2}\Big) \quad ; \quad \mathscr{F}^{-1}\Big(\frac{1}{(\omega\pm i)^3}\Big) \quad ; \quad \mathscr{F}^{-1}\Big(\frac{\sin\omega}{(\omega\pm i)^4}\Big)$$

(3.4)
(1) Consider the first-order linear nonhomogeneous ordinary differential equation (ODE) for the unknown $x = x(t)$ $\big($where $f(t) \in L^2(\mathbf{R})$ is given$\big)$

$$\dot{x} + x = f(t)\,, \qquad x = x(t)$$

Introducing the Fourier transforms $\widehat{f}(\omega)$ and $\widehat{x}(\omega)$, obtain the solution $x(t)$ in the form of convolution product with a Green function $G(t)$, where $f(t)$ is the input and $x(t)$ the output: $x(t) = \big(G * f\big)(t)$. Find and draw the function $G(t)$. Explain why one obtains only one solution, and not ∞^1 solutions, as expected from general the theory of first-order ODEs.

(2) The same questions for the equation

$$\dot{x} - x = f(t)\,, \qquad x = x(t)$$

(3) The equation of an electric series circuit of a resistance R and an inductance L

$$L\frac{dI}{dt} + RI = V(t)$$

where $I = I(t)$ is the current and $V = V(t)$ the applied voltage, has the same form as the equation in (1). The Fourier transform $\widehat{G}(\omega)$ has here an obvious physical interpretation ..., but there is, apparently, a “wrong” sign compared with the usual impedance formula $Z(\omega) = R + i\omega L$: why?

(3.5)
Using Fourier transform, find explicitly the solution $x(t)$ of the equations given in questions (1) and (2) of the above problem in the cases[1]

$$f(t) = \theta(t)\exp(-ct) \qquad \text{and} \qquad f(t) = \exp(-c|t|)\,, \qquad c > 0$$

with $c \neq 1$ and with $c = 1$.

(3.6)
(1) Using Fourier transform, find and draw the Green function $G(t)$ of the ODE

$$a\ddot{x} + b\dot{x} + cx = f(t)\,, \qquad x = x(t)$$

for different values of the real constants a, b, c. Notice that the equation of the motion of a particle, subjected to an elastic force, to a viscous damping and to an external time-dependent force is exactly of this form, with all coefficients $a, b, c > 0$.

(2) In the case $a, b, c > 0$, show that all the singularities in the complex plane ω lie in the inferior half-plane $\mathtt{Im}\,\omega < 0$, with 3 possibilities: 2 complex conjugate single poles, 2 single poles in the imaginary axis, and a double pole. Find and draw $G(t)$ in each one of these cases. Verify also that in all cases the Green function is causal: $G(t) = 0$ when $t < 0$. (For the case $b = 0$, see Problem 3.78; for the case $c = 0$, see Problem 3.80; for the case $b = c = 0$, see Problem 3.79).

(3.7)
Find and draw the Green function $G(t)$ by calculating the inverse Fourier transform in each one of the following four cases. What of these are causal, i.e., $G(t) = 0$ for $t < 0$?

$$\widehat{G}(\omega) = \frac{\exp(\pm i\omega)}{1 \pm i\omega}$$

(3.8)
(1) Let $f(x) \in L^1(\mathbf{R}) \cap L^2(\mathbf{R})$. Using Fourier transform, give another proof (see Problem 1.4) that the subspace of the functions satisfying

---

[1] Direct calculation of the convolution product $\big(G * a\big)(t)$ is often not easy; it may be preferable to obtain first $\mathscr{F}\big(f(t)\big) = \widehat{f}(\omega)$ and then calculate $x(t)$ by inverse Fourier transform: $x(t) = \mathscr{F}^{-1}\big(\widehat{G}(\omega)\widehat{f}(\omega)\big)$.

$$\int_{-\infty}^{+\infty} f(x)\, dx = 0$$

is dense in $L^2(\mathbf{R})$. The same for the functions such that

$$\int_{-\infty}^{+\infty} f(x) \sin qx\, dx = 0 \qquad \text{for any } q \in \mathbf{R}$$

Hint: Observe that $\int_{-\infty}^{+\infty} f(x)\, dx = \widehat{f}(0)$, etc.
(2) Let $f(x)$ be such that $x^n f(x) \in L^1(\mathbf{R}) \cap L^2(\mathbf{R})$ for all integer $n \geq 0$: using Fourier transform show that the subspace of the these functions is dense in $L^2(\mathbf{R})$ (cf. Problem 1.3).
(3) Let $f(x)$ be such that $x^n f(x) \in L^1(\mathbf{R}) \cap L^2(\mathbf{R})$ for all integer $n \geq 0$ and

$$\int_{-\infty}^{+\infty} x^n f(x)\, dx = 0\,, \qquad \forall n \geq 0$$

using Fourier transform show that the set of the these functions is dense in $L^2(\mathbf{R})$.

(3.9)
To solve the following questions, recall that $\mathscr{F}\big((\sin x)/x\big) = \ldots$ and use Parseval identity for Fourier transform.
(1) Calculate (for a different procedure, based on integration in the complex plane, see Problem 2.30)

$$\int_{-\infty}^{+\infty} \frac{\sin^2 x}{x^2}\, dx$$

(2) Calculate the norm $\|f_n(x)\|_{L^2}$, where

$$f_n(x) = \frac{d^n}{dx^n} \frac{\sin x}{x}\,, \qquad n = 0, 1, 2, \ldots\,,\ x \in \mathbf{R}$$

(3) Show that the sequence of functions

$$f_n(x) = \frac{\sin x}{x - n\pi}\,, \qquad n \in \mathbf{Z},\ x \in \mathbf{R}$$

is an orthogonal but not a complete set in $L^2(\mathbf{R})$. Characterize the functions $h(x) \in L^2(\mathbf{R})$ such that $(h, f_n) = 0,\ \forall n \in \mathbf{Z}$.

(3.10)
(1) Consider the following integral $I(a)$ as a scalar product in $L^2(\mathbf{R})$, as indicated

$$I(a) = \int_{-\infty}^{+\infty} \frac{\exp(i\omega a) - 1}{i\omega(1 + i\omega)}\, d\omega = \Big(\frac{1}{1 - i\omega}\,,\ \frac{\exp(i\omega a) - 1}{i\omega}\Big)\,, \qquad a > 0$$

recalling the inverse Fourier transforms of the functions appearing in the scalar product and applying Parseval identity, evaluate $I(a)$. Evaluate then the limit $\lim_{a\to+\infty} I(a)$.

(2) Using Parseval identity as before, evaluate

$$\lim_{a\to+\infty}\int_{-\infty}^{+\infty}\frac{\exp(i\omega a)-1}{i\omega}\exp(-\omega^2)\,d\omega$$

(3.11)
(1) Calculate the two inverse Fourier transforms

$$\mathscr{F}^{-1}\Big(\frac{\exp(i\omega)}{1-i\omega}\Big) \quad\text{and}\quad \mathscr{F}^{-1}\Big(\frac{\exp(-i\omega)}{1+i\omega}\Big)$$

(2) Show that the second inverse Fourier transform in the above question can be immediately obtained from the first one observing that $\widehat{f}(-\omega)=\ldots$.

(3) By integration in the complex plane, evaluate the integral

$$I = \int_{-\infty}^{\infty}\frac{\exp(2i\omega)}{(1-i\omega)^2}\,d\omega$$

(4) Check the result obtained above observing that $I = (\widehat{f_2},\widehat{f_1})$ where $\widehat{f_1} = \exp(i\omega)/(1-i\omega)$ and $\widehat{f_2} = \ldots$, using Parseval identity and the results in (1).

(3.12)
A particle of mass $m = 1$ is subjected to an external force $f(t)$ and to a viscous damping; denoting by $v = v(t)$ its velocity, the equation of motion is then

$$\dot v + \beta v = f(t)\,, \qquad t\in\mathbf{R},\ \beta>0$$

(1) Let $f(t) = \theta(t)\exp(-t)$ and $\beta \neq 1$. Using Fourier transform, find $\widehat{v}(\omega) = \mathscr{F}\big(v(t)\big)$ and $v = v(t)$.

(2) One has from (1) that $\lim_{t\to+\infty} v(t) = 0$, then the final kinetic energy $v^2(+\infty)/2$ of the particle is zero. This implies that the work W_f done by the force, i.e.,

$$W_f = \int_L f\,dx = \int_{-\infty}^{+\infty} f(t)v(t)\,dt$$

(where L is the space covered by the particle) is entirely absorbed by the work W_β done by friction, which is

$$W_\beta = \int_L \beta v\,dx = \beta\int_{-\infty}^{+\infty} v^2(t)\,dt$$

Evaluate W_f and W_β using the given $f(t)$ and the expression of $v(t)$ obtained in (1), and verify that indeed $W_f = W_\beta$.

(3) Repeat the above check using Fourier transform: precisely, thanks to the Parseval identity, write $W_f = \big(f, v\big)$ and $W_\beta = \beta\big(v, v\big)$ as integrals in the variable ω in terms of the expressions of $\widehat{f}(\omega)$ and $\widehat{v}(\omega)$ obtained above, evaluate these integrals by integration in the complex plane ω and verify again that $W_f = W_\beta$.

(4) The same questions in the case $\beta = 1$.

The following problem extends this result to the case of general forces $f(t) \in L^1(\mathbf{R}) \cap L^2(\mathbf{R})$. Problems 3.75 and 3.76 deal with the case of vanishing damping $\beta \to 0^+$.

(3.13)
This is the same problem as the previous one, now with a *general* force $f(t) \in L^1(\mathbf{R}) \cap L^2(\mathbf{R})$ $\big($and $f(t)$ real, of course$\big)$:

(1) Find the Fourier transform $\widehat{G}(\omega)$ of the Green function of the equation. Show that $\widehat{v}(\omega) = \mathscr{F}\big(v(t)\big) \in L^1(\mathbf{R})$, which implies that $\lim_{t\to+\infty} v(t) = 0$ and then that the final kinetic energy $v^2(+\infty)/2$ of the particle is zero.

(2) (*a*) Using Parseval identity, write the work done by the force

$$W_f = \int_L f\,dx = \int_{-\infty}^{+\infty} f(t)v(t)\,dt = \big(f, v\big)$$

(where L is the space covered by the particle) as an integral in the variable ω in terms of $\widehat{v}(\omega)$ and $\widehat{G}(\omega)$.

(*b*) Proceeding as in (*a*), write the work W_β done by friction

$$W_\beta = \int_L \beta v\,dx = \beta\int_{-\infty}^{+\infty} v^2(t)\,dt = \beta\big(v, v\big)$$

as an integral containing $\widehat{v}(\omega)$; show then that $W_f = W_\beta$, as expected $\big($it is clearly understood that $v(-\infty) = 0\big)$. *Hint*: recall that $f(t)$ is real, and the same is for $v(t)$, therefore $\widehat{v}^*(\omega) = \ldots$; use also the obvious fact that an expression as $F(x)F(-x)$ is an *even* function of the real variable x.

(3.14)
Consider a linear system described by a Green function $G(t)$, with input $a(t)$ and output $b(t)$ related by the usual rule $b(t) = \big(G * a\big)(t)$. Let $a(t) \in L^2(\mathbf{R})$.

(1) Assume $G(t) \in L^1(\mathbf{R})$. Show that $b(t) \in L^2(\mathbf{R})$ and find a constant C such that $\|b(t)\|_{L^2(\mathbf{R})} \le C\,\|a(t)\|_{L^2(\mathbf{R})}$.

(2) Assume $G(t) \in L^2(\mathbf{R})$. Show that $b(t)$ is a continuous function vanishing as $|t| \to \infty$.

(3) Assume that the Fourier transform $\widehat{G}(\omega) = \mathscr{F}\big(G(t)\big) \in L^2(\mathbf{R})$ and has a compact support. Show that $b(t)$ is continuous and differentiable (how many times?). Can one expect that also $b(t)$ has compact support?

(3.15)
The "classical uncertainty principle" states that, given any function $f(t)$ $\big($under some obvious regularity assumptions, see (1) below$\big)$, one has

$$\Delta t\, \Delta\omega \geq 1/2$$

where the quantities Δt and $\Delta\omega$ are defined by

$$\Delta t^2 = \frac{1}{\|f(t)\|^2}\int_{-\infty}^{+\infty}\big(t-\overline{t}\big)^2|f(t)|^2\,dt \quad , \quad \overline{t} = \frac{1}{\|f(t)\|^2}\int_{-\infty}^{+\infty} t\,|f(t)|^2\,dt$$

$$\Delta\omega^2 = \frac{1}{\|\widehat{f}(\omega)\|^2}\int_{-\infty}^{+\infty}\big(\omega-\overline{\omega}\big)^2|\widehat{f}(\omega)|^2\,d\omega \quad , \quad \overline{\omega} = \frac{1}{\|\widehat{f}(\omega)\|^2}\int_{-\infty}^{+\infty}\omega\,|\widehat{f}(\omega)|^2\,d\omega$$

(1) To show this result, assume for simplicity $\overline{t} = \overline{\omega} = 0$ (this is not restrictive), assume $f(t) \in L^2(\mathbf{R})$, $tf(t) \in L^2(\mathbf{R})$, $\omega\widehat{f}(\omega) \in L^2(\mathbf{R})$ and finally that $\omega\,\widehat{f}(\omega)$ vanishes as $|\omega| \to \infty$; then verify and complete the steps of the following calculation:

$$0 \leq \left\| \frac{\omega}{2\Delta\omega^2}\widehat{f}(\omega) + \frac{d\widehat{f}(\omega)}{d\omega}\right\|^2 = \ldots$$

$$= \frac{\|\widehat{f}(\omega)\|^2}{4\Delta\omega^2} + \|\mathscr{F}\big(itf(t)\big)\|^2 + \frac{1}{2\Delta\omega^2}\int_{-\infty}^{+\infty}\omega\,\frac{d}{d\omega}\Big(\widehat{f}(\omega)\widehat{f}^*(\omega)\Big)\,d\omega = \ldots$$

$$= 2\pi\,\|f(t)\|^2\Big(\Delta t^2 - \frac{1}{4\Delta\omega^2}\Big)$$

(2) Observing that the "minimum uncertainty", i.e., $\Delta t\,\Delta\omega = 1/2$, occurs when the quantity appearing at the beginning of the first line above is zero, show that the minimum is verified when $\widehat{f}(\omega) = \ldots$ and then (now with generic $\overline{t}$ and $\overline{\omega}$, not necessarily zero)

$$f(t) = c\,\exp\big(-\alpha^2(t-\overline{t})^2/2\big)\exp(-i\overline{\omega}t)\,, \qquad \alpha = \sqrt{2}\,\Delta\omega = \frac{1}{\sqrt{2}\Delta t}$$

(3) Give an estimation of the spatial length $\Delta x = c\,\Delta t$ of the wave packet of a red light with $\lambda \simeq 7000$ Å (e.g., an atomic emission) and $\Delta\omega/\omega = \Delta\lambda/\lambda \simeq 10^{-6}$, and of a red laser wave with $\Delta\omega/\omega \simeq 10^{-12}$.

(4) Changing the variables t and ω into x and $k = 2\pi/\lambda$ and using de Broglie principle $\lambda = h/p$, deduce the well-known Heisenberg uncertainty principle in quantum mechanics $\Delta x\,\Delta p \geq \hbar/2$.

3.1.2 Fourier Transform and Linear Operators in $L^2(\mathbf{R})$

This subsection is devoted to considering first examples of linear operators $T : L^2(\mathbf{R}) \to L^2(\mathbf{R})$ which can be conveniently examined introducing their "Fourier transform" $\widehat{T}$ defined in this way: if

$$Tf(x) = g(x)\,, \qquad f(x),\ g(x) \in L^2(\mathbf{R})$$

then $\widehat{T}$ is the operator such that

$$\widehat{T}\,\widehat{f}(\omega) \;=\; \widehat{g}(\omega)\,, \qquad \text{i.e.,} \quad \widehat{T} = \mathscr{F}\,T\,\mathscr{F}^{-1}.$$

Other more general examples of operators in the context of Fourier transforms and distributions will be considered in Sect. 3.2.2

(3.16)
(1) Show that $\|\widehat{T}\| = \|T\|$.

(2) Assume that $\widehat{T}$ is a projection $\big($on some subspace $H_1 \subset L^2(\mathbf{R})\big)$; is the same true for T? on what subspace?

(3) Assume that $\widehat{T}$ admits an eigenvector $\varphi = \varphi(\omega)$ with eigenvalue λ; what information can be deduced for T?

(4) Assume that $\widehat{T}$ admits a orthonormal complete system of eigenvectors; does the same hold for T?

(3.17)
Find the operator $\widehat{T}$ in each one of the following cases:

$$T\ f(x) = f(x-a) \quad \text{with} \quad a \in \mathbf{R} \quad ; \quad T\ f(x) = \frac{df}{dx} \quad ;$$

$$T\ f(x) = \int_{-\infty}^{+\infty} f(x-y)\,g(y)\,dy \equiv \big(f * g\big)(x) \quad \text{with} \quad g(x) \in L^1(\mathbf{R}) \cap L^2(\mathbf{R})$$

(3.18)
Consider the operator defined in $L^2(\mathbf{R})$

$$T\ f(t) \;=\; \int_{-\infty}^{+\infty} f(\tau)\frac{\sin(t-\tau)}{\pi(t-\tau)}\,d\tau$$

Introducing Fourier transform,

(1) Show that T is a projection: on what subspace?

(2) T has a clear physical interpretation in terms of the variable ω: explain!

(3) What differentiability properties can be deduced for the function $g(t) = T\,f(t)$? and about its behavior as $|t| \to \infty$?

(4) Study the convergence as $n \to \infty$ of the sequence of operators

$$T_n\,f(t) = \int_{-\infty}^{+\infty} f(\tau)\frac{\sin\big(n(t-\tau)\big)}{\pi(t-\tau)}\,d\tau$$

(3.19)
Consider the operator defined in $L^2(\mathbf{R})$

$$T\,f(x) = \int_{-\infty}^{+\infty} f(y)\frac{1}{1+(x-y)^2}\,dy$$

Introducing Fourier transform,

(1) Find $\|T\|$

(2) Find $\mathtt{Ran}\,T$, specifying if $\mathtt{Ran}\,T = L^2(\mathbf{R})$ or at least is dense in it.

(3) For what $\rho \in \mathbf{R}$ does the operator $T - \rho I$ admit bounded inverse?

(4) What differentiability properties for the functions $g(x) = Tf(x)$ can be expected?

(5) If $\{f_n(x)\}$ is a complete set in $L^2(\mathbf{R})$, is the same true for the set $g_n = Tf_n$?

(6) Study the convergence as $a \to 0$ of the family of operators

$$T_a\,f(x) = \int_{-\infty}^{+\infty} f(y)\frac{a}{a^2+(x-y)^2}\,dy\,, \qquad a > 0$$

(3.20)
(1) Using Fourier transform, show that the family of operators T_a

$$T_a f(x) = f(x-a)\,, \qquad a \in \mathbf{R}$$

converges weakly to zero as $a \to \infty$.

(2) Show that T_a converges strongly to the identity operator I as $a \to 0$.

(3.21)
Consider the following, slightly different definition (in the factors $1/\sqrt{2\pi}$) of the Fourier transform (and inverse Fourier transform):

$$\widetilde{\mathscr{F}} = \frac{1}{\sqrt{2\pi}}\mathscr{F} \quad \text{, i.e.,} \quad \widetilde{\mathscr{F}}\big(f(x)\big) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(x)\exp(ixy)\,dx = g(y)$$

and

$$\widetilde{\mathscr{F}}^{-1} = \sqrt{2\pi}\,\mathscr{F}^{-1} \quad \text{, i.e.,} \quad \widetilde{\mathscr{F}}^{-1}\big(g(y)\big) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} g(y)\exp(-ixy)\,dy = f(x)$$

(1) Show that $\widetilde{\mathscr{F}}$ (and $\widetilde{\mathscr{F}}^{-1}$ of course) are unitary operators in the Hilbert space $L^2(\mathbf{R})$.
(2) Show that

$$\widetilde{\mathscr{F}}^2 = S \quad \text{where } S \text{ is the parity operator : } S\,f(x) = f(-x)$$

and then $\widetilde{\mathscr{F}}^4 = I$
(3) Let T be the Hermite operator:

$$T = -\frac{d^2}{dx^2} + x^2$$

Show that $\widehat{T} = T$ and then $\widetilde{\mathscr{F}}T = T\widetilde{\mathscr{F}}$.
(4) Recall that the eigenfunctions of T are the Hermite functions $u_n = \exp(-x^2/2)$ $H_n(x)$ (where $H_n(x)$ are polynomials, $n = 0, 1, 2, \ldots$), and that the corresponding eigenvalues $\lambda_n = 2n + 1$ are non-degenerate: using then the result seen in (2) and (3), find the eigenfunctions and the eigenvalues of the operator $\widetilde{\mathscr{F}}$.
(5) Conclude showing that the Fourier operator $\widetilde{\mathscr{F}}$ is a linear combination of 4 projections; on what subspaces?

3.2 Tempered Distributions and Fourier Transforms

We will be almost exclusively concerned with the space of "tempered" distributions $\mathscr{S}'$, which can be considered as the "largest" space where Fourier transforms are introduced in a completely natural and well-defined way.
As in the introduction to Sect. 1.2, where sequences of linear operators were concerned, also in this Section questions as "Study the convergence" or "Find the limit" of sequences (or families) of functions/distributions are "cumulative" questions, which require first, as obvious, to conjecture the possible limit, but also to specify in what sense the limit exists. There is in $\mathscr{S}'$ the notion of convergence "in the sense of distributions": given a sequence $T_n \in \mathscr{S}'$ (or a family $T_a \in \mathscr{S}'$), one says that $T_n \to T$ in $\mathscr{S}'$ if for any test function $\varphi(x) \in \mathscr{S}$ one has $< T_n, \varphi > \to < T, \varphi >$. There are distributions which are associated to ordinary functions $u = u(x)$; in this case distributions and functions can be "identified", writing, e.g., $< u, \varphi >$, with $\varphi \in \mathscr{S}$, instead of the more correct notation $< T_u, \varphi >$. Accordingly, one can consider the convergence in $\mathscr{S}'$ of a sequence of functions $u_n(x)$, meaning the convergence of the distributions T_{u_n}. This is a new notion of convergence for sequences (or families) of functions, in addition (and to be compared) to the "old" notions, as, e.g., pointwise or uniform convergence, or—in the case $u_n(x) \in L^2(\mathbf{R})$ —the convergence in the L^2 norm or in the weak L^2 sense. Recall that $u_n(x) \to u(x)$

in the sense of weak L^2 convergence if $\forall g \in L^2$, one has $(g, u_n) \to (g, u)$. By the way, also in this section, Lebesgue theorem will be a useful tool for examining convergence properties: see the introductory remarks to Sect. 1.2. The notion of limit in $\mathscr{S}'$ is particularly relevant because it involves, for instance, the "approximation" of the Dirac delta $\delta(x)$ by means of "regular" (possibly C^∞) functions. The same holds for other distributions, as the derivatives of the delta, and, e.g., the important notion of Cauchy "Principal Part" $\mathtt{P}(1/x)$, defined by

$$< \mathtt{P}\frac{1}{x}, \varphi > = \lim_{\varepsilon \to 0^+} \Big(\int_{-\infty}^{-\varepsilon} + \int_{\varepsilon}^{+\infty} \frac{\varphi(x)}{x}\, dx \Big), \qquad \varepsilon > 0$$

A function (or distribution) which will be used in the following is the "sign of x", i.e.,

$$\mathtt{sgn}\, x = \begin{cases} -1 & \text{for } x < 0 \\ 1 & \text{for } x > 0 \end{cases} = \theta(x) - \theta(-x)$$

E.g., one has $\mathscr{F}(\mathtt{sgn}\, x) = 2i\, \mathtt{P}\big(1/\omega\big)$ and $\mathscr{F}\big(\mathtt{P}(1/x)\big) = \pi i\, \mathtt{sgn}(\omega)$.

3.2.1 General Properties

(3.22)
Find the pointwise limit as $n \to \infty$ of the sequence of functions

$$f_n(x) = \begin{cases} n \sin nx & \text{for } 0 \le x \le \pi/n \\ 0 & \text{elsewhere} \end{cases}, \qquad n = 1, 2, \dots;\ x \in \mathbf{R}$$

and then their limit in $\mathscr{S}'$. *Hint*: consider $f_n(x)$ as distributions, apply them to a generic test function $\varphi(x) \in \mathscr{S}$, i.e., $< f_n, \varphi > = \int_{-\infty}^{+\infty} \dots$, perform a change of variable ….

(3.23)
(1) (a) Find the limits in $\mathscr{S}'$ as $n \to \infty$ of the sequences of functions, with $x \in \mathbf{R}$,

$$f_n(x) = \theta(x) \exp(-nx)\,, \qquad n = 1, 2, \dots$$

or, which is the same,

$$f_\varepsilon(x) = \frac{1}{\varepsilon}\theta(x) \exp(-x/\varepsilon)\,, \qquad \varepsilon > 0$$

(as in previous problem, consider the functions as distributions, apply them to a generic test function, etc.).

(*b*) Obtain again the limit using Fourier transform. *Hint*: the pointwise limit of the Fourier transforms can be easily obtained, and coincides with the $\mathscr{S}'$-limit (why?), then ....

(2) The same questions (*a*) and (*b*) for the sequence of functions

$$f_n(x) = n \exp(-n|x|)$$

(3) and for the sequence of functions

$$f_n(x) = n \exp(-x^2 n^2)$$

(4) and for the sequence of functions

$$f_n(x) = \frac{n}{1 + n^2 x^2}$$

(5) Find $\lim_{n\to\infty} g_n(x)$ in $\mathscr{S}'$ where

$$g_n(x) = n^3 x \exp(-x^2 n^2)$$

both using the limit obtained in (3) $\big($notice that $g_n(x) \propto f_n'(x)$, where $f_n(x)$ is the sequence given in (3)$\big)$, and using Fourier transform.

(3.24)
To calculate the limit in $\mathscr{S}'$ as $n \to \infty$ of the sequence of functions

$$f_n(x) = \frac{\sin nx}{x}, \qquad x \in \mathbf{R}$$

recall that $\mathscr{F}\big((\sin nx)/x\big) = \ldots$, then find the limit (pointwise and $\mathscr{S}'$) of the Fourier transforms; therefore ....

(3.25)
(1) Find the limit as $t_0 \to \infty$ of the "quasi-monochromatic" wave seen in Problem 3.1

$$f(t) = \begin{cases} \exp(-i\omega_0 t) & \text{for} \quad |t| < t_0 \\ 0 & \text{for} \quad |t| > t_0 \end{cases}$$

and of its Fourier transform $\widehat{f}(\omega)$ $\big($put $\omega - \omega_0 = x$ in $\widehat{f}(\omega)$, cf. also Problem 3.24$\big)$. The physical interpretation is clear!

(2) The same questions for the limit as $a \to +\infty$ of the "quasi-monochromatic" waves seen in Problem 3.2 $\big($cf. also Problem 3.23, q. (2) and q. (3)$\big)$

(3.26)
(1) Let

$$u(t) = \begin{cases} 1 & \text{for} \quad 0 < t < 1 \\ 0 & \text{elsewhere} \end{cases}$$

The Fourier transform $\widehat{u}(\omega) = \mathscr{F}\big(u(t)\big)$ can be trivially obtained by elementary integration. Observing that one can write $u(t) = \theta(t) - \theta(t-1)$, one could also calculate $\widehat{u}(\omega)$ using the formula for $\mathscr{F}\big(\theta(t)\big)$: verify that the two results (seemingly different at first sight), after some simplifications, actually coincide, as expected!

(2) Find the Fourier transform of

$$v(t) = \begin{cases} 0 \ \text{for}\ 0 < t < 1 \\ 1 \ \text{elsewhere} \end{cases} = 1 - u(t) = \theta(-t) + \theta(t-1)$$

(3.27)
Starting from the (well-known) Fourier transform of $1/(1+x^2)$, calculate the Fourier transform of $x^2/(1+x^2)$ in these two different ways:

(a) writing

$$\frac{x^2}{1+x^2} = 1 - \frac{1}{1+x^2}$$

(b) using the rule $\mathscr{F}\big(x^2 f(x)\big) = -d^2 \widehat{f}(k)/dk^2$.

(3.28)
Find the following Fourier and inverse Fourier transforms:

$$\mathscr{F}\big(|t|\big)\ ;\ \mathscr{F}^{-1}\big(|\omega|\big)\ ;\ \mathscr{F}\big(t\,\theta(t)\big)\ ;\ \mathscr{F}\big(\theta(t)\ \cos t\big)\ ;\ \mathscr{F}\Big(\mathrm{P}\Big(\frac{1}{x}\Big)\frac{1}{1+x^2}\Big)\ ;$$

$$\mathscr{F}^{-1}\Big(D\,\mathrm{P}\Big(\frac{1}{\omega}\Big)\Big)\ ;\ \mathscr{F}^{-1}\Big(\mathrm{P}\frac{\omega}{\omega-1}\Big)\ ;\ \mathscr{F}^{-1}\Big(\frac{\exp(-i\omega)}{\omega+i}\mathrm{P}\Big(\frac{1}{\omega}\Big)\Big)\ ;$$

$$\mathscr{F}\big(\exp(i|t|)\big) = \mathscr{F}\big(\theta(t)\ \exp(it) + \ldots\big)\ ;\ \mathscr{F}^{-1}\big(\omega^2 \sin|\omega|\big)\ ;\ \mathscr{F}^{-1}\Big(\Big(\mathrm{P}\frac{1}{\omega}\Big)\frac{1}{\omega \pm i}\Big)\ ;$$

$$\mathscr{F}^{-1}\Big(\mathrm{P}\frac{1}{\omega(\omega-1)}\Big) = \mathscr{F}^{-1}\Big(\mathrm{P}\Big(\frac{1}{\omega-1}\Big) - \mathrm{P}\Big(\frac{1}{\omega}\Big)\Big)\ ;\ \mathscr{F}^{-1}\Big(\mathrm{P}\Big(\frac{\exp(i\omega)-1}{\omega^2}\Big)\Big)$$

(3.29)
(1) The Fourier transform $\widehat{f}(k)$ of the function

$$f(x) = \frac{1-\cos x}{x}\,, \qquad x \in \mathbf{R}$$

can be evaluated by integration in the complex plane using (with some care) Jordan lemma. Here, two alternative ways are proposed:

(a) put $g(x) = 1 - \cos x$ and evaluate first $\mathscr{F}\big(g(x)\big) = \widehat{g}(k)$; notice on the other hand that $\widehat{g}(k) = \mathscr{F}\big(xf(x)\big) = -i\, d\widehat{f}(k)/dk$, which gives $\widehat{f}(k)$ $\big($one constant must then be fixed ..., recall that $\widehat{f}(k)$ must belong to $L^2(\mathbf{R})\big)$;

(b) write

$$\mathscr{F}\big(f(x)\big) = \mathscr{F}\Big(\mathrm{P}\Big(\frac{1}{x}\Big) - \frac{1}{2}\exp(ix)\mathrm{P}\Big(\frac{1}{x}\Big) - \frac{1}{2}\exp(-ix)\mathrm{P}\Big(\frac{1}{x}\Big)\Big)$$

and use $\mathscr{F}\big(\mathrm{P}(1/x)\big) = \ldots$.

(2) The same for the function

$$F(x) = \frac{1 - \cos x}{x^2}\,, \qquad x \in \mathbf{R}$$

(a) with $g(x) = 1 - \cos x$ as before, now $\widehat{g}(k) = \mathscr{F}\big(x^2F(x)\big) = -d^2\widehat{F}(k)/dk^2$ $\big($two constants must be fixed to determine $\widehat{F}(k)$ ..., recall that as before $\widehat{F}(k)$ must belong to $L^2(\mathbf{R})\big)$;

(b) write

$$\mathscr{F}\big(f(x)\big) = \mathscr{F}\Big(-D\,\mathrm{P}\Big(\frac{1}{x}\Big) + \frac{1}{2}\exp(ix)D\,\mathrm{P}\Big(\frac{1}{x}\Big) + \frac{1}{2}\exp(-ix)D\,\mathrm{P}\Big(\frac{1}{x}\Big)\Big)$$

and use $\mathscr{F}\big(D\,\mathrm{P}(1/x)\big) = \ldots$ (clearly, $D = d/dx$ is the derivative of distributions).

(3) Observing that $f(x) = x\,F(x)$, verify that $d\widehat{F}(k)/dk = \ldots$.

(3.30)
Find the Fourier transform of the following distribution

$$\mathrm{P}\Big(\frac{\sin x}{x - a}\Big)\,, \qquad a \in \mathbf{R}$$

For what values of $a \in \mathbf{R}$ can the symbol P be omitted? for these values of a, what is the support of the Fourier transform?

(3.31)
(1) Using Fourier transform, find the "fundamental limits"

$$\lim_{\varepsilon \to 0^+} \frac{1}{x \pm i\varepsilon}\,, \qquad \varepsilon > 0$$

and deduce

$$\lim_{\varepsilon \to 0^+} \frac{x}{x^2 + \varepsilon^2}\,, \quad \lim_{\varepsilon \to 0^+} \frac{\varepsilon}{x^2 + \varepsilon^2}$$

(2) Find then

$$\lim_{\varepsilon\to 0^+}\int_{-\infty}^{\infty}\frac{\exp(-x^4)}{x\pm i\varepsilon}\,dx$$

(3.32)
Let

$$u_a(\omega) = \frac{\exp(ia\omega)-1}{i\omega}, \qquad a>0$$

(1) Find the inverse Fourier transform $\mathscr{F}^{-1}\big(u_a(\omega)\big)$ and $\lim_{a\to\infty} u_a(\omega)$.

(2) Using the above result, find

$$\lim_{a\to\infty}\int_{-\infty}^{+\infty} u_a(\omega)\cos\omega\ \exp(-\omega^2)\,d\omega$$

(3) Considering $\varphi(\omega)=\omega/(1+\omega^2)$ as a test function, use again the result obtained in (1) to find

$$\lim_{a\to\infty}\int_{-\infty}^{+\infty} u_a(\omega)\frac{\omega}{1+\omega^2}\,d\omega$$

(4) Check the above result: evaluate the integral (either by integration in the complex plane or using inverse Fourier transform)

$$I_a = \int_{-\infty}^{+\infty}\frac{\exp(ia\omega)-1}{1+\omega^2}\,d\omega, \qquad a>0$$

then find $\lim_{a\to\infty} I_a$ and compare with the result in (3).

(3.33)
Let

$$u_a(x) = \begin{cases} -1 \text{ for } x<a \\ 1 \quad \text{for } x>a \end{cases}, \qquad x\in\mathbf{R},\ a\in\mathbf{R}$$

(1) Find $\lim_{a\to+\infty} u_a(x)$.

(2) Find the Fourier transform $\widehat{u}_a(y)=\mathscr{F}\big(u_a(x)\big)$ and $\lim_{a\to+\infty}\widehat{u}_a(y)$.

(3) Using the result in (2), find

$$\lim_{a\to+\infty}\mathrm{P}\int_{-\infty}^{+\infty}\frac{\exp(iay)}{y}\exp(-y^2)\,dy$$

(4) Considering $\varphi(y)=1/(y-2i)$ as a test function, use again the result obtained in (2) to find

$$\lim_{a\to+\infty} \mathrm{P}\int_{-\infty}^{+\infty} \frac{\exp(iay)}{y(y-2i)}\,dy$$

(5) To check the above result evaluate now the integral by integration in the complex plane

$$I_a = \mathrm{P}\int_{-\infty}^{+\infty} \frac{\exp(iay)}{y(y-2i)}\,dy$$

then find $\lim\limits_{a\to+\infty} I_a$ and compare with the result in (4).

(3.34)
(1) Observing that

$$u_\varepsilon(x) = \frac{1}{(x-i\varepsilon)^2} = -\frac{d}{dx}\frac{1}{x-i\varepsilon}\,, \qquad \varepsilon > 0$$

find the Fourier transform $\widehat{u}_\varepsilon(\omega) = \mathscr{F}\big(u_\varepsilon(x)\big)$ and $\lim\limits_{\varepsilon\to 0^+} \widehat{u}_\varepsilon(\omega)$.

(2) Find $\lim\limits_{\varepsilon\to 0^+} u_\varepsilon(x)$ either from (1) via inverse Fourier transform, or applying the result in Problem 3.31, q. (1).

(3) Using the result obtained in (2), evaluate the limits

$$\lim_{\varepsilon\to 0^+} \int_{-\infty}^{+\infty} \frac{\exp(-x^2)}{(x-i\varepsilon)^2} \quad \text{and} \quad \lim_{\varepsilon\to 0^+} \int_{-\infty}^{+\infty} \frac{x\exp(-x^2)}{(x-i\varepsilon)^2}$$

(3.35)
Let T_a be the distribution

$$T_a = \frac{1}{a}\Big(\mathrm{P}\frac{1}{x-a} - \mathrm{P}\frac{1}{x}\Big)\,, \qquad a > 0$$

(1) Find the Fourier transform $\widehat{T_a}$ of T_a.

(2) Find the limit $\widehat{T} = \lim\limits_{a\to 0} \widehat{T_a}$ and find $T = \mathscr{F}^{-1}\widehat{T}$.

(3) Evaluate $< T, \exp(-x^2) >$ and $< T, \sin x \exp(-x^4) >$.

(3.36)
Using Fourier transform:

(1) Evaluate the convolution product

$$C(x) = \delta'(x) * \big(x\theta(x)\big)$$

(2) Show that for any $n = 1, 2, \ldots$ one has

$$\delta^{(n)}(x) * T = D^{(n)}T$$

(3.37)
Let $C(f)$ be the convolution product

$$C(f) = \int_{-\infty}^{+\infty} \exp(-|x-y|)\,\mathtt{sgn}(x-y)\, f(y)\, dy$$

Using Fourier transform, find the most general solution $f(x)$ of the following equations

(a) $C(f) = \delta(x)$

(b) $C(f) = x\exp(-x^2)$

(c) $C(f) = (i/2)\, f$

(d) $C(f) = 2i\, f$

(3.38)
Let $h_a(x) = \sin ax/(\pi x),\ a > 0,\ x \in \mathbf{R}$, and consider the convolution product

$$g_a(x) = \big(h_a * f\big)(x) = \int_{-\infty}^{+\infty} f(x-y) h_a(y)\, dy$$

Use Fourier transform.

(1) Let $f(x) \in L^2(\mathbf{R})$: show that $g_a(x)$ is infinitely differentiable and $\in L^2(\mathbf{R})$; find $\lim\limits_{a\to\infty} g_a(x)$

(2) Let $f(x) = \mathtt{P}(1/x)$: find $\mathscr{F}\big(g_a(x)\big)$ and $\lim\limits_{a\to\infty} g_a(x)$.

(3) Let $f(x) = \delta'(x)$: find $\mathscr{F}\big(g_a(x)\big)$ and $\lim\limits_{a\to\infty} g_a(x)$.

(3.39)
(1) By integration in the complex plane evaluate

$$I_a = \mathtt{P}\int_{-\infty}^{+\infty} \frac{\exp(iat)}{t(t-x)}\, dt\,, \qquad x \in \mathbf{R},\ a > 0$$

(2) Find the Fourier transform $\widehat{F}_a(\omega)$ of the convolution product

$$F_a(x) = \mathtt{P}\frac{1}{x} * \frac{\sin ax}{\pi x}\,, \qquad a > 0$$

What properties of $F_a(x)$ can be deduced from its transform: is $F_a(x)$ a bounded function? continuous and differentiable (how many times?), is $F_a(x) \in L^1(\mathbf{R}) \cap L^2(\mathbf{R})$?

(3) (a) check the answers given in (2) observing that $F_a(x) = -(1/\pi)\mathtt{Im}(I_a)$;

(b) it is also easy to obtain $F_a(x)$ directly, evaluating the inverse Fourier transform of the function $\widehat{F}_a(\omega)$ obtained in (2).

(4) Find $\lim_{a\to+\infty} F_a(x)$ and conclude observing that the above results provide another approximation of the distribution $\mathrm{P}(1/x)$ with a C^∞ function $\big($for a simpler approximation see Problem 3.31, q. (1)$\big)$.

(3.40)
(1) Find the inverse Fourier transform $f_\varepsilon(t)$ of

$$\widehat{f_\varepsilon}(\omega) = \frac{1}{\omega + i\varepsilon}\frac{1}{\omega - i}\,, \qquad \varepsilon > 0$$

and then evaluate the limit (in $\mathscr{S}'$) $\lim_{\varepsilon\to 0^+} f_\varepsilon(t)$.

(2) Exchange the operations: first evaluate $\lim_{\varepsilon\to 0^+} \widehat{f_\varepsilon}(\omega)$ and then find the inverse Fourier transform. The results should coincide! (why?)

(3.41)
(1) (*a*) Find the inverse Fourier transform $f_\varepsilon^{(+)}(t)$ of

$$g_\varepsilon^{(+)}(\omega) = \frac{1}{1 - (\omega + i\varepsilon)^2}\,, \qquad \varepsilon > 0$$

and then the limit (in $\mathscr{S}'$, of course) $F^{(+)}(t) = \lim_{\varepsilon\to 0^+} f_\varepsilon^{(+)}(t)$;

(*b*) the same for the inverse Fourier transform $f_\varepsilon^{(-)}(t)$ of

$$g_\varepsilon^{(-)}(\omega) = \frac{1}{1 - (\omega - i\varepsilon)^2}\,, \qquad \varepsilon > 0$$

and for $F^{(-)}(t) = \lim_{\varepsilon\to 0^+} f_\varepsilon^{(-)}(t)$.

(2) Check the results in (1) evaluating now $G^{(\pm)}(\omega) = \lim_{\varepsilon\to 0^+} g_\varepsilon^{(\pm)}(\omega)$ and then the inverse Fourier transforms $F^{(\pm)}(t)$ of $G^{(\pm)}(\omega)$. *Hint*: find first the 4 limits $\big($see Problem 3.31, q. (1)$\big)$

$$\lim_{\varepsilon\to 0^+} \frac{1}{1 \pm \omega \pm i\varepsilon}$$

(3) Do the functions $F^{(+)}(t)$ and $F^{(-)}(t)$ coincide?

(3.42)
(1) The same questions as in the above problem for the functions (apparently similar to those in the problem above)

$$g_\varepsilon^{(\pm)}(\omega) = \frac{1}{(1 \pm i\varepsilon)^2 - \omega^2}\,, \qquad \varepsilon > 0$$

(2) Do the functions $F^{\pm}(t)$ coincide with those obtained in the problem above?

(3.43)
(1) Consider the sequence of functions

$$f_n(x) = n\,\theta(x)\exp(-xn) \quad \text{and} \quad F_n(x) = \theta(x)\big(1-\exp(-xn)\big)\,, \quad n=1,2,\dots$$

Find $\widehat{f_n}(\omega) = \mathscr{F}\big(f_n(x)\big)$ and $\widehat{F_n}(\omega) = \mathscr{F}\big(F_n(x)\big)$.
(2) Show that $F_n'(x) = f_n(x)$ and, using the formula $\mathscr{F}\big(F'(x)\big) = \dots$, obtain again $\widehat{f_n}(\omega)$ from $\widehat{F_n}(\omega)$.
(3) Find $\lim_{n\to\infty} f_n(x)$ and $\lim_{n\to\infty} F_n(x)$; verify that $\lim_{n\to\infty} f_n(x) = \dfrac{d}{dx}\Big(\lim_{n\to\infty} F_n(x)\Big)$.

(3.44)
Consider the family of functions

$$\widehat{f_a}(\omega) \;=\; i\,\frac{\exp(ia\omega)+\exp(-ia\omega)-2}{a^2\omega}\,, \qquad a>0$$

(1) Find the pointwise limit $\lim_{a\to 0}\widehat{f_a}(\omega)$. Does the limit exist in $L^2(\mathbf{R})$, in $\mathscr{S}'$?
(2) Recalling that $\mathscr{F}^{-1}\big(\mathtt{P}(1/\omega)\big) = \dots$, find and draw the inverse Fourier transform

$$f_a(x) \;=\; \mathscr{F}^{-1}\big(\widehat{f_a}(\omega)\big)$$

Why can the symbol $\mathtt{P}$ be omitted in the above $\widehat{f_a}(\omega)$?
(3) Using (1), find $\lim_{a\to 0} f_a(x)$.
(4) Considering $f_a(x)$ as distributions, choose $\varphi(x) = x\exp(-x^2)$ as test function: calculate $< f_a, \varphi >$ and $\lim_{a\to 0} < f_a, \varphi >$. Then check the answer given in (3).

(3.45)
Let $F_L(x)$ be the function

$$F_L(x) \;=\; \begin{cases} -L & \text{for} \quad x\le -L \\ x & \text{for} \quad |x|\le L \\ L & \text{for} \quad x\ge L \end{cases}\,, \qquad L>0$$

(1) Find $\widehat{F_L}(\omega) = \mathscr{F}\big(F_L(x)\big)$.
(2) Find $f_L(x) \;=\; F_L'(x)$ and $\widehat{f_L}(\omega) \;=\; \mathscr{F}\big(f_L(x)\big)$; check the results obtained here and in (1) using the property $\widehat{f_L}(\omega) = \mathscr{F}\big(F_L'(x)\big) = -i\omega\mathscr{F}\big(F_L(x)\big)$.
(3) Find $\lim_{L\to\infty}\widehat{F_L}(\omega)$.

(3.46)
(1) Let $g(\omega)$ be the (given) Fourier transform of a function $f(x)$ and let $F(x)$ be such that $F'(x) = f(x)$. Show that, expectedly, the Fourier transform $G(\omega) = \mathscr{F}\big(F(x)\big)$ of $F(x)$ is determined by $g(\omega)$ apart from an additional term.

(2) Let

$$F(x) = \begin{cases} x & \text{for} \quad 0 < x < 1 \\ 0 & \text{elsewhere} \end{cases}$$

Find $f(x) = F'(x)$ $\big($notice that $F(x)$ is not continuous, then $F'(x) = \ldots\big)$ and find $g(\omega) = \mathscr{F}\big(f(x)\big)$; then deduce $G(\omega) = \mathscr{F}\big(F(x)\big)$ observing that the additional term can now be fixed thanks to the property that $G(\omega)$ must be a C^∞ function (why?).

(3) Confirm the above result observing that $F(x)$ can be written as $F(x) = x\,u(x)$ where

$$u(x) = \begin{cases} 1 & \text{for} \quad 0 < x < 1 \\ 0 & \text{elsewhere} \end{cases}$$

and applying the rule $\mathscr{F}\big(xf(x)\big) = \ldots$.

(4) Let now

$$F_1(x) = \begin{cases} 0 & \text{for} \quad x < 0 \\ x & \text{for} \quad 0 < x < 1 \\ 1 & \text{for} \quad x > 1 \end{cases}$$

Observing that $F_1'(x) = u(x)$ $\big($*not* $F'(x) = u(x)$, cf. question (1)$\big)$, deduce $\mathscr{F}\big(F_1(x)\big)$: now, the additional term can be fixed, e.g., using $F_1(x) = F(x) + \theta(x-1)$.

(3.47)
(1) Observing that $\dfrac{d}{dx}\arctan x = \dfrac{1}{1+x^2}$ and that $\mathscr{F}\big(1/(1+x^2)\big) = \ldots$, use question (1) of the above problem to deduce $\mathscr{F}(\arctan x)$; the additional term can be fixed observing that the function $\arctan x$ is an *odd* function and $\delta(x)$ is *even*, therefore .... For a different way to obtain this result, see next problem.

(2) Find $\lim\limits_{n\to\infty} \arctan nx$ and $\lim\limits_{n\to\infty} \mathscr{F}(\arctan nx)$

(3.48)
Let the output $b(t)$ produced by a linear system, when $a(t)$ is the applied input, be given by

$$b(t) = \int_{-\infty}^{t} a(t')\,dt'$$

(1) Verify that the Green function of this system is $G(t) = \theta(t)$.

(2) Using the result $\mathscr{F}(\theta * f) = \ldots$, and choosing $a(t) = \exp(-t^2)$, find $\mathscr{F}\big(\mathrm{erf}(t)\big)$, where $\mathrm{erf}(t) = \int_{-\infty}^{t} \exp(-x^2)dx$.

(3) The same as in (2) choosing $a(t) = 1/(1+t^2)$: find again $\mathscr{F}(\arctan t)$ (cf. previous problem).

(3.49)
Let

$$f_0(t) = \begin{cases} 1 & \text{for} \quad 0 < t < 1 \\ 0 & \text{elsewhere} \end{cases} , \qquad t \in \mathbf{R}$$

and let $\widehat{f_0}(\omega)$ be its Fourier transform. Consider the series

$$\widehat{F_\varepsilon}(\omega) = \widehat{f_0}(\omega)\Big(1 + \exp(-\varepsilon)\, \exp(i\omega) + \exp(-2\varepsilon)\, \exp(2i\omega) + \dots\Big)$$

$$= \widehat{f_0}(\omega) \sum_{n=0}^{\infty} \exp\big(-n(\varepsilon - i\omega)\big)\,, \qquad \varepsilon > 0$$

(1) Find and draw the inverse Fourier transform $F_\varepsilon(t) = \mathscr{F}^{-1}\big(\widehat{F_\varepsilon}(\omega)\big)$. *Hint*: use a translation theorem.
(2) Find the sum $\widehat{F_\varepsilon}(\omega)$ of the series.
(3) Is $F_\varepsilon(t) \in L^2(\mathbf{R})$?
(4) Find $\lim_{\varepsilon\to 0} \widehat{F_\varepsilon}(\omega)$.

(3.50)
(1) It is well known that in general the product of distributions cannot be defined. For example, one could try to define $\delta^2(x)$ starting from the product of some sequences of "regular" functions $u_n(x)$ which approximate $\delta(x)$. Several examples of such functions are proposed in the first problems of this subsection, another well- known family of functions approximating $\delta(x)$ is

$$u_\varepsilon(x) = \begin{cases} 1/(2\varepsilon) & \text{for} \quad |x| < \varepsilon \\ 0 & \text{for} \quad |x| > \varepsilon \end{cases} , \qquad x \in \mathbf{R}\,,\ \varepsilon > 0$$

Show that all the sequences $u_n^2(x)$ $\big(\text{or } u_\varepsilon^2(x)\big)$ have no limit.
(2) Verify that no result is obtained also considering sequences as $u_n(x)\,\delta(x)$ $\big($the limit depends on the approximating sequence $u_n(x)$ which has been chosen$\big)$.

(3.51)
As the product of distributions, the convolution product is in general not defined. Consider

$$\sin x * \frac{\sin ax}{x} = \int_{-\infty}^{+\infty} \sin(x-y) \frac{\sin ay}{y}\, dy\,, \qquad a > 0$$

and use Fourier transform. Is this convolution product defined for all a?

(3.52)
(1) Let

$$u_\varepsilon(x) = \exp(-\varepsilon|x|)\,, \qquad \varepsilon > 0$$

Calculate the convolution product

$$v_\varepsilon(x) = u_\varepsilon(x) * \mathtt{sgn}\, x$$

(find first the Fourier transform $\widehat{v_\varepsilon}(\omega) = \ldots$ and then calculate its inverse Fourier transform).

(2) Find the limits in $\mathscr{S}'$ as $\varepsilon \to 0^+$ of $u_\varepsilon(x)$ and of $v_\varepsilon(x)$.

(3) One could conjecture to give a definition of the convolution $1 * \mathtt{sgn}\, x$ (or equivalently, apart from some factor, of the product $\delta(\omega)\mathtt{P}(1/\omega)$) approximating in $\mathscr{S}'$ the constant function 1 with $u_\varepsilon(x)$ and then passing to the limit as $\varepsilon \to 0^+$ using the above results. Explain why this is not correct: repeat calculations in (1) and (2) now approximating 1 with

$$u_{\varepsilon_1,\varepsilon_2}(x) = \begin{cases} \exp(\varepsilon_1 x) & \text{for } x < 0 \\ \exp(-\varepsilon_2 x) & \text{for } x > 0 \end{cases}\,, \qquad \varepsilon_1,\ \varepsilon_2 > 0,\ \varepsilon_1 \neq \varepsilon_2$$

(the definition cannot depend on the approximation chosen ...!)

(3.53)
(1) As well known, $x\,\delta'(x) = -\delta(x)$. Show that

$$x^2\delta(x) = x^2\delta'(x) = 0 \quad , \quad x^2\delta''(x) = 2\delta(x)$$

and verify that the Fourier transforms of these identities produce obvious results.

(2) Generalize: find

$$x^2\delta'''(x) = \ldots \quad , \quad \ldots, \quad x^m\delta^{(n)}(x) = \ldots$$

(3) (a) Verify that

$$h(x)\,\delta(x - x_0) = h(x_0)\,\delta(x - x_0)$$

where $h(x)$ is any function continuous in a neighborhood of x_0.

(b) Extend to

$$h(x)\,\delta'(x - x_0) = \ldots$$

(c) The Fourier transform $\mathscr{F}(x-1)$ can be evaluated in two ways:

$$\mathscr{F}(x-1) = \mathscr{F}(x) - \mathscr{F}(1) \quad \text{and} \quad \mathscr{F}(x-1) = \exp(i\omega)\,\mathscr{F}(x)$$

as an application of (b), verify that the two results coincide.

From Problems 3.54–3.57, the independent variable is denoted by y, to avoid confusion with the notations used in other problems where the results obtained here are applied.

(3.54)
(1) Without using Fourier transform, find the most general distributions $T \in \mathscr{S}'$ which satisfy each one of the following equations:

$$(a)\ \ y\,T = 0\,; \quad (b)\ \ y\,T = 1\,; \quad (c)\ \ y^2 T = 0\,; \quad (d)\ y^2 T = 1\,;$$

$$(e)\ y^3 T = 0\,; \quad (f)\ y^3 T = 1\,; \quad (g)\ y\,T = \sin y; \quad (h)\ y\,T = \cos y$$

(2) Specify if there is some solution, among those found in (1), which belongs to $L^2(\mathbf{R})$ $\big($clearly and more precisely: specify if there is some distribution $T = T_u$ which is associated to a function $u(y) \in L^2(\mathbf{R})\big)$.

(3.55)
The same questions as in the above problem for the equations:

$$(a)\ (y-1)\,T = 0\,; \quad (b)\ (y-1)\,T = 1\,; \quad (c)\ (y \pm i)T = 0\,; \quad (d)\ (y \pm i)T = 1;$$

$$(e)\ (y-\alpha)\,T = 0\,; \quad (f)\ (y-\alpha)\,T = 1 \quad \text{with } \alpha = a + ib,\ a, b \text{ real } \neq 0$$

(3.56)
The same questions as in Problem 3.54 for the equations:

$$(a)\ y(y-1)\,T = 0;\ (b)\ y(y-1)\,T = 1;\ (c)\ (y^2-1)\,T = 0;\ (d)\ (y^2-1)\,T = 1;$$

$$(e)\ y(y \pm i)\,T = 0;\ (f)\ y(y \pm i)\,T = 1;\ (g)\ (y^2+1)\,T = 0;\ (h)\ (y^2+1)\,T = 1$$

(3.57)
The same questions as in Problem 3.54 for the equations:

$$(a)\ \ (\sin y)\,T = 0\,; \quad (b)\ \ \big(1 + \exp(iy)\big)\,T = 0\,;$$

$$(c)\ (1-\cos y)\,T = 0\,; \quad (d)\ (y - \sin y)\,T = 0\,; \quad (e)\ \exp(-1/y^2)\,T = 0$$

(3.58)
Using Fourier transform, solve the following equations for the unknown function $u(x)$:

$$(a)\ u'(x) + u(x - \pi/2) = 0\,; \quad (b)\ u''(x) + u(x) = u(x - \pi) + u(x + \pi)\,;$$
$$(c)\ u(x - a) + 2u(x + 2a) = 3u(x)\,, \qquad a > 0\,;$$
$$(d)\ 2u'(x) + u(x - 1) - u(x + 1) = 0$$

(3.59)
Using Fourier transform, find the distributions T which solve the following equations:

$$(a)\ \ x\,T = \delta(x)\,; \quad (b)\ \ (x - 1)\,T = \delta(x)\,; \quad (c)\ \ x^2\,T = \delta(x)$$

(3.60)
Using Fourier transform, find the distributions T which solve the equation:

$$x\,DT + T = 0$$

(clearly DT is the derivative of T).

(3.61)
Let $f_\varepsilon(x)$ be the function in $L^2(-\pi, \pi)$ defined by

$$f_\varepsilon(x) = \begin{cases} 1/(2\varepsilon) & \text{for} \quad |x| < \varepsilon \\ 0 & \text{for} \quad |x| > \varepsilon \end{cases}\,, \qquad 0 < \varepsilon < \pi$$

and evaluate its Fourier expansion in terms of the complete set $\{\exp(inx)\,, n \in \mathbf{Z}\}$. Consider then the periodic prolongation $\widetilde{f}_\varepsilon(x)$ to all $x \in \mathbf{R}$ of $f_\varepsilon(x)$ with period 2π. Evaluate the limit as $\varepsilon \to 0^+$ of $\widetilde{f}_\varepsilon(x)$ and of its Fourier expansion (notice that this expansion is automatically periodic; explain why these limits can be safely performed) to obtain the Fourier expansion of the “Dirac comb”

$$\sum_{m\in\mathbf{Z}} \delta(x - 2m\pi) = \frac{1}{2\pi} \sum_{n\in\mathbf{Z}} \exp(inx)$$

(3.62)
(1) Write the Fourier expansion in $L^2(-\pi, \pi)$ of the function $\big($see Problem 1.20, q. (1)$\big)$

$$f(x) = \begin{cases} -1 & \text{for} \quad -\pi < x < 0 \\ 1 & \text{for} \quad 0 < x < \pi \end{cases}$$

in terms of the orthogonal complete set $\{1,\ \cos nx,\ \sin nx\ , n=1,2,\ldots\}$ $\big($actually, the subset $\{\sin nx\ n=1,2,\ldots\}$ is enough$\big)$. Consider then the periodic prolongation $\widetilde{f}(x)$ to all $x \in \mathbf{R}$ of $f(x)$ with period 2π: evaluate the first derivative of $\widetilde{f}(x)$ and of its Fourier expansion (the expansion is automatically periodic) to obtain the identity

$$\frac{2}{\pi}\sum_{n\in\mathbf{Z}}\cos\big((2n+1)x\big) = \sum_{m\in\mathbf{Z}}\delta\big(x-2m\pi\big)-\sum_{m\in\mathbf{Z}}\delta\big(x-(2m+1)\pi\big)$$

(2) Proceed as in (1) for the function (cf. Problem 1.21)

$$f(x) = \begin{cases} x+\pi & \text{for} \quad -\pi < x < 0 \\ x-\pi & \text{for} \quad\ \ 0 < x < \pi \end{cases}$$

to obtain again the Fourier expansion of the "Dirac comb" as in the problem above $\big($what is the derivative of $f(x)$?$\big)$.

3.2.2 *Fourier Transform, Distributions, and Linear Operators*

In this subsection, we will consider some examples of linear operators whose action is extended from the Hilbert space $L^2(\mathbf{R})$ to the linear space of distributions $\mathscr{S}'$. Also many of the problems proposed in the following subsections can be stated in terms of operators of this type; indeed, whenever there is a linear relationship between an "input" $a(t)$ and the corresponding "output" $b(t)$, one can define a linear operator T simply putting $T\big(a(t)\big)=b(t)$.

(3.63)
(1) (a) Using Fourier transform, study the convergence as $n\to\infty$ of the sequence of operators $T_n : L^2(\mathbf{R})\to L^2(\mathbf{R})$ $\big($already considered in Problem 3.18, q. (4)$\big)$

$$T_n\, f(x) = \int_{-\infty}^{+\infty} f(x)\frac{\sin\big(n(x-y)\big)}{\pi(x-y)}\, dy$$

Recognize that this type of convergence can be viewed as the statement, in the language of the operators in Hilbert space, of the property of the sequence of functions $g_n(x)=\sin nx/(\pi x)$ of converging to $\delta(x)$ (in $\mathscr{S}'$, of course).

(b) Changing the variable x into the time variable t, the operators T_n can be interpreted as ideal filters of "low frequencies" $|\omega|\le n$ (see Problem 3.18). Construct the operators S_n of the filters for "high frequencies" $|\omega|\ge n$. What is $\lim\limits_{n\to\infty} S_n$?

(2) Consider now the family of operators studied in Problem 3.19, q. (6)

$$T_a f(x) = \int_{-\infty}^{+\infty} f(y) \frac{a}{a^2 + (x-y)^2} \, dy$$

and study their convergence as $a \to 0^+$. Verify that the same remark holds as for the sequence of operators seen in (1) (a). Construct similar examples of sequences (or families) of operators replacing $g_n(x)$ with other sequences (or families) of functions tending to $\delta(x)$.

(3.64)
It is well known that the operator $T f(x) = x f(x)$ in $L^2(\mathbf{R})$ has no eigenvectors. However, looking for "eigenvectors" in the vector space $\mathscr{S}'$, one easily sees that for each "eigenvalue" $\lambda \in \mathbf{R}$, there is the "eigenvector" $\delta(x - \lambda)$. Using this fact:

(1) Look for "eigenvalues" and "eigenvectors" of the operator

$$T f(x) = \sin x \, f(x), \qquad x \in \mathbf{R}$$

(2) (a) Using Fourier transform, look for "eigenvalues" and "eigenvectors" of the operator

$$T f(x) = f(x-1), \qquad x \in \mathbf{R}$$

Look in particular for the "eigenvectors" corresponding to the "eigenvalue" $\lambda = 1$.

(b) Essentially the same question as in (a): what is the most general form of the Fourier transform $\widehat{f}(\omega)$ of a periodic function $f(x)$ of period 1 $\big($or period τ: $f(x) = f(x-\tau)$, $\forall x \in \mathbf{R}\big)$?

(3.65)
Consider the operator defined in $L^2(\mathbf{R})$

$$T f(x) = f(x) - f(x-1)$$

Using Fourier transform:

(1) Find $\|T\|$. Is there any $f_0(x) \in L^2(\mathbf{R})$ such that $\|T f_0\| = \|T\| \|f_0\|$?

(2) Are there eigenvectors of T in $L^2(\mathbf{R})$? and in $\mathscr{S}'$?

(3) Find $\mathtt{Ker}\, T$ and $\mathtt{Ran}\, T$, specifying if $\mathtt{Ran}\, T = L^2(\mathbf{R})$ or at least is dense in it.

(4) More in general, the same questions for the operator

$$T f = \alpha f(x) + \beta f(x-1), \qquad \alpha, \beta \in \mathbf{C}$$

(3.66)
(1) In a given linear system, if the input is $a(t) = \theta(t) \exp(-t)$ then the corresponding output is $b(t) = \exp(-|t|)$. Using Fourier transform, show that this uniquely defines

the Green function $G(t) \in L^2(\mathbf{R})$ of the system $\big($with the usual notations $b(t) = \big(G * a\big)(t)\big)$.

(2) Consider the linear operator $T : L^2(\mathbf{R}) \to L^2(\mathbf{R})$ defined by

$$T\,\big(a(t)\big) = b(t) = \big(G * a\big)(t)$$

where $a(t)$ and $b(t)$ are given in (1). Find $\|T\|$ and check if there is some input $a(t) \in L^2(\mathbf{R})$ such that $b(t) = \lambda\, a(t)$.

(3) It may appear surprising that the operator T can be completely defined in $L^2(\mathbf{R})$ giving only *one* information, namely the result obtained when T is applied to the *single* function $a(t)$, according to (1). Give an explanation of this fact (see, however, also next problem).

(3.67)

(1) Differently from the case considered above, it can happen that giving a single input $a(t)$ with the corresponding output $b(t) = \big(G * a\big)(t)$ is *not* enough to define the Green function of the system. Discuss the following cases:

$(a)\ a(t) = \exp(-|t|),\ b(t) = \theta(t)\exp(-t)$

$(b)\ a(t) = \dfrac{\sin t}{t},\ b(t) = 0$

$(c)\ a(t) = \dfrac{\sin 2t}{t},\ b(t) = \dfrac{\sin t}{t}$

$(d)\ a(t) = \dfrac{\sin t}{t},\ b(t) = \dfrac{\sin 2t}{t}$

$(e)\ a(t) = \begin{cases} -\exp(t) & \text{for } t < 0 \\ \exp(-t) & \text{for } t > 0 \end{cases},\ b(t) = \exp(-|t-1|) - \exp(-|t+1|)$

$(f)\ a(t) = \begin{cases} -\exp(t) & \text{for } \ t < 0 \\ \exp(-t) & \text{for } t > 0 \end{cases},\ b(t) = \exp(-|t-1|) + \exp(-|t+1|)$

$(g)\ a(t) = \begin{cases} 1 & \text{for } |t| < 1 \\ 0 & \text{elsewhere} \end{cases},\ b(t) = \exp(-|t-1|) - \exp(-|t+1|)$

$(h)\ a(t) = \begin{cases} 1 & \text{for } |t| < 1 \\ 0 & \text{elsewhere} \end{cases},\ b(t) = 0$

(2) For completeness and comparison, consider also the following cases, where both $a(t)$ and $b(t) \notin L^2(\mathbf{R})$:

$$a(t) = \delta'(t),\ b(t) = \delta(t+1) \pm \delta(t-1)$$

(3.68)

Consider a linear system described by a Green function $G(t)$ as usual, and introduce the linear operator $T : L^2(\mathbf{R}) \to L^2(\mathbf{R})$ defined by $T\,\big(a(t)\big) = b(t) = \big(G * a\big)(t)$. Assume that $G(t) \in L^1(\mathbf{R}) \cap L^2(\mathbf{R})$.

(1) Using Fourier transform, show that if the input $a(t) \in L^2(\mathbf{R})$ then also the output $b(t) \in L^2(\mathbf{R})$.

(2) Find $\|T\|$.

(3) Assume that some input $a(t)$ is "a good approximation", in the sense of the norm $L^2(\mathbf{R})$, of another input $\widetilde{a}(t)$ $\big($i.e., $\|a(t) - \widetilde{a}(t)\|_{L^2(\mathbf{R})} < \varepsilon$ for some "small" $\varepsilon > 0\big)$. Show that also the corresponding output $b(t)$ approximates $\widetilde{b}(t)$ and find some constant C such that $\|b(t) - \widetilde{b}(t)\|_{L^2(\mathbf{R})} < C\varepsilon$.

(4) Show that the properties in (1), (2), and (3) are still true under the only hypothesis that the Fourier transform $\widehat{G}(\omega)$ is a bounded function.

(5) What changes if $\widehat{G}(\omega)$ is unbounded?

3.2.3 Applications to ODEs and Related Green Functions

(3.69)
(1) Consider the differential equation

$$\dot{x} + ax = f(t)\,, \qquad x = x(t)\,,\ a > 0$$

Find the Fourier transform $\widehat{G}(\omega)$ of the Green function of this equation and then the function $G(t)$ (see also Problem 3.4). Find also *the most general* Green function $G(t)$.

(2) Obtain again the most general Green function $G(t)$ directly "by hand": start from the solution $G_-(t)$ obtained integrating the equation $\dot{G}_- + aG_- = 0$ for $t < 0$, and the solution $G_+(t)$ obtained integrating the equation $\dot{G}_+ + aG_+ = 0$ for $t > 0$; deduce then the "global" solution of the equation $\dot{G} + aG = \delta(t)$ imposing the suitable discontinuity condition at the point $t = 0$ $\big($to $G_-(t)$ as $t \to 0^-$ and to $G_+(t)$ as $t \to 0^+\big)$.

(3) The same questions (1) and (2) for the equation

$$\dot{x} - ax = f(t)\,, \qquad x = x(t)\,,\ a > 0$$

(4) Among the Green functions obtained for the equation in (1), does a Green function exist which is causal and belongs to $L^2(\mathbf{R})$ (or to $\mathscr{S}'$)? And among the Green functions obtained for the equation in (3)?

(3.70)
Consider again the two equations

$$\dot{x} \pm ax = f(t)\,, \qquad x = x(t)\,,\ a > 0$$

(1) Consider the Green functions obtained *using Fourier transform* of these equations:

(a) find their limits as $a \to 0^+$ specifying in what sense these limits exist;

(b) do these limits coincide?

(c) do these limits solve the equation $\dot{x} = \delta(t)$?

(2) Find the most general solution of the equation

$$\dot{x} = \delta(t)$$

Is there any relationship between these solutions and the limits obtained in $(1)\,(a)$?

(3.71)
Consider the equation

$$\dot{x} + x = f(t)\,, \qquad x = x(t)$$

(1) Let $f(t) = \theta(t)\exp(-\alpha t)$, $\alpha \neq 1$.

(a) Solve the equation by means of Fourier transform. Only one solution is obtained; considering the solutions of the homogeneous equation $\dot{x}_0 + x_0 = 0$, write then the most general solution of the equation, and find the solution which satisfies the condition $x(1) = 1$.

(b) The same with $\alpha = 1$.

(2) The same as in (1) with $f(t) = \mathtt{sgn}\, t$.

(3.72)
Consider the equation

$$\dot{x} + ix = f(t)\,, \qquad x = x(t)$$

(1) Using Fourier transform find the most general Green function of this equation.

(2) Find the most general solution of the equation if $f(t) = \theta(t)\exp(-t)$, and then the solution which satisfies the condition $x(1) = 1$.

(3) The same as in (2) if $f(t) = \mathtt{sgn}\, t$.

(4) Discuss the main difference between this and the previous problem: how many Green function are obtained in each case using Fourier transform?

(3.73)
Consider the equation (use Fourier transform in all questions)

$$\dot{x} + ax = f(t)\,, \qquad x = x(t)\,,\ a > 0$$

(1) Find the solution $x(t)$ if $f(t) = \sin t$.

$(2)\,(a)$ Let $f(t) = h(t-1) - h(t+1)$, where $h(t) \in L^2(\mathbf{R})$. Show that the corresponding solution $x = x_a(t) \in L^2(\mathbf{R})$ and find some constant C such that $\|x(t)\|_{L^2(\mathbf{R})} \leq C\|h(t)\|_{L^2(\mathbf{R})}$.

(b) Let $\widetilde{x}(t) = \lim\limits_{a\to 0^+} x_a(t)$: show that also $\widetilde{x}(t) \in L^2(\mathbf{R})$ and find a constant C such that $\|\widetilde{x}(t)\|_{L^2(\mathbf{R})} \leq C\|h(t)\|_{L^2(\mathbf{R})}$.

(3) The same as in (2) (a) and (b) if $f(t) = h(t-1) + h(t+1)$.

(3.74)

Consider the following ODE for the unknown function $x = x(t)$, where $f = f(t)$ is given; use Fourier transform:

$$a\,\dot{x} + b\,x \;=\; \dot{f}\,, \qquad a, b > 0$$

(1) Show that if $f(t) \in L^2(\mathbf{R})$ then also $x(t) \in L^2(\mathbf{R})$ and find a constant C such that $\|x\|_{L^2} \leq C\,\|f\|_{L^2}$.

(2) Put $a = b = 1$. Show that the solution $x(t)$ can be written as $x(t) = f(t) + x_1(t)$ where $x_1(t)$ satisfies the ODE …

(3) Put $a = b = 1$ and find $x(t)$ in the cases:

$$f(t) \;=\; \delta(t) \quad \text{and} \quad f(t) \;=\; \theta(t)$$

(3.75)

The equation of the motion of a particle of mass $m = 1$ subjected to an external force $f(t)$ and to a viscous damping $\big($cf. Problem 3.12$\big)$ is

$$\dot{v} + \beta v \;=\; f(t)\,, \qquad t \in \mathbf{R},\ \beta > 0$$

having denoted by $v = v(t)$ its velocity.

(1) Let $f(t) = \theta(t)\exp(-t)$ and $\beta \neq 1$. Find the Fourier transform $\widehat{v}(\omega) = \mathscr{F}\big(v(t)\big)$, and then $v(t)$. Find $v^{(0)}(t) = \lim\limits_{\beta\to 0^+} v(t)$. Does this limit exist in $L^2(\mathbf{R})$?

(2) When $\beta = 0$, the solution $v(t)$ of the equation $\dot{v} = \theta(t)\exp(-t)$ can be obtained by direct integration (with no use of Fourier transform): find $v^{(0)}(t)$ $\big($under the condition $v^{(0)}(t) = 0$ if $t \leq 0\big)$ and compare with the result obtained in (1). Find then $v^{(0)}(+\infty)$ and the final kinetic energy $\big(v^{(0)}(+\infty)\big)^2/2$ of the particle.

(3) (a) Evaluate explicitly the work W_f done by the force (with generic $\beta > 0$)

$$W_f \;=\; \int_L f\,dx \;=\; \int_{-\infty}^{+\infty} f(t)v(t)\,dt$$

(where L is the space covered by the particle) using the given $f(t)$ and the expression of $v(t)$ obtained in (1); find then the limit

$$W_f^{(0)} = \lim_{\beta\to 0^+} W_f$$

Is this limit equal to $\displaystyle\int_{-\infty}^{+\infty} f(t)v^{(0)}(t)\,dt$? (why?). Show that $W_f^{(0)}$ is equal to the final kinetic energy of the particle when $\beta = 0$.

(b) Find again W_f now using Parseval identity: write $W_f = \big(f, v\big)$ as an integral in the variable ω in terms of the expressions of $\widehat{f}(\omega)$ and $\widehat{G}(\omega)$ obtained above, evaluate explicitly this integral by integration in the complex plane ω, and then find $W_f^{(0)} = \lim_{\beta\to 0^+} W_f$. Check the result with (a) before.

(c) A third possibility for finding the limit $W_f^{(0)} = \lim_{\beta\to 0^+} W_f$: write W_f as an integral in the variable ω as in (b) and, *before* evaluating this integral, perform first the $\lim_{\beta\to 0^+}$ $\big($recall that $\lim_{\varepsilon\to 0^+} 1/(x + i\varepsilon) \to \ldots\big)$, etc. The result should coincide with the previous one! (These three ways for obtaining the same result propose some different relevant aspects and provide useful exercises!)

(3.76)
Consider now the same problem as the above one but extending to the case of a *general* force $f(t) \in L^1(\mathbf{R}) \cap L^2(\mathbf{R})$ $\big($and $f(t)$ real, of course$\big)$.
(1) Let $\beta > 0$; write the Fourier transform $\widehat{v}(\omega) = \mathscr{F}\big(v(t)\big)$ in terms of the Fourier transform of the Green function $\widehat{G}(\omega)$ of the equation and of the applied force $\widehat{f}(\omega) = \mathscr{F}\big(f(t)\big)$.
(2) Write the work W_f done by the force $\big($cf. Problems 3.12, 3.13$\big)$

$$W_f = \int_L f\,dx = \int_{-\infty}^{+\infty} f(t)v(t)\,dt = \big(f, v\big)$$

(where L is the space covered by the particle) using Parseval relation as an integral in the variable ω in terms of $\widehat{f}(\omega)$ and $\widehat{G}(\omega)$. Evaluate then the limit $W_f^{(0)} = \lim_{\beta\to 0^+} W_f$.

Hint: recall that $f(t)$ is real, therefore $\widehat{f^*}(\omega) = \ldots$.

(3) When $\beta = 0$, the equation $\dot{v} = f(t)$ can be integrated directly (with no use of Fourier transform): show that $v^{(0)}(+\infty) = \widehat{f}(0)$ $\big($under the condition $v(-\infty) = 0\big)$, and that, as expected, $W_f^{(0)}$ is equal to the final kinetic energy of the particle.

(4) When $\beta = 0$, under what condition on the applied force $f(t)$, does the work $W_f^{(0)}$ done by $f(t)$ vanish?

(3.77)
Consider the following equation

$$\dot{x} + iax = \delta(t) - \delta(t-1)\,, \qquad x = x(t)\,, a \in \mathbf{R}$$

Use Fourier transform.

(1) For what values of $a \in \mathbf{R}$ are there solutions $x(t) \in L^2(\mathbf{R})$?

(2) Find the most general solution of this equation in the cases $a = 0$, $a = \pi$, $a = 2\pi$.

(3) The same questions (1) and (2) for the equation

$$\dot{x} + iax = \delta(t) + \delta(t-1)$$

(3.78)
(1) Using Fourier transform, find the Green function(s) of the ODE

$$\ddot{x} + x = f(t), \qquad x = x(t)$$

How many Green functions are obtained using Fourier transform? and does one obtain in this case the most general Green function ? Write in particular the causal Green function: does it belong to $L^2(\mathbf{R})$? to $\mathscr{S}'$? (see also Problem 3.56 (*d*)).

(2) Repeat calculations proceeding "by hand", as in Problem 3.69: here, one has to impose the continuity condition at $t = 0$ to $G_\pm(t)$, and a suitable discontinuity condition to $\dot{G}_\pm(t)$ ....

(3.79)
The same questions (1) and (2) as in the above problem for the equation

$$\ddot{x} = f(t), \qquad x = x(t)$$

(see also Problem 3.54 (*d*)).

(3.80)
Using Fourier transform, find the Green function of each one of the following equations

$$\ddot{x} - x = f(t) \quad ; \quad \ddot{x} \pm \dot{x} = f(t), \qquad x = x(t)$$

How many Green functions are obtained proceeding through Fourier transform? Explain why one does not obtain the expected ∞^2 solutions. Find then the most general Green function (see also Problem 3.56 (*h*), (*f*))

(3.81)
(1) By means of Fourier transform, find the most general solution of the following equation and find, in particular, the solution which "respects causality"[2]

$$\ddot{x} = \theta(t)\exp(-t), \qquad x = x(t),\ t \in \mathbf{R}$$

Hint: cf. Problem 3.54 (*d*) and use a decomposition as $\dfrac{1}{y^2(y-c)} = \dfrac{a_1}{y} + \dfrac{a_2}{y^2} + \dfrac{b}{y-c}$.

[2] Assuming that the "input" $f(t) = 0$ for $t < t_0$, the solution respecting causality is requested to be zero for $t < t_0$ (in the present case $x(t) = 0$ for $t < 0$).

(2) Obtain again the general solution by direct repeated integration (without using Fourier transform; impose continuity at $x = 0$ of $x(t)$ and $\dot{x}(t)$).

(3.82)
Solve by means of Fourier transform the equation

$$\ddot{x} - x \;=\; \theta(t)\exp(-2t)\,, \qquad x = x(t),\; t \in \mathbf{R}$$

How many solutions are obtained in this way (cf. Problem 3.56 (h))? Write then the most general solution. Find, in particular, the solution respecting causality (see previous problem). Is this solution in $L^2(\mathbf{R})$? in $\mathscr{S}'$? Is there a solution in $L^2(\mathbf{R})$?

(3.83)
The same questions as in the above problem for the equation (cf. Problem 3.56 (f))

$$\ddot{x} + \dot{x} \;=\; \theta(t)\exp(-t)\,, \qquad x = x(t),\; t \in \mathbf{R}$$

(3.84)
Solve by means of Fourier transform the equation (cf. Problem 3.56 (d))

$$\ddot{x} + x \;=\; \theta(t)\exp(-t)\,, \qquad x = x(t),\; t \in \mathbf{R}$$

Does one obtain in this way the most general solution? Find in particular the solution respecting causality. Is this solution in $L^2(\mathbf{R})$? in $\mathscr{S}'$?

(3.85)
(1) (a) Find the most general solution $x(t)$ of the equation

$$\dot{x} + i\,x \;=\; \exp(-i\alpha t)\,, \qquad t \in \mathbf{R},\; \alpha \neq 1$$

both by means of direct elementary integration, and by means of Fourier transform.
(b) The same questions in the “resonant case” $\alpha = 1$, i.e.,:

$$\dot{x} + i\,x \;=\; \exp(-it)$$

Hint: recall that

$$(y-1)\delta'(y-1) \;=\; \dots$$

(2) The same as in (1) for the equation

$$\ddot{x} + x \;=\; \sin\alpha t$$

with $\alpha \neq 1$ and $\alpha = 1$. *Hint*: in the resonant case $\alpha = 1$, show first that

$$(y^2-1)\big(\delta'(y-1) \pm \delta'(y+1)\big) \;=\; A\big(\delta(y-1) \mp \delta(y+1)\big) \quad \text{where } A = \dots$$

(3.86)
Let $f_n(t)$ be the functions

$$f_n(t) \;=\; \begin{cases} \sin t & \text{for} \quad |t| \le 2n\pi \\ 0 & \text{for} \quad |t| \ge 2n\pi \end{cases} \quad , \qquad n = 1, 2, \ldots, \; t \in \mathbf{R}$$

(1) (a) Calculate the second derivative $\ddot{f}_n(t)$.

(b) Without using Fourier transform but using the result in (a), find the most general solution of the equation

$$\ddot{x} + x \;=\; \delta(t + 2n\pi) - \delta(t - 2n\pi)$$

(2) Let

$$\widehat{g}_n(\omega) \;=\; \frac{\exp(-2in\pi\omega) - \exp(+2in\pi\omega)}{1 - \omega^2}$$

Find $g_n(t) = \mathscr{F}^{-1}\big(\widehat{g}_n(\omega)\big)$. *Hint*: notice that $(-\omega^2 + 1)\widehat{g}_n(\omega) = \ldots$, therefore $g_n(t)$ satisfies the differential equation ....

(3) Find the limits (in $\mathscr{S}'$, of course)

$$\lim_{n\to+\infty} \Big(\exp(-2in\pi\omega) - \exp(+2in\pi\omega)\Big) \;\text{ and }\; \lim_{n\to+\infty} \frac{\exp(-2in\pi\omega) - \exp(+2in\pi\omega)}{1 - \omega^2}$$

In all the above problems of this subsection, the independent variable has been the time t, and in most cases the notion of causality has been introduced. In the following problems of this subsection, we will introduce as independent variable the "position" $x \in \mathbf{R}$, and the unknown variable will be denoted by $u = u(x)$. The procedure for solving differential equations is clearly exactly the same as before, but the physical interpretation changes. For instance, there is no reason to impose a condition similar to the causality: indeed, assuming that there is some point x_0 such that the applied term $f(x)$ is zero for all $x < x_0$, one expects that the effect propagates also to the points "at the left" of x_0, i.e., also to the points $x < x_0$. Similarly, there is no reason to impose to the Green function to be zero for $x < 0$. So, instead of causality conditions, one can have to impose different conditions depending on the physical situations (e.g., vanishing of the solution at $x \to +\infty$, some boundary conditions, suitable continuity or discontinuity properties, etc.).

(3.87)
Consider the equation

$$-u''(x) = \delta(x - x_1)\,, \qquad 0 \le x \le 1$$

where x_1 is any fixed point with $0 < x_1 < 1$, and with boundary conditions

$$u(0) = u(1) = 0$$

Notice that this problem amounts, e.g., to looking for *stationary* (i.e., time-independent) solutions $u = u(x)$ of the d'Alembert equation describing an elastic string with fixed end points in $x = 0$ and $x = 1$, in the presence of an applied force $f(x)$

$$u_{tt} - u_{xx} = f(x)$$

when the force is "concentrated" at the point x_1. Equivalently, this amounts to finding the Green function of this problem. Use three different procedures:

(1) Apply Fourier transform to the equation $-u'' = \delta(x - x_1)$, assuming here $x \in \mathbf{R}$, find $\widehat{u}(k) = \mathscr{F}\big(u(x)\big)$ $\big($see Problem 3.54$(d)\big)$ and evaluate the inverse Fourier transform $u(x)$; impose then to this $u(x)$ the boundary conditions $u(0) = u(1) = 0$.

(2) Solve the equation $-u'' = \delta(x - x_1)$ "by hand": find the solution $u_-(x)$ of the equation $u''_- = 0$ for $0 < x < x_1$, and the solution $u_+(x)$ of the equation $u''_+ = 0$ for $x_1 < x < 1$; impose the boundary conditions $u_-(0) = 0$ and $u_+(1) = 0$, impose finally the continuity condition at $x = x_1$ to $u_\pm(x)$ and the suitable discontinuity condition at $x = x_1$ to $u'_\pm(x)$.

(3) Solve the equation $-u'' = \delta(x - x_1)$ by direct integration $\big($without using Fourier transform, recall that $\frac{d}{dx}\theta(x - x_1) = \ldots\big)$, etc.

(3.88)
By means of Fourier transform, find the most general solution of the equation (cf. Problem 3.81)

$$u_{xx} = \theta(x)\exp(-x)\,, \qquad u = u(x),\ x \in \mathbf{R}$$

Find, in particular, the solution vanishing when $x \to +\infty$, and the solution satisfying the boundary conditions $u(-1) = u(1) = 0$.

(3.89)
Solve by means of Fourier transform the equation (cf. Problem 3.84)

$$u_{xx} + u = \theta(x)\exp(-x)\,, \qquad u = u(x),\ x \in \mathbf{R}$$

Find, in particular, the solution satisfying the boundary conditions $u(0) = u(\pi/2) = 0$. Show that it is impossible to have a solution satisfying the boundary conditions $u(0) = u(\pi) = 0$: why?

3.2.4 Applications to General Linear Systems and Green Functions

(3.90)
(1) Consider any linear system defined by a Green function $G(t)$, where input $a(t)$ and output $b(t)$ are related by $b(t) = \big(G * a\big)(t)$ as usual. Let the input be a monochromatic wave $a(t) = \exp(-i\omega_0 t)$, $\forall t \in \mathbf{R}$. Using Fourier transform, show that also the output (if not zero) is a monochromatic wave differing from the input in presenting an "amplification" and a "phase shift".

(2) Find the output $b(t)$ if the input is the superposition of two monochromatic waves as $a(t) = \exp(-i\omega_1 t) + \exp(-i\omega_2 t)$, $\forall t \in \mathbf{R}$, with $\omega_1 \neq \omega_2$.

(3.91)
(1) By integration in the complex plane, evaluate the inverse Fourier transform

$$\mathscr{F}^{-1}\Big(\mathtt{P}\Big(\frac{1}{\omega - 1}\Big)\frac{1}{(\omega - i)^2}\Big)$$

(2) The Fourier transform of the Green function $G = G_n(t)$ of a linear system is given by

$$\widehat{G}_n(\omega) = \frac{1}{(\omega - i)^n}, \qquad n = 1, 2, \ldots$$

Without calculating $G_n(t)$, specify for what n the Green function $G_n(t)$ is a *real* function.

(3) Let now $n = 2$ in the above Green function and let $a(t) = \sin t$ be the input applied to the system. Find $\widehat{a}(\omega)$ and the corresponding output $b(t)$. Is the output a real function? $\big($cf. (2)$\big)$.

(4) The same as in (3) if $a(t) = \sin|t| = \mathtt{sgn}\, t\, \sin t$ $\big($see (1)$\big)$.

(3.92)
Consider a linear system described by a Green function $G(t)$ with the usual notations.

(1) Let the Fourier transform $\widehat{G} = \widehat{G}_0(\omega)$ of the Green function be given by

$$\widehat{G}_0(\omega) = -i\omega$$

What is the relationship between the input $a(t)$ and the output $b(t)$? Find $b(t)$ if $a(t) = \theta(t)$.

(2) Consider now the following "approximation" $\widehat{G}_\varepsilon(\omega) \in L^2(\mathbf{R})$ of $\widehat{G}_0(\omega)$ given by

$$\widehat{G}_\varepsilon(\omega) = \frac{-i\omega}{(1 - i\varepsilon\omega)^2}, \qquad \varepsilon > 0$$

(verify that indeed $\widehat{G}_\varepsilon(\omega) \to \widehat{G}_0(\omega)$ in $\mathscr{S}'$). Let $a(t) = \theta(t)$. Find the corresponding output $b_\varepsilon(t)$ if the Green function is given by $\widehat{G}_\varepsilon(\omega)$ and show that indeed $b_\varepsilon(t)$ is an approximation of ....

(3.93)
A linear system is defined by a Green function , whose Fourier transform is

$$\widehat{G}_\tau(\omega) = \frac{\omega^4}{(1+\omega^2)^2} \exp(i\omega\tau) , \qquad \tau \in \mathbf{R}$$

(1) If the input is

$$a(t) = \begin{cases} 1 & \text{for} \quad t < 0 \\ 2 & \text{for} \quad t > 0 \end{cases}$$

is it possible to say if the corresponding output $b = b_\tau(t)$ belongs to $L^2(\mathbf{R})$ (without calculating it)?

(2) What properties (continuity, differentiability, behavior as $|t| \to \infty$) can be expected for $b_\tau(t)$?

(3) Study the convergence as $\tau \to \infty$ of $b_\tau(t)$.

(3.94)
Consider a linear system defined by the Green function

$$G = G_T(t) = \begin{cases} 1 & \text{for} \quad 0 < t < T \\ 0 & \text{elsewhere} \end{cases} , \qquad T > 0$$

with input $a(t)$ and output $b = b_T(t)$ related by the usual rule $b = G * a$.

(1) Using Fourier transform find $b = b_T(t)$ if $a(t) = t \exp(-t^2)$

(2) Find $\lim_{T\to\infty} b_T(t)$. Does this limit exist in the norm $L^2(\mathbf{R})$? in the sense of $\mathscr{S}'$? Confirm the conclusion examining $\lim_{T\to\infty} \widehat{b}_T(\omega)$.

(3) Let now $G(t) = \theta(t)$: using Fourier transform find $b(t)$ if $a(t) = t \exp(-t^2)$; does this $b(t)$ coincide with the limit obtained in (2)?

(3.95)
Consider a linear system defined by the Green function (use Fourier transform)

$$G = G_c(t) = \frac{\sin ct}{\pi t} , \qquad c > 0$$

with input $a(t)$ and output $b = b_c(t)$ related by the usual rule $b_c = G_c * a$.

(1) Is there some input $a(t) \neq 0$ such that $b_c(t) = 0$ for all $t \in \mathbf{R}$?

(2) Let $a(t) \in L^2(\mathbf{R})$: what properties of the output $b_c(t)$ can be expected? (specify if $b_c(t) \in L^1(\mathbf{R}) \cap L^2(\mathbf{R})$, discuss its continuity and differentiability, boundedness, behavior at $|t| \to \infty$).

(3) Find $\lim_{c\to\infty} G_c(t)$. If $a(t) \in L^2(\mathbf{R})$, find $\lim_{c\to\infty} b_c(t)$; does this limit exist also in the sense of the norm $L^2(\mathbf{R})$?

(3.96)
Consider a linear system defined by the Green function

$$G = G_T(t) = \begin{cases} 1 & \text{for} \quad |t| < T \\ 0 & \text{elsewhere} \end{cases} , \qquad T > 0$$

with input $a = a(t)$ and output $b = b_T(t)$ related by the usual rule $b_T = G_T * a$.

(1) Is there some nonzero input $a(t) \in L^2(\mathbf{R})$ such that $b_T(t) = 0$ for all $t \in \mathbf{R}$? and if $a(t) \in \mathscr{S}'$?

(2) Let $a(t) \in L^2(\mathbf{R})$: what properties of the output $b_T(t)$ can be expected? (specify if $b_T(t) \in L^1(\mathbf{R}) \cap L^2(\mathbf{R})$, discuss its continuity and differentiability, boundedness, behavior at $|t| \to \infty$).

(3) Find $\lim_{T\to+\infty} G_T(t)$ and $\lim_{T\to+\infty} \widehat{G}_T(\omega)$.

(4) Let $a(t) \in L^1(\mathbf{R}) \cap L^2(\mathbf{R})$: find $\lim_{T\to+\infty} \widehat{b}_T(\omega)$ and $\lim_{T\to+\infty} b_T(t)$.

(3.97)
Assume that the input $a(t)$ and the corresponding output $b(t)$ of a system are related by the rule

$$a(t) = b(t) + \alpha \int_{-\infty}^{t} \exp\big(-(t - t')\big) b(t')\, dt'$$

(1) Using Fourier transform, find the Green functions in the following cases:

$$\alpha = 1 \quad ; \quad \alpha = -1 \quad ; \quad \alpha = -2$$

(2) Verify (*not* using in this question Fourier transform) that the given equation with $a(t) = 0$ and $\alpha = -2$ admits a solution of the form $b_0(t) \propto \exp(\beta t)$, with $\beta \in \mathbf{R}$ to be determined.

(3) Assuming that the only solutions of the equation with $\alpha = -2$ in the "homogeneous" case $a(t) = 0$ are those found in (2), write the most general Green function of the equation in the case $\alpha = -2$ and specify if there is a *causal* Green function.

(3.98)
The input $a(t)$ and the output $b(t)$ of a linear system are related by the rule

$$\dot{b}(t) = a(t) - a(t-1)$$

(1) Using Fourier transform, find the most general Green function $G(t)$ of this problem.

(2) Both by direct integration (without using Fourier transform), and using Fourier transform, find $b(t)$ in the cases

$$a(t) = t\exp(-t^2) \quad \text{and} \quad a(t) = \theta(t)$$

(3) The same questions (1) and (2) if the equation is

$$\dot{b}(t) = a(t) + a(t-1)$$

(3.99)
The input $a(t)$ and the output $b(t)$ of a linear system are related by the rule

$$\ddot{b}(t) = 2a(t) - a(t-1) - a(t+1)$$

(1) Using Fourier transform, find the most general Green function $G(t)$ of the problem (see Problem 3.29, q. (2)), and specify if there is a Green function belonging to $L^2(\mathbf{R})$.
(2) Find the most general solution $b(t)$ in the cases

$$a(t) = \dot{\delta}(t)\,; \quad a(t) = t\,; \quad a(t) = \theta(t)\,\exp(-t)$$

(3.100)
(1) Assume that in a linear system the output $b(t)$ is related to the input $a(t)$ by the rule

$$b(t) = \int_{-\infty}^{+\infty} G(t+t')\,a(t')\,dt'$$

where $G(t)$ is given.
(a) Find $b(t)$ in the special case $G = \delta(t+t')$.

(b) Clearly, the above relation between $a(t)$ and $b(t)$ cannot be expressed in the form $b = G * a$ and, as a consequence, also $\widehat{b}(\omega) = \widehat{G}(\omega)\,\widehat{a}(\omega)$ is no longer true. However, introducing the change of variable $t'' = -t'$, the above relation can be written as $b = G * a^{(-)}$ where $a^{(-)} = \ldots$. Using this trick, evaluate $b(t)$ using Fourier transform in the case

$$G(t) = \theta(t)\,\exp(-t) \ \text{ and } \ a(t) = \theta(t)\,\exp(-\alpha t)\,, \qquad \alpha > 0$$

(2) As another example of a linear system where the Green function is not of the form $G(t-t')$, consider the "multiplicative" case, where the output $b(t)$ is simply given by $b(t) = M(t)\,a(t)$. Show that also this case can be written in the general form $b(t) = \int_{-\infty}^{+\infty} G(t,t')a(t')\,dt'$, putting $G(t,t') = \ldots$.

(3.101)
(1) Let $g(\omega) \in L^2(\mathbf{R})$ be a function which satisfies the following property

$$\mathrm{P}\int_{-\infty}^{+\infty} \frac{g(y)}{y-\omega}\,dy \equiv -\mathrm{P}\frac{1}{\omega} * g \;=\; \pi i\, g(\omega)\,, \qquad \omega \in \mathbf{R}$$

What information can be deduced about the support of the inverse Fourier transform $f(t) = \mathscr{F}^{-1}\big(g(\omega)\big)$? $\big($in our notations, $\mathscr{F}^{-1}(g_1 * g_2) = 2\pi\, \mathscr{F}^{-1}(g_1)\mathscr{F}^{-1}(g_2)\big)$.

(2) By integration in the complex plane, calculate the four integrals

$$\mathrm{P}\int_{-\infty}^{+\infty} \frac{\exp(\pm i y)}{(y \pm i)(y-\omega)}\,dy\,, \qquad \omega \in \mathbf{R}$$

(3) What among the four functions $g(y) = \exp(\pm i y)/(y \pm i)$ satisfies the property given in (1)? Confirm the result: find the inverse Fourier transforms of the four functions $g(y)$; what of these has support in $t \geq 0$? This is a special case of the Kramers–Kronig relations, or —more in general—of the dispersion rules, which connect causality with analyticity and "good behavior" of the Fourier transform in the complex plane ω.

(4) As a simple application, assume that Fourier transform $g = \widehat{G}(\omega)$ of a Green function $G(t) \in L^2$ satisfies the property stated by the equation in (1), and therefore is causal. Separating real and imaginary parts $\widehat{G}(\omega) = \widehat{G}_1(\omega) + i\,\widehat{G}_2(\omega)$, the real part of this equation gives

$$\mathrm{P}\int_{-\infty}^{+\infty} \frac{\widehat{G}_1(y)}{y-\omega}\,dy \;=\; -\pi\,\widehat{G}_2(\omega)$$

and a similar equation taking the imaginary part. Let for instance

$$\widehat{G}_1(\omega) \;=\; \frac{1}{1+\omega^2}$$

using the above equation, deduce $\widehat{G}_2(\omega)$ and $G(t)$. This shows that real and imaginary parts of $\widehat{G}(\omega)$ are not independent, but that the imaginary part is completely determined by the real part, and conversely. As another consequence, writing $\widehat{G}(\omega) = A(\omega)\,\exp\big(i\Phi(\omega)\big)$, one concludes that the modulus $A(\omega)$ and the phase $\Phi(\omega)$ (which clearly admit quite different physical meaning resp. as "amplification" and "phase distortion") are reciprocally connected.

3.2.5 Applications to PDE's

(3.102)
Introduce and use the Fourier transform $\widehat{u} = \widehat{u}(k,t)$ with respect to the variable x of the function $u = u(x,t)$, defined as

$$\widehat{u}(k,t) = \int_{-\infty}^{+\infty} u(x,t)\, \exp(ikx)\, dx$$

(1) Show that the heat equation (also called diffusion equation)

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \qquad u = u(x,t)\,;\; x \in \mathbf{R},\; t \geq 0$$

is transformed into an ODE for $\widehat{u}(k,t)$ and find its most general solution.

(2) Show that the solution $u(x,t)$ of the heat equation with the initial condition

$$u(x,0) = f(x)$$

may be written in the form

$$u(x,t) = f(x) * G(x,t)$$

and find the "Green function" $G(x,t)$.

(3) Find $\lim_{t \to 0^+} G(x,t)$. Explain why this is the result to be expected.

(4) Is there any $u(x,0) = f(x) \in L^2(\mathbf{R})$ (or $\in \mathscr{S}'$) such that $u(x,t=1) = e^{-1}u(x,0)$?

(3.103)
Consider the heat equation and use the Fourier transform and the same notations as in the previous problem. Assume that the initial condition $u(x,0) \in L^2(\mathbf{R})$, and let T_t be the time-evolution operator $(t > 0)$

$$T_t \,:\, u(x,0) \to u(x,t)$$

(1) Find $\|T_t\|$

(2) Study the limit as $t \to +\infty$ of the operator T_t.

(3) Study the limit as $t \to 0^+$ of the operator T_t.

(3.104)
Use the same notations for the heat equation as before.

(1) Let

$$u(x,0) = f(x) = \frac{\sin x}{x}, \qquad x \in \mathbf{R}$$

$\big($then $\widehat{f}(k) = \ldots\big)$. For fixed $t > 0$, is it possible to establish, without calculating the corresponding solution $u(x,t)$, if

(a) $u(x,t) \in L^2(\mathbf{R})$?

(b) $\lim_{x \to \pm\infty} u(x,t) = 0$, and $u(x,t)$ is rapidly vanishing as $x \to \pm\infty$?

(c) $u(x,t) \in L^1(\mathbf{R})$?

(2) Show that if the initial condition $u(x,0) = f(x) \in L^2(\mathbf{R})$, then for any $t > 0$ the solution $u(x,t) \in L^2(\mathbf{R})$ and is infinitely differentiable with respect to x and to t. Show that the same is also true if $f(x)$ is a combination of delta functions $\delta(x-a)$ (for any $a \in \mathbf{R}$) and derivatives thereof.

(3) Find the solution $u(x,t)$ in these cases

$$f(x) = 1 \quad ; \quad f(x) = x \quad ; \quad f(x) = x^2$$

(Calculations by means of Fourier transform need some care with the coefficients 2π, $\pm i$, etc., but the results are disappointingly obvious …)

(3.105)
Consider the equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - a\frac{\partial u}{\partial x}\,, \qquad u = u(x,t),\ a > 0$$

which can be viewed as a "perturbation" (for "small" a) of the heat equation. Let

$$f(x) = u(x,0)$$

be the given initial condition.

(1) Show that the solution $u(x,t)$ can be written in the form $u(x,t) = f(x) * G(x,t)$ and find $G(x,t)$.

(2) Denote by $u_a(x,t)$ the solution of the given equation and resp. by $u_0(x,t)$ the solution of the heat equation with the same initial condition $f(x)$. Let $f(x) \in L^2(\mathbf{R})$: show that, for fixed t (put $t = 1$, for simplicity), $u_a(x,1)$ and $u_0(x,1)$ remain "near" in the $L^2(\mathbf{R})$ norm, i.e., that $\|u_a(x,1) - u_0(x,1)\|_{L^2(\mathbf{R})} \to 0$ as $a \to 0$. *Hint*: use Lebesgue theorem or the elementary property (especially useful for "small" y)

$$|\exp(iy) - 1| = |y|\left|\frac{\exp(iy)-1}{y}\right| \le |y|\,, \qquad \forall y \in \mathbf{R}$$

(3) Is the same true as in (2) if $f(x) = \delta''(x)$?

(4) Show that if $f(x) = \mathtt{sgn}\,x$ then neither $u_a(x,1)$ nor $u_0(x,1)$ belong to $L^2(\mathbf{R})$, however verify that $u_a(x,1) - u_0(x,1) \in L^2(\mathbf{R})$ and the property seen in (2) still holds.

(3.106)
Use the same Fourier transform $\widehat{u}(k,t)$ as in the above problems but now for the d'Alembert equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}\,, \qquad u = u(x,t)\,;\ x,\ t \in \mathbf{R}$$

(1) Show that the d'Alembert equation is transformed into an ODE for $\widehat{u}(k,t)$ and find its most general solution.

(2) Let the initial conditions be given by

$$u(x,0) = f(x) \quad , \quad u_t(x,0) = 0$$

show that the solution $u(x,t)$ is a superposition of two waves.

(3) Let $u(x,0) = f(x) \in L^2(\mathbf{R})$ and $u_t(x,0) = 0$: study the limit as $t \to +\infty$ of the above solution $u(x,t)$.

(4) Let the initial conditions be as before: study the limit as $t \to 0^+$ of the above solution $u(x,t)$.

(3.107)
Proceeding by means of Fourier transform as in the above problem for the d'Alembert equation, consider now the initial conditions

$$u(x,0) = 0 \quad , \quad u_t(x,0) = g(x)$$

(1) Show that the solution $u(x,t)$ can be written in the form

$$u(x,t) = g(x) * G(x,t)$$

and find $\widehat{G}(k,t) = \mathscr{F}\big(G(x,t)\big)$ and $G(x,t)$.

(2) Find $G_t(x,t) = \partial G(x,t)/\partial t$; verify that $G_t(x,0) = \delta(x)$ and explain why this result should be expected.

(3) Let $g(x) = \theta(-x)\exp(x)$: write the Fourier transform $\widehat{u}(k,t)$ of the corresponding solution; without evaluating $u(x,t)$, but only recalling the statement of Jordan lemma, show that $u(x,t) = 0$ for $x > t > 0$ (this also should be expected: why?)

(3.108)
Consider the nonhomogeneous d'Alembert equation

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \delta(x - vt)\,, \qquad u = u(x,t),\ x,t \in \mathbf{R}$$

describing (e.g.,) an infinite elastic string subjected to a "delta" force traveling with velocity v.

(1) Introducing the Fourier transform $\widehat{u} = \widehat{u}(k,t)$ find the most general solution in the case $v \neq c$.

(2) The same in the case $v = c$.

(3.109)
Consider the PDE

$$\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial x \partial t} + \frac{\partial^2 u}{\partial t^2} = 0\,, \qquad u = u(x,t)\,;\ x \in \mathbf{R},\ t \in \mathbf{R}$$

with initial conditions

$$u(x,0) = f(x)\,,\ u_t(x,0) = g(x)$$

(1) Introducing the Fourier transform $\widehat{u}(k,t)$ as in previous problems, transform the given PDE into an ODE for $\widehat{u}(k,t)$ and find its solution in terms of the Fourier transforms $\widehat{f}(k)$ and $\widehat{g}(k)$ of the given initial conditions.

(2) If $f(x),\ g(x) \in L^2(\mathbf{R})$, is the same true, in general, for the solution $u(x,t)$ for any fixed $t \in \mathbf{R}$?

(3) Find the solution $u(x,t)$ of the PDE in terms of the given initial conditions $f(x)$ and $g(x)$.

(4) Find the solution $u(x,t)$ of the given PDE in the case

$$f(x) = \theta(x)\exp(-x)\,,\ g(x) = 0$$

(3.110)
Consider the Laplace equation

$$\varDelta_2 u \equiv \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0\,, \qquad u = u(x,y)$$

on the half-plane $y \geq 0$, with the boundary condition

$$u(x,0) = f(x)\,, \qquad x \in \mathbf{R}$$

Introduce and use the Fourier transform $\widehat{u} = \widehat{u}(k,y)$ with respect to the variable x of the function $u = u(x,y)$, defined as

$$\widehat{u}(k,y) = \int_{-\infty}^{+\infty} u(x,y)\ \exp(ikx)\,dx$$

(1) Show that the Laplace equation is transformed into an ODE for $\widehat{u}(k,y)$, whose most general solution is

$$\widehat{u}(k,y) = A(k)\exp(ky) + B(k)\exp(-ky)$$

which can be more conveniently written

$$\widehat{u}(k,y) = A'(k)\exp(-|k|y) + B'(k)\exp(|k|y)$$

(2) Show that, imposing a boundedness condition $\big($therefore $B'(k) = 0\big)$ and the given boundary condition, the solution can be written as a convolution product

$$u(x, y) = f(x) * G(x, y)$$

and give the expression of the Green function $G(x, y)$.

(3) Either from the expression of $\widehat{G}(k, y)$ or of $G(x, y)$, find $\lim_{y\to 0} G(x, y)$ (which is just the expected result: why?)

(3.111)

(1) With the same notations and assumptions as in the problem above, show that if $u(x, 0) = f(x) \in L^2(\mathbf{R})$, then for $y > 0$ the solution $u(x, y)$ is infinitely differentiable with respect to x and to y. This should be expected: way?

(2) Let $f(x) = 1/(1 + x^2)$. Find $\widehat{u}(k, y)$ and then $u(x, y)$. Compare with Problem 2.45, where the same result is obtained by means of a completely different procedure.

(3) The same as in (2) if $f(x) = x^2/(1 + x^2)^2$.

(3.112)

The (non-)uniqueness of the solutions of the Laplace equation in the half-plane $y \geq 0$ with given boundary condition (see previous problems) has been discussed in Problem 2.48. Indeed, the nonuniqueness is due to the presence of nonzero solutions of the equation $\Delta_2 u = 0$ with *vanishing* boundary condition. This can be reconsidered by means of Fourier transform.

(1) With the same procedure as in the above problems, show that the Fourier transform of the most general solution of the Laplace equation on the half-plane $y \geq 0$ with vanishing boundary condition $u(x, 0) = 0$ has the form

$$\widehat{u}(k, y) = C(k)\big(\exp(ky) - \exp(-ky)\big)$$

(2) Observing that the above $\widehat{u}(k, y)$ can belong to $\mathscr{S}'$ *only if* $C(k)$ has support in the single point $k = 0$, find some solutions $u(x, y)$: choose, e.g.,

$$C(k) = \delta'(k),\ \delta''(k),\ \text{etc.}$$

(3.113)

(1) Find the three-dimensional Fourier transform $\widehat{f}(k_1, k_2, k_3)$ of $f(x, y, z) = y \exp(-|x|)$.

(2) Find the three-dimensional Fourier transform $\widehat{f}(k_1, k_2, k_3)$ of the functions (in spherical coordinates $r,\ \theta,\ \varphi$)

$$f_1 = 1/r^2 \quad \text{and} \quad f_2 = \exp(-r)$$

Hint: Write $\exp(i\,\mathbf{k}\cdot\mathbf{x})d^3\mathbf{x} = \exp(ikr\cos\theta)\,r^2 \sin\theta\, d\theta\, d\varphi\, dr$ (with $k = |\mathbf{k}|,\ r = |\mathbf{x}|$) and perform fist the (trivial) integration in $d\varphi$, then in $d\theta$, ...

(3.114)
(1) Find the inverse Fourier transform in $\mathbf{R}^3$ of $g(k) = 1/k^2$ (similar calculations as in the previous problem) and deduce $\mathscr{F}(1/r)$.

(2) Using Fourier transform verify that the Green function for the Poisson equation

$$\Delta V = -4\pi\rho(\mathbf{x}), \qquad V = V(\mathbf{x}),\ \mathbf{x} \in \mathbf{R}^3$$

(where Δ is the three-dimensional Laplacian) is just $G = 1/r$. Deduce the elementary rule for finding the electric potential $V(x)$ produced by a distribution of charges $\rho = \rho(\mathbf{x})$, i.e., $V = (1/r) * \rho = \ldots$.

(3.115)
Verify, using Fourier transform, that the harmonic function $u(x, y) = x^2 - y^2$ satisfies the equation

$$\Delta_2 u = 0, \qquad u = u(x, y)$$

where Δ_2 is clearly the two-dimensional Laplacian. The same question for $u(x, y) = x^3 - 3xy^2$. *Warning*: in 2 dimensions, one has $\mathscr{F}\big(f(x)g(y)\big) = \widehat{f}(k_1)\widehat{g}(k_2)$ with clear notations, therefore, e.g., $\mathscr{F}\big(f(x)\big) = \widehat{f}(k_1)2\pi\delta(k_2)$, see also Problem 3.113, q. (1).

3.3 Laplace Transforms

The notations for the Laplace transform of a (locally summable) function $f = f(x)$ will be

$$\mathscr{L}\big(f(x)\big) = \int_0^{+\infty} f(x)\exp(-sx)\,dx = g(s) = \widetilde{f}(s)$$

where $s \in \mathbf{C}$ and with $\mathrm{Re}\, s > \lambda$, the summability abscissa *of the Laplace transform. According to this definition, the functions $f(x)$ to be transformed are defined only for $x \geq 0$ (or must be put equal to zero for $x < 0$); so that, e.g., the transform $\mathscr{L}(\sin x) = 1/(1+s^2)$ should be more correctly written $\mathscr{L}\big(\theta(x)\sin x\big) = 1/(1+s^2)$. The function $\theta(x)$ is usually understood and omitted in this context; it will be explicitly introduced only when possible misunderstanding can occur, especially when comparing Fourier and Laplace transforms.*

All problems in this subsection, apart from Problems 3.129 and 3.130, can be solved using elementary Laplace transforms $\big($as, e.g., $\mathscr{L}\big(\exp(kx)\big) = 1/(s-k)$ with $\lambda = \mathrm{Re}\, k\big)$ and standard properties of Laplace transform, i.e., no need of the general Laplace inversion formula (also known as Bromwich or Riemann–Fourier formula)

$$f(x) = \mathscr{L}^{-1}\big(\widetilde{f}(s)\big) = \frac{1}{2\pi i}\int_{\ell} \widetilde{f}(s)\exp(sx)\,ds$$

where ℓ is any "vertical" line in the complex plane s from $a - i\infty$ to $a + i\infty$ with $a > \lambda$.

(3.116)
(1) Without trying to evaluate the Laplace transform of the following functions, specify their summability abscissas:

$$x^{\alpha}\,;\; x^{\alpha}\exp(-x^2)\,;\; x^{\alpha}\exp(\beta x)\,,\ \text{with}\,\alpha > -1 \text{ (why this limitation?) and } \beta \in \mathbf{R}$$

(2) The same question for the functions

$$\frac{1}{x+c}\,; \quad \frac{\exp(\gamma x)}{x+c}\,; \quad \frac{\sin(x^2)}{(x+c)^2}\,,\ \text{with } c > 0 \text{ (why this limitation?) and } \gamma \in \mathbf{C}$$

(3.117)
(1) Study the singularities in the complex plane s of the following Laplace transform

$$g(s) = \frac{\exp(-s) + s - 1}{s^2(s+1)}$$

(2) Find and draw the inverse Laplace transform $f(x) = \mathscr{L}^{-1}\big(g(s)\big)$.
(3) Deduce from (1) and/or (2) the summability abscissa λ of $g(s)$.

(3.118)
The same questions as in the previous problem for the Laplace transform

$$g(s) = \frac{1 + \exp(-\pi s)}{s^2 + 1}$$

(3.119)
(1) Show that, if the abscissa λ of a Laplace transform $\widetilde{f}(s) = \mathscr{L}\big(f(x)\big)$ satisfies $\lambda < 0$, then the Fourier transform $\widehat{f}(\omega)$ of $f(x)$ can be obtained from $\widetilde{f}(s)$ by a simple substitution. Compare e.g., the Laplace and Fourier transforms of $f(x) = \theta(x)\exp(-x)$.
(2) Find and draw the inverse Laplace transform $f(x) = \mathscr{L}^{-1}\big(\widetilde{f}(s)\big)$ of

$$\widetilde{f}(s) = \frac{1 - \exp(-s) - s\,\exp(-s)}{s^2}$$

and find its abscissa λ. Compare this Laplace transform with the Fourier transform $\widehat{f}(\omega) = \mathscr{F}\big(f(x)\big)$.

(3) This and the following question deal with the "critical" case $\lambda = 0$ (see also next problem). Obtain $\mathscr{F}\big(\theta(x)\big)$ as $\lim_{\varepsilon\to 0^+} \mathscr{F}\big(\theta(x)\exp(-\varepsilon x)\big)$, $\varepsilon > 0$, and compare with $\mathscr{L}\big(\theta(x)\big)$.

(4) Find and draw the inverse Laplace transform $f(x) = \mathscr{L}^{-1}\big(\widetilde{f}(s)\big)$ of

$$\widetilde{f}(s) = \frac{1-\exp(-s)}{s^2}$$

Find then the Fourier transform $\widehat{f}(\omega) = \mathscr{F}\big(f(x)\big)$ and compare with $\widetilde{f}(s)$.

(3.120)
Without trying to evaluate their Laplace and Fourier transforms, show that the following functions admit $\mathscr{L}$-transform with abscissa $\lambda = 0$; do these functions also admit $\mathscr{F}$-transform $\big($in $L^1(\mathbf{R})$, $L^2(\mathbf{R})$ or $\mathscr{S}'\big)$? $\big($see also the examples in questions (3) and (4) of the problem above$\big)$

$$\theta(x)\frac{1}{1+x^2} \quad ; \quad \theta(x)\frac{x}{1+x^2} \quad ; \quad \theta(x)\frac{x^2}{1+x^2} \quad ; \quad \theta(x)\exp(\pm\sqrt{x})$$

(3.121)
Using the result obtained in q. (1) of Problem 3.119:

(1) Show that if $f(x) \in L^2(I)$ where $I \subset \mathbf{R}$ is a compact interval, then its Fourier transform $\widehat{f}(\omega)$ is an analytic function for all $\omega \in \mathbf{C}$.

(2) Deduce that, in the same assumption, $\widehat{f}(\omega)$ can have at most isolated zeroes in the complex plane; in particular cannot have compact support on the real axis.

(3) Show that properties (1) and (2) are also shared, e.g., by the functions $\theta(x)\exp(-x^2)$, $\theta(x)\exp(-x^4)$ and also $\exp(-x^4)$ (the argument can be easily extended to $x < 0$).

(3.122)
Consider the Laplace transform

$$\widetilde{f}(s) = \frac{\exp(-as)-\exp(-bs)}{s^2+1}\,, \qquad a,\ b \in \mathbf{R}$$

(1) How can one choose $a,\ b \in \mathbf{R}$ in order to have abscissa $\lambda = -\infty$?

(2) Find and draw the inverse Laplace transform $f(x) = \mathscr{L}^{-1}\big(\widetilde{f}(s)\big)$ if $a = 0$ and in the cases $b = \pi,\ 2\pi\ ,\ 4\pi$.

(3) Use the above results to obtain the inverse Fourier transform of the function

$$g_{n,m}(\omega) = \frac{\exp\big(i\omega(1+2\pi m)\big)-\exp\big(i\omega(1+2\pi n)\big)}{1-\omega^2}$$

where $n, m \in \mathbf{Z}$ with $n > m$. Show that $g_{n,m}(\omega)$ is a C^∞ function. What is the support of this inverse Fourier transform?

(3.123)
The equation of an electric series circuit of a resistance R, an inductance L, and a capacitor C is

$$\frac{1}{C}\int_0^t I(t')\,dt' + L\frac{d}{dt}I + RI \;=\; V(t)\,, \qquad I = I(t)$$

(differently from the previous problems, here the independent variable is the time t), with usual notations. Transform this equation by means of Laplace transform and show that, putting $I(0) = 0$, the Laplace transforms $\widetilde{V}(s) = \mathscr{L}\big(V(t)\big)$ and $\widetilde{I}(s) = \mathscr{L}\big(I(t)\big)$ are related by the rule

$$\widetilde{V}(s) = \widetilde{G}(s)\,\widetilde{I}(s)$$

where $\widetilde{G}(s)$ is the Laplace transform of a Green function $G(t)$. Find and draw $G(t)$ for different values of R, L, C (cf. Problem 3.6).

(3.124)
Consider the equation of a harmonic oscillator subjected to an external force $f(t)$

$$\ddot{y} + y \;=\; f(t)\,, \qquad y = y(t)$$

(as in the problem above, the independent variable is the time t), with given initial values $y(0) = a,\ \dot{y}(0) = b$.

(1) Put $f(t) \equiv 0$ and solve by means of Laplace transform the equation (the solution is trivial and well known!)

(2) Put $a = b = 0$ and

$$f(t) \;=\; \theta(t) - \theta(t-c)\,, \qquad c > 0$$

Write the Laplace transform $\widetilde{f}(s) = \mathscr{L}\big(f(t)\big)$. Find and draw the solution $y(t)$ if $c = \pi$ and if $c = 2\pi$. For what values of c one has $y(t) = 0$ for any t larger than some $t_0 > 0$?

(3) The same questions as in (2) with

$$f(t) \;=\; \delta(t) + \delta(t-c)$$

where $\delta(t)$ is the Dirac delta.

(3.125)
Using $\mathscr{L}\big(xf(x)\big) = \ldots$ and $\lim_{\mathrm{Re}\, s\to+\infty}\mathscr{L}\big(f(x)\big) = 0$, find

$$\mathscr{L}\Big(\frac{\exp(ax)-\exp(bx)}{x}\Big) \quad ; \quad \mathscr{L}\Big(\frac{\sin x}{x}\Big)$$

(3.126)
(1) The Bessel function of zero-order $y = J_0(x)$ satisfies the ODE

$$xy'' + y' + xy = 0$$

Applying Laplace transform to this equation, find the Laplace transform $\widetilde{J}_0(s) = \mathscr{L}\big(J_0(x)\big)$. *Hint*: use the rules $\mathscr{L}\big(xf(x)\big) = \ldots$ and $\mathscr{L}\big(f'(x)\big) = \ldots$ and obtain a first-order ODE for $\widetilde{J}_0(s)$ which can be directly solved; recall that $J_0(0) = 1$ and the "initial value theorem":

$$f(0^+) = \lim_{x\to 0^+} f(x) = \lim_{\mathrm{Re}\, s\to+\infty} s\,\mathscr{L}\big(f(x)\big)$$

(2) As an application, find the convolution product $J_0(x) * J_0(x)$.

(3.127)
Use a translation rule for Laplace transforms and the known formula

$$\sum_{n=0}^{\infty} \exp(-nas) = \frac{1}{1-\exp(-as)}\,, \qquad a > 0,\ \mathrm{Re}\, s > 0$$

to evaluate the following transforms:
(a) $\mathscr{L}\big(f(x)\big)$ where $f(x)$ is the square wave function for $x > 0$:

$$f(x) = \begin{cases} 1 \quad \text{for } 0 < x < 1,\ 2 < x < 3,\ \ldots,\ 2n < x < 2n+1, \ldots \\ -1 \quad \text{for } 1 < x < 2,\ 3 < x < 4, \ldots \end{cases}$$

(b) $\mathscr{L}^{-1}\Big(\dfrac{1}{s(1-\exp(-s))}\Big)$

(c) $\mathscr{L}^{-1}\big(g(s)\big)$ where (see Problem 3.118)

$$g(s) = \frac{1}{s^2+1}\frac{1+\exp(-\pi s)}{1-\exp(-\pi s)} = \frac{1}{s^2+1}\mathrm{ctanh}\big(\pi s/2)\big)$$

(3.128)
Consider the d'Alembert equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}\,, \qquad u = u(x,t)$$

with given initial conditions

$$u(x,0) = f(x)\,, \quad u_t(x,0) = g(x)$$

Introduce and use the Laplace transform $\widetilde{u}(x,s)$ with respect to t, i.e.,

$$\widetilde{u}(x,s) = \int_0^{+\infty} u(x,t)\exp(-st)\,dt$$

(1) Show that the d'Alembert equation is transformed into an ODE for $\widetilde{u}(x,s)$.

(2) In the case of vanishing initial conditions $f(x) = g(x) = 0$, solve the ODE obtained in (1) to obtain the Laplace transform $\widetilde{u}(x,s)$ of the most general solution of the equation.

(3) In the same conditions as in (2), assume that the equation describes an elastic semi-infinite string (in $x \geq 0$), and that extremum at $x = 0$ of the string is subjected to a given transversal displacement $u(0,t) = \varphi(t)$. Find the solution $u(x,t)$ imposing the condition that $u(x,t)$ is bounded for all $x \geq 0$.

(3.129)
Using the general Laplace inversion formula (see the introduction to this subsection):

(1) Show that the result $f(x)$ is independent of the choice of the abscissa a of the line ℓ of integration (provided that $a > \lambda$) and that $f(x) = 0$ if $x < 0$, as expected.

(2) Verify, by means of an integration in the complex plane s, the well-known formula

$$\mathscr{L}^{-1}\Big(\frac{1}{s^2+1}\Big) = \theta(x)\,\sin x$$

(3.130)
Use the general Laplace inversion formula (see the introduction of this subsection) to calculate

$$f(x) = \mathscr{L}^{-1}\Big(\frac{1}{\sqrt{s}}\Big)$$

Hint: An integration in the complex plane s is requested, in the presence of a cut along the negative part of the real axis x.

(3.131)
Using Laplace transform, show that the set $\{u_n(x) = x^n\exp(-x),\ n = 0, 1, 2, \ldots\}$ is a complete set in the Hilbert space $L^2(0, +\infty)$. *Hint*: show first that the Laplace transform of a function $f(x) \in L^2(0, +\infty)$ has abscissa $\lambda \leq 0$. Verify then that the completeness condition $(u_n, f) = 0$, $\forall n = 0, 1, 2, \ldots$ becomes a condition on the derivatives of the Laplace transform $\widetilde{f}(s)$ evaluated at $s = \ldots$.

Chapter 4
Groups, Lie Algebras, Symmetries in Physics

4.1 Basic Properties of Groups and of Group Representations

(4.1)
Consider a group G of finite order N (the order is the number of the elements contained in the group).

(1) Show that the order of all subgroups of G is a divisor of N. *Hint*: let H be a subgroup; consider the "cosets" gH with $g \in G, \ldots$.

(2) Deduce that if N is a prime number then G has no subgroups (apart from, obviously, the identity and G itself), and therefore is simple.

(3) Show that if N is prime, then G is cyclic and Abelian.

(4.2)
Show that if $\mathscr{R}$ is an *unitary* representation of a group G acting on a basis space V (V may be either finite-dimensional or a Hilbert space) and $\mathscr{R}$ admits an invariant subspace $V_1 \subset V$, then also the orthogonal complementary subspace V_2 (i.e., $V_1 \oplus V_2 = V$) is invariant under $\mathscr{R}$. Therefore, $\mathscr{R}$ is completely reducible: $\mathscr{R} = \mathscr{R}_1 \oplus \mathscr{R}_2$ with $\mathscr{R}_i : V_i \to V_i$.

(4.3)
Let $\mathscr{R}$ be any representation of a group G on a basis space V (V may be either finite-dimensional or a Hilbert space) and let T be any operator defined in V and commuting with $\mathscr{R}$, i.e.,

$$T\,\mathscr{R}(g) = \mathscr{R}(g)\,T\,, \qquad \forall g \in G$$

Let λ be an eigenvalue of T and assume that the subspace V_λ of the eigenvectors of T with eigenvalue λ has dimension >1. Show that V_λ is invariant under $\mathscr{R}$, i.e., that

G. Cicogna, *Exercises and Problems in Mathematical Methods of Physics*, Undergraduate Lecture Notes in Physics,
https://doi.org/10.1007/978-3-319-76165-7_4

if $v_\lambda \in V_\lambda$ then also $v' \equiv \mathscr{R}(g)\, v_\lambda \in V_\lambda$, $\forall g \in G$, or $Tv' = \lambda v'$. In particular, if $\mathscr{R}$ is irreducible, deduce the Schur lemma (in its simplest version).

(4.4)

(1) Show that if a group G is Abelian then all its irreducible representations are one-dimensional.

(2) Show that if a group G is simple then all its representations are faithful (apart, obviously, the trivial representation $\mathscr{R} : g \to 1,\ \forall g \in G$).

(4.5)

Find the group describing the symmetry of the equilateral triangle and all its inequivalent irreducible representations. What degeneracy can be expected for the vibrational energy levels of a physical system which exhibits this symmetry (e.g., the molecule of ammonia NH_3)?

(4.6)

Consider a system consisting of three particles with the same mass m, placed at the vertices of an equilateral triangle having side ℓ, and subjected to elastic forces produced by three equal springs situated along the sides of the triangle. The length at rest of the springs is ℓ. To describe the small displacements (in the plane) of the particles from their equilibrium positions one clearly needs six-dimensional vectors $\mathbf{x} \in \mathbf{R}^6$. The action of the symmetry group of the triangle on this space produces a six-dimensional representation. Write first the characters of the inequivalent irreducible representations of the symmetry group of the equilateral triangle (see the previous problem), and show that this six-dimensional representation has the following characters

$$(6;\, 0, 0, 0;\, 0, 0)$$

written in the order: identity; reflections; rotations. Decompose then this representation as a direct sum of irreducible representations. These irreducible representations describe different displacements of the particles: recognize that two of these displacements are trivial (correspond to rigid displacements), and other two, having dimension 1 and 2, correspond to oscillations of the system.

(4.7)

Consider the group describing the symmetry of the square and determine in particular the dimension of its inequivalent irreducible representations (the use of Burnside theorem can help). Compare with the result obtained in Problem 1.94 concerning the degeneracy of the eigenvalues of the Laplace operator $T = \partial^2/\partial x^2 + \partial^2/\partial y^2$ for the square with vanishing boundary conditions. Show that "accidental degeneracies" are present in this case.

(4.8)

(1) Find all the inequivalent irreducible representations of the additive group Z_7 of the integers *mod 7*. *Hint*: see Problems 4.1, q. (3), 4.4, q. (1) and recall Burnside theorem.

(2) The same for the group Z_6. Are there not-faithful representations? Compare with the case of Z_7 and with that of Problem 4.5.

(4.9)
The symmetry group $\mathscr{O}$ of the cube contains a subgroup (denoted here by $\mathscr{O}_1$) of 24 transformations not involving reflections, and other 24 transformations including reflections. Show that the group can be written as a direct product $\mathscr{O} = \mathscr{O}_1 \times Z_2$, where Z_2 is (isomorphic to) $\{1, -1\}$. Knowing that there are 5 inequivalent irreducible representations of $\mathscr{O}_1$ and using Burnside theorem, find the dimensions of these representations (and of those of $\mathscr{O}$, of course), and the degeneracy which can be expected for the vibrational energy levels of a system which exhibits the symmetry of the cube.

(4.10)
(1) Let Φ be the homomorphism $\Phi : GL_n(\mathbf{C}) \to \mathbf{C}$ of the general linear complex group $GL_n(\mathbf{C})$ of the invertible $n \times n$ complex matrices M into the multiplicative group of nonzero complex numbers $\mathbf{C}$, defined by

$$\Phi(M) = \det M$$

Specify the kernel $K = \mathtt{Ker}\,\Phi$ and the quotient group $Q = GL_n(\mathbf{C})/K$. It is true that $GL_n(\mathbf{C})$ is the direct product $GL_n(\mathbf{C}) = K \times Q$?

(2) The same questions for the group U_n of the unitary matrices.

(3) The same questions for the group $GL_n(\mathbf{R})$ of the invertible real matrices, and for the group O_n of the orthogonal matrices.

(4.11)
(1) The "basic" representation of the rotation group SO_2 in the $\mathbf{R}^2$ plane is, as well known,

$$R(\varphi) = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$$

There is no eigenvector of $R(\varphi)$ in the *real* space $\mathbf{R}^2$ (apart from the case $\varphi = \pi$), but there *must* exist eigenvectors in $\mathbf{C}^2$ (SO_2 is Abelian, then its irreducible representations are one-dimensional): find the eigenvectors of $R(\varphi)$ and then decompose the representation. What can be a physical meaning of these eigenvectors? $\big($for a possible interpretation, see question (3)$\big)$.

(2) What are the other inequivalent irreducible representations of SO_2? What of these are faithful?

(3) The electric field $\mathbf{E}$ of a planar electromagnetic wave propagating along the z-axis can be written, with usual and clear notations,

$$\mathbf{E}(x, y, z, t) = \begin{pmatrix} E_x \\ E_y \end{pmatrix} =$$

$$= \begin{pmatrix} E_1 \cos(kz - \omega t + \varphi_1) \\ E_2 \cos(kz - \omega t + \varphi_2) \end{pmatrix} = \mathrm{Re}\left\{\exp\left(i(kz - \omega t)\right)\begin{pmatrix} E_1 \exp(i\varphi_1) \\ E_2 \exp(i\varphi_2) \end{pmatrix}\right\}$$

Then the vector $\mathbf{p} \in \mathbf{C}^2$ defined by

$$\mathbf{p} = \begin{pmatrix} E_1 \exp(i\varphi_1) \\ E_2 \exp(i\varphi_2) \end{pmatrix}$$

shows the *state of polarization* of the e.m. wave. For example, $(1,\ 0)$ describes the linear polarization along the x-axis, etc. What types of polarization are described by $(1,\ \pm i)$?

(4.12)
Verify that the Lorentz boost with velocity v involving one spatial variable x and the time t can be written in the form

$$L(\alpha) = \begin{pmatrix} \cosh\alpha & -\sinh\alpha \\ -\sinh\alpha & \cosh\alpha \end{pmatrix}$$

where $\alpha = \arctan(v/c)$. This (not unitary!) representation of the "pure" (i.e., without space and time inversions) one-dimensional Lorentz group is reducible (the group is indeed Abelian): perform this reduction; what is the physical meaning of the eigenvectors of $L(\alpha)$?

4.2 Lie Groups and Lie Algebras

(4.13)
(1) Show that the matrices $M \in SU_n$ can be put in the form

$$M = \exp A$$

where the matrices A are anti-Hermitian (i.e., $A^+ = -A$) and traceless. Show that the space $\mathscr{A}$ of these matrices is a vector space and that for any $A_1,\ A_2 \in \mathscr{A}$ then also the commutator $[A_1, A_2] \in \mathscr{A}$. Find the dimension of the space $\mathscr{A}$, which is the Lie algebra of SU_n, as vector space on the reals (which is, by definition, the dimension of SU_n); find then the dimension of U_n.
(2) Show that the matrices $M \in SO_n$ can be put in the form

$$M = \exp A$$

where the matrices A are real antisymmetric. Notice that, in this case, imposing *only* the orthogonality of M, one gets automatically $\mathrm{Tr}\,A = 0$ and then $\det M = 1$: why cannot this procedure be extended to the whole group O_n, i.e., to the matrices with $\det = -1$? Find the dimension of the groups SO_n. In the case of SO_3, find a basis for the vector space $\mathscr{A}$ of the 3×3 real antisymmetric matrices.

(4.14)
Consider a group G of matrices M and the neighborhood of the identity[1] where one can write (see previous problem)

$$M = \exp A$$

where the matrices A describe a vector space $\mathscr{A}$, the Lie algebra of G, of dimension r (over the reals).

(1) Choose a matrix $A_1 \in \mathscr{A}$ and consider $M_1 = \exp(a_1 A_1)$ where $a_1 \in \mathbf{R}$. Show that M_1 is an Abelian one-parameter subgroup of G. Let M_1 and $M_2 = \exp(a_2 A_2)$ two of these subgroups: in general, do these subgroups commute?

(2) Let $A_1, \ldots, A_r$ be a basis for the vector space $\mathscr{A}$: any $A \in \mathscr{A}$ can then be written as $A = \sum_{i=1}^{r} a_i A_i$ with $\mathbf{a} \in \mathbf{R}^r$; show that

$$A_i = \left.\frac{\partial M}{\partial a_i}\right|_{\mathbf{a}=0}$$

(4.15)
(1) Find the expression of the generator A of the Lie algebra of the rotation group SO_2 in its "basic" representation given in Problem 4.11, q. (1).

(2) The same for the pure Lorentz group given in Problem 4.12.

(3) Evaluate

$$\exp(aA) = \sum_{n=0}^{\infty} \frac{(aA)^n}{n!}$$

in the cases

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} ; \; A = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} ; \; A = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} ; \; A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Hint: $A^2 = \ldots$. Notice that this is a partial converse of questions (1) and (2).

(4.16)
Describe (or give a geometrical or physical interpretation of) the groups generated by the following (one-dimensional) Lie algebras:

$$A = 1 ; \quad A = i ; \quad A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} ; \quad A = \begin{pmatrix} i & 0 \\ 0 & 2i \end{pmatrix} ; \quad A = \begin{pmatrix} i & 0 \\ 0 & i\sqrt{2} \end{pmatrix} ;$$

[1] It can be useful to recall that, according to Ado theorem, all finite-dimensional Lie algebras admit a faithful representation by means of matrices.

$$A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \; ; \quad A = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \; ; \quad A = \text{any } n \times n \text{ matrix}$$

(4.17)
The elements of the Euclidean group E_3 in $\mathbf{R}^3$ (or in general E_n in $\mathbf{R}^n$) are the transformations $\mathbf{x} \to \mathbf{x}' = \mathscr{O}\mathbf{x}+\mathbf{b}$, where $\mathscr{O} \in O_3$ and $\mathbf{b} \in \mathbf{R}^3$. Write the composition rule in E_3 and show that the mapping

$$(\mathscr{O}, \mathbf{b}) \to \begin{pmatrix} & & & b_1 \\ & \mathscr{O} & & b_2 \\ & & & b_3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

is a faithful representation of E_3 in the group of 4×4 matrices. Find the expression of the Lie generators $A_1,\ A_2,\ A_3$ and $B_1,\ B_2,\ B_3$ of E_3 in this representation (choose for simplicity as matrix $\mathscr{O}$ a rotation around the z-axis).

(4.18)
(1) Let T_1 be the group of translations along $\mathbf{R}$ and let the parameter $a \in \mathbf{R}$ denote the translation (clearly T_1 is isomorphic to $\mathbf{R}$, the additive group of reals). Consider the representation of T_1 acting on the functions $f(x) \in L^2(\mathbf{R})$ according to

$$a \to U_a \quad \text{where} \quad (U_a f)(x) = f(x-a)$$

Show that, as suggested by the notation, this representation is unitary. Assume the parameter a "infinitesimal" and $f(x)$ regular enough (expandable); expand then

$$f(x-a) = f(x) - \ldots$$

and obtain the differential expression A of the Lie generator of the translation group.[2] Extend to the group of translations in $\mathbf{R}^3$, i.e., $\mathbf{x} \to \mathbf{x}' = \mathbf{x} + \mathbf{a},\ \mathbf{a} \in \mathbf{R}^3$.

(2) Consider the representation $\mathscr{R} = \mathscr{R}_\varphi$ of the rotation group SO_2 on the plane $\mathbf{x} \equiv (x, y)$ acting on the space of functions $f(\mathbf{x}) \in L^2(\mathbf{R}^2)$ according to the usual rule

$$f(\mathbf{x}) \to \big(\mathscr{R}_\varphi f\big)(\mathbf{x}) = f\big(R_\varphi^{-1}\mathbf{x}\big) = f(\mathbf{x}')$$

where R_φ is given in Problem 4.11, q. (1). Consider an "infinitesimal" rotation $R_\varphi^{-1}\mathbf{x} = \mathbf{x}' \equiv (x + \varphi y + \ldots,\ y - \varphi x + \ldots)$ and assume $f(\mathbf{x})$ regular; expand then

$$f(\mathbf{x}') \equiv f(x', y') = f(x, y) + \ldots$$

[2]It is customary in physics to introduce a factor i in the definition of these generators, in order to have *Hermitian* operators. For example, $A = -i\, d/dx$, which is proportional to the momentum operator $P = -i\hbar\, d/dx$ in quantum mechanics, as well known.

and obtain the differential expression A of the Lie generator of the rotation group.

(3) Repeat the calculations considering a function $f(x,t)$ and an "infinitesimal" Lorentz boost $L(\alpha)$ (see Problem 4.12) in the plane x, t.

(4.19)
(1) Let $x \in \mathbf{R}$ and consider a dilation $x \to x' = (\exp a)x$, with $a \in \mathbf{R}$; introduce the representation of the group of dilations acting on functions $f(x) \in L^2(\mathbf{R})$ according to the usual rule (cf. Problem 4.18)

$$a \to T_a \;\text{ where }\; \big(T_a f\big)(x) = f\big(\exp(-a)x\big) = f(x - ax + \ldots) = f(x) - \ldots$$

Assuming $f(x)$ regular (expandable), find the Lie generator D of the dilation group in this representation T_a. Show that this representation is not unitary and that D is not anti-Hermitian (nor Hermitian).

(2) Consider the group of transformations $a \to S_a$, depending on a parameter $a \in \mathbf{R}$, where

$$\big(S_a f\big)(x) = \exp(-a/2) f\big(\exp(-a)x\big) = (1-a/2+\ldots) f(x-ax+\ldots) = f(x)-\ldots$$

and find the generator $\widetilde{D}$ of this transformation. Show that S_a is unitary and that $\widetilde{D}$ is anti-Hermitian (in agreement with Stone theorem). (This is a reformulation of the Problem 1.93).

(4.20)
(1) Show that all two-dimensional algebras can be put in one of the two forms $[A_1, A_2] = 0$ or $[A_1, A_2] = A_2$.

(2) Study the two-dimensional algebra generated by D, where D is the generator of dilations considered in the above problem, q. (1) and the generator $P = -d/dx$ of the translations along x, and verify that this algebra admits a faithful representation by means of the two matrices (cfr. also Problem 4.17).

$$D' = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad , \quad P' = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

(4.21)
Let D_1, D_2 be the Lie generators of the dilations along x and y in the $\mathbf{R}^2$ plane, and let A be the generator of the rotations in this plane. Using either the 2×2 matrix representation or the differential representation (see Problems 4.15, 4.18, 4.19, q. (1), and 4.20) of these generators, construct the Lie algebra (i.e., obtain the commutation rules) of these operators. Show that this algebra is semisimple (what is the invariant subalgebra?). Compare this algebra with the following three-dimensional algebras:

(i) the algebra of SO_3, *(ii)* that of the Euclidean group E_2, *(iii)* that of the Poincaré group consisting of the Lorentz boosts involving x and t and of the two translations

along x and t. Is there any isomorphism between these algebras? and between the groups?

(4.22)
(1) Using, e.g., the differential representation (see Problem 4.18), construct the Lie algebra (i.e., obtain the commutation rules) of the three-dimensional group consisting of the Lorentz boosts involving t with two real variables $x,\ y$ and of the rotations on the plane x, y.

(2) The same question for the six-dimensional "pure" Lorentz group[3] $\mathscr{L}$ of the Lorentz transformations involving t with the space variables in $\mathbf{R}^3$ and of the rotations in $\mathbf{R}^3$ (the latter is clearly the subgroup $SO_3 \subset \mathscr{L}$).

(3) The same question for the 10-dimensional Poincaré group, consisting of the Lorentz group and the four translations along $\mathbf{x}$ and t.

(4.23)
(1) Show that the operators

$$D_0 = \frac{d}{dx} \quad , \quad D_1 = x\frac{d}{dx} \quad , \quad D_2 = x^2\frac{d}{dx}$$

generate a three-dimensional Lie algebra. Verify also that no (finite-dimensional) algebra can be generated by two or more operators of the form $x^n(d/dx)$ if at least one of the (integer) exponents n is > 2.

(2) Put

$$A_1 = \frac{1}{\sqrt{2}}\big(-D_0 + D_2/2\big)\ ,\ A_2 = D_1\ ,\ A_3\frac{1}{\sqrt{2}}\big(D_0 + D_2/2\big)$$

find the commutation rules $[A_i,\ A_j]$ and compare with the commutation rules of the three-dimensional algebras mentioned in the two previous problems.

(4.24)
The representations of the non-compact group T_1 of the one-dimensional translations along $\mathbf{R}$ cannot be simultaneously unitary, irreducible, and faithful. Specify what among the above properties are (or are not) satisfied by each one of the four representations of T_1 listed below. Denote by a the translation specified by the parameter $a \in \mathbf{R}$ (T_1 is clearly isomorphic to the additive group of reals).

(a) Consider the representation by means of 2×2 matrices according to

$$a \to \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$$

[3] $\mathscr{L}$ is the subgroup "connected to the identity" of the full Lorentz group, usually denoted by $O(3, 1)$, which includes also space inversions and time reversal. The same remark holds for the group considered in q. (1), which is a subgroup of $O(2, 1)$.

(verify that this is indeed a representation of T_1: see Problem 4.17).

(b) Consider the representation acting on the functions $f(x) \in L^2(\mathbf{R})$ according to $\big($see Problem 4.18, q. (1)$\big)$

$$a \to U_a \quad \text{where} \quad (U_a f)(x) = f(x - a)$$

In this case, introduce the Fourier transform $\mathscr{F}\big(f(x)\big) = \widehat{f}(k)$ and show that *any* subspace of functions $f(x)$ having Fourier transform $\widehat{f}(k)$ with support $J \subset \mathbf{R}$, properly contained in $\mathbf{R}$, is a (∞-dimensional) invariant subspace for this representation. Therefore, to have one-dimensional representations, one must enlarge the choice of the basis space and consider "functions" having Fourier transform $\widehat{f}(k)$ with support in a single point, i.e., $\widehat{f}(k) = \delta(k - \lambda)$. This leads to the representation for T_1 proposed in c).

(c) Consider the representation

$$a \to \exp(i\lambda a)\,, \qquad \lambda \in \mathbf{R}$$

(d) Consider the representation

$$a \to \exp(\mu a)\,, \qquad \mu \in \mathbf{R}.$$

(4.25)

(1) Extend the representations considered in the previous problem to the group of translations including reflections (i.e., $Sx = -x$; is this group Abelian?).

(2) The same for the group of translations in $\mathbf{R}^3$ without reflections and for the group which includes reflections.

(4.26)

Consider the three-parameter group (called Heisenberg group) of the 3×3 matrices defined by

$$M(\mathbf{a}) = \begin{pmatrix} 1 & a_1 & a_3 \\ 0 & 1 & a_2 \\ 0 & 0 & 1 \end{pmatrix}$$

where $\mathbf{a} \equiv (a_1, a_2, a_3) \in \mathbf{R}^3$.

(1) Find the generators $A_1,\ A_2,\ A_3$ of this group and verify that the commutation rules are

$$[A_1,\ A_2] = A_3 \quad , \quad [A_1,\ A_3] = [A_2,\ A_3] = 0$$

(which are the same—apart from some factor $i\hbar$—as those of the quantum mechanical operators p, q and the identity I), and construct the one-parameter subgroups generated by $A_1,\ A_2$ and A_3.

(2) Show that the map

$$M(\mathbf{a}) \to \exp(-ia_3)\exp(ixa_2)\, f(x-a_1)$$

where $f(x) \in L^2(\mathbf{R})$, is a *unitary* representation of the Heisenberg group. Find the generators of the group in this representation and verify that (expectedly!) they satisfy the same commutation rules as seen in (1).

(3) Show that the representation given in (2) is irreducible. *Hint*: introduce Fourier transform and use an argument similar to that in Problem 4.24.

4.3 The Groups SO_3, SU_2, SU_3

(4.27)
(1) Let $\mathbf{x} \equiv (x, y, z) \in \mathbf{R}^3$ and consider the vector space generated by the quadratic monomials $x^2, y^2, z^2, xy, xz, yz$. This space is clearly invariant under the rotations $\mathbf{x} \to \mathbf{x}' = R\,\mathbf{x}$ with $R \in SO_3$. Decompose this space in a direct sum of subspaces which are the basis of irreducible representations of SO_3. What is the relationship between these subspaces and the spherical harmonics $Y_{\ell,m}(\theta, \varphi)$?

(2) The same for the space generated by the cubic monomials $x^3, x^2y, \ldots$. What is the dimension of this space?

(4.28)
(1) Given two Lie algebras $\mathscr{A}$ and $\mathscr{A}'$ with the same dimension r, and their generators $A_1, \ldots, A_r$ and resp. $A'_1, \ldots, A'_r$, assume that these generators satisfy the same commutation rules apart from a multiplicative nonzero constant in their structure constants, i.e.,

$$[\, A_i\, ,\; A_j\,] = c_{ijk} A_k\ , \quad [\, A'_i\, ,\; A'_j\,] = \lambda\, c_{ijk} A'_k\, , \qquad \lambda \neq 0$$

Show that the generators A'_i can be redefined, simply introducing generators A''_i proportional to A'_i, in such a way that the structure constants of A_i and A''_i are the same.

(2) A commonly used choice for the generators of the algebras of SO_3 and SU_2 are

$$A_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}\ ;\ A_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}\ ;\ A_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and resp. (the well-known Pauli matrices σ_j are given by $\sigma_j = iA_j$)

$$A'_1 = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} \ ; \ A'_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \ ; \ A'_3 = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$$

Show that this is an example of the situation seen in (1) with $\lambda = \ldots$. Calculate the quantities (Casimir operators) $C = A_1^2 + A_2^2 + A_3^2$ and $C' = A_1'^2 + A_2'^2 + A_3'^2$: the results should be expected from the quantum mechanical interpretation, ...

(3) Denote by $A''_1,\ A''_2,\ A''_3$ the 2×2 matrices having the same structure constants as the 3×3 matrices $A_1,\ A_2,\ A_3$, according to (1). Evaluate $\exp(aA_3)$ and $\exp(a''A''_3)$: for what values of a one has $\exp(aA_3) = I$ and resp. for what values of a'' one has $\exp(a''A''_3) = I$?

(4.29)
(1) Show that all groups SU_n $(n \geq 2)$ admit a nontrivial center. Conclude: the SU_n are not simple groups.

(2) Consider the group SU_2: show that its center is Z_2 (the quotient SU_2/Z_2 is isomorphic to SO_3, as well known). The presence of not-faithful representations of SU_2 is then expected: what are the representations of SU_2 which are not faithful?

(4.30)
(1) Start considering the vectors $\mathbf{v} \in \mathbf{R}^3$ as basis space for the "basic" representation $\mathscr{R}$ by means of 3×3 orthogonal matrices of SO_3, and then consider the tensor product $\mathbf{R}^3 \otimes \mathbf{R}^3$ (isomorphic, of course, to the nine-dimensional vector space of the real 3×3 matrices $\mathbf{M}$), together with the resulting direct product representation $\mathscr{R}\otimes\mathscr{R}$ of SO_3 acting on this space according to:

$$M_{ij} \to M'_{ij} = R_{ir}R_{js}M_{rs} = (RMR^t)_{ij}$$

(where ^{t} means matrix transposition). Show that there are invariant subspaces for this representation and obtain its decomposition into irreducible representations. According to the quantum mechanical interpretation, the vectors $\mathbf{v}$ correspond to the angular momentum $\ell = 1$; give then the interpretation of the above decomposition as a result of the combination of two angular momenta $\ell = 1$, which is often written, with evident and convenient notations, $(\ell = 1) \otimes (\ell = 1)$ or also $\underline{3} \otimes \underline{3}$.

(2) Generalize, e.g., to tensors with three indices: M_{ijk}, i.e., $(\ell = 1) \otimes (\ell = 1) \otimes (\ell = 1)$ or $\underline{3} \otimes \underline{3} \otimes \underline{3}$.

(4.31)
(1) Start considering the "basic" representation $\mathscr{R}$ of SU_3 by means of 3×3 unitary matrices U with determinant equal to 1, and the "contragredient" representation $\mathscr{R}^*$ obtained by means of matrices $U^* = (U^t)^{-1}$ where * means complex conjugation and ^{t} matrix transposition. Writing $z^i = (z_i)^*$ in such a way that upper indices are transformed by $\mathscr{R}^*$, introduce the convenient notation $\underline{3}$ and $\underline{3}^*$ to denote the basis

spaces for the (inequivalent) representations[4] $\mathscr{R}$ and $\mathscr{R}^*$. Consider then the tensor products $\underline{3} \otimes \underline{3}$ and $\underline{3} \otimes \underline{3}^*$ with the resulting direct products of the representations $\mathscr{R} \otimes \mathscr{R}$ and $\mathscr{R} \otimes \mathscr{R}^*$, i.e., respectively

$$T_{ij} \to T'_{ij} = U_{ir} U_{js} T_{rs} = (UTU^t)_{ij}$$

and

$$T_i^j \to T_i'^j = U_{ir} (U^*)^{js} T_r^s = (UTU^+)_i^j$$

Decompose these representations into irreducible representations of SU_3. It can be useful to introduce here the shorthand notation $T_{(\dots)}$ and $T_{[\dots]}$ to denote resp. symmetric and antisymmetric tensors, and recall that, using the totally antisymmetric tensor ε_{ijk}, the three-dimensional representation on the antisymmetric tensors $T_{[ij]}$ is equivalent to the representation on the vectors $z^k = \varepsilon^{ijk} T_{[ij]}$.

(2) Decompose into irreducible representations the tensor products $\underline{3}^* \otimes \underline{3}^*$ and $\underline{3} \otimes \underline{3} \otimes \underline{3}$.

4.4 Other Relevant Applications of Symmetries to Physics

The following four problems give some simple examples of the application of symmetry properties to the study of differential equations. Actually, this technique can be greatly developed, introducing "less evident" symmetries of the differential equations; denoting by $u = u(x_1, x_2, \dots)$ the unknown function, one can look, e.g., for transformations generated by infinitesimal operators of the form

$$A = \xi_1(x_1, x_2, \dots, u) \frac{\partial}{\partial x_1} + \xi_2(x_1, x_2, \dots, u) \frac{\partial}{\partial x_2} + \dots + \varphi(x_1, x_2, \dots, u) \frac{\partial}{\partial u}$$

where the functions ξ_i and φ can be arbitrary functions (with some obvious regularity assumptions) and not only constants or linear functions of the independent variables x_i as in usual elementary cases, and in all the examples considered in this book. In addition, also the dependent variable u is assumed to be subjected to a transformation, as shown by the presence of the term φ in the expression of the generator A. A presentation of these methods and of their applications is given, e.g., in the book by P.J. Olver, see Bibliography.

[4] It can be useful to point out that, differently from all the groups SU_n with $n > 2$, the "basic" irreducible representations $\mathscr{R}$ and $\mathscr{R}^*$ of SU_2 by means of 2×2 unitary matrices are equivalent.

(4.32)
The Laplace equation in $\mathbf{R}^2$

$$u_{xx} + u_{yy} = 0\,, \qquad u = u(x, y)$$

is clearly symmetric under the transformations of the group O_2 in the plane x, y (the equation remains unaltered under the transformations of O_2). This implies that if $u = u(x, y)$ is a solution to this equation, then also $\widetilde{u} = u(x', y')$, with $x' = x\cos\varphi - y\sin\varphi$, etc., is a solution for any φ. Observing that, for instance, $u = \exp x \cos y = \mathtt{Re}\,(\exp z)$, with $z = x + iy$, solves the Laplace equation, construct a family of other solutions of the equation. Verify that this new family of solutions is the same one can obtain by means of the transformation $z \to z' = \exp(i\varphi)z$. Using the same procedure, construct other families of solutions starting from a known one.

(4.33)
The nonhomogeneous Laplace equation for $u = u(x, y)$ and with $r^2 = x^2 + y^2$

$$u_{xx} + u_{yy} = r^n\,, \qquad n = 0, 1, \ldots$$

is clearly symmetric under the transformations of the group O_2 in the plane x, y. One then can look for the existence of solutions which are *invariant* under O_2, i.e., for solutions of the form $u = f(r)$. Transform then the given equation (or recall the well-known expression of the Laplace equation in terms of r, φ) into an ODE for the unknown $f(r)$ and solve this equation.

(4.34)
(1) The heat equation

$$u_t = u_{xx}\,, \qquad u = u(x, t)$$

is symmetric under independent translations of x and t, and therefore under any combination of these translations. Look then for solutions which are functions only of $x - vt$ ("traveling wave solutions" with arbitrary velocity v). Transform the PDE into an ODE for the unknown function $u = f(s)$ of the independent variable $s = x - vt$ and obtain the most general solution of this form.

(2) Do the same for the d'Alembert equation

$$u_{xx} = u_{tt}\,, \qquad u = u(x, t)$$

What traveling waves are obtained?

(4.35)
(1) The d'Alembert equation

$$u_{xx} = u_{tt}\,, \qquad u = u(x, t)$$

is symmetric under the Lorentz transformations. Starting, for instance, from the solution $u = x^3 + 3xt^2$, construct a family of other solutions to the d'Alembert equation.

(2) Consider the nonhomogeneous d'Alembert equation

$$u_{xx} - u_{tt} = (x^2 - t^2)^n , \qquad n = 0, 1, \ldots$$

Observing that the equation is symmetric under Lorentz transformations, one can look for the existence of solutions which are Lorentz-invariant, i.e., for solutions of the form $u = f(s)$ where $s = x^2 - t^2$. Transform then the given equation into an ODE for the unknown $f(s)$ and solve this equation.

(4.36)
Consider a quantum mechanical system exhibiting spherical symmetry, e.g., a particle placed in a spherically symmetric (radial) potential $V(r)$. The eigenvalues of its Hamiltonian (Schrödinger equation), i.e., the energy levels, are expected to have a degeneracy …, as a consequence of the symmetry under the group SO_3 and the Schur lemma. If the system is placed into a (weak) uniform magnetic field, the SO_3 symmetry is broken and the surviving symmetry is SO_2 (*not* O_2: way?). What consequence can be expected about the degeneracy of the energy levels? What changes if instead the system is placed into an uniform electric field, where the residual symmetry is O_2? This is a qualitative description, based only on symmetry arguments and in particular on Schur lemma, of the Zeeman and respectively the Stark effects.

The following problems will be concerned with groups SO_3, SO_4, SU_3 and their Lie algebras. In view of the applications to physical problems (the hydrogen atom and the harmonic oscillators in quantum mechanics), it is convenient to define the generators of the algebras as Hermitian *operators, multiplying by a factor i the definitions used in all previous problems (see, e.g., Problems* 4.13 and 4.18*). Accordingly, for instance, the generator A_3* of *SO_3 (with clear notation) will be, in matrix form,*

$$A_3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

with real eigenvalues ± 1 *and 0, or, in differential form,*

$$A_3 = i\Big(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\Big) = -i\frac{\partial}{\partial \varphi}$$

(4.37)
This problem is devoted to the study of the group SO_4 and its algebra. Consider the space $\mathbf{R}^4$ with Cartesian coordinates x_1, x_2, x_3, x_4.

(1) Show that SO_4 has six parameters and then its algebra is six-dimensional. Denote by A_1, A_2, A_3 the Hermitian generators of the subgroup SO_3 of the rotations in the subspace $\mathbf{R}^3$ with basis x_1, x_2, x_3, and by B_1, B_2, B_3 the generators of the rotations in the subspaces $(x_1, x_4),\ (x_2, x_4),\ (x_3, x_4)$. Either using a 4×4 matrix representation or the differential representation, show that the algebra of SO_4 is described by the following commutation rules, with $i, j = 1, 2, 3$,

$$[\, A_i\,,\, A_j\,] = i\,\varepsilon_{ijk} A_k\,, \quad [\, B_i,\, B_j\,] = i\,\varepsilon_{ijk} A_k\,, \quad [\, A_i\,,\, B_j\,] = i\,\varepsilon_{ijk} B_k$$

(2) Put

$$M_i = \frac{1}{2}(A_i + B_i)\,, \quad N_i = \frac{1}{2}(A_i - B_i)$$

and show that the operators $M_i,\ N_i$ satisfy

$$[\, M_i\,,\, M_j\,] = i\,\varepsilon_{ijk} M_k\,, \quad [\, N_i\,,\, N_j\,] = i\,\varepsilon_{ijk} N_k\,, \quad [\, M_i\,,\, N_j\,] = 0$$

so that the Lie algebra of SO_4 (*not* the group) is isomorphic to the algebra of the direct product $SU_2 \times SU_2$.

(3) Recalling that the irreducible representations of SU_2 are specified by an integer or half-integer number $j = 0,\ 1/2,\ 1, \ldots$, and have dimension $N = 2j+1 = 1, 2, \ldots$, describe the irreducible representations of $SU_2 \times SU_2$ and find their dimensions.

(4) Show that SO_4 has rank 2; verify then that the quantities (Casimir operators)

$$C_1 = \mathbf{A}^2 + \mathbf{B}^2 = 2\big(\mathbf{M}^2 + \mathbf{N}^2\big) \quad \text{and} \quad C_2 = \mathbf{A} \cdot \mathbf{B} = \mathbf{M}^2 - \mathbf{N}^2$$

commute with the algebra of SO_4 and find the values of C_1 and C_2 in the various representations.

(4.38)
Consider the Schrödinger equation for the hydrogen atom, with usual notations,

$$\mathtt{H}\, u \equiv -\frac{\hbar^2}{2m}\Delta\, u - \frac{e^2}{r}\, u = E\, u$$

As any other Hamiltonian exhibiting spherical symmetry, this Hamiltonian commutes with the three operators A_i, where e.g., $A_3 = i(y\,\partial/\partial x - x\,\partial/\partial y) = x p_y - y p_x$, etc., proportional to the operators $L_i = \hbar A_i$, describing, as well known, the angular momentum:

$$[\, \mathtt{H},\, L_i\,] = 0$$

It is known from classical mechanics that the three components of the Runge–Lenz vector

$$\widetilde{\mathbf{R}} = \mathbf{p} \times \mathbf{L} - me^2 \frac{\mathbf{r}}{r}$$

are constants of motion for the Kepler Problem. This has a precise counterpart in quantum mechanics: putting indeed, in a correct quantum mechanical form,

$$\mathbf{R} = \frac{1}{2}\big(\mathbf{p} \times \mathbf{L} - \mathbf{L} \times \mathbf{p}\big) - me^2 \frac{\mathbf{r}}{r}$$

one can find

$$[\mathtt{H}, R_i] = 0$$

Introducing the dimensionless Hermitian operators

$$B_i = \frac{1}{\hbar\sqrt{-2mH}} R_i\,, \qquad i = 1, 2, 3$$

(only bound states will be considered, then the energy is negative) it can be shown, after some tedious (not requested) calculus, that the six operators A_i, B_i satisfy precisely the commutation rules of the algebra of SO_4, isomorphic to the algebra $SU_2 \times SU_2$ (obtained in the previous problem).

(1) With the notations of the previous problem, i.e., $\mathbf{M} = (\mathbf{A}+\mathbf{B})/2$, $\mathbf{N} = (\mathbf{A}-\mathbf{B})/2$, show that

$$\mathbf{L} \cdot \mathbf{R} = \mathbf{A} \cdot \mathbf{B} = \mathbf{M}^2 - \mathbf{N}^2 = 0$$

Conclude from the last equality: what representations of $SU_2 \times SU_2$ are involved in the hydrogen atom? and what degeneracy can be expected for the eigenvalues of the energy of the hydrogen atom?

(2) It can be also shown that

$$(\mathbf{A}^2 + \mathbf{B}^2 + 1)\, H = -\frac{me^4}{2\hbar^2}$$

Observing that $\mathbf{A}^2 + \mathbf{B}^2 = 4\mathbf{M}^2$ and recalling that the eigenvalues of $\mathbf{M}^2$ are ..., deduce the eigenvalues of the energy of the hydrogen atom.

(3) Decompose the n^2-dimensional irreducible representation corresponding to the nth energy level as a sum of the irreducible representations of the rotation subgroup SO_3 (i.e., find the angular momenta ℓ contained in the nth level); to this aim, observe that $\mathbf{L} = \hbar(\mathbf{M} + \mathbf{N})$, therefore, the angular momentum ℓ can be obtained as a superposition of the two "spin" j_M and $j_N = j_M$, and therefore $\ell = 0, 1, \ldots, 2j_M = \ell_{Max} = n - 1$. Show that

$$\sum_{\ell=0}^{\ell_{Max}}(2\ell+1) \;=\; n^2\,, \qquad n=1,2,\ldots$$

(4.39)
As in the case of the hydrogen atom, also the three-dimensional harmonic isotropic oscillator admits, besides the SO_3 rotational symmetry, an additional symmetry.

(1) Consider first the three-dimensional harmonic isotropic equation in classical mechanics: its Hamiltonian is (with $m=k=1$ and with standard notations)

$$H \;=\; \frac{1}{2}\mathbf{p}^2+\frac{1}{2}\mathbf{r}^2$$

Put

$$\zeta_j \;=\; \frac{1}{\sqrt{2}}\big(x_j+ip_j\big)\,, \qquad j=1,2,3$$

and show that H can be written as $H=(\boldsymbol{\zeta},\boldsymbol{\zeta})$, i.e., as a scalar product in $\mathbf{C}^3$. Show also that the 9 quantities $\zeta_j^*\zeta_k$ are constants of motion: $(d/dt)\zeta_j^*\zeta_k=0$.

(2) (*a*) Write the Schrödinger equation for the oscillator (with $m=k=1$)

$$\mathtt{H}\,u \equiv -\frac{\hbar^2}{2}\Delta\,u+\frac{1}{2}\mathbf{r}^2\,u \;=\; E\,u$$

Put now (with $\hbar=1$)

$$\eta_j \;=\; \frac{1}{\sqrt{2}}\Big(x_j+\frac{\partial}{\partial x_j}\Big)\,, \qquad j=1,2,3$$

and show that the nine operators

$$A^j_k \;=\; \eta^+_j\eta_k\,, \qquad j,k=1,2,3$$

commute with the Hamiltonian operator $\mathtt{H}$.

(*b*) Verify that the following Hermitian operators

$$i(A^1_2-A^2_1)\,,\; i(A^2_3-A^3_2)\,,\; i(A^3_1-A^1_3)$$

$$A^1_2+A^2_1\,,\; A^2_3+A^3_2\,,\; A^3_1+A^1_3\,,\; A^1_1-A^2_2\,,\; A^1_1+A^2_2-2A^3_3\,,\; A^1_1+A^2_2+A^3_3$$

generate the Lie algebra of the group $U_3=SU_3\times U_1$. Verify that the first three operators are just the generators of the rotation subgroup SO_3, that the last operator, i.e., $\mathtt{Tr}\,\mathbf{A}$, generates U_1 and satisfies $\mathtt{H}=\mathtt{Tr}\,\mathbf{A}+(3/2)$.

(*c*) Show that the irreducible representations of U_3 involved in the energy eigenstates of the harmonic oscillator have the symmetric tensors $T_{(i_1,i_2,\ldots)}$ as basis space (see Problem 4.31 for what concerns the irreducible representations of SU_3 and the notations). *Hint*: consider for instance the second excited state, which is given by $u = x_j x_k \exp(-r^2/2)$, with degeneracy 6, ..., the third excited state with degeneracy 10, ...

(d) Decompose the six-dimensional representation considered in the previous question *c)* as a sum of irreducible representations of SO_3 (i.e., find the angular momenta ℓ contained in the second excited level). Generalize to the other levels.

(e) Generalize to the N-dimensional isotropic oscillators (with $N > 1$).

Answers and Solutions

Problems of Chap. 1

(1.2)
(1) E.g., $f(x) = 1/\sqrt{x} \in L^1(0, 1)$ but $\notin L^2(0, 1)$.

$$\|f\|_{L^1} = \int_I |f| dx = (\chi_I, |f|) \leq \sqrt{\mu(I)} \|f\|_{L^2}$$

where $\chi_I(x)$ is the characteristic function of the interval I
(2) E.g., $f(x) = 1/\sqrt{x}$ if $0 < x < 1$ (and $= 0$ elsewhere) $\in L^1(\mathbf{R})$ but $\notin L^2(\mathbf{R})$.
E.g., $f(x) = x/(1 + x^2) \in L^2(\mathbf{R})$ but $\notin L^1(\mathbf{R})$
(3) No. Yes
(4) Yes. Yes

(1.3)
(1) Yes. Yes. Their *closure* is the space $L^2(\mathbf{R})$; as a consequence, they are not Hilbert subspaces
(2) Yes, as in (1): recall, e.g., that the Hermite functions $u_n(x) = \exp(-x^2/2) H_n(x)$, where $H_n(x)$ are polynomials of degree $n = 0, 1, 2, \ldots$, are a complete set in $L^2(\mathbf{R})$

(1.4)
(2) Yes: if $f \in L^1(\mathbf{R}) \cap L^2(\mathbf{R})$, the suggested procedure is enough; if $f \in L^2(\mathbf{R})$ but $\notin L^1(\mathbf{R})$, then first approximate f with a function $g \in L^1(\mathbf{R}) \cap L^2(\mathbf{R})$, etc.

(1.5)
(1) This is a closed (and therefore a *Hilbert*) subspace. The orthogonal complement is the one-dimensional subspace of the constant functions
(2) The subspace of zero mean-valued functions is *dense* in $L^2(\mathbf{R})$: see previous problem

(1.6)
(2) This should be intuitively obvious, recalling that the approximation of a function

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$f(x) \in L^2$ with a function $g(x)$ belonging to these subspaces is to be intended not pointwise, but in L^2-norm, i.e., in mean-square. It is clear that the procedure used for (1) can be easily extended to any $f(x) \in L^2(I)$, etc. Another argument: one can observe that, given any orthonormal complete system $\{e_n(x)\}$ in L^2 of C^∞ functions, then (it is not restrictive to choose for simplicity $x_0 = 0$) $u_n(x) = xe_n(x)$ is a complete system in L^2, and that the same is true for $v_n(x) = x^k e_n(x)$ for any integer k, and also for $w_n(x) = \exp(-1/x^2)e_n(x)$. Recall then that the finite linear combinations of these complete systems provide dense subspaces with the required properties

(1.7)
(1) Yes: $\lim \ldots \le \|f\|/\sqrt{2N} \to 0,\ \forall f(x)$
(2) This limit is trivially zero for even functions, and is zero in the dense set of functions with compact support; it is equal to 1 if, e.g., $f(x) = x/(1+x^2)$, is $+\infty$ if, e.g., $f(x) = 1/x^{2/3}$ if $x > 1$ (and $= 0$ if $x < 1$); in the last case the integral $\int_{-N}^{+N} \ldots$ behaves for large N as $N \cos(N^{1/3})$

(1.11)
(1) Yes. No
(2) It is enough to refer to the *even*, respectively, *odd*, prolongation of the given function to the full interval $(-1, 1)$
(3) Look for a function $g(x)$ such that $(x^{N+m}, g) = 0,\ \forall m \ge 0$: one has $0 = (x^{N+m}, g) = (x^m, x^N g),\ \forall m \ge 0$, then $x^N g(x) = 0 \ldots$

(1.12)
(1) Yes: look for $z \in H$ such that $(v_n, z) = 0,\ \forall n = 2, 3, \ldots$: this implies $(e_n, z) = (e_1, z),\ \forall n$, then necessarily $z = 0$, recalling that $\|z\|^2 = \sum_n |(e_n, z)|^2$
(3) $\alpha_n \in \ell^2$
(4) $|\alpha| \ge |\beta|$

(1.13)
(1) Yes. Yes. Use an argument as in the above problem, but now $n \in \mathbf{Z}$ and this makes the difference from q. (4) of the above problem

(1.14)
(1) $(a), (b), (e), (f), (g)$ are complete. For the cases $(d), (e), (g), (h)$, recall that the only functions $\in L^2(-\pi, \pi)$ orthogonal to $\{\cos nx,\ \sin nx\}$ are the constants
(2) Apart from the obvious condition $h(x)e_n(x) \in L^2(I)$, the condition concerns only the zeroes of $h(x)$: this function must have at most isolated zeroes

(1.16)
(1) $(w_n, z) = 0\ \forall n$ with $z \in \ell^2$ implies ...

(1.17)
(1) Only (γ) is correct; (β) is wrong because $\sin 2\pi x \notin L^2(0, \infty)$
(2) No
(3) (a) Yes. (b) The set is not complete: e.g., $f(x) = \sin x$ if $\pi \le x \le 2\pi$ and $= 0$ out of this interval is orthogonal to all v_n

(1.18)
(1) Yes. No: the function $(\sin x)/x \in L^2(0, \pi)$ is orthogonal to the subset $\{x \ \sin nx\}$ if $n = 2, 3, \ldots$
(2) Yes. Yes: there are no functions in $L^2(0, \pi)$ orthogonal to the subset $\{x^2 \ \sin nx\}$ with $n = 2, 3, \ldots$

(1.19)
(*a*) Yes ; (*b*) No ; (*c*) Yes ; (*d*) No ; (*e*) No
(*f*) No: the function

$$h(x) = \begin{cases} -\exp(x^2) & \text{for } -2\pi < x < 0 \\ \exp(x^2) & \text{for } \ 0 < x < 2\pi \end{cases}$$

is orthogonal to all the functions in the set

(1.20)
(2) The series converges to the square wave

$$S(x) = \begin{cases} 1 & \text{for } \ 0 < x < \pi \\ -1 & \text{for } -\pi < x < 0 \end{cases}$$

with its periodic prolongation with period 2π to all $x \in \mathbf{R}$.
At $x = \pi$ the series converges to zero, at $x = 3\pi/2$ converges to -1

(1.22)
$\widetilde{f_1}(x) = x$ for $0 < x < a$, $\widetilde{f_1}(x) = a/2$ at $x = 0$ and $x = a$, and is periodic with period a.
$\widetilde{f_2}(x) = |x|$ for all $x \in \mathbf{R}$; in this case, the series converges uniformly.
$\widetilde{f_3}(x) = x$ for $-a < x < a$, $\widetilde{f_3}(x) = 0$ at $x = \pm a$. $\widetilde{f_2}$ and $\widetilde{f_3}$ are periodic with period $2a$

(1.23)
(1) $\widetilde{f}(x, y) = 0$ along the lines $x = n\pi$, $y = m\pi$, $n, m \in \mathbf{Z}$; $\widetilde{f}(x, y) = 1$ in the interior of Q; $\widetilde{f}(x, y) = -1$ in the interior of the four squares adjacent to Q; $\widetilde{f}(x, y) = 1$, etc.
(2) $\widetilde{f}(x, y) = \begin{cases} \sin x & \forall x \in \mathbf{R},\ 0 < y < \pi \\ -\sin x & \forall x \in \mathbf{R},\ \pi < y < 2\pi \end{cases}$ etc.

(1.24)
(1) The condition $\sum_n |a_n| < \infty$ ensures that a series of the form $\sum_n a_n \exp(inx)$ is uniformly convergent, and then converges to a continuous function

(1.25)
(1) The function is even, continuously differentiable, its second derivative $f''(x) \in L^2(-\pi, \pi)$, but is expected to be not continuous

(2) Recall that a series of the form $\sum_n \alpha_n \sin nx$ is certainly uniformly convergent (and then converges to a continuous function) if $\sum_n |\alpha_n| < \infty$, see the above problem, q.(1). This condition is satisfied in this case; indeed, we have $\alpha_n = a_n/n$ where $a_n \in \ell^2$, and $\sum_n |a_n|/n$ can be viewed as the scalar product between two sequences $\in \ell^2$

(1.26)
(1) (a) $\pi\sqrt{\pi}/2$ and, respectively, 0 (remember that the Fourier expansion produces a periodic prolongation of the function …);
(b) No, the periodic prolongation of $f(x)$ is discontinuous, see Problem 1.24, q.(1)
(2) No, $f'(x) \notin L^2(-\pi, \pi)$
(3) Yes. No, the periodic prolongation of $f(x)$ is not continuously differentiable, see Problem 1.24, q.(2)

(1.27)
(1) The Fourier expansion converges just to the function $f_1(x)$; notice that the functions v_{2m+1} are *even* with respect to the point $x = \pi/2$
(2) The Fourier expansion gives zero
(3) The Fourier expansion converges to the constant function $\pi/2$; notice that $f_3 = \pi/2 + f_2$
(4) Yes

(1.29)
The expansion of f_1 converges to the constant function $= 1/2$ in $0 < x < 4\pi$.
The expansion of f_2 converges just to f_2

(1.30)
The expansion of f_1 converges to f_1. The expansion of f_2 converges to the function
$$g(x) = \begin{cases} (4/\pi)\sin x & \text{for} \quad 0 < x < \pi \\ 0 \quad \text{for} & x > \pi \end{cases}$$

(1.31)
(1) The Fourier expansion gives a piecewise constant function; in each interval $(n-1, n)$ this function takes the constant value c_n given by the mean value of the function $f(x)$ in that interval: $c_n = \int_{n-1}^{n} f\,dx$
(2) Pointwise, not uniform convergence to 0. No Cauchy property in the $L^2(0, \infty)$ norm. Weak L^2-convergence to 0: indeed, due to the Bessel inequality $\sum_n |(u_n, g)|^2 \le \|g\|^2$, one has $(u_n, g) \to 0$; an alternative argument: $|(u_n, g)| \le \|u_n\|\|g_n\| = \|g_n\|$ where $g_n = g_n(x)$ is the restriction of $g(x)$ to the interval $(n-1, n)$, and $\|g_n\| \to 0$ because $\sum_n \|g_n\|^2 = \|g\|^2$

(1.32)
(1) No solution if the coefficients $g_{\pm 1} \neq 0$. If $g_{\pm 1} = 0$ the solution exists but is not unique
(2) The solution exists and is unique for any $g(t)$

(1.33)
(1) The solution exists and is unique for any $g(t)$
(2) Use the same argument as in Problem 1.25, q.(2)

(1.34)
The answer to this apparent contradiction is simply that in these problems, we are using Fourier expansions in spaces as $L^2(0, 2\pi)$, and therefore, we are "forced" to look for *periodic* solutions with *fixed* period 2π. Then, if one wants to have the complete solution, one has to add "by hand" the "extraneous" solutions

(1.38)
(2) $g_0 = 0$ is guaranteed by the Gauss law: the integral $\int_0^{2\pi} G(\varphi)\, d\varphi$ is the flux across the circumference of the electric (radial) field which is given just by $\partial U/\partial r$; this flux is zero because $\Delta U = 0$ means that there are no charges at the interior of the circle; the nonuniqueness of the solution corresponds to the property that the potential is defined apart from an additive constant

(1.39)
(3) Only three boundary conditions must be given; the solution is as in (1) but $a_n = 0, \ \forall n$

(1.43)
(2) Strong convergence

(1.44)
(3) $S_\infty f(x) = f(-x)$

(1.45)
(1) Yes. No
(2) (*a*) Yes. No ; (*b*) the dimension is ∞ ; (*c*) no eigenvectors

(1.46)
(2) T injective, not surjective; the opposite for S
(3) $\|T\| = \|S\| = 1$
(6) There is an eigenvector of S with eigenvalue λ for *any* $|\lambda| < 1$!
(7) No

(1.47)
(1) T (and T^+, of course) is unitary. T and S do not possess eigenvectors; it can be useful to recall that if an operator preserves norms then its possible eigenvalues λ (if any) satisfy necessarily the condition $|\lambda| = 1$

(1.48)
(2) T and S are unitary operators and do not possess eigenvectors. See remark in the solution of the above problem

(1.49)
For each λ such that $|\lambda| < 1$ there are infinitely many eigenvectors!

(1.50)
T has no eigenvectors.
$\|T\| = \|S\| = 1/\sqrt{2}$; recall that $x^\alpha \in L^2(0,1)$ only if $\alpha > -1/2$

(1.51)
(1) For any f, one has

$$\|S^N f\|^2 = \int_0^\infty |f(x+N)|^2 dx = \int_N^\infty |f(x)|^2 dx \to 0$$

being $f \in L^2$; then $S^N \to 0$ strongly, not in norm because $\|S^N\| = 1$. Instead, T^N converges weakly to 0:

$$|(g, T^N f)| = \left| \int_N^\infty g^*(x) f(x-N) dx \right| \le \|f\| \|g^{(N)}\| \to 0$$

where $g^{(N)} = g^{(N)}(x)$ is the "queue" of the function $g(x)$, i.e., $g^{(N)}(x) = g(x)$ for $x > N$ and $g^{(N)}(x) = 0$ for $0 < x < N$. On the other hand, $\|T^N f\| = \|f\|$, then T^N does not converge strongly
(2) T^N converges weakly to 0:

$$(g, T^N f) = \int_{-\infty}^{+\infty} g^*(x) f(x-N)\, dx = (2\pi)^{-1} \int_{-\infty}^{+\infty} \widehat{g}^*(y) \widehat{f}(y) \exp(iNy)\, dy \to 0$$

where $\widehat{f}$, $\widehat{g}$ are the Fourier transforms of f, g, thanks to the Riemann–Lebesgue theorem, because $\widehat{g}^*(y)\widehat{f}(y) \in L^1(\mathbf{R})$. No strong convergence, indeed $\|T^N f\| = \|f\|$. Obviously, the same results hold for S^N
(3) $S^N \to 0$ strongly, indeed, for any $x = \sum_{n=1}^\infty a_n e_n$ one has

$$\|S^N x\|^2 = \sum_{n>N} |a_n|^2 \to 0$$

being $\{a_n\} \in \ell^2$; $T^N \to 0$ weakly: given $x = \sum a_n e_n,\ y = \sum b_n e_n$, one has

$$|(y, T^N x)| = \left| \sum_{n>N} b^*_{n+N} a_n \right| \le \|x\| \|y^{(N)}\| \to 0$$

where $y^{(N)}$ is the "queue" of the vector y, i.e., $y^{(N)} = \sum_{n>N} b_n e_n$; no strong convergence
(4) $T^N \to 0$ weakly:

$$(g, T^N f) = \int_0^{2\pi} g^*(x) f(x) \exp(iNx) dx \to 0$$

indeed this can be viewed as the Nth Fourier coefficient of the function $g^*(x)f(x) \in L^1(0, 2\pi)$. A Riemann–Lebesgue-type theorem ensures that these coefficients vanish as $N \to \infty$. Clearly, no strong convergence. The same for S^N
(5) Exactly the same argument and result as in the previous case of q. (4): given $a \equiv (\ldots, a_{-1}, a_0, a_1, a_2, \ldots)$ and $b \equiv (\ldots, b_{-1}, b_0, b_1, b_2, \ldots)$ and introducing the functions $f = \sum_{n\in\mathbf{Z}} a_n e_n$, $g = \sum_{n\in\mathbf{Z}} b_n e_n$ where $e_n = \exp(inx)(2\pi)^{-1/2}$ the scalar product $(b, T^N a) = \sum_{n\in\mathbf{Z}} b^*_{n+N} a_n$ can be thought as the scalar product of two functions $\in L^2(0, 2\pi)$, etc.

(1.52)
(1) (a) $\mathtt{Ker}\, T = \{0\}$; (c) T^{-1} unbounded
(2) (a) Yes. Yes ; (b) no eigenvectors
(3) (a) $\|T\| = |\alpha| + |\beta|$; (c) $|\alpha/\beta| \neq 1$; (d) No. Yes

(1.53)
(2) Only if $(x_0, c) = 0$, where $c = \sum_n c_n e_n$, the eigenvectors do *not* give a complete set
(3) $\|T\| = 1 + \|x_0\|^2$

(1.54)
(1) (a) $|\alpha_n| = 1$; (b) $\sup_n |\alpha_n| < \infty$; (c) $T^2 e_n \propto e_{n+2}$ then for no α_n the operator T can be a projection (apart from the case $T = 0$!). Or: $T \neq T^+$, see (2)
(2) $T^+ e_n = \alpha^*_{n-1} e_{n-1}$
(3) (a) $\|T\| = 2$;
(b) $\mathtt{Ker}\, T$ is generated by e_k with $k = 4n - 1$; e.g., e_3 belongs to $\mathtt{Ker}\, T \cap \mathtt{Ran}\, T$

(1.55)
(1) (b) If $x = \sum_{n=1}^N a_n e_n$ then $Tx = (\sum_{n=1}^N a_n) x_0$ and T is unbounded
(2) $\mathtt{Ker}\, T = \ell^{(0)}$, dense in H !
(3) E.g., $z_n = e_1 - (e_2 + \ldots e_{n+1})/n \in \mathtt{Ker}\, T$ and $w_n = e_1 + e_n/n$
(4) No

(1.56)
(2) $\alpha = -\beta$, dimension $= 1$
(3) Put $T' = T - \alpha I$ and compare with Problem 1.55

(1.57)
(1) (a) $\sup_n |c_n| < \infty$, (b) $|c_n|^2 = |c_{-n}|^2$, (c) $c_n = c^*_{-n}$, (d) $c_n c_{-n} = 1$
(2) The two-dimensional subspaces are generated by $e_{\pm n}$. For any $n \neq 0$ one finds two independent eigenvectors, then the eigenvectors form a complete set in H, not necessarily orthogonal
(3) (a) eigenvectors orthogonal, eigenvalues not necessarily real;
(b) eigenvalues $\pm 1/\sqrt{n}$ for $n \neq 0$; T not normal

(1.58)
(1) The eigenvectors are a complete set
(3) $c \neq \pm n^2$
(4) $1/4, 1, 1/\sqrt{2}$

(1.59)
(1) e_0 with zero eigenvalue; $e_n \pm e_{-n}$ with eigenvalues $\pm 1/n^2$, not degenerate
(2) $\mathtt{Ran}\, T$ is orthogonal to the subspace of constants, but is dense in (not coinciding with) the complementary subspace, which is just the closure $\overline{\mathtt{Ran}\, T}$; e.g., $g(x) = \sum_{n \neq 0}(1/n)\exp(inx) \notin \mathtt{Ran}\ T$, see also q. (3c) below
(3) $Te_0 = 0$ implies that the equation does not admit solution if $g_0 = (e_0, g) \neq 0$; the presence of the coefficients $1/n^2$ implies that g must be at least two times differentiable with $g'' \in L^2$. Then: (a) (not unique) solution $f = 2\cos(2x)+$ const. (b) $\cos^4(x) \geq 0$, then $g_0 \neq 0$, no solution; (c) $g_0 = 0$ but g is *not* two times differentiable, then no solution
(4) T^N converges to the exchange operator S_1 defined by $S_1\, e_{\pm 1} = e_{\mp 1}$ and $S_1 = 0$ on the other e_n. Indeed $\|T^N - S_1\| = \max_{n \neq 0} |\text{eigenvalues}| = 1/2^{2N} \to 0$

(1.60)
(1) One has $\|z\| = 1/\sqrt{3}$ and $Tx = (e_1 + e_2)(z, x)$, then: (a) $\|T\| = \sqrt{2/3}$,
(b) $(e_1 + e_2)$ with eigenvalue $3/4$ and any vector orthogonal to z with zero eigenvalue (notice that $\sqrt{2/3} > 3/4$),
(c) $T^+ x = z\big((e_1, x) + (e_2, x)\big)$
(2) If $\alpha_n \in \ell^2$, $\beta_n \in \ell^2$, put $\widehat{\alpha} = \sum_n \alpha_n^* e_n$ and $\widehat{\beta} = \sum_n \beta_n^* e_n$: then $Tx = (\widehat{\alpha}, x)e_1 + (\widehat{\beta}, x)e_2$ is bounded (and conversely)
(3) (a) If $\widehat{\alpha}$ and $\widehat{\beta}$ are linearly dependent, e.g., $\widehat{\alpha} = c\widehat{\beta}$, then the range of T is one-dimensional and is given by the multiples of $e_1 + ce_2$.
(b) If $\mathtt{Ran}\ T$ is one-dimensional, $\mathtt{Ker}\ T$ is given by the vectors orthogonal to $\widehat{\alpha}$, then ...; if $\widehat{\alpha}$ and $\widehat{\beta}$ are linearly independent, $\mathtt{Ran}\ T$ is generated by e_1, e_2, $\mathtt{Ker}\ T$ is orthogonal to $\mathtt{Ran}\ T$ if both $\widehat{\alpha}$ and $\widehat{\beta}$ are combinations of e_1 and e_2
(4) One has $\|T_N x\| = \|e_1 + e_2\||(z_N, x)| = \sqrt{2}|(z_N, x_N)| \to 0$ where x_N is the "queue" of the vector x, i.e., $x_N = \sum_{n>N} a_n e_n$. But $\|T_N\| = \sqrt{2}\|z_N\| = \sqrt{2/3}$, then strong convergence to zero. Instead, only weak convergence to zero for T_N^+: indeed $\|T_N^+ x\| = \|z_N\||a_1 + a_2|$, on the other hand $(y, T_N^+ x) = (a_1 + a_2)(y, z_N) \to 0$; alternatively: $(y, T_N^+ x) = (T_N y, x) \ldots$

(1.61)
(3) $T_N \to I$, etc., in strong sense

(1.62)
(3) $\sqrt{2a}\|h\|$
(4) $T^+ f = (h, f)\chi_a(x)$ where $\chi_a(x)$ is the characteristic function of the interval $(-a, a)$

(1.65)
(1) Strong convergence to the identity operator
(2) (a) The identity ! (b) The derivative ; (c) $g(x) = Cf(x) \in C^\infty$

(1.66)
(1) Only $A_n^{(-)}$ and $C_n^{(-)}$ are projections
(3) Strong convergence to 0

(1.68)
(4) Apart from the eigenvalue zero, there is the eigenvalue $2/3$ with eigenvector x, and the eigenvalues $(18 \pm 2\sqrt{61})/15$ with eigenvectors $5 + (-6 \pm \sqrt{61})x^2$

(1.69)
(3) The eigenvalues are 0 and 4 for both operators, the eigenvectors are different
(4) Yes, clearly!

(1.70)
(1) $|c_n| = |(\chi_n, f)| = |(\chi_n, f_n)| \leq \|f_n\|$, where $f_n = f_n(x)$ is the restriction of $f(x)$ to the interval $(n-1, n)$, then $\sum_n \|f_n\|^2 = \|f\|^2$, see Problem 1.31
(2) The N-dimensional subspace generated by $\chi_1, \ldots, \chi_N$ with eigenvalue 1 and the orthogonal space with zero eigenvalue; T_N is a projection for each N
(3) (b) $\|T_\infty\| = 1$;
(c) Not compact because the eigenvalue 1 is infinitely degenerate; alternatively, because the weakly convergent sequence χ_n is mapped into itself by T_∞
(4) Strong convergence: $\|(T_\infty - T_N)f\|^2 = \sum_{n>N} |c_n|^2 \to 0$, but $\|T_\infty - T_N\| = 1$
(5) (a) Norm-convergence; (b) yes

(1.71)
(4) If $g(x) \in \mathtt{Ran}\ T$ then $g(x)$ is a continuous function. Indeed, recall that a series of the form $\sum_n \alpha_n \sin nx$ is certainly uniformly convergent (and then converges to a continuous function) if $\sum_n |\alpha_n| < \infty$, see Problem 1.24. This condition is satisfied in this case, see the argument used in Problem 1.25. Conversely, e.g., $g(x) = \sum_n n^{-4/3} \sin nx$ is continuous but $\notin \mathtt{Ran}\ T$. The closure $\overline{\mathtt{Ran}\ T}$ coincides with the Hilbert subspace of the odd functions

(1.72)
(3) Yes. No. Yes. Yes (see previous problem)
(4) $\mathtt{Ran}\ T$ contains only even and zero mean-valued functions, then $\beta = 0$, $\alpha = -\pi/2$

(1.73)
(1) $T_n f = (\pi/2) e_n (e_n - e_0, f)$; $\|T_n\| = \pi/\sqrt{2}$
(3) The eigenvalues are 0 and $\pi/2$
(4) No: $T_n f = 0$ implies $(e_0, f) = (e_n, f)$, $\forall n$, then only $f = 0$
(5) $T_n \to 0$ only in weak sense

(1.74)
(1) (a) $\|T\| = a$; (b) $f_\varepsilon(x) = \begin{cases} 1 \text{ for } a - \varepsilon < x < a \\ 0 \text{ elsewhere} \end{cases}$
(2) No eigenvectors and $\mathtt{Ker}\ T = \{0\}$, trivial
(3) The spectrum is the closed interval $[0, a]$

(1.75)
(1) E.g., $f(x) = x/(1 + x^2) \notin D(T)$. $D(T)$ is however dense in H, indeed $D(T)$

certainly contains functions which behave as $1/|x|^\alpha$ for any $\alpha > 3/2$ when $|x| \to \infty$; but it is known that the set of functions rapidly going to zero at the infinity is dense in H. This implies that also operators with $\varphi = x^a$ for any $a > 0$ possess dense domain

(2) $f_n(x) = \begin{cases} 1 \text{ for } n < x < n+1 \\ 0 \text{ elsewhere} \end{cases}$

(3) Any (continuous) function $f(x) \neq 0$ in a neighborhood of $x = 0$ does not belong to $\mathtt{Ran}\, T$; $\mathtt{Ran}\, T$ is however dense in H (the closure $\overline{\mathtt{Ran}\, T} = H$). This can be shown in several ways: $\mathtt{Ran}\, T$ contains functions which behave as $|x|^\alpha$ for any $\alpha > 1/2$ when $x \to 0$, and these functions can certainly approximate in norm L^2 any function belonging to L^2 (see also Problem 1.6). Recall also that if $\{e_n(x)\}$ is an orthonormal complete system in H, then $u_n(x) = x\, e_n(x)$ is a complete system in H, and $u_n(x) \in \mathtt{Ran}\, T$, therefore .... As another useful argument, notice that any function $g(x) \in L^2$ can be approximated (in the norm L^2, of course) by a "truncated" function

$$g_\varepsilon(x) = \begin{cases} 0 & \text{for } |x| < \varepsilon \\ g(x) & \text{for } |x| > \varepsilon \end{cases}$$

for ε "small" enough, and observing that functions as $g_\varepsilon(x)$ clearly belong to $\mathtt{Ran}\, T$. All these arguments also hold for operators with $\varphi(x) = x^a$ for any $a > 0$: question (4).

(1.76)

(1) $f_n(x) = \begin{cases} 1 \text{ for } 1/n < x < 1 \\ 0 \text{ elsewhere} \end{cases}$

(2) The role of domain and that of range of this operator are exchanged with respect to the ones of the operator in the previous problem

(1.78)

(a) $\|T\| = \sup_{x\in\mathbf{R}} |\varphi(x)|$; (b) $\varphi(x) = 1$ for x in any subset J (even if not connected) $\subset \mathbf{R}$, and $= 0$ in $\mathbf{R} \setminus J$; (c) let $\inf_{x\in\mathbf{R}} |\varphi(x)| = m > 0$, then $\|T^{-1}\| = 1/m$; (d) $|\varphi(x)| = 1$

(1.79)

(1) $\|T\| = 1$, no function $f_0(x)$

(2) No. No

(3) $\mathtt{Ran}\, T$ dense in H but $\neq H$, or $H = \overline{\mathtt{Ran}\, T}$; $D(T) = H$

(4) $T^N \to 0$ in strong sense, thanks to Lebesgue theorem: $x^{2N}/(1 + x^2)^N \to 0$ pointwise, and $|f(x)|^2 \in L^1$; but $\|T^N\| = 1$

(1.80)

T admits the eigenvalues 0 and 1 infinitely degenerate. E.g., $\theta(x)/(1+x) \notin \mathtt{Ran}\, T$, but $\mathtt{Ran}\, T$ dense in $L^2(\mathbf{R})$, see Problem 1.75. $\mathtt{Ran}\, T$ is orthogonal to $\mathtt{Ker}$, but it is not a Hilbert space; one has $L^2(\mathbf{R}) = \mathtt{Ker}\, T \oplus \overline{\mathtt{Ran}\, T}$. $T^N \to P$, where P is the projection on the subspace $L^2(0, 1)$, in strong sense, see previous problem

(1.81)
(3) $\|(T+2iI)^{-1}\| = 1/\sqrt{5}$ in case (1)*(b)*, and $= 1$ in cases 2*(b)* and 3*(b)*. Weak convergence to 0 in all cases, see Problem 1.47

(1.82)
(1) $T^+ f = \big(1-\alpha^* \exp(-ix)\big) f$, yes
(2) $\|T\| = 1+|\alpha|$. $|\alpha| \neq 1$

(1.83)
(1) $|\alpha| = 1$
(2) In general not orthogonal, but complete system for all α
(3) Yes

(1.84)
(2) $\alpha \neq ik,\ k \in \mathbf{Z}$, and under this condition the solution is unique
(3) $g_{-2} \propto \big(\exp(-2ix), g\big) = 0$
(4) $\|T^{-1}\| = 1,\ 1/2,\ 1,\ 2$

(1.86)
(3) The condition is $g_0 = (e_0, g) = 0$ $\big($i.e., $g(x)$ must be zero mean-valued$\big)$

(1.87)
(3) Yes, S is a bounded operator
(4) $C_N = \big(1+(N+1)\big)^{-1/2}$

(1.88)
(3) The solution exists if the Fourier coefficients $g_{\pm 1} \propto \big(\exp(\pm ix), g\big) = 0$.
Yes. No
(4) $\alpha \neq n^2$
(5) $\|T_\alpha^{-1}\| = 1/|a|$

(1.89)
(1) T is Hermitian (in a dense domain in H). Its eigenfunctions $\cos(nx/2)$, $n = 0, 1, 2, \ldots$ are orthogonal complete system in H. $\mathtt{Ker}\ T$ is the one-dimensional subspace of the constants and $\mathtt{Ran}\ T$ its orthogonal complementary subspace
(2) $g(x)$ must be zero mean-valued; in this case, written $g = \sum_{n=1}^{\infty} g_n \cos(nx/2)$, the most general solution $f(x)$ is

$$f(x) = f_0 - 4\sum_{n=1}^{\infty} \frac{1}{n^2} g_n \cos\frac{nx}{2}$$

where f_0 is an arbitrary constant
(3) $\left|\frac{df}{dx}\right| \le 2\sum_{n\neq 0}(1/n)|g_n| \le K\|g\|_{L^2}$ with $K = \frac{2}{\sqrt{\pi}}\sqrt{\sum_{n\neq 0}(1/n^2)}$

(1.90)
(1) $C = 1$
(2) $G = (2/\pi) \sum_{n \geq 1} \sin nx/n^2$
(3) $f_\alpha(x)$ converges to $h(x)$ in the norm L^2 and also uniformly. Notice that f_α and h are continuous functions

(1.91)
(1) Yes
(2) $\|T^{-1}\| = 1/\big(1 + (c^2/4)\big)$

(1.92)
(1) $u_n(x) = \exp(in\pi x^2),\ n \in \mathbf{Z}$
(2) $u_n(x)$ given in (1) are orthonormal with respect to the scalar product $(f,\ g)_\rho$, and are a complete set
(3) $c_n = (u_n, f)_\rho/(u_n, u_n)_\rho = (u_n, f)_\rho = \dots$
(4) $\sum_n |c_n|^2 = (f, f)_\rho = 2$
(5) No periodic. At the points $\sqrt{9/4 + 2m},\ m \in \mathbf{Z}$

(1.93)
(2) (b) $c(\alpha) = \exp(\alpha/2)$
(3) One has $B = A$ and $\widetilde{B} = \widetilde{A}$; these operators are the infinitesimals Lie generators of the transformations defined in (2). According to Stone theorem, the generator of a unitary transformation is anti-Hermitian

(1.94)
(1) Eigenvectors $\{e_{nm} = \sin(nx)\sin(my);\ n, m = 1, 2, \dots\}$, $\lambda = -(n^2 + m^2)$, degeneracy 1 or 2 (exceptionally 3, see Problem (4.7))
(2) $\|T^{-1}\| = 1/2$
(3) Yes. Yes
(5) $\{e_{nm} \pm e_{mn}\}$ is an orthogonal complete system

(1.95)
(1) α, β purely imaginary
(2) $\lambda = \lambda_{k_1,k_2} = (ik_1 - \alpha)(ik_2 - \beta)$, the eigenvectors

$$u_{k_1,k_2} = \exp(ik_1x)\exp(ik_2y)\,, \qquad k_1, k_2 \in \mathbf{Z}$$

are an orthogonal complete system for $L^2(Q)$
(3) $\texttt{Ker}\,T \neq \{0\}$ if $\alpha = ik_1$ or $\beta = ik_2$
(4) The solution exists (not unique) if the Fourier coefficients of g with respect to the orthogonal complete system obtained in (2) satisfy $g_{1,-1} = g_{-1,1} = 0$. The proposed equation is solved by $f = -(1/5)\sin 2(x + y) + c_1 \sin(x - y) + c_2 \cos(x - y)$, where c_1, c_2 are arbitrary constants

(1.96)
(1) $\lambda = \lambda_{k_1,k_2} = i(k_1 + ak_2)$, the eigenvectors $u_{k_1,k_2} = \exp(ik_1x)\exp(ik_2y),\ k_1, k_2 \in \mathbf{Z}$, are an orthogonal complete system for $L^2(Q)$

(2) (a) In $\mathtt{Ker}\, T$ one has $k_1 = -k_2$; an orthogonal complete system for $\mathtt{Ker}\, T$ is $v_k = \exp\big(ik(x-y)\big),\ k \in \mathbf{Z}$; (b) $k_1 = k_2 = 0$, dim of $\mathtt{Ker}\, T = 1$
(3) $u(x, \pi) = 0$ if $0 < x < \pi$, and $u(x, \pi) = 1$ if $\pi < x < 2\pi$
(4) $u(x, y) = \varphi(x - y)$ where now φ is periodically prolonged with period 2π

(1.97)
(1) (b) $c_n \in \ell^2$

(1.98)
Strong convergence to 0

(1.101)
(2)

$$\|u(x,t)\|^2 = (\pi/2)\sum_n |f_n|^2 \exp(-2n^2 t) \le \exp(-2n_1^2 t)\|f(x)\|^2$$

where f_n are the coefficients of the Fourier expansion of $f(x)$ and n_1 the first nonzero coefficient of this expansion

(1.102)
(3)

$$\|E_t - I\|^2 = \sup_n \frac{\sum_n |f_n|^2\big(\exp(-n^2 t) - 1\big)^2}{\sum |f_n|^2} = 1$$

In strong sense:

$$\sum_{n=1}^{\infty} |f_n|^2\big(\exp(-n^2 t) - 1\big)^2 \le \sum_{n=1}^{N} |f_n|^2 + \sum_{n=N+1}^{\infty} |f_n|^2 \to 0$$

Indeed, given $\varepsilon > 0$, there is N such that

$$\sum_{n>N} |f_n|^2\big(\exp(-n^2 t) - 1\big)^2 \le \sum_{n>N} |f_n|^2 < \varepsilon$$

on the other hand, being $\exp(-n^2 t) \to 1$ as $t \to 0^+$, there is t such that

$$\sum_{n=1}^{N} |f_n|^2\big(\exp(-n^2 t) - 1\big)^2 < \varepsilon$$

then …

(1.103)
(2) $u(x, t)$ tends to a constant
(3) Yes

(4) $E_t \to P_0$ in norm as $t \to +\infty$, where P_0 is the projection on the constants; $E_t \to I$ strongly as $t \to 0^+$, see previous problem

(1.104)
$u(x,t) = \sum \big(f_n \exp(-n^2 t) + (F_n/n^2)(1 - \exp(-n^2 t)\big) \exp(inx)$

(1.106)
(1) $v(t) = e_0 \exp(t)$
(2) $v(t) = e_1 \cosh t + e_{-1} \sinh t$

(1.107)
(1) $v(t) = e_0$
(2) $v(t) = e_n \cos nt + e_{-n} \sin nt$

(1.108)
(1) $u_n = \exp(ix(n + 1/2)),\ n \in \mathbf{Z}$
(3) Period $T = 4\pi$; notice that the solution is simply $f_0(x + t)$
(5) No convergence in norm:

$$\|E_t - I\| = \sup_n \big| \exp\big(it(n + 1/2)\big) - 1 \big| = 2$$

this $\sup_n$ turns out to be independent of t; strong convergence follows, e.g., from Lebesgue theorem

(1.109)
(2) (b) $u(t) = (\exp t)(e_1 + t\, e_2)$

(1.110)
(1) Only if $f(x) = 0$
(2) If, e.g., $f(x) = \sin nx$, and then for any finite linear combination thereof
(3) $f(x) = \sum_n (1/n^2) \sin nx$
(4) No: e.g., $f_N(x) = (1/N) \sin Nx$

(1.109)
(2) $u_1(x, 0) = (1/2)|\sin 2x|$;

$$u(x,\pi) = \begin{cases} 0 & \text{for } 0 \le \pi/2 \le \pi \\ -\sin 2x & \text{for } \pi/2 \le x \le \pi \end{cases}$$

(1.113)
(2) The convergence is ensured by the property $\sum |(e_n, x)|^2 < \infty$; the series converges to the vector Tx

(1.115)
(1) The subspace on which the projection projects must be finite-dimensional
(2) No, no: consider, e.g., the case where T is an unbounded functional

(1.117)
(1) If A is bounded, the conditions are $A^+ = A$ and A positive definite. If A is unbounded, the obvious conditions about its domain must be included
(2) A Cauchy sequence u_n with respect to the norm induced by the scalar product $<\, ,\, >$ may not be a Cauchy sequence with respect to the norm induced by the scalar product $(\ ,\ )$: let, e.g., A be defined by $A\, e_n = e_n/n$ where $\{e_n,\ n = 1, 2, \ldots\}$ is an orthonormal complete set, and let $u_n = e_n, \ldots$

(1.118)
(2) Strong convergence to $-I$
(4) Yes

(1.119)
(3) Yes
(4) $\|T^N\| = \sup_{n=1,3,5,\ldots}\{1, \big(2/n\pi\big)^N\} = 1$
(5) T^N is norm-convergent to P_0, i.e., the projection on the one-dimensional subspace of constants: $\|T^N - P_0\| = \sup_{n=1,3,5,\ldots} \big(2/n\pi\big)^N = (2/\pi)^N \to 0$

(1.120)
(1) $\|T\| = \max\{|\alpha+\beta|, |\alpha-\beta|\}$
(3) $|\alpha+\beta| = |\alpha-\beta| = 1\,;\ \beta = \pm i/\sqrt{2}$
(4) T^n is norm-convergent to the projection on the even functions in $L^2(\mathbf{R})$

(1.121)
(2) (b) $\mathtt{Ran}\, T_n$ is not a Hilbert subspace; one has $H = \mathtt{Ker}\, T_n \oplus \overline{\mathtt{Ran}\, T_n}$, where $\overline{\mathtt{Ran}\, T_n}$ is the closure of $\mathtt{Ran}\, T_n$
(3) The solution exists, not unique, for any $g \in H$
(4) T_n is norm-convergent to the multiplication operator $(x/1+x^2)\, I$, and S_n converges strongly, not in norm, to the multiplication operator $\sin \pi x\, I$

(1.122)
(1) $T^4 = I$
(2) The eigenvectors are $\{\exp(inx)\, ,\, n \in \mathbf{Z}\}$; $|\sin 2x|$ has period $\pi/2$ and then is eigenvector of T

(1.123)
(3) $T = P_+ - P_-$ where P_+ projects on the subspace generated by e_k with $k = 4m+1, m \in \mathbf{Z}$, etc.
(4) $1,\ 2\sqrt{2},\ 1/2$
(5) eigenvalues 0 and 2; norm=2

(1.124)
(1) An orthonormal complete system for $\mathtt{Ran}\, T$ is $e_0 = 1/\sqrt{2},\ e_1 = \sqrt{3/2}\, x$
(2) $A = 1/2,\ B = 0$; $A = 0, B = 2/3$. With $A = 1/2\, , B = 2/3$, then $T_{AB} = P_0 + P_1$, with clear notations, is a projection, see (1), cf. Problem 1.114, q.(1)
(3) Apart from the eigenvalue $\lambda = 0$, T admits the eigenvalues 2 and 2/3 with eigenvectors, respectively, e_0 and e_1

(4) $Tf = 2e_0(e_0, f) + (2/3)e_1(e_1, f)$ then

$$\|T\|^2 = 4|f_0|^2 + (4/9)|f_1|^2 \leq 4(|f_0|^2 + |f_1|^2) \leq 4\|f\|^2$$

and $\|T\| = 2$. Notice that in $Tf = \int_{-1}^{1} f\,dx + x(x, f)$ the term $\int_{-1}^{1} f\,dx = (1, f)$ is to be *consistently* intended as a *constant function* (not as a functional), then, e.g., $\|1\| = \sqrt{2}$ and

$$\|\int_{-1}^{1} f\,dx\| = \|1(1, f)\| = \sqrt{2}|(1, f)| \leq 2\|f\|$$

The result $\|T\| = 2$ is confirmed by $\|T\| = \max|eigenvalues| = 2$, thanks to the fact that the eigenvectors of T provide an orthogonal complete system
(5) An orthonormal complete system for $\mathtt{Ran}\,T$ is $e_0 = 1/\sqrt{2}$ and $e' = \sqrt{5/8}(3x^2 - 1)$. For no A, B both different from zero, the operator T can be a projection

(1.125)
(1) (*b*) The eigenvalues are: α with eigenvectorsthe zero mean-valued functions, and $\alpha + 2\beta\pi$ with eigenvector the one-dimensional subspace of constant functions. The eigenvectors provide an orthogonal complete system for $L^2(-\pi, \pi)$. Then, $\|T\| = \max\{|\alpha|,\ |\alpha + 2\pi\beta|\}$. See previous problem for what concerns the term $\int_{-\pi}^{\pi} f(y)\,dy$
(2) $\beta = -1/(2\pi)$, see (1)
(3) T_n converges weakly to the operator αI:

$$\big(g, (T_n - \alpha I)f\big) = 2\pi\beta(g, e_n)(e_0, f) \to\ 0$$

where $e_n = \exp(inx)/\sqrt{2\pi}$; no strong convergence:

$$\|T_n f - \alpha f\| = 2\pi|\beta|\,\|e_n\|\,|(e_0, f)| = 2\pi|\beta|\,|f_0|$$

(1.126)
(2) eigenvalues of T: $\lambda_k = \sin ka/(ka),\ k \in \mathbf{Z}$ with eigenvectors $\exp(ikx)$, doubly degenerate (apart from the case $k = 0$) for "generic" a; if, e.g., $a = \pi/2$ then …
(3) T is Hermitian, because its eigenvalues are real and its eigenvectors are an orthogonal complete system; $\|T\| = \sup_k |\lambda_k| = 1$
(4) Yes

(1.127)
(2) $\mathtt{Ran}\,T_a$ is one-dimensional and $T_a f(x) = c_0$, the mean value of f
(4) Yes, no, T_a^{-1} exists unbounded
(5) $T_a \to I$ strongly, it is possible to use an argument similar to that seen in Problem 1.102; no norm-convergence: $\|T_a - I\| = \sup_n \left|\dfrac{\sin na}{na} - 1\right| = 1$

(1.128)
(2) eigenvalues infinitely degenerate if $a = \pi/2$, not degenerate if $a = 1$
(3) $\|T_a\| = \sup|\text{eigenvalues}| = 2$
(4) For any a; $\|(T_a - iI)^{-1}\| = 1$ if $a = \pi/2$; and, if $a = 1$:

$$\|(T_a - iI)^{-1}\| = \sup_n \frac{1}{|\exp(ian) - 1 - i|}$$
$$= \frac{1}{\text{distance of the point } 1+i \text{ from the circle of radius } 1} = \frac{1}{\sqrt{2}-1}$$

(1.129)
(2) Yes

(1.131)
(1) (c) No eigenvectors
(2) One eigenvector for any λ such that $\mathtt{Re}\,\lambda > 0$
(3) No eigenvectors.

Problems of Chap. 2

(2.1)
(2) All these equations admit infinite solutions

(2.4)
(1) $f(z) = c_0 + c_1 z + c_2 z^2, \ c_i = \text{const}$
(2) $f(z) = c_0 \exp(c_1 z)$

(2.5)
(a) Taylor series converging in $|z| < 1$; (b) Taylor series converging in $|z-2i| < \sqrt{5}$;
(c) $f(z) = \dfrac{1}{z(1/z - 1)} = -\dfrac{1}{z}\displaystyle\sum_{n=0}^{\infty}\dfrac{1}{z^n} = \ldots$ this series converges in $|z| > 1$;
(d) no need of expansion in powers of $(z-1)^n$: $f(z)$ is itself a Taylor–Laurent series containing the only term $-1/(z-1)$

(2.6)
(1) $R = 1$
(2) $z/(1-z)^2$
(3) $2 < |z| < 3$

(2.7)
The point $z = \infty$ is an accumulation point of simple poles

(2.8)
Essential singularity at $z = \infty$; $\mathtt{R}(\infty) = 0$

(2.10)
$R = 3\pi$

(2.14)
Pole at $z = 0$ of order $n - 6$ if $n > 6$, etc.

(2.15)
Branch points at $z = 0$ and $z = \infty$; analytic $\forall\, z$ and essential singularity at $z = \infty$; analytic $\forall\, z$ and essential singularity at $z = \infty$ for the functions in the first line; etc.

(2.16)
$z = 0$ and $z = 1$; $z = 0, 1, \infty$; $z = 0\,, 1$; $z = 0\,, 1\,, \infty$; $z = 0, 1$

(2.17)
(1) $n \leq 5$; essential singularity at $z = \infty$
(2) Nothing changes: the branch points at $z = \pm 1$ do not prevent the expansion near $z = 0$, it is enough to place the cut line …

(2.18)
(1) $\alpha = n\pi i\ , n \in \mathbf{Z}$; essential singularity at $z = \infty$ for any α
(2) For all α the points $z = \pm 1$ are branch points and $z = \infty$ an essential singularity

(2.19)
(1) Not true: choose, e.g., $z = -i$
(2) apart from the cut: simple pole at $z = -1$; no singularity for the second function
(3) apart from the cut: simple pole at $z = -i$; no singularity for the second function

(2.22)
$\pi/2\,;\ i\pi/18$

(2.23)
$(\pi i/12)\sin(1)\,;\ 2\pi i(1+\pi)\,; -8\pi i$

(2.24)
$\pi i/9\,;\ 2\pi i$

(2.25)
$\pi\sqrt{2}\,; \pi\big(2 - (4/\sqrt{3})\big)\,;\ -2\pi/(3\sqrt{3})$

(2.26)
$-2\pi a \exp(-a)$ with the sign $+$, and 0 with the sign $-$; π/e

(2.27)
$\pi/e\,;\ -(2\pi/\sqrt{3})\exp(-\sqrt{3}/2)\sin(1/2)$

(2.28)
$(\pi/e)(e-1)\,;\ \pi\,;\ (\pi/2)\big(1+\exp(-\pi/2)\big)\,;\ 2\pi$. Using notations of Fig. 2.1, which refers to the first one of the integrals, one has

$$\int_{-R}^{-r} + \int_{-\gamma} + \int_{r}^{R} + \int_{\Gamma} = 2\pi i\, \mathrm{R}(i)$$

then

$$\lim_{r\to 0}\int_{-\infty}^{+\infty} = \mathtt{P}_0 \int_{-\infty}^{+\infty} = 2\pi i\, \mathtt{R}(i) + \pi i\, \mathtt{R}(0) = \text{etc.}$$

where $\mathtt{P}_0$ is the Cauchy principal part of the integral (with respect to the singular point at $x = 0$). Taking then the imaginary part of the result, the symbol $\mathtt{P}_0$ is unnecessary

(2.29)
$(\pi i/e)(e-1)\,;\ (2\pi/e)(1-e)$

(2.30)
π

(2.31)
$\pi/(1-i)\,;\ \pi i\left(\frac{1}{2}-\frac{2}{3e}\right)$

(2.32)
$\pi/\cosh(a\pi/2)$

(2.33)
$\sqrt{\pi}(1+i)/2\sqrt{2};\ \sqrt{\pi}/2\sqrt{2}$

(2.34)
(1) (a) $\mathtt{R}(-i) = \exp(3i\pi/4) = (-1+i)/\sqrt{2}$; ($b$) $\mathtt{R}(-i) = 3i\pi/2$; (c) $\mathtt{R}\,(-1) = i$;
(d) $\mathtt{R}(-i) = 4i/3$; (e) $\mathtt{R}(-i) = -i\sinh\pi$; ($f$) $\mathtt{R}(-i) = -\exp(-\pi)$
(2) (a) $\log 2 + i\pi$; (b) $i\sqrt{2}$;
(c) $\mathtt{R}\,(i) = (i/4)\sqrt[4]{2}\exp(3i\pi/8) = (i/8)\sqrt[4]{2}\left(\sqrt{2-\sqrt{2}} + i\sqrt{2+\sqrt{2}}\right)$;
$\mathtt{R}\,(-i) = (i/4)\sqrt[4]{2}\exp(5i\pi/8) = (i/8)\sqrt[4]{2}\left(-\sqrt{2-\sqrt{2}} + i\sqrt{2+\sqrt{2}}\right)$;
$\mathtt{R}\,(-1) = -i/2\sqrt{2}$
(3) (a) $2i\sqrt{1-x^2},\ |x|\le 1$; ($b$) $(x-1)^\alpha\big(1-\exp(2\pi i\alpha)\big),\ x\ge 1$; ($c$) $-2\pi i$;
(d) $-2\pi i$; (e) $+2\pi i$

(2.35)
$\pi/\sqrt{2}\,;\ \pi/\sqrt{3}\,;\ (\pi/2\sqrt{2})(1-i)$

(2.36)
$a\pi/\sin a\pi\ ;\ (\pi/3)/\sin\big(\pi(b+1)/3\big)$

(2.37)
(1) $2\pi/3\sqrt{3}\,;\ 2\pi/3\sqrt{3}$
(2) $0\,;\ \pi/2$

(2.38)
$0\,;\ -\pi^2/8\sqrt{2}$

(2.39)
$\pi/\sin a\pi\ ;\ \pi/\sin b\pi$

(2.40)
$\pi^3/8$

(2.41)
$\pi\ ;\ \pi(2-\sqrt{3})\ ;\ 11\pi/8$

(2.42)
$2\pi \log(a+1)$

(2.44)
(1) $u'(x', y') \propto y'$ where $y' = -i\,\mathtt{Re}\,(z') = -i\,\mathtt{Re}\,(z^2)$, then …
(2) $u'(x', y') = (u_0/\alpha)y' = (u_0/\alpha)\mathtt{Re}\,(-iz')$, then …

(2.45)
(3) $u(x, y) = (1+y)/\big(x^2 + (1+y)^2\big)$

(2.47)
The region D is transformed into the strip $0 \le y' \le 1$ where the solution is $\widetilde{u}(x', y') = y' = \mathtt{Re}\,(-iz')$; then, $u(x, y) = \mathtt{Re}\big((1-z)/(1+z)\big) = \dots$

(2.48)
(2) Notice that the map $\Psi(z')$ diverges at $z' = -1$, i.e., $r' = 1$, $\varphi' = \pm\pi$, which corresponds to $z = \infty$. As expected, the solution $\widetilde{u}(r', \varphi')$ is a harmonic function in the interior of the circle $r' < 1$ and vanishes on the circumference $r' = 1$ (which corresponds to the boundary $y = 0$) *apart from the singularity at* $\varphi' = \pm\pi$. So, the uniqueness of solutions of the Dirichlet Problem can be recovered excluding solutions as iz, iz^2, etc., by the introduction of a boundedness condition at $z = \infty$ (the so-called "normal conditions" at the infinity).

Problems of Chap. 3

(3.4)
(1–2) The solutions $x_0(t) = A\exp(\mp t)$ of the homogeneous equations $\dot{x}_0 \pm x_0 = 0$ do not admit Fourier transform if $A \neq 0$, not even as distributions in $\mathscr{S}'$, see Problem 3.69.
(3) $\widehat{G}(\omega) = 1/(R - i\omega L)$; the difference in the sign depends on the initial choice in the definition of the Fourier transform. Clearly, there are no consequences if one uses consistently the rules for the Fourier transform and for its inverse.

(3.7)
Only one is causal

(3.9)
(1) π
(2) $\sqrt{\pi/(2n+1)}$
(3) Any $h(x)$ such that the support of its Fourier transform $\widehat{h}(\omega)$ has no intersection with the interval $|\omega| < 1$ satisfies $(h, f_n) = 0$

(3.10)
(1) $\lim_{a\to+\infty} I(a) = 2\pi$
(2) π

(3.11)
(4) $I = 0$

(3.12)
(1) $v(t) = \theta(t)\dfrac{\exp(-\beta t) - \exp(-t)}{1-\beta}$
(3) $W_f = W_\beta = 1/(2+2\beta)$

(3.13)
(2)

$$(f, v) = (1/2\pi)\int_{-\infty}^{+\infty} (\beta + i\omega)\widehat{v}^*(\omega)\widehat{v}(\omega)\, d\omega = (\beta/2\pi)\int_{-\infty}^{+\infty} \widehat{v}(-\omega)\widehat{v}(\omega)\, d\omega = \beta(v, v)$$

(3.14)
(1) $C = \sup_{\omega\in\mathbf{R}} \widehat{G}(\omega)| \le \|G(t)\|_{L^1(\mathbf{R})}$
(2) The product of two functions $\in L^2(\mathbf{R})$ belongs to …

(3.18)
(1) The subspace of the functions $f(t)$ such that the support of their Fourier transform $\widehat{f}(\omega)$ is contained in the interval $|\omega| \le 1$
(2) T is the ideal "filter" for "low frequencies", $|\omega| \le 1$
(3) $g(t) \in C^\infty,\ g(t) \to 0$ as $|t| \to \infty$
(4) Strong convergence to the identity operator

(3.19)
(1) π
(2) $\mathtt{Ran}\, T \ne L^2(\mathbf{R})$, but dense in it: $\overline{\mathtt{Ran}\, T} = L^2(\mathbf{R})$
(3) $\rho \notin [0, \pi]$
(4) $g(x) \in C^\infty$
(5) Yes
(6) Strong convergence to πI. Use Lebesgue theorem

(3.20)
(1)

$$\big(g, T_a f\big) \propto \int_{-\infty}^{+\infty} \exp(ia\omega)\widehat{g}^*(\omega)\widehat{f}(\omega)\, d\omega \to 0$$

thanks to Riemann–Lebesgue lemma for the Fourier transform in $L^1(\mathbf{R})$ $\big($recall that the product of two functions $\in L^2(\mathbf{R})$ belongs to …$\big)$
(2) Use Lebesgue theorem

(3.21)
(5) The coefficients of the combination are ± 1 and $\pm i$

(3.22)
The pointwise limit is zero, *for all* $x \in \mathbf{R}$; the $\mathscr{S}'$-limit is $2\delta(x)$

(3.23)
(1) The $\mathscr{S}'$-limit of the Fourier transforms is the constant function $= 1$, then the $\mathscr{S}'$-limit of the given sequences is $\delta(x)$

(2) $\widehat{f_n}(y) \to 2;\ f_n(x) \to 2\,\delta(x)$

(3) $\widehat{f_n}(y) \to \sqrt{\pi};\ f_n(x) \to \sqrt{\pi}\,\delta(x)$

(4) $\widehat{f_n}(y) \to \pi;\ f_n(x) \to \pi\,\delta(x)$

(5) $g_n(x) \to -(\sqrt{\pi}/2)\,\delta'(x)$

(3.24)
$\widehat{f_n}(y) \to \pi;\ f_n(x) \to \pi\,\delta(x)$

(3.25)
(1) $f(t) \to \exp(-i\omega_0 t);\ \widehat{f}(\omega) = 2\dfrac{\sin(\omega-\omega_0)t_0}{\omega-\omega_0} \to 2\pi\,\delta(\omega-\omega_0)$ as $t_0 \to \infty$, i.e., the frequency contribution is "concentrated" in $\omega = \omega_0$

(3.29)

$$(1)\ \widehat{f}(k) = \begin{cases} -\pi i & \text{for } -1 < k < 0 \\ \pi i & \text{for } \ 0 < k < 1 \\ 0 & \text{elsewhere} \end{cases}$$

(3.30)
$a = n\pi,\ n \in \mathbf{Z}$, the support is $|\omega| \le 1$

(3.31)
(1) The limits in the first line are $\mathrm{P}(1/x) \mp \pi i \delta(x)$, etc.
(2) $\mp i\pi$

(3.32)
(1) $\lim\limits_{a\to\infty} u_a(\omega) = \mathscr{F}\big(\theta(t)\big) = i\,\mathrm{P}(1/\omega) + \pi\,\delta(\omega)$
(2) π. Notice that this limit cannot be obtained by means of usual Lebesgue theorem or by integration in the complex plane
(3) $i\pi$

(3.33)
(2) $\widehat{u}_a(y) = 2i\exp(iay)\mathrm{P}(1/y) \to -2\pi\,\delta(y)$
(3) $i\pi$
(4) $-\pi/2$

(3.34)
(3) $-2\sqrt{\pi}$ and $i\pi$

(3.35)
(2) $T = -D\,\mathtt{P}(1/x)$
(3) $-2\sqrt{\pi}$ and, respectively, 0

(3.37)
(a) $f(x) = -(1/4)\mathtt{sgn}\,x + (1/2)\delta'(x) + \text{const}$; (b) $\big((3/4) - x^2\big)\exp(-x^2) + \text{const}$;
(c) $f(x) = c_1\cos(\omega_1 x) + c_2\sin(\omega_2 x)$ where $c_1,\ c_2 = $const and $\omega_{1,2} = 2 \pm \sqrt{3}$;
(d) $f(x) = 0$

(3.38)
(1) $g_a(x) \to f(x)$ in the $L^2(\mathbf{R})$ norm
(2) $g_a(x) \to \mathtt{P}(1/x)$ in $\mathscr{S}'$
(3) $g_a(x) \to \delta'(x)$ in $\mathscr{S}'$

(3.39)
(1) $\pi i\big(\exp(iax) - 1\big)/x$
(2) $|F_a(x)| \le \|\widehat{F}_a(\omega)\|_{L^1(\mathbf{R})} = a$; $F_a(x) \in L^2(\mathbf{R}),\ \notin L^1(\mathbf{R})$
(3) (b) $F_a(x) = (\cos ax - 1)/x$
(4) $(1 - \cos ax)/x \to \mathtt{P}(1/x)$

(3.41)
(1) $F^{(+)}(t) = \theta(t)\sin t$
(3) No

(3.42)
(1) $F^{(+)}(t) = \big(1/(2i)\big)\big(\theta(-t)\exp(-it) + \theta(t)\exp(it)\big)$
$F^{(+)}(t)$ and $F^{(-)}(t)$ do not coincide

(3.43)
(1–3) $\widehat{F}_n(\omega) = \pi\delta(\omega) + i\,\mathtt{P}(1/\omega) - 1/(n - i\omega) \to \mathscr{F}\big(\theta(x)\big)$, etc.

(3.44)
(1) $\lim = -i\omega$, in $\mathscr{S}'$

(2) $$f_a(x) = (1/a^2)\begin{cases} 1 & \text{for } -a < x < 0 \\ -1 & \text{for } 0 < x < a \\ 0 & \text{for } |x| > a \end{cases}$$

(3) $\lim = \delta'(x)$

(3.45)
(3) $-2\pi i\delta'(\omega)$

(3.46)
(1) $\widehat{G}(\omega) = ig(\omega)\mathtt{P}(1/\omega) + \text{const} \times \delta(\omega)$

(2) $$\mathscr{F}\big(F(x)\big) = \frac{\exp(i\omega) - 1 - i\omega\exp(i\omega)}{\omega^2}$$

(3.47)
(1) $\mathscr{F}(\arctan x) = i\pi \exp(-|\omega|)\mathtt{P}(1/\omega)$
(2) $(\pi/2)\,\mathtt{sgn}\,x$, and ...

(3.48)
(2) $\mathscr{F}\big(\mathrm{erf}(t)\big) = i\sqrt{\pi}\exp(-\omega^2/4)\mathtt{P}(1/\omega) + \pi^{3/2}\delta(\omega)$
(3)

$$\mathscr{F}\Big(\int_{-\infty}^{t} \dots\Big) = \mathscr{F}(\arctan t + \pi/2) = \mathscr{F}(\arctan t) + \pi^2\delta(\omega)$$

$$= \big(i\,\mathtt{P}(1/\omega) + \pi\delta(\omega)\big)\big(\pi\exp(-|\omega|)\big)$$

etc.

(3.49)
(3) Yes
(4) $i\,\mathtt{P}(1/\omega) + \pi\delta(\omega)$

(3.51)
The convolution product is not defined for $a = \pm 1$

(3.52)
(1) $v_\varepsilon(x) = 2\begin{cases}\big(-1 + \exp(\varepsilon x)\big)/\varepsilon & \text{for } x < 0\\ \big(1 - \exp(-\varepsilon x)\big)/\varepsilon & \text{for } x > 0\end{cases}$
(2) $u_\varepsilon(x) \to 1,\ v_\varepsilon(x) \to 2x$

(3.53)
(1) $d^2(1)/dy^2 = d^2(y)/dy^2 = 0;\ \ d^2(y^2)/dy^2 = 2$

(3.54)
$(d)\ T = -\dfrac{d}{dy}\mathtt{P}(1/y) + c_0\delta(y) + c_1\delta'(y)\,;\ (g)\ T = \dfrac{\sin y}{y} + c\delta(y)$
(2) only for the equation (g) there is a (nonzero) solution $\in L^2(\mathbf{R})$, obtained choosing the arbitrary constant $c = 0$

(3.55)
$(b)\ T = \mathtt{P}\big(1/(y-1)\big) + c\,\delta(y-1)\,;\ (c)\ T = 0\,;\ (d)\ T = 1/(y \pm i)$

(3.56)
$(b)\ \ T = \Big(\mathtt{P}\frac{1}{y-1} - \mathtt{P}\frac{1}{y}\Big) + c_0\delta(y) + c_1\delta(y-1);$

$(d)\ T = \frac{1}{2}\Big(\mathtt{P}\frac{1}{y-1} - \mathtt{P}\frac{1}{y+1}\Big) + c_1\delta(y-1) + c_2\delta(y+1)\,;\ \ (h)\ T = 1/(1+y^2)$
only for the last equation there is a (nonzero) solution $\in L^2(\mathbf{R})$

(3.57)
$(a)\ T = \sum_{n\in\mathbf{Z}} c_n\delta(y - n\pi)$, where c_n can be "almost arbitrary", not only $\in \ell^2$ but also, e.g., polynomially divergent;

(c)
$$T = \sum_{n\in\mathbf{Z}} c_n \delta(y - 2n\pi) + \sum_{m\in\mathbf{Z}} c_m \delta'(y - 2m\pi)$$

(e) $T =$ any finite combination of $\delta^{(n)}(y),\ n = 0, 1, 2, \ldots$

(3.58)
(a) and (b) $u(x) = c_1 \cos x + c_2 \sin x$; (c) any combination of $\exp(2\pi i n x/a)$;

(d) $u = c_0 + c_1 x + c_2 x^2$

(3.59)
(a) $T = -\delta'(x) + c\,\delta(x)$; ($b$) $T = -\delta(x) + c\,\delta(x - 1)$;

(c) $T = (1/2)\delta''(x) + c_0\delta(x) + c_1\delta'(x)$

(3.60)
$T = c\,\delta(x)$ and $T = i\,\mathrm{P}(1/x) + c\,\delta(x)$

(3.64)
(2) $\widehat{T}\,\widehat{f}(\omega) = \exp(i\omega)\widehat{f}(\omega) = \lambda\widehat{f}(\omega)$, etc. If $\lambda = 1$ then $\widehat{f}(\omega)$ are combinations of $\delta(\omega - 2n\pi)$, $n \in \mathbf{Z}$, and the "eigenvectors" of T are (expectedly!) the periodic functions with period 1, see also Problem 3.57, (a)

(3.65)
(2) No eigenvectors in $L^2(\mathbf{R})$, but there are "eigenvectors" in $\mathscr{S}'$ (see problem above) for all $|\lambda| \le 2$
(3) $\mathtt{Ker}\,T = \{0\}$, $\mathtt{Ran}\,T \ne L^2(\mathbf{R})$, but dense in it: $\overline{\mathtt{Ran}\,T} = L^2(\mathbf{R})$

(3.66)
(1) $G(t) = 2\,\theta(-t)\,\exp(t)$, not causal
(2) $\|T\| = 2$, no eigenfunctions $\in L^2(\mathbf{R})$
(3) The expression $b(t) = \big(G * a\big)(t)$ ensures "time invariance" of the system, i.e., knowing $T\big(a(t)\big) = b(t)$ one *also* knows $T\big(a(t+\tau)\big) = b(t+\tau)$ for any $\tau \in \mathbf{R}$, then it is enough that the set of the functions $a(t+\tau)$ contains a complete set in $L^2(\mathbf{R})$, ...

(3.67)
(a) $G(t)$ uniquely defined but $\in \mathscr{S}'$, $\notin L^2(\mathbf{R})$.

(b) The only information is $\widehat{G}(\omega) = 0$ for $|\omega| < 1$.

(c) $\widehat{G}(\omega) = 1$ for $|\omega| < 1$ and $\widehat{G}(\omega) = 0$ for $1 < |\omega| < 2$, undetermined for $|\omega| > 2$.
(d) Impossible.
(e) $G(t)$ is determined apart from an arbitrary additive constant; there is *one* $G(t) \in L^2(\mathbf{R})$.
(f) The same arbitrariness as in (e), but there are no $G(t) \in L^2(\mathbf{R})$.
(g) $G(t)$ is uniquely defined in the hypothesis $G(t) \in L^2(\mathbf{R})$, otherwise $\widehat{G}(\omega)$ contains arbitrary combinations of delta functions, then $G(t)\ldots$, see Problem 3.57 (a).
(h) The same as in (g), now $G(t) = 0$ is the only Green function $\in L^2(\mathbf{R})$.

(3.68)
(2) $\|T\| = \sup_{\omega\in\mathbf{R}} |\widehat{G}(\omega)| \le \|G(t)\|_{L^1(\mathbf{R})}$, see also Problem 3.14
(3) $C = \|T\|$
(5) The operator T is unbounded, then ...

(3.69)
(1) $G(t) = \theta(t)\exp(-at) + c\exp(-at)$; $\exp(-at) \notin \mathscr{S}'$ but is a Schwartz distribution $\mathscr{D}'$
(4) Yes for the equation in (1). The Green function belonging to $L^2(\mathbf{R})$ for the equation in (3) is "anticausal", i.e., $G(t) = 0$ if $t > 0$

(3.70)
(1) The limits are, respectively, $\theta(t)$ and $-\theta(-t) \in \mathscr{S}'$ and solve the equation in (1) (c).

(2) The most general solution is $x(t) = \theta(t) + c$; choosing suitably the arbitrary constant c one obtains the solutions in (1) (a)

(3.73)
(2) (a) A possible constant is, e.g., $C = 2/a$; (b) $C = 2$
(3) $x_a(t) \in L^2(\mathbf{R})$, with, e.g., $C = 2/a$, but its limit as $a \to 0^+$ does not belong, in general, to $L^2(\mathbf{R})$

(3.74)
(1) $C = 1/a$
(3) $x(t) = \delta(t) - \theta(t)\ \exp(-t)$; $x(t) = \theta(t)\ \exp(-t)$

(3.75)
(1) $v^{(0)}(t) = \theta(t)\big(1 - \exp(-t)\big)$. The limit $\notin L^2(\mathbf{R})$, but $\in \mathscr{S}'$
(2) $v^{(0)}(+\infty) = 1$
(3) $W_f^{(0)} = 1/2$

(3.76)
(2) $(f, v) = (1/2\pi)\displaystyle\int_{-\infty}^{+\infty} \frac{\widehat{f}^*(\omega)\widehat{f}(\omega)}{\beta - i\omega}\, d\omega \to \widehat{f}^2(0)/2 = W_f^{(0)}$
(3) $v^{(0)}(+\infty) = \int_{-\infty}^{+\infty} f(t)\, dt = \widehat{f}(0)$
(4) $f(t)$ must be zero mean-valued

(3.77)
(1) $a = 2n\pi,\ n \in \mathbf{Z}$
(3) $a = (2n+1)\pi,\ n \in \mathbf{Z}$

(3.78)
(1) The causal Green function is $G(t) = \theta(t)\sin t \in \mathscr{S}'$

(3.79)
$G(t) = c + c_1 t + (1/2)|t|$, Green function causal $G(t) = t\,\theta(t) \in \mathscr{S}'$

(3.80)
Using Fourier transform one obtains *one* Green function for the first equation and ∞^1 Green functions for the second one. The most general Green function is obtained considering also the solutions of the homogeneous equation. For the first equation, these are $A\exp(t) + B\exp(-t)$, which belong to $\mathscr{S}'$ only if $A = B = 0$. For the second equation, these solutions are $A + B\exp(\mp t)$, which belong to $\mathscr{S}'$ only if $B = 0$ $\big($indeed, $\exp(\pm t) \in \mathscr{D}'$, the Schwartz distributions$\big)$

(3.81)
(2) The solution respecting causality is

$$x(t) = \theta(t)\big(-1 + t + \exp(-t)\big)$$

(3.82)
The solution respecting causality is

$$x(t) = \theta(t)\big(-(1/2)\exp(-t) + (1/3)\exp(-2t) + (1/6)\exp(+t)\big)$$

$\big(\notin L^2(\mathbf{R})$ and $\notin \mathscr{S}'\big)$; there is a solution in $L^2(\mathbf{R})$ given by

$$x(t) = \theta(-t)\big(-(1/6)\exp(t)\big) + \theta(t)\big(-(1/2)\exp(-t) + (1/3)\exp(-2t)\big)$$

(3.83)
The solution respecting causality is

$$x(t) = \theta(t)\big(1 - \exp(-t) - t\exp(-t)\big) \notin L^2(\mathbf{R})\,, \in \mathscr{S}'$$

(3.84)
The solution respecting causality is

$$x(t) = (1/2)\theta(t)\big(\exp(-t) - \cos t + \sin t\big) \notin L^2(\mathbf{R}),\ \in \mathscr{S}'$$

(3.85)
(1) (a) $x(t) = c\ \exp(-it) + \exp(-i\alpha t)/(\alpha - 1)$
(2) In the case $\alpha = 1,\ x(t) = c_1\cos t + c_2\sin t - (t/2)\cos t$

(3.86)
(3) 0 and $i\pi\big(\delta(\omega - 1) - \delta(\omega + 1)\big)$

(3.87)

$$u(x) = \begin{cases} x(1 - x_1) & \text{for} \quad 0 \le x \le x_1 \\ x_1(1 - x) & \text{for} \quad x_1 \le x \le 1 \end{cases}$$

(3.88)
The most general solution is $u = (-1/2)\,\mathrm{sgn}\,x + (1/2)|x| + \theta(x)\exp(-x) + c_1 + c_2 x$; the solution vanishing at $x \to +\infty$ is $u = \theta(-x)(1 - x) + \theta(x)\exp(-x)$

(3.89)
The solution satisfying the given boundary conditions is

$$u = (1/2)\theta(x)\big(\exp(-x) - \cos x + \sin x\big) - (1/2)(e^{-\pi/2} + 1)\sin x$$

(3.90)
(1) Writing $\mathscr{F}\big(G(t)\big) = \widehat{G}(\omega) = A(\omega)\ \exp\big(i\Phi(\omega)\big)$, one has

$$\widehat{b}(\omega) \ = \ 2\pi\widehat{G}(\omega)\,\delta(\omega-\omega_0) = 2\pi\ A(\omega_0)\exp(i\Phi(\omega_0))\,\delta(\omega-\omega_0)$$

then …
(2) The output is the superposition of two waves with different amplitudes and a "phase distortion" (in general):

$$b(t) \ = \ \exp(i\Phi_1)\Big(A_1\ \exp(-i\omega_1 t) + A_2\exp\big(-i\omega_2 t + i(\Phi_2-\Phi_1)\big)\Big)$$

where $\Phi_1 = \Phi(\omega_1),\ A_1 = A(\omega_1)$, etc.; in general, $A_1 \neq A_2,\ \Phi_1 \neq \Phi_2$

(3.91)
(2) n even

(3.92)
(1) $b(t) = \delta(t)$
(2) $b_\varepsilon(t) \to \delta(t)$

(3.93)
(1) $b_\tau(t) \in L^2(\mathbf{R})$
(2) $b_\tau(t)$ is rapidly vanishing as $|t| \to \infty$
(3) Considering $b_\tau(t)$ as distributions, $b_\tau(t)$ converges to 0 in $\mathscr{S}'$ and in the weak $L^2(\mathbf{R})$ sense, not in norm $L^2(\mathbf{R})$

(3.94)

$$b_T(t) \ = \ (1/2)\big(\exp\big(-(t-T)^2\big) - \exp(-t^2)\big) \to -(1/2)\exp(-t^2)$$

and

$$\widehat{b_T}(\omega) = (1/2)\big((\exp(i\omega T) - 1\big)\widehat{a}(\omega) \to (-1/2)\mathscr{F}\big(a(t)\big)$$

in $\mathscr{S}'$, not $L^2(\mathbf{R})$. Expectedly, the limit coincides with the result in (3)

(3.95)
(1) Yes: $b_c(t) = 0,\ \forall t$, if the support of the $\mathscr{F}$-transform $\widehat{a}(\omega)$ has no intersection with the interval $|\omega| < c$
(2) In general $\widehat{b_c}(\omega)$ is not a continuous function, then in general $b_c(t) \notin L^1(\mathbf{R})$, but $\in L^2(\mathbf{R})$, is bounded, (possibly not rapidly) vanishing at $|t| \to \infty$, infinitely differentiable
(3) $b_c(t) \to a(t)$, also in the $L^2(\mathbf{R})$ norm

(3.96)
(1) No. Yes, with $\widehat{a}(\omega) = \sum_{0\neq n\in\mathbf{Z}} c_n\delta(\omega - \frac{\pi n}{T})$, see Problem 3.57 (a)
(2) In general $\widehat{b_T}(\omega)$ is not a continuous function, then in general $b_T(t) \notin L^1(\mathbf{R})$, but $\in L^2(\mathbf{R})$, is bounded, (possibly not rapidly) vanishing at $|t| \to \infty$, continuous but in general not differentiable
(4) $\widehat{b_T}(\omega) \to 2\pi\delta(\omega)\widehat{a}(\omega) = 2\pi\widehat{a}(0)\delta(\omega),\ b_T(t) \to \widehat{a}(0) = \int_{-\infty}^{+\infty} a(t)\,dt$

(3.97)
(1) $G(t) = \delta(t) - \theta(t)\exp(-2t)\,;\ G(t) = \delta(t) + \theta(t) + \text{const};$
$G(t) = \delta(t) - 2\theta(-t)\exp(+t)$
(2) $\beta = 1$
(3) The causal Green function is $G(t) = \delta(t) + 2\theta(t)\exp(+t) \notin \mathscr{S}'$

(3.98)
(1) $G(t) = \begin{cases} 1 & \text{for} \quad 0 < t < 1 \\ 0 & \text{elsewhere} \end{cases} \quad + \text{const}$

(3.99)
(1) The Green function belonging to $L^2(\mathbf{R})$ is $G(t) = \begin{cases} -t-1 & \text{for} \ -1 < t < 0 \\ t-1 & \text{for} \quad 0 < t < 1 \\ 0 & \text{elsewhere} \end{cases}$

(3.100)
(2) $G(t, t') = M(t')\delta(t - t')$

(3.101)
(1) $\mathscr{F}^{-1}(-\mathrm{P}(1/\omega) * g) = \pi i(\mathrm{sgn}\,t)f(t) = \pi i f(t)$, then $f(t) = 0$ if $t < 0$
(3) Only one satisfies the property given in (1), see Problem 3.7
(4) $\widehat{G}_2(\omega) = \dfrac{\omega}{1+\omega^2}$ and $G(t) = \theta(t)\ \exp(-t)$. It can be noted that, in this example, the existence of a connection between $\widehat{G}_1(\omega)$ and $\widehat{G}_2(\omega)$ can be deduced in a different way, observing that $\widehat{G}(\omega) = 1/(1 - i\omega)$ is analytic $\forall\,\omega \neq -i$ (in particular along the real axis), and the relationship between real and imaginary parts of an analytic function is well known ….

(3.102)
(3) The limit is $\delta(x)$, indeed $f(x) * \delta(x) = f(x)$
(4) $f(x) = c_1\cos x + c_2\sin x \in \mathscr{S}'$

(3.103)
(2) Strong convergence (not norm-convergence) to zero
(3) Strong convergence (not norm-convergence) to the identity operator

(3.104)
(1) $u(x, t) \in L^2(\mathbf{R})$ vanishes as $x \to \pm\infty$ but not rapidly, and $\notin L^1(\mathbf{R})$, indeed $\widehat{u}(k, t)$ is not a continuous function
(3) $u(x, t) = 1\,;\ u(x, t) = x\,;\ u(x, t) = x^2 + 2t$

(3.105)
(3) Yes

(3.106)
(3) The limit is zero in the sense of $\mathscr{S}'$ and in the weak $L^2(\mathbf{R})$ convergence
(4) The limit is $u(x, 0)$ in the sense of the $L^2(\mathbf{R})$ norm:

$$\|\widehat{f}(k)(\cos kt - 1)\|_{L^2(\mathbf{R})} \to 0$$

thanks to Lebesgue theorem

(3.107)
(1) $G(x,t) = \begin{cases} 1/2 & \text{for} \quad |x| < t \\ 0 & \text{for} \quad |x| > t \end{cases}$
(2) $\widehat{G}_t(k,t) = \cos kt,\ \ G_t(x,t) = \ldots$; $G(x,t)$ is indeed the solution if $g(x) = \delta(x)$
(3) What is the velocity of the wave propagation?

(3.108)
(1) $u(x,t) = |x - vt|$ plus arbitrary functions $f(x - ct) + g(x + ct)$
(2) $u(x,t) = (1/2)\mathtt{sgn}(x - t)$ +, etc., as in (1)

(3.109)
(1) $\widehat{u}(k,t) = \big(\widehat{f}(k)(1 - ikt) + t\,\widehat{g}(k)\big)\exp(ikt)$
(2) No
(4) $u(x,t) = (1 - t)\theta(x - t)\exp(t - x) + t\,\delta(x - t)$

(3.110)
(2) $G(x,y) = (y/\pi)/(x^2 + y^2)$
(3) The limit is $\delta(x)$

(3.112)
(2) The solutions are proportional to $y,\ xy$, etc.

(3.113)
(1) $\widehat{f}(k_1, k_2, k_3) = -8i\pi^2\big(1/(1 + k_1^2)\big)\delta'(k_2)\delta(k_3)$
(2) $\mathscr{F}(1/r^2) = 2\pi^2/k$

(3.114)
(1) $\mathscr{F}(1/r) = 4\pi/k^2$

(3.116)
(1) $0,\ -\infty,\ \beta$; if $\alpha \le -1$ the functions are not (locally) summable
(2) $0,\ \mathtt{Re}\,\gamma,\ 0$

(3.117)
(3) $\lambda = -1$

(3.118)
$f(x) = \sin x$ for $0 \le x \le \pi$ and $= 0$ elsewhere; $\lambda = -\infty$

(3.119)
(1) $\widehat{f}(\omega)$ is the function of $\omega \in \mathbf{R}$ obtained from $\widetilde{f}(s)$ replacing s with $-i\omega$.
(2) $f(x) = x$ if $0 \le x < 1$ and $= 0$ elsewhere, with $\lambda = -\infty$.
(4)

$$f(x) = \begin{cases} x & \text{for } 0 \le x \le 1 \\ 1 & \text{for } x \ge 1 \end{cases}$$

$\big($and $f(x) = 0$ if $x \le 0$, of course$\big)$; with $\lambda = 0$. The Fourier transform is

$$\widehat{f}(\omega) = \mathtt{P}\Big(\frac{\exp(i\omega) - 1}{\omega^2}\Big) + \pi\delta(\omega)$$

(3.122)
(1) $a - b = 2n\pi,\ n \in \mathbf{Z}$
(3) The support is $1 + 2\pi m \le x \le 1 + 2\pi n$

(3.124)
(2) If $c = 2n\pi$, $n = 1, 2, \ldots$, then $y(t) = 0$ for any $t \ge c$
(3) If $c = (2n-1)\pi$, $n = 1, 2, \ldots$, then $y(t) = 0$ for any $t \ge c$

(3.125)
$\widetilde{f}(s) = \log\big((s-b)/(s-a)\big)$; $\widetilde{f}(s) = (\pi/2) - \arctan s$

(3.126)
(1) $\widetilde{J}_0(s) = 1/\sqrt{1+s^2}$

(3.127)
$(a)\ \mathscr{L}\big(f(x)\big) = \dfrac{\big(1-\exp(-s)\big)^2}{s\big(1-\exp(-2s)\big)} = \dfrac{1}{s}\tanh(s/2)$

$$(b)\ f(x) = \begin{cases} 1 & \text{for} \quad 0 < x < 1 \\ 2 & \text{for} \quad 1 < x < 2 \\ \vdots & \\ n & \text{for} \quad n-1 < x < n \\ \vdots & \end{cases}$$

and $f(x) = 0$ for $x < 0$, of course

$(c)\ f(x) = \theta(x)|\sin x\,|$

(3.128)
(3) $\widetilde{u}(x,s) = \widetilde{\varphi}(s)\exp(-sx)$ and then $u(x,t) = \theta(t-x)\varphi(t-x)$

(3.130)
$f(x) = 1/\sqrt{\pi x}$.

Problems of Chap. 4

(4.5)
The group contains six elements; there are three inequivalent irreducible representations, two of dimension 1, and one of dimension 2, in agreement with Burnside theorem: $6 = 1^2 + 1^2 + 2^2$, only one is faithful. The expected degeneracies are then 1 and 2

(4.7)
The degeneracies are 1 and 2. But, e.g., the eigenvalue $\lambda = -50$ has degeneracy 3, being obtained when $n = 1,\ m = 7;\ n = 7,\ m = 1$ and $n = m = 5$, with the notations of Problem 1.94

(4.8)
(1) Finding the seven one-dimensional inequivalent irreducible representations of Z_7 amounts essentially to finding the solutions of the equation $\alpha^7 = 1,\ \alpha \in \mathbf{C}$

(4.9)
There are two one-dimensional representations, one of dimension 2, and two of dimension 3 of $\mathscr{O}_1$, in agreement with Burnside theorem: $24 = 1^2+1^2+2^2+3^2+3^2$ (respectively, four of dimension 1, two of dimension 2, and four of dimension 3 of $\mathscr{O}$). The expected degeneracies are 1, 2, and 3

(4.10)
(1) $GL_n(\mathbf{C}) = SL_n(\mathbf{C}) \times \mathbf{C}$, where $SL_n(\mathbf{C})$ is the "special" subgroup of matrices M with $\det M = 1$.
(2) $U_n = SU_n \times U_1$.
(3) One has $GL_n(\mathbf{R}) = SL_n(\mathbf{R}) \times \mathbf{R}$ *only if* n is odd (indeed, if n is even, there are no real matrices commuting with $SL_n(\mathbf{R})$, and then of the form λI, such that $\lambda^n < 0$). Similarly, $O_n = SO_n \times Z_2$, where Z_2 is (isomorphic to) $\{1, -1\}$, *only if* n is odd

(4.11)
(2) $\exp(im\varphi)$, $m \in \mathbf{Z}$; only $\exp(\pm i\varphi)$ are faithful
(3) The circular polarizations

(4.12)
The eigenvectors lie along the light "cones", or better, in the space x, t, the light "lines" $x = \pm ct$

(4.13)
(1) The dimensions are, respectively, $n^2 - 1$ and n^2

(2) In general, the correspondence between the group G and the Lie algebra $\mathscr{A}$ is continuous and one-to-one in a neighborhood of the identity of the group and the "zero" of the algebra (the origin of $\mathscr{A}$ viewed as linear space). The matrices in O_n have determinant either $+1$ or -1; those with det $= -1$ cannot be continuously connected with the matrices with det $= +1$; the former belong to a manifold not connected to the manifold of the latter, which contains the identity. The dimension of SO_n is $n(n-1)/2$

(4.14)
(1) No

(4.16)
The first group is a one-dimensional dilation; the second a rotation; the next generates dilations in $\mathbf{R}^2$; the two last algebras of the first line can be viewed, for instance, as the generators of a periodic motion (closed orbit) and, respectively, of a nonperiodic motion (dense orbit) on a torus. The first two algebras of the second line, considered as transformations on $\mathbf{R}^2$ (respectively, $\mathbf{R}^3$), describe a diverging spiral in the plane and a spiral on a cylinder. The last case can be interpreted, e.g., as the group of time-evolution $u(t) = \exp(At)u_0$ of the dynamical system $\dot{u} = Au$, where $u = u(t) \in \mathbf{R}^n$, with a given initial condition $u(0) = u_0 \in \mathbf{R}^n$, and where the time t plays the role of Lie parameter

(4.17)
$$A_3 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \text{ etc. ; } B_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \text{ etc.}$$

(4.18)
(1) $A = -d/dx$
(2) $A = y(\partial/\partial x) - x(\partial/\partial y)$
(3) $A = t(\partial/\partial x) + x(\partial/\partial t)$

(4.19)
(2) $D = -x\, d/dx;\ \widetilde{D} = -(x\, d/dx + 1/2)$

(4.24)
The representation (a) is not unitary, partially reducible, faithful.
(b) is unitary, reducible, faithful. (c) is unitary, irreducible, not faithful.
(d) is not unitary, irreducible, faithful

(4.27)
(1) There is a one-dimensional invariant subspace $r^2 = x^2+y^2+z^2$; the other invariant subspace is the five-dimensional space generated by the five spherical harmonics $Y_{\ell,m}(\theta, \varphi)$ with $\ell = 2$: e.g., $z^2 \propto Y_{2,0} \propto \cos^2\theta$; $x^2 - y^2 \propto Y_{2,2} + Y_{2,-2} \propto \sin^2\theta \cos 2\varphi$

(4.28)
(2) Apart from a factor i, in quantum mechanics the matrices A_i correspond to the components of the angular momentum $\ell = 1$ and $C = i^2\ell(\ell + 1)$; similarly, apart from a factor $2i$, the matrices A'_i correspond to the components of the spin $j = 1/2$ and $C' = (2i)^2 j(j + 1)$
(3) $a = 0, \pm 2\pi, \ldots$; $a'' = 0, \pm 4\pi, \ldots$

(4.29)
(1) The center of SU_n contains n elements. The center is in particular an invariant subgroup
(2) It is known that for any integer N there is an irreducible representation of SU_2 of dimension N which is related to an angular momentum or spin $j = 0,\ 1/2,\ 1, \ldots$ with $N = 2j + 1$, and that the odd-dimensional representations are also (faithful) irreducible representations of SO_3. Therefore, the odd-dimensional representations of SU_2 are not-faithful representations of SU_2. Notice in particular that SO_3 has no center

(4.30)
(1) The invariant subspaces are given by the antisymmetric matrices (three-dimensional), the traceless symmetric matrices (five-dimensional), and the "traces", i.e., the multiples of the identity (one-dimensional). In symbols: $(\ell = 1) \otimes (\ell = 1) = (\ell = 0) \oplus (\ell = 1) \oplus (\ell = 2) = \underline{1} \oplus \underline{3} \oplus \underline{5}$

(4.31)
(1) $\underline{3} \otimes \underline{3}$ decomposes into the six-dimensional representation acting on the symmetric tensors $T_{(ij)}$ and the three-dimensional representation on the antisymmetric tensors $T_{[ij]}$, equivalent to the vectors $z^k = \varepsilon^{ijk} T_{[ij]}$, or: $\underline{3} \otimes \underline{3} = \underline{6} \oplus \underline{3}^*$. The tensor product $\underline{3} \otimes \underline{3}^*$ decomposes into the eight-dimensional representation of the traceless tensors T^i_j and the one-dimensional "traces": $\underline{3} \otimes \underline{3} = \underline{8} \oplus \underline{1}$

(2) $\underline{3} \otimes \underline{3} \otimes \underline{3} = (\underline{3} \otimes \underline{3}) \otimes \underline{3} = (\underline{3}^* \oplus \underline{6}) \otimes \underline{3} = \underline{1} \oplus \underline{8} \oplus \underline{8} \oplus \underline{10}$ where $\underline{10}$ is the 10-dimensional representation acting on the symmetric tensors $T_{(ijk)}$

(4.33)
$f(r) = c_1 \log r + r^{n+2}/(n + 2)^2 + c_2$, where c_i are constants

(4.34)
(1) The ODE is $f_{ss} + v f_s = 0$, with solution $u(x, t) = c_1 \exp\big(-v(x - vt)\big) + c_2$, for any v
(2) Apart from the trivial solution $u = x - vt$ with arbitrary v, the only admitted velocities are $v = \pm 1$, with the well-known solutions $u = f_1(x - t) + f_2(x + t)$, with arbitrary f_1, f_2

(4.35)
(2) The ODE is $4s f_{ss} + 4 f_s = s^n$, with solution
$u(x, t) = c_1 \log|x^2 - t^2| + (x^2 - t^2)^{n+1}/\big(4(n + 1)^2\big) + c_2$

(4.37)
(3) The irreducible representations are described by the couple $j_M,\ j_N$ with dimension $(2j_M+1)(2j_N+1)$
(4) $C_1 = 2j_M(j_M+1) + 2j_M(j_M+1);\ C_2 = \dots$

(4.38)
(1) The representations involved are those with $j_M = j_N$. The degeneracies are then $(2j_M+1)^2 = n^2,\ n = 1, 2, \dots$
(2) $E_n = -me^4/2\hbar^2 n^2$, $n = 1, 2, \dots$, as well known

(4.39)
2 (*d*) The six-dimensional representation contains $\ell = 0$ (one-dimensional) and $\ell = 2$ (five-dimensional).

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