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Elements of Cosmological Thermodynamics

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Dedicated to my parents
Aditya Saha & Manasi Saha
and to my beloved brother
Soumyadeep (Piku)

Foreword

This book is well equipped to serve as a nice introductory text to the application of horizon thermodynamics in the context of Cosmology. The implications of modified and generalized Hawking temperatures in the formulation of cosmological thermodynamics are carefully explained. The book also discusses various drawbacks within this research field and lists some open issues for the readers to deal with. I hope their endeavours will enrich this research field as well as open up new frontiers for future generations to work with.

Kolkata, India
December 2017

Subenoy Chakraborty

Preface

The study of cosmological horizons has gained much attention in recent literature. Consequently, the thermodynamics of horizons or, more precisely, cosmological thermodynamics (gravitational thermodynamics is also used sometimes) has become a very popular research field and is emerging very rapidly. The apparent reason for this popularity is the discovery of the intimate relationship between gravity and the laws of thermodynamics which resulted in the establishment of the laws of black hole thermodynamics. Attempts were then undertaken to generalize these laws in the context of Cosmology. Nowadays, cosmological models, which, in addition to passing the observational tests, are consistent with the laws of thermodynamics, particularly the first law, the generalized second law and thermodynamic equilibrium, are more acceptable as compared to those which only pass the observational tests.

This book in the SpringerBriefs Series in Physics serves as a concise introduction to the vibrant research field of cosmological thermodynamics. The first two chapters focus entirely on establishing the basic features of relativistic cosmology and equilibrium thermodynamics respectively. The third chapter discusses the origin and provides a brief history of cosmological thermodynamics and also explains how this particular field of research was gradually developed from the mathematical point of view. In this regard, the modified and generalized forms of the Hawking temperature are introduced and their implications explained. The next three chapters discuss its application in determining the thermodynamic viability of certain well-known and widely accepted cosmological models, particularly, the flat, homogeneous and isotropic Friedmann–Lemaitre–Robertson–Walker model, the isotropic but inhomogeneous Lemaitre–Tolman–Bondi model, and the gravitationally induced adiabatic particle creation model. The work described in this book (in Chaps. 4–6) is the result of research undertaken while I was a Ph.D. student under the able supervision of Prof. S. Chakraborty at the Department of Mathematics of Jadavpur University and later a Postdoctoral Fellow under the mentorship of Prof. N. Banerjee at the Department of Physical Sciences of Indian Institute of Science Education and Research (IISER), Kolkata.

The book is most suitable for graduate level students and researchers who have a basic understanding of Differential Geometry. Some familiarity with General Relativity will prove useful but is not necessary. A brief overview of the important and relevant concepts of Cosmology and Thermodynamics in the first two chapters of the book enables even the first year undergraduate students in Applied Mathematics, Physics, as well as allied subjects to gain some insights into this field. The final chapter of this book enlists few (of many) shortcomings in this field and also outlines prospective open issues which may impart a sense of challenge into the mind of the reader. I sincerely hope that the lucid language used throughout the book and the rich bibliography at the end of each chapter will be identified as important assets by the readers and at least a few of them will be inspired to find new directions of research in this field.

This book has been checked several times with extreme care to free it from all discrepancies and typos. Even then the vigilant readers may find mistakes and several portions of this book may seem to be unwarranted or irrelevant. I take sole responsibility for these errors which may have resulted from my inadequate knowledge in the subject or escaped my notice. Any comments or suggestions for improvement are welcome and should be directed to my email id subhajit1729@gmail.com.

Kolkata, India Subhajit Saha
December 2017

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I begin by offering my heartfelt thanks and warmest regards to my Ph.D. thesis supervisor Prof. Subenoy Chakraborty without whose guidance, support, understanding, patience and professionalism, this book would not exist. I am also indebted to him for agreeing to write a foreword to this book. I also thank *Kakima* (Mrs. Archana Chakraborty) for her delicious cuisines and warm hospitality, and Sumanta for his support and academic help occasionally. I extend my sincere gratitude to the Inter-University Centre for Astronomy and Astrophysics (IUCAA), Pune for their vibrant Visiting Program as well as their extensive research facilities and warm hospitality which helped in writing down a significant part of this book. I should specially mention the role of IUCAA library which helped in checking the manuscript of this book for plagiarism through their URKUND anti-plagiarism software. I also wish to thank my friends and fellow researchers, particularly Abdulla, Abhijit, Saugata, Sujoy *da* and Sunandan *da* as well as my colleagues at Panihati Mahavidyalaya (where I am presently teaching) for their constant support and encouragement. I also express my indebtedness to my high school physics tutor Mrs. Rita Neogi and her family for their well wishes and deep concern for my well-being. For discussion or clarification on certain topics within the subject, I thank Profs. Narayan Banerjee, Christian Corda, Diego Pavon, Stefano Viaggiu, Alexis Helou and Valerio Faraoni. I am also grateful to Herma and Liesbeth (of Springer) for their prompt help with my technical queries during the writing of this book. Special thanks are also due to Mr. Subodh Kumar and Ms. Komala Jaishankar for their extensive help extended during the pre-production stage. Above all, I express my deepest gratitude and indebtedness to my parents and my beloved brother for their sacrifices, well wishes, encouragement, mental support and utmost love and care.

About the Book

Based on the author's own work and results obtained by renowned cosmologists across the globe, this short book provides a concise introduction to the relatively new research field of cosmological thermodynamics. Starting with a brief overview of basic cosmology and thermodynamics, this text gives an interesting account of the application of horizon thermodynamics to the homogeneous and isotropic Friedmann-Lemaitre-Robertson-Walker model, the inhomogeneous Lemaitre-Tolman-Bondi model, and the gravitationally induced adiabatic particle creation scenario. Both seasoned and new researchers in this field will appreciate the lucid presentation and the rich bibliography.

Contents

About the Author

Dr. Subhajit Saha is an Assistant Professor of Mathematics at Panihati Mahavidyalaya, a West Bengal Govt. aided college in Sodepur, India. He is actively engaged in research in the field of Cosmological Thermodynamics and is currently exploring its applications in the gravitationally induced particle creation scenario as well as other areas of Theoretical Cosmology. He obtained his PhD degree from Jadavpur University in 2015. He worked in the field of "Horizon Thermodynamics with Modified Hawking Temperature" under the supervision of Prof. Subenoy Chakraborty. Prior to that, he obtained his B.Sc. in Mathematics and M.Sc. in Applied Mathematics from Jadavpur University in 2009 and 2011 respectively. He was awarded the prestigious National Postdoctoral Fellowship by the Science and Engineering Research Board of the Govt. of India in 2016. He pursued postdoctoral research at Indian Institute of Science Education and Research (IISER) Kolkata under the mentorship of Prof. Narayan Banerjee. Dr. Saha has published around 25 articles in peer-reviewed international journals, several of them in reputed journals published by Springer, Elsevier, World Scientific, and the American Physical Society. He is a Life Member of the Indian Association for General Relativity and Gravitation and a Member of the International Astrostatistics Association.

Chapter 1
Fundamentals of Relativistic Cosmology

Abstract This chapter provides a concise introduction to the basic notions of Cosmology. Starting from the Cosmological Principle, the Weyl's Postulate, and the Einstein's equations, this chapter goes on to explain the relevant concepts of Cosmology required to gain the necessary insights into the subject of cosmological thermodynamics. It gives a brief, yet an effective description of the Friedmann–Lemaitre–Robertson–Walker metric, the observational parameters, the cosmological horizons, particularly, the apparent, event, and particle horizons, and the inexplicable dark energy.

Keywords Cosmology · Cosmological principle · Weyl's postulate · Einstein's field equations · FLRW metric · Friedmann equation · Raychaudhuri equation Hubble parameter · Dark energy · Cosmological constant

Cosmology is the scientific study of the dynamical and the large scale structure of the Universe in its entirety. Most of the early civilizations have recorded scientific observations related to Cosmology, and the thirst for understanding has continued for over five millenniums. Cosmologists ask questions such as "Where did the Universe come from? How and why did it begin? Will it come to an end some time in the future? What are the matter/energy constituents of the Universe and how were they made? Is it infinite in spatial extent or does it have boundaries? Was there a beginning of time? Could time run backwards? What is the geometry of the Universe?" Cosmology tries to answer these questions by describing the past, explaining the present, and predicting the future of the Universe. It was only after Albert Einstein discovered General Relativity (GR) in 1915 that Cosmology came into existence as a separate scientific study. The subject has seen spectacular progress since the beginning of the 1970s which commenced with a sudden development in understanding the physical processes of the early Universe through theoretical means and aided by a string of observational discoveries. Prior to the discovery of cosmic acceleration, the *standard cosmological model* represented an almost universal consensus amongst cosmologists as to the best description of the Universe. Also known as the *hot Big Bang model*, it states that our Universe started with a small singularity, then expanded over the last 14 billion years, and that the (decelerated) expansion is still going on today. It is speculated that the Universe was initially very hot and dense,

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and gradually, it evolved into a relatively cool and slender state. In this chapter, we discuss the basic features (within the scope of this book) of the hot big bang model as well as the presently observed acceleration of the Universe. Most of the discussions in Sect. 1.1–1.4 have been inspired from the books by d'Inverno (1998), Ryden (2002), Padmanabhan (2002), Liddle (2003), Carroll (2004), and Cheng (2005), and the extensive review by Trodden and Carroll (2004).

1.1 Homogeneity and Isotropy: The FLRW Metric

The central theme of Cosmology is the fact that our place in this Universe is nothing special. In 1952, Baade's observation that the Milky Way is a fairly typical galaxy, led to the modern view, referred to as the *cosmological principle* which is believed to be a generalization of the Copernican principle that our Earth does not lie at the center of the solar system. The formal statement of the cosmological principle can be given in the following manner — *"over sufficiently large*[1] *distance scales, the Universe presents itself as a homogeneous and isotropic entity at every epoch."* It must be noted that the cosmological principle is not exact but is an approximation which holds better and better with larger distance scales. It is therefore a property associated with the global Universe and it breaks down if one considers local phenomenon. At this point, the reader may feel that the words "homogeneous" and "isotropic" in the above statement need further clarification. Now, *homogeneity* is the claim that there is no special point in the Universe, while *isotropy* is the claim that all directions in the Universe are the same, i.e., there is no preferred direction. Note that the concepts of homogeneity and isotropy are not equivalent at all. For instance, a universe with a uniform magnetic field is homogeneous but not isotropic since directions along the field lines are distinguishable from those perpendicular to it. Alternatively, a spherically-symmetric distribution is isotropic but not necessarily homogeneous when viewed from its central point. However, isotropy about every point does enforce homogeneity. It is difficult to test homogeneity directly, although the number counts of galaxies and the linearity of the *Hubble law*[2] provide some evidence. The greatest support for isotropy came in the year 1965 when the *cosmic microwave background* (CMB) was discovered by Penzias and Wilson. It is a bath of thermal radiation pervading the Universe with a temperature of 2.7 K and is isotropic to fractions of a percent as measured by the WMAP and the PLANCK satellites. This radiation is presumed to be the thermal remnant of the hot big bang.

The cosmological principle paints a picture of the Universe as a physical system of "cosmic fluid". The galaxies move like fundamental particles in the fluid. Thus, the motion of a cosmic fluid element is the smeared-out motion of the constituent galaxies. In other words, each particle follows geodesic world lines. This leads to the

[1] Generally considering scales larger than 100 Mpc.

[2] It states that the radial velocities of recession of galaxies are directly proportional to their distances from us, i.e., $v = H_0 d$ where H_0 is the constant of proportionality known as the Hubble constant. It was the first observational evidence that our Universe is expanding.

assumption that there is a privileged class of observers (called *fundamental observers*) in the Universe, associated with such a special motion of these particles. This is contained in the *Weyl's postulate* (due to H. Weyl 1923) which states that *the particles of the cosmic fluid (or "substratum") lie in spacetime on a congruence of timelike geodesics diverging from a point in the finite or infinite past*. Therefore, there is one and only one geodesic passing through each point of spacetime (except at a singular point in the past where they all intersect) and consequently, the matter at any point possesses a unique velocity, thereby providing a hint that we may consider the cosmic fluid to be a perfect fluid.

Such a picture of the Universe allows us to pick a privileged coordinate frame — the *comoving coordinate system*, where t denotes the proper time of each fluid element and x^i ($i = 1, 2, 3$) denote the spatial coordinates carried by each fluid element. The comoving coordinate is also the cosmic rest frame since each fluid element's (or each fundamental observer's) position coordinates are unchanged with time. The Universe will be anisotropic to any other observer who moves with a uniform velocity relative to this fundamental class of observers. We can get the simplest description of the laws of physics if we use the coordinate system appropriate to this fundamental class of observers.

The form of the spacetime metric[3] in such a coordinate system (ct, r, θ, ϕ), (c is the velocity of light) which incorporates homogeneity and isotropy will be given by[4]

$$ds^2 = -c^2dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right]. \tag{1.1}$$

Here, the factor[5] $a(t)$ determines the overall scale of the spatial metric and can be identified as the *scale factor*[6] of the Universe. This metric, which describes a universe that is compatible with the cosmological principle, is called the *Robertson–Walker* (RW) *metric*. The Universe, described by the above metric is known as the *Friedmann–Robertson–Walker* (FRW) Universe or sometimes the *Friedmann–Lemaitre–Robertson–Walker* (FLRW) Universe. The value of k sets the *curvature,* and therefore the size of the spatial surfaces. It can take the values -1, 0 or $+1$ and accordingly, the Universe is said to be *open*, *flat* or *closed*.

1.2 The Friedmann and the Acceleration Equations

The FLRW metric is obtained from the cosmological principle by purely kinematic means. We now focus on the dynamics by analyzing differential equations governing

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[3]The metric in Eq. (1.1) bears the signature (−,+,+,+) which is in fact the most widely used in the literature. However, it can also be derived with the signature (+,−,−,−),

[4]For a simple proof, see p. 162 of T. Padmanabhan, Theoretical Astrophysics Volume III: Galaxies and Cosmology, Cambridge University Press (2002).

[5]The *proper distance* or the true distance is determined by multiplying the scale factor $a(t)$ to the comoving distance.

[6]The *redshift parameter* z is connected to the scale factor by the relation $a = \frac{1}{1+z}$.

the evolution of the scale factor $a(t)$. Such equations of motion can be determined by taking into account firstly, the Weyl's postulate which requires the cosmic fluid to be a perfect fluid having energy-momentum (EM) tensor

$$T_{\mu\nu} = (\rho c^2 + p)u_\mu u_\nu + p g_{\mu\nu}, \quad u_\mu u^\mu = -1, \tag{1.2}$$

where $g_{\mu\nu}$ is the metric tensor, u_μ is the fluid four-velocity, and ρ and p are the energy density and pressure in the rest frame of the fluid, and secondly, applying Einstein's equation (also known as Einstein field equations (EFE))

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} \tag{1.3}$$

to the RW metric. In Eq. (1.3), $G_{\mu\nu}$ is the Einstein tensor, $R_{\mu\nu}$ is the Ricci tensor and R is the Ricci scalar, all of which are dependent on the metric and its derivatives. $G = 6.673 \times 10^{-11}\,\mathrm{Nm^2\,kg^{-2}}$ is the Newton's universal gravitational constant. Now, since the fluid elements are comoving, the normalized four-velocity in the coordinates of (1.1) is

$$u_\mu = (1, 0, 0, 0). \tag{1.4}$$

Thus, using Eqs. (1.1), (1.2), and (1.4), one obtains two independent equations from Eq. (1.3) given by (choosing $c = 1$)

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{kc^2}{a^2} = \frac{8\pi G}{3}\rho \tag{1.5}$$

and

$$2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{kc^2}{a^2} = -8\pi G \frac{p}{c^2}, \tag{1.6}$$

where the 'dot' denotes differentiation with respect to (w.r.t.) time t. It is often useful to combine Eqs. (1.5) and (1.6) to obtain the *acceleration equation*[7]

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right). \tag{1.7}$$

Equation (1.5) is known as the *Friedmann equation*.

The *energy conservation equation* is obtained by the vanishing of the covariant divergence of the EM tensor, i.e.,

$$\nabla_\mu T^{\mu\nu} = 0, \tag{1.8}$$

which yields

[7] This equation is also known in the literature as the *Raychaudhuri equation*.

$$\dot{\rho} + 3\frac{\dot{a}}{a}\left(\rho + \frac{p}{c^2}\right) = 0. \tag{1.9}$$

Note that the above conservation equation can also be obtained by differentiating Eq. (1.5) w.r.t. t, then multiplying through by $\frac{1}{8\pi G}$ and adding the result to Eq. (1.7) multiplied through by $-\frac{3}{8\pi G}\left(\frac{\dot{a}}{a}\right)$. Thus, the field equations of the theory incorporates the equation for the conservation of energy. Note that we shall frequently work with natural units for G and the speed of light c, i.e., $G = 1 = c$ or any other relevant unit systems (mentioned at appropriate places) in order to simplify expressions or calculations, without any loss of generality.

It is important to note that the EFE can be derived mathematically from the Einstein–Hilbert action (R is the Ricci scalar and S_{matter} is the matter action)

$$S = \int \frac{R}{16\pi G}\sqrt{-g}\,d^4x + S_{matter} \tag{1.10}$$

by using the principle of least action.

1.3 Observable Parameters

It is a standard practice to specify cosmological models via a few parameters, which are then determined by analyzing observational data so as to identify the best model of the Universe. We now define and discuss the implications of three most important observable parameters of our Universe — Hubble parameter, deceleration parameter, and density parameter.

The Hubble Parameter *H*

The rate at which the Universe expands is measured by the *Hubble parameter H* which is defined as

$$H = \frac{\dot{a}(t)}{a(t)}. \tag{1.11}$$

The *Hubble constant*, denoted by H_0, is interpreted as the value assumed by the Hubble parameter at the present epoch a_0. Observations show that $H_0 = 70 \pm 10$ km/sec/Mpc (Mpc stands for megaparsec and 1 Mpc $= 3.09 \times 10^{24}$ cm). As the value of H_0 is still a bit uncertain, we often parameterize it as

$$H_0 = 100\,\mathrm{h\,km/s/Mpc} \tag{1.12}$$

so that $h \approx 0.7$. It is an easy exercise to note that in terms of the Hubble parameter H and its first order time-derivative $\dot{H}$, the Friedmann equation (1.5) and the acceleration Eq. (1.7) can be written as (choosing $c = 1$)

$$H^2 + \frac{k}{a^2} = \frac{8\pi G}{3}\rho, \tag{1.13}$$

and

$$\dot{H} + H^2 = -\frac{4\pi G}{3}(\rho + 3p) \tag{1.14}$$

respectively, while the energy conservation Eq. (1.9) becomes

$$\dot{\rho} + 3H(\rho + p) = 0. \tag{1.15}$$

The Deceleration Parameter *q*

Another important observable parameter of our Universe is the *deceleration parameter* q which is defined as

$$q = -\frac{a\ddot{a}}{\dot{a}^2} \tag{1.16}$$

such that a positive q implies deceleration, while a negative q implies acceleration. Thus, the deceleration parameter measures the rate of change of the rate of expansion of the Universe.

The Density Parameter *Ω*

The *density parameter* Ω is a dimensionless quantity which is defined as (choosing $c = 1$)

$$\Omega = \frac{8\pi G}{3H^2}\rho = \frac{\rho}{\rho_c}, \tag{1.17}$$

where the *critical density* ρ_c is defined by

$$\rho_c = \frac{3H^2}{8\pi G}. \tag{1.18}$$

The Friedmann equation (1.13) is expressed in terms of the density parameter as

$$\Omega - 1 = \frac{k}{H^2 a^2}. \tag{1.19}$$

The sign of κ is therefore determined by Ω and we have

$$\rho < \rho_c \Leftrightarrow \Omega < 1 \Leftrightarrow k < 0$$
$$\rho = \rho_c \Leftrightarrow \Omega = 1 \Leftrightarrow k = 0$$
$$\rho > \rho_c \Leftrightarrow \Omega > 1 \Leftrightarrow k > 0.$$

The density parameter, then, determines which of the three RW geometries describes our Universe. Thus, it is important to determine it observationally. Measurements of the CMB anisotropy by de Bernardis et al. (2000) in the BOOMERANG collaboration and Spergel et al. (2003) in the WMAP collaboration lead us to believe that Ω is very close to unity and hence our Universe can be considered to be spatially flat (at large scales).

1.4 Solving the Equations

In order to solve the Eqs. (1.5), (1.7), and (1.9) i.e., to determine how the scale factor $a(t)$ evolves, it is necessary to know the relation between the pressure p and the energy density ρ. The fluid approximation used here allows us to assume that p is a single-valued function of ρ given by $p = p(\rho)$ which is termed as the *equation of state* (EoS). Consequently, an *equation of state parameter* w may be introduced such that

$$p = w\rho. \tag{1.20}$$

Many useful cosmic fluids obey this relation with a constant w. In fact, a constant w simplifies our equations to a great extent. We also observe that on employing the energy conservation Eq. (1.9), the energy density ρ is found to be proportional to the scale factor in a power law fashion as

$$\rho \propto \frac{1}{a^{3(1+w)}}. \tag{1.21}$$

Now, assuming a flat universe and a constant EoS parameter $w \neq -1$, the Friedmann equation (1.5) can be solved exactly—

$$a(t) = a_0 \left(\frac{t}{t_0}\right)^{\frac{2}{3(1+w)}}, \tag{1.22}$$

where $a_0 = a(t_0)$ is the scale factor at the present epoch. Note that for $w = -1$, one obtains $a(t) \propto e^{Ht}$. We shall conclude this section by considering two kinds of fluids which have a constant EoS parameter w and determining how the energy density and the scale factor evolve in each case.

Dust

Dust is non-relativistic matter which exerts a negligible pressure so that we can assume $p = 0$ (which implies $w = 0$). A pressureless universe is a good approximation for the atoms in the Universe once it has cooled down owing to the fact that they are quite well-separated and rarely interact. It also provides a nice description of a collection of galaxies in the Universe as they are subjected to gravitational interactions only. Therefore, in this case, Eqs. (1.21) and (1.22) respectively become

$$\rho \propto \frac{1}{a^3} \qquad \text{and} \qquad a(t) = a_0 \left(\frac{t}{t_0}\right)^{\frac{2}{3}}. \tag{1.23}$$

Radiation

Particles moving at highly relativistic speeds, such as neutrinos, fall in this category. Their kinetic energy gives rise to a pressure force, known as the radiation pressure, which can be shown to be $p = \frac{\rho}{3}$ using the standard theory of radiation. So, in this case, $w = \frac{1}{3}$ and we obtain

$$\rho \propto \frac{1}{a^4} \qquad \text{and} \qquad a(t) = a_0 \left(\frac{t}{t_0}\right)^{\frac{1}{2}} \tag{1.24}$$

from Eqs. (1.21) and (1.22) respectively.

1.5 Cosmological Horizons: Particle, Event, and Apparent

One of the most crucial feature of FLRW models is the existence of different cosmological horizons. There is a fundamental limit to our vision, since no particle has velocity greater than the velocity of light. The finite speed of light leads to the concept of horizons and limits our ability to comprehend the entire Universe. The term "horizon" is used in different contexts in the literature, often without clear definition, however in layman terms, a horizon is a one-sided membrane which blocks information from a family of observers. The three most common cosmological horizons found in the literature are particle horizon, event horizon, and apparent horizon which we shall discuss in this section. For a diagrammatic representation of horizons, the reader may see Chap. 2 of V. Mukhanov (2005). For a comprehensive study of horizons, the reader is referred to the book by V. Faraoni (2015), and references therein.

Particle Horizon

A *particle horizon* is defined as the maximum proper distance over which a comoving observer can have causal communication at any epoch t. Thus, the particle horizon encompasses every signal accessible to a comoving observer at $r = 0$ between the time of beginning of the Universe at $t = 0$ and the time t. The age of the Universe is roughly 14 billion years which gives a naive estimate for the particle horizon to be 14 billion light years. Mathematically, the particle horizon at any time t as seen by a comoving observer at $r = 0$ is the spherical surface with center at $r = 0$ and having proper radius (considering radial null geodesics $r = r(t)$ in a flat FLRW spacetime)

$$R_P = a(t) \int_0^t \frac{dt'}{a(t')}. \tag{1.25}$$

Thus, the particle horizon exists only when this improper integral converges. Note that the particle horizon is physically relevant in the perspective of cosmic inflation, a phenomenon of the early Universe first conceived by Guth (1981) with a view to solve the *horizon* and *flatness* problems.

Event Horizon

An *event horizon* is defined as the maximum proper distance over which a comoving observer at any epoch t can have causal communication at a sufficiently large cosmic time. In other words, it is the boundary of the spacetime region which comprises all the events accessible to a comoving observer at $r = 0$ between the present time t and the future infinity $t = \infty$. The proper radius of an event horizon is given by (considering radial null geodesics $r = r(t)$ in a flat FLRW spacetime)

$$R_E = a(t) \int_t^\infty \frac{dt'}{a(t')}. \tag{1.26}$$

In other words, R_E is the proper distance to the most distant event the observer will ever see. If the improper integral in Eq. (1.26) is convergent, then the observer at $r = 0$ will be totally unaware of the events beyond R_E, whereas if it is divergent, the observer will be aware of events arbitrarily far away from her (may be at a sufficiently large time). Note that the cosmological event horizon is observer dependent in contrast to BH event horizon which is absolute in nature.

Apparent Horizon

An *apparent horizon*[8] is defined geometrically as an imaginary surface beyond which null geodesic congruences[9] recede from the observer. Note that the FLRW metric can be locally expressed in the form

$$ds^2 = h_{ij}(x^i)dx^i dx^j + R^2 d\Omega_2^2, \tag{1.27}$$

where i, j can take values 0 and 1. The two dimensional metric

$$d\gamma^2 = h_{ij}(x^i)dx^i dx^j, \tag{1.28}$$

where

$$h_{ij} = \text{diag}\left(-1, \frac{a^2}{1 - kr^2}\right) \tag{1.29}$$

is known as the normal metric, while $R = ar$ is interpreted as the area radius. If we construct another scalar associated with this normal space as

$$\chi(t) = h^{ij}(x^i)\partial_i R \partial_j R = 1 - \left(H^2 + \frac{k}{a^2}\right) R^2, \tag{1.30}$$

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[8] For an extensive discussion on its physical reality and its relevance in Cosmology, the readers may see the recent paper by Melia (2018).

[9] I suggest the readers to go through the first three chapters of the book by E. Poisson (2004) to have a basic understanding of Differential Geometry and related concepts which are relevant to this book.

then the apparent horizon of a comoving observer is defined in mathematical terms as a spherical surface located at the vanishing of this scalar, i.e., $\chi(t) = 0$, and we obtain

$$R_A = \frac{1}{\sqrt{H^2 + \frac{k}{a^2}}}, \tag{1.31}$$

which reduces to

$$R_A = \frac{1}{H} \tag{1.32}$$

in a flat universe. It is also worthwhile to mention that for a flat universe. the apparent horizon coincides with the *Hubble horizon*, which roughly measures the distance that light would travel if space were not expanding. Unlike the particle and event horizons, the apparent horizon is not, in general, a null surface. For a universe filled with a perfect fluid with a constant EoS $p = w\rho$, the apparent horizon will be null if and only if either $p = -\rho$ (phantom fluid) or $p = \frac{1}{3}\rho$ (radiation) (for a proof, see p. 80 of the book by V. Faraoni).

1.6 The Present Universe: Dark Energy

The study of Cosmology has taken a new turn in recent years since the research teams led by Riess et al. and Perlmutter et al. independently reported in 1998 that our Universe is accelerating. The responsible factor behind this late-time cosmic acceleration has been named[10] *"dark energy"* (DE). It is a startling fact that DE is distributed evenly throughout the Universe, not only in space but also in time. In other words, the expansion of the Universe does not dilute its effect. Consequently, DE does not have any local gravitational effects, but rather a global effect on the Universe as a whole. This leads to a repulsive force which acts as a mechanism to accelerate the expansion of the Universe. However, in spite of extensive research over the last two decades, its nature and origin has still been a mystery to cosmologists. Independent observational data from sources such as Supernovae Type Ia (SNe Ia), CMB, and Baryon Acoustic Oscillations (BAO) have shown that about 68% of the present energy of the Universe consists of DE, the other ingredients being ordinary matter or baryons (approximately 5%) and dark matter[11] (DM) (about 27%). Thus, we are aware of only 5% of the Universe, the rest 95% is invisible.

The acceleration Eq. (1.7) tells us that the Universe consisting of only a single fluid component will accelerate if $\rho + 3p < 0$, i.e., if $w < -\frac{1}{3}$. So it is evident that such

[10]The term "dark energy" was coined by Michael Turner in 1998 motivated by Fritz Zwicky's "dark matter" from the 1930s.

[11]Dark matter is an exotic, invisible substance which interacts neither with baryonic matter nor with electromagnetic radiation, thereby making it impossible to detect with current instruments. Its existence can be understood by the gravitational effects it appears to have on galaxies and galaxy clusters and also by the effects of gravitational lensing.

an acceleration requires the pressure to be sufficiently negative. Thus, the EoS for DE can be written as $p = w\rho$, where w is a constant and $w < -\frac{1}{3}$. Fluids constrained by $\rho + 3p \geq 0$ or $w \geq -\frac{1}{3}$ are said to satisfy the *strong energy condition* (SEC). We therefore conclude that DE must violate the SEC in order to allow the Universe to accelerate.

At this point, it is worth mentioning that there exist alternative approaches to explain the cosmic acceleration. The most widely accepted one among them is by modifying the geometric (i.e, l.h.s.) part of the Einstein equations (known as "modified gravity models" (MGMs)), however, in this book, we shall stick to Einstein gravity (GR) only. Other interesting approaches include gravitationally induced particle creation models (discussed in Chap. 5) and considering exact inhomogeneous cosmological models by abandoning the cosmological principle (discussed in Chap. 6). For the benefit of the readers, I wish to mention here that most of the discussions in Sect. 1.6.1–1.6.5 have been inspired from the books by Amendola and Tsujikawa (2010) and Matarrese et al. (eds.) (2011), and the extensive reviews by Carroll (2001), Copeland et al. (2006), Frieman et al. (2008), and Sahni (2004).

1.6.1 The Simplest Candidate: The Cosmological Constant

The simplest candidate which can play the role of DE is the *cosmological constant* Λ for which the EoS parameter w takes the value -1. It was originally introduced in 1917 by Einstein in his field equations with an aim to obtain a static universe. Realizing it to be the "biggest blunder" of his life, he dropped it in 1929, when Hubble discovered that the Universe is expanding. There was nothing to regret after all as the cosmological constant was revived again as an entity responsible for the late-time acceleration of the Universe. The cosmological constant is sometimes physically interpreted as the energy density of 'empty' space. This is related to the 'zero-point energy' (in quantum physics) which is relevant even without the presence of particles. Unfortunately, particle physics theories predict a much larger value for the cosmological constant than observations allow (its value as obtained from observations is one divided by one followed by 123 zeroes!). This discrepancy is referred to as the *cosmological constant problem* (CCP) and is one among many unsolved problems in elementary particle physics. Then there is the *cosmic coincidence problem* which deals with the intriguing question of "why the energy densities of pressureless matter and DE are of the same order precisely at the present epoch although the individual evolutionary properties of these fluid sources are quite different?" Several models such as decaying vacuum models, interacting scalar field descriptions of DE, and a single fluid model with an antifriction dynamics have been proposed with a view to alleviate such problems.

The EFE (1.3), in presence of a positive Λ, become (choosing $c = 1$)

$$G_{\mu\nu} - \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}, \tag{1.33}$$

so that the Friedmann equation (1.5) and the acceleration Eq. (1.7) take the forms

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} = \frac{8\pi G}{3}\rho + \frac{\Lambda}{3} \tag{1.34}$$

and

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) + \frac{\Lambda}{3} \tag{1.35}$$

respectively. Since $w = -1$ for cosmological constant, so we must have $p = -\rho$. In that case, one can evaluate the scale factor as

$$a(t) = \begin{cases} a_0 \cosh\left(\sqrt{\frac{\Lambda}{3}}t\right), & k = +1 \\ a_0 \exp\left(\sqrt{\frac{\Lambda}{3}}t\right), & k = 0 \\ a_0 \sinh\left(\sqrt{\frac{\Lambda}{3}}t\right), & k = -1 \end{cases} . \tag{1.36}$$

All the solutions stated above expand exponentially in the limit $t \to \infty$, independently of the spatial curvature. In fact, all the solutions correspond to *de Sitter*[12] *spacetime*, in different coordinate systems. It is interesting to note that, in a flat de Sitter universe, the event and apparent horizons coincide. For an extensive study on the cosmological constant, the readers may see the reviews by Padmanabhan (2003) and Peebles and Ratra (2003).

The following subsections deal with alternative models of DE, assuming that the CCP is resolved in a way such that Λ vanishes completely. We shall discuss "modified matter models" (MMMs) in which the EM tensor $T_{\mu\nu}$ on the r.h.s. of the EFE contain an exotic matter source having a negative pressure. The most interesting models under this classification include quintessence, K-essence, phantom (ghost) field, coupled DE, and Chaplygin gas (and its variations).

1.6.2 Quintessence

A DE candidate described by a canonical scalar field ϕ is known as *quintessence*. Unlike the cosmological constant, the EoS of quintessence is dynamic in nature. The action for quintessence is described by

$$S = \int d^4x\sqrt{-g}\left[\frac{R}{16\pi G} - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right] + S_{matter}, \tag{1.37}$$

where R is the Ricci scalar and ϕ is a scalar field with a potential $V(\phi)$, while S_{matter} is the matter action. The energy density ρ_ϕ and the pressure p_ϕ associated with the field are given by

[12] A negative cosmological constant ($\Lambda < 0$) leads to *anti de Sitter* (AdS) spacetime. Such a spacetime is possible only for $k = -1$.

$$\rho_\phi = \frac{\dot{\phi}^2}{2} + V(\phi) \quad \text{and} \quad p_\phi = \frac{\dot{\phi}^2}{2} - V(\phi), \tag{1.38}$$

respectively. The conservation equation, $\dot{\rho}_\phi + 3H(\rho_\phi + p_\phi) = 0$ reduces to

$$\ddot{\phi} + 3H\dot{\phi} + \frac{dV}{d\phi} = 0. \tag{1.39}$$

The EoS of the scalar field is

$$w_\phi = \frac{p_\phi}{\rho_\phi} = \frac{\dot{\phi}^2 - 2V(\phi)}{\dot{\phi}^2 + 2V(\phi)}. \tag{1.40}$$

The Friedmann equation (1.5) and the acceleration Eq. (1.7) become

$$H^2 = \frac{8\pi G}{3}\left[\frac{\dot{\phi}^2}{2} + V(\phi) + \rho_{\text{matter}}\right] \tag{1.41}$$

and

$$\dot{H} + H^2 = -\frac{8\pi G}{3}\left[\dot{\phi}^2 - V(\phi) + \rho_{\text{matter}}\right], \tag{1.42}$$

respectively.

Note that at early times, $\rho_{\text{matter}} \gg \rho_\phi$, however, ρ_ϕ needs to dominate at late times so as to explain the late-time cosmic acceleration. The latter happens if $w_\phi < -\frac{1}{3}$, which from Eq. (1.40) implies $\dot{\phi}^2 < V(\phi)$. Thus, the scalar potential should be flat enough so that the scalar field evolves slowly. If the slowly rolling scalar field satisfying the condition $\dot{\phi}^2 \ll V(\phi)$ is the dominant contribution to the energy density of the Universe, we obtain the approximate relations $3H\dot{\phi} + \frac{dV}{d\phi} \simeq 0$ and $3H^2 \simeq 8\pi G V(\phi)$ from Eqs. (1.39) and (1.41) respectively. Therefore, the field EoS in Eq. (1.40) can be approximated as

$$w_\phi \simeq -1 + \frac{2\varepsilon_s}{3}, \tag{1.43}$$

where $\varepsilon_s = \frac{1}{16\pi G}\left(\frac{dV/d\phi}{V}\right)^2$ is the *slow-roll parameter*. ε_s is much less than unity during the accelerated expansion of the Universe because the potential is sufficiently flat. Thus, $-1 < w_\phi < -\frac{1}{3}$.

1.6.3 Phantom Field

Current observational data, such as from the PLANCK satellite, reveal the possibility of the DE EoS to be slightly below -1. These DE fluids are generally classified as *phantom* fluids. Phantom DE is associated with a scalar field governed by a negative kinetic energy which rolls up the potential. The energy density of the scalar field grows indefinitely if the potential is unbounded from above. Phantom scalar fields were first introduced by Hoyle in his steady state theory.

The action of the phantom field minimally coupled to gravity is given by

$$S = \int d^4x\sqrt{-g}\left[\frac{R}{16\pi G} + \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right] + S_{matter}. \tag{1.44}$$

Note that the kinetic term is opposite in sign as compared to the action (1.37) for an ordinary scalar field. As the energy density and the pressure have the expressions given by

$$\rho_\phi = -\frac{\dot{\phi}^2}{2} + V(\phi) \quad \text{and} \quad p_\phi = -\frac{\dot{\phi}^2}{2} - V(\phi) \tag{1.45}$$

respectively, the EoS parameter becomes

$$w_\phi = \frac{p_\phi}{\rho_\phi} = \frac{\dot{\phi}^2 + 2V(\phi)}{\dot{\phi}^2 - 2V(\phi)}. \tag{1.46}$$

Then, one has $w_\phi < -1$ for $\frac{\dot{\phi}^2}{2} < V(\phi)$.

1.6.4 Holographic Dark Energy

As the energy densities of DE and DM are of the same order[13] at the present epoch, it may be expected that DE may have some relation (terms like *interaction* or *coupling* can also be used) with DM. The most widely accepted approach is to introduce an interaction term on the r.h.s. of the conservation equations for DE and DM. Such an interaction term contains a dimensionless coupling parameter. Interaction between the dark sectors is presumed to be a possible way to resolve the cosmic coincidence problem. For instance, one may see the paper by del Campo et al. (2009).

The form of coupled DE which we shall discuss in this subsection under the name of *holographic DE* (HDE). This model is motivated by the *holographic principle* which states that the number of degrees of freedom of a bounded physical system is proportional to the area of its boundary. In this context, Cohen et al. (1999) established a relationship between a short distance (ultraviolet) cutoff[14] and a long distance

[13]The matter era should have dominated for a sufficient amount of time so as to allow structure formation which means that the DE could not have begun to dominate very early. The fact that it is relevant exactly at the present time is what we have already referred to as the *cosmic coincidence problem* (in Sect. 1.6.1) and is one of the greatest mysteries of contemporary Cosmology.

[14]Generally, a cutoff is referred to as a threshold value for a physical quantity such as energy, momentum, or length. Cutoffs are introduced in order to prevent singularities from appearing during calculations. Note that, in the context of HDE, the traditional terms “infrared” and “ultraviolet” do not literally refer to specific regions of the spectrum.

(infrared) cutoff in quantum field theory (QFT) based on the limit set by the formation of a black hole (BH). This leads to an upper bound on the zero-point energy density. Subsequently, Hsu (2004) interpreted this energy density as the HDE density satisfying $\rho_d = \frac{3c^2}{L^2}$, where c (the readers should be careful with this notation, which is similar to that of the speed of light) is a dimensionless parameter which is determinable from observational data and L is an infra-red (IR) cutoff in units $M_p^2 = 1$ (M_p is the Planck mass). It was demonstrated by Li (2004) that the precise EoS for DE can be obtained and the late-time accelerating Universe can be incorporated into the HDE only if L is assumed to be the radius of the future event horizon. Further, Das et al. (2006) and Amendola et al. (2006) showed that, unlike noninteraction models, an interaction (between dust form of DM and HDE) model of the Universe is consistent with the observationally measured phantom EoS. The interaction models are also favored by a range of observational data, most importantly by the CMB and matter distribution at large scales. Moreover, they incorporate the transition of DE across the phantom barrier of $w_d = -1$ as demonstrated by Wang et al. (2005). The total energy density is $\rho = \rho_m + \rho_d$, where ρ_m and ρ_d are the energy densities of DM and DE respectively. The total pressure is given by $p = p_m + p_d$, where p_m and p_d are the pressures of DM and DE respectively. Now as DM is considered to be in the form of dust, so we have $p = p_d$. Since the IR cutoff is chosen as the radius of the future event horizon, $R_E = a\int_a^\infty \frac{dx}{Hx^2}$, so we have $\rho_d = 3c^2 R_E^{-2}$. The conservation equations for DM and DE are written as

$$\dot{\rho}_m + 3H\rho_m = Q \tag{1.47}$$

and

$$\dot{\rho}_d + 3H(1 + w_d)\rho_d = -Q \tag{1.48}$$

respectively, where w_d is the EoS of DE and Q denotes the interaction term which is usually assumed to be of the form $3b^2 H\rho$ with b^2 as the coupling constant. Taking $y = \frac{\rho_m}{\rho_d}$ as the ratio[15] of the energy densities, from (1.47) and (1.48), we have

$$\dot{y} = 3b^2 H(1 + y)^2 + 3Hyw_d. \tag{1.49}$$

Now, choosing $8\pi G = 1$ and $\kappa = 0$ (i.e., a flat universe), the Friedmann equation (1.13) can be written as $\Omega_m + \Omega_d = 1$, where $\Omega_m = \frac{\rho_m}{3H^2}$ and $\Omega_d = \frac{\rho_d}{3H^2}$. Then we have $y = \frac{1-\Omega_d}{\Omega_d}$ and $\dot{y} = -\frac{\dot{\Omega}_d}{\Omega_d^2}$. Comparing with (1.49), we get the EoS of DE as

$$w_d = -\frac{\Omega_d'}{3\Omega_d(1 - \Omega_d)} - \frac{b^2}{\Omega_d(1 - \Omega_d)}, \tag{1.50}$$

where the 'dot' denotes the derivative with w.r.t. time t and the 'prime' denotes the derivative w.r.t. $\ln a$. In this model, R_E can also be expressed as $R_E = \frac{c}{\sqrt{\Omega_d}H}$. Taking its derivative w.r.t. t and using the expressions for y and $\dot{y}$, as well as Eq. (1.49), we

[15] This ratio is a constant if Hubble radius is chosen as the IR cutoff.

have

$$\frac{\Omega_d'}{\Omega_d^2} = (1 - \Omega_d)\left[\frac{1}{\Omega_d} + \frac{2}{c\sqrt{\Omega_d}} - \frac{3b^2}{\Omega_d(1 - \Omega_d)}\right]. \tag{1.51}$$

The EoS of DE now reduces to

$$w_d = -\frac{1}{3} - \frac{2}{3c}\sqrt{\Omega_d} - \frac{b^2}{\Omega_d}. \tag{1.52}$$

The effective EoS (w_{eff}) of HDE is given by

$$w_{eff} = \frac{p}{\rho} = \frac{p_d}{\rho_m + \rho_d} = w_d \Omega_d. \tag{1.53}$$

The deceleration parameter in this model takes the form

$$q = \frac{1}{2} - \frac{3b^2}{2} - \frac{\Omega_d}{2} - \frac{\Omega_d^{\frac{3}{2}}}{c}. \tag{1.54}$$

It must be noted that although the interacting HDE model successfully describes the late-time accelerating Universe with the transition EoS of DE, it is inappropriate to describe the past deceleration phase of the Universe.

1.6.5 Chaplygin Gas Models

There exist another interesting class of DE models associated with a fluid known as the *Chaplygin gas*, named after the Russian scientist Sergey Chaplygin. Kamenshchik et al. (2001) were the first to investigate its role in understanding the late-time acceleration of the Universe. The simplest form of its EoS is

$$p = -\frac{A}{\rho}, \tag{1.55}$$

where A is a positive constant. With this EoS, the conservation Eq. (1.15) can be integrated to give

$$\rho = \sqrt{A + \frac{D}{a^6}}, \tag{1.56}$$

where D is an arbitrary constant. Then for small values of a, $\rho \sim \frac{\sqrt{D}}{a^3}$ and $p \sim -\frac{A}{\sqrt{D}}a^3$ which implies that the gas behaves as a pressureless dust (DM) at early times. For large values of a, $\rho \sim A$ and $p \sim -A$, i.e., it acts as a cosmological constant at late times which eventually leads to an accelerating Universe.

The advantages of the Chaplygin class of cosmological models is three-fold. Firstly, they describe a smooth transition from the decelerating phase of the Universe to the present phase of cosmic acceleration and such a transition is achieved with only one fluid. Secondly, they provide an interesting possibility for the unification of DE and DM. Finally, they represent the simplest deformations of the ΛCDM[16] model. However it was shown by Amendola et al. that the Chaplygin gas models face a huge challenge from observed CMB anisotropies. This problem can be overcome if the *generalized Chaplygin gas* (GCG) proposed by Bento et al. (2002) is considered which has the EoS

$$p = -\frac{A}{\rho^{\alpha}}, \quad 0 < \alpha < 1. \tag{1.57}$$

Nevertheless, the parameter α is severely constrained ($0 < \alpha < 0.2$) at the 95% confidence level WMAP observational data as shown by Amendola et al. (2003).

The GCG model was later modified by Benaoum (2002) and named the *modified Chaplygin gas* (MCG) model having EoS

$$p = \gamma\rho - \frac{B}{\rho^{n}}, \tag{1.58}$$

where $\gamma \leq 1$, B and n are positive constants. The MCG model can be reduced to the GCG model by a suitable choice of the parameters involved. Debnath et al. (2004) have shown that in this model the Universe decelerates for small values of a and accelerates for large values of a. Also, this model can describe the evolution of the Universe starting from the radiation era ($\gamma = \frac{1}{3}$ and ρ is very large) up to the ΛCDM era (ρ is a constant having a small magnitude). Thus, the MCG is able to explain the evolution of the Universe to a larger extent than the Chaplygin gas or the GCG models. Wu et al. (2007) studied the dynamics of the MCG model, while Bedran et al. (2008) investigated the evolution of the temperature function in the presence of a MCG. The model has also been found to be consistent with perturbative studies (Costa et al. 2008) and the spherical collapse problem (Debnath and Chakraborty 2008). Various other modifications of Chaplygin gas have appeared in the literature such as variable Chaplygin gas, holographic and interacting holographic Chaplygin gas, viscous Chaplygin gas models mongst others, however, each one of them comes with both merits and demerits as far as Cosmology is concerned. Chaplygin gas models have also been considered in modified as well as higher-dimensional gravity theories.

1.7 Final Remarks

Finally, it would be nice to mention that the term *concordance model* has been introduced in Cosmology in order to identify the most widely accepted cosmological model in present time. A concordance model is important because the measurement

[16] See Sect. 1.7 for explanation.

of many astronomical quantities such as distance, radius, luminosity, and surface brightness is dependent upon the cosmological model considered. Currently, the concordance model is the ΛCDM model in which the Universe contains a cosmological constant Λ, associated with DE, and cold (i.e., pressureless) DM (abbreviated CDM). Despite Λ having serious problems as discussed earlier, recent observational data prefers the ΛCDM model over alternative cosmological models such as MMMs, MGMs, and inhomogeneous cosmologies, which challenge the assumptions of the ΛCDM model. Therefore, the general picture of evolution of the Universe can be understood by the following flowchart: big bang $\longrightarrow$ inflationary era (early accelerating phase) $\longrightarrow$ radiation dominated era (decelerating phase) $\longrightarrow$ matter dominated era (decelerating phase) $\longrightarrow$ ΛCDM era (late-time accelerating phase). The ultimate fate of the Universe is determined by its density (or curvature) and its final state may be *big freeze* (also known as *heat death*), *big crunch*, or *big rip*.

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Chapter 2
A Brief Overview of Equilibrium Thermodynamics

Abstract In this chapter, the fundamentals of equilibrium thermodynamics are briefly examined. The basic terminology is followed by the discussion of the thermodynamic concepts of temperature, internal energy, entropy, absolute entropy, and enthalpy. Finally, the four laws of thermodynamics are explained in detail.

Keywords Thermodynamics · Isolated system · Thermodynamic equilibrium · Temperature · Entropy · Internal energy · Zeroth law · First law · Second law · Third law

Thermodynamics, in the narrow sense, means the physics of matter in equilibrium (or nonequilibrium), with regard to changes in temperature, pressure, and chemical composition. In other words, it is the study of energy and its transformations. Thermodynamics deals with the interaction of energy with matter and the transformation of energy from one form to another. Its purely deductive conclusions about the nature of the interactions of energy and matter mimic the real energy and matter interactions in all physical and chemical phenomenon in which such interactions can be tested. Thus, thermodynamics can be considered as a powerful tool to describe as well as predict the course of real phenomenon. This field of study is extremely general in the sense that no hypotheses are assumed regarding the nature and structure of matter to be dealt with. Thermodynamics has unlimited practical uses such as in computation of the maximum efficiency of engines, prediction of the amount of heat required to carry out physical and chemical processes, estimation of the temperature of flames and the heat effects of chemical reactions, calculation of the conditions of chemical and phase equilibrium, establishment of relationships between properties of materials, and many more. In this chapter, a brief overview of the basic concepts and the laws of (equilibrium) thermodynamics are presented. Most of the discussions undertaken in this chapter are inspired from the extensive lecture notes by Prof. J.J. Kelly, and to a lesser extent from the books by Callen (1960), Sussman (1972), and Zemansky and Dittman (1997). Later, in Chap. 5, nonequilibrium thermodynamics in the context of matter creation will be discussed.

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2.1 Definitions and Basic Concepts

System, Boundary, and Surroundings

A closed surface which separates a restricted region of space or a finite portion of matter from its surroundings is called the *boundary*. The region within the arbitrary boundary which forms the center of our study, is called the *system*. Everything outside the system that has a direct influence on the behavior of the system is called the *surroundings*. The surroundings may be considered as an environment which imposes certain conditions the system of interest (such as constant temperature, pressure, chemical potential etc.).

Open and Closed Systems

A system which has an interaction of both matter and energy[1] with its surroundings is said to be an *open* system. On the other hand, a *closed* system is one which allows an exchange of energy but no exchange of matter with its surroundings. Closed systems, therefore have a fixed mass and composition but variable energy levels.

Isolated System

A system which has absolutely no interaction (of either matter or energy) with its surroundings is said to be an *isolated* system. These systems have rigid, perfectly insulating walls or boundaries that are impermeable to matter. Their mass and energy content, therefore remains constant over time.

After a system has been chosen, it must be described in terms of quantities (or properties) related to the behavior of the system or its interactions with the surroundings, or both. These properties are generally of two types which are defined below.

Macroscopic and Microscopic Properties

The *macroscopic* properties refer to variables or parameters of a system at approximately the human scale, or larger, such as mass, composition, temperature, pressure, and volume. They are few in number and can be directly measured. The *microscopic* properties refer to variables or parameters of a system at approximately the molecular scale, or smaller, like the structure of its constituent particles, their motions, their energy states, their interactions, etc. Unlike macroscopic properties, they are many in number and cannot be directly measured, but must be calculated.

Any physical system composed of matter or radiation large enough to be described by macroscopic parameters without taking its microscopic constituents into consideration is said to be a *thermodynamic system*. Such a system always includes the notion of temperature, which distinguishes thermodynamics from other macroscopic branches of science, such as geometrical optics, mechanics, or electricity and magnetism. Thermodynamic transformations can be either *reversible* or *irreversible*

[1] Here, energy generally refers to heat energy.

depending upon whether the thermodynamic system returns or does not return to its initial state when external constraints are applied to its surroundings.

Intensive and Extensive Properties

Intensive properties of a thermodynamic system (such as temperature) are those that are independent of the mass of the system, while those properties (such as internal energy[2]) that are proportional to the mass of the system are said to be *extensive*.

Homogeneous and Inhomogeneous Systems

A system within which the intensive thermodynamic properties are the same everywhere is said to be a *homogeneous* system, while an *inhomogeneous* system is one which shows spatial variations in one or several of its intensive properties.

Thermodynamic Equilibrium

A system is said to be in a state of *thermodynamic*[3] *equilibrium* if the conditions for all three types of equilibrium, namely, mechanical, chemical, and thermal, are satisfied. It is apparent that, in thermodynamic equilibrium, there will be no tendency whatsoever, either of the system or of the surroundings to change their state. When the conditions for any one of the three types of equilibrium (mentioned above) are violated, the system is said to be in a *nonequilibrium* state.

Thermodynamic Equation of State and Equilibrium Surface

A functional relationship among the properties[4] of an equilibrium system is said to be its *thermodynamic equation of state*. If it is possible to describe completely the state of a particular system by the three parameters, namely, pressure (p), volume (V), and temperature (T), then the thermodynamics EoS becomes

$$f(p, V, T) = 0, \tag{2.1}$$

which reduces the number of independent variables by one. An equilibrium state can be represented as a point on the surface described by the EoS which is known as the *equilibrium surface*. Any point not a part of this surface is a nonequilibrium state of the system.

However, it must be noted that it is not possible for a general theory like thermodynamics, based on general laws of nature, to produce an EoS of a system. Such an EoS is generally an experimental addition to thermodynamics.

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[2]See Sect. 2.2.2 for explanation.

[3]This is different from *thermal equilibrium* which is a state achieved by two (or more) systems which have been in communication with each other through a *diathermic* wall (a rigid wall which allows the transmission of heat only).

[4]Terms like *variables* or *parameters* or *coordinates* can also be used.

2.2 The Laws of Thermodynamics

As with all branches of science, experimental observation forms the basis of thermodynamics too, and these observations have been incorporated into certain basic laws which are known as the zeroth, the first, the second, and the third laws of thermodynamics. The strength of this discipline lies in its ability to derive general relationships with the help of these elementary laws (or axioms) and a relatively small amount of empirical information without taking into account its microscopic structure.

2.2.1 Temperature: The Zeroth Law

Temperature can be regarded as an entity having equal magnitude in systems which are in thermal equilibrium. In other words, the temperature of a system is a property which determines whether a system is in thermal equilibrium with other systems or not.

The concepts of temperature and thermal equilibrium now lead us to the *zeroth law of thermodynamics* which states that *"if system A is in thermal equilibrium with system B and system B is in thermal equilibrium with system C, then systems A and C are in thermal equilibrium with each other."* The law is equivalent to the fundamental mathematical principle that "things equal to the same thing are equal to each other." R.H. Fowler, in 1931 realized that it was necessary to define thermal equilibrium before the first law of thermodynamics (FLT) could be postulated. Consequently, he was forced to adopt 'zero' as the number of his law.

2.2.2 Energy Conservation: The First Law

The *first law of thermodynamics*, which is basically an energy conservation law was formulated in 1848 by H. Helmholtz and W. Thomson based on experimental data collected by J. Joule and insight provided by J. Mayer. The law can be formally stated as *"the work required to change the state of an otherwise isolated system depends solely upon the initial and final states involved and is independent of the method used to accomplish this change."* The first law (FL) enables us to assign a property called *internal energy*[5] (E) to a system, which can be defined as the energy contained within the system with the exception of the kinetic and potential energies of the system as a whole due to motion of the system and external force fields respectively.

[5]The value of internal energy is dependent only on the present state of the system and not on the path taken or the processes undergone to prepare it. Hence, it is considered as a state function of a system.

It is a measure of the amount of energy gained or lost by the system. In an *adiabatic* process, $\delta Q = 0$, so the change in its internal energy equals the work (W) done on the system by its surroundings. However, for a closed system, the energy conservation is expressed as

$$\triangle E = Q - W, \tag{2.2}$$

where Q and W are, respectively, the amount of heat transferred to the system from its surroundings and that of work done by the system on its surroundings. The sign convention is due to Clausius and is relevant in studying heat engines, which perform useful work that is treated as positive. However, the IUPAC convention (due to physicists such as Max Planck) is mostly followed in recent times which formulates the FL in terms of the work done on the system. So, in this case, the FL for a closed system is written as

$$\triangle E = Q + W. \tag{2.3}$$

This convention assumes all net energy transfers to and from the system as positive and negative respectively, regardless of whether the system is used as an engine or some other device. Now, in a *quasistatic* (or adiabatic)[6] process, the work done by the system on its surroundings is the product pdV, while the work done on the system is $-pdV$. Either of the above two sign conventions show that the change in internal energy of the system is[7] given by

$$dE = \delta Q - pdV, \tag{2.4}$$

where δQ denotes the infinitesimal increase in the heat transferred to the system by its surroundings.

Another important quantity is the *enthalpy* $\bar{H}$ defined by[8]

$$\bar{H} = E + pV \tag{2.5}$$

and considered to be equivalent to the total heat content of a thermodynamic system. The total derivative of (2.5) is given by

$$d\bar{H} = dE + pdV + Vdp. \tag{2.6}$$

Now, on substituting dE with the expression in Eq. (2.4), we obtain

[6] A quasistatic process occurs so slowly that the system can always be assumed to be arbitrarily close to equilibrium. A reversible thermodynamic process should be necessarily quasistatic.

[7] In thermodynamics, d indicates a proper differential which is dependent only on the change of state, whereas δ indicates an improper differential which is also dependent on the process used to change the state.

[8] We shall use the notation $\bar{H}$ instead of the standard notation H for enthalpy so that it does not get mistaken for the Hubble parameter H.

$$d\bar{H} = \delta Q + Vdp \tag{2.7}$$

for a quasistatic process.

2.2.3 Entropy: The Second Law

Although the FLT guarantees energy conservation, many processes occur in nature which are inconsistent with this conservation law. For example, if an ice cube is placed in a glass of warm water, then the FL allows some internal energy of the ice to be transferred to the water such that the water is warmed while the ice cools, we never see such an event to happen in reality. Similarly, if equal parts of two gases are mixed, it is natural to see that the two gases will never separate spontaneously at a later time. The fact that these types of transformations are never spontaneous[9] forms the basis of the *second law of thermodynamics* (SLT), There are several equivalent statements of SLT. The statement due to Clausius is that *"there exists no thermodynamic process whose only effect is to extract heat from a colder reservoir and transfer it to a hotter one"*, whereas Kelvin stated that *"there exists no thermodynamic process whose only effect is to extract heat from a reservoir and to transform it entirely into work."* Thus, although heat and work are equivalent forms, they are not completely interconvertible. Other abstract statements of the second law, like the one by Caratheodory, have also been formulated, although it has less obvious connection to physical phenomena. The SLT leads us to the thermodynamic concept of *entropy*[10] (S), which is defined to be a state function of the extensive parameters of a system such that A. their values maximize S consistent with certain external constraints imposed on the system and B. the entropy of a composite system is additive over its constituent subsystems. The former is sometimes called the *maximum entropy principle* and it can be shown to be equivalent to the Clausius and Kelvin statements. Let us treat this principle to be the primary statement of the second law. Mathematically, the second law can be expressed as

$$dS \geq 0. \tag{2.8}$$

If a system undergoes an infinitesimal reversible change of state, then we have

$$dS = \frac{\delta Q}{T}. \tag{2.9}$$

[9] A *spontaneous* process is one which can occur in an isolated system, or between interacting systems at a finite rate without effects on, or help from, or interaction with, any other part of the Universe.

[10] In physical terms, entropy can be interpreted as the minimum number of bits needed to specify completely the state of a physical system.

Here, δQ is the quantity of heat received by the system and T is the absolute temperature. Thus, the entropy of an adiabatic system can never decrease and is a constant iff all processes are reversible.

2.2.4 Absolute Entropy: The Third Law

In 1906, W. Nernst proposed that *"the entropy of a pure perfect crystal can be taken to be zero at absolute zero temperature."* Later, this proposition was generalized as Nernst–Simon statement of the *third law of thermodynamics* as *"the change in entropy that results from any isothermal*[11] *reversible transformation of a condensed system approaches zero as the temperature approaches zero."* Mathematically, the Nernst–Simon statement is written as

$$\lim_{T \to 0} (\Delta S)_T = 0. \tag{2.10}$$

Alternatively, the statement implies that, at absolute zero, the entropy of any system in thermal equilibrium must approach a unique constant , called the *absolute entropy*. This observation leads to the Planck statement of the third law—*"As $T \to 0$, the entropy of any system in equilibrium approaches a constant value that is independent of all other thermodynamic variables."* Further, the third law can also be interpreted as unattainability theorem —*"there exists no process, no matter how idealized, capable of reducing the temperature of any system to absolute zero in a finite number of steps."*

2.3 Final Remarks

This chapter provides a concise overview of the relevant concepts as well the four laws of thermodynamics. The ability of thermodynamics in explaining real phenomenon provides strong evidence that the "real Universe" can be considered to be as logical and precise as the "thermodynamic Universe." The importance of thermodynamics can be understood by the following quote of Einstein in his Autobiographical Notes, *"It is the only physical theory of universal content, concerning which I am convinced that within the framework of the applicability of its basic concepts, it will never be overthrown."*

[11] An *isothermal* process is one which occurs at a constant temperature.

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Suggested Further Reading

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Chapter 3
Cosmological Thermodynamics

Abstract This chapter starts with the origin and a brief history of the application of the thermodynamic laws to (static) black holes and its subsequent generalization to Cosmology. The concepts of the generalized second law of thermodynamics, surface gravity and its connection with Hawking temperature, and Bekenstein entropy are carefully explained. Finally, the implications of the modified and the generalized Hawking temperatures are discussed with reference to the cosmological event horizon.

Keywords Generalized second law · Bekenstein entropy · Hawking temperature
Holographic principle · Surface gravity · Bekenstein system

3.1 Hawking Temperature and Bekenstein Entropy

The study of BHs during the last four decades has revealed evidence for an intimate connection among the theories of quantum mechanics, gravitation, and thermodynamics. This has led to the establishment of BH thermodynamics Bardeen et al. (1973) which shows that, classically, certain laws of BH mechanics bear a remarkable mathematical resemblance to the ordinary laws of thermodynamics applied to a system containing a BH. Classically, BHs are perfect absorbers but they do not emit anything. However, semiclassical description of BH physics indicates that a BH behaves like a black body and emits thermal radiation.

Stephen Hawking (1972), used GR to deduce the *horizon area theorem* which states that *the total horizon area in a closed system containing BHs never decreases*. A year later, Jacob Bekenstein (1973) identified the striking equivalence between Hawking's area theorem and the SLT which states that *the total entropy of a closed system never decreases*. He argued that the area of BHs strongly correlate with their entropy. This resemblance as well as the idea of information loss when an object falls into a BH led Bekenstein to propose that the entropy of a BH is proportional to the area of its event horizon.[1] Consequently, a *generalized second law* (GSL) should

[1]The event horizon of a BH is the boundary which marks the limits of the BH. Anything (even light) that falls within the event horizon can never escape.

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hold which states that *the entropy of matter located inside a BH plus a suitable multiple of the area of the BH never decreases*. However, Hawking was reluctant to acknowledge that this observation was more than just a mere coincidence. Hawking's argument was that a BH must have a temperature if it has entropy, and that implies that it must radiate like a black body. But if nothing can escape from a BH, how can it radiate? The Hawking–Bekenstein debate was finally settled when Hawking (1975) discovered a mechanism by which BHs radiate. He found that the quantum nature of particle creation allows a BH to radiate like a black body.

In mathematical terms, the entropy (henceforth *Bekenstein entropy*) and the temperature (henceforth *Hawking temperature*) associated with the event horizon of a BH are expressed as

$$S_{BH} = \left(\frac{c^3}{G\hbar}\right)\frac{A_h}{4} \quad \text{and} \quad T_{BH} = \left(\frac{\hbar c}{k_B}\right)\frac{\kappa_h}{2\pi}, \tag{3.1}$$

where $\hbar$ is the reduced Planck's constant and k_B is the Boltzmann's constant. κ_h and A_h are the *surface gravity*[2] on the horizon and the area bounded by the horizon respectively. The first law of BH thermodynamics connects the Hawking temperature and the Bekenstein entropy with the mass of the BH. Since temperature and entropy are manifestations of the spacetime geometry, i.e., the EFE, there could be a speculation about some relationship between BH thermodynamics and the EFE. As a matter of fact, Jacobson (1995) derived the Einstein equations from the first law of BH thermodynamics for all local Rindler[3] causal horizons, assuming the temperature to be the Unruh temperature measured by an accelerated observer located just inside the horizon. Subsequently, starting from the EFE, Padmanabhan (2002a, b) derived the first law of BH thermodynamics on the horizon for a general static spherically symmetric spacetime. For a detailed discussion on BH thermodynamics, interested readers may see the article by R.M. Wald (2001) and the book by E. Poisson (2004).

3.2 Thermodynamics in Cosmology

Inspired by early proposals made by 't Hooft (1993) and Susskind (1995), over the last one and a half decades or so, cosmologists have attempted to generalize this nice equivalence between the thermodynamic laws and the Einstein equations in the context of Cosmology. Indeed, spacetimes admitting horizons exhibit a

[2]The *surface gravity* can be defined as the local gravitational field strength experienced by a test particle at the surface of an astronomical body. This definition can be extended to include cosmological horizons as well. The surface gravity for a static or a stationary BH is a constant thanks to the zeroth law of BH thermodynamics, while in the case of a dynamic horizon relevant in the context of Cosmology, the surface gravity depends on the radius of the horizon as well as on its time derivative.

[3]Rindler horizons and Unruh temperatures are generally associated with accelerated observers in Minkowski spacetime of Special Theory of Relativity.

resemblance to thermodynamic systems and the notions of temperature and entropy can be associated with them, similar to the case of BHs. This analogy between thermodynamics and the gravitational dynamics of horizons is intriguing, nevertheless, it is not understood very clearly yet. For a review, one may look into the articles by T. Padmanabhan (2002a, 2005) published in Modern Physics Letters A (2002) and Physics Reports (2005). A possible way to interpret these results is to hypothesize that spacetime corresponds to an elastic solid and its equations of motion are analogous to those of elasticity. One would find a detailed discussion on this concept in the first para of the paper by Paranjape et al. (2006).

The GSL was extended in the context of cosmological spacetimes by Davies (1988) and Brustein (2000). Owing to the remarkable universality of the laws of thermodynamics, it is generally presumed that those cosmological models which are consistent with the laws of thermodynamics in addition to being supported by observational data, are more acceptable (physically realistic to be precise!) as compared to those models which only pass the observational tests. Now, in a flat FLRW universe, the apparent horizon always exists, while the event horizon is relevant only when the cosmic fluid violates the SEC (i.e. when $\rho + 3p < 0$).[4] Taking advantage of the universal existence of the apparent horizon (and also due to a high degree of mathematical simplicity!), its thermodynamic properties have been studied extensively in the literature, including in a quasi-de Sitter geometry of inflationary universe (Frolov and Kofman 2003).

Now, using the expressions in Eq. (3.1), the Bekenstein entropy and the Hawking temperature on the apparent horizon R_A of a FLRW universe can be evaluated as

$$S_A = \pi R_A^2 \qquad \text{and} \qquad T_A = \frac{|\kappa_A|}{2\pi} = \frac{1}{2\pi R_A} \tag{3.2}$$

respectively, assuming $\hbar = k_B = c = G = 1$ without any loss of generality. In the second equation above, the expression of surface gravity on the apparent horizon given by

$$\kappa_A = -\frac{1}{2}\frac{\partial \chi}{\partial R}\bigg|_{R=R_A} \tag{3.3}$$

has been taken into account, which turns out to be $-\frac{1}{R_A}\left(1 - \frac{\dot{R}_A}{2HR_A}\right)$. Note that χ is the scalar introduced in Eq. (1.30). This is basically the consequence of the formulation of surface gravity by Hayward (1998) and Kodama (1980). Some remarks regarding the Hawking temperature on the apparent horizon are in order. Recall that while deriving the Friedman equation from the Clausius relation, one needs to evaluate the amount of energy crossing the apparent horizon in an infinitesimal time interval. The apparent horizon is assumed to be slowly evolving during the infinitesimal time interval so that $\dot{R}_A = 0$. This leads to $T_A = \frac{1}{2\pi R_A}$. It is also worth noting that Cai et al. (2009) established that the cosmological apparent horizon is associated with a

[4] A proof of this important fact can be found in Chap. 3 of the book by V. Faraoni (2015).

Hawking temperature having the same form as in Eq. (3.2) by applying the tunneling approach proposed by Parikh and Wilczek.

Cai and Kim (2005) were the first to establish the equivalence of FLT and the Friedmann equation on the apparent horizon in Einstein gravity as well as in the Gauss–Bonnet and Lovelock gravity theories. Later, Akbar and Cai (2006) extended the study to scalar-tensor gravity and $f(R)$ gravity theories. Note that the entropy of BHs does not obey the area formula in these higher order derivative gravity theories. Later, Gong and Wang (2007), by introducing a mass-like function having the dimensions of energy and equal to the Misner–Sharp–Hernandez[5] (MSH) mass on the apparent horizon, the FLT was obtained from the Friedmann equation on the apparent horizon in different theories of gravity, including in the Einstein, Lovelock, nonlinear, and scalar-tensor theories.

The remaining portion of this chapter will focus on the basic features of *cosmological thermodynamics* (also goes by the names *horizon thermodynamics* and *universal thermodynamics*) which are necessary for further investigations to be undertaken within the scope of this book. Recall that, according to thermodynamics, an isolated macroscopic physical system evolves toward thermodynamic equilibrium (TE) (consistent with the constraints imposed on the system). Now if S be the total entropy of the system, then the following restrictions should hold (*'dot'* denotes derivative w.r.t. time)—(1) $\dot{S} \geq 0$ (i.e., the entropy function cannot decrease) and (2) $\ddot{S} < 0$ (i.e., the entropy function attains a maximum). Our task will be to test the validity of these two laws for a local portion of the Universe assumed to be isolated by a cosmological horizon[6] and filled with some physically acceptable cosmic fluid. In order to meet our purpose, the above two inequalities should be generalized as

$$\dot{S}_h + \dot{S}_{fh} \geq 0 \quad \text{(GSL)}, \tag{3.4}$$

$$\ddot{S}_h + \ddot{S}_{fh} < 0 \quad \text{(TE)}, \tag{3.5}$$

where S_h and S_{fh}, respectively, denote the entropy of the horizon and that of the fluid inside it. Note that, while the former should be satisfied at every epoch of evolution, the latter should hold at least during the final phases of evolution. Some researchers prefer to evaluate the above derivatives with w.r.t. the parameter $x = \ln a$. The reader may easily verify that the transformation from t-derivatives (*'dot'*) to x-derivatives (*'dash'*) is described by the following equations:

$$S' = \left(\frac{1}{H}\right)\dot{S} \qquad \text{and} \qquad S'' = \left(\frac{1}{H}\right)\ddot{S} - \left(\frac{\dot{H}}{H^2}\right)\dot{S}. \tag{3.6}$$

[5]The reader may go through Chap. 3 of the book by V. Faraoni for a detailed account of the MSH mass and its connection with the cosmological apparent horizon. This formalism was extended from BHs to the cosmological context by Bak and Rey (2000).

[6]We shall mostly consider the cosmological apparent and event horizons, however, arbitrary horizons may also be considered whenever necessary or relevant.

Now, the variations in the entropies of the horizon and the fluid inside it can be determined by the Clausius relation

$$T_h dS_h = \delta Q_h = -dE_h \tag{3.7}$$

and the Gibbs relation

$$T_{fh} dS_{fh} = dE_f + pdV_h \tag{3.8}$$

respectively. In Eqs. (3.7) and (3.8), dE_h is the amount of energy flowing across the horizon, which, during the infinitesimal time interval dt, has the expression[7]

$$-dE_h = 4\pi R_h^3(\rho + p)Hdt, \tag{3.9}$$

$E_f = \rho V_h$ is the total energy of the fluid inside, $V_h = \frac{4}{3}\pi R_h^3$ is the volume of the fluid, and (T_h, T_{fh}) are the temperatures of the horizon and the fluid bounded by the horizon, respectively. We shall assume that the temperature of the horizon is same as that of the fluid inside it, i.e., $T_h = T_{fh}$. Finally, it should be noted that we shall not deal with the zeroth and the third laws of thermodynamics because the former is a trivial one and the latter is inconsistent with classical statistical mechanics and require the prior establishment of quantum statistics in order that it could be properly appreciated.

3.3 Modifications of Hawking Temperature

The FL on the cosmological apparent horizon was first established by Bousso (2005). Observe that from Eq. (3.9), we get,

$$-dE_A = 4\pi R_A^2(\rho + p)dt = \dot{R}_A dt, \tag{3.10}$$

while the first order differential of S_A is given by

$$dS_A = (2\pi R_A)\dot{R}_A dt. \tag{3.11}$$

Noting that the term inside the parenthesis is the inverse of T_A, we obtain the FL on the apparent horizon

$$-dE_A = T_A dS_A. \tag{3.12}$$

In the following year, B. Wang, Y. Gong, and E. Abdalla published a paper (2006) wherein they argued that the FL on the event horizon does not hold if one considers the entropy and the temperature on it as

[7]The tensorial form of Eq. (3.9) is given by $-dE_h = 4\pi R_h^2 T_{ab}\kappa^a\kappa^b dt$, where κ^μ is a null vector.

$$S_E = \pi R_E^2 \qquad \text{and} \qquad T_E = \frac{1}{2\pi R_E} \tag{3.13}$$

respectively, in analogy to the expressions considered for the apparent horizon. In other words, the only modification they did was to replace the subscript $'A'$ on the thermodynamic quantities with $'E'$. Following the same analogy, they replaced the first part of Eq. (3.10) with

$$-\,dE_E = 4\pi R_E^2(\rho + p)dt. \tag{3.14}$$

Moreover, it was argued that the FL might only be applicable to variations between nearby states of local TE, whereas the event horizon reflects the global properties of spacetimes. Now, although the holographic principle might be speculated to support the definition of S_E proposed in Eq. (3.13), but one must keep in mind that the Hawking temperature on a horizon is obtained by calculating the surface gravity on that horizon. Consequently, the expression for T_E can not be written in analogy with the expression for T_A. Rather, it should be obtained from the definition of the Hawking temperature in terms of the surface gravity on the horizon. This important rectification was provided by Chakraborty (2012) and he later coined the term *modified Hawking temperature* (MHT) to describe the modification to the original form of the Hawking temperature (inverse of the horizon radius divided by 2π) which is given by

$$T_E = \frac{|\kappa_E|}{2\pi} = \frac{R_E}{2\pi R_A^2} \tag{3.15}$$

with

$$\kappa_E = -\frac{1}{2}\frac{\partial \chi}{\partial R}\bigg|_{R=R_E}. \tag{3.16}$$

In a flat FLRW universe, Eq. (3.15) reduces to

$$T_E = \frac{H^2 R_E}{2\pi}. \tag{3.17}$$

Moreover, in a flat FLRW universe, the ratio of the temperatures of event and apparent horizons is equal the ratio of their radii, i.e.,

$$\frac{T_E}{T_A} = \frac{R_E}{R_A}. \tag{3.18}$$

Now, despite this remarkable modification, it is surprising to learn that the FL does not hold true on the cosmological event horizon except in a few cases as shown by Chakraborty (2012). He demonstrated that the FL holds on the event horizon for a perfect fluid with a constant EoS $p = w\rho$ ($w < -\frac{1}{3}$), while it is violated in the noninteracting HDE model with R_E as the IR cutoff. The reason for this discrepancy can be somewhat understood by the following arguments. Bousso (2002) showed that the apparent horizon forms the largest surface whose interior can be considered as a Bekenstein system.[8] The Bekenstein entropy-mass bound is satisfied in the region lying outside the surface of the apparent horizon but the Bekenstein entropy-area bound is violated in that region, thus rendering it to be a non-Bekenstein system. In other words, the entropy-area bound is saturated at the apparent horizon and the portion of the flat FLRW Universe bounded by the cosmological event horizon may not always form a Bekenstein system owing to the fact the event horizon may not always be contained within the apparent horizon. Consequently, the entropy on the event horizon may not always equal πR_E^2.

With a view to rescue the FLT on the event horizon, Chakraborty and Saha (2014) further modified the MHT by multiplying a time-dependent dimensionless parameter α to it. This new form of Hawking temperature will be called the *generalized Hawking temperature* (GHT) and it can be mathematically represented as

$$T_E^{(g)} = \alpha T_E = \frac{\alpha R_E}{2\pi R_A^2}. \tag{3.19}$$

It is an easy exercise to verify that the FL will be satisfied on the cosmological event horizon if the parameter α is interpreted as the ratio of the velocity of the apparent horizon to that of the event horizon. Thus,

$$\alpha = \frac{\left(\frac{\dot{R}_A}{R_A}\right)}{\left(\frac{\dot{R}_E}{R_E}\right)}. \tag{3.20}$$

Nevertheless, this approach to establish the FLT on the event horizon may not be physically reasonable as such due to the uncertainty in identifying the Universe bounded by an event horizon as a Bekenstein system. The correct approach would be to redefine the entropy on the event horizon so that this new expression for entropy together with the MHT satisfies the Clausius relation on the horizon. However, such a redefined entropy is not available yet and we hope that future endeavours in this field may provide us with such an expression, consistent with the thermodynamics of the event horizon.

3.4 Final Remarks

This chapter underlines the basic elements of cosmological thermodynamics and introduces the concepts of MHT and GHT. Chapters 4–6, respectively, deal with certain applications of the subject in studying the thermodynamic viability of some

[8] A *Bekenstein system* is one which satisfies (in natural units) the Bekenstein entropy-mass bound $S \leq 2\pi RE$ as well as the Bekenstein entropy-area bound $S \leq \frac{A}{4}$, where S, R, E, and A are respectively the entropy, radius, energy, and area of the system.

well-known cosmological models, particularly, the flat, homogeneous and isotropic FLRW model, the gravitationally induced particle creation model, and the isotropic but inhomogeneous Lemaitre–Tolman–Bondi model. The final chapter enlists few (of many) shortcomings and also discusses some prospective open issues within this (rapidly) emerging research field.

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Suggested Further Reading

BLACK HOLE THERMODYNAMICS: Wald, R.M. 2001. The thermodynamics of black holes. Living Reviews in Relativity 4: 6; Carlip, S. 2014. Black hole thermodynamics. International Journal of Modern Physics D 23: 1430023.

Chapter 4
Cosmological Thermodynamics in FLRW Model

Abstract This chapter is concerned with a thermodynamic analysis on a flat FLRW universe admitting both apparent and event horizons. As the physical system within the cosmological apparent horizon forms a Bekenstein system, so we have considered the Bekenstein entropy and the Hawking temperature on the apparent horizon. However, since the system bounded by the event horizon may not be a Bekenstein system, we have assumed the Clausius relation on the event horizon to determine its entropy variation. Moreover, we have assumed the modified Hawking temperature on the event horizon. Three types of dark energy cosmic fluids have been considered—a perfect fluid with a constant equation of state, an interacting holographic dark energy, and a Chaplygin gas.

Keywords Apparent horizon · Event horizon · Gibbs equation · Modified Hawking temperature · Generalized Hawking temperature · Holographic dark energy · Modified Chaplygin gas

This chapter undertakes a thorough investigation of the viability of cosmological thermodynamics, more precisely, of the GSL and TE, in the flat FLRW model of the Universe, admitting both apparent and event horizons. Three types of DE fluids are considered for the purpose, a perfect DE fluid with a constant EoS, the interacting HDE, and the MCG. The apparent horizon is endowed with the Bekenstein entropy and the Hawking temperature. However, on the event horizon, we assume that the Clausius relation is satisfied so that we can make use of Eqs. (3.7) and (3.9) to determine its entropy variation. The temperature on the horizon is considered to be the MHT. As the thermodynamic system bounded by the event horizon may or may not form a Bekenstein system, so we do not resort to the Bekenstein entropy-GHT formalism for studying cosmological thermodynamics on the event horizon. The reader should note that we closely follow the approach adopted by Saha and Chakraborty (2012, 2014), although, we write most of our equations in more convenient forms, and consequently, our deductions seem to be much simpler and easy to anticipate.

S. Saha, *Elements of Cosmological Thermodynamics*,
SpringerBriefs in Physics, https://doi.org/10.1007/978-3-319-74706-4_4

4.1 GSL and TE: General Formulation

In this section, we derive general expressions for the total entropy of our thermodynamic system, namely, the (local) Universe bounded by a horizon (either an apparent or an event horizon), and calculate their first and second order time derivatives in order to test the validity of the GSL and TE respectively. The MHT will be employed in the case of an event horizon. Moreover, we assume that the temperatures of the horizon and the fluid inside are equal. We simplify our calculations by assuming units where 8π, G, c are unity. The total entropy of the thermodynamic system on the horizon R_h will be denoted by S_{Th} (i.e., $S_{Th} = S_h + S_{fh}$), where $h \to A, E$.

First of all, we evaluate the velocity and the acceleration of the apparent horizon which are given by

$$\dot{R}_A = -\frac{\dot{H}}{H^2} = \frac{3}{2}\left(1 + \frac{p}{\rho}\right) \tag{4.1}$$

and

$$\ddot{R}_A = \frac{9H}{2}\left(1 + \frac{p}{\rho}\right)\left(\frac{p}{\rho} - \frac{\dot{p}}{\dot{\rho}}\right) \tag{4.2}$$

respectively. Next, we evaluate the velocity[1] and the acceleration of the event horizon as

$$\dot{R}_E = HR_E - 1 \tag{4.3}$$

and

$$\ddot{R}_E = -\frac{1}{2}H^2 R_E\left(1 + 3\frac{p}{\rho}\right) - H \tag{4.4}$$

respectively.

Now, the first and the second order time derivatives of the entropy on the apparent horizon are evaluated as

$$\begin{aligned} \dot{S}_A &= \frac{1}{4}R_A\dot{R}_A \\ &= \frac{3}{8H}\left(1 + \frac{p}{\rho}\right) \end{aligned} \tag{4.5}$$

and

$$\begin{aligned} \ddot{S}_A &= \frac{1}{4}(\dot{R}_A^2 + R_A\ddot{R}_A) \\ &= \frac{9}{16}\left(1 + \frac{p}{\rho}\right)\left[\left(1 + \frac{p}{\rho}\right) + 2\left(\frac{p}{\rho} - \frac{\dot{p}}{\dot{\rho}}\right)\right] \end{aligned} \tag{4.6}$$

[1] The velocity of the event horizon is obtained by applying the Leibniz's rule for differentiation under the integral sign to Eq. (1.26).

respectively. The corresponding derivatives on the event horizon (the entropy variation of the horizon being given by Eqs. (3.7) and (3.9)) are derived as

$$\dot{S}_E = \frac{3}{8} H R_E^2 \left(1 + \frac{p}{\rho}\right) \tag{4.7}$$

and

$$\ddot{S}_E = \frac{3}{8} H^2 R_E \left(1 + \frac{p}{\rho}\right) \left[\left(1 + \frac{p}{\rho}\right) - 3\left(1 + \frac{\dot{p}}{\dot{\rho}}\right) + 2\left(R_E - \frac{1}{H}\right)\right] \tag{4.8}$$

respectively, where we have used the MHT given by Eq. (3.15).

Finally, from Eq. (3.8), the general expressions for the first and the second order time derivatives of the entropy of the fluid bounded by a horizon R_h become

$$\dot{S}_{fh} = \frac{3H^2 R_h^2}{2T_{fh}} \left(1 + \frac{p}{\rho}\right) \left(\dot{R}_h - H R_h\right) \tag{4.9}$$

and

$$\begin{aligned}\ddot{S}_{fh} = \frac{3H^2 R_h}{2T_{fh}} \left(1 + \frac{p}{\rho}\right) \Big[&\left(2\dot{R}_h - 3H R_h\right)\left(\dot{R}_h - H R_h\right) \\ &+ R_h \left\{\ddot{R}_h - \left(R_h \dot{H} + \dot{R}_h H\right)\right\}\Big]\end{aligned} \tag{4.10}$$

respectively, where we have used the energy conservation equation given by Eq. (1.15).

Therefore, assuming $T_A = T_{fA}$, the first and the second time derivatives of the total entropy on the apparent horizon R_A are evaluated as

$$\begin{aligned}\dot{S}_{TA} &= \dot{S}_A + \dot{S}_{fA} \\ &= \frac{9}{16H}\left(1 + \frac{p}{\rho}\right)^2\end{aligned} \tag{4.11}$$

and

$$\begin{aligned}\ddot{S}_{TA} &= \ddot{S}_A + \ddot{S}_{fA} \\ &= \frac{9}{16}\left(1 + \frac{p}{\rho}\right)\left[\left(1 + 6\frac{p}{\rho}\right)\left(1 + \frac{p}{\rho}\right) - \left(5 + 3\frac{p}{\rho}\right)\frac{\dot{p}}{\dot{\rho}}\right]\end{aligned} \tag{4.12}$$

respectively. It is clear from Eq. (4.11) that the GSL holds on the apparent horizon irrespective of the nature of the fluid.

Again, assuming that the MHT $T_E = T_{fE}$, the first and the second time derivatives of the total entropy on the event horizon R_E are obtained as

$$\begin{aligned}\dot{S}_{TE} &= \dot{S}_E + \dot{S}_{fE} \\ &= \frac{3}{8} R_E (H R_E - 1)\left(1 + \frac{p}{\rho}\right) \end{aligned} \tag{4.13}$$

and

$$\begin{aligned}\ddot{S}_{TE} &= \ddot{S}_E + \ddot{S}_{fE} \\ &= -\frac{3}{8} H R_E (H R_E - 1)\left(1 + \frac{p}{\rho}\right)\left[\frac{1}{H R_E} + \frac{3}{2}\left(\frac{1 + \frac{p}{\rho}}{H R_E - 1}\right)\right. \\ &\quad \left. - \frac{1}{2}\left(1 + 3\frac{p}{\rho} - 6\frac{\dot{p}}{\dot{\rho}}\right)\right] \end{aligned} \tag{4.14}$$

respectively.

4.2 GSL and TE: Model Specific Formulation

We now undertake a comparative study on the apparent and event horizons for three well-known DE fluids, namely, a perfect fluid having a constant EoS $w < -\frac{1}{3}$, the interacting HDE model with the event horizon as the IR cut-off and endowed with the standard interaction term $Q = 3b^2 H\rho$, and the MCG, in the context of cosmological thermodynamics. Recall that the fluid of the first kind is further classified as quintessence ($-1 < w < -\frac{1}{3}$), phantom ($w < -1$), and the cosmological constant ($w = -1$).

4.2.1 Perfect Fluid with a Constant EoS w, $w < -\frac{1}{3}$

Apparent Horizon

Equations (4.11) and (4.12) become

$$\dot{S}_{TA} = \frac{9}{16H}(1 + w)^2 \tag{4.15}$$

and

$$\ddot{S}_{TA} = \frac{9}{16}(1 + w)[(1 + w)^2 + 2w^2] \tag{4.16}$$

$$= \frac{1}{6}\dot{R}_A\left[\dot{R}_A^2 + 2\left(\dot{R}_A - \frac{3}{2}\right)^2\right]. \tag{4.17}$$

respectively. Thus, although the GSL holds true for all values of w, the TE is valid only for a phantom fluid ($w < -1$) for which $\dot{R}_A < 0$.

Event Horizon

Equations (4.13) and (4.14) become

$$\dot{S}_{TE} = \frac{3}{8} R_E (1+w)(HR_E - 1) \tag{4.18}$$
$$= \frac{R_E}{4} \dot{R}_A \dot{R}_E, \tag{4.19}$$

and

$$\ddot{S}_{TE} = -\frac{3}{8} HR_E (1+w)(HR_E - 1)\left[\frac{1}{HR_E} + \frac{3}{2}\left(\frac{1+w}{HR_E - 1}\right) - \frac{1}{2}(1-3w)\right] \tag{4.20}$$
$$= -\frac{1}{4}\dot{R}_A \dot{R}_E \left[\frac{\dot{R}_A}{\dot{R}_E}(1+\dot{R}_E)^2 - (1+2\dot{R}_E)\right]. \tag{4.21}$$

respectively. It can be readily seen that the GSL is trivially true for a cosmological constant because $\dot{R}_A = 0$ in that case. However, the GSL is satisfied for a quintessence fluid ($\dot{R}_A > 0$) if $\dot{R}_E \geq 0$ (i.e., $R_E \geq R_A$) while it holds for a phantom fluid ($\dot{R}_A < 0$) if $\dot{R}_E \leq 0$ (i.e., $R_E \leq R_A$). In other words, the GSL holds good whenever the product of the two velocities are non-negative.

As far as the validity of TE is concerned, the above analysis suggests that the expression inside the square bracket in Eq. (4.20) must be positive. Now, for a quintessence fluid, $\dot{R}_A > 0$, so the GSL implies that $\dot{R}_E$ must be positive,[2] i.e., $\frac{R_E}{R_A} > 1$. Hence, the TE holds in this case if[3] $1 < \frac{R_E}{R_A} \leq \frac{4}{1-3w}$. For a phantom fluid, $\dot{R}_A < 0$, so $\dot{R}_E < 0$ in accordance with the GSL which in turn implies that $\frac{R_E}{R_A} < 1$. So, in this case, the TE is valid if[4] $\frac{R_E}{R_A} < \min\left\{1, \frac{2}{1-3w}\right\}$. The reader must note that the inequalities obtained here are merely sufficient conditions and are not at all necessary for the attainment of TE. These inequalities can be easily realized from Fig. 4.1 where we show[5] the variation of $\ddot{S}_{TE}$ against $\frac{R_E}{R_A}$ for $w = -0.4, 0.9$ (quintessence) and $w = -1.5$ (phantom).

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[2] Note that if either or both the horizons are static or if the fluid is equivalent to a cosmological constant, then the GSL holds but the TE does not.

[3] The right inequality is obtained by taking into account the fact that the sum of the last two terms inside the square bracket in Eq. (4.20) must be positive.

[4] The second inequality inside the braces is obtained by taking into account the fact that the sum of the first and the last terms inside the square bracket in Eq. (4.20) must be positive.

[5] All the figures in this chapter have been plotted with the help of Maple plotting software.

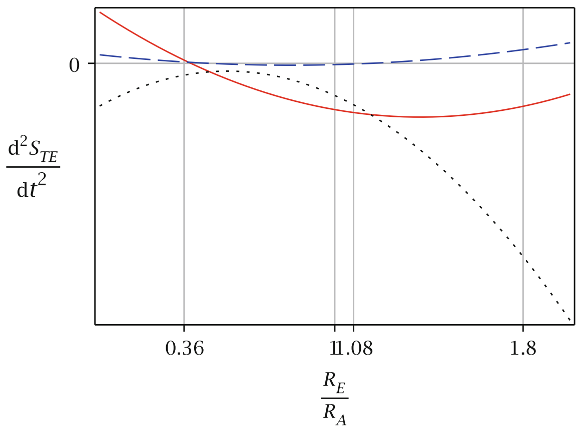

Fig. 4.1 The variation of $\ddot{S}_{TE}$ against $\frac{R_E}{R_A}$ for $w = -0.4$ (solid line), $w = -0.9$ (dashed curve), $w = -1.5$ (dotted curve). The inequalities respectively become $1 < \frac{R_E}{R_A} \leq 1.8$, $1 < \frac{R_E}{R_A} \leq 1.08$, and $\frac{R_E}{R_A} < \min\{1, 0.36\} = 0.36$

4.2.2 Holographic Dark Energy

In an interacting HDE model with event horizon as the IR cut-off and an interaction term $Q = 3b^2 H\rho$, the velocities of the apparent and the event horizons become

$$\dot{R}_A = \frac{3}{2}\left[(1-b^2) - \frac{\Omega_d}{3}\left(1 + \frac{2}{c}\sqrt{\Omega_d}\right)\right] \tag{4.22}$$

and

$$\dot{R}_E = \frac{c}{\sqrt{\Omega_d}} - 1 \tag{4.23}$$

respectively.

Apparent Horizon

Equations (4.11) and (4.12) are evaluated as (recall that $w = w_d\Omega_d$ here)

$$\dot{S}_{TA} = \frac{9}{16H}\left[(1-b^2) - \frac{\Omega_d}{3}\left(1 + \frac{2}{c}\sqrt{\Omega_d}\right)\right]^2 \tag{4.24}$$

and

$$\ddot{S}_{TA} = \frac{1}{4}\dot{R}_A\left[\dot{R}_A(4\dot{R}_A - 5) - 3(1 + \dot{R}_A)c_s^2\right] \tag{4.25}$$

respectively. The reader should note that the r.h.s. of Eq. (4.25) is the general form for $\ddot{S}_{TA}$ when expressed in terms of the horizon velocities and the adiabatic sound speed[6] c_s^2. Now, as c_s^2 is always positive, so the system will be

[6]At the level of inhomogeneities, the adiabatic sound speed of a cosmological fluid relates the pressure and density perturbations as $c_s^2 = \frac{\partial p}{\partial \rho}$, which, in our case, reduces to $c_s^2 = \frac{\dot{p}}{\dot{\rho}}$. It is generally

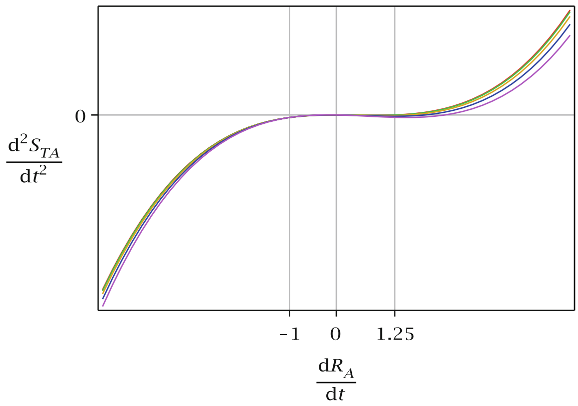

Fig. 4.2 The variation of $\ddot{S}_{TA}$ against $\dot{R}_A$ for $c_s^2 = 0, 0.25, 0.5, 0.75, 1$

in TE either if $0 < \dot{R}_A < 1.25$ or if $\dot{R}_A \leq -1$. Note that these inequalities constrain the parameter b^2 as $\frac{1}{6}\left[1 - 2\Omega_d\left(1 + \frac{2}{c}\sqrt{\Omega_d}\right)\right] < b^2 < 1 - \frac{\Omega_d}{3}\left(1 + \frac{2}{c}\sqrt{\Omega_d}\right)$ and $b^2 \geq \frac{1}{3}\left[5 - \Omega_d\left(1 + \frac{2}{c}\sqrt{\Omega_d}\right)\right]$ respectively. Again, these are merely sufficient conditions and are not at all necessary for the attainment of TE. To demonstrate this fact, we show, in Fig. 4.2, the variation of $\ddot{S}_{TA}$ against $\dot{R}_A$ for five different values of c_s^2. Finally, we see that, if the apparent horizon is static, i.e., $\dot{R}_A = 0$, then the GSL is satisfied but the TE is not.

Event Horizon:

Equations (4.13) and (4.14) are evaluated as

$$\dot{S}_{TE} = \frac{3}{8}\left(\frac{c}{H\sqrt{\Omega_d}}\right)\left(\frac{c}{\sqrt{\Omega_d}} - 1\right)(1 + w_d\Omega_d) \tag{4.26}$$

$$= \frac{R_E}{4}\dot{R}_A\dot{R}_E, \tag{4.27}$$

and

$$\ddot{S}_{TE} = -\frac{1}{4}\dot{R}_A\dot{R}_E\left[\frac{\dot{R}_A}{\dot{R}_E}(1 + \dot{R}_E)^2 + (1 + \dot{R}_E)\left\{3(1 + c_s^2) - 2\dot{R}_A - 2\right\} + 1\right] \tag{4.28}$$

respectively. Note that the r.h.s. of Eq. (4.28) is the general form for $\ddot{S}_{TE}$ when expressed in terms of the horizon velocities and the adiabatic sound speed. This equation will reduce to Eq. (4.21) when the EoS parameter w is assumed to be constant. Now, from Eq. (4.27), we find that the GSL holds if the product $\dot{R}_A\dot{R}_E \geq 0$, i.e., if the velocities of both the horizons are either non-negative or non-positive. The system will attain TE if $0 < \dot{R}_A \leq \frac{3}{2}(1 + c_s^2) - 1 \leq 2$. This inequality is obtained by assum-

restricted as $0 \leq c_s^2 \leq 1$. The lower bound does not allow DE fluctuations to grow exponentially, thus preventing unphysical situations, while the upper bound prevents propagation at superluminal speeds.

ing both $\dot{R}_A$ and $\dot{R}_E$ to be positive, which is one of the possibilities ensured by the GSL. Consequently, the parameter b^2 is constrained as $-\frac{1}{3}\left[1+\Omega_d\left(1+\frac{2}{c}\sqrt{\Omega_d}\right)\right] \leq b^2 < 1-\frac{\Omega_d}{3}\left(1+\frac{2}{c}\sqrt{\Omega_d}\right)$. The other case, where both the velocities are negative, does not lead to any realistic physical restriction for the attainment of TE as far as the form of Eq. (4.28) is concerned. Finally, note that if either or both the horizons are static, then the GSL holds but the TE is not attained.

4.2.3 Modified Chaplygin Gas

In the MCG model, the energy conservation Eq. (1.9) gives

$$\rho = \left[\frac{B}{1+\gamma} + \frac{C}{a^\mu}\right]^{\frac{1}{1+n}}, \tag{4.29}$$

where $\mu = 3(1+n)(1+\gamma)$. Then, the radius and the velocity of the apparent horizon are evaluated as

$$R_A = R_0\left[\frac{a^\mu}{Ba^\mu + C(1+\gamma)}\right]^{\frac{1}{2(1+n)}} \tag{4.30}$$

and

$$\dot{R}_A = \frac{3}{2}C(1+\gamma)^2\left[\frac{1}{Ba^\mu + C(1+\gamma)}\right] \tag{4.31}$$

respectively, where a is the scale factor, C is the constant of integration, and $R_0 = \sqrt{3}(1+\gamma)^{\frac{1}{2(1+n)}}$. The radius and the velocity of the event horizon are evaluated in terms of the hypergeometric ${}_2F_1$ function as

$$R_E = R_1\, {}_2F_1\left[\frac{1}{2(1+n)}, \frac{1}{\mu}, 1+\frac{1}{\mu}, \frac{-C}{Ba^\mu}\right] \tag{4.32}$$

and

$$\dot{R}_E = R_1\frac{1}{2(1+n)(1+\frac{1}{\mu})}\left(\frac{C}{B}\right)\left(\frac{H}{a^\mu}\right) {}_2F_1\left[1+\frac{1}{2(1+n)}, 1+\frac{1}{\mu}, 2+\frac{1}{\mu}, \frac{-C}{Ba^\mu}\right] \tag{4.33}$$

respectively, with $R_1 = \frac{R_0}{B^{\frac{1}{2(1+n)}}}$. Also, the square of the adiabatic sound speed is given by

$$c_s^2 = \frac{\partial p}{\partial \rho} = \gamma_0 - \frac{2}{3}n\dot{R}_A, \tag{4.34}$$

where $\gamma_0 = \gamma + n(1+\gamma) > 0$. It must be noted that the restriction $0 \leq c_s^2 \leq 1$ constrains $\dot{R}_A$ as

$$\frac{3}{2n}\{\gamma(1+n)+n-1\} \leq \dot{R}_A \leq \frac{3}{2n}\{\gamma(1+n)+n\}. \tag{4.35}$$

Apparent Horizon:

Equations (4.11) and (4.12) reduce to

$$\dot{S}_{TA} = \frac{9}{16H}\left[\frac{C(1+\gamma)}{Ba^{\mu}+C}\right]^2 \tag{4.36}$$

and

$$\ddot{S}_{TA} = \frac{1}{4}\dot{R}_A\left[\dot{R}_A\{(4+2n)\dot{R}_A-(1+n)(1+3\gamma)-4\}-3\{\gamma+n(1+\gamma)\}\right] \tag{4.37}$$

respectively. We find that the TE is attained either if $0 < \dot{R}_A \leq \frac{4+(1+n)(1+3\gamma)}{4+2n}$ and $\gamma > \max\left\{-\frac{5+n}{3(1+n)}, -\frac{n}{1+n}\right\}$ or if $\dot{R}_A < \min\left\{0, -\frac{3n+3\gamma(1+n)}{4+(1+n)(1+3\gamma)}\right\}$ for any value of γ. These inequalities are illustrated in Fig. 4.3 where we show the variation of $\ddot{S}_{TA}$ against $\dot{R}_A$ for $n=\frac{9}{2}$. We have chosen the values of γ as $\frac{1}{3}$ and $-\frac{1}{2}$ in accordance with the constraint imposed by the second inequality obtained above. However, it is very important to note that, the choices of γ and n should be such that these inequalities are consistent with the admissible values of $\dot{R}_A$ guaranteed by Eq. (4.35). Finally, if the apparent horizon is static, then, although the GSL is satisfied, the TE does not hold.

Event Horizon:

Equations (4.13) and (4.14) reduce to

$$\dot{S}_{TE} = \frac{3}{8}\left(1+\gamma-\frac{B}{\rho^{1+n}}\right)R_E(HR_E-1) \tag{4.38}$$

$$= \frac{R_E}{4}\dot{R}_A\dot{R}_E, \tag{4.39}$$

and

$$\ddot{S}_{TE} = -\frac{1}{4}\dot{R}_A\dot{R}_E\left[\frac{\dot{R}_A}{\dot{R}_E}(1+\dot{R}_E)^2+(1+\dot{R}_E)\left\{(1+n)(3(1+\gamma)-2\dot{R}_A)-2\right\}+1\right]. \tag{4.40}$$

respectively. As in the previous two cases, in this case also, the GSL holds whenever both the horizons have either non-negative velocities or non-positive velocities. Furthermore, after careful observation, the reader should agree that the system is consistent with TE either if $\dot{R}_E > 0$ and $\dot{R}_A > \max\left\{0, \frac{3(1+\gamma)(1+n)-2}{2(1+n)}\right\}$ or if $\dot{R}_E \leq -1$ and $\dot{R}_A < \min\left\{0, \frac{3(1+\gamma)(1+n)-2}{2(1+n)}\right\}$. The TE is also achieved for $-1 < \dot{R}_E < 0$

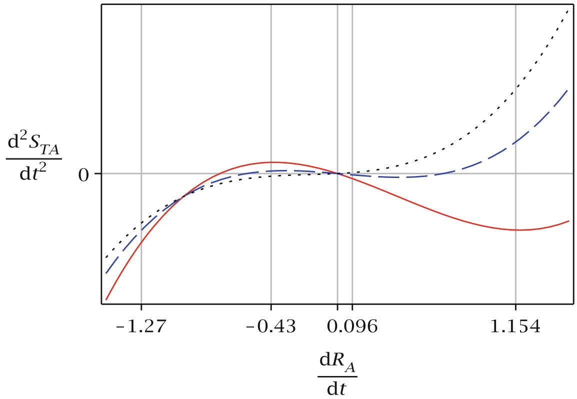

Fig. 4.3 The variation of $\ddot{S}_{TA}$ against $\dot{R}_A$ for $n = \frac{9}{2}$. This choice of n reduces the second inequality to $\gamma > \max\{-0.57, -0.81\}$, i.e., $\gamma > -0.57$. This justifies our choices of $\gamma = \frac{1}{3}$ (solid curve) and $\gamma = -\frac{1}{2}$ (dashed curve). These choices of γ, respectively, lead to the inequalities $0 < \dot{R}_A \leq 1.154$ and $0 < \dot{R}_A \leq 0.096$ for which TE is attained. In order to illustrate the constraint imposed by the third inequality, we consider $\gamma = \frac{1}{3}$ (solid curve) and $\gamma = -1$ (dotted curve). These choices of γ reduce the inequality to $\dot{R}_A < -1.27$ and $\dot{R}_A < -0.43$ respectively. The reader should, however, always keep in mind that these constraints should be consistent with the bounds imposed on $\dot{R}_A$ by c_s^2 as obtained in Eq. (4.35)

provided $\frac{3(1+\gamma)(1+n)-2}{2(1+n)}$ is negative. However, it should again be noted that the constraints obtained on $\dot{R}_A$ should be consistent with the bounds in Eq. (4.35). Finally, if either or both the horizons are static, then the GSL holds but TE does not.

4.3 Chapter Summary

In this chapter, we considered a flat FLRW universe and examined the viability of the GSL and the TE for the Universe admitting both apparent and event horizons. As the physical system within the cosmological apparent horizon forms a Bekenstein system, so we considered the Bekenstein entropy and the Hawking temperature on the horizon. However, owing to the fact that the system bounded by the event horizon may not be a Bekenstein system, we assumed the Clausius relation on the horizon to determine its entropy variation. In other words, we assumed that the FLT is satisfied on the event horizon. Note that we considered the MHT on the event horizon. Next, for the cosmic fluid, we considered three well-known and extensively studied DE fluids, namely, fluid 1: a perfect fluid having a constant EoS w ($w < -\frac{1}{3}$), fluid 2: an interacting HDE fluid (with the event horizon as the IR cut-off), and fluid 3: MCG. We used the Gibbs relation to determine variation of the fluid entropy. Consequently, we determined the differential of the total entropy of the system as well as the first and second time derivatives of the total entropy. In doing so, we assumed that the temperature of the horizon is equal to that of the fluid. Finally, we made a comparative

Table 4.1 Viability of GSL and TE for different fluid types in the case of an apparent horizon

Fluid type	GSL	Equilibrium
Fluid 1	Always holds	If the fluid is of the phantom type, i.e., $w < -1$, $\dot{R}_A < 0$
Fluid 2	Always holds	Either if $0 < \dot{R}_A < 1.25$ or if $\dot{R}_A \leq -1$ Alternatively, either if $\frac{1}{6}(1-2m) < b^2 < 1 - \frac{m}{3}$ or if $b^2 \geq \frac{1}{3}(5-m)$, where $m = \Omega_d \left(1 + \frac{2}{c}\sqrt{\Omega_d}\right)$
Fluid 3	Always holds	Either if $0 < \dot{R}_A \leq \frac{4+(1+n)(1+3\gamma)}{4+2n}$ and $\gamma > \max\left\{-\frac{5+n}{3(1+n)}, -\frac{n}{1+n}\right\}$ or if $\dot{R}_A < \min\left\{0, -\frac{3n+3\gamma(1+n)}{4+(1+n)(1+3\gamma)}\right\}$ for any γ The above constraints should also be consistent with $\frac{3}{2n}\{\gamma(1+n)+n-1\} \leq \dot{R}_A \leq \frac{3}{2n}\{\gamma(1+n)+n\}$

Table 4.2 Viability of GSL and TE for different fluid types in the case of an event horizon

Fluid type	GSL	Equilibrium
Fluid 1	If $\dot{R}_A \dot{R}_E \geq 0$	Attained for a quintessence fluid if $1 < \frac{R_E}{R_A} \leq \frac{4}{1-3w}$ Attained for a phantom fluid if $\frac{R_E}{R_A} < \min\left\{1, \frac{2}{1-3w}\right\}$
Fluid 2	If $\dot{R}_A \dot{R}_E \geq 0$	Attained if $0 < \dot{R}_A \leq \frac{3}{2}(1+c_s^2) - 1 \leq 2$ Alternatively, attained if $-\frac{1}{3}(1+m) \leq b^2 < 1 - \frac{m}{3}$, where $m = \Omega_d \left(1 + \frac{2}{c}\sqrt{\Omega_d}\right)$
Fluid 3	If $\dot{R}_A \dot{R}_E \geq 0$	Attained either if $\dot{R}_E > 0$ and $\dot{R}_A > \max\left\{0, \frac{3(1+\gamma)(1+n)-2}{2(1+n)}\right\}$ or if $\dot{R}_E \leq -1$ and $\dot{R}_A < \min\left\{0, \frac{3(1+\gamma)(1+n)-2}{2(1+n)}\right\}$ Also attained if $-1 < \dot{R}_E < 0$ provided $\frac{3(1+\gamma)(1+n)-2}{2(1+n)}$ is negative The above constraints should also be consistent with $\frac{3}{2n}\{\gamma(1+n)+n-1\} \leq \dot{R}_A \leq \frac{3}{2n}\{\gamma(1+n)+n\}$

study of cosmological thermodynamics on both the horizons and obtained several physical restrictions either on the horizon velocities or on the model parameters so that the GSL and TE are viable in each model. The reader should, however, note that such constraints are sufficient only and are not at all necessary for the viability of the GSL and the TE. In Tables 4.1 and 4.2, respectively, we tabulate the results obtained for the apparent and event horizons.

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Chapter 5
Cosmological Thermodynamics in the Gravitationally Induced Particle Creation Mechanism

Abstract This chapter deals with a thermodynamic analysis on the apparent horizon of a spatially flat Friedmann–Lemaitre–Robertson–Walker universe endowed with the gravitationally induced adiabatic particle creation described by an arbitrary particle creation rate. Assuming a perfect fluid with a (constrained) constant equation of state, the validity of the first law, the generalized second law, and thermodynamic equilibrium have been tested. It is evident that our analysis may help to constrain various parameters of phenomenological particle creation models that have been considered in recent literature.

Keywords Dissipative process · Nonequilibrium thermodynamics · Eckart theory · Israel–Stewart theory · Bulk viscosity · Creation pressure · Specific entropy · Isentropic particle creation

5.1 Dissipative Processes and Nonequilibrium Thermodynamics

Most cosmological models can be described with the consideration of a perfect fluid alone. However, real fluids are dissipative and there are many processes in Cosmology which require a relativistic theory of dissipative fluids. Examples include several early universe processes such as Grand Unified Theory (GUT) phase transition, reheating at the end of inflation, neutrino decoupling from the cosmic plasma, nucleosynthesis, and decoupling of photons from matter during recombination. One needs to employ *nonequilibrium* or *irreversible thermodynamics* in order to model such processes. Perhaps the most satisfactory approach to irreversible thermodynamics is via non equilibrium kinetic theory although it is quite complicated.

Schrodinger (1939) pioneered the microscopic description of the gravitationally induced particle creation mechanism (PCM) in an expanding universe. Three decades later, Parker (1968, 1969) discussed this issue in the context of QFT in curved space-time. Later on, Prigogine et al. (1989) provided a macroscopic description of PCM induced by the gravitational field. Eckart (1940) was the first to extend the concept of *standard* (alternative terms include *classical* and *quasi-stationary*) irreversible

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thermodynamics from Newtonian to relativistic fluids. Then, a variation of this theory was put forward by Landau and Lifshitz (1958). Nevertheless, this *first order theory*, like its Newtonian version, suffers from serious shortcomings associated with stability and causality which, respectively, imply that, the equilibrium states in the theory are unstable and dissipative perturbations propagate at infinite speeds. These problems can be overcome by considering *extended* irreversible thermodynamics in which the set required to describe non equilibrium states is extended to contain the dissipative variables as well. Consequently, causal and stable behaviour can be assured under a wide range of conditions due to this feature. Müller (1967) developed a non-relativistic version of this extended thermodynamic theory, and independently, Israel (1976) and Israel and Stewart (1979a, b) developed a relativistic one. Extended irreversible thermodynamics also goes by the names *causal* thermodynamics, *second order theory* (because the entropy contains terms of second order in the dissipative variables unlike the first order theory), and *transient* thermodynamics. For a nice and concise review, one may go through the lectures by Maartens (1996). A covariant description of irreversible thermodynamics has also been developed and investigated by Pavón et al. (1982) and later by Calvao et al. (1992). In this chapter, we discuss the horizon thermodynamics of a flat FLRW universe in the perspective of the gravitationally induced PCM. It is important to note that we do not need to resort to either the first order Eckart theory or the second order Israel-Stewart theory for undertaking such a study as long as we deal with adiabatic matter creation.

5.2 Thermodynamics of Adiabatic Open Systems with Particle Creation

In this section, following Balfagon (2015), we discuss the thermodynamics of adiabatic open systems with a varying particle number. As we shall see, the assumption of adiabaticity brings in rich insights into the study of irreversible thermodynamics in the perspective of PCM.

We shall consider a volume V having N number of particles. Since N is constant for a closed system, so the FLT (Gibbs equation) for closed systems is expressed as

$$dE = dQ - pdV, \tag{5.1}$$

where dQ is the amount of heat received by the system during time dt. Equation (5.1) can be written in an alternative form as

$$d\left(\frac{\rho}{n}\right) = dq - pd\left(\frac{1}{n}\right), \tag{5.2}$$

with $\rho = \frac{E}{V}, n = \frac{N}{V}, dq = \frac{dQ}{N}$.

Equation (5.2) is applicable to open systems if we consider N to be time dependent. Defining $h = \frac{\bar{H}}{V}$ as the enthalpy density, the Gibbs equation for open systems is written as

$$d(\rho V) = dQ - pdV + \frac{h}{n}d(nV). \tag{5.3}$$

In adiabatic systems, where $dQ = 0$, Eq. (5.3) reduces to

$$d(\rho V) + pdV - \frac{h}{n}d(nV) = 0. \tag{5.4}$$

The heat transferred to the system in such a scenario is solely due to the change in the particle number. As suggested by Prigogine (1988), in cosmological perspective, this change corresponds to energy transfer from gravitational field to the created matter constituents. Thus, matter creation acts as a source of internal energy. Expanding Eq. (5.4) and dividing the equation by Vdt, we obtain

$$\dot{\rho} = \frac{h}{n}\dot{n}. \tag{5.5}$$

We shall write Eq. (5.4) in a different way so as to determine the condition which must be satisfied by an adiabatic FLRW universe with a varying particle number. Defining a comoving volume $V = ka^3$, where a is the scale factor of the FLRW Universe and k is a constant, it is evident that

$$\frac{\dot{n}}{n} = \frac{\dot{N}}{N} - 3H, \tag{5.6}$$

where H is the Hubble parameter and $\frac{\dot{N}}{N}$ can be interpreted as the rate of change of the number of particles, which we shall denote by Γ. Substituting Eq. (5.6) into (5.4), we obtain

$$\dot{\rho} + 3H(\rho + p) - \frac{\dot{N}}{N}(\rho + p) = 0. \tag{5.7}$$

Useful information can be extracted by studying the variation in the entropy of the Universe. As a matter of fact, variation in the entropy is possible due to two reasons — either due to the variation in the heat content of the system or due to the variation in the particle number. Thus, using the definition of entropy, we have

$$TdS = dQ + T\frac{S}{N}dN, \tag{5.8}$$

where T is the temperature of the system and the ratio $\frac{S}{N}$ is the *specific entropy* (or the entropy per particle) σ. Assuming the Universe to be adiabatic, Eq. (5.8) becomes

$$dS = \sigma dN. \tag{5.9}$$

Equation (5.9) looks simple but encapsulates two very important properties of an adiabatic FLRW universe with a varying particle number—

• *The SLT inhibits the annihilation of particles in the Universe.* Dividing Eq. (5.9) by dt, we obtain $\dot{S} = \sigma \dot{N}$. As $\sigma > 0$, so SLT ($\dot{S} \geq 0$) implies $\dot{N} \geq 0$.
• *Specific entropy is constant.* Consider the expression that defines specific entropy, $S = \sigma N$. The total derivative is given by $dS = \sigma dN + N d\sigma$ which on comparing with Eq. (5.9), one gets $d\sigma = 0$ which implies that σ is constant. Note that if σ is the same constant for every particle, then the fluid is said to be *isentropic*.

5.3 PCM in Cosmology: A Natural Alternative for DE?

Having formulated the basic equations of nonequilibrium thermodynamics of an adiabatic open system with a varying particle number, we now discuss the implications of PCM in the cosmological context. It is important to note that, in a homogeneous and isotropic Universe, bulk viscous pressure[1] is the sole dissipative phenomenon that can occur. This kind of dissipation can arise if either the various components of the cosmic substratum are coupled or the particle number is not conserved. As the Universe expands, the cosmic subfluids cool at different rates since their internal EoS are different. This phenomenon allows the cosmic system to depart from equilibrium. As a consequence, a bulk viscous pressure of the whole cosmic medium is generated. Various aspects of this phenomenon were investigated by Weinberg (1971), Straumann (1976), Schweizer (1982), Udey and Israel (1982), and Zimdahl (1996). On the other hand, nonconservation of particle number may be thought of as the effect of quantum particle production from the gravitational field. As demonstrated later, this bulk viscosity will be described by a backreaction term in the EFE whose negative pressure may provide a self-sustained mechanism of cosmic acceleration. Indeed, many phenomenological particle creation models exist in the literature and most, if not all, can incorporate the late-time cosmic acceleration as well as provide a viable alternative to the concordance ΛCDM model. Moreover, PCM is consistent with the SLT.[2] However, the exact form of the rate of particle creation (Γ) is still unknown and research in this particular area has largely been phenomenological. Fortunately, there are certain guiding principles which provide us with a hint on the possible forms of Γ during different evolutionary stages of the Universe. Gunzig et al. (1998) argued that most of the particles were created when the Universe was very young. To be precise, the following thermodynamic conditions can be imposed:

[1]The physical difference between particle creation and bulk viscosity was clarified in a paper by Lima and Germano (1992).

[2]Balfagon (2015) verified that destruction of baryonic and/or DM particles is also consistent with the SLT due to a particle exchange with DE.

(a) The early inflation should begin with a maximal entropy production rate, thereby implying a maximal particle creation rate. Thus, the Universe starts with a nonequilibrium thermodynamic state and gradually transforms into an equilibrium state.
(b) A true[3] vacuum for radiation should satisfy $\rho \to 0$ as $a \to 0$.
(c) The very early Universe should have $\Gamma > H$ so that the created radiation behaves as a thermalized heat bath. As the expansion proceeds, the rate of creation should fall behind the rate of expansion.

The above conditions will be satisfied if the particle creation rate Γ is considered to be proportional to the total energy density ρ (i.e., $\Gamma \propto H^2$). For a clarification, one may take a look at the paper by Gunzig et al. (1998).

A simple choice for Γ during the deceleration phase is $\Gamma \propto H$. Finally, for the late-time accelerating phase, the following thermodynamic requirements can be prescribed:
(A) The late-time accelerated expansion should experience a minimal entropy production rate, thereby implying a minimal particle creation rate. This allows the Universe to re-enter an era of nonequilibrium thermodynamics.
(B) The false vacuum at late-time should satisfy $\rho \to 0$ as $a \to \infty$.
(C) The rate of creation should be faster as compared to the rate of expansion.

One can easily verify that $\Gamma \propto \frac{1}{H}$ will be consistent with the above requirements. For a clarification, one may see the paper by Saha and Chakraborty (2015).

Although the above three choices for Γ are found to be consistent[4] with the evolutionary phases of the Universe, i.e., from inflation at early times to cosmic acceleration at late times, these choices are not unique at all. In other words, the choices of Γ, though thermodynamically motivated, are entirely phenomenological. For instance, $\Gamma \propto H^k$ for several $k > 2$ (satisfying conditions (a), (b), and (c)) may also explain the inflationary phase with better consistency as compared to $\Gamma \propto H^2$.

5.4 PCM in Cosmology: Basic Equations

This section deals with the basic equations that govern a flat FLRW universe endowed with isentropic particle production. We start with the modified form of the EM tensor $T_{\mu\nu}$ of a relativistic fluid having bulk viscosity which is given by[5]

$$T_{\mu\nu} = (\rho + p + \Pi)u_\mu u_\nu + (p + \Pi)g_{\mu\nu}, \quad u_\mu u^\mu = -1, \tag{5.10}$$

[3]QFT suggests that a *false vacuum* is one which exists at a local minimum of energy and hence it is not truly stable, whereas a *true vacuum* exists at a global minimum and is stable. The readers may see the books by Birrell and Davies (1982), Mukhanov and Winitzki (2007), and Parker and Toms (2009) for further study on QFT.

[4]One may take a look at the paper by Chakraborty (2014) for an explicit demonstration.

[5]In this chapter, we assume without any loss of generality that the physical constants, namely, c, G, $\hbar$, and κ_B, as well as 8π are unity.

where Π is the bulk viscous pressure[6] associated with the creation of particles; all other terms have their usual meanings. The Einstein's equations $G_{\mu\nu} = T_{\mu\nu}$ yield

$$3H^2 = \rho, \tag{5.11}$$

$$\dot{H} = -\frac{1}{2}(\rho + p + \Pi). \tag{5.12}$$

Here, the equilibrium pressure p and the energy density ρ are assumed to be related by the EoS $p = (\gamma - 1)\rho$, with $\frac{2}{3} \leq \gamma \leq 2$. The lower bound on γ ensures that the fluid does not become exotic, or equivalently, the SEC remains valid. The EM conservation law becomes

$$\dot{\rho} + 3H(\rho + p + \Pi) = 0. \tag{5.13}$$

We also have an equation accounting for the nonconservation of the number of particles (given by Eq. (5.6))—

$$\dot{n} + 3Hn = n\Gamma. \tag{5.14}$$

Now, using Eqs. (5.5), (5.13) and (5.14), we obtain

$$\Pi = -\frac{\Gamma}{3H}(\rho + p), \tag{5.15}$$

Note that this relation is valid only for an adiabatic (or isentropic) fluid. Thus, the creation pressure Π is linearly related to the creation rate Γ under the condition of adiabaticity. In other words, a dissipative fluid is equivalent to a perfect fluid with a varying particle number. From Eqs. (5.11) and (5.12) and using Eq. (5.15), we get

$$\frac{\dot{H}}{H^2} = -\frac{3\gamma}{2}\left(1 - \frac{\Gamma}{3H}\right) \tag{5.16}$$

The deceleration parameter q in this model takes the form

$$\begin{aligned} q &= -\frac{\dot{H}}{H^2} - 1 \\ &= \frac{3\gamma}{2}\left(1 - \frac{\Gamma}{3H}\right) - 1, \end{aligned} \tag{5.17}$$

and the effective EoS parameter for this model (denoted by w_{eff}) becomes

[6]Some authors prefer to use the notation p_c interpreted as the creation pressure associated with the creation of particles.

$$\begin{aligned} w_{eff} &= \frac{p+\Pi}{\rho} \\ &= \gamma\left(1-\frac{\Gamma}{3H}\right)-1, \end{aligned} \tag{5.18}$$

which, in a DE dominated universe, corresponds to quintessence era for $\Gamma < 3H$ and phantom era for $\Gamma > 3H$, while $\Gamma = 3H$ corresponds to a cosmological constant.

5.5 PCM in Cosmology: Thermodynamic Analysis

In the subsections that follow, we study the validity of the FLT, the GSL, and TE in the gravitationally induced particle creation scenario with an arbitrary particle creation rate Γ, following the work undertaken by Saha and Mondal (2017, 2018). We consider an apparent horizon as our thermodynamic boundary, since, unlike the event horizon, a cosmic apparent horizon always exists. To be precise, thermodynamic study of the gravitationally induced PCM scenario with other cosmological horizons has not been undertaken yet. It is not even clear whether such studies will be viable.

Now, recall that the cosmological apparent horizon is located at $R_A = \frac{1}{H}$ and its first order derivative w.r.t. the cosmic time t can be evaluated as

$$\begin{aligned} \dot{R}_A &= -\frac{\dot{H}}{H^2} \\ &= \frac{3\gamma}{2}\left(1-\frac{\Gamma}{3H}\right). \end{aligned} \tag{5.19}$$

The (Bekenstein) entropy and (Hawking) temperature associated with the apparent horizon are given by

$$S_A = \left(\frac{c^3}{G\hbar}\right)\frac{4\pi R_A^2}{4} = \frac{1}{8}R_A^2, \tag{5.20}$$

and

$$T_A = \left(\frac{\hbar c}{\kappa_B}\right)\frac{1}{2\pi R_A} = \frac{4}{R_A} \tag{5.21}$$

respectively.

5.5.1 First Law

Again, recall that the FLT on the apparent horizon is associated with the Clausius relation

$$-dE_A = T_A dS_A. \tag{5.22}$$

The differential dE_A of the amount of energy crossing the apparent horizon can be evaluated as

$$\begin{aligned} -dE_A &= \frac{1}{2} R_A^3 (\rho + p + \Pi) H dt \\ &= \frac{3\gamma}{2}\left(1 - \frac{\Gamma}{3H}\right) dt, \end{aligned} \tag{5.23}$$

while using the expressions of T_A and S_A given in Eqs. (5.20) and (5.21), the expression $T_A dS_A$ becomes

$$T_A dS_A = \frac{3\gamma}{2}\left(1 - \frac{\Gamma}{3H}\right) dt, \tag{5.24}$$

where we have used relation (5.19).

We find that the FL holds on the apparent horizon. This provides us with a consistency check for the applicability of horizon thermodynamics within the framework of the gravitationally induced PCM in a flat FLRW universe.

5.5.2 Generalized Second Law: An Expression for Total Entropy

The Gibbs equation for the fluid within the portion of the Universe bounded by the apparent horizon can be written as

$$T_{fA} dS_{fA} = dE_f + p dV_A, \tag{5.25}$$

where all the symbols have their usual meanings as discussed in Chap. 3.

Now, the assumption of adiabaticity gives an evolution equation for the fluid temperature T_{fA} given by (see Eq. (11) and the second relation of Eq. (35) in the paper by Zimdahl 2000)

$$\frac{\dot{T}_{fA}}{T_{fA}} = (\Gamma - 3H)\frac{\partial p}{\partial \rho}. \tag{5.26}$$

Noting from Eq. (5.16) that $\Gamma - 3H = \frac{2}{\gamma}\left(\frac{\dot{H}}{H}\right)$, the above equation leads to the integral

$$\ln\left(\frac{T_{fA}}{T_0}\right) = \frac{2(\gamma - 1)}{\gamma}\int \frac{dH}{H}. \tag{5.27}$$

On integration, we obtain,

$$T_{fA} = T_0 H^{\frac{2(\gamma-1)}{\gamma}}, \tag{5.28}$$

where T_0 is the constant of integration. Note that Γ does not appear explicitly in the equation.

The first order time derivative of the fluid entropy can be obtained from Eq. (5.25) as

$$\dot{S}_{fA} = \frac{3\gamma}{2}\left(\frac{3\gamma}{2} - 1\right) T_0^{-1}\left(1 - \frac{\Gamma}{3H}\right) H^{\frac{2(1-\gamma)}{\gamma}}. \tag{5.29}$$

Now, using Eqs. (5.21), (5.24), and (5.29), the time derivative of the total entropy can be obtained as

$$\dot{S}_{TA} = \frac{3\gamma}{8H}\left(1 - \frac{\Gamma}{3H}\right)\left[1 + 4\left(\frac{3\gamma}{2} - 1\right) T_0^{-1} H^{\frac{2}{\gamma}-1}\right]. \tag{5.30}$$

It can be easily verified from the last equation that GSL holds if $\Gamma \leq 3H$, or equivalently, if $\frac{\Gamma}{3H} \leq 1$. Therefore, the GSL is not consistent with the phantom fluid. Furthermore, S_{TA} is a constant of motion when $\Gamma = 3H$, i.e., when $w_{eff} = -1$, a cosmological constant.

Another remarkable fact is that Eq. (5.30) gives us an opportunity (by replacing dt by $\frac{dH}{\dot{H}}$ and using Eq. (5.16)) to derive an expression for the total entropy in terms of the Hubble parameter H in the form

$$\begin{aligned} S_{TA} &= S_A + S_{fA} \\ &= \frac{1}{8H^2}\left[1 - 8\left(\frac{\frac{3\gamma}{2} - 1}{\frac{2}{\gamma} - 1}\right) T_0^{-1} H^{\frac{2}{\gamma}}\right]. \end{aligned} \tag{5.31}$$

The essence of Eq. (5.31) lies in the fact that the particle creation rate Γ does not occur explicitly in the equation. Requiring that the total entropy be always positive, we can, in principle, obtain a lower bound on T_0 given by

$$T_0 \geq 8\left(\frac{\frac{3\gamma}{2} - 1}{\frac{2}{\gamma} - 1}\right) H^{\frac{2}{\gamma}}. \tag{5.32}$$

Equation (5.32) implies that we can also impose a lower bound on the fluid temperature T_{fA} as

$$T_{fA} \geq 8\left(\frac{\frac{3\gamma}{2} - 1}{\frac{2}{\gamma} - 1}\right) H^2. \tag{5.33}$$

For radiation era (i.e., $\gamma = \frac{4}{3}$) and matter dominated era (i.e., $\gamma = 1$), the lower bounds on T_{fA} become $T_{fA} \geq 16H^2$ and $T_{fA} \geq 4H^2$ respectively.

5.5.3 Thermodynamic Equilibrium

Case I: Γ is constant — If the particle creation rate Γ is assumed to be constant, then the second order time derivative of the total entropy can be obtained from Eq. (5.30) as

$$\begin{aligned}\ddot{S}_{TA} &= \frac{d^2}{dt^2}(S_A + S_{fA}) \\ &= \frac{9\gamma^2}{16}\left(1-\frac{\Gamma}{3H}\right)\left(1-\frac{2\Gamma}{3H}\right)\left[1+4\left(\frac{3\gamma}{2}-1\right)T_0^{-1}H^{\frac{2}{\gamma}-1}\left\{1-\left(\frac{2}{\gamma}-1\right)\right.\right. \\ &\quad \times \left.\left.\left(\frac{1-\frac{\Gamma}{3H}}{1-\frac{2\Gamma}{3H}}\right)\right\}\right]. \end{aligned} \tag{5.34}$$

In Table 5.1, we have explored relevant subintervals of Γ in order to test the validity of TE. For our convenience, let us introduce the notations X and Y for the expressions $\left(1-\frac{\Gamma}{3H}\right)\left(1-\frac{2\Gamma}{3H}\right)$ and $\left\{1-\left(\frac{2}{\gamma}-1\right)\left(\frac{1-\frac{\Gamma}{3H}}{1-\frac{2\Gamma}{3H}}\right)\right\}$ respectively. From the table, it is evident that TE holds unconditionally for $\frac{1}{2} < \frac{\Gamma}{3H} < 1$, while it never holds for $\frac{\Gamma}{3H} \leq \min\left\{\frac{1}{2}, \frac{2\gamma-2}{3\gamma-2}\right\}$ and $\frac{\Gamma}{3H} \geq 1$. Thus, in this case, TE is inconsistent with the cosmological constant as well as the phantom fluid.

As discussed in Lima et al. (2014), from different observational sources, it has been well established that the radiation phase was followed by a matter dominated era which eventually transited to a second de Sitter phase (or the late-time acceleration phase). Accordingly, it can be expected that in the radiation dominated era, the entropy increased but TE was not achieved. If this were not true, the Universe would have attained a state of maximum entropy and would have stayed in it forever unless acted upon by some "external agent." However, it is a well known fact that the production of particles was suppressed during the radiation phase, so in this model, there would be no external agent to remove the system from TE. Therefore, our present analysis leads us to conclude that during the radiation phase, if Γ is constant, then $\frac{\Gamma}{3H} \leq \frac{1}{3}$, or equivalently, $\Gamma \leq H$.

Case II: Γ is not constant — For a variable Γ, Eq. (5.34) can be generalized as

Table 5.1 Equilibrium configuration for different subintervals of Γ

Subintervals of Γ	Sign of X	Sign of Y	Equilibrium?
$\Gamma \leq \frac{3H}{2}$	Non-negative	Non-negative for $\frac{\Gamma}{3H} < \frac{2\gamma-2}{3\gamma-2}$	Never for $\frac{\Gamma}{3H} \leq \min\left\{\frac{1}{2}, \frac{2\gamma-2}{3\gamma-2}\right\}$
$\frac{3H}{2} < \Gamma < 3H$	Negative	Positive	Always
$\Gamma \geq 3H$	Non-negative	Positive	Never

$$
\begin{aligned}
\ddot{S}_{TA} &= \frac{d^2}{dt^2}(S_A + S_{fA}) \\
&= \frac{27\gamma^2}{16}\left[\left\{\left(1-\frac{\Gamma}{3H}\right)^2 - \left(\frac{4}{27\gamma^2}\right)\frac{\ddot{H}}{H^3}\right\}\left\{1+4\left(\frac{3\gamma}{2}-1\right)T_0^{-1}H^{\frac{2}{\gamma}-1}\right\}\right. \\
&\quad \left. -\frac{4}{3}\left(\frac{3\gamma}{2}-1\right)\left(\frac{2}{\gamma}-1\right)T_0^{-1}H^{\frac{2}{\gamma}-1}\left(1-\frac{\Gamma}{3H}\right)^2\right],
\end{aligned}
\tag{5.35}
$$

where we have substituted the value of $\dot{\Gamma}$ evaluated as

$$
\dot{\Gamma} = (6H - \Gamma)\frac{\dot{H}}{H} + \left(\frac{2}{\gamma}\right)\frac{\ddot{H}}{H}.
$$

It is evident from Eq. (5.35) that it is quite difficult to perform an analysis similar to the one that we have done in the previous case. The only definite conclusion which can be made here is that TE holds if $\ddot{H} \geq \frac{27}{4}\gamma^2 H^3\left(1-\frac{\Gamma}{3H}\right)^2$.

5.6 Chapter Summary

This chapter dealt with a thermodynamic analysis on the apparent horizon of a spatially flat FLRW universe with gravitationally induced adiabatic particle creation described by an arbitrary particle creation rate Γ. Assuming a perfect fluid EoS $p = (\gamma - 1)\rho$ with $\frac{2}{3} \leq \gamma \leq 2$, the validity of the FL, the GSL, and TE have been tested and the following results have been deduced:

- The FL holds on the apparent horizon without any restriction, i.e., irrespective of Γ and γ.
- The GSL holds if $\Gamma \leq 3H$, or equivalently, if $\frac{\Gamma}{3H} \leq 1$, which implies that the GSL is not consistent with the phantom fluid in a DE dominated universe.
- For a constant particle creation rate, TE always holds for $\frac{1}{2} < \frac{\Gamma}{3H} < 1$, while it never holds for $\frac{\Gamma}{3H} \leq \min\left\{\frac{1}{2}, \frac{2\gamma-2}{3\gamma-2}\right\}$ and $\frac{\Gamma}{3H} \geq 1$. Thus, in this case, TE is inconsistent with the cosmological constant as well as the phantom fluid.
- When Γ is not constant, the only definite conclusion which can be made is that TE holds if $\ddot{H} \geq \frac{27}{4}\gamma^2 H^3\left(1-\frac{\Gamma}{3H}\right)^2$, however, such a condition is by no means necessary for the attainment of equilibrium.

Furthermore, an expression for the total entropy with no explicit dependence on Γ has been found. Such an expression suggests that for $\Gamma = 3H$ (corresponding to a cosmological constant in a DE dominated universe), the total entropy is a constant of motion. Also, imposing the condition that the total entropy is always positive, a lower bound on the fluid temperature T_{fA} has been obtained. It is evident that $T_{fA} \geq 16H^2$ and $T_{fA} \geq 4H^2$ for radiation and matter dominated eras respectively.

Our thermodynamic analysis also shows that if Γ is a constant, then $\Gamma \leq H$ during the radiation phase.

Finally, it should be noted that our thermodynamic analysis may help to constrain various parameters of phenomenological particle creation models that have been considered in recent literature.[7]

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Chapter 6
Possible Applications in Inhomogeneous Cosmological Models

Abstract This chapter deals with a speculative application of cosmological thermodynamics in the inhomogeneous Lemaitre–Tolman–Bondi model. We have assumed a (inhomogeneous) perfect fluid for the purpose. Starting from the basic equations of the model, we have defined the surface gravity at an arbitrary horizon and obtained three equivalent expressions for it. Consequently, the Hawking temperature has been obtained. Finally, by employing the Bekenstein entropy and the generalized Hawking temperature at the horizon, we have undertaken a thermodynamic study of the model.

Keywords Backreaction · LTB metric · Mass function · Unified first law
Bekenstein-Hawking formalism

6.1 Inhomogeneous Cosmological Models

Several observations suggest that there are inhomogeneities on scales less than 150 Mpc. The impact of these inhomogeneities on the expansion of the Universe can be understood by studying other solutions of Einstein's equations which do not assume homogeneity. As stated in Bolejko et al. (2011), *Inhomogeneous cosmological models* are defined as those exact solutions of the EFE that contain at least a subclass of nonvacuum and nonstatic FLRW solutions as a limit. In short, they are generalizations (or perturbations) of the homogeneous and isotropic FLRW models of the Universe. Contrary to the FLRW models, the physical quantities in these inhomogeneous models vary from point to point in space. The assumption of homogeneity based on the philosophically motivated "Cosmological Principle" is simply an approximation employed with a view to simplify equations. This approximation has worked very well so far but inhomogeneities should be taken into account in order to analyze future and more precise observations. Further, local inhomogeneities have been anticipated as a reason for the observed accelerated expansion of the Universe due to their so-called *backreaction*[1] on the metric. The Universe can be

[1]For a discussion on backreaction in LTB model, one may take a look at the paper by Räsänen (2004).

S. Saha, *Elements of Cosmological Thermodynamics*,
SpringerBriefs in Physics, https://doi.org/10.1007/978-3-319-74706-4_6

effectively described by averaging out the inhomogeneities in order to obtain effective EFE which contain new terms together with the ones found in the homogeneous case. It is also worthwhile to note that inhomogeneities as an alternative to DE was first discussed by Pascual-Sanchez (1999). There exists many exact inhomogeneous solutions to the Einstein equations, however, only Lemaitre–Tolman–Bondi (LTB) models which are credited to Lemaitre (1933), Tolman (1934), and Bondi (1947), Szekeres–Szafron models, and Stephani-Barnes models have been studied in the context of Cosmology. Interested readers may go through the nice review by Bolejko et al. (2011) in order to have a brief overview on these inhomogeneous cosmological models. In this chapter, we devote our attention solely to the LTB model since it provides an exact toy model for an inhomogeneous universe. It may not be realistic as such due to its high degree of symmetry, however, it is interesting due to two reasons, as pointed out by Enqvist (2008). Firstly, it can be easily tested for the effects of inhomogeneities when fitting observational data without DE and secondly, due to unambiguity in data fitting, the nature of the effective acceleration in the models can be understood by comparing the averaged and exact models, after averaging out the spatial degrees of freedom. Section 6.2 chalks out the basic equations involved in the LTB model, while the subsequent sections explore the possible applications of horizon thermodynamics in such a model.

6.2 The Lemaitre–Tolman–Bondi Model: Basic Equations

We consider a spherically symmetric universe having radial inhomogeneities viewed from our location at the center. Choosing comoving spatial coordinates with the spatial origin as the symmetry center, the line element or metric for the LTB spacetime can be written as[2]

$$ds^2 = -dt^2 + \frac{R'^2}{1+f(r)}dr^2 + R^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{6.1}$$

Here, $R \equiv R(r,t)$ is the area radius and $R'(r,t) = \frac{\partial R}{\partial r}$, while the curvature scalar $f(r)$ (corresponding to κ in FLRW model) determines whether the spacetime is *bounded*, *marginally bounded*, *unbounded* according as $-1 < f(r) < 0$, $f(r) = 0$, $f(r) > 0$ respectively. Note that the transformations

$$R(r,t) \to a(t)r, \quad f(r) \to -kr^2 \tag{6.2}$$

gives us back the metric associated with the homogeneous FLRW Universe.

[2] *Overdot* and *overdash* refer to partial derivatives w.r.t. t and r respectively.

If the Universe is assumed to contain a perfect fluid[3] having EM tensor

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + p g_{\mu\nu}, \tag{6.3}$$

where $\rho(r,t)$ and $p(r,t)$ are respectively the energy density and pressure, and the fluid 4-velocity u^μ is normalized by $u_\mu u^\nu = -1$, then by introducing the *mass function* $F(r,t)$ (related to the mass within the comoving radius r) as

$$F(r,t) = R(\dot{R}^2 - f(r)), \tag{6.4}$$

the EFE read (assuming G and c to be unity)

$$\left.\begin{aligned} \tfrac{F'(r,t)}{R^2 R'} &= 8\pi\rho, \\ \tfrac{\dot{F}(r,t)}{R^2 \dot{R}} &= -8\pi p. \end{aligned}\right\} \tag{6.5}$$

Differentiating Eq. (6.4) w.r.t. t and using the second relation in Eq. (6.5), the evolution equation for R can be obtained as

$$2R\ddot{R} + \dot{R}^2 + 8\pi p R^2 = f(r). \tag{6.6}$$

Further, the EM conservation relation, $T^{\nu}_{\mu\,;\nu} = 0$, gives

$$\left.\begin{aligned} \dot{\rho} + 3H(\rho + p) = 0 \\ p' = 0 \end{aligned}\right\} \tag{6.7}$$

where $H = \frac{1}{3}\left(\frac{\dot{R}'}{R'} + 2\frac{\dot{R}}{R}\right)$ is the (averaged) Hubble parameter in the LTB model.

At this point, the reader may note that the physical interpretation of perfect fluid LTB models is quite difficult owing to the fact that $p = p(t)$, but $\rho = \rho(r,t)$, making these models very restrictive. This difficulty can be alleviated if the fluid is understood to be an inhomogeneous dust (assumed as CDM) interacting with a homogeneous DE fluid. Indeed, this approach proves to be useful to fit observations as demonstrated by Sussman et al. (2005). On the contrary, dust sources with the LTB metric are physically well motivated and are well suited to fit observations. Clarkson and Maartens (2010) have provided a justification for these models using perturbative analysis. However, Mishra and Singh (2014) showed that special types of density voids constructed with these models are not consistent with the Bekenstein-Hawking

[3]Here, fluid is assumed to consist of successive shells leveled by r having a time-dependent local density ρ. Also $R(r,t)$ is interpreted as the location of the shell leveled by r at time t with initial condition (by proper rescaling) $R(r,0) = r$. Furthermore, the proper area of the mass shells is determined by the quantity $4\pi R^2(t,r)$.

formalism. In the rest of this chapter, following the work of Saha and Chakraborty (2015), we determine whether a perfect fluid LTB model endowed with Bekenstein entropy and GHT at an arbitrary horizon R_h can be considered as a Bekenstein-Hawking thermodynamic system or not.

6.3 Surface Gravity and Hawking Temperature in the LTB Model

Let us recall the scalar quantity $\chi(r, t)$, which in the LTB model, becomes

$$\begin{aligned}\chi(r, t) &= h^{ab}\partial_a R \partial_b R \\ &= 1 - \frac{F}{R}.\end{aligned} \tag{6.8}$$

The surface gravity κ_h at an arbitrary horizon R_h is written as ($h = \det(h_{ab})$)

$$\kappa_h = \frac{1}{2\sqrt{-h}}\partial_a\left(\sqrt{-h}h^{ab}\partial_b R\right)\Big|_{R=R_h}, \tag{6.9}$$

which for the present LTB model, can be evaluated as[4]

$$\kappa_h = -\frac{1}{2}\left(\ddot{R} + \frac{\dot{R}'\dot{R}}{R'} - \frac{f'}{2R'}\right)\Big|_{R=R_h}. \tag{6.10}$$

κ_h can also be written in terms of the mass function F and its partial derivatives in the following equivalent forms:

$$\kappa_h = -\frac{1}{4R}\left(\frac{\dot{F}}{\dot{R}} + \frac{F'}{R'} - \frac{2F}{R}\right)\Big|_{R=R_h} \tag{6.11}$$

$$= -\frac{1}{4R}\left\{8\pi R^2(\rho - p) - \frac{2F}{R}\right\}\Big|_{R=R_h}. \tag{6.12}$$

Then, the Hawking temperature on the horizon R_h given by

$$T_h = \frac{|\kappa_h|}{2\pi}. \tag{6.13}$$

[4]We assume that $(\dot{R})' = (R')^{\cdot}$.

6.4 Thermodynamic Analysis with GHT

Since we are dealing with an inhomogeneous model of the Universe where physical quantities are functions of both r and t, the definitions of the cosmological horizons and the study of thermodynamics with them may not be as straightforward as we have done in the homogeneous FLRW model. In other words, thermodynamic study in the LTB model is much speculative. Keeping this in mind, we do not resort to any particular horizon but choose an arbitrary horizon located at $R = R_h$ which is assumed to be endowed with the Bekenstein entropy and the GHT (introduced in Eq. (3.19)). Thus, on the horizon R_h,

$$S_h = \pi R_h^2 \qquad \text{and} \qquad T_h^{(g)} = \beta T_h \tag{6.14}$$

with T_h obtained from Eq. (6.13). Note that the (positive) dimensionless parameter β is a function of both r and t. These are the most generalized expressions which one may consider in horizon thermodynamics as far as our study is concerned.

In order to test the validity of the Clausius relation $-dE_h = T_h^{(g)} dS_h$ on the horizon R_h, we need to determine the infinitesimal amount of energy flow across the horizon, dE_h, for which we employ the *unified first law of thermodynamics* (UFLT[5]) given by

$$dE_h = A_h \Psi + W dV_h, \tag{6.15}$$

where $A_h = 4\pi R_h^2$, $V_h = \frac{4}{3}\pi R_h^3$ are respectively the area and volume bounded by the horizon, $\Psi = \psi_a dx^a$ with $\psi_a = T_a{}^b \partial_b R + W \partial_a R$ as the energy flux, while $W = -\frac{1}{2}\text{Trace}(T)$ is the work function. The trace is taken over the 2D space normal to the spheres of symmetry. In our model, $\Psi = -\frac{1}{2}(\rho + p)(\dot{R}dt - R'dr)$ and $W = \frac{1}{2}(\rho - p)$, so,

$$\begin{aligned} dE_h &= 4\pi R^2 (\rho R' dr - p\dot{R}dt)\Big|_{R=R_h} \\ &= \frac{1}{2} dF_h. \end{aligned} \tag{6.16}$$

Note that we have used the EFE (in Eq. (6.5)) to obtain the last equality. Thus, in order for the Clausius relation to hold on R_h, we must have

$$\frac{F'}{R'} = -\frac{\beta}{2}\left\{8\pi R^2(\rho - p) - \frac{2F}{R}\right\} \tag{6.17}$$

$$\frac{\dot{F}}{\dot{R}} = -\frac{\beta}{2}\left\{8\pi R^2(\rho - p) - \frac{2F}{R}\right\} \tag{6.18}$$

[5]The UFLT is equivalent to the EFE at any spherical surface of symmetry. For a detailed discussion on UFLT, one may see the papers by Cai and Cao (2006) and Chakraborty et al. (2010).

It is easy to see that the above pair of equations imply

$$\frac{F'}{R'} = \frac{\dot{F}}{\dot{R}}. \tag{6.19}$$

which reduces to[6]

$$\rho + p = 0. \tag{6.20}$$

Thus, irrespective of β, the FL holds on R_h whenever Eq. (6.20) is satisfied. In other words, the FL does not hold true for general perfect fluid LTB models.

Our next task is to test the validity of the GSL. To that effect, let us recall that the entropy variation of the fluid inside the horizon is determined by the Gibbs equation which is expressed as

$$T_{fh} dS_{fh} = 4\pi R_h^3 (\rho + p) \left[\left\{ \frac{R_h^{'}}{R_h} - \frac{w'}{3w(1+w)} \right\} dr + \left\{ \frac{\dot{R}_h}{R_h} - H \right\} dt \right], \tag{6.21}$$

where $w(r,t) = \frac{p(t)}{\rho(r,t)}$ can be interpreted as the EoS of the fluid distribution. Again, the FL on the horizon gives

$$T_h^{(g)} dS_h = 4\pi R_h^3 \rho \left\{ -\frac{R_h^{'}}{R_h} dr + w \frac{\dot{R}_h}{R_h} dt \right\}. \tag{6.22}$$

Now, assuming $T_h^{(g)} = T_{fh}$, we obtain[7]

$$T_h^{(g)} dS_{Th} = 4\pi R_h^3 \rho \left[\left\{ w \frac{R_h^{'}}{R_h} - \frac{w^{'}}{3w} \right\} dr + \left\{ (1+2w) \frac{\dot{R}_h}{R_h} - (1+w) H \right\} dt \right], \tag{6.23}$$

by taking the sum of the last two equations. $S_{Th} = S_h + S_{fh}$ is the total entropy of the thermodynamic system. We find that the GSL will hold, provided $(1+2w)\frac{\dot{R}_h}{R_h} \geq (1+w)H$ and $w\frac{R_h^{'}}{R_h} \geq \frac{w'}{3w}$. Applying the condition obtained in Eq. (6.20), these inequalities reduce to $dR_h \leq 0$. Thus, GSL will hold if the horizon radius is a decreasing function of both the radial as well as the temporal coordinate. However, this conclusion is true only when the condition $\rho + p = 0$ holds. In other words, for general perfect fluid LTB models, the GSL may not always hold even with appropriate physical restrictions. Thus, such models may not be considered as Bekenstein-Hawking systems.

[6] Note that Eq. (6.20) is valid for any smooth function $\Omega(r,t)$ such that the one-form $d\Omega = RdF$ is integrable. The integrability of the one-form is ensured by Eq. (6.19).

[7] We assume that Eq. (6.23) is integrable, i.e., $\frac{\partial^2 S_{TA}}{\partial r \partial t} = \frac{\partial^2 S_{TA}}{\partial t \partial r}$.

6.5 Chapter Summary

This chapter dealt with a speculative application of cosmological thermodynamics in the inhomogeneous LTB model. We have assumed a perfect fluid with EoS $w(r,t) = \frac{p(t)}{\rho(r,t)}$. Starting from the basic equations of the model, we have defined the surface gravity at an arbitrary horizon R_h and obtained three equivalent expressions for it. Consequently, the Hawking temperature has been obtained. Finally, by employing the Bekenstein entropy and the GHT at R_h, we have undertaken a thermodynamic study of the model. Note that the variation of the fluid entropy has been derived from the UFLT at R_h. We have deduced the following:

- The FL, or equivalently, the Clausius relation at the horizon R_h holds true if $\rho + p = 0$, irrespective of the form of the parameter β in GHT.
- The GSL holds true if the inequalities $(1+2w)\frac{\dot{R}_h}{R_h} \geq (1+w)H$ and $w\frac{R_h^{'}}{R_h} \geq \frac{w'}{3w}$ are satisfied. Applying the condition $\rho + p = 0$, these inequalities reduce to $dR_h \leq 0$. Thus, the GSL requires the horizon radius to be decreasing in both the radial as well as the temporal coordinate.

In a nutshell, thermodynamic analysis in the LTB model suggests that general perfect fluid LTB models may not be consistent with the Bekenstein-Hawking formulation. Only certain specific models such as those governed by an EoS $w = -1$ may behave as Bekenstein-Hawking systems as far as the FL and the GSL are concerned.

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Chapter 7
Controversies and Open Issues in Cosmological Thermodynamics

Abstract This chapter serves as an epilogue to this book. It enlists several shortcomings as well as certain open issues in the research field of cosmological thermodynamics. These problems are quite intriguing and are far from being settled. Among them, the problem due to dynamic cosmological horizons is quite serious and as a result, many Cosmologists still remain sceptical about the thermodynamic formulation of our Universe based on cosmic horizons.

Keywords Dynamic cosmological horizons · Flat FLRW universe · Cosmic fluid

As promised, this chapter enlists few of the most intriguing (in my opinion!) problems within this interesting research field which are yet to be settled.

1. **The problem due to dynamic cosmological horizons**: Early proposals by 't Hooft (2003) and Susskind (1995) hinted at a striking similarity between the dynamics of our Universe and that of the BHs. However, it must be noted that the laws of BH mechanics as well as their thermodynamics apply to static BHs only, but the horizons of our Universe (in other words, cosmological horizons) are dynamic in nature. It is not evident if equilibrium thermodynamics should play a role in such highly dynamical situations. In spite of this uncertainty, we formulated such a thermodynamic study by (customarily) assuming that the cosmological apparent horizon is slowly evolving, i.e., $\dot{R}_A = 0$, during an infinitesimal time interval. Extensive research is needed to address this important issue and the correct concept of surface gravity (and hence, horizon temperature) in dynamical[1] situations should be established. In these regard, the advancements made by Binétruy and Helou (2015), Helou (2015), and Viaggiu (2015) are noteworthy.
2. **Assumption of a flat FLRW universe**: Recall that the apparent horizon in a FLRW universe has a radius $R_A = \frac{1}{\sqrt{H^2+\frac{\kappa}{a^2}}}$ for an arbitrary spatial curvature κ. However, owing to strong observational evidences, we assumed a flat universe, i.e., $\kappa = 0$, and the above equation reduced to a much simpler one, $R_A = \frac{1}{H}$. Of course, this simplified our calculations to a great extent. However, based on the most recent observational evidence obtained with the PLANCK satellite,

[1]For an interesting account of dynamical horizons, one may see the paper by Nielsen and Visser (2006).

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Ade et al. (2016) determined that $|\Omega| < 0.0005$. The curvature is indeed very much close to zero, nevertheless, it may not be identically zero. So, in future, it would not harm to study the effects of a non-zero spatial curvature κ on the cosmological thermodynamics of a FLRW universe. Binétruy and Helou (2015) have made some progress in that direction.

3. **Temperature of the cosmic fluid bounded by the horizon**, S_{fh}: While studying the cosmological thermodynamics of the FLRW (Chap. 4) and LTB (Chap. 6) models, we assumed that the temperatures of the horizon and the fluid inside are equal. The main reason behind this assumption is our apparent ignorance regarding the temperature of the fluid inside the horizon, T_{fh}, in these models. Mimoso and Pavón (2016) tried to determine upto what extent this customary assumption is justified and they found a startling result that relativistic matter or radiation can never achieve TE with the horizon, while non-relativistic matter and DE might achieve the same, although approximately. Assuming that the corresponding scalar field of DE is not in a pure quantum state, they established that the horizon and DE temperatures will remain almost the same for most part of the evolutionary history of the Universe provided they are equal or sufficiently close at some point of expansion of the Universe. They, therefore, argued that the assumption that the horizon and the DE temperatures are equal, is not unjustified. Since we considered DE fluids for the thermodynamic analysis, our assumption in this regard is fine. However, for non-DE type fluids, the equality of the two temperatures should not be taken for granted.

I conclude this book with the hope that although the research field related to the study of dynamical horizons is far from being settled and cosmologists are sceptical about many of its aspects, yet the simple presentation and the extensive bibliography presented in this book will allow the readers to grasp the basic concepts of the subject without much difficulty. Upon reading this book, the readers also become aware of the various shortcomings within this field. I am certain that most of them will feel the urge to deal with these open questions themselves which will not only make this subject more transparent and physically concrete but also unfurl new directions of research.

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