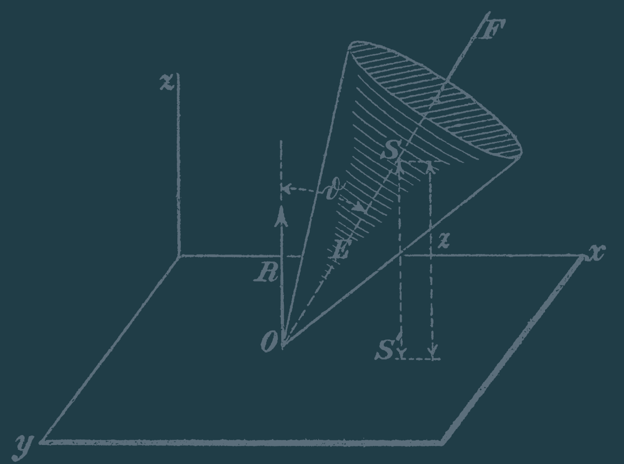
BIRKHÄUSER

Felix Klein
Arnold Sommerfeld

The Theory of the Top
Volume II

Development of the Theory in the Case of the Heavy Symmetric Top



Translated by
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Advertisement of Volume II of the Theory of the Top

(from the notices of the B. G. Teubner publishing company in Leipzig).

As a continuation of the first volume of the Theory of the Top that appeared in the summer of 1897, the second volume now follows. The first volume established the general kinematic and kinetic foundations of the theory; this volume poses, above all, the exercise of discussing the motion of the symmetric top with a fixed support point, under the influence of gravity, in all its details. Some related problems—the motion of the general top under the influence of gravity and the Poinsot motion of the force-free top for a general mass distribution—are more considered in passing and used for comparison than treated exhaustively.

The presentation is given a somewhat broader basis only for the discussion of questions concerning the stability of motion, since this currently developing theory may claim a special interest at the present time. In this part of the work, the definitions and formulations retain sufficient generality to encompass arbitrary mechanical systems. The top appears here only as a particularly instructive example, or, if one will, an "idea-forming motive." Moreover, the concept of the stability of motion is conceived here in an essentially different manner than in the relevant textbooks (of Thomson and Tait or Routh), but naturally in a way that subsumes the generally accepted concept of the stability of equilibrium.

For what concerns the actual subject of the present volume, the motion of the heavy symmetric top, the greatest possible comprehensiveness is sought in the treatment of the problem. It is therefore not sufficient to present a general formal treatment of the problem; we also seek—in the sense of the principles set out in the Introduction—to open the way for full geometric and mechanical understanding of the motion, which is, without question, a not less important goal for the treatment of a mechanical problem than the analytic command of the subject.

On this basis, Chapter IV commences with a qualitative description of the trajectory of the apex of the top, which only later is confirmed through a precise quantitative discussion of the motion. The integration of the differential equations is first accomplished in a geometric manner, whereby certain known first integrals of the motion are constructed as simple properties of the impulse vector. From the same point of view, the unquestionably best analytic method for the calculation of the motion of the top, its representation by elliptic functions, is reserved until the last chapter of this volume, in favor of the representation by elliptic integrals, which indeed is less complete, but may at first lie nearer to geometric intuition and mechanical interpretation.

On the other hand, it is necessary, if one can speak of a truly complete treatment of the problem, to pursue the analytic developments to an actual numerical calculation of the motion of the top. The conclusion of the fourth chapter thus presents an introduction to numerical calculation on the basis of the Legendre integral tables, as well as a method for the derivation of approximation formulas by which one can, in the practically most important cases, directly replace the exact formulas. In Chapter VI, the question of numerical calculation is taken up once again, and answered in the most satisfactory manner with the help of the ϑ-series (including the estimation of the error bound).

Chapter V treats of some particular and particularly distinctive types of motion. Two such motions are emphasized, which are designated as pseudoregular precession and the upright motion of the top.

Pseudoregular precession is the motion that occurs under the usual experimental conditions of a sufficiently large eigenrotation. It hardly differs, considered externally, from actual regular precession. The paradox that is associated with this motion is extensively discussed, and is reduced to an imprecision of observation. Since most popular attempts to explain the motion of the top view regular precession as the most practically important phenomenon, there follows a short summary and critique of the popular top literature.

The upright motion of the top is a uniform rotation about the vertically positioned figure axis. This motion is well known as stable for a sufficiently large rotational velocity, and as labile for a smaller velocity (where the meaning of these words is still to be discussed). Among the motions that result from a disturbance of the labile state of motion, an asymptotic case occurs that is of particular importance with respect to the associated general stability considerations.

In the representation of the motion through elliptic functions in Chapter VI, the fundamental meaning of the rotation parameters α, β, γ, δ appears in full light. In these parameters, the representation of the motion attains a simplicity and clarity that is not otherwise possible.

Moreover, all necessary developments from the theory of elliptic functions are reproduced in the book with some completeness, so that the relevant parts of the presentation can be directly regarded as an introduction to this theory. It does not appear improper, from a didactic point of view, to attach such an introduction to a specific example. The present presentation differs from others in that the connection to the general theory is brought out in a particularly clear manner through the detailed use of geometric relations.

The concluding section of this volume may be of particular interest. Here the integration problem of the motion of the top is taken up once more, and indeed on the basis of the general Lagrange equations with α, β, γ, δ themselves as the coordinates. It is shown that these equations are the so-called Hermite–Lamé differential equations, and that their integrals can be directly written in the form of elliptic functions without any intermediate calculations worth mentioning. At the same time, there follows from the form of these equations the remarkable fact that the motion of the top can be identified with the motion of a spherical pendulum in a space of four dimensions.

After the pure theory of the motion of the top has thus been brought to a certain conclusion, it will be shown in the third and final volume of the book to what extent this theory coincides with experience, or what modifications must be made so that it can be applied to a series of facts from physics and astronomy. Further, the point of view acquired in the specific example of the top will be applied to the conception of mechanics in general, and finally some detailed excursions into the domain of modern theoretical physics will be undertaken.

Volume II

Development of the Theory in the Case of the Heavy Symmetric Top

Chapter IV.

The general motion of the heavy symmetric top. Introduction to elliptic integrals.

§1. Intuitive discussion of the expected forms of motion; preliminary agreements.

We turn in this chapter to the definitive treatment of the *heavy* symmetric top with three degrees of freedom, and thus assume throughout that the center of gravity is different from the support point.

We have investigated until now only an entirely particular case of the motion of the heavy top; namely, its regular precession (cf. §6 of the previous chapter). We now treat, in contrast, of its general motion, for arbitrary choice of the initial state. But before we enter into the somewhat extensive *quantitative* discussion, which can be carried out completely only with the help of elliptic integrals, we first wish to conceive our problem *qualitatively*, and seek to acquire, in an intuitive way, an initial overview of the expected forms of the motion. A corresponding procedure is always necessary, especially for complicated problems in mechanics, since one otherwise runs the danger of losing oneself in details, and forgetting the subject itself in the formulas.*)

We first strike some simplifying agreements.

1. We always refer in the following to the *spherical top.* This is all the more permitted, as we will soon learn to reduce the general motion of the heavy symmetric top to that of the heavy spherical top. The general existence of a heavy spherical top—that is, a body with a spherical ellipsoid of inertia whose center of gravity is different from

*) Cf. here the noteworthy remarks with which Mr. Poincaré prefaces his important "qualitative" investigations of differential equations, Journal de Liouville, sér. III t. 7 and 8, 1881, 1882.[108]

F. Klein and A. Sommerfeld, *The Theory of the Top: Volume II. Development of the Theory in the Case of the Heavy Symmetric Top*, DOI 10.1007/978-0-8176-4827-5_1,

the center of the sphere of inertia—has already been illustrated by an example on page 106. We denote the common value of the moments of inertia of our top by A.

2. We will assume that the center of gravity S lies *beneath the support point* O ($P < 0$) for a vertically erected figure axis. This assumption obviously signifies no restriction of generality, since it is indeed in our hands to designate as the figure axis one or the other of the half-lines into which the line OS is divided by the point O.

3. We must further choose the elements of the motion of the top to which we will direct our primary attention. According to the Poinsot theory of rotation, we are obliged to make clear, in the first place, the locus of the rotation vector in the body and in space. The forms of the polhode and herpolhode curves would then provide a complete image of the motion. It is not easy, however, to envision the rolling of these curves distinctly; moreover, the locus of the rotation vector is poorly visible in experiments, and is brought to perception only with special devices (cf. page 14). Much more evident, for the usual construction of our models, is the locus of the figure axis in space. As a consequence, we prefer to seek, instead of the polhode and herpolhode curves, the curve described during the motion by some point on the figure axis; the point, for example, that has distance 1 from O. In that we imagine that the part of the figure axis endowed with mass (cf., for example, the figure on page 1) has length 1, we will henceforth designate the named point as the apex of the top. The curve that this "apex of the top" traces on a unit sphere described about O then provides an intuitively characteristic, if not entirely complete, image of the course of the motion; the latter condition because our curve indeed expresses only the motion of the figure axis in space, but not the rotation of the top about the figure axis.

In order to be able to reproduce the curve of the apex of the top graphically, we must project the unit sphere on which it runs onto an appropriate drawing plane. We choose for this plane the equatorial plane of the sphere; that is, the horizontal plane passing through the support point. As for the type of projection, it is perhaps most natural to choose an orthogonal parallel projection (briefly called an *"orthographic projection"*), and thus to draw the curve as it would appear to a viewer looking from an infinite distance above the sphere. This would, however, bring with it certain undesirable circumstances that will soon be pointed out. It is better to employ a *stereographic projection*, in which we use the lowest point of the sphere, the "south pole," as the center of projection. The drawing then gives the image of the curve received by an eye at the south pole of the sphere. The equator of the sphere appears in the drawing as the unit circle, whose midpoint corresponds to the highest point of the sphere, the "north pole," and whose interior corresponds to the "northern hemisphere."

Much more complete than the named orthographic projection, or also the stereographic projection employed in the following, are, however, the *stereoscopic images* of the motion of the top that Mr. Greenhill and Mr. Dewar*) have published. Here, two appropriate central projections of the trajectory are presented, which, observed through the stereoscope, bring out the complete impression of the spherical curve. The typographic difficulty of giving an absolutely adequate image of a space curve is therefore happily resolved by the employment of the stereoscope. We abstain from this reproduction of the trajectory only because we do not wish to assume that the reader is possessed of a stereoscope.[109]

4. We can further simplify, without restricting the generality of our problem, if we suitably choose the *initial time* from which we follow the motion of the top. The initial time will always be chosen so that the apex of the top is found at a highest or lowest point of its trajectory. The curve of the apex of the top will thus have an initial horizontal tangent, if it does not, in particular, form a vertically directed cusp. At the same time, the initial position of the instantaneous rotation axis is also predetermined by our choice of the initial time. The rotation axis and the coinciding impulse axis then lie, evidently, in a vertical plane passing through the initial position of the figure axis.

5. The initial state and the character of the resulting motion are now essentially determined by three data: the initial inclination of the figure axis with respect to the vertical, the initial inclination of the impulse vector with respect to the vertical, and the length of this vector. The other data that come into consideration—for example, the azimuth at which our figure axis appears as seen from above—are inessential, and

*) Proceedings of the London Math. Soc., Vol. 27, pp. 587 and ff., 1896, Engineering, Vol. 64, p. 311, 1897.

have, in particular, no influence on the form of the figures to be drawn in the following. We measure the initial inclination of the figure axis with respect to the vertical, as usual, by the angle ϑ; the position and length of the impulse vector are known if we give, for example, its perpendicular projections onto the vertical and the figure axis. Since we denote the components of the impulse vector in the X, Y, Z frame by L, M, N, the projection of the impulse vector onto the figure axis is to be denoted by the letter N. If we further introduce the components of the impulse vector in the x, y, z frame, whose z-axis, as agreed previously, should coincide with the vertical, then the projection of the impulse onto the vertical is to be assigned the letter n. The impulse component N determines the velocity r with which the top turns about its own axis. Thus N will be designated concisely as the *eigenimpulse*, and the velocity component r as the *eigenrotation*. On the other hand, the impulse component n represents a turning-impact about the vertical axis that is equivalent to a certain horizontally directed ordinary impact exerted on the apex of the top. This impact determines the velocity with which the apex of the top progresses laterally (that is, in the horizontal direction) in its initial position. As a consequence, the impulse component n will be designated concisely as the *lateral impact*. In summary, we can say that the character of the general motion of the top depends essentially on only three constants: the initial inclination of the figure axis with respect to the vertical, the eigenimpulse, and the lateral impulse at the beginning of the motion; that is, on the values of the quantities ϑ, N, and n at the time $t = 0$. It will be shown in the third section, moreover, that the impulse components n and N retain their initial values for the entire course of the motion, so that we can speak simply of the "constants" n and N instead of the "initial values" of these quantities, which circumstance we will, for simplification of the manner of expression, already make use of now.

6. While we will bestow on the constant n *all possible values*, we wish to assume, in this section, that the constant N is *positive*; we will thus assume that the rotation of the top about the figure axis occurs in the clockwise sense. Further, we wish to fix the initial value of the angle ϑ in a special manner from the outset. Namely, we specify that the figure axis is initially *horizontal*, so that the angle ϑ is equal to $\frac{\pi}{2}$.

The curve of the apex of the top will thus proceed from the equator of the unit sphere, having at the equator a highest or lowest point. The extent to which the following considerations are specialized by these stipulations $\left(N > 0 \text{ and } \vartheta = \frac{\pi}{2}\right)$ will be examined later. —

After these preliminary agreements, we can proceed to classify all possible forms of motion of our spherical top that are given by arbitrary values of the constants n and N between a few particularly simple special cases, and avail ourselves, moreover, of a kind of continuity principle, which we formulate thus: *for continuous changes of the initial state (the values of n and N), the motion of the top will also change continuously; discontinuous transitions, which mechanically speaking would be interpreted as unstable forms of motion, will first be regarded by us as excluded.* This principle demands an exact quantitative verification, as does our entire qualitative manner of deduction. In fact, the following considerations are set forth not as absolutely rigorous, but rather as only plausible; we will later have to supplement them in various directions. However, the figures that are drawn for the curves of the apex of the top are not only qualitatively, but also quantitatively correct.

The totality of cases that we have to review are represented, according to the infinitely many values that the constants N and n may have, by a twofold infinite multiplicity. (Cf. here the schema in Fig. 36 on page 215.) We must naturally be satisfied, in the following, with extracting a series of individual characteristic types from this multiplicity.

We consider first the special cases that we have indicated. These are, on the one hand, *regular precession*, and, on the other hand, *pendulum motion.*

According to pages 178 and 179, there are, for a given mass distribution of the top and for given values of the (constant) angle ϑ and the velocity component μ, two possible values of the corresponding velocity component ν that give rise to a regular precession. In the case of the spherical top, these values are both real, and are given by the equations

$$\text{a)}\quad \nu = \frac{P}{A\mu}, \qquad \text{b)}\quad \nu = \pm\infty.$$

It is easy, especially in the present case $\vartheta = \frac{\pi}{2}$, to go over from the precession constants ν and μ to our impulse components n and N.

The quantities ν and μ signify (cf., for example, the figures of page 48) the parallel projections of the rotation vector onto the vertical and the figure axis, respectively. If the figure axis now stands perpendicular to the vertical, these parallel projections are identical with the respective normal projections of the rotation vector that we have previously denoted by ϱ and r. From ϱ and r, however, the impulse components n and N for our spherical top are calculated by multiplication with the value of the moment of inertia, which we denote by A. We therefore replace, in a) and b), ν and μ by $\frac{n}{A}$ and $\frac{N}{A}$, respectively. As a result, the magnitude of the lateral impulse n that gives rise, together with the eigenimpulse N, to a regular precession $\vartheta = \frac{\pi}{2}$ is

$$\text{a)} \quad n = \frac{AP}{N} \quad \text{or} \quad \text{b)} \quad n = \pm\infty.$$

Case a), the previously so-called "slow precession," occurs for a lateral impact that acts in the counterclockwise sense as seen from the vertical ($n < 0$), since we assumed above that $P < 0$ and $N > 0$; in case b), the so-called "fast precession," the sense of the lateral impact is undetermined ($n = \pm\infty$).

We have next to acquaint ourselves with the second of the above-named special cases, the *pendulum motion.* The top moves as an ordinary pendulum if we impart to it, in the initial position, neither an eigenimpulse nor a lateral impact ($N = n = 0$), and therefore, concisely said, if the impulse vector initially has length zero. Although this statement is self-evident, we nevertheless wish to prove it in detail from our considerations of the impulse.

If, for a horizontally placed figure axis, the top is abandoned to the influence of gravity, this influence generates, during the first moment of time dt, an infinitesimal impulse vector of magnitude $P \sin\vartheta\, dt = P\, dt$ that has the line of nodes OK (cf. Fig. 24) as its axis. The top therefore begins to turn about this axis, so that the angle ϑ is diminished. The position of the line of nodes will not be changed by this rotation. In the next moment, the additional impulse of gravity acts about the same axis OK and is added to the previous impulse algebraically; the rotational velocity of the top about this axis accelerates correspondingly; the line of nodes retains its original position. Through repetition of this consideration, one recognizes that the impulse of the top continuously falls in the direction OK; the motion consists at each instant of a rotation about this axis; the figure axis moves in a fixed vertical plane, and the curve of the apex of the top is a vertically positioned circular arc. The velocity of the apex of the top along the trajectory is calculated by the condition that the rate of change of the impulse at any time must equal the exterior turning-force $P \sin \vartheta$, or, if we wish to avail ourselves of the manner of expression of D'Alembert's principle, that the turning-force must maintain equilibrium with the inertial resistance of our motion.

It remains only to show that the apex of the top descends, after overrunning the highest point of the sphere, just as much as it previously ascended; that is, falls to a point of the equator and then reverses. Let us conceive, for this purpose, that moment at which the apex of the top passes the highest point of the sphere. If we were to reverse the sense of the impulse vector present at this moment, and therefore the velocities of the collected points of the top, the apex of the top would traverse, according to a general fundamental theorem of mechanics, its previous path in the reverse sense, therefore describing a circular quadrant. The path that the apex of the top describes in the continuation of its original motion results from this reversed path, however, by reflection in the vertical plane OK, as follows from the symmetry of the force system that influences the motion. The continuation of the path consequently consists again of a circular quadrant that is descended until the equatorial point B diametrically opposed to the equatorial point A is reached. Since the apex of the top arrives at B with velocity zero, we now have exactly the same conditions as at the beginning of the motion at A. As a result, the continuation of the trajectory consists of the semicircle AB traversed in the reverse sense, and so forth. The motion is thus characterized as a simple pendulum motion.

In the stereographic projection, our trajectory appears (cf. the adjacent figure) simply as a diameter (AB) of the unit circle that is traversed alternately in the sense of the upper or lower arrow.

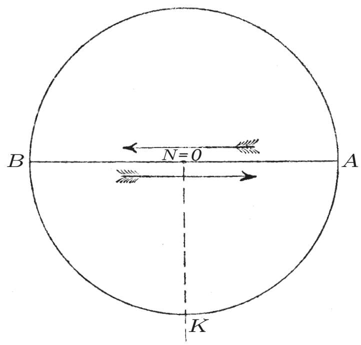


Fig. 24.

We begin with this first figure, characterized by the values $n = N = 0$, in order to make clear the more general cases of the motion. We will next give a series of figures in which the lateral impact is continually equal to zero ($n = 0$), while the eigenimpulse (N) will successively increase.

§2. Intuitive discussion of the expected forms of motion; continuation and conclusion.

As we now go over to the actual carrying out of our qualitative discussion, we first take, as agreed, $n = 0$, and moreover assume, in the first figure to be developed (Fig. 25), that the eigenimpulse is relatively small compared to the change in the length of the impulse vector that the action of gravity produces during a unit time, so that continuity with the preceding figure is preserved. In order to be able to assess the resulting change of the trajectory, we wish beforehand to lead the top artificially along the previous path with the previous velocity, and at the same time take care that the eigenimpulse N retains its initial value. This would be realized practically, for example, if one constructed a groove with the form and position of the previously described trajectory, in which the apex of the top could slide without friction.

In this enforced motion, the equilibrium between the force of gravity and the inertial resistance that existed for the free pendulum motion $N = 0$ is no longer present. In fact, only the changes in the horizontal component of the impulse will be compensated by the force of gravity, while the changes in the eigenimpulse, which, as we assume, retains its length but changes its direction in space simultaneously with the figure axis, remain unbalanced. As a result, there arises a resistance that we can designate, since it stands perpendicular to the instantaneous rotation axis, as a deviation resistance (cf. §5 of the preceding chapter). In our example, this resistance would be manifested as a lateral pressure on the groove.

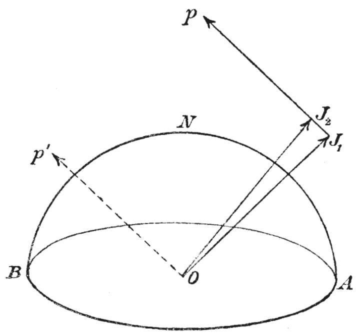


Fig. 24 a.

The sense of this pressure is easy to see from the auxiliary figure above. The figure shows the eigenimpulse in two neighboring positions OJ_1, OJ_2 during the first phase of the motion; that is, while the apex of the top oscillates on the vertical circle ANB from A to B. The sense of the change of the impulse is given by the arrow p. The deviation resistance, which is opposed to the sense of the impulse change, acts about the axis of this arrow, or the parallel arrow p' extended through O, in the counterclockwise sense as seen from p or p'. It therefore seeks to deflect the apex of the top to the rear of our figure during the entire first phase of the motion. One recognizes, in the same way, that during the second phase of the motion—that is, while the apex of the top oscillates from B to A—a deviation resistance appears that strives to overturn the apex of the top to the fore of the figure. In general, we can say that *the apex of the top seeks, as a result of the deviation resistance, to deflect to the right as calculated from its direction of progression*; the right wall of the groove would therefore have to bear, for the given sense of the motion, a certain pressure. The magnitude of this pressure, which is equal to the uncompensated part of the rate of change of the impulse, is, as likewise follows from our latter figure, directly proportional to the magnitude of the eigenimpulse.

We now set the figure axis free, in that we remove the guidance of the apex of the top. The consequence is this: *the apex of the top will evade the corresponding deviation resistance of the trajectory to the right.* The straight line by which we represented the pendulum oscillation in Fig. 24 will go over, using the previous graphical manner of presentation, into an arc that opens toward the side to which the deflecting force of the deviation resistance acts. Since we have assumed for the present that the eigenimpulse is relatively small, the deviation of our arc from the straight line will also be relatively small (cf. Fig. 25).

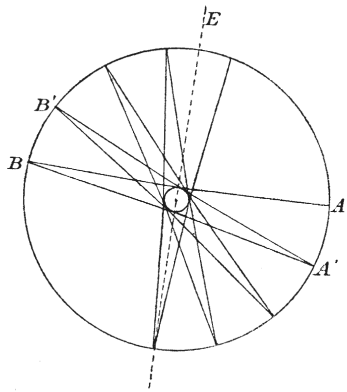


Fig. 25.

It is easy to see that the apex of the top must progress from its initial position A perpendicularly with respect to the equator, both in space and in our stereographic image. We need only make clear, for this purpose, the approximate position of the impulse vector. The impulse vector initially lies, since we assumed $n = 0$, horizontally. The following positions of our vector result from this initial position if we successively add the supplemental impulse of gravity geometrically. The axis of this supplemental impulse likewise lies, however, always horizontally. The impulse vector must therefore have a horizontal axis during the entire duration of the motion (in contrast to the enforced motion represented in Fig. 24a, where the impulse vector is gradually elevated). In this consideration already lies, as we note in passing, the proof of our previous assertion that the impulse component n always retains its initial value $n = 0$, to which we will return in the next section under more general assumptions. The rotation axis of our spherical top, however, coincides with the impulse axis, and is therefore also horizontal. Since the apex of the top progresses at each moment perpendicularly with respect to the direction of the rotation axis, the apex of the top must progress from its assumed initial position on the equator in the vertical direction. The trajectory of the apex of the top is thus perpendicular to the equator at the beginning of the motion, as claimed. Moreover, the velocity of progression is zero at the first moment, since the rotation axis initially passes through the apex of the top itself.

To survey the entire course of our trajectory, we must only further consider its symmetry properties. We conceive, for this purpose, that moment at which the apex of the top has attained its highest position (H) on the sphere. According to our continuity principle, this highest point deviates only slightly from the highest point that the apex of the top attains in the pendulum motion, the north pole. The stereographic image of this highest point will therefore be not far removed from the midpoint of the figure. Through O and H we lay the vertical plane E, whose intersection with the equator gives us the position of the instantaneous impulse vector. We now argue just as above for the simple pendulum.

If we were to reverse the sense of the impulse vector at the considered moment, the apex of the top would traverse its previous path AH in the reverse sense, and arrive at A with velocity zero. At the same time, we consider that in each of the two positions of the top that are symmetric with respect to the plane E, the turning-moment of gravity is the same. It is thus to be concluded that the apex of the top will describe, in the continuation of its original direction of motion from H, the arc HB that is symmetric to AH with respect to the vertical plane E, and that it will arrive at B with the velocity 0. The arc AB of our trajectory thus consists of two equal mirror-image halves.

Arriving at B, the apex of the top finds itself under exactly the same conditions as at A. As a result, the further course of the trajectory must be the same as at the beginning of the motion. A new arc BA' is therefore attached at point B, an arc that consists, in its turn, of two symmetric halves, and is congruent to the arc AB. It can be generated from the latter by a rotation about the vertical. A cusp evidently arises in this manner at B. The same consideration is valid for point A' as for point B. The trajectory also forms a cusp at A', and progresses with an arc $A'B'$ that is congruent to the arc AB. All these arcs AB, BA', $A'B'$, ... will, because of their congruent form and similar position, be tangent at their respective highest points H to a certain parallel circle on the sphere, which in our case narrowly encompasses the north pole. In summary, we can describe the course of our trajectory in the following manner.

The trajectory of the apex of the top represents, in our case, a zigzag curve that circulates about the vertical in the counterclockwise sense, without, in general, closing; it consists of a series of congruent arcs, or, if we wish, a series of half-arcs that are alternately symmetrically equal and congruent to each other. The curve is entirely contained within two parallel circles; namely, in our case, the equator and a parallel circle in the neighborhood of the north pole. Our curve is tangent to the latter circle where it strikes it; it touches the former with cusps.

A word here about the advantage of our chosen method of projection. If we had used the orthographic projection instead of the stereographic projection, apparently regular arcs would appear instead of cusps, as shown in the orthographic projection of Fig. 25a. In fact, every space curve that terminates in a cusp presents, viewed from the tangent direction of the cusp, an aspect that in no way betrays to the eye the presence of a singularity. Instead of the cusps present in the original, we have only a so-called "masked singularity" in the image. It is thus clear that the use of the orthographic projection would efface the character of the trajectory. We will, therefore, always produce the following figures in the stereographic projection.[110]

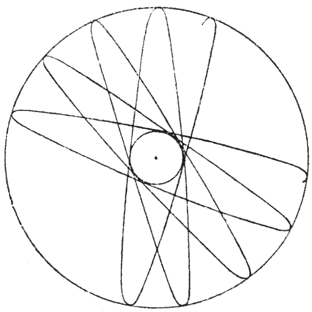


Fig. 25a.

We now wish to let the magnitude of the eigenimpulse successively increase, and at the same time take the lateral impact, as previously, equal to zero. If we again make the orienting experiment of page 204, in which we lead the apex of the top along the path of the pendulum motion, then we will notice a now stronger deviation resistance that strives to deflect the figure axis to the right of its direction of motion. In fact, it follows from Fig. 24a that this resistance is proportional to the magnitude of the eigenimpulse. If we go over from the enforced pendulum motion to the unconstrained top motion, then the curvature of the individual arcs of which the trajectory is composed will become greater, and their span width smaller, as we increase the value of the eigenimpulse N. At the same time, the highest point of the individual arcs is successively removed from the north pole; the bounding parallel circle that contains the collected highest points of the arcs must therefore broaden with increasing N.

The following figures express these phenomena in three steps. In Fig. 26, the eigenimpulse is chosen approximately three times as great as

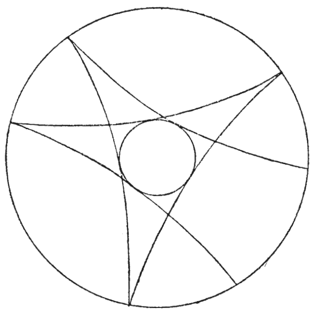


Fig. 26.

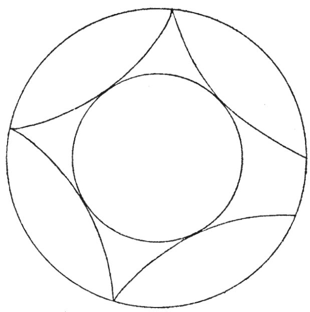


Fig. 27.

in Fig. 25, and in Fig. 27 nine times as great. Fig. 28 represents the limiting case of a very large N. While in Figs. 26 and 27 the relation with the figure of the pendulum may still be recognized clearly, Fig. 28 shows a trajectory that differs only microscopically, so to speak, from a continuously traversed circle. It has the smallest conceivable similarity with the figure of the pendulum motion, sooner appearing, on imprecise inspection, to coincide with the second of the previously mentioned special cases, the regular precession. This coincidence, however, pertains, as we must emphasize, only to the *location* of the apex of the top, and not to its *velocity and velocity direction.* While in regular precession the velocity of progression of the apex of the top is constant along the entire circle, it varies for our case, in a very short time interval, between the value zero that occurs at the equator and a maximum value that is attained at the tangent point with the second bounding parallel circle. We will be able to fittingly designate this most highly noteworthy form of motion as a *pseudoregular precession.*

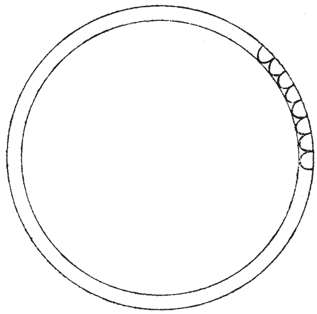


Fig. 28.

In experiments, this latter case is even the rule, since the devices for winding the top usually produce a very strong eigenimpulse compared to the action of gravity. If one thus investigates the motion of the top only experimentally, one easily comes to the paradoxical conception that the apex of the top initially moves perpendicularly to the direction of the acting force, a conception that naturally runs directly counter to the principles of mechanics, but which is nonetheless frequently advocated in the literature.*) We therefore emphasize expressly that *under the previously assumed initial conditions* $(n = 0)$, *our trajectory is always a cusped curve; an actual regular precession is completely impossible.*

For the arrangement of experiments, we infer from our latter figure the rule that we must choose the eigenimpulse of the top as small as possible if we at all wish to observe an accurately confirmable trajectory; it is thus recommended that the top be set into rotation by the hand instead of with the cord, so that the strength of the impulse can be conveniently regulated. —

While we have thus far allowed the eigenimpulse of the top to be increased stepwise with the lateral impact constantly assumed to be zero, we will now, conversely, vary the lateral impact and fix the eigenimpulse. Thus an entire series of new figures develops from each of the previous.

From the ordinary pendulum motion in Fig. 24, for example, there always arises, with the addition of a lateral impact, a case of the *motion of the so-called spherical pendulum*, in which the apex of the top behaves just like a heavy mass particle that is fixed to the end

*) Cf. our criticism of the popular top literature in the following chapter (§3).

of a rigid and massless rod pivoted at O, to which a horizontally directed impulse is imparted in its initial position. We need not delay ourselves here with this well-known and easily observed motion. We only mention, for the sake of the following, that the reversion points (cusps) that appear at the equator in our figure for the ordinary pendulum motion are resolved into flattened arcs that are tangent to the equator, and that the span width of the arcs, which originally amounted to two right angles, is somewhat broadened. In fact, the apex of the top must, since it is subject to the lateral impact n in its initial position and in each of the following positions where it reaches the equator, progress instantaneously in these positions in the direction of the equator, and indeed, according to the sign of n, in the clockwise or counterclockwise sense as seen from the vertical.

If we begin, on the other hand, from the pseudoregular precession in Fig. 28, there then arises, with the addition of any lateral impact, a motion that, observed coarsely, is not very different from the previous, and that (in a broadened sense) may again be designated as a pseudoregular precession. Here our microscopic cusped arcs will resolve into small loops or flattened arcs, according to whether we let the initial horizontal impulse act in the clockwise or counterclockwise sense about the vertical. Observation naturally gives no clear account of this modification of the trajectory.

Essentially new types result, in contrast, from Figs. 25–27. We wish, in particular, to consider in detail the case of a relatively small eigenimpulse, and therefore develop the series of figures that results from Fig. 25 by variation of n. We first let n decrease from the value zero in Fig. 25, therefore imparting to the apex of the top in the initial position an impact that seeks to turn it in the counterclockwise sense.

We recall that for a certain negative value $n = \dfrac{AP}{N}$ calculated above, regular precession must occur. The trajectory of the apex of the top simply becomes, in this case, the equator traversed in the counterclockwise sense (cf. Fig. 31 below). The comparison of Figs. 25 and 31 now provides a clue that enables us to assess the form of the trajectory for the intermediate values of n $\left[0 > n > \dfrac{AP}{N}\right]$. We may presume on the basis of our continuity principle, namely, that these trajectories must always be classified between Figs. 25 and 31. Relying on this principle, or on our earlier deliberations, we will further assume that the general symmetry relations of trajectory 25, the congruence of the component arcs, etc., persist for the addition of a lateral impact.

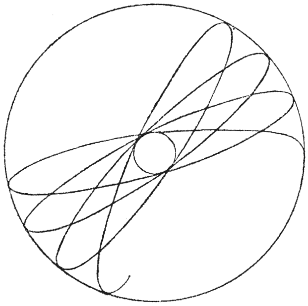


Fig. 29.

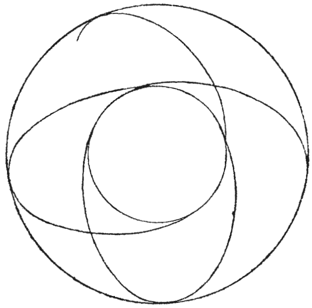


Fig. 30.

In Fig. 25, the individual congruent arcs were contained between the equator and a smaller parallel circle in the neighborhood of the north pole; in Fig. 31, we can say, this second parallel circle coincides with the equator, since the trajectory itself has gone over into the equator. We thus conclude, for intermediate values of n, that the second bounding parallel circle is always enlarged. As a result, the component arcs of the trajectory will bulge toward the equator, and at the same time must be successively stretched in length for decreasing n. Moreover, it is clear, just as in the passage from the ordinary to the spherical pendulum, that the cusps in Fig. 25 will be resolved into flattened arcs, since indeed wherever the apex of the top reaches the equator, the impact n acts in a direction tangent to the equator.

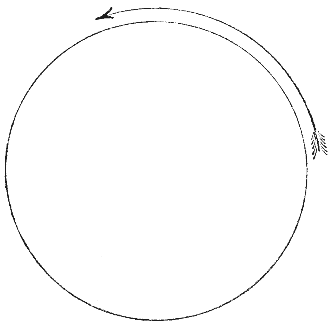


Fig. 31.

We thus draw the three Figs. 29, 30, and 31, of which the first corresponds to a small value of the lateral impact, so that the continuity with Fig. 25 is evident (the particular value $n = -N$ was chosen in the figure), and the second to a greater value of n (and indeed a value fivefold greater than in the previous case); the third figure is the special case of slow regular precession $n = \dfrac{AP}{N}$ (under the proportions of the drawing, this case occurs for a value of n again fivefold greater than that in Fig. 30.)

We can thus describe the collected character of the trajectories, under the present conditions of a constant N and a value of n that decreases from zero to $\dfrac{AP}{N}$, in the following manner:

The trajectory progresses in the counterclockwise sense about the vertical without, in general, closing; it always consists of a series of symmetrically equal or congruent half-arcs, whose span width successively increases with decreasing n*, and which are pressed more and more to the equator. The trajectory is entirely within two parallel circles, to which it is always tangent where it strikes them; namely, at the equator, on the one hand, and, on the other hand, at a parallel circle that is always enlarged with decreasing* n*.*

We return once more to Fig. 25, and now let n increase in the positive sense for a fixed value of N; we therefore impart to the apex of the top, in its initial state, a lateral impact in the clockwise sense. The result will thus be, to a certain degree, the reverse of the previous. While the second bounding parallel circle broadens, as we saw, with decreasing n, it will first diminish with increasing n; while the span width of the individual component arcs was earlier increased, it now initially decreases. Otherwise, the general character of the motion remains similar to Fig. 25; in particular, the trajectory must circulate, considered as a whole, in the counterclockwise sense about the vertical for sufficiently small positive n. On the other hand, however, the apex of the top moves, because of the positive sign of n, along the equator in the clockwise sense in its initial position, and also at each later moment when it reaches the equator. We thus conclude that there is a point on each component arc where the apex of the top, proceeding in the radial direction, changes its rotation sense about the vertical; we thus recognize the necessity of the appearance of loops. The cusps of Fig. 25 that went over into flattened arcs in Fig. 29 now resolve into loops, as is indeed a frequent phenomenon in geometry. All these remarks will be confirmed by the following Fig. 32, which corresponds to a very small positive value of n (we have chosen $n = 0{,}4\,N$).

If we let n increase further, we soon arrive at a value where the inner parallel circle has contracted to the north pole of the unit sphere (cf. Fig. 33). It is noted in passing that this case corresponds, as we will later see, to the value $n = N$. The points in the previous figure at which the rotation sense about the vertical changes now come together in the north pole of the sphere.

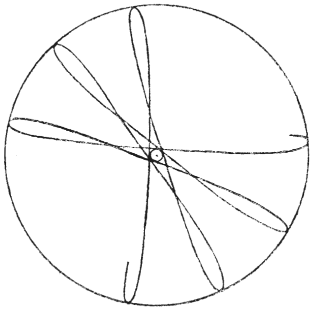


Fig. 32.

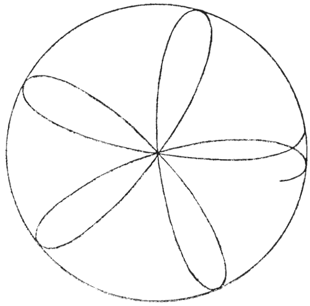


Fig. 33.

To envision the configuration of the trajectories with progressively increasing n, we finally draw upon the limiting case $n = \infty$, for which, according to page 202, the motion again becomes a regular precession. While the inner parallel circle is reduced to a point in the case $n = N$, it coincides in the case $n = \infty$ with the exterior bounding circle. We will again presume, on the basis of our continuity principle, that the parallel circle in question continuously widens with increasing values of n $(N < n < \infty)$. While it was previously tangent to the trajectory from the exterior, it will now be enclosed by the trajectory. The curve now runs throughout in the clockwise sense, and indeed with increasing velocity about the vertical as n increases;

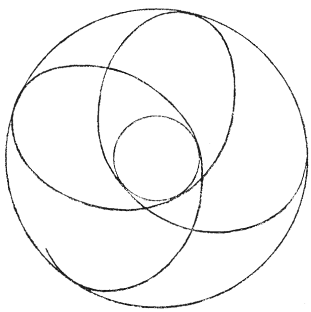


Fig. 34.

it will be pressed by the widening inner circle always more to the equator; the span width of the congruent arcs of which it is composed becomes larger and larger. An example of this form of the trajectory is given in Fig. 34, in which, incidentally, $n = 5N$ is assumed. The limiting case $n = \infty$, the fast regular precession, may be indicated schematically by Fig. 35. Corresponding to the twofold sense of the running arrows, we can conceive the latter figure just as well as the limiting case of the trajectory for infinitely increasing positive as for infinitely decreasing negative n.

If we wish to summarily describe the behavior of the trajectory for increasing positive n, we can speak, for example, in the following manner:

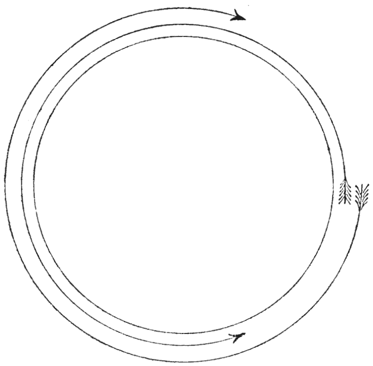


Fig. 35.

For positive n, the trajectory always runs in the clockwise sense in the neighborhood of the equator, but twice reverses its rotation sense, for not too large values of n, within each of the congruent arcs of which it is composed. The appearance of loops is characteristic for these trajectories. The inner bounding circle is initially excluded from the trajectory, but is enclosed from the exterior after the highest point of the sphere is once crossed. From then on, the span width of the individual component arcs always increases, and becomes infinitely large in the limiting case $n = \infty$.

It remains only to investigate the passage from the slow precession $n = \dfrac{AP}{N}$ in Fig. 31 to the fast precession $n = \pm\infty$ in Fig. 35. We saw that the moving parallel circle successively nears the equator with decreasing $n < \dfrac{AP}{N}$, and coincides with the equator in the case of regular precession. For further decrease in n, it will first retain its direction of motion, and therefore again be removed from the equator, going over into the southern half of the unit sphere or the exterior of the unit circle in the stereographic projection. Correspondingly, the equator in the image would henceforth become the inner, and the moving parallel circle the outer bounding circle. This tendency of the motion does not, however, last long; there is, namely, a deepest (or in the image, most exterior) position for our moving parallel circle. After this position is attained for a certain value of n, the parallel circle again tends, for further decrease of n, toward the equator, with which it indeed must coincide in the case $n = -\infty$. Under the assumed proportions of our figures, however, the extreme position of our parallel circle lies so near the equator that it would be completely indistinguishable from the equator by eye. Correspondingly, the trajectory of the apex of the top will also always remain extraordinarily near the equator, so that we must forgo its graphical reproduction. In the stereographic image, it would enclose the equator in nearly circular windings, which, corresponding to the large negative value of n, would be traversed in the counterclockwise sense with great velocity.

The circle of possibilities that are offered for fixed N and variable n has thus been closed.

We could now develop, by adding a lateral impact, the corresponding series of figures from Figs. 26 and 27, just as we have done for Fig. 25. We must be content, however, to point out later (cf. §7) the differences from the preceding series that would thus appear.

In conclusion, we wish to localize the preceding figures through the entry of their numbers in a schema, in which n is assigned as the abscissa and N as the ordinate.

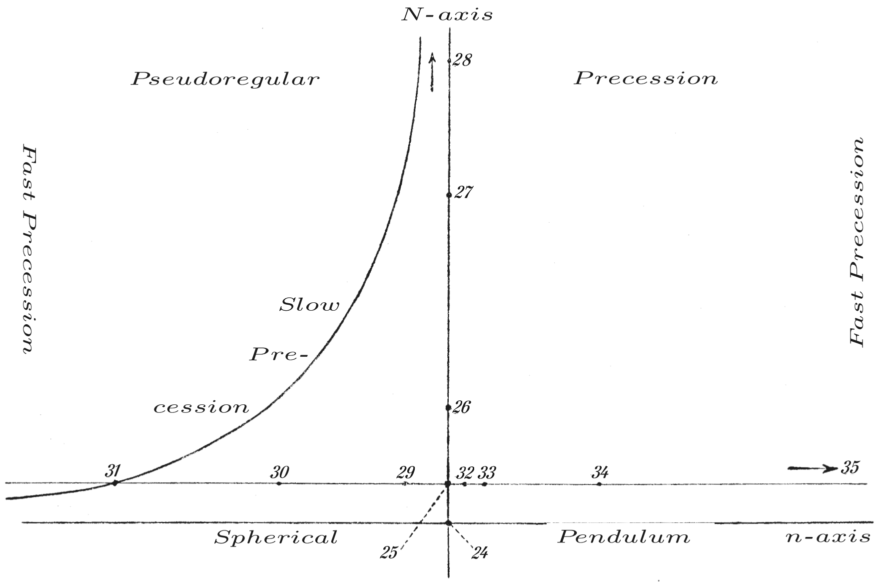


Fig. 36.

The ordinate axis of our schema contains Figs. 24 through 28; the abscissa axis corresponds to the collected cases of the ordinary or spherical pendulum motion. Figs. 29 through 35 lie on a line parallel to the abscissa axis and slightly removed from it. If we go parallel to the ordinate axis to infinity, we always arrive in the domain of pseudoregular precession; if we proceed in the direction of the abscissa to infinity, we find the case of fast regular precession. The slow regular precession determines, as follows from the formula $nN = PA$, an equilateral hyperbola in the drawn position. The doubly extended multiplicity that is formed by the totality of all trajectories with a horizontal initial position of the figure axis and a positive value of the eigenimpulse thus finds an intuitive representation in our schema.

§3. Quantitative treatment of the general motion of the heavy symmetric top. Execution of the six required integrations.

We now go over from the approximate qualitative discussion to an exact quantitative treatment, in which we first remove the restriction to the spherical top and consider an arbitrary symmetric top with moments of inertia A and C. The final goal is to integrate the system of differential equations that governs the motion of the top.

We first employ some impulse considerations of a geometric character that allow the execution of the intended integration to be, in part, replaced. We rely essentially on the fundamental theorem IIa of page 115, according to which the rate of change of the impulse in space is equal to the turning-force produced by the external forces; that is, in our case, to the turning-moment of gravity. We seek above all to express, as on page 123 for the force-free top, the detailed form of the two "impulse curves"; that is, the curves that the endpoint of the impulse vector describes relative to space and relative to the body.

We first note that the turning-force of gravity, whose axis indeed falls along the line of nodes, is constantly perpendicular to the vertical. According to our impulse theorem, the endpoint of the impulse in space therefore proceeds in a direction that is also constantly perpendicular to the vertical. We thus see that

The endpoint of the impulse moves relative to space in a horizontal plane; our first impulse curve is therefore a plane curve; the projection of the impulse onto the vertical—that is, the quantity earlier designated as the "lateral impulse"—has an invariable length.

If, as agreed on page 200, we denote the components of the impulse (the coordinates of its endpoint) in the xyz frame by l, m, n, we then have

(1) $$n = \text{const.}$$

This result was already mentioned in the previous section, and was established for a special case in a similar manner.

Secondly, we consider the curve that the endpoint of the impulse describes relative to the top. Here we begin, instead of from the impulse theorem IIa, which determines the rate of change of the impulse in space, from the impulse theorem IIb of page 145, which establishes the change of the impulse with respect to the top. According to this theorem, the rate of change of the impulse with respect to the top is equal in direction and magnitude to the turning-force of the external forces augmented by the so-called centrifugal turning-force. The latter was found in the same place to be equal to the vector product of the impulse and rotation vectors; its axis thus stands perpendicular to these two vectors, and therefore stands, since for the symmetric top these two vectors lie in a plane with the figure axis, perpendicular to the figure axis. Since, moreover, the turning-force of gravity also stands perpendicular to the figure axis, we see that the endpoint of the impulse must constantly progress in the top perpendicularly to the figure axis. We thus have the following theorem:

The endpoint of the impulse moves relative to the top in a plane parallel to the equatorial plane; our second impulse curve is also a plane curve; the projection of the impulse onto the figure axis—that is, the quantity introduced above as the eigenimpulse—has an invariable length.

If we denote, as previously, the impulse components in the XYZ system by L, M, N, then the equation

(2) $$N = \text{const.}$$

obtains. We have thus derived in general a result likewise already mentioned in the previous section.

In addition to the impulse curves, we can consider the curves that the endpoint of the rotation vector describes relative to space and relative to the top; that is, the *herpolhode* and *polhode* curves. Of these, however, only the polhode curve behaves as simply as the impulse curve. *In fact, the polhode curve lies in a plane perpendicular to the figure axis;* its points have the fixed distance from the equatorial plane

$$r = \frac{N}{C}.$$

If we seek, in contrast, to make the corresponding passage from our first impulse curve to the herpolhode curve, which consists, as we know, in a deformation with respect to the axes X, Y, Z, then we obtain, because of the changing spatial position of these axes, no such simple result: *the herpolhode curve is* (disregarding the case of the spherical top) *not a plane curve, but rather*, as we will soon show, *a generally spherical curve.*

This remark can once again serve to place in the correct light the advantage that the impulse vector offers over the rotation vector in kinetic respects. It obviously leads to intractabilities if one places the kinematically defined rotation vector before the impulse in kinetic questions. The impulse vector, in kinetics, is the simplest motion-regulating element.

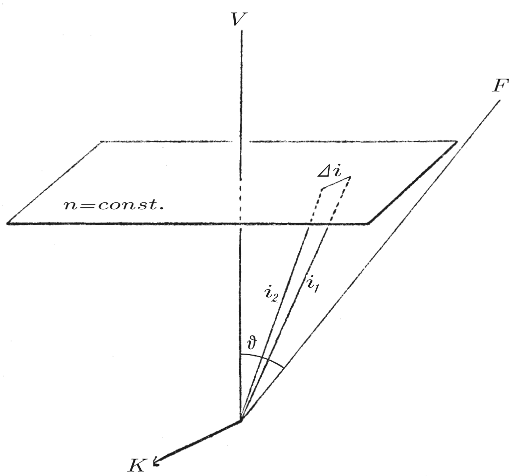


Fig. 37.

While we have thus far drawn our conclusions from the direction of the impulse change, we now wish to consider its magnitude as well. We will thus be led to a further property of the general motion of the top.

Consider the two neighboring positions i_1 and i_2 of the impulse vector at the beginning and end of the time interval Δt. The endpoints of i_1 and i_2 give a binding line Δi that is parallel, according to our impulse theorem, to the line of nodes, and, for sufficiently small Δt, has the length

$$\text{(a)} \qquad |\Delta i| = P \sin\vartheta \, \Delta t.$$

At the same time, according to the Pythagorean theorem (cf. the figure),

$$\text{(b)} \qquad \begin{aligned} |i_2|^2 - |i_1|^2 &= |\Delta i|^2 - 2|i_1||\Delta i| \cos(i_1, \Delta i) \\ &= |\Delta i|^2 + 2|i_1||\Delta i| \cos(i_1, K). \end{aligned}$$

Now $|i_1| \cos(i_1, K)$ signifies the projection of the impulse vector onto the line of nodes, which coincides, up to the factor A, with the projection of the rotation vector onto this line. A rotation about the line of nodes produces in the time interval Δt, however, a change in the angle ϑ of such value that $\Delta\vartheta : \Delta t$ will equal the magnitude of the named rotation. We thus have

$$|i_1|\cos(i_1, K) = A\,\frac{\Delta\vartheta}{\Delta t},$$

and, because of (a),

$$\text{(c)} \qquad |\Delta i||i_1|\cos(i_1, K) = AP\sin\vartheta\,\Delta\vartheta.$$

If we pass to the limit $\Delta t = 0$ in equation (b), then the term of the second order $|\Delta i|^2$ falls away, and we obtain, with consideration of (c),

$$d(|i|^2) = 2AP\sin\vartheta\,d\vartheta = -d(2AP\cos\vartheta).$$

Thus the expression

$$|i|^2 + 2AP\cos\vartheta$$

retains its original magnitude during the motion. If we denote this fixed quantity by k, we then have

$$\text{(3)} \qquad |i|^2 + 2AP\cos\vartheta = k.$$

We could have written down this relation without further ado if we had only invoked our general impulse theorem and the theorem of the *vis viva*, which was indeed deduced, in its turn, from our impulse theorem in the second chapter. In fact, we see immediately that equation (3) goes over, with consideration of (2), into the theorem of the *vis viva.* Instead of (3), namely, we can write

$$L^2 + M^2 + N^2 + 2AP\cos\vartheta = k.$$

If we now divide by $2A$ and introduce on the right a new constant h that is related to the previous constant k by the formula

$$\text{(3a)} \qquad h = \frac{k}{2A} + \frac{N^2}{2}\left(\frac{1}{C} - \frac{1}{A}\right),$$

there follows

$$\text{(3b)} \qquad \frac{1}{2}\Big(\frac{L^2 + M^2}{A} + \frac{N^2}{C}\Big) + P\cos\vartheta = h.$$

The first term on the left-hand side of (3b) signifies the kinetic energy

$$T = \frac{1}{2}\Big(\frac{L^2 + M^2}{A} + \frac{N^2}{C}\Big) = \frac{1}{2}(A(p^2 + q^2) + Cr^2)$$

of the top; the second term represents the potential energy U in the case of the action of gravity. According to the definition of pages 117 and 118, the second term is, namely, $dU = -dA$, where dA is the work that the external forces perform on our system for an infinitesimal displacement; that is, in our case,

$$dA = P\sin\vartheta\,d\vartheta.$$

There follows, in fact,

$$U = P\cos\vartheta,$$

so that we can again write, instead of (3b),

$$T + U = h. \tag{3'}$$

We must therefore conceive the preceding geometric consideration as a *new proof of the theorem of the* vis viva *for the heavy symmetric top.*

We now wish to make clear the analytic meaning of our results thus far. We will see that we have found, in equations (1), (2), and (3), three first integrals of the differential equations of the heavy symmetric top.

For this purpose, we will derive these equations anew by integration of the differential equations. For greatest convenience, we pose the latter in the form of the general Lagrange equations, as given on page 154. Here we must insert for T the expression

$$T = \frac{A}{2}(\sin^2\vartheta\cdot\psi'^2 + \vartheta'^2) + \frac{C}{2}(\varphi' + \cos\vartheta\cdot\psi')^2$$

of page 156, from which follow

$$\frac{\partial T}{\partial\varphi} = \frac{\partial T}{\partial\psi} = 0.$$

On the other hand, the work that gravity performs in the infinitesimal displacement $d\varphi$, $d\psi$, $d\vartheta$ is, as was just used,

$$dA = \Phi\,d\varphi + \Psi\,d\psi + \Theta\,d\vartheta = P\sin\vartheta\,d\,\theta.$$

The components Φ, Ψ, Θ of the external force in the directions of the coordinates φ, ψ, ϑ are therefore

$$\Phi = \Psi = 0, \quad \Theta = P\sin\vartheta.$$

Thus the first two Lagrange equations become

$$\frac{d[\Phi]}{dt} = \frac{d[\Psi]}{dt} = 0;$$

we therefore have, for the entire duration of the motion,

$$[\Phi] = \text{const.}, \quad [\Psi] = \text{const.}$$

Now according to page 109, however, the impulse components $[\Phi]$ and $[\Psi]$ are nothing other than the perpendicular projections of the impulse vector onto the figure axis and the vertical, and are therefore equal, respectively, to N and n. Our preceding equations are therefore identical with (1) and (2). We will thus be able to designate the fixed values of our two impulse components as two first integration constants.

We can, naturally, also derive these first integrals from the Euler equations of pages 141 and 142. We must set $A = B$, and insert for $\mathsf{\Lambda}$, M, N the respective values

$$\mathsf{\Lambda} = P \sin\vartheta \cos(K, X), \quad \mathsf{M} = P \sin\vartheta \cos(K, Y), \quad \mathsf{N} = P \sin\vartheta \cos(K, Z).$$

First,

$$\cos(K, Z) = 0,$$

since the line of nodes stands perpendicular to the figure axis; since we further denote (cf. Fig. 3 on page 18) the angle that the line of nodes forms with the positive X-axis by φ,

$$\cos(K, X) = \cos\varphi, \quad \cos(K, Y) = -\sin\varphi.$$

Thus there follow

$$\mathsf{\Lambda} = P \sin\vartheta \cos\varphi, \quad \mathsf{M} = -P \sin\vartheta \sin\varphi, \quad \mathsf{N} = 0.$$

The third of equations $(3'')$ of page 142 is therefore simply

$$C\frac{dr}{dt} = 0, \quad \text{or} \quad \frac{dN}{dt} = 0,$$

so that equation (2) again follows.

The derivation of (1) from the Euler equations would require a somewhat longer calculation that we wish to suppress here.

We can find our equation (3), finally, according to the usual analytic method of proof of the theorem of the *vis viva* from the Lagrange equations or the Euler equations, in that we multiply these equations by appropriate factors and add; here there appears as the third constant of integration our above quantity h, the constant of the *vis viva* that is equivalent to our above constant k.

In addition to the three integrals of the motion given by equations (1) to (3), the complete integration of our problem demands the construction of three further relations between the position coordinates of the top and time, each with an arbitrary constant of integration. These relations cannot, as can our first three integrals, be given in elementary form; correspondingly, it is hardly possible to derive them through direct geometric considerations. Their analytic character, nevertheless, is very simple; they may be represented, namely, by mere quadratures.

We must now consider, in addition to the Euler equations, the so-called kinematic equations (9) of page 45, or, more conveniently, consider, in addition to the Lagrange equations (1) of page 154, the equations (2),

$$[\Phi] = \frac{\partial T}{\partial \varphi'}, \quad [\Psi] = \frac{\partial T}{\partial \psi'}, \quad [\Theta] = \frac{\partial T}{\partial \vartheta'},$$

of the same page. In the first two of these equations, we insert the constant values N and n on the left-hand sides; we calculate the right-hand sides according to the expression for T given above. We thus find

$$C(\varphi' + \cos\vartheta \cdot \psi') = N,$$
$$C\cos\vartheta \cdot \varphi' + (A\sin^2\vartheta + C\cos^2\vartheta)\psi' = n.$$

Through the combination of these two equations there first follows

$$\psi' = \frac{n - N\cos\vartheta}{A\sin^2\vartheta}; \tag{4}$$

there then follows from the first equation

$$\varphi' = N\left(\frac{1}{C} - \frac{1}{A}\right) + \frac{N - n\cos\vartheta}{A\sin^2\vartheta}. \tag{5}$$

The integration with respect to t can naturally not yet be executed in this form. But if we enter equation (3′) of the *vis viva* with the found values of φ' and ψ', there follows

$$\frac{A}{2}\left\{\vartheta'^2 + \left(\frac{N\cos\vartheta - n}{A\sin\vartheta}\right)^2\right\} + \frac{C}{2}\left\{\frac{N^2}{C^2}\right\} + P\cos\vartheta = h.$$

Here we introduce the important auxiliary variable

$$u = \cos\vartheta;$$

the previous equation is then written, after we have multiplied it by $2A\sin^2\vartheta$, as

$$A^2u'^2 + (Nu - n)^2 + \frac{A}{C}N^2(1 - u^2) + 2APu(1 - u^2) = 2Ah(1 - u^2).$$

Thus

$$\frac{du}{dt} = \sqrt{U}, \tag{6}$$

where U signifies an abbreviation for the somewhat complicated expression

$$U = \frac{1}{A^2}\left[2Ah(1-u^2) - (Nu-n)^2 - \frac{A}{C}N^2(1-u^2) - 2APu(1-u^2)\right]. \tag{7}$$

We introduce in U our previous quantity k instead of the constant h; since, according to equation (3a),

$$2Ah - \frac{A}{C}N^2 = k - N^2,$$

there follows

$$U = \frac{1}{A^2}[-(Nu - n)^2 + (k - N^2 - 2APu)(1 - u^2)], \tag{7'}$$

or, ordered according to powers of u,

$$(7'') \qquad U = \frac{1}{A^2}[2APu^3 - ku^2 + 2(nN - AP)u + (k - N^2 - n^2)].$$

Since U depends solely on our auxiliary variable u, the integration in (6) can be executed immediately. We have only to write

$$(6') \qquad dt = \frac{du}{\sqrt{U}}.$$

If we insert this value of dt in (4) and (5), then the latter equations become

$$(4') \qquad d\psi = \frac{n - Nu}{A(1-u^2)} \frac{du}{\sqrt{U}}$$

and

$$(5') \qquad d\varphi = N\Big(\frac{1}{C} - \frac{1}{A}\Big)\, dt + \frac{N - nu}{A(1-u^2)} \frac{du}{\sqrt{U}}.$$

We now carry out the quadratures in (4′), (5′), and (6′) and obtain

$$(8) \qquad \begin{cases} t = \displaystyle\int \frac{du}{\sqrt{U}}, \\ \psi = \displaystyle\int \frac{n - Nu}{A(1-u^2)} \frac{du}{\sqrt{U}}, \\ \varphi = \displaystyle\int \frac{N - nu}{A(1-u^2)} \frac{du}{\sqrt{U}} + N\left(\frac{1}{C} - \frac{1}{A}\right) t, \end{cases}$$

where U is defined by (7), (7′), or (7″).

We have not written the three additive constants of integration, since they are irrelevant for the geometric character of the corresponding motion. These integration constants are, for example, the values t_0, ψ_0, φ_0 that we may assign to the value of u at the lower limits of the integrals. They can be dispatched by a suitable choice of the point of time from which we measure t and a suitable specification of the axes x and X from which we measure ψ and φ. Nevertheless, we state that we have obtained in the three essential constants n, N, and k (or h), as well as in the three inessential constants t_0, ψ_0 and φ_0, the required number of *six arbitrary quantities* that correspond to the general motion of a system with three degrees of freedom.

In this place we will regard the attainment of our goal by mere quadratures as a stroke of luck; in the following, however, this will not be an isolated occurrence. In fact, we will become acquainted in a later chapter with a comprehensive class of important mechanical problems, the motion of the so-called cyclic systems, that may be treated, exactly as our top problem, in whole or in part by mere quadratures. The top will appear there as an instructive example for the theory of cyclic systems. At the same time, the integration method followed here will receive a new illumination through the comparison with more general problems.[111]

As a concluding historical remark, we note that the formation of the above integral formulas is due to Lagrange,*) who considered the general problem of the heavy symmetric top for the first time.[112]

§4. General periodicity properties of the motion. Preliminaries on the behavior of the elliptic integrals for a circulation of the integration segment. Integral representation of $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\gamma}$, $\boldsymbol{\delta}$.

One designates the integrals assembled in the previous section for t, ψ, and φ as *elliptic integrals*, since they lead to the square root of an expression of the third degree under the integral sign. As is well known, such integrals may not be replaced, in general, by elementary functions. They define, rather, a class of transcendental functions that has been investigated by mathematicians with predilection for 100 years.

Our next exercise is to read the most general and most evident properties of the motion of the heavy top from the form of these functions. In particular, we wish to demonstrate in an analytic manner that the apex of the top progresses to and fro between two parallel circles of the unit sphere, and that its trajectory consists purely of congruent or symmetrically equal segments. These facts were already attained in the first sections of this chapter in a mechanical-geometric manner. Disregarding the more precise demonstration of the somewhat uncertain earlier conclusions, we will, in the following, attain the possibility of confirming the form of the trajectory in detail by numerical calculation.

We must therefore employ the same fundamental considerations that have been developed elsewhere in the theory of elliptic functions. In that we tie these considerations to our concrete example, we hope to give a comfortable first introduction to this theory, which we will not assume as known.

*) In the *Mécanique analytique sec. partie*, sect. IX, Nr. 35. Ges. W. Bd. XII.

We place ourselves, at first, completely in the standpoint of the older authors—for example, Legendre—and consider only real values of the integration variable u, which we will represent on a line, the "u-axis." Because of the geometric meaning of u ($u = \cos\vartheta$), only the portion of this axis between -1 and $+1$ comes directly into consideration for mechanics.

In addition, this segment will be swept, in general, only in part for the motion of the top. Since, namely, the increment of t must certainly be a real quantity, u may take, because of the relation

$$dt = \frac{du}{\sqrt{U}},$$

only values for which $\sqrt{U}$ is real, and therefore U is positive. Now U, however, will in general be negative and never positive at the points $u = \pm 1$. We have, according to equation (7′) of the preceding section,

$$\text{for } u = +1 \cdots\cdot U = -\frac{1}{A^2}(N-n)^2,$$
$$\text{for } u = -1 \cdots\cdot U = -\frac{1}{A^2}(N+n)^2;$$

that is, $U < 0$ except when $N = \pm n$, and therefore $U = 0$. The variable u is thus restricted to a certain interval between -1 and $+1$ in which U is positive. Such an interval must always be present; were it not present, we would have to say that the integration constants n, N, and k that appear in U were chosen impermissibly in the sense of mechanics, since they correspond to no real motion. The boundaries of our interval will be formed by two points at which U vanishes. Let these points be $u = e$ and $u = e'$.

In addition to these two roots, the equation $U = 0$ has a third root that is necessarily real and lies outside the interval from -1 to $+1$. We denote this root by e'',*) and easily show that $e'' > +1$ or $e'' < -1$ according to whether $P > 0$ or $P < 0$. In fact, U takes the sign of the highest term in u for infinitely large u, and therefore, according to equation (7′) of the preceding section, the sign of Pu^3. This sign is positive in the case $P > 0$ for $u = +\infty$ and in the case $P < 0$ for $u = -\infty$; U therefore changes its sign in the former case between $+\infty$ and $+1$, and

*) Weierstrass consistently denotes the three roots of the expression of the third degree under the integral sign by e_1, e_2, e_3. We have chosen the different notation above, since for us the Weierstrass normalization of the integral does not come into consideration.

in the latter case between $-\infty$ and -1. Thus the third root in question certainly lies on our u-axis, and indeed to the right of $u = +1$ in the case $P > 0$ and to the left of $u = -1$ in the case $P < 0$.

We must further distinguish well between the two signs of $\sqrt{U}$. It is convenient to assume, instead of a simple u-axis, a *double axis*, or, as we may also say, to imagine the u-axis to be *doubly covered.* Each two overlying points of this double axis (cf. Fig. 38) then represent the same values of u and U, but opposite values of $\sqrt{U}$. At the vanishing points of U, these two opposite values are not different. The same holds for the position $u = \pm\infty$, where $\sqrt{U}$ takes the two values $\pm\infty$, which are not different in function theory. We express this in the figure by allowing the two coverings of the axis to come together in a point at these positions. We also designate the four points e, e', e'', ∞, if we prepare for the usual terminology of *Riemann surfaces*, as *branch points.* The method of the double covering of our u-axis already belongs, in general, to the circle of ideas introduced by Riemann in function theory, which we can approach more closely only in the sixth chapter.

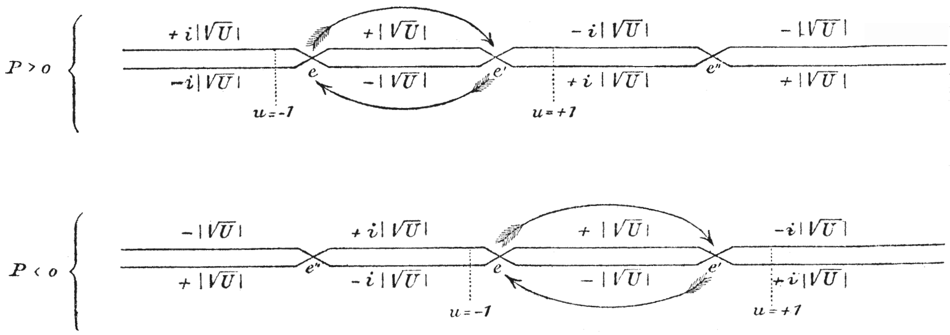


Fig. 38.

From the standpoint of the real integration variable adopted here, the distribution of positive and negative values of the square root to the two coverings is, to a certain extent, at our pleasure. We will naturally arrange the sequence of values of $\sqrt{U}$ in the two coverings so that they form a continuous sequence, and, therefore, so that a change of sign occurs only at the branch points ($U = 0$ or $U = \infty$). The choice of sign in each interval ee', $e'e''$, etc., remains arbitrary. Only later, when we speak of complex values of the quantity u, will a specific rule be given. For the present, we will regard the specification of the signs in the preceding figure as an arbitrary convention.

We now investigate the elliptic integral constructed for t, in its dependence on u, in somewhat more detail. Here we use the evident fact that in mechanics time signifies not only a real, but also a continuously increasing quantity, and that dt must therefore be necessarily positive.

As the lower limit of the integral, we take the smaller of the two roots of $U = 0$ between -1 and $+1$, which we denote (as in the figure) by e. Starting from this point, we must let u occur only in the mechanically useful interval between e and e', and indeed, according to the previously established principle, in the upper covering of this interval, so that we obtain for

$$dt = \frac{du}{\sqrt{U}}$$

a positive value. We must then let u increase, continuously remaining in the upper covering, to the point $u = e'$. Arriving at this point, we must reverse direction, since dt is real, and must pass into the lower covering, so that dt remains positive. If we arrive back at e, we must, on the same grounds, pass into the upper covering. The process then repeats, progressing in the same fashion.

The path that we assign to the variable u must therefore consist of the continuous circulation of the segment ee' in the specified sense. (Cf. the arrow in the preceding figure.)

This statement immediately yields a first important property of the general motion of the top. It asserts, namely, that

The trajectory of the apex of the top continuously oscillates to and fro between two parallel circles $\cos\vartheta = e$ and $\cos\vartheta = e'$ on the unit sphere.

At the same time, we learn to calculate the position of the two parallel circles. We are obliged, for this purpose, to solve the cubic equation $U = 0$; *its two roots between -1 and $+1$ provide the two values in question of $\cos\vartheta$.*

Further, we can now give the time that elapses while the apex of the top goes over from its lowest to its next highest position. We designate this time by ω, and have

$$\omega = \int_e^{e'} \frac{du}{\sqrt{U}}. \tag{1}$$

Just as great is the time duration in which the apex of the top returns back to its lowest position; this time duration is, because of the stipulated negative sign of $\sqrt{U}$ on the lower covering,

$$\int_{e'}^{e} \frac{du}{-|\sqrt{U}|} = \omega.$$

An interval 2ω thus passes each time that u executes a complete circuit of the integration segment ee'. This is the basis on which one designates 2ω as the *period of the elliptic integral* ($\pi\varepsilon\varrho\acute{\iota}o\delta o\varsigma$ = circuit).

We obtain the same time interval 2ω if we let u, beginning from *any* value of the integration interval, circulate the segment ee' and return to the starting point. Conversely, to any two points of time that differ by 2ω, or any multiple of this quantity, there corresponds the same value of u; that is, the same vertical elevation of the apex of the top above the equatorial plane of the unit sphere. *The motion of the apex of the top therefore represents, with respect to its vertical component, a periodic process in time.*

The same also holds, however, for the horizontal component of this motion. The latter is determined by the changing value of the angle ψ. The angle ψ originally denoted the angle measured from the x-axis to the line of nodes. The line of nodes, however, always stands perpendicular to the figure axis and to its (orthographic or stereographic) projection in the equatorial plane. Since the magnitude of ψ in our integral representation of page 223 is defined, in any case, only up to an additive constant of integration, we can, if we choose this constant specifically, also conceive ψ directly as the angle that the projection of the figure axis forms with the x-axis. The equation

$$\psi = \psi(u),$$

in which the function ψ signifies the elliptic integral given above, then directly yields the equation of the trajectory in polar coordinates. We need only express u in terms of the radius vector ϱ from O to the image of the apex of the top, and thus set, according to whether we use the orthographic or the stereographic image,

$$u = \sqrt{1-\varrho^2} \quad \text{or} \quad u = \frac{1-\varrho^2}{1+\varrho^2}.$$

It is now clear that the integral $\psi = \psi(u)$ will exhibit periodicity properties that are entirely similar to those of the just investigated integral for t. We must first determine, because of the factor $\dfrac{du}{\sqrt{U}}$ and its meaning as the increment of time, the integration path of the variable u in the same manner as above. If we begin, correspondingly, from the lower limit e, then the integration, extended once to the point e', yields the characteristic increase of ψ during the time ω. We denote this increase by ψ_ω, and have

$$\psi_\omega = \int_e^{e'} \frac{n - Nu}{A(1-u^2)} \frac{du}{\sqrt{U}}. \tag{2}$$

We obtain the double of this increase if we integrate further beyond the point e' in the lower covering and return to the initial point. In fact, because of the specified sign of $\sqrt{U}$ for the lower covering,

$$\int_{e'}^{e} \frac{n - Nu}{A(1-u^2)} \cdot \frac{du}{-|\sqrt{U}|} = \psi_\omega.$$

The same value $2\psi_\omega$ naturally results if, beginning at *any* point of our segment, we let the integration variable circulate this segment completely one time. The quantity $2\psi_\omega$ thus represents the increase of the azimuth of the apex of the top while it returns, after crossing a highest and a lowest position, from any point to a point lying at the same height above the equatorial plane. This increase is thus the same for all points of the trajectory. In other words,

The trajectory of the apex of the top coincides with itself if we rotate it about the vertical through the angle $2\psi_\omega$. It thus consists of a (generally infinite) series of congruent arcs, each of which is traversed in the time 2ω. The motion of the apex of the top represents, considered spatially as well as temporally, a periodic process.

We have already recognized geometrically the just stated congruence of the individual arcs; our present analytical supplement now teaches us, furthermore, to calculate the span width of the arcs by means of equation (2).

We next wish to verify analytically what we likewise recognized earlier; namely, that each arc of the trajectory is divided into two symmetrically equal half-arcs. We state, for this purpose, understanding by u any value between e and e', that the two integrals

$$\int_u^{e'} d\psi \quad \text{and} \quad \int_{e'}^{u} d\psi,$$

of which we imagine the first carried out in the upper, and the second in the lower covering, have the same value (say ψ_0). On a specified congruent arc, each value of u corresponds to two points that have the respective azimuth values ψ_1 and ψ_2, where

$$\psi_1 = \psi_\omega - \psi_0, \quad \psi_2 = \psi_\omega + \psi_0.$$

We thus see that each two successive points of our trajectory corresponding to the same value of u are mirror images with respect to the line $\psi = \psi_\omega$. *Each of our individual congruent arcs is thus divided*, as claimed, *into two mirror-formed half-arcs.*

Finally, exactly the same conclusions are valid with respect to the integral through which we have represented φ. We can directly say that the φ-coordinate also increases, for a complete circuit of the variable u about the integration segment, by a determined additive quantity $2\varphi_\omega$. *As a result, the motion of the top about the figure axis also has a periodic character. This motion is repeated in the same tempo at which the trajectory is periodically reproduced.* The φ-coordinate is indeed less important for the geometric character of the motion than the ψ-coordinate; in particular, it is not at all expressed in our previous graphical representation of the motion of the top.

We have associated our discussion thus far with the expressions for the Euler angles ψ and φ, which, because of their intuitive meaning, are in fact most convenient for geometric questions. We remark, however, that for the purpose of a thorough analytic treatment, our parameters α, β, γ, δ offer a decided advantage. This will be made clear in the sixth chapter. Here we satisfy ourselves with deriving, from the above expressions for t, ψ, and φ, the corresponding integral representations for the logarithms of our parameters.

We begin from the original definitions of α, β, γ, δ in equations (8) of page 21, where, for example,

$$\alpha = \cos\frac{\vartheta}{2} \cdot e^{\frac{i(\varphi+\psi)}{2}}$$

was defined. We calculate $\lg\alpha$, replace $\cos\frac{\vartheta}{2}$ by its value $\sqrt{\frac{u+1}{2}}$ in terms of u, and obtain

$$\lg\alpha = \frac{1}{2}\lg(u+1) + \frac{i}{2}(\varphi+\psi) - \frac{1}{2}\lg 2.$$

We now insert the integrals for φ and ψ of page 223, write $\lg(u+1)$ as an integral, and permit ourselves to neglect the additive constant on the right, in that we imagine it combined with the unwritten arbitrary constant of integration. After reduction, there results for $\lg\alpha$ the following expression, to which we immediately adjoin the representations acquired in the corresponding manner for $\lg\beta$, $\lg\gamma$, $\lg\delta$:

$$(3)\qquad \begin{cases} \lg\alpha = \displaystyle\int\left\{\frac{A\sqrt{U}+i(n+N)}{2A(u+1)} + \frac{iN}{2}\left(\frac{1}{C}-\frac{1}{A}\right)\right\}\frac{du}{\sqrt{U}}, \\[2ex] \lg\beta = \displaystyle\int\left\{\frac{A\sqrt{U}-i(n-N)}{2A(u-1)} - \frac{iN}{2}\left(\frac{1}{C}-\frac{1}{A}\right)\right\}\frac{du}{\sqrt{U}}, \\[2ex] \lg\gamma = \displaystyle\int\left\{\frac{A\sqrt{U}+i(n-N)}{2A(u-1)} + \frac{iN}{2}\left(\frac{1}{C}-\frac{1}{A}\right)\right\}\frac{du}{\sqrt{U}}, \\[2ex] \lg\delta = \displaystyle\int\left\{\frac{A\sqrt{U}-i(n+N)}{2A(u+1)} - \frac{iN}{2}\left(\frac{1}{C}-\frac{1}{A}\right)\right\}\frac{du}{\sqrt{U}}. \end{cases}$$

It may at first appear that these expressions are more complicated than the integral representations for φ and ψ. In reality, however, they exhibit a much simpler function-theoretic behavior. We will later see that the quantities $\lg\alpha$, $\lg\beta$, $\lg\gamma$, $\lg\delta$ are so-called normal integrals of the third kind, while ψ and φ may only be composed additively from such simplest elements.

§5. On the relation between the motions of different tops that yield the same impulse curve, and on the motion of the spherical top.

We will show in this section that we need henceforth consider, as already mentioned previously in passing, only the motion of the spherical top. For this purpose, we first employ a somewhat more general deliberation.

We consider a specific "first" symmetric top, and consider the curve that the endpoint of the impulse describes in space for an arbitrary natural motion. We then ask ourselves *whether we can conceive this curve as an impulse curve in a multifold manner*; that is, does this curve correspond, as the locus of the endpoint of the impulse vector in space, to the motion of a suitably chosen "second" symmetric top? We will see that this question is to be affirmed.

The constants of the mass distribution and the motion of our first and second tops may be distinguished by the indices 1 and 2, so that A_1, C_1, P_1 and A_2, C_2, P_2 denote the moments of inertia and the gravitational moments of our first and second tops, respectively. We wish to suppose, specifically, that the second top possesses the same equatorial principal moment of inertia and the same gravitational turning-moment as the first, so that

$$A_1 = A_2, \quad P_1 = P_2;$$

the moment of inertia about the figure axis can, in contrast, be different. Further, we wish to specify the initial position of our second top so that its figure axis coincides with that of the first top, and thus

$$\vartheta_1 = \vartheta_2$$

at the beginning of the motion. Finally, we must also specify the initial position and magnitude of the impulse as equal in the two cases, since we indeed wish that the two tops give rise to the same impulse curve. We therefore set

$$n_1 = n_2, \quad N_1 = N_2, \quad k_1 = k_2,$$

and can omit, moreover, the indices of the assumed equal constants A, P, n, N, k.

The question posed above may now be decided, if we refer to our explicit integral formulas, very easily. We need only note that the expression for U in equation $(7')$ of page 222, just as the formulas for t and ψ in equations (8) of page 223, depend merely on the assumed equal quantities n, N, k, A and P, and are independent, in contrast, of the assumed unequal principal moment of inertia C. As a result, t and ψ will be the same functions of u for the two tops; that is, the trajectories of the apices of the two tops will be identical with respect to their spatial form and temporal progression. That the impulse curves will also be identical then follows directly from the equality of the impulse components n and N, as well as the equality of the length of the impulse, which for corresponding equal values of $\cos\vartheta$ is stated by the theorem of the *vis viva* in the form of equation (3) of page 219.

Our question above is thus to be answered in the following sense: *To a specific possible impulse curve correspond the infinitely many motions* *of all those symmetric tops with the same initial impulse, the same initial position of the figure axis, the same gravitational turning-moment, and the same equatorial principal moment of inertia. In addition to the impulse curves, the trajectories of the apices of the tops will also be identical for all such motions.*

We can also, however, give an immediate geometric account of this characteristic coincidence between the motions of the different tops.

We compare, for this purpose, the course of the impulse curves of our first and second tops, in that we compose the initial impulse successively with the infinitesimal turning-impact of gravity. There follows, because of the equality of P and the equality of the initial impulse and the initial position of the figure axis, the same change of the impulse at the first moment for the two tops. In particular, we can say that the difference $|i_1|^2 - |i_2|^2$ retains its initial value zero at the first moment, or, more precisely, that the differential quotient of this difference with respect to time is equal to zero in the initial position.

We consider, further, that the invariable relations (3) of page 219,

$$|i_1|^2 + 2AP\cos\vartheta_1 = k, \quad |i_2|^2 + 2AP\cos\vartheta_2 = k,$$

obtain between the length of the impulse and the inclination of the figure axis for the motions of the two tops.

From these two equations, we conclude that the inclination of the figure axis with respect to the vertical initially changes in the same manner for the two cases. If we form, namely, the difference of the preceding equations, there follows, by differentiation with respect to t,

$$2AP\frac{d}{dt}(\cos\vartheta_1 - \cos\vartheta_2) = -\frac{d}{dt}(|i_1|^2 - |i_2|^2) = 0.$$

If we also add, in addition to the equality of the inclination angle ϑ at the first moment, the equality of the projection N of the impulse vector onto the figure axis, it follows with necessity that the initially unified figure axes remain unified at the next moment.

We are thus led back to the same conditions that held at the beginning of the two motions. Through repetition of our conclusion, we see that each two symmetric tops that have the same initial position of the figure axis and the same initial impulse must always have, for equal values of A and P, the same impulse and the same position of the figure axis. This is, however, again the theorem stated above.

Each two of the motions compared here are naturally not completely identical. They differ, for example, in the form of the herpolhode curve, as we will show in more detail below. The position and magnitude of the rotation vectors will obviously be different for equal impulse curves and equal trajectories of the apex of the top but unequal values of the principal moments of inertia C. This difference of the instantaneous rotation can be expressed, however, only in the respective motion of the top about its figure axis—that is, in the value of the angle φ—since indeed the motion of the figure axis itself, as we saw, must be the same. In fact, C also enters explicitly in the integral formula for φ (see equations (8) of page 223).

According to this same equation, however, we can say that

$$\varphi - N\left(\frac{1}{C} - \frac{1}{A}\right)t \tag{1}$$

always has the same value for each two tops of our series. The rotation components φ' for two such tops therefore differ only by a constant, whose magnitude depends on the moment of inertia C.

In particular, a *spherical top* is found among our series of tops with equal impulse curves and trajectories. We will draw upon this spherical top with predilection when any "first" symmetric top is given. According to the preceding, we must determine its moment of inertia, which we denote by A, so that $A = A_1$.

If we take, further, the quantities P, n, N, k and the initial position of the figure axis equal to the corresponding quantities of the given symmetric top, then we are certain that the trajectories of the spherical top and the symmetric top become identical, while, at the same time, the two angular velocities φ' differ only by a constant. *The spherical and the symmetric tops are thus, for our purpose, not essentially different.* If one has treated generally of the motion of the former, the motion of the latter may be given immediately.

The possibility of the reduction of the general top problem to the spherical top was first noted by Mr. D a r b o u x.*)

As a first application of this reduction, we may prove a theorem, already mentioned on page 218, on the herpolhode curve of the sym-

*) Mouvement d'un corps pesant de révolution, Journ. de Liouville, sér IV, t. 1, 1885.[113]

metric top. We wish to show that *the herpolhode curve of the symmetric top is a spherical curve.*

We begin from the fact that the herpolhode curve of the corresponding spherical top is *planar.* In fact, the herpolhode curve for the spherical top is similar to the impulse curve; that is, the curve that the impulse vector describes in space. That this latter is a plane curve was shown explicitly on page 216.

We have represented the coordinates of the herpolhode curve, which we denote, as previously, by π, κ, ϱ, in terms of the values of φ, ψ, ϑ and their differential quotients with respect to time on page 45; in particular, there was given for the third coordinate

$$\varrho = \psi' + \cos\vartheta \cdot \varphi'.$$

Now ϱ has, for the spherical top with moment of inertia A, the constant value $\dfrac{n}{A}$. Further, the values of ψ and ϑ for the symmetric top are, as we just saw, equal to the corresponding values for the spherical top, while the angular velocity φ' is calculated, according to (1), from that of the corresponding quantity for the spherical top by the addition of $N\Big(\dfrac{1}{C} - \dfrac{1}{A}\Big)$. We thus have, understanding by φ' the value of this angular velocity for the spherical top and by ϱ the value of the third herpolhode coordinate for the symmetric top,

$$\begin{aligned}\frac{n}{A} &= \psi' + \cos\vartheta \cdot \varphi',\\ \varrho\ &= \psi' + \cos\vartheta \cdot \varphi' + N\left(\frac{1}{C} - \frac{1}{A}\right)\cos\vartheta,\end{aligned}$$

or

$$\varrho = \frac{n}{A} + N\left(\frac{1}{C} - \frac{1}{A}\right)\cos\vartheta. \tag{2}$$

We express, further, the length of the rotation vector for the motion of the symmetric top once in terms of its coordinates π, κ, ϱ, and once again in terms of the coordinates p, q, r, where we can also write $\dfrac{N}{C}$ instead of r. We thus obtain

$$\pi^2 + \kappa^2 + \varrho^2 = p^2 + q^2 + \frac{N^2}{C^2}. \tag{3}$$

The theorem of the *vis viva* in the form of equation (3) of page 219 then permits us to calculate $p^2 + q^2$ in yet another manner. Namely, if we place in the named equation

$$|i|^2 = L^2 + M^2 + N^2 = A^2(p^2 + q^2) + N^2,$$

then there follows

$$A^2(p^2+q^2)+N^2+2AP\cos\vartheta=k,$$

or

$$p^2+q^2=\frac{k-N^2-2AP\cos\vartheta}{A^2}.$$

Equation (3) thus becomes

$$\pi^2+\kappa^2+\varrho^2=\frac{k-2AP\cos\vartheta}{A^2}+N^2\left(\frac{1}{C^2}-\frac{1}{A^2}\right). \tag{4}$$

Finally, we eliminate $\cos\vartheta$ from (2) and (4), and find an equation in which, except for π, κ, ϱ, only constants appear; namely,

$$\pi^2+\kappa^2+\varrho^2=\frac{k}{A^2}-\frac{2CP(A\varrho-n)}{AN(A-C)}+N^2\left(\frac{1}{C^2}-\frac{1}{A^2}\right), \tag{5}$$

or

$$\pi^2+\kappa^2+\left(\varrho+\frac{CP}{N(A-C)}\right)^2=\frac{k}{A^2}+\frac{2CPn}{AN(A-C)}+\frac{C^2P^2}{N^2(A-C)^2}+N^2\left(\frac{1}{C^2}-\frac{1}{A^2}\right). \tag{5$'$}$$

This, however, is the equation of a sphere. Its midpoint lies on the vertical at the distance $\dfrac{CP}{N(A-C)}$ from the support point; its radius is equal to the square root of the right-hand side of (5′). *The herpolhode curve is therefore, in fact, a spherical curve.*

The difference between the herpolhode curves for the different tops of our series also follows immediately from the dependence of the position and size of the sphere on the moment of inertia C. In particular, the radius of the spherical top in our series becomes, because of the denominator $A-C$, infinitely large; at the same time, its midpoint is removed to infinity. The spherical curve is thus transformed, for this special case, into a plane curve, as it must be.

The introduction of the spherical top is particularly recommended because of a characteristic *reciprocity law* that obtains for the motion of the spherical top. We claim that

The inverse motion of the spherical top—that is, the rotation of space with respect to the imagined fixed top—is again a top motion.

We see the correctness of this theorem geometrically if we ponder, in detail, that the figure axis and the vertical play the same roles, for the direct motion of the spherical top, as the vertical and the figure axis play for the inverse motion, or, more precisely said, as the half-lines diametrically opposed to the vertical and the figure axis.

The analytic proof consists in the following. We set $C = A$ in equations ($7'$) and (8) of pages 222 and 223, and change n into $-N$ and N into $-n$, which corresponds to an interchange of the vertical with the half-line diametrically opposed to the figure axis, and so forth, while we leave our third integration constant k unchanged. Then the expression U, and thus also the function t of u, remain unchanged. At the same time, ψ goes over into $-\varphi$ and φ into $-\psi$. We know from the first chapter (see pages 30 and 31), however, that the change of ϑ, ψ, φ into ϑ, $-\varphi$, $-\psi$ corresponds to the passage from the direct to the inverse rotation. The inverse motion is therefore, in fact, again a top motion; it is characterized by the essential constants $-N$, $-n$, k if the corresponding constants of the direct motion are n, N, k.

This reciprocity law is naturally limited by the special symmetry relations of the spherical top. For more general systems, the inverse motion has an entirely different kinetic character than the direct, as was pointed out previously on page 12. Our reciprocity law already loses its validity for the symmetric top, since, in this case, the figure axis and the vertical do not appear with equal significance in the construction of the rotation vector from the impulse vector. Analytically, this is expressed by the fact that the term

$$N\left(\frac{1}{C} - \frac{1}{A}\right)t,$$

which does not remain unchanged with the interchange of n and N with $-N$ and $-n$, appears in the expression for φ in equations (8) of page 223.

We will later draw a considerable advantage from the established reciprocity law of the spherical top in the calculation of our impulse curves or our polhode and herpolhode curves. When we have somehow found, for example, the polhode curve, or, what is the same for the spherical top, the "second impulse curve," we can immediately construct the equation of the herpolhode curve, or that of the "first impulse curve." Namely, the polhode curve of the direct motion lies diametrically opposed with respect to the support point, as noted on page 14, to the herpolhode curve of the inverse motion. From the herpolhode curve of the inverse motion, however, the herpolhode curve of the direct motion follows on the basis of our reciprocity law through the interchange of n and N with $-N$ and $-n$. Thus we can state the following rule for the derivation of the herpolhode curve from the assumed known polhode curve:

One reverses the signs of the coordinates p, q, r of the polhode curve, which may be found as functions of time and the integration constants n, N, k, and writes $-N$, $-n$ in place of n, N. Then the coordinates p, q, r of the polhode curve go over into the coordinates π, κ, ϱ of the herpolhode curve. The coordinates l, m, n of the first impulse curve result in the same manner from the coordinates L, M, N of the second.

We place below, for easier use, the most important formulas acquired thus far for the special case of the spherical top.

From equations (4) and (5) of page 222, there follow, for a spherical top of moment of inertia A,

$$\psi' = \frac{n - Nu}{A(1-u^2)}, \quad \varphi' = \frac{N - nu}{A(1-u^2)}, \quad u = \cos\vartheta; \tag{6}$$

equations (8) of page 223 become

$$\begin{cases} t \ = \displaystyle\int \frac{du}{\sqrt{U}}, \\ \psi = \displaystyle\int \frac{n - Nu}{A(1-u^2)} \, \frac{du}{\sqrt{U}}, \\ \varphi = \displaystyle\int \frac{N - nu}{A(1-u^2)} \, \frac{du}{\sqrt{U}}, \end{cases} \tag{7}$$

$$A^2 U = -(Nu - n)^2 + (k - N^2 - 2APu)(1 - u^2); \tag{7'}$$

the integral representations of α, β, γ, δ, now run, finally, according to page 231,

$$\begin{cases} \lg\alpha = \displaystyle\int \frac{A\sqrt{U} + i(n+N)}{2A(u+1)} \, \frac{du}{\sqrt{U}}, \\ \lg\beta = \displaystyle\int \frac{A\sqrt{U} - i(n-N)}{2A(u-1)} \, \frac{du}{\sqrt{U}}, \\ \lg\gamma = \displaystyle\int \frac{A\sqrt{U} + i(n-N)}{2A(u-1)} \, \frac{du}{\sqrt{U}}, \\ \lg\delta \ = \displaystyle\int \frac{A\sqrt{U} - i(n+N)}{2A(u+1)} \, \frac{du}{\sqrt{U}}. \end{cases} \tag{8}$$

The passage to a symmetric top with moment of inertia $C \gtrless A$ is simply accomplished afterward if we add to the preceding value of φ or $\lg\alpha$, $\lg\beta$, $\lg\gamma$, $\lg\delta$ the term

$$N\left(\frac{1}{C} - \frac{1}{A}\right)t \quad \text{or} \quad \pm\frac{i}{2}N\left(\frac{1}{C} - \frac{1}{A}\right)t, \tag{9}$$

respectively.

§6. Confirmation of the forms of motion of the spherical top developed in the first sections; the characteristic curves of the third order in the case $e = 0$.

After having established the general periodicity properties of the motion in the fourth section, and thus having obtained a first confirmation of the intuitive developments of the first sections, we now seek to acquire a more detailed insight into the form of the trajectories, in so far as this is possible in an analytic-algebraic manner without further entrance into the theory of elliptic integrals. We will investigate, for this purpose, how the form of the trajectory depends on the constants n, N, and k. We can restrict ourselves, according to the previous section, to the case of a spherical top, whose moment of inertia we denote by A.

We first wish to make a change in the choice of the integration constants. We wish to introduce, instead of the constant k, which has no sufficiently simple geometric meaning, and which, moreover, is subject to certain rather complicated inequalities in order that the corresponding motion be real, a new constant that indicates the initial position of the figure axis with respect to the vertical. In particular, we choose the initial time from which we follow the motion, as already agreed on page 199, so that the apex of the top initially occupies a highest or lowest point of its trajectory on the unit sphere. The initial inclination (ϑ_0) of the figure axis is then, according to page 227, determined by one of the roots of the cubic equation $U = 0$ between -1 and $+1$. If we denote this root by e, we then have $\cos\vartheta_0 = e$.

We can now eliminate the constant k from U, and introduce e instead. This is accomplished in the following manner. We have, according to equation (7′) of the previous section,

$$\frac{A^2U}{1-u^2} = -\frac{(Nu-n)^2}{1-u^2} + k - N^2 - 2APu.$$

If we set $u = e$, then the left-hand side vanishes; thus the further equation

$$0 = -\frac{(Ne-n)^2}{1-e^2} + k - N^2 - 2APe$$

obtains. We eliminate k by taking the difference of these two equations; there follows

$$A^2U = -(Nu-n)^2 + \frac{(Ne-n)^2(1-u^2)}{1-e^2} - 2AP(u-e)(1-u^2).$$

Since the right-hand side must vanish for $u = e$, we can extract $u - e$ as a factor. We correspondingly set, if we introduce a common denominator,

$$U = \frac{u-e}{A^2(1-e^2)} U_1, \tag{1}$$

where U_1, as one easily calculates, has the value

$$U_1 = -(u+e)(n^2+N^2) + 2Nn(1+eu) - 2AP(1-e^2)(1-u^2). \tag{2}$$

Since we regard one of the roots of our cubic equation $U = 0$ as prescribed (in that we adjudicate, as one expresses it, this root), the determination of the remaining roots now depends only on a quadratic equation $U_1 = 0$. Of its two roots, the one that lies between -1 and $+1$ determines for us the second parallel circle $u = e'$ that bounds, together with the known parallel circle $u = e$, the trajectory of the apex of the top on the unit sphere.

The introduction of the constant e therefore brings a double advantage: 1) the rather unintuitive constant k is replaced by a quantity that is expressed immediately in the initial position of the figure axis and the form of the trajectory; 2) the cubic equation $U = 0$ is replaced by an easily solvable quadratic equation $U_1 = 0$. We will thus, in the following, regard n, N, and e as the essential elements of the motion of the top, and occasionally refer to them as constants of integration.

As we now proceed to a more exact investigation of the form of the trajectory, we wish to know, above all, how the position of the second bounding circle $u = e'$ depends on the choice of the constants of integration. Since our next object is to subject Figs. 24–35 to a detailed confirmation, we assume throughout, as in the first sections, that the figure axis stands initially horizontal, and correspondingly first set $e = 0$.

In Figs. 24 to 28, n had the fixed value zero, while N varied. If we therefore set $e = n = 0$ and, for example, $N = v$, then the equation $U_1 = 0$ represents for us the dependence in question between u and v; that is, the position of the bounding circle $u = e'$ and the magnitude of the eigenimpulse $v = N$. This equation now runs

$$uv^2 = -2AP(1-u^2). \tag{3}$$

While this equation is of the second degree in u, its degree is again raised to 3 as soon as we regard, as we now must, u and v as simultaneous variables. In order to be able to survey the dependence between these quantities conveniently, we interpret them as rectangular coordinates in a uv-plane, with u as the abscissa and v as the ordinate, and thus obtain as the image of the equation $U_1 = 0$ *a curve of the third order* ("a C_3").

The form of this C_3 is easy to see. To each abscissa u there corresponds a pair of oppositely equal (not necessarily real) values $\pm v$; the line $u =$ const. therefore cuts the curve in two points that are mirror images with respect to the axis of the abscissa; the curve itself lies symmetrically with respect to this axis. Two symmetric points can move together only if $\pm v = 0$ or ∞, in which case the considered line $u =$ const. is tangent to our C_3. Now we have, according to equation (3), $v = 0$ if $u = \pm 1$, and $v = \infty$ if $u = 0$ or $u = \infty$. Our curve thus has a vertical tangent at each of the points $u = \pm 1$, $v = 0$, and has the ordinate axis as an asymptote.

We further note that the left-hand side of our equation (3) is positive to the right of the ordinate axis. The right-hand side, however, is pospositive only as long as $u < 1$, since we wished to assume $P < 0$ in §1. As a result, there are no real points of the curve to the right of the line $u = +1$. In a corresponding manner, one sees that no points of the curve can lie inside the strip to the left of the ordinate axis and to the right of the line $u = -1$. The curve must therefore open (cf. Fig. 39) to the left at the position $u = +1$, $v = 0$, and will then approach the line $u = 0$ asymptotically. The curve will likewise open to the left at the position $u = -1$, $v = 0$, from which it runs like a parabola to infinity. Our C_3 therefore consists of two disconnected

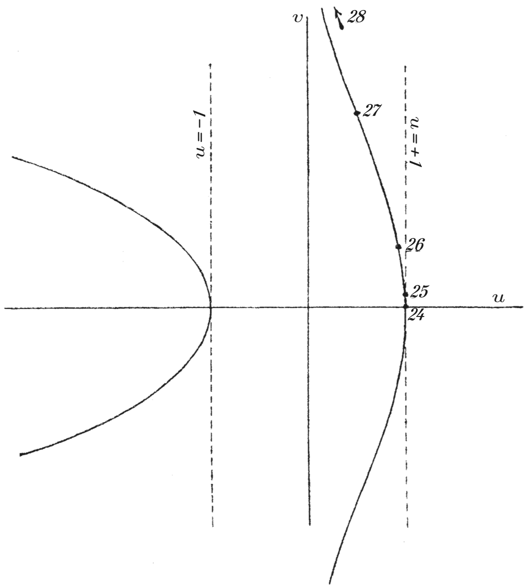


Fig. 39.

branches that we distinguish as the "even" and the "odd" branch. The odd branch is that contained in the strip $0 < u < 1$, and the even branch is our parabola-like branch.

On the basis of this figure, we can completely confirm our previous conclusions about the change of the parallel circle $u = e'$ for increasing N. Naturally, only such values of the abscissa u that are contained between -1 and $+1$ come into consideration in mechanics. If we therefore disregard the isolated point $u = -1$, which, as we will see in the following chapter, corresponds to an entirely particular type of motion, then we must concern ourselves with the odd branch. We draw the line $v = N$ parallel to the u-axis and let N increase from zero. The abscissa of the intersection point of this line with the odd branch gives us the value of e' that corresponds to N. The case $N = 0$ corresponds to the ordinary pendulum motion, for which $e' = 1$, and for which the parallel circle e' therefore reduces to the north pole of the unit sphere. For increasing N, e' successively decreases, as the figure shows, and our parallel circle therefore widens and asymptotically approaches the equator for $N = \infty$. The motion thus goes over, in conformity with Fig. 28, into our pseudoregular precession.

It is also easy to verify the appearance of the cusps in Figs. 25–28. According to equations (7) of the previous section, we have, in the case $n = 0$,

$$\frac{d\psi}{du} = \frac{-Nu}{A^2(1-u^2)}\frac{1}{\sqrt{U}}.$$

On the equator $u = 0$, the right-hand side becomes zero, since $\sqrt{U}$ vanishes to a lower order than u. The trajectory must therefore run radially in the stereographic projection; but since it cannot cross over the equator, it must also run back in the radial direction, so that, in fact, a cusp is formed.

A few statements may be added regarding the numerical values that form the basis of Figs. 25 through 28, in so far as they refer to the position of the bounding parallel circles. We have assumed $A = 1$ and $P = -1$ for the production of these figures, values that may always be attained, moreover, by the choice of appropriate units for the measurement of space and time. The numerical value of N and the corresponding value of e' are given by the following table or by our curve above, in which the numbers of the respective figures are inserted at the representative positions (u, v) of the odd branch.

Fig. 24	$e' = 1$	$N = 0$
„ 25	$\dfrac{99}{100}$	$\sqrt{\dfrac{398}{9900}} = 0{,}20$
„ 26	$\dfrac{9}{10}$	$\sqrt{\dfrac{19}{45}} = 0{,}65$
„ 27	$\dfrac{1}{2}$	$\sqrt{3} = 1{,}73$
„ 28	0	∞

For a negative value of N, there results, because of the symmetric position of our C_3 with respect to the axis of the abscissa, precisely the same magnitude of the parallel circle and the same trajectory as for positive N, which is, moreover, evident mechanically. The restriction to positive values of N established in the first section was thus well grounded.

Our considerations thus far give us, at the same time, information on the motion of the spherical top in the case $N = 0$ and variable n. According to equation (2), in fact, the position of the parallel circle in this case is again determined—in conformity with the reciprocity law of the previous section—by equation (3) or our Fig. 39, in which v is interpreted as the impulse component n. We have designated the trajectories of the spherical top characterized by $N = 0$ as trajectories of the spherical pendulum. We can justify this designation after the fact on the basis of the preceding section.

A pendulum is not, however, a spherical top, but rather a symmetric top with a special property. Its moment of inertia about the figure axis—that is, about the axis of the rod on whose end the mass particle is fixed—is equal to zero, while the moment of inertia about an axis perpendicular to the rod is equal to ml^2, where m is the mass of the particle and l is the length of the rod. At the same time as the moment of inertia about the figure axis, the eigenimpulse of the pendulum is naturally necessarily equal to zero. We saw, however, that a symmetric top describes the same trajectory as a spherical top of equal gravitational moment and equatorial moment of inertia and equal impulse constants N, n, k. As a result, the trajectory of our spherical top is actually equal, in the case $N = 0$, to the trajectory of a certain spherical pendulum.

We thus wish to regard the figures on the axes $n = 0$ and $N = 0$ in the schema 36 as settled, and turn now to the cases in which neither of our two impulse components n, N vanishes, and, in particular, to Figs. 29–35.

In these figures we have fixed the value of N and varied that of n. We wish, correspondingly, to set $n = v$ in equation (2), and moreover set, as previously, $e = 0$. The mutual dependence between the position of the bounding circle $u = e'$ and the lateral impact $v = n$ is then given by the equation

$$u(v^2 + N^2) - 2Nv + 2AP(1 - u^2) = 0, \tag{4}$$

which again represents a *curve of the third order.*

To determine its form, we seek, as above, its vertical tangents. Two of these have the equation $u = \pm 1$; in fact, these lines intersect our C_3 in two coinciding points, since for $u = \pm 1$ equation (4) reduces to

$$(v \mp N)^2 = 0.$$

The tangent points thus lie at the positions $u = \pm 1$, $v = \pm N$. In addition to these two (coinciding) intersection points, however, the lines $u = \pm 1$ must have a third intersection point with our C_3, which can only lie at infinity. The curve therefore extends to infinity in the vertical direction. The line $u = 0$ will be an asymptote. Namely, if we insert this value into (4), there results only one corresponding ordinate value

$$v_1 = \frac{AP}{N}.$$

The line $u = 0$ must therefore be tangent to the C_3 at infinity.

We next ask whether there are still further vertical tangents in addition to the three found. The general criterion for the appearance of a vertical tangent is that the equation $U_1 = 0$, conceived as an equation in v, give a double root when the abscissa value of a vertical tangent is inserted for u. The equation $U_1 = 0$ is quadratic in v. The condition for the appearance of a double root in the quadratic equation $av^2 + 2bv + c = 0$ is obtained, however, by setting the discriminant $ac - b^2$ to zero. The explicit calculation of this condition gives, in our case,

$$(1 - u^2)(N^2 - 2APu) = 0.$$

We therefore see that in addition to the lines $u = \pm 1$,

$$u_2 = \frac{N^2}{2AP}$$

is also a vertical tangent. Its tangent point has the ordinate

$$v_2 = -\frac{b}{a} = \frac{N}{u_2} = \frac{2AP}{N}.$$

The position of this tangent with respect to those found previously is now essential. Since we assume $P < 0$, the tangent u_2 necessarily lies to the left of the asymptote $u = 0$. It is further asked, however, whether it runs to the left or the right of the tangent $u = -1$. Since it concerns us here only to confirm the figures of the first sections, we assume the values $A = -P = 1$, $N = 0{,}20$ given on pages 242, 243, and reserve the investigation of the form of the C_3 under more general assumptions for the following section. The numerical value of the abscissa of our fourth tangent then becomes

$$u_2 = -0{,}02 > -1.$$

We can now insert the curve of the third order into the framework formed by our four vertical tangents. We begin the drawing at the point $u = 1$, $v = N$. The curve runs vertically here, and asymptotically approaches the ordinate axis above. If we progress with it downward, it crosses the ordinate axis at the point $u = 0$, $v = v_1$, is tangent to the line $u = u_2$ at the point $v = v_2$, and then asymptotically approaches the negative ordinate axis. In addition, a second parabola-like branch of the curve is attached at the point $u = -1$, $v = -N$. The curve therefore consists again of an "even" and an "odd" branch of vertical extent, of which the latter alone is mechanically important. Moreover, we have had to choose the unit of measure in the drawing (cf. Fig. 40), because of the rather small value of N ($N = 0{,}20$), five times as small on the vertical as on the horizontal.

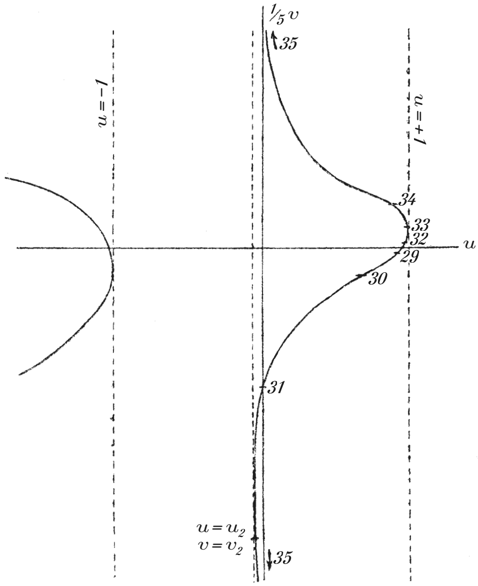


Fig. 40.

We now have before us, in this C_3, a complete image of the variation of the second bounding circle for the variation of the lateral impact. If we draw, namely, the line $v = n$ parallel to the u-axis, then this line strikes the odd branch at a point whose abscissa gives the magnitude of the circle $u = e'$. If we give the line $v = n$ all possible positions between $v = -\infty$ and $v = +\infty$, then the intersection point runs through the entire odd branch. We wish to consider this process in the same order in which Figs. 29–35 follow one another.

We first let n decrease from the case $n = 0$ represented in Fig. 25, therefore displacing our parallel to the u-axis downward. The value of e' is thus diminished; that is, the second bounding circle is widened, until it coincides (for $e' = e = 0$) with the equator. The motion then goes over into the case of slow precession; the corresponding point of our C_3 is its intersection with the ordinate axis, which, as we saw above, has the ordinate

$$v_1 = n = \frac{AP}{N};$$

the same value of n was already derived on page 202 for the slow precession.

We now return again to the position $v = 0$ of our line, and displace it upward, in that we let n increase. The value of e' then increases for a time (that is, our bounding circle diminishes) until it has contracted to the north pole of the unit sphere. The corresponding value of n is, as we infer from our C_3,

$$n = N.$$

The value of e' then decreases with further increasing n; the bounding circle broadens and is transformed asymptotically into the equator for $n = \infty$; the trajectory approaches more and more the fast regular precession represented in Fig. 35.

There yet remain the cases that form the passage between the slow and the fast precession for negative n. The second bounding circle remains very near the equator for all these cases, and lies, as our C_3 shows, in the southern hemisphere of the unit sphere ($e' < 0$). It first decreases slightly to the extremal value u_2, which, under the proportions of our figure, is equal to $-0{,}02$ and corresponds to the value $v_2 = n = -10$. It then decreases again, and is transformed into the equator for $v = -\infty$. In the circumstance that this entire portion of the C_3 lies extraordinarily near the ordinate axis, we recognize the reason that we could not previously (cf. page 214) draw the trajectories clearly for the corresponding cases of the motion of the top.

We assemble the numerical data that are the basis of Figs. 29–35 in the following table; they are also to be seen in the positioning of the respective numbers on our C_3.

$$A = -P = 1, \quad N = 0{,}20.$$

Fig. 29	$e' = 0{,}96$	$n = -\,0{,}20$
„ 30	0,67	$-\,1$
„ 31	0	$-\,5 = \dfrac{AP}{N}$
„ 32	0,9964	$+\,0{,}08$
„ 33	1	$+\,0{,}20 = N$
„ 34	0,87	$+\,1$
„ 35	0	$\pm\,\infty$

Note well that our C_3 corresponds only to an entirely determined and indeed a small value of N, as do our Figs. 29–35. If one lets this value increase, the form of the C_3 and the series of the trajectories also change. For increasing N, the portion of the C_3 lying above the abscissa axis will be stretched in length; the lower part will become rounder, in that the vertical tangent $u_2 = \dfrac{N^2}{2AP}$ wanders to the left and at the same time its tangent point $v_2 = \dfrac{2AP}{N}$ moves nearer to the abscissa axis. An essential qualitative change in the course of the trajectories will first occur, however, when the named vertical tangent first moves over the line $u = -1$ to the left; we will discuss this change in detail in the following section.

We note here that for a sign reversal of N, the C_3 is reflected in the axis of the abscissa. In fact, the equation $U_1 = 0$ remains unchanged if we simultaneously change N into $-N$ and n into $-n$. As a consequence, the same series of trajectories results for negative values of N as for the corresponding positive values, only in the reverse sequence.

We mention, finally, that an approach to the geometric discussion of the equation of the third degree $U = 0$ is also given in the previously cited work of Routh.*)

§7. The characteristic curves of the third order for arbitrary position of the initial circle e; distinction between strong and weak tops.

We must now employ the deliberations of the previous section once again in greater generality, and, in particular, see to what extent the series of figures developed in the second section are specialized by the

*) Rigid dynamics, advanced part, p. 114, art. 204.[114]

specific assumptions (for example, $e = 0$) adopted there. We therefore let the initial circle e be arbitrary, and base our considerations, correspondingly, on the equation $U_1 = 0$ in the form of page 240. Since this equation remains unchanged if we interchange n and N, we can restrict ourselves to the examination of the dependence of the trajectories on one of these quantities—for example, n—and regard N as a fixed parameter whose magnitude, however, is not inessential. We again set $n = v$; the dependence between u and v is then represented by the curve of the third order

$$-(u+e)(v^2+N^2) + 2Nv(1+eu) - 2AP(1-e^2)(1-u^2) = 0. \tag{1}$$

If we again ask for the vertical tangents to the curve, we must employ the deliberations of page 244 anew. Two of these tangents (I and II) are given, as previously, by $u = \pm 1$; their tangent points are $v = \pm N$. Further, there is again a vertical line (III) that is tangent to the curve at infinity. It has the equation $u = -e$, and intersects the curve at the point

$$v = \frac{AP}{N}(1-e^2).$$

A fourth vertical tangent (IV) follows, as on page 244, by setting to zero the "discriminant," which by a small calculation is given here in the form

$$(N^2 - 2AP(u+e))(1-e^2)(1-u^2).$$

Our fourth vertical tangent is therefore the line

$$u = -e + \frac{N^2}{2AP}, \tag{2a}$$

with the tangent point

$$v = Ne + \frac{2AP}{N}(1-e^2). \tag{2b}$$

Because of (2a), this line always lies to the left of the asymptote $u = -e$ for negative P, and always lies to the right of the asymptote $u = -e$ for positive P. We must, however, distinguish two subcases, according to whether this line for negative (positive) P also lies to the left (right) of the line $u = -1$ ($u = +1$) or to the right (left) of that line. The conditions thus run, respectively,

$$P < 0 \quad \begin{cases} -e + \dfrac{N^2}{2AP} < -1, \\ -e + \dfrac{N^2}{2AP} > -1, \end{cases} \qquad P > 0 \quad \begin{cases} -e + \dfrac{N^2}{2AP} > +1, \\ -e + \dfrac{N^2}{2AP} < +1. \end{cases}$$

Whether one or the other of these inequalities is fulfilled depends, for a given mass distribution and a given initial position of the top, on the strength of its eigenimpulse. *We thus distinguish two kinds of tops that are designated as s t r o n g and w e a k tops*; these are defined, in the cases $P < 0$ and $P > 0$, by the preceding inequalities

$$(3) \qquad P < 0 \quad \begin{cases} N^2 > -2AP(1-e) \ldots \text{strong top}, \\ N^2 < -2AP(1-e) \ldots \quad \text{weak top}; \end{cases}$$

$$(3') \qquad P > 0 \quad \begin{cases} N^2 > +2AP(1+e) \ldots \text{strong top}, \\ N^2 < +2AP(1+e) \ldots \quad \text{weak top}. \end{cases}$$

It is noted that our distinction is not absolute, but rather depends on the initial position e of the top. For example, every top is a strong top for positive P in the case $e = -1$, where the trajectory begins at the south pole of the unit sphere.

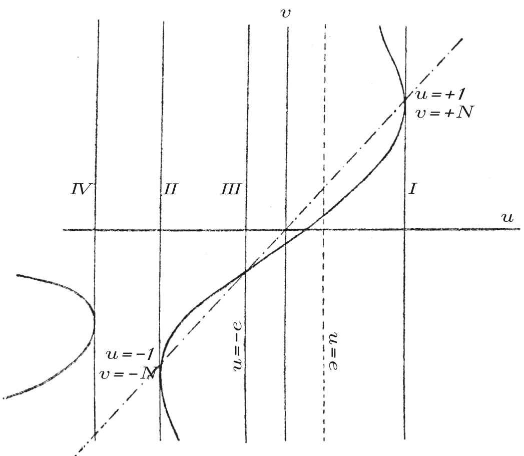
The form of the C_3 is now different according to whether a strong or a weak top is at hand. In both cases, the curve consists of an even and an odd branch. For the strong top, however, the odd branch crosses the entire vertical strip between $u = -1$ and $u = +1$; for the weak top, the odd branch is restricted to the portion of this strip that is bounded by the lines

$$u = -\,1 \;\text{ and }\; u = -\,e + \frac{N^2}{2AP} \quad (P < 0), \quad \text{or}$$

$$u = +\,1 \;\text{ and }\; u = -\,e + \frac{N^2}{2AP} \quad (P > 0).$$

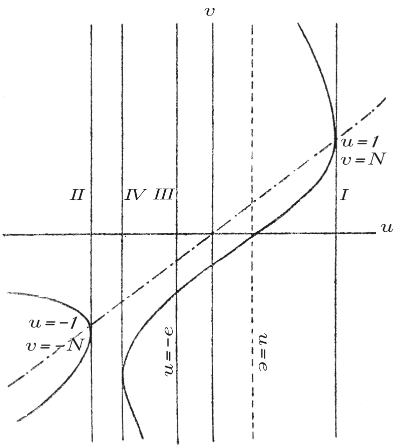
(Only in the boundary case between the strong and the weak top, where the = sign appears in (3) and (3′) instead of the $\gtrless$ signs, do the two parts of our C_3 merge into a single curve by means of a double point at $u = \mp 1$, $v = \mp N$. It will not be necessary to speak of this limiting case explicitly in the following. It naturally mediates the continuous passage between the motions of the strong and weak tops. Only in the next chapter (cf., namely, §8) will we have to return to the special motion of this limiting case that is characterized by the double point in the C_3, and which may claim a special interest with respect to the theory of small oscillations.)

In order not to have too many different cases, we will assume, as in the first section, that $P < 0$. Figs. 41 and 42 correspond to this assumption. The case $P > 0$ may be reduced, according to page 198, to the case $P < 0$ if we carry over the designation "figure axis" from one to the other of the two half-lines into which the symmetry axis is divided by the support point. The signs of N and P then evidently change, while the impulse component n remains unchanged. At the same time,



Strong Top

Fig. 41.



Weak Top

Fig. 42.

the angle ϑ goes over into $\pi - \vartheta$. The quantities u, e, e', e'' will therefore also be reversed in sign. Thus it is clear that we obtain the characteristic curve of the third order in the case $P > 0$ from the case drawn by reflecting the latter in the ordinate axis. We can reasonably forgo this repetition.

We now make our construction for the determination of the bounding circle $u = e'$, and thus displace the line $v = n$ parallel to the u-axis from $v = -\infty$ to $v = +\infty$ and seek the abscissa value of its intersection point with the odd branch of the C_3. Here the following characteristic distinction between the strong and the weak top appears: for the strong top, the projection of the intersection point onto the axis of the abscissa sweeps through the entire interval between -1 and $+1$; for the weak top, it runs only through the portion of this interval that is bounded by $u = +1$ and $u = -e + \dfrac{N^2}{2AP}$. In both cases, moreover, the value that is attained is attained two times. *This has the consequence that the parallel circle e' for the strong top can assume every position on the sphere, and indeed does so for two different values of n;*

the parallel circle for the weak top, in contrast, is excluded from a certain spherical calotte that surrounds the south pole of the unit sphere and is bounded by the circle $u = -e + \dfrac{N^2}{2AP}$.

We thus see that Figs. 29–35 are based on the assumption of a weak top, since for these figures the greatest part of the southern hemisphere generally remains free of the trajectory curves. In fact, the second of the two criteria (3) is fulfilled for the previously assumed values

$$N = 0{,}20, \quad A = 1, \quad P = -1, \quad e = 0.$$

Our figures now allow a convenient further investigation of the various previously drawn cases that can occur for the motion of the top: the appearance of regular precession, cusp formation, etc. This is done under the following enumeration.

1) We first see what our curves state about the possibility of *regular precession.* Regular precession will occur when $e' = e$. We thus draw the line $u = e$; its intersection points with the curve of the third order, if such points are at hand, give the values of n that are required for regular precession. Thus the strong and the weak tops are again distinguished:

For the strong top, there are always two real intersection points with the line $u = e$, *and thus two (generally different) possible cases of regular precession.*

For the weak top, in contrast, the intersection points are real only if the line $u = e$ lies to the right of the tangent $u = -e + \dfrac{N^2}{2AP}$, and therefore if the inequality

$$4APe < N^2$$

obtains.

For the weak top, there are either two cases of regular precession or none, according to whether

$$4APe < N^2 \quad \text{or} \quad 4APe > N^2 \tag{4}$$

obtains.

We note that for the horizontal initial position of the figure axis ($e = 0$), the first of our inequalities is fulfilled identically, in the case that N is not directly zero. Correspondingly, we always had two real cases of precession in the second section of this chapter for $N > 0$, even for a weak top.

The values of n that correspond to the two cases of regular precession are naturally given by (1) if we set $u = e$ and solve the resulting quadratic equation

$$(v - Ne)(N - ve) = AP(1 - e^2)^2; \tag{5}$$

we find

$$v = \begin{Bmatrix} n_1 \\ n_2 \end{Bmatrix} = \frac{(1 + e^2)N \pm (1 - e^2)\sqrt{N^2 - 4APe}}{2e}. \tag{6}$$

(It will be well to relate this result to the developments of pages 178 and 179. We saw there that there are, for a given mass distribution, a given value of $\cos\vartheta = e$, and *a given velocity component* μ, two values of the velocity component ν that give rise, in the case of a spherical top, to a regular precession; namely,

$$\text{(a)}\ \ \nu = \frac{P}{A\mu}, \qquad \text{(b)}\ \ \nu = \pm\infty.$$

Our current result apparently differs from the previous in the following manner: for a given mass distribution, a given value of the parallel circle e, and *a given impulse component* N, a regular precession occurs for the two values of the impulse component n that are given in (6). The discrepancy obviously rests on the fact that we have fixed the velocity component μ in one case and the impulse coordinate N in the other; we cannot wonder that we obtain different values of the corresponding precession constants ν and n in the two cases.

To pursue the connection between the roots n_1, n_2 and the previously distinguished cases (a) and (b) more precisely, we note that both values n_1 and n_2 correspond to case (a). In fact, it is easily shown that our equation (5) is identical with the equation $P = A\mu\nu$, acquired from the theory of the deviation resistance, that yields the root (a). If we use, namely, the values of ψ' and φ' given in equation (6) on page 238 and consider that these values are equal for regular precession to the constants ν and μ, respectively, then we have

$$\nu = \frac{n - Ne}{A(1 - e^2)}, \quad \mu = \frac{N - ne}{A(1 - e^2)}. \tag{7}$$

As a result, equation (5) is actually transformed into the relation $P = A\mu\nu$, which states that the turning-force of gravity is in equilibrium with the inertial resistance of the spherical top.

We convince ourselves further that the precession case (b), for which μ has a given finite value and ν is infinitely large, corresponds to an infinitely large value of N that was disregarded by assumption. The adjacent figure is of service here. The endpoint of the rotation vector is imagined to be at infinity on the parallel to the vertical through the endpoint of the component μ. If we project this vector perpendicularly onto the figure axis, then, understanding by Ω the length of the rotation vector, the length r of the orthogonal projection onto the figure axis is $r = \Omega \cos\vartheta = \infty$, so that the impulse component N also becomes infinitely large.

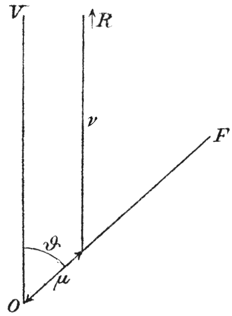


Fig. 43.

The indicated exceptional case is the case $e = 0$, in which the roots (a) and (b) are equal, respectively, to the roots n_1, n_2 (divided by A). In fact, the parallel components ν and μ are respectively equal, under the assumption $e = 0$, to the orthogonal components with respect to the vertical and the figure axis. We then find once again from (6) the roots (a) and (b) through the passage to the limit $e = 0$; namely,

$$\text{(a)}\ \frac{n_1}{A} = \nu = \frac{P}{A\mu}, \quad \text{(b)}\ \frac{n_2}{A} = \nu = \pm\infty.$$

Despite the recognized distinction, it is permissible to designate the two cases of regular precession that are possible for a given value of N, just as well as the two cases that correspond to a given value of μ, as *slow* and *fast* precession. Since we will have no essential future recourse to the precession constants μ and ν, no misunderstanding will arise from this duplicity of nomenclature.)

2) We next wish to investigate *the limiting case of the motion of the top for infinitely increasing n.* While we know that this limiting case coincides, under the assumption $e = 0$, with fast regular precession, which then, in turn, degenerates into an infinitely fast precession, it is essentially different from regular precession for a more general initial position.

For this limiting case $n = \infty$, we can first conclude from our curve of the third order that the second bounding circle e' coincides, for the strong as well as the weak top, with the parallel circle $-e$; in fact, $-e$ is the abscissa of the infinitely distant point of the C_3 on the odd branch.

In the limiting case $n = \infty$, the apex of the top therefore oscillates about the equator as a mean position, in that the trajectory fluctuates back and forth between the circles $+e$ and $-e$.

The form of the trajectory thus becomes extremely simple. For orientation, we consider in advance a top for which $P = N = 0$, a weightless top without eigenimpulse. The considered motion corresponds (because $N = 0$) to the motion of the spherical pendulum; this is the special case (because $P = 0$) in which the action of gravity is nullified. The trajectory must therefore be the same as that of a single mass particle that is subjected to no external forces and is constrained to remain on the surface of a sphere. This particle evidently describes, however, a great circle on the sphere with constant velocity.

If P and N are now set unequal to zero, with, however, the lateral impact n taken to be infinitely large, then the path of the apex of the top remains the same as before. The influence of the initial impact, namely, will completely overwhelm that of gravity and the eigenimpulse.

The result of calculation is in conformity with this deliberation. Namely, if we let n become infinitely large, there follows from equations (1) and (2) of page 240, in the first approximation,

$$U = -\frac{n^2}{A^2}\frac{u^2 - e^2}{1 - e^2}.$$

The integral for t of page 238 simplifies to

$$t = \frac{A\sqrt{1-e^2}}{n}\int\frac{du}{\sqrt{e^2 - u^2}} = \frac{A\sqrt{1-e^2}}{n}\,\text{arc sin}\,\frac{u}{e}; \tag{8}$$

thus

$$u = e\sin\frac{nt}{A\sqrt{1-e^2}}.$$

The rate of change of u therefore becomes infinitely large. At the same time, the integral for ψ of page 238 is transformed, approximately, into

$$\psi = \sqrt{1-e^2}\int\frac{du}{(1-u^2)\sqrt{e^2-u^2}};$$

the value of ψ becomes, as one easily verifies,

$$\psi = \text{arc sin}\left(\frac{\sqrt{1-e^2}}{e}\frac{u}{\sqrt{1-u^2}}\right). \tag{8$'$}$$

If we set $u = \cos\vartheta$ and $e = \cos\vartheta_0$, we can write this equation more simply as

$$\sin\psi\,\mathrm{tg}\,\vartheta = \mathrm{tg}\,\vartheta_0; \tag{9}$$

interpreted correctly, this states *that the apex of the top describes a great circle.*

The apex of the top, namely, has coordinates

$$X = Y = 0, \quad Z = 1$$

in the XYZ frame; its coordinates in the xyz frame thus become, according to the transformation formulas (5) of page 19,

$$x = \sin\vartheta \sin\psi, \quad y = -\sin\vartheta\cos\psi, \quad z = \cos\vartheta.$$

Thus we can also write equation (9) of our trajectory as

$$x = \operatorname{tg}\vartheta_0\, z.$$

This, however, is the equation of a plane through O that intersects the unit sphere in the previously named great circle.

If we place, in particular, the figure axis horizontally in the initial position, then our great circle is transformed into the equator, and we again have the infinitely fast regular precession of the first section, which is now confounded with our limiting case.

The conditions of the first section were therefore chosen, to an extent, too particularly. *For a more general initial position of the figure axis, there is a sequence of transitions between the fast precession and our limiting case that eluded us in the first section.* The necessary supplement, however, can easily be supplied.

We represent in Figs. 44 and 45 the stereographic projections of the trajectories of the fast precession and the limiting case. The trajectory of the former is the circle $u = e$, and of the latter the more boldly drawn circle tangent to the circles $u = +e$ and $u = -e$.

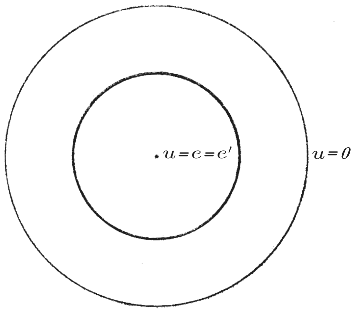


Fig. 44.

The transition curves between the two have the following character. The bounding circle e', starting from its position in Fig. 44, gradually widens with increasing n, in that it crosses the equator and asymptotically approaches the circle $u = -e$. In the stereographic projection, the trajectory that must run to and fro between the circles e and e' surrounds the former circle, while it is enclosed by the latter. As the prototype of this trajectory we can regard, for example, Fig. 30, where the inner circle would now be interpreted as the fixed initial circle e, and the outer as the bounding circle e'. The span width of the individual component arcs decreases with increasing n, until it is reduced to zero for $n = \infty$, so that the trajectory simply runs back into itself.

Next to the limiting case $n = \infty$ for finite N is placed the limiting case $N = \infty$ for fixed n, which may be treated similarly. As mentioned, the equation $U_1 = 0$ remains unchanged by the interchange of n and N, so that we can bestow the interpretation n just as

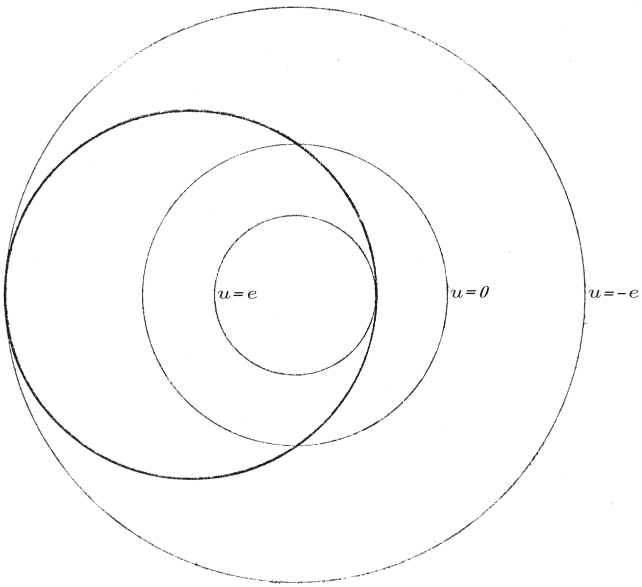


Fig. 45.

well as N to the ordinate v in Figs. 41 and 42. It follows that for increasing N and fixed n, the bounding circle e' is also transformed asymptotically into the position $-e$. The apex of the top therefore also oscillates, in this limiting case, back and forth between the two circles e and $-e$ with infinite speed. The two circles coincide only for a horizontal position of the figure axis; the amplitude of the oscillation in this limit becomes vanishingly small, and we have the type of pseudoregular precession represented in Fig. 28. In all other cases, in contrast, the trajectory in the limit $N = \infty$ will present an essentially different image.

The circumstances under which the particularly interesting case of pseudoregular precession occurs for a more general initial position of the figure axis will be presented in detail in the next chapter.

3) We wish, finally, to investigate the possibility of *cusp and loop*

formation. The trajectory can touch the parallel circle e or e' with cusps only when $\dfrac{d\psi}{du} = 0$ for $u = e$ or $u = e'$. We thus conclude, just as on page 242, that the conditions for the appearance of cusps on the circles e and e' are, respectively,

$$n - Ne = 0, \quad n - Ne' = 0.$$

For the initial circle e, it thus appears that cusp formation will always occur for a specific ratio $n : N$. In §2 only the special case in which this ratio is zero was present, and cusp formation therefore always occurred if the lateral impact was equal to zero, independent of the value of the eigenimpulse.

In order to be able to conveniently envision the appearance of cusps on the boundary circle e', we again revert to our curve of the third order. If we imagine N fixed and n variable ($n = v$), then our condition above is represented in the uv-plane by the line

$$v - Nu = 0$$

that joins the tangent points of the tangents $u = \pm 1$ and is drawn as a dotted line in Figs. 41 and 42. The question is whether this line intersects the C_3 inside the mechanically valid interval or not.

Two intersection points fall at the points $u = \pm 1$, $v = \pm N$. They do not correspond, however, to actual cusp formation, since the boundary circle e' has contracted in this case to a single point, the north or south pole. As for the third intersection point, a glance at our curve shows immediately that it lies on the odd branch in the case of the strong top and on the even branch in the case of the weak top.

For the strong top, there is a specific parallel circle e', for fixed N and appropriately chosen n, that is touched by the trajectory with cusps, and for the weak top there is no such circle.

As we saw, Figs. 29–35 in the second section correspond to a weak top, so that cusp formation on the circle e' cannot occur in these figures. It now appears, moreover, that the same occurrence is also excluded in the case of the weak top for an arbitrary position of the initial circle e. The previous series of figures thus offers, for the weak top, a sufficiently general image of the sequence of the trajectories.

The exercise still remains, however, to clarify the continuous ordering of the cusped trajectory curves that were just found for the strong top. This is done at the end of this section. In advance, we wish to extract from our curve of the third order a criterion for the appearance of *loops.*

As we just saw, cusp formation occurs if the equation $n - Nu = 0$ is fulfilled for $u = e$ or e'. If, however, this equation is fulfilled for a value of u between e and e', then the stereographic image of the trajectory runs in the radial direction each time it crosses the determined parallel circle u. There follows, as on page 242, the existence of loops. We thus recognize the appearance of loops geometrically in the following manner: we draw our line $v = n$ parallel to the abscissa axis and intersect this line with the line $v - Nu = 0$. If the abscissa of the intersection point lies between e and e', then loops appear; if it lies outside this interval, then loops are impossible.

If one applies this rule to the C_3 of the weak top, one sees immediately that loop formation can appear only in the interval between the curve that touches the initial circle e with cusps and the trajectory that passes through the highest point of the sphere. An example is offered in Fig. 32 of page 213.

The same interval is also distinguished by loop formation for the strong top. Here, however, there is a second interval that extends from the intersection point of the C_3 with the line $v - Nu = 0$ to the tangent point of the C_3 with the line $u = -1$. An intersection point of $v = n$ with the line $v - Nu = 0$ is then found (cf. Fig. 41) to the right of the C_3; its projection onto the axis of the abscissa falls in the region ee'. The corresponding looped curves are continuously joined on one side to the trajectory that touches the circle e' with cusps, and on the other side to the curve that passes through the south pole of the unit sphere.

For the weak top we therefore have one, and for the strong top two intervals with loop formation. —

In conclusion, we wish to supplement, as already proposed, the series of trajectories of the first sections for the case of the strong top, in that we follow the passage from the slow regular precession to the limiting case $n = \infty$, through the cases of loop and cusp formation.

We begin from the slow regular precession, give N a fixed positive value, and let n decrease. The value e' first decreases uniformly with n, as is evident from Fig. 41.

The stereographic image of the trajectory is thus tangent to the initial circle from the exterior, in that the trajectory encloses it, and is tangent to the second bounding circle from the interior. For a certain value of n constructed above, cusps replace the tangents to the circle e'. For further decrease of n, the cusps resolve into loops. This character of the trajectory persists until n has reached the value $-N$, where the circle e' contracts to the south pole, and its stereographic projection correspondingly becomes infinitely large. From now on the circle e' widens again (that is, it diminishes in the stereographic image) and tends asymptotically to the parallel circle $u = -e$. The trajectory thus assumes more and more the simple form of Fig. 45.

§8. On the numerical calculation of the elliptic integrals for t and ψ.

In a problem with applications, as is present here, we may not be satisfied with presenting the possibility of calculation in a general schema. We must seek, rather, to advance to actual numerical implementation. While the older mathematicians, to Gauſs and Jacobi inclusive, always strove to present their results through not only convergent, but also well convergent and practical processes, the current development of mathematics often neglects the duty of numerical execution. We wish, in contrast, to regard the numerical implementation of a theory as the capstone of the edifice, to which we attribute no smaller importance and no smaller interest than any other part of the whole. In the particular problems that lead to elliptic functions, we are in the pleasant position, thanks to the high development of this theory, of being able to effect numerical evaluation without any difficulty, as will be shown in this section.

We first consider an integral of the form of our

$$t = \int \frac{du}{\sqrt{U}}, \tag{1}$$

in which U signifies any polynomial of the third or fourth degree in u. We assume only that the roots of $U = 0$ are real. Such an integral is designated as an *elliptic integral of the first kind*, since it may always be brought into the normal form that Legendre has introduced as the "fonction de première espèce." The designation "everywhere finite integral," which is associated with the behavior of t in the complex plane and thus characterizes the integral of the first kind in function-theoretic respects, can be explained only in the sixth chapter.

The Legendre normal form of the integral of the first kind is, in Legendre's notation,

$$F(k,\varphi) = \int_0^\varphi \frac{d\varphi}{\sqrt{1-k^2\sin^2\varphi}}; \tag{2}$$

here φ is called the amplitude and k the modulus of the integral; it is assumed that $0 \leq \varphi \leq \pi/2$, $0 < k < 1$. If we set $\sin^2\varphi = x$, then we can also write

$$F(k,\varphi) = \frac{1}{2}\int_0^x \frac{dx}{\sqrt{x(1-x)(1-k^2x)}}. \tag{2$'$}$$

Almost all methods for the evaluation of elliptic integrals of the first kind require the transformation of the given integral to the Legendre normal form. No exception is made even by those authors, such as Schwarz*) and Halphen,*) who begin from the Weierstrass theory and translate the formulas of the older theory into the Weierstrass notation. As important as the Weierstrass theory is in function-theoretic respects, it appears to have made no actual advance over the older theory on the numerical side. We thus prefer to revert directly to the Legendre notation and conception for numerical questions, instead of rewriting them each time in the Weierstrass notation.

In order to be able to carry out the transformation of the integral (1) into the Legendre normal form, one must seek the roots of the equation $U = 0$. If we restrict ourselves to the present case of the top, in which U is a polynomial of the third degree, then we have only to solve a cubic equation. This equation even reduces, since we regard the root e as known (cf. page 239), to the quadratic equation $U_1 = 0$ with roots e'

*) Cf. H. A. Schwarz: Formeln und Lehrsätze zum Gebrauch der elliptischen Funktionen, and Halphen: Théorie des fonctions elliptiques, Bd. I. Kap. 8.

and e''. To the thus determined roots e, e', e'', we must add, according to page 226, the equally valid "fourth branch point" ∞.

We wish to assume, for example, $P > 0$, and choose the designation of the roots e, e', e'' so that their order of succession, as in the schema $P > 0$ of page 226, becomes

$$-1 < e < e' < +1 < e'' < \infty.$$

The transference of the integral (1) into the form (2′) may now always be effected by a *linear transformation*; that is, by setting the new integration variable x equal to a linear function of the original variable u. At the same time, it may always be attained that the quantities x and k in (2′) will be real numbers between 0 and 1. The transformation formulas are different according to whether the original integration interval lies in the domain ee', $e'e''$, $\ldots$.

To treat of an integral in the interval ee' with lower limit e and upper limit u, for example, we can arrange our transformation so that the values e, e', ∞ go over into the values 0, 1, ∞, respectively. The point e'' on the u-axis between e' and ∞ is then transformed into a point on the x-axis between 1 and ∞ that we call $1/k^2$, so that k^2 signifies a positive proper fraction. At the same time, the upper limit of the original integral between e and e' is transformed into the upper limit of the new integral between 0 and 1.

The required linear transformation is now evidently

$$\frac{u-e}{e'-e} = x,$$

from which follows

$$\frac{e''-e}{e'-e} = \frac{1}{k^2}.$$

Our polynomial U, which we can give the form

$$U = c^2(u-e)(e'-u)(e''-u),$$

where c^2 is the coefficient $\dfrac{2P}{A}$ of u^3, is transformed with the introduction of x into the expression

$$U = c^2(e'-e)^3 x(1-x)\left(\frac{1}{k^2} - x\right) = \frac{c^2(e'-e)^3}{k^2} x(1-x)(1-k^2x).$$

The original integral

$$t = \int_e^u \frac{du}{\sqrt{U}} = \frac{1}{c}\int_e^u \frac{du}{\sqrt{(u-e)(e'-u)(e''-u)}}$$

thus takes the form

(3)
$$t = \pm\frac{k}{c\sqrt{e'-e}}\int_0^x \frac{dx}{\sqrt{x(1-x)(1-k^2x)}} = \pm\sqrt{\frac{2A}{P(e''-e)}}F(k,\varphi),$$

where the amplitude φ and the modulus k have the meanings

(3′)
$$\varphi = \text{arc sin}\sqrt{\frac{u-e}{e'-e}}, \quad k = \sqrt{\frac{e'-e}{e''-e}},$$

and $F(k,\varphi)$ is the Legendre integral defined in (2). The sign of t depends upon which of the schematically represented coverings in Fig. 38 of page 226 we wish to choose for the integration.

To treat, on the other hand, of an integral whose upper and lower limits lie in the region $(-\infty\ e)$, we arrange the transformation equation between u and x so that the points $-\infty$, e, e'' will be transformed into the points 0, 1, ∞, respectively. The point e' between e and e'' corresponds to a value of x between 1 and ∞ that we call $1/k'^2$, so that k'^2 also denotes a positive proper fraction.

The linear transformation that yields the desired transference is obviously
$$\frac{e''-e}{e''-u} = x,$$
so that we obtain for k'^2 the value
$$\frac{e''-e}{e''-e'} = \frac{1}{k'^2}.$$
If we now replace u in the expression for U by x, then
$$U = -c^2(e-u)(e'-u)(e''-u) = -c^2(e''-e)^3\frac{x(1-x)(1-k'^2x)}{x^4}.$$
There follows, for example, if $-\infty$ is the lower limit and $u<e$ is the upper limit of the original integral,
$$t = \frac{\pm i}{c\sqrt{e''-e}}\int_0^x \frac{dx}{\sqrt{x(1-x)(1-k'^2x)}} = \pm i\sqrt{\frac{2A}{P(e''-e)}}F(k',\varphi);$$
the amplitude φ and the modulus k' are thus, according to the preceding, determined as
$$\varphi = \text{arc sin}\sqrt{\frac{e''-e}{e''-u}}, \quad k' = \sqrt{\frac{e''-e'}{e''-e}}.$$
These two quantities again satisfy the above conditions
$$0<\varphi<\frac{\pi}{2}, \quad 0<k'<1.$$

The modulus k', which is related to the modulus k defined in (3′) by the equation

$$k^2 + k'^2 = 1,$$

is called, moreover, the "complementary modulus to k."

One always achieves the goal in a similar manner whenever the original integration interval lies between the points e, e', e'', ∞, where we assume only that the interval contains none of these points in its interior; in that case we must divide the interval into component intervals. The general rule for the construction of the appropriate transformation formula is the following:

One establishes a specific directional sense on the u-axis, and assigns the two branch points inside which the original integration interval lies, in the order that corresponds to this direction, to the points 0 *and* $+1$. *One then proceeds in the established sense beyond the integration region on the u-axis, which one imagines to be closed at infinity, and assigns the next-to-nearest branch point that one subsequently strikes as the point ∞. Then there is always a linear transformation between u and x that yields the named ordering. This ordering necessarily changes the fourth branch point, whose ordering we can no longer choose, into a point that lies on the x-axis between* $+1$ *and* $+\infty$*; all points of the original integration region correspond, at the same time, to values of x that are contained between* 0 *and* 1.

Moreover, the ordering of the u- and x-axes may always be constructed in two ways, in that the directional sense of the u-axis in our rule indeed remains arbitrary.

We wish to write out the transformation to the Legendre normal form for four particular integrals t that will play an essential role in the sixth chapter. These are the integrals

$$\omega = \int_e^{e'} \frac{du}{\sqrt{U}}, \quad i\omega' = \int_e^{-\infty} \frac{du}{\sqrt{U}}, \quad ia = \int_{-1}^{e} \frac{du}{\sqrt{U}}, \quad ib = \int_{e'}^{+1} \frac{du}{\sqrt{U}}.$$

We have already considered the first of these integrals in the third section; it gives the time that is required for the top to traverse a half-arc of its trajectory. The remaining integrals have no mechanical meaning in the elementary sense.

There now follow from equations (3) and (4), and from our general rule, the following expressions for our four integrals:

$$(5) \qquad \omega = MF\left(k, \frac{\pi}{2}\right), \quad \omega' = MF\left(k', \frac{\pi}{2}\right), \quad a = MF(k', \varphi_a),$$

$$b = MF(k', \varphi_b),$$

where the symbols M, k, k', φ_a, φ_b have the meanings

$$(5') \quad \begin{cases} M = \sqrt{\dfrac{2A}{P(e''-e)}}, \ \ k = \sqrt{\dfrac{e'-e}{e''-e}}, \ \ k' = \sqrt{\dfrac{e''-e'}{e''-e}} = \sqrt{1-k^2}, \\ \varphi_a = \text{arc sin}\sqrt{\dfrac{1+e}{1+e'}}, \quad \varphi_b = \text{arc sin}\sqrt{\dfrac{e''-e}{e''-e'}\cdot\dfrac{1-e'}{1-e}}. \end{cases}$$

The exercise of numerically evaluating an arbitrary elliptic integral of the first kind is thus reduced to the simpler exercise of finding the value of the Legendre integral $F(k, \varphi)$. The different paths that lead to this end should be named briefly.

1. The most obvious path would be to transform the square root under the integral sign into a *series* according to the binomial theorem, and execute the integration term by term. The series that one then obtains are not, however, sufficiently convenient when k is somewhat different from zero. To improve their convergence, one must combine this method with the immediately following second method, as has been thoroughly done, in fact, by Schwarz.*)

2. A method of equal theoretical and practical beauty consists in subjecting the integration variable to a quadratic transformation of such a nature that the integral of the first kind is transformed into itself, only with a modified modulus and a transformed amplitude. The so-called *Landen transformation* is to be cited in the first place here. The transformed modulus k_1 is simply equal to the ratio of the geometric to the arithmetic mean of the modulus k and the number 1; one thus has $k_1 = 2\sqrt{k}/(1+k)$. Through an appropriately continued application of this transformation, one is led to a series of moduli k_1, k_2, k_3, ... (a "module ladder"), whose individual terms continuously increase and approach the value 1. In the reverse sense, the Landen transformation therefore yields a module ladder that decreases toward 0. If, however, the modulus of the elliptic integral is made sufficiently small in this manner, the integration becomes executable in the simplest way. In fact, we directly have, understanding by k_n a sufficiently small modulus

*) Cf., for example, Art. 48 of the formula collection.

and by φ_n the corresponding transformed value of the amplitude, $F(k_n, \varphi_n) = \varphi_n$. This method has been employed to great effect by Legendre in the calculation of his tables.[115]

The so-called *Gaußian method of the arithmetic–geometric mean**) does not differ essentially from the preceding; it is distinguished only by formally greater elegance.

Instead of a *quadratic*, one can also exploit *a higher-order transformation* for the numerical computation of elliptic integrals, as was developed for the first time by Jacobi.**)

3. A third method is based on the *inversion of the elliptical integral* and the introduction of the *ϑ-functions*. It leads, just as the previous method, very quickly to the goal, but cannot yet be discussed in this place.***)

4. One can further think of evaluating elliptic integrals directly by *mechanical quadrature*, with the possible aid of an integration apparatus. This procedure offers the advantage of being directly applicable to an arbitrary elliptic integral, and makes the transformation to the normal form superfluous. On the other hand, however, this method requires the calculation or delineation of the quantity $\dfrac{1}{\sqrt{1-k^2\sin^2\varphi}}$ or $\dfrac{1}{\sqrt{U}}$ for a large sequence of points of the integration interval. Thus the named advantage will be amply offset, so that this method can hardly compete with the others.

5. A last method that we wish to recommend most particularly consists in not calculating at all, but rather using the *Legendre tables*.†) In fact, we would dispense with this beautiful means of help no more than we would find the logarithm of a number other than from the logarithm tables. The use of the Legendre tables is very convenient. One need only pass from the modulus k to an angle Θ by means of the trigonometric table that is defined by the equation $k = \sin\Theta$. One then

*) Gauſs: Ges. Werke, Bd. III, pp. 361 and ff.[116]

**) Jacobi: Ges. Werke, Bd. I, p. 31;[117] cf. also Klein–Fricke, Modulfunktionen, II p. 111.

***) Presented in detail by Schellbach: Die Lehre von den elliptischen Integralen und den Thetafunktionen, Berlin 1864, particularly 1. Abteilung, 4. Abschnitt.[118]

†) Bd. II of the traité des fonctions elliptiques, Paris 1826, pp. 284–363 and pp. 222–245. It is much to be desired that this now rather rare table be made of easier access through a new printing.[119]

finds, for all full degrees of Θ and φ between 0 and 90, the value of $F(\sin\Theta, \varphi)$ with 9 decimal place precision in the tables. The so-called complete integrals of the first kind—that is, the values of $F\left(\sin\Theta, \frac{\pi}{2}\right)$—are calculated even more precisely by Legendre. In addition to the integrals of the first kind, the tables give also the so-called integral of the second kind $E(k, \varphi)$, into whose definition we need not enter here.

Thus one will be compelled to revert to one of the previous methods only if one must evaluate an integral of the first kind with complex limits or a complex modulus.

As an example, we calculate, in this sense, the time that the apex of the top requires in Figs. 24–28 to arrive from a lowest point of its trajectory to the next following highest point; that is, the value of the half-period ω.

While we previously assumed $P = -1$ in those figures, we now take, so as to be able to apply our latter formulas directly, $P = +1$, and therefore must (according to page 250) reverse in sign the value of e' given on page 243. Moreover, we will now denote this value, since it represents the smallest of the roots of $U = 0$ under consideration, by e. The second root corresponds to the equator that appears in all the figures as the boundary circle, so that we have $e' = 0$. The still wanting root e'' results from the quadratic equation $U_1 = 0$, which in our case (cf. page 240) takes the simple form

$$uN^2 + 2AP(1 - u^2) = 0.$$

If we set $P = 1$ and also, as previously, $A = 1$, there follows

$$u^2 - \frac{1}{2}uN^2 - 1 = 0.$$

This equation shows that e'' simply becomes the reciprocal of the root given on page 243. The quantity M that appears in equation (5), as well as k and the corresponding angle Θ, are thus very easy to compute. We take the value of $\lg F\left(k, \frac{\pi}{2}\right)$ from Table I of Legendre, and compute the desired quantity ω through equation (5). We place the results in the following table.[120]

$$A = 1, \quad P = -1, \quad n = 0, \quad e' = 0.$$

Fig.	e''	$\lg_{10} M$	k	Θ	$\lg_{10} F\left(k, \frac{\pi}{2}\right)$	ω
24	1	0	$\sqrt{\frac{1}{2}}$	45°	0,26813	1,854
25	$\frac{100}{99}$	0 (circa)	$\sqrt{\frac{9801}{19801}}$	44,7	0,26709	1,848
26	$\frac{10}{9}$	0,99879 − 1	$\sqrt{\frac{81}{181}}$	42,0	0,25820	1,807
27	2	0,95154 − 1	$\sqrt{\frac{1}{5}}$	26,6	0,22007	1,484
28	∞	$-\infty$	undet.	undet.	undet.	0

We thus see that the transit time of the single half-arc decreases with increasing N until it attains the value 0 in Fig. 28. If we agree that the values of A, P, n, and N are to be interpreted in the absolute system of measure, then the given value of ω signifies seconds.

Still more important for us than the relation between t and u is the dependence between ψ and u, since this relation directly provides the form of the trajectory. We must therefore orient ourselves further on the calculation of the elliptic integral for ψ.

In order to be able to connect again with Legendre, we wish to express ψ in terms of the so-called *Legendre normal integral of the third kind.* Legendre defined his normal integral of the third kind as

$$\Pi(k, \varphi, p) = \int_0^\varphi \frac{1}{1 - p\sin^2\varphi} \frac{d\varphi}{\sqrt{1 - k^2\sin^2\varphi}}.$$

The quantity p, which is assumed by Legendre as real and which furthermore may not lie between $+1$ and $+\infty$, so that the integral taken on a real path is meaningful, is called the *parameter* of the integral.

We wish to show that ψ may be represented as a linear combination of two normal integrals of the third kind.

For this purpose, we first separate the factor $\dfrac{n - Nu}{A(1 - u^2)}$ under the integral sign into partial fractions; that is, we set

$$\frac{n - Nu}{A(1 - u^2)} = \frac{1}{2A}\left(\frac{n + N}{1 + u} + \frac{n - N}{1 - u}\right),$$

so that we obtain

$$\psi = \frac{n+N}{2A}\int \frac{1}{1+u}\frac{du}{\sqrt{U}} + \frac{n-N}{2A}\int \frac{1}{1-u}\frac{du}{\sqrt{U}}.$$

Further, we bring the quantity $\dfrac{du}{\sqrt{U}}$ into the form $\dfrac{dx}{\sqrt{x(1-x)(1-k^2x)}}$ through one of the transformations given above. We can restrict ourselves here to the case in which the original integration variable runs in the domain ee'. We must then apply the transformation that led to equation (3); we thus define the new integration variable x as

$$\frac{u-e}{e'-e} = x.$$

There follows

$$1 \pm u = (1 \pm e) \pm (e'-e)x,$$

while the quantity $\dfrac{du}{\sqrt{U}}$ is simultaneously transformed as in (3). The expression for ψ thus becomes[121]

$$\begin{aligned}\psi = {} & \frac{n+N}{A}\sqrt{\frac{A}{2P(e''-e)}}\int \frac{1}{1+e+(e'-e)x}\frac{dx}{\sqrt{x(1-x)(1-k^2x)}} \\ & + \frac{n-N}{A}\sqrt{\frac{A}{2P(e''-e)}}\int \frac{1}{1-e-(e'-e)x}\frac{dx}{\sqrt{x(1-x)(1-k^2x)}}.\end{aligned}$$

In order to produce the Legendre normal form, we must only set $x = \sin^2\varphi$ and extract $1+e$ and $1-e$ from the first and second integrals, respectively. *Then ψ follows directly as a linear combination of two normal integrals in the form*

$$\psi = C_1\Pi(k,\varphi,p_1) + C_2\Pi(k,\varphi,p_2),$$

where the quantities k, φ are defined by the previous equations (3′), and C_1, C_2, p_1, p_2 have the meanings[122]

$$\begin{aligned} C_1 &= \frac{n+N}{1+e}\sqrt{\frac{2}{AP(e''-e)}}, \quad & C_2 &= \frac{n-N}{1-e}\sqrt{\frac{2}{AP(e''-e)}}, \\ p_1 &= \frac{e'-e}{1+e}, & p_2 &= -\frac{e'-e}{1-e}. \end{aligned}$$

It would be further necessary only to find the numerical values of the Legendre normal integrals of the third kind in the simplest possible way. Unfortunately, there are and can be no tables for this purpose. Since the value of the integral $\Pi(k,\varphi,p)$ depends on three different quantities, the tables in question must be tables with a threefold entry. Such tables, however, may be calculated only with disproportionate labor, and are generally not printed.

Nevertheless, we can also draw upon the Legendre tables here if we restrict ourselves to the calculation of the so-called "complete integral of the third kind" $\Pi\left(k, \frac{\pi}{2}, p\right)$, which for our trajectory would mean that we ask only for the span width $2\psi_\omega$ of the individual component arcs, and forswear the construction of the individual points of the curve. As Legendre*) has shown, his complete integrals of the third kind may always be reduced to integrals of the first and second kinds that contain the parameter p in their upper limits and have as moduli partly the modulus k of the integral Π and partly the complementary modulus. Since we can look up the values of the integrals of the first and second kinds directly in the tables, these reduction formulas allow the complete integral of the third kind, and thus also the magnitude of ψ_ω, upon which the form of the trajectory primarily depends, to be found relatively quickly.[123]

The span widths of the component arcs in the figures of the first sections were computed in this manner.**)[125] We will not, however, enter here into the execution of this computation or the true meaning of the reduction formulas, since in the sixth chapter we will treat in detail of the above method (3), which teaches us to find arbitrarily many points of the trajectory in the shortest way.[126]

§9. On the approximate calculation of the top trajectories.

The contrast between approximate and exact calculation is generally not sharp. Every numerical calculation is carried out, in so far as it does not by chance treat of rational numbers, only to a certain degree of precision. The contrast should not be called *"exact and approximate calculation,"* but rather *"calculation with arbitrary and with bounded approximation."* While the calculation of elliptic integrals according to the methods of the previous section (in so far as the Legendre tables are not directly used) can be driven to any arbitrary degree of precision, we will not carry the methods of this section

*) Cf. the first volume of the *Traité*, Chap. 23, where three different reduction formulas are constructed according to the value of the parameter, as well as the previously cited book of Schellbach, Abt. 1, Abschn. 10.

**) I must mention with thanks that I have been extensively supported in this calculation by the mathematics student Mr. Blumenthal.[124]

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so far that they allow, without further development, an arbitrary sharpening—a procedure that very often occurs in practical applications.

If such methods of bounded precision are to have a true value, we must require, above all, the ability to estimate the committed error. We will satisfy this requirement in the following. If it is shown that the error lies beneath the allowable error bound for the purpose at hand, then our approximate calculation will provide the same service as one formed with arbitrary precision. In fact, we will later treat of the most interesting cases of the motion of the top directly through the method of bounded precision to be considered now.

We begin with the elliptic integral of the first kind

$$t = \int_e^u \frac{du}{\sqrt{U}},$$

which we can give (cf. page 261) the form

$$\sqrt{\frac{2P}{A}}\, t = \int_e^u \frac{du}{\sqrt{(u-e)(e'-u)(e''-u)}}.$$

We assume, as in the preceding section, that

$$P > 0 \text{ and } -1 < e < e' < +1 < e''.$$

The variable u is restricted in the integration between the bounds e and e'. We thus have, in every case,

$$e'' - e' < e'' - u < e'' - e.$$

The integrand is positive as long as u is in the upper covering of the u-axis. If we now insert for $e'' - u$ the smaller value $e'' - e'$ or the larger value $e'' - e$, the value of the integral becomes larger or smaller, respectively. We thus have, as long as we do not let the integration variable cross the branch point e',

(1)
$$\frac{1}{\sqrt{e''-e}} \int_e^u \frac{du}{\sqrt{(u-e)(e'-u)}} < \sqrt{\frac{2P}{A}} t < \frac{1}{\sqrt{e''-e'}} \int_e^u \frac{du}{\sqrt{(u-e)(e'-u)}}.$$

These last integrals can easily be evaluated trigonometrically. We set, for this purpose,

$$\text{(2)} \qquad e' + e = 2u_0, \quad e' - e = 2\varepsilon, \quad u - u_0 = \delta.$$

Here ε signifies half the vertical distance between the two bounding circles inside which the trajectory runs; u_0 determines the parallel circle whose plane lies midway between the planes of the bounding circles, or, as we wish to say briefly, the "mean parallel circle of the trajectory." The quantity δ measures the distance of the apex of the top from the plane of this mean parallel circle. We obtain

$$u - e = \varepsilon + \delta, \quad e' - u = \varepsilon - \delta, \tag{3}$$

and

$$\int_e^u \frac{du}{\sqrt{(u-e)(e'-u)}} = \int_{-\varepsilon}^{\delta} \frac{d\delta}{\sqrt{\varepsilon^2 - \delta^2}} = \text{arc sin}\left(\frac{\delta}{\varepsilon}\right) + \frac{\pi}{2}.$$

Thus, according to (1),

$$\frac{1}{\sqrt{e'' - e}}\,\text{arc sin}\,\frac{\delta}{\varepsilon} < \sqrt{\frac{2P}{A}}(t - t_0) < \frac{1}{\sqrt{e'' - e'}}\,\text{arc sin}\,\frac{\delta}{\varepsilon},$$

where $t_0 = \frac{\pi}{2}\sqrt{\frac{A}{2P}}$. If we reckon time from the moment at which the apex of the top passes through the mean parallel circle u_0, then we can simply write t instead of $t - t_0$.

We have thus found two bounds between which the (so-reckoned) time t must lie; namely, the lower bound

$$\sqrt{\frac{A}{2P(e'' - e)}}\,\text{arc sin}\,\frac{\delta}{\varepsilon}$$

and the upper bound

$$\sqrt{\frac{A}{2P(e'' - e')}}\,\text{arc sin}\,\frac{\delta}{\varepsilon}.$$

Our approximation formula is now obtained by simply substituting for t a mean value between these two limits.

We replace, for example, $\sqrt{e'' - e}$ and $\sqrt{e'' - e'}$ in the preceding formulas by the mean value $\sqrt{e'' - u_0}$, and write

$$t = \sqrt{\frac{A}{2P(e'' - u_0)}}\,\text{arc sin}\,\frac{\delta}{\varepsilon}. \tag{4}$$

We wish, above all, to estimate the error that we commit here. This error is called τ, and will be calculated as a fraction of the entire value of t. The error τ will certainly be, disregarding the sign, smaller than the difference of our two bounding values divided by the smaller of them. We thus have

$$|\tau| < \frac{\sqrt{\dfrac{1}{e''-e'}} - \sqrt{\dfrac{1}{e''-e}}}{\sqrt{\dfrac{1}{e''-e}}},$$

or

$$(5) \qquad |\tau| < \sqrt{\frac{e''-e}{e''-e'}} - 1.$$

This bound for the "relative" error $|\tau|$ found in this manner depends intimately on the Legendre modulus k of the elliptic integral for t. According to equation $(5')$ of the preceding section, namely,

$$\sqrt{\frac{e''-e}{e''-e'}} = \frac{1}{k'} = \sqrt{\frac{1}{1-k^2}};$$

thus

$$(6) \qquad |\tau| < \frac{1-k'}{k'}.$$

By way of example, we can use our approximation formula to calculate the time duration ω that the apex of the top requires to arrive from the lower parallel circle $u = e$ or $\delta = -\varepsilon$ to the upper circle $u = e'$ or $\delta = \varepsilon$. There follows from (4) the approximate value

$$(7) \qquad \omega = \sqrt{\frac{A}{2P(e''-u_0)}}\,\pi.$$

If we simultaneously wish to express the degree of accuracy of this formula, we can write, understanding by ϑ an unknown proper fraction,

$$(7') \qquad \sqrt{\frac{A}{2P(e''-u_0)}}\,\pi\left\{1 \pm \vartheta\left(\sqrt{\frac{e''-e}{e''-e'}} - 1\right)\right\}.$$

We wish to familiarize ourselves here with a thought that will come to full value only in the sixth chapter. It is obviously convenient, from an analytic standpoint, to go over in equation (4) from the (infinitely many-valued) arc-sine function to the (single-valued) sine function. This also corresponds completely to the spirit of mechanical problems, in which one will wish to calculate the position of the top as a function of time instead of the time as a function of the position of the apex of the top. We will thus *invert* equation (4), in that we express δ or u as an explicit function of t. We obtain

$$(8) \qquad \delta = \varepsilon \sin\left\{\sqrt{\frac{2P(e''-u_0)}{A}}\,t\right\},$$

or, according to (2),

$$(8') \qquad u = u_0 + \delta = u_0 + \varepsilon \sin\left\{\sqrt{\frac{2P(e''-u_0)}{A}}\,t\right\}.$$

We will later carry out the corresponding *inversion* from the originally acquired infinitely-many-valued integrals to our elliptic formulas.

Under particular circumstances, it can occur that the error τ determined above will be very small. Our approximation formulas then provide the same service as the previous exact equations. We read the circumstances under which this occurs from the inequality (5): either e must be approximately equal to e', or e'' must be very large. Summarizing the two possibilities, we can say that either the first two or the last two of the four branch points e, e', e'', ∞ must move very close to each other.

The first possibility occurs if we go over from regular precession, for which e' exactly equals e, to a slightly different motion by a small change of the integration constants. Such *"neighboring motions to regular precession"* are treated in the first section of the next chapter, and will be represented by approximation formulas in the sense of this section. The same also applies to the *motion of the "upright top" in the stable case* for a sufficiently small external disturbance (cf. §4 and §5 of the following chapter).

In order to determine when the second case of a very large value of e'' occurs, we wish to express e'' in terms of our integration constants n, N, etc.

Since e' and e'' are determined as roots of the quadratic equation $U_1 = 0$, we obtain the value $e' + e''$ if we divide the negatively taken coefficient of u in this equation by the coefficient of u^2. We thus find, from equation (2) of page 240,

$$e'' = \frac{n^2 + N^2 - 2nNe}{2AP(1-e^2)} - e'. \tag{9}$$

This value increases, in general, with increasing n and N, as well as with decreasing P. It may at first appear that e'' also becomes infinite or very large in the case that e is equal or approximately equal to ± 1. This is, however, not so, since the numerator then vanishes at the same time as the denominator. Namely, the numerator signifies geometrically the square of the length of the binding line between the endpoint of the impulse component n and the impulse component N in the initial position $u = e$. If $e = \pm 1$, the length of this binding line is evidently zero.[127]

Thus e'' will become very large only if one of the impulse components n and N becomes very large, or, more precisely said, if the square of one of these quantities represents a very considerable number in proportion to the quantity AP. This was the case on page 253 and ff. for the investigation of the *limit cases* $n = \infty$, $N = \infty$. We thus see the basis on which we could dispense with the representation of the motion through elliptic integrals in these limiting cases; it would now be easy to estimate the error τ in the earlier approximation formulas more precisely.

We can assign our designation "*pseudoregular precession*" to the first as well as the second of the different possibilities on the previous page. For this motion, which likewise will be investigated in the next chapter, the application of our approximation formulas will therefore also give only a very small error.

According to (5), all these individual cases are characterized, from the standpoint of the elliptic integrals, by the complementary modulus k' being approximately equal to 1; that is, the Legendre modulus itself being approximately zero. It is clear in advance from the previous section that we may dispense with the theory of the elliptic integrals in such cases, and can represent the motion with great accuracy in terms of elementary functions. For vanishing modulus k, namely, the Legendre normal integral $F(k, \varphi)$ (see equation (2) of page 260) is transformed directly into the value of the amplitude φ, where φ is expressed in terms of the original variable u or δ as an arc sine. This corresponds precisely to the approximate representation of the motion given in the preceding. The advance of the current consideration consists merely in our ability to now estimate, for nonvanishing k, the magnitude of the error of our approximation formula, according to (6), in terms of the magnitude of k.

We may recall once again the previously sketched calculation of the elliptic integral according to the methods of Legendre or Gauſs. As mentioned, these methods rest on the repeated application of a certain quadratic transformation that has the consequence of successively diminishing the modulus of the integral. In one of the cases where our approximation formulas give only a small error, the application of this transformation becomes superfluous, in that the modulus is so small from the beginning that we can evaluate the integral directly in an elementary way without considerable error.

In the above-named special cases of pseudoregular precession, the upright top, etc., the limiting case that Gauſs and Legendre strove to attain through sufficiently frequent application of their transformation methods is present in itself.

In addition to the approximation formula for u, we will have use of such a formula for ψ. We begin from the expression

$$\psi' = \frac{n - Nu}{A(1-u^2)} \tag{10}$$

given by equation (6) of page 238. We separate the right-hand side into partial fractions, as was already done on page 267, and obtain

$$\psi' = \frac{n+N}{2A(1+u)} + \frac{n-N}{2A(1-u)}. \tag{11}$$

We set here, as in equation (2) of page 270, $u = u_0 + \delta$, and carry out the identity transformations

$$\begin{aligned}
\frac{1}{1+u} &= \frac{1}{1+u_0} - \frac{\delta}{(1+u_0)^2} + R_+, \quad R_+ = \frac{\delta^2}{(1+u_0)^2}\frac{1}{1+u},\\
\frac{1}{1-u} &= \frac{1}{1-u_0} + \frac{\delta}{(1-u_0)^2} + R_-, \quad R_- = \frac{\delta^2}{(1-u_0)^2}\frac{1}{1-u}
\end{aligned}$$

on the expressions $(1 \pm (u_0+\delta))^{-1}$. Equation (11) thus becomes

$$\begin{aligned}
\psi' = \frac{n+N}{2A(1+u_0)} + \frac{n-N}{2A(1-u_0)} - \delta\Big(\frac{n+N}{2A(1+u_0)^2} - \frac{n-N}{2A(1-u_0)^2}\Big)\\
+ \left(\frac{n+N}{2A}R_+ + \frac{n-N}{2A}R_-\right).
\end{aligned} \tag{12}$$

We then introduce for δ the approximate value from equation (8). We have, if we express the precision bound τ in our formula,

$$\delta = \varepsilon \sin\left\{\sqrt{\frac{2P(e''-u_0)}{A}}(1 \pm \vartheta\tau)t\right\}.$$

We thus write, on the basis of the mean value theorem or the Taylor series truncated at the first term,

$$\begin{aligned}
\delta &= \varepsilon \sin\left\{\sqrt{\frac{2P(e''-u_0)}{A}}t\right\} + r,\\
r &= \varepsilon\sqrt{\frac{2P(e''-u_0)}{A}}\,\vartheta\cdot\tau\cdot t\cdot\cos\left\{\sqrt{\frac{2P(e''-u_0)}{A}}(1\pm\vartheta'\tau)t\right\},
\end{aligned} \tag{13}$$

where ϑ', just as the previous ϑ, signifies a proper fraction.

Equation (12) now takes the form, if we group the terms appropriately,

$$\begin{aligned}
\psi' &= \frac{n - Nu_0}{A(1-u_0^2)} + \frac{2nu_0 - N(1+u_0^2)}{A(1-u_0^2)^2}\,\varepsilon\sin\left\{\sqrt{\frac{2P(e''-u_0)}{A}}t\right\} + R,\\
R &= \frac{n+N}{2A}R_+ + \frac{n-N}{2A}R_- + \frac{2nu_0 - N(1+u_0^2)}{A(1-u_0^2)^2}\,r.
\end{aligned} \tag{14}$$

We now carry out the integration with respect to t; we obtain, if we disregard an inessential constant of integration that determines the value of ψ at $t = 0$, a term that increases in proportion to t, a second term that changes periodically, and finally a remainder term.

The approximate representation of ψ in question now follows simply by suppressing the remainder term in the obtained equation and setting (14′)

$$\psi = \frac{n - Nu_0}{A(1-u_0^2)}\,t + \frac{N(1+u_0^2) - 2nu_0}{(1-u_0^2)^2\sqrt{2AP(e''-u_0)}}\,\varepsilon\cos\left\{\sqrt{\frac{2P(e''-u_0)}{A}}\,t\right\}.$$

We will have to establish the degree of the approximation afterward through a discussion of the remainder term.

Equations (8′) *and* (14′), *taken together, provide an approximate representation for the trajectory of the apex of the top that will indeed be burdened, in general, with a considerable error, but which may, under certain circumstances, replace the exact formula with advantage.*

This representation permits of a very intuitive interpretation. We first wish to consider the two component motions individually; they are represented, respectively, by the two first or the two second terms of the named equations. The two first terms are

$$u_1 = u_0, \quad \psi_1 = \frac{n - Nu_0}{A(1-u_0^2)}\,t.$$

They define a *regular precession* in which the mean parallel circle u_0 is traversed with the constant angular velocity $\dfrac{n - Nu_0}{A(1-u_0^2)}$. The two second terms

$$u_2 = \varepsilon\sin\left\{\sqrt{\frac{2P(e''-u_0)}{A}}t\right\},$$

$$\psi_2 = \frac{N(1+u_0^2) - 2nu_0}{(1-u_0^2)^2\sqrt{2AP(e''-u_0)}}\,\varepsilon\cos\left\{\sqrt{\frac{2P(e''-u_0)}{A}}t\right\}$$

are harmonically changing quantities of the same period and unequal amplitude; they represent, considered in themselves, an *elliptical oscillation.* We also designate the latter, in that we adopt the usual expression in astronomy, as the *nutation* of the apex of the top.

The complete motion, as it is described by our approximation formulas, consists of the superposition of the just described component motions. *Our formulas thus represent the motion of the apex of the top as the superposition of a regular precession and a periodically repeating nutation.* We must imagine that the apex of the top is led along the mean parallel circle u_0 with constant angular velocity, and at the same time executes its nutational oscillation relative to this motion. One may reconsider the figures of §2 in this sense, and envisage the nature of the required precession and nutation in each case. The representation is particularly suggestive and fruitful for the pseudoregular precession represented in Fig. 28.

The main question of the *degree of precision of our approximate formulas* now remains to be discussed. The degree of precision of our formula for u has been determined above in a completely satisfactory manner. In every individually determined numerical case, there is also no difficulty in the error determination for our formula for ψ. Under general assumptions, however, this error is not so smoothly estimated. We must distinguish, rather, a series of special cases according to the signs of the quantities n, N, u_0, etc. A few remarks are sufficient here.

For the error determination in (14′), we must begin with the value of the remainder R in (14), in terms of which the error f is calculated as

$$f = \int R\,dt;$$

the quantity f thus signifies (in contrast to the above error τ) not the relative, but rather the absolute error.

We will consider in detail the special case in which the two parallel circles e and e' lie sufficiently near to one another, so that $2\varepsilon = e' - e$ is a small quantity. In this case, the error τ in our error estimation above is also very small, and in particular, according to (5), vanishes to the first order for vanishing ε. Now of the three terms from which R in equation (14) is composed, the two that are multiplied by R_+ and R_- contain the factor ε^2, since δ contains the factor ε; the third term (cf. the above expression for r) possesses the factor $\varepsilon\tau$. We can therefore say that R vanishes to the second order for vanishing ε, while the remaining terms in our approximation formula become zero to the first order in ε. *For sufficiently small ε, our error f thus represents an arbitrarily small fraction of the right-hand side of* (14′). *In this case, the bounded approximation of our formulas* (8′) *and* (14′) *goes over into an arbitrary approximation.*[128]

It is well to note an exception. The term $1 - u_0^2$ or $1 \pm u_0$ is present in the denominator of R. If one of these factors decreases in the same measure that the parallel circles e and e' come together, then the smallness of the numerator in R will be balanced by that of the denominator. The previous statement is therefore valid only if the trajectory of the apex of the top does not run in the immediate neighborhood of the north or south pole of the unit sphere. In such a case, our approximation formulas can yield, even for arbitrarily small ε, an entirely false image of the motion. We will therefore undertake, in the following chapter (cf. §5), a special consideration of the motions occurring in the neighborhood of the poles.

In conclusion, a word on the relation of our current manner of calculation of the elliptic integrals of the first kind to the methods in the previous section.

If, in the derivation of formula (4), we replace the factor $(e'' - u)^{-\frac{1}{2}}$ by the constant quantity $(e'' - u_0)^{-\frac{1}{2}}$, this is equivalent to expanding this term in increasing powers of $u - u_0$ and truncating the series at the constant term. This now suggests the retention of more terms or the entire series in the calculation of t. In the latter case, there results a convergent infinite series that may be expressed in terms of cyclometric functions. If we consider a sufficient number of these terms, then we can improve, in complete generality, the degree of the approximation at our pleasure. One thus sees that *our approximation procedure of bounded precision, formed in this manner, reverts to the series method of arbitrary precision named on page* 264 *under* (1).

Chapter V.

On special forms of motion of the heavy symmetric top, particularly pseudoregular precession, and on the stability of motion.

§1. Regular precession and its neighboring forms of motion.

In this chapter we wish to investigate more precisely some special motions of the top: regular precession, for example, and, in particular, the motion designated by us as pseudoregular precession. The great question of the *stability of motion* will stand in the foreground of our interest, a question that has often been considered in recent times, but which does not yet appear to have been formulated with the necessary sharpness and clarity.

We begin with the investigation of regular precession for the spherical top with moment of inertia A. We obtain this motion as a limiting case of the general motion of the top if we let the two parallel circles $u = e$ and $u = e'$ that enclose the trajectory of the apex of the top come together. If we consider, further, that e and e' are roots of the cubic equation $U = 0$, then we can say that *regular precession is characterized analytically by a double root of the equation* $U = 0$ *between* -1 *and* $+1$. The differential quotient $\dfrac{dU}{du}$ must therefore vanish as well as U for $u = e$. If we form this equation according to equations (1) and (2) of page 240, we obtain the condition

$$A\frac{n - Ne}{A(1-e^2)}\frac{N - ne}{A(1-e^2)} = P, \tag{1}$$

which, as has already been remarked on page 252, is identical with the equation $A\mu\nu = P$ that results from the theory of the deviation resistance.

Our general method of integration fails in a characteristic manner for this simplest case of the motion of the top. If $e = e'$, namely, the integration path for u in the expressions for t, ψ, and φ contracts to a

F. Klein and A. Sommerfeld, *The Theory of the Top: Volume II. Development of the Theory in the Case of the Heavy Symmetric Top*, DOI 10.1007/978-0-8176-4827-5_2,

single point, and our integrals lose their meaning. We thus ascend to the nonintegrated equations

$$(2) \qquad \frac{du}{dt} = \sqrt{U}, \quad \frac{d\psi}{dt} = \frac{n - Nu}{A(1-u^2)}, \quad \frac{d\varphi}{dt} = \frac{N - nu}{A(1-u^2)},$$

and verify directly that they are satisfied if we let

$$(3) \qquad \cos\vartheta = u = e, \quad \psi = \nu t, \quad \varphi = \mu t.$$

In fact, the first equation is transformed, for $u = e =$ const., into $0 = 0$; the latter two equations will likewise be satisfied if we specify the quantities ν and μ in terms of the integration constants n, N, and e as

$$(4) \qquad \nu = \frac{n - Ne}{A(1-e^2)}, \quad \mu = \frac{N - ne}{A(1-e^2)}.$$

Here we must call attention to a remarkable contradiction to our previous results that is, however, only of a formal nature. Our last consideration shows that equations (2) are satisfied by a completely arbitrary choice of the integration constants e, n, and N and the corresponding determination of the constants ϑ, μ, and ν in equations (3) and (4). It may thus appear that regular precession represents a possible motion for arbitrary initial conditions, while it was claimed previously, and follows from our earlier developments, that regular precession is possible only if the condition (1) among the integration constants obtains.

To confirm the necessity of the latter condition directly, we return to the original differential equations of motion, which we pose in the Lagrangian form. According to page 154, these equations are, if we insert for the components Θ, Φ, Ψ of the external force the values given on page 220 and use for T the expression (6) of page 156 with $C = A$,

$$(5) \qquad \begin{cases} \dfrac{d[\Theta]}{dt} = -A\varphi'\psi'\sin\vartheta + P\sin\vartheta, \quad \dfrac{d[\Psi]}{dt} = 0, \quad \dfrac{d[\Phi]}{dt} = 0. \\ [\Theta] \quad = A\vartheta', \quad [\Psi] = A(\psi' + \varphi'\cos\vartheta), \quad [\Phi] = A(\varphi' + \psi'\cos\vartheta). \end{cases}$$

If we insert the values $\vartheta' = 0$, $\psi' = \nu$, $\varphi' = \mu$ corresponding to equations (3), the second line gives $[\Theta] = 0$, $[\Psi] =$ const., $[\Phi] =$ const.; the latter two equations of the first line are identically satisfied, while the first equation yields our previous condition

$$A\mu\nu = P.$$

According to our original equations, consequently, only a certain class of regular precession that is characterized by equation (1) *can in fact occur.*

We wish, however, to pursue further the basis on which equations (2) possess integrals that are not comprised by those of the general differential equations (5). For this purpose, we must broaden the geometric conception of the meaning of our differential equations.

It may thus be permitted, for brevity, to speak only of the differential equations for u and ψ.

The differential equations (2) determine, for each point (u, ψ) of the trajectory, a certain direction of progression $\left(\frac{d\psi}{du}\right)$, or, if we wish, a certain velocity $\left(\frac{du}{dt}, \frac{d\psi}{dt}\right)$. We wish to imagine the respective direction of progression marked as a kind of signpost at each point of the stereographic image of the unit sphere. We designate the embodiment of the individual point and the corresponding signpost, in association with a currently common means of expression, as a *line element.*[129]

To integrate the differential equations now means to give a curve that is purely composed of such line elements, or to describe a path that always runs in the direction of the signposts.

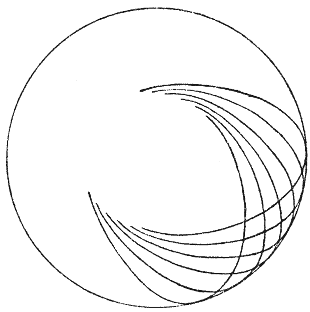

Fig. 46.

On the basis of this definition, one sees immediately that each regular precession (n, N, e) obtained by choosing N, n arbitrarily and e so that the equation $U = 0$ is fulfilled must satisfy the differential equations (2). We consider, namely, the general trajectory that corresponds to the integration constants n, N, and e. The character of this curve was described in the previous chapter. We construct, in addition, the entire ensemble of trajectories that result if we rotate each first curve about the midpoint (the image of the north pole) of the figure (cf. Fig. 46). All these curves are naturally integral curves of (2); they are all tangent, moreover, to the parallel circle $u = e$. As a result, each smallest segment of the parallel circle $u = e$ represents a line element that corresponds to our differential equation.

The parallel circle itself is thus an integral curve of the equations (2), whether the condition (1) between e, n, and N is fulfilled or not.

Our consideration may be generalized immediately to arbitrary differential equations of the first order. If we know an ensemble of integral curves of such equations and construct the envelope of the ensemble, this envelope likewise satisfies the differential equations. One designates these particular types of integral curves as *singular solutions*, since they do not result from the general solutions by specialization of the constants.

With the use of this terminology, we can thus say that *regular precession is indeed a solution of the differential equations* (2) *for an arbitrary choice of the constants* e, n, *and* N, *but a singular solution.*

One now easily grasps that the singular solutions of (2) are not also to be used as solutions of the differential equations (5). If we make, namely, the analogous consideration for equations (5), then we must speak not simply of line elements, but rather, for example, of *line elements of the second order.* After a point and a direction of progression passing through it are chosen, the corresponding value of the second differential quotient is now determined by the differential equations. Our signposts are now, so to speak, conditional signposts; they prescribe, if we go forward from a point in a certain direction, that we should move further on the trajectory with a certain curvature. To integrate the equations (5), we must therefore assemble these line elements of the second order into a curve, or construct the curvature of the trajectory as prescribed by our conditional signposts. The integral curves that we thus attain must, in every case, be contained among the integral curves of (2). The latter equations can possibly permit, however, of other integrals. For we cannot conclude, from only the fact that the directions of progression of a curve satisfy the equations (2), that its curvatures are in conformity with the equations (5). For singular solutions this is in fact, as we saw, not the case.

We can further claim, however, that all the general solutions of (2) must also satisfy equations (5). For these solutions form a continuous multiplicity of trajectories, and since some of them must certainly be integral curves of (5), so must they all. *The general solutions of* (2) *therefore have the curvature prescribed by* (5), *but not the singular solutions.*[130]

We see in this example how it is always necessary to return to the true meaning of the formulas (here, to the consideration of the line elements), and never to trust blindly in the formal correctness of calculational operations. —

We now go over to a new type of motion of the top, the "*neighboring motions to regular precession*" that were already mentioned on page 273. To this end, we impart a small impact to the top while it is conceived to be in a regular precession motion. The direction of the impact may be arbitrary, but the magnitude will be chosen as arbitrarily small. Our impact is composed with the impulse corresponding to the regular precession according to the parallelogram rule. The components of the original impulse at a certain point of time, which we can call the "initial time," will therefore be augmented by arbitrarily small increments, the components of our impact. The question is, what motion corresponds to the changed initial impulse?

We decompose the impulse most conveniently into its components with respect to the three distinguished axes of our problem, the figure axis, the vertical, and the line of nodes; that is, into the perpendicular projections $[\Psi]$, $[\Phi]$, and $[\Theta]$ of the impulse vector onto these axes. Of these projections, the first two are invariable during every natural motion of the heavy top, and are identical with the integration constants N and n. Let these letters specifically denote the characteristic values of the components $[\Phi]$ and $[\Psi]$ for the regular precession; the increments of these components due to our impulse are called N' and n'. The third impulse component $[\Theta]$ is, in general, variable during the motion. Only for regular precession do we have the special case $[\Theta] = 0$, since, according to (5), $[\Theta] = A\vartheta'$ and $\vartheta' = 0$. The increment that is added through the impact thus signifies the total value of the $[\Theta]$-component at the beginning of the motion. We designate it as $[\Theta_0]$, to indicate that this value represents the $[\Theta]$-component only at the time $t = 0$.

In the following, we will investigate the effects of the impulse increments n', N', and $[\Theta_0]$ individually. In this spirit, we first ask for the displacement of the two parallel circles e and e' due to the exclusive augmentation of the impulse component $[\Psi]$ by n'.

It is first clear that one of the parallel circles e and e' coincides with the precession circle e. Since $[\Theta_0] = 0$, we have $\vartheta' = 0$ and therefore also $u' = 0$ at the beginning of the motion, where $u = e$. One root of the equation $U = 0$ is therefore equal to e after the impact as well as before. The second root e', which in the case $n' = 0$ coincides with e, will be changed by our impact. We designate the increment of e' by 2ε, and therefore set, as on page 270, $e' - e = 2\varepsilon$. The quantity ε is therefore a number that vanishes with n', as follows immediately from the continuity of our C_3 on page 250. If we take n' sufficiently small, then we can also make ε arbitrarily small.

The magnitude of e', or that of 2ε, is calculated from the equation $U_1 = 0$ of page 240, or, as we wish to write in more detail,

$$U_1(u, v) = 0.$$

This equation must be satisfied, on the one hand, in the case of regular precession (that is, for $u = e$, $v = n$), and, on the other hand, for the neighboring motion to regular precession, and therefore for $u = e + 2\varepsilon$, $v = n + n'$. If we expand $U_1(e + 2\varepsilon, n + n')$ according to Taylor's theorem in the neighborhood of the pair of values (e, n), there follows, since $U_1(e, n) = 0$,

$$U_1(e + 2\varepsilon, u + n') = 2\varepsilon\frac{\partial U_1}{\partial u} + n'\frac{\partial U_1}{\partial v} + \cdots ;$$

here the left-hand side vanishes; on the right-hand side, we neglect all the unwritten higher powers because of the smallness of ε and n'. There follows

$$2\varepsilon\,\frac{\partial U_1}{\partial u} + n'\,\frac{\partial U_1}{\partial v} = 0,$$

or

$$2\varepsilon = -n'\,\frac{\partial U_1}{\partial v}\bigg/\frac{\partial U_1}{\partial u},$$

where we must insert $v = n$, $u = e$ on the right-hand side. Without calculating the right-hand side more precisely, we are satisfied to have shown that ε is determined in this manner as a quantity that vanishes with n' in every case.

The matter is no different if we increase N by N', but hold fixed the original values $[\Psi] = n$, $[\Theta_0] = 0$. One parallel circle is again e; the displacement 2ε of the other parallel circle is calculated as previously; since, namely, the equation $U_1 = 0$ is formed symmetrically in n and N, we have only to interchange n and N in the final formula for ε, and write N' instead of n'.

In the third case, where we add the impulse $[\Theta_0]$ and take $n' = N' = 0$, both parallel circles e and e' change. Namely, $\vartheta_0' = 0$ no longer obtains at the beginning of the motion, but rather $A\vartheta_0' = [\Theta_0]$, and thus the initial value of u, which we denote by u_0, is no longer a root of $U = 0$. For the determination of e and e' we must therefore begin from the cubic equation and not from the quadratic $U_1 = 0$. It will now be shown that the two parallel circles e and e' are removed equally far, in the first approximation, from the original precession circle u_0, so that u_0 signifies, as in the previous section, the mean parallel circle

$$u_0 = \frac{e+e'}{2}.$$

The original form of the expression U was given in equation ($7'$) of page 238 as

(6) $$A^2U = -(Nu-n)^2 + (k - N^2 - 2APu)(1-u^2).$$

At the beginning of the motion ($u = u_0 = \cos\vartheta_0$), the left-hand side of this equation is known. Since, in general,

$$U = \left(\frac{du}{dt}\right)^2 = \sin^2\vartheta \cdot \vartheta'^2,$$

A^2U becomes, for $u = u_0$,

$$A^2U = \sin^2\vartheta_0 (A\vartheta_0')^2 = (1-u_0^2)[\Theta_0]^2.$$

Thus follows the equation

(7) $$(1-u_0^2)[\Theta_0]^2 = -(Nu_0-n)^2 + (k - N^2 - 2APu_0)(1-u_0^2).$$

We wish to eliminate k from (6) and (7) after we have set $U = 0$ in (6). We thus find for the desired values $u = e$ and $u = e'$ the equation

(8) $$(1-u^2)[\Theta_0]^2 = (Nu-n)^2 - (Nu_0-n)^2 \frac{1-u^2}{1-u_0^2} + 2AP(u-u_0)(1-u^2).$$

The polynomial of the third degree on the right-hand side can easily be resolved into linear factors. If we set, namely, the right-hand side equal to zero, then we must again find the roots of the equation $U = 0$ in the case of regular precession, since $[\Theta_0] = 0$ for this motion. The linear factors of the right-hand side are therefore $u - u_0$, $u - u_0$, and $u - e''$. The additionally occurring factor independent of u is equal to the coefficient of u^3 in equation (8), and the right-hand side of this equation is equal to

$$-2AP(u-u_0)^2(u-e'').$$

Thus we can write more simply, in place of (8),

$$(1-u^2)[\Theta_0]^2 = 2AP(u-u_0)^2(e''-u).$$

We may now assume $[\Theta_0]$ to be arbitrarily small, so that the right-hand side will also become extraordinarily small. The desired roots e and e' therefore lie extraordinarily close to u_0. We set $u = u_0 + \varepsilon$, and obtain for the determination of ε the equation

(9) $$\varepsilon^2 = \frac{(1-(u_0+\varepsilon)^2)[\Theta_0]^2}{2AP(e''-u_0-\varepsilon)}.$$

We can expand the right-hand side in a convergent series in powers of ε. Since we can assume, however, that ε as well as $[\Theta_0]$ is arbitrarily small, we need only the first term. From this there follow two oppositely equal values for ε; namely,

$$\varepsilon = \pm\sqrt{\frac{1-u_0^2}{2AP(e''-u_0)}}\,[\Theta_0].$$

Thus the desired values become

$$e = u_0 - \varepsilon \quad \text{and} \quad e' = u_0 + \varepsilon,$$

where ε vanishes with $[\Theta_0]$.

The parallel circles e and e' therefore stand, in the first approximation, equally far from the initial circle u_0; this represents, as claimed, the "mean parallel circle of the motion" in the sense of the previous section.

We return, after this preparation, to our distinction on page 283 of the three cases that were characterized, respectively, by the three values of the additional impulses n', N', $[\Theta_0]$. It is common for all three cases, as we saw, that the parallel circles e and e' lie ever nearer to each other as the impact is chosen smaller. *The trajectories in question therefore run arbitrarily near to the original precession circle*, as was already expressed in our appellation.

We next recall the results of the previous section. The approximation formulas (8′) and (14′) there, which generally provide a bounded approximation, directly give, in the present special case, an arbitrarily good approximation; the error will be arbitrarily small for sufficiently small ε in the formula for u as well as in that for ψ. We will therefore apply those formulas without misgiving*) to the present three cases, and may write

$$(10) \qquad \begin{cases} u = u_0 + \varepsilon \sin \dfrac{\pi t}{\omega}, \\ \psi = \nu t + \nu_1 \varepsilon \cos \dfrac{\pi t}{\omega}; \end{cases}$$

the abbreviations used here have the meanings

$$(11) \qquad \omega = \pi\sqrt{\frac{A}{2P(e''-u_0)}}, \quad \nu = \frac{n - Nu_0}{A(1-u_0^2)},$$

$$\nu_1 = \frac{N(1+u_0^2) - 2nu_0}{\pi A(1-u_0^2)^2}\,\omega,$$

*) One notes, however, the condition of page 277 that u_0 may not be equal, or approximately equal, to ± 1. Some remarks on this case of precession follow in §5.

where n and N denote the total value of the impulse components $[\Psi]$ and $[\Phi]$; that is, the values of these components for the original precession possibly augmented by n' and N'.

In order to obtain a clear representation of our neighboring trajectory, we consider separately the *component motions of regular precession and nutation* into which the motion of the apex of the top appears to be decomposed by (10).

The trajectory of the first component motion is the precession circle $u = u_0$. *This coincides with the original precession circle only when the axis of the impact is the line of nodes* ($n' = N' = 0$, $[\Theta_0] \neq 0$). In the two other cases, in contrast, it is displaced vertically, in comparison with the original precession circle, by the small increment ε. More precisely, we can say that for the impact $[\Theta_0]$, the deviation between the mean parallel circle u_0 and the circle of the original precession vanishes to an order higher than the first with vanishing magnitude of the impact, and vanishes only to the first order, in contrast, for the impacts n' and N'.

The precessional velocity of our first component motion is given, in all cases, by the quantity ν in equation (11). *This quantity again coincides with the original precessional velocity only in the case of the impact* $[\Theta_0]$, since the impulse components n and N then remain unchanged, and the value of u_0 then coincides, in the more precise sense just discussed, with the original value of u for the regular precession. In the two other cases, the value of the precessional velocity deviates from the original value by quantities that are of the same order of magnitude as n' and N'.

The second component of the motion, the nutation, is, according to the preceding, a harmonic oscillation with unequal amplitudes in the coordinates u and ψ, or, as we say more briefly, an elliptical oscillation relative to the circle of the precession. The quantity ω determined by (11) gives the half-period of the oscillation; that is, the time interval during which the apex of the top passes from e to e'. The vertical oscillation amplitude is measured by ε, and the horizontal by $\nu_1\varepsilon$. One sees without further ado *that the oscillation period generally remains finite with vanishing impact, while the two oscillation amplitudes vanish.* If we express ω, namely, in terms of our usual integration constants n, N, etc., in that we insert for e'' the value from equation (9) of page 273, there follows, in the limit of vanishing magnitude of the impact,

$$\omega = \pi\sqrt{\frac{A^2(1-e^2)}{n^2+N^2-2nNe-4APe(1-e^2)}};$$

here n, N, e signify the constants of the original regular precession. This limiting value of ω is obviously different from zero. The same holds for the limiting value of the angle ψ_ω by which the azimuth of the apex of the top increases during a nutation. According to equation (11), it simply becomes

$$\psi_\omega = \nu\omega.$$

It is now easy to complete the series of figures of the previous chapter by inserting a figure in the immediate neighborhood of Fig. 31 ("slow precession"). In Fig. 31, we assumed $A = -P = 1$, $e = 0$, $N = 0{,}2$, $n = -5$. If we now give n, by way of example, a value somewhat different from -5, then an oscillation of half-period

$$\omega = \pi\sqrt{\frac{1}{n^2+N^2}} = \frac{\pi}{5} \text{ circa}$$

is superposed on the regular precession; at the same time, the azimuthal amplitude during the time ω becomes

$$\psi_\omega = n\omega = -\pi \text{ circa.}$$

The figure to be inserted in the immediate neighborhood of Fig. 31 is thus drawn schematically in the following manner (cf. Fig. 47). One notices the characteristic fact that our nutation, which has, as we know, vanishing dimensions in itself, is extended by the superposition with the finite precessional velocity into a nearly complete revolution about the vertical.

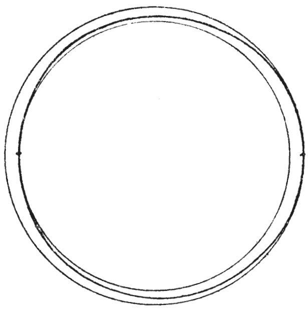

Fig. 47.

Entirely the same figure can also serve as a representation of the neighboring trajectory to the infinitely fast precession (Fig. 35). Only here, because of the special value $n = \infty$, ω becomes equal to zero and ψ_ω becomes exactly equal to $\pm\pi$. —

The essential purpose that we have pursued in the consideration of the neighboring solutions to regular precession consists, however, not so much in the knowledge of the motions themselves, but rather in the fact that we can now make conclusions regarding the *stability of regular precession.* We claim, on the basis of our investigation of the neighboring motions, that *regular precession is certainly a stable motion of the top.*

We have already explained on page 129 what we wish to understand by the word stability, if not with sufficient sharpness. We reserve a thorough definition until the sixth section of the current chapter; for the present case, our earlier explanation suffices. We therefore repeat: a motion is called stable if, for the addition of a sufficiently small impact of arbitrary direction, the character of the motion changes *continuously.*

If we wish to apply this criterion to our present case, then we must complete the preceding considerations in two directions.

By "motion," namely, we must understand not only the motion of the apex of the top along its trajectory, but rather the totality of the position of the top; that is, the embodiment of the values that, for example, the coordinates φ, ψ, ϑ take during the course of time. But we now know that the value of the φ-coordinate can be obtained from that of the ψ-coordinate through the interchange of n and N with $-N$ and $-n$ (cf. the reciprocity law of the spherical top in §5 of the preceding chapter). We thus need not make a new development for the φ-coordinate to fully command the "motion," but rather can claim that the φ-coordinate behaves qualitatively just like the ψ-coordinate.

Further, our definition of stability corresponds to an arbitrary impact; that is, to a simultaneous augmentation of the impulse components $[\Phi]$, $[\Psi]$, $[\Theta]$ by small increments. But it is clear that the effect of an arbitrary impact may be composed by the direct superposition of the effects of the special impacts n', N', $[\Theta_0]$ considered above, if only those increments are sufficiently small. The resulting formulas for u, ψ, and φ are therefore of the same character as those for u and ψ above.

Now a glance at equations (10) shows that these formulas are transformed continuously into the equations of regular precession if we continuously diminish the magnitude of the impulse increments. The same therefore holds for an entirely arbitrary disturbance with respect to the total character of the motion. This is also transformed continuously into regular precession if we let the disturbance diminish continuously to zero.

Thus the stability of regular precession is ensured.

Our definition of stability used here is different from the commonly given definition (cf. §6 of this chapter). While we demand only that the change of the motion be continuous (that is, ever smaller as the impact is taken smaller), a motion is often said to be stable elsewhere only if the deviation for the disturbed motion remains *always* very small (or arbitrarily small). We wish, in general, not to join in this use of language, since it brings with it, as will later be shown, an improper restriction of the concept of stability. We remark, however, that regular precession is also to be considered as stable according to this narrower concept of stability, in so far, namely, as we direct our attention merely to the geometric form of the trajectory of the apex of the top and disregard its temporal course. In fact, the trajectory changed by an additional impact is entirely enclosed in a spherical zone of breadth (measured in the vertical direction) 2ε, and can, through the diminishment of ε, and thus through the diminishment of the disturbance, be brought arbitrarily near to the precession circle e in its entire extent. That the analogue is not the case for every motion is shown, among other examples, by the force-free motion of a single mass particle according to the Galilean law of inertia. With the addition of an impact, the original linear path is transformed into another straight line that is removed arbitrarily far from the original in the course of time, however small the impact is chosen to be. We are therefore obliged to speak, with respect to the trajectory of regular precession, of a *particularly high degree of stability.*

The situation is already different if we take into consideration, in addition to the form of the trajectory, its temporal course, or the entire character of the motion with inclusion of the φ-coordinate.

We cannot generally claim, namely, that the distance from the apex of the top in the motion altered by a small impact to its position at the corresponding time in the original motion always remains small. In fact, we saw that the mean value of the angular velocity ψ' will be changed by the addition of the impact n' or N', so that the apex of the top will traverse its path after the disturbance with a velocity different from before. In the course of time, therefore, the positions of the apex of the top in the compared motions will differ by an arbitrary finite amount. The corresponding result obtains, according to our reciprocity law, for the coordinate φ, in so far as the impact changes the values of n and N.

The precessional velocity ψ', and therefore also the value of φ', retain their original values only for the special case in which the impact effects no change of $[\Phi]$ and $[\Psi]$, and consists only of $[\Theta_0]$. The altered trajectory then oscillates with equal amplitude above and below the original precession circle. Only through the restriction to this special impact will the deviation of the position of the apex of the top, and the position of the top in general, always remain small; regular precession is to be called stable in the usual sense only under this constraint. We summarize the latter remarks once more as follows:

Under the usual definition of stability, regular precession must be consistently designated as unstable. It can be called stable only if one of two restrictions is added. Either one considers only the geometric form of the trajectory, and not the motion on the trajectory or the motion of the top in general, or one directs the disturbance so that it merely effects a change of ϑ', with φ' and ψ', in contrast, unchanged. From the standpoint of our concept of stability, on the other hand, we must declare regular precession as stable without restriction. We can state that the geometric form of the trajectory, and, for the special disturbances named above, the motion in general have a particularly high degree of stability.

§2. Pseudoregular precession; resolution of the paradoxes of the motion of the top.

We now come to the most important point of the entire theory of the top. It has attracted, because of its paradoxical nature and frequency of actual occurrence, the highest interest of theoretical and experimental natural philosophers. We refer to the motion of the top that we have designated as *pseudoregular precession.*

We will first depict the characteristic nature of this motion by comparison with *regular precession.*

As we have seen, regular precession occurs only under the special circumstances that have been given in detail in the previous section and in the sixth section of the third chapter. From the experimental standpoint, however, one can easily come to the conception that regular precession is the general motion of the heavy top, and that it occurs for an arbitrary choice of the initial conditions. In fact, if we wind the top with a cord, as is usually done, and then abandon it to the influence of gravity without the addition of a rather considerable impact, it appears that the figure axis describes a circular cone about the vertical with uniform velocity. This result must naturally appear as paradoxical in the highest degree. For it is inconceivable that the vertically acting weight should generally produce a motion in which, for example, all points of the figure axis continuously progress in the horizontal direction, and therefore exactly perpendicular to the direction of the external force.

Against this conception, it is now to be noted *first* that the named observational result is not exact. The motion has only an external similarity with regular precession. If we examine it more precisely, we note that the figure axis executes small periodic oscillations about the circular cone of the regular precession, which for very large rotational velocity are indeed hardly noticeable, and most likely manifest themselves in a periodic trembling of the support. It is on this basis that we have bestowed upon the motion in question the name of *pseudoregular precession.*

The illusion is further strengthened by the fact that the deviation from regular precession is quickly absorbed by all sorts of secondary circumstances that are usually not considered in abstract mechanics, such as friction and the elasticity of the support. These secondary circumstances, however, will presently remain out of consideration.

Second, the phenomena usually present in experiments are not general, but rather are specialized in a certain manner. For the winding always produces an impulse that falls precisely or approximately in the direction of the figure axis, and has, moreover, a very considerable length.

We are thus led, proceeding first from the experimental standpoint, to formulate the conditions for the possibility of pseudoregular precession in the following manner:

The motion will be a pseudoregular precession if the impulse vector initially falls in the approximate direction of the figure axis and has a considerable length.

The words "approximate" and "considerable" must naturally be made more precise. We wish to say, for example, that two directions "approximately" coincide if we can no longer distinguish their intersection points on the unit sphere with the naked eye.

To attach an exact representation to the word "considerable," we must compare the length of the impulse vector with the magnitude of the force of gravity. A direct comparison of these two quantities is doubtful, since they have different dimensions, and their quotient is therefore not an absolute number. In fact, $|i|$ has the dimension $\left[\frac{ml^2}{t}\right]$, and P the dimension $\left[\frac{ml^2}{t^2}\right]$ (cf. pp. 88 and 84). In contrast, $|i|^2$ and AP, for example, are compatible quantities, in that both have the dimension $\left[\frac{m^2l^4}{t^2}\right]$. As a result, we can directly compare these two quantities numerically. We now wish to establish that the length of the impulse should qualify as considerable if its square is at least 100 times (measured in equal units) the product AP. Since the primary component of $|i|$ is formed, according to the preceding stipulation, by the eigenimpulse N, and since certainly $|i| \geq N$, we can also conceive our explanation of the word "considerable" as the following: the length of the impulse vector qualifies as considerable if

$$N^2 > 100AP.$$

This condition for the occurrence of pseudoregular precession obviously differs essentially from that given in the previous section for regular precession. While the condition for regular procession was a quantitative condition and required an entirely determined relation of the integration constants, our present condition is of a qualitative nature; it imposes only certain inequalities on the constants. Correspondingly, pseudoregular precession is also only a qualitative concept; according to whether we place the emphasis on the first or last syllables of the word, we postulate a greater or lesser similarity with regular precession.

We note, moreover, that the conditions just given were fulfilled when we spoke of pseudoregular precession in the previous chapter (cf. Fig. 28). The initial positions of the impulse and the figure axis were horizontal ($n = 0$, $e = 0$), and the length of the impulse was assumed to be infinite ($N = \infty$).

Because of the importance of our motion, it is well to make the treatment as elementary as possible, as is often attempted in other works to which we will return in the next section. Thus we will first disregard our knowledge of the general forms of motion, and only later relate our results to the representation of the motion of the top by elliptic integrals.

A purely elementary treatment is naturally possible, in our case, only on the basis of more or less plausible omissions that may be justified rigorously only by rigorous theory and analytic means. Such a treatment is nevertheless instructive, since it forces us to return to the simplest basis of explanation, which is contained only obscurely in the final formulas.

We assume, for the moment, that gravity does not act at all. Our top, which we assume to be a spherical top, then rotates, as we know, with constant velocity about the impulse axis, which is fixed in space. The figure axis describes a circular cone that will be very narrow, since, by assumption, the initial difference in direction between the figure axis and the impulse axis is very small for our motion. The apex of the top progresses through a small circle on the unit sphere. *We assume, moreover, that the impulse axis does not coincide with and is not near the vertical*; the meridian plane through the figure axis will then always deviate only slightly from the meridian plane through the impulse axis during the motion, and the two planes can be treated, in the first approximation, as coincident. Further, the angle ϑ between the vertical and the figure axis will change only very little in our rotation and can be taken, in the first approximation, as constant.

We now consider the action of gravity. Under its influence, the impulse does not remain constant, but rather is composed, at each moment, with the turning-impact of gravity $P \sin\vartheta$. It must be our next exercise to supply an image of the trajectory of the endpoint of the impulse.

The motion of the top naturally consists, now as previously, of a rotation about the (now no longer fixed) axis of the impulse. If we consider only a sufficiently short interval of time—for example, that of only one rotation of the figure axis about the impulse—then we can make the same assumptions for our present motion that were just made for the rotation about the fixed impulse axis. We can say, namely, that *the change of the impulse stands perpendicular to the meridian plane through the impulse* (instead of that through the figure axis). And, *the rate of change has the constant magnitude* $P \sin\vartheta_0$ (instead of the variable magnitude $P \sin\vartheta$), *where* ϑ_0 *signifies any mean value of the angle* ϑ. Through these statements, however, the trajectory of the endpoint of the impulse is determined in the simplest manner. *It is simply a circular arc about the vertical, and will be traversed with constant velocity.* The intersection point of the impulse vector with the unit sphere, which we wish to denote by J, thus moves *on a parallel* *circle*, or, if we consider only a sufficiently small part of the unit sphere and replace the sphere at the considered location by its tangent plane, *on a straight line.*

We can easily give the progressional velocity v of the point J. This is evidently in the same proportion to the progressional velocity $P \sin \vartheta_0$ of the impulse endpoint as 1 is to $|i|$, where $|i|$ is understood as the length of the impulse vector. In this proportion we may simply replace $|i|$ by the projection N of the impulse vector onto the figure axis, since this projection amounts, by assumption, to the primary component of the impulse. We thus find for the velocity v the value

$$v = \frac{P \sin \vartheta_0}{N}. \tag{1}$$

The motion of the figure axis and the trajectory of the apex of the top are now easy to determine. The apex of the top F—that is, the intersection point of the figure axis with the unit sphere—must always progress, since the instantaneous motion consists of a rotation about the impulse vector, perpendicularly to the binding line JF. Further, the angular velocity w with which F turns about J is simply equal to $\frac{|i|}{A}$. If we replace, as above, $|i|$ by the principal component N of the impulse, then there results for w the constant value

$$w = \frac{N}{A}. \tag{2}$$

Through the latter statements, however, the trajectory of the apex of the top is characterized as a cycloid.

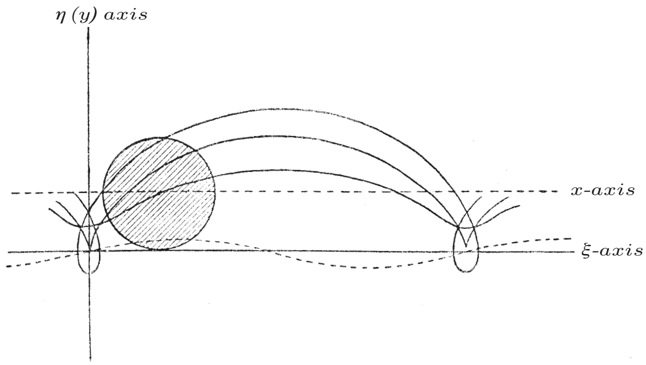

Fig. 48.

In fact, if we generate a cycloid in the usual manner (cf. Fig. 48) by rolling a wheel on a line, then each point fixed to the wheel turns with constant angular velocity about the instantaneous tangent point of the wheel, and always proceeds perpendicularly to the binding line with this tangent point, while the tangent point itself travels with constant velocity on its line. We can thus directly identify the trajectory of the apex of the top F with the trajectory of a point on a rolling wheel (that is, with a cycloid), and the path of the impulse point J with the linear path of the tangent point.

Thus the equation of the trajectory may be written down immediately. We use rectangular coordinates ξ, η, in that we take the line on which our wheel rolls as the ξ-axis. Let r be the radius of the wheel, a the distance from the point F that describes the cycloid to the midpoint of the wheel, and w the angular velocity of the rolling, where we must choose w corresponding to equation (2). We assume, for $t = 0$, that the point F lies perpendicularly above the tangent point J of the wheel and on the η-axis, so that its distance from this axis will be $\eta_0 = r + a$. Then the equations of the cycloid are

$$\left\{\begin{aligned} \xi &= rwt + a\sin wt, \\ \eta &= \quad\; r + a\cos wt. \end{aligned}\right. \tag{3}$$

The quantities r and a are determined in terms of the constants of the top in the following manner. Since rw signifies the velocity with which the tangent point J progresses on the ξ-axis, we must have $rw = v$ and therefore, with consideration of (1) and (2),

$$r = \frac{AP\sin\vartheta_0}{N^2}. \tag{4}$$

The quantity a is determined from the initial distance η_0 of the points J and F by the equation

$$a = \eta_0 - r.$$

Now η_0 measures the deviation of the impulse vector from the figure axis in the initial position. If we denote the perpendicular from the endpoint of the impulse vector to the figure axis by p and its projection onto the vertical by n', then we have (cf. Fig. 49)

$$\eta_0 = \frac{p}{N} = \frac{n'}{N\sin\vartheta_0}. \tag{5}$$

Further, as follows from the figure,

$$n' = n - N\cos\vartheta_0. \tag{5'}$$

The value of a follows from (4) and (5) as

$$a = \frac{n'}{N\sin\vartheta_0} - \frac{AP\sin\vartheta_0}{N^2}. \tag{6}$$

If we insert the found values of r and a into equation (3) and express w according to (2), we obtain *the equation of the trajectory in the definitive form*

$$(7)\qquad \begin{cases} \xi = \dfrac{P\sin\vartheta_0}{N}t + \left(\dfrac{n'}{N\sin\vartheta_0} - \dfrac{AP\sin\vartheta_0}{N^2}\right)\sin\dfrac{N}{A}t, \\ \eta = \dfrac{AP\sin\vartheta_0}{N^2} + \left(\dfrac{n'}{N\sin\vartheta_0} - \dfrac{AP\sin\vartheta_0}{N^2}\right)\cos\dfrac{N}{A}t. \end{cases}$$

According to whether the distance a is larger, equal to, or smaller than the radius r, we have a *prolate*, a *common*, or a *curtate cycloid*, which three types are indicated in Fig. 48.

We must particularly emphasize a further point. In the determination of the impulse curve, we assumed that the figure axis was always only slightly removed from the impulse axis. The permissibility of this assumption is immediately evident only for the beginning of the motion; it follows at this time from the initial conditions. It now follows generally from the periodic behavior of our trajectory, however, that the initially present conditions will be exactly present again after each traversal of one complete arc of the cycloid. As a result, our deliberation is valid for the following phases of the motion just as well as for the first.

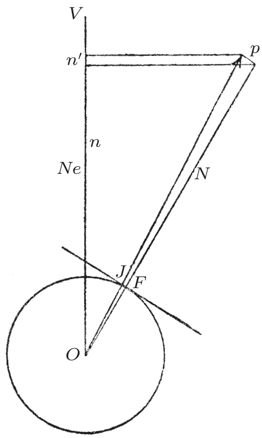

Fig. 49.

We naturally obtain only an approximate representation of the motion from the preceding consideration. Strictly speaking, we must not say that *the trajectory of the apex of the top is a cycloid under the given initial conditions*, but rather that *the trajectory always deviates less from a cycloid as the initial impulse is larger and its direction coincides more exactly with the figure axis.*

It is easy to see the type of deviation that will occur. Since the meridian plane through the impulse does not coincide exactly with the meridian plane through the figure axis, the impulse curve will not be exactly a parallel circle or a straight line; it will rather exhibit, according to which side the one meridian plane is removed from the other during the rotation of the figure axis, a small lateral buckling upward or downward, as indicated by the dotted line in Fig. 48. The trajectory of the apex of the top that corresponds to this undulating impulse curve will also exhibit small periodically recurring distortions with respect to the cycloidal form. These deviations are not contained, however, in the figure or our formulas; they would correspond, in the latter, to neglected terms of higher order.

Moreover, we could extend our cycloid so that it also renders these terms of second and higher order correctly. We must, for this purpose, let another circle roll on the rolling circle, on this circle another, and so forth. Through the free choice of the radii and the rotational velocities, we obtain a schema that is sufficiently general to render an arbitrary motion with arbitrary precision. Our approximate manner of representation of the trajectory above appears, from this point of view, as the first term of an infinite series of approximations.*)

We must conceive Fig. 48 as an extraordinarily enlarged rendering of the actual proportions. In experiments, the individual cycloidal arcs will be so small and so rapidly successive that the eye cannot perceive them, and the impression of an ordinary regular precession is obtained. As witness to this we calculate, for example, the time of passage and the span width of an individual cycloidal arc. The time during which an individual arc will be traversed is called 2ω; it amounts, according to our formula, to

$$2\omega = \frac{2\pi}{w} = 2\pi\frac{A}{N};$$

it will thus become zero with increasing N. The span width of the cycloidal arc—that is, the segment through which the figure axis progresses in the horizontal direction during one period—is equal to

$$\frac{P\sin\vartheta_0}{N}\,2\omega = \frac{AP}{N^2}\sin\vartheta_0\cdot 2\pi;$$

this quantity contains the factor $\dfrac{AP}{N^2}$, which we assumed above as very small $\left(<\dfrac{1}{100}\right)$. Simultaneously with the span width, the height of the cycloidal arc will also become vanishingly small for increasing N, corresponding to the formula

$$a = \frac{n'}{N\sin\vartheta_0} - \frac{AP\sin\vartheta_0}{N^2}.$$

The collected dimensions of the cycloid will therefore become, so to speak, microscopically small. The eye perceives, of the entire play of the

*) The "*principle of cycloidal approximation*" indicated in the text also provides the mathematical foundation on which rests the conception of celestial mechanics in the *Ptolemaic system of the world.* Cf. Möbius, Elemente der Mechanik des Himmels, 1843, Kap. III, Theorie der epicykloidischen Bewegung. Ges. W. Bd. IV.[131]

apex of the top, only an undetermined mean that consists of an apparent regular precession.

A numerical example may illustrate this. We consider a rotor whose mass forms a torus of square cross section. Let the side of the cross-sectional square be 2 cm, and the distance of its midpoint from the figure axis 5 cm. Let the support point have the distance 2,5 cm from the center of gravity of the rotor. For the calculation of the moment of inertia, we permit ourselves to imagine the mass of an individual cross section to be concentrated at the midpoint of the cross section. One then easily finds, in the absolute system of measure,

$$C = 1000\varrho\pi, \quad A = 750\varrho\pi, \quad P = 100\varrho\pi g,$$

understanding by ϱ the density of the material.

The eigenrotation of the rotor amounts approximately (as on page 135) to 20 rotations per second. Its angular velocity about the figure axis is then 40π; thus

$$N = 40\,000\varrho\pi^2 \quad \text{and} \quad \frac{N^2}{AP} = \frac{(40\,000\varrho\pi^2)^2}{75\,000(\varrho\pi)^2 g} = \frac{64\,000\pi^2}{3g}.$$

The fraction $\dfrac{\pi^2}{g}$ can be replaced approximately by $\dfrac{1}{100}$; then

$$\frac{N^2}{AP} = \frac{640}{3} > 200.$$

The top considered here is, to be sure, not a spherical top. We know, however, that a top with unequal moments of inertia A and C describes the same trajectory at the same tempo as a spherical top with moment of inertia A and the same impulse constants n, N, etc. As a result, the above formulas may be carried over to our case.

If the initial inclination ϑ_0 of the figure axis with respect to the vertical is now approximately 30° and we abandon our top to the influence of gravity without adding a considerable lateral impact, then there results for the transit time of a single cycloidal arc, according to the preceding,

$$2\omega = \frac{1500\varrho\pi^2}{40000\varrho\pi^2} < 0{,}04 \text{ sec.}$$

At the same time, the height of the arc will become, if we take, for example, n' directly equal to zero,

$$|2a| = 2\frac{AP}{N^2}\sin\vartheta_0 = \frac{AP}{N^2} < 0{,}05 \text{ mm.}$$

It is clear that these small quantities will all but completely elude observation.

As we see, the explanation of pseudoregular precession presents no difficulty at all on our adopted path. If one otherwise finds this motion astonishing and paradoxical, this rests, in good measure, on the fact that one's conception of mechanical processes usually begins from particle mechanics, and thus one thinks exclusively, in our case, of the downward effect of gravity on a freely moving mass particle. An explanation of pseudoregular precession is naturally also possible on the basis of particle mechanics, as we will discuss in more detail in the following section. The path that leads from particle mechanics to the understanding of our top motion is, however, naturally rather long. It is essentially shortened if we operate from the outset with the concepts of the moment of inertia, the instantaneous rotation, and, in particular, the concept of the impulse, and thus begin, as was done here, from the conception of a rigid body as a unified mechanical system. These concepts are naturally derived in the end from particle mechanics, but this derivation is undertaken in advance, and need not be interjected afterward.

Partly to confirm the preceding considerations, and partly to relate them to the general representation of the motion of the top, we now wish to conceive our problem once again analytically. We are in the pleasant position of being able to manage with the approximation formulas at the conclusion of the previous chapter, whose bounded precision goes over in the present limiting case to an arbitrary precision. We must first estimate the magnitude of the error in the application of those approximation formulas.

Under the assumption that the impulse vector falls nearly in the direction of the figure axis, the impulse component n (cf. Fig. 49) will be nearly equal to Ne. We denote the difference $n - Ne$, as in equation $(5')$, by n', so that $n' : N$ is assumed to be a small number. In addition, we will have use of our assumption that $AP : N^2$ is a small number.

We regard the initial parallel circle e, as previously, as known; the second parallel circle e' is then easily determined approximately from the equation $U_1 = 0$. If we set, namely, $n = Ne + n'$ in this equation and neglect the square of n' compared with that of N, then we obtain, according to equation (2) of page 240,

$$U_1 = N^2(1 - e^2)(e - u) + 2n'N(1 - e^2) - 2AP(1 - e^2)(1 - u^2).$$

We assume, just as in the geometric consideration above, that the initial position of the figure axis does not coincide with and is not near the vertical ($e \neq \pm 1$), so that we can divide by $N^2(1-e^2)$ and obtain for the determination of e' the equation

$$e - u + \frac{2n'}{N} - \frac{2AP}{N^2}(1-u^2) = 0. \tag{8}$$

Since the two latter terms of this equation are by assumption small numbers, we already see that one root (e') is approximately equal to e, and the second (e'') must become very large. We obtain a more precise value of e' if we replace u by e in the latter small terms, or, preferably, by $u_0 = \dfrac{e+e'}{2}$; there then follows

$$e' = e + 2\varepsilon, \quad \varepsilon = \frac{n'}{N} - \frac{AP}{N^2}(1-u_0^2). \tag{9}$$

On page 273, we distinguished two cases in which the approximation formulas for the calculation of u give an arbitrarily small error. The first case was that e and e' differ sufficiently little, and the second that e'' becomes sufficiently large. The first case occurs, as we see, for pseudoregular precession. In fact, the vertical distance between the two bounding parallel circles e and e' amounts only to the very small quantity 2ε. For redundancy, however, the second criterion also obtains in our case: e'' is, as already mentioned, a very large number. In fact, if we calculate e'' according to equation (9) of page 273, there follows, if we set $n = Ne + n'$ and neglect n'^2 compared with N^2,

$$e'' = \frac{N^2}{2AP} - e'.$$

This value is, according to our assumption on the ratio $N^2 : AP$, a very large number; we will even be able to omit the proper fraction e' in relation to the first term. We thus set

$$e'' = \frac{N^2}{2AP},$$

and, with the same degree of approximation,

$$\begin{cases} e'' - u_0 = \dfrac{N^2}{2AP}, \\[2ex] \dfrac{2P(e''-u_0)}{A} = \dfrac{N^2}{A^2}. \end{cases} \tag{10}$$

A somewhat more precise value for the latter quantity would be

$$\frac{2P(e''-u_0)}{A} = \frac{N^2 - 4APu_0}{A^2}; \tag{10'}$$

we prefer to satisfy ourselves in the following, however, with the simpler value from equation (10).

Because of both the smallness of ε and the bigness of e'', the estimated error τ on page 272 will be very small in our case, and indeed always smaller as our conditions adopted as the basis of the motion are more completely fulfilled. We can thus represent u with arbitrary approximation through equation (8′) of page 272. With consideration of equations (9) and (10) of the previous page, there follows

$$u = u_0 + \left(\frac{n'}{N} - \frac{AP}{N^2}(1-u_0^2)\right)\sin\frac{N}{A}t.$$

The smallness of ε justifies us, further, in also representing ψ by the approximation formula (14′) of page 276. We saw, namely, that the error in this representation vanishes with vanishing ε.

The coefficients of that equation may be simplified in our case. If we set $n = Ne + n'$, then

$$n - Nu_0 = N(e-u_0) + n' = N\frac{e-e'}{2} + n' = -N\varepsilon + n',$$

and therefore, with consideration of (9),

$$\frac{n - Nu_0}{A(1-u_0^2)} = \frac{P}{N}.$$

The second coefficient

$$\frac{N(1+u_0^2) - 2nu_0}{(1-u_0^2)^2\sqrt{2AP(e''-u_0)}}$$

simplifies in a similar manner. According to (10), we can write

$$\frac{N(1-u_0^2) - 2(n'-N\varepsilon)u_0}{N(1-u_0^2)^2};$$

since this expression appears in equation (14′) multiplied by the small quantity ε, we can further simplify it, in that we neglect $(n' - N\varepsilon)$ with respect to N; the imprecision effected in the result is of the order ε^2, and would therefore influence only the form of the remainder term. Our second coefficient can therefore be set equal to

$$\frac{1}{1-u_0^2}.$$

We thus obtain from the reduced equation the simple approximate value

$$\psi = \frac{P}{N}t + \frac{\varepsilon}{1-u_0^2}\cos\frac{N}{A}t.$$

The trajectory of the apex of the top is now represented, if we set

$u = \cos\vartheta$, $u_0 = \cos\vartheta_0$ and insert for ε the value from (9), by the two equations

$$(11)\qquad \begin{cases} \cos\vartheta = \cos\vartheta_0 + \left(\dfrac{n'}{N} - \dfrac{AP}{N^2}\sin^2\vartheta_0\right)\ \sin\dfrac{N}{A}t, \\ \quad\ \psi = \dfrac{P}{N}t \quad + \left(\dfrac{n'}{N\sin^2\vartheta} \ - \ \dfrac{AP}{N^2}\right)\cos\dfrac{N}{A}t, \end{cases}$$

in precise agreement with the cycloid theory, as we will immediately show in more detail.

The relation of these formulas to the general representation of the motion of the top by elliptic integrals is, according to the discussion of pages 274 and 277, clear. The smallness of the modulus k, or, equivalently, that of the error τ, satisfactorily explains why the elliptic integrals can be replaced with good approximation, in our case, by trigonometric functions.

These remarks become partially invalid if our original assumptions are only partially fulfilled. If, for example, the initial impulse nearly coincides with the figure axis and is extraordinarily large, but is not large in comparison with the action of gravity—that is, if n'/N but not AP/N^2 is a small number—then the motion will differ essentially from pseudoregular precession; the error in our approximation formulas can be very considerable. In fact, a top with large N and P will behave just like a top with correspondingly diminished values of N and P; it can thus describe the general motions depicted in the previous chapter that can be represented with arbitrary precision only in terms of elliptic integrals. If, on the other hand, the eigenimpulse is very large, the gravity moment is not very large, and the impulse axis deviates essentially from the figure axis in the initial position—that is, if AP/N^2 but not n'/N is small—then the parallel circles e and e' need no longer be neighboring. Nevertheless, if the value of e'' (see equation (9) of page 273) is still very large, the error in the trigonometric representation will still be very small. This case, we can say, is the case of a top that is neighboring to a force-free top (of moderate N and vanishing P). Just as the motion of the latter can be described approximately by trigonometric terms, so can that of the former.

One recognizes that our equations (11) are identical with the earlier formulas (7) in the following manner: we first replace the spherical surface by its tangent plane at the considered point on the trajectory, which is permitted because of the extraordinarily small dimensions of the latter. We choose the mean direction of progression $u = u_0$ of the apex of the top as the x-axis of a rectangular coordinate system (xy) whose origin coincides with the position of the apex of the top at the time $t = 0$. Then the rectangular coordinates are easily expressed in terms of the previous coordinates ψ, u. Since, namely, ψ is (disregarding an additive constant) the azimuth to which the projection of the figure axis has advanced in the equatorial plane from its initial position in the time t, and since, on the other hand, x signifies the horizontal displacement of the apex of the top on the spherical surface (or in its tangent plane) during the same time, ψ is to x as the radius of the equator is to the radius of the parallel circle passing through the apex of the top; that is, approximately as 1 to $\sin\vartheta_0$. Therefore

$$x = \psi \sin\vartheta_0.$$

Further, $u - u_0 = \cos\vartheta - \cos\vartheta_0$ signifies the vertical projection of the meridional deviation of the apex of the top from the mean parallel u_0. This meridional deviation itself is, however, our coordinate y. We thus have

$$y = \frac{u - u_0}{\sin\vartheta_0} = \frac{\cos\vartheta - \cos\vartheta_0}{\sin\vartheta_0}.$$

As a result, equations (9) become

$$(12) \qquad \begin{cases} x = \dfrac{P\sin\vartheta_0}{N} t + \left(\dfrac{n'}{N\sin\vartheta_0} - \dfrac{AP\sin\vartheta_0}{N^2}\right)\cos\dfrac{N}{A}t, \\[2ex] y = \qquad\qquad\quad \left(\dfrac{n'}{N\sin\vartheta_0} - \dfrac{AP\sin\vartheta_0}{N^2}\right)\sin\dfrac{N}{A}t. \end{cases}$$

These formulas now differ from equations (7) only by a displacement of the coordinate system. While our ξ-axis previously coincided with the impulse curve and (cf. Fig. 48) lay asymmetrically with respect to the trajectory of the apex of the top, we have chosen our x-axis so that it is identical with the mean position u_0 of the apex of the top. We bring equations (7) and (12) into formal coincidence if we set

$$x = \xi, \quad y = \eta - \frac{AP\sin\vartheta_0}{N^2}$$

and shift, moreover, the origin of time by $\dfrac{\pi A}{2N}$.

We can thus generate our trajectory (12) again geometrically as a *cycloid* by the rolling of a circle on a straight line, or also, as is equivalent in the present case, conceive our motion as the *superposition of a regular precession and a nutation* in the sense of page 276.

We take the equations of the regular precession from the first terms on the right-hand side of (11); they are

$$\cos\vartheta = \cos\vartheta_0, \quad \psi = \frac{P}{N}t. \tag{13}$$

We best describe the nutation in terms of its equation in x, y coordinates, which we take from the second terms on the right-hand sides of (12). We have

$$x = \varepsilon\cos\frac{Nt}{A}, \quad y = \varepsilon\sin\frac{Nt}{A}, \quad \varepsilon = \frac{n'}{N\sin\vartheta_0} - \frac{AP\sin\vartheta_0}{N^2}. \tag{14}$$

Under the special assumptions that form the basis of pseudoregular precession, the nutation therefore becomes a *circular oscillation*; its horizontal and its meridional amplitudes are both equal to ε. In contrast, we saw that under the general assumptions at the end of the previous chapter, as well as for the trajectories neighboring to regular precession, the nutation was an *elliptical oscillation.* Further, the *nutation period*

$$2\omega = \frac{2\pi A}{N}, \tag{15}$$

or, if we assume instead of (10) the somewhat more precise equation

$$2\omega = \frac{2\pi A}{\sqrt{N^2 - 4APu_0}} \tag{15'}$$

in (10′), becomes infinitely small with increasing N for pseudoregular precession, while it remains finite, for example, for the curves neighboring to regular precession. Finally, the *precessional velocity* $\dfrac{P}{N}$ now also becomes infinitely small, while it likewise has, in general, a finite value.

Of the two component motions into which we have divided the pseudoregular precession, the eye perceives clearly only the first. Indeed, this motion is, as we just emphasized, exceedingly slow for sufficiently large N. But the factor t explicitly enters its equation. *In spite of the exceedingly small value of the angular velocity, we will note, if only we extend the observation time long enough, a distinct precession of the figure axis.* In the equations of our second component motion, in contrast, t appears only as the argument of trigonometric functions. *These rapidly changing terms of small absolute value thus escape observation.* We can also formulate this difference using common astronomical terminology in the following manner:

Our first component motion represents a secular, and our second a periodic perturbation of the stationary position.

While our first component motion is decisive for the description of the trajectory or the depiction of observational results, the second term is the more important for the mechanical explanation of the process.

The mechanical explanation is associated, according to the fundamental laws of dynamics, not so much with the position as with the velocity and the acceleration of the mass elements. But if we differentiate the equations of our trajectory with respect to t, then the relative magnitudes of the individual terms are changed. The periodic term, namely, is multiplied each time by the (very large) factor $\frac{N}{A}$, while the secular term loses the factor t, or (by a second differentiation) vanishes altogether. As a result, the explanation of the process of the motion must also take essential account of the second component motion. If we would consider, on the basis of an imprecise observation, the motion as an actual regular precession, it must appear, in fact, as incomprehensible and paradoxical. *In our case the mechanical explanation must be based, rather, directly on the element of the motion that is all but lost in observation.*

In these developments we see the complete solution of the paradoxes of the motion of the top. We recognize, in particular, *why the description of the motion of the top as regular precession indeed reproduces observations very well under the usual experimental conditions, but can still be insufficient for the mechanical explanation.*

As a final historical remark, we note that pseudoregular precession, if not under this name, was first derived from the general differential equations of the motion by Poisson.*) In Poisson, however, as well as in the later analysts,**) the geometric and mechanical essentials are not as explicitly pared from the formulas as here. The reader runs the danger, in the study of these purely analytic presentations, of directly overlooking or inadequately grasping the essential.[132]

*) Cf. Traité de Mécanique, t. II, Nr. 432, p. 175 of the second edition.

**) For example, Kirchhoff, Mechanik, 7 Vorlesung, §5.

§3. Popular explanations of the phenomena of the top in the literature.

Elementary presentations of the theory of the top are concerned almost exclusively with regular precession, since this motion is of primary importance in experiments, and since the general motion of the top may not at all be represented by elementary means. We give, in this place, an overview of the more important popular explanations, without making any claim of completeness. The overall picture that emerges here is not very pleasant. We will meet many indefensible or incomplete attempted explanations. This circumstance, moreover, was directly the original motive for the composition of the present detailed monograph.

1. A first category of presentations is satisfied with a plain depiction of the processes. The following experiment is emphasized above all others. If one sets the top into a strong rotation and then applies a force to the figure axis—by pulling, for example, the apex of the top to one side with an encircling thread—then the axis apparently deflects perpendicularly to the direction of the thread. This and similar things are vividly set forth in the previously cited*) interesting work of P e r r y, where the top is compared directly to an obstinate beast that is goaded in one direction and always runs in another.[133]

The named experiment can also serve to illustrate the behavior of the figure axis under the influence of gravity. In fact, we can compare gravity with a pull that strives to move the center of gravity, and therefore also the figure axis, downward at each moment. Corresponding to the experiment, we will thus expect that the apex of the heavy top apparently deflects perpendicularly with respect to the gravitational force; that is, in the horizontal direction.

As a mere observational result, one must allow such a representation of the phenomena as valid. Its validity, however, lies only within the imprecise limits of the observation. In fact, we know that the initial direction of the motion of the apex of the top, if we abandon it to the pull of a thread or the influence of gravity without the addition of a lateral impact, is not perpendicular to the pull, but in the direction

*) p. 134.

of the pull (cf. the adjacent lobed curve [common cycloid]), and that only the smallness of the arcs of which the curve is composed produces the impression of the experiment.

2. It is occasionally attempted to explain the described imprecise observation by an incorrect conclusion from the principles of mechanics in the following manner. The top initially rotates about the figure axis, which is somehow inclined to the vertical. The figure axis then represents, at the same time, the rotation axis and the impulse axis. Now the continuous pull of gravity comes into effect. This corresponds to a turning-impulse that is directed perpendicularly to the meridian plane passing through the figure axis and the vertical, and is composed with the original turning-impulse according to the parallelogram of forces. One now says that the diagonal of the parallelogram gives the changed position of the "axis." That is correct with respect to the impulse axis, and, for the spherical top, also for the rotation axis. In the explanation that we have in mind, however, the "axis" is further understood, tacitly, to include the figure axis, to which the statement regarding the diagonal of the parallelogram in no way applies; it is therefore concluded that the figure axis must always progress perpendicularly with respect to the cited meridian plane; that is, on a circular cone about the vertical!— In reality, the figure axis naturally moves on a circular cone about the changing instantaneous rotation axis, which in its turn is determined by the position of the impulse. The consequence is that the initial deflection of the figure axis is in no way perpendicular to the direction of the pull, but rather is vertically downward. If, as assumed here, the impulse axis and the figure axis initially coincide, an actual regular precession is simply impossible. The condition for the latter consists, as we have seen previously, in a certain separation of the impulse axis and the figure axis; that is, that the apex of the top is given, in addition to the pull of gravity, an entirely determined lateral impact.

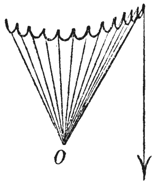

Fig. 50.

The entire process that we have just discussed can serve as an excellent example of a "quaternio terminorum."[134] The error is simply that the word "axis" is used with two different meanings; this is all the more remarkable, since one must necessarily ask, what will become of the motion if the velocity of the initial rotation is decreased, in which case the observer may clearly recognize a departure from regular precession?

The named error befell no less than the famous French experimentalist F o u c a u l t and his competitor S i r e, and has since been commonly found in the literature. For more precise details, we refer to a noteworthy work of G i l b e r t*): Etude historique et critique sur le problème de la rotation.

3. We now proceed to the so-called *Airy explanation*, which is likewise doubtful in an essential point. As an astronomer, Airy**) is particularly interested in the problem of the precession and nutation of the Earth; to this subject he prepends his elementary theory of the motion of the top as an introduction.

Airy first derives the theorem of the parallelogram of rotation vectors, but does not emphasize that this theorem has merely a kinematic significance. Airy has not the concept of the impulse vector and the parallelogram of impulse vectors, which is solely decisive in kinetic respects, and regulates the course of the motion. The mass distribution of the body allegedly remains entirely general.[137]

Airy then treats of a rotation problem that has only a distant similarity with that of the heavy top. He assumes, namely, that a body is subject to a force that continuously strives to turn it about an axis $O\Delta$ perpendicular to the *rotation axis* OD and in a fixed plane ΔOD. The magnitude of the force is invariable. (It is to be noted here that in the actual rotation problem of the heavy top, the axis of the additional turning (the line of nodes) stands perpendicular not to the rotation axis, but rather to the figure axis, and is also generally not constant, a circumstance of which Airy is obviously completely aware. Airy first treats only of a fictitious problem.) Through successive application of the theorem of the parallelogram of rotation vectors, Airy concludes that the rotation vector remains constant in magnitude, and that its direction rotates in the plane ΔOD with constant velocity.[138]

A fundamental error lies in this conclusion, however, even if we accept the fictitious law concerning the direction and magnitude of the additional rotation. No consideration, namely, is given to the possibility of the "eigenmotion" of the rotation vector. Even if the external turning-force that produces the rotation about the axis $O\Delta$ did not

*) Annales de la société scientifique de Bruxelles, 1878.[135] Mr. F r a n k e has called attention to a similar error: Ztschr. f. d. mathem. u. naturw. Unterricht, Bd. 17, 1886.[136]

**) A i r y, Mathematical Tracts, Cambridge, 1831. Cf. the chapter: Precession of the equinoxes, no. 1–15.

act, the rotation axis OD would generally change in the body and in space. In fact, a force-free top does indeed generally describe a regular precession in which the rotation vector is led on a circular cone around the impulse axis. Airy, in contrast, tacitly assumes that the rotation axis would remain in its instantaneous position if the external force suddenly ceased to act.

The named assumption is fulfilled only in the special case of the spherical top, for which, as we emphasized, each axis can be called a permanent rotation axis. As a result, we must say that *the Airy theorems hold not for the general case of a rotating body, but rather, in contrast, only for the most special case, the case of the spherical top.*

We may take this occasion to warn of an overestimation of the kinetic significance of the rotation vector.

The rotation vector fundamentally recognizes only the instantaneous kinematic state of the motion. Kinetics depends not on the rotation vector, but rather on the impulse vector. The impulse vector is composed with the turning-moment of the external forces in the simplest manner (according to the parallelogram law) and thus determines, together with the mass distribution of the body, the course of the motion. The rotation vector then follows from the position of the impulse vector, and moves exactly as prescribed by the position of the impulse vector and the mass distribution of the body. The parallelogram of rotation vectors is indeed correct kinematically, but is kinetically meaningless, since the rotation vector can progress in the body and in space even without the addition of an external rotation-causing force.

(A simple example to which Mr. K o p p e (cf. below) has drawn attention may show how one can be led to false results, in kinetic questions, by the parallelogram of rotations.

We ask for the turning-moment that is required to turn the figure axis of a (symmetric) top. More precisely, we formulate the question in the following manner. The top initially rotates about its figure axis OF, and is free of the influence of external forces. Let its rotational velocity be r, and its impulse be $Cr = N$. We then turn the figure axis by force through the small angle $d\vartheta$, in such a way that if we release the axis, the top rotates permanently with the original velocity r about the altered and henceforth spatially stationary axis OF_1 (cf. Fig. 51). It is asked for the required turning-moment.

According to the impulse theory, two things are necessary in order to bring about the named state of affairs: (1) One must impart to the figure axis a rotational velocity about the axis OH that is perpendicular to OF and OF_1, and must annihilate this velocity when the position OF_1 is attained. The corresponding impulses, which likewise occur about the mutual perpendicular to OF and OF_1, cancel oppositely ($OH = -OH_1$). (2) One must arrange, in addition, for the change of position of the impulse vector from the position OF to the position OF_1. If one does not do this, namely, then the figure axis would begin, after having reached its position OF_1, a regular precession about the unchanged position OF of the impulse vector instead of standing, as we demanded, stationary in space. The required additional impulse for the change of position is $di = Nd\vartheta$; its axis OG lies in the plane OFF_1, and is perpendicular to the infinitesimally differing axes OF and OF_1. The time rate of change of the impulse $\frac{di}{dt}$ gives, in axis and magnitude, the turning-moment that must be applied to turn the figure axis. The correct answer to our question, as follows from the parallelogram of the impulse vectors, is thus

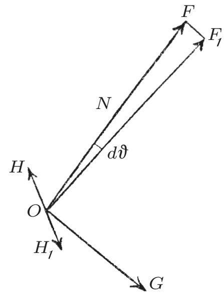

Fig. 51.

$$N\vartheta' = Cr\vartheta'.$$

The same question may now be answered according to the parallelogram of the rotation vectors. The change of position and subsequent fixing of the figure axis again requires, in total, no turning-moment. To change the position of the rotation axis, one must, according to the parallelogram of rotations, add the rotation $r\vartheta'$ about the axis OG. One finds the turning-moment corresponding to this rotation in a known manner by multiplication of the named angular velocity by the moment of inertia A corresponding to the axis OG. The turning-moment that is sought would thus be

$$Ar\vartheta'.$$

The false and the correct values coincide, as one sees, only in the case of the spherical top. In every other case, the application of the parallelogram of rotations in kinetics can be delusive.[139])

Correspondingly, the Airy consideration for the case of the general or the symmetric top is thus to be amended by speaking throughout of the impulse vector instead of the rotation vector. This is done by P o i n s o t in his *Théorie des équinoxes.**) He considers, in addition, a somewhat more general case than Airy, in that the additional impulse is assumed to be perpendicular to the instantaneous impulse, and, moreover, in a fixed plane that need not pass through the instantaneous impulse (while Airy, as mentioned, assumes an additional rotation in a plane passing through the instantaneous rotation axis). If one carries out, in this case, the successive parallelogram constructions with the impulse vector, one sees that the impulse vector describes a circular cone about the normal to the fixed plane, and therefore, in particular, a vertically positioned circular cone if one imagines the fixed plane as horizontal. As Poinsot himself explicitly emphasizes, his problem does not completely coincide, because of the given assumption on the direction of the additional impulse, with the problem of the heavy top. Correspondingly, the given result that the impulse cone is a circular cone is only approximately correct for the heavy top.

Our own popular explanation of pseudoregular precession at the beginning of the present chapter directly represents an extension of the Poinsot presentation. We have there established, on the basis of the Poinsot impulse principle, the successive positions of the figure axis in space, and have also established the sense of the deviations that the actual motion will exhibit in comparison to our always only approximate construction.

The Airy explanation is completed in another direction (namely, by the consideration of nutation) by Mr. A. S c h m i d t in his stimulating work "Die elementare Behandlung des Kreiselproblems."**) However, one must also here add the restriction to the spherical top, since the author operates throughout with the parallelogram of rotations, instead, as is generally irremissible in kinetic questions, of the parallelogram of impulse vectors.

As emphasized, neither the Airy nor the Poinsot assumptions will be realized for the general motion of the heavy top. The additional impulse or the additional rotation for the top stands perpendicular neither to the rotation axis nor to the impulse vector, but rather to the figure axis. The assumptions of the Airy or the Poinsot considerations are exactly fulfilled only for the specific mutual positions of the vertical, the rotation axis, and the figure axis in an exact regular precession.

*) Connaissance des temps, Paris 1857, Introduction, Nr. 1–10.[140]

**) Mathem.-naturw.-Mitteilungen von Böklen, 1886, Heft III.[141]

The practical application of the Airy consideration to the case of the heavy top is sought in the *Theoretical Physics* of Mr. v. Lang.*) In order to have a problem corresponding to the Airy assumptions, v. Lang assumes, as an initial state, a simple rotation about the figure axis, so that the incremental rotation corresponding to gravity initially stands perpendicular to the rotation axis. This state of affairs must, however, be altered immediately by the action of gravity. If, nevertheless, it is assumed that the additional rotation (better the rotation impulse) always stands perpendicular to the instantaneous rotation axis (better the impulse axis), then there again appears to be a duplicity in the use of the word "axis." Correspondingly, the result that v. Lang attains is not correct. According to his derivation, the rotation axis must *exactly* describe a circular cone about the vertical, which, as we know, is correct for the assumed initial state *only approximately*, and this *only for very large rotational velocities.* Moreover, one must, if one would maintain the v. Lang use of the rotation vector, add the explicit restriction to the spherical top.

A correct presentation in the style of Poinsot is given by De Jonquières**), who derives the appearance of the cusped curve.

4. The well-known *Poggendorff explanation****) is based on essentially different principles from the latterly named explanations or our own elementary consideration at the beginning of the previous chapter. While we treated the top as a unified mechanical system, Poggendorff returns to the motion of the individual mass particles. Because of the brevity and the completely elementary character of the demonstration, the Poggendorff explanation is in no way complete, and easily gives occasion for errors. We reproduce the Poggendorff explanation in a somewhat free manner, without being able to claim with certainty that we exactly reproduce the meaning of the author, which, from his words, is not completely clearly established.

We consider, with Poggendorff, a horizontally positioned rotor that has been given a specific rotational velocity about its figure axis, and which is free to turn about a point O of its axis.[144] We imagine that the free end of the axis moves downward in the vertical plane by a small

*) §55.[142]

**) Théorie élémentaire du mouvement de la toupie, Revue maritime et coloniale, 1886.[143]

***) "Noch ein Wort über die Fesselsche Rotationsmaschine." Poggendorffs Annalen, Bd. 90, p. 348.

amount, which corresponds to the semblance of the action of gravity on the rotor. We designate this motion concisely as motion I. The velocity vectors of the individual mass particles will evidently be displaced partly parallel to themselves and partly away from their directions. The latter requires, for each particle, a force that is equal in magnitude and direction to the time rate of change of the impulse of the respective mass particle. If one composes all these forces into a turning-force, one easily finds a turning-force with a vertical axis. We must exert this turning-force if we wish to enforce motion I. If we do not exert it, but nevertheless imagine that the rotor attains to motion I, then there remains an equal and oppositely directed turning-force, which, if it alone acted, would effect a motion of the figure axis in the horizontal direction. The latter is designated as motion II. The motion II would now, in turn, cause a change of direction in the individual impulses of the mass particles. The required forces are composed into a turning-force with a horizontal axis. This must again be added externally if the motion II should be possible. Otherwise, there remains an oppositely directed equal horizontal turning-force about the horizontal axis, which effects a motion III, in consequence of which the figure axis is turned vertically upward, and therefore opposes the motion I. The appearance of the motion III now explains why the motion tendency I given by the action of gravity does not continue, but rather can be overcome by the gradually increasing motion tendency III. The appearance of motion II shows, at the same time, that the points of the figure axis can meanwhile acquire a horizontal velocity component. The actual motion, we must imagine, consists of a combination of the motions I, II, and III (and indeed, as we can add according to the preceding, in such a combination that the required turning-forces for the production of these motions are composed at every instant into a turning-moment exactly equal to the gravitational moment).

As one sees, only a very approximate image of the resulting motion is acquired through this rather rough consideration. The strengths with which the different motion components I, II, III occur remain completely undetermined. Nothing more detailed regarding the form of the trajectory that the apex of the top describes on the spherical surface can be stated merely on the basis of the above consideration.

The Poggendorff manner of expression is, as said, somewhat different from the preceding. It suggests the error that motions I and III could cancel one another, so that a purely horizontal motion of the apex of the top would remain. This is naturally entirely impossible, in so far as the rotor has, in its initial position, no horizontal component of progression.

In the textbooks*) that repeat the Poggendorff explanation, the named error is often committed explicitly.

The Poggendorff explanation is completed by Mr. Koppe**) and carried out to a quantitative determination of the motion. Koppe introduces the concept of the Coriolis force for the single mass particle (cf. Chap. III, §7), apparently encompassed by Poggendorff, and describes the trajectory of the apex of the top in a thoroughly correct manner as a cycloid. We emphasize, in particular, the worthy critical remarks at the beginning of his work, which were very useful to us in the formulation of the preceding, without, however, wishing to subscribe to the censure raised by Mr. Koppe with respect to analytic treatment in general.[146] The latter may be made more precise by the remarks at the conclusion of the previous section.

The essentially correct work of Jouffret cited above***) likewise operates with the Coriolis force.

F. Heinen also follows Poggendorff in the description of his rotation apparatus.†) The Heinen presentation is, however, very much more detailed than the Poggendorff, and also gives no more than a general qualitative representation of the expected motion.[147]

All in all, we do not wish to recommend, for the reasons given on page 300, the return to particle mechanics that is common to the last group of explanations. (One may compare in this respect, for example, Mr. Koppe's††) indeed correct but extremely detailed derivation of the turning-moment $Cr\vartheta'$ discussed above (p. 311) with our determination of this turning-moment, which in any case leaves nothing to be desired in simplicity.[148])

5. Explanations†††) that would derive the experimentally observed elevation of the axis of the top from the principles of abstract dynamics

*) For example, Müller-Pouillet, Bd. I, §74.[145]

**) Über die Bewegung des Kreisels. Ztschr. f. d. phys. u. chem. Unterricht, 4. Jahrg., 1890.

***) Page 190.

†) Braunschweig 1857.

††) Zur Kreiselbewegung, Ztschr. f. d. phys. u. chem. Unterricht, 9. Jahrg., 1896.

†††) Cf., for example, Munter, Ztschr. f. d. mathem. u. phys. Unterricht, Bd. 26, p. 565.[149]

must be held as particularly mistaken. We know that the presently considered ideal, frictionless top can in no way elevate itself, but always maintains the same mean inclination to the vertical. The elevation of the axis results, if at all, only through friction at the support point, which we will later cover in detail.

§4. On the stability of the upright top. Geometric discussion.

A particular case of regular precession is that in which the trajectory contracts to a single point, the highest or lowest point of the unit sphere. The top then rotates with uniform velocity about the vertically positioned figure axis. We wish to treat of this interesting motion in detail in order to form our concept of the stability of motion, and thus prepare the more general investigations of the sixth section.

The figure axis can be directed vertically upward as well as vertically downward in this motion. We will restrict ourselves to the former case, which can be done without loss of generality if only we interchange, if necessary, the half-line designated as the figure axis with its opposite.

We will therefore reckon $\vartheta = 0$, and have two subcases to distinguish, according to whether $P < 0$ or $P > 0$. We speak here of a spherical top with moment of inertia A.

We designate the considered motion concisely as the *motion of the upright top.*

We first note that our coordinates φ, ψ, ϑ are inappropriate for the present case. Namely, the line of nodes in the equatorial plane is obviously undetermined for the upright figure axis. The angles φ and ψ (the angles of the X- and x-axes, respectively, with respect to the line of nodes) thus have no independent meaning. The angle $\varphi+\psi = \chi$ that directly represents the angle between the X- and x-axes, and therefore measures the rotation of the top with respect to space, is, however, well defined. With the use of this coordinate, our motion is simply characterized by the two equations

$$(1) \qquad \vartheta = 0, \quad \chi' = \text{const.}$$

We now easily convince ourselves *that the motion of the upright top is possible and compatible with the fundamental impulse laws for an arbitrary value of the rotational velocity* χ'. For the motion represented by (1), namely, the impulse always coincides with the vertical and has a constant length. The change of the impulse vector is therefore zero at all times. In addition, the effect of gravity $P \sin \vartheta$ is always zero for the vertically positioned figure axis. The equilibrium that is required by our impulse theorem between the change of the impulse and the additional impulse of the external force thus obtains. Equations (1), in fact, represent a possible motion of the top, whatever value the rotational velocity χ' may have.

The relation

$$n = N \tag{2}$$

is evidently valid for the upright top. Since, namely, the figure axis and the vertical always coincide, the vertical projection of the impulse is directly identical with the projection onto the figure axis, as well as with the length of the impulse vector.

The condition (2), moreover, is characteristic not only for the uniform rotation of the upright top, but also, more generally, for each trajectory that passes through the highest point of the sphere. In fact, the projection of the impulse onto the figure axis will always be identical with the projection onto the vertical in a passage through the north pole, since the two directions coincide at such a moment. Since, moreover, the impulse components n and N are, as we know, constant, the given relation must hold generally for such motions.

We now go over to the stability question, and impart, for this purpose, an impact to the top during its rotation about the vertical. We characterize the impact by the corresponding turning-impact with respect to the support point, and represent it by a vector. It is assumed that the length of this vector does not exceed an arbitrarily given quantity. For the sake of generality, we might first assume nothing about the direction of the turning-impact vector. However, it is apparent that we may take this direction as horizontal. If, namely, an obliquely directed turning-impact vector is present, we may decompose it into a vertical and a horizontal component. The vertical component merely effects a change of the rotational velocity of the top, and leaves the character of the motion unchanged. We can thus disregard the vertical component and assume a turning-impact about a horizontal axis.

The originally present impulse is naturally composed with this turning-impact according to the parallelogram law. We denote the additional impulse by $[\Theta_0]$, since it gives us, at the same time, the perpendicular components of the total impulse with respect to the figure axis and the vertical at time $t = 0$. The relation between our impulse component $[\Theta_0]$ and the rotation component generated by it thus becomes, according to the general dependence between the impulse and the rotation vectors,

$$[\Theta_0] = A\vartheta_0', \tag{3}$$

where ϑ_0' signifies the initial value of the angular velocity ϑ'. The length $|i|$ of the total impulse, which for the undisturbed motion was constant and equal to N, now becomes variable. In particular, the initial value $|i_0|$ is given, according to Pythagoras, by the equation

$$|i_0|^2 = N^2 + [\Theta_0]^2.$$

Finally, we calculate the impulse constant k by means of equation (3) on page 219. Since $\vartheta = 0$ for $t = 0$, there follows

$$k = |i_0|^2 + 2AP = N^2 + [\Theta_0]^2 + 2AP. \tag{4}$$

In order to survey the character of the motion produced by our impact, we ask, above all, how deeply the apex of the top descends on the unit sphere. We seek, therefore, the manner in which the root e' of the equation $U = 0$ depends on $[\Theta_0]$. We must return to the original form of this equation on page 238, since the derived equation $U_1 = 0$ is now inutile because of the prefixed factor $1 : (1 - e^2)$ that becomes infinitely large in our case ($e = 1$). We have, according to the indicated place, with consideration of the condition $n = N$,

$$A^2U = -N^2(1-u)^2 + (k - N^2 - 2APu)(1-u^2), \tag{5}$$

or, if we express k corresponding to equation (4),

$$A^2U = -N^2(1-u)^2 + \big([\Theta_0]^2 + 2AP(1-u)\big)(1-u^2). \tag{5'}$$

The factor $1 - u$ that corresponds to the known root $e = 1$ stands, as it must, on the right-hand side. We detach this factor, and obtain for the two remaining roots the quadratic equation

$$-N^2(1-u) + \big([\Theta_0]^2 + 2AP(1-u)\big)(1+u) = 0.$$

Here we may set $[\Theta_0] = v$, and interpret the resulting equation

$$v^2(1+u) - (1-u)\big(N^2 - 2AP(1+u)\big) = 0 \tag{6}$$

geometrically in the u, v plane.

The resulting curve is again of the third order. It lies symmetrically with respect to the axis of the abscissa (u-axis), and consists of an even and an odd branch. We seek the form of the curve on the basis of its vertical tangents, which are particularly easy to determine here.

We summarize the equations of the vertical tangents, as well as the positions of their tangent points, as follows:

Equation:	$u = +1$	Tangent point:	$v = 0$
„	$u = -1$	„	$v = \infty$
„	$u = -1 + \dfrac{N^2}{2AP}$	„	$v = 0.$

The second of these tangents is thus an asymptote. It is essential for us how the third tangent lies with respect to the others. We distinguish, in this respect, two cases, according to whether $P < 0$ or $P > 0$.

First case: $P < 0$.

The third tangent lies to the left of the asymptote. Real values of v result for the domains

$$-1 < u < +1 \text{ and } -\infty < u < -1 + \frac{N^2}{2AP}.$$

The odd branch runs in the strip between $u = -1$ and $u = +1$, and the even branch extends from the tangent $u = -1 + \dfrac{N^2}{2AP}$ toward the left to infinity (cf. Fig. 52). We draw the parallel $v = [\Theta_0]$ in an arbitrary neighborhood of the abscissa axis, and note the intersection of the parallel with the odd branch. The abscissa of this intersection point yields the parallel circle $u = e'$. As one sees, the quantity e' always comes nearer to unity as we take $[\Theta_0]$ smaller. (Thus, as we note for the sake of the following, the difference $1 - e'$ will be of the order $[\Theta_0]^2$; it represents "an infinitesimal quantity of the second order" if we let the impact $[\Theta_0]$ be "an infinitesimal of the first order.") As a result, we can attain, by the choice of $[\Theta_0]$, that the trajectory

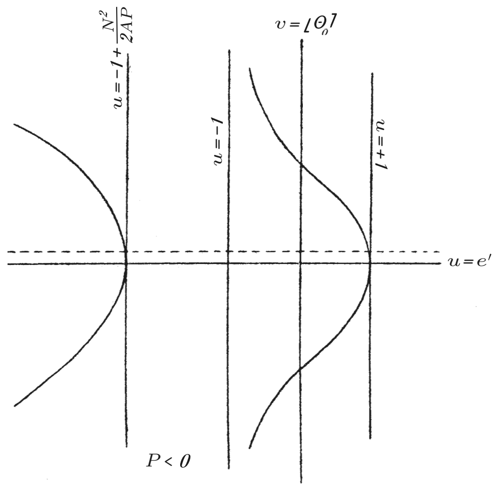

Fig. 52.

of the originally upright top runs, after the addition of our impact, in an arbitrarily small neighborhood of the original trajectory, the north pole. *With respect to the trajectory, there certainly exists, in the case* $P < 0$*, a continuous passage from the original to the altered motion.*

The trajectory of the apex of the top does not, as we know, completely express the motion, in that it gives no information concerning the rotation of the top about the figure axis.

For the undisturbed motion, this rotation is measured by the angle χ; since for the upright motion the angular velocity χ' is expressed in terms of the impulse N as

$$\chi' = \frac{N}{A},$$

we have (for a special choice of the initial time)

$$\chi = \frac{N}{A} t.$$

We wish to measure the rotation after the addition of the impact $[\Theta_0]$, in order to have a comparable quantity, by the corresponding angle χ. We thus have, with consideration of (2),

$$\chi' = \varphi' + \psi' = \frac{n - Nu}{A(1-u^2)} + \frac{N - nu}{A(1-u^2)} = \frac{2N}{A(1+u)}.$$

Now since u remains arbitrarily near 1, we expand in powers of $u-1$ and neglect all higher powers. There follows

$$\chi' = \frac{N}{A} - \frac{N}{A}\frac{u-1}{2}.$$

The second term on the right-hand side is, as emphasized in one of the remarks above, an infinitesimal of the second order in relation to the impact $[\Theta_0]$. One is not required, however, to retain such quantities according to the usual treatment of stability considerations in the literature. For the neighboring motions of regular precession in the first section, we also have suppressed the terms of the second order (terms with the factor ε^2), in that we neglected the remainder R in the expression for ψ. If we also restrict ourselves now to terms of the first order, then we will again be led back, if we carry out the integration with respect to t, to the original formula

$$\chi = \frac{N}{A} t.$$

(It is noted, moreover, that the not unobjectionable neglect of the terms of the second order is only provisional, and that it does not at all come into question in the definitive conception of the stability criterion to be developed in §6.)

Our consideration shows that the angle χ of the changed motion always remains, up to terms of the second order, in the neighborhood of the angle χ of the original motion. This statement refers, however, only to the case in which the added turning-impact has a purely horizontal axis. For a general impact, in which not only a certain value of $[\Theta_0]$ is added but the original impulse component N is also changed, the situation is naturally different.

We consider, for example, the simplest case in which $[\Theta_0] = 0$ is assumed and N is increased by the small quantity N'. The motion then remains, after as before, that of the upright top. The angle χ for the altered motion is determined by the equation

$$\chi = \frac{N + N'}{A} t.$$

As one sees, this deviates from the angle χ of the undisturbed motion by a term that is proportional to the first power of the impact N'. We can nevertheless interpolate between the original and the altered motions, by the diminishment of N', motions that mediate a continuous passage from the angle χ of the one to the other motion.

After all this, we will without doubt, and indeed independently of whether we allow the neglect of the terms of the second order or not, be able to say that

The motion of the upright top in the case $P < 0$ is certainly a stable motion.

This result is naturally in complete conformity with the well-known fact that the equilibrium position of the unwound top ($N = 0$) is stable if the center of gravity lies beneath the support point ($P < 0$).

Second case: $P > 0$.

Much more interesting is the second case $P > 0$. The third of the vertical tangents given on page 319 now lies to the right of the asymptote $u = -1$. According to whether this tangent lies to the right or the left of the tangent $u = +1$, there arise two subcases a) and b).

The first subcase occurs if

$$\text{(a)} \qquad +1 < -1 + \frac{N^2}{2AP}; \;\; \text{that is, } N^2 > 4AP;$$

the second if

$$\text{(b)} \qquad +1 > -1 + \frac{N^2}{2AP}; \;\; \text{that is, } N^2 < 4AP.$$

We compare these conditions to the previous distinction between the weak and the strong top. Since we have assumed $P > 0$, we must draw upon the criterion $(3')$ of page 249.

If we insert there the value $e = +1$ corresponding to the upright initial position, then the inequalities of that criterion are transformed directly into the inequalities (a) and (b). *The subcase* (a) *therefore corresponds to a strong top, and the subcase* (b) *to a weak top.* (It is noted, in passing, that the case of a negative P is always to be reckoned as a *strong* top, since the inequality (a) will then be satisfied in a self-evident manner.)

We next display again our curves of the third order that correspond to equation (5) for our two cases $P > 0$. For the strong top of case (a), the odd branch passes through the strip -1 to $+1$, in that it is tangent to $u = -1$ at infinity and to $u = +1$ on the axis of the abscissa. For the weak top, in contrast, the odd branch is enclosed in the strip from -1 to $-1 + \dfrac{N^2}{2AP}$. The even branch lies in both cases to the right of the odd, and touches either the tangent $-1 + \dfrac{N^2}{2AP}$ or the tangent $+1$ at its intersection point with the axis of the abscissa.

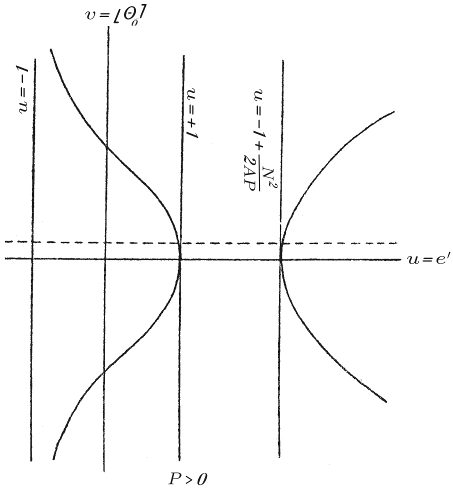

Fig. 53. Strong Top.

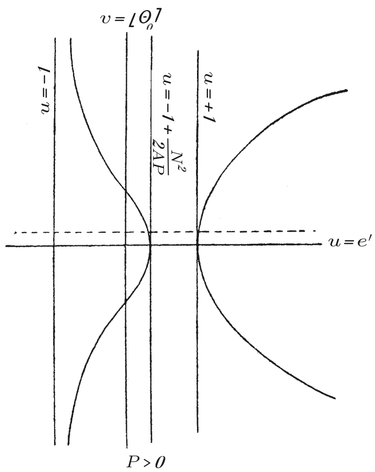

Fig. 54. Weak Top.

The character of the trajectory produced by a disturbance $[\Theta_0]$ now depends on the form of these curves.*) We again draw the line $v = [\Theta_0]$ parallel to the axis of the abscissa; the abscissa of the intersection point of this line with the odd branch determines the size of the second

*) Cf. F. Klein, On the stability of the sleeping top. American Bulletin, 1896.[150]

parallel circle. There now results a very interesting contrast between the strong and the weak top.

In Fig. 53, namely, the cited intersection point lies in the immediate neighborhood of the point $u = 1$, $v = 0$.

For the strong top, the parallel circle e' always becomes smaller as the impact $[\Theta_0]$ is chosen smaller, and is transformed continuously, for vanishing $[\Theta_0]$, into the north pole $e = 1$. The trajectory of the apex of the top remains in the immediate neighborhood of the original point-shaped trajectory; its dimensions can, through the diminishment of the impact, be arbitrarily diminished.

In Fig. 54, in contrast, the abscissa of the constructed intersection point always differs from unity by a finite quantity that cannot be suppressed below $1 + 1 - \dfrac{N^2}{2AP}$.

For the weak top, the position of the second parallel circle changes in a discontinuous manner. For the smallest disturbance, this parallel circle, which is reduced to the north pole for the undisturbed motion, jumps immediately onto a circle for which e' is smaller than $-1+\dfrac{N^2}{2AP}$. The dimensions of the trajectory cannot be arbitrarily diminished by the diminishment of $[\Theta_0]$. The upright rotation of the weak top therefore occupies an isolated position in the system of the trajectories.

The latter remarks already show that the weak top is unstable in the upright position. For what concerns the strong top in the case $P > 0$, we can employ the same deliberations as above in the case $P < 0$. We thus state the following general theorem:

The strong top is stable in the upright position, the weak top labile.

We must evidently reckon the boundary case between the strong and weak tops—that is, the top with $N^2 = 4AP$—as a stable case. We can directly measure, namely, the impact $[\Theta_0]$ so small that e' differs arbitrarily little from

$$-1 + \frac{N^2}{2AP} = -1 + 2 = +1.$$

We thus add:

The top that stands on the boundary between the strong and the weak tops is likewise stable in the upright position.

Moreover, the trajectory in the stable as well as the unstable cases has the form of a *rosette* that progresses at regular time intervals (because of the general periodicity property of the motion) through the north pole of the sphere and consists purely of congruent loops. The difference between the two cases is revealed only in the *magnitude of the rosette*, or, more precisely said, in the *change of its magnitude for a diminishment of the impact.* The rosette in the labile case is a no less regular, periodically recurring curve than that of the stable case; it is indeed no different to the eye, under particular circumstances to be indicated immediately, from the rosette of the stable case. We particularly emphasize this point, since many false representations may be promulgated here.

The English designate the upright motion of the top in the most intuitive manner as the motion of the "sleeping top." If, as we wish to assume, not only mechanical but also geometric rotational symmetry about the figure axis is present, then the top appears to the eye to be at rest in the upright position. That this rest, however, is only apparent, is shown if the top is awakened to a certain extent by an impact. Its originally hidden motion will then become apparent externally. We judge the stability or lability according to its behavior in awakening. If the awakening is gentle, we call the upright motion stable; if, in contrast, the least disturbance produces disproportionally large elongations, the motion is called labile.

That stability is generally possible in the upright position for the case $P > 0$ represents a fact of peculiar interest, which at first may again appear paradoxical. While the nonrotating top in the upright position is naturally entirely unstable in the case $P > 0$ and reacts to the smallest impact with a full pendulum oscillation, it will, set into sufficiently strong ($N^2 > 4AP$) rotation, be enabled to afford a certain degree of resistance to the influence of gravity. The weak top thus mediates the passage between the top with zero eigenrotation and the strong top.

The position of the parallel circle e' to which the figure axis most deeply descends for any impact can be regarded as a measure of the greater or lesser weakness of the top. The position of the parallel circle e' that corresponds to the impact $[\Theta_0] = 0$ is, as we saw, given by

$$e' = -1 + \frac{N^2}{2AP}.$$

For the top of zero eigenrotation, this value will equal -1; the figure axis then describes, as just mentioned, a great circle starting from the north pole that passes through the south pole. The generally appearing rosette is degenerate here. For increasing N, the value of e' increases continuously and approaches, for $N^2 = 4AP$, the value $e = +1$.

The passage between the labile and stable cases of the upright top motion is itself, in this manner, continuous to a certain extent. If, namely, N^2 is smaller than $4AP$ but still differs very slightly from $4AP$, the elongation of the trajectory for an arbitrarily small impact will indeed be different from zero but still inconsiderable; it can even, through the appropriate assumption of N, be reduced beneath any given value. Nevertheless, there remains the characteristic property of the labile case that, after we have once disposed of N, the dimensions of the trajectory cannot be arbitrarily diminished through the diminishment of $[\Theta_0]$.

Theoretically, the motion in this case (where $\dfrac{4AP}{N^2} - 1$ is a negative number with a small absolute value) is always labile; experimentally, in contrast, such a motion would not differ markedly from a theoretically stable motion. In both cases we have a rosette of qualitatively similar course and extraordinarily small dimensions. We may thus, for example, speak of *theoretical and practical lability and stability*, and say that *in the case where N^2 is only very little smaller than $4AP$, the motion is theoretically labile, but practically still always stable.*

In the sixth section of this chapter we will become acquainted with other simpler examples of theoretical lability and practical stability, as well as theoretical stability and practical lability. —

We may ask in general, finally, for such motions of the heavy *symmetric* top that consist of a simple rotation about an axis fixed in space.

It follows on the basis of symmetry that such an axis can be none other than the vertical, and that the rotation about this axis must proceed uniformly. The motion then belongs to the class of regular precession, and indeed is a regular precession for which the herpolhode cone is infinitely thin, and for which, therefore, μ has the value zero. The other precession constant ν, which here directly indicates the magnitude of the angular velocity, is thus determined. The theory of the deviation resistance, namely (see equation (3) of page 77), gives for ν the equation

$$P = (C - A)\nu^2 \cos\vartheta. \tag{15}$$

Only in the case $\vartheta = 0$ does this equation lose its validity, since we have canceled the factor $\sin\vartheta$ from the equation that originally appears in the indicated place, so that this equation is identically satisfied in the case $\vartheta = 0$. *Thus the upright figure axis is the single line about which the body can turn permanently with an arbitrary velocity. Every other axis requires, in case it should appear as a permanent axis, a determined (up to the sign) value of the angular velocity.*

According to whether this value is real or imaginary, we will designate the corresponding axis as a "permissible" or an "impermissible rotation axis." To reach a decision here, we wish to divide the entire bundle of the half-rays extended from O into two half-bundles by means of the equatorial plane of the top. If we imagine that the figure axis is chosen so that the center of gravity lies above the support point for the vertically directed figure axis ($P > 0$), we designate the half-bundle that contains the center of gravity as the upper, and the other as the lower. Equation (15) then shows that

For the prolate top ($C < A$) *all half-rays of the lower half-bundle, and for the oblate top all half-rays of the upper, are permissible permanent rotation axes. Each of these axes receives two oppositely equal values of the angular velocity. In particular, the angular velocities for the spherical top are always* $\pm\infty$.

§5. Continuation. Analytic treatment of the motion of the upright top altered by an impact.—Formulas for pseudoregular precession with small precession circle.

We will now supplement the qualitative discussion of the motion of the upright top in the preceding section by a detailed quantitative discussion, with the view of acquiring a precise basis for our later criticism of the method of small oscillations, a method that plays a well-known important role in modern dynamics. We begin from the approximation formulas of the ninth section of the preceding chapter. The approximation formulas there for u can be carried over directly to the present case. In contrast, the formulas for ψ require a modification, since the term $1 - u_0^2$ appears in the denominator in those formulas; this term now vanishes, at least in the stable cases, for a vanishing impact.

We first write the approximation formula for u in the present case. To remain in consonance with the notation of the named §9, we understand by e the lower, and by e' the upper of the two bounding parallel circles, so that $e' = 1$. The quantity ε in our approximation formula signifies half the vertical distance between the circles e and e', so that now

$$\varepsilon = \frac{1-e}{2}; \tag{1}$$

the quantity u_0 was formerly the "mean parallel circle" whose plane had equal vertical distance ε from the planes of the parallel circles e and e', so that in the present case there follows

$$u_0 = 1 - \varepsilon. \tag{2}$$

We can thus write equation ($8'$) of page 227, with the use of the abbreviation introduced in (7) of page 272, as

$$1 - u = \frac{1-e}{2}\left(1 - \cos\frac{\pi t}{\omega}\right); \quad \omega = \sqrt{\frac{A}{2P(e'' - u_0)}}\,\pi. \tag{3}$$

We have adopted, at the same time, a convenient displacement of the initial point of time, in that we have used the cosine instead of the sine, which has the consequence that the figure axis is vertical at the beginning of the motion ($u = 1$ for $t = 0$).

We introduce in (3), instead of u and e, the corresponding angles

$$u = \cos\vartheta, \quad e = \cos\eta,$$

and go over from the whole to the half-angles. There follows, if we take the square root of the right- and left-hand sides,

$$\sin\frac{\vartheta}{2} = \sin\frac{\eta}{2}\sin\frac{\pi t}{2\omega}. \tag{4}$$

The uncertainty in this formula is determined by the magnitude τ of the relative error in t. According to equation (5) of page 272,

$$|\tau| < \sqrt{\frac{e'' - e}{e'' - e'}} - 1, \tag{5}$$

where, according to (1), we set $e = 1 - 2\varepsilon$, $e' = 1$, and calculate e'' from equation (9) of page 273. Since $n = N$ for the upright motion, we can cancel the factor $2(1-e)$ from the numerator and denominator of the named equation, and obtain

$$e'' = \frac{N^2}{AP(1+e)} - 1 = \frac{N^2 - 2AP(1-\varepsilon)}{2AP(1-\varepsilon)}. \tag{6}$$

As a result, there follows

$$|\tau| < \sqrt{\frac{N^2 - 4AP(1-\varepsilon)^2}{N^2 - 4AP(1-\varepsilon)}} - 1. \tag{$5'$}$$

This estimation provides, in every case, a basis for the determination of the degree of precision of equation (4).

We ask, in particular, for those cases in which the bounded precision of equation (4) goes over into an arbitrary precision, and in which, therefore, $|\tau|$ can be made arbitrarily small. This obviously occurs, to speak generally, *in the stable cases*, where we can attain ε arbitrarily small through the choice of the impact. The value of the square root in ($5'$) then differs arbitrarily little from 1.

In the labile cases, in contrast, the magnitude of ε is not in our power. The right-hand side of ($5'$) will then be, even for an arbitrarily small impact, a quantity different from zero; there exists no basis for the assumption that our representation (4) would also be arbitrarily precise in this case.

We must next consider, in particular, the *boundary case* ($N^2 = 4AP$) *between the stable and unstable cases.* We saw that this case must generally be ordered under the stable cases with respect to the behavior of the trajectory, since the motion of the apex of the top runs, for a sufficiently small impact, in the immediate vicinity of the north pole. It now appears, in contrast, that this boundary case stands on the side of the labile cases with respect to the degree of precision of our approximation formulas. If we set, namely, $N^2 = 4AP$ in ($5'$), then the right-hand side becomes

$$\sqrt{\frac{2\varepsilon - \varepsilon^2}{\varepsilon}} - 1;$$

that is, if we make ε sufficiently small,

$$\sqrt{2} - 1.$$

We therefore have, in spite of the stable character and in spite of the possibility of an arbitrary diminishment of the trajectory, a case before us in which we can expect only a bounded precision of our approximation formula.

The situation is similar in the *theoretically labile but practically stable cases*, where $N^2 - 4AP$ is indeed smaller than zero, but differs only extremely little from zero, and where, at the same time, ε is also extremely small for a sufficiently small impact. In these cases as well, the smallness of ε is not sufficient to reduce the magnitude of the error arbitrarily.

It would be necessary, in the latter cases, that ε be not simply small, but rather also small compared with $(4AP - N^2)/4AP$, which is not the case, since, as we saw on page 323, the difference 2ε of the values of e and e' is at least equal to $\dfrac{4AP - N^2}{2AP}$. Thus we must expect in this case, just as in the theoretically and practically labile cases, only bounded precision.

A related remark applies to those stable cases in which $N^2 - 4AP$ is indeed greater than zero, but differs only very slightly from zero. In these cases it is certainly possible to arbitrarily diminish the dimensions of the trajectory and the value of ε by the choice of the impact; it is therefore possible to make ε not only small, but also small with respect to $(N^2 - 4AP)/4AP$. Our approximation formula could be held as arbitrarily precise for *so* small an impact. As soon, however, as we take the impact only a little greater, so that ε is no longer small compared with $(N^2 - 4AP)/4AP$, the error can immediately increase considerably. As a result, the precision of our formula for all not very small impacts would still be only bounded. We will, in this case, *speak of a theoretically arbitrary, but practically bounded approximation.*

We wish to formulate the latter somewhat subtle distinctions once again in summary:

Our approximation formula possesses a theoretically and practically bounded precision in all unstable cases, and in those stable cases that are found on the boundary between stability and lability. It possesses an arbitrary precision in theoretical and practical respects in those stable cases that are sufficiently far removed from the unstable cases. In contrast, we have theoretically arbitrary, but practically bounded precision in those cases that are indeed stable, but lie near the boundary of lability.

We next derive the approximation formula for ψ. We thus prefer, instead of relying on the general formulas of page 275 and ff., to begin the investigation anew, since it can be led further and formed more simply in the present case than was possible in general.

We therefore begin from the equation

$$\psi' = \frac{n - Nu}{A(1 - u^2)}.$$

Since $n = N$ for the upright motion, we can cancel the factor $1 - u$ on the right-hand side. If we then apply an identity transformation on the right-hand side, we obtain

$$\psi' = \frac{N}{A}\frac{1}{1 + u} = \frac{N}{2A}\left(1 + \frac{1 - u}{1 + u}\right).$$

To go over to an approximate representation, we restrict ourselves to the first term in the parentheses. The second term then provides the required error estimation. Our approximate representation therefore runs, if we immediately carry out the integration with respect to t and set the inessential constant of integration equal to zero,

$$\psi = \frac{N}{2A}t. \tag{7}$$

The error is given exactly by

$$f = \int \frac{N}{2A}\frac{1-u}{1+u}\,dt.$$

We can assume $N > 0$, in which case the integrand is always positive. If we insert for u its smallest value $e = 1 - 2\varepsilon$, we simultaneously diminish the denominator and enlarge the numerator of the integrand. As as result, there certainly follows

$$f < \int \frac{N}{2A}\frac{2\varepsilon}{2-2\varepsilon}\,dt;$$

that is,

$$f < \frac{N}{2A}\frac{\varepsilon}{1-\varepsilon}t. \tag{8}$$

We have thus determined an upper bound for the absolute error of the approximation formula (7). The relative error $\frac{f}{\psi}$ will, correspondingly, be smaller than

$$\frac{\varepsilon}{1-\varepsilon}. \tag{8'}$$

The discussion of the degree of precision of equation (7) is just as simple as the error estimation. The precision will obviously be arbitrary if we can attain that ε will be arbitrarily small; it will presumably be bounded if we cannot arbitrarily diminish ε, or, equivalently, the dimensions of the trajectory. We must therefore say:

Equation (7) *gives an arbitrarily good approximation in all stable cases (with inclusion of the boundary case between stability and lability), as well as in the practically stable and theoretically labile cases; it gives, in contrast, only a bounded approximation in the cases of actual (practical) lability.*

One notes that the results here are essentially different from those in the above investigation of the degree of precision of equation (4). —

After this preparatory discussion of the approximation formulas, we investigate the character of the various motions, as given on the basis of our approximation formulas. We first look more closely at the cases that are represented arbitrarily well by (4) and (7); that is, the stable cases that are, for sufficient smallness of the impact $[\Theta_0]$, sufficiently far removed from the boundary of lability.

In these cases we can obviously replace, since the dimensions of the trajectory are indeed vanishingly small, $\sin\frac{\eta}{2}$ and $\sin\frac{\vartheta}{2}$ by $\frac{\eta}{2}$ and $\frac{\vartheta}{2}$ in (4). At the same time, we wish to simplify the value of ω in equation (3), in that we set, in an approximate manner, $\varepsilon = 0$, $u_0 = 1$. There then follow from (6) and (3)

$$e'' = \frac{N^2 - 2AP}{2AP}, \quad \omega = \frac{A\pi}{\sqrt{N^2 - 4AP}}. \tag{9}$$

The equations for the trajectory then take the form

$$\begin{cases} \vartheta = \eta \sin\left\{\sqrt{\dfrac{N^2 - 4AP}{4A^2}}t\right\}, \\ \psi = \dfrac{N}{2A}\, t. \end{cases} \tag{10}$$

For presentation and drawing, it is convenient to project the trajectory onto the equatorial plane. Here we can use an orthogonal projection (the same image would result for the stereographic projection, only in half scale). If x and y denote the rectangular coordinates of the projection point with respect to a coordinate frame placed at the midpoint of the unit sphere, then we have

$$\begin{cases} x = \sin\vartheta\cos\psi = \eta\sin\left\{\sqrt{\dfrac{N^2 - 4AP}{4A^2}}t\right\} \cdot \cos\dfrac{N}{2A}\, t, \\ y = \sin\vartheta\sin\psi = \eta\sin\left\{\sqrt{\dfrac{N^2 - 4AP}{4A^2}}t\right\} \cdot \sin\dfrac{N}{2A}\, t. \end{cases} \tag{11}$$

We can describe the represented motion in words in the following manner:

The motion of the horizontal projection of the apex of the top consists of an ordinary harmonic oscillation (represented by the first factors in x and y) with amplitude η and quarter oscillation period ω, combined with a rotation of the oscillation direction (represented by the second factors) with angular velocity $N\!:\!2A$.

The form of the trajectory is determined essentially by the angle through which the azimuth ψ increases, for example, during the time ω. We denote this angle, as earlier, by ψ_ω, and have, according to (7) and (9),

$$\psi_\omega = \frac{\pi}{2}\frac{N}{\sqrt{N^2 - 4AP}}. \tag{12}$$

Evidently,

$$\psi_\omega > \frac{\pi}{2} \quad \text{in the case} \quad P > 0,$$

$$\psi_\omega < \frac{\pi}{2} \quad \text{in the case} \quad P < 0.$$

In the boundary case $P = 0$, ψ_ω will be directly equal to $\frac{\pi}{2}$. The trajectory of the apex of the top then becomes simply a circle, the precession circle of force-free motion; the axis of precession is that of the initial impulse N altered by the impact $[\Theta_0]$. The latter determination of the trajectory is also valid, moreover, for the case $N = \infty$, as follows from the already repeatedly used principle that a top with an infinitely large eigenimpulse and a finite gravity moment P behaves just as a weightless top with a finite N.

The form of the trajectories is thus easy to envision in all the stable cases for which our two approximation formulas are arbitrarily precise. The following characteristic figures, which one will recover for all analogous oscillation processes, illustrate the three types $P > 0$, $P = 0$ or $N = \infty$, and $P < 0$. The trajectory in Fig. 55 closes on itself, accidentally, almost completely; Fig. 56 represents the passage between Figs. 55 and 57.

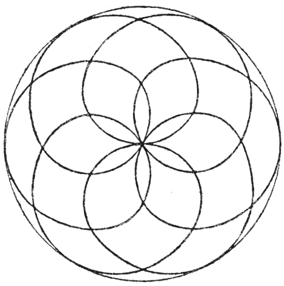

Fig. 55.

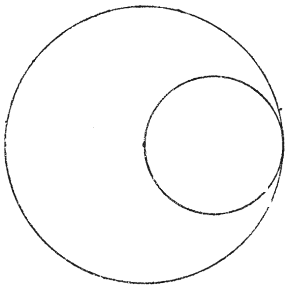

Fig. 56.

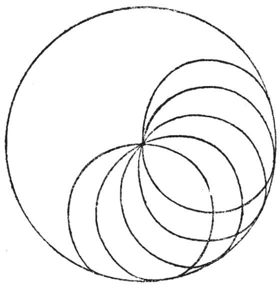

Fig. 57.

As we see, our oscillations in the stable case are tautochronous; that is, their time duration is, in the first approximation, independent of the magnitude of the impact $[\Theta_0]$ and the resulting magnitude of the amplitude η, assuming that both quantities are taken as sufficiently small. Our oscillations thus exhibit the same behavior that is well known for the so-called small pendulum oscillations, and that is characteristic of so-called "small oscillations" in general.

The small pendulum oscillations must naturally fall under the oscillations of the upright top as a special case, and indeed are classified under a top of vanishing moment of inertia C. Since the pendulum is stable in the vertical position only if the mass particle lies beneath the support point, we must assume, in the preceding, $P < 0$. And indeed we have, understanding by m the oscillating mass and l the length of the pendulum,

$$P = -mgl, \quad A = ml^2.$$

Formula (9) thus yields, since $N = Cr = 0$,

$$4\omega = 2\pi\sqrt{\frac{A}{-P}} = 2\pi\sqrt{\frac{l}{g}};$$

that is, the well-known equation for the period of the complete pendulum oscillation.

We now turn to the unstable cases and to the theoretically stable but practically unstable cases. The preceding approximation formulas can also be of use for the judgment of these cases; we must only bear in mind that the precision of this representation is no longer arbitrary, and must estimate the magnitude of the possible error according to the inequalities ($5'$) and ($8'$) in each case. In any case, we may conclude from this representation (although burdened with finite error) that the qualitative character of the trajectory will be generally similar to that of the stable cases; the dimensions, according to the degree of lability, are only enlarged, and, in detail, quantitative deviations from the simple sine law occur. By and large, the trajectories in the labile case will also be, as already emphasized on page 324, rosettes of a form similar to the preceding figures. If we wish, in contrast, to calculate the trajectories in the labile cases with arbitrary precision, then we must obviously revert to the elliptic integrals.

We must naturally consider not the simplified approximation formula (10) for the stable top, but rather the general formula (4). In particular, it would be a gross error if we were to take the oscillation period ω from equation (9) instead of equation (3). The latter equation will yield, in the labile cases, a reasonable result that is more usable as the error τ becomes smaller. Equation (9), in contrast, yields an entirely senseless result. It would give, namely, an imaginary value of ω for a labile top, since $N^2 - 4AP < 0$. It would obviously be a violent distortion of the true state of affairs if one would conclude from this calculation of ω that the motion of the unstable top proceeds aperiodically, and that the apex of the top is correspondingly removed ever more from the north pole. Nevertheless, one often finds this entirely false conclusion. What is still more remarkable, however, is to elevate this distortion to a principle, a very fruitful instrument, as we will later see, for acquiring a preliminary judgment of the stability of motion; one calls this procedure the method of small oscillations!

The motion of the upright weak top in the limiting case $[\Theta_0] = 0$ may claim an entirely special interest. We wish to imagine that we impart to the initially upright figure axis a series of gradually decreasing impacts, and wish to investigate the limit that the trajectory approaches for $[\Theta_0] = 0$. This motion will play a principal role, as we already remark now, in the general investigations of the following section.

We first wish to ask ourselves to what extent we can accurately describe this motion through our approximation formulas. We therefore determine the error τ for the present case, and must form, for this purpose, a judgment of the positions of the roots e, e', and e''. The smallest root e will be given by the parallel circle to which the apex of the top most deeply descends in the limit of a vanishing additional impact. This parallel circle is, according to Fig. 54 of page 332, the circle $u = e = -1 + \dfrac{N^2}{2AP}$. The next largest root will be, because of the upright position, $e' = 1$. In order to find the third root e'', we return to the expression for U in equation (5′) (cf. page 318). If we set $[\Theta_0] = 0$, there follows

$$(13) \qquad \begin{cases} A^2U = -N^2(1-u)^2 + 2AP(1-u)(1-u^2) \\ = 2AP(u-e)(1-u)^2. \end{cases}$$

The third root therefore becomes identical with the second root in this special case; we have

$$e'' = e' = 1.$$

The inequality (5) then yields, however, because of the vanishing denominator, the value ∞ as the upper bound for the error τ! *In this special case of the unstable top motion (but also not only in this case), our approximate representation will be, in so far as it allows an estimation of the error, entirely useless, in that the given upper bound of the error can attain any arbitrarily large magnitude.*

Fortunately, however, the exact representation in this special case is so simple that we can easily dispense with our approximation formulas. It happens, namely, that the elliptic integrals degenerate into executable elementary integrals, corresponding to the circumstance that two of our branch points e' and e'' coalesce. According to equation (13), namely,

$$t = \int \frac{du}{\sqrt{U}} = \sqrt{\frac{A}{2P}} \int \frac{1}{1-u} \frac{du}{\sqrt{u-e}};$$

the integral on the right is expressed in terms of a logarithm as

$$\int \frac{1}{1-u} \frac{du}{\sqrt{u-e}} = \frac{1}{1-e} \lg \frac{\sqrt{1-e}+\sqrt{u-e}}{\sqrt{1-e}-\sqrt{u-e}}.$$

There follows, with consideration of the value given above for e,

$$t = \frac{A}{\sqrt{4AP-N^2}} \lg \frac{\sqrt{1-e}+\sqrt{u-e}}{\sqrt{1-e}-\sqrt{u-e}}. \tag{14}$$

The suppression of the constant of integration in this formula amounts to a special stipulation of the initial time. Since the right-hand side vanishes for $u = e$, the initial time $t = 0$ signifies the moment at which the apex of the top passes its lowest point. The time in equation (14) (and indeed with good reason) is therefore calculated not, as previously, from the occurrence of the disturbance in the upright position, but rather from the lowest position.

The impact $[\Theta_0] = 0$ corresponds to the initial velocity $\vartheta' = 0$ of the apex of the top at the north pole (we indeed have, in general, $[\Theta] = A\vartheta'$). *In conformity, the time during which the apex of the top descends from the north pole to the parallel circle e becomes infinitely large.* We have previously denoted this time by ω. In fact, equation (14) yields for this time, or for the time measured in the reversed sense during which u increases from e to 1, the value

$$\omega = \infty.$$

The explicit representation of the trajectory further demands the calculation of the integral for ψ. We set, for this purpose, $n = N$ in the general formula

$$\psi = \int \frac{n - Nu}{A(1-u^2)} \frac{du}{\sqrt{U}},$$

and insert the value of U from equation (13). There follows

$$\psi = \frac{N}{\sqrt{2AP}} \int \frac{1}{1-u^2} \frac{du}{\sqrt{u-e}}.$$

The integration may once again be executed in an elementary way. We have, namely,

$$\int \frac{1}{1-u^2}\frac{du}{\sqrt{u-e}} = \frac{1}{2}\int \frac{1}{1-u}\frac{du}{\sqrt{u-e}} + \frac{1}{2}\int \frac{1}{1+u}\frac{du}{\sqrt{u-e}}.$$

The first integral on the right has already been given above; the second yields

$$\int \frac{1}{1+u}\frac{du}{\sqrt{u-e}} = \frac{2}{\sqrt{1+e}}\,\mathrm{arctg}\frac{\sqrt{u-e}}{\sqrt{1+e}}.$$

Thus

$$\psi = \frac{1}{2}\frac{N}{\sqrt{4AP-N^2}}\,\lg\frac{\sqrt{1-e}+\sqrt{u-e}}{\sqrt{1-e}-\sqrt{u-e}} + \mathrm{arctg}\,\frac{\sqrt{u-e}}{\sqrt{1+e}}. \tag{15}$$

Since the entire right-hand side vanishes for $u = e$, our formula directly gives the increase of the angle ψ as the apex of the top proceeds from the lower parallel circle e to the general position u.

Equation (15) yields the desired representation of the trajectory. We must direct our attention, in particular, to the first term of the right-hand side. This term continuously increases as u approaches 1, and becomes logarithmically infinite for $u = 1$. The second term is essentially irrelevant in relation to the first, since it remains finite for $u = 1$. The form of the trajectory is thus clear: *our curve winds continuously about the north pole, in that it always approaches the pole without ever attaining it.* We have before us, in essence, a *logarithmic spiral.*

In the case of the weak top, the undisturbed upright motion thus represents, as one says in association with Poincaré's investigations of celestial mechanics, an *asymptotic solution of the top problem*, since there is an ensemble of motions that approach it asymptotically.

If we follow the trajectory from the parallel circle e to the other side, then we obtain a mirrored branch equal to that just described, which likewise strives asymptotically to the north pole. In our case the complete trajectory therefore consists not, as in general, of infinitely many arcs, but rather of two equal mirror-image component arcs.

Nevertheless, our aperiodic spiral curve is continuously associated, in a certain sense, with the periodic trajectory corresponding to an impulse different from zero. We must imagine that for decreasing $[\Theta_0]$ the traversal time of the span width of each individual of the infinitely many component arcs becomes larger and larger, and that the number of times that the component arcs circumscribe the north pole simultaneously increases without bound. *From the other side—that is, from the side of the upright top itself—the passage from our spiral boundary curve to the unstable uniform rotation is naturally completely discontinuous.* In fact, the point-formed trajectory of the upright top motion jumps abruptly to the limiting spiral curve if we first perturb with an impact and then let this impact decrease to zero.

A special case of the usual pendulum motion is also included here. If we apply an impact to an unwound ($N = 0$) top—that is, a pendulum—with its center of gravity lying perpendicularly above the support point, then the apex swings from the highest to the lowest point of the sphere and describes a great circle. If we let the impact decrease more and more toward zero, the trajectory itself remains unchanged. The velocity at the highest point will only, in the limit, become zero, and the oscillation period infinite. Correspondingly, our formulas yield, in this case,

$$e = -1, \quad \psi = \text{const.}$$

The following Fig. 58 is drawn in orthographic projection for the particular values

$$A = P = 1, \quad N = \sqrt{2},$$

in which case $e = 0$ and the equation of the trajectory may be written as

$$\psi = \frac{1}{2} \lg \frac{1 + \sqrt{u}}{1 - \sqrt{u}} + \operatorname{arctg} \sqrt{u}.$$

A corresponding asymptotic motion for the force-free three-axis body has been known since the time of Poinsot (cf. the note on page 132). That the heavy symmetric top is capable of such a motion, however, appears not to have been noted in a characteristic manner until now.[151]

In association with the motion of the upright strong top, we supply an addendum to the earlier treatment of *pseudoregular precession* that remained unsettled for the case in which the trajectory of the apex of the top runs in the immediate vicinity of the north pole. We now take up this case; the parallel circles e and e' ($e < e'$) thus differ very little from one another and from 1. Furthermore, the characteristic conditions for pseudoregular precession obtain, so that the impulse falls nearly in the direction of the figure axis and has a considerable length (N^2 large compared with AP). We may presume that the motion will be similar to the motion of the upright strong top for a sufficiently small impact, which we will confirm analytically.

Since we again wish to apply our approximation formulas, we first examine their degree of precision. The formula for u gives, as we know, an arbitrary precision if the bound

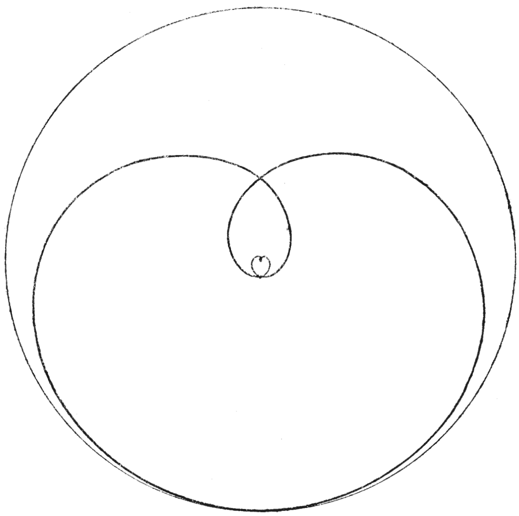
Fig. 58.

$$\sqrt{\frac{e''-e}{e''-e'}}-1$$

for the error τ becomes arbitrarily small. Here it is not sufficient that the difference between e and e' be sufficiently small; there is added the further condition that e'' may not lie very near to the values of e and e'. We must therefore form a judgment of the magnitude of e''.

According to equation (9) of page 273,

$$e''=\frac{n^2+N^2-2nNe}{2AP(1-e^2)}-e';$$

as e approaches 1, the values n and N approach each other; the numerator and denominator vanish simultaneously, so that a particular investigation becomes necessary. We return to the original expression for U (see equation (7′) of page 238). If we insert there $u=e$, then

$$(Ne-n)^2=(k-N^2-2APe)(1-e^2),$$

or

$$\frac{n^2+N^2-2nNe}{1-e^2}=k-2APe.$$

The right-hand side of this equation has a simple mechanical meaning. According to equation (3) of page 219,

$$k=|i|^2+2AP\cos\vartheta;$$

the right-hand side in question is therefore directly equal to $|i|_e^2$; that is, equal to the square of the length of the impulse in the initial position e.

This quantity is, by assumption, a very large number in relation to AP, and differs very little from N^2. It follows that e'' likewise has a large numerical value and that we can set in an approximate manner, just as for the upright motion,

$$e'' = \frac{N^2 - 2AP}{2AP}, \tag{16}$$

where $|i|_e^2$ has been replaced by N^2 and e' by 1.

The approximation formula (8) of page 272 for u therefore possesses an arbitrarily high degree of precision in this case of pseudoregular precession. We write, correspondingly,

$$\left\{\begin{aligned} &u = u_0 + \varepsilon \sin \frac{\pi t}{\omega} = u_0 + \delta, \\ &u_0 = \frac{e' + e}{2}, \quad \varepsilon = \frac{e' - e}{2}, \quad \omega = \pi\sqrt{\frac{A^2}{N^2 - 4AP}}, \quad \delta = \varepsilon \sin \frac{\pi t}{\omega}. \end{aligned}\right. \tag{17}$$

The construction of an appropriate approximation formula for ψ causes greater complications. We have

$$\psi = \int \frac{n - Nu}{A(1-u^2)}\, dt = \int \frac{n-N}{2A(1-u)}\, dt + \int \frac{n+N}{2A(1+u)}\, dt = \psi_1 + \psi_2.$$

The calculation of ψ_2 causes no difficulty. If we use the identity

$$\frac{1}{1+u} = \frac{1}{1+u_0+\delta} = \frac{1}{1+u_0} - \frac{\delta}{(1+u_0)^2} + \frac{\delta^2}{(1+u_0)^2(1+u)},$$

then, approximately,

$$\psi_2 = \frac{n+N}{2A(1+u_0)} t + \frac{(n+N)}{2A(1+u_0)^2} \cdot \frac{\omega\varepsilon}{\pi} \cdot \cos \frac{\pi t}{\omega}. \tag{18}$$

The omitted remainder term is, as one is easily convinced, always arbitrarily small in relation to the retained terms if ε is sufficiently small.

For the calculation of ψ_1, the preceding expansion is of no use, since the assumed small quantity $1 - u_0$ would appear in the denominator and make the estimation of the error illusory. We are thus dependent on the actual execution of the integration.

If we use for u the value from equation (17), ψ_1 has the form

$$\psi_1 = c \int \frac{dt}{a - \sin \alpha t}, \quad \alpha = \frac{\pi}{\omega}, \quad a = \frac{1-u_0}{\varepsilon} > 1, \quad c = \frac{n-N}{2A\varepsilon}.$$

Integration yields, as one can verify,

$$\psi_1 = \frac{-c}{\alpha\sqrt{a^2-1}} \operatorname{arctg} \frac{1 - a \sin \alpha t}{\sqrt{a^2-1} \cos \alpha t}. \tag{19}$$

We are led here, therefore, to a complicated dependence on t, which is not immediately decomposed, as in the given formula for ψ_2, into a term proportional to time and a periodic term. For the sake of brevity, we will not enter into the error estimation here.

We first wish to relate our latter formula to the earlier representation of pseudoregular precession and to the motion of the upright top. For pseudoregular precession with nonvanishing precession circle, $1 - u_0$ differs from zero, and a, because of the small denominator ε, is very large. As a result, the argument of the arc tangent goes over into $\operatorname{tg}\alpha t$, and the formula for ψ_1 becomes

$$\frac{c}{\sqrt{a^2-1}}\,t = \frac{n-N}{2A(1-u_0)}\,t.$$

This term, together with the first term in equation (17), determines the mean precessional velocity for pseudoregular precession in the previously (page 302) given manner.

We see, on the other hand, what equation (19) gives for the upright motion, where the trajectory passes through the north pole of the unit sphere. Here $e' = 1$ and $1 - u_0 = \varepsilon$. We thus have $a = 1$. The argument of the arc tangent assumes, in this case, only the three values $+\infty$, $-\infty$, and 0. In general, its value is $\pm\infty$, according to the sign of $\cos\alpha t$; at those moments, however, when $\alpha t = (4n+1)\dfrac{\pi}{2}$—that is (cf. equation (17)), when the apex of the top passes through the north pole—the argument jumps from $+\infty$ through 0 to $-\infty$; the value of the arc tangent thus increases jumpwise by -2π. (To see this, however, one must consider not the limiting case $a = 1$ itself, but rather $a > 1$.) In equation (7) of page 330, through which we represented the ψ-coordinate for the upright top, this jumpwise change was not expressed. Rather, this formula gives only the single component ψ_2 (and indeed, only the first term of this component). On the other hand, a glance at Figs. 55–57 explains the meaning of the term in question. In passing the north pole, ψ in fact necessarily increases instantaneously, and indeed by π, since the trajectory progresses through the north pole with a continuous tangent. At the same time, we conclude that the (otherwise not entirely easy to determine) limiting value of the factor

$$\frac{c}{\alpha\sqrt{a^2-1}} = \frac{(n-N)\pi}{2\pi A\sqrt{(1-u_0)^2-\varepsilon^2}}$$

for $1 - u_0 = \varepsilon$ and $n = N$ is equal to $\frac{1}{2}$. We wish to use this limiting value also in the case that the trajectory does not pass exactly through the north pole, and for (19) write more simply

$$(19') \qquad \psi_1 = \frac{1}{2} \operatorname{arctg} \frac{\varepsilon - (1-u_0)\sin\frac{\pi}{\omega}t}{\sqrt{(1-u_0)^2 - \varepsilon^2}\,\cos\frac{\pi}{\omega}t}.$$

The definitive formulas for the description of pseudoregular precession with a very small precession circle thus become, according to (17), (18), and (19′),

$$(20) \qquad \begin{cases} u = u_0 + \varepsilon \sin\dfrac{\pi t}{\omega}, \\ \psi = \dfrac{n+N}{2A(1+u_0)}t + \dfrac{n+N}{2A(1+u_0)^2}\dfrac{\omega\varepsilon}{\pi}\cos\dfrac{\pi t}{\omega} \\ \qquad\qquad + \dfrac{1}{2}\operatorname{arctg}\dfrac{\varepsilon - (1-u_0)\sin\frac{\pi}{\omega}t}{\sqrt{(1-u_0)^2-\varepsilon^2}\cos\frac{\pi}{\omega}t}. \end{cases}$$

The comparison of our motion with that of the upright top motion gives us a clear image of the origin of the term ψ_1. The sudden jump of the ψ-coordinate in the case of the upright top must be resolved for our pseudoregular precession into a continuous but possibly very rapid change that occurs each time the apex of the top approaches the north pole. It is clear that this exceptional change of the azimuth does not conform to the general schema of precession and nutation. Correspondingly, we see that the general character of the equations for the representation of pseudoregular precession is changed considerably compared to the previous, and see, in particular, *that we cannot resolve, as we could previously, the motion into a regular precession and a simple harmonic oscillation.*[152]

A similar modification would naturally also be required for the equations with which we have represented the neighboring motions to regular precession in the case that the latter occurs in the immediate vicinity of the north pole. *Here again the motion may no longer be smoothly divided into a mean precession and an overlying nutation.*

Notwithstanding the formal difference in the structure of the equations, the essential properties of the general pseudoregular precession are still retained.

We already saw that the time duration ω of a passage from one to the other boundary circle may be represented by the approximation formula

$$\omega = \sqrt{\frac{A^2}{N^2 - 4AP}}\,\pi,$$

of equation (17), which is identical with equation (15′) of page 305.

We further calculate, according to (20), the quantity $2\psi_\omega$; that is, the change of the azimuth ψ during two successive passages. The arc tangent increases during this interval by -2π; the angle ψ thus increases by

$$2\psi_\omega = \frac{n+N}{2A(1+u_0)}\,2\omega - \pi.$$

Here we pass to the limit $u_0 = 1$, $n = N$, which corresponds to the upright top, and obtain

$$2\psi_\omega = \frac{N}{A}\omega - \pi.$$

If we insert for ω the just given value and expand, in that we retain only the first power of $\dfrac{AP}{N^2}$, as previously for pseudoregular precession, then there results, finally,

$$2\psi_\omega = \left\{\left(1 - \frac{4AP}{N^2}\right)^{-\frac{1}{2}} - 1\right\}\pi = \frac{2AP}{N^2}\pi.$$

This value agrees precisely with that which we would calculate from equation (11) of page 303 for the usual pseudoregular precession.

Pseudoregular precession with a very small precession circle has a certain significance in applications (particularly in ballistics), for which reason its belated settlement appeared necessary in this place.

§6. Generalities on the stability and lability of motion.

It is the exercise of this section to sharpen our already repeatedly applied definition of stable and labile motions, and weigh it against other definitions of this concept. The example of the top will provide an appropriate starting point for more general considerations.

The concept of stability for moving systems first appeared in astronomical mechanics. It already played a well-known important role for L a p l a c e, who gave his specious proof of the stability of the planetary system.*) And indeed one calls a system of particles stable, according

*) Cf., for example, J a c o b i's fourth lecture on *Dynamik*.[153]

to Laplace, if, in the course of time, no particle can be removed to infinity. As one sees, this stability concept is specifically tailored to the requirements of astronomy and the phenomena of a nonrigid aggregate of particles. It has very little to do with what we denote as stability in the following.

The question of stability was first considered from the physical standpoint by Lord Kelvin.*) There is, according to him, "scarcely any question in dynamics more important for Natural Philosophy than the stability or instability of motion."**) Since then this subject has been considered, in particular, by numerous English authors. We take the usual definition of stability from the prize work of Mr. E. J. Routh, *On the stability of a given state of motion.****) According to Routh, the motion of a system is called *stable if, for an arbitrary but small disturbance, the deviation between the position coordinates of the system in the altered and the original motions at equal points of time always remains small.* (A "small" quantity is understood as one "whose square can be neglected.")

We must first make the word "disturbance" more precise. We initially understand by a disturbance the *totality of the differences between the initial values of the impulse coordinates of the original and the altered motions.* Here we must appeal, however, to later developments with respect to what should be understood by the impulse coordinates of an arbitrary mechanical system.

A modification required by the demands of modern rigor is now suggested for the given definition. We prefer to speak of *arbitrarily small* instead of *small* disturbances and deviations. We thus connect with the well-grounded fundamental concepts of differential calculus, especially the limit concept.

At the same time we separate, by this modification, the cases denoted on page 325 as *practically stable* from the *theoretically stable* cases. For the practically stable, theoretically labile cases of the upright top motion, the deviation of the apex of the top from its initial position was (as a result of the particular choice of the constants A, P, and N) always small; but it could not (through the diminishment of

*) In the first edition of the Natural Philosophy, 1867.

**) Cf. Thompson and Tait: Natural Philosophy, art. 346, Vol. I, p. 416.

***) London 1877,[154] cf. Chap. I, art. 1, as well as the textbook *Rigid Dynamics* of the same author, Part II, art. 256 and 257.

the disturbance) be made arbitrarily small. According to the given definition of Routh, these cases would be stable; after the proposed modification, however, they are to be reckoned as unstable. Whether the latter or the former is to be preferred remains to be seen. In any case, the precise treatment of the stability definition is facilitated by the requirement of *arbitrary* smallness of the disturbance and the deviation; that is, by the restriction to *theoretical* stability. We reserve for the end of the section the return to the treatment of the practically stable, theoretically labile cases.

We immediately mention a second modification that we wish to propose. The above definition demands that the deviation between the position *coordinates* of the system be small (or can be made arbitrarily small). But the choice of the coordinate system is, from the standpoint of general Lagrangian mechanics, entirely at our discretion. It is very well possible that the deviation between the position coordinates for a certain choice of the coordinate system remains small in the course of the motion, and for another choice becomes arbitrarily large—in so far as we do not subject the choice of the coordinate system to certain restrictions, into which we cannot enter in this place. *In the above conception, the stability definition thus pertains, strictly speaking, to a mechanically meaningless property of the motion that is not independent of the coordinate system.*

It is easy to correct this undesirable circumstance. We must speak not of the *deviation of the position coordinates*, but rather of the *deviation of the position of the system.* We call this deviation arbitrarily small if the distances between the positions of each individual point of the system for the one and the other motion are arbitrarily small at the corresponding moments of time. The distance between two points, however, is a concept that is independent of the choice of coordinates.

In order that the estimation of the magnitude of the disturbance be also independent of the coordinate system, we can imagine the total impulse of the system resolved into the corresponding impulse of each individual mass particle, which can be determined at any time from the mass and velocity of the particle. The disturbance will then be called arbitrarily small if the deviations between the corresponding individual impulses of the system at the beginning of the original and perturbed motions lie beneath an arbitrarily prescribed bound.

We are thus able to make the Routh definition of stability more precise in the following manner: *A motion is called stable if the deviations between the corresponding positions of the system that result from a disturbance can always be suppressed beneath a given bound by choosing the magnitude of the disturbance beneath an appropriately determined bound.*

We wish to examine the suitability of this definition in detail.

From a purely logical standpoint, any definition that is not in contradiction with itself and generally corresponds to some object in reality is naturally permissible. From the standpoint of natural science, however, we must demand more of a stability definition than its mere internal or external want of contradiction. One generally associates, namely, the word "unstable" with the conception of an exceptional and turbulent process. We must therefore demand of our definition, in order for this conception to be justified, that no generally regular and ordinary motion fall under the concept of the unstable processes, and no apparently irregular motion fall under the stable processes.

A series of examples will now show that the above definition of stability is, from this point of view, inappropriate. We take the relevant examples partly from the theory of the top, and partly from the simplest problems of particle mechanics.

We first consider regular precession and its neighboring motions, as in §1 of this chapter. We verify from the aspect of Fig. 47 that the altered trajectory always runs in the vicinity of the original; and indeed this state of affairs obtains for an arbitrary type of disturbance.

The situation is otherwise if we consider not only the spatial form of the trajectory, but also, as the above stability definition demands, the time in which the apex of the top traverses the trajectory. We saw on page 287 that the mean angular velocity

$$\frac{n - Nu_0}{A(1 - u_0^2)}$$

of the altered motion generally differs from the precessional velocity

$$\frac{n - Ne}{A(1 - e^2)}$$

of the original motion whenever the impact has not the line of nodes as its axis, although by always less as the impact is chosen smaller.

This difference is sufficient, however, to effect a finite difference in the two compared motions over the course of time. We can directly determine a time interval t (that naturally increases with a decreasing disturbance) after which the difference of the ψ-values for our two motions will be greater, for example, than $\frac{\pi}{2}$.

If we would avoid this, we must subject the character of the disturbance to the special condition that merely the $[\Theta]$-component of the impulse is altered by the disturbance. This would be, however, an arbitrary stipulation that is not provided in the definition thus far.*) Moreover, the altered precessional velocity would still coincide with the original (cf. page 320) only up to quantities that are proportional to the second power of the disturbance $[\Theta_0]$. For a corresponding enlargement of the time interval t, we can claim in this case as well an arbitrary finite difference in the simultaneous values of the ψ-coordinates for the original and the altered motions.

Thus it is clear, according to the wording of our above definition, that the simplest motion of the top, regular precession, is to be designated as unstable.

The corresponding holds in elevated measure for the general motion of the top. Here the form of the trajectory of the apex of the top never remains, for a sufficiently small impact, arbitrarily near the original. In fact, both the span width of the component arcs and the time in which the arcs are traversed are generally altered by a change of the impulse (cf. Figs. 29–35 of the previous chapter). These changes can indeed be made arbitrarily small if the impulse change is chosen as sufficiently small. But if a sufficiently large time interval is taken into consideration, arbitrary finite differences between the corresponding positions of the apex of the top result from such arbitrarily small changes. *If we would adopt the above definition, then we must simply declare the collected motions of the top to be unstable.* This also applies, in particular, to the upright motion of the strong top if we consider the coordinate χ and allow a change of the impulse component N, or in entire generality if we retain in the calculation

*) In actuality, however, such a stipulation is generally introduced after the fact by the English authors. Cf. the following paragraphs.

such deviations that are proportional to the second power of the disturbance (cf. page 320).

To cite an example from the mechanics of a single mass particle, we consider, with Mr. Appell,*) the circular motion of a mass particle in a fixed plane under the influence of an applied central force with the action law r^n. It is shown that the altered trajectory due to a disturbance *always* remains near the original circle for a sufficiently small impact in the case $n > -3$. It is otherwise with the position of the particle on the trajectory. This will obviously be slightly changed for the duration of the motion only if the added impact has no component in the direction of the original path, and therefore if the added impact leaves the velocity of the particle unchanged. Mr. Appell thus sees it necessary, on the basis of the above definition, to declare that the motion of the particle is unstable also in the case $n > -3$.[155]

One can avoid the named difficulties, in part, if one allows only such impulse changes that do not change the energy of the system. In the work of Thomson and Tait, such a disturbance is designated as "conservative," and the restriction to conservative disturbances is immediately adopted in the stability definition.**) With this modification of the stability concept, the circular path in the last example is to be declared as stable in the case $n > -3$;***) one will thus generally avoid the appearance of "secular disturbances," as one can name deviations from the original to the altered position that increase with time (at least in so far as these secular terms are only of the order of the second power of the impulse change).

But there remain enough other undesirable circumstances. We consider, for example, the force-free motion of an individual mass particle according to the Galilean law of inertia. Are we able to conclude that this most regular, so to speak, of all motions is to be declared as unstable? According to the wording of the usual definition, we must do so. For the altered motion of the particle due to an added impulse, which for want of external forces is again linear and uniform, is removed from the original path more and more, as small as we may measure the disturbance.

*) Mécanique rationelle, t. II, art. 458.
**) Natural Philosophy, art. 346, 347.
***) Natural Philosophy, art. 350.

We further call upon the interesting example of geodesic lines; that is, the force-free paths of an individual mass particle that is somehow constrained to remain on a curved surface. Here we must distinguish two cases, according to whether the curvature of the surface (in the Gaussian sense) is positive or negative. If we let our mass particle run on a surface of negative curvature (for example, on a hyperboloid of one sheet) and impart to it a small impulse, then the disturbed trajectory is removed more and more from the original; we can give no bound beneath which the magnitude of the impulse must lie, so that the distance from the particle in the altered motion to the corresponding position in the original motion remains beneath a given bound. Thus we must designate, with Thomson and Tait,*) all geodesic trajectories on surfaces of negative curvature as unstable trajectories.[156] On a surface of positive curvature, on the other hand, geodesic trajectories that are to be named as stable in the sense of the above definition are in any case imaginable. It can be shown, namely, that if one constructs from any point on such a surface two geodesic lines that differ infinitesimally in their initial directions, these lines must continually intersect one another at intervals that differ according to the magnitude of the curvature. If we therefore consider one of these two lines as the original trajectory of our mass particle and the other as that altered by a disturbance, then the former will constantly *oscillate about* the latter. If we can further demonstrate that the amplitude of the oscillation does not increase systematically with increasing time, then we can, with the restriction to conservative disturbances, declare the trajectory as stable according to the above definition.**)

All in all, however, we must say that *the above stability definition, according to which a permanent smallness of the deviation is demanded,*

*) l. c. art. 355, where one may also refer to the extraordinarily simple proofs of the theorems on geodesic lines cited in the text.

**) Thomson and Tait, l. c. art. 355. The conclusion regarding the stability (in the intended sense of the authors) of the trajectory appears premature, however, without an investigation of the oscillation amplitude. In fact, the meridians on the ellipsoid of revolution, for example, are unstable trajectories in the Thomson sense: with addition of a lateral impact they are transformed into unclosed curves that are alternately tangent to a parallel circle in the vicinity of the north pole and a parallel circle in the vicinity of the south pole, and wind around the ellipsoid with a span-width that differs from 2π. If we follow such a curve sufficiently far, it is removed more and more from the original meridian.[157]

is too narrow. It refers the simplest and most regular motion (Galilean inertial motion!), among others, to the class of unstable motions, which contradicts the natural conception of the word.

One can seek to essentially retain the above definition, and change it only by demanding the smallness of the deviation not for an arbitrary, but rather for a bounded time duration. One would then define stability by the following postulate:

It should be possible to choose the disturbance so small that the deviations between corresponding positions of the system in the original and the altered motions remain beneath a prescribed bound for a given time interval $t < T$.

On the basis of this definition, the Galilean inertial motion, the geodesic trajectories on the hyperboloid, the general motion of the top, etc., would be inserted immediately into the category of stable motions. It would produce, however, an undesirable circumstance of another type. It must declare as stable, namely, motions of such doubtlessly irregular character as the upright rotation of the weak top.

We first recall the behavior of the weak top in the limiting case of an infinitely decreasing impact ($\lim [\Theta_0] = 0$). Our previous investigation shows that in this limiting case the apex of the top does not remain in an arbitrary neighborhood of the north pole, but that its velocity at the north pole is equal to zero. And indeed, the time ω at which the apex of the top in Fig. 53 arrives at the north pole, starting from an arbitrary point of the spiral, becomes infinitely large.

One then considers the behavior of the apex of the top for a nonzero but extraordinarily small impact $[\Theta_0]$.

The trajectory is then no spiral, but will nevertheless encircle the north pole a few times in its immediate vicinity; the velocity ϑ' with which it departs from the north pole is not equal to zero, but is always extraordinarily small.

We can certainly now establish an upper bound for the impact $[\Theta_0]$ so that for each smaller impact and for each time $t < T$ (T, for example, equal to one year), the deviation between the position coordinates ϑ in the altered and the original motions will be smaller than ε (ε, for example, equal to one arc second).

In other words: *the motion of the upright weak top, on the basis of our current definition, would be stable!*

Similar considerations may be employed for another case that is likewise generally recognized as unstable. In the rotation of the asymmetric weightless top about its intermediate principal axis, the velocity with which the rotation axis leaves the intermediate principal axis will also be equal, in the limit, to zero for an impact that decreases to zero. For a bounded time $t < T$, the altered motion therefore remains in an arbitrary neighborhood of the original for an appropriate choice of the impact.

We thus conclude that *our current stability definition, according to which the smallness of the deviation is demanded only for a bounded time interval, is t o o b r o a d. It allows motions to pass as stable that we must reasonably regard as entirely labile.*

We have thus fallen into a characteristic dilemma, from which we can escape only if we again take up the stability definition that we have already applied many times. We wish to call a motion stable if a *continuous passage* is possible between it and the altered motion due to an arbitrary impact. It is then necessary only to make this somewhat undetermined continuity concept more precise, and to characterize it through an analytic criterion. We wish, for this purpose, to proceed in the following manner. We alter the motion in question by means of a finite impact of an arbitrary character. We then let the magnitude of the impact decrease to zero, and seek the limit to which the altered motion thus tends. *If this limit exists and coincides with the given motion, we call the passage between the original and the altered motions continuous.*

Our final stability definition, stated for an arbitrary mechanical system, is thus the following:

A motion is called stable if it coincides with the limit to which the motion tends as the result of an arbitrary impulse change, as the magnitude of this change decreases to zero. It will be called labile, in contrast, if it is different from this limit, if different limits result for different types of impulse change, or if a limit does not in general exist.

According to this definition, for example, regular precession, the general motion of the top in which the trajectory runs to and fro between two parallel circles, the Galilean inertial trajectory, etc., are obviously to be reckoned as stable motions, while the upright motion of the weak top, the rotation of the three-axis top about its intermediate principal inertial axis, etc., are to be reckoned, as is fair, as labile motions.

In fact, we saw for the upright weak top, for example, that as the value of the impact $[\Theta_0]$ decreases to zero, there remained an entirely determined motion (a spiral trajectory) that was different from the simple rotation about the vertical. As for the geodesic trajectories on the hyperboloid of revolution, these would generally be stable, with the exception of the throat-circle, which is unstable and represents an asymptotic solution that we do not wish to develop in more detail.[158]

A further point in which our stability definition is evidently superior to the otherwise common definition deserves to be especially emphasized. If one judges the stability of motion in the usual sense from the smallness of the deviation that results from a small impact, and if one represents this deviation by approximation formulas, then one usually neglects (cf. page 320) all those terms that vanish as the second or higher power of the disturbance. If this neglect, however, is applied to a secular term (a term, for example, multiplied by t), then the imprecision of the approximation formula always increases with time; while the approximation formula thus allows the conclusion of a permanently small deviation, it can occur in reality that the deviation between the perturbed and the original motions attains any arbitrary value. In this case, a motion would appear as stable according to the usual method, while a finite deviation still appears with time for a disturbance. Our treatment of the stability definition, in contrast, is completely free of such difficulties. For us it is a question not of deviations of the first or second order, but rather the direct *equality* between the original motion and the limit of the perturbed motion. This equality, compared with the smallness of the deviation in the usual definition, allows a judgment of not only greater sharpness, but also greater ease.

We can also recommend our new definition by showing that *in the case of equilibrium, where the concepts stable and labile have long been established, it coincides with the generally accepted meaning of these words.* A simple example suffices in this respect. A heavy mass particle on a spherical surface is in an unstable equilibrium at the highest point (the north pole) of the sphere, and in a stable equilibrium at the lowest point (the south pole). This follows from our definition of stability for motion, of which the definition of stability for equilibrium is a special case, and coincides with the usual conception. If we give, namely, an impact to the particle at the north pole, it describes a great circle on the sphere; if we let the magnitude of the impact continuously decrease to zero, then the trajectory of the particle remains the same, and only the velocity decreases; the velocity at the north pole is zero in the limit, and differs from zero at all other points of the great circle. The limit here is therefore a specific well-defined motion that is different from the original state of rest. Moreover, there is another limit for each direction of the impact. If we let, in contrast, an impact act on the particle resting at the south pole, then the particle oscillates to and fro on a great circle; the magnitude of the amplitude depends on the magnitude of the impact and decreases to zero with it. The limit of which our definition speaks is therefore the original rest position itself.

We could consider, finally, a *weightless* particle resting on a sphere. If we strike this particle, it describes a great circle with constant velocity; if we let the magnitude of the impact become zero, then the velocity at each point of the trajectory will also be zero. One can be in doubt whether this limit should be designated as rest or as motion. In any case, one must say that a determined limit does not exist, since the position of the great circle depends on the direction of the impact. For such cases, the otherwise applied designation of "indifferent equilibrium" appears appropriate.

We would by no means deny that the stability concept posed at the beginning of this section is also worthy of investigation, especially if it is improved by the restriction to conservative impacts. We would only speak, in that case, not simply of stability, but rather, for example, of *absolute stability*. Therefore:

If a motion is of such a nature that it is transformed, for a sufficiently small conservative change of the impulse coordinates, into a motion for which the positions of the system a l w a y s remain arbitrarily near the corresponding positions of the original motion, then we call the motion absolutely stable. That such a motion also satisfies our definitive stability declaration is self-evident. The circular trajectory cited on page 347 is, under the condition $n > -3$, absolutely stable in this sense. We must, however, remark again that the usually adopted method of investigation in the literature (for example, by Routh as well as by Thomson and Tait), according to which absolute stability is judged from the terms of the first order with the neglect of the higher powers, is, from our standpoint, incomplete. According to the preceding definition, we can call a motion absolutely stable only if the complete altered motion, and not simply the terms of the first order, always remains arbitrarily near to the original.

We can make a series of other distinctions. If we wish to emphasize that an altered stable motion arrives at the same position as the original after a certain time interval, then we can call the motion *periodically stable.* The opposite would be *divergingly stable.* The geodesic trajectories on surfaces of positive curvature are, in so far as they are in general stable, always periodically stable, and those on negatively curved surfaces divergingly stable. The Galilean inertial motion provides a further example of diverging stability. Absolute and periodic stability need not coincide, as the example of the geodesic lines on the ellipsoid can show (cf. the footnote on page 348).

We can further distinguish between *partial* and *total stability.* We would speak of partial stability if our stability criterion is fulfilled only for certain impacts, and of total stability if it is fulfilled entirely; that is, for all possible impacts. Our stability definition thus far refers to total stability. If, in contrast, we restrict with Thomson and Tait to conservative impacts, then we ask for a type of partial stability. As we will see in the following section, one is mostly interested in the literature only in partial stability, especially in the case of so-called cyclic systems.

Finally, we emphasize once more the contrast between *theoretical and practical stability and lability.*[159] Our developments thus far in this section refer completely to theoretical stability. It can occur, however, as we already saw in the previous section, that a motion does not satisfy our stability criterion but is still, for practical purposes, as good as stable. This will occur if the limit in question of the original motion indeed differs from, but is only so little different that it nearly coincides with, the original motion.

The opposite will be the case if the limit of the changed motion is indeed identical with the original, but the changed motion differs essentially from the undisturbed shortly before the passage to the limit; that is, for very small values of the impulse coordinate changes. An example of a *practically labile, theoretically stable equilibrium* in this sense is given by a mass particle in a very small frictionless cavity on the summit of a mountain. A *theoretically labile and practically stable equilibrium* is represented by a particle that lies on a slight elevation in the bottom of a valley.

We wish to remark, in conclusion, that our stability definition can also be modified, if desired, by considering small changes of the position coordinates in addition to small changes of the impulse coordinates, or small changes in the position and velocity coordinates instead of small changes of the impulse coordinates, as is often done, in fact, for equilibrium investigations.[160] This modification, however, appears to have no important consequences. Moreover, our change of the impulse coordinates may best correspond to the physical concept of a disturbance, and may deserve preference over a change of the velocity coordinates, which would indeed be equally worthy mathematically, but whose physical sense would be less meaningful.

§7. Energy criteria for the stability of equilibrium and motion.

The application of our stability definition of the previous section assumes the general knowledge of the trajectories, and particularly the knowledge of the limit to which the motion tends for a vanishing disturbance. The judgment of whether a motion is stable or unstable is thus rather troublesome. One will wish to simplify this judgment and will seek, in particular, criteria that lead to the goal without the knowledge of the general motion, and therefore without the integration of the mechanical differential equations. We will, in this respect, hardly have anything new to offer; our exercise is, rather, of an essentially critical nature. We intend to show in the following section, namely, that the most practical criterion, which follows from the so-called method of small oscillations, gives occasion for many objections.

The first investigation to be employed must be modeled after the well-known *criterion for the stability of equilibrium* that was first stated by L a g r a n g e, and was formulated precisely and proven in a short but meaningful work of D i r i c h l e t.*) Dirichlet considered an arbitrary mechanical system whose constraints are independent of

*) Crelle's Journal, Bd. 32, pp. 85–88, 1846.

time, and which is subject only to "conservative forces"; that is, forces whose work can be represented by a function of the coordinates whose negatively taken value is the potential energy V. For such a system, the theorem of the *vis viva* obtains in the form

$$T + V = h.$$

The well-known result of the Dirichlet investigation now runs thus: *the equilibrium is certainly stable if V is an actual minimum in the equilibrium position.*

We conduct the proof, with consideration of the immediately following generalization of the criterion, somewhat differently from Dirichlet in the following manner.

In the equilibrium position, $T = 0$; the value of V can, since it is defined only up to an additive constant, likewise be set to zero. Thus $h = 0$ in the equilibrium position. If V is an actual minimum, we can give limits for the coordinates that determine the position of the system so that V is greater than a (sufficiently small) positive quantity k as soon as one or more of the position coordinates are equal to the given limit values, while, at the same time, the values of the remaining position coordinates remain inside these limits. To enable us to express this concisely, we wish to speak of the totality of the values of our position coordinates that lie inside the given limits as a "domain," and wish to denote those coordinate values for which at least one coordinate coincides with the established limit value for this coordinate as the "boundary of the domain." We then have, for the boundary of our domain,

$$V > k.$$

All the more, therefore, since T is necessarily positive, is

$$T + V > k, \tag{1}$$

and indeed independent of the values that we may assign to the velocity coordinates in T.

We now impart a disturbance to the system. The theorem of the *vis viva* again obtains for the resulting motion. We can measure the disturbance so small that the constant h of the *vis viva* will be less than k. Thus

$$T + V < k \tag{2}$$

for the disturbed motion.

It is now clear that this motion runs entirely and always inside the previously given domain. In the opposite case, namely, it would occur that one position coordinate (or possibly more simultaneously) would attain the limit given above, while the remaining position coordinates would still possess values that correspond to the interior of the region. At this moment, however, the inequality (1) must hold, which is incompatible with the simultaneously valid inequality (2).

The boundaries of the region can now be narrowed arbitrarily, and the preceding conclusion, for a corresponding diminishment of the disturbance, will be repeated.

Therefore the system always remains, for the altered motion, in an arbitrarily small neighborhood of the original equilibrium position. Thus the equilibrium is certainly, according to our and every other definition of the word, stable.

The proven criterion therefore *suffices* to guarantee the stability of the equilibrium. There arises, however, the further question of whether the requirement of this criterion is also *necessary*, or, otherwise expressed, whether the Lagrange–Dirichlet theorem may be reversed, in the sense that an equilibrium is certainly unstable for nonexistence of a minimum (or perhaps only the presence of a maximum). There is still no conclusive result here. At least, Mr. L y a p u n o v and Mr. H a d a m a r d, in their relevant works,[*]) can state the converse of the theorem in question only under special assumptions on the nature of V (for example, under the assumption that the nonexistence of a minimum in the quadratic terms of the power series of V can be recognized, or the presence of a maximum to the terms of the lowest order.)[161]

We now come to an interesting transference of the preceding "energy criterion" ("energy test of stability") from the case of equilibrium to that of motion. This transference has been accomplished by Routh.[**])

We begin, with Routh, just as for the equilibrium criterion (under the given assumptions on the nature of the constraints of the system and the forces acting on the system), from the energy expression

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[*]) Cf. L y a p u n o v, Journal de Liouville, sér. V, t. 3 (Sur la stabilité de l'équilibre), where further literature citations to the works of the author are to be found, and H a d a m a r d, ibid., (Sur certaines trajectoires en dynamique); cf. especially page 365.

[**]) Cf. Rigid dynamics, Part II, Chap. III, art. 95 and ff. Stability of motion, Chap. VI, art. 1–3.

$$T + V = h.$$

The left-hand side, the total energy of the system, is a known function of the position and velocity coordinates that has the constant known numerical value h for all stages of the given motion. One can now show that if the total energy is an extremum for the given motion (that is, a maximum or a minimum) with respect to all the appearing position and velocity coordinates, the given motion must be absolutely stable.

However, the situation can certainly not occur in the manner just expressed. We consider, namely, the dependence of the total energy on the velocity coordinates. Since the *vis viva* represents a positive quadratic function of the velocity coordinates, it will in general increase with increasing, and decrease with decreasing values of the velocity coordinates. If, however, the total energy is to be an actual extremum, then T must either only increase or only decrease for the increase or decrease of the velocity coordinates.

As a result, one must necessarily, with Routh, pose the stability question more particularly. One will ask, in order to be able to extract an actual use from the energy criterion, not for total, but rather for partial stability of some kind (cf. the conclusion of the previous section). One will thus fix the initial values of individual impulse coordinates (we wish to name them N, n, ...) and arrange the impact so that it effects only a change of the remaining impulse coordinates. Moreover, we wish, for the sake of simplicity, to assume that the impulse coordinates N, n, ..., as in the case of the top, also remain constant in the course of the motion, so that we will speak in the following of the "impulse constants" N, n, ...

The impulse coordinates are, however, as will be shown later in general, simple (and indeed linear) functions of the velocity coordinates, where the position coordinates can enter in the coefficients. If we denote the velocity coordinates by ϑ', φ', ..., then we have equations of the form

$$(3) \qquad f_1(\vartheta', \varphi', \ldots) = N, \quad f_2(\vartheta', \varphi', \ldots) = n, \ldots$$

which obtain for the altered motion just as for the original. (Routh more generally considers, instead of such impulse equations, any "first integral equations" of the problem, whose left-hand sides are an aggregate of the position and velocity coordinates, and whose right-hand sides are constants. The impact must then be chosen so that it does not change the constants of the right-hand sides.)

We are thus able to eliminate from the expression for the total energy as many velocity coordinates as we have constraint equations of the given form, whereby the impulse constants N, n, ... (or the integration constants N, n, ...) will enter into the energy expression. The demand, that the resulting energy expression represent an actual extremum with respect to the collected position and velocity coordinates explicitly contained in it, evidently amounts to less than the earlier demand that it be a simple extremum (with respect to all coordinates). We will see immediately that our current demand is actually fulfilled, for example, for certain motions of the top.

This arranged in advance, we state the Routh energy criterion as follows:

The given motion is, in all the uneliminated position and velocity coordinates, absolutely stable if the energy expression is, after the required elimination, an actual extremum with respect to just these position and velocity coordinates, and indeed partially stable with respect to all disturbances that do not change the constants N, n, ... in the energy expression.

The proof is formed as in the case of equilibrium: if $T + V$ is an actual minimum for the given motion (the case of a maximum is treated similarly), then we can give positive and negative increments for each individual uneliminated position and velocity coordinate so that the current value of $T + V$ is increased as soon as we apply the respective increment to at least one of the position and velocity coordinates, while at the same time the remaining coordinates remain unchanged or are changed only by less than the established increments. And indeed, the resulting increase of $T + V$ may, at each moment of the motion, become greater than the positive (to be chosen as sufficiently small) quantity k, so that

$$(1') \qquad T + V > h + k.$$

For the motion altered by a single disturbance, the theorem of the *vis viva* likewise obtains. The disturbance can be chosen so small that the original value of h will be increased less than k. We therefore have, along the entire altered motion,

$$(2') \qquad T + V < h + k.$$

Here we assume a partial disturbance that leaves the values of the constants N, n, ... unchanged.

From $(1')$ and $(2')$ one concludes, as above, that the differences between the coordinates for the original and the altered motions can never attain the magnitude of the previously given increments. Since these increments, however, can be chosen as arbitrarily small, there follows immediately the absolute stability of the given motion with respect to all the uneliminated coordinates.

First, the requirement of the Routh criterion should be sharpened somewhat more precisely. It is actually not enough to demand that the energy function $T + V$ be simply an extremum. It is assumed in the proof, rather, *that for all values of time, or, equivalently, for all positions on the original trajectory, one and the same positive (or negative) number k may be given*, above (or below) which lies the change in the energy function for the increase of one or more of its arguments by a certain nonzero increment. The latter requirement says more than the requirement that $T+V$ should be an extremum at each individual position. It can very well be, for example, that we can give, for each value of t, a number k of the named character, but that this value will be always smaller (or larger) for increasing t, and in the limit $t = \infty$ will no longer differ from zero. We wish to call the type of extremum that is assumed for the proof of the Routh theorem, in association with the usual designation in function theory, a *uniform extremum*, where the word "uniform" refers to the dependence of the energy function on time, and signifies nothing other than that the nonzero value of k can be fixed independently of the value of time. The Routh criterion would thus be more precisely stated as *the motion is certainly absolutely stable if the energy function is an extremum with respect to all its uneliminated arguments, the position and velocity coordinates, and indeed uniformly for all values of t.*

The relation of this motion criterion to the previous equilibrium criterion is clear. If V is a minimum with respect to the collected position coordinates, then $T + V$ is also a minimum in the equilibrium position $T = 0$ with respect to all the position and velocity coordinates. If, conversely, $T + V$ is a minimum with respect to all the position and velocity coordinates, then one need only set all the velocity coordinates equal to zero to see that V must be, at the same time, a minimum with respect to the collected position coordinates. *In the case of equilibrium, the Routh criterion, in so far as it refers to a minimum of $T+V$, is therefore transformed into the Lagrange–Dirichlet criterion, and vice versa.* A specialization of the impact of the type considered above by means of equation (3) thus becomes superfluous in this special case.

The other statement of the Routh theorem, that a motion is also absolutely stable if $T+V$ represents a maximum for this motion, obviously does not come into question in the case of equilibrium. Since, namely, T is certainly a minimum in the case of equilibrium, $T+V$ cannot be a maximum. In fact, T and also $T+V$ indeed increase, if, for example, we fix the values of the position coordinates, but change the velocity coordinates in any way.

The application of the Routh motion criterion is naturally less convenient, and its importance less encompassing, than that of the equilibrium criterion, since we must assume for the former more about the character of the motion than for the latter about the character of the equilibrium, and since the occurrence of an extremal energy value for motion can generally be attained, so to speak, only by a restriction of the mobility of the system in the sense of equations (3). If we in fact demand, with Routh, that the total energy should be an extremum not only with respect to the position coordinates, but also with respect to a number of (uneliminated) velocity coordinates, then we pose, in contrast to the case of equilibrium, as many more conditions as the number of uneliminated velocity coordinates. There remains, for the more precise formulation of the motion criterion, the troublesome addition of the uniformity of the extremum for all values of t, an addition that will evidently be superfluous for the equilibrium criterion.

Correspondingly, the domain of applicability of the Routh criterion will be rather restricted. The examples that Routh gives in the cited works do not essentially differ, after the required elimination, from equilibrium problems. The motion whose stability is to be investigated is generally chosen, namely, so that it can be characterized by equating to zero all the velocity coordinates that explicitly enter into the energy expression.*) In this case, the expression for the *vis viva* that concerns

*) This corresponds, in particular, to the so-called cyclical motions, which we will later cover in detail.

the velocity coordinates is itself a minimum, directly as in the equilibrium case. It remains only to investigate the extremal properties of the energy expression with respect to the position coordinates, which is then no more difficult than the investigation of the potential energy in the case of equilibrium. The requirement of the uniformity of the extremum also becomes superfluous for such cases of motion.

The following two examples, which we take from the theory of the top, are of this special nature. They concern the repeatedly discussed cases of *upright motion* and *regular precession.* We wish to assume the top specifically as a spherical top with moment of inertia A. The energy expression then runs

$$(4)\qquad T+V=\frac{A}{2}(\vartheta'^2+\sin^2\vartheta\cdot\psi'^2+(\varphi'+\cos\vartheta\cdot\psi')^2)+P\cos\vartheta.$$

The invariability of the impulse components n and N for the specified motion implies, according to page 222, the relations

$$(5)\qquad A(\psi'+\cos\vartheta\,\varphi')=n,\quad A(\varphi'+\cos\vartheta\,\psi')=N.$$

The disturbance should be chosen in the partial sense, so that these impulse constants are not altered. The impact should thus change merely the $[\Theta]$-component of the impulse; that is, have the line of nodes as its axis.

By means of equations (5), we now eliminate φ' and ψ' from (4). There follows, since the right-hand side of (4) and the coefficients in (5) do not contain the position coordinates φ and ψ, an expression that depends only on ϑ and ϑ'; namely,

$$(4')\qquad T+V=\frac{1}{2A}\left[A^2\vartheta'^2+\frac{(n-N\cos\vartheta)^2}{\sin^2\vartheta}+N^2+2AP\cos\vartheta\right].$$

We must examine the extremal properties of this function of two variables.

For the *upright motion* ($\vartheta=\vartheta'=0$), $n=N$. Since this relation is not altered by the impact, the simplified energy expression for the disturbed motion follows from (4′) as

$$2A(T+V)=A^2\vartheta'^2+\frac{N^2}{\cos^2\vartheta/2}+2AP\cos\vartheta.$$

Applying the well-known rule for seeking the maxima and minima of a function of two variables, we expand the previous expression about the position $\vartheta = \vartheta' = 0$ according to Taylor's theorem, and obtain

$$2A(T+V) = N^2 + 2AP + A^2\vartheta'^2 + \frac{1}{4}(N^2 - 4AP)\vartheta^2 + \cdots .$$

The linear terms in ϑ and ϑ' vanish; the quadratic terms form a positive definite quadratic form if

$$N^2 - 4AP > 0. \tag{6}$$

In this case, therefore, $T+V$ is an actual minimum for $\vartheta = \vartheta' = 0$. The motion of the upright top is therefore absolutely stable, under the condition (6), with respect to partial disturbances that leave the values of N and n unchanged.

We have thus rediscovered our earlier stability criterion, indeed in a less sharp form, in that the symbol $>$ appears instead of the symbol $\geq$. That the value of ϑ in the altered motion always lies in the neighborhood of the original value $\vartheta = 0$ under the condition (6) is sufficiently known to us from §§4 and 5. Our earlier considerations show, moreover, that the upright motion in the case $N^2 - 4AP \geq 0$ is also to be designated as stable, if not as absolutely stable, with respect to the coordinates φ and ψ and for a change of N, and that the upright motion is labile in the case $N^2 - 4AP < 0$. Our current consideration naturally gives no conclusion on the latter points.

We next consider the example of *regular precession.* This motion is characterized by $\vartheta' = 0$ and ϑ equal to the constant value ϑ_0 that is determined from the equation $A\mu\nu = P$, or (cf. page 279)

$$\frac{n - N\cos\vartheta_0}{\sin^2\vartheta_0} \cdot \frac{N - n\cos\vartheta_0}{\sin^2\vartheta_0} = AP. \tag{7}$$

We imagine this motion again disturbed by an impact that leaves the impulse constants N and n unchanged, and merely influences the impulse component $[\Theta]$ that does not explicitly appear in the energy function. We expand the expression $2A(T+V)$ about the position $\vartheta = \vartheta_0$, $\vartheta' = 0$ according to Taylor's theorem. The constant term, which signifies the amount of energy for the regular precession, is, according to equation $(4')$,

$$a_0 = \left(\frac{n - N\cos\vartheta_0}{\sin\vartheta_0}\right)^2 + N^2 + 2AP\cos\vartheta_0;$$

this is inessential for the following. We next seek the terms of the first order in $\vartheta - \vartheta_0$ and ϑ', which are of the form

$$a_1(\vartheta - \vartheta_0) + a_2\vartheta'.$$

The coefficient a_2 obviously vanishes. The coefficient a_1 is equal to the value of

$$\frac{\partial\{2A(T+V)\}}{\partial\vartheta}$$

for $\vartheta = \vartheta_0$. If one calculates this differential quotient, one finds, without trouble,

$$2\sin\vartheta\left\{\frac{n-N\cos\vartheta}{\sin^2\vartheta}\cdot\frac{N-n\cos\vartheta}{\sin^2\vartheta}-AP\right\}. \tag{8}$$

This expression, however, vanishes because of equation (7). The terms of the first order therefore vanish, as required for the occurrence of a maximum or minimum.

The terms of the second order then have the form

$$a_{11}(\vartheta-\vartheta_0)^2 + 2a_{12}(\vartheta-\vartheta_0)\vartheta' + a_{22}\vartheta'^2.$$

Here one sees immediately that $a_{22} = A^2$ and $a_{12} = 0$. It remains, therefore, only to calculate

$$a_{11} = \frac{1}{2}\left(\frac{\partial^2\{2A(T+V)\}}{\partial\vartheta^2}\right)_{\vartheta=\vartheta_0}.$$

If we carry out a repeated differentiation with respect to ϑ in (8) and set $\vartheta = \vartheta_0$, there follows

$$\begin{aligned} a_{11} &= \sin\vartheta_0\frac{d}{d\vartheta_0}\left(\frac{n-N\cos\vartheta_0}{\sin^2\vartheta_0}\cdot\frac{N-n\cos\vartheta_0}{\sin^2\vartheta_0}\right) \\ &= \frac{(N^2+n^2-2Nn\cos\vartheta_0)(1+3\cos^2\vartheta_0)}{\sin^4\vartheta_0}. \end{aligned}$$

This expression is certainly positive. Namely, the first factor of the numerator signifies the square of the line segment that we obtain if we join the endpoint of the impulse component N with the endpoint of the impulse component n; the remaining factors are obviously likewise positive. Thus the terms of the second order

$$\frac{(N^2+n^2-2Nn\cos\vartheta_0)(1+3\cos^2\vartheta_0)}{\sin^4\vartheta_0}(\vartheta-\vartheta_0)^2 + A^2\vartheta'^2$$

represent a positive quadratic form. The existence of a minimum is therefore proven. It follows from the Routh criterion that regular precession is absolutely stable with respect to the coordinate ϑ for all disturbances that leave the impulse components n and N unchanged—in conformity with the results of §1. The behavior of the trajectory for changes of n and N and with respect to the coordinates φ and ψ escapes our last consideration; we have previously seen that for such general disturbances the motion is indeed stable, but no longer absolutely stable.

The question of the converse of the Routh criterion again arises. Can we claim, for example, that the motion cannot be absolutely stable in the case that $T + V$ is not an actual minimum or maximum? This is presently not known with certainty. The comparison with the Dirichlet criterion and the reported difficulties that confront the converse of the latter make a possible converse of the Routh criterion appear as not directly promising. —

§8. On the method of small oscillations.

We now enter into the best-known method of investigation for the stability question, the so-called *method of small oscillations.* The preceding considerations for the top provide the means to understand the inner value of these important expansions that continually recur in the literature. The method of small oscillations developed historically from the consideration of the pendulum, whose small oscillations have long been studied, and have a wide-ranging theoretical and practical significance.

We will, however, have to conceive the method of small oscillations more broadly here than in its use for the pendulum. Pendulum oscillations, namely, are *oscillations about an equilibrium position*; in contrast, we will generally speak, since the question for us is the stability of motion, of *oscillations about a state of motion.*

First, a pair of words about the method in general.

If one regards the motion whose stability is to be examined as completely known, then one imagines the position coordinates for this motion as known functions of time. One now alters the motion by an impact, and considers the differences between the position coordinates of the original and the altered motions; one assumes that these differences, together with their differential quotients with respect to time, are small quantities, since one asks for small oscillations of the system. One then expands the differential equations for these coordinate differences and simplifies the equations through the neglect of the higher powers of the assumed small quantities. It occurs, in certain rather general cases, that the simplified differential equations can easily be integrated. From their solution one judges the character of the altered motion, and thus draws a conclusion on the stability, or actually the absolute stability, of the originally given state of motion. To be more precise, we wish to discuss this method for the previously treated *problem of the upright top*, which can indeed be regarded as a direct generalization of the usual pendulum problem.

We first remark, as on page 316, that the Euler angles φ, ψ, ϑ are not very appropriate for the treatment of the upright top, since the coordinates φ and ψ lose an obvious meaning in the original upright position. We therefore again use the combination

$$\varphi + \psi = \chi. \tag{1}$$

The quantities

$$\begin{cases} x = \sin\vartheta\cos\psi, \\ y = \sin\vartheta\sin\psi \end{cases} \tag{2}$$

of page 331 may serve as further position coordinates; these quantities signify the rectangular coordinates of the projection of the apex of the top onto the equatorial plane. In these coordinates, the original motion is characterized by the equations

$$x = y = 0, \quad C\chi' = N.$$

We now imagine that the motion is altered by a disturbance, where we again wish (as on page 361) to disregard a change of the angular velocity χ'. Then the values of x and y are themselves the differences between the position coordinates of the original and the altered motions. It is now a matter, above all, of establishing the differential equations for x and y; that is, for the altered motion. We use the schema of the general Lagrange equations. It is thus required to know the expression for the *vis viva* and the components of gravity in terms of the coordinates x, y, and χ.

According to equation (6) of page 156, we have for the symmetric top, which we prefer here over the spherical top so as to conveniently comprise the pendulum in the calculation,

$$T = \frac{A}{2}(\sin^2\vartheta\cdot\psi'^2 + \vartheta'^2) + \frac{C}{2}(\varphi' + \cos\vartheta\cdot\psi')^2.$$

From (1) and (2) now follow

$$\sin\vartheta = \sqrt{x^2+y^2}, \quad \operatorname{tg}\psi = \frac{y}{x},$$

$$\vartheta' = \frac{xx' + yy'}{\sqrt{(x^2+y^2)(1-x^2-y^2)}}, \quad \psi' = \frac{xy' - yx'}{x^2+y^2}, \quad \varphi' = \chi' - \frac{xy' - yx'}{x^2+y^2}.$$

Thus

$$T = \frac{A}{2}\Big(\frac{(xy'-yx')^2}{x^2+y^2} + \frac{(xx'+yy')^2}{(x^2+y^2)(1-x^2-y^2)}\Big)$$
$$+ \frac{C}{2}\Big(\chi' - (1-\sqrt{1-x^2-y^2})\frac{xy'-yx'}{x^2+y^2}\Big)^2.$$

On the other hand, we calculate the potential energy V in terms of the new coordinates x and y. We have

$$V = P\cos\vartheta = P\sqrt{1-x^2-y^2}.$$

The components of the gravitational force with respect to the coordinates x and y are given in a well-known manner as partial differential quotients of this value of V.

In the construction of the differential equations, we wish to neglect all higher powers of the assumed small quantities x, y, and the differential quotients x' and y'. *This is equivalent to expanding the expressions for T and V with respect to x, x', y, and y', and retaining only the quadratic terms in the expansion.* If we do this, then the denominators cancel, and the expressions simplify to

$$\text{(3)}\qquad \begin{cases} T = \dfrac{A}{2}(x'^2+y'^2) + \dfrac{C}{2}\big(\chi'^2 + \chi'(xy'-yx')\big), \\ V = P - \dfrac{P}{2}(x^2+y^2). \end{cases}$$

From these expressions we calculate the impulse components $[X]$, $[Y]$, $[\chi]$ and the gravity components X and Y. (The χ-component is evidently equal to zero.) There follow

$$\begin{aligned} [X] &= \frac{\partial T}{\partial x'} = Ax' - \frac{C}{2}\chi' y, \\ [Y] &= \frac{\partial T}{\partial y'} = Ay' + \frac{C}{2}\chi' x, \\ [\chi] &= \frac{\partial T}{\partial \chi'} = C\big(\chi' + \frac{1}{2}(xy'-yx')\big), \\ X &= -\frac{\partial V}{\partial x} = Px, \\ Y &= -\frac{\partial V}{\partial y} = Py, \end{aligned}$$

where we will strike out, in the given value for $[\chi]$, the terms of the second order in x and y compared to χ' in a consistent manner.

The Lagrange equation for the χ-coordinate is now simply

$$\text{(4)}\qquad \frac{d[\chi]}{dt} = 0, \quad \text{or} \quad C\chi' = \text{const.} = N.$$

If we further calculate

$$\frac{\partial T}{\partial x} = \frac{C}{2}\chi' y' \quad \text{and} \quad \frac{\partial T}{\partial y} = -\frac{C}{2}\chi' x',$$

then the other two Lagrange equations

$$\frac{d[X]}{dt} - \frac{\partial T}{\partial x} = X, \quad \frac{d[Y]}{dt} - \frac{\partial T}{\partial y} = Y$$

take, with consideration of (4), the form

$$\tag{5} \begin{cases} Ax'' - Ny' = Px, \\ Ay'' + Nx' = Py. \end{cases}$$

The differential equations have, as we see, an extremely simple structure; they are *homogeneous linear differential equations with constant coefficients.*

We may first recall the special case of the pendulum. If we set $C = 0$ and therefore also $N = 0$, and further set $A = ml^2$, $P = -mgl$, in that we imagine a pendulum of length l whose mass particle m is perpendicularly beneath the support point in its rest position, there follow from (5)

$$\tag{5'} \begin{cases} Ax'' = Px, \qquad x'' = -\dfrac{g}{l}x, \\ \qquad\qquad\text{or} \\ Ay'' = Py, \qquad y'' = -\dfrac{g}{l}y. \end{cases}$$

Integration immediately yields the well-known oscillation law of the pendulum for sufficiently small amplitude.

Equations (5) differ from (5′), as one sees, through the appearance of terms in x' and y'. These terms, we can say, show us the existence of a rotation ("gyration" about the axis of the top); they are thus designated, in the work of Thomson and Tait, as "*gyroscopic terms.*" We later intend to enter in detail into the interesting theory of these terms. It is noted here that the coefficients in (5), namely $\begin{vmatrix} 0 & -N \\ +N & 0 \end{vmatrix}$, form a so-called skew determinant.

Our differential equations (5) are solved according to a well-known rule in the following manner.*) One first combines the two equations

*) The following calculations are typical for the integration of an arbitrary system of linear differential equations with constant coefficients. The technique of such integrations is very broadly developed in the second volume of the Rigid Dynamics of Routh.

(5) into one, in that one multiplies by 1 and i, respectively, and adds. The complex combination $x + iy$ is called z. There follows

$$Az'' + iNz' = Pz. \tag{6}$$

One now sets, by way of trial,

$$z = ae^{i\lambda t}; \tag{7}$$

equation (6) then yields the condition

$$A\lambda^2 + N\lambda + P = 0 \tag{8}$$

for the quantity λ. There follow two values $\lambda = \mu$ and $\lambda = \mu'$, where

$$\left.\begin{matrix}\mu\\ \mu'\end{matrix}\right\} = \frac{-N \pm \sqrt{N^2 - 4AP}}{2A}, \tag{9}$$

and two particular solutions

$$z = ae^{i\mu t} \quad \text{and} \quad z = a'e^{i\mu' t}$$

of (6), each with an arbitrary constant a or a'; the general solution is composed from these two solutions by superposition. The general solution of (6) thus runs, if we separate a and a' into real and imaginary parts ($a = \alpha - i\beta$, $a' = \alpha' - i\beta'$),

$$z = (\alpha - i\beta)e^{i\mu t} + (\alpha' - i\beta')e^{i\mu' t}. \tag{10}$$

One now has two principal cases to distinguish, according to whether μ and μ' are real or complex. *The former occurs if* $N^2 - 4AP > 0$, *the latter if* $N^2 - 4AP < 0$.

I) If $N^2 - 4AP > 0$, we write instead of (10), in that we resolve z into its real and imaginary parts,

$$\begin{cases} x = \alpha\cos\mu t + \beta\sin\mu t + \alpha'\cos\mu' t + \beta'\sin\mu' t, \\ y = \beta\cos\mu t - \alpha\sin\mu t + \beta'\cos\mu' t - \alpha'\sin\mu' t. \end{cases} \tag{11}$$

As initial conditions we prescribe, for example,

$$x = y = y' = 0.$$

There then follow from (11)

$$\alpha + \alpha' = 0, \quad \beta + \beta' = 0, \quad \mu\alpha + \mu'\alpha' = 0;$$

that is,

$$\alpha = \alpha' = 0, \quad \beta = -\beta' = \frac{\eta}{2}.$$

The introduced and still disposable quantity η corresponds to the undetermined magnitude of the initial velocity x'.

There now follow from (11)

$$\begin{cases} x = \dfrac{\eta}{2}\,(\sin\mu t - \sin\mu' t) = \quad \eta\cos\dfrac{\mu + \mu'}{2}\,t \cdot \sin\dfrac{\mu - \mu'}{2}\,t, \\ y = \dfrac{\eta}{2}\,(\cos\mu t - \cos\mu' t) = -\,\eta\sin\dfrac{\mu + \mu'}{2}\,t \cdot \sin\dfrac{\mu - \mu'}{2}\,t. \end{cases} \tag{12}$$

If we calculate, finally, $\dfrac{\mu+\mu'}{2}$ and $\dfrac{\mu-\mu'}{2}$ from (9), then we can write for (12)

$$(13)\qquad \begin{cases} x = \eta \cos \dfrac{N}{2A}t \cdot \sin \dfrac{\sqrt{N^2-4AP}}{2A}t, \\ y = \eta \sin \dfrac{N}{2A}t \cdot \sin \dfrac{\sqrt{N^2-4AP}}{2A}t. \end{cases}$$

Thus we are led back exactly to equations (11) on page 331.

II) We consider now the second principal case $N^2-4AP<0$, where μ and μ' become complex. We set

$$\mu = \nu + i\nu', \quad \mu' = \nu - i\nu',$$

resolve (10) into real and imaginary parts, and obtain

$$(14)\qquad \begin{cases} x = +\,\{\alpha e^{-\nu' t} + \alpha' e^{+\nu' t}\}\cos \nu t + \{\beta e^{-\nu' t} + \beta' e^{+\nu' t}\}\sin \nu t, \\ y = -\,\{\beta e^{-\nu' t} + \beta' e^{+\nu' t}\}\cos \nu t + \{\alpha e^{-\nu' t} + \alpha' e^{+\nu' t}\}\sin \nu t. \end{cases}$$

From the initial conditions

$$x = y = y' = 0$$

there follow

$$\alpha + \alpha' = \beta + \beta' = -\beta + \beta' = 0;$$

that is,

$$\beta = \beta' = 0, \quad \alpha = -\alpha' = \frac{\eta}{2}.$$

Thus

$$(15)\qquad \begin{cases} x = \eta\Big(\dfrac{e^{-\nu' t} - e^{+\nu' t}}{2}\Big)\cos \nu t, \\ y = \eta\Big(\dfrac{e^{-\nu' t} - e^{+\nu' t}}{2}\Big)\sin \nu t. \end{cases} \qquad \begin{cases} \nu \;= -\dfrac{N}{2A}, \\ \nu' = \dfrac{\sqrt{4AP-N^2}}{2A}. \end{cases}$$

III) Finally, we must also complete the calculation for the boundary case $N^2-4AP=0$, in which μ and μ' coincide. One must assume, in this case, the complete solution of the differential equation (6) with the required number of arbitrary constants in the form

$$z = (a + a't)e^{i\mu t}.$$

For the real and imaginary parts of z there now result, if we again set $a = \alpha - i\beta$, $a' = \alpha' - i\beta'$,

$$x = +\,(\alpha + \alpha' t)\cos \mu t + (\beta + \beta' t)\sin \mu t,$$

$$y = -\,(\beta + \beta' t)\cos \mu t + (\alpha + \alpha' t)\sin \mu t.$$

Under the previous initial conditions there follow, further,

$$\alpha = \beta = \beta' = 0,$$

and, if we replace α' by η,

$$\begin{cases} x = \eta t \cos \mu t, \\ y = \eta t \sin \mu t, \end{cases} \quad \mu = -\frac{N}{2A}. \tag{16}$$

We now wish to discuss the results in equations (13), (15), *and* (16), *acquired in the spirit of the method of small oscillations.*

The motion of the apex of the top represented by (13) is completely periodic in its temporal course. It therefore represents an oscillation, and indeed a small oscillation, since the maximum distance η of the apex of the top from its original position is always smaller as we choose the initial value of x'—that is, the initial disturbance—smaller. According to (12), moreover, we can conceive this oscillatory process as the superposition of two simple harmonic oscillations ("fundamental oscillations") of periods $\dfrac{2\pi}{\mu}$ and $\dfrac{2\pi}{\mu'}$.

It is different with the motions given by (15) and (16). These motions consist of a periodic and an aperiodic component. The latter causes the distance

$$\sqrt{x^2 + y^2} = \eta \, \frac{e^{-\nu' t} - e^{+\nu' t}}{2} \quad \text{or} = \eta t, \text{ respectively,}$$

of the point x, y from the stationary position to become larger and larger with increasing time; more precisely said, this distance exceeds, for sufficiently large t, any arbitrary bound, as small as the original impact may be. The motion is then no oscillation, and certainly no small oscillation. The trajectory has, rather, a spiral form. Equation (15) represents, in essence, a logarithmic spiral, and equation (16) an Archimedean spiral.

What conclusion is now to be drawn with respect to the stability of the upright motion of the top? We first wish to take a fully naive standpoint, from which the neglect of the higher powers, generally common in scientific calculations, is accepted without misgiving. From this standpoint, we will pronounce equations (13), (15), and (16) as approximate, if not exact, descriptions of the actual trajectories, and will directly say that

In the first principal case $N^2 - 4AP > 0$, the motion of the apex of the top is stable, and indeed, in our terminology, absolutely stable. In the second principal case $N^2 - 4AP < 0$ and the boundary case $N^2 - 4AP = 0$, in contrast, the motion is unstable.

We must now take a position with respect to this manner of conclusion.

If we first judge the manner of conclusion according to its results, then we must say that *the distinction between the stable and labile cases is generally correct, but also only generally.* In fact, the inequality $N^2 - 4AP \gtrless 0$ is indeed our well-known stability criterion. The boundary case $N^2 - 4AP = 0$, however, is falsely classified here; it appears to belong to the labile cases, while it is to be reckoned by more rigorous methods (cf. page 323) entirely in the stable cases.

We see, further, what form of the trajectory is given: *in the first principal case the motion of the apex of the top is given by our present method correctly; that is, with ever greater approximation as the original impulse is smaller. In the second principal case and in the boundary case, in contrast, our present formulas provide an entirely false image of the motion.* In fact, for example, the qualitative character of the trajectory for $N^2 - 4AP = 0$ is not at all different in a more rigorous treatment from the trajectory for $N^2 - 4AP > 0$. Also, the trajectories in the case $N^2 - 4AP < 0$ have, in general, the same periodicity properties as the stable trajectories in the case $N^2 - 4AP > 0$; the spiral character that these trajectories should in general have, according to the present formulas, exists in actuality only in a particular special case. One can also recall that the quantities x and y must certainly be, according to their geometric meaning, smaller than 1, while they are capable of arbitrary values as a result of the formulas (15) and (16).

The situation is still worse if we examine our procedure according to its inner correctness. We consider first the alleged evidence, from the method of small oscillations, that the upright motion of the top is stable in the case $N^2 - 4AP > 0$.

In that we allowed omissions in the differential equations of the motion, or in the expressions for T and V, that are only (or at most) correct in the stable case, we make from the outset the assumption that the motion is stable. We then carry out the calculation given above, and find that in the case $N^2 - 4AP > 0$ the result of the calculation does not directly contradict our original assumption. If we would now conclude, in reverse, the correctness of that assumption, then we would be guilty of an obvious "circulus vitiosus." Nevertheless, this conclusion is regularly made in the method of small oscillations.

Our criticism formulated here is not something new. We wish to cite, in this respect, a few words of Dirichlet that occur at the beginning of the previously cited work on the stability of equilibrium. The equilibrium criterion discussed in the previous section was originally based by Lagrange directly on the method of small oscillations. Dirichlet argues the indefensibility of this basis and remarks, amidst other comments, that "It can be doubted, with justification, whether quantities for which one finds small bounds under the *assumption* that they always remain small—for only in this lies the authority to neglect the higher terms—will actually be enclosed after an arbitrary time in these bounds, or generally even in narrow bounds."*) Exactly the same objection may be raised against most modern works in which the method of small oscillations is employed for the investigation of the stability of motion.

One see in this example, moreover, how long it takes until the results of rigorous mathematical research find entrance and consideration in the applied sciences.

Apparently more favorable are the prospects for the alleged evidence from our method that the motion is unstable in the case $N^2 - 4AP \leq 0$. We make at the beginning of the calculation, by the neglect of the higher terms, the explicit assumption that the values of x and y always remain small, or, more correctly said, can be made arbitrarily small through the choice of the impact. This assumption is now led in an unambiguous manner ad absurdum by the result of the calculation in the case $N^2 - 4AP \leq 0$. The conclusion thus appears justified that x and y do not always remain small, and that the motion in this case is unstable.

But strictly speaking, it is not the inadmissibility of the assumption of small x and y that is shown, but rather the inadmissibility of the omissions that were made. It can very well be that x and y, and therefore also the higher terms in the relevant expansions, always remain arbitrarily small, but cannot be made arbitrarily small with respect to the first retained terms, in so far as, namely, the latter vanish identically for particular values of the constants. In this case, the neglect of the higher terms would obviously be unjustified; the motion can appear as

*) Jacobi is extremely similar in the fourth Vorlesung über Dynamik. Cf. Ges. W. Supplementb. p. 30.[162]

unstable according to the method of small oscillations, while it can in reality be stable.

Such a case is directly present in the boundary case $N^2 - 4AP = 0$, for which the stability judgment, as it is provided by the method of small oscillations, is indeed delusional.

Thus we will formulate our criticism of the method in the following summary manner: *Neither the cases appearing as stable in this method are proven as stable, nor are the cases appearing as labile in actuality always labile. The method therefore says, strictly speaking, nothing about the stability and lability of motion.*

The usefulness of the method, judged from this rigorous standpoint, is merely that it provides a convenient path to approximation formulas for many stable cases (in our example, the cases $N^2 - 4AP > 0$), where, however, the degree of approximation and the validity of the formulas first remain unconfirmable.

From a more practical standpoint, however, one is admittedly obliged to essentially modify this verdict. As long as one has no general method that is free of exceptions, one must adopt a nonrigorous method, especially for the problems of the greatest interest that have until now been treated with the method of small oscillations, and cannot be simply passed over.

Our criticism naturally applies only to the current state of the method, and not to the method itself. This has, without question, a valuable kernel of truth, which, freed of scoria, promises not only provisional, but also reliable conclusions on the interesting questions of modern mechanics. It will presumably only be necessary to add some restriction and sharpening to the method.

The direction in which this sharpening is to be sought cannot be doubtful after the preceding. *One must derive the process of the motion from the exact differential equations, at least in outline, and seek to estimate, on the basis of such general knowledge of the motion, the error that one commits in the method of small oscillations through the neglect of the higher terms.* We have acquired our formulas for the approximate calculation of the motion of the top at the end of the previous chapter in such a manner; these formulas indeed prove to be identical with the approximation formulas following from the method of small oscillations in all cases in which the latter are useful.

Numerous efforts in the indicated direction are already present in the literature. We mention the works of Mr. L y a p u n o v,*) and particularly P o i n c a r é's**) investigations of celestial mechanics, as well as the modern contributions to perturbation calculations, in which, for the most part, the expansions that occur in the method of small oscillations are treated with mathematical precision. Starting points for the mathematical sharpening of the method are also found in the works of Mr. Routh,***) which are the richest repository for stability questions.

§9. On the motion of the heavy asymmetric top.

We now wish to report on the sparse results that have been acquired until now regarding the motion of the general heavy top.

The differential equations for the general top can be formed, for example, according to the schema of the Euler equations (see page 141, equation (3)). We do not wish to rewrite these equations, but only repeat their mechanical meaning: they state that the impulse change at each moment is equal to the infinitesimal turning-impact of gravity. To determine the axis and the magnitude of this turning-impact, we mark in the body the center of gravity S; it has, in the XYZ frame, the coordinates ξ, η, ζ. We assume, for the sake of generality, that the direction OS does not coincide with one of the principal axes, which are chosen as the coordinate axes. The axis of the turning-moment of gravity is then simultaneously perpendicular to OS and the vertical. Its magnitude is equal to $P\sin\vartheta$, where $P = mgE$ is the product of the weight of the top and the distance between the points O and S, and ϑ is the angle between the vertical and the half-line OS.

As in the third section of the preceding chapter, we can immediately make a statement regarding the behavior of the impulse that yields a

*) Cf. p. 356.

**) Méthodes nouvelles de la Mécanique céleste, cf., for example, Chap. IV, p. 177, where, however, the definition of stability is still formulated entirely in the sense of the method of small oscillations.[163]

***) Rigid dynamics, Part II, Chap. VII, Stability of motion, Chap. VII. The development given here already provides information about why the judgment of stability on the basis of merely the linear expansion terms can be false in our example of the boundary case $N^2 - 4AP = 0$, where the periods of the two fundamental oscillations coincide.

first integral of the equations of motion. Because the axis of the gravity moment is horizontal, the endpoint of the impulse necessarily progresses in the horizontal direction. *The vertical component of the impulse is therefore constant*, and we again have

$$n = \text{const.}$$

In addition, the theorem of the *vis viva*

$$T + V = h,$$

which we have indeed derived in the second chapter for any rigid body and any conservative force system, naturally obtains here as well.

If we wish to formulate this equation in words, then we must recall the geometric meaning of the expression for the *vis viva* (cf. page 96), and must further interpret the potential energy $V = mg \cdot E \cos\vartheta$ in an obvious manner. We can then say:

The half scalar product of the impulse vector and the rotation vector, increased by the product of the weight and the vertical elevation of the center of gravity, remains constant during the motion of the general top.

In contrast, the well-known equation $N = \text{const.}$ for the symmetric top loses its validity in our case; this was, evidently, solely a consequence of the symmetric mass distribution.

The complete analytic command of the motion of the asymmetric top is, however, not yet possible on the basis of the two integrals above. Before we speak of further investigation in this direction, it is well to take the desired goal more sharply in view. The goal must obviously be this: to acquire a clear representation of the process of the motion. That way will be best, which leads most directly to this goal.

In contrast, the goal appears to be essentially displaced in many works on mechanics (for example, those to be cited directly). One acquires the impression that the most important exercise of analytic mechanics consists in reducing a problem to quadratures, or in discovering such problems that can be solved by quadratures. In reality, however, the reduction to quadratures is but a means to the goal that is applicable in the rarest cases, and which in itself, where it is applicable, does not completely achieve its purpose if the obtained integrals have a complicated manner of construction. The one-sided emphasis on quadraturability or nonquadraturability corresponds, without question, only to the scholastic habits of mathematicians, and is not based on the matter itself.

If it occurs, in a particular case, that a problem leads to quadratures, or, more generally, to known functions, one will naturally be happy to draw use from this circumstance. One must, however, bear in mind that with the closed analytic representation of the integrals only the first step has been taken, and that the primary exercise must consist in attaining a complete geometric and mechanical understanding of the motion on the basis of this representation.

In all other cases in which a reduction to known functions is not possible, one must, in contrast, adopt another procedure—a procedure that is generally required for the integration of differential equations: one seeks, first of all, to form a representation of the *qualitative* course of the trajectories defined by the differential equations, in that one studies, for example, the singular points of the differential equations, the unstable cases of the motion, the possible periodic and asymptotic trajectories, etc. Only then does one develop, from this preliminary knowledge, appropriate convergent or nonconvergent *approximation methods* that enable the *quantitative* calculation of the trajectories with arbitrary or bounded precision. The investigations of P o i n c a r é on the three-body problem can serve here as a model; his great results are due directly to the just sketched free and generalized conception of the integration process.

In this sense, little has been accomplished for the treatment of the asymmetric top. The first works to be cited begin, rather, exclusively from cases of closed analytic representability.

Mrs. S. K o w a l e v s k i*) finds that in addition to the integrals given above, a further integral can be given in a rather simple algebraic form if the mass distribution of the top satisfies the following conditions: the ellipsoid of inertia is again an ellipsoid of rotation ($A = B$), and the center of gravity lies not on the figure axis, but rather in the equatorial plane ($\zeta = 0$); in addition,

$$2C = A\ (= B).$$

Under these assumptions, it is possible to treat of the general motion completely.

Mrs. Kowalevski**) expresses the position and velocity coordinates

*) Sur le probléme de la rotation d'un corps solide autour d'un point fixe, Acta Mathematica, Bd. 12. 1888.[164]

**) Cf. Mr. F. K ö t t e r: *Sur le cas traité par M^{me}. Kowalevski* etc. Acta Mathem. 17, 1893.[165]

of the top in terms of two auxiliary quantities, which, in turn, are related to time through integrals in which the square root of an expression of the fifth degree appears. Such integrals, which represent the nearest generalization of elliptic integrals, are designated as *hyperelliptic. In the case of Mrs. Kowalevski, the general motion of an asymmetric top with a special mass distribution may be completely represented by hyperelliptic integrals.*

The required geometric discussion was later added to this general analytic schema by Mr. Joukowsky.*) Mr. Joukowsky succeeds in describing the process of the motion through a few simple geometric theorems, and even illustrates it with a model.[166]

The general question regarding all cases of the heavy top in which a third algebraic integral is present, in addition to the two given, has recently been considered by Mr. R. Liouville.**)[167]

The investigations of Mr. Levi-Cività***) and Mr. Liebmann†) begin from another standpoint, that of the general Lie group theory. These authors ask how the mass distribution (the kinetic energy) and the force distribution (the potential energy) must be constituted if two first integrals that are linear in the velocity coordinates should be possible. In this case one is certain that the problem may be dispatched by quadratures.

The posing of the problem is broader here than in the previously cited cases, since the question concerns not only the heavy top, but also the suitable determination of the law of the exterior force. The investigation gives a total of 25 possible cases, of which, however, most are imaginary (that is, correspond to a mechanically nonrealizable mass distribution). That the question of the totality of the integrable cases is not settled is already shown by the fact that the Liebmann tabulation does not contain the case of Mrs. Kowalevski, where one reaches the goal by means of quadratures without the presence of two integrals that are linear in the velocity coordinates.

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*) Cf. Jahresbericht der deutschen Mathematikervereinigung, Bd. IV, 1895.

**) Cf. Acta Mathematica, Bd. XX, 1897.

***) Sul moto di un corpo rigido intorno ad un punto fisso. Accademia dei Lincei, 1896.

†) Cf. p. 161.

As one already recognizes from the indications given here, the investigations of Levi-Cività and Liebmann proceed entirely from the abstract mathematical side. —

While the question in the cases considered so far is to discover and analytically represent the *general* motion of the asymmetric heavy top, we now wish to report in somewhat more detail on two *particular* cases of the motion of the top that one completely commands in analytic and geometric respects. One of these cases was noted by Mr. W. H e ſ s;*) his study was later deepened by a number of Russian mathematicians.**) The other case is treated by Mr. O. S t a u d e.***) In the Heſs case, the character of the induced motion is subject to a certain simple constraint; at the same time, a particular assumption is made about the position of the center of gravity in the body. In the motions investigated by Mr. S t a u d e, in contrast, the mass distribution remains entirely arbitrary, while the character of the motion is specialized in a more extensive manner.

It is remarked in advance, moreover, that the degree of particularization in these two cases is no higher than in the Kowalevski case, the case of the weightless asymmetric top, or the case of the heavy symmetric top. Namely, *three* restricting conditions are always posed; in the three latter cases these conditions refer purely to the mass distribution, in the case of Heſs partly to the mass distribution and partly to the motion, and in the Staude case purely to the nature of the motion.[169]

To present the Heſs case of the motion of the top in a natural way,†) we wish to ask for the circumstances under which it can occur for a general top *that the impulse is always contained in a plane that is fixed in the body and passes through the support point O.*††) Analytically

*) Über die Eulerschen Bewegungsgleichungen u.s.w. Math. Ann. Bd. 37, 1890.[168]

**) Cf., in analytic respects, P. N e k r a s s o f f: *Recherches analytiques sur un cas de rotation d'un solide pesant autor d'un point fixe*, Math. Ann. Bd. 47, 1896, where further literature citations are to be found, and, in geometric respects, N. J o u k o w s k y, Jahresbericht der deutschen Mathematikervereinigung Bd. III 1892/93.

***) *Über permanente Rotationsaxen*, Crelle's Journal, Bd. 113, 1894. The same subject is treated in a Russian work of Mr. B. M l o d z i e i e w s k i, Moskau 1894. Cf. also R o u t h, Rigid dynamics, Bd. II, art. 214.

†) Cf. A. S o m m e r f e l d, Bemerkungen zum Heſs'schen Falle der Kreiselbewegung. Göttinger Nachrichten 1898.[170]

††) We first exclude the case in which the impulse vector is absolutely fixed in the body. We will treat of this case later. It leads directly to the rotations investigated by Mr. S t a u d e.

expressed, this signifies *whether and when a linear homogeneous integral in the impulse coordinates L, M, N with constant coefficients is possible.*

The position of the impulse is given at any time if we compose the currently present impulse with the infinitesimal turning-impact of gravity. The axis of the latter stands perpendicular to the vertical plane through the center of gravity S and the support point O. The endpoint of the impulse therefore always progresses perpendicularly to the axis OS, which we wish to name the "center of gravity axis."

If the impulse is always to lie in a fixed plane of the body, this plane can be none other than the normal plane to the center of gravity axis through O. This plane will be denoted by e; its equation runs, if we denote, as earlier, the coordinates of the center gravity in the principal inertial frame XYZ by ξ, η, ζ,

$$\xi L + \eta M + \zeta N = 0.$$

This is already the form of the particular integral found by Mr. Heſs.[171]

Not only, however, does the position of the impulse axis change as a result of gravity, but the position of the body, and in particular the position of the center of gravity axis, also changes as a result of the instantaneous rotation corresponding to the impulse. If the angle between the impulse axis and the center of gravity axis is always to be a right angle, then not only must the impulse endpoint progress perpendicularly to the center of gravity axis as a result of gravity, but the center of gravity must also progress perpendicularly to the impulse axis as a result of the instantaneous rotation. This occurs, however, only *if the rotation axis is continuously contained in the plane determined by the impulse and the center of gravity axes.*

Through our latter requirement, a condition will be imposed on the mass distribution of the body, which we must now investigate further. We recall, for this purpose, the geometric relation between the positions of the impulse and the rotation vectors. According to page 102, we can find the direction of the impulse vector from that of the rotation vector if we construct the tangent plane to the ellipsoid of inertia

$$AX^2 + BY^2 + CZ^2 = 1$$

at one of the intersection points with the rotation axis, and drop the perpendicular from this tangent plane; this perpendicular then gives the direction of the impulse vector. In our case it is convenient, however, to use instead of the ellipsoid of inertia, whose left-hand side gives the *expression for twice the* vis viva *in the velocity coordinates*, an ellipsoid that one calls the "reciprocal ellipsoid of inertia," whose left-hand side represents the *expression for twice the* vis viva *in the impulse coordinates.* The equation of this ellipsoid is

$$\frac{X^2}{A} + \frac{Y^2}{B} + \frac{Z^2}{C} = 1.$$

One then recognizes, exactly as on pages 101 and 102, the correctness of the following construction. To find the direction of the rotation vector from that of the impulse vector, one marks on the reciprocal ellipsoid of inertia the intersection point of the impulse axis, and constructs at this point the tangent plane to our ellipsoid; the perpendicular to this plane through O then yields the direction of the rotation vector.

We now imagine the center of gravity axis OS drawn in the reciprocal ellipsoid of inertia, and the normal plane e placed through O; the impulse vector should be contained in this plane. The plane e intersects the ellipsoid in a conical cut, which, as we will now show, must be a *circle.*

Let, namely, P be any point of the conical cut and t its tangent at P. Then OP gives a possible direction of the impulse axis. The tangent plane e' to the ellipsoid will lie through t, and the perpendicular OQ that determines the direction of the rotation axis corresponding to the impulse axis OP falls from O to this plane. According to the above condition, the lines OP, OQ, and OS must always lie in a plane. Our tangent t, however, stands perpendicular to OQ as well as OS, since it represents the intersection line of the normal planes e' and e erected to OQ and OS at Q and O. Thus t stands also perpendicular to OP. Our conical cut therefore has the property that the tangent at each of its points stands perpendicular to the radius vector from the midpoint. Our conical cut is therefore, in fact, a circle.[172]

The condition to be established for the mass distribution of the top is thus found. *If the impulse vector is always to be contained in the normal plane e to the center of gravity axis, then this plane must cut the reciprocal ellipsoid of inertia in a circle*; or, *the center of gravity must lie on the perpendicular erected at O to a circular intersection plane of the reciprocal ellipsoid of inertia.*

This is the geometric formulation of the condition in question in the form given by Mr. J o u k o w s k y (cf. the citation on page 378). Mr. H e f s expresses this fact in an analytic formulation. We arrive at the latter if we note that the two circular intersection planes through O of the reciprocal ellipsoid of inertia (under the assumption $A > B > C$) are given by the equation

$$\left(\frac{1}{A} - \frac{1}{B}\right) X^2 = \left(\frac{1}{B} - \frac{1}{C}\right) Z^2.$$

Since the center of gravity (ξ, η, ζ) should now lie on the normal to one of these planes, there follows the proportion

$$\xi^2 : \eta^2 : \zeta^2 = \frac{1}{A} - \frac{1}{B} : 0 : \frac{1}{B} - \frac{1}{C};$$

that is,

$$\eta = 0, \quad \xi^2 \left(\frac{1}{B} - \frac{1}{C}\right) = \zeta^2 \left(\frac{1}{A} - \frac{1}{B}\right).$$

These are the analytical conditions given by Mr. Heſs for the possibility of his case of motion.

The mechanical character of the Heſs case of motion is quite simple. We will see that it is a direct generalization of the well-known *pendulum motion of the symmetric top.*

We imagine, for this purpose, that the body is initially in a stable equilibrium position; the center of gravity axis is therefore directed perpendicularly downward and the body is without rotation. We then turn the center of gravity axis through any angle from the vertical and abandon the body to the influence of gravity, taking care at the beginning of the motion that either no rotation at all is present, or only such a rotational impulse is added whose component about the center of gravity axis is zero, and whose axis therefore lies in the normal plane e to the center of gravity axis. Then, as we have seen, the impulse vector always remains in the plane e if the condition is fulfilled that the center of gravity lies on a normal to the circular cut of the reciprocal ellipsoid of inertia.

If we now go over from the asymmetric to the symmetric top, then we will be led through the just given procedure to the ordinary or spherical pendulum motion. The condition for the occurrence of the pendulum motion for the symmetric top (cf. page 215, Fig. 36) is that the component N of the impulse in the direction of the figure axis (which for the symmetric top is likewise the center of gravity axis) be initially equal to zero. And indeed, the motion is the "ordinary" or the "spherical" pendulum according to whether, for an arbitrary position of the figure axis, the stationary top is abandoned to the influence of gravity without an impact or with the addition of a purely lateral impact; that is, a turning-impulse whose axis is directed perpendicularly to the figure axis, and therefore lies in the normal plane to the center of gravity axis—that is, in the equatorial plane of the symmetric top. Then the impulse always remains in the equatorial plane, since the impulse component N always retains its initial value 0. At the same time, the condition that we must add in the case of the asymmetric top with respect to the position of the center of gravity is fulfilled of itself for the symmetric top, since here the circular cut of the reciprocal ellipsoid of inertia coincides with the equatorial plane, and the center of gravity axis coincides with the figure axis that is perpendicular to the equatorial plane.

We can thus say that *the Heſs case of the motion of the top may claim a special interest, since it represents the direct generalization of the very well known pendulum motion.* Correspondingly, we will henceforth designate this case concisely as the *case of the Heſs pendulum.*

A direct identity in a more quantitative respect also obtains, in part, between the motion of the Heſs pendulum and the pendulum motion of the symmetric top, or, equivalently, the pendulum motion of an individual mass particle. *We will show, namely, that the center of gravity of the Heſs pendulum, generally speaking, moves exactly as the mass particle of a spherical pendulum.*

For the proof, we write the two generally valid theorems, the theorem of the *vis viva* and the impulse theorem $n =$ const., in an appropriate form.

We first express the *vis viva* T in a suitable manner in terms of the impulse coordinates. We choose as coordinate axes X, Y, Z not, as previously, the principal inertial axes, but rather the following three lines: the Z-axis as the center of gravity axis, the Y-axis as coinciding with the intermediate principal axis, and the X-axis as the line perpendicular to these two. In these coordinates, the expression for the *vis viva* must take the form

$$2T = \frac{L^2 + M^2}{B} + 2\lambda LN + \frac{N^2}{C'},$$

where B, λ, and C' are given by the mass distribution of the body, and, in particular, B denotes the magnitude of the intermediate principal moment of inertia. In fact, the surface $2T = 1$ is indeed our reciprocal ellipsoid of inertia; the equation of a circle must therefore follow if we set $N = 0$ in the equation of this surface. On this basis, the terms with L^2 and M^2 have the same coefficient, and the term with LM falls away. Further, the term with MN also falls away, since the

Y-axis is chosen as a principal axis, and thus the "products of inertia" with respect to this axis must equal zero (cf. page 100).

The components of the rotation vector with respect to the axes X, Y, Z, which should be denoted, according to the general rule of page 98, by p, q, r, follow from the expression for T as

$$p = \frac{\partial T}{\partial L} = \frac{L}{B} + \lambda N, \quad q = \frac{\partial T}{\partial M} = \frac{M}{B}, \quad r = \frac{\partial T}{\partial N} = \lambda L + \frac{N}{C'}.$$

For the Heſs pendulum ($N = 0$), we therefore have, in particular,

$$\text{(1)} \qquad \begin{cases} 2T = \dfrac{L^2 + M^2}{B}, \\ \quad p = \dfrac{L}{B}, \quad q = \dfrac{M}{B}, \quad r = \lambda L. \end{cases}$$

The theorem of the *vis viva* therefore takes, with the notation introduced at the beginning of this section, the form

$$\text{(2)} \qquad \frac{L^2 + M^2}{B} + 2P\cos\vartheta = 2h.$$

We further calculate the vertical projection n of the impulse. If we denote, as on page 17, the direction cosines of the vertical with respect to the coordinate axes by c, c', c'', then we have

$$\text{(3)} \qquad n = Lc + Mc' + Nc'',$$

where, according to page 19, c, c', c'' are expressed in terms of the Euler angles φ, ψ, ϑ as

$$c = \sin\vartheta\sin\varphi, \quad c' = \sin\vartheta\cos\varphi, \quad c'' = \cos\vartheta.$$

Since, moreover, $N = 0$ for the Heſs pendulum, equation (3) becomes

$$\text{(4)} \qquad n = \sin\vartheta(L\sin\varphi + M\cos\varphi).$$

This quantity is, according to our general impulse theorem, a constant.

We next draw upon the two kinematic equations (9) for ψ' and ϑ' on page 45. If we insert for p and q the values from (1), then those equations become

$$\text{(5)} \qquad \begin{cases} B\psi'\sin\vartheta = L\sin\varphi + M\cos\varphi, \\ B\vartheta' \qquad\; = L\cos\varphi - M\sin\varphi. \end{cases}$$

The first of these equations yields, with consideration of (4),

$$\text{(4')} \qquad B\psi'\sin^2\vartheta = n.$$

We further conclude from the two equations

$$L\cos\varphi - M\sin\varphi = B\vartheta',$$
$$L\sin\varphi + M\cos\varphi = \frac{n}{\sin\vartheta},$$

which are identical with (5), if we multiply them by 1 and $-i$ and add, that

$$(L - iM)e^{-i\varphi} = \frac{B\sin\vartheta\,\vartheta' - in}{\sin\vartheta}, \tag{6}$$

and, if we square them and add, that

$$L^2 + M^2 = \frac{B\sin^2\vartheta\,\vartheta'^2 + n^2}{\sin^2\vartheta}. \tag{7}$$

With the latter values we enter equation (2) for the *vis viva.* If we use the abbreviation $u = \cos\vartheta$, then we obtain

$$(Bu')^2 = 2hB(1-u^2) - 2Pu(1-u^2) - n^2. \tag{8}$$

At the same time, equation (6) becomes

$$B\psi' = \frac{n}{1-u^2}. \tag{9}$$

The two equations (8) and (9) already contain the proof of our previous claim. They are, namely, identical with the equations of the pendulum motion that one obtains from the general integrals (4), (6), (7) for the symmetric top on page 222, if one sets there $N = 0$ (and, moreover, to bring the constants into conformity, changes A into B).

The motion of the center of gravity is completely determined by the angles ψ and ϑ. This can therefore be regarded, according to equations (8) and (9), as known. To completely command the motion of the Heſs pendulum, we must still investigate the rotation of the body about the center of gravity axis. This rotation is given, according to equation (1), by $r = \lambda L$. The following elegant path now presents itself for the study of the quantity r.

We know that the impulse J is always contained in the plane e, the normal plane to the center of gravity axis. In this plane we imagine extending, in the Gauſsian manner, the complex variable

$$J = L + iM.$$

The behavior of this variable will be governed by the Euler equations, which can be combined into a single differential equation for our complex variable.

Since our axes X, Y, Z are not principal axes, we use the form of the Euler equations that is valid for arbitrary axes; that is, the equations

$$(10)\qquad \begin{cases} \dfrac{dL}{dt} = \quad rM - qN + \mathsf{\Lambda}, \\ \dfrac{dM}{dt} = -rL + pN + \mathsf{M}, \\ \dfrac{dN}{dt} = \quad qL - pM + \mathsf{N} \end{cases}$$

given by (3′) on page 141.

Here we must insert for $\mathsf{\Lambda}$, M, N the components of the gravity moment with respect to our coordinate axes X, Y, Z. The magnitude of this moment is $P\sin\vartheta$, and its direction falls on the line of nodes. Since the latter encloses the angle φ with the X-axis and the angle $\frac{\pi}{2}+\varphi$ with the Y-axis (cf. Fig. 3 of page 18), we have

$$\begin{aligned} \mathsf{\Lambda} &= \quad P\sin\vartheta\cos\varphi = \quad P\sqrt{1-u^2}\,\cos\varphi, \\ \mathsf{M} &= -P\sin\vartheta\sin\varphi = -P\sqrt{1-u^2}\,\sin\varphi, \\ \mathsf{N} &= \quad 0. \end{aligned}$$

Equations (10) thus take, if we insert for p, q, r the values from (1) and furthermore set $N=0$, as corresponds to the Heſs case, the form

$$(11)\qquad \begin{cases} \dfrac{dL}{dt} = \quad \lambda LM + P\sqrt{1-u^2}\,\cos\varphi, \\ \dfrac{dM}{dt} = -\lambda L^2 - P\sqrt{1-u^2}\,\sin\varphi, \end{cases}$$

while the third equation is fulfilled identically.

We now multiply equations (11) by 1 and i and add; there follows

$$(12)\qquad \frac{dJ}{dt} + i\lambda LJ - P\sqrt{1-u^2}\,e^{-i\varphi} = 0.$$

Here we write

$$\begin{aligned} L &= \frac{1}{2}(L+iM+L-iM) = \frac{1}{2}\Big(L+iM+\frac{L^2+M^2}{L+iM}\Big) \\ &= \frac{1}{2}\Big(J+\frac{L^2+M^2}{J}\Big), \end{aligned}$$

and replace L^2+M^2, according to the theorem of the *vis viva* (eqn. (2)) by $2B(h-Pu)$. In a similar manner, there results from (6)

$$\begin{aligned} e^{-i\varphi} &= \frac{-Bu'-in}{\sqrt{1-u^2}}\,\frac{1}{L-iM} = \frac{-Bu'-in}{\sqrt{1-u^2}}\,\frac{J}{L^2+M^2} \\ &= \frac{-Bu'-in}{\sqrt{1-u^2}}\,\frac{J}{2B(h-Pu)}. \end{aligned}$$

Thus equation (12) goes over into the definitive form

$$\frac{dJ}{dt} + \frac{i\lambda}{2}J^2 + \frac{P}{2B}\frac{Bu' + in}{h - Pu}J + i\lambda B(h - Pu) = 0. \tag{13}$$

This is a so-called *Riccati equation*, a differential equation of the first order and the second degree. The coefficients of this equation are regarded as known functions of t. Through equation (8), namely, $\frac{du}{dt}$ is represented as

$$\frac{du}{dt} = \sqrt{U},$$

where U is a polynomial of third degree in u. Thus t follows as the elliptic integral

$$t = \int \frac{du}{\sqrt{U}}.$$

In reverse, u and u' are therefore also determined as functions of t, and indeed, as we will see in the next chapter (cf. §3), as so-called doubly periodic functions of t.

Finally, is is convenient to go over from our differential equation of the first order and the second degree to a differential equation of the second order and the first degree, as is usual in the treatment of Riccati equations. This is accomplished in our case by the substitution

$$J = \frac{i\lambda}{2}\frac{d\log w}{dt},$$

which is typical for the transformation in question. Written in terms of the new complex variable w, our differential equation is

$$\frac{d^2w}{dt^2} + \frac{P}{2B}\frac{Bu' + in}{h - Pu}\frac{dw}{dt} + i\lambda B(h - Pu)w = 0. \tag{14}$$

We thus arrive at a so-called *linear homogeneous differential equation of the second order with doubly periodic coefficients.* A thorough analytic investigation of this equation has been made. We cannot develop this in detail, but rather refer, in this respect, to the previously cited work of Mr. N e k r a s s o f f.

Without the use of equation (14), the Heſs case is studied very thoroughly in a purely geometric manner and illustrated convincingly by a model by Mr. J o u k o w s k y in the previously cited work.[173] The transmitted theorem on the motion of the center of gravity of the Heſs pendulum, in addition to other beautiful results, is due to this author.

We now come to the second of the previously mentioned particular cases of motion, that treated by Mr. S t a u d e. Mr. Staude finds that a heavy top of arbitrary mass distribution can always be given a simple infinity of motions that consist of a uniform rotation about a vertically placed axis fixed in the body.

We wish to derive the result of Mr. Staude geometrically from the impulse theory. *The question is to determine which axes, vertically directed, can act as permanent rotation axes for the generalized top.*[174]

We direct our attention to the position of the impulse in the body. If the rotation vector is to remain constant in direction and magnitude in space and therefore also in the body, then the impulse must also retain its magnitude as well as its position with respect to the top, since indeed the vector of the impulse can be derived in an unambiguous way from the rotation vector. The criterion for the possibility of the simple rotation about a vertical axis will therefore be *that the endpoint of the impulse vector has a fixed position in the body.* (In space this point then describes, naturally, a circle about the vertical.)

The endpoint of the impulse vector is called J, and that of the rotation vector R. The possible change of position of the point J with respect to the body now depends, as already remarked for the Heſs pendulum, on two circumstances: the action of gravity, on the one hand, and the instantaneous rotation of the top on the other hand. These two circumstances must cancel one another if the motion is to have the assumed nature.

Because of the action of gravity, the displacement of the point J in space is, as emphasized above, simultaneously perpendicular to the center of gravity axis OS and the vertical, or, as we can also say in the present case, simultaneously perpendicular to OS and OR. Because of the instantaneous rotation of the top, the point J would be led, if it were fixed in space, relative to the body in a circle about the vertical; the displacement of point J due to this circumstance is simultaneously perpendicular to OR and OJ. If the two displacements are to cancel, then their directions, above all, must coincide. The three lines OR, OS, and OJ must therefore have a common perpendicular; that is, *the three named directions must lie in a plane.* We are thus led to the same condition as in the case of the Heſs pendulum (cf. page 379). The conclusions that we now draw, however, are different, due to the changed starting point compared to the previous. While the requirement that the axes OR, OS, and OJ should lie in a plane for the Heſs pendulum was to be fulfilled only through the specialization of the mass distribution, the same requirement now, since the points R and J lie fixed in the body, is to be satisfied for an arbitrary mass distribution by the appropriate choice of R.

If we denote, as usual, the coordinates of R and J in the principal coordinate frame by p, q, r and L, M, N, respectively, then our condition states analytically that the equation

$$\begin{vmatrix} p, & q, & r \\ L, & M, & N \\ \xi, & \eta, & \zeta \end{vmatrix} = 0 \tag{15}$$

must obtain. We now conceive this equation as a condition for the position of the rotation axis. If we express L, M, N in the well-known manner in terms of p, q, r, then we see that the possible permanent rotation axes are lines that lie on a *cone of the second degree*, whose equation we can write as

$$(A - B)\zeta pq + (B - C)\xi qr + (C - A)\eta rp = 0. \tag{16}$$

Geometrically, a cone of the second degree is determined if we know five of its rays. Five such rays are easily found in our case. The three directions OR, OJ, OS certainly lie in a plane if two of them coincide. If OR and OJ coincide, then their common direction is a principal axis. Thus the three principal axes lie on our cone of the second degree. If OJ is identical with OS, then OR lies in an entirely determined direction OS', which, with the help of the ellipsoid of inertia, can be easily constructed, and passes through the point with coordinates $\frac{\xi}{A}, \frac{\eta}{B}, \frac{\zeta}{C}$. Therefore this line also lies on our cone, as does, naturally, the center of gravity axis itself. Our cone can therefore be constructed from the three principal axes and the lines OS and OS'; *it is known if the mass distribution of the body is given.*

In addition to the *directions* of the two previously named component displacements of the point J, the *magnitudes* of these displacements must also coincide, and their senses be opposite, if our rotation axis is to be permanent. From this condition follows the magnitude of the angular velocity with which the body can rotate about the axis in question. Through the action of gravity, the point J will attain, as we know, the displacement

$$P \sin\vartheta \; dt \tag{17}$$

in the time dt. As a consequence of the instantaneous rotation, which, calculated in the clockwise sense, may have the angular velocity Ω, J will progress with the velocity

$$-\Omega |J| \sin\psi$$

with respect to the body, where ψ denotes the angle between the rotation axis and the impulse axis, $|J|$ the length of the impulse, and thus $|J|\sin\psi$ the distance of the point J from the vertical. From this velocity there results, during the time dt, the displacement

$$-\Omega |J| \sin\psi \, dt. \tag{17$'$}$$

Through the comparison of expressions (17) and (17′) follows the further constraint equation

$$\Omega |J| \sin\psi = P \sin\vartheta. \tag{18}$$

If we have now chosen any generator of our cone as the rotation axis and positioned it vertically upward, then the position of the corresponding impulse J (that is, the angle ψ) is determined and is independent of the rotational speed Ω imparted to the top. The magnitude of the impulse, in contrast, depends, according to the construction described on page 101, on the magnitude of the rotation, and indeed will simply be $|J| = \lambda\Omega$, where λ signifies a positive proportionality factor independent of Ω. Equation (18) thus becomes

$$\lambda\Omega^2 \sin\psi = P \sin\vartheta. \tag{18$'$}$$

From this equation, Ω^2 is to be determined for any axis of our cone.

Thus Ω^2 can have either a positive or a negative value; that is, Ω can have a real or an imaginary value. The rays (or, more correctly, half-rays) of our cone are thus divided into two classes, the "permissible" and "impermissible" rotation axes. *Only the "permissible" half-rays, for which Ω is real, can, directed vertically upward, be actual permanent rotation axes.*

One easily sees that two opposing half-rays of our cone, taken as rotation axes, give opposite signs of Ω^2. In fact, the direction of the impulse will also be changed to the opposite direction in the passage from one half-ray to its opposite, Therefore $\sin\psi$ remains unchanged, while the effect of gravity $P\sin\vartheta$ will be reversed in sign. *If one half-ray is a permissible axis, then the opposite is an impermissible axis.* (An exception would occur only for the particular values $\Omega^2 = 0$ and ∞.)

We next imagine that the half-ray specified as the rotation axis is led successively along one of the two half-cones. The sign of Ω^2 changes only if $\sin\vartheta$ or $\sin\psi$ reverses in sign. The former occurs if we let the half-ray become the center of gravity axis OS, and the latter if we let the half-ray become one of the three principal inertial axes. *Thus one and the other of the half-cones are divided into four domains by the principal inertial axes and the center of gravity axis; these domains alternately contain the permissible and impermissible rotation axes.*

For what concerns the transition positions ($\psi = 0$ and $\vartheta = 0$), equation (18′) implies that $\Omega = \infty$ in the former case and $\Omega = 0$ in the latter. The zero rotation for the upright center of gravity axis naturally signifies nothing other than the (stable or unstable) *equilibrium position* of the body. The infinite velocities about the three principal inertial axes are likewise evident.

The various degeneracies of the cone (16) will not be completely discussed here. We refer, in this respect, to the original work of Staude, and moreover recall, for what concerns the symmetric top, the remarks of page 335. We saw there that for the symmetric top not ∞^1, but rather ∞^2 permanent rotation axes are possible, of which the permissible, according to whether a prolate or oblate top is at hand, fill the "upper" or the "lower half-bundle." Correspondingly, our above equation (16) is identically fulfilled in the case of the symmetric top ($A = B$, $\xi = \eta = 0$). That the figure axis remains a permanent rotation axis for every value of Ω follows, in particular, from (18), since this equation is identically satisfied (because $\vartheta = 0$, $\psi = 0$) for the upright figure axis. —

In conclusion, we wish to indicate how one could advance, from our point of view, from the presently solved special cases to a general qualitative understanding of the motion of the asymmetric top.

Through the investigation of the named special cases, as well as through the knowledge of the motion of the symmetric top, individual paths are paved, so to speak, into the unknown territory of the generalized top, paths that traverse it in various directions. One should now seek to expand laterally from these passable roads, if only a short distance, by investigating a nearly symmetric top instead of a symmetric top, a nearly Kowalevski top instead of a Kowalevski top, etc. These neighboring cases may unquestionably be treated with arbitrary precision by appropriate approximation processes if the deviation from the known cases is not too large and time is restricted to a bounded interval.

It is by no means demanded that the approximation method be always sought in a power series expansion (for example, in a power series expansion in the assumed small difference of the principal moments of inertia A and B for the nearly symmetric top). Rather, an approximation method appropriate and adequate to the purpose is to be used in each case.

If we have thus acquired a judgment of the sense in which a small deviation from the solvable cases acts, it would be further necessary to construct the connections between the different known motions or their neighboring cases. Here one must undertake a type of *interpolation*; one must insert intermediate cases of the motion under the generally unquestionable assumption of a continuous passage between each two motions. As a schema for this interpolation process, we can recommend our intuitive treatment of the symmetric top at the beginning of the preceding chapter.

Chapter VI.

Representation of the motion of the top by elliptic functions.

§1. The Riemann surface $(u, \sqrt{U})$.

While our primary interest has been directed until now toward the geometric and mechanical understanding of the motion of the top, the tone of this chapter will be set by the analytical side of our problem. It is inevitable that the corresponding developments will be abstract, and will first appear to lie farther than the previous from the reality of the mechanical processes, even if we renounce, as we must for the sake of brevity, a thorough rigor and completeness in the function-theoretical considerations. In the following, mathematics should not exclusively serve the interest of mechanics, but rather mechanics should, at the same time, be used for the illustration of a mathematical theory, the theory of elliptic functions.

That this is possible, and that a *mutual* fructification between application and theory in fact occurs, are very noteworthy circumstances to which we first wish to turn our attention.

It is unquestionably natural, from a mechanical standpoint, to describe the motion of the top by representing its position coordinates—in particular, the quantity u—as functions of time, instead of conceiving, as until now, time and the remaining position coordinates as functions of u. If we thus pose for ourselves, from the mechanical point of view, the exercise of "inverting" the dependence between t and u, we will be most highly astonished to see how this same exercise is also of the greatest interest from the standpoint of pure mathematics. The older work on the *theory of elliptic integrals*, particularly the work of Legendre, operates on our current level, where we calculate t as a function of u. But the great advance that has been achieved in this field by Abel and Jacobi depends essentially on the indicated *concept of inversion*,

F. Klein and A. Sommerfeld, *The Theory of the Top: Volume II. Development of the Theory in the Case of the Heavy Symmetric Top*, DOI 10.1007/978-0-8176-4827-5_3,

on the *passage from elliptic integrals to the so-called elliptic functions.* We encounter here a remarkable concurrence of theory and practice, a *preordained harmony*, so to speak, between pure and applied mathematics that has continually worked to the welfare of both in the history of our science.

Under the same point of the view, we next wish to emphasize the significance of our rotation parameters α, β, γ, δ for the following developments. These quantities were originally introduced in the interest of kinematics, in order to simplify as far as possible the formulas of the rotation transformation. It will now be shown that the same quantities bring at least as great a simplification to the analytic advancement of the problem, and that it is directly these parameters that must, in the theory of elliptic functions, be favored above all others.

On the other hand, the practical side of our problem will also be advanced by elliptic functions, in so far as (cf. §6) the most complete and simplest formulas for the numerical calculation of the trajectory of the top will be taken from this theory; these formulas permit us to manage, for not excessive precision, with a few trigonometric terms.

The first step toward the analytic deepening of our problem is the assignment of generally *complex values* to the previous quantities. It is true here, as in so many cases, that analytic relations which appear obscure when restricted to real variables are immediately clarified if we pass into the complex domain.

We thus set

$$u = u_1 + iu_2$$

and represent the value of u not on a line, but rather, after the example of G a u ſ s or A r g a n d), in a plane. Since one regards $u = \infty$ as *one* value in function theory, whether it represents an infinity of the real part, of the imaginary part, or of both simultaneously, one also conceives, as is well known, the infinite remoteness of the Gauſsian plane as *a single point.**) One thus attains a single-valued invertible correspon-

*) Cf., for example, H. B u r k h a r d t: *Einführung in die Theorie der analytischen Funktionen*, Leipzig 1897. We refer the reader to this book with respect to all those function-theoretical questions that cannot be explained in sufficient detail in the text.[175]

dence between the values of u and the points of the plane.

The representation of the *Gaußian plane*, however, does not yet suffice for our purpose; we must broaden this representation to the image of a so-called *Riemann surface.*

We must indeed always consider, in our integral expressions, the two values of u and $\sqrt{U}$ simultaneously. To each value of u correspond two values of $\sqrt{U}$ that differ in their signs. In order to keep these values well separated, we imagine the Gaußian plane to be *doubly covered*, just as we earlier considered the u-axis to be doubled. According to whether we calculate $\sqrt{U}$ with one or the other sign, we find ourselves at the corresponding point of one or the other realization of the u-plane. We distinguish the two realizations as the *upper* and the *lower sheets.*

The two opposite values of $\sqrt{U}$ coincide only if $U = 0$ or $U = \infty$. We have already learnt in the fourth chapter to recognize the positions where this coincidence occurs. They are the points $u = e$, e', e'', and ∞ on the real axis. If we imagine that the two sheets are stitched together at these four positions, then only one point of the double plane is associated to each pair of corresponding values of u and $\sqrt{U}$, and vice versa. The points of the double plane and the pairs of values $(u, \sqrt{U})$ are therefore related in the same *one-to-one* manner as the points of the Gaußian plane and the values of u.

The type of connection between the upper and lower sheets at the positions e, e', e'', and ∞ requires, however, a more particular investigation. One may not, apparently, attach the two sheets to each other at these four points and then simply let the two sheets run over one another, at least not if one demands that the relation between the points of the double plane and the pair of values $(u, \sqrt{U})$ should be *continuous.*

We consider, for example, the point $u = e$, and proceed around it on a small circle in the counterclockwise sense. The attachment of the two sheets would produce the previously named continuous relation only if we return, after beginning with a pair of values $(u, \sqrt{U})$ and traversing the circle, to the same pair of values $(u, \sqrt{U})$. We will show through the following consideration, however, that we will end with the opposing pair of values $(u, -\sqrt{U})$.

The traversal of our circle signifies analytically that we set

$$u - e = \varrho e^{i\varphi}$$

and let the angle φ increase by 2π for a fixed value of ϱ. Now we have

$$U = c(u-e)(u-e')(u-e''), \quad c = \frac{2P}{A},$$

and therefore

$$(1) \qquad \sqrt{U} = \sqrt{\varrho}\, e^{\frac{i\varphi}{2}} \sqrt{c(u-e')(u-e'')}.$$

Here we can imagine ϱ chosen as so small that the quantity under the square root in the last equation changes arbitrarily little for the traversal of the circle. In particular, our circle should naturally contain neither of the points e', e'' in its interior. If we now let φ increase by 2π, then $\sqrt{U}$ changes its sign. We therefore arrive, by means of our circuit, from one sheet to the other.

In order to account for this fact, we must form the following representation of the connection of the two sheets. We must imagine that a line extends from the point $u = e$, and that in crossing this line one is transported from one sheet to the other; the two sheets *interpenetrate* along this line. The same holds for the points e', e'', ∞. Correspondingly, we best describe the connection between the two sheets in the following manner. We first imagine the two sheets laid simply over one another, and then cut them both along the real axis, for example, from e to e' and from e'' to ∞. We stitch the free edges together alternately, so that an edge of the upper sheet will always be joined with the oppositely lying edge of the lower sheet. We then have, in fact, the desired connection.

We must, however, take into the bargain the not entirely convenient fact that the two realizations of the u-plane interpenetrate along the segments ee' and $e''\infty$. It is noted, however, that this imperfection of the geometric image is due only to the restrictiveness of our three-dimensional space representation. Had we one more dimension at our disposal, we could let the stitched sheets run next to one another in the required way without interpenetration, so that they would have only the branch points in common.

The interpenetration curves, whose form is not essential—in our procedure they are segments of straight lines—are called *branch lines*, just as their endpoints, the positions e, e', e'', ∞, were already designated previously as *branch points*. The complete image of our geometric representation is called, after its creator, a *Riemann surface.* We speak concisely of the Riemann surface $(u, \sqrt{U})$.

We must now orient ourselves more precisely on our Riemann surface. We first establish, concerning the positions of the points e, e', e'', that

$$-1 < e < e' < +1 < e'' < +\infty,$$

which corresponds, according to Fig. 38 of page 226, to the assumption $P > 0$. We will then distinguish, just as one divides the Gaußian plane into a positive and a negative half-plane according to the sign of

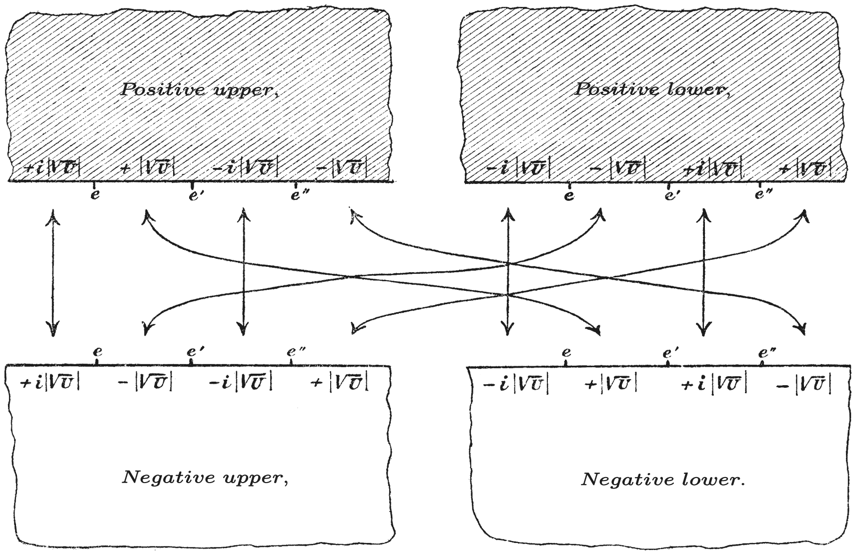


Fig. 59.

the imaginary part of u, four such half-planes on our Riemann surface, which we draw schematically and name in Fig. 59. The two positive half-planes are made recognizable from the negative by hatching. The arrows indicate how the individual segments of the real axis in our four half-planes are connected to each other by the stitching.

We insert in this figure the values of $\sqrt{U}$ along the real axis, where we can choose the sign of $\sqrt{U}$ at one point of the surface arbitrarily. In the remaining positions, $\sqrt{U}$ is then determined uniquely by the requirement of continuity. We wish, for example, to establish that for any one point between e and e', where $\sqrt{U}$ indeed signifies a real number, the positive value of the square root should correspond to the positive upper half-plane. The same sign then obtains throughout between e and e' on the boundary of the positive upper half-plane. We therefore write here the value $+|\sqrt{U}|$. The negative lower half-plane between e and e' receives the same value, since this segment ee' is indeed connected to the positive upper half-plane. The value $-|\sqrt{U}|$ is then applied to the segment ee' of the negative upper and the positive lower half-planes.

In order to be able to make the corresponding assignments for the other segments of the real axis, we proceed, for example, around the point e in the positive upper half-plane on a half-circle in the counterclockwise sense, so that we arrive, beginning from a point between e and e', at a point between $-\infty$ and e. According to equation (1), $\sqrt{U}$ then takes on the factor $e^{\frac{i\pi}{2}} = +i$. We therefore write the value $+i|\sqrt{U}|$ on the real axis between $-\infty$ and e in the positive upper sheet, as well as on the negative upper sheet connected to it. Proceeding in such a manner, we complete the naming of the individual intervals.

A comparison with Fig. 38 ($P > 0$) shows that our previous assignments coincide with the current results for the boundaries of the positive upper half-planes. The previous figure simply represents a cut through the Riemann surface parallel to the real axis, displaced slightly toward the side of the positive half-planes. The difference is only that the values of $\sqrt{U}$ previously appeared, for the restriction to a real variable, as an arbitrary stipulation, while they now follow, after the arbitrary stipulation of the sign at one point of the surface, for all other points with necessity.

§2. Behavior of the elliptic integrals on the Riemann surface.

We must now examine the behavior of the elliptic integrals on our Riemann surface. We will assume a general knowledge of the meaning of an integral on a complex path; we will also be unable to dispense, on occasion, with the Cauchy theorem that states the circumstances under which two different integration paths with the same initial and final points give the same integral value.*

*) Cf. H. Burkhardt, l. c. §35.

We first consider the "integral of the first kind"

$$t = \int \frac{du}{\sqrt{U}},$$

and wish to show that it befits R i e m a n n's designation of an "*everywhere finite integral.*"

It is well known that an integral over an integration path of finite length can be infinite only when the path is extended across a singularity of the integrand and the order of the singularity is not smaller than 1. In our case, the singular points of the integrand are identical with the null values of U; that is, with the branch points e, e', e''. The order of the singularities of the integrand is $\frac{1}{2}$. As a result, our integral remains finite even if we let its upper or lower limit coincide with one of these points.

Further, it is well known from the integral calculus that an integration path can be extended to infinity without the integral losing its finite sense if the integrand vanishes at infinity to an order higher than the first. In our case, however, $\sqrt{U}$ vanishes as $u^{-3/2}$ for $u = \infty$. Thus our integral also remains finite at infinity.

If we therefore choose any point on the Riemann surface as the lower limit, any point as the upper limit, and join the two by any integration path that may possibly encircle the branch points any (finite) number of times, the resulting integral always has a finite value. *The designation "everywhere finite integral" is therefore justified.*

In particular, we wish to consider the integral values that correspond to a complete circuit around a pair of branch points. *We designate these values concisely as the periods of the elliptic integral.* We somehow proceed, for example, around the branch points ee' on the upper sheet of the Riemann surface, and in such a sense that we keep the branch line to the right. (Cf. here, and in the following, Fig. 60.) According to C a u c h y's theorem, all such circuits yield the same value of the integral. In particular, we can contract the integration path to the branch line ee'. If we do this, then we must first integrate from e to e' on the boundary of the positive upper half-plane, and then integrate from e' to e on the boundary of the negative upper half plane, which leads both times to the same integral value; namely, the value earlier known to us as ω. Each single complete circuit about the branch points ee', executed in the given sense on the upper sheet, thus corresponds to an increase in t by "the first period" 2ω. It is clear that for a reversal of the circulation sense or for an integration path on the lower sheet, the resulting integral value will be -2ω.

Each integration path that encloses the points ee' can, however, also be conceived as a circuit about the two other branch points $e''\infty$. Each such circuit (and therefore also, in particular, a contracted path on the real axis from e'' to ∞) thus corresponds, for the proper establishment of the sense, to the same integral value 2ω.

We can further consider a circuit about the points ∞e or $e'e''$. Each two such circuits likewise give, for the proper establishment of the sense of progression, the same integration result. It is enough, for example, to consider the circuit about the points ∞e, which we imagine taken partly in the negative lower, and partly, after crossing the branch cut ee', in the positive upper half-plane, and in such a sense that the line ∞e always lies to the left. The value of this integral is evidently twice the integral value that one obtains if one progresses from the branch point e to $-\infty$ on the boundary of the positive upper half-plane. This value was designated on page 263 as $i\omega'$; its calculation may be effected just as the calculation of ω. Thus the "second period" of our integral, which corresponds to a circuit about the pair of points ∞e or $e'e''$, is equal to $2i\omega'$.

In the following figure, we schematically represent the integration paths and the corresponding integral values considered thus far. The

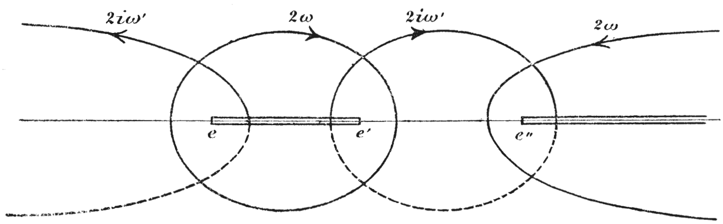


Fig. 60.

sense in each case is marked by an arrow; the integration paths are dotted where they run on the lower, and solid where they run on the upper sheet.

Integrals that correspond to arbitrary closed integration paths may be reduced to the two periods 2ω and $2i\omega'$. *The general value of such integrals, which is also called "the general period" of t, is thus*

$$2m\omega + 2im'\omega'.$$

The (positive or negative) whole numbers m and m' simply signify the number of (right-traveling or left-traveling) circuits of the integration path around the segments ee' and $e''\infty$.

If we specify only the upper and lower limits of our integral as points of the Riemann surface, in that we leave the form of the integration path undetermined, then the value of t, corresponding to the given value of the general period, is *determined only up to multiples of* 2ω *and* $2i\omega$. If we consider, in contrast, only points on one of our four half-planes and add the restriction that the integration path should run entirely in this half-plane, then the value of t is determined (according to the Cauchy theorem) uniquely by the specification of the upper and lower limits. —

We now go over to the function-theoretical investigation of the elliptic integrals that we encountered previously for the quantities ψ, φ, and, in particular, α, β, γ, δ.

In contrast to t, one sees that ψ is not everywhere finite on the Riemann surface. The branch points e, e', e'', ∞ indeed give also here, on the same basis as above, no occasion for singularities. The integrand for ψ will, however, further become infinite for $u = \pm 1$, and, in particular, infinite to the first order. There follows, as we will see, a logarithmic discontinuity of the integral.

We consider, for example, the value $u = +1$. In the integral

$$\psi = \int \frac{n - Nu}{A(1-u^2)} \frac{du}{\sqrt{U}},$$

we set $u = 1$ everywhere except in the singular factor $1 - u$. We must use for U the value $-\frac{1}{A^2}(N-n)^2$ given on page 225; then

$$\psi = \pm\frac{i}{2} \int \frac{du}{1-u} = \mp\frac{i}{2} \log(1-u).$$

This equation gives an approximate representation for the behavior of the quantity ψ in the vicinity of the two positions on the Riemann surface where $u = 1$; it shows that ψ becomes, in fact, logarithmically infinite at these positions. The "multiplier of the singularity," as we wish to call the constant that multiplies the logarithm, will thus be equal to $-\dfrac{i}{2}$ on one sheet and to $+\dfrac{i}{2}$ on the other.

Entirely the same deliberation shows that ψ also becomes logarithmically infinite at the two positions $u = -1$, with the multipliers $\pm\dfrac{i}{2}$. If we consider, moreover, that the angle φ for the spherical top behaves, according to page 237, just like the angle ψ, then we can say that

The Euler angles φ and ψ become logarithmically infinite at four points on our Riemann surface; namely, at the positions $u = \pm 1$ on the upper and lower sheets, and, in particular, with multipliers that assume opposite signs for the points lying over one another on the two sheets, and generally have the absolute value $\dfrac{1}{2}$.

The theory of elliptic integrals is acquainted, however, with still simpler logarithmically infinite integrals; namely, integrals with only *two* logarithmically singular points. One designates these as *integrals of the third kind*, and uses them to linearly compose such integrals with more singularities. We have, in fact, already represented our ψ on page 268 as a sum of two integrals of the third kind, the so-called *Legendre normal integral* Π.

For further analytic treatment, the Euler angles are thus not, in any case, the simplest analytic elements. It will be shown, however, that our parameters α, β, γ, δ yield such elements in the simplest manner; *namely, that the quantities* $\log\alpha$, $\log\beta$, $\log\gamma$, $\log\delta$ *are directly elliptic integrals of the third kind.*

We first consider the expression

$$\log\alpha = \int \frac{A\sqrt{U} + i(n+N)}{2A(u+1)}\,\frac{du}{\sqrt{U}}$$

in equations (8) of page 238. Here the factor $(u-1)$ that appeared in ψ has vanished in the denominator. Also, the factor $(u+1)$ causes a singularity of $\log\alpha$ on only one of the two sheets. As we saw above, there follows, for $u = -1$,

$$A^2 U = -(N+n)^2,$$

and therefore, according to the stipulation contained in Fig. 59,

$$A\sqrt{U} = +\,i|N+n| \text{ on the upper sheet,}$$

$$A\sqrt{U} = -\,i|N+n| \text{ on the lower sheet.}$$

For convenience, we wish to assume in the following that

$$N > n > 0.$$

Then the absolute values $|N+n|$ and $|N-n|$ in the given values of $A\sqrt{U}$ are the same as $(N+n)$ and $(N-n)$, respectively. The numerator of the integrand of $\log\alpha$ thus becomes, for $u=-1$,

$$+\,i(n+N)+i(n+N) = \;2i(n+N) \text{ on the upper sheet,}$$
$$-\,i(n+N)+i(n+N) = \;0 \text{ on the lower sheet.}$$

On the lower sheet, therefore, the vanishing of the numerator cancels the simultaneous vanishing of the denominator, so that $\log\alpha$ remains finite here. On the upper sheet, in contrast, there follows approximately, if we set $u=-1$ throughout except in the factor $(u+1)$,

$$\log\alpha = \int\frac{du}{1+u} = \log(1+u).$$

On the upper sheet, $\log\alpha$ therefore possesses a logarithmic singular point.

Since, according to a general rule for integrals of algebraic functions, logarithmic singularities must always appear in pairs, we will seek yet a second singular point of $\log\alpha$. This lies at $u=\infty$. If we make the substitution $v=\dfrac{1}{u}$, as is usual for the investigation of infinity, then there results for $v=0$, approximately,

$$\log\alpha = -\int\frac{dv}{2v} = -\log\sqrt{v} = \log\sqrt{u}.$$

The second logarithmic singular point of $\log\alpha$ therefore lies at infinity. At the remaining branch points, in contrast, $\log\alpha$ again remains finite.

The same deliberations show that each of the quantities $\log\beta$, $\log\gamma$, $\log\delta$ also has only two logarithmic singular points, one at infinity, and

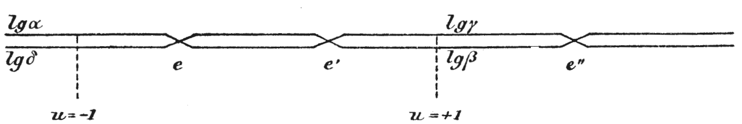


Fig. 61a.

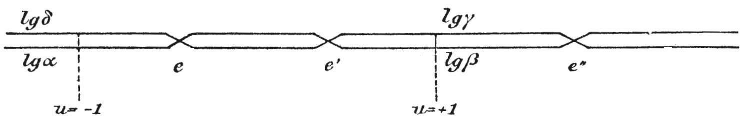


Fig. 61b.

the second at $u=\pm 1$ in the upper or lower sheet. We represent how the singular points are distributed at the four points $u=\pm1$ by the schema of Fig. 61a, which again represents a cut of our Riemann surface, and which is drawn on the basis of the above agreement $N > n > 0$. For a different assumption of the signs and relative magnitudes of N and n, our four singular points are interchanged in an easily assignable manner. The schema b corresponds, for example, to the assumption $0 > N > n$.

We have thus proven that

The logarithms of our parameters α, β, γ, δ are elliptic integrals that become logarithmically infinite at only two positions of the Riemann surface; namely, at one of the four points $u = \pm 1$ and the point ∞. These logarithms are therefore directly elliptic integrals of the third kind.

The characteristic advantages of our parameters, however, are still not exhausted. We already directed our attention in the integral for ψ to the multiplier with which the logarithmic term is burdened. The increase by which the integral increases additively for a circuit about the relevant singular point depends on this multiplier. Now among integrals of the third kind, those integrals for which this increase, calculated in a manner to be made more precise directly, is equal to $\pm 2\pi i$ possess a particularly simple function-theoretical character. It is thus justified to distinguish such integrals by the special name of *normal integrals of the third kind.**) We note that Legendre reached the normalization of his integrals of the third kind from another more formal point of view, and that the Legendre normal integrals mentioned on page 267 are therefore not normal integrals in our current sense.

In order to be able to give the definition of the normal integral of the third kind precisely, we must first explain what we wish to understand by a "positive" and by a "closed" circuit about a point on our Riemann surface.

We set, according to whether we treat of a finite point $u = a$ or the infinitely distant point $u = \infty$, either $u - a = \varrho e^{i\varphi}$ or $u = \dfrac{1}{v}$ and $v = \varrho e^{i\varphi}$. We then let, for fixed sufficiently small ϱ, the angle φ increase in both cases from zero toward the positive side. There results in the u-plane and on the Riemann surface a path that we say surrounds

*) The importance of this normalization is argued by Jacobi directly in the example of the top. Cf. his Ausführungen über den Divisor des Integrals dritter Gattung. Ges. W. B. II p. 477 and ff.[176]

the point $u = a$ or $u = \infty$ in the *positive sense.* We continue this positive circulation until the angle φ has attained the value 2π or 4π, according to whether we treat of an ordinary point of the Riemann surface or a branch point. We then encircle the relevant point in the u-plane once in the first case, and twice in the second case. On the Riemann surface, however, we encircle the point only once in both cases, since for a branch point two circular paths in the u-plane first lead back to the initial point on the Riemann surface. In both cases, we thus speak of a *single, closed circuit on the Riemann surface.*

The precise definition of the normal integral of the third kind now runs in the following manner.

By a normal integral of the third kind we understand an integral with two logarithmic singular points that increases by $2\pi i$ for a single closed circuit on the Riemann surface about one of the singular points; for a corresponding circuit about the other singular point, it increases, in consequence of a general rule, by $-2\pi i$.

We now show immediately that our integrals $\log\alpha$, $\log\beta$, $\log\gamma$, $\log\delta$ are normal integrals in this sense. We consider, for example, $\log\alpha$.

For the investigation of the singular point $u = -1$ of $\log\alpha$, we set, according to the just given directive, $u + 1 = \varrho e^{i\varphi}$ and let, for a sufficiently small value of ϱ, the angle φ increase from 0 to 2π. Since, according to the above, $\log\alpha$ behaves in the neighborhood of the point $u = -1$ as $\log(u + 1)$, we have

$$\log\alpha = \log(u + 1) + \cdots = \log\varrho + i\varphi + \cdots .$$

We thus see that $\log\alpha$ increases by $2\pi i$ for a positive, closed encirclement of the point $u = -1$. To treat of $u = \infty$, on the other hand, we set $v = \dfrac{1}{u} = \varrho e^{i\varphi}$ and let φ once again vary for sufficiently small ϱ, and indeed now from 0 to 4π. Since $\log\alpha$ goes over at infinity into $-\log\sqrt{v}$,

$$\log\alpha = -\log\sqrt{v} + \cdots = -\log\sqrt{\varrho} - \frac{i\varphi}{2} + \cdots .$$

As one sees, $\log\alpha$ increases for the just defined positive closed encirclement of the position $u = \infty$ directly by $-2\pi i$.

In the same manner, we convince ourselves that $\log\beta$, $\log\gamma$, $\log\delta$ also increase by $\pm 2\pi i$ for circular passages about their singular points.

We can thus state the noteworthy theorem that

The logarithms of our parameters are not only integrals of the third kind, but are indeed normal integrals.

This circumstance will become of striking importance if we go over from the logarithms to the values of our parameters themselves. The latter quantities, namely, evidently remain completely unchanged for a circular passage about the singular points of their logarithms. For example, α behaves in the vicinity of the position $u = -1$ directly as

$$C(u+1).$$

If, in contrast, the increase for a circuit about the position $u = -1$ were not equal to $2\pi i$, but rather equal to $2\pi i\lambda$ and, correspondingly, the multiplier of the singularity were not equal to 1 but rather to λ, then α would behave as

$$C(u+1)^\lambda,$$

and therefore this position would be a branch point if λ were not a whole number. We can thus say that

Thanks to the normality property of our integral of the third kind, our parameters are completely u n b r a n c h e d on the Riemann surface.

This is not to say that these parameters are also *single-valued* on the Riemann surface. Their logarithms, corresponding to their representation as integrals of the third kind, increase additively for a circuit about a pair of branch points by certain characteristic increments, "the periods of the integral of the third kind," directly as we have depicted for the integral of the first kind in Fig. 60. *The value of the parameters themselves will thus be multiplied by certain characteristic factors for a circuit about the branch points.* We need not enter here into the calculation of these increases or these factors from the integral representation, since we will give an explicit representation of α, β, γ, δ in the fourth section through which the named calculation will be accomplished of itself.

In any case, all these remarks illuminate the superiority in function-theoretical simplicity of our parameters α, β, γ, δ over the Euler angles. It will be directly shown that *the parameters α, β, γ, δ represent the simplest analytic building stones from which the general formulas for the motion of the top may be composed.*

§3. The image of the Riemann surface $(u, \sqrt{U})$ in the t-plane.

In this section, we must carry out the inversion of the elliptic integral of the first kind. While we have thus far conceived t as a function of its upper limit u, or better yet as a function of the position $(u, \sqrt{U})$ on the Riemann surface, we will later wish to regard the pair of values $(u, \sqrt{U})$ as a function of t. In a preliminary stage, we first treat of $(u, \sqrt{U})$ and the corresponding value of t as equally entitled variables. We thus represent $t = t_1 + it_2$, in its turn, in a complex plane, the *t-plane.* We lay off t_1 as the abscissa and t_2 as the ordinate, and ask what path or what region t describes in its plane while the variable u sweeps through an arbitrary path or an arbitrary region of the Riemann surface. One designates this question function-theoretically as the *question of the image of the Riemann surface $(u, \sqrt{U})$ in the t-plane.*

As the lower limit of the integral we take, as previously, the branch point e, and thus consider

$$t = \int_e^u \frac{du}{\sqrt{U}}.$$

We begin with the boundary of the positive upper half-plane of our Riemann surface. *What path does t describe as we let u run from left to right on the real axis of the positive upper half-plane*?

For the answer to this question, we rely essentially on Fig. 59, where we have registered, according to reality and sign, the effectual values of $\sqrt{U}$ for the individual intervals of the real axis. The corresponding increments of t—that is, the quantities $dt = \dfrac{du}{\sqrt{U}}$ that correspond to the positive increments du of u—are thus given according to reality and sign. We summarize them in the following table:

If u moves				the increment dt is
from	e	to	e',	positive real,
from	e'	to	e'',	positive imaginary,
from	e''	to	$+\infty$,	negative real,
from	$-\infty$	to	e,	negative imaginary.

At our initial point e, the lower limit of the integral, t is naturally equal to zero. The path that t describes therefore begins at the origin of the t-plane. If u advances from e to e', the corresponding point t progresses, according to the preceding table, along the positive real axis. If u attains the value e', the path bends through a right angle and first runs parallel to the positive imaginary t-axis.

If we come to e'' on the Riemann surface, the path once again makes a rectangular bend; it then runs again parallel to the real axis, but in the negative sense. If we let u go to positive infinity and return from negative infinity on the real axis, then we have another rectangular bend in the t-plane; the motion of the representative point, which previously occurred in the sense of the negative real axis, now runs in the sense of the negative imaginary axis. *In total, t therefore describes a rectangular path of straight lines as u runs through the real axis.*

We easily find that this line path must close on itself, and calculate, moreover, the lengths of its edges, if we consider the values of the full period circuits given in Fig. 60. If we contract, namely, the drawn integration paths to the respective segments of the real axis, then the full circuit is divided into two congruent straight line halves, each of which yields as the integral value of t half the entire period 2ω or $2i\omega'$. It follows that *the rectangular line path in the t-plane is the contour of an ordinary rectangle* (cf. Fig. 62); *the length of the horizontal sides is ω, and the length of the vertical sides is ω'. The corners of the rectangle are, in correspondence with the sequence*

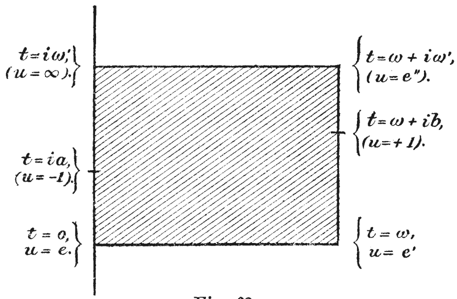


Fig. 62.

$$u = e, \;\; e', \;\; e'', \;\; \infty$$

of the branch points, given by the values

$$t = 0, \;\; \omega, \;\; \omega + i\omega', \;\; i\omega'.$$

We now enter with the variable u into the interior of the positive upper half-plane, and convince ourselves that the variable t then passes into the interior of our rectangle.

Since, namely, we keep that half-plane to the left if we run through the real u-axis from $-\infty$ to $+\infty$, the corresponding region in the t-plane must also, as a consequence of a general function-theoretical principle, be attached to the left of the rectangle traversed in the direction $i\omega'$, 0, $\omega, \ldots$. Further, we must make clear to ourselves that the image in the t-plane has no vacancies, branch points, or foldings. We can, for this purpose, use the integral formula to produce series expansions that permit the value of t to be calculated as a convergent power series from the value of u, and vice versa. If one carries out this suggestion in more detail, one recognizes that points of the positive upper half-plane can correspond only to points in the interior of our rectangle, and that these points must cover the space between our rectangular borders simply and without gaps. We will therefore be able to say that

The area of our rectangle represents the image in the t-plane of the upper half-plane of the Riemann surface.

In order to conceive the imaging process as concretely as possible, we can imagine, for example, that the considered half-plane is spanned by an elastic membrane that is fixed to the real axis. We represent the segments of the real axis between the branch points as connected to each other by joints. We now rotate these segments with respect to each other and deform them until they have gone over into the rectangular borders of the t-plane. At the same time, the original membrane is deformed under the retention of continuity into a membrane that spans the rectangle. The one membrane is thus represented as the *strain* of the other. This procedure naturally gives, at first, only a very approximate qualitative representation of the mathematical dependence between the variables t and u. We can, however, repeat our image with quantitative correctness if we ascribe the property of actual elastic force to our membrane, and permit only such strains that preserve the similarity of the smallest elements, so that any two curves emanating from a point enclose the same angle after the deformation as before. Through this stipulation, the type of strain, as one can easily check, is completely specified, if only one adds the further condition, for example, that the three points e, e', e'' of the half-plane should correspond to the three successive corners 0, ω, $\omega + i\omega'$ of our rectangular boundary. As is well known, one calls such a mapping of two regions, similar in the smallest elements, a *conformal* or *angle-preserving mapping.*

Without any formulas, the analytic dependence between the variables t and u may be stated purely geometrically in the following manner:

The dependence corresponds to that of every two variables t and u whose representative points are transformed into one another by the conformal mapping of a half-plane onto the area of a rectangle (under the assignment of the branch points to the corners of the rectangle). —

A very beautiful apparatus that effects the conformality of the mapping automatically has recently been constructed by Mr. S. Finsterwalder.*) Mr. Finsterwalder constructs a network of flexible wires in which he connects every three wires by a three-bored bushing, where it is convenient in the construction to arrange the bores at angles that are generally equal to one another. The bushing midpoints then form, in our original position, the corners of a regular hexagonal tiling of the plane. Since the wires can freely slide to and fro in their guides and can moreover bend, our apparatus has a very high degree of mobility. The number of degrees of freedom will even be infinitely large if we imagine, as must actually be done in the present application to conformal mapping, the collected dimensions of the network to be infinitesimal and the bushings to be infinitely numerous and infinitely dense.

One is convinced by experiment alone that it is possible to give the boundary of the network any arbitrary form; that is, to form any other region from the region initially occupied by the network, where three boundary points of one region will still be able to correspond arbitrarily to three boundary points of the other. That this mapping is conformal follows immediately, for the angles with which the wires come together, from the rigidity of the bushings; if, however, three angles at each point remain unchanged for a continuous mapping (or, more precisely said, for a mapping provided by an analytic function), then the same holds, according to general principles, for all angles.

Through the Finsterwalder apparatus we would therefore be able to realize the mapping effected by our elliptic integral in a purely experimental manner. —[177]

*) Cf. Jahresbericht der deutschen Mathematiker-Vereinigung, Bd. 6, 1897.

We can carry out the same consideration for the negative upper half-plane as for the positive upper, and for the positive lower, etc. *These half-planes also map in the t-plane into rectangles of the just depicted form.*

The position of these rectangles with respect to the image of the positive upper half-plane depends on the integration path, and, in particular, on the interval in which we cross over the real axis in order to arrive from the positive upper to the other half-planes. The positive upper half-plane is connected, for example, to the negative upper along the line $-\infty e$. If, beginning from e, we cross over this segment in the integration, then we come to values of t that correspond to the negative upper half-plane. In the t-plane, these points fill a rectangle that lies to the left of the side $t = 0$ to $t = i\omega'$ of the original rectangle. If we step, on the other hand, through the branch line ee' into the negative lower sheet, then we arrive at the representative point of the t-plane in the interior of a rectangle that represents an image of the negative lower half-plane. This rectangle lies beneath the image of the line ee'; that is, beneath the side $t = 0$ to $t = \omega$ of the rectangle. If we then go on the Riemann surface from the negative lower into the positive lower half-plane, in that we again cross over the line from $-\infty$ to e, there corresponds to this passage a new rectangle that has the side from $-i\omega'$ to 0 in common with the image of the negative lower half-plane. In total, we thus have acquired an image of our four half-planes; that is, an image of the entire Riemann surface. *This consists of a large rectangle that has the origin of the t-plane as its midpoint, and is composed of our four small rectangles.* We hatch the two rectangles that correspond to the positive half-planes, and arrive at the following complete image of the Riemann surface (cf. Fig. 63), in which these rectangles appear next to each other in the most highly transparent manner.

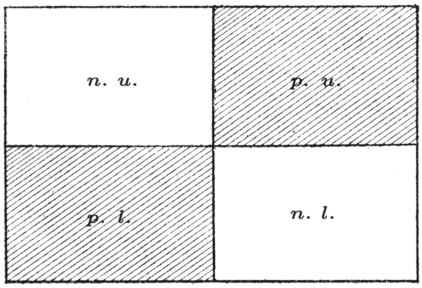


Fig. 63.

We designate such a rectangle, whose sides are the periods of the elliptic integral, as a “period rectangle.”

A few clarifications are still required regarding the boundary points of our period rectangle. These correspond, as one has seen, to the points of the two segments $e'e''$ and $e''\infty$ in the four half-planes. In these segments the continuity of our image is evidently interrupted. While, for example, the segment $e'e''$ of the positive upper sheet directly coincides with the segment $e'e''$ of the negative upper sheet on the closed Riemann surface, according to the conventional connection of the boundaries of our branch lines, the images of these segments in the t-plane are on opposite vertical sides of the period rectangle; correspondingly, the images of the coinciding segments $e''\infty$ on the Riemann surface lie in the t-plane on the opposite horizontal sides. We must, in order to have a completely continuous and completely single-valued image of the Riemann surface, assume that the opposite boundaries of the rectangle are assigned in such a manner that each two opposite boundary points can be held as identical in the image. However, we need not go into more detail here, since this defect of discontinuity in the image will vanish of itself in the following if we further complete our figure.

In fact, our image figure is not yet finished. We have previously gone from the positive upper to the negative upper half-plane through the segment $-\infty e$. We can just as well arrive there, however, through $e'e''$. If we do the latter, then we obtain as the image of the negative upper half-plane a rectangle that lies next to the image of the segment $e'e''$; that is, next to the side ω to $\omega + i\omega'$ of our initially drawn rectangle. In general, there are two different passages on the Riemann surface from each positive half-plane to each of the two negative half-planes, and from each negative half-plane to each of the two positive half-planes. Correspondingly, we must complete the figure so that a further rectangle lies next to each free rectangle side, where each hatched rectangle will be surrounded by four unhatched, and each unhatched surrounded by four hatched. As the completed image figure we thus obtain a *tessellated pattern*, as represented in Fig. 64. The individual horizontal strips contain images of the half-planes of either only the upper or only the lower sheet. A system of period rectangles is singled out from the system of smaller rectangles by a somewhat stronger drawing of the boundaries.

This rectangular partition, together with the representation of the Riemann surface, gives the simplest and most complete conception of the analytic relation between the quantities t and u or $\sqrt{U}$.

We first consider t as a function of u, in that we imagine the upper limit u given as a specified point of the Riemann surface; that is, we imagine that the sign of $\sqrt{U}$ is chosen in a specified manner in addition to the value of u. (The lower limit should, as previously, lie at the branch point e.) We already saw on page 400 that the value of t is not completely determined by the specification of the upper limit. Depending on the form of the integration path, one obtains infinitely many values of t that differ by multiples of the periods 2ω and $2i\omega'$. This state of affairs is expressed with particular clarity by our rectangular partition. Since, namely, each half-plane of the Riemann surface is mapped into infinitely many rectangles in the t-plane, there are, for each point of the surface, infinitely many corresponding points of the t-plane, and indeed one finds such a point in each period rectangle. If we displace our entire figure parallel to itself by 2ω in the direction of the real axis or by $2\omega'$ in the direction of the imaginary axis, then it always comes into coincidence with itself; each rectangle goes over into an identically designated rectangle, and each point into a point that always corresponds to the same point on the Riemann surface. We wish to designate all these points as *equivalent points*. If t is any point in the t-plane that corresponds to the position $u, \sqrt{U}$ on the Riemann surface, then the equivalent points will be represented by

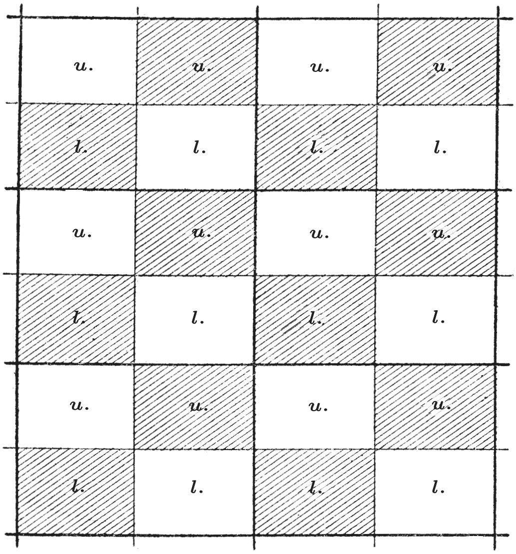


Fig. 64.

$$t + 2m\omega + 2m'i\omega',$$

where m and m' signify any positive or negative whole numbers. All these values of t correspond to the same upper limit $u, \sqrt{U}$ of the integral. In analytic respects, namely, we draw the conclusion that

Conceived as a function of the upper limit, t is an infinitely-many-valued function.

We now give, conversely, the value of t, and ask for the corresponding value of u. Each point in the t-plane corresponds, on the Riemann surface and all the more so in the u-plane, to one and only one entirely determined point. It follows immediately that

Conceived as a function of t, u is a single-valued function.

Now it is certainly more advantageous analytically to operate with single-valued rather than multivalued functions. At the same time, we remarked at the beginning of the previous section that it is desirable from the standpoint of mechanics to represent the direct time dependence of the elements of the motion. *Both grounds dispose us to "invert" our elliptic integrals; that is, to regard t in the future as the independent variable, and to represent the quantity u as a function of t.*

Concerning the properties of the function u of t (we write concisely $u = u(t)$), we can immediately add a still more precise determination. If we increase, namely, the value of the argument by multiples of the two periods 2ω and $2i\omega'$, then we arrive at a point in the t-plane that corresponds to the same position of the Riemann surface. Thus u remains unchanged by the increase of its argument by one of the two periods. We say that *u is a doubly periodic function of t*, or that *u is an elliptic function.*

Actually, we must also prove explicitly that *u is an analytic function, a function of the complex argument t.* However, we wish in this respect to call upon the theorem of general function theory that through inversion of an analytic function there always results again an analytic function. Since t is, through its integral representation, certainly a function of the complex argument u, we conclude that the function $u(t)$ must also be an analytic function.

We need not enter in detail into the actual calculation of the function $u(t)$, since we again forsake this function in the following, and will become acquainted with still simpler functions (the so-called ϑ-functions) from which the doubly periodic function u, among others, may be composed in the most convenient manner.

We now wish to show that the introduction of the everywhere finite integral t as the independent variable will be of essential advantage for a large class of further functions. *We will see, namely, that many functions that are multivalued in their dependence on u will be made single-valued by the introduction of t, or, as we wish to say concisely, will be "uniformized."*

These are, in the first place, the *single-valued functions on the Riemann surface*, and therefore, in particular, the rational functions of u and $\sqrt{U}$. In fact, our rectangular partition shows that each point of the t-plane corresponds not only to one and only one point of the u-plane, but also to one and only one position on the Riemann surface. The location on the Riemann surface and all singled-valued functions of this location therefore depend on the position in the t-plane—that is, on the value of the everywhere finite integral—in a single-valued manner.

As the simplest example, we consider the double-valued function $\sqrt{U}$ in the u-plane, which is single-valued on the Riemann surface. That this function is also single-valued in the t-plane may be verified immediately. Since, namely,

$$t = \int \frac{du}{\sqrt{U}},$$

there follows

$$\sqrt{U} = \frac{du}{dt} = u'(t);$$

through the introduction of the variable t, the double-valued function $\sqrt{U}$ will be uniformized.

The uniformizing effect of t reaches, however, still further: *not only will single-valued quantities on the Riemann surface be single-valued functions of t, but also, rather, all multivalued quantities on the surface whose multivaluedness is of the same nature as the multivaluedness of the everywhere finite integral itself; that is, multivalued functions that remain unchanged for all those circuits that leave the value of t unchanged.* (In particular, the quantities in question must naturally be unbranched relative to the Riemann surface.)

As a proof, one considers that each circuit on the Riemann surface for which the value of the function to be uniformized changes will also change, by assumption, the value of the variable t, and will thus lead into another region of the t-plane. The different values of the function in question, which possibly correspond to the same point of the Riemann surface, will therefore lie in truly different places in the t-plane.

This important principle finds an immediate application to our parameters α, β, γ, δ. We indeed saw on page 405 that α, β, γ, δ are unbranched on the Riemann surface, and that they are changed only for those circular passages (and then only by certain characteristic factors) that also change the value of the everywhere finite integral (and, in particular, by the additive periods 2ω and $2i\omega'$).

We can therefore say that

Our parameters α, β, γ, δ, infinitely-many-valued but unbranched on the Riemann surface, become single-valued in the t-plane.

The resulting single-valued, and, as we can add, analytic functions $\alpha(t)$, $\beta(t)$, $\gamma(t)$, $\delta(t)$ are, as said, not doubly periodic, since they each change by a constant factor for a passage from one period rectangle to another; we nevertheless likewise designate them, with Hermite (see below), as *elliptic functions*, and indeed more precisely as *elliptic functions of the second kind*, in distinction to the purely doubly periodic functions, which we call *elliptic functions of the first kind.*

We ask, furthermore, for the null and singular points of our elliptic functions $\alpha(t)$, $\beta(t)$, $\gamma(t)$, $\delta(t)$ in the t-plane, since the later analytic representation of our parameters is based on the locations of these points. The null and singular points of the quantities $\alpha(t)$, $\beta(t)$, $\gamma(t)$, $\delta(t)$ are naturally identical with the logarithmic singular points of $\log\alpha$, $\log\beta$, $\log\gamma$, and $\log\delta$. We have clearly represented the distribution of the latter in Fig. 61 under the assumption

$$N > n > 0.$$

This same assumption will be made in the following.

We investigate in particular, for example, the function $\alpha(t)$. According to Fig. 61a, $\log\alpha$ becomes infinitely large for $u = -1$ and $u = \infty$, and indeed $\log\alpha$ behaves, according to page 402,

$$\text{as } \log(u+1) \text{ for } u = -1, \text{ and as } \log\sqrt{u} \text{ for } u = \infty.$$

If we go over from the logarithm to the antilogarithm, we see that the parameter α vanishes for $u = -1$, and becomes infinite for $u = \infty$.

In the t-plane, the value $u = \infty$ corresponds, according to Fig. 62, to the point $t = i\omega'$, or to one of the equivalent points

$$\text{(I)} \qquad t = i\omega' + 2m\omega + 2m'i\omega'.$$

Further, the position $u = -1$ of the upper sheet maps, as likewise follows from Fig. 62, into a point that lies on the imaginary axis between $t = 0$ and $t = i\omega'$, or into one of the equivalent points. We denote the corresponding t-value by ia; a is then determined by the integral

$$ia = \int_e^{-1} \frac{du}{\sqrt{U}}$$

already given on page 263.

The totality of the equivalent positions—that is, those points of the t-plane that represent the position $u = -1$ on the upper sheet of the Riemann surface—is thus given by

$$\text{(II)} \qquad t = ia + 2m\omega + 2m'i\omega'.$$

We already know that $\alpha(t)$ becomes infinitely large at the points (I), and vanishes at the positions (II). We wish, in addition, to establish the order of the singularity and the order of the nullity. For this purpose, we recall that $\log\alpha$ changes by $\pm 2\pi i$ for a single closed circuit on the Riemann surface about the positions $u = -1$ and $u = \infty$ (cf. page 404). Under the conformal mapping to the t-plane, however, a single closed circuit on the Riemann surface is transformed into just such a circuit in the t-plane, as follows from the concept of the conformal map. The null and singular points of α are therefore constituted so that $\log\alpha$ takes on the same increase $\pm 2\pi i$ for a single circuit. This directly signifies, however, that the order of the nullity and the singularity is equal to 1. Thus we can say that

The points (I) *are simple singular points, and the points* (II) *are simple null points of the function $\alpha(t)$. The function α has no other null or singular points.*

The null and singular points of β, γ, and δ in the t-plane follow in a similar manner. We single out, for example, γ. On the Riemann surface, $\log\gamma$ becomes logarithmically infinite for $u = \infty$ and (cf. Fig. 61) $u = +1$ on the upper sheet. The singularities on the Riemann surface correspond in the t-plane to the position $t = i\omega'$ and the equivalent positions. The totality of these positions is again represented by

$$\text{(I)} \qquad t = i\omega' + 2m\omega + 2m'i\omega'.$$

The point $u = +1$ of the upper sheet corresponds in the t-plane, according to Fig. 62, to a point on the rectangle side between $t = \omega$ and $t = \omega + i\omega'$. The distance of this point from the real axis is called b; one calculates b, as already given on page 263, if one extends, for example, the everywhere finite integral

$$ib = \int_{e'}^{1} \frac{du}{\sqrt{U}}$$

on the upper sheet from the branch point e' to the point 1. The value of t at the named point will then be $t = \omega + ib$. In addition to this point, we naturally have the collected equivalent points

$$\text{(II)} \qquad t = \omega + ib + 2m\omega + 2m'i\omega'$$

to consider. We conclude, just as above, *that the points* (I) *are simple singular points and the points* (II) *are simple null points, and indeed the sole null and singular points of the function* $\gamma(t)$.

The still wanting null and singular points of β and δ follow from the remark that α and δ, on the one hand, and β and $-\gamma$, on the other hand, are conjugate imaginary quantities. This follows for real values of t from the original definitions of our parameters (cf. page 21); it is also valid, however, for complex conjugate values of time, as follows immediately from the integral representation of the logarithms of α, β, γ, δ. Thus the null points of β and δ will be conjugate to those of α and γ; we obtain these null points from the values given under (II) if we simply exchange $+ia$, $+ib$ with $-ia$, $-ib$. Further, the singular points of β and δ coincide directly with those of α and γ, since the points given under (I) are in their totality conjugate to themselves. The complete table of the null and singular points of our four parameters thus appears as follows.

	Null points.	Singular points.
α	$+ ia + 2m\omega + 2m'i\omega'$	$i\omega' + 2m\omega + 2m'i\omega'$.
β	$+ \omega - ib + 2m\omega + 2m'i\omega'$	
γ	$- \omega + ib + 2m\omega + 2m'i\omega'$	
δ	$- ia + 2m\omega + 2m'i\omega'$	

We are now in a position to develop the explicit representations of our parameters as functions of time. We base these representations on the so-called ϑ-functions, which have played an entirely fundamental role in the theory of elliptic transcendentals since the time of J a c o b i.

§4. Representation of α, β, γ, δ by ϑ-quotients.

The ϑ-functions are single-valued functions of their argument, which we denote by t, that become infinite only at infinity. They have, like α, β, γ, δ, a system of equivalent null points, of which one null point falls in each period rectangle; for a passage from one period rectangle to another, they are each multiplied by a characteristic factor, which, however, is not independent of t, as it is for α, β, γ, δ.

A further property of the ϑ-functions that is particularly valuable for our purpose is that they converge extraordinarily well, and enjoy suitable series representations for numerical calculation.*)

A great disparity unfortunately reigns among different authors for the notation of the ϑ-functions. Here, we will retain none of the usual notations precisely, which is harmless, in so far as our presentation is understandable in itself.

While Jacobi considered four not essentially different ϑ-functions, we will manage with only one. We denote it by $\vartheta(t)$, and arrange the definition so that ϑ is an odd function of t, and that ϑ therefore vanishes only at the origin and all the equivalent points of the t-plane. Formally, our ϑ-function is given by the series

$$\vartheta(t) = \sum_{-\infty}^{+\infty}{}_{(n)}\, e^{-\frac{\omega'}{\omega}\left(\frac{2n-1}{2}\right)^2\pi + \frac{t+\omega}{\omega}\frac{2n-1}{2}\pi i}. \tag{1}$$

With the use of the abbreviations

$$q = e^{-\frac{\omega'}{\omega}\pi}, \quad s = \frac{t\pi}{2\omega}, \tag{2}$$

we can, as will be convenient on occasion, also write[178]

$$\vartheta(t) = 2q^{1/4}\sin s - 2q^{9/4}\sin 3s + 2q^{25/4}\sin 5s - \cdots . \tag{3}$$

The Jacobi notation for our function would be[179]

$$\mathsf{H}(t).$$

The properties of the ϑ-function that were already mentioned in general above may now be easily verified, on the basis of equation (1), for our special choice of the function.

One first sees that our function is finite and single-valued for all finite values of t. Namely, the series (1) converges, as one easily checks, in the entire t-plane.

*) Instead of the ϑ-function, the σ-function introduced by Weierstrass is now generally used in the literature. The σ-function differs from the ϑ-function only by an exponential factor. We prefer the ϑ-function for our purpose, since its employment requires less preparation, and the specific advantages of the σ-function do not come into play here. Moreover, the ϑ-function is still indispensable afterward for numerical calculation, which we must always keep in mind.

We next determine the behavior of the ϑ-function for an increase of its argument by multiples of the periods. An addition of the real period 2ω obviously changes each individual term of the series (3) only in sign. We therefore have

$$\vartheta(t+2\omega) = -\vartheta(t). \tag{4}$$

If we add the imaginary period $2i\omega'$, and therefore write $t+2i\omega'$ instead of t, the exponent of the general term in (1) becomes

$$-\frac{\omega'}{\omega}\left(\frac{2n-1}{2}\right)^2\pi - \frac{\omega'}{\omega}(2n-1)\pi + \frac{t+\omega}{\omega}\frac{2n-1}{2}\pi i =$$
$$-\frac{\omega'}{\omega}\left(\frac{2n+1}{2}\right)^2\pi + \frac{\omega'}{\omega}\pi + \frac{t+\omega}{\omega}\frac{2n+1}{2}\pi i - \frac{t+\omega}{\omega}\pi i.$$

This exponent therefore has the n-independent increase

$$\frac{\omega'}{\omega}\pi - \frac{t+\omega}{\omega}\pi i,$$

and, in addition, is increased in the index n by one. Through the latter circumstance, however, the value of the series is not changed, since n runs from $-\infty$ to ∞. We thus find that

$$\vartheta(t+2i\omega') = -e^{\frac{\omega'}{\omega}\pi - \frac{t\pi i}{\omega}}\vartheta(t). \tag{5}$$

Conversely, the functional properties (4) and (5) can serve, together with the requirement that ϑ should be nowhere infinite, to define the ϑ-function up to a constant factor, as we mention only historically.

In order to obtain the intended representation of our parameters, we now consider the quotient

$$\frac{\vartheta(t-ia)}{\vartheta(t-i\omega')}.$$

This quotient becomes zero or infinitely large at the location $t = ia$ or $t = i\omega'$, respectively, and at the equivalent positions, and indeed to the first order in both cases. This expression therefore has the same null and singular points as our function $\alpha(t)$ (cf. the previous section). It therefore differs from the latter only by a factor that neither vanishes nor becomes infinitely large for any finite value of t, and whose logarithm can therefore become infinitely large for no finite value of t. Such a factor can always be given in the form

$$e^{G(t)},$$

where $G(t)$ is a function that is nowhere infinite in the finite domain, a so-called entire transcendental function. We therefore have the equation

$$\alpha(t) = e^{G(t)}\frac{\vartheta(t-ia)}{\vartheta(t-i\omega')}. \tag{6}$$

In our case, it follows from the properties of the parameter $\alpha(t)$, on the one hand, and of the ϑ-function, on the other hand, that the *transcendental entire* function must reduce to a *linear entire* function. According to the just proven functional equations for the ϑ-function, the ϑ-quotient on the right-hand side is multiplied, for an increase of t by one of the two periods, by a constant factor; namely, by

$$+1$$

for an addition of 2ω, and by

$$e^{-\frac{a-\omega'}{\omega}\pi}$$

for an addition of $2i\omega'$. Further, we know (cf. page 414) that $\alpha(t)$ takes on a factor that is independent of t for an increase of t by one of the two periods. The factors on the right and left sides in equation (6) must coincide; we therefore have, understanding by c and c' two constants that are composed from the named factors,

$$\begin{aligned} G(t+2\omega) \quad - G(t) &= c, \\ G(t+2i\omega') - G(t) &= c', \end{aligned}$$

from which there follows, by differentiation with respect to t,

$$\begin{aligned} G'(t+2\omega) \quad &= G'(t), \\ G'(t+2i\omega') &= G'(t). \end{aligned}$$

Thus $G'(t)$ is a doubly periodic function that becomes infinite nowhere in the finite domain. It is shown in function theory, however, that such a function is necessarily a constant. We therefore have

$$G'(t) = l,$$

and

$$e^{G(t)} = ke^{lt}, \tag{7}$$

where k and l are certain quantities that are independent of t, and that will be given more precisely directly.

The explicit form of the function $\alpha(t)$ is now known on the basis of equations (6) and (7). If we add the analogously constructed and likewise derived expressions for β, γ, and δ, then we obtain

$$\left\{ \begin{aligned} \alpha &= k_1\, e^{l_1 t}\, \frac{\vartheta(t-ia)}{\vartheta(t-i\omega')}, \\ \beta &= k_2\, e^{l_2 t}\, \frac{\vartheta(t-\omega+ib)}{\vartheta(t-i\omega')}, \\ \gamma &= k_3\, e^{l_3 t}\, \frac{\vartheta(t+\omega-ib)}{\vartheta(t+i\omega')}, \\ \delta &= k_4\, e^{l_4 t}\, \frac{\vartheta(t+ia)}{\vartheta(t+i\omega')}. \end{aligned} \right. \tag{8}$$

We have thus reached the following result: *our parameters α, β, γ, δ are represented by simple ϑ-quotients, to which are added an exponential quantity and a constant as factors.*

Moreover, it is necessary to calculate only two of these expressions, say α and β, since the two other parameters are then determined as conjugate imaginary quantities.

It is noted that the method adopted here corresponds quite properly to the beautiful principle that Riemann has emphasized in all his investigations: first discuss the properties of the functions to be treated, and suppress all formulaic matters to the conclusion, where they must, so to speak, follow of themselves from the established properties. Thus we have, in fact, acquired our representation (8) as a necessary consequence of the preceding investigation of the single-valuedness of our functions and the positions of their null and singular points.

In order to place the fundamental meaning of the expressions (8) in the correct light, we add a few historical remarks with regard to elliptic functions.

One originally understood by an elliptic function, since the time of Jacobi, only a function that remained *completely unchanged* for an increase of its argument by multiples of the periods.

It was then shown*) that a so-defined elliptic function assumes every value, and in particular the values zero and infinity, the same number of times in the rectangle (or, more generally, the parallelogram) formed by the two periods 2ω and $2i\omega'$. This number (n) is called the degree of the elliptic function. One can further prove that the relation

$$\Sigma a_\nu - \Sigma b_\nu = 2\mu\omega + 2i\mu'\omega',$$

where μ and μ' are two whole numbers, obtains between the arguments of the null points (a_ν) and the arguments of the singular points (b_ν) that lie in a single period rectangle. It is now always possible to express an elliptic function in terms of a ϑ-quotient in the form

$$ke^{lt} \prod_{\nu=1,..n} \frac{\vartheta(t-a_\nu)}{\vartheta(t-b_\nu)}.$$

In fact, this expression possesses, thanks to the functional properties of the ϑ-function given above, the required periodicity if one chooses

*) These general theorems were first recognized by Liouville. Cf. his note Sur les fonctions elliptiques. Liouville's Journal, Bd. XX, 1855.[180]

the quantity l equal to $\frac{i\pi\mu'}{\omega}$. In particular, we note that a pure ϑ-quotient ($l = 0$) represents, in any case, a doubly periodic function if the argument sum of the numerator is equal to that of the denominator.

H e r m i t e*) later noted that one is led, particularly in mechanical applications, to more general ϑ-quotients, between whose null and singular points the given relation does not hold, and that it is worth the effort to introduce these quotients as independent elements of the theory. He bestowed them with the name, already used on page 415, of *elliptic functions of the second kind*, and distinguished the purely periodic functions from them as *elliptic functions of the first kind*. An elliptic function of the second kind changes, if one lets the argument t increase by a period, by a constant factor; it behaves, as we say, *m u l t i p l i c a t i v e l y*.

The number of the ϑ-functions in the numerator (or the denominator) always gives the degree of the function. Thus the following distinction obtains between elliptic functions of the first and second kinds: *there are no elliptic functions of the first kind and of the first degree; in contrast, elliptic functions of the second kind and first degree are very well possible.*

If, namely, $n = 1$ for an elliptic function of the first kind, then we would have only one null point in the period parallelogram, and, because of the relation between the a_ν and b_ν, one coinciding singular point. One could then cancel the ϑ-function in the numerator against that in the denominator, so that the function must reduce to a constant.

This remark finds no application to elliptic functions of the second kind, since for them the relation between the a_ν and b_ν does not obtain. Functions of the first degree are naturally the simplest and most important among the elliptic functions of the second kind. We now see that

Our parameters α, β, γ, δ are, in the Hermite terminology, such elliptic functions of the second kind and the first degree. The simplest elements of the motion of the top in kinematic respects are also as simple as possible in the analytical representation.

Not as simple—and on this directly depends the preeminence of our parameters—are the explicit representations of the Euler angles φ, ψ, ϑ,

*) In the work cited on page 151.

or, more correctly, their trigonometric functions.

We have already considered the function $\cos\vartheta = u(t)$ in the previous section. This function is, since it remains entirely unchanged for the increase of its argument by the periods, an elliptic function of the first kind, not of the first degree (doubly periodic functions of the first degree are indeed, according to the above, not possible), but rather of the second degree. In fact, one sees immediately that $u(t)$ becomes infinite and zero at two (different or coinciding) positions of the period rectangle. The point $u = 0$ of the u-plane corresponds, namely, to two different positions on the Riemann surface, one in the lower and one in the upper sheet, and thus also to two different points in each period rectangle of the t-plane. The position $u = \infty$, on the other hand, is a branch point; its image in the t-plane ($t = i\omega'$) is thus to be counted twice. Correspondingly, four ϑ-functions will appear in the representation of the function $u(t)$, two in the numerator and two (equal to each other, with argument $t - i\omega'$) in the denominator.

The matter is similar for the angle ψ. We first prefer, since the multipliers of the logarithmic singularities of ψ are, according to page 400, $\pm\frac{i}{2}$, to consider $2i\psi$ instead of ψ. Thus we return to the multipliers ± 1. If we now go over to the corresponding exponential function $e^{2i\psi}$, then we can show, just as for α, β, γ, δ, that this function is unbranched on the Riemann surface and is thus single-valued in the t-plane. Its null and singular points are known from the preceding. Since ψ becomes logarithmically infinite for $u = \pm 1$, $e^{2i\psi}$ vanishes on the Riemann surface at these two positions on one of the two sheets, and will become infinite to the first order on the other sheet.

The corresponding positions of the t-plane are the points $\pm ia$ and $\pm ib$ (or the equivalent positions). The null and singular points of ψ are therefore distributed at these positions. Correspondingly, the analytic representation will be composed of the ϑ-functions $\vartheta(t + ia)$, $\vartheta(t - ia)$, $\vartheta(t + ib)$, $\vartheta(t - ib)$, in the sense that two of these functions appear in the numerator and two in the denominator. We therefore have, in every case, an *elliptic function of the second degree.* From the properties of the elliptic integral for ψ, it further follows that $e^{2i\psi}$ does not remain unchanged for a passage to another period rectangle of the t-plane, but rather is multiplied by a factor that is independent of t.

The quantity $e^{2i\psi}$ therefore again represents an *elliptic function of the second kind.*

If we consider that the angle φ behaves similarly to ψ, then we can say in summary that

The trigonometric functions

$$\cos\vartheta, \quad \cos 2\psi + i\sin 2\psi, \quad \cos 2\varphi + i\sin 2\varphi$$

of the Euler angles are not, as are our parameters α, β, γ, δ, elliptic functions of the first degree, but rather of the second degree (and indeed partly of the first, and partly of the second kind).

Moreover, we can also directly reduce the representation of the named trigonometric functions to the representations of α, β, γ, δ by ϑ-quotients. We need only compare, for this purpose, the schemata (7) and (9) of pages 20 and 21. From the last horizontal or vertical rows there follow, namely,

$$(8')\qquad \begin{cases} \cos\vartheta = \alpha\delta + \beta\gamma, & \sin\vartheta = \sqrt{-4\alpha\beta\gamma\delta}, \\ e^{2i\psi} = \dfrac{\alpha\beta}{\gamma\delta}, & e^{2i\varphi} = \dfrac{\alpha\gamma}{\beta\delta}. \end{cases}$$

With consideration of (8), we have before us in these equations the *explicit representation of the Euler angles as functions of time.*

To conclude these considerations, we must append the determination of the constants k_i and l_i in equations (8).

We first show that these constants may be reduced to one another pairwise; we have, namely,

$$(9)\qquad \begin{cases} l_4 = -\,l_1, & l_3 = -\,l_2, \\ k_4 = \,k_1, & k_3 = \,k_2. \end{cases}$$

The two latter relations result in the following manner. We set $t = 0$ in (8) and first obtain, understanding by α_0, β_0, γ_0, δ_0 the initial values of α, β, γ, δ,

$$(10)\qquad \begin{cases} k_1 = \alpha_0 \dfrac{\vartheta(-i\omega')}{\vartheta(-ia)}, \quad k_2 = \beta_0 \dfrac{\vartheta(-i\omega')}{\vartheta(-\omega+ib)}, \quad k_3 = \gamma_0 \dfrac{\vartheta(+i\omega')}{\vartheta(+\omega-ib)}, \\ \qquad\qquad k_4 = \delta_0 \dfrac{\vartheta(+i\omega')}{\vartheta(+ia)}. \end{cases}$$

We can express α_0, β_0, γ_0, δ_0, according to the definitions (8) of page 21, in terms of the initial values ϑ_0, φ_0, ψ_0 of the Euler angles. Of these values, however, the quantities φ_0 and ψ_0 that give the initial position of the X- and x-axes with respect to the line of nodes are entirely arbitrary. In fact, the character of the motion in no way depends upon how we orient the X-axis in the equatorial plane of the top and the x-axis in the horizontal plane. Without restriction of generality, we can therefore let, for example, the X- and x-axes initially coincide with the line of nodes; that is, take $\varphi_0 = \psi_0 = 0.$*) The cited equations of page 21 then show, however, that $\alpha_0 = \delta_0$ and $\beta_0 = \gamma_0$. At the same time, according to (10), with consideration that the appearing ϑ-quotients now become pairwise equal, there follow, as claimed,

$$k_1 = k_4, \quad k_2 = k_3.$$

We deduce the two first relations (9) from the fact that the products $\alpha\delta$ and $\beta\gamma$ are easily assignable doubly periodic functions. We have (again according to equations (8) of page 21)

$$\alpha\delta = \frac{u+1}{2}, \quad \beta\gamma = \frac{u-1}{2}. \tag{11}$$

The two quantities on the right-hand side are, however, just like u itself, doubly periodic functions.

We wish to give their representations by ϑ-functions. The quantity $\dfrac{u+1}{2}$ vanishes if $u = -1$; that is, if $t = \pm ia$. Similarly, $\dfrac{u-1}{2}$ vanishes if $u = +1$; that is, if $t = \pm(\omega - ib)$. Further, $\dfrac{u+1}{2}$ and $\dfrac{u-1}{2}$ become infinite if $u = \infty$; that is, if $t = \pm i\omega'$. We now form the ϑ-quotients

$$\frac{\vartheta(t+ia)\vartheta(t-ia)}{\vartheta(t+i\omega')\vartheta(t-i\omega')} \quad \text{and} \quad \frac{\vartheta(t+\omega-ib)\vartheta(t-\omega+ib)}{\vartheta(t+i\omega')\vartheta(t-i\omega')}.$$

These quotients are (since in both cases the argument sum of the numerator is equal to that of the denominator), directly doubly periodic functions with the same null and singular points as $\dfrac{u+1}{2}$ and $\dfrac{u-1}{2}$. Our ϑ-quotients can differ from these expressions only by a constant. We thus write

$$\begin{cases} \dfrac{u+1}{2} = k^2 \dfrac{\vartheta(t+ia)\vartheta(t-ia)}{\vartheta(t+i\omega')\vartheta(t-i\omega')}, \\[2ex] \dfrac{u-1}{2} = k'^2 \dfrac{\vartheta(t+\omega-ib)\vartheta(t-\omega+ib)}{\vartheta(t+i\omega')\vartheta(t-i\omega')}. \end{cases} \tag{12}$$

*) If we would not make this simplifying assumption, then α, β, γ, δ would each be burdened in the final formulas by a factor of absolute value 1; namely, with

$$e^{\frac{i(\varphi_0+\psi_0)}{2}}, \quad e^{\frac{i(-\varphi_0+\psi_0)}{2}}, \quad e^{\frac{i(\varphi_0-\psi_0)}{2}}, \quad e^{\frac{i(-\varphi_0-\psi_0)}{2}},$$

respectively, which necessarily remain undetermined, and are irrelevant for all that follows.

The constants k^2 and k'^2 introduced here follow easily if we set, for example, $t = \omega + ib$ and correspondingly $u = +1$ in the first of equations (12), and $t = ia$ and $u = -1$ in the second. We then obtain, namely,

$$(13) \qquad \begin{cases} k^2 = \dfrac{\vartheta(\omega + i\omega' + ib)\vartheta(\omega - i\omega' + ib)}{\vartheta(\omega + ia + ib)\vartheta(\omega - ia + ib)}, \\[2ex] k'^2 = \dfrac{\vartheta(ia + i\omega')\vartheta(ia - i\omega')}{\vartheta(\omega + ia + ib)\vartheta(\omega - ia + ib)}. \end{cases}$$

We now enter these values of $\dfrac{u+1}{2}$ and $\dfrac{u-1}{2}$ in (11) and insert, at the same time, the expressions for $\alpha\delta$ and $\beta\gamma$ that result from (8). The ϑ-quotients that depend on t then cancel, and we obtain

$$k_1 k_4 e^{(l_1+l_4)t} = k^2,$$
$$k_2 k_3 e^{(l_2+l_3)t} = k'^2.$$

Thus

$$l_1 + l_4 = 0, \quad l_2 + l_3 = 0,$$
$$k_1 k_4 = k^2, \quad k_2 k_3 = k'^2.$$

The first row yields the affirmation of the claim stated in equations (9). The second row gives, with consideration of these same equations,

$$(14) \qquad k_1 = k_4 = k, \quad k_2 = k_3 = k'.$$

The determination of the constants k_i is thus accomplished; it remains only to say a word about the signs with which the square roots in k and k' should be calculated. These signs follow from the initial values of α, β, γ, δ. Since φ_0 and ψ_0 were taken as equal to zero and $\dfrac{\vartheta_0}{2}$ signifies in every case an acute angle, $\cos\dfrac{\vartheta_0}{2}$ and $\sin\dfrac{\vartheta_0}{2}$ are positive quantities; thus α_0 becomes, according to the defining equations for α, β, γ, δ on page 21, positive real, and β_0 positive imaginary. The signs of k and k' are therefore to be chosen so that for $t = 0$ a positive value of α and a positive imaginary value of β result from equations (8) on page 420. If one now considers that, understanding by τ a positive real number $< 2\omega'$, $\vartheta(+i\tau)$ is positive imaginary, $\vartheta(-i\tau)$ is negative imaginary, $\vartheta(\omega \pm i\tau)$ is positive real, and $\vartheta(-\omega \pm i\tau)$ is negative real, as is easily concluded from the series (3), one recognizes *that k and k' are both real, and that k is to be calculated with a positive sign and k' with a negative sign.* One should keep this determination of sign in mind in the following, where we will not explicitly return to it.

Finally, we have still to find the common values of the constants l_1 and $-l_4$, and l_2 and $-l_3$.

For this purpose, we differentiate the first of equations (8) logarithmically with respect to t and find

$$l_1 = \frac{d \log \alpha}{dt} - \frac{\vartheta'(t - ia)}{\vartheta(t - ia)} + \frac{\vartheta'(t - i\omega')}{\vartheta(t - i\omega')}. \tag{15}$$

We set here for t any particular value, say $t = ia$. Then the first two terms on the right-hand side cancel, as we immediately verify, and there follows

$$l_1 = \frac{\vartheta'(ia - i\omega')}{\vartheta(ia - i\omega')}. \tag{16}$$

For $t = ia$, namely, $u = -1$, and so we have, according to Taylor's theorem,

$$u + 1 = c(t - ia) + \cdots, \quad c = \left(\frac{du}{dt}\right)_{t=ia}. \tag{17}$$

We write, further,

$$\frac{d \log \alpha}{dt} = \frac{d \log \alpha}{du} \cdot \frac{du}{dt}.$$

According to page 402, however,

$$\frac{d \log \alpha}{du} = \frac{1}{u + 1}$$

for $u = -1$, up to terms that vanish with $u + 1$. We thus have, with consideration of (17),

$$\left(\frac{d \log \alpha}{dt}\right)_{t=ia} = \frac{1}{u + 1}\left(\frac{du}{dt}\right)_{t=ia} = \frac{c}{u + 1} = \frac{1}{t - ia}.$$

At the same time, since the ϑ-function is an odd function of its argument that vanishes with its argument to the first order,

$$\left(\frac{\vartheta'(t - ia)}{\vartheta(t - ia)}\right)_{t=ia} = \frac{1}{t - ia},$$

once more up to terms that vanish with $t - ia$ or $u + 1$. The two singular first terms in equation (15) therefore cancel, in fact, for $t = ia$, and there results for l_1 the simple value given in (16).

In an entirely corresponding manner one finds for l_2 the value

$$l_2 = \frac{\vartheta'(\omega - ib - i\omega')}{\vartheta(\omega - ib - i\omega')}. \tag{16$'$}$$

The quantities l_i determined in this manner are, as one is easily convinced, all purely imaginary. From the defining equation (3) of the ϑ-function it follows, namely, that this function is itself purely imaginary for a purely imaginary argument, and that, at the same time, its differential quotient becomes real. The same occurs, as is likewise evident from equation (3), if the real part of the argument (as in the expression for l_2) is equal to $-\omega$. We thus write, in that we understand by l and l' two real quantities, $l_1 = il$, $l_2 = il'$.

The complete tabulation of the constants l_i and k_i, through which our latter results are summarized, is now

$$(18)\begin{cases} k_1 = k, \quad k_2 = k', \quad k_3 = \quad k', \quad k_4 = \quad k \\ l_1 = il, \quad l_2 = il', \quad l_3 = -\, il', \quad l_4 = -\, il \\ k^2 = \dfrac{\vartheta(\omega+i\omega'+ib)\vartheta(\omega-i\omega'+ib)}{\vartheta(\omega+ia+ib)\vartheta(\omega-ia+ib)}, \; k'^2 = \dfrac{\vartheta(ia+i\omega')\vartheta(ia-i\omega')}{\vartheta(\omega+ia+ib)\vartheta(\omega-ia+ib)} \\ l = i\dfrac{\vartheta'(i\omega' - ia)}{\vartheta(i\omega' - ia)}, \quad l' = i\dfrac{\vartheta'(i\omega' - \omega + ib)}{\vartheta(i\omega' - \omega + ib)}. \end{cases}$$

If we insert these values of the constants into the formulas (8) *of page 420, then our parameters α, β, γ, δ are represented in a very transparent manner as functions of t.*

With respect to the constants of the problem, one can take a twofold standpoint.

First, one can regard, as was always done in the previous developments, the quantities that give the initial position, initial motion, and the mass distribution of the top as the fundamental constants of the problem. These were the quantities e, n, N, P, and A, where, however, as one easily recognizes, only the ratio $n : N : P : A$ is of importance for the latter four quantities. These four quantities therefore represent, together with e, only four essential numerical data. We wish to call these four data the "*elementary constants of the problem.*" From this first standpoint one must, before one can apply our final formulas, calculate from the elementary constants the values of the integral of the first kind denoted by ω, ω', a, and b, for which appropriate methods are developed in the fourth chapter.

Second, however, one can also regard these four integral values as the fundamental characteristic data of the spherical top, and can prescribe these four quantities in an entirely arbitrary manner. We call these four quantities concisely the "*transcendental constants of the problem.*" From this second standpoint, the knowledge of the elementary constants is superfluous for the command of the motion, since only the given transcendental constants are present in the final formulas. Moreover, those constants may be calculated from these at any time with the help of the ϑ-series.

The elementary constants are indeed more natural in geometric and mechanical respects. Nevertheless, the preference for the transcendental constants brings with it, in analytical respects, the advantages of greater symmetry and simplicity, so that we would designate, of the two named standpoints, the second as the higher and analytically more satisfying. It will, in particular, be decisive for us in the later sections of this chapter.

In conclusion, a few historical notes.

Jacobi*) was the first to consider the representation of the motion of the heavy top in terms of elliptic functions. It was not given to him, however, to publish his results. In the literature, the subject was treated for the first time after the manner of Jacobi by Lottner**), who also edited the relevant part of Jacobi's posthumous papers.[181] Both authors begin by expressing the nine direction cosines between the axes of the moving frame and axes of fixed frame in terms of ϑ-functions.***) We have the representation of the nine direction cosines, on the basis of our representation of the parameters α, β, γ, δ, directly in hand. We need only insert the values of the latter into the schema (9) of page 21, so that elliptic functions of the second kind and the second degree would result, and separate the real and imaginary parts. Since this procedure, however, would signify a passage from the simpler to the more complex, we can with good reason forgo its execution.

Mr. W. Heſs†) comes quite close to the introduction of our parameters α, β, γ, δ in his work "Über das Gyroskop." Toward the end of this work is found, as a result of a rather extensive calculation associated with the Lottner representation, the remark that the "elements of the Euler rotation," which in our notation are the quaternion quantities

$$A = \frac{\beta+\gamma}{2i}, \quad B = \frac{-\beta+\gamma}{2}, \quad C = \frac{\alpha-\delta}{2i}, \quad D = \frac{\alpha+\delta}{2},$$

*) Nouvelle théorie de la rotation d'un corps de révolution grave etc. and Sur la rotation d'un corp etc. Ges. W. Bd. II, pp. 477 and 493.

**) Reduktion eines schweren, um einen festen Punkt rotierenden Revolutionskörpers auf die elliptischen Transcendenten, Crelle's Journ. Bd. 50, 1855.

***) More precisely, Jacobi and Lottner considered two coordinate frames that rotate relative to the named frames with uniform velocity about the Z- and z-axes, respectively. The introduction of these coordinate frames corresponds partly to the reduction of the heavy symmetric top to the spherical top, and partly to the separation of a precessional component from the purely periodic nutational component of the motion.

†) Math. Annalen Bd. 29, 1887, cf., in particular, the last two pages.

exhibit a simpler aspect than the direction cosines considered by Jacobi and Lottner, "in that the former each depend on only one constant parameter, while the latter, in contrast, depend on two"; that is, in our terminology, the former are in essence elliptic functions of the first degree, while the latter are elliptic functions of the second degree. However, the author draws no further consequences from this remark; the quaternions appear there only incidentally, and in no way appear as the basis of the theory.

§5. The trajectory of the apex of the top, the polhode and herpolhode curves, etc., represented by ϑ-quotients.

The formulas given in the preceding section implicitly contain the answers to all questions that concern the motion of the top. A further treatment can only explicitly emphasize the consequences of the general analytic representation with respect to a few particular points.

We have previously directed our main interest to the depiction of the "trajectory." We thus wish to ask again here for the trajectory of the apex of the top. We will see that its equation emerges very elegantly with the help of the ϑ-functions.

We must first reconsider the geometric function-theoretical methods of the first chapter (cf. §3 of the same).

We considered there (cf. page 29) two commonly situated unit spheres with centers at O, one fixed in space and one fixed in the top, that were, in the Riemannian sense, bearers of the complex variables λ and Λ. The variable λ that was assigned to the points of the unit sphere fixed in space is associated with the rectangular coordinates xyz of those points through the equation (cf. page 28, equation (11))

$$\lambda = \frac{x+iy}{1-z}. \tag{1}$$

In the same manner, the relation between the variable Λ and the rectangular coordinates XYZ of the points of the unit sphere fixed in the top is

$$\Lambda = \frac{X+iY}{1-Z}. \tag{1'}$$

Finally, however, the simple relation

$$\lambda = \frac{\alpha\Lambda+\beta}{\gamma\Lambda+\delta} \tag{2}$$

obtains between the variables λ and Λ that correspond to two momentarily coinciding points of the spheres. The coefficients α, β, γ, δ

are the same quantities that we represented as functions of time in the preceding section.

In order to obtain the trajectory of the apex of the top, we substitute in ($1'$) the coordinates $X = 0$, $Y = 0$, $Z = +\sqrt{1 - X^2 - Y^2}$ of the apex of the top, so that $\Lambda = \infty$. Because of this value, equation (2) becomes

$$\lambda = \frac{\alpha}{\gamma}.$$

We insert here the values of α and γ from the previous section, and first obtain

$$\lambda = \frac{k}{k'} e^{i(l+l')t} \frac{\vartheta(t - ia)}{\vartheta(t + \omega - ib)} \frac{\vartheta(t + i\omega')}{\vartheta(t - i\omega')}.$$

If we consider the functional equations of the ϑ-function, then we can set

$$(3) \qquad \frac{\vartheta(t + i\omega')}{\vartheta(t - i\omega')} = -e^{-\frac{i\pi t}{\omega}}, \quad \frac{k^2}{k'^2} = \frac{\vartheta^2(\omega - i\omega' + ib)}{\vartheta^2(ia - i\omega')} e^{-\frac{i\pi}{\omega}(\omega + ib - ia)}.$$

Thus, for the correct determination of the signs (cf. page 426 above),

$$(4) \qquad \begin{cases} \lambda = K e^{iLt} \dfrac{\vartheta(t - ia)}{\vartheta(t + \omega - ib)}; \\[2ex] K = \dfrac{\vartheta(\omega - i\omega' + ib)}{\vartheta(ia - i\omega')} e^{-\frac{i\pi}{2\omega}(\omega + ib - ia)}, \quad L = l + l' - \dfrac{\pi}{\omega}. \end{cases}$$

This is the desired equation of the trajectory. *As we see, the trajectory of the apex of the top is again determined by an elliptic function of the second kind and the first degree.*

In order to understand the simple geometric meaning of our manner of representation, we recall the geometric meaning of the complex variable λ. We referred the unit sphere, whose points were distinguished through the variable λ, to its equatorial plane by means of the stereographic projection from the north pole. The quantity λ was then the complex value that the stereographic image of an individual point of the sphere receives according to the usual Gaufsian interpretation of complex quantities. We need only resolve λ into real and imaginary parts in order to obtain the rectangular coordinates of the stereographic image point in the equatorial plane.

Equation (4) *thus directly provides the plane stereographic image of the spatial trajectory; it can immediately serve as the basis for the graphical representation of the trajectory in the stereographic projection.*

We further recall (cf. page 207) that the stereographic projection has, for the purpose of depiction, certain advantages over the otherwise common orthographic projection. It is pleasing that our analytic representation of the trajectory conforms directly with the practical requirements of depiction.

We next compare with the representation of the trajectory that corresponds to the *orthographic* projection onto the equatorial plane.

We arrive at this representation if we ask for the rectangular coordinates x, y, z of the apex of the top in space; if we disregard the third coordinate z, the two remaining coordinates determine the desired orthographic projection. Moreover, it is recommended to go over from x and y to the complex combination $\xi = x + iy$ (or $\eta = x - iy$) in the sense of page 20, and therefore establish the orthographic projection, like the previous stereographic, in terms of a complex variable in the equatorial plane.

Now the coordinates of the apex of the top in the XYZ frame fixed in the top are $X = Y = 0$, $Z = 1$. The corresponding complex combinations defined on page 20 therefore become $\mathsf{\Xi} = \mathsf{H} = 0$, $\mathsf{Z} = -1$. We extract the desired value of ξ, expressed in terms of our α, β, γ, δ, from the schema (9) on page 21; it is

$$\xi = -2\alpha\beta.$$

If we insert here the values of α and β, then we obtain

$$(5) \qquad \xi = Ke^{iLt}\frac{\vartheta(t-ia)\vartheta(t-\omega+ib)}{\vartheta^2(t-i\omega')}, \quad K = -2kk', \quad L = l + l'$$

as the equation of the trajectory in the orthographic projection. This representation is evidently inferior to the previous in simplicity. *The trajectory in the orthographic projection is determined by an elliptic function of the second kind and the second degree, while it is given in the stereographic projection by an elliptic function of the first degree, a simple ϑ-quotient.*

That this circumstance is not to be undervalued will become clear in the following section, when we turn to the numerical calculation of the trajectory. Since we have need in (5) of four (or three different) ϑ-values, and in (4) of only two such values, the calculation of the orthographic projection of the trajectory requires nearly twice the work of the calculation of the stereographic projection.

The representation (5) was chosen by Hermite for the treatment of the spherical pendulum in his work cited on page 151.

Moreover, we chose not the north pole of the fixed unit sphere as the center of projection for the stereographic images in the series of figures in Chap. IV, but rather the south pole. This is always recommended if the trajectory runs completely or primarily in the northern hemisphere, since the stereographic image otherwise appears excessively enlarged and distorted. We can, however, easily go over from one projection to the other. We achieve this geometrically through a so-called inversion of the unit circle in the xy-plane; this corresponds analytically to replacing the value of the complex variable λ by the conjugate reciprocal value $1:\overline{\lambda}$. If we make this transformation in equation (4), there results for the stereographic projection from the south pole the representation

$$(4') \qquad \begin{cases} \lambda = Ke^{iLt}\dfrac{\vartheta(t+\omega+ib)}{\vartheta(t+ia)}; \\ K = \dfrac{\vartheta(i\omega'-ia)}{\vartheta(\omega+i\omega'-ib)}e^{-\frac{i\pi}{2\omega}(\omega-ib+ia)}, \quad L = l+l'-\dfrac{\pi}{\omega}. \end{cases}$$

Without the least effort, we can now also give the trajectory that an entirely arbitrary point of the top describes during the motion. We wish to assume, for the sake of brevity, that the relevant point has distance 1 from O, so that it always coincides with a point of the moving sphere. (In other cases, we need only multiply the formula to be given by the distance of the point from O.) We then characterize the position of our point on the moving sphere by the complex value $\Lambda = \Lambda_0$ in the previously described manner. The variable position of the point in space—that is, the desired trajectory or its plane image obtained by the stereographic projection from the north pole—will then be given, according to equation (2), by

$$(6) \qquad \lambda = \frac{\alpha\Lambda_0+\beta}{\gamma\Lambda_0+\delta}.$$

The expressions contained in (6) for arbitrary Λ_0 are no longer direct elliptic functions of the first or second kind, but only linear combinations of such functions. —

The preceding developments are at first valid, as are all the results of this chapter, only for the case of the spherical top. We can, however, very easily go over, according to §5 of the fourth chapter, to a symmetric top that has the same equatorial moment of inertia A as the spherical top and an arbitrary moment of inertia C; this will actually be carried out for the equations of the trajectory.

For this purpose, we must increase, according to page 234, the velocity coordinate φ' of the spherical top by the constant quantity

$$N\left(\frac{1}{C}-\frac{1}{A}\right),$$

which we wish to denote by c, while ϑ and ψ remain unchanged. The corresponding changes of α, β, γ, δ (cf. the original definition of these quantities on page 21) consist in multiplying

$$(7)\qquad \left\{\begin{array}{c|c|c|c|c} & \alpha & \beta & \gamma & \delta \\ \hline \text{by} & e^{+\frac{ic}{2}t} & e^{-\frac{ic}{2}t} & e^{+\frac{ic}{2}t} & e^{-\frac{ic}{2}t}. \end{array}\right.$$

As a result, the equation for the trajectory of a point Λ_0 of the symmetric top is

$$\lambda = \frac{\alpha e^{ict}\Lambda_0 + \beta}{\gamma e^{ict}\Lambda_0 + \delta},$$

where α, β, γ, δ signify the values of these parameters for the spherical top. In particular, exactly the same equation results for the trajectory of the apex ($\Lambda_0 = \infty$) of the symmetric top as for the spherical top—as is self-evident according to §5 of the fourth chapter. —

We now proceed to derive, in a similar manner, the *equations of the polhode and herpolhode curves of the spherical top*; that is, the curves that the endpoint of the rotation vector describes in the body and in space. We denote the coordinates of this point, as previously, by

$$p,\ q,\ r \ \text{ or } \ \pi,\ \kappa,\ \varrho,$$

according to whether we refer it to the coordinate frame fixed in space or in the top. The third coordinates r and ϱ are naturally constant for the spherical top, since they result from impulse components N and n through division by the moment of inertia A. We join the first two coordinates into the complex combinations $p+iq$, $\pi+i\kappa$; the expressions for these quantities and for r and ϱ in terms of α, β, γ, δ are, according to equations (5) and (6) of pages 43 and 44,

$$(8)\qquad \begin{cases} p+iq = 2i\Big(+\beta\dfrac{d\delta}{dt} - \delta\dfrac{d\beta}{dt}\Big), \\ \quad -r = 2i\Big(-\alpha\dfrac{d\delta}{dt} + \gamma\dfrac{d\beta}{dt}\Big). \end{cases}$$

$$(8') \qquad \begin{cases} \pi + i\kappa = 2i\Big(+\beta\dfrac{d\alpha}{dt} - \alpha\dfrac{d\beta}{dt}\Big), \\ \quad -\varrho = 2i\Big(+\delta\dfrac{d\alpha}{dt} - \gamma\dfrac{d\beta}{dt}\Big). \end{cases}$$

If we insert here the values of α, β, γ, δ, then we have before us the explicit representation of the coordinates of the polhode and herpolhode curves. The first two equations, considered in themselves, yield the orthographic projection of the polhode curve onto the equatorial plane of the top or the orthographic projection of the herpolhode curve onto the horizontal plane. The two last equations determine, at the same time, the height at which our curves run above the equatorial plane or the horizontal plane.

We wish to show that the preceding equations simplify in a very noteworthy manner. We consider, for example, $p + iq$.

We write

$$(9) \qquad p + iq = 2i\beta\delta\Theta,$$

and first convince ourselves that the quantity

$$\Theta = \frac{d\log\delta}{dt} - \frac{d\log\beta}{dt}$$

is a doubly periodic function of the second degree. We note in general, namely, that the function

$$\frac{d\log\vartheta(t-t_0)}{dt},$$

according to the functional equations of the ϑ-function, remains entirely unchanged or is increased additively by $-\dfrac{\pi i}{\omega}$ if we add 2ω or $2i\omega'$ to its argument. Thus the difference of two such functions is, in every case, a doubly periodic function. From such differences and constant terms, however, is composed our quantity Θ.

The explicit expression for Θ may, as a result of equations (8) and (18) of the previous section, be written after a few easy reductions as

$$\Theta = -\frac{\vartheta'(ia - i\omega')}{\vartheta(ia - i\omega')} + \frac{\vartheta'(\omega + ib - i\omega')}{\vartheta(\omega + ib - i\omega')} + \frac{\vartheta'(t + ia)}{\vartheta(t + ia)} - \frac{\vartheta'(t - \omega + ib)}{\vartheta(t - \omega + ib)}.$$

The singular points of Θ in the period rectangle are thus evidently $t = -ia$ and $t = \omega - ib$. Further, the null points become

$$t = -i\omega' \ \text{ and } \ t = \omega + i\omega' - ia - ib;$$

in the former case, namely, the first and second terms cancel the third and fourth, and in the latter case the first and second terms cancel the fourth and third.

Thus we can also give our quantity Θ the form

$$\Theta = C\frac{\vartheta(t+i\omega')\vartheta(t-\omega-i\omega'+ia+ib)}{\vartheta(t+ia)\vartheta(t-\omega+ib)}; \tag{10}$$

the expression on the right-hand side is, since the argument sum of the numerator is equal to that of the denominator, a doubly periodic function with the same null and singular points as Θ. The additional quantity C is a constant.

Equation (9) now takes the simple form

$$p+iq = Ke^{i(l'-l)t}\,\frac{\vartheta(t-\omega-i\omega'+ia+ib)}{\vartheta(t-i\omega')}. \tag{11}$$

In order to determine the constant K, in which the previously used and still unknown quantity C enters, we compare the values of $p+iq$ from (11) and (8) for an appropriately chosen point of time t. We have, for example, for $t=-ia$,

$$\delta = 0,\quad \frac{d\delta}{dt} = ke^{-la}\,\frac{\vartheta'(0)}{\vartheta(-\,ia+i\omega')},\quad \beta = k'e^{l'a}\frac{\vartheta(ia+\omega-ib)}{\vartheta(ia+i\omega')};$$

therefore, according to (8),

$$p+iq = 2ikk'e^{(l'-l)a}\,\frac{\vartheta'(0)\vartheta(ia+\omega-ib)}{\vartheta(-\,ia+i\omega')\vartheta(ia+i\omega')};$$

on the other hand, there follows from (11)

$$p+iq = Ke^{(l'-l)a}\,\frac{\vartheta(+\,\omega+i\omega'-ib)}{\vartheta(ia+i\omega')};$$

thus

$$K = 2ikk'\frac{\vartheta'(0)\vartheta(ia+\omega-ib)}{\vartheta(-\,ia+i\omega')\vartheta(\omega+i\omega'-ib)}.$$

If we insert, finally, the values of k and k' from equations (18) on page 428, then we obtain simply

$$K = \frac{-2\vartheta'(0)}{\vartheta(\omega+ia+ib)}e^{\frac{\pi(a+b)}{2\omega}}. \tag{11'}$$

The polhode curve is represented in orthographic projection by equations (11) and (11′). We particularly emphasize the simplicity of this representation;

The complex variable that determines the perpendicular projection of the rotation vector onto the equatorial plane of the top is again directly an elliptic function of the second kind and the first degree.

In order to express the constant third component $r = \dfrac{N}{A}$ in terms of our transcendental constants ω, ω', a, and b, we can insert a particular value of t in equation (8). We choose, for example, $t = \omega + ib$, so that $\gamma = 0$ and (because $\alpha\delta - \beta\gamma = 1$) $\alpha\delta = 1$. There follows

$$\begin{aligned}-r &= -2i\alpha\delta\frac{d\log\delta}{dt} = -2i\frac{d\log\delta}{dt}\\ &= 2i\Big(il - \frac{\vartheta'(\omega+ia+ib)}{\vartheta(\omega+ia+ib)} + \frac{\vartheta'(\omega+i\omega'+ib)}{\vartheta(\omega+i\omega'+ib)}\Big),\end{aligned}$$

and therefore, if we insert the value of l,

$$(12)\qquad -r = 2i\Big(-\frac{\vartheta'(i\omega'-ia)}{\vartheta(i\omega'-ia)} + \frac{\vartheta'(\omega+i\omega'+ib)}{\vartheta(\omega+i\omega'+ib)} - \frac{\vartheta'(\omega+ia+ib)}{\vartheta(\omega+ia+ib)}\Big).$$

We can immediately extract the corresponding *representation of the herpolhode curve* from the preceding equations for the polhode curve. We obtain, according to page 44, the coordinates $-\pi$, $-\kappa$, $-\varrho$ from the coordinates p, q, r if we exchange α and δ and reverse the signs of β and γ.

We can effect this exchange and change of sign, however, as a precise inspection of equations (8) and (18) of the previous section shows, simply by writing $-a$ instead of $+a$, so that $-l$ goes over into $l - \dfrac{\pi}{\omega}$. *We can thus write the equations of the herpolhode curve as*

$$(13)\qquad \pi + i\kappa = K'e^{i\left(l+l'-\frac{\pi}{\omega}\right)t}\,\frac{\vartheta(t-\omega-i\omega'-ia+ib)}{\vartheta(t-i\omega')},$$

$$(13')\qquad K' = \frac{2\vartheta'(0)}{\vartheta(\omega-ia+ib)}\,e^{\frac{\pi(b-a)}{2\omega}},$$

$$(14)\qquad -\varrho = 2i\Big(\frac{\vartheta'(i\omega'+ia)}{\vartheta(i\omega'+ia)} - \frac{\vartheta'(\omega+i\omega'+ib)}{\vartheta(\omega+i\omega'+ib)} + \frac{\vartheta'(\omega-ia+ib)}{\vartheta(\omega-ia+ib)}\Big).$$

We also arrive at the same equations from the principle developed on page 238, according to which we need only replace, in order to go over from the polhode curve to the herpolhode curve, the values of n and N by $-N$ and $-n$, and reverse the signs of the coordinates. If we do this, then the inequality

$$N > n > 0,$$

that was assumed until now as the basis of the representation of α, β, γ, δ by ϑ-quotients is changed, in the sense that for the values $N = -n$, $n = -N$ the condition is

$$0 > N > n.$$

The distribution of the logarithmic singular points of α, β, γ, δ at the points ± 1 of the Riemann surface in this case was represented in Fig. 61b. We see from this figure that the null points of β and γ are not changed by the named exchange, but that those of α and δ—that is, the positions $\pm ia$ of the t-plane—are interchanged. We therefore again have the sign reversal of a, and thus the passage from (11), (12) to (13), (14).

The preceding equations can also serve immediately for the representation of the impulse curves; that is, the curves that the endpoint of the impulse vector describes in space and in the body. Since, namely, the latter curves for the spherical top are geometrically similar to the curves of the polhode and the herpolhode, we need only multiply equations (11), (12), (13), and (14) on the right-hand sides by the value of the moment of inertia A in order to obtain the representation of the impulse coordinates $L+iM$, N, $l+im$, n. We thus state the summarizing theorem that

The polhode, herpolhode, and impulse curves for the spherical top may all be calculated in terms of elliptic functions of the second kind and the first degree.

This result also remains valid for the passage to the symmetric top, in so far as it refers to the impulse curve and the polhode curve.

We can, namely, derive the *polhode curve of the symmetric top* from that of the spherical top if we multiply the parameters α, β, γ, δ in the first of equations (8) by the factors given in (7). Correspondingly, we must insert

$$e^{-\frac{ic}{2}t}\left(\frac{d\beta}{dt} - \frac{ic}{2}\beta\right) \text{ and } e^{-\frac{ic}{2}t}\left(\frac{d\delta}{dt} - \frac{ic}{2}\delta\right)$$

for $\dfrac{d\beta}{dt}$ and $\dfrac{d\delta}{dt}$, respectively, understanding by β, δ, $\dfrac{d\beta}{dt}$, $\dfrac{d\delta}{dt}$ the values of these quantities for the spherical top. Since the additional terms $-\dfrac{ic}{2}\beta$ and $-\dfrac{ic}{2}\delta$ cancel in the difference on the right-hand of (8), we need only multiply the right-hand side of (11) by the factor e^{-ict} to obtain the quantity $p+iq$ for the symmetric top. This quantity will therefore likewise be given by an elliptic function of the first degree.

The coordinates of the impulse curve that represents the *locus of the impulse endpoint in the body* differ from the coordinates of the polhode curve only by the constant factors of the moments of inertia. This curve will therefore be described, in essence, by the same equations as the polhode curve.

As for the *trajectory of the impulse endpoint in space*, this curve for the symmetric top is not at all different from the same curve for the spherical top. In fact, we saw in Chap. IV, §5 that this curve is exactly the same for all tops of the series considered there. Equations (13) and (14) thus directly give (after multiplication by A) the impulse curve in question for a symmetric top whose one moment of inertia A is equal to that of our spherical top, and whose other moment of inertia C is arbitrary.

Less simple is the representation of the *herpolhode curve for the symmetric top.* In order to make use here of the preceding calculations, we express, in the generally valid equation ($8'$), the parameters α, β, γ, δ of the symmetric top in terms of those of the spherical top according to the table (7), and obtain

$$\pi + i\kappa = 2i\Big(\alpha\frac{d\beta}{dt} - \beta\frac{d\alpha}{dt}\Big) + 2c\alpha\beta.$$

We have brought the first term on the right to its simplest form in equation (13) above. In the second term, we insert the known values of α and β, and denote the multiplicative constant concisely by K. Then

$$\pi+i\kappa = e^{i(l+l')t}\Big(K'e^{-\frac{i\pi t}{\omega}}\frac{\vartheta(t-\omega-i\omega'-ia+ib)}{\vartheta(t-i\omega')} + K\frac{\vartheta(t-ia)\vartheta(t-\omega+ib)}{\vartheta^2(t-i\omega')}\Big),$$

or

$$(15)\quad \begin{cases} \pi + i\kappa = \\ e^{i(l+l')t}\,\dfrac{\vartheta(t-ia)\vartheta(t-\omega+ib)}{\vartheta^2(t-i\omega')}\Big(K+K'e^{-\frac{i\pi t}{\omega}}\dfrac{\vartheta(t-i\omega')\vartheta(t-\omega-i\omega'-ia+ib)}{\vartheta(t-ia)(t-\omega+ib)}\Big). \end{cases}$$

We claim that this expression can once again be written as a ϑ-quotient, where, however, two ϑ-functions appear in the numerator and two in the denominator. There first follows from the functional equations of the ϑ-function that the quantity in the parentheses is completely unchanged through the increase of t by one of the periods 2ω or $2i\omega'$. This quantity is therefore an elliptic function of the first kind and the second degree, with the singular points $t = ia$ and $t = \omega - ib$. Such a function necessarily has, as mentioned on page 421, two null points in the single period rectangle, which we denote by c_1 and c_2, and can be represented by the ϑ-quotient

$$K_1\frac{\vartheta(t-c_1)\vartheta(t-c_2)}{\vartheta(t-ia)\vartheta(t-\omega+ib)}.$$

But if we insert this value into the parentheses of (15), there follows, in fact,

$$(16)\qquad \pi + i\kappa = K_1 e^{i(l+l')t}\,\frac{\vartheta(t-c_1)\vartheta(t-c_2)}{\vartheta^2(t-i\omega')}.$$

We do not wish to enter into the precise determination of c_1, c_2, and K_1. We only state that

The horizontal projection of the herpolhode curve for the symmetric top is given by an elliptic function of the second kind and the second degree.[*])

In a similar manner, we can determine, starting from the second of equations (8), the *vertical projection ϱ of the herpolhode curve.* We find for this projection *an elliptic function of the first kind and second degree*; namely, a linear function of the doubly periodic quantity $\cos\vartheta = u(t)$, as already follows from equation (2) of page 235. —

In conclusion, a remark of more general content. We are repeatedly led, in the preceding, to elliptic curves of the second kind. Such curves appear not only for the heavy symmetric top, but also for the force-free asymmetric top (cf. §8 of this chapter) and for numerous other geometric and mechanical problems (the spherical catenary, the so-called elastic curves, etc., etc.). They form a large class among the related transcendental curves, which, in geometric respects, and also particularly with regard to applications, are not inferior in interest to algebraic curves. It would thus be well worth the labor to construct a *geometric theory of these transcendental curves*, from the same points of view that are decisive for the theory of algebraic curves. One would then have, for example, to investigate the possible singularities of such curves, to explore the intersection point theorems, to discuss the form relations, etc. Our elliptic curves of the first degree would naturally play a particularly important role in this theory. Without question, this geometric research would open a field with the promise of beautiful and relatively easy results.

§6. Numerical calculation of the motion by ϑ-series.

One of the final goals that we must always bear in mind for every problem in mechanics is this: to command the motion to the extent that the position of the moving system can be determined numerically at each instant. It is shown in this section that this goal is

[*]) Should it not also be possible to represent this curve, which indeed, according to page 235, runs on a certain sphere, by an elliptic function of the *first* degree, in that one projects it stereographically onto the equatorial plane of this sphere, and asks for the equation for the complex variable of the stereographic image point?

conveniently attained through the preceding theory. The procedure to be pursued may first be sketched in general.

If any real top is given, the first step consists in seeking, by experiment or calculation, to establish its mass distribution; that is, the values of A, C, and P with respect to the support point.

We must then know the initial position and the initial motion of the top. The initial position is sufficiently described by the inclination angle ϑ_0 of the figure axis with respect to the vertical; we will take the initial values of the angles φ and ψ, which are irrelevant for the character of the motion, directly equal to zero, as on page 425.

We best characterize the initial motion by the position and magnitude of the impulse vector. If we permit ourselves the agreed simplification on page 199 sub 4 that the impulse initially lies in the same vertical plane that contains the figure axis, then our vector is established by its two components n and N. We then know, at the same time, that the parallel circle $u = \cos\vartheta_0 = e$ must be one of the bounding circles for the trajectory, and that the apex of the top must initially progress in the horizontal direction.

With the constants n, N, and e, we form the quadratic equation $U_1 = 0$ of page 240, whose roots define the second bounding circle e', as well as the quantity e''.

We are now in a position to exploit the Legendre theory of elliptic integrals and the Legendre tables for our purpose. We calculate, above all, the Legendre modulus k, the complementary modulus k', and the auxiliary quantities M, φ_a, and φ_b of page 264. We then look up the values of $F\left(k, \frac{\pi}{2}\right)$ and $F\left(k', \frac{\pi}{2}\right)$ in Table I of Legendre, and the values of $F(k', \varphi_a)$ and $F(k', \varphi_b)$ in Table IX. The values of ω, ω', a, and b follow through multiplication by M. The motion of the apex of the top and the motion of the impulse endpoint in space are just as completely determined by these latter transcendental constants as by the original constants A, C, P, n, N, e. In fact, we by no means need, for the calculation of the named trajectory and impulse curves, to attend any longer to the mass distribution of the top, its initial motion, or its initial position. Our entire exercise consists in the calculation of certain ϑ-series, in whose coefficients the quantities ω and ω' enter, and in whose argument, moreover, the quantities a and b enter. The schema according to which we must calculate is the same for all tops and for all motions of the top. The entire multiplicity of the forms of motion depends solely on the variety of the values of the four transcendental constants that are inserted into our schema.

(We need return to the values of the original constants only if we wish to calculate further, for example, the direct values of α, β, γ, δ for the symmetric top, the trajectory of a point different from the apex of the top, or the polhode and herpolhode curves, in whose equations the quantity $N\left(\frac{1}{C}-\frac{1}{A}\right)$ is present.)

The consideration of the greatest possible convenience of the calculation can dispose us, however, to modify our procedure under certain circumstances. We note that the ϑ-series

$$\vartheta(t) = 2q^{1/4}\sin s - 2q^{9/4}\sin 3s + 2q^{25/4}\sin 5s - \cdots, \tag{1}$$

$$q = e^{-\frac{\omega'\pi}{\omega}}, \quad s = \frac{t\pi}{2\omega} \tag{2}$$

of page 418 converges faster as q is smaller, and therefore as the ratio $\frac{\omega'}{\omega}$ is larger. In particular, the calculation is more convenient in a period rectangle of greater height than breadth ($\omega' > \omega$) than in a period rectangle of greater breadth than height ($\omega > \omega'$). It is therefore important to know a transformation of the ϑ-function that always permits the reduction of the latter case to the former.

One is led to the intended transformation if one convinces oneself that the ϑ-function $\vartheta(t,\omega,\omega')$ takes on exactly the same factor, for an increase of the argument t by multiples of the periods, as the product of the ϑ-function $\vartheta(it,\omega',\omega)$ and the exponential quantity $e^{-\frac{\pi t^2}{4\omega\omega'}}$. Since the ϑ-function, according to page 419, is determined up to a t-independent factor by its behavior for the increase of its argument by multiples of the periods, one concludes that the ϑ-function $\vartheta(t,\omega,\omega')$ must be equal to the named product up to a multiplicative constant. We will not enter here into the determination of this constant, which causes some complications. The definitive formula is

$$\vartheta(t,\omega,\omega') = -i\sqrt{\frac{\omega}{\omega'}}\,e^{-\frac{\pi t^2}{4\omega\omega'}}\,\vartheta(it,\omega',\omega). \tag{3}$$

The ϑ-function on the right-hand side is evidently to be calculated by the series

$$(4) \qquad \vartheta(it, \omega', \omega) = 2q'^{1/4} \sin s' - 2q'^{9/4} \sin 3s' + 2q'^{25/4} \sin 5s' - \cdots ,$$

$$(5) \qquad q' = e^{-\frac{\omega\pi}{\omega'}}, \quad s = \frac{it\pi}{2\omega'}.$$

The usefulness of formula (3) is evident. If we must calculate a ϑ-function $\vartheta(t, \omega, \omega')$ in which $\omega > \omega'$, and thus q is a relatively large quantity, we will first calculate the series $\vartheta(it, \omega', \omega)$, which, because $q' < q$, will converge better. Equation (3) then permits us to return to the originally desired ϑ-function with light labor.

That the equation in question has, in addition to this practical interest, a more general theoretical interest in the subject of the transformations of ϑ-functions, may only be mentioned briefly here.

It can occur, however, that the depicted advantage that lies in the diminishment of q will be partly offset by the possible enlargement of the trigonometric functions in the ϑ-series. In fact, the factor i is introduced into the argument of the sine function in the passage to the function $\vartheta(it, \omega', \omega)$, which can, especially for real t, degrade the convergence considerably. The advantages and disadvantages of the transformation (3) are thus to be weighed against one another in an individual case.

Partly in order to illustrate the rapidity with which the ϑ-series converge, and partly in order to prepare for the working through of an example, we now wish to form a judgment of how many terms of the ϑ-series we must consider for our purpose. This naturally depends on the precision that we wish to achieve.

In our case, there would be no purpose in driving the precision as far, for example, as is usual in the mechanical problems of astronomy. In fact, the execution of a numerical example can serve only one of two goals: to enliven the geometric conception of the mechanical processes through the drawing of a quantitatively correct figure, or to compare the theoretically found process of the motion with experiments. In the former case, a moderate precision obviously suffices, since the drawing can be executed with only relatively little precision. In the latter case, it is to be remarked that experimental phenomena will be distorted by secondary circumstances—particularly by frictional processes—to the extent that a precise agreement with the abstract theory is generally not to be expected. A precision of $\frac{1}{1000}$, for example, will therefore suffice in our case; *we will correspondingly allow* $\frac{1}{1000}$ *of the total value as the permissible error bound*; that is, we will neglect quantities whose absolute value, divided by the absolute value of the total, is smaller than $\frac{1}{1000}$.

After this agreement, we show once and for all that we need always consider, for an appropriate disposition of the calculation, *only the first two terms of the ϑ-series.* We thus assume 1) that $\omega' \geq \omega$ and 2) that the argument of the ϑ-series belongs to the period rectangle that encompasses the origin. Were $\omega' < \omega$, namely, we could go over, according to equation (3), to a series in which the values of the periods are interchanged; further, if the argument of the representative point lay in one of the other period rectangles, we could draw upon the functional equations for the ϑ-function, in that we exclude appropriate multiples of the periods to reduce the point to the initial rectangle.

For a proof of the preceding claim, we replace each term in equation (1) by its absolute value. For the remainder R of the series beginning with the third term, we obtain

$$|R| < 2q^{25/4}|\sin 5s| + 2q^{49/4}|\sin 7s| + \cdots . \tag{6}$$

We first show, in general, that

$$|\sin(a+ib)| \leq e^{|b|}. \tag{7}$$

In fact, we have

$$|\sin(a+ib)|^2 = \frac{e^{-2b}+e^{+2b}-2\cos 2a}{4} \leq \left(\frac{e^{-b}+e^{+b}}{2}\right)^2,$$

and therefore

$$|\sin(a+ib)| \leq \frac{e^{-b}+e^{+b}}{2} \leq e^{|b|}.$$

We can now restrict ourselves, according to the above, to values of t whose imaginary part, taken in absolute value, is not larger than ω'. If we denote the imaginary part of s by b, then, according to equation (2),

$$|b| \leq \frac{\omega'\pi}{2\omega},$$

and thus, because of (7),

$$|\sin 5s| \leq e^{\frac{5}{2}\frac{\omega'}{\omega}\pi}, \quad |\sin 7s| \leq e^{\frac{7}{2}\frac{\omega'}{\omega}\pi}, \quad \ldots .$$

The right-hand sides of these inequalities can be denoted, according to (2), by $q^{-\frac{5}{2}}$, $q^{-\frac{7}{2}}$, ... The inequality (6) thus becomes

$$|R| < 2(q^{15/4} + q^{35/4} + \cdots).$$

One easily convinces oneself that the terms on the right-hand side decrease more strongly than the terms of the geometric series

$$2(q^{15/4} + q^{30/4} + q^{45/4} + \cdots) = \frac{2q^{15/4}}{1 - q^{15/4}}.$$

As a result,

$$|R| < \frac{2q^{15/4}}{1 - q^{15/4}}. \tag{8}$$

In order to obtain an upper bound for the relative error, we must divide the absolute value of this remainder R, or the just established upper bound of the same, by the absolute value of the entire value $\vartheta(t)$, or by any lower bound of the same. It is to be considered that the ϑ-function vanishes for vanishing t, so that we would evidently obtain a very unfavorable result for our relative error for very small values of $|t|$. We therefore add the explicit restriction, if we wish to apply our procedure, that the value of $|t|$ be not too small; say, for example, not smaller than $\frac{2\omega}{100}$. Under this assumption, $|s| > \frac{\pi}{100}$; at the same time, there obtains for all points in the interior of our period rectangle

$$|\sin s| > \sin\frac{\pi}{100} > 0{,}03. \tag{9}$$

We must now establish, under the assumption $|t| \geq \frac{2\omega}{100}$, a lower bound for the value of $|\vartheta(t)|$.

We first write, with the use of the abbreviation R introduced above,

$$|\vartheta(t)| = 2q^{1/4}|\sin s| \cdot \left|1 - q^2\,\frac{\sin 3s}{\sin s} + \frac{R}{2q^{1/4}\sin s}\right|.$$

We next use the theorem that the absolute value of a sum is greater than or equal to the difference of the absolute values of the summands. Thus

$$|\vartheta(t)| \geq 2q^{1/4}|\sin s| \left\{\left|1 - q^2\,\frac{\sin 3s}{\sin s}\right| - \left|\frac{R}{2q^{1/4}\sin s}\right|\right\}. \tag{10}$$

We will now further diminish the quantity in the braces, in that we make the first term smaller and the second term larger. If we set for $|R|$ the value in (8) found as too large, and for $|\sin s|$ the value in (9) given as too small, then

$$\left|\frac{R}{2q^{1/4}\sin s}\right| < \frac{q^{14/4}}{1-q^{15/4}}\frac{1}{0{,}03}.$$

If we consider, further, that we wish to restrict ourselves to the case $\omega' \geq \omega$ (that is, $q \leq e^{-\pi}$), then we obtain, even in the most unfavorable case $\omega' = \omega$ ($q = e^{-\pi}$),

$$\left|\frac{R}{2q^{1/4}\sin s}\right| < 0{,}0006. \tag{11}$$

On the other hand, we wish to divide out the fraction $\sin 3s : \sin s$ in the first term of the braces of (10). If we once again replace the absolute value of the resulting sum by the difference of the absolute values, we find

$$\left|1 - q^2\,\frac{\sin 3s}{\sin s}\right| = |1 - 3q^2 + 4q^2\sin^2 s| > 1 - 3q^2 - 4q^2|\sin^2 s|.$$

According to (7), however,

$$|\sin^2 s| < e^{\frac{2\omega'\pi}{2\omega}}, \quad \text{or } |\sin^2 s| < q^{-1}$$

for all points in our period rectangle. Thus we have

$$\left|1 - q^2\,\frac{\sin 3s}{\sin s}\right| > 1 - 4q - 3q^2,$$

or, if we once again go over to the most unfavorable case $q = e^{-\pi}$,

$$\left|1 - q^2\,\frac{\sin 3s}{\sin s}\right| > 1 - 0{,}1724 - 0{,}0054; \quad \text{that is,} \quad > 0{,}8222. \tag{12}$$

From (10), (11), and (12), there follows for $|\vartheta(t)|$ the lower bound

$$|\vartheta(t)| > 2q^{1/4}|\sin s| \cdot 0{,}8214,$$

or, with consideration of (9),

$$|\vartheta(t)| > 2q^{1/4} \cdot 0{,}0246. \tag{13}$$

With the help of the inequalities (8) and (13), the upper bound for the relative error $|R| : |\vartheta(t)|$ can be calculated immediately. We have, namely,

$$\frac{|R|}{|\vartheta(t)|} < \frac{q^{14/4}}{1-q^{15/4}}\frac{1}{0{,}0246};$$

if one evaluates this expression for the most unfavorable case $q = e^{-\pi}$, one finds that

$$\frac{|R|}{|\vartheta(t)|} < \frac{1{,}68}{2{,}46}\,10^{-3} < \frac{1}{1000}.$$

We are thus able to say that

For the required precision $\frac{1}{1000}$, *it always suffices to retain the first two terms of the* ϑ*-series, unless the argument of the* ϑ*-series differs very little from zero* $\left(|t| < \frac{2\omega}{100}\right)$.

The latter exception, moreover, is due only to the steps of our calculation, and not to the nature of the matter. Through more particular deliberations, which we do not, however, wish to carry out here, the exception may be eliminated, so that our theorem would obtain general validity.

It is still to be noted that we have made quite rough approximations in our estimation, so that the situation will be more favorable in actuality. This circumstance may justify the subsequent calculation of the series for the differential quotient $\vartheta'(t)$, which converges only slightly more poorly than than the ϑ-series itself, without more than the second term. Moreover, an error estimation for $\vartheta'(t)$ would not be difficult, and could be carried out almost exactly as above. Further, we will be justified on the same basis if each individual series is truncated at the second term in quotients or products, even though the error bound for a combination of n ϑ-series must at first be taken as n times the previously established error bound. —

We now proceed to the actual execution of a numerical example.

We adopt as a basis, for example, the top considered on page 299, which consists of a rotor with a square cross section. The side length of the square cross section is 2 cm, the distance from its midpoint to the figure axis is 5 cm, and the support point lies $\frac{5}{2}$ cm beneath the center of gravity. For the moment of inertia and the turning-moment of gravity, we found in the same place, understanding by ϱ the density of the material, the values

$$C = 1000\,\varrho\pi, \quad A = 750\,\varrho\pi, \quad P = 100\,\varrho\pi g$$

in the absolute measurement system.

Concerning the initial position of the top, we stipulate, for example, that the angle ϑ_0 between the figure axis and the vertical is equal to $60°$ at the beginning of the motion. We then have

$$e = \cos\vartheta_0 = \frac{1}{2}.$$

We next establish the state of the initial motion in terms of the impulse components N and n. We wish to choose N so that we have a strong top. According to page 249, this requires, in the present case $P > 0$,

$$N^2 > 2AP(1+e);$$

that is, for our top, since g is approximately $100\pi^2$,

$$N^2 > 3 \cdot 750 \cdot (100)^2 \varrho^2 \pi^4.$$

We satisfy this inequality if we take, for example,

$$N = 4800\,\varrho\pi^2.$$

The corresponding value of the rotation component r will then be

$$r = \frac{N}{C} = 4{,}8\,\pi;$$

that is (cf. page 11), 2,4 rotations per second. Further, we wish to prescribe the value of the impulse component n so that n will be smaller than N and positive, in which case we can apply the results on the null points of α, β, γ, δ (cf. page 402) and their associated representation by ϑ-functions in exactly the previous form. We choose, for example,

$$n = 4200\,\varrho\pi^2.$$

We next calculate the values of e' and e'' from the equation $U_1 = 0$ of page 240, which in our case runs

$$-\left(u + \frac{1}{2}\right)(4200^2 + 4800^2)\varrho^2\pi^4 + 2\left(1 + \frac{u}{2}\right)4200 \cdot 4800\,\varrho^2\pi^4$$
$$-\frac{3}{2}(1-u^2)750 \cdot 100\,\varrho^2\pi^2 g = 0.$$

If we again use for g the value $100\pi^2$, there follows

$$125u^2 - 228u + 97 = 0,$$

or

$$u^2 - 1{,}824\,u + 0{,}776 = 0.$$

The roots of this equation are 0,6759 and 1,1481. We thus have, in the sequence established on page 261,

$$e = 0{,}5000, \quad e' = 0{,}6759, \quad e'' = 1{,}1481. \tag{14}$$

We now calculate the Legendre modulus

$$k = \sqrt{\frac{e'-e}{e''-e}} = 0{,}5210,$$

and go over from this modulus to the angle

$$\Theta = \text{arc}\sin k = 31{,}40°.$$

The angle Θ' associated with the complementary modulus k' in the corresponding manner is therefore

$$\Theta' = 90^\circ - \Theta = 58{,}60^\circ.$$

The auxiliary quantities M, φ_a, φ_b of page 264 become

$$\log M = 0{,}1851 - 1, \quad \varphi_a = 71{,}10^\circ, \quad \varphi_b = 70{,}60^\circ.$$

We now find the values of $\log F\left(k, \frac{\pi}{2}\right)$ and $\log F\left(k', \frac{\pi}{2}\right)$ in the Legendre Table I. We find from page 228 and page 233 of this table

$$\log F\left(k, \frac{\pi}{2}\right) = 0{,}2298, \quad \log F\left(k', \frac{\pi}{2}\right) = 0{,}3263.$$

Thus we have

$$\begin{array}{lll|lll}
 & \log F\left(k, \frac{\pi}{2}\right) & = 0{,}2298 & & \log F\left(k', \frac{\pi}{2}\right) & = 0{,}3263 \\
 & \log M & = 0{,}1851 - 1 & & \log M & = 0{,}1851 - 1 \\
\hline
 & \log \omega & = 0{,}4149 - 1 & & \log \omega' & = 0{,}5114 - 1 \\
(15) & \omega & = 0{,}2600 & & \omega' & = 0{,}3246.
\end{array}$$

The favorable case for the calculation of the ϑ-function is therefore present in our example, since the height of the period rectangle is greater than the breadth. We thus have no reason to take up the transformation of the ϑ-series given in equation (3).

From $\log \omega$ and $\log \omega'$ we form, according to equation (2), the quantity q. There follows

$$\log q = 0{,}2959 - 2, \quad q = 0{,}0198.$$

We must further determine the quantities a and b from the Legendre Table IX, where a small interpolation is necessary. There follows, according to page 339 of this table,

$$F(k', \varphi_a) = 1{,}5129, \quad F(k', \varphi_b) = 1{,}4988.$$

Thus

$$\begin{array}{lll|lll}
 & \log F(k', \varphi_a) & = 0{,}1798 & & \log F(k', \varphi_b) & = 0{,}1757 \\
 & \log M & = 0{,}1851 - 1 & & \log M & = 0{,}1851 - 1 \\
\hline
 & \log a & = 0{,}3649 - 1 & & \log b & = 0{,}3608 - 1 \\
(16) & a & = 0{,}2317 & & b & = 0{,}2295.
\end{array}$$

We immediately compile a few of the quantities often present in the calculation of our ϑ-series in a small table:

$$e^{\frac{a\pi}{2\omega}} = 4{,}0563,\quad e^{-\frac{a\pi}{2\omega}} = 0{,}2465,\quad e^{\frac{3a\pi}{2\omega}} = 66{,}74,\quad e^{-\frac{3a\pi}{2\omega}} = 0{,}02,$$
$$-i\sin\frac{ia\pi}{2\omega} = 1{,}9049,\quad \cos\frac{ia\pi}{2\omega} = 2{,}1514,$$
$$-iq^2\sin\frac{3ia\pi}{2\omega} = 0{,}0130,\quad q^2\cos\frac{3ia\pi}{2\omega} = 0{,}0130;$$
$$e^{\frac{b\pi}{2\omega}} = 4{,}0031,\quad e^{-\frac{b\pi}{2\omega}} = 0{,}2498,\quad e^{\frac{3b\pi}{2\omega}} = 64{,}15,\quad e^{-\frac{3b\pi}{2\omega}} = 0{,}02,$$
$$-i\sin\frac{ib\pi}{2\omega} = 1{,}8766,\quad \cos\frac{ib\pi}{2\omega} = 2{,}1265,$$
$$-iq^2\sin\frac{3ib\pi}{2\omega} = 0{,}0125,\quad q^2\cos\frac{3ib\pi}{2\omega} = 0{,}0125;$$
$$e^{\frac{(\omega'-a)\pi}{2\omega}} = 1{,}7531,\; e^{-\frac{(\omega'-a)\pi}{2\omega}} = 0{,}5704,\; e^{\frac{3(\omega'-a)\pi}{2\omega}} = 5{,}39,\; e^{-\frac{3(\omega'-a)\pi}{2\omega}} = 0{,}19,$$
$$-i\sin\frac{i(\omega'-a)\pi}{2\omega} = 0{,}5913,\quad \cos\frac{i(\omega'-a)\pi}{2\omega} = 1{,}1617,$$
$$-iq^2\sin\frac{3i(\omega'-a)\pi}{2\omega} = 0{,}0011,\quad q^2\cos\frac{3i(\omega'-a)\pi}{2\omega} = 0{,}0011;$$
$$e^{\frac{(\omega'-b)\pi}{2\omega}} = 1{,}7766,\; e^{-\frac{(\omega'-b)\pi}{2\omega}} = 0{,}5629,\; e^{\frac{3(\omega'-b)\pi}{2\omega}} = 5{,}60,\; e^{-\frac{3(\omega'-b)\pi}{2\omega}} = 0{,}18,$$
$$-i\sin\frac{i(\omega'-b)\pi}{2\omega} = 0{,}6068,\quad \cos\frac{i(\omega'-b)\pi}{2\omega} = 1{,}1697,$$
$$-iq^2\sin\frac{3i(\omega'-b)\pi}{2\omega} = 0{,}0011,\quad q^2\cos\frac{3i(\omega'-b)\pi}{2\omega} = 0{,}0011.$$

We can now proceed to the calculation of the trajectory that the apex of the top describes on the unit sphere. Since e and e' are both positive, the trajectory runs entirely in the northern hemisphere, so that we will draw the curve as it appears in the stereographic projection from the *south pole*. Correspondingly, we select equation (4′) of page 433 as the analytic representation of the trajectory, and have

$$(17)\qquad \begin{cases} \lambda = Ke^{iLt}\dfrac{\vartheta(t+\omega+ib)}{\vartheta(t+ia)}; \\[2ex] K = \dfrac{\vartheta(i\omega'-ia)}{\vartheta(\omega+i\omega'-ib)}e^{-\frac{i\pi}{2\omega}(\omega-ib+ia)}, \quad L = l + l' - \dfrac{\pi}{\omega}. \end{cases}$$

The quantity λ then directly signifies, as we know, the complex variable that corresponds in the usual Gaußian sense to the image point of the apex of the top in the equatorial plane of the unit sphere for the stereographic projection from the south pole.

Here the constants K and L are first to be found.

We have, if we use the ϑ-series truncated at the second term and the values given in the preceding table,

$$K = \frac{-i\sin\dfrac{i(\omega'-a)\pi}{2\omega} + iq^2\sin\dfrac{3i(\omega'-a)\pi}{2\omega}}{\cos\dfrac{i(\omega'-b)\pi}{2\omega} + q^2\cos\dfrac{3i(\omega'-b)\pi}{2\omega}}\, e^{\frac{\pi}{2\omega}(a-b)}$$
$$= \frac{0{,}5913 - 0{,}0011}{1{,}1697 + 0{,}0011}\,\frac{4{,}0563}{4{,}0031} = 0{,}511.$$

For the determination of L, we next calculate the quantities

$$l = i\,\frac{\vartheta'(i\omega' - ia)}{\vartheta(i\omega' - ia)}, \quad l' = i\,\frac{\vartheta'(i\omega' - \omega + ib)}{\vartheta(i\omega' - \omega + ib)}$$

according to equations (18) of page 428. Here we wish to transform l' slightly, in that we write

$$l' = i\left(\frac{\vartheta'(-i\omega' - \omega + ib)}{\vartheta(-i\omega' - \omega + ib)} - \frac{i\pi}{\omega}\right) = -i\,\frac{\vartheta'(\omega + i\omega' - ib)}{\vartheta(\omega + i\omega' - ib)} + \frac{\pi}{\omega};$$

therefore

$$l' - \frac{\pi}{\omega} = -i\,\frac{\vartheta'(\omega + i\omega' - ib)}{\vartheta(\omega + i\omega' - ib)}.$$

Since we also wish to truncate the series for ϑ' at the second term, there follow, with consideration of our table,

$$l = \frac{\pi}{2\omega}\,\frac{\cos\dfrac{i(\omega'-a)\pi}{2\omega} - 3q^2\cos\dfrac{3i(\omega'-a)\pi}{2\omega}}{-i\sin\dfrac{i(\omega'-a)\pi}{2\omega} + iq^2\sin\dfrac{3i(\omega'-a)\pi}{2\omega}} =$$
$$= \frac{\pi}{2\omega}\,\frac{1{,}1617 - 0{,}0033}{0{,}5913 - 0{,}0011} = \frac{\pi}{2\omega}\,1{,}963,$$
$$l' - \frac{\pi}{\omega} = \frac{\pi}{2\omega}\,\frac{-i\sin\dfrac{i(\omega'-b)\pi}{2\omega} - 3iq^2\sin\dfrac{3i(\omega'-b)\pi}{2\omega}}{\cos\dfrac{i(\omega'-b)\pi}{2\omega} + q^2\cos\dfrac{3i(\omega'-b)\pi}{2\omega}} =$$
$$= -\frac{\pi}{2\omega}\,\frac{0{,}6068 + 0{,}0033}{1{,}1697 + 0{,}0011} = -\frac{\pi}{2\omega}\,0{,}521.$$

Thus

$$L = \frac{\pi}{2\omega}(1{,}963 - 0{,}521) = \frac{\pi}{2\omega}\,1{,}442.$$

We insert these value of the constants into equation (11), truncate the ϑ-series once more at the second term, and resolve the trigonometric functions into real and imaginary parts. If we set as an abbreviation $\dfrac{t\pi}{2\omega} = s$ and consult our table on page 450, there follows

$$(18)\quad \lambda = 0{,}511 e^{1{,}442\, is}\,\frac{(2{,}1265\cos s + 0{,}0125\cos 3s) - i(1{,}8766\sin s + 0{,}0125\sin 3s)}{(2{,}1514\sin s - 0{,}0130\sin 3s) + i(1{,}9049\cos s - 0{,}0130\cos 3s)}.$$

We wish to calculate ten points on the single half-arc between the parallel circles e and e' that correspond to the equally spaced time values $t = 0, \frac{\omega}{9}, \frac{2\omega}{9}, \dots, \frac{9\omega}{9}$. The corresponding values of s are $s = 0, \frac{\pi}{18}, \frac{2\pi}{18}, \dots, \frac{\pi}{2}$, or $s = 0, 10°, 20°, \dots, 90°$. The corresponding values of λ may be denoted by $\lambda_0, \lambda_1, \lambda_2, \dots, \lambda_9$. One first calculates, in a practical manner, the absolute values of these quantities; the ratio of the real and imaginary parts then results by trigonometric means. The result of the relatively convenient calculation is shown in the following table:

$$\begin{aligned}
\lambda_0 &= \qquad\quad - 0{,}577\, i\\
\lambda_1 &= 0{,}164 - 0{,}549\, i\\
\lambda_2 &= 0{,}304 - 0{,}470\, i\\
\lambda_3 &= 0{,}406 - 0{,}356\, i\\
\lambda_4 &= 0{,}464 - 0{,}227\, i\\
\lambda_5 &= 0{,}480 - 0{,}108\, i\\
\lambda_6 &= 0{,}471 + 0{,}017\, i\\
\lambda_7 &= 0{,}439 + 0{,}118\, i\\
\lambda_8 &= 0{,}393 + 0{,}205\, i\\
\lambda_9 &= 0{,}338 + 0{,}281\, i.
\end{aligned}$$

The last value provides a desirable confirmation of our calculation; its absolute value, namely, must equal the radius of the parallel circle e' projected stereographically into the equatorial plane; that is, $\operatorname{tang}\left(\frac{1}{2}\operatorname{arc}\cos e'\right)$. In fact,

$$\log|\lambda_9| = 0{,}6434 - 1 = \log\operatorname{tang}(23°45'),$$

and in conformity with equation (14), up to an error that amounts to less than $\frac{1}{1000}$ of the total value,

$$\cos(47°30') = 0{,}6756 = e'.$$

The preceding values of λ give us, for the drawing of the trajectory in the stereographic projection, the rectangular coordinates of ten of its points, which are marked in the following figure by the numbers $0, 1, 2, \dots$. One has only to bear in mind that the positive real and imaginary axes of the λ-plane result by stereographic projection from the two meridians of the unit circle that pass through the positive x- and y-axes, respectively. Since the former is transformed into the latter by a rotation *in* the clockwise sense as seen from the vertical (cf. the stipulation contained in Fig. 4 of page 18), the corresponding holds for the positive real and imaginary axes of the λ-plane. If we reckon the real axis positive toward the right, then we must reckon the imaginary axis positive *downward*, and therefore opposite to its usual direction in the Gaußian plane. The arrows in the figure are drawn correspondingly.

The trajectory begins, as the figure shows, from the negative imaginary axis, and encircles the vertical in the clockwise sense. The differences in the lengths of the arc segments 01, 12, 23, ..., which are traversed in the same time interval $\frac{\omega}{9}$, give us, at the same time, an image of the changing velocity of the apex of the top. The further course of the trajectory beyond point 9 has been completed in correspondence with the symmetry property of the trajectory.

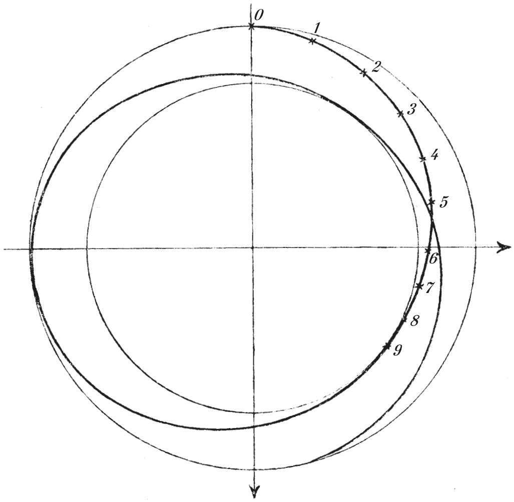


Fig. 65.

We note, in particular, that the span width ψ_ω of the individual half-arc of our curve, whose calculation was already discussed on page 269, is generally given by the value of the exponent Lt for $t = \omega$. Our trajectory equation now yields for ψ_ω the value

$$\psi_\omega = L\omega = (l + l')\omega - \pi.$$

In our example, this is, specifically,

$$\psi_\omega = 1{,}442\,\frac{\pi}{2} = 129^\circ\,46'\,48''.$$

The calculation of the remaining elements of the motion of the top—for example, the calculation of the horizontal projections of our two impulse curves—now offers no difficulty at all. The similarity in the analytic representation of these curves to the representation of our trajectory will effect a qualitative similarity between the courses of these curves and the curve just drawn.

In conclusion, we would like to point out most explicitly the remarkable simplicity and the entirely elementary character of the acquired manner of numerical calculation for the trajectory of the top. The analytic apparatus of the elliptic functions not only provides us with the theoretically precise trajectory equation (17), but also, which is no less remarkable, with a most highly elementary, practically sufficient approximation formula (in our example, equation (18)) that is far more complete and convenient than the approximate methods developed at the conclusion of the fourth chapter. The permissibility of the approximation formula in question is naturally bound to the condition that one reduces the argument of the ϑ-function in advance to the period parallelogram surrounding the origin, and that one has transformed, if necessary, a period rectangle of greater breadth than height into one of greater height than breadth. The ease with which this reduction (through the functional equations of the ϑ-function) or this transformation (through the transformation equation (3)) can be carried out for the ϑ-functions forms one of the advantages that distinguishes calculation with ϑ-functions over the direct evaluation of elliptic integrals.

The presence of the transcendental constants ω, ω', a, and b in our approximation formula can impair the elementary character of the approximation method just as little as the presence of exponential or trigonometric quantities, since those quantities, just as these, can be taken without trouble from the relevant tables. Thus if one desires an elementary treatment of the motion of the top, this is to be sought directly in the domain of elliptic functions, and realized by the truncation of the ϑ-series.[182]

One can even say, if we state the results of this section somewhat more generally, that *a ϑ-series practically signifies nothing other than a sum of two trigonometric terms. Every formula with elliptic functions may be replaced, for the purpose of numerical calculation, by one with a few trigonometric functions.*

§7. Representation of the motion of the force-free top by elliptic functions.

In this section, we wish to investigate more thoroughly the motion of the force-free top that was previously (cf. page 149 ff.) pursued only to the formation of elliptic integrals, and represent this motion explicitly by elliptic functions, according to the model of the theory of the heavy top.

We may restrict ourselves here to the analytical side of the problem, since we have extensively discussed the geometric properties of this motion previously (Chap. III, §2) in association with P o i n s o t. We wish to show, in the first place, how our parameters α, β, γ, δ effect here also the simplest and most complete description of the process of the motion.

In order to have a concise designation, we bestow upon the entire class of motions to be considered here the previously used name of *Poinsot motion.*

We first present the previously (pp. 148–150) acquired formulas that are of primary importance in the following. There are, first, the two algebraic integrals of the force-free motion of the top; namely, the theorems

$$(1)\qquad \begin{cases} Ap^2 + \ \ Bq^2 + \ \ Cr^2 = 2h, \\ A^2p^2 + B^2q^2 + C^2r^2 = G^2 \end{cases}$$

of the conservation of the *vis viva* and the conservation of the length of the impulse. If, further,

$$u = p^2 + q^2 + r^2$$

denotes the square of the length of the rotation vector, then p^2, q^2, and r^2 are calculated as linear functions of u, where, in particular,

$$(2)\qquad r^2 = \frac{uAB - 2h(A+B) + G^2}{(C-A)(C-B)};$$

the time t, however, is the elliptic integral

$$(3)\qquad t = \frac{1}{m}\int \frac{du}{pqr},$$

in which the constant m (which was denoted by c on page 149) has the value

$$(4)\qquad m = 2\left(\frac{B-C}{A} + \frac{C-A}{B} + \frac{A-B}{C}\right) = -2\frac{(A-B)(B-C)(C-A)}{ABC}.$$

We choose the spatially fixed impulse axis in a convenient manner as the third coordinate axis of the spatially fixed coordinate frame x, y, z. The direction cosines c, c', c'' of this axis with respect to the principal inertial frame X, Y, Z fixed in the body are then expressed simply by the equations

$$(5)\qquad c = \frac{Ap}{G}, \quad c' = \frac{Bq}{G}, \quad c'' = \frac{Cr}{G}.$$

In order to associate the treatment of the force-free top as closely as possible with that of the heavy top, we now wish to introduce, instead of the integration variable u, a new variable v that is equal to $\cos\vartheta = c''$. We then have, according to (5),

$$(6)\qquad v = \frac{Cr}{G};$$

the relation between u and v is, according to (2),

$$\frac{G^2 v^2}{C^2} = \frac{uAB - 2h(A+B) + G^2}{(C-A)(C-B)}. \tag{7}$$

In order to express t in terms of v, we calculate from (6) and (7)

$$\frac{du}{r} = \frac{G(C-A)(C-B)}{ABC}\,dv. \tag{8}$$

It further follows from (1), by elimination of q^2 or p^2, that

$$A(B-A)p^2 = (2hC - G^2v^2)\frac{B}{C} - G^2 + G^2v^2,$$

$$B(A-B)q^2 = (2hC - G^2v^2)\frac{A}{C} - G^2 + G^2v^2,$$

so that

$$\begin{cases} p^2 = \dfrac{G^2(B-C)}{AC(B-A)}\left(\dfrac{(2hB-G^2)C}{(B-C)G^2} - v^2\right), \\[2ex] q^2 = \dfrac{G^2(A-C)}{BC(A-B)}\left(\dfrac{(2hA-G^2)C}{(A-C)G^2} - v^2\right). \end{cases} \tag{9}$$

If we introduce the abbreviations

$$\begin{cases} e^2 = \dfrac{(2hA-G^2)C}{(A-C)G^2}, \quad e'^2 = \dfrac{(2hB-G^2)C}{(B-C)G^2}, \\[2ex] V = \dfrac{(A-C)(B-C)}{AB}\dfrac{G^2}{C^2}(e^2-v^2)(v^2-e'^2), \end{cases} \tag{9'}$$

then we can write

$$pq = \frac{G}{A-B}\sqrt{V}. \tag{10}$$

Transformed into the new variable v, our integral (3) thus runs, with consideration of (8) and (10) and the value of m given in (4),

$$t = \int \frac{dv}{\sqrt{V}}.$$

The polynomial of the fourth degree V now stands beneath the square root. Its null points $v = \pm e$ and $v = \pm e'$ give the branch points of the *two-sheeted Riemann surface* $(v, \sqrt{V})$ with which we must operate in the following. Each two overlying points of the Riemann surface are characterized by equal values of v and opposite values of $\sqrt{V}$.

In order to be able to judge the relative positions of the branch points, we first calculate from (9)

$$e'^2 - e^2 = \frac{(2hC - G^2)(A-B)C}{(A-C)(B-C)G^2}; \tag{10'}$$

we further assume for the principal moments of inertia A, B, C that

$$A > B > C.$$

The signs of e^2, e'^2, and $e'^2 - e^2$ will then be equal, according to (9′) and (10′), to the signs of the quantities

$$2hA - G^2, \quad 2hB - G^2, \quad 2hC - G^2.$$

The latter are given, however, if we successively eliminate p^2, q^2, and r^2 from equations (1). There follow, namely,

$$(11) \qquad \begin{cases} 2hA - G^2 = B(A-B)q^2 + C(A-C)r^2, \\ 2hB - G^2 = A(B-A)p^2 + C(B-C)r^2, \\ 2hC - G^2 = A(C-A)p^2 + B(C-B)q^2. \end{cases}$$

The right-hand side of the first equation is always positive, since p^2, q^2, r^2 are real positive quantities for the actual motion of the top, and because of the agreed inequality for the principal moments of inertia; the right-hand side of the last equation is certainly negative. The right-hand side of the second equation, in contrast, can (because $B - A < 0$ and $B - C > 0$) be positive or negative. We thus separate the motion of the force-free top into the separate classes $2hB - G^2 > 0$ and $2hB - G^2 < 0$; the unstable rotation about the intermediate principal inertial axis with $p = 0$, $r = 0$ belongs, among other examples, to the boundary cases $2hB - G^2 = 0$ that are common to both classes. We wish, however, to consider for the time being only motions that belong to the class $2hB - G^2 > 0$. We then have

$$2hA - G^2 > 0, \quad 2hB - G^2 > 0, \quad 2hC - G^2 < 0,$$

and, correspondingly,

$$e^2 > 0, \quad e'^2 > 0, \quad e^2 > e'^2.$$

The four branch points $\pm e$, $\pm e'$ therefore lie on the real axis, and follow one another in the sequence

$$-\infty < -e < -e' < +e' < e < +\infty.$$

The following figure represents a cut of the Riemann surface $(v, \sqrt{V})$ through the real axis. On the resulting double line, we distinguish

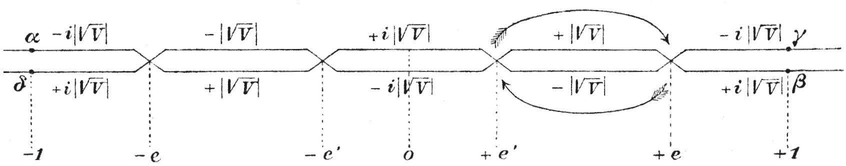


Fig. 66.

the four segments $(-e, -e')$, $(-e', +e')$, $(+e', +e)$, and the segment $(+e, \infty, -e)$ that is closed at infinity. According to equation (9′), V is positive in the first and third of these segments, and negative in the second and fourth. In order that t be real, the integration variable v is to be restricted to one of the first named segments, say the segment $(+e', +e)$. In order that dt be positive, moreover, we progress in the sense of the arrow in the figure. We place the lower limit of the integral at the branch point $v = e$.

In the following, we denote by 2ω the increase of t for a complete circuit about our integration segment ee', and therefore set

$$\omega = \int_e^{e'} \frac{dv}{\sqrt{V}}. \tag{12}$$

The same increase 2ω then results for a circuit about the segment $(-e, -e')$ (directed in the appropriate sense). On the other hand, we denote (for reasons that will later be evident) the increase of t for a circuit about one of the other two segments $(-e', e')$ or $(e, \infty, -e)$ by $4i\omega'$, so that, for example,

$$2i\omega' = \int_{+e}^{\infty} \frac{dv}{\sqrt{V}} + \int_{-\infty}^{-e} \frac{dv}{\sqrt{V}}. \tag{12$'$}$$

The quantities 2ω and $4i\omega'$ are called *the periods of our everywhere finite integral t.*

We must now concern ourselves, above all, with our parameters α, β, γ, δ. We first derive integral expressions for the logarithms of these quantities from the equations (4) on page 43.

We have, according to the first of these equations,

$$\frac{d \log \alpha}{dt} = \frac{ir}{2} + \frac{q + ip}{2} \frac{\beta}{\alpha}. \tag{13}$$

In order to express the quotient $\dfrac{\beta}{\alpha}$ in terms of known quantities, we compare the third horizontal rows in the schemata (3) and (9) of pages 17 and 21. We then find

$$\alpha\gamma = \frac{ic' - c}{2}, \quad \beta\delta = \frac{ic' + c}{2}, \quad \alpha\delta + \beta\gamma = c''. \tag{14}$$

If one combines the last of these equations with the identity $\alpha\delta - \beta\gamma = 1$, there follows

$$\alpha\delta = \frac{c'' + 1}{2}, \quad \beta\gamma = \frac{c'' - 1}{2}. \tag{14$'$}$$

From (14) and (14′) now follows, by division,

$$\frac{\beta}{\alpha} = \frac{ic' + c}{c'' + 1} = \frac{c'' - 1}{ic' - c}.$$

We insert here the values of c, c', c'' given in (5), and obtain

$$\frac{\beta}{\alpha} = \frac{Ap + iBq}{Cr + G} = \frac{G - Cr}{Ap - iBq}. \tag{15}$$

Thus equation (13) becomes

$$\frac{d\log\alpha}{dt} = \frac{ir}{2} + \frac{q + ip}{2}\frac{Ap + iBq}{Cr + G}. \tag{16}$$

On the right-hand side, we wish to express p, q, r in terms of v. If we introduce a common denominator, the right-hand side gives

$$\frac{i(Ap^2 + Bq^2 + Cr^2) + iGr + (A - B)pq}{2(Cr + G)}.$$

The first term in the numerator is, according to the theorem of the *vis viva*, equal to the constant $2ih$; the last term is equal, according to equation (10), to $G\sqrt{V}$. Thus (16) takes the form, if we substitute for r the value from equation (6),

$$\frac{d\log\alpha}{dt} = \frac{\dfrac{2ih}{G} + \dfrac{iG}{C}v + \sqrt{V}}{2(v + 1)}. \tag{16$'$}$$

If we replace, finally, the differentiation with respect to t by differentiation with respect to v, there follows

$$\frac{d\log\alpha}{dv} = \frac{\dfrac{2ih}{G} + \dfrac{iG}{C}v + \sqrt{V}}{2(v + 1)}\,\frac{1}{\sqrt{V}}. \tag{16$''$}$$

The integral representation of $\log\alpha$ *thus runs*, if we neglect the inessential constant of integration,

$$\log\alpha = \int \frac{\dfrac{2ih}{G} + \dfrac{iG}{C}v + \sqrt{V}}{2(v + 1)}\,\frac{dv}{\sqrt{V}}; \tag{17}$$

in a corresponding manner, one finds that

$$\left\{\begin{aligned} \log\beta &= \int \frac{-\dfrac{2ih}{G} + \dfrac{iG}{C}v + \sqrt{V}}{2(v - 1)}\,\frac{dv}{\sqrt{V}},\\ \log\gamma &= \int \frac{\dfrac{2ih}{G} - \dfrac{iG}{C}v + \sqrt{V}}{2(v - 1)}\,\frac{dv}{\sqrt{V}},\\ \log\delta &= \int \frac{-\dfrac{2ih}{G} - \dfrac{iG}{C}v + \sqrt{V}}{2(v + 1)}\,\frac{dv}{\sqrt{V}}. \end{aligned}\right. \tag{17}$$

We must now investigate the behavior of these integrals on our Riemann surface $(v, \sqrt{V})$, and, in particular, seek the positions of their singularities.

We consider, for example, $\log\alpha$. Among the singularities of the integrand—that is, the positions $v = \pm e$, $\pm e'$; $v = -1$, $v = \infty$—the branch points $\pm e$ and $\pm e'$ do not come into consideration as singularities of the integral, since the order of the infinity here is less than 1. The two other positions $v = -1$ and $v = \infty$ each represent two overlying points on the Riemann surface. We will now see that $\log\alpha$ becomes logarithmically infinite at the *two* overlying points $v = \infty$, but at only one of the points $v = -1$.

For the investigation of the position $v = \infty$, we can neglect the first two terms in the numerator of the integral representation (17) in comparison to the third term $\sqrt{V}$, since the first terms become infinite to a lower order than the third. If we also neglect 1 in the denominator compared with v, then we obtain simply

$$\log\alpha = \int \frac{dv}{2v} = \frac{1}{2}\log v. \tag{18}$$

This calculation is valid for both points $v = \infty$ of the Riemann surface, and for all four parameters α, β, γ, δ. We therefore see that

For $v = \infty$, the logarithms of all four parameters will become logarithmically infinite on both sheets of the surface.

We add, on the basis of the explanations of page 404 and on the representation of $\log\alpha$ in equation (18), that

For a positive, closed, single circuit about one of the positions $v = \infty$ on the Riemann surface, the logarithms of our four parameters increase by $-\pi i$.

As we go over to the position $v = -1$, we first remark that if $v + 1 = 0$—that is, $Cr + G = 0$—then, according to the latter of equations (15), either $Ap + iBq$ or $Ap - iBq$ vanishes at the same time. One case occurs in one, and the other in the other sheet of the Riemann surface.

On the sheet where $Ap + iBq$ vanishes, $\dfrac{d\log\alpha}{dt}$ and therefore also $\log\alpha$ remain, according to equation (16), finite, since here the null value of the denominator will be canceled by that of the numerator. We note, in particular, the value of $\sqrt{V}$, which is given from the just proven vanishing of the numerator in equation (17) as

$$\sqrt{V} = -\left(\frac{2ih}{G} + \frac{iG}{C}v\right).$$

On the other sheet, in contrast, where $Ap - iBq$ vanishes, a singularity of $\log\alpha$ occurs. The type of singularity is determined in the following manner. Since the values of $\sqrt{V}$ at overlying points of the Riemann surface differ only in sign, the value of $\sqrt{V}$ at the point in question will be opposite to that just given; we now have

$$\sqrt{V} = +\left(\frac{2ih}{G} + \frac{iG}{C}v\right).$$

The numerator in equation (17) will therefore be simply equal to $2\sqrt{V}$, so that

$$(18') \qquad \log\alpha = \int \frac{dv}{v+1} = \log(v+1).$$

If one investigates, in a corresponding manner, the behavior of $\log\beta$, $\log\gamma$, $\log\delta$ at the positions $v = \pm 1$, one arrives at the summarizing result that

The logarithms of our parameters α, β, γ, δ will each become logarithmically infinite at one of the four points of the Riemann surface $(v, \sqrt{V})$ that correspond to the values $v = \pm 1$. The increase of these logarithms for a single positive circuit about the relevant singularity is equal, as follows from equation (18′), *to* $+2\pi i$.

The distribution of the singularities on the two sheets of the Riemann surface is indicated in Fig. 66 of page 457 by the annexation of the letters α, β, γ, δ.

The behavior of our parameters on the Riemann surface in the present case is thus not entirely as simple as in the case of the heavy symmetric top. Their logarithms are now not, as they previously were, normal integrals of the third kind, since they become infinite at three positions of the Riemann surface; namely, at each of the two points $v = \infty$ and one of the points $v = \pm 1$. This circumstance has important consequences for the later representation by ϑ-functions.

We now go over from the logarithms to the parameters themselves. The logarithmic discontinuity (18′) changes into a simple null point; the discontinuity (18), however, changes into a singularity in whose neighborhood α (and, correspondingly, β, γ, and δ) behaves as $C\sqrt{v}$. It follows *that our parameters are branched at infinity of the Riemann surface*; they change in sign, namely, for a closed circuit on the surface about one of the two points $v = \infty$. They are, in general, otherwise unbranched relative to the Riemann surface, and remain, correspondingly, unchanged for all circuits that leave the value of t unchanged. For the "period circuits" in which t is increased by 2ω or $4i\omega'$, they take on certain characteristic factors whose logarithms can be calculated as the periods of the integrals (17).

After this preparation, we take up the concept of inversion from §3 of this chapter. We therefore consider t as the independent variable and interpret it in a complex t-plane. In this plane, we draw the previously described chessboard pattern (cf. Fig. 64) that separates the plane into infinitely many rectangles, with the difference that the lengths of the horizontal and vertical rectangle sides are now equal to 2ω and $4\omega'$. Each of these rectangles represents a conformal mapping of our Riemann surface $(v, \sqrt{V})$.

It is now enough (as previously) to pursue the distribution of the values of the functions $\alpha(t)$, $\beta(t)$, $\gamma(t)$, $\delta(t)$ in one of these rectangles, since the passage to the neighboring rectangles can be realized through multiplication by the previously cited constants. We consider, for example (cf. Fig. 67), the rectangle with the corners

$$\omega + 2i\omega', \quad -\omega + 2i\omega', \quad -\omega - 2i\omega', \quad \omega - 2i\omega'.$$

Since each parameter on the Riemann surface will be 0 at one position and ∞ at two positions, there must also be one null point and two singular points for each parameter in our rectangle, which is indeed a single-valued image of the surface. Where do these positions lie?

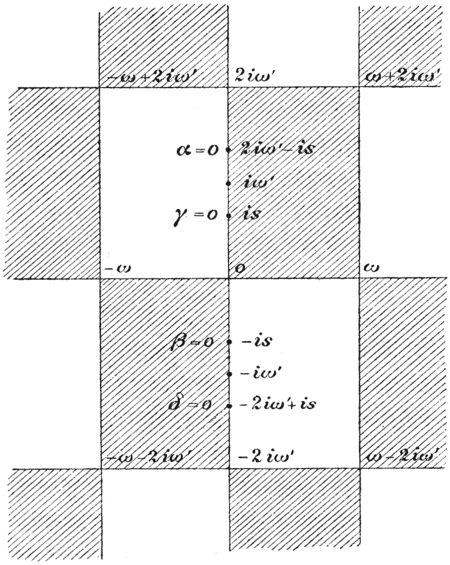


Fig. 67.

In order to discover the null points, we calculate the value of the integral of the first kind t from e to 1. Since e signifies a cosine (the cosine of the smallest angle that the Z-axis forms with the z-axis during the motion of the top), $e \leq 1$ in every case, and the value of the named integral is imaginary. We set it equal to is, and have, since $\sqrt{V}$ contains only the square of v,

$$is = \int_e^1 \frac{dv}{\sqrt{V}} = -\int_{-e}^{-1} \frac{dv}{\sqrt{V}}. \tag{19}$$

Thus the null points of γ and β are given directly by $t = +is$ and $t = -is$. In order to further obtain the null points of α and δ, we go, on either the upper or lower sheet, from e to infinity through $-e$, and from there to the point $v = -1$. The integral t thus assumes one of the values $\pm 2i\omega' \pm is$. Of these values, the values $2i\omega' - is$ and $-2i\omega' + is$ lie, interpreted as points in the t-plane, in our period rectangle. *One of them yields the null point of α, and the other that of δ.* It is not difficult to decide how the null points of γ and β are distributed in detail to the points $\pm is$ and the null points of δ and α to the points $\pm(2i\omega' - is)$. We do not, however, enter into this, but rather refer to Fig. 67.

The positions of the singular points, further, are characterized by the integral

$$\int_e^\infty \frac{dv}{\sqrt{V}}.$$

Here we can also write, since V contains only even powers of v,

$$\frac{1}{2}\left(\int_e^\infty \frac{dv}{\sqrt{V}} + \int_{-\infty}^{-e} \frac{dv}{\sqrt{V}}\right).$$

The value of this expression is known, however, from equation $(12')$; it is equal, namely, to $i\omega'$. If we displace the integration path into the other sheet, there results, evidently, $-i\omega'$ instead of $+i\omega'$. *The points $t = \pm i\omega'$ in the t-plane therefore correspond to the two overlying points $v = \infty$, the singular points of our parameters.*

In addition to the points given and drawn in Fig. 67, the collected equivalent points, which differ from those given by the period multiples $2m\omega + 4m'i\omega'$, are naturally likewise null and singular points. The order of the nullities and infinities is the same as on the Riemann surface. The order of the null points is always equal to 1, and the order of the singular points is equal to $\frac{1}{2}$.

We now base the analytic representation of our parameters in terms of ϑ-functions on the positions of the null and singular points. It is well to remark that a small change is to be made to the definition of the ϑ-function (cf. page 418), since the periods of our integral of the first kind, which were previously called 2ω and $2i\omega'$, are now denoted by 2ω and $4i\omega'$. Correspondingly, $2\omega'$ is to be replaced throughout by $4\omega'$ in the series representation, the functional equations of the ϑ-function, etc. In order to avoid ambiguity, we wish to denote this ϑ-function by $\Theta(t)$—though very much in contradiction to the meaning of this symbol introduced by Jacobi—while we will reserve the notation $\vartheta(t)$ for the series formed with the quantities 2ω, $2i\omega'$. Written in detail, the definitions of the two functions $\vartheta(t)$ and $\Theta(t)$ to be used in the following are[183]

$$\begin{aligned}\vartheta(t) &= \vartheta(t, 2\omega, 2i\omega') = e^{-\frac{\omega'\pi}{4\omega}} \sin\frac{\pi t}{2\omega} - e^{-\frac{9\omega'\pi}{4\omega}} \sin\frac{3\pi t}{2\omega} + \cdots,\\ \Theta(t) &= \vartheta(t, 2\omega, 4i\omega') = e^{-\frac{2\omega'\pi}{4\omega}} \sin\frac{\pi t}{2\omega} - e^{-\frac{18\omega'\pi}{4\omega}} \sin\frac{3\pi t}{2\omega} + \cdots.\end{aligned}$$

In order to arrive at the representation of α, we now consider the quotient

$$\frac{\Theta(t - 2i\omega' + is)}{\sqrt{\Theta(t - i\omega')\Theta(t + i\omega')}}.$$

This quotient will be zero and infinite at the same positions and to the same order as our parameter α. Moreover, it changes, just like α, only by a constant factor for an increase of t by 2ω or $4i\omega'$. If we divide α by this quotient, we therefore obtain a single-valued function that vanishes nowhere in the finite domain and is nowhere infinite, and that is multiplied by a constant factor if t is increased by one of the periods. One recognizes, however, exactly as on page 420, that such a function must have the form ke^{lt}. *There follows*

$$\alpha = k_1 e^{l_1 t} \frac{\Theta(t - 2i\omega' + is)}{\sqrt{\Theta(t - i\omega') \cdot \Theta(t + i\omega')}}. \tag{20}$$

In a corresponding manner there follow

$$\left\{\begin{aligned}\beta &= k_2 e^{l_2 t}\, \frac{\Theta(t + is)}{\sqrt{\Theta(t - i\omega') \cdot \Theta(t + i\omega')}},\\ \gamma &= k_3 e^{l_3 t}\, \frac{\Theta(t - is)}{\sqrt{\Theta(t - i\omega') \cdot \Theta(t + i\omega')}},\\ \delta &= k_4 e^{l_4 t}\, \frac{\Theta(t + 2i\omega' - is)}{\sqrt{\Theta(t - i\omega') \cdot \Theta(t + i\omega')}}.\end{aligned}\right. \tag{20}$$

Here the constants k and l are still to be determined.

With respect to the constants l, we first show that

$$l_4 = -l_1 \quad l_3 = -l_2. \tag{21}$$

From equations (14′), namely, it follows that the products

$$\alpha\delta = \frac{v+1}{2}, \quad \beta\gamma = \frac{v-1}{2}$$

are, just like v itself, doubly periodic functions with periods 2ω and $4i\omega'$. If we now increase t by 2ω, then $\alpha\delta$ and $\beta\gamma$ change by the factors

$$e^{(l_1+l_4)2\omega} \text{ and } e^{(l_2+l_3)2\omega},$$

respectively, since the Θ-functions in the quotients for $\alpha\delta$ and $\beta\gamma$ remain unchanged. On the other hand, these same quantities change, if we increase t by $4i\omega'$, by the respective factors

$$e^{(l_1+l_4)4i\omega'} \text{ and } e^{(l_2+l_3)4i\omega'}.$$

All four of these factors are therefore to be set to 1, which is achieved only if we take, as given above, $l_4 = -l_1$, $l_3 = -l_2$.

We further wish to show that

$$l_2 = l_1 + \frac{\pi i}{2\omega}. \tag{22}$$

For this purpose, we begin from one of the equations (14), which we can write, with consideration of (5), as

$$\alpha\gamma = \frac{iBq - Ap}{2G}. \tag{23}$$

Now, however, the values for p^2 and q^2 computed in equation (9) show that p and q change in sign for a single circuit about the branch points $v = e$ and $v = e'$, for which t increases by 2ω, but p and q remain unchanged for a circuit about the points $v = e$ and $v = -e$, for which t is increased by $4i\omega'$. On the other hand, according to equations (20) and (21), $\alpha\gamma$ takes on, if t is increased by 2ω or $4i\omega'$, the respective factor

$$e^{(l_1-l_2)2\omega} \text{ or } e^{(l_1-l_2)4i\omega'}e^{-\frac{2\pi\omega'}{\omega}}.$$

The first of these factors is therefore to be set equal to -1, and the second to $+1$. This leads with necessity to the given relation (22).

There remains the one quantity l_1. By logarithmic differentiation of (20), its value results as

$$l_1 = \frac{d\log\alpha}{dt} - \frac{d\log\Theta(t-2i\omega'+is)}{dt} + \frac{1}{2}\left\{\frac{d\log\Theta(t+i\omega')}{dt} + \frac{d\log\Theta(t-i\omega')}{dt}\right\}. \tag{24}$$

Here we can insert for t any special value for which the corresponding value of v and therefore also, according to (16′), the value of $\dfrac{d\log\alpha}{dt}$ is known. One can take, for example, $t = is$, whereby

$$v = +1, \quad \sqrt{V} = -\frac{2ih}{G} + \frac{iG}{C}, \quad \frac{d\log\alpha}{dt} = \frac{iG}{2C}.$$

It then follows that

$$(24') \qquad l_1 = \frac{iG}{2C} + \frac{\Theta'(2i\omega')}{\Theta(2i\omega')} + \frac{1}{2}\left\{\frac{\Theta'(i\omega' - is)}{\Theta(i\omega' - is)} - \frac{\Theta'(i\omega' + is)}{\Theta(i\omega' + is)}\right\}.$$

One recognizes from this expression that l_1 is purely imaginary, so that we prefer to set l_1 equal to il, where l is real.

One can easily simplify the expression for l_1 or l; we place no value on this, however, since we will regard *the quantity l itself*, in addition to the quantities ω, ω', and s that alone do not suffice for the specification of the motion, *as one of the characteristic constants of the Poinsot motion.*

The complete tabulation of our constants l_i is now

$$(25) \qquad l_1 = il, \quad l_2 = i\left(l + \frac{\pi}{2\omega}\right), \quad l_3 = -i\left(l + \frac{\pi}{2\omega}\right), \quad l_4 = -il.$$

In order to determine the constants k_i, we return to the initial values of α, β, γ, δ at $t = 0$, and express these values, according to the original definitions of page 21, in terms of the initial values of the Euler angles φ, ψ, ϑ. We then have

$$\alpha_0 = \cos\frac{\vartheta_0}{2}\, e^{\frac{i(\varphi_0+\psi_0)}{2}} \quad \text{etc.}$$

The initial time $t = 0$ is now chosen so that $v = e$ at $t = 0$, and therefore, according to equations (9), $q = 0$. From (5) it follows, however, that the direction cosine c' between the Y-axis and the z-axis also vanishes with q. The angle between these two axes is therefore a right angle. We designated, however, the line of the XY-plane that stands perpendicular to the z-axis as the line of nodes. The line of nodes and the Y-axis therefore coincide at $t = 0$. Thus $\varphi_0 = -\dfrac{\pi}{2}$ is to be assumed. The angle ψ_0, further, is completely at our pleasure. In fact, the course of the motion can in no way depend on how we orient the xyz-frame in space. We can, in particular, let the x-axis coincide with the line of nodes in the initial position, and correspondingly choose $\psi_0 = 0$. Then, however, the cited equations of page 21 show that

$$(26) \qquad \alpha_0 = -\delta_0, \quad \beta_0 = -\gamma_0.$$

If we set, on the other hand, $t = 0$ in equations (20), then we recognize that these equations would imply

$$\frac{\alpha_0}{k_1} = -\frac{\delta_0}{k_4}, \quad \frac{\beta_0}{k_2} = -\frac{\gamma_0}{k_3}. \tag{26'}$$

If the latter equations are to be compatible with the preceding, then

$$k_1 = k_4, \quad k_2 = k_3 \tag{27}$$

must necessarily follow.

In order, finally, to determine the value of k_1 and k_4, on the one hand, and of k_2 and k_3, on the other hand, we return to the products

$$\alpha\delta = \frac{v+1}{2}, \quad \beta\gamma = \frac{v-1}{2} \tag{28}$$

in equations (14′). Here we wish to represent the doubly periodic functions of t,

$$\frac{v \pm 1}{2},$$

in terms of t. Since $v+1$ vanishes for $t = \pm(2i\omega' - is)$ and becomes infinite for $t = \pm i\omega'$, the expression

$$\frac{\Theta(t - 2i\omega' + is)\Theta(t + 2i\omega' - is)}{\Theta(t - i\omega')\Theta(t + i\omega')}$$

has the same null and singular points in the t-plane as $v+1$. It is, moreover, a doubly periodic function, and can thus differ from $\frac{v+1}{2}$ only by a constant factor. We thus have

$$\frac{v+1}{2} = C\frac{\Theta(t - 2i\omega' + is)\Theta(t + 2i\omega' - is)}{\Theta(t - i\omega')\Theta(t + i\omega')}, \tag{29}$$

and, correspondingly,

$$\frac{v-1}{2} = C'\frac{\Theta(t - is)\Theta(t + is)}{\Theta(t - i\omega')\Theta(t + i\omega')}. \tag{29'}$$

The values of the introduced constants C and C', which naturally will not be confounded with the moment of inertia C, follow easily if we insert, for example, $t = is$ in (29) and $t = 2i\omega' - is$ in (29′). Then the left-hand sides will become $+1$ and -1, respectively, and the Θ-quotients on the right-hand sides will be equal to

$$\frac{\Theta(2i\omega' - 2is) \cdot \Theta(2i\omega')}{\Theta(i\omega' - is) \cdot \Theta(i\omega' + is)}$$

and

$$\frac{\Theta(2i\omega' - 2is) \cdot \Theta(2i\omega')}{\Theta(i\omega' - is) \cdot \Theta(3i\omega' - is)} = \frac{\Theta(2i\omega' - 2is) \cdot \Theta(2i\omega')}{\Theta(i\omega' - is) \cdot \Theta(i\omega' + is)}\, e^{-\frac{(\omega' - s)\pi}{\omega}},$$

respectively. We thus have

$$C = \frac{\Theta(i\omega' - is)\Theta(i\omega' + is)}{\Theta(2i\omega' - 2is)\Theta(2i\omega')}, \quad C' = -e^{\frac{(\omega' - s)\pi}{\omega}}C. \tag{30}$$

If we now insert the values of α, β, γ, δ from (20) in the left-hand sides of equations (28) and express the right-hand sides according to (29), the t-dependent portions cancel, and we immediately obtain

$$k_1 k_4 = C, \quad k_2 k_3 = C',$$

and therefore, with consideration of (27),

$$k_1 = \sqrt{C}, \quad k_2 = \sqrt{C'}, \quad k_3 = \sqrt{C'}, \quad k_4 = \sqrt{C}. \tag{31}$$

The signs of these square roots, as well as those appearing in (20), are to be chosen so that values of α_0, β_0, γ_0, δ_0 arising from the latter coincide with the previously discussed initial conditions.

Summarizing all the preceding, we can represent our parameters through the elegant system of equations

$$\left\{\begin{aligned} \alpha &= \sqrt{C}e^{ilt}\frac{\Theta(t-2i\omega'+is)}{\sqrt{\Theta(t-i\omega')\Theta(t+i\omega')}},\\ \beta &= i\sqrt{C}e^{ilt}e^{\frac{i\pi}{2\omega}(t-i\omega'+is)}\frac{\Theta(t+is)}{\sqrt{\Theta(t-i\omega')\Theta(t+i\omega')}},\\ \gamma &= i\sqrt{C}e^{-ilt}e^{-\frac{i\pi}{2\omega}(t+i\omega'-is)}\frac{\Theta(t-is)}{\sqrt{\Theta(t-i\omega')\Theta(t+i\omega')}},\\ \delta &= \sqrt{C}e^{-ilt}\frac{\Theta(t+2i\omega'-is)}{\sqrt{\Theta(t-i\omega')\Theta(t+i\omega')}};\\ C &= \frac{\Theta(i\omega'-is)\Theta(i\omega'+is)}{\Theta(2i\omega'-2is)\Theta(2i\omega')}. \end{aligned}\right. \tag{32}$$

Through these equations, the collected Poinsot motions are brought into a unified analytic schema. If we insert for the four appearing constants ω, ω', s, and l all possible real values, all possible motions of the force-free top must follow. One is easily convinced by an enumeration that our four constants are actually independent of each other, and that their number cannot be further reduced. All further theorems that we will construct in the following are simple consequences of this analytic schema.

We first direct our attention to the trajectory described by a point of the top that has distance 1 from O. (The trajectories of all other points of the top, whose distance from O is not equal to 1, are naturally geometrically similar to these trajectories.) We characterize the position of the relevant point with respect to the top, as previously, by the complex quantity Λ, and the position with respect to space by λ.

The previously known relation

$$\lambda = \frac{\alpha\Lambda + \beta}{\gamma\Lambda + \delta} \tag{33}$$

then obtains between λ and Λ. If we insert here the found values of α, β, γ, δ, then λ is known as a function of time. This function then directly yields, in the previously described manner, the stereographic projection of the trajectory onto the xy-plane.

The equations for the trajectories of the points on the Z-axis with distance 1 *from O will be particularly simple.* We choose the point on the positive Z-axis; for the point on the negative Z-axis, the following is valid mutatis mutandis. This point corresponds, according to equation $(1')$ on page 430, to the value $\Lambda = \infty$; the corresponding value of λ will be denoted by λ_Z. The equation for the trajectory in the stereographic projection then runs simply, according to (33),

$$\lambda_Z = \frac{\alpha}{\gamma},$$

or, if we substitute from (32),

$$\lambda_Z = -ie^{2ilt + \frac{i\pi}{2\omega}(t + i\omega' - is)} \frac{\Theta(t - 2i\omega' + is)}{\Theta(t - is)}. \tag{34}$$

It must be possible, however, to construct entirely corresponding representations for the trajectories of points on the other principal axes, since one can indeed interchange the designation of the axes. These representations may also be derived directly from equation (33). We insert for Λ the values of the points at distance 1 from O on the positive Y- and X-axes. These are (see equation $(1')$ of page 430) the values $\Lambda = i$ and $\Lambda = 1$. The equations for the trajectories of these points thus run

$$\lambda_Y = \frac{\alpha i + \beta}{\gamma i + \delta}, \quad \lambda_X = \frac{\alpha + \beta}{\gamma + \delta}.$$

One is now convinced, if one substitutes from (32), that these expressions can also be written as simple Θ-quotients. The periods, however, of the Θ-functions that occur here are not, as up to now, 2ω and $4i\omega'$, but rather 4ω and $2i\omega'$ for λ_Y, and 4ω and $2\omega + 2i\omega'$ for λ_X. The basis for this dissimiltude of periods is obviously, as one can examine in more detail, that we have used the relevant data for our three axes XYZ (p, q, r, A, B, C, etc.) in an asymmetric manner in the choice of the integration variable v and in the calculation of t.

The conversion of the preceding Θ-expressions into the new with half or twice the period belongs to a theoretically important domain often cultivated by mathematicians, the so-called *transformation theory of elliptic functions.* It is very interesting that this theory, which was developed from a purely abstract point of view and which we will encounter repeatedly in the following, finds a concrete application in our treatment of the Poinsot motion, It is unfortunately not possible for us to enter into this theory in detail here; we must restrict ourselves, in so far as it is required for the present purpose, to convey and derive it ad hoc.

Through the indicated deliberations, one now recognizes the correctness of the following summarizing statement:

The trajectory that is described in the Poinsot motion by a point lying at the distance 1 *from* O *on one of the three principal axes may always be written in a remarkably simple form by an elliptic function of the second kind and the first degree. The periods of the elliptic functions are, according to whether we consider the* Z-, Y-, *or* X-*axis,* 2ω *and* $4i\omega'$, *or* 4ω *and* $2i\omega'$, *or, finally,* 4ω *and* $2\omega + 2i\omega'$.

In contrast, the expressions for λ in equation (33) that correspond to other points of the top are not purely multiplicative for an increase of t by one of the periods 2ω or $4i\omega'$; they are thus not to be designated as elliptic functions.

We next consider, for the sake of completeness, the nine direction cosines a, b, c, a', b', c', a'', b'', c'' that the axes of the moving XYZ frame form with the axes of the fixed xyz frame. We wish to show, with the help of conversions that once again belong to the transformation theory of elliptic functions, that these quantities may also be represented in a very simple manner; *namely, as elliptic functions of the second kind and the first degree with periods* 2ω *and* $2i\omega'$. More precisely said, this holds not for the direction cosines themselves, but rather, on the one hand, for the complex combinations

$$a + ib, \quad a' + ib', \quad a'' + ib''$$

(as well as the conjugate quantities), and, on the other hand, for the cosines

$$c, \quad c', \quad c''.$$

We first write the expressions for these quantities in terms of α, β, γ, δ, which follow immediately from comparison of the schemata (3) and (9) of pages 17 and 21, and verify the correctness of the stated result for the constructed expressions. The expressions in question are

$$(35)\quad \begin{cases} a + ib = \alpha^2 - \beta^2, & a' + ib' = \quad i(\alpha^2 + \beta^2), \quad a'' + ib'' = -2\alpha\beta, \\ \qquad c = -\alpha\gamma + \beta\delta, & \qquad c' = -\, i(\alpha\gamma + \beta\delta), \qquad\quad c'' = \alpha\delta + \beta\gamma. \end{cases}$$

We consider, for example, the first of these quantities, $a+ib$. According to equation (32), we have

$$a+ib = Ce^{2ilt}\,\frac{\Theta^2(t-2i\omega'+is)+e^{\frac{i\pi}{\omega}(t-i\omega'+is)}\Theta^2(t+is)}{\Theta(t-i\omega')\Theta(t+i\omega')}. \tag{36}$$

The denominator vanishes for $t = i\omega' + 2m\omega + 4m'i\omega'$ and for $t = -i\omega' + 2m\omega + 4m'i\omega'$. In their totality, therefore, the null points of the denominator are given by

$$t = -i\omega' + 2m\omega + 2m'i\omega'.$$

Further, the numerator vanishes, as one easily concludes from the properties of the Θ-function, for $t = \omega + i\omega' - is + 2m\omega + 4m'i\omega'$ and $t = \omega - i\omega' - is + 2m\omega + 4m'i\omega'$. The totality of these t-values may be written as

$$t = \omega + i\omega' - is + 2m\omega + 2m'i\omega'.$$

For an increase of t by 2ω or $2i\omega'$, moreover, the numerator and the denominator change, after one has multiplied them by the common factor $e^{-\frac{t\pi i}{2\omega}}$, by the factors

$$\begin{aligned} &-1,\ +e^{\frac{\pi\omega'}{\omega}-\frac{i\pi}{\omega}(t+is-i\omega')} \quad &&\text{(numerator)},\\ &-1,\ -e^{\frac{\pi\omega'}{\omega}-\frac{i\pi}{\omega}(t+i\omega')} \quad &&\text{(denominator)}, \end{aligned}$$

respectively. These, however, are exactly the factors by which the numerator and denominator of the ϑ-quotient

$$\frac{\vartheta(t-\omega+is-i\omega')}{\vartheta(t+i\omega')}$$

would change; the periods of this ϑ-quotient are 2ω and $2i\omega'$, and its null and singular points coincide with those of $a+ib$.

This quotient can differ from the above combination of Θ-series only by a constant factor. We thus obtain the following relation between the Θ-functions of periods 2ω, $4i\omega'$ and the ϑ-series of periods 2ω, $2i\omega'$:

$$\begin{cases} C\dfrac{\Theta^2(t-2i\omega'+is)+e^{\frac{i\pi}{\omega}(t-i\omega'+is)}\Theta^2(t+is)}{\Theta(t-i\omega')\Theta(t+i\omega')} = \\[2ex] C_1\dfrac{\vartheta(t-\omega+is-i\omega')}{\vartheta(t+i\omega')}. \end{cases} \tag{37}$$

For the determination of C_1, we set $t=-is$, whereby the left-hand side, because of the value of C given in (32), becomes

$$\frac{\Theta(2i\omega')}{\Theta(2is-2i\omega')}.$$

We therefore first have

$$C_1 = \frac{\Theta(2i\omega')}{\Theta(2is - 2i\omega')} \cdot \frac{\vartheta(is - i\omega')}{\vartheta(\omega + i\omega')}. \tag{37'}$$

This value may be simplified still further, and reduced entirely to ϑ-functions. We prepend, for this purpose, the general relation*)

$$\frac{\Theta(2t)}{\Theta(2t_1)} = \frac{\vartheta(t)\vartheta(t+\omega)}{\vartheta(t_1)\vartheta(t_1+\omega)}. \tag{38}$$

In order to realize its correctness, one notes that the numerators of the right- and left-hand sides change, for an increase of t by 2ω or $2i\omega'$, by exactly the same factor. The numerators are therefore equal to one another up to a multiplicative constant. If one sets $t = t_1$, one see that this constant is correctly chosen.

We now wish to insert into equation (38) the particular values $t = i\omega'$, $t_1 = is - i\omega'$. Then

$$\frac{\Theta(2i\omega')}{\Theta(2is - 2i\omega')} = \frac{\vartheta(i\omega')\vartheta(\omega + i\omega')}{\vartheta(is - i\omega')\vartheta(\omega + is - i\omega')}.$$

Thus we can write more simply, instead of (37′),

$$C_1 = \frac{\vartheta(i\omega')}{\vartheta(\omega + is - i\omega')}. \tag{39}$$

Equation (36) for the desired quantity $a + ib$ thus takes, because of (37) and (39), the definitive form

$$a + ib = \frac{\vartheta(i\omega')}{\vartheta(\omega + is - i\omega')} \frac{\vartheta(t - \omega + is - i\omega')}{\vartheta(t + i\omega')} e^{2ilt}.$$

One can convert all the expressions given in (35) in an entirely corresponding manner, and reduce them to ϑ-functions of periods 2ω, $2i\omega'$. *We summarize the results in the following table:*[184]

$$\left\{\begin{aligned}
a + ib &= \frac{\vartheta(i\omega')\vartheta(t - \omega + is - i\omega')}{\vartheta(is - i\omega' + \omega)\vartheta(t + i\omega')}\, e^{2ilt},\\
a' + ib' &= i\,\frac{\vartheta(\omega + i\omega')\vartheta(t + is - i\omega')}{\vartheta(is - i\omega' + \omega)\vartheta(t + i\omega')}\, e^{2ilt},\\
a'' + ib'' &= i\,\frac{\vartheta(\omega)\vartheta(t + is)}{\vartheta(is - i\omega' + \omega)\vartheta(t + i\omega')}\, e^{2ilt} e^{\frac{i\pi}{2\omega}(t - i\omega' + is)},\\
c &= \frac{\vartheta(is)\vartheta(t - \omega)}{\vartheta(is - i\omega' + \omega)\vartheta(t + i\omega')}\, e^{-\frac{i\pi}{2\omega}(t + i\omega' - is)},\\
c' &= i\,\frac{\vartheta(\omega + is)\vartheta(t)}{\vartheta(is - i\omega' + \omega)\vartheta(t + i\omega')}\, e^{-\frac{i\pi}{2\omega}(t + i\omega' - is)},\\
c'' &= \frac{\vartheta(is + i\omega')\vartheta(t + i\omega' + \omega)}{\vartheta(is - i\omega' + \omega)\vartheta(t + i\omega')}\, e^{-\frac{\pi s}{\omega}}.
\end{aligned}\right. \tag{40}$$

*) This formula, as well as equation (37), etc., are developed systematically in the theory of the transformations of elliptic functions.

We have derived, in these equations, the essential result of a famous work of J a c o b i*) (Sur la rotation d'un corps etc.) in a new manner. This result runs, in our terminology, in the following manner:

The nine direction cosines between the axes of the moving and the fixed coordinate systems (or, more correctly, the given complex combinations of these direction cosines) are all elliptic functions of the first degree with periods 2ω and $2i\omega'$.

We proceed, finally, to the consideration of the *polhode* and *herpolhode curves***) *of the Poinsot motion.*

Concerning the polhode curve, only a few words for the present. According to equation (5), the coordinates p, q, r of the polhode curve are proportional to the direction cosines c, c', c''; they are thus represented, just like these direction cosines, by *elliptic functions of the first degree.* The factors by which p, q, r are multiplied for an increase of t by the periods 2ω, $2i\omega'$ will be particularly simple; namely, ± 1, as is evident from equations (40). This also follows from the circumstance that the quantities p^2, q^2, and r^2 must be doubly periodic as entire functions of v; that is, must have the factor $+1$ for period increases. For the square roots p, q, r, only the factors ± 1 can occur.

We enter in more detail into the herpolhode curve. We write its equations in the form of page 44 as

$$\pi + i\kappa = 2i\Big(\beta\frac{d\alpha}{dt} - \alpha\frac{d\beta}{dt}\Big),$$
$$-\varrho = 2i\Big(\delta\frac{d\alpha}{dt} - \gamma\frac{d\beta}{dt}\Big).$$

If we insert the values of α and β from (32) into the first equation, we obtain

$$\pi + i\kappa = 2i\alpha\beta\Big(\frac{d\log\alpha}{dt} - \frac{d\log\beta}{dt}\Big)$$
$$= -2C\frac{\Theta(t-2i\omega'+is)\Theta(t+is)}{\Theta(t-i\omega')\Theta(t+i\omega')}e^{2ilt+\frac{i\pi}{2\omega}(t-i\omega'+is)}\cdot\Big\{\frac{\Theta'(t-2i\omega'+is)}{\Theta(t-2i\omega'+is)} - \frac{\Theta'(t+is)}{\Theta(t+is)} - \frac{i\pi}{2\omega}\Big\}.$$

Here we wish to go over once again to ϑ-functions of periods 2ω and $2i\omega'$. We first note that the expression in the braces changes by the factor $+1$ or -1 for the increase of t by 2ω or $2i\omega'$, respectively. The same changes occur, however, for the quotient

*) Cf. Ges. Werke Bd. 2 page 293 or Crelle's Journal Bd. 39.

**) The representation of the polhode curve, and also, in part, that of the herpolhode curve, were given in terms of elliptic functions for first time by R u e b in his dissertation, Utrecht 1834.

$$\frac{\vartheta(t-\omega+is)}{\vartheta(t+is)},$$

whose null and singular points also coincide with the null and singular points of our braces. We therefore have

$$\frac{\Theta'(t-2i\omega'+is)}{\Theta(t-2i\omega'+is)}-\frac{\Theta'(t+is)}{\Theta(t+is)}-\frac{i\pi}{2\omega}=C_1\frac{\vartheta(t-\omega'+is)}{\vartheta(t+is)}.$$

If we set $t=-is$, we determine the constant C_1 as

$$C_1=\frac{\vartheta'(0)}{\vartheta(\omega)}.$$

The Θ-quotient before the braces may likewise be converted to ϑ-functions of periods 2ω and $2i\omega'$. We have, obviously,

$$C\frac{\Theta(t-2i\omega'+is)\Theta(t+is)}{\Theta(t-i\omega')\Theta(t+i\omega')}=C_2\frac{\vartheta(t+is)}{\vartheta(t+i\omega')}.$$

We again determine the introduced constant C_2 if we set $t=-is$. With consideration of the value of C from (32), there first follows

$$C_2=\frac{\Theta'(0)\vartheta(i\omega'-is)}{\Theta(2i\omega'-2is)\vartheta'(0)}.$$

For the further simplification of this quotient, we return to equation (38). If we set there $t=0$, $t_1=i\omega'-is$, then we obtain

$$\frac{2\Theta'(0)}{\Theta(2i\omega'-2is)}=\frac{\vartheta'(0)\vartheta(\omega)}{\vartheta(i\omega'-is)\vartheta(\omega+i\omega'-is)}.$$

Therefore

$$C_2=\frac{1}{2}\frac{\vartheta(\omega)}{\vartheta(\omega+i\omega'-is)}.$$

Thus there follows for $\pi+i\kappa$ the definitive form

$$\pi+i\kappa=-\frac{\vartheta'(0)}{\vartheta(\omega+i\omega'-is)}e^{2ilt+\frac{i\pi}{2\omega}(t-i\omega'+is)}\frac{\vartheta(t-\omega+is)}{\vartheta(t+i\omega')}. \tag{41}$$

This expression exhibits the greatest analogy with the previous expression for the herpolhode curve of the heavy symmetric top (equations (13) and (13′) of page 437). It is not only likewise an *elliptic function of the first degree*, but can also, if we make an appropriate substitution for s, be directly transformed into that one. In the following section, we will have important consequences to extract from this remark.

We can form the expression for ϱ in a corresponding manner. According to what has gone before (cf. page 124), we know that ϱ is a constant $\left(=\dfrac{2h}{G}\right)$. We can therefore insert a special value of t (for example, $t=+is$) in the general value of ϱ given above. Then γ vanishes, and, at the same time, $\alpha\delta = 1$ (because $\alpha\delta - \beta\gamma = 1$). We thus obtain

$$-\varrho = 2i\alpha\delta\left(\frac{d\log\alpha}{dt}\right) = 2i\frac{d\log\alpha}{dt};$$

that is,

$$-\varrho = 2i\Big(il + \frac{\Theta'(2is - 2i\omega')}{\Theta(2is - 2i\omega')} - \frac{1}{2}\Big\{\frac{\Theta'(is - i\omega')}{\Theta(is - i\omega')} + \frac{\Theta'(is + i\omega')}{\Theta(is + i\omega')}\Big\}\Big).$$

We wish to go over again from Θ- to ϑ-functions. We achieve this by the substitutions

$$\frac{\Theta'(2is - 2i\omega')}{\Theta(2is - 2i\omega')} = \frac{1}{2}\Big\{\frac{\vartheta'(is - i\omega')}{\vartheta(is - i\omega')} + \frac{\vartheta'(\omega + is - i\omega')}{\vartheta(\omega + is - i\omega')}\Big\}$$

and

$$\frac{\Theta'(is - i\omega')}{\Theta(is - i\omega')} + \frac{\Theta'(is + i\omega')}{\Theta(is + i\omega')} = \frac{\vartheta'(is - i\omega')}{\vartheta(is - i\omega')} - \frac{i\pi}{2\omega}.$$

The first of these equations follows from (38) by logarithmic differentiation; one verifies the second if one compares the behaviors of the right- and left-hand sides for period increases and correctly determines an additive constant.

Instead of the original value of ϱ, we can now write

$$-\varrho = i\Big(2il + \frac{3i\pi}{2\omega} + \frac{\vartheta'(\omega + is + i\omega')}{\vartheta(\omega + is + i\omega')}\Big). \tag{42}$$

It follows as the collective result of this section that our parameters α, β, γ, δ provide a very appropriate instrument for the treatment of the Poinsot motion. If their expressions in terms of Θ-series are once found, then we have in hand all the remaining elements of the motion. These expressions are indeed not entirely as simple as in the theory of the heavy spherical top; in addition, we could not avoid somewhat detailed conversions for the passage from Θ-functions to ϑ-functions. This is based, however, on the nature of the matter, and has a certain interest in itself with respect to the transformation theory of elliptic functions. If we had avoided the use of Θ-series altogether and had begun, with Jacobi, directly from the expressions for the nine direction cosines, then the completeness of the development would have suffered. In particular, the beautiful result on the trajectories described by points of the principal inertial axes would have escaped us.

§8. Conjugate Poinsot motions. Jacobi's theorem on the relation between the motion of the force-free asymmetric top and the heavy spherical top.

Having dispatched the treatment of the single Poinsot motion, we now describe the relation of two such motions arranged in a specific manner with respect to one another, the so-called "conjugate" Poinsot motions. There will follow an often cited theorem of J a c o b i that states a remarkable relation between the motion of the heavy spherical top and the theory of the conjugate Poinsot motions.

For this purpose, we return once more to the polhode curve, and show that one and the same polhode curve can always be conceived as the polhode curve of a Poinsot motion in a twofold manner; it simultaneously represents, namely, the polhode curve for two different real force-free tops.

A polhode curve always consists (cf. the figure of page 131) of two symmetrically equal branches. If a point with the coordinates p, q, r describes one branch, then the point $-p$, $-q$, $-r$ traverses the other branch in the opposite direction. For an individual Poinsot motion, the endpoint of the rotation vector will naturally sweep through only one branch.

We ask whether the other branch plays the same role for another Poinsot motion.

The answer follows immediately from the Euler equations. By assumption, the coordinates p, q, r satisfy the equations

$$\frac{dp}{dt} = \frac{B-C}{A}qr, \quad \frac{dq}{dt} = \frac{C-A}{B}rp, \quad \frac{dr}{dt} = \frac{A-B}{C}pq.$$

The coordinates of the diametral point $p' = -p$, $q' = -q$, $r' = -r$, however, evidently satisfy the equations

$$(1) \qquad \frac{dp'}{dt} = -\frac{B-C}{A}q'r', \quad \frac{dq'}{dt} = -\frac{C-A}{B}r'p', \quad \frac{dr'}{dt} = -\frac{A-B}{C}p'q'.$$

The quantities p', q', r' therefore correspond as rotation components to another top whose moments of inertia—we wish to denote them by A', B', C'—are related to the moments of inertia of the original top by the equations

$$(2) \quad \frac{B'-C'}{A'} = -\frac{B-C}{A}, \quad \frac{C'-A'}{B'} = -\frac{C-A}{B}, \quad \frac{A'-B'}{C'} = -\frac{A-B}{C}.$$

We are first obliged to show that a *real* top is defined by these equations; that is, that a mass distribution with the principal moments of inertia A', B', C' is possible. For this purpose, it is enough to convince oneself I) that the quantities A', B', C' are all positive, and II) that they satisfy the well-known inequalities of page 100 (the same inequalities that also obtain between the sides of an ordinary triangle).

I) Equations (2) represent three linear, homogeneous equations for the three unknowns A', B', C'; naturally only the ratios $A' : B' : C'$ are determined by these equations. And indeed, we easily find by the solution of these equations

$$A' : B' : C' = A(B + C - A) : B(C + A - B) : C(A + B - C). \tag{3}$$

The quantities on the right-hand side, however, are all positive, since A, B, C indeed satisfy the required equations for the reality of the original top. If we therefore choose, as is permitted, one of the quantities A', B', C' as positive, then, according to the preceding proportion, both of the other quantities must be positive.

II) If we had solved, conversely, equations (2) for A, B, C, then we would obviously have obtained the proportion

$$A : B : C = A'(B' + C' - A') : B'(C' + A' - B') : C'(A' + B' - C'). \tag{3'}$$

Thus the three quantities $B' + C' - A'$, $C' + A' - B'$, $A' + B' - C'$ also behave as three positive numbers. Since at least one of these must be positive, the other two are as well.

Thus our top A', B', C' is proven as real.

We wish, further, to relate the constants $2h'$ and G' of our second top to the constants $2h$ and G of the first. That the three components p', q', r' satisfy two integrals of the form

$$\begin{aligned} A'p'^2 + B'q'^2 + C'r'^2 &= 2h', \\ A'^2p'^2 + B'^2q'^2 + C'^2r'^2 &= G'^2 \end{aligned}$$

is clear from the outset, since these equations are a direct analytic consequence of the Euler equations (1) (written in terms of A', B', C'). The relation in question is thus given immediately from equations (11) of page 457. We write these equations in the form

$$\left\{ \begin{aligned} \frac{2hA - G^2}{BC} &= \frac{A - B}{C} q^2 + \frac{A - C}{B} r^2, \\ \frac{2hB - G^2}{CA} &= \frac{B - A}{C} p^2 + \frac{B - C}{A} r^2, \\ \frac{2hC - G^2}{AB} &= \frac{C - A}{B} p^2 + \frac{C - B}{A} q^2. \end{aligned} \right. \tag{4}$$

$$(4')\qquad \begin{cases} \dfrac{2h'A'-G'^2}{B'C'} = \dfrac{A'-B'}{C'}q'^2 + \dfrac{A'-C'}{B'}r'^2, \\ \dfrac{2h'B'-G'^2}{C'A'} = \dfrac{B'-A'}{C'}p'^2 + \dfrac{B'-C'}{A'}r'^2, \\ \dfrac{2h'C'-G'^2}{A'B'} = \dfrac{C'-A'}{B'}p'^2 + \dfrac{C'-B'}{A'}q'^2. \end{cases}$$

In these two triplets of equations, the right-hand sides are opposite to each other. Thus follow the relations

$$(5)\qquad \begin{cases} \dfrac{2h'A'-G'^2}{B'C'} = -\dfrac{2hA-G^2}{BC}, \quad \dfrac{2h'B'-G'^2}{C'A'} = -\dfrac{2hB-G^2}{CA}, \\ \qquad\qquad \dfrac{2h'C'-G'^2}{A'B'} = -\dfrac{2hC-G^2}{AB}, \end{cases}$$

of which the third, because of (2), is a consequence of the first two. Two of them can thus be used for the determination of the relations of h' and G' to the moments of inertia A', B', C'. Thus the five constants A', B', C', h', G' are known up to a factor of proportionality. The motion, however, in no way depends on this factor, which necessarily remains undetermined.

The Poinsot motion defined by the ratios $A' : B' : C' : h' : G'$ is the previously mentioned *conjugate* to the motion $A : B : C : h : G$. Conversely, the latter motion is, as follows immediately from the symmetry of the equations, the conjugate to the former. One notes that if the relation

$$A > B > C$$

obtains for one of the two conjugate tops, as we assume, then the inequality

$$A' < B' < C'$$

follows for the other top; further, if one of the two conjugate top motions, as we assume, belongs to the class

$$2hB - G^2 > 0,$$

then the other belongs to the class

$$2h'B' - G'^2 < 0.$$

We must further establish the relation between the "transcendental" constants ω, ω', s, and l of the two conjugate tops, which are more important for us than the "elementary" constants $A{:}B{:}C{:}h{:}G$.

One first sees that the constants ω and ω' for the two tops are the same. The equality of ω follows directly from the meaning of this quantity. The quantity ω, we can say, signifies the time during which the rotation vector passes across the arc of the polhode curve that spans the coordinate planes $q = 0$ and $p = 0$. In fact, according to equations (9) of the previous paragraph, $q = 0$ for $v = e$ (that is, for $t = 0$), and $p = 0$ for $v = e'$ (that is, for $t = \omega$). The two diametral branches of the polhode curve, and, in particular, the just named arc (from which the entire polhode curve is composed by congruent and symmetrically equal repetition), will, however, be traversed by the rotation vectors of the conjugate tops in the same time. Therefore ω must, in fact, be equal for the two tops.

We can, further, similarly prove the equality of ω' if we permit the preceding conclusion for imaginary values of time. We can also, however, proceed as follows. According to equations (6) and (10) of pages 455 and 456,

$$t = \int \frac{dv}{\sqrt{V}} = \frac{C}{A - B} \int \frac{dr}{pq} = \frac{C'}{A' - B'} \int \frac{dr'}{p'q'}. \tag{6}$$

Now the value of $i\omega'$ results for the one and the other motion of the top if we take as the lower limit of the relevant integral for t the value of r or r' for which $q = 0$ or $q' = 0$, and as the upper limit the value $r = \infty$ or $r' = \infty$. The two resulting integrals are thus, according to the previous equation, identical, since the named upper and lower limits correspond due to the relations $p' = -p$, $q' = -q$, $r' = -r$.

The situation is otherwise for the constants s of the two motions, which we distinguish as s and s'. We had

$$is = \int_e^1 \frac{dv}{\sqrt{V}}.$$

If we introduce, as in (6), the integration variable r, then

$$is = \frac{C}{A - B} \int \frac{dr}{pq}, \tag{7}$$

where the lower limit is the value of r for which q vanishes, and the upper limit is the value $r = \dfrac{G}{C}$. In the corresponding manner, is' is defined as

$$is' = \frac{C'}{A' - B'} \int \frac{dr'}{p'q'}, \tag{7'}$$

where the lower limit is imagined as the value of r' for which q' vanishes, and the upper limit is the value $r' = \dfrac{G'}{C'}$. In the integrals (7) and (7′), therefore, only the lower limits are corresponding values. The upper limits are generally different, and *thus the constants s and s' will also be different.*

We consider, finally, the constants l and l'. Since the individual Poinsot motion depends on four arbitrary quantities, and since the conjugate Poinsot motion is completely determined by the original, it is possible to express the constants l and l' individually in terms of ω, ω', s, and s'. We achieve this in the following manner. According to equations (5) of page 43, we have

$$p + iq = 2i\Big(\beta\frac{d\delta}{dt} - \delta\frac{d\beta}{dt}\Big)$$

for one of the two conjugate motions. If α', β', γ', δ' denote the values of α, β, γ, δ constructed according to the schema of equations (32) for the other motion, there obtains, at the same time,

$$p' + iq' = 2i\Big(\beta'\frac{d\delta'}{dt} - \delta'\frac{d\beta'}{dt}\Big).$$

The relation between the two conjugate polhode curves thus yields

$$\Big(\beta\frac{d\delta}{dt} - \delta\frac{d\beta}{dt}\Big) = -\Big(\beta'\frac{d\delta'}{dt} - \delta'\frac{d\beta'}{dt}\Big),$$

or

$$\beta\delta\Big(\frac{d\log\delta}{dt} - \frac{d\log\beta}{dt}\Big) = -\beta'\delta'\Big(\frac{d\log\delta'}{dt} - \frac{d\log\beta'}{dt}\Big).$$

In this equation, we insert for t the special values $t = -is'$ and $t = -is$. In the first case l', and in the second case l vanishes from our equation, so that we obtain in the first case l, and in the second case l' individually. *There follows for l the value*

(8)
$$\begin{cases} 2il + \dfrac{i\pi}{2\omega} = \dfrac{\Theta'(-is-is'+2i\omega')}{\Theta(-is-is'+2i\omega')} - \dfrac{\Theta'(is-is')}{\Theta(is-is')} \\ -e^{-\frac{i\pi}{2\omega}(is-is')}\dfrac{\Theta'(0)\Theta(2i\omega'-2is)}{\Theta(is-is')\Theta(-is-is'+2i\omega')}\dfrac{\Theta(i\omega'-is')\Theta(i\omega'+is')}{\Theta(i\omega'-is)\Theta(i\omega'+is)}. \end{cases}$$

The corresponding value of l' follows by the interchange of s and s'.

In order to simplify the right-hand side, we temporarily imagine the quantity $-is'$ as variable, and ask for the singular points of the right-hand expression in this variable, which we wish to denote by t'.

The first term will evidently become infinitely large for the values

(I) $$t' = is - 2i\omega' + 2m\omega + 4m'i\omega',$$

and indeed always with the residuum $+1$. In the same manner, the second term will become infinite for

(II) $$t' = -is + 2m\omega + 4m'i\omega',$$

and indeed with the residuum -1. The third term will likewise become infinite at all the named positions (I) and (II), and only at these positions. In order the compute the residuum, we remark (1) that for $t' = is - 2i\omega'$ and for $t' = -is$, the residuum is -1, and (2) that our third term takes on the factor -1 or $+1$ for the increase of t' by 2ω or $4i\omega'$, respectively. Thus the residuum of this third term will generally be equal to $(-1)^{m+1}$ at the positions (I) and (II). The singularity of the first term for even m, and that of the second term for odd m, will thus be canceled by the third term. Thus there remain only the singular points

$$\begin{aligned} &(\mathrm{I}') \quad t' = is + 2\omega + 2i\omega' + 4m\omega + 4m'i\omega', \quad \text{(residuum } +2\text{)},\\ &(\mathrm{II}') \quad t' = -is \qquad\qquad\qquad + 4m\omega + 4m'i\omega', \quad \text{(residuum } -2\text{)}. \end{aligned}$$

This is a disposition of singular points that corresponds to a doubly periodic function of t' with periods 4ω and $4i\omega'$. In fact, the right-hand side of (8) also remains entirely unchanged for the increase of t' by 4ω and $4i\omega'$.

We can now easily express the doubly periodic function in question in terms of ϑ-functions of period 2ω and $2i\omega'$. We consider, namely,

$$\frac{\vartheta'\left(\dfrac{t' - is - 2\omega - 2i\omega'}{2}\right)}{\vartheta\left(\dfrac{t' - is - 2\omega - 2i\omega'}{2}\right)} - \frac{\vartheta'\left(\dfrac{t' + is}{2}\right)}{\vartheta\left(\dfrac{t' + is}{2}\right)}.$$

This quantity directly has, conceived as a function of t', the singular points (I$'$) and (II$'$) with the correct residua, and is doubly periodic with periods 4ω and $4i\omega'$. It can thus differ from the right-hand side of equation (8) only by an additive constant c; that is, by a quantity independent of t' that is determined by insertion of a special value (for example, $t' = -i\omega'$); and indeed one finds, in this manner, $c = -\dfrac{i\pi}{\omega}$.

If we further introduce the abbreviations

$$\text{(9)} \qquad \frac{s' - s}{2} = \omega' - a, \quad \frac{s' + s}{2} = b,$$

and therefore set

$$\text{(9}'\text{)} \qquad s = -\,\omega' + a + b, \quad s' = \omega' - a + b,$$

then we can write the latterly determined value of $2il$ *as*

$$\text{(10)} \qquad 2il + \frac{3i\pi}{2\omega} = \frac{\vartheta'(i\omega' - ia)}{\vartheta(i\omega' - ia)} - \frac{\vartheta'(\omega + i\omega' + ib)}{\vartheta(\omega + i\omega' + ib)}.$$

The corresponding value of l' results, as mentioned, if we exchange s and s' in (8). *One thus obtains*

$$(10') \qquad 2il' + \frac{3i\pi}{2\omega} = -\frac{\vartheta'(i\omega' - ia)}{\vartheta(i\omega' - ia)} - \frac{\vartheta'(\omega + i\omega' + ib)}{\vartheta(\omega + i\omega' + ib)}.$$

We further note the formulas

$$(11) \qquad \begin{cases} i(l + l') = -\dfrac{\vartheta'(\omega + i\omega' + ib)}{\vartheta(\omega + i\omega' + ib)} - \dfrac{3i\pi}{2\omega}, \\ i(l - l') = \dfrac{\vartheta'(i\omega' - ia)}{\vartheta(i\omega' - ia)} \end{cases}$$

that follow from (10) and (10′).

The relation between the constants s, s', l, l' is thus found.

For the consideration of the conjugate Poinsot motion, we can now adopt the point of view that it would be more practical, in part for reasons of symmetry, to establish the individual Poinsot motion in terms of the four quantities ω, ω', s, and s' or ω, ω', a, and b instead of the quantities ω, ω', s, and l.

We wish, in particular, to rewrite the *herpolhode curve* equations (41) and (42) of the previous section in terms of these constants. These equations become, if we immediately apply some reductions,

$$\pi + i\kappa = \frac{\vartheta'(0)}{\vartheta(\omega+ia+ib)} e^{\frac{\pi(a+b)}{2\omega}} e^{\left\{\frac{\vartheta'(i\omega'-ia)}{\vartheta(i\omega'-ia)} - \frac{\vartheta'(\omega+i\omega'+ib)}{\vartheta(\omega+i\omega'+ib)}\right\}t} \frac{\vartheta(t-\omega-i\omega'+ia+ib)}{\vartheta(t-i\omega')},$$

$$-\varrho = i\left(\frac{\vartheta'(i\omega' - ia)}{\vartheta(i\omega' - ia)} - \frac{\vartheta'(\omega + i\omega' + ib)}{\vartheta(\omega + i\omega' + ib)} + \frac{\vartheta'(\omega + ia + ib)}{\vartheta(\omega + ia + ib)}\right).$$

These are, however, precisely the formulas for the polhode curve of the heavy spherical top, as we have constructed them on pages 436 and 437, with the single difference that the additional factor of $-\dfrac{1}{2}$ occurs in our present formulas.

If we write the equations of the herpolhode curve of the conjugate top in the same manner, then we obtain formulas that differ from the preceding only by the exchange of $+a$ with $-a$; this directly produces, however, our earlier equations of the herpolhode curve of the heavy spherical top, with the difference that now the factor $+\dfrac{1}{2}$ occurs compared to the previous.

We thus have the remarkable result that

The coordinates of the herpolhode curves of our two conjugate tops with constants ω, ω', a, and b are, at every moment, identical with the *coordinates of the polhode or herpolhode curves of the heavy spherical top with the same constants, divided by* -2 *or* $+2$*, respectively.*

A deeper relation between our conjugate Poinsot motions and the motion of the heavy spherical top obviously lies hidden here, a relation that we propose to clarify through the following deliberations.

We imagine the two conjugate tops rotating about the common point O with coinciding rotation axes and a common initial position, and ask for the relative motion of the two corresponding inverse motions; that is, the two motions that the surrounding space appears to execute to an observer who takes his standpoint once in one, and once again in the other of the two conjugate tops.

The two original direct motions are illustrated to us, according to the prescription of Poinsot, if we let the relevant polhode cones roll on the herpolhode cones without sliding. We obtain the inverse (reversed) motions, conversely, if we fix the polhode cones and roll the herpolhode cones on the polhode cones without sliding. The relative motion of the two herpolhode cones then represents to us the Poinsot image of the relative motion of the inverse Poinsot motions.

Now the conjugate top possesses, by definition, a diametrically opposed polhode curve, and consequently a congruent polhode cone. In our case, therefore, the two herpolhode cones roll on one and the same polhode cone. And indeed these herpolhode cones are continually tangent to one and the same generator, which is given by the instantaneous values of the ratios $p:q:r = p':q':r'$, and turn about the generator with one and the same angular velocity

$$\sqrt{p^2+q^2+r^2} = \sqrt{p'^2+q'^2+r'^2}$$

in the opposite sense. (We are obliged to imagine that one herpolhode cone is tangent to the polhode cone on the interior and the other on the exterior, so that they can always remain in contact during their rotations that occur in the opposite sense.) If, however, two cones roll without sliding on a single third cone and always remain in contact, then they also roll without sliding on one another. We can thus, in order to obtain the relative motion in question, completely discard the polhode cone, and roll the one herpolhode cone directly upon the other. In this manner, we acquire, from the Poinsot image of the original motions of the top, a simultaneous Poinsot image of the above named relative motion.

If we fix, at our pleasure, one of the two herpolhode cones, then this fixed cone plays the role of the herpolhode cone for the relative motion, while the other herpolhode cone would be designated as the polhode cone of the relative motion. We thus see that

The herpolhode and polhode cones of our relative motion are identical with the two herpolhode cones of the original individual motions.

The situation is somewhat different if we take into consideration, in addition to the cones, the herpolhode and polhode curves of the relative motion that run on the cones. These curves are not directly identical with the herpolhode curves of the conjugate Poinsot motions. For, first of all, these latter curves indeed roll on the polhode curves of the Poinsot motions that lie diametrically opposed with respect to O. Thus the two points of the herpolhode curves that give the endpoints of the rotation vectors at each moment in the Poinsot motions must also be found on opposite sides of O. In order to have two curves rolling on each other, we must replace one herpolhode curve by its diametral image with respect to O. But even the resulting two curves, the herpolhode curve of one and the diametral image of the other Poinsot motion, are not yet directly the polhode and herpolhode curves of the relative motion. Namely, we easily convince ourselves that the rotational velocity of the polhode cone in the relative motion is twice as great as the rotational velocity in the original individual motions. In fact, we must first rotate the cone posing as the polhode cone of the relative motion into the position of the polhode cone for the Poinsot motion, which is accomplished by the rotation $(-p\,dt, -q\,dt, -r\,dt)$, and then rotate it into the position of the cone serving as the herpolhode cone, which requires the rotation $(p'\,dt, q'\,dt, r'\,dt) = (-p\,dt, -q\,dt, -r\,dt)$. The total rotation that occurs in the time dt for the relative motion therefore amounts to $(-2p\,dt, -2q\,dt, -2r\,dt)$; the angular velocity is thus, for the correct choice of sign, $2\sqrt{p^2+q^2+r^2}$; that is, twice as great as for the original Poinsot motions.

Now we obtain the herpolhode and polhode curves if we lay off the magnitude of the angular velocity on the relevant cones in the direction of the instantaneous rotation axis. We obtain, evidently, curves that are geometrically similar to and twice the scale of the herpolhode curve of the one and the diametral image of the herpolhode curve of the other of our conjugate Poinsot motions. Thus it follows that

The coordinates of the herpolhode and polhode curves of our relative motion result from the coordinates of the herpolhode curves of the two conjugate Poinsot motions if these coordinates are multiplied by $+2$ *and* -2.

We have, however, calculated above that the coordinates of the herpolhode curves for the Poinsot motions, multiplied by $+2$ and -2, coincide exactly with the coordinates of the herpolhode and polhode curves of the heavy spherical top. If we consider that a motion is entirely determined by its herpolhode and polhode curves, we acquire the remarkable result that

The motion of the heavy spherical top is identical with the relative motion of the two inverse motions corresponding to the conjugate Poinsot motions.

This is the famous Jacobi theorem mentioned at the onset—although in a formulation that deviates not inessentially from the original Jacobi formulation. The relation that has been discussed here between two different rotation problems is, in fact, an astonishing one, and lies in no way on the surface of the matter.

In order to understand this relation still more clearly, we can ask, in particular, *what corresponds, in the Poinsot motions, to the figure axis of the spherical top, and what to the vertical?* The answer is simply this: *the impulse axis of one Poinsot motion gives the figure axis, and that of the other the vertical.* We remark, namely, that the transcendental cones in question are all periodic, in the sense that they coincide with themselves when turned by a certain angle about a certain axis. This "periodicity axis" for the herpolhode cone of the Poinsot motions is the spatially fixed impulse axis, and is the figure axis or the vertical, respectively, for the polhode or the herpolhode cone of the spherical top motion. Now since the cones coincide alternately, their "periodicity axes" must also coincide. The axis of the impulse of one Poinsot motion is therefore identical with the vertical, and that of the other identical with the figure axis of the spherical top.

There follows from the Jacobi theorem, further, a remarkably simple construction for the relative positions of the vertical and the figure axis for the motion of the heavy spherical top. We imagine the ellipsoids of inertia of the two conjugate tops constructed and placed so that their principal axes coincide. If we project the intersection curve of these two ellipsoids from O, then we obtain the common polhode cone of the conjugate tops. We then construct the impulse axes corresponding to any one generator $p:q:r$ of the cone in one and the other top—that is, the lines $Ap : Bq : Cr$ and $A'p : B'q : C'r$. We find these lines purely geometrically, in that we lay the tangent planes to the ellipsoids of inertia at the intersection points of the generators, and drop the perpendiculars from these planes to O. These two perpendiculars then directly give the mutual positions of the vertical and the figure axis for the motion of the spherical top. Their inclination cosine, as one sees, is simply equal to

$$\frac{AA'p^2 + BB'q^2 + CC'r^2}{GG'}. —$$

For supererogation, we wish to give yet a second proof. *We now wish, namely, to verify the Jacobi theorem again from the composition formulas for α, β, γ, δ.*[185]

Let our two Poinsot motions be determined, as previously, by the parameters α, β, γ, δ and α', β', γ', δ'. The corresponding inverse motions of space are then determined, according to page 30, by the parameter values

$$\delta, -\beta, -\gamma, \alpha \ \text{ and } \ \delta', -\beta', -\gamma', \alpha'.$$

We obtain the relative motion of our two conjugate tops—that is, the rotation that transforms one top from its position at time t into the position of the other top—by (1) bringing back one of the two tops from its position at time t to its original position, and (2) rotating from this original position to the position of the other top. The relative motion of the two inverse motions results in the same manner, in that we (1) make in reverse one of the two inverted motions, say the one given by δ', $-\beta'$, $-\gamma'$, α', which is accomplished by a rotation with the parameters α', β', γ', δ', and then execute the other inverse motion δ, $-\beta$, $-\gamma$, α. In order to obtain the relative motion of the inverse motions, we must, therefore, sequentially execute the two rotations

$$(1) \ \alpha', \beta', \gamma', \delta' \quad \text{and} \quad (2) \ \delta, -\beta, -\gamma, \alpha.$$

According to the composition formulas of page 32, the resultant rotation that is equal in its effect to the sequential rotations (1) and (2) now has the parameters

$$\alpha'' = \alpha'\delta - \beta'\gamma, \quad \beta'' = -\alpha'\beta + \beta'\alpha,$$
$$\gamma'' = \gamma'\delta - \delta'\gamma, \quad \delta'' = -\gamma'\beta + \delta'\alpha.$$

We wish to show that these are, in fact, the parameters of the spherical top.

For this purpose, we must enter for α, β, γ, δ; α', β', γ', δ' their expressions in terms of Θ-functions on the basis of the constants ω, ω', s, and s' or ω, ω', a, and b, and must once more go over from Θ- to ϑ-functions. We need only calculate, for example, α'' and β'', since the two other parameters may be derived from these through the interchange of $+i$ with $-i$.

We obtain for α'', according to equations (32) of page 468, the representation

$$\alpha'' = \sqrt{CC'}e^{i(l'-l)t}$$
$$\frac{\Theta(t-2i\omega'+is')\Theta(t+2i\omega'-is') + e^{\frac{i\pi}{2\omega}(-2i\omega'+is'+is)}\Theta(t+is')\Theta(t-is)}{\Theta(t-i\omega')\Theta(t+i\omega')}.$$

It is first apparent that the exponential factor here conforms to the exponential factor that appears for the motion of the heavy spherical top in the expression for α. In fact, we have, according to equation (11),

$$e^{i(l'-l)t} = e^{-\frac{\vartheta'(i\omega'-ia)}{\vartheta(i\omega'-ia)}t}.$$

Further, one easily checks that our Θ-quotient takes on, for the increase of t by 2ω and $2i\omega'$, the factors $+1$ and

$$e^{-\frac{i\pi}{\omega}\frac{is'-is}{2}} = e^{-\frac{i\pi}{\omega}(i\omega'-ia)},$$

respectively. These are, however, the same factors by which the ϑ-quotient

$$\frac{\vartheta(t-ia)}{\vartheta(t-i\omega')}$$

is multiplied for the same increases. Since the singular points in the t-plane also coincide, our above expression must be equal, up to a constant, to this ϑ-quotient. This same ϑ-quotient also occurred, however, in the expression for the parameter α in the motion of the heavy spherical top. That, finally, the multiplying constants in the two compared expressions also coincide, we wish to mention without proof.

For what concerns β'', we have

$$\beta'' = \sqrt{CC'}e^{i\left(l+l'+\frac{\pi}{2\omega}\right)t}$$
$$\cdot\frac{-e^{\frac{i\pi}{2\omega}(-i\omega'+is)}\Theta(t-2i\omega'+is')\Theta(t+is) + e^{\frac{i\pi}{2\omega}(-i\omega'+is')}\Theta(t+is')\Theta(t-2i\omega'+is)}{\Theta(t-i\omega')\Theta(t+i\omega')}.$$

The fraction on the right-hand side again changes only by a certain constant factor if we add one of the periods to t; namely, by the factors $+1$ and

$$-e^{-\frac{i\pi}{\omega}\left(-i\omega'+\frac{is+is'}{2}\right)} = e^{-\frac{i\pi}{\omega}(-\omega+ib-i\omega')}.$$

These same factors, however, are also taken on by the ϑ-quotient

$$\frac{\vartheta(t-\omega+ib)}{\vartheta(t+i\omega')} = -e^{\frac{i\pi t}{\omega}}\,\frac{\vartheta(t-\omega+ib)}{\vartheta(t-i\omega')}.$$

Thus β'' will be proportional, with consideration of equations (11), to the expression

$$e^{i\left(l+l'+\frac{3\pi}{2\omega}\right)t}\,\frac{\vartheta(t-\omega+ib)}{\vartheta(t-i\omega')} = e^{-\frac{\vartheta'(\omega+i\omega'+ib)}{\vartheta(\omega+i\omega'+ib)}t}\,\frac{\vartheta(t-\omega+ib)}{\vartheta(t-i\omega')}.$$

This is, however, the variable component of the value of β for the motion of the heavy spherical top. Finally, the constant factor of proportionality also coincides, which we do not wish, however, to prove explicitly.

Thus the identity of the parameters α'', β'', γ'', δ'' of our relative motion with the parameters α, β, γ, δ of the heavy spherical top is shown, wherein lies a repeated and, indeed, the most direct conceivable proof of the Jacobi theorem. —

The formulation of the Jacobi theorem given here is, as mentioned, somewhat different from the original formulation of Jacobi.*) Jacobi, namely, decomposes the individual Poinsot motion into a periodic and an aperiodic component, or, as we may perhaps say in analogy with the foregoing, into a nutation and a precession component. The precession component, considered in itself, leads the body about the impulse axis with uniform angular velocity, and is arranged so that the additional nutation component represents a purely periodically recurring motion. The precessional velocity will thus be given, in essence, by the constant l, and the nutational motion by the ϑ-quotient appearing in the representation of the Poinsot motion. Jacobi now imagines that the precessional motion is extracted, in that he refers the motion to a spatially moving frame xyz that rotates uniformly with the precessional velocity l about the z-axis that coincides with the impulse axis. Measured in this frame, the motion is a pure, periodically recurring nutation. Since Jacobi frees the conjugate Poinsot motion in the same manner from its precessional component, he asks for the relative motion of the

*) Ges. Werke Bd. II page 480.

two conjugate nutations, and finds that this relative motion is identical with the nutational motion of the heavy symmetric top. (Jacobi can replace the spherical top here by the symmetric top, since the motions of the two differ, as we know (cf. page 234), only by an exponential factor; that is, only in their precessional components, while they are equal, in contrast, in their nutational components.) Our formulation of the Jacobi theorem is obviously simpler and more extensive than Jacobi's, since the nutation and precession are considered simultaneously. We wish to concisely name the theorem, as we have stated it, the "completed Jacobi theorem."

We have also deviated considerably from Jacobi in the proof. Jacobi describes the position of the XYZ coordinate frame with respect to the xyx frame in the two conjugate Poinsot motions in terms of the nine direction cosines $a, \ldots, c''$; that is, in terms of the coefficients of the ternary substitutions that transform the coordinates XYZ into the coordinates xyz. For the investigation of the relative motion, he must therefore compose *two ternary* substitutions, and calculate the $3 \times 3 = 9$ coefficients of the resultant substitution. In contrast, the simplification in our proof is that we compose, instead of two ternary, *two binary* substitutions, and had need only of the $2 \times 2 = 4$ coefficients of the resultant rotation. Moreover, the proof of Jacobi himself is only indicated in his posthumous papers; it was first executed after his death by Lottner.*)

The possibility of completing the Jacobi theorem in the sense intended here—that is, of composing the Poinsot motions directly without separating the precessional components in advance—was first noted by Halphen.**)

A very simple elementary proof of the Jacobi theorem is given by Mr. Darboux.***) Darboux asks directly for the force that is required for the kinetic realization of our relative motion, and finds that this is identical with the force of gravity. A similar train of thought is pursued slightly later by Mr. Routh.†)

*) Cf. Jacobi's ges. Werke, Bd. II, pp. 510 and ff.

**) Comptes Rendues, Bd. 100, pp. 1065–1068.

***) Journ. de Liouville 1885 in the work cited on page 234; cf. also the Notes XVIII and XIX to the Cours de Mécanique of Despeyrous-Darboux Bd. II.

†) Quarterly Journal of Mathem. vol. XXIII 1888. On a theorem of Jacobi in dynamics. Cf. also Vol. II of the Rigid Dynamics by the same author, art. 174, 175 and 206.

We cannot enter here into the other works (Padova in the Atti d. Acad. di Torino, vol. XIX 1884 and Atti d. R. Istit. di Veneto, vol. III 1892; Halphen, Fonctions Elliptiques, Bd. II, Cap. 2 and 3, A. de Saint-Germain in the monograph cited on page 115) that are concerned with the Jacobi theorem.

In conclusion, a few remarks on the purpose and the meaning of the Jacobi theorem.

We must emphasize above all, in this respect, *that this theorem has a purely kinematic character.* In fact, the concept of relative motion is merely a kinematic concept. We know nothing in advance of the kinetic realization of a relative motion. The relative motion must, in general, follow under conditions entirely different from those under which the relevant individual motions proceed. Correspondingly, the Jacobi theorem depends directly on the kinematically defined polhode and herpolhode curves, while the kinetically more important impulse curves withdraw from our consideration. We must therefore say that *the Jacobi theorem states not a mechanical, but rather a geometric relation between the Poinsot motion and the motion of the spherical top.*[186]

Further, one can doubt whether the Poinsot motion is actually so much simpler and more transparent than the motion of the heavy spherical top that it is worth the trouble to reduce the latter motion to the former. The Poinsot motion may indeed be described, to a certain degree of completeness, by elementary means (by the rolling of an ellipsoid on a plane). One should not, however, overestimate the contrast between elementary and transcendental dependence. It can very well be that a motion given in transcendental form is not more complicated for numerical calculation and for the intuition than an algebraically represented motion.

But even if we assume that the individual Poinsot motion is fully commanded in all its details, we still possess no clear representation of the relative motion of two such motions, or of the corresponding inverse motions. In fact, the image of a relative motion is very difficult to grasp intuitively. It is hardly possible, without lengthy developments, to advance from the Jacobi theorem to a clear representation, for example, of the trajectory of the apex or the polhode and herpolhode curves of the heavy spherical top. We therefore prefer not to attach to the Jacobi point of view that one possesses, in the simultaneous consideration of the conjugate Poinsot motions, a means to understand the actual basis of the motion of the heavy top.

Nevertheless, the kinematic relation unlocked by Jacobi between the Poinsot motion and the motion of the heavy spherical top is so remarkable and interesting that we could not pass over it in silence here.

§9. The Lagrange equations for α, β, γ, δ of the heavy spherical top and their direct integration. Relation between the motion of the spherical top and a problem in particle mechanics.

After our parameters α, β, γ, δ have proven themselves so variously distinguished in the preceding, we will ask, in this concluding section, whether we cannot bestow upon them a still more central position in the theory of the heavy spherical top, in that we adopt them from the onset as a basis for the construction of the differential equations and their integration. It is indeed somewhat dissatisfying that we have used these parameters extensively only toward the end of the theory (in this chapter), while we accomplished the original integration with the Euler angles φ, ψ, ϑ. We now wish to show, in contrast, that the differential equations of the motion of the heavy spherical top assume an astonishingly simple form when written in terms of α, β, γ, δ. The integration process is fashioned much more elegantly and concisely by the consistent use of our parameters than by the previous method, and leads with one blow to the definitive representation of the motion by ϑ-quotients.

The circumstance that we only now present these developments, and that we have been satisfied until now with clumsy analytical methods, depends merely on the disposition of the material, and not on the nature of the matter. The following considerations assume, namely, some previous knowledge from the theory of elliptic functions that could be prepared only in this chapter. Only for this reason have we until now forsworn the consistent use of the parameters α, β, γ, δ.

The following developments are noteworthy in yet another direction. We will relate the motion of the heavy spherical top, namely, with the motion of a single mass particle in a space of four dimensions, just as we have compared it in the previous section with the force-free motion of the asymmetric top. The relation proposed here is deeper, valid also in kinetic respects, while that discovered by Jacobi was only of a kinematic nature.

At the same time, we will have occasion to take a glance at a characteristic, possibly multidimensional conception of mechanical problems in the sense of particle mechanics, for which our treatment of the spherical top serves as an exquisitely simple example.

The general Lagrange equations of the heavy spherical top form the starting point of our considerations. The wonderful fact that the form of these equations remains the same for all possible coordinates through which we may describe the position of a mechanical system was already emphasized on page 155. The equations of motion are always derived from the expression T of the *vis viva* and the expression dA of the work for an infinitesimal displacement (or from the potential energy V) according to one and the same rule. As mentioned on page 158, the Lagrange schema remains valid in essence even if one establishes the position of the system in terms of supernumerary coordinates; that is, in terms of quantities that are bound by one or more relations $F=$ const. or $F_1, F_2, \ldots =$ const. One then has only to use, instead of T, the expression $T+\lambda F$ or $T+\lambda_1 F_1+\lambda_2 F_2+\cdots$, where the "Lagrange multipliers" λ are chosen so that the solutions of the Lagrange equations will be compatible with the constraint equations.

We now wish to use this rule in order to write the equations of motion of the heavy spherical top in terms of the parameters α, β, γ, δ, which was already alluded to on page 158. If we understand by $[\mathsf{A}]$, $[\mathsf{B}]$, $[\Gamma]$, $[\Delta]$ the components of the impulse, and by A, B, Γ, Δ the components of the external force that correspond to α, β, γ, δ, then we have, without further ado,

$$(1)\quad \begin{cases} [\mathsf{A}] = \dfrac{\partial(T+\lambda F)}{\partial\alpha'}, & [\mathsf{B}] = \dfrac{\partial(T+\lambda F)}{\partial\beta'}, \quad [\Gamma] = \dfrac{\partial(T+\lambda F)}{\partial\gamma'}, \quad [\Delta] = \dfrac{\partial(T+\lambda F)}{\partial\delta'}, \\ \mathsf{A} = -\dfrac{\partial V}{\partial\alpha}, & \mathsf{B} = -\dfrac{\partial V}{\partial\beta}, \quad \Gamma = -\dfrac{\partial V}{\partial\gamma}, \quad \Delta = -\dfrac{\partial V}{\partial\delta}. \end{cases}$$

$$(2)\quad \begin{cases} \dfrac{d[\mathsf{A}]}{dt} - \dfrac{\partial(T+\lambda F)}{\partial\alpha} = \mathsf{A}, \\ \dfrac{d[\mathsf{B}]}{dt} - \dfrac{\partial(T+\lambda F)}{\partial\beta} = \mathsf{B}, \\ \dfrac{d[\Gamma]}{dt} - \dfrac{\partial(T+\lambda F)}{\partial\gamma} = \Gamma, \\ \dfrac{d[\Delta]}{dt} - \dfrac{\partial(T+\lambda F)}{\partial\delta} = \Delta. \end{cases}$$

The meaning of the quantities used here is the following. The constraint equation that binds α, β, γ, δ is, as we know,

$$F = \alpha\delta - \beta\gamma = 1. \tag{3}$$

The potential energy is then $V = P\cos\vartheta$. Since, according to the defining equations (8) of page 21, $\cos\vartheta = \alpha\delta + \beta\gamma$, we have, written in terms of α, β, γ, δ,

$$V = P(\alpha\delta + \beta\gamma). \tag{4}$$

For the calculation of the *vis viva* T of the spherical top, we begin from the expression

$$T = \frac{A}{2}(p^2 + q^2 + r^2),$$

but wish to write, in order to avoid duplicity of notation in the following, $\frac{M}{2}$ instead of A. If we use for $p + iq$, $-p + iq$, and r the values from equations (5) of page 43, there follows

$$\begin{aligned} T &= M\{(\beta\delta' - \delta\beta')(\alpha\gamma' - \gamma\alpha') - (\gamma\beta' - \alpha\delta')(\delta\alpha' - \beta\gamma')\}, \\ &= M\{(\alpha\delta - \beta\gamma)(\alpha'\delta' - \beta'\gamma')\}. \end{aligned}$$

With consideration of the constraint equation $F = 1$, we therefore acquire the extraordinarily simple value

$$T = M(\alpha'\delta' - \beta'\gamma'). \tag{5}$$

Because of the given values of F, V and T, equations (1) and (2) become

(6)
$$\begin{cases} [\mathsf{A}] = M\delta', & [\mathsf{B}] = -M\gamma', & [\Gamma] = -M\beta', & [\Delta] = M\alpha', \\ \mathsf{A} = -P\delta, & \mathsf{B} = -P\gamma, & \Gamma = -P\beta, & \Delta = -P\alpha, \\ \dfrac{d[\mathsf{A}]}{dt} - \lambda\delta = \mathsf{A}, & \dfrac{d[\mathsf{B}]}{dt} + \lambda\gamma = \mathsf{B}, & \dfrac{d[\Gamma]}{dt} + \lambda\beta = \Gamma, & \dfrac{d[\Delta]}{dt} - \lambda\alpha = \Delta. \end{cases}$$

If we eliminate the impulse and force components from (6), then we have, if we write the equations in the reverse sequence,

$$\begin{cases} M\alpha'' - \lambda\alpha = -P\alpha, \\ M\beta'' - \lambda\beta = +P\beta, \\ M\gamma'' - \lambda\gamma = +P\gamma, \\ M\delta'' - \lambda\delta = -P\delta. \end{cases} \qquad \alpha\delta - \beta\gamma = 1. \tag{7}$$

These are the exceedingly simple and symmetric equations of motion of the heavy spherical top in terms of α, β, γ, δ.

We will return to these differential equations in detail later. We first wish to go a step further, and resolve the preceding equations into their real and imaginary parts, in that we go over from α, β, γ, δ to the quaternion quantities A, B, C, D. Since, according to page 21,

$$\begin{aligned} \alpha &= D + iC, \quad & \beta &= -B + iA, \\ \gamma &= B + iA, \quad & \delta &= D - iC, \end{aligned}$$

there follow

$$(8) \qquad \begin{cases} MA'' - \lambda A = +PA, \\ MB'' - \lambda B = +PB, \\ MC'' - \lambda C = -PC, \\ MD'' - \lambda D = -PD, \end{cases} \qquad A^2 + B^2 + C^2 + D^2 = 1.$$

These *Lagrange differential equations of the heavy spherical top in terms of A, B, C, D* are, as one sees, not different in their manner of construction from the preceding equations; only the form of the constraint equation appears to be altered.

These equations now suggest an extraordinary interpretation of the motion of the top in the sense of particle mechanics in four-dimensional space.

We wish to conceive the quantities A, B, C, D, according to their definition as real quantities, as the usual rectangular coordinates in a space of four dimensions, and indeed imagine a four-dimensional space that is the exact generalization of our usual Euclidean three-dimensional space. We will, in particular, apply the Pythagorean theorem in our four-dimensional space, and correspondingly measure the distance between the two points A_1, B_1, C_1, D_1; A_2, B_2, C_2, D_2 through the expression

$$\sqrt{(A_2 - A_1)^2 + (B_2 - B_1)^2 + (C_2 - C_1)^2 + (D_2 - D_1)^2}.$$

The "motion" of a particle of mass M is now to be investigated in this space. We will set the square of the velocity of this particle, according to the just stated measure of distance, equal to

$$A'^2 + B'^2 + C'^2 + D'^2;$$

its kinetic energy will thus be

$$(9) \qquad T = \frac{M}{2}(A'^2 + B'^2 + C'^2 + D'^2).$$

We further wish to assume that our point is subject to a force whose potential energy at the position A, B, C, D of the four-dimensional space is equal to

(10) $$V = \frac{P}{2}(-A^2 - B^2 + C^2 + D^2).$$

Finally, the point should be constrained to remain on the "unit sphere" centered at the origin of the coordinates; that is, its distance from the origin should be constantly equal to 1. This signifies, in consequence of our above stipulation of the measure of distance in the four-dimensional space, that the constraint equation

(11) $$F = A^2 + B^2 + C^2 + D^2 = 1$$

should constantly be fulfilled.

By the "motion" of a particle in a space of four dimensions, we understand nothing other than the embodiment of such coordinate changes that satisfy the differential equations of motion for a particle in a space of three dimensions, augmented by one row.

We can concisely designate the four-dimensional problem that is defined here as *the motion of a spherical pendulum in a space of four dimensions, under the influence of a force system characterized by* (10).*)

If we now form the differential equations of this spherical pendulum in analogy to the well-known Lagrange equations of the first kind in the case of three-dimensional particle mechanics, there result precisely the equations (8). We can therefore say that

The motion of the heavy spherical top is identical with the motion of a spherical pendulum in a space of four dimensions under the influence of the previously given force system.

In order to be able to express ourselves concisely, we wish to name the particle on the four-dimensional sphere, whose coordinates at any time are equal to the quaternion parameters of the spherical top, the *representative* of the motion of the top, and seek to provide the clearest

*) The *precise analogue* of the three-dimensional spherical pendulum in a space of four dimensions would obviously be the motion of a mass particle on the unit sphere in a force field whose potential energy is proportional to one of the coordinates A, B, C, D, or, somewhat more generally, that depends on only one of these coordinates. The level surfaces of this force field consist of a system of parallel (threefold extended) "planes" of the four-dimensional space, just as the level surfaces of gravity in a space of three dimensions consist of the system of the collected horizontal planes, while the level surfaces of our force field (10) represent a system of surfaces of the second degree.

The top analogous to the spherical pendulum in this narrower sense is naturally integrable. It corresponds, in the given list of Mr. Liebmann (Math. Ann. Bd. 50, p. 65), to case (5), emphasized as real, for which

$$V = f(\beta - \gamma) = f(2B)$$

is assumed.

possible image of the motion of this representative. We thus consider, in the first place, the interpretation of the previously known laws of motion of the top, the impulse theorems $n = \text{const.}$, $N = \text{const.}$, and the theorem of the *vis viva*, as properties of the motion of the representative. These laws are now derived anew on the basis of equations (8). And we will indeed proceed precisely as we would for the treatment of the three-dimensional spherical pendulum on the basis of the Lagrange equations of the first kind written in terms of the rectangular coordinates x, y, z.

We first multiply equations (8) sequentially by B, A, D, C, and take the difference of the first two and the last two equations. There follow

$$M\,(A''B - B''A) = 0,$$
$$M\,(C''D - D''C) = 0.$$

The left-hand sides are obviously perfect differential quotients with respect to time. We can thus integrate, and obtain, if we denote the constants of integration by $\dfrac{n+N}{2}$ and $\dfrac{n-N}{2}$,

$$\text{(12)} \qquad \begin{cases} M\,(A'B - B'A) = \dfrac{n+N}{2}, \\[2ex] M\,(C'D - D'C) = \dfrac{n-N}{2}. \end{cases}$$

The choice of the notation for the constants indicates the manner in which these equations are related to our previous integrals $n = \text{const.}$ and $N = \text{const.}$ In order to conceive these equations in words, we remark that the left-hand sides are proportional to the areas of certain infinitesimal triangles. The area of the triangle with corners $0,0,0,0$; $A,B,0,0$; $A+dA, B+dB, 0, 0$, for example, is $\frac{1}{2}(A'B - B'A)\,dt$. One is thus convinced of the correctness of the following statement:

The representative of the spherical top moves so that the radius vector from the coordinate origin to the projection point of the representative on the (twofold extended) planes $C = D = 0$ or $A = B = 0$ describes equal areas in equal times.

Our impulse theorems $n = \text{const.}$ and $N = \text{const.}$ are thus most intimately related to the area theorems of the usual particle mechanics, since the derivation of equations (12) runs precisely parallel to the usual derivation of the area theorems.

In order to acquire the theorem of the *vis viva* anew, we multiply equations (8) sequentially by A', B', C', D' and add.

If we consider, according to equation (11), that

$$AA' + BB' + CC' + DD' = 0, \tag{13}$$

there follows

$$M(A'A'' + B'B'' + C'C'' + D'D'') = P(AA' + BB' - CC' - DD').$$

Perfect differential quotients again stand on the right- and left-hand sides. We thus integrate, and find, understanding by h the constant of integration,

$$\frac{M}{2}(A'^2 + B'^2 + C'^2 + D'^2) = \frac{P}{2}(A^2 + B^2 - C^2 - D^2) + h, \tag{14}$$

or, if we use the abbreviations in equations (9) and (10),

$$T + V = h.$$

We thus arrive at the theorem of the *vis viva*, and indeed exactly through the process that one usually applies in the mechanics of single particles when calculating with rectangular coordinates. In order to interpret this theorem in the sense of four-dimensional particle mechanics, we consider that T is proportional to the square of the velocity of the representative; we can thus say, for example, that

In its motion, our representative always passes the individual level surfaces $V = const.$ with the same velocity, which may be easily calculated according to the latter formula in terms of the constant h, the mass M, and the value of the potential V corresponding to the considered level surface.

Finally, we wish to calculate the magnitude of the Lagrange multiplier λ. This gives us a "pressure" that our mass particle exerts in the radial direction on the spherical surface that guides it, or, if we wish, the tension of the arm to whose end our mass particle is fixed. We multiply, for this purpose, equations (8) sequentially by A, B, C, D and add. There follows, because of the constraint equation $A^2+B^2+C^2+D^2 = 1$,

$$\lambda = M(AA'' + BB'' + CC'' + DD'') - P(A^2 + B^2 - C^2 - D^2).$$

According to equation (13), however,

$$AA'' + BB'' + CC'' + DD'' = -(A'^2 + B'^2 + C'^2 + D'^2),$$

and therefore, with consideration of (14),

$$M(AA'' + BB'' + CC'' + DD'') = -P(A^2 + B^2 - C^2 - D^2) - 2h.$$

The given value of λ therefore reduces to

$$\lambda = -2P(A^2 + B^2 - C^2 - D^2) - 2h = 4V - 2h. \tag{15}$$

We wish to state this result as a theorem in the following manner:

In the passage through a given level surface $V =$ const., *the representative always presses perpendicularly against the spherical surface that bears it with same strength* $\lambda = 4V - 2h$.

We now give the previously promised revision of our earlier integration process. We will directly develop with the greatest brevity, in that we begin from the differential equations for A, B, C, D or α, β, γ, δ, a new and complete analytic theory of the motion of the top, without assuming the earlier results as known.

The beginning of the integration process in terms of the quaternion quantities A, B, C, D has already been made in equations (12)–(15). For further execution, we use the auxiliary quantity

$$u = -A^2 - B^2 + C^2 + D^2, \tag{16}$$

and seek to express this quantity as a function of t. This purpose is served by the following calculations.

We form

$$u' = 2(-AA' - BB' + CC' + DD')$$

and combine this expression with equation (13), from which follow

$$\begin{cases} AA' + BB' = -\dfrac{u'}{4}, \\ CC' + DD' = +\dfrac{u'}{4}. \end{cases} \tag{17}$$

We then square and sum the first of these equations and equations (12), and find

$$(A^2 + B^2)(A'^2 + B'^2) = \frac{M^2 u'^2 + 4(n + N)^2}{16M^2}. \tag{18}$$

In the same manner, there follows from the second of equations (17) and equations (12)

$$(C^2 + D^2)(C'^2 + D'^2) = \frac{M^2 u'^2 + 4(n - N)^2}{16M^2}. \tag{18'}$$

We further have, because of the definition of u and the constraint equation $A^2 + B^2 + C^2 + D^2 = 1$,

$$\begin{cases} A^2 + B^2 = \dfrac{1 - u}{2}, \\ C^2 + D^2 = \dfrac{1 + u}{2}, \end{cases} \tag{19}$$

so that instead of (18) we can also write

$$A'^2 + B'^2 = \frac{M^2u'^2 + 4(n+N)^2}{8M^2(1-u)},$$
$$C'^2 + D'^2 = \frac{M^2u'^2 + 4(n-N)^2}{8M^2(1+u)},$$

and

$$A'^2 + B'^2 + C'^2 + D'^2 = \frac{M^2u'^2 + 4(n^2+N^2) + 8nNu}{4M^2(1-u^2)}.$$

With this value, we enter the equation of the *vis viva.*

We then obtain

$$u'^2 = U,$$

where U has the meaning

$$U = \frac{4}{M^2}\{2Mh(1-u^2) - (n^2+N^2+2nNu) - MPu(1-u^2)\}; \tag{20}$$

thus t is determined by the elliptic integral

$$t = \int_e^u \frac{du}{\sqrt{U}}. \tag{21}$$

We imagine the lower limit e of the integral to be one of the roots of $U = 0$. Let the other two roots be e' and e''. We immediately introduce the following designations for a few characteristic values of t:

(21′)
$$\omega = \int_e^{e'} \frac{du}{\sqrt{U}}, \quad i\omega' = \int_e^{-\infty} \frac{du}{\sqrt{U}}, \quad ia = \int_e^{-1} \frac{du}{\sqrt{U}}, \quad ib = \int_{e'}^{+1} \frac{du}{\sqrt{U}}.$$

If we invert the relation between t and u in (21), then u results as a doubly periodic function of t with periods 2ω and $2i\omega'$. We have thus adapted our previous developments to the new notation.

From now on we return to the original differential equations (7) for α, β, γ, δ, which, as will later be shown, are more convenient for the following than those written in equations (8) for the quaternion quantities. We insert the value of λ from equation (15), and imagine u calculated as a doubly periodic function of t. We thus obtain

$$\begin{cases} \dfrac{d^2\alpha}{dt^2} = \left(\dfrac{2P}{M}u(t) - \dfrac{2h-P}{M}\right)\alpha, \\[2ex] \dfrac{d^2\beta}{dt^2} = \left(\dfrac{2P}{M}u(t) - \dfrac{2h+P}{M}\right)\beta. \end{cases} \tag{22}$$

The differential equations for the two other parameters γ and δ are exactly equal to those given.

Our problem thus depends on the solution of these *linear differential equations of the second order with doubly periodic coefficients.* Concerning such equations, a place may first be found for a few historical notes.

The preceding differential equations belong to a class of equations that has often been studied in the literature. We designate them as *Lamé equations*, since they represent generalizations of the differential equations that were first treated by Lamé in a problem of heat conduction. In contrast to the most general equations that bear the name of Lamé, our differential equations are distinguished by the important property that their integrals are single-valued functions of t. Such equations are particularly associated with Hermite, and have indeed been investigated directly in connection with rotation problems (see below). It is thus justified to bestow on these equations the name of Hermite, and to designate them as the *Hermite case of the Lamé equations.*

From the form of the differential equation, one can determine the occurrence of the Hermite case according to general rules in the following manner. One seeks—we first treat of the equation for α—the singular points of the differential equation; that is, those points of the t-plane at which nonintegral powers occur in the expansion of the integral. These singular points are, in our case, identical with the singular points of the coefficient of α, which are none other than the singular points of the function $u(t)$; that is, the points

$$t = i\omega' + 2m\omega + 2m'i\omega'.$$

A power series expansion is then developed at one of these positions. At the position $t = i\omega'$, for example, one sets

$$\alpha = \alpha_0(t - i\omega')^{-n} + \alpha_1(t - i\omega')^{-n+1} + \cdots,$$

where the coefficients of the expansion and the exponent n are to be determined by the differential equation. For $\dfrac{\alpha''}{\alpha}$ there results a series that begins with the term

$$\frac{n(n+1)}{(t - i\omega')^2}.$$

One likewise expands the coefficient of α in (22) in a series that progresses in powers of $t - i\omega'$, which will begin with the $(-2)^{\mathrm{nd}}$ power. The first term of this series will be written as

$$\frac{m}{(t - i\omega')^2},$$

where m is the "multiplier of the singularity". The quantity n is then determined from the equation

$$n(n+1) = m. \tag{23}$$

If the integral of our differential equation is to be a single-valued function of t, n must obviously be a whole number. The preceding equation thus gives us, read in reverse, a condition that the multiplier m must satisfy in the Hermite case. We therefore see that

The occurrence of the Hermite case may be determined from the differential equation by seeking the multiplier m with which the coefficient of α becomes infinite at the position $t = i\omega'$ and the equivalent points. This multiplier must have the form $n(n+1)$, understanding by n a positive whole number.

This condition is derived here only as a necessary condition for the single-valuedness of the integral; it is not difficult to see, although we do not pursue this here, that it also represents, on the basis of the parallelogram tiling of the t-plane, a sufficient condition.

We now easily convince ourselves that our criterion is fulfilled for equations (22). We consider, for this purpose, the integral

$$t - i\omega' = \int_\infty^u \frac{du}{\sqrt{U}};$$

if we substitute $u = \dfrac{1}{v}$ and expand $U^{-\frac{1}{2}}$ in ascending powers of v, there follows in the first approximation, after the execution of the integration,

$$t - i\omega' = -\sqrt{\frac{Mv}{P}} = -\sqrt{\frac{M}{Pu}},$$

and therefore, in reverse,

$$u = \frac{M}{P}\frac{1}{(t-i\omega')^2}. \tag{24}$$

If one carries out the calculation one term further, one obtains in the same manner, as we add for the sake of the following,

$$u = \frac{M}{P}\frac{1}{(t-i\omega')^2}\left(1 + \frac{2}{3}\frac{h}{M}(t-i\omega')^2\right). \tag{24$'$}$$

The multiplier m that was discussed above is now determined immediately from equation (24). We have simply

$$m = \frac{2P}{M}\cdot\frac{M}{P} = 2;$$

from equation (23) there follows

$$n = 1.$$

We can therefore say that

The Hermite case of the Lamé equation is actually present in equations (22), *and indeed in the simplest incidence of this case, the subcase* $n = 1$.

The integrals of the Hermite–Lamé equation are now, especially in the simplest case $n = 1$, written down immediately. One is first convinced, since the differential equation remains completely unchanged for the increase of t by 2ω and $2i\omega'$, that its integrals must also exhibit a very simple behavior with respect to changes of the argument by periods. If one designates, namely, two particular solutions of the differential equation as $z_1(t)$ and $z_2(t)$, then $z_1(t+2\omega)$ and $z_1(t+2i\omega')$ (as well as $z_2(t+2\omega)$ and $z_2(t+2i\omega')$) must be composed linearly from z_1 and z_2. Through a special choice of the particular solutions z_1 and z_2, one can even attain that $z_1(t+2\omega)$ and $z_1(t+2i\omega')$ will be directly proportional to $z_1(t)$, so that

$$z_1(t + 2\omega) = \varrho z_1(t), \quad z_1(t + 2i\omega') = \sigma z_1(t).$$

The other particular solution z_2 may be chosen correspondingly. The behavior of these particular solutions with respect to repeated period increases is thus clear.

We generally designated single-valued functions of t that behave "multiplicatively" in this manner for increases of the argument by periods as elliptic functions of the second kind. These functions can, as we know, be represented as a product of an exponential factor and a ϑ-quotient, where as many ϑ-functions appear in the numerator and denominator as the degree of the function; that is, the number of the singular points in an individual period rectangle. This number is, moreover, known in advance in the case of the Hermite–Lamé equation. We saw, namely, that if $m = n(n + 1)$, one and only one n-fold singular point was present at $t = i\omega$ (and the equivalent points). The number n determined from the multiplier m therefore directly gives the degree of the elliptic function. We thus acquire the following result:

The Hermite–Lamé equation will generally be integrated by elliptic functions of the second kind and n^{th} degree. In the simplest case $n = 1$ present here, we manage with elliptic functions of the second kind and the first degree.

We remark further that our differential equation remains completely unchanged by the exchange of t with $-t$. It thus follows that $z(t)$ and

$z(-t)$ are always simultaneous integrals of the differential equation. If, in particular, $z(t)$ is chosen as one of the multiplicative particular solutions, then $z(-t)$ is also a multiplicative solution that differs, in general, from $z(t)$. *We can therefore set the previously named solutions* z_1 *and* z_2 *equal to*

$$z_1 = z(t), \quad z_2 = z(-t).$$

Moreover, we add the self-evident remark that

The general solution follows from our particular solutions in the form

$$c_1 z(t) + c_2 z(-t), \tag{25}$$

where c_1 *and* c_2 *are arbitrary constants of integration.*

In our case $n = 1$, $z(t)$ has the simple form

$$z(t) = e^{\lambda t} \frac{\vartheta(t - t_1)}{\vartheta(t - i\omega')}, \tag{26}$$

where the constants λ and t_1 are to be determined from the differential equation. The following calculation that serves for the determination of these constants shows at the same time that $z(t)$ actually satisfies the differential equation in question for the correct choice of the constants, and yields an explicit proof of all the preceding remarks.

We first consider the quantity

$$\tag{27} \left\{ \begin{aligned} \frac{z''}{z} &= \frac{d^2 \log z}{dt^2} + \Big(\frac{d \log z}{dt}\Big)^2 = \\ &= \frac{d^2 \log \vartheta(t-t_1)}{dt^2} - \frac{d^2 \log \vartheta(t-i\omega')}{dt^2} + \Big(\lambda + \frac{d \log \vartheta(t-t_1)}{dt} - \frac{d \log \vartheta(t-i\omega')}{dt}\Big)^2. \end{aligned} \right.$$

This quantity will be infinite only at the positions $t = t_1$ and $t = i\omega'$, as well as the equivalent points, and, in particular, to the first order at the former and the second order at the latter. If we expand $\dfrac{d \log \vartheta(t)}{dt}$ according to Taylor's theorem about $t = 0$, namely, there follows

$$\frac{d \log \vartheta(t)}{dt} = \quad \frac{1}{t} + \frac{\vartheta'''(0)}{3\vartheta'(0)}\, t + \cdots, \tag{28}$$

and therefore

$$\frac{d^2 \log \vartheta(t)}{dt^2} = -\frac{1}{t^2} + \frac{\vartheta'''(0)}{3\vartheta'(0)} \quad + \cdots. \tag{28'}$$

Thus the expansions of our above expression at the positions $t = t_1$ and $t = i\omega'$ run, if we write only the singular terms,

$$\frac{z''}{z} = \frac{2}{t - t_1}\Big(\lambda - \frac{d \log \vartheta(t_1 - i\omega')}{dt_1}\Big) + \cdots \quad (\text{for } t = t_1),$$

$$\frac{z''}{z} = \frac{2}{(t - i\omega')^2} + \frac{2}{t - i\omega'}\Big(\lambda + \frac{d \log \vartheta(i\omega' - t_1)}{dt_1}\Big) + \cdots \quad (\text{for } t = i\omega').$$

We can now remove the singular point at $t = t_1$ by the choice of λ; we need only set

$$\lambda = \frac{d \log \vartheta(t_1 - i\omega')}{dt_1};$$

at the same time, the term with $(t - i\omega')^{-1}$ in our second expansion also vanishes. This expansion now runs

$$\frac{z''}{z} = \frac{2}{(t - i\omega')^2} + \cdots \,, \tag{29}$$

or, if we wish to include the constant term in the power series,

$$\frac{z''}{z} = \frac{2}{(t - i\omega')^2} - \left(\frac{d^2 \log \vartheta(t - t_1)}{dt^2}\right)_{t = i\omega'} + \frac{\vartheta'''(0)}{3\vartheta'(0)} + \cdots \,. \tag{29$'$}$$

One immediately sees from the representation (27) that $\dfrac{z''}{z}$ is a doubly periodic function with periods 2ω and $2i\omega'$. Thus the expansions (28) and (29) will also be valid for all the positions equivalent to $t = t_1$ and $t = i\omega'$. The singular points at the positions equivalent to $t = t_1$ will also be removed, and equation (29) will be valid not only at the position $t = i\omega'$, but also at all the equivalent positions. We have thus proven that

For our choice of λ, $\dfrac{z''}{z}$ is a doubly periodic function of t that becomes infinite to the second order with the multiplier 2 at the position $t = i\omega'$ and the equivalent positions, and only at these positions.

Such a function, however, is, according to the above,

$$\frac{2P}{M} u(t).$$

The difference of the two would therefore be a doubly periodic function that becomes infinite at no point of the t-plane. Such a function necessarily reduces, however, to a constant c. We therefore have

$$\frac{z''}{z} - \frac{2P}{M} u(t) = c. \tag{30}$$

This is directly a Lamé equation. We thus see that

For the above choice of λ, our elliptic function of the first degree $z(t)$ satisfies the Hermite–Lamé equation

$$z'' = \left(\frac{2P}{M} u(t) + c\right) z. \tag{30$'$}$$

The quantity c in this equation depends on the still to be disposed constant t_1. Through the appropriate choice of this constant, it is possible to attain an arbitrary value for the constant c, and, in particular, to arrange that the preceding equation goes over directly into the first or second of equations (22).

To achieve this, we first express the quantity c in terms of t_1 in a manner that is convenient for the following. We set, for this purpose, $t = i\omega'$ in (30) and write

$$c = \left\{ \frac{z''}{z} - \frac{3P}{M} u(t) + \frac{P}{M} u(t) \right\}_{t=i\omega'}. \tag{31}$$

We then consider the expression

$$\frac{P}{M} u(t) + \frac{d^2 \log \vartheta(t - i\omega')}{dt^2};$$

this is a doubly periodic function with periods 2ω and $2i\omega'$ that becomes infinite at no point of the t-plane. In fact, the singular points of the first and second terms directly cancel, according to equations (24) and (28′), for $t = i\omega'$ and the equivalent points. Our expression is therefore a constant, so that we can write[187]

$$\frac{P}{M} u(t) = -\frac{d^2 \log \vartheta(t - i\omega')}{dt^2} + c_1.$$

We determine the value of c_1 in a twofold manner, in that we once set $t = ia$ and $u = -1$, and once again set $t = \omega - ib$ and $u = +1$. We thus obtain

$$c_1 = \left(\frac{d^2 \log \vartheta(t - i\omega')}{dt^2} \right)_{t=ia} - \frac{P}{M}$$

and

$$c_1 = \left(\frac{d^2 \log \vartheta(t - i\omega')}{dt^2} \right)_{t=\omega-ib} + \frac{P}{M}.$$

Correspondingly, there follows for $u(t)$ the twofold expression

$$\frac{P}{M} u(t) = -\frac{d^2 \log \vartheta(t - i\omega')}{dt^2} + \left(\frac{d^2 \log \vartheta(t - i\omega')}{dt^2} \right)_{t=ia} - \frac{P}{M}$$

and

$$\frac{P}{M} u(t) = -\frac{d^2 \log \vartheta(t - i\omega')}{dt^2} + \left(\frac{d^2 \log \vartheta(t - i\omega')}{dt^2} \right)_{t=\omega-ib} + \frac{P}{M}.$$

We write, in particular, the resulting expansions at $t = i\omega'$, which we calculate with the constant term; they run, according to (28′),

$$\frac{P}{M}u(t) = \frac{1}{(t-i\omega')^2} - \frac{\vartheta'''(0)}{3\vartheta'(0)} + \left(\frac{d^2 \log \vartheta(t-i\omega')}{dt^2}\right)_{t=ia} - \frac{P}{M}$$

and

$$\frac{P}{M}u(t) = \frac{1}{(t-i\omega')^2} - \frac{\vartheta'''(0)}{3\vartheta'(0)} + \left(\frac{d^2 \log \vartheta(t-i\omega')}{dt^2}\right)_{t=\omega-ib} + \frac{P}{M}.$$

We insert these expansions for the third term on the right-hand side of equation (31). At the same time, we replace the first and second terms by the expansions (29′) and (24′). Then for $t = i\omega'$ the singular terms cancel, as they must, and we obtain

$$(32) \quad c = -\left(\frac{d^2 \log \vartheta(t-t_1)}{dt^2}\right)_{t=i\omega'} + \left(\frac{d^2 \log \vartheta(t-i\omega')}{dt^2}\right)_{t=ia} - \frac{2h+P}{M}$$

and

$$(32') \quad c = -\left(\frac{d^2 \log \vartheta(t-t_1)}{dt^2}\right)_{t=i\omega'} + \left(\frac{d^2 \log \vartheta(t-i\omega')}{dt^2}\right)_{t=\omega-ib} - \frac{2h-P}{M}.$$

We have thus acquired two different representations for the constant c in equation (30′). We use them to determine the quantity t_1 in such a way that equation (30′) goes over into the first or second of equations (22). This is attained for the first of equations (22) if we set

$$(33) \qquad t_1 = ia,$$

and for the second if we set

$$(33') \qquad t_1 = \omega - ib,$$

so that, according to (32) and (32′), c will become, in fact,

$$c = -\frac{2h+P}{M}$$

or

$$c = -\frac{2h-P}{M}.$$

If we choose the disposable constant t_1 in our elliptic function $z(t)$ as in equations (33) *and* (33′)*, the resulting functions are particular solutions of the two equations* (22).

It now remains only to show that our parameters α and β are equal, up to a constant factor, to these particular solutions.

We must first assume the value of α, according to the schema of the general solution (25), in the form

$$(34) \qquad \alpha = c_1 z(t) + c_2 z(-t).$$

The parameter δ that is conjugate to α then runs, since our solutions $z(t)$ and $z(-t)$ are conjugate imaginary quantities because of the particular form of $z(t)$,

$$\delta = \overline{c}_2 z(t) + \overline{c}_1 z(-t), \tag{34$'$}$$

understanding by $\overline{c}_1$ and $\overline{c}_2$ the constants conjugate to c_1 and c_2.

We will now see that we must set either c_2 (and therefore also $\overline{c}_2$) or c_1 (and therefore also $\overline{c}_1$) equal to zero.

We use, for this purpose, the equations

$$\alpha\delta = \frac{u+1}{2}, \quad \beta\gamma = \frac{u-1}{2}, \tag{35}$$

which follow from the defining equations for α, β, γ, δ of page 21. The first of these shows that the null positions of α and δ must coincide with those of $u+1$; that is, with the positions $t = \pm ia + 2m\omega + 2m'i\omega'$. If we therefore set $t = ia$, then either α or δ must vanish. In equation (34), we must therefore take either $c_2 = 0$ or $\overline{c}_1 = 0$. If we set, on the other hand, $t = -ia$, then it follows, in the same manner, that we must take either $c_1 = 0$ or $\overline{c}_2 = 0$. We therefore have, in fact, the two possibilities identified above, that either c_1 or c_2 must equal zero. Whether we choose one or the other makes no great difference. *In each case, it follows that our parameters α and δ are directly proportional to the particular solutions $z(t)$ and $z(-t)$.*

If we choose, for example, $c_2 = 0$, then we obtain, if we insert the found expressions for the constants that appear in $z(t)$,

$$\alpha = c_1 e^{\frac{\vartheta'(ia-i\omega')}{\vartheta(ia-i\omega')}t}\, \frac{\vartheta(t-ia)}{\vartheta(t-i\omega')},$$

$$\delta = \overline{c}_1 e^{-\frac{\vartheta'(ia-i\omega')}{\vartheta(ia-i\omega')}t}\, \frac{\vartheta(t+ia)}{\vartheta(t+i\omega')}.$$

(If we had chosen the other possibility $c_1 = 0$, the expressions for α and δ would only have been interchanged.)

In an entirely similar manner, we find from the second of equations (35), if we once set $t = \omega - ib$ and once again set $t = -\omega + ib$, that β and γ are also directly equal, up to a factor, to the multiplicative particular solutions of our second equation (22). The expressions for β and γ will be, in this manner,

$$\beta = c_2 e^{\frac{\vartheta'(\omega-ib-i\omega')}{\vartheta(\omega-ib-i\omega')}t}\, \frac{\vartheta(t-\omega+ib)}{\vartheta(t-i\omega')},$$

$$\gamma = \overline{c}_2 e^{-\frac{\vartheta'(\omega-ib-i\omega')}{\vartheta(\omega-ib-i\omega')}t}\, \frac{\vartheta(t+\omega-ib)}{\vartheta(t+i\omega')}.$$

We have thus reacquired our previous values of α, β, γ, δ (cf. pages 420 and 428) in the shortest and most direct way. There remains only the determination of the multiplicative constants c_1 and c_2, which may be effected exactly in the manner given on pages 425 and 426. We need not enter into this again here.

The differential equations for A, B, C, D, which are likewise Lamé equations and do not at all differ from those for α, β, γ, δ, may naturally be integrated in precisely the same manner. The simplification that α, β, γ, δ provide compared to A, B, C, D is important only in the final result, where it is shown that α, β, γ, δ are directly proportional to the multiplicative particular solutions of our Lamé equation, while A, B, C, D are composed from them linearly. —

The principal purpose that we have pursued in this supplement, to show that our parameters α, β, γ, δ are of use not only for the formulation of the final results, but also for the direct integration of the problem of the top, is thus achieved.

We note that the integration procedure given here corresponds precisely to the intentions of Hermite in his famous investigations on the application of elliptic functions (the Applications des fonctions elliptiques cited on page 151). While Hermite considered for the problem of the heavy top only the special case of the ordinary spherical pendulum, his method extends, thanks to the introduction of the parameters α, β, γ, δ, to the treatment of the heavy spherical top (our four-dimensional spherical pendulum), from which the passage to an arbitrary symmetric top is possible at any time according to the previous rule. And while Hermite finds elliptic functions of the second kind and second degree for the rectangular coordinates x, y, z of the ordinary spherical pendulum, or for their complex combinations $x+iy$, $x-iy$, there follow for the rectangular coordinates A, B, C, D of our four-dimensional spherical pendulum, or rather for their complex combinations α, β, γ, δ, elliptic functions of the first degree, so that the Hermite results themselves are simplified through the preceding. In order to appreciate the complete parallelism of the Hermite and the present developments, one compares, in particular, the cited work on page 109 and ff.

At the same time, the preceding considerations contribute to still another approach already present in the literature.

In the work cited on page 142, namely, Mr. Tait poses the exercise of treating the general problem of rotation on the basis of the quaternion theory. His results in the kinematic part are particularly elegant; but remarkable approaches that are intimately related to our latter considerations are also found in the kinetic part. Tait composes our four quaternion quantities A, B, C, D, as is usual in the quaternion theory, into one complex quantity

$$q = iA + jB + kC + D,$$

and forms the differential equation of the second order that this quantity satisfies (cf., in particular, Art. 30 of the cited work), and indeed immediately for the most general case of an arbitrary asymmetric mass distribution and an arbitrary external force system. He does not, however, succeed in advancing to a general integration of this differential equation, but rather he declares this exercise, as is hardly otherwise possible for the attempted generality, to be "inextricably complicated."[188]

Our above considerations now show, in the indeed entirely special case of the heavy spherical top, that the differential equation for the quaternion q, or, equivalently, the four differential equations for the quaternion components A, B, C, D, assume an extraordinarily simple form and admit of a very elegant integration procedure. At the same time, however, we recognize that it is practical for the analytic execution of the integration to go over again from the quaternion quantities A, B, C, D to our parameters α, β, γ, δ, which are of unsurpassable simplicity in analytic respects. In any case, we may regard our present integration procedure as a special realization of the uncompleted ideal of the advocates of the quaternion theory in the matter of the rotation problem.

For what concerns the relation of the motion of the top to particle mechanics, we remark that the above arguments can be regarded as a special example of a general mathematical method, according to which one can conceive, in a certain sense, every arbitrarily complicated mechanical problem as a problem of particle mechanics. Namely, one assigns to the mechanical system, as done above, a single mass particle, a "representative," in that one interprets the position coordinates of the system as the coordinates of the representative. Thus one will be led, corresponding to the number of position coordinates used, into a space of possibly higher dimension. One further adopts in this space, in association with the expression for the *vis viva*, an appropriate determination of the measure of distance; one calculates, namely, the distance between two infinitely near points, or, as one expresses it more concisely, the line element of the relevant space, through the equation

$$ds^2 = 2T dt^2,$$

where the right-hand side evidently becomes a definite quadratic form of the infinitesimal coordinate differences of the two points, and again defines, moreover, the motion of the mass particle through the statement of the Lagrange equations.

The positions that the representative assumes in its so-defined motion then correspond sequentially to the positions of the original mechanical system. The motion of the representative will be a precise image of the motion of the system.

It is clear that the multidimensional particle representation sketched here is, fundamentally, only a reinterpretation of the original statement of the problem. It imparts no actual new knowledge, but rather only allows, in many cases, a convenient formulation of the same state of affairs.

The usefulness of this particle mechanics conception is nevertheless shown directly and most clearly in our example of the spherical top. Here the analogy with a moving mass particle on a four-dimensional unit sphere leads us to an essential simplification of the integration process, and permits us to conceive and state the previously known integral theorems of the motion of the top in a new and very intuitive form.

The particle-mechanical interpretation does not, however, generally turn out as simply as in the present case. One must, in general, adopt an unusual determination of the measure of distance in the space of the representative, and must correspondingly postulate a complicated and arbitrary kind of geometry and mechanics. In contrast, the noteworthiness of our developments above consists directly in the fact that we manage here with the elementary Euclidean geometry, in that all properties of real three-dimensional space carry over directly to the four-dimensional space of our representative.

The multidimensional conception of mechanical problems has been familiar to mathematicians since the important works of Beltrami*) in the year 1869 and Lipschitz**) in the year 1872. It may have first found entrance into broader circles, however, through the beautiful work of Hertz on the Principles of Mechanics in the year 1894, which depends completely and entirely on this multidimensional particle mechanics representation. —

We conclude this chapter by completing the historical notes of pages 429 and 430 through some new and very interesting information.

When the printing of this volume was essentially completed, we were made aware by Mr. O. Bolza of Chicago that Weierstrass had already utilized, on the occasion of a lecture *über die Anwendungen der elliptischen Funktionen* in the year 1879, our parameters α, β, γ, δ to represent the motion of the heavy symmetric top. An elaboration of this lecture has been placed at our disposal in the most amicable manner by Mr. J. Hänlein of Berlin. We wish to express in this place our most sincere thanks to both of these gentlemen.[189]

In his lecture, Weierstrass first emphasizes that the consideration of the nine direction cosines for the problem of the heavy symmetric top entails calculational complications that are avoided if one uses the three Euler symmetric rotation parameters λ, μ, ν (cf. the footnote of page 60) for the specification of an individual rotation. From these parameters, he goes over by means of the proportion $\lambda : \mu : \nu : 1 = A : B : C : D$ to our four quaternion quantities, which are determined up to a common change of sign through the addition of the constraint equation $A^2 + B^2 + C^2 + D^2 = 1$. The geometric interpretation of A, B, C, D is illustrated by means of the rotation axis and the rotation angle, in the sense of equations (14) of page 38. Finally, Weierstrass forms the complex combinations $A + iB$, $C + iD$—that is, in essence, two of our four parameters α, β, γ, δ—and determines these as elliptic functions of the second kind and first degree by means of certain formulas that coincide exactly, specialized to the case of the spherical top, with our representation of pages 420 and 428. (A purely superficial difference is that Weierstrass uses the σ-function instead of the ϑ-function, and writes, moreover, λ, μ, ν, ϱ instead of A, B, C, D.)

*) Mem. dell' Istituto di Bologna, ser. II, t. VIII, Teorica generale dei parametri differenziali.

**) Crelle's Journal Bd. 74, Untersuchung eines Problems der Variationsrechnung, in which the Problem der Mechanik is contained.

On the other hand, Weierstrass lacks (disregarding the developments of the last section) the relation of the parameters α, β, γ, δ to the complex variable that we imagined as extended on the Riemann spherical surface, and the equation for the linear transformation

$$\lambda = \frac{\alpha\Lambda + \beta}{\gamma\Lambda + \delta}$$

of this variable for the execution of the rotation $(\alpha, \beta, \gamma, \delta)$.

Specifically concerning this latter point, we are now able to furnish a still older and much more interesting historical note. In the recent review of the posthumous papers of Gauſs, it has been shown, namely, that this entire manner of representation was already completely familiar to Gauſs, and, further, that the foundations of the quaternion theory are contained explicitly in the occasional notes of Gauſs. We cite, with respect to this astonishing discovery, a few sentences from a preliminary notice "Über den Stand der Herausgabe von Gauſs' Werken," Nachr. der Kgl. Ges. der Wiss. zu Göttingen, Heft 1, 1898:[190]

"Gauſs had already interpreted, exactly as Riemann did later, a complex variable $z = x + iy$ on the sphere, and knew that the rotations of the sphere about its midpoint are represented by a linear transformation of this z with a certain simple manner of construction! And what is still more astonishing, in 1819 he represented the combination of a spatial rotation about the origin and an arbitrary similarity transformation (a "mutation of space," as he said) by means of the same four parameters that the quaternion theory later used; he designated the embodiment of these four parameters as a "mutation scale," and gave the explicit formula for the composition of two scales (and therefore the multiplication of two quaternions), for which he used the symbolic notation $(a\,b\,c\,d) \cdot (\alpha\,\beta\,\gamma\,\delta) = (A\,B\,C\,D)$; he explicitly remarked that this involved a noncommutative process!"

To return once again to the cited lecture of Weierstrass, we remark that Weierstrass did not apply the parameters α, β, γ, δ to the force-free top. Rather, he began here from the Jacobi direction cosines, which, after the previous integration of the Euler equations, he was able to write out rather directly as elliptic functions of the first degree. A short account of this method is found in the Mathematical Dictionary of H o f f m a n n – N a t a n i, Bd. VI, page 273, under "Rotation."[191]

Appendix to Chapter VI.[192]

§10. The top on the horizontal plane.

As a complement to the theory of the top with a fixed support point, this appendix will consider the motion of the top with a horizontally mobile support point, and therefore the motion that adults and children imagine, in the first place, by the words "top motion." We do not intend to go as far in analytic respects as in the previous problem, but rather will be satisfied if we can formulate a clear image of the qualitative character of the motion. This is achieved, with the circumvention of all the analytic difficulties that would otherwise appear, if we calculate *with bounded precision*, as has been recommended in Chap. IV, §9. We wish to establish the motion only with the sharpness that would be employed by the naked eye of an observer of an ordinary toy top, without particular refinement of the means of observation and without careful exclusion of causes of disturbance, so that we can actually be content with a very low precision. From the mathematical standpoint, a rough approximation is also satisfactory *in so far as we can estimate the allowed error in our calculation.* We will place particular value on this point in the following. In contrast, it would appear to us worthless, on the basis of the just named criterion for the aspired accuracy, to refine the degree of precision of the calculation further through the use of higher analytic means.

We will assume that the plane that bears the top is *perfectly smooth*, and therefore frictionless, since we will return to the effect of friction in the next chapter. The counterpressure of the plane, the reaction R against the top, is then perpendicular to the plane, and therefore directed vertically. Since the support point O is no longer, as it was in the previous problem, a geometrically distinguished point, we will choose not this point, but rather the mechanically distinguished center of gravity S as the reference point in the sense of Chap. II, §2.

The line OS is called, as previously, the figure axis, and its angle with respect to the vertical is called ϑ. The distance OS will be denoted by E, and the total mass of the top by M. We set, for conciseness, $P = MgE$, so that P, as previously, signifies the moment of gravity about the horizontal axis through O that is perpendicular to the figure axis when the figure axis is horizontal. We assume for brevity, with no essential specialization, that the ellipsoid of inertia constructed at the center of gravity is a *sphere*; the common value of the moments of inertia about all axes through S is called A.

Since we neglect friction, the only external forces that come into consideration are the weight Mg and the reaction force R, whose magnitude will be determined in the following. The potential of gravity is, up to an arbitrary constant,

$$V = Mgz = P\cos\vartheta, \tag{1}$$

where $z = E\cos\vartheta$ is the vertical coordinate of the center of gravity in the fixed coordinate system indicated in Fig. 68. In contrast, the reaction force R gives no contribution to the potential energy, since it does no work, but rather is perpendicular to the motion of its application point. On the other hand, gravity produces no turning-moment at our reference point S, while the reaction force gives rise to a turning-moment whose axis is the "line of nodes," and therefore the line that stands perpendicular to the vertical as well as the figure axis.

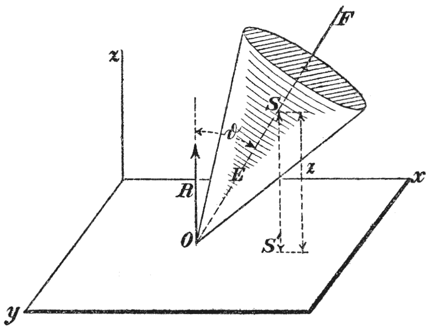


Fig. 68.

We compose, as usual, the external forces into a *single-force* and a *turning-force* (force-pair) with respect to the reference point S. According to the preceding, *the single-force is constantly vertical, and is equal to* $R - mg$. *The axis of the turning-force is always horizontal and perpendicular to the figure axis; it is equal, in magnitude, to the moment of* R *about* S.

The most important information about the course of the motion is provided to us here, as generally, by our impulse theorem of Chap. II, §5, which comprises the center of gravity and area theorems in an intuitive manner. The impulse is decomposed here into two components, the *single-impulse* (or pushing-impact) and the *turning-impulse* (or turning-impact). The components of the former with respect to the spatially fixed coordinate axes x, y, z are denoted, as previously, by $[X]$, $[Y]$, $[Z]$, and the components of the latter by l, m, n. We will also have need of the components L, M, N of the turning-impulse with respect to the coordinate system fixed in the top, whose origin is the center of gravity and whose Z-axis is the figure axis.

According to the cited impulse theorem, the rate of change of each component of the single-impulse and the turning-impulse is equal to the instantaneous value of the corresponding component of the single-force and the turning-force. From what we have noted about the direction and axis of the latter, however, it follows *that the horizontal component of the single-impact and the vertical component of the turning-impact remain constant during the motion; the external forces influence only the vertical component of the pushing-impact and the horizontal component of the turning-impact.*

In symbols, this is

$$[X] = \text{const.}, \quad [Y] = \text{const.}, \quad n = \text{const.} \tag{2}$$

Since, according to page 102, $[X]$, $[Y]$, $[Z]$ are proportional to the respective center of gravity velocities ($[X] = Mx'$, etc.), the first two of equations (2) state that the horizontal projection of the center of gravity progresses in a straight line with constant velocity. This result naturally stands and falls with the supposition of a frictionless support point.

If we add the impulse theorem for the third component of the pushing-impact (or the corresponding center of gravity theorem), then we obtain

$$\frac{d[Z]}{dt} = R - Mg.$$

We can conceive this equation as the *determining equation for the reaction force*; if we insert for $[Z]$ the value Mz', there follows

$$R = Mz'' + Mg. \tag{3}$$

In addition to the equation $n = \text{const.}$, the equation

$$N = \text{const.} \tag{4}$$

also obtains here, as we conclude from the "modified impulse theorem" IIb of page 145. According to this theorem, namely, the rate of change of the turning-impact *relative to the body* is equal in axis and magnitude to the turning-force of the external forces augmented by the resultant centrifugal turning-force (cf. page 144). The latter simply vanishes for the spherical top, and the axis of the former stands perpendicular not only to the vertical, but also to the figure axis. As a result, the rate of change of the turning-impact component N in the direction of the figure axis is equal to zero, and this component itself is constant.

Finally, we add the theorem of the *vis viva* $T+V=h$. T is composed here of two parts, the *vis viva* T_1 of the turning motion and that of the progressing motion T_2. The latter is expressed in terms of the velocity of the center of mass, and is

$$T_2 = \frac{M}{2}(x'^2 + y'^2 + z'^2).$$

We introduce, as for the top with a fixed point, the abbreviation

$$\cos\vartheta = u, \tag{5}$$

and have, according to Fig. 68,

$$z = Eu, \quad z' = Eu'.$$

If we bear in mind that the velocity components x' and y' are constant, then we can imagine that the first two terms of T_2 are combined with the constant h and suppress these terms. We thus write, more simply,

$$T_2 = \frac{ME^2}{2}u'^2.$$

The *vis viva* T_1 of the turning motion differs not from the *vis viva* of the top with a fixed support point. The expression for T_1 in terms of the quantity $u = \cos\vartheta$ has already been developed on page 222. It runs, specialized to the spherical top,

$$T_1 = \frac{A}{2}\left\{\frac{u'^2}{1-u^2} + \frac{(Nu-n)^2}{A^2(1-u^2)} + \frac{N^2}{A^2}\right\}.$$

If we use for V the expression $V = Pu$ in (1), then we can write the equation for the *vis visa* as

$$\frac{A}{2}\left\{\frac{u'^2}{1-u^2} + \frac{(Nu-n)^2}{A^2(1-u^2)} + \frac{N^2}{A^2}\right\} + \frac{M}{2}E^2u'^2 + Pu = h. \tag{6}$$

Since, according to (2) and (4), n and N are constants, the preceding equation provides a relation between u and u'; that is, between u and t, from which the changing inclination of the figure axis with respect to the vertical can be extracted as a function of time. We first calculate u'^2 by introducing a common denominator, and obtain $u'^2 = U$, where

$$U = \frac{2Ah(1-u^2) - 2APu(1-u^2) + 2nNu - N^2 - n^2}{A^2 + Aa(1-u^2)}; \tag{7}$$

here $a = ME^2$ signifies the moment of inertia of the total mass M, imagined as concentrated at the center of gravity, about a line through O perpendicular to the figure axis. There now follows

$$
(8)\qquad \begin{cases} \dfrac{du}{dt} = \sqrt{U}, \quad \text{or} \quad dt = \dfrac{du}{\sqrt{U}}, \\ \qquad t = \displaystyle\int \frac{du}{\sqrt{U}}. \end{cases}
$$

We wish to immediately give a corresponding representation for the angle ψ that the nodal line of the top forms with an arbitrary fixed horizontal line. (We will not have explicit use in the following for the third Euler angle φ.) The angle ψ is determined, according to equation (4) of page 222, by

$$
\frac{d\psi}{dt} = \frac{n - N\cos\vartheta}{A\sin^2\vartheta},
$$

and therefore, if we set $\cos\vartheta = u$, by

$$
\frac{d\psi}{dt} = \frac{n - Nu}{A(1-u^2)}.
$$

We integrate, in that we express dt in terms of du through (8), and obtain

$$
(8')\qquad \psi = \int \frac{n - Nu}{A(1-u^2)} \frac{du}{\sqrt{U}}.
$$

We first compare the integral representation of the motion in equations (8) and (8′), which are attributable to Poisson,*) with the corresponding representation that was developed previously (page 223) for the top with a fixed point. There is, on the one hand, a strong analogy between the formulas, but we note, on the other hand, that the expression for U is now somewhat more complicated than previously.

While the previous expression for U possessed three null points and one singular point, our current U has three null points (the values of u for which the numerator vanishes) and three singular points (namely, the value $u = \infty$ and the values for which the denominator vanishes).

From the analogy of the current and previous formulas, it follows immediately that part of our previous results remain unchanged. We can immediately carry over to the present case, for example, the proof of the "periodicity properties of the motion" of which we spoke previously, in that we repeat word for word the conclusions of Chap. IV, §4. Thus t and ψ are each changed by a specific increment, a so-called "period,"

*) Cf. Poisson, Traité de Mécanique II, Nr. 434 ff.[193]

as the integration variable u is led to and fro between the two roots e_0 and e_1 of U that lie between -1 and $+1$.

This periodicity emerges particularly clearly in the trajectory that the support point describes on the horizontal plane, which now appropriately takes the place of the trajectories considered in Chap. IV. We can, without restriction of generality, assume that the projection S' of the center of gravity S on the horizontal plane is fixed, and that the constant horizontal velocity of the center of gravity therefore has, in particular, the value zero. (In other cases, we need only adopt a coordinate frame that progresses with the horizontal velocity of the center of gravity; the figure to be drawn in the following would then be stretched in the direction of this velocity in an easily evident manner.) Under this assumption, the center of gravity wanders up and down on the fixed vertical erected at the point S', and the trajectory is pulled to and fro between two concentric circles described about the point S' that correspond to the two inclinations $\cos\vartheta = e_0$ and $\cos\vartheta = e_1$ of the figure axis. From the named periodicity properties, it then follows that the sequential arcs of the trajectory between the circles e_0 and e_1 are alternately symmetric and congruent. For the qualitative visualization of the trajectory, the previous figures from Chap. IV, §1 and §2 can serve with corresponding modifications; for a particular and particularly important case, we will calculate the trajectory in the following more precisely, and carry out the just indicated deliberations in detail.

On the other hand, the different manner of construction of our U in the present and previous cases would also bring with it many differences in a further analytic treatment. One designates the integrals in (8) and (8$'$), since they are more complicated by a degree than the previous elliptic integrals, as *hyperelliptic*. The difference between the two is particularly apparent in the complex domain, if we would seek to represent our quantities u and ψ as functions of time for all (also complex) values of t.*) It is not our intention, however, to enter in any way into the difficulties that appear here. For the numerical command of the motion

*) u and ψ are, in the complex domain, no longer single-valued functions of t, but u, ψ, and t may well be represented single-valuedly in the entire realm of complex values through an auxiliary variable, as is shown in the theory of the so-called automorphic functions. Cf. F. K l e i n: The mathematical theory of the Top. Princeton Lectures. New York, 1897, particularly the last section.[194]

of the top in the real domain, which can alone be our purpose, nothing would be achieved in this manner for the present state of the theory. We would sooner recommend a procedure that Weierstraſs*) has devised along general lines for all similar problems, and that is aimed at calculating u as function of t by a trigonometric series with arbitrary precision.[195] However, we can also disregard the rather detailed calculations entailed here, since we will be satisfied, as agreed above, with a low precision.

We will be especially interested, for the analogy with the previously treated cases, in *pseudoregular precession*, since this is realized, as a rule, through the ordinary actuation devices. We will therefore assume that the eigenimpulse N "is very large." This should signify (cf. p. 293) that the square of N is large compared with the compatible quantity AP. We can assume, for example, that

$$N^2 > 100\,AP.$$

We wish to assume, further, that the top receives no lateral impact at the beginning of the motion ($t = t_0$), and that its initial motion therefore consists of a pure rotation about the figure axis. The initial turning-impulse then lies in the direction of the figure axis and is to be denoted by N. If the initial inclination of the figure axis with respect to the vertical is ϑ_0 and one sets $\cos\vartheta_0 = e_0$, then the projection of the turning-impulse onto the vertical will be

$$n = Ne_0. \tag{9}$$

Moreover, one recognizes that $u' = 0$ must necessarily obtain for $t = t_0$. The figure axis is, by assumption, the initial rotation axis, and therefore cannot change its location in space instantaneously. From $u' = 0$ there follows, according to equation (8), $U = 0$; that is,

$$2Ah(1 - e_0^2) - 2APe_0(1 - e_0^2) + 2nNe_0 - N^2 - n^2 = 0.$$

We thus extract the value of h for our motion if we express n in terms of N through (9); we obtain, namely,

$$2Ah = 2APe_0 + N^2. \tag{10}$$

We next wish to separate U into its linear factors. The factor $e_0 - u$ must be present in the numerator of U, since U indeed vanishes

*) Über eine Gattung reell periodischer Funktionen. Monatsberichte der Berliner Akademie 1866, p. 97.

for $u = e_0$. In fact, it follows from (7), (9), and (10), that

$$U = \frac{(e_0 - u)(2AP(1-u^2) - N^2(e_0 - u))}{A^2 + Aa(1-u^2)}. \tag{11}$$

The additional vanishing points of the numerator and denominator may be called e_1, e_2, and $\pm e$, respectively. If one sets the denominator equal to zero, one finds

$$e^2 = 1 + \frac{A}{a}; \tag{12}$$

there follows, further, from setting the numerator to zero and solving a quadratic equation,

$$\left\{\begin{aligned} e_1 &= \frac{N^2}{4AP}\left(1 - \sqrt{1 - \frac{8APe_0}{N^2} + \frac{16A^2P^2}{N^4}}\right), \\ e_2 &= \frac{N^2}{4AP}\left(1 + \sqrt{1 - \frac{8APe_0}{N^2} + \frac{16A^2P^2}{N^4}}\right). \end{aligned}\right. \tag{13}$$

The quantity $|e_0|$, according to its geometric meaning, is < 1, and $|e|$, according to equation (12), is > 1. Since we assume that $AP : N^2$ is a small number, we can establish the order of magnitude of e_1 and e_2 if we expand the square root in (13) in terms of this quantity and obtain

$$\left\{\begin{aligned} e_1 &= e_0 - \frac{2AP}{N^2}(1 - e_0^2) + \cdots, \\ e_2 &= \frac{N^2}{2AP} + \cdots. \end{aligned}\right. \tag{14}$$

The quantity e_1 is therefore slightly smaller than e_0, and e_2 is very large. The relative position of the five locations e_0, e_1, e_2, $\pm e$ is illustrated in Fig. 69.

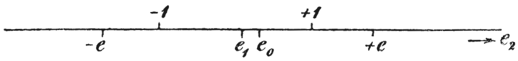


Fig. 69.

Represented in terms of its linear factors, U thus takes the form

$$U = \frac{2P}{a}\frac{(e_0 - u)(u - e_1)(e_2 - u)}{(e^2 - u^2)}; \tag{15}$$

the factor $\frac{2P}{a}$ is calculated from the fact that U must behave, according to the earlier representation (11), as $-\frac{2P}{a}u$ for $u = \infty$.

We now consider the integral (8). In this integral u must necessarily lie between the bounds -1 and $+1$, and dt must be real and positive. If we begin the integration with the initial value $u = e_0$, then u must first decrease until e_1, then increase until e_0, etc., since undershooting e_1 or overshooting e_0 would correspond to an imaginary value of dt. (To ensure that dt is positive, it is only necessary to reverse the undetermined sign of $\sqrt{U}$ for each reversal of the integration variable.) The integration variable u is therefore restricted to the narrow domain between e_0 and e_1. We thus conclude that the factor $e_2 - u$ changes only very little during the integration, since its maximum $e_2 - e_1$ and its minimum $e_2 - e_0$ nearly coincide because of the bigness of e_2 and the smallness of $e_0 - e_1$. We could thus approximate the factor $e_2 - u$, for example, as the constant

$$e_2 - u_0, \text{ where } u_0 = \frac{e_0 + e_1}{2}.$$

According to (8) and (15), the *exact* time is given by

$$t - t_0 = \sqrt{\frac{a}{2P}} \int_{e_0}^{u} \sqrt{\frac{e^2 - u^2}{(e_0 - u)(u - e_1)(e_2 - u)}}\, du,$$

where t_0 denotes the time that corresponds to the initial position $u = e_0$. We introduce, in addition, an *approximate* time t', in that we replace $e_2 - u$ by the given approximate value $e_2 - u_0$, and therefore write

$$(16) \qquad t' - t_0 = \sqrt{\frac{a}{2P(e_2 - e_0)}} \int_{e_0}^{u} \sqrt{\frac{e^2 - u^2}{(e_0 - u)(u - e_1)}}\, du.$$

The two times t and t' are always related to each other, as one easily realizes, by

$$(17) \qquad (t' - t_0)\sqrt{\frac{e_2 - u_0}{e_2 - e_1}} < t - t_0 < (t' - t_0)\sqrt{\frac{e_2 - u_0}{e_2 - e_0}}.$$

If we have determined the approximate value of time, the true value is thus known within very narrow bounds. In order to establish these bounds in still more detail, we write, according to (14),

$$\sqrt{\frac{e_2 - u_0}{e_2 - e_1}} = \left(1 - \frac{1}{2}\frac{u_0}{e_2} + \cdots\right)\left(1 + \frac{1}{2}\frac{e_1}{e_2} + \cdots\right) = 1 - \frac{u_0 - e_1}{2e_2} + \cdots$$
$$= 1 - \frac{e_0 - e_1}{4e_2} + \cdots = 1 - \frac{A^2P^2}{N^4}(1 - e_0^2),$$
$$\sqrt{\frac{e_2 - u_0}{e_2 - e_0}} = \left(1 - \frac{1}{2}\frac{u_0}{e_2} + \cdots\right)\left(1 + \frac{1}{2}\frac{e_0}{e_2} + \cdots\right) = 1 - \frac{u_0 - e_0}{2e_2} + \cdots$$
$$= 1 + \frac{e_0 - e_1}{4e_2} + \cdots = 1 + \frac{A^2P^2}{N^4}(1 - e_0^2).$$

If, as we assumed on page 519, $N^2 > 100AP$, then these quantities deviate from unity by less that 10^{-4} on one or the other side, and our approximate time differs from the true time by less than $\frac{1}{100}\%$. For practical purposes, this deviation is generally of no importance.

Through the introduction of our approximate time, however, the problem is displaced from the domain of hyperelliptic integrals into the domain of elliptic integrals. We could immediately apply the previous methods to the integral (16), and *represent u as an elliptic function of the approximate time t'*. The necessary error estimation may then be taken from the inequalities (17). We wish, however, to go still one step further, and reduce the calculation to elementary functions. We note, for this purpose, that the factor $e^2 - u^2$ also changes very little, because of the smallness of the integration interval $e_0 - e_1$, and that it can be set approximately equal to

$$e^2 - u_0^2.$$

Correspondingly, we introduce a second approximate time t'', in that we write

$$t'' - t_0 = \sqrt{\frac{a(e^2 - u_0^2)}{2P(e_2 - u_0)}} \int_{e_0}^{u} \frac{du}{\sqrt{(e_0 - u)(u - e_1)}}. \tag{18}$$

The inequality

$$(t'' - t_0)\sqrt{\frac{e_2 - u_0}{e_2 - e_1}\frac{e^2 - e_0^2}{e^2 - u_0^2}} < t - t_0 < (t'' - t_0)\sqrt{\frac{e_2 - u_0}{e_2 - e_0}\frac{e^2 - e_1^2}{e^2 - u_0^2}} \tag{19}$$

then obtains between t and t''

The bounds for the true time value that result here are no longer as narrow as the previous. One calculates from (14), namely, that

$$\begin{aligned}
\sqrt{\frac{e^2 - e_0^2}{e^2 - u_0^2}} &= \left(1 - \frac{1}{2}\frac{e_0^2}{e^2} + \cdots\right)\left(1 + \frac{1}{2}\frac{u_0^2}{e^2} + \cdots\right) = 1 - \frac{e_0^2 - u_0^2}{2e^2} + \cdots \\
&= 1 - \frac{(e_0 - e_1)e_0}{2e^2} + \cdots = 1 - \frac{AP}{N^2}\frac{e_0(1 - e_0^2)}{e^2} + \cdots, \\
\sqrt{\frac{e^2 - e_1^2}{e^2 - u_0^2}} &= \left(1 - \frac{1}{2}\frac{e_1^2}{e^2} + \cdots\right)\left(1 + \frac{1}{2}\frac{u_0^2}{e^2} + \cdots\right) = 1 + \frac{u_0^2 - e_1^2}{2e^2} + \cdots \\
&= 1 + \frac{(e_0 - e_1)e_0}{2e^2} + \cdots = 1 + \frac{AP}{N^2}\frac{e_0(1 - e_0^2)}{e^2} + \cdots.
\end{aligned}$$

These factors always deviate from unity by less than $\frac{AP}{N^2} = 10^{-2}$; since the additional factors $\sqrt{\frac{e_2 - u_0}{e_2 - e_1}}$ and $\sqrt{\frac{e_2 - u_0}{e_2 - e_0}}$ in (19) were equal to 1 to a higher order, the deviation of the true time t from the approximate time t'' is thus established. The deviation amounts to less than 1% of the true time; it is therefore no longer as small as previously, but is sufficiently small for our purpose.

The integral (18) leads to cyclometric functions. If we introduce, for the sake of convenience, the abbreviations ε, v, w defined by

$$\begin{cases} \varepsilon = \dfrac{e_0 - e_1}{2} = \dfrac{AP}{N^2}(1 - e_0^2) + \cdots, \quad v = u - u_0, \\ \omega = \pi\sqrt{\dfrac{a}{2P}\dfrac{e^2 - u_0^2}{e_2 - u_0}}, \end{cases} \tag{20}$$

then

$$\int_{e_0}^{u} \frac{du}{\sqrt{(e_0 - u)(u - e_1)}} = \int_{\varepsilon}^{v} \frac{dv}{\sqrt{\varepsilon^2 - v^2}} = \text{arc sin}\,\frac{v}{\varepsilon} - \frac{\pi}{2},$$

and therefore

$$t'' - t_0 = \frac{\omega}{\pi}\left(\text{arc sin}\,\frac{v}{\varepsilon} - \frac{\pi}{2}\right). \tag{21}$$

The quantity ω signifies the approximate time during which u traverses the integration interval from e_0 to e_1, or during which v traverses the corresponding domain from $+\varepsilon$ to $-\varepsilon$. We call ω the "half period of the motion of the top." It is equal to the time during which the figure axis returns from one of the extreme inclinations e_0 to the other extreme inclination e_1, or from the mean inclination u_0 to the very same inclination along the shortest path. After the time 2ω, the inclination of the figure axis is repeated periodically. If the yet undetermined point of time t_0 is taken, for the sake of convenience, to be equal to $\dfrac{\omega}{2}$, then the null point of the time t'' coincides with the mean inclination of the figure axis ($v = 0$ or $u = u_0$). Equation (21) then becomes

$$t'' = \frac{\omega}{\pi}\,\text{arc}\sin\frac{v}{\varepsilon}. \tag{22}$$

It is now evident that we will prefer to "invert" this relation, and write it in the form

$$v = \varepsilon\sin\frac{\pi t''}{\omega} \quad \text{or} \quad u = u_0 + \varepsilon\sin\frac{\pi t''}{\omega}. \tag{23}$$

The changing inclination of the figure axis is thus written *in an explicit manner as a function of the approximate time* t''. To estimate the allowed error in this calculation—that is, to bound the value of u that corresponds to the value of the true time t—we need only return, in a certain manner, to the inequality (19). This inequality tells us the extent to which the true and approximate times that correspond to the same value of u are most different from one another. We can therefore write

$$\sqrt{\frac{e_2-e_0}{e_2-u_0}\frac{e^2-u_0^2}{e^2-e_1^2}}\,(t-t_0) < t''-t_0 < \sqrt{\frac{e_2-e_1}{e_2-u_0}\frac{e^2-u_0^2}{e^2-e_0^2}}\,(t-t_0).$$

The true value of u—that is, the value of u at the time t—must therefore be equal to one of the values that are determined by the formula (23) for the preceding interval of the approximate time. Since (disregarding the vicinities of the integration limits e_0 and e_1) u always increases or always decreases with increasing t, we can also say that *the true value of u is contained between the values that are calculated, according to equation* (23), *for the approximate time points*

$$(24)\qquad \begin{cases} t''-t_0 = \sqrt{\dfrac{e_2-e_0}{e_2-u_0}\dfrac{e^2-u_0^2}{e^2-e_1^2}}\,(t-t_0) \quad \text{and} \\ t''-t_0 = \sqrt{\dfrac{e_2-e_1}{e_2-u_0}\dfrac{e^2-u_0^2}{e^2-e_0^2}}\,(t-t_0). \end{cases}$$

The desired error estimation is thus achieved.

We proceed in a corresponding manner for the angle ψ in equation ($8'$). If we insert for n and U the values from (9) and (15), then there first follows

$$(25)\qquad \psi = \frac{N}{A}\sqrt{\frac{a}{2P}}\int\frac{(e_0-u)}{1-u^2}\sqrt{\frac{e^2-u^2}{(e_0-u)(u-e_1)(e_2-u)}}\,du.$$

We could again introduce an approximate azimuth ψ' that may be calculated by an elliptic integral, in that we replace the factor e_2-u by e_2-u_0. However, we prefer to take a step further and write

$$(26)\qquad \psi'' = \frac{N}{A}\frac{1}{1-u_0^2}\sqrt{\frac{a}{2P}\frac{e^2-u_0^2}{e_2-u_0}}\int\frac{(e_0-u)\,du}{\sqrt{(e_0-u)(u-e_1)}},$$

so that the further calculation passes into the trigonometric domain. We substitute as the integration variable the approximate time t''; we have, according to equations (18) and (23),

$$dt'' = \sqrt{\frac{a}{2P}\frac{e^2-u_0^2}{e_2-u_0}}\frac{du}{\sqrt{(e_0-u)(u-e_1)}},$$

$$u = u_0+\varepsilon\sin\frac{\pi t''}{\omega},\quad e_0-u = \varepsilon\Big(1-\sin\frac{\pi t''}{\omega}\Big)$$

and therefore

$$(27)\qquad \begin{aligned} \psi'' &= \frac{N}{A}\frac{\varepsilon}{1-u_0^2}\int\Big(1-\sin\frac{\pi t''}{\omega}\Big)\,dt'' \\ &= \frac{N}{A}\frac{\varepsilon}{1-u_0^2}\frac{\omega}{\pi}\Big(\frac{\pi}{\omega}(t''-t_0)+\cos\frac{\pi t''}{\omega}\Big). \end{aligned}$$

The constant of integration has been chosen here so that ψ'' vanishes for $t'' = t_0 = \omega/2$. *The approximate value of ψ is again represented as an explicit function of the approximate time t''.*

It follows by comparison of (26) and (25) that the relation

$$\text{(28)} \quad \frac{1-u_0^2}{1-e_1^2}\sqrt{\frac{e^2-e_0^2}{e^2-u_0^2}\frac{e_2-u_0}{e_2-e_1}}\,\psi'' < \psi < \frac{1-u_0^2}{1-e_0^2}\sqrt{\frac{e^2-e_1^2}{e^2-u_0^2}\frac{e_2-u_0}{e_2-e_0}}\,\psi''$$

obtains between the true azimuth ψ and the approximate value ψ''. Since the two factors by which ψ'' is multiplied differ only slightly from 1, the true value of ψ is thus enclosed in narrow bounds with the help of the approximate value ψ''.

The remaining quantities that refer to the motion of the top may be represented in terms of ψ and u. We are particularly interested in the *trajectory of the support point* in the horizontal plane, since this element of the motion is most evident to the eye, and may also be conveniently registered experimentally (cf. the following chapter, §10). We extend a complex variable ξ in the horizontal plane, so that the origin of this variable coincides with the point S'. The changing positions of the support point O then correspond to a sequence of ξ-values, and the trajectory of the support point, as well as the time in which it is traversed, are completely known if we have represented this ξ-value as a function of time in the form $\xi = f(t)$.

First, the absolute value of ξ is easily given from Fig. 68; it is, namely,

$$\text{(29)} \qquad |\xi| = OS' = E\sin\vartheta = E\sqrt{1-u^2}.$$

The angle of the ray OS' (the projection of the figure axis onto the horizontal plane) with respect to an arbitrary fixed ray of this plane is then equal, up to a constant, to the angle ψ that the line of nodes forms with respect to an arbitrary fixed horizontal line.

The equation of the trajectory is therefore written in the form

$$\text{(30)} \qquad \xi = E\sqrt{1-u^2}\,e^{i\psi},$$

where the calculation of u and ψ must follow from equations (23) and (27), and the error determination from the inequalities (24) and (28).

We will illustrate the calculational procedure by a numerical example. It is well to first proceed less precisely, in order to visualize the character of the trajectory. We wish to start by neglecting the difference between the true and the approximate times, and also allow some simplifications that follow from the expansion in the assumed small quantity

$$\varepsilon = \frac{AP}{N^2}(1 - e_0^2) + \cdots = \frac{AP}{N^2}(1 - u_0^2) + \cdots .$$

We write, in this sense,

$$u = u_0 + \varepsilon \sin \frac{\pi t}{\omega}, \quad \sqrt{1-u^2} = \sqrt{1-u_0^2}\left(1 - \frac{\varepsilon u_0}{1-u_0^2} \sin \frac{\pi t}{\omega}\right)$$
$$= \sqrt{1-u_0^2}\left(1 - \frac{APu_0}{N^2} \sin \frac{\pi t}{\omega}\right), \quad \psi = \frac{P}{N}\left(t - t_0 + \frac{\omega}{\pi} \cos \frac{\pi t}{\omega}\right).$$

The simplified equation of the trajectory thus becomes

$$(31) \qquad \xi = E\sqrt{1-u_0^2}\left(1 - \frac{APu_0}{N^2} \sin \frac{\pi t}{\omega}\right) e^{\frac{iP}{N}\left(t - t_0 + \frac{\omega}{\pi} \cos \frac{\pi t}{\omega}\right)}.$$

This suggests the following interpretation: in the mean, the support point moves about the fixed point S' on a circle of radius $E\sqrt{1-u_0^2}$ with constant angular velocity $\frac{P}{N}$. This component of the motion is to be designated as a regular *precession* of the figure axis. The precession is overlaid with an oscillation or *nutation* of the figure axis with period 2ω, due to which the radius $S'O$ is changed periodically in magnitude as well as in direction. The time duration and the magnitude of the nutation are generally small, as is the angular velocity of the precession. *The trajectory of the support point thus consists of a finely scalloped circle; the nutational oscillations, because of their smallness and the rapidity with which they are traversed, escape gross observation, and the motion appears, in the first place, as a regular precession.* The trajectory has entirely the same character as the previous pseudoregular precession of the top with a fixed support point, and also conforms with this motion in the magnitude $\frac{P}{N}$ of the precessional velocity.

We now come to the precise execution of a numerical example. If our top is a homogeneous cone of revolution with height $h = 8$ cm, its center of gravity S lies at the distance $E = \frac{3}{4}h = 6$ cm from its vertex O. If the ellipsoid of inertia constructed at the center of gravity is to be a sphere, then the radius r of the circular base must be equal to half the height. In fact, one finds without difficulty for the moments of inertia C, A_1, A about the figure axis, an axis through O perpendicular to the figure axis, and such an axis through S, respectively, the values

$$C = \frac{3}{10}Mr^2, \quad A_1 = \frac{3}{20}M(r^2 + 4h^2), \quad A = \frac{3}{20}M\left(r^2 + \frac{1}{4}h^2\right),$$

where M is the total mass of the top. By equating the values of A and C there follows

$$r = \frac{1}{2}h = 4 \text{ cm.}$$

Thus

$$A = C = \frac{48}{10}M, \quad a = ME^2 = 36M,$$

and

$$\frac{A}{a} = \frac{4}{30}, \quad e^2 = 1 + \frac{A}{a} = 1{,}133.$$

The initial rotation about the figure axis may be assumed to have the value of 20 revolutions per second, so that the angular velocity amounts to $2\pi \cdot 20$. Then

$$N^2 = 4\pi^2 \cdot 400 \cdot A^2$$

and

$$\frac{N^2}{AP} = \frac{4\pi^2 \cdot 40 \cdot 48\, M}{MgE} = \frac{4\pi^2 \cdot 40 \cdot 48}{981{,}0 \cdot 6} = 12{,}878.$$

We have therefore chosen this ratio to be considerably smaller than was previously assumed, since otherwise the figure to be drawn would be uncharacteristic, and the trajectory would hardly differ from a circle. The degree of approximation, which is indeed essentially determined by the magnitude of this ratio, will correspondingly be somewhat less favorable in the following than for the general consideration. This harms nothing, however, since not the smallness of the error, but rather its estimation, is our primary interest in this place. We may let the figure axis form the angle 45° with respect to the vertical in the initial position (at time t_0). Then

$$e_0 = \sqrt{\frac{1}{2}} = 0{,}707.$$

The quantities e_1 and e_2 result from equating the numerator of U to zero, and therefore, according to (11), from the solution of the quadratic equation

$$2AP(1-u^2) = N^2(e_0 - u).$$

If one inserts the given numerical values for $\frac{N^2}{AP}$ and e_0, then one obtains

$$u^2 - 6{,}439u = -3{,}552, \quad e_1 = 0{,}609, \quad e_2 = 5{,}830.$$

Thus

$$\varepsilon = \frac{e_0 - e_1}{2} = 0{,}049, \quad u_0 = \frac{e_0 + e_1}{2} = 0{,}658, \quad u_0^2 = 0{,}433,$$

$$\omega = \sqrt{\frac{a(e^2 - u_0^2)}{2P(e_2 - u_0)}}\,\pi = 6{,}38 \cdot 10^{-2} \text{ sec.},$$

and

$$\frac{N}{A}\frac{\varepsilon}{1-u_0^2}\frac{\omega}{\pi} = 0{,}220.$$

We next calculate the following factors, which, according to (24) and (28), are essential for the estimation of the error:

$$\sqrt{\frac{e_2-e_0}{e_2-u_0}\frac{e^2-u_0^2}{e^2-e_1^2}} = 1-4{,}6\cdot 10^{-2}, \quad \sqrt{\frac{e_2-e_1}{e_2-u_0}\frac{e^2-u_0^2}{e^2-e_0^2}} = 1+5{,}7\cdot 10^{-2},$$

$$\frac{1-u_0^2}{1-e_1^2}\sqrt{\frac{e^2-e_0^2}{e^2-u_0^2}\frac{e_2-u_0}{e_2-e_1}} = 1-1{,}6\cdot 10^{-1},$$

$$\frac{1-u_0^2}{1-e_0^2}\sqrt{\frac{e^2-e_1^2}{e^2-u_0^2}\frac{e_2-u_0}{e_2-e_0}} = 1+1{,}9\cdot 10^{-1}.$$

We now determine the trajectory, and, in particular, the segment of the trajectory that is traversed from the initial time $t = t_0 = \frac{\omega}{2}$ until the time $t = t_6 = \frac{3\omega}{2}$. Between these two points of time we interpolate five equally spaced times t_1, t_2, t_3, t_4, t_5. The time interval between the points is then $\frac{\omega}{6}$. We assign to each of these "true" time points, according to equation (24), two "approximate" time points.

There correspond to t_1, for example, the two values

$$t'' - t_0 = (1-4{,}6\cdot 10^{-2})\frac{\omega}{6} \text{ and } t'' - t_0 = (1+5{,}7\cdot 10^{-2})\frac{\omega}{6}.$$

If we form the product $\frac{\pi t''}{\omega}$ that appears in our formulas as the argument of the trigonometric functions and express this product in degree measure, there follows

$$\frac{\pi t''}{\omega} = \frac{\pi}{2} + (1-4{,}6\cdot 10^{-2})\frac{\pi}{6} = 90^\circ + 28^\circ 37'$$

and

$$\frac{\pi t''}{\omega} = \frac{\pi}{2} + (1+5{,}7\cdot 10^{-2})\frac{\pi}{6} = 90^\circ + 31^\circ 43'.$$

According to equation (23), the values of u that correspond to these two approximate times are

$$u = 0{,}658 + 0{,}049\cos(28^\circ 37') = 0{,}701$$

and

$$u = 0{,}658 + 0{,}049\cos(31^\circ 43') = 0{,}699.$$

Thus the true value of u at the time t_1 can be set equal to

$$u = 0{,}700 \pm \vartheta\cdot 0{,}001,$$

where ϑ (just as ϑ', ϑ'', ... below) is an unknown proper fraction.

The bounds for the length of the vector $S'O = E\sqrt{1-u^2}$ follow from the bounds for u. If we express the former in mm, we find the two extreme values 42,78 and 42,90 mm. We thus write, for $t = t_1$, $E\sqrt{1-u^2} = 42{,}84 \pm \vartheta' \cdot 0{,}06$ mm.

We next consider the direction of $S'O$ at the time t_1, and thus the angle ψ. We first calculate the approximate values ψ'' that correspond, according to equation (27), to the given bounding values of t''; namely,

$$\frac{\pi t''}{\omega} = 90^\circ + 28^\circ 37', \quad \psi'' = 0{,}220\left(28^\circ 37' - \frac{180^\circ}{\pi}\sin 28^\circ 37'\right) = 15',$$
$$\frac{\pi t''}{\omega} = 90^\circ + 31^\circ 43', \quad \psi'' = 0{,}220\left(31^\circ 43' - \frac{180^\circ}{\pi}\sin 31^\circ 43'\right) = 21'.$$

The first of these values corresponds to the inclination $u = 0{,}701$ of the figure axis; the uncertainty of this value follows from the inequality (28), which states that the true value of ψ at the given inclination satisfies

$$(1 - 1{,}6 \cdot 10^{-1})\, 15' < \psi < (1 + 1{,}9 \cdot 10^{-1})\, 15',$$

or

$$13' < \psi < 18'.$$

In the same manner, (28) gives

$$(1 - 1{,}6 \cdot 10^{-1})\, 21' < \psi < (1 + 1{,}9 \cdot 10^{-1})\, 21',$$

or

$$18' < \psi < 25'.$$

for the true value of ψ at the inclination $u = 0{,}699$. Since the quantity u must lie between $u = 0{,}701$ and $u = 0{,}609$ at the time $t = t_1$, the angle ψ must lie between $13'$ and $25'$. We thus write, for $t = t_1$,

$$\psi = 19' \pm \vartheta'' \cdot 6'.$$

The quantities u, $E\sqrt{1-u^2}$, and ψ are determined in the same manner for the times $t = t_2, t_3, \ldots$. (For the initial time $t = t_0$ there obviously follows, and indeed exactly, $u = e_0 = 0{,}707$, $E\sqrt{1-u^2} = 42{,}43$ mm, and $\psi = 0$.) The results of the calculation are contained in the table of page 530.

	t_0	t_1	t_2	t_3	t_4	t_5	t_6
u	0,707	$0{,}700 \pm 0{,}001$	$0{,}682 \pm 0{,}002$	$0{,}658 \pm 0{,}004$	$0{,}632 \pm 0{,}005$	$0{,}616 \pm 0{,}003$	$0{,}609 \pm 0{,}000$
$E\sqrt{1-u^2}$	42,4	$42{,}8 \pm 0{,}1$	$44{,}0 \pm 0{,}1$	$45{,}2 \pm 0{,}2$	$46{,}4 \pm 0{,}3$	$47{,}2 \pm 0{,}2$	$47{,}6 \pm 0{,}0$
ψ	0	$19' \pm 6'$	$2^0\,26' \pm 46'$	$7^0\,38' \pm 2^0\,20'$	$16^0\,21' \pm 4^0\,50'$	$28^0\,3' \pm 7^0\,58'$	$41^0\,21' \pm 11^0\,8'$

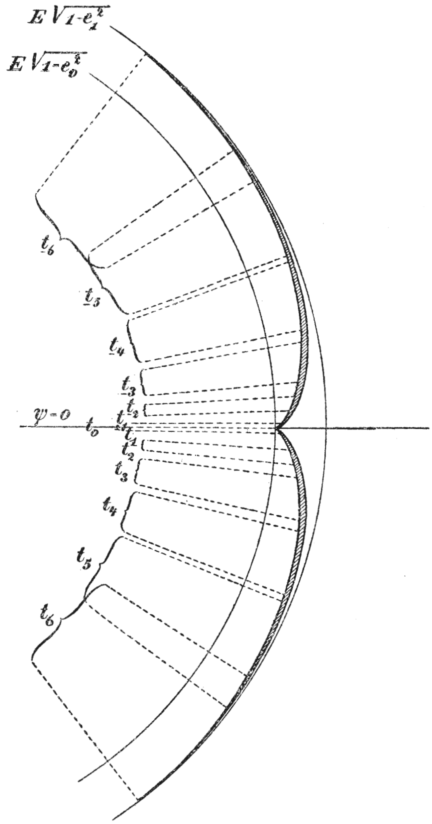


Fig. 70.

On the basis of these numbers, Fig. 70 is drawn. The figure shows how the trajectory is pulled to and fro between the two bounding circles of radii $E\sqrt{1-e_0^2} = 42{,}4$ mm and $E\sqrt{1-e_1^2} = 47{,}6$ mm. It is tangent to the larger of the two circles, and touches the smaller with cusps.

The bounded precision of our calculation is expressed by drawing the trajectory not as a mathematical line, but rather as a strip of changing width; further, the time points t_i correspond not to a determined point of the line, but rather to a certain band of the strip that is made recognizable in the figure. This band will become always larger, and the certainty of our calculation always smaller, as we move farther from the initial time $t = t_0$. The time points $t_{-1}, t_{-2}, \ldots$ precede the initial time t_0 at the same distances as the time points $t_1, t_2, \ldots$ follow it. The trajectory for $t < t_0$ results from that for $t > t_0$ simply by a reflection about the ray $\psi = 0$.

One will admit, after a glance at our figure, that our approximate calculation satisfies all the requirements, with respect to the *form* of the trajectory, that one can pose from the scientific standpoint for the solution of the present exercise; with respect to the *time* in which our trajectory is traversed, one can doubt whether our approximation is satisfactory; at $t = \pm t_5$, for example, the band inside which the position of the support point is uncertain becomes rather large. We could plead in this respect, however, that the precision of our calculation may go somewhat in parallel with the precision with which the position of the support point may be determined at a given time by a not particularly refined method of observation.

We must bear in mind, above all, that the actual course of the motion will be distorted to a high degree by friction at the support point, which we have disregarded in this section. Thus if we would sharpen the preceding calculation without taking friction into consideration, this would signify dwelling on trifles and missing the main point.

Addenda and Supplements.[196]

To Chap. V.

To p. 315. The moment $Cr\varphi'$, which we designate as the deviation resistance in §5 of Chap. III and as the top-effect in later applications, is called the "induced force" by Mr. K o p p e in the work cited in the footnote ††) of p. 315, where the importance of this moment for the elementary understanding of the phenomena of the top is indicated. Koppe's indication is in full consonance with our own conception (cf. the heading of §1 of Chap. IX, where the formula in question is designated concisely as the most important in the theory of the top, as well as all the following technical applications, where this formula, or its generalization, is used throughout). In a review of the first two volumes of our book (Ztschr. f. d. phys. u. chem. Unterricht, Nov. 1898) and in the Berichten der Berliner mathematischen Gesellschaft, 1, 1902, Mr. Koppe has given a newly simplified derivation of this moment that essentially coincides with ours, and with which we therefore agree completely.[197]

To p. 337 It is stated that the asymptotic motion of the "upright" top, which is essential for our subsequent stability criterion, appears not to have been previously noted in a characteristic manner. On the contrary, this motion is indicated in A. G. G r e e n h i l l, Applications of elliptic functions, London, 1892, p. 243, §226 E; the possibility of an elementary calculation of the relevant trajectory is presented there, corresponding to the occurrence of a so-called pseudo-elliptic case. The asymptotic character of the motion is not, however, discussed in detail.

To p. 341. As Mr. K o p p e notes in the cited review, and as also follows from the equations (10) on p. 368 that are derived according to the method of small oscillations, the trajectories represented in formula (20) can also be conceived, in the stable cases $N^2 - 4AP > 0$, as epi- or hypocycloids (namely, as the superposition of two undamped circular oscillations). They take this simple form, however, only if one neglects a term of the order ε, which is equal to the order $1 - u$, compared to terms of the order $\sqrt{\varepsilon}$ or $\sqrt{1-u}$. With this restriction, we can therefore agree again with the remark of Mr. K o p p e. The velocities of the precession and nutation are different from those that occur for an arbitrary inclination of the figure axis.

(The superposition may be recognized if the curve represented in (20) is mapped by perpendicular projection (which, in the present case where the curve runs entirely in the neighborhood of the north pole, coincides with the stereographic projection from the south pole) onto the horizontal plane. In the mapping,

$$r^2 = \sin^2\vartheta = 1 - u^2 = (1+u)(1-u),$$

or, neglecting the terms of higher order, since u is approximately equal to 1,

$$r^2 = 2(1-u) = 2\Big(1 - u_0 - \varepsilon\sin\frac{\pi t}{\omega}\Big).$$

It is first evident that $1-u_0$ and ε are to be conceived as small quantities of the second order in the linear dimensions. We thus write the second equation of (20), with the suppression of all terms that are recognizable as of the second order, as

$$2\psi = \frac{n+N}{2A}t + \operatorname{arctg}\frac{w_1}{w_2},$$

where

$$w_1 = \varepsilon - (1-u_0)\sin\frac{\pi t}{\omega},$$
$$w_2 = \sqrt{(1-u_0)^2 - \varepsilon^2}\cos\frac{\pi t}{\omega}.$$

We now form

$$(x+iy)^2 = r^2 e^{2i\psi},$$

where x and y signify the coordinates of the apex of the top in the horizontal projection, by means of the identity

$$e^{i\operatorname{arctg}\frac{w_1}{w_2}} = \frac{w_2 + iw_1}{\sqrt{w_1^2 + w_2^2}}.$$

Here

$$\begin{aligned}\sqrt{w_1^2+w_2^2} &= \sqrt{\varepsilon^2 - 2\varepsilon(1-u_0)\sin\frac{\pi t}{\omega} + (1-u_0)^2\sin^2\frac{\pi t}{\omega} + ((1-u_0)^2-\varepsilon^2)\cos\frac{\pi t}{\omega}}\\ &= \sqrt{\varepsilon^2\sin^2\frac{\pi t}{\omega} - 2\varepsilon(1-u_0)\sin\frac{\pi t}{\omega} + (1-u_0)^2}\\ &= 1 - u_0 - \varepsilon\sin\frac{\pi t}{\omega} = \frac{r^2}{2}.\end{aligned}$$

Thus

$$r^2 e^{2i\psi} = 2e^{i\frac{n+N}{2A}t}\Big[\sqrt{(1-u_0)^2-\varepsilon^2}\cos\frac{\pi t}{\omega} + i\Big(\varepsilon - (1-u_0)\sin\frac{\pi t}{\omega}\Big)\Big].$$

Through the introduction of the argument $\pi/2 - \pi t/\omega$ and the passage to the half of this argument, the bracket [—] in this equation is

$$
\begin{aligned}
&2\sqrt{(1-u_0)^2-\varepsilon^2}\,\sin\frac{\pi(\omega-2t)}{4\omega}\cos\frac{\pi(\omega-2t)}{4\omega}\\
&+i\Big[\varepsilon-(1-u_0)\Big(\cos^2\frac{\pi(\omega-2t)}{4\omega}-\sin^2\frac{\pi(\omega-2t)}{4\omega}\Big)\Big]\\
&=i\Big(\sqrt{1-u_0+\varepsilon}\,\sin\pi\frac{\omega-2t}{4\omega}-i\sqrt{1-u_0-\varepsilon}\,\cos\pi\frac{\omega-2t}{4\omega}\Big)^2.
\end{aligned}
$$

Finally,

$$
re^{i\psi}=2\sqrt{i}\,e^{i\frac{n+N}{4A}t}\left(\sqrt{1-u_0+\varepsilon}\,\sin\pi\frac{2t-\omega}{4\omega}+i\sqrt{1-u_0-\varepsilon}\,\cos\pi\frac{2t-\omega}{4\omega}\right),
$$

instead of which we may also write

$$
\begin{aligned}
x+iy=\ &(\sqrt{1-u_0+\varepsilon}+\sqrt{1-u_0-\varepsilon}\,)\,e^{i\left(\frac{n+N}{4A}+\frac{\pi}{2\omega}\right)t}\\
&+(\sqrt{1-u_0+\varepsilon}-\sqrt{1-u_0-\varepsilon}\,)\,e^{i\left(\frac{n+N}{4A}-\frac{\pi}{2\omega}\right)t}.
\end{aligned}
$$

In the stable cases $N^2-4AP>0$, ω is real, according to equation (17) of page 339, and thus the above equation for the trajectory of the apex of the top represents, in fact, the superposition of two purely periodic circular oscillations; that is, an epi- or hypocycloid.)

To §6. The concept of stability is treated in an important work of D. J. Korteweg: Über Stabilität periodisch ebener Bahnen, Sitzungsber. d. Wiener Akad., May 1886, which the authors had, at the time of writing, unfortunately overlooked.[198]

The Korteweg definition coincides with that of Routh. Korteweg makes the same objection to the definition of Thomson and Tait that is raised here in the footnote of p. 348. The necessity of considering the higher-order terms is likewise strongly emphasized by Korteweg, and, more importantly, these terms are actually considered, and the judgment of stability or instability is not based on the terms of the first order alone. The concept of practical lability (for theoretical stability) is also indicated by Korteweg; cf. §24 of his work.

To p. 350. We do not wish to give the impression that the judgment of stability or lability is always natural and unambiguous if one accepts our definition. Mr. Korteweg has called attention to an example in which one would speak of instability (or at least indifferent stability) according to the geometric aspect of the perturbed trajectories, but where the analytic test gives, according to our definition, stability. The example treats of the central motion of a single mass particle for the case $n=-3$ (cf. p. 347). The corresponding attractive force is $-f/r^3$. The motion of the mass particle is determined by the two equations

$$r^2\varphi' = c, \tag{1}$$

$$r^2\varphi'^2 + r'^2 - \frac{f}{r^2} = 2h, \tag{2}$$

where r and φ are polar coordinates with the pole at the center of attraction, c is the constant of the area theorem, h is the constant of the *vis viva*, and the mass of the particle is taken as 1. From (1) and (2) there follows

$$r'^2 = 2h + \frac{c^2 - f}{r^2}. \tag{3}$$

Circular motion ($r' = 0$) occurs as a special case. If the radius of the circle is a, then the centrifugal force is $a\varphi'^2 = c^2/a^3$. This must be canceled by the attractive force $-f/a^3$. Therefore $f = c^2$, and it follows from (3) that $h = 0$. We now consider, for example, a perturbed motion for which h is likewise zero (a conservative perturbation), but $f \neq c^2$, and therefore $f - c^2 = \varepsilon^2$, say. Equation (3) then gives

$$rr' = \varepsilon. \tag{4}$$

By comparison with (1), one concludes that

$$\frac{1}{r}\frac{dr}{d\varphi} = \frac{\varepsilon}{c}, \quad \text{or} \quad r = Ce^{\frac{\varepsilon}{c}\varphi}, \tag{5}$$

where the constant of integration C can be taken as equal to a. The trajectories are then logarithmic spirals that begin from point O, intersect the circle of radius a, and continue to infinity. The circle $r = a$ is not at all distinguished among these trajectories. This is true for every value of ε, no matter how small; for decreasing ε, the spiral windings only become increasingly tight. For $\varepsilon = 0$, however, there follows $r = C$, and therefore a circular trajectory, and indeed the original circle, since we naturally let the perturbation of the position of the particle also decrease to zero.

The limit of the perturbed trajectory is thus analytically identical with the original (the equality would also obtain for an arbitrary non-conservative impact). Geometrically, however, one will hardly allow the single circle as the limit of the successively tightening logarithmic spirals; one is led to the circle only in the analytic limit process, since the totality of the spirals no longer represents, in the limit, a single analytic curve. We must, nevertheless, designate the circular trajectory $r = a$ as stable if we adopt our definition literally.

The situation is different in the case $n < -3$; here asymptotic trajectories occur (spirals whose windings, beginning from the origin or from infinity, approach the circle $r = a$ from both sides; in the case $n = -3$ the corresponding spiral trajectories were not to be designated as asymptotic, since they were no longer analytic curves in the limit, and did not have a fixed circle as a limit). The limit passage for $n < -3$ can be arranged so that the limit of the perturbed path also becomes, in analytic respects, not the original trajectory, but rather such an asymptotic trajectory.

In the domain of equilibrium problems, the concept of "indifferent equilibrium" is generally common, while it has not, until now, been usual in the domain of motion problems. It is possible to designate, in a corresponding manner, the above case of central motion for $n = -3$, and similar cases in general, as "indifferent states of motion." In fact, this case coincides in many respects with the case of the force-free stationary particle, if one refers the motion to a coordinate system that rotates with a uniform velocity φ'.

If we consider our developments in §§6–8 together with the above example, we come to the conclusion that the multiplicity of possible trajectories and states of motion is much too large to be classified without force into specific categories such as stable, labile, or indifferent. Each such classification will contradict the natural conception in certain cases; we would not exempt our own stability criterion in §6.

To page 353. The expression "practical instability" was first used by W. Gibbs in 1876; cf. the German edition of some originally separate treatises in Thermodynamische Studien, Leipzig 1892, p. 94.

To page 376. In the discussion of the particular cases of the heavy asymmetric top for which full analytic treatment is available, our presentation of the case of Mrs. Kowalevski is somewhat too brief (as Mr. F. Kötter emphasizes in his work Bermerkungen zu F. Kleins und A. Sommerfelds Buch über die Theorie des Kreisels, Berlin 1899, Mayer & Müller[199]). This is not only because we had not given a proper development for this case, but also because the execution of the integration cannot, it appears, be realized through simple geometric deliberations, as in the other cases considered. The complete treatment of the case would have required too lengthy an analytic development.

In work of F. Kötter cited in the footnote, it is particularly emphasized that the author has recognized, after a correct presentation of the Kowalevski formulas, the composition of the motion of the Kowalevski top from two simple rotations: a rotation of a new coordinate frame with a constant vertical component of rotational velocity, and a rotation about an axis of the introduced frame.

To page 377 ff. A large series of works has recently been published on the integration problem for the heavy asymmetric top, as well as on the motion of the asymmetric top for a generalized law of force, in which integrable exceptional cases are treated. With respect to all these works, we refer to the recent summary report of Stäckel: Rotation starrer Körper und Verwandtes, Encyklopädie der math. Wiss. IV, 13.

An investigation of Schiff (Moscow, math. Sammlung, Bd. 24, 1903), which asks for all motions in which the impulse vector retains a constant length, appears to be more closely associated with the type of problem formulation in §9. Its contention, however, that such motions exist in greater generality, has proven to be erroneous; as Stäckel has shown (Die reduzierten Differentialgleichung der Bewegung des schweren unsymmetrischen Kreisels, Math. Annalen 7, pp. 399, 1909), the named question leads only to the Staude permanent rotations, for which not only the length of the impulse vector, but also its position relative to the body, are preserved. Thus these permanent rotations remain as the single known motions of the general heavy asymmetric top.

On the other hand, however, the question of special cases of the mass distribution that admit of a third algebraic integral, in addition to the two expressed by the conservation of energy and the conservation of the vertical component of the impulse, has been carried further. While the investigations of R. Liouville (cf. p. 377) were still incomplete, E. Husson (Annales de Toulouse 2^{e} série, VIII, 1906: Recherches des intégrales algébriques ... and Acta math. XXXI, 1908: Sur un théorème de M. Poincaré ...) proved that in addition to the already known cases, no further cases exist in which a third integral is possible. Thus there remain, in fact, the Euler case (center of gravity at the point of suspension), the Lagrange case (symmetric top) and the case of Mrs. Kowalevski as the only cases that are thus far available for complete treatment. While the proof of Husson uses transcendental methods similar to those that Poincaré has developed for celestial mechanics, P. Burgatti has recently given an elementary proof of the theorem with only algebraic means (Dimostrazione della non esistenza d'integrali ..., Rend. del circolo matem. di Palermo, t. XXIX, 1910).

The two proofs, moreover, rely on a previously established theorem of Poincaré (Les nouvelles méthodes de la mécanique céleste, t. I), which, applied to the case of the force-free top, gives the equality of two principal moments of inertia as the condition for the existence of the third algebraic integral.

Since there is thus no prospect of advancing deeper into the unknown domain of the motion of the heavy top through the integration of exceptional cases, the pursuit of the qualitative discussion that is recommended on p. 391 is all the more important. The present approaches in this direction are restricted to the treatment of small oscillations, which, generated by a disturbance, are superposed on known motions. The small oscillations of an asymmetric body about its equilibrium position (for a vertically downward directed center of gravity axis), which can naturally be treated without difficulty with the restriction to terms of the first order, are investigated by M. Lecornu (Sur les petits mouvements d'un corps pesant, Bull. de la Soc. math. de France, 30, 1902). We mention this investigation because the author also attains (independently of the Staude work) a formulation of the condition for a permanent upright rotation, and indeed in the following differing conception: an axis through the support point can be a permanent rotation axis if it can be a principal axis with respect to one of its points. The equivalence of this condition with the Staude condition is easily verified.

Motions of a nearly symmetric top are treated by M. Winkelmann (Zur Theorie des Maxwell'schen Kreisels, Diss. Göttingen 1904) through the formation of "perturbations of the first order" (in the astronomical terminology), where the motion of the symmetric top is taken as the undisturbed motion. Winkelmann does not, however, advance to an error estimation or an investigation of the duration of validity of the constructed equations. We wish to point out, finally, the thorough numerical treatment of the top problem in the astronomical literature, where particular approaches to the approximate treatment of the heavy asymmetric top are also found (for example, Charlier, Eine neue Methode zur Behandlung des Rotationsproblems, Arkiv för Matematik, Kopenhagen, IV, 1908).

To page 379. Our formulation of the Hess case (namely, the question of the conditions under which is it possible that the impulse vector is always perpendicular to the center of gravity vector) contains, as Mr. Stäckel remarks (Math. Annalen 67, p. 423), the simple pendulum-like oscillations of bodies with a suitably chosen mass distribution that Mlodzjejewskij (cf. the footnote of p. 378) has mentioned. A pendulum-like oscillation is understood here as a case in which the body oscillates as a physical pendulum about a horizontal axis, while, however, only one point of this axis is supported. It is evident on the basis of symmetry that this motion always occurs if the center of gravity lies in a principal plane of the body and the body is initially given only a rotation about the horizontal axis perpendicular to this plane. The motion therefore requires one condition for the mass distribution (the center of gravity in one of the principal planes), and three conditions for the constants of the motion. The impulse vector in this case lies on the principal axis perpendicular to the center of gravity but has a variable length, while for the Staude permanent rotation axes the impulse vector lies on a fixed axis of the body and also retains a constant length. In the derivation of the Hess condition this case was excluded, since it was assumed there that the impulse vector is actually distributed during the motion on an arbitrary ray of the body-fixed plane normal to the center of gravity vector, while here the impulse vector always remains on a distinguished line of this plane.

To Chap. VI.

To p. 429. The mathematical side of the theory of the symmetric top has been pursued further, from the standpoint of the theory of elliptic functions, by Greenhill through the investigation of particular cases in which the top curves represented by ϑ-quotients (the trajectories of the apex of the top, or the herpolhode or polhode curves) become algebraic curves (for example, Annals of Mathematics, 5, 1904).

To p. 472. In the work cited in the note to p. 376, Mr. F. Kötter remarks that formulas (40), which represent the nine direction cosines for the motion of the force-free top in terms of ϑ-functions, contain transpositions. The factor $-i$ must appear in the second equation of the system before the fraction instead of i, and, in addition, the formulas given for c and c' are to be interchanged, and therefore to be replaced by

$$c = i\frac{\vartheta(\omega+is)\vartheta(t)}{\vartheta(is-i\omega'+\omega)\vartheta(t+i\omega')}e^{-\frac{i\pi}{2\omega}(t+i\omega'-is)},$$

$$c' = \frac{\vartheta(is)\vartheta(t-\omega)}{\vartheta(is-i\omega'+\omega)\vartheta(t+i\omega')}e^{-\frac{i\pi}{2\omega}(t+i\omega'-is)}.$$

In order to orient the corrected system with respect to the Jacobi formulas (cf. the footnote on page 473), we give the following tabular summary:

Jacobi's notation	Our notation
$\mathsf{H}(t)$	$\vartheta(t)$
$K,\ K'$	$\omega,\ \omega'$
a	$\omega' - s$
$\alpha + i\alpha'$	$(a' + ib')e^{-i\left(2l+\frac{\pi}{2\omega}\right)t}$
$\beta + i\beta'$	$(a + ib)e^{-i\left(2l+\frac{\pi}{2\omega}\right)t}$
$\gamma + i\gamma'$	$(a'' + ib'')e^{-i\left(2l+\frac{\pi}{2\omega}\right)t}$
$\alpha'',\ \beta'',\ \gamma''$	$c',\ c,\ c''.$

The Jacobi formulas have a still simpler form through the use of the function symbols Θ, H_1, and Θ_1, which are related to H by

$$\Theta(t) = -ie^{\frac{i\pi}{4\omega}(2t+i\omega')}\mathsf{H}(t + i\omega'),$$

$$\mathsf{H}_1(t) = \mathsf{H}(t + \omega), \qquad \Theta_1(t) = \Theta(t + \omega).$$

To p. 486. The second proof of the Jacobi theorem that begins here had already been given previously by F. Caspary. Cf. Darboux Bulletin (2) 13, 1899: Sur les expressions des angles d'Euler, de leurs fonctions trigonométriques et des neuf coefficients d'une substitution orthogonal au moyen des fonctions thêta. (See also the note to p. 511 below.)

To p. 490. Concerning our view of the kinematic character of the Jacobi theorem, compare the opposing conception of Mr. F. Kötter in his repeatedly cited work.

To p. 505. It is to be noted that c_1 is used here with a different meaning than on p. 503 and on p. 506 below, and on the following pages.

To p. 511. Since the Weierstrass lecture (1879) was not published, the first application in the literature of the parameters α, β, γ, δ to the analytic treatment of the top was made by F. Caspary, for both the force-free top and the heavy spherical top. Caspary was led in an analytic way to consider certain theta-quotients that correspond to our α, β, γ, δ, and are composed bilinearly in such a manner that the resulting expressions identically satisfy both the orthogonality conditions for the direction cosines and the Euler equations for the direction cosines and the rotation components. In addition to the work cited on page 486, we mention Comptes Rendus, 107 (1888), p. 859, 901, 937; 112 (1891), p. 1120; cf. also E. Jahnke, ibid. 126 (1898), p. 1126. Concerning the relation of the parameters α, β, γ, δ to the complex variables extended on the unit sphere, which was already known to Gauſs (cf. p. 512), was found anew by Riemann (cf. p. 30), and was applied in 1872 by Schwarz in the theory of hypergeometric functions (Crelle's Journal 75) and by Klein for the transformations of the icosahedron (cf. p. 30), the necessary references are to be found in the given places in the text itself. Klein then applied the parameters to the theory of the top (cf. Göttinger Nachrichten 1896, p. 3 and the Princeton Lectures cited on p. 518, where the parameters are also applied to the theory of the top on the horizontal plane, and lead to simple results in the context of the theory of automorphic functions, into which we could not enter in our presentation).

Translators' Notes.

108. (page 197) The title of Poincaré's two-part paper, in which he is styled as *Ingénieur des Mines*, is *Mémoire sur les courbes définies par une équation différentielle* [Poincaré 1881/82]. In the introduction to the paper, Poincaré compares the solution of a differential equation to the solution of an algebraic equation and the construction of an algebraic curve:

> The complete study of a function comprises two parts:
>
> 1° The qualitative part (so to speak), or the geometric study of the curve defined by the function;
>
> 2° The quantitative part, or the numerical calculation of the values of the function.
>
> Thus, for example, one begins the study of an algebraic equation by seeking, with the aid of Sturm's theorem, the number of real roots. This is the qualitative part. One then calculates the numerical value of the roots, which constitutes the quantitative study of the equation. In the same manner, one begins the study of an algebraic curve by *constructing* this curve, as one says in the course of special mathematics; that is to say, one seeks the closed branches of the curve, the infinite branches, etc. After this qualitative study of the curve, one can determine a certain number of points exactly.
>
> One must naturally begin the theory of any function with the qualitative part, and that is why the problem that presents itself in the first place is the following:
>
> *To construct the curves defined by differential equations.*

Poincaré restricts himself in this paper to the particular differential equation

$$\frac{dx}{X} = \frac{dy}{Y},$$

where $X(x, y)$ and $Y(x, y)$ are polynomials in x and y. He maps the x, y plane onto a sphere by a gnomic projection, and studies the resulting geometry of the solution curves.

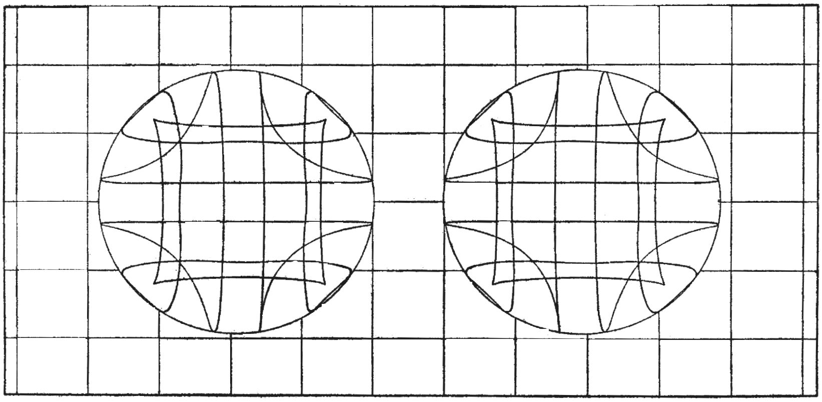
Roland K. W. Roeder has written a detailed commentary on Poincaré's paper, including an interesting discussion of several errors in the examples that Poincaré uses to illustrate his theory [Roeder 2003].

109. (page 199) In 1898, Greenhill wrote to Sommerfeld to thank him for a copy of Vol. II of the *Theorie des Kreisels*. Greenhill wrote that he "felt much flattered at the kind way in which you mention, on p. 199, the stereoscopic views drawn by Mr. Dewar. But I think it ought to be mentioned that, after a little practice, it is soon possible to dispense with the stereoscope and to obtain the solid effect with the unassisted eyes" [Greenhill 1898].

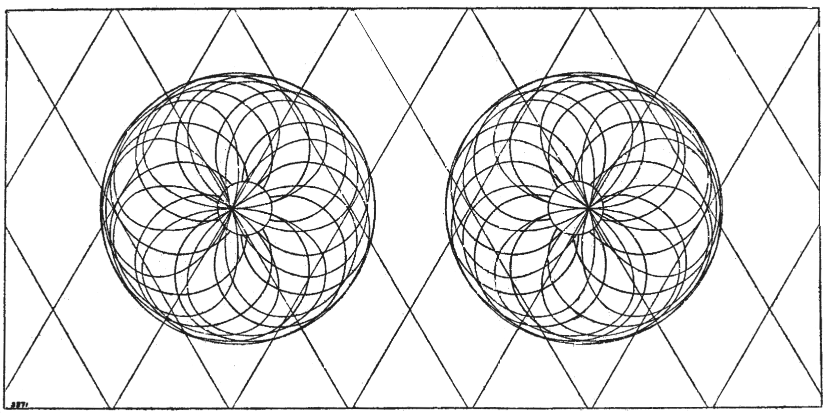
Greenhill's claim may be tested on the stereoscopic representations reproduced in Fig. 164. Fig. 164(a) shows trajectories of the apex of the top for three special cases in which the trajectory curves are closed. Fig. 164(b) is more complicated. It contains two closed trajectory curves, an upper curve that passes through the north pole and a lower curve that passes through the south pole. Greenhill describes the two curves as follows [Greenhill 1897]:

> The upper curve has an apsidal angle of 144 deg., with a tenfold rosette, and closes in on itself after three revolutions in azimuth; the lower curve has an apsidal angle of 72 deg., similarly with a tenfold rosette, and closes in on itself after one revolution in azimuth. If the two curves were drawn complete from the point of view chosen (the zenith), the parts in the neighbourhood of the upper and lower poles would overlap and obscure each other, and for this reason the polar part of the upper curve has been omitted, giving the appearance of a hole cut in a transparent spherical shell which rests on a tesselated pavement. The point of the top is supposed to be fixed at the centre of the sphere. It is easy to follow the lower rosette through a complete course of its convolutions, but the upper curve is not so easily traced, because it crosses the circle of maximum angular diameter, the bounding circle of the diagram, and makes its way indistinguishably for some distance beyond that.

The stereographic representations in Greenhill's papers were made by T. I. Dewar, about whom we have found no biographical information. It seems, however, that Dewar had a personal interest in stereoscopic images. In 1894, he published an article in *Nature* that describes his recovery from a railway accident that left him with double vision



(a)



(b)

Fig. 164. Stereoscopic images of the trajectory of the apex of the top. (a): [Greenhill 1914, Fig. 80]; (b): [Greenhill 1897, p. 311].

for a period of several years [Dewar 1894]. Some of Dewar's stereoscopic drawings were exhibited at the Soirée of the Royal Society of London in June of 1895.

In 1914, Greenhill published a *Report on Gyroscopic Theory*, a comprehensive work that "is intended to have the same scope" as the "*Theorie des Kreisels* of Klein and Sommerfeld, where no mathematical difficulty is passed over or ignored" [Greenhill 1914, p. ii]. Reading Klein and Sommerfeld or Greenhill, one feels transported to a time when intellectual giants walked the Earth. May we be worthy bearers of the treasures that have been handed down to us!

110. (page 207) Fig. 165 shows three representations of the trajectory corresponding to Fig. 25. The curves in Fig. 165 were computed with the ϑ-quotient formulas given by Klein and Sommerfeld in Chapter VI. The rounding of the cusps at the equator in the orthographic projection onto the equatorial plane is exaggerated in Fig. 25a on page 207, but the remarkable qualitative difference between the orthographic and stereographic projections onto the equatorial plane is clearly visible in Fig. 165.

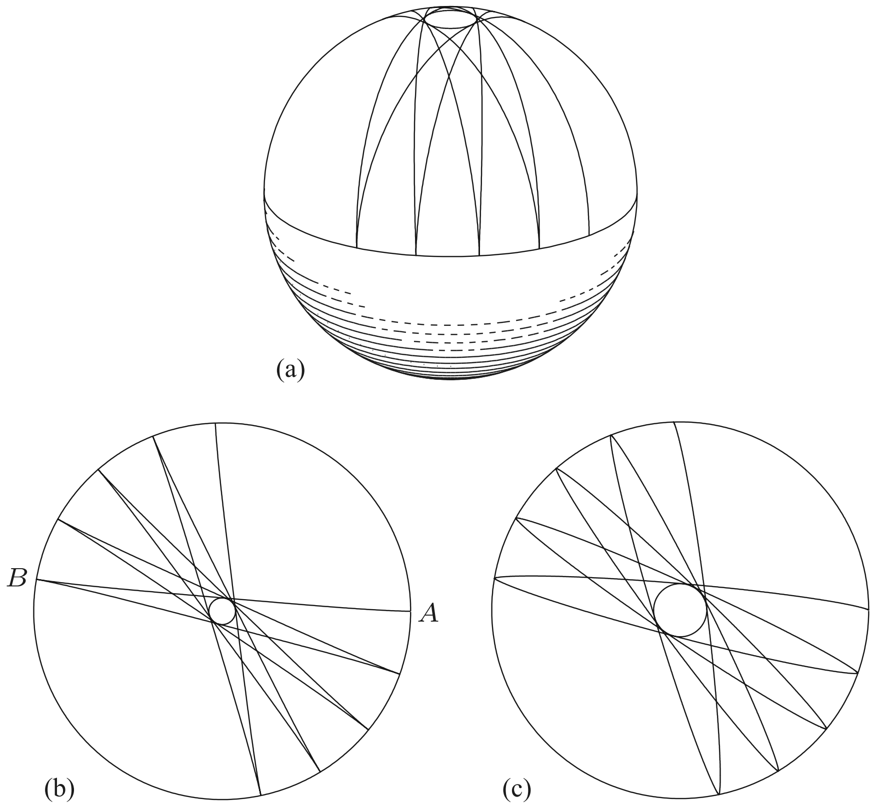


Fig. 165. Representations of the trajectory of the apex of the top for $A = 1$, $P = -1$, $n = 0$, $N = 0.20$, $e = 0$.
(a) orthographic projection onto a nonequatorial plane;
(b) stereographic projection from the south pole onto the equatorial plane;
(c) orthographic projection onto the equatorial plane.

A conceptual mechanical realization of the motion corresponding to Fig. 165 in shown in Fig. 166 (see the following two pages). The top in Fig. 166 consists of two rotors that are rigidly fixed to a single shaft; the dimensions are chosen so that the top is approximately spherical with respect to its fixed point at the center of the larger rotor. The configurations in Fig. 166 correspond to equal time increments along the trajectory from A to B in Fig. 165(b), with Fig. 166(a) corresponding to point A and Fig. 166(g) corresponding to point B.

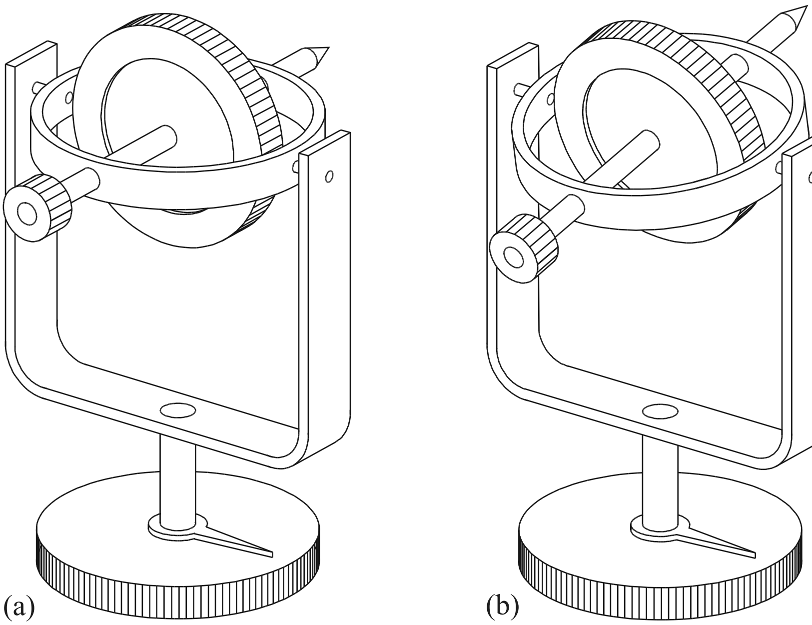
It is difficult to build a mechanical device that can execute the motion represented in Figs. 165 and 166. As the apex of the top passes near the vertical in Figs. 166(c)–(e), the fork that carries the circular ring must rotate relatively quickly through an angle of almost 180°. Since the rotational velocity of the top about its figure axis is assumed to be small, the inertia of the fork and ring can have a significant dynamic effect that is not included in the analysis corresponding to Fig. 165. The Rozé top (Vol. I, p. 2) and the Maxwell top (Vol. I, note 27) behave more ideally, but their range of motion is limited; they cannot take the initial position $\vartheta = \dfrac{\pi}{2}$ assumed by Klein and Sommerfeld in sections 1 and 2 of Chap. IV.

A description of some experiments with a mechanical device similar to that shown in Fig. 166 is given by Kurt Magnus (1912–2003) in his very fine book *Der Kreisel* [Magnus 1945].

111. (page 224) Sadly, Klein and Sommerfeld never return to the discussion of cyclic systems.

112. (page 224) The cited section of the *Mécanique Analytique* is in the second volume of the second edition [Lagrange 1815]. After deriving the equations for dt, $d\psi$, and $d\varphi$ in terms of u and du, Lagrange remarks only that the variables in these equations are separated, and that their integration "depends, in general, on the rectification of conic sections." He then proceeds to the consideration of small oscillations of a heavy rigid body that is suspended at a point and may spin arbitrarily about the line through the suspension point and the center of gravity.

113. (page 234) The stated purpose of Darboux's paper [Darboux 1885] is to give a "direct and elementary" demonstration of the Jacobi theorem that is discussed by Klein and Sommerfeld in Chap. VI, §8. The relation between the motions of the symmetric and the spherical tops and the reciprocity theorem for the inverse motion of the spherical top are stated as preliminary results in the first section of the paper.



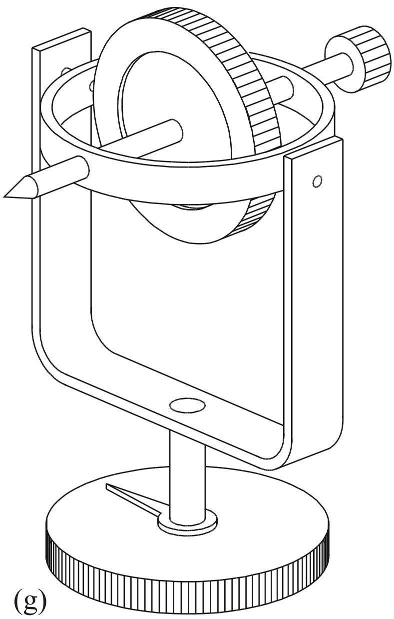
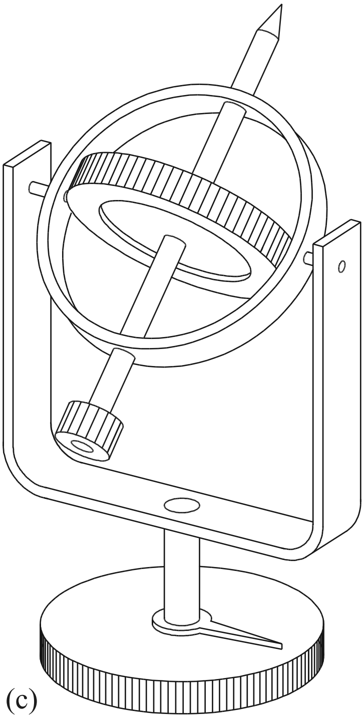
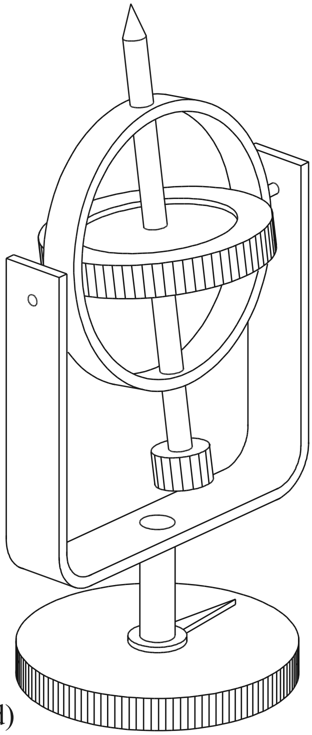


Fig. 166. Conceptual mechanical realization of the motion of the top along the trajectory from *A* to *B* in Fig. 165 (clockwise from the upper left).



(c)



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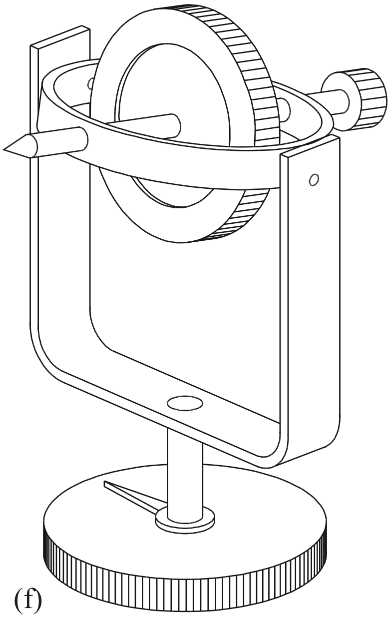
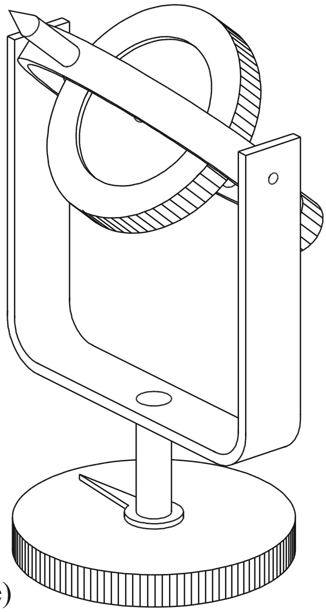


(f)



(e)

114. (page 247) Routh's geometric construction for the determination of the parallel circles that bound the trajectory of the top may be summarized as follows. In Fig. 167, O is the support point of the top, OV is the vertical, and OQ is a line segment on the figure axis. The length OQ is called l, and is defined, in the notation of Klein and Sommerfeld, as $l = A/ME$. Point Q is then the *center of oscillation* of the top. Routh defines the lengths OG and OH as

$$OG = a = \frac{A}{ME}\frac{n}{N}, \quad OH = b = \frac{A}{ME}\Big(\frac{h}{P} - \frac{N^2}{2CP}\Big).$$

An x, y coordinate system is defined at G, with the x-axis downward and the y-axis to the right. The coordinates of the point Q are then

$$x = a - l\cos\vartheta, \quad y = l\sin\vartheta,$$

and the equation $U = 0$, where U is given by equation (7) page 222, can be written as

$$\frac{N^2}{2PME}x^2 = y^2(x + c), \tag{1}$$

where

$$c = b - a = \frac{A}{ME}\Big(\frac{h}{P} - \frac{N^2}{2CP} - \frac{n}{N}\Big).$$

The angle ϑ between the vertical and the figure axis therefore oscillates between the two values for which the point Q lies on the cubic curve (1).

The form of the cubic (1) is shown by the dashed curves in Fig. 167(a) for the case $c > 0$. For $c < 0$, the curve consists of an isolated point at the origin and two separated branches, as shown in Fig. 167(b). The case $c = 0$ corresponds to a top with initial conditions $\dot\vartheta = \dot\psi = 0$. In this case, the cubic curve reduces to the line $x = 0$ and the parabola $N^2x = 2PMEy^2$.

Routh does not give a systematic discussion of how the initial conditions of the top influence the shape of the cubic (1) or the positions of the bounding parallel circles.

115. (page 265) A pleasantly direct account of the transformation theory for elliptic integrals is given by Cayley [Cayley 1876]. A form of the Landen transformation that applies to the algebraic integral

$$\int \frac{dx}{\sqrt{x(1-x)(1-k^2x)}}$$

may be based on the quadratic change of variables

$$y = \frac{(1+k)^2x}{(1+kx)^2}.$$

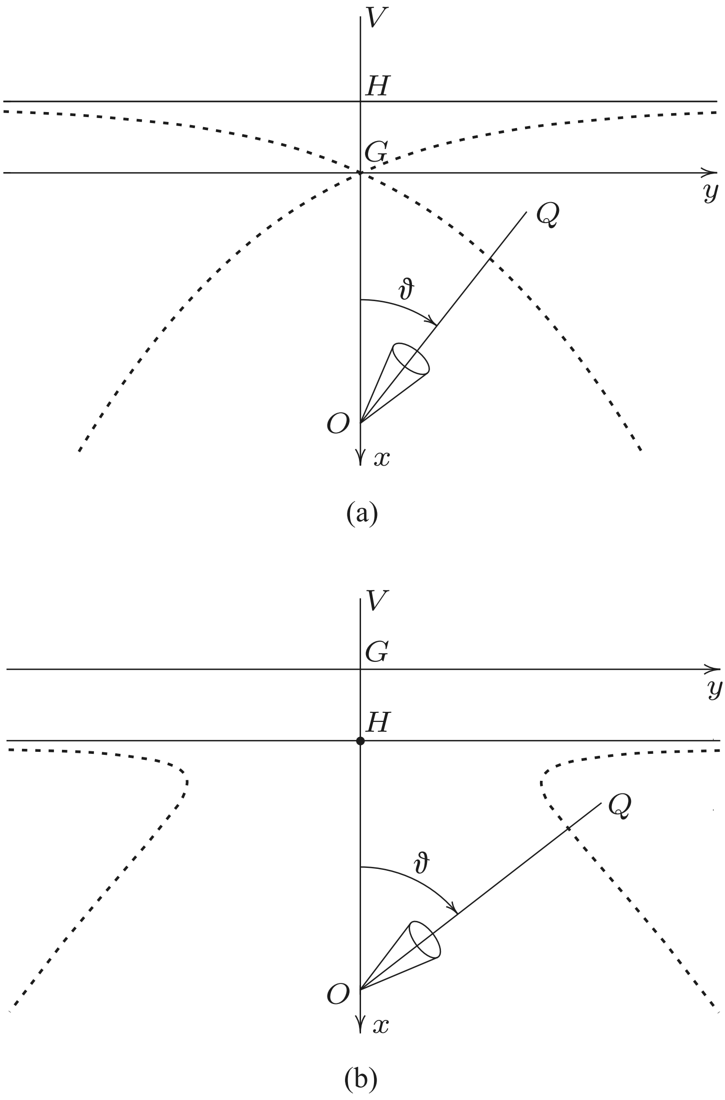


Fig. 167. Routh's geometric construction for determining the bounding positions of the figure axis of the top. (a) $c > 0$; (b) $c < 0$. "To avoid confusion in the figure, the body, which is represented by a top, is drawn smaller than it should be" [Routh 1884, pp. 113–115].

If

$$k_1 = \frac{2\sqrt{k}}{1+k},$$

then

$$dy = (1+k)^2 \frac{1-kx}{(1+kx)^3}\, dx,$$

$$1-y = \frac{(1-x)(1-k^2x)}{(1+kx)^2},$$

$$1-k_1^2y = \frac{(1-kx)^2}{(1+kx)^2},$$

so that

$$\int \frac{dy}{\sqrt{y(1-y)(1-k_1^2y)}} = (1+k)\int \frac{dx}{\sqrt{x(1-x)(1-k^2x)}}.$$

The reverse transformation for the trigonometric integral

$$F(k,\varphi) = \int \frac{d\varphi}{\sqrt{1-k^2\sin^2\varphi}}$$

is accomplished by the change of variables

$$\Big(\frac{1-k'}{1+k'}\Big)\sin\varphi_1 = \sin(2\varphi-\varphi_1),$$

or, equivalently,

$$\sin\varphi_1 = \frac{(1+k')\sin 2\varphi}{2\sqrt{1-k^2\sin^2\varphi}}.$$

For $0<\varphi<\dfrac{\pi}{2}$, the angle φ_1 should be chosen so that

$$0<\varphi_1<\frac{\pi}{2} \;\text{ if }\; 0<\varphi<\tan^{-1}\sqrt{\frac{1}{k'}}$$

and

$$\frac{\pi}{2}<\varphi_1<\pi \quad\text{if}\quad \tan^{-1}\sqrt{\frac{1}{k'}}<\varphi<\frac{\pi}{2},$$

where

$$\frac{\pi}{4}<\tan^{-1}\sqrt{\frac{1}{k'}}<\frac{\pi}{2}.$$

With this change of variables,

$$F(k,\varphi) = \frac{1}{1+k'}F(k_1,\varphi_1),$$

where

$$k_1 = \frac{1-k'}{1+k'}.$$

The English mathematician John Landen (1719–1790) developed his transformation while studying integral expressions for the lengths of elliptic and hyperbolic arcs. George Neville Watson (1886–1965) has written an interesting historical account of Landen's work and its role in the development of the theory of elliptic functions [Watson 1933].

116. (page 265) The cited place in the complete works of Gauss is the beginning of two posthumous papers [Gauss 1866] in which Gauss investigates the iterative equations

$$a_{n+1} = \frac{1}{2}(a_n + b_n), \quad b_{n+1} = \sqrt{a_n b_n}.$$

For given initial values a_1 and b_1, Gauss proves that a_n and b_n approach a common limit as $n \to \infty$; this limit is denoted by $M(a,b)$, and is called the arithmetic–geometric mean of a and b. The convergence to the limit $M(a,b)$ is generally very rapid. For $a_1 = \sqrt{2}$ and $b_1 = 1$, for example, Gauss gives the numerical values

$$\begin{array}{ll}
a_1 = 1.41421\ 35623\ 73095\ 04880\ 2 & b_1 = 1.00000\ 00000\ 00000\ 00000\ 0 \\
a_2 = 1.20710\ 67811\ 86547\ 52440\ 1 & b_2 = 1.18920\ 71150\ 02721\ 06671\ 7 \\
a_3 = 1.19815\ 69480\ 94634\ 29555\ 9 & b_3 = 1.19812\ 35214\ 93120\ 12260\ 7 \\
a_4 = 1.19814\ 02347\ 93877\ 20908\ 3 & b_4 = 1.19814\ 02346\ 77307\ 20579\ 8 \\
a_5 = 1.19814\ 02347\ 35592\ 20744\ 1 & b_5 = 1.19814\ 02347\ 35592\ 20743\ 9.
\end{array}$$

By applying the sequential changes of variables

$$\begin{aligned}
\sin\varphi_1 &= \frac{(1+k')\sin 2\varphi}{2\sqrt{1-k^2\sin^2\varphi}}, \\
\sin\varphi_2 &= \frac{(1+k_1')\sin 2\varphi_1}{2\sqrt{1-k_1^2\sin^2\varphi_1}}, \quad k_1 = \frac{1-k'}{1+k'}, \\
\sin\varphi_3 &= \frac{(1+k_2')\sin 2\varphi_2}{2\sqrt{1-k_2^2\sin^2\varphi_2}}, \quad k_2 = \frac{1-k_1'}{1+k_1'}, \quad \text{etc.}
\end{aligned}$$

to the elliptic integral

$$F(k,\varphi) = \int_0^{\varphi} \frac{d\varphi}{\sqrt{1-k^2\sin^2\varphi}},$$

it can be shown [Cayley 1876, pp. 324–332] that

$$F(k,\varphi) = \frac{1}{M(1,k')}\,\Phi, \quad \text{where} \quad \Phi = \lim_{n\to\infty} \frac{\varphi_n}{2^n}.$$

The angle φ_{n+1} should be chosen so that the pair of values $(\varphi_{n+1}, \varphi_n)$ lies in one of the hatched regions of Fig. 168. The value of k_n rapidly approaches 0; even for $k = 0.9999$, $k_4 = 0.0037$. When $k_n \approx 0$ (and therefore $k_n' \approx 1$), the iteration equation for φ is very nearly $\varphi_{n+1} = 2\varphi_n$. Cayley repeats a calculation of Legendre [Legendre 1825, p. 91] for the case $k = \sin 75°$, $k' = \cos 75°$, $\tan\varphi = \dfrac{\sqrt{2}}{\sqrt[4]{3}}$. Cayley presents his results in the following table [Cayley 1876, p. 335]:

	a	b	k	k'	φ
(0)	1.000 0000	0.258 8190	0.965 9258	0.258 8190	47° 3′ 31″
(1)	0.629 4095	0.508 7426	0.588 7908	0.808 2856	62° 36′ 3″
(2)	0.569 0761	0.565 8688	0.106 0200	0.994 3636	119° 55′ 48″
(3)	0.567 4724	0.567 4701	0.002 8260	0.999 9959	240° 0′ 0″
(4)	0.567 4713	0.567 4713	0.000 0020	0.999 9999	480° 0′ 0″

Thus Cayley concludes that

$$F(k,\varphi) \approx \frac{\varphi_4}{a_4 2^4} = 0.9226877,$$

which agrees with Legendre's value to seven decimal places.

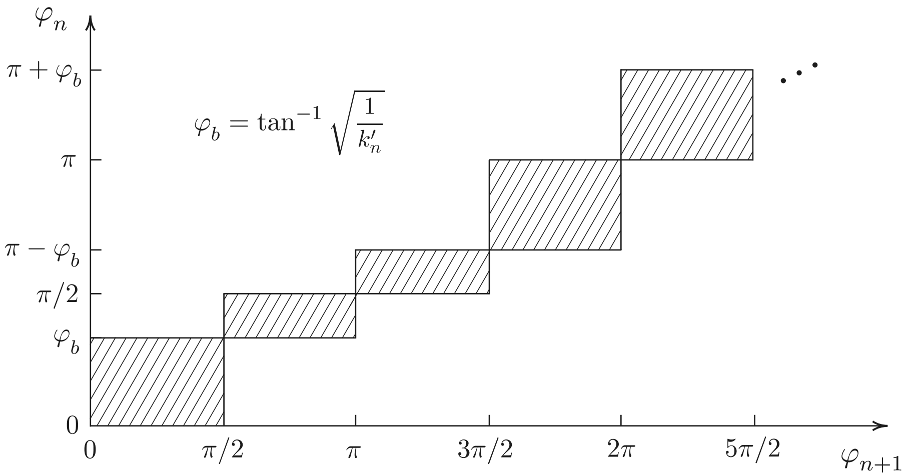


Fig. 168. Allowable regions for the pair of values $(\varphi_n, \varphi_{n+1})$ in the Landen transformation.

Gauss compiled a seven-decimal-place table of $M(1, \sin\theta)$ (and $\log_{10} M(1,\sin\theta) + 10$) for values of θ from $0°$ to $90°$ at intervals of $30'$ [Gauss 1866, p. 403]. Since $\Phi = \pi/2$ for $\varphi = \pi/2$, the complete elliptic integral of the first kind is given simply by

$$F(k, \frac{\pi}{2}) = \int_0^{\pi/2} \frac{d\varphi}{\sqrt{1 - k^2 \sin^2\varphi}} = \frac{\pi}{2M(1, k')}.$$

117. (page 265) Jacobi's paper [Jacobi 1881, pp. 31–36] is an extract from a letter written to Heinrich Christian Schumacher (1780–1850), the founder of the still active journal *Astronomische Nachrichten.* Jacobi begins with the elliptic integral of the first kind in the trigonometric form

$$\int \frac{d\varphi}{\sqrt{1 - cc \sin^2\varphi}}.$$

For any prime number n, he considers the change of variables

$$\sin\varphi = \frac{U}{V},$$

where U is a linear combination of odd powers of $\sin\psi$ up to the n^{th} degree, and V is a linear combination of even powers of $\sin\psi$ up the $(n-1)^{\text{st}}$ degree. Jacobi poses the problem of finding the coefficients in U and V so that

$$\int \frac{d\varphi}{\sqrt{1 - cc \sin^2\varphi}} = m \int \frac{d\psi}{\sqrt{1 - kk \sin^2\psi}},$$

and gives explicit results for the cases $n = 3$ and $n = 5$. Jacobi, who at the time had not seen Legendre's tables, also gives a method for determining the incomplete function $F(k, \varphi)$ in terms of the complete function $F\left(k, \dfrac{\pi}{2}\right)$ and certain angles that divide the complete function evenly. Jacobi denotes these angles by φ', φ'', …; they satisfy the equations

$$F(k, \varphi^{(m)}) = \frac{m}{p} F(k, 90°), \;\; m = 1, 2, \ldots, p-1,$$

where p is any odd number. Legendre shows how these angles can be computed in a simple way [Legendre, 1825, pp. 19–31]. Jacobi then defines the angle ψ by the equation

$$\operatorname{tg}(45° - \tfrac{1}{2}\psi) = \frac{\operatorname{tg}\frac{1}{2}(\varphi' - \varphi)}{\operatorname{tg}\frac{1}{2}(\varphi' + \varphi)} \cdot \frac{\operatorname{tg}\frac{1}{2}(\varphi''' + \varphi)}{\operatorname{tg}\frac{1}{2}(\varphi''' - \varphi)} \cdots \frac{\operatorname{tg}\frac{1}{2}(\varphi^{(p-2)} \pm \varphi)}{\operatorname{tg}\frac{1}{2}(\varphi^{(p-2)} \mp \varphi)} \cdot \operatorname{tg}(45° \mp \tfrac{1}{2}\varphi),$$

and states (Jacobi's papers seem to be full of such astounding statements) that

$$F(k, \varphi) = \mu F(\lambda, \psi),$$

where

$$\mu = \frac{1}{2(\operatorname{cosec}\varphi' - \operatorname{cosec}\varphi''' + \cdots \mp \operatorname{cosec}\varphi^{(p-2)} \pm \frac{1}{2})},$$

$$\lambda = 2k\mu(\sin\varphi' - \sin\varphi''' + \cdots \mp \sin\varphi^{(p-2)} \pm \tfrac{1}{2}).$$

The upper signs in the equations for ψ, μ, and λ should be chosen if p is of the form $4n+1$, and the lower signs if p is of the form $4n-1$. The angle ψ should be taken between $\dfrac{m\pi}{2}$ and $\dfrac{(m+1)\pi}{2}$ if φ lies between $\varphi^{(m)}$ and $\varphi^{(m+1)}$. The modulus λ is always very small compared to k, so that

$$F(k, \varphi) \approx \mu\psi,$$

and the constant μ can be approximated by $\mu = \dfrac{2}{p\pi}F(k, 90^\circ)$, with the correction $\dfrac{\mu\lambda\lambda}{8}\sin 2\psi$. Expressing ψ in seconds and putting $\mu' = \dfrac{F(k, 90^\circ)}{324000 \cdot p}$, the formula for the calculation of $F(k, \varphi)$ becomes

$$F(k, \varphi) \approx \mu'\psi.$$

For the case $k = \sin 45^\circ$ and $p = 5$, Jacobi quotes Legendre's values

$$\begin{aligned} \varphi' &= 21^\circ\ \ 0'\ 36'', 02754\ 43 \\ \varphi''' &= 58^\circ\ 38'\ 10'', 31402\ 70 \\ F(k, 90^\circ) &= 1{,}85407\ 46773\ 01, \end{aligned}$$

and uses these values to derive the formulas

$$\begin{aligned} \mu' &= 0{,}00000\ 11444\ 90541\ 544 \\ \operatorname{tg}\tfrac{1}{2}(90^\circ - \psi) &= \frac{\operatorname{tg}(10^\circ 30' 18'', 01 - \frac{1}{2}\varphi)}{\operatorname{tg}(10^\circ 30' 18'', 01 + \frac{1}{2}\varphi)} \cdot \frac{\operatorname{tg}(29^\circ 19' 5'', 16 + \frac{1}{2}\varphi)}{\operatorname{tg}(29^\circ 19' 5'', 16 - \frac{1}{2}\varphi)} \cdot \operatorname{tg}(45^\circ - \tfrac{1}{2}\varphi) \\ F(k, \varphi) &= 0{,}00000\ 11444\ 90541 \cdot \psi \\ \text{correction} &= -0{,}00000\ 007 \cdot \sin 2\psi. \end{aligned}$$

Jacobi then completes the calculation for $\varphi = 30°$ in the following manner:

$$\begin{array}{rl}
\log\operatorname{tg}\ 4°\,29'\,41,''99 &= 8{,}89549\ \ 90\ n\\
\log\operatorname{tg} 44°\,19'\ \ 5,''16 &= 9{,}98966\ \ 16\\
\text{Compl.}\log\operatorname{tg} 25°\,30'\,18,''01 &= 0{,}32140\ \ 63\\
\text{Compl.}\log\operatorname{tg} 14°\,19'\ \ 5,''16 &= 0{,}59306\ \ 27\\
\log\operatorname{tg} 30°\ \ 0'\ \ 0,''00 &= \underline{9{,}76143\ \ 94}\\
\log\operatorname{tg}(45° - \tfrac{1}{2}\psi) &= 9{,}56106\ \ 90\ n\\
45° - \tfrac{1}{2}\psi &= -20°0'0,''47\\
\psi &= 468000,''95\\
\mu'\psi &= 0{,}53562\ \ 266\\
\text{Correction} & \underline{\qquad\qquad\ +7}\\
F(k,\varphi) &= 0{,}53562\ \ 273\\
\text{M. Legendre finds} & \ 0{,}53562\ \ 27328\ 22
\end{array}$$

Working through the detailed calculations of Gauss, Legendre, Cayley, and Jacobi, one obtains a sense of the awe-inspiring intellectual tradition that could make possible a book such as the *Theorie des Kreisels*.

118. (page 265) Karl Heinrich Schellbach (1805–1892) was professor of mathematics in the Friedrich-Wilhelm-Gymnasium and the *Kriegsakademie* in Berlin, and was a founder, with Ernst Werner von Siemens (1816–1892) and Hermann von Helmholtz (1821–1894), of the *Physikalisch-Technischen Reichsanstalt*, the German national standards laboratory now called the *Physikalisch-Technischen Bundesanstalt*. In his *Lehre von den Elliptischen Integralen und den Theta-Functionen*, Schellbach writes the elliptic integral of the first kind as

$$u = \int_0^\varphi \frac{d\varphi}{\sqrt{1 - k^2\sin^2\varphi}},$$

and derives expressions for φ as a function of u in terms of the Jacobi theta-functions. One of his results, written in terms of the ϑ-function used by Klein and Sommerfeld in Chap. VI, is [Schellbach 1864, p. 63]

$$\begin{aligned}
\frac{\sqrt{1 - k^2\sin^2\varphi}}{\sqrt{k'}} &= -i\frac{\vartheta(u + K - K'i)}{\vartheta(u - K'i)}\\
&= \frac{1 + 2q\cos 2x + 2q^4\cos 4x + 2q^9\cos 6x + \cdots}{1 - 2q\cos 2x + 2q^4\cos 4x - 2q^9\cos 6x + \cdots},
\end{aligned}$$

where

$$q = e^{-\frac{K'}{K}\pi}, \quad x = \frac{\pi u}{2K},$$

$$K = \int_0^{\pi/2} \frac{d\varphi}{\sqrt{1 - k^2 \sin^2 \varphi}}, \quad K' = \int_0^{\pi/2} \frac{d\varphi}{\sqrt{1 - k'^2 \sin^2 \varphi}}.$$

By truncating the series expansions of the ϑ-functions at the q^9 terms and performing a series of algebraic and trigonometric transformations, Schellbach derives the remarkably simple approximation formula

$$\cos \frac{\pi u}{K} = \frac{1}{2\lambda} \mathrm{tg}\left(\tfrac{1}{4}\pi - \delta\right) - 2(\lambda^3 + 5\lambda^7) \mathrm{tg}\left(\tfrac{1}{4}\pi - \delta\right) \sin^2 \frac{\pi u}{K}, \tag{2}$$

where

$$\lambda = \tfrac{1}{2} \mathrm{tg}^2 \tfrac{1}{2}\beta, \quad k = \sin \alpha,$$

$$\sqrt{k'} = \cos \beta, \quad \sin \alpha \sin \varphi = \sin \gamma, \quad \frac{\cos \beta}{\cos \gamma} = \mathrm{tg}\, \delta.$$

Retaining only the first term on the right-hand side of equation (2) gives six-decimal-place accuracy for u if k is less than about 0.7. For larger values of k, Schellbach gives an improved solution of equation (2) in the form

$$u = u_1 + \frac{2K}{\pi}(\lambda^3 + 5\lambda^7) \mathrm{tg}\left(\tfrac{1}{4}\pi - \delta\right) \sin \frac{\pi u_1}{K},$$

where u_1 is the value obtained by retaining only the first term on the right-hand side of (2). We have verified that this improved approximation gives six-decimal-place accuracy for u when k is as large as 0.97. Schellbach also develops specific alternative methods to calculate u for values of k near 1.

119. (page 265) The Legendre tables were not reprinted until 1931 [Emde 1931] and 1934 [Pearson 1934]. A review of tables of elliptic integrals and elliptic functions that appeared before the year 1948 is given by Alan Fletcher [Fletcher 1948].

120. (page 266) Since the values of e'' in the table on p. 267 are greater than or equal to 1, the table should, according to Fig. 38 on page 226, be labeled $P = +1$.

121. (page 268) Factors of two are missing in the denominators of the two terms on the right-hand side of the following equation for ψ. These factors of two should also be included in the definitions of C_1 and C_2 below.

122. (page 268) According to the definition of the normal integral of the third kind on page 267, the quantities p_1 and p_2 should be reversed in sign.

123. (page 269) Legendre defines his elliptic integrals of the first, second, and third kinds as

$$F(c,\varphi) = \int \frac{d\varphi}{\sqrt{1 - c^2 \sin^2 \varphi}},$$

$$E(c,\varphi) = \int \sqrt{1 - c^2 \sin^2 \varphi}\, d\varphi$$

$$\Pi(n,c,\varphi) = \int \frac{d\varphi}{(1 + n \sin^2 \varphi)\sqrt{1 - c^2 \sin^2 \varphi}}.$$

The corresponding complete integrals (the integrals with lower limit 0 and upper limit $\frac{\pi}{2}$) are denoted by $F^1(c)$, $E^1(c)$, and $\Pi^1(n,c)$. Legendre also uses the abbreviations $b = \sqrt{1 - c^2}$ and $\Delta(c,\varphi) = \sqrt{1 - c^2 \sin^2 \varphi}$. Legendre derives the following expressions for the complete integral of the third kind in terms of the integrals of the first and second kind [Legendre 1825, Chap. XXIII]:

First case: $n > 0$.

If $n > 0$, let $n = \cot^2 \theta$, $0 < \theta < \pi/2$. Then

$$\frac{\Delta(b,\theta)}{\sin\theta\cos\theta}\Pi^1(n,c) = \frac{\pi}{2} + \frac{\sin\theta}{\cos\theta}\Delta(b,\theta)F^1(c) + F^1(c)F(b,\theta) - F^1(c)E(b,\theta) - E^1(c)F(b,\theta).$$

Second case: $-1 < n < -c^2$.

If $-1 < n < -c^2$, let $n = -1 + b^2 \sin^2 \theta$, $0 < \theta < \pi/2$. Then

$$\frac{b^2 \sin\theta\cos\theta}{\Delta(b,\theta)}[\Pi^1(n,c) - F^1(c)] = \frac{\pi}{2} + F^1(c)F(b,\theta) - E^1(c)F(b,\theta) - F^1(c)E(b,\theta).$$

Third case: $-c^2 < n < 0$.

If $-c^2 < n < 0$, let $n = -c^2 \sin^2 \theta$, $0 < \theta < \pi/2$. Then

$$\Pi^1(n,c) = F^1(c) + \frac{\tan\theta}{\Delta(c,\theta)}[F^1(c)E(c,\theta) - E^1(c)F(c,\theta)].$$

We are not sure that these formulas would allow us to find the azimuth angle ψ_ω "relatively quickly."

124. (page 269) It is likely that "Herr stud. math. Blumenthal," as he is called in the original, is Ludwig Otto Blumenthal (1876–1944), who studied mathematics in Göttingen under Hilbert, Klein, and Sommerfeld from 1894 to 1898. Blumenthal was professor of mathematics in Aachen from 1905 until his expulsion in 1933. He died in the concentration camp at Theresienstadt in 1944. In 1951, Sommerfeld wrote a memorial in which he describes, in very warm terms, his association with Blumenthal in Göttingen and Aachen [Sommerfeld 1951]. The Aachen mathematicians Paul Butzer and Lutz Volkmann have recently published a detailed review of Blumenthal's mathematical work [Butzer 2006].

125. (page 269) The trajectories in Sections 1 and 2 of Chap. IV were apparently constructed in the following manner. After setting $e = 0$ and finding the roots e' and e'' of the quadratic U_1, a unit circle is first drawn to represent the equator of the unit sphere, and a concentric circle with radius

$$r' = \tan\left(\frac{1}{2}\cos^{-1} e'\right) = \sqrt{\frac{1-e'}{1+e'}}$$

is then drawn to represent the second bounding parallel circle of the apex of the top. The angle ψ_ω is next calculated from the formula

$$\begin{aligned}\psi_\omega &= \int_0^{e'} \frac{n - Nu}{A(1-u^2)} \frac{du}{\sqrt{U}} \\ &= C_1 \Pi\left(k, \frac{\pi}{2}, p_1\right) + C_2 \Pi\left(k, \frac{\pi}{2}, p_2\right),\end{aligned}$$

where C_1, C_2, p_1, p_2 are defined (see the corrections in notes 121 and 122) on page 268. Having calculated the angle ψ_ω, one can plot the initial equatorial point A, the first point H on the inner bounding circle, and the second equatorial point B, as in Fig. 169. The first segment AHB of the trajectory curve can then be sketched in. The segment AHB should be tangent to the inner circle at H and symmetric with respect to the line $\psi = \psi_\omega$. The character of the curve at the points A and B depends on the values of n and N. In Fig. 169, which is drawn for the numerical values corresponding to Fig. 27, the curve has cusps that are perpendicular to the outer circle at A and B. The continuation arcs BA', $A'B'$, and $B'A''$ are all congruent to the initial arc AB.

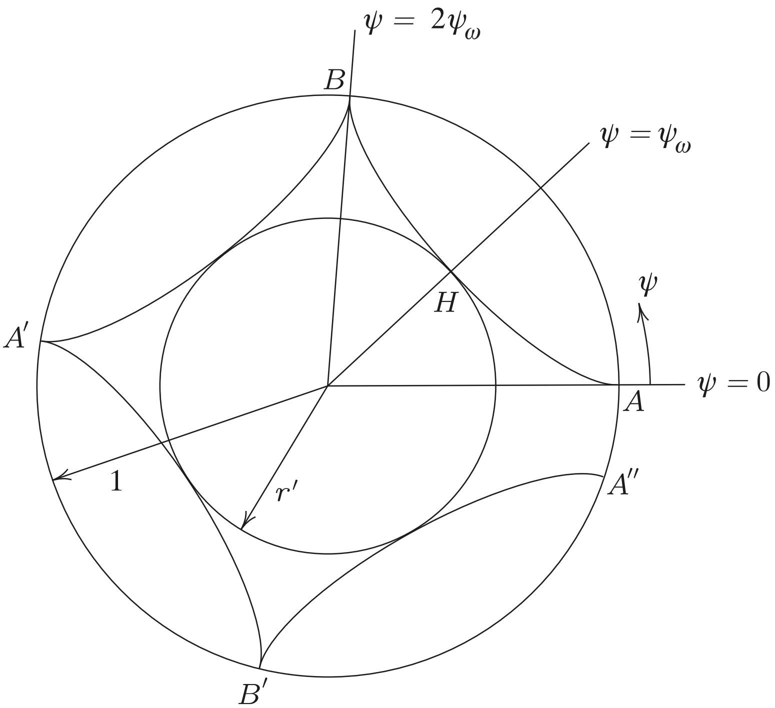


Fig. 169. Construction of the trajectory of the apex of the top.

The construction for the trajectory of the apex of the top is a good example of the ingenious graphical procedures that are often found in nineteenth-century works on mechanics.

126. (page 269) We have used the ϑ-quotient formulas given in Chap. VI, together with the numerical data on pages 243 and 247, to check the figures of Sections 1 and 2 in detail. The figures are generally quite accurate. The largest error occurs in Fig. 26 on page 208, in which the span width angle ψ_ω is noticeably too small. A small discrepancy in the angle ψ_ω can also be seen in the comparison of Fig. 25 on page 205 and our recomputation in Fig. 165(b).

127. (page 273) The value of the fraction

$$\frac{n^2 + N^2 - 2nNe}{2AP(1 - e^2)}$$

in the limit $e \to \pm 1$ is discussed further on pp. 338–339.

128. (page 277) As an illustration of Klein and Sommerfeld's approximate solution for the trajectory of the apex of the top, we show in Fig. 170 the exact (computed from the ϑ-function representation in Chap. VI) and the approximate (computed from equation (8′) on p. 272 and equation (14′) on p. 276) trajectories for the case $A = 1$, $P = 1$, $n = -0.1$, $N = 3.5$, and $e' = 0$. The trajectories are shown in the stereographic projection from the north pole of the unit sphere. The initial time is chosen so that the apex of the top is at the equator at $t = 0$. For the given numerical values,

$$e = -0.2128, \quad \varepsilon = \frac{e' - e}{2} = 0.1064.$$

The locations of the bounding circles and the qualitative form of the trajectory are captured accurately by the approximate solution. The fractional error in the azimuth angle ψ is proportional to ε, but the absolute error grows linearly with time due to the approximate coefficient of t in equation (14′).

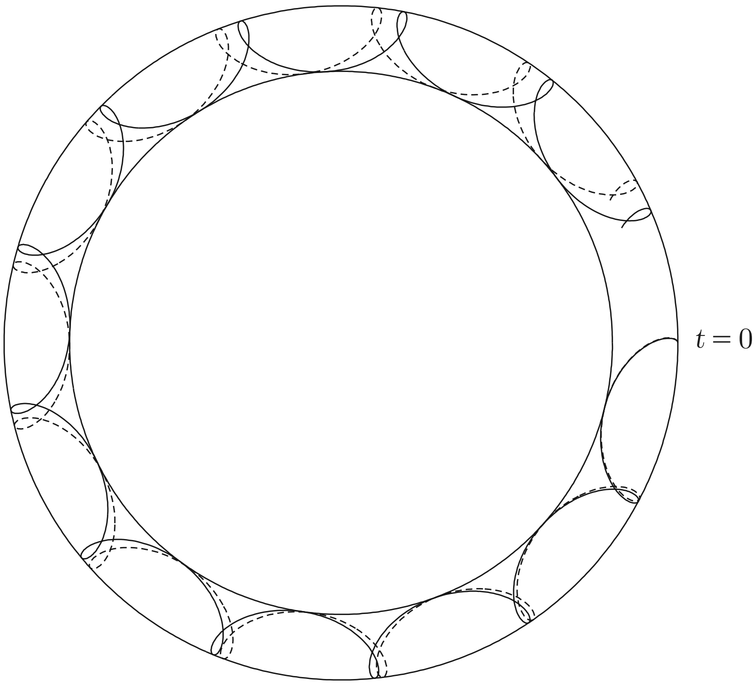


Fig. 170. Exact and approximate trajectories of the apex of the top for the case $A = 1$, $P = 1$, $n = -0.1$, $N = 3.5$, and $e' = 0$.
—— : exact; – – – : approximate.

129. (page 281) The theory of line elements, which is derived from Hermann Grassmann's *Ausdehnungslehre* [Grassmann 1844] and which contributed to the development of modern vector analysis, is discussed by Klein in the second volume of his *Elementarmathematik vom höheren Standpunkte aus* [Klein 1914, Chaps. I–III].

130. (page 282) Singular solutions also exist for the energy equation

$$\dot{x}^2 + x^2 = 2E, \tag{3}$$

which corresponds to the linear oscillator equation

$$\ddot{x} + x = 0. \tag{4}$$

Equation (3) has the constant solutions $x = \pm\sqrt{2E}$ that are not solutions of (4). These constant solutions are the envelopes of the continuous family of sinusoidal solutions of (3) and (4) that have energy E; that is, the solutions

$$x = \sqrt{2E}\sin(t + \phi),$$

where ϕ is an arbitrary phase angle. In fact, the energy equation (3) also has singular solutions of the type illustrated in Fig. 171, in which constant segments $x = \pm\sqrt{2E}$ are joined to the sinusoidal solutions of (3) and (4). The possibility of these composite singular solutions was pointed out to one of the translators (GS) by Sommerfeld's student Werner Heisenberg (1901–1976).

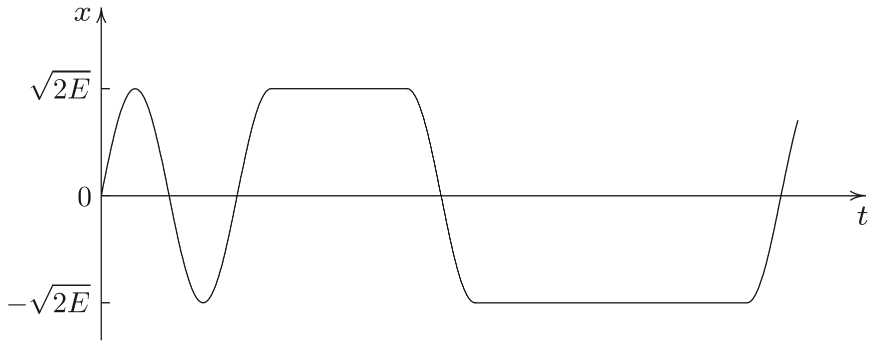


Fig. 171. Singular solution of the energy equation $\dot{x}^2 + x^2 = 2E$.

131. (page 298) Möbius considers the following class of epicycloidal motions [Möbius 1887, pp. 60–74]. A point A moves on a circle of

radius a about the fixed point O with constant angular velocity n. In the plane of this circle, a second point A_1 moves on a circle of radius a_1 about the moving point A with constant angular velocity n_1. A third point A_2 moves in the same plane on a circle of radius a_2 about the moving point A_1 with constant angular velocity n_2, etc. Möbius considers the trajectory of the point A_n, and shows, for example, that it is easy to choose the parameters of the motion so that the point A_2 moves on an *ellipse*. He then considers small higher-order terms in the epicycloidal series, and later uses these terms to analyze the small orbital perturbations caused by the gravitational interactions among the planets.

132. (page 306) After deriving the differential equations for the Euler angles ϑ, ψ, and φ in the case of the heavy symmetric top, Poisson simplifies and solves these differential equations for the two special cases in which (1) the angle ϑ is assumed to be always very small; and (2) the angle ϑ is assumed to be always very close to its initial value. As an illustration of the second case, Poisson mentions the Bohnenberger machine (cf. Vol. I, p. 2), which, as Poisson says, "faithfully represents all circumstances of this rotational motion" [Poisson 1833, p. 178].

Kirchhoff solves the differential equations for the top for the special case in which the initial angular velocity consists only of the component r about the figure axis, and then derives approximate expressions for the Euler angles ψ, ϑ, and φ by letting the initial value of r become infinitely large [Kirchhoff 1883, pp. 72–74].

No attempt at an error estimation is made by either Poisson or Kirchhoff. Klein and Sommerfeld's criticism of purely analytic presentations seems particularly applicable, in this case, to Kirchhoff. He derives his results systematically, but with no discussion or interpretation.

133. (page 307) Perry compares the top to both a pig and a crab: "If I try to make a very quickly spinning body change the direction of its axis, the direction of the axis will change, but not in the way I intended. It is even more curious than my countryman's pig, for when he wanted the pig to go to Cork, he had to pretend he was driving the pig home. His rule was a very simple one, and we must find a rule for our spinning body, which is rather like a crab, that will only go along the road when you push it sidewise" [Perry 1957, pp. 17–18].

134. (page 308) A *quaternio terminorum* is a logical error in which four terms (instead of the required three) are used in a categorical

syllogism [Howard 1970, p. 98]. The error often depends, as in this case, on the equivocal use of a word.

135. (page 309) Louis-Philippe Gilbert (1832–1892) was professor of mathematics and mechanics in the Catholic University of Leuven. His ninety-five page paper [Gilbert 1878] is a delightful account of theoretical and experimental investigations of rigid-body rotation, beginning with D'Alembert and ending with a reference to Hermite. Gilbert describes the experiments of Foucault and the horologist Etienne-Georges Sire (1826–1906), and discusses the papers in which Foucault and Sire are guilty of a "grave confusion" between the rotation axis and the figure axis. Gilbert's own instrument, the baryscope, is described by Klein and Sommerfeld in Chap. VIII, §9.

136. (page 309) The note by Oberlehrer Dr. Franke of Schleusingen (possibly Hermann Franke (1847–1932)) in the *Zeitschrift für mathematischen und naturwissenschaftlichen Unterricht* [Franke 1886] was written in response to a previous paper [Hauck 1886a] by Privy Councillor Prof. Dr. Guido Hauck (1845–1905), professor of descriptive geometry and eventual rector of the *Technische Hochschule* in Berlin. After deriving the conical motion of the figure axis of the top by a specious application of the parallelogram law, Hauck places his false result into the context of a duality between the motion of the top and the circular motion of a particle in a central force field. This duality, in turn, is cited as an example of a more general "reciprocity law of space."

In a reply to Franke's note, Hauck admits his error, but states that his treatment of the motion of the top is sufficiently accurate under the usual conditions [Hauck 1886b]. He does not tell us how the usual conditions affect his general reciprocity law of space.

Hauck's more successful works include a study of subjective perspective in ancient Greek architecture [Hauck 1879] and the editing of a posthumously published theory of scientific knowledge by Paul du Bois-Reymond [du Bois-Reymond 1890].

137. (page 309) In the fourth (1858) edition of Airy's *Mathematical Tracts*, a restriction to a mechanically spherical or symmetric body is added. Klein and Sommerfeld refer to the (then sixty-six-year-old) second edition [Airy 1831].

138. (page 309) Airy poses and (correctly) solves the equation that would be written in modern notation as

$$\frac{d\boldsymbol{\omega}}{dt} = \mathbf{a} \times \boldsymbol{\omega},$$

where $\mathbf{a}$ is a constant vector that is perpendicular to the angular velocity $\boldsymbol{\omega}$. But this equation has no basis in the kinetics of the top.

139. (page 311) The problem of turning the axis of the symmetric top was originally posed and solved incorrectly by J. Wanka of Fiume (now Rijeka, Croatia). The correct result was given by Max Koppe (1853–??), a teacher in the Andreas-Gymnasium in Berlin [Wanka 1896; Koppe 1896].

A detailed derivation of the solution described in words by Klein and Sommerfeld may be carried out as follows. In Fig. 172, the xyz-axes are fixed in space, and the XYZ-axes are the principal inertial axes of the top. The Z-axis coincides with the figure axis OF. The Euler angles ψ, ϑ, φ define the orientation of the XYZ-axes with respect to the xyz-axes. For the given problem, ψ and $\dot{\varphi}$ are constant. The angular velocity $\boldsymbol{\omega}$ of the top is then (cf. Vol. I, p. 45)

$$\boldsymbol{\omega} = \dot{\vartheta}\cos\varphi\,\mathbf{u}_X - \dot{\vartheta}\sin\varphi\,\mathbf{u}_Y + \dot{\varphi}\,\mathbf{u}_Z,$$

and the angular momentum $\mathbf{H}$ of the top is

$$\mathbf{H} = A\dot{\vartheta}\cos\varphi\,\mathbf{u}_X - A\dot{\vartheta}\sin\varphi\,\mathbf{u}_Y + C\dot{\varphi}\,\mathbf{u}_Z.$$

The moment $\mathbf{M}$ that is necessary to produce the motion is therefore

$$\begin{aligned}\mathbf{M} &= \frac{d\mathbf{H}}{dt} = \left(\frac{d\mathbf{H}}{dt}\right)_{XYZ} + \boldsymbol{\omega}\times\mathbf{H}\\ &= A\ddot{\vartheta}(\cos\varphi\,\mathbf{u}_X - \sin\varphi\,\mathbf{u}_Y) + C\dot{\varphi}\dot{\vartheta}(-\sin\varphi\,\mathbf{u}_X - \cos\varphi\,\mathbf{u}_Y)\\ &= A\ddot{\vartheta}\,\mathbf{u}_{OH_1} + C\dot{\varphi}\dot{\vartheta}\,\mathbf{u}_{OG}.\end{aligned}$$

The two terms in this expression for $\mathbf{M}$ correspond to the two items listed by Klein and Sommerfeld at the top of page 311. If, beginning and ending at rest, the angle ϑ changes by $d\vartheta$ in the time interval dt, then the time integral of $\ddot{\vartheta}$ over the interval dt vanishes, and there is no net moment about the axis OH_1. The second term in the direction of OG is then the required moment.

140. (page 312) The *Connaissance des Temps* was first published in 1679 by the French priest and astronomer Jean Picard (1620–1682), who measured the arc length of one degree of the Earth meridian through Paris to an accuracy of 100 feet. In the nineteenth century, the *Connaissance des Temps* was primarily an ephemeris, a table of the positions of the heavenly bodies in the sky at specified times. Poinsot's long article on the precession of the rotation axis of the Earth [Poinsot 1858] was published in the *Additions à la Connaissance des Temps* for the year 1858.

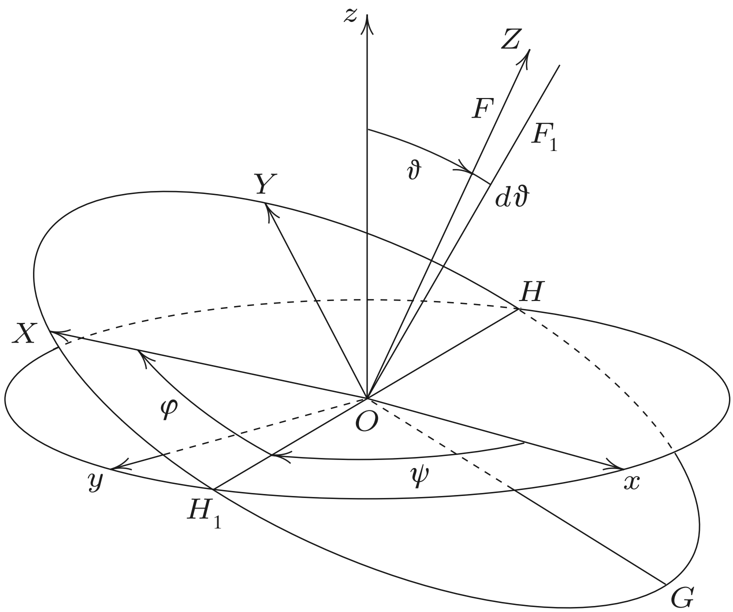


Fig. 172. Euler angles ψ, ϑ, φ for the specification of the orientation of the body axes XYZ with respect to the spatial axes xyz.

141. (page 312) Carl August von Schmidt (1840–1929) was professor of physics in the Realgymnasium in Stuttgart. The purpose of his paper [Schmidt 1886] is to develop a presentation of the motion of the top that is accessible to students of elementary physics. Schmidt discusses the alleged reciprocity law of his "very esteemed friend" Guido Hauck, and concludes that the circular motion of a particle under a central force and the regular precession of a top under the action of gravity are not actually reciprocal, but only analogous.

142. (page 313) Viktor von Lang (1838–1921) was professor of physics in the University of Vienna. He cites and closely follows the Airy explanation, and gives a very brief qualitative discussion of its application to the Bohnenberger (cf. Vol. I, p. 2) and the Fessel (cf. note 144 below) machines. In a remarkable act of conscious or unconscious self-contradiction, he then states the evident experimental fact that "the rotation axis begins to describe a wavy surface only if the rotation [presumably the rotation about the figure axis] becomes very slow, or the gravitational imbalance very large" [von Lang 1867, p. 65].

143. (page 313) Ernest Jean Philippe de Fauque de Jonquières (1820–1901) was a student of Chasles who became a vice-admiral in the French navy. He spent several years as part of the French colonial occupation force in Vietnam. His paper [de Jonquières 1886] uses Poinsot's concept of the *couple d'impulsion* to give a clear, correct, and primarily qualitative description of the motion of the top.

144. (page 313) The mechanical device considered by Johann Christian Poggendorff (1796–1877) is called the Fessel machine. Its original form is shown in Fig. 173. When the disk A does not rotate and the ring C is released from a stationary position in which the axis B is horizontal, the ring C rotates downward about a horizontal axis through the hinge D. When the disk A is given a rapid rotation, however, the axis B remains essentially horizontal when the ring C is released, and the ring C begins to rotate about the vertical axis EG. If the disk A spins rapidly but the axis B is fixed in a horizontal position by locking the hinge D, the released ring C does *not* rotate about the vertical axis EG.

Friedrich Fessel (1821–??) was once an assistant in the physical laboratory of Julius Plücker in Bonn [Schubring 1989, p. 79]. In a paper that appeared shortly before Poggendorff's, Plücker writes that Fessel "was formerly a teacher in the provincial technical school, but is now, through the favor or disfavor of circumstances, wholly dependent on his art as a mechanic" [Plücker 1853].

Plücker suggested some improvements to the Fessel machine; the new configuration, which quickly became a common instrument for demonstrating the properties of rigid-body motion, is shown in Fig. 174. Plücker also quotes a letter in which Fessel describes the original development and subsequent modification of his machine:

> I have let the new apparatus rotate, according to your advice, on a fixed point. The experiment succeeded *most magnificently.* Should you wish to say something about the origin of the device in the notice for the Poggendorff journal, please use the following suggestions. Two years ago, I let the 24-inch-diameter flywheel of a steam engine rotate between my hands, in order to see whether the assistant had fashioned the wheel correctly.
>
> I then felt that the plane of the wheel was fixed during the rotation, and that one could remove one hand without the wheel (now supported only on a pivot) falling. I

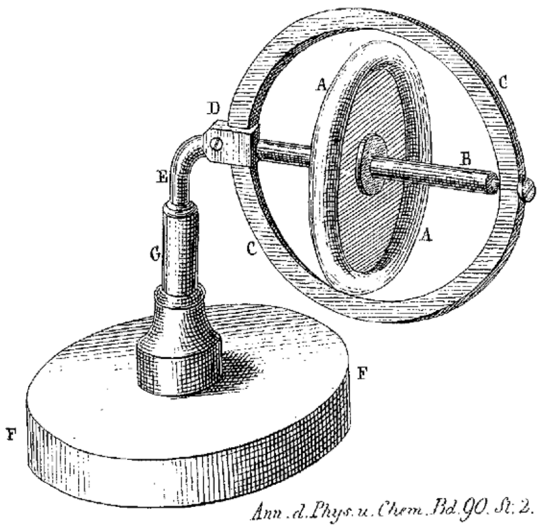


Fig. 173. Original form of the Fessel machine
[Poggendorff 1853, Taf. II].

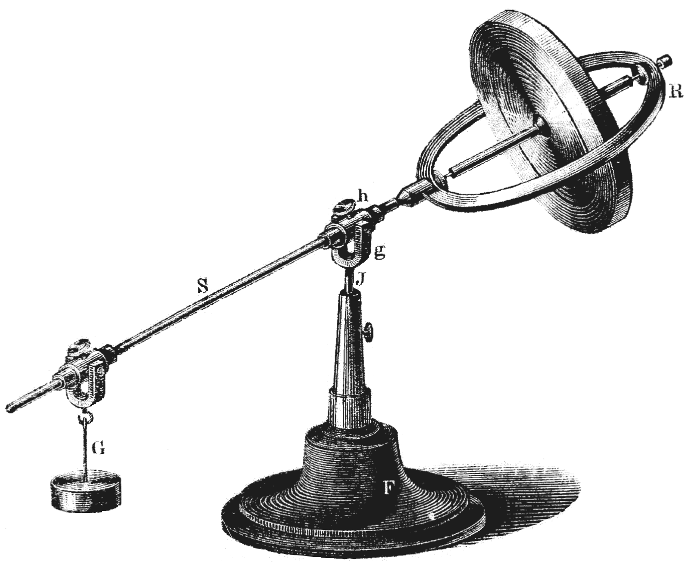


Fig. 174. Plücker's modification of the Fessel machine
[Heinen 1857, p. 5].

> ascribed the apparent rotation in a horizontal plane to the elevation of the wooden axle in the palm of my hand. To prevent this elevation, I let an ellipsoid rotate in the ring of a precisely constructed apparatus, and supported the ellipsoid alternately by two projecting pins that formed an elongation of its axis, and therefore did not themselves rotate. But it soon became apparent that the rotation in a horizontal plane was not *incidental*, but rather *essential*. Various accidents that the apparatus suffered in these experiments caused it to be set aside until I very recently assigned to a momentarily unoccupied worker its renovation and simultaneous alteration, which succeeded completely.

145. (page 315) Claude Servais Mathias Pouillet (1791–1868) published his *Éléments de physique expérimentale et de météorologie* in 1827. A German edition by Johann Heinrich Jacob Müller (1809–1875) was published in 1842, and a five-volume eleventh edition of the Müller–Pouillet *Lehrbuch der Physik* appeared as late as 1929.

146. (page 315) Koppe's comments on Von Lang, Poggendorff, and Poinsot are very similar to those of Klein and Sommerfeld. He mentions the solution of the differential equations of the top in terms of elliptic functions, and states that "a clear insight into the basis of the motion cannot be obtained in this way, for even if the differential equations permit of a brief interpretation, the passage to their integrals, which depends on all possible properties of elliptic functions, cannot be followed intuitively. It is indeed possible, through apparently allowable operations, to arrive at analytic results that impede the mechanical comprehension" [Koppe 1890].

147. (page 315) Franz Heinen (1807–1870) was director of the *Realschule* in Düsseldorf. The most interesting parts of his attractive little book [Heinen 1857] are the descriptions of experiments with the Bohnenberger and Fessel machines. Heinen added a rotational degree of freedom to the rotor of the Fessel machine, as illustrated in Fig. 175. A similarly modified Fessel machine was presented to the Royal Society of London in 1854 by Charles Wheatstone (1802–1875) [Wheatstone 1855].

148. (page 315) See the supplementary note by Klein and Sommerfeld on p. 533.

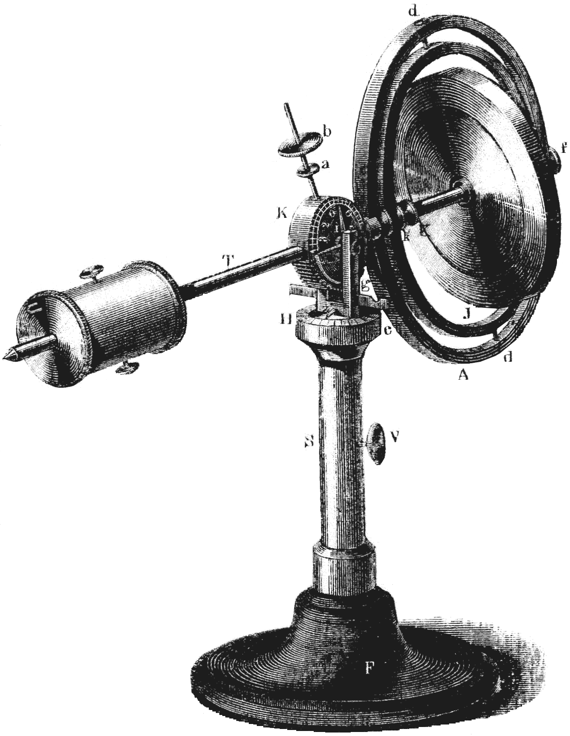


Fig. 175. Heinen's modification of the Fessel machine
[Heinen 1857, p. 7].

149. (page 315) The creative explanation by Dr. Munter of Herford, in which the rigidity of the top is blatantly disregarded, was immediately criticized by Franke and Schmidt [Munter 1895; Franke 1895; Schmidt 1895].

150. (page 322) Klein's paper on the stability of the sleeping top [Klein 1897] was presented in an address to a meeting of the American Mathematical Society in Princeton, New Jersey, on Saturday, October 17, 1896. Earlier in the same week, Klein had delivered four lectures on the theory of the top as part of the sesquicentennial celebration of Princeton University (cf. note 194 below).

151. (page 337) See the supplementary note by Klein and Sommerfeld on p. 533.

152. (page 341) See the supplementary note by Klein and Sommerfeld on p. 533.

153. (page 342) The fourth lecture of Jacobi's *Dynamik* is devoted to the principle of the conservation of energy for a system of particles. Jacobi shows how the principle may be used to deduce some general properties of the motion of the planets in the solar system, and then discusses the stability analysis of Laplace [Jacobi 1884, pp. 29–31]:

> In this and similar considerations lies the kernel of the famous investigations of *Laplace*, *Lagrange*, and *Poisson* on the stability of the system of the world. There exists, namely, the theorem that if one assumes the elements of one planetary trajectory to be variable and expands the major axis in terms of the time, then the time appears only as the argument of periodic functions; there are no terms proportional to the time. *Laplace* first proved this theorem only for small eccentricities and the first power of the [planetary] masses. *Lagrange* extended it [Lagrange 1808] with a stroke of the pen to arbitrary eccentricities. *Poisson*, finally, proved that it is also valid if the second power of the masses is considered [Poisson 1809]; this work is one of his most beautiful. With consideration of the third power of the masses, the time already appears outside the periodic functions, but is still multiplied by these functions; if the fourth power is considered, then t actually appears without being multiplied by periodic functions. The result for the third power therefore gives oscillations about a mean value, but oscillations that become infinitely large for $t = \infty$; with the consideration of the fourth power, however, such oscillations are no longer present. One arrives at a similar result for small vibrations; with consideration of the higher powers of the displacements, one reaches the conclusion that a small impulse leads, with increasing t, to always larger oscillations.
>
> Strictly speaking, however, all these results prove absolutely nothing. For if one neglects the higher powers of the displacements, one assumes that the time is small, and cannot make any conclusion for large values of t. One should therefore not wonder if the time were also to appear outside the periodic functions for the first and second powers of the masses; for the justification of the expansion and the neglect of the higher powers of the masses lies only

> in the assumption that t does not exceed a certain bound. Thus one moves in a circle.
>
> An illustrative example is given by the pendulum. The position in which the bob is perpendicularly above the suspension point is a labile equilibrium of the pendulum. Here one obtains the time outside the sine and cosine, and thus concludes, with validity, that an infinitesimal impulse gives a finite motion; but it would be very false to conclude, from the circumstance that the time appears outside the periodic functions, that the motion of the pendulum is not periodic, for in this case the bob rotates periodically about its suspension point. It would be just as false to conclude, from the result that is given by the consideration of the higher powers of the masses in the solar system, that it is not stable.

154. (page 343) Routh's work on the stability of motion was the winner of the 1877 Adams Prize, an award endowed in 1848 by the members of St. John's College, Cambridge, in order to commemorate the discovery of the planet Neptune by John Couch Adams (1819–1892). The preface of Routh's work gives an excerpt from the 1877 prize notice [Routh 1877]:

> The University having accepted a Fund raised by several members of St. John's College for the purpose of founding a Prize to be called the *Adams Prize*, for the best essay on some subject of Pure Mathematics, Astronomy or other branch of Natural Philosophy, the Prize to be given once in two years, and to be open to the competition of all persons who have at any time been admitted to a degree in this University—
>
> The Examiners give notice that the following is the subject of the Prize to be adjudged in 1877: *The Criterion of Dynamical Stability.*
>
> To illustrate the meaning of the question imagine a particle to slide down inside a smooth inclined cylinder along the lowest generating line, or to slide down outside along the highest generating line. In the former case a slight derangement of the motion would merely cause the particle to oscillate about the generating line, while in the latter case the particle would depart from the generating line al-

together. The motion in the former case would be, in the sense of the question, stable, in the latter unstable.

The criterion of the stability of equilibrium of a system is, that its potential energy should be a minimum; what is desired is, a corresponding condition enabling us to decide when a dynamically possible motion of a system is such, that if slightly deranged the motion shall continue to be only slightly departed from.

The essays must be sent in to the Vice-Chancellor on or before the 16th December 1876, &c., &c.

S. G. PHEAR, *Vice-Chancellor.*
J. CHALLIS.
G. G. STOKES.
J. CLERK MAXWELL.

The Adams Prize is now awarded annually, and is open to any resident of the United Kingdom under the age of 40. The 2009 prize (£13,000) was awarded to Raphaël Rouquier for achievements in research on representation theory.

155. (page 347) In both the first (1896) and second (1904) editions of his *Traité de mécanique rationelle*, Appell gives the following definition of a stable motion. Let a particular motion of a system be described by the generalized coordinates $q_1(t)$, $q_2(t), \ldots$, $q_k(t)$, with corresponding initial conditions $q_1(0)$, $q_2(0), \ldots$, $q_k(0)$ and $\dot{q}_1(0)$, $\dot{q}_2(0), \ldots$, $\dot{q}_k(0)$. "One says," according to Appell, "that this motion is stable if, when given *arbitrary* initial conditions that are infinitely close to the preceding, the motion of the system is infinitely close to the particular motion considered" [Appell 1896, p. 371].

After his example of the circular orbit, which is treated only by the method of small oscillations, Appell writes that the scope of his (five-volume) work does not permit of further discussion of the stability question, and refers the reader to Routh.

156. (page 348) Some geodesic curves on a hyperboloid of one sheet are shown in Fig. 176. The curves were obtained by integrating the equations of motion of a particle that moves freely on the surface of the hyperboloid. Since the only force on such a particle is the constraint force normal to the surface, the osculating plane of the trajectory contains the normal to the surface at every point, and the trajectory is thus a geodesic curve [Struik 1988, p. 131].

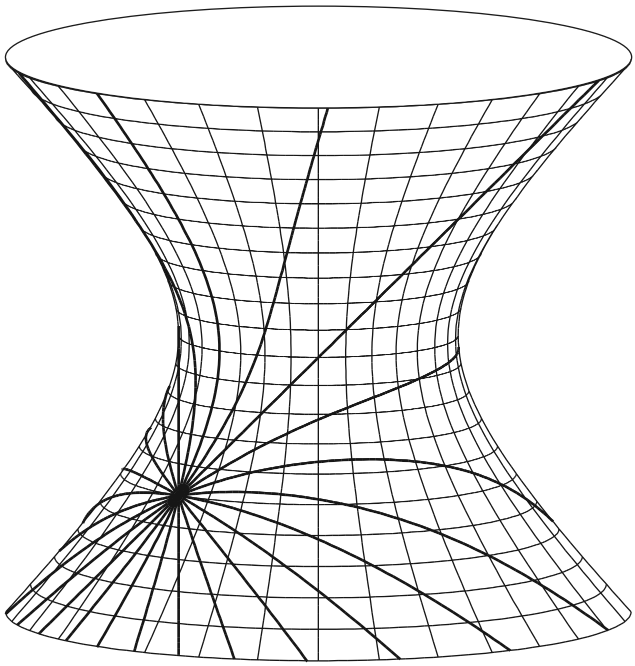


Fig. 176. Geodesic trajectories on a hyperboloid of one sheet.

To derive the differential equations for the particle trajectories on the hyperboloid, we may use the three-dimensional Cartesian coordinates x, y, z of the particle, and proceed as in Chap. VI, §9. The Lagrangian for the motion of the particle on the hyperboloid $x^2 + y^2 - z^2 = 1$ is

$$L = \frac{m}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - m\lambda(x^2 + y^2 - z^2 - 1),$$

where m is the mass of the particle and λ is a Lagrange multiplier. The Lagrange equations

$$\ddot{x} + 2\lambda x = 0, \quad \ddot{y} + 2\lambda y = 0, \quad \ddot{z} - 2\lambda z = 0$$

and the constraint $x^2 + y^2 - z^2 = 1$ imply the conservation laws

$$\dot{x}^2 + \dot{y}^2 + \dot{z}^2 = 2T = \text{const.}, \quad (x\dot{y} - \dot{x}y) = H_3 = \text{const.}$$

The coordinate z satisfies the differential equation

(5) $$\dot{z}^2 = \frac{2Tz^2 + (2T - H_3^2)}{(1 + 2z^2)},$$

and the Lagrange multiplier λ can be expressed as

$$\lambda = \frac{H_3^2 - T}{(1+2z^2)^2}.$$

The special case $H_3^2 = T$ ($\lambda = 0$) corresponds to the straight-line geodesics on the hyperboloid.

157. (page 348) The ellipsoidal geodesic described by Klein and Sommerfeld is illustrated in Fig. 177. It begins at point E on the equator, and its initial direction is nearly the direction of the meridian through E. The geodesic was obtained by integrating the equations of motion of a constrained particle on the ellipsoid, as described in note 156 for the hyperboloid.

Geodesics on the ellipsoid have been studied in great detail. The method of using constrained Cartesian coordinates to derive and solve the differential equations for the geodesics has been generalized by Askold Perelomov to an ellipsoid of dimension $n-1$ in an n-dimensional Euclidean space [Perelomov 2000]. Perelomov's paper contains references to early work on the case $n = 3$ by Jacobi and Weierstrass.

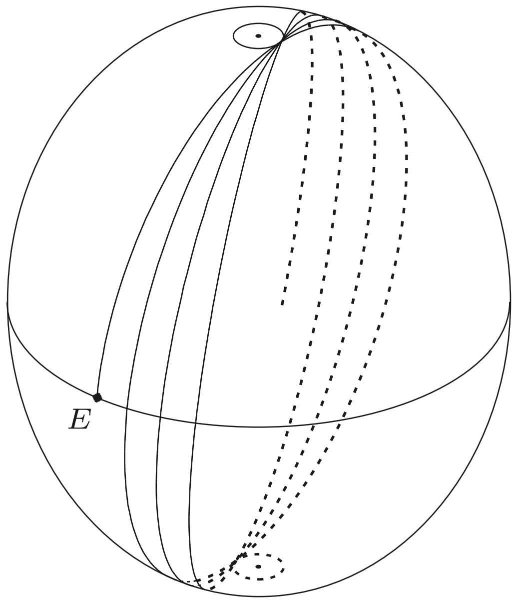


Fig. 177. Geodesic trajectory on an ellipsoid of revolution.

158. (page 351) If a particle moving on the throat-circle of the hyperboloid of one sheet is given a small vertical impact, it begins to spiral upward. If one diminishes the magnitude of the impact to zero, one obtains a trajectory that, in the reverse direction, approaches the throat-circle asymptotically, as shown in Fig. 178. This trajectory may be obtained by integrating the differential equations in note 156 for the special case $2T = H_3^2$ and $z(0) > 0$. In this case, the differential equation (5) for $z(t)$ can be integrated in terms of elementary functions. The geodesic trajectory in Fig. 178 is the analogue of Klein and Sommerfeld's top trajectory in Fig. 58 on page 338.

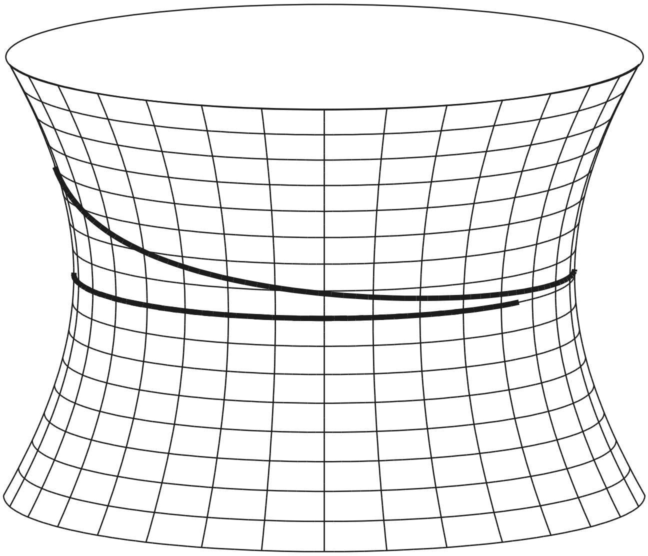


Fig. 178. Geodesic trajectory on a hyperboloid of one sheet that asymptotically approaches the throat-circle.

159. (page 353) See the supplementary notes by Klein and Sommerfeld on p. 535 and p. 537.

160. (page 354) The stability property defined by Klein and Sommerfeld is now referred to more narrowly as *continuous dependence on initial conditions*. It seems impossible to formulate a general definition of stability that can be reasonably applied to an arbitrary motion of an arbitrary mechanical system; Klein and Sommerfeld come to this conclusion at the end of their supplementary note on pp. 535–537.

161. (page 356) According to V. I. Arnold (b. 1937), "It seems likely that in an analytic system with n degrees of freedom, an equilibrium position which is not a minimum point is unstable, but this has never been proved for $n > 2$" [Arnold 1989, p. 100].

162. (page 372) The comments of Jacobi are included in the quotation of note 153.

163. (page 374) In the fourth chapter of the first volume of *Les méthodes nouvelles de la mécanique céleste*, Poincaré considers the system of differential equations [Poincaré 1892, p. 176]

$$\frac{dx_i}{dt} = X_i \quad (i = 1, 2, \ldots, n), \tag{6}$$

where the quantities X_i are given functions of the variables x_i, and are either independent of the time t or periodic in t. Poincaré supposes that the system (6) admits of a periodic solution

$$x_i = \varphi_i(t).$$

He then considers a *variation* of this solution in the form

$$x_i = \varphi_i(t) + \xi_i,$$

where ξ_i are assumed to be small quantities whose squares may be neglected. The differential equations for ξ_i are then

$$\frac{d\xi_i}{dt} = \frac{dX_i}{dx_1}\xi_1 + \frac{dX_i}{dx_2}\xi_2 + \cdots + \frac{dX_i}{dx_n}\xi_n.$$

The coefficients $\dfrac{dX_i}{dx_k}$ in these linear differential equations are, when one has replaced x_i by $\varphi_i(t)$, periodic in t. Poincaré states that the differential equations for ξ_i therefore have particular solutions of the form

$$\begin{cases} \xi_1 = e^{\alpha_1 t} S_{11}, & \xi_2 = e^{\alpha_1 t} S_{21}, \quad \xi_n = e^{\alpha_1 t} S_{n1}, \\ \xi_1 = e^{\alpha_2 t} S_{12}, & \xi_2 = e^{\alpha_2 t} S_{22}, \quad \xi_n = e^{\alpha_2 t} S_{n2}, \\ \ldots\ldots\ldots\ldots, & \ldots\ldots\ldots\ldots, \quad \ldots\ldots\ldots\ldots, \\ \xi_1 = e^{\alpha_n t} S_{1n}, & \xi_2 = e^{\alpha_n t} S_{2n}, \quad \xi_n = e^{\alpha_n t} S_{nn}, \end{cases}$$

where α_i are constants and S_{ik} are periodic functions of time. The constants α_i are called the *characteristic exponents* of the periodic solution. If the squares of α_i are all real and negative, then the quantities ξ_i remain finite for all values of t from $-\infty$ to ∞. In this case, Poincaré calls the original solution $x_i = \varphi_i(t)$ *stable*; otherwise, he calls this solution *unstable*.

164. (page 376) Sophie Kowalevski (1850–1891) writes the differential equations for the general heavy top in the form [Kowalevski 1889]

$$\tag{7}\begin{aligned} A\frac{dp}{dt} &= (B-C)qr + Mg(y_0\gamma'' - z_0\gamma'), & \frac{d\gamma}{dt} &= r\gamma' - q\gamma'', \\ B\frac{dq}{dt} &= (C-A)rp + Mg(z_0\gamma - x_0\gamma''), & \frac{d\gamma'}{dt} &= p\gamma'' - r\gamma, \\ C\frac{dr}{dt} &= (A-B)pq + Mg(x_0\gamma' - y_0\gamma), & \frac{d\gamma''}{dt} &= q\gamma - p\gamma', \end{aligned}$$

where p, q, r are the components of the angular velocity with respect to the principal inertial frame xyz at the fixed support point, A, B, C are the moments of inertia with respect to the xyz frame, M is the mass of the top, g is the acceleration of gravity, x_0, y_0, z_0 are the coordinates of the center of gravity of the top in the xyz frame, and γ, γ', γ'' are the cosines of the angles between the vertical and the x, y, and z axes, respectively. Kowalevski notes that in the two known cases for which these equations can be integrated (the torque-free case $x_0 = y_0 = z_0 = 0$ and the symmetric case $A = B$, $x_0 = y_0 = 0$), the quantities $p(t)$, $q(t)$, $r(t)$, $\gamma(t)$, $\gamma'(t)$, $\gamma''(t)$ are uniform functions that have no singularities other than poles for all finite values of (complex) time. To seek new solutions with this property, she assumes the series

$$\tag{8}\begin{aligned} p &= t^{-n_1}(p_0 + p_1 t + p_2 t^2 + \cdots), \\ q &= t^{-n_2}(q_0 + q_1 t + q_2 t^2 + \cdots), \\ r &= t^{-n_3}(r_0 + r_1 t + r_2 t^2 + \cdots), \\ \gamma &= t^{-m_1}(f_0 + f_1 t + f_2 t^2 + \cdots), \\ \gamma' &= t^{-m_2}(g_0 + g_1 t + g_2 t^2 + \cdots), \\ \gamma'' &= t^{-m_3}(h_0 + h_1 t + h_2 t^2 + \cdots), \end{aligned}$$

where n_1, n_2, n_3, m_1, m_2, m_3 are integers. She finds that a new solution of this form with a sufficient number of arbitrary constants is possible if

$$n_1 = n_2 = n_3 = 1, \quad m_1 = m_2 = m_3 = 2,$$

and

$$A = B = 2C, \quad z_0 = 0.$$

For this new case, she chooses the xyz coordinate system so that $y_0 = 0$, and derives the first integral

$$\{(p+qi)^2 + c_0(\gamma + i\gamma')\}\{(p-qi)^2 + c_0(\gamma - i\gamma')\} = k^2,$$

where k is a constant, $c_0 = Mgx_0$, and the unit of length has been chosen so that $C = 1$. This first integral, together with the known integrals corresponding to the conservation of energy and the conservation of angular momentum about the vertical, allow her to solve the original system of differential equations by quadratures.

In a second paper [Kowalevski 1890], Kowalevski shows that the series solution (8) with integer values of n_i and m_i can represent the general solution of equations (7) only in one of the following four cases:

(i) $$A = B = C,$$

(ii) $$x_0 = y_0 = z_0 = 0,$$

(iii) $$A = B, \quad x_0 = y_0 = 0,$$

(iv) $$A = B = 2C, \quad z_0 = 0.$$

It is not obvious that the series solutions (8) with integer values of n_i and m_i will necessarily lead to a new algebraic first integral of (7) and a solution by quadratures. This coincidence "still seems to be rather mysterious" [Audin 1996, p. 21].

165. (page 376) See the supplementary note of Klein and Sommerfeld on p. 537.

166. (page 377) In the cited paper [Joukowksy 1894–95], Nikolai Egorovich Joukowsky (1847–1921) gives a rather involved geometric construction for the motion of the Kowalevski top. No description or discussion of a physical model is given .

167. (page 377) Roger Liouville (1856–??) showed that the differential equations of the heavy asymmetric top have a third algebraic integral (the first two being the conservation of energy and the conservation of angular momentum about the vertical) only when the ellipsoid of inertia with respect to the fixed point O is an ellipsoid of revolution, the center of gravity is in the equatorial plane of the ellipsoid of inertia, and the principal moments of inertia of the top are A, $B = A$, and $C = mA/2$, where m is an arbitrary integer [Liouville 1897]. In an interesting paper on particular motions of the asymmetric

top, the American mathematician John J. Corliss (1896–??) adds that the physical constraint $C \leq A + B$ implies that m cannot exceed 4 [Corliss 1932]. In the case $m = 4$, a third algebraic integral exists only when the (constant) angular momentum about the vertical axis vanishes [Whittaker 1904, p. 163]. Some further references to the problem of finding algebraic integrals for the motion of the asymmetric top are given in the supplementary note by Klein and Sommerfeld on p. 538.

168. (page 378) Hess writes the six differential equations of the heavy asymmetric top in the form given at the beginning of note 164 [Hess 1890]. He then introduces the three new variables ν, ϱ, and μ, where ν is the square of the magnitude of the angular momentum, ϱ is the projection of the angular momentum onto the figure axis, and μ is the cosine of the angle between figure axis and the vertical. (Stäckel later used the related variables T, U, and S, where T is the kinetic energy of the top, U is half the square of the magnitude of the angular momentum, and S is the dot product of the angular momentum vector and the vector from the support point to the center of gravity [Stäckel 1908; Stäckel 1909].) Hess shows that the original six differential equations may be reduced to three. The new system of three differential equations has singular solutions if (1) the magnitude of the angular momentum is constant during the entire motion, or (2) the projection of the angular momentum onto the figure axis is constant during the entire motion. For a general mass distribution, these two singular solutions do not lead to solutions of the original six differential equations. Hess shows that the singular solution (2) does lead to a solution of the original six differential equations in a special case; this is the Hess case that is described geometrically by Klein and Sommerfeld.

169. (page 378) In the list of corrections that were published in Vol. IV of the *Theorie des Kreisels* in 1910, Klein and Sommerfeld state that the degree of particularization is one higher in the Staude case than in the other cases mentioned. In the Staude case, four requirements are imposed: a ray of the Staude cone defined on p. 388 must be initially vertical (one condition for the initial orientation of the body), and the angular velocity vector must be initially vertical with its magnitude determined according to pp. 388 and 389 (three conditions for the initial angular velocity of the body).

170. (page 378) Sommerfeld's paper is available in his *Gesammelte Schriften* [Sommerfeld 1968, Vol. I, pp. 417–420]. At the end of this

paper, Sommerfeld states that his discussion of the Hess case can be extended with small modifications to the top whose support point is free to move on a horizontal plane.

171. (page 379) See the supplementary note by Klein and Sommerfeld on p. 539.

172. (page 380) The geometric configuration discussed by Klein and Sommerfeld is illustrated in Fig. 179. The plane e' is tangent to the reciprocal ellipsoid of inertia at the point P, which lies on the circular planar cut e through the ellipsoid. The point Q lies in the plane e', and the line OQ is perpendicular to e'. The impulse vector is in the direction OP, and the angular velocity vector is in the direction OQ.

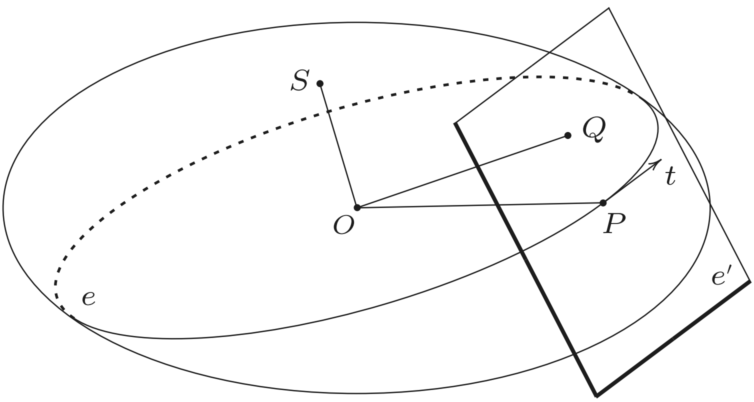


Fig. 179. Geometric configuration of the ellipsoid of inertia for the Hess case of the motion of the asymmetric top.

173. (page 386) Joukowsky's model of the Hess pendulum is shown in Fig. 180. The pendulum is represented by the disk $ABA'B'$ that is fixed to the axis OC. The disk is supported by a sharp point O on the underside of the disk. Four posts are attached to the disk at the points A, B, A', B'. The disk diameters AA' and BB' are perpendicular, and the points A, B, A', B' are equally distant from the center O of the disk. The posts carry the weights P, P', Q, Q'. By bringing the weights P', Q, Q' to the lower surface of the disk and displacing the weight P downward by an appropriate distance, the lower surface of the disk can be made to represent the circular intersection plane (plane

e in Fig. 179) of the ellipsoid of inertia of the top with respect to the point O. With this arrangement of the weights, the disk will move as a Hess top if the disk is set into motion by an instantaneous force whose moment about O lies in the plane of the disk. For the special cases in which this instantaneous force is either zero or horizontally directed, the motion will proceed in such a manner that the edge of the disk always passes through a fixed point g, conveniently indicated by the bracket F. These special initial conditions, Joukowsky says, can be used to find the correct location of the weight P.

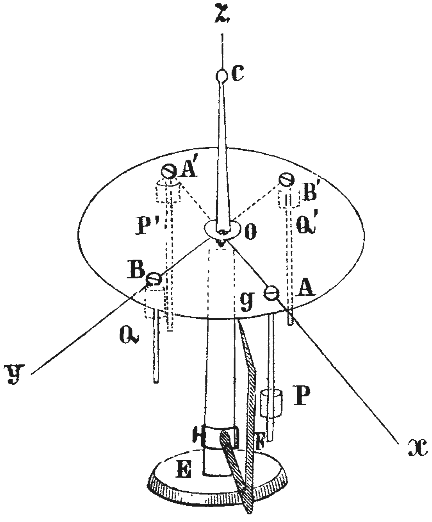


Fig. 180. Joukowsky's model of the Hess top [Joukowsky 1892–93, p. 63].

174. (page 387) Otto Staude (1857–1928) studied mathematics under Felix Klein at the University of Leipzig. He taught at Dorpat (now Tartu, Estonia) and Rostock, and was rector of the University of Rostock in 1901–02 and 1918–19. Staude begins by writing the differential equations of the top in the form of note 164, and seeks solutions in which γ, γ', γ'' are constant [Staude 1894].

175. (page 393) Heinrich Friedrich Karl Ludwig Burkhardt (1861–1914) wrote his textbook on complex variables [Burkhardt 1897] while he was professor of mathematics in the University of Zurich. In 1905, Burkhardt was one of the reviewers of Albert Einstein's doctoral the-

sis on the dimensions of molecules. The fourth (1912) edition of Burkhardt's text was translated into English by Samuel Eugene Rasor (1873–1950), professor of mathematics in the Ohio State University.

176. (page 403) Jacobi's discussion of the elliptic integral of the third kind is a posthumously published fragment of an uncompleted work on the rotation of a symmetric rigid body about a fixed point [Jacobi 1882, pp. 477–492].

177. (page 409) Sebastian Finsterwalder (1862–1951) was professor of mathematics in the *Technische Hochschule* in Munich. He worked in many fields of application, including geodesy and photogrammetry. Finsterwalder's three-bored bushing for the mechanical realization of a plane conformal map is shown in Fig. 181, and the conformal map between the Riemann surface $(u, \sqrt{U})$ and the t-plane is illustrated in Fig. 182. Special bushings are required at the branch points $u = e$, $u = e'$, and $u = e''$, where the angles in the t-plane are half the corresponding angles in the Riemann surface.

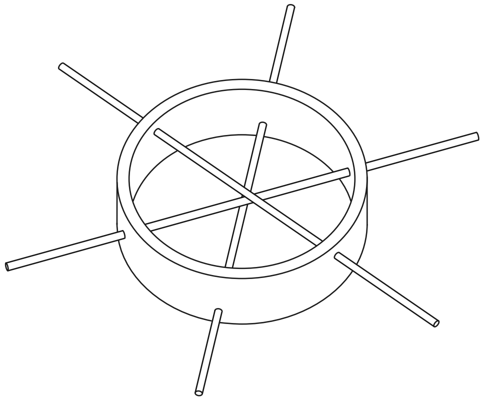


Fig. 181. Finsterwalder's three-bored bushing for the mechanical realization of a conformal map.

Finsterwalder's remarkable paper [Finsterwalder 1897] also describes mechanical models for surfaces defined by many other types of two- and three-dimensional mappings.

One may reflect on the fact the entire theory of elliptic functions behind Fig. 182 is motivated by a mechanical problem in which the variable u is restricted to the segment of the real axis between 0 and 1.

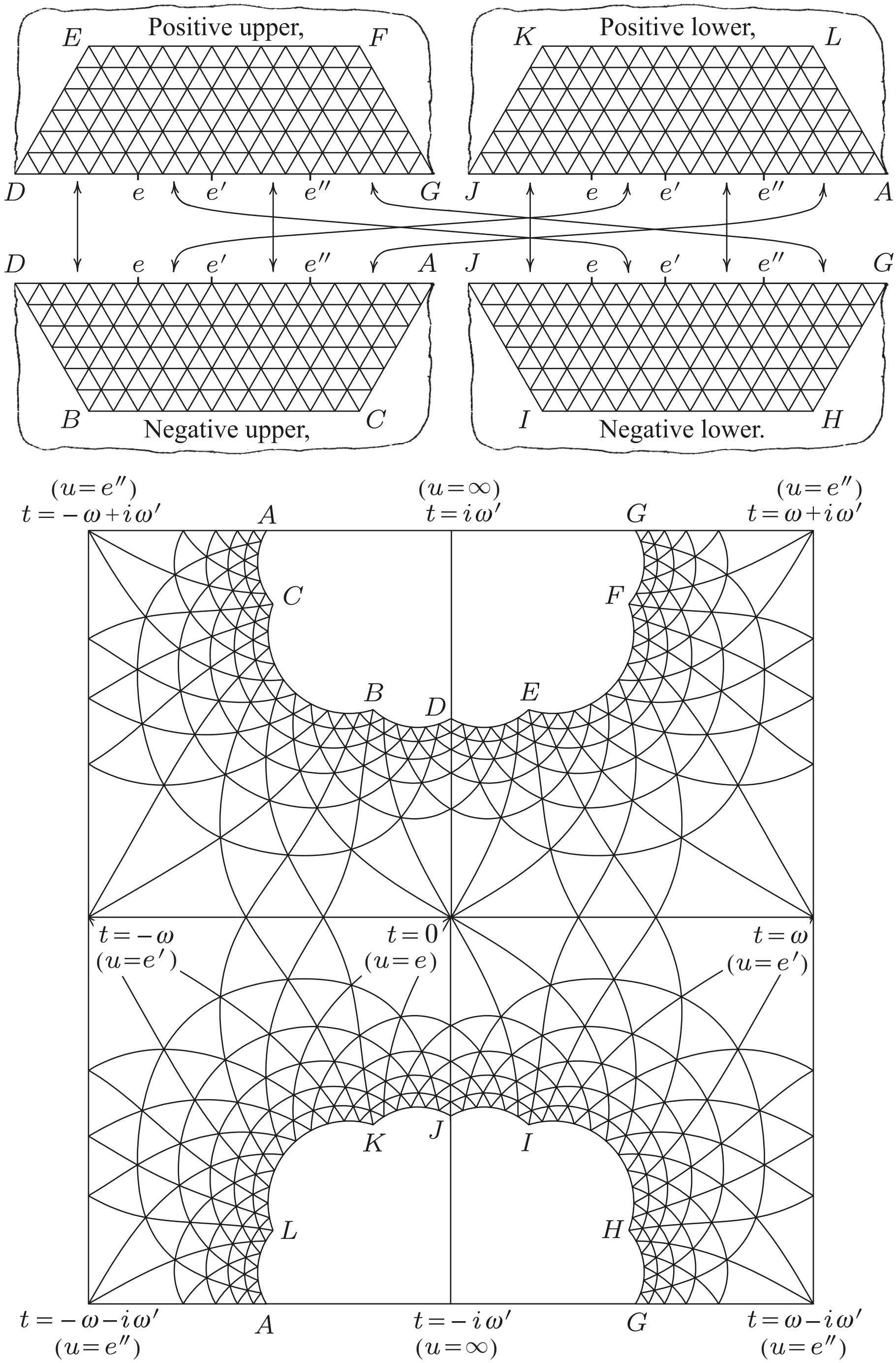


Fig. 182. Illustration of the conformal map between the Riemann surface $(u, \sqrt{U})$ and a period rectangle in the t-plane.

178. (page 418) If equation (1) on p. 418 is taken as the definition of the ϑ-function, then the right-hand side of equation (3) on the same page should be multiplied by -1. All expressions involving ϑ-quotients are unaffected by this change.

179. (page 418) We have corrected the argument of the H-function according to the list of corrections that appeared in Vol. IV in 1910. A further discussion of the theta-functions used by Jacobi is given in the supplementary note by Klein and Sommerfeld on pp. 540–541.

180. (page 421) The theorems of Joseph Liouville (1809–1882) are proved in the *Theory of functions of a complex variable* by Andrew R. Forsyth [Forsyth 1893, §115 and ff.].

181. (page 429) Jacobi died of smallpox in 1851 at the age of 47. Karl Leopold Eduard Lottner (1826–1887) was professor and prorector in the *Realschule* in Lippstadt, and was an assistant to Weierstrass in the editing and publication of the complete works of Jacobi. Lottner was the editor of the fragmentary and unpublished manuscript *Sur la rotation d'un corps* that Jacobi wrote during the last two years of his life [Jacobi 1882, pp. 425–512, 525]. Lottner found that the results in his own 1855 paper on the representation of the motion of the heavy top by elliptic functions [Lottner 1855] had been anticipated by Jacobi, and that Jacobi had further used these results to state the theorem discussed by Klein and Sommerfeld in Chap. VI, §8.

In the list of corrections that were published in Vol. IV of the *Theorie des Kreisels* in 1910, Klein and Sommerfeld state that Jacobi's representation of the motion of the heavy top by elliptic functions was anticipated in 1834 by Adolf Stephanus Rueb (1806–1854) in his doctoral thesis cited in the footnote of page 473 [Rueb 1834].

182. (page 454) Klein and Sommerfeld seem to have convinced no one of the utility and elementary character of their ϑ-quotients for the representation of the motion of the top. In 1917, for example, Edwin Bidwell Wilson, professor of mathematics in the Massachusetts Institute of Technology (alma mater floreat!), wrote a review of Greenhill's *Report on Gyroscopic Theory* (cf. note 109 above). Wilson makes a comparison of Greenhill with Klein and Sommerfeld [Wilson 1917]:

> We are indeed fortunate that the Committee got Greenhill to prepare the report. His long-continued investigations on the top have made him a world-recognized authority on the subject. The fact that we have available for study

> the great work of Klein and Sommerfeld on the same subject does not in the slightest detract from the value of the present work. The merest glance will convince any reader that the two treatises are conceived in very different ways and that they rather supplement than overlap one another. Besides Greenhill's is decidely shorter and more concerned with apparatus. It will be of more interest to the student of mechanics and engineering, though of less to the student of the theory of functions of a complex variable. And here we might remark that a beautiful theory and a practical analysis susceptible to ready calculation are unfortunately not always to be combined. The theta function is a thing of beauty, but I have always found the solution of the problem of the motion of the top by theta functions with complex arguments anything but satisfactory from the point of view of calculation. Perhaps after all Legendre knew what he was about when he introduced his third elliptic integral,—at any rate we judge that Greenhill thinks so.

Could Wilson have been completely insensible to the magnificent numerical example of Klein and Sommerfeld? But perhaps Wilson's case is explained by the last sentence of his review: "We have now in English a great reference book on the top, a worthy companion and counterpart to that of Klein and Sommerfeld in German, one decidedly more compact and cheaper in price."

183. (page 464) According to equation (3) on page 418, the right-hand sides of the following equations for $\vartheta(t)$ and $\Theta(t)$ should be multiplied by 2. All Θ-quotients, and the expressions for α, β, γ, δ on page 464, are unaffected by this change.

184. (page 472) Corrections to equations (40) on p. 472 are given in the supplementary note of Klein and Sommerfeld on p. 540.

185. (page 486) See the supplementary note by Klein and Sommerfeld on p. 541.

186. (page 490) See the supplementary note by Klein and Sommerfeld on p. 541.

187. (page 505) See the supplementary note by Klein and Sommerfeld on p. 541.

188. (page 509) Tait's quaternion equation for the motion of a general rigid body with a fixed point is [Tait 1898, p. 110]

$$\psi = 4V.\, \dot{q}\phi(q^{-1}\dot{q})q^{-1} + 2q\phi(V.\, q^{-1}\ddot{q})q^{-1}. \tag{9}$$

In this equation, $q(t)$ is the quaternion that corresponds to the rotation of the body from its initial position to its position at time t, and $\psi(t)$ is the net (vector) moment on the body with respect to the fixed point. The symbol V denotes an operator that extracts the vector part of the quaternion that follows. The lower point after V is Tait's peculiar form of parenthesization; equation (9) might also be written as

$$\psi = 4V(\dot{q}\phi(q^{-1}\dot{q})q^{-1}) + 2q\phi(V(q^{-1}\ddot{q}))q^{-1}.$$

The symbol ϕ denotes an operator whose action on any vector ϱ is defined by

$$\phi(\varrho) = \Sigma\,.\, m(\alpha S\alpha\varrho - \alpha^2\varrho),$$

where α represents the position vector of the mass element m of the body in the initial position, Σ denotes the sum over all the mass elements of the body, and S is an operator that extracts the scalar part of the quaternion that follows.

Tait says that though his quaternion equation is "remarkably simple," it must, "in the present state of the development of quaternions, be looked upon as intractable, except in certain very particular cases."

189. (page 511) Oskar Bolza (1857–1942) studied mathematics under Felix Klein in Göttingen, where he received the doctoral degree in 1886. He spent several years in the United States, teaching at Johns Hopkins University, Clark University, and the University of Chicago. A memoriam to Bolza was published in the *Bulletin of the American Mathematical Society* in 1944 [Bliss 1944]. J. Hänlein may be Jakob Hänlein, who was a candidate for the doctoral degree in the University of Berlin in 1881.

The reaction of Sommerfeld to the discovery of the Weierstrass lecture is described by Sommerfeld's biographer Michael Eckert as follows [Eckert 2000, p. 88]:

> When the printing of the second volume of the *Theorie des Kreisels* was almost completed, Klein received the notes of a lecture by Weierstrass on the "Applications of elliptic functions," and sent them to Sommerfeld for examination. "What is to be done? I am in favor of writing an appendix at the end of volume 2: 'Weierstrass knew all this,' " answered Sommerfeld. A few days later he sent the

> lecture back to Klein, together with an appendix in which the merits of the Weierstrass theory were acknowledged.

Some additional references to the history of the Cayley–Klein parameters α, β, γ, δ are given in the supplementary note of Klein and Sommerfeld on p. 541.

190. (page 512) The quotation from the *Nachrichten der Königliche Gesellschaft der Wissenschaften zu Göttingen, Geschäftliche Mittheilungen* is taken from a report by Klein that was also published in the *Mathematische Annalen* [Klein 1898]. Gauss's notes on the mutations of space are printed in Vol. 8 of his *Werke* [Gauss 1900].

191. (page 513) The *Mathematisches Wörterbuch* of Ludwig Hoffmann (described on the title page as "former building-master in Berlin") and Leopold Natani (1819–1905) was published in seven volumes between 1858 and 1867. The entry "Rotation of a body about a fixed point" in Vol. VI begins with the parallelogram law for angular velocity vectors and ends with the solution for the motion of a force-free body in terms of elliptic functions [Hoffmann 1867, pp. 364–381].

192. (page 513) This masterful appendix originally appeared at the beginning of Vol. III of the *Theorie des Kriesels* in 1903.

193. (page 517) In Chap. VI of Vol. II of the second edition of the *Traité de Mécanique*, Poisson derives the differential equations for the motion of a rigid body that remains in contact at one point with a given plane [Poisson 1833]. The contact point may vary both in the plane and in the body, and the plane may have an arbitrarily prescribed motion. The special case treated by Sommerfeld is considered only very briefly by Poisson, who merely writes the differential equations for the angles ϑ and ψ with the comment that they may be solved by means of elliptic functions.

194. (page 518) Klein's Princeton lectures on the theory of the top were published in 1897, and reprinted in 2004 [Klein 1897/2004]. The lectures were part of the sesquicentennial celebration of Princeton University, and were given on October 12–15, 1896, at 11:00 a.m. At 9:00 a.m. on the same days, Joseph John Thomson (1856–1940), Cavendish Professor of Physics in the University of Cambridge, spoke on "The Discharge of Electricity in Gases" [Thomson 1903]. The celebration also included a torchlight procession, speeches by President Grover Cleveland and then Princeton University Professor Woodrow Wilson, and a football game in which Princeton defeated the University of Virginia,

48–0. The football game was attended by many of the European guests; it is not known whether Klein was present. It is recorded, however, that the toast to Mathematics at the Farewell Dinner on October 22 was answered by Professor Felix Klein of the University of Göttingen [Princeton University 1898, p. 174].

195. (page 519) Weierstrass considers the real periodic functions $x(t)$ that satisfy the differential equation [Weierstrass 1866]

$$\left(\frac{dx}{dt}\right)^2 = F(x),$$

where

(1) $F(x)$ vanishes for two real values a and b of x;

(2) the quotient $\dfrac{(x-a)(x-b)}{F(x)}$ does not change in sign and does not become infinite in the interval from a to b;

(3) for any specified value of t, the corresponding value of x is contained in this interval.

"Geometric and mechanical problems," writes Weierstrass, "not seldom lead to such a differential equation."

196. (page 533) These addenda and supplements by Klein and Sommerfeld were added when Vol. IV appeared in 1910.

197. (page 533) Like many other reviewers, Koppe uses his review of the first two volumes of the *Theorie des Kreisels* as an opportunity to cite and promote his own work, especially his concept of the "induced force" supposedly created by the motion of the top. He even makes the astounding statement that "I am happy to have received, through the present book, the stimulus to develop this [induced] force once again in a shorter form from a somewhat different point of view" [Koppe 1898, p. 300].

Klein and Sommerfeld always refer to Koppe respectfully, but Sommerfeld's opinion of Koppe is more honestly expressed in an 1898 letter to Klein [Eckert 2000, pp. 95–96]:

> I have just written, according to your wishes, a few polite and insignificant words to Ball, and many significant but still polite words to Koppe. I find what Koppe says to be noteworthy only in point of arrogance, and otherwise, with a few exceptions, superficial or erroneous. Today I found his works, as I again looked through them for the composition of the letter, to be much weaker than

> previously. The "induced force" that Koppe praises to the heavens is useful only for pseudoregular precession, and is self-evident from the standpoint of the impulse theory. I have naturally restrained myself greatly in the letter to him, so as not to become involved in a tiresome literary feud. I attach the letter and ask you to take note of it, if possible, and then send it.
>
> I would like to keep the works of Koppe here for the time being; I attach the letter (I apologize for the marginal notes, which are only for orientation). The letters of Koppe's opponents (Schmidt, etc.) will be much more impolite; at least they have a reason for it. The correct answer to K.[oppe] would be from you according to the famous example: "Furthermore, I have tolerated the tone of your letter for the last time."

The concluding quotation may be a reference to a telegram from Kaiser Wilhelm II to Graf-Regenten Ernst zu Lippe-Biesterfeld [Röhl 2001, p. 938].

198. (page 535) Diederik Johannes Korteweg (1848–1941) was professor of mathematics in the University of Amsterdam. In addition to his original research in applied mathematics, he worked for sixteen years on the editing of the complete works of his countryman Christiaan Huygens (1629–1695).

199. (page 537) Fritz Wilhelm Ferdinand Kötter (1857–1912) was professor of mechanics in the *Bergakademie* in Berlin and the *Technische Hochschule* in Charlottenburg. He was an unsuccessful competitor of Sommerfeld for a professorial chair in Aachen. In his twenty-six page booklet *Remarks on F. Klein's and A. Sommerfeld's book on the theory of the top* [Kötter 1899], Kötter criticizes Klein and Sommerfeld's treatment of the Kowalevski top, and takes personal offense to their comments about quadratures and the scholastic habits of mathematicians:

> As already said, the book is concerned almost exclusively with the theory of the symmetric top. The motion of the asymmetric top is treated only incidentally. Except for the rather complete treatment of the force-free top, only the Hess case and the von Staude case of permanent rotation about a vertical axis of the body are discussed in detail. In contrast, the integrable case, through whose discovery Mrs.

> von Kowalevksi has enriched science in such a high degree, is touched only in passing.
>
> In a comparatively large space, the occasion is taken to pronounce a judgment on the efforts of some mathematicians, which I cannot regard as correct. If it were merely a question of the opinions of Mr. Sommerfeld, one could pass over the matter in silence, leaving the final judgment confidently to time and one's colleagues. But since a third party may not separate what is to be ascribed to one or the other of the two authors, the well-deserved great prestige of Mr. Privy Councillor Klein makes it necessary for me to stand up against these deprecatory statements of opinion, and to reduce their content to its true value.

Kötter goes on at some length about the importance of particular integrable cases in mechanics, and about his own work on the Kowalevski top. He also objects to Klein and Sommerfeld's treatment of the Jacobi theorem in Chap. VI, §8, in which he finds a kinetic as well as a kinematic significance. Kötter ends his remarks with great pathos:

> I am at the conclusion of my discussion. Its purpose is unmistakable to all: the affirmation of the true scientific standpoint against the oppressive weight of authority. It is human nature to wish that goals which are finally acquired by long and wearisome labor will also be regarded by others as valuable. And it is unpleasant to see the results of one's own activity rejected offhand by those whose judgment one is inclined to value. Thus, however, it is only with difficulty that one acquires an impartial opinion of one's own affairs; a harsh judgment will be felt as unjustified, when it is in fact justified. If we are to arrive at an impartial estimation of the value of a judgment of our own affairs, there is often no choice but to examine the manner of forming a judgment in affairs to which we personally stand distant. It was on this basis that I was obliged to go so far into the estimation of efforts to which I have no other connection than the admiration with which I regard them.

It is instructive to note Klein and Sommerfeld's public response to Koppe and Kötter. Their minor contributions are generously acknowledged, and their petty criticisms are completely ignored.

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